way to cool the atmosphere as required by the thickness tendency is by adiabatic cooling through the vertical motion field. Thus, the vertical motion maintains a hydrostatic temperature field (i.e., a field in which temperature and thickness are proportional) in the presence of differential vorticity advection. Without this compensating vertical motion, either the vorticity changes at 500 hPa could not remain geostrophic or the temperature changes in the 500- to 1000-hPa layer could not remain hydrostatic.

To summarize, we have shown as a result of scaling arguments that for synoptic-scale motions where vorticity is constrained to be geostrophic and temperature is constrained to be hydrostatic, the vertical motion field is determined uniquely by the geopotential field. Further, we have shown that this vertical motion field is just that required to ensure that changes in vorticity will be geostrophic and changes in temperature will be hydrostatic. These constraints, whose importance can hardly be overemphasized, are elaborated in the next subsection.

6.4.2 The Q Vector

In order to better appreciate the essential role of the divergent ageostrophic motion in quasi-geostrophic flow, it is useful to examine separately the rates of change, following the geostrophic wind, of the vertical shear of the geostrophic wind and of the horizontal temperature gradient.

On the midlatitude β-plane the quasi-geostrophic prediction equations may be expressed simply as

$$\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0 \tag{6.38}$$

$$\frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0 \tag{6.39}$$

$$\frac{D_g T}{Dt} - \frac{\sigma p}{R}\omega = \frac{J}{c_p} \tag{6.40}$$

These are coupled by the thermal wind relationship

$$f_0 \frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y}, \qquad f_0 \frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x} \tag{6.41a,b}$$

or in vector form:

$$\left(f_0 \mathbf{k} \times \frac{\partial \mathbf{V}_g}{\partial p}\right) = \frac{R}{p}\boldsymbol{\nabla} T \tag{6.42}$$

Equations for the evolution of the thermal wind components are obtained by taking partial derivatives with respect to p in (6.38) and (6.39), multiplying through by f_0, and applying the chain rule of differentiation in the advective part of the total derivative to obtain

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] + f_0^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \quad (6.43a)$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = -f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] - f_0^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \quad (6.43b)$$

However, by the thermal wind relations (6.41a) and (6.41b), the first terms on the right-hand side in each of these may be expressed, respectively, as

$$-f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] = -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right]$$

$$-f_0\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\right] = -\frac{R}{p}\left[\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right]$$

Using the fact that the divergence of the geostrophic wind vanishes,

$$\partial u_g/\partial x + \partial v_g/\partial y = 0 \quad (6.44)$$

The above terms can be expressed, respectively, as

$$Q_2 \equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right] = -\frac{R}{p}\frac{\partial \mathbf{V}_g}{\partial y}\bullet\nabla T \quad (6.45a)$$

$$Q_1 \equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] = -\frac{R}{p}\frac{\partial \mathbf{V}_g}{\partial x}\bullet\nabla T \quad (6.45b)$$

If we now take partial derivatives of (6.40) with respect to x and y, multiply the results by Rp^{-1}, and again apply the chain rule of differentiation to the advection terms, we obtain

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial x}\right) = -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right) + \sigma\frac{\partial \omega}{\partial x} + \frac{\kappa}{p}\frac{\partial J}{\partial x} \quad (6.46a)$$

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} \quad (6.46b)$$

Using the definitions of (6.45a,b), we can rewrite (6.43a,b) and (6.46a,b) as

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -Q_2 + f_0^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p} \quad (6.47)$$

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 + \sigma\frac{\partial \omega}{\partial y} + \frac{\kappa}{p}\frac{\partial J}{\partial y} \quad (6.48)$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = Q_1 - f_0^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} \tag{6.49}$$

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial x}\right) = Q_1 + \sigma\frac{\partial \omega}{\partial x} + \frac{\kappa}{p}\frac{\partial J}{\partial x} \tag{6.50}$$

Suppose that $Q_2 > 0$ and that the thermal wind is westerly ($\partial u_g/\partial p < 0$ and $\partial T/\partial y < 0$). Then from (6.47), Q_2 forces an increase in the westerly shear following the geostrophic motion ($\partial u_g/\partial p$ becomes more negative). However, from (6.48), $Q_2 > 0$ forces a positive change in the meridional temperature gradient following the geostrophic motion ($\partial T/\partial y$ becomes less negative). Q_2 thus tends to destroy the thermal wind balance between the vertical shear of the zonal wind and the meridional temperature gradient. Similarly, Q_1 destroys the thermal wind balance between vertical shear of the meridional wind and the zonal temperature gradient. An ageostrophic circulation is thus required to keep the flow in approximate thermal wind balance.

Subtracting (6.47) from (6.48) and using (6.41a) to eliminate the total derivative gives

$$\sigma\frac{\partial \omega}{\partial y} - f_0^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{\partial v_g}{\partial p} = -2Q_2 - \frac{\kappa}{p}\frac{\partial J}{\partial y} \tag{6.51}$$

Similarly, adding (6.50) to (6.49) and using (6.41b) to eliminate the total derivative gives

$$\sigma\frac{\partial \omega}{\partial x} - f_0^2\frac{\partial u_a}{\partial p} - f_0\beta y\frac{\partial u_g}{\partial p} = -2Q_1 - \frac{\kappa}{p}\frac{\partial J}{\partial x} \tag{6.52}$$

If we now take $\partial(6.52)/\partial x + \partial(6.49)/\partial y$ and use (6.12) to eliminate the ageostrophic wind, we obtain the **Q** vector form of the omega equation:

$$\sigma\,\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} = -2\nabla\bullet\mathbf{Q} + f_0\beta\frac{\partial v_g}{\partial p} - \frac{\kappa}{p}\nabla^2 J \tag{6.53}$$

where

$$\mathbf{Q} \equiv (Q_1, Q_2) = \left(-\frac{R}{p}\frac{\partial \mathbf{V}_g}{\partial x}\bullet\nabla T, -\frac{R}{p}\frac{\partial \mathbf{V}_g}{\partial y}\bullet\nabla T\right) \tag{6.54}$$

Equation (6.54) shows that vertical motion is forced by the sum of the divergence of **Q**, the Laplacian of the diabatic heating, and a term related to the β effect that is generally small for synoptic-scale motions. Unlike the traditional form of the omega equation, the **Q** vector form does not have forcing terms that partly cancel. The forcing of ω for adiabatic flow can be represented simply by the pattern of the **Q** vector. By the arguments of the last subsection, the left-hand side in (6.54) is proportional to the vertical velocity. Hence, a convergent **Q** forces ascent, and a divergent **Q** forces descent.