5. zadaća, 4. zadatak Neka je $D : \mathcal{P}_n \to \mathcal{P}_n$ linearan operator t.d. je $D(p) = p'' - p$. Odredite jezgru operatora D^2.

Rj: $p(t) = a_0 + a_1 t + \dots + a_k t^k$

$p'(t) = a_1 + 2a_2 t + \dots + k a_k t^{k-1}$

$p''(t) = 2a_2 + \dots + k(k-1) a_k t^{k-2}$

$$D(p) = p'' - p = 2a_2 + \dots + k(k-1) a_k t^{k-2}$$
$$- a_0 - a_1 t - \dots - a_k t^k$$

Ako je $\text{st}\, p = k$; $p \neq 0$, onda je $a_k \neq 0$ i

$\text{st}\, D(p) = k \Rightarrow \text{st}\, D(D(p)) = k$

- Dakle, ako je $\text{st}\, p = k > 0 \Rightarrow \text{st}\, D^2(p) = k > 0 \Rightarrow$

$p \notin \operatorname{Ker} D^2$

- Ako je $\text{st}\, p = 0 \Rightarrow p(t) = a_0 \Rightarrow D(p) = 0 - a_0 = -a_0$

$\Rightarrow D^2(p) = a_0 \quad p \in \operatorname{Ker} D^2 \Leftrightarrow a_0 = 0$

Dakle, $\operatorname{Ker} D^2 = \{0\}$

2. način: $D(p) = p'' - p$

$$D^2(p) = D(p''-p) = (p''-p)'' - (p''-p)$$

$$= p'''' - 2p'' + p$$

$$D^2(1) = 1$$

$$D^2(t) = t$$

$$D^2(t^2) = -4 + t^2$$

$$\vdots$$

$$D^2(t^k) = k(k-1)(k-2)(k-3)t^{k-4} - 2k(k-1)t^{k-2} + t^k$$

$$\vdots$$

$$D^2(t^n) = n(n-1)(n-2)(n-3)t^{n-4} - 2n(n-1)t^{n-2} + t^n$$

$D^2(1), D^2(t), \dots, D^2(t^n)$ imaju svi različite stupnjeve

$\Rightarrow \{D^2(1), D^2(t), \dots, D^2(t^n)\}$ je lin. nez. skup, ali i

s. i. za $\quad \operatorname{Im} D^2 \quad \Rightarrow r(D^2) = n+1$

$\Rightarrow d(D^2) = \dim \mathcal{P}_n - (n+1) = 0$

$\Rightarrow \operatorname{Ker} D^2 = \{0\}$

5. zadaća, 9. zadatak Odredite linearan operator $R : V^3(O) \to V^3(O)$ takav da je $R(\vec{k}) = -\vec{i}$ i R je zrcaljenje s obzirom na neku ravninu.

Rj: Neka je R zrcaljenje s obzirom na ravninu Π s vektorom normale $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$. Tada je

$$R(\vec{v}) = \vec{v} - 2\langle \vec{v} | \vec{n} \rangle \vec{n} \quad \text{i} \quad |\vec{n}| = 1$$

$$\Rightarrow \quad R(\vec{k}) = \vec{k} - 2\left(\vec{k} \cdot (a\vec{i} + b\vec{j} + c\vec{k})\right) \cdot (a\vec{i} + b\vec{j} + c\vec{k})$$

$$= \vec{k} - 2c\,(a\vec{i} + b\vec{j} + c\vec{k})$$

$$= -2ac\,\vec{i} - 2bc\,\vec{j} + (1 - 2c^2)\vec{k}$$

Kako je $R(\vec{k}) = -\vec{i} \Rightarrow -2ac\vec{i} - 2bc\vec{j} + (1 - 2c^2)\vec{k} = -\vec{i}$

$$\Rightarrow -2ac = -1, \quad -2bc = 0, \quad 1 - 2c^2 = 0$$

$$\Rightarrow \underset{(1)}{2ac = 1}, \quad \underset{(2)}{2bc = 0}, \quad \underset{(3)}{1 - 2c^2 = 0}$$

Iz (2) $\Rightarrow bc = 0 \Rightarrow b = 0$ ili $c = 0$.

Ako je $c = 0$, onda zbog (3) slijedi $1 - 2 \cdot 0^2 = 0 \Rightarrow \Leftarrow$

Dakle, $c \neq 0$, a onda je $\boxed{b = 0}$.

$$\left.\begin{array}{l} \text{Iz (3)} \Rightarrow c = \pm \frac{1}{\sqrt{2}} \\ \text{Iz (1)} \Rightarrow a = \frac{1}{2c} = \pm \frac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow \vec{n} = \pm \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad R(\vec{v}) = \vec{v} - 2\left(\vec{v}\cdot\left(\frac{1}{\sqrt{2}}\vec{\imath} + \frac{1}{\sqrt{2}}\vec{u}\right)\right)\cdot\left(\frac{1}{\sqrt{2}}\vec{\imath} + \frac{1}{\sqrt{2}}\vec{u}\right)$$

$$= \vec{v} - \left(\vec{v}\cdot(\vec{\imath}+\vec{u})\right)\cdot(\vec{\imath}+\vec{u})$$

6. zadaća, 1. zadatak Postoji li monomorfizam / epimorfizam / izomorfizam

a) $A: \mathbb{R}^2 \to M_2(\mathbb{R})$

- $\dim \mathbb{R}^2 = 2$, $\dim M_2(\mathbb{R}) = 4$

  $2 \neq 4 \Rightarrow \mathbb{R}^2$ i $M_2(\mathbb{R})$ nisu izomorfni pa ne postoji izomorfizam $A: \mathbb{R}^2 \to M_2(\mathbb{R})$

- $d(A) + r(A) = \dim \mathbb{R}^2 = 2 \Rightarrow r(A) \leq 2 \Rightarrow r(A) \neq \dim M_2(\mathbb{R})$

  $\Rightarrow \operatorname{Im} A \neq M_2(\mathbb{R}) \Rightarrow A: \mathbb{R}^2 \to M_2(\mathbb{R})$ ne može biti epimorfizam

- Konstruirajmo monomorfizam $A: \mathbb{R}^2 \to M_2(\mathbb{R})$. Npr.

$$A(1,0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(0,1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Proširimo po linearnosti

$$A(x,y) = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \quad x, y \in \mathbb{R}$$

$$A(x,y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff x = y = 0 \iff (x,y) = (0,0)$$

$\Rightarrow d(A) = 0 \Rightarrow A$ je monomorfizam

b) $B: \mathcal{P}_3(\mathbb{C}) \to \mathbb{C}^3$

- $\dim \mathcal{P}_3(\mathbb{C}) = 4, \quad \dim \mathbb{C}^3 = 3$

  $3 \neq 4 \Rightarrow \mathcal{P}_3(\mathbb{C})$ i $\mathbb{C}^3$ nisu izomorfni

  $\Rightarrow$ Ne postoji izomorfizam $B: \mathcal{P}_3(\mathbb{C}) \to \mathbb{C}^3$

- $d(B) + r(B) = \dim \mathcal{P}_3(\mathbb{C}) = 4 \qquad \operatorname{Im} B \leq \mathbb{C}^3 \Rightarrow r(B) \leq 3$

  $\Rightarrow d(B) = 4 - r(B) \geq 4 - 3 = 1 \Rightarrow d(B) \neq 0$

  $B: \mathcal{P}_3(\mathbb{C}) \to \mathbb{C}^3$ ne može biti monomorfizam.

- Konstruirajmo epimorfizam $B: \mathcal{P}_3(\mathbb{C}) \to \mathbb{C}^3$. Npr.

$$B(1) = (1,0,0)$$
$$B(t) = (0,1,0)$$
$$B(t^2) = (0,0,1)$$
$$B(t^3) = \text{bilo što, npr. } (0,0,0)$$

Proširimo po linearnosti $B(a_0 + a_1 t + a_2 t^2 + a_3 t^3) =$
$= (a_0, a_1, a_2), \quad a_0, a_1, a_2, a_3 \in \mathbb{C}$

$\operatorname{Im} B = \mathbb{C}^3 \Rightarrow B$ je epimorfizam

c) $C: V^3(O) \to \mathcal{P}_2(\mathbb{R}) \qquad \dim V^3(O) = 3 = \dim \mathcal{P}_2(\mathbb{R})$

Konstruiramo izomorfizam $C: V^3(O) \to \mathcal{P}_2(\mathbb{R})$

$$C(\vec{i}) = 1, \quad C(\vec{j}) = t, \quad C(\vec{k}) = t^2$$

Proširimo po linearnosti $C(a\vec{i} + b\vec{j} + c\vec{k}) = a + bt + ct^2$
C je izomorfizam jer baza za $V^3(O)$ preslikana u bazu za $\mathcal{P}_2(\mathbb{R})$
C je ujedno i monomorfizam i epimorfizam.

6 zadaća, 3. zadatak: Odredite djelovanje ortogonalne projekcije $P: \mathbb{R}^3 \to \mathbb{R}^3$ na proizvoljnom vektoru $x \in \mathbb{R}^3$ ako je $\operatorname{Ker} P = [\{(1,0,1), (-3,1,0)\}]$

Rj: P = ortogonalna projekcija na potprostor M

$P(x) = x_M$, gdje je $x = x_M + x_{M^\perp}$, $x_M \in M$, $x_{M^\perp} \in M^\perp$

$P(x) = 0 \iff x_M = 0 \iff x = x_{M^\perp} \iff x \in M^\perp$

$\Rightarrow \operatorname{Ker} P = M^\perp \Rightarrow M = (M^\perp)^\perp = (\operatorname{Ker} P)^\perp$

$(x_1, x_2, x_3) \in (\operatorname{Ker} P)^\perp \iff \begin{matrix} (x_1, x_2, x_3) \cdot (1,0,1) = 0 \\ (x_1, x_2, x_3) \cdot (-3,1,0) = 0 \end{matrix}$

$\iff x_1 + x_3 = 0 \;\&\; -3x_1 + x_2 = 0$

$\iff x = (x_1, 3x_1, -x_1)$

$M = (\operatorname{Ker} P)^\perp = [\{(1,3,-1)\}]$

$$e_1 = \frac{(1,3,-1)}{\sqrt{1^2+3^2+1^2}} = \frac{1}{\sqrt{11}}(1,3,-1)$$

$$x_M = \langle x | e_1 \rangle e_1 = \frac{1}{11}(x_1 + 3x_2 - x_3)(1,3,-1)$$

$$\boxed{P(x) = \frac{1}{11}(x_1 + 3x_2 - x_3)(1,3,-1)}$$

6. zadaća, 4. zadatak $x = x_M + x_{M^\perp}$, $x_M \in M$, $x_{M^\perp} \in M^\perp$

$$P(x) = x_M$$

a) $(P \circ P)(x) = P(P(x)) = P(x_M) = P(\underbrace{x_M}_{\in M} + \underbrace{0}_{\in M^\perp}) = x_M = P(x) \quad \forall x \in V$

$\Rightarrow P \circ P = P$

b) $\|x\| = \|x_M + x_{M^\perp}\| = \sqrt{\langle x_M + x_{M^\perp} \mid x_M + x_{M^\perp} \rangle} = \sqrt{\|x_M\|^2 + \underbrace{\|x_{M^\perp}\|^2}_{\geq 0}} \geq \sqrt{\|x_M\|^2}$

$= \|x_M\|$

$\Rightarrow \|x\| \geq \|x_M\| = \|Px\|$

Jednakost vrijedi $\Longleftrightarrow \|x_{M^\perp}\|^2 = 0 \Leftrightarrow x_{M^\perp} = 0 \Leftrightarrow \boxed{x \in M}$

6. zadaća, 5. zadatak a) $A: \mathbb{R}^3 \to \mathbb{R}^3$ $A(x_1, x_2, x_3) = (2x_1 + x_2, -2x_1 - x_2, x_3)$

$$(e) = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$$

$$[A]_{(e)} = \begin{array}{c} \begin{array}{ccc} A(e_1) & A(e_2) & A(e_3) \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$[A^2]_{(e)} = [A \circ A]_{(e)} = [A]_{(e)} \cdot [A]_{(e)} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [A]_{(e)}$$

$\Rightarrow A^2 = A$

$$A(x_1,x_2,x_3)=(0,0,0) \iff (2x_1+x_2,-2x_1-x_2,x_3)=(0,0,0)$$

$$\iff 2x_1+x_2=0,\ -2x_1-x_2=0,\ x_3=0$$

$$\iff x_2=-2x_1,\ x_3=0$$

$$\mathrm{Ker}\,A=\{(x_1,-2x_1,0): x_1\in\mathbb{R}\}=[\{(1,-2,0)\}]$$

$$A(x_1,x_2,x_3)=(2x_1+x_2,-2x_1-x_2,x_3)=$$

$$=x_1(2,-2,0)+x_2\cdot(1,-1,0)+x_3\cdot(0,0,1)$$

$$=[\{(2,-2,0),(1,-1,0),(0,0,1)\}]=$$

$$=[\{(1,-1,0),(0,0,1)\}]$$

Kad bi A bila ortogonalna projekcija na svoju sliku $M=\mathrm{Im}\,A$, onda bi vrijedilo $M^\perp=\mathrm{Ker}\,A$.

($x=\underbrace{x_M}_{\in M}+\underbrace{x_{M^\perp}}_{\in M^\perp}$ $\quad P(x)=x_M \quad x\in\mathrm{Ker}\,P \iff x_M=0 \iff x\in M^\perp$

$$\left.\begin{array}{l}\mathrm{Ker}\,P=M^\perp\\ \mathrm{Im}\,P=M\end{array}\right\}\Rightarrow \mathrm{Ker}\,P\oplus\mathrm{Im}\,P=V$$)

Uočimo da je $(1,-1,0)\in\mathrm{Im}\,A$, $(1,-2,0)\in\mathrm{Ker}\,A$, a $\langle(1,-1,0)|(1,-2,0)\rangle=3\neq 0 \Rightarrow (1,-1,0)\not\perp(1,-2,0)$

$\Rightarrow$ A nije ortogonalna projekcija na svoju sliku.