2.2 Metoda supstitucije i metoda parcijalne integracije

Zadatak 2.11 Izračunajte integrale koristeći metodu supstitucije:

$$\text{(a)} \int x^2(2x^3+4)^4\,dx \qquad \text{(b)} \int_0^{\ln 2} \frac{e^x}{\sqrt{e^x+1}}\,dx.$$

Rješenje.

(a) $\displaystyle\int x^2(2x^3+4)^4\,dx = \begin{bmatrix} t = 2x^3+4 \\ dt = 6x^2\,dx \end{bmatrix} = \int \frac{t^4}{6}\,dt = \frac{t^5}{30} + C = \frac{(2x^3+4)^5}{30} + C$

(b) $\displaystyle\int_0^{\ln 2} \frac{e^x}{\sqrt{e^x+1}}\,dx = \begin{bmatrix} t = e^x+1 & 0 \mapsto e^0+1 = 2 \\ dt = e^x\,dx & \ln 2 \mapsto e^{\ln 2}+1 = 3 \end{bmatrix} = \int_2^3 \frac{dt}{\sqrt{t}} = 2\sqrt{t}\Big|_2^3 = 2(\sqrt{3} - \sqrt{2})$

△

Zadatak 2.12 Izračunajte integrale:

$$\text{(a)} \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \qquad \text{(b)} \int_2^3 x\sqrt{x^2-4}\,dx \qquad \text{(c)} \int \frac{x}{\sqrt{x+1}}\,dx \qquad \text{(d)} \int_0^{\frac{\pi}{2}} \cos^4 x \sin^3 x\,dx$$

$$\text{(e)} \int \sqrt{e^x-1}\,dx \qquad \text{(f)} \int_0^{\frac{\pi}{2}} \sin 2x\sqrt{1+\sin^2 x}\,dx \qquad \text{(g)}^\star \int_{-\pi}^{\pi} \frac{x}{2+\cos x}\,dx.$$

Rješenje.

(a) (a) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \begin{bmatrix} t = \sqrt{x} \\ dt = \frac{dx}{2\sqrt{x}} \end{bmatrix} = 2\int \cos t\,dt = 2\sin t + C = 2\sin\sqrt{x} + C$

(b) $\displaystyle\int_2^3 x\sqrt{x^2-4}\,dx = \begin{bmatrix} t = x^2-4 & 2 \mapsto 0 \\ dt = 2x\,dx & 3 \mapsto 5 \end{bmatrix} = \frac{1}{2}\int_0^5 \sqrt{t}\,dt = \frac{1}{3}t^{3/2}\Big|_0^5 = \frac{\sqrt{125}}{3}$

(c) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx = \begin{bmatrix} t = \sqrt{x+1} & x = t^2-1 \\ dt = \frac{dx}{2\sqrt{x+1}} & \end{bmatrix} = 2\int (t^2-1)\,dt = \frac{2}{3}t^3 - 2t + C =$
$\displaystyle= \frac{2}{3}(x-2)\sqrt{x+1} + C$

(d) $\int\limits_0^{\frac{\pi}{2}} \cos^4 x \sin^3 x \, dx = \left[\begin{array}{ll} t = cosx & 0 \mapsto 1 \\ dt = -\sin x \, dx & \frac{\pi}{2} \mapsto 0 \end{array} \right] = \int\limits_0^1 t^4(1-t^2) \, dt = \left(\frac{t^5}{5} - \frac{t^7}{7} \right) \Big|_0^1 =$
$= \frac{2}{35}$

(e) $\int \sqrt{e^x - 1} \, dx = \left[\begin{array}{ll} t = \sqrt{e^x - 1} & x = \ln(t^2+1) \\ & dx = \frac{2t \, dt}{t^2+1} \end{array} \right] = 2 \int \frac{t^2}{t^2+1} \, dt =$
$= 2 \int dt - 2 \int \frac{dt}{t^2+1} = 2t - 2 \operatorname{arctg} t + C = 2\sqrt{e^x - 1} - 2 \operatorname{arctg} \sqrt{e^x - 1} + C$

(f) $\int\limits_0^{\frac{\pi}{2}} \sin 2x \sqrt{1 + \sin^2 x} \, dx = \left[\begin{array}{ll} t = 1 + \sin^2 x & 0 \mapsto 1 \\ dt = 2 \sin x \cos x \, dx & \frac{\pi}{2} \mapsto 2 \end{array} \right] = \int\limits_1^2 \sqrt{t} \, dt = \frac{2}{3} t^{3/2} \Big|_1^2 =$
$= \frac{2}{3}(2\sqrt{2} - 1)$

(g) $\int\limits_{-\pi}^{\pi} \frac{x}{2 + \cos x} \, dx =$ neparna funkcija na simetričnoj domeni $= 0$

△

Zadatak 2.13 Neka je $f \colon [-a, a] \to \mathbb{R}$ Riemann-integrabilna funkcija.

(a) Ako je f neparna, dokažite da je

$$\int\limits_{-a}^{a} f(x) \, dx = 0.$$

(b) Ako je f parna, dokažite da je

$$\int\limits_{-a}^{a} f(x) \, dx = 2 \int\limits_{0}^{a} f(x) \, dx.$$

Rješenje.

(a) Vrijedi $\int\limits_{-a}^{a} f(x) \, dx = \left[\begin{array}{ll} t = -x & -a \mapsto a \\ dt = -dx & a \mapsto -a \end{array} \right] = \int\limits_{-a}^{a} f(-t) \, dt = -\int\limits_{-a}^{a} f(t) \, dt$, odakle je $2 \int\limits_{-a}^{a} f(x) \, dx = 0$ pa slijedi tvrdnja.

(b) $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \begin{bmatrix} t=-x & -a\mapsto a \\ dt=-dx & 0\mapsto 0 \end{bmatrix} =$$
$$= \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

△

Zadatak 2.14 Neka je $f\colon \mathbb{R}\to\mathbb{R}$ neprekidna periodična funkcija s periodom $\tau > 0$. Dokažite da za sve $a\in\mathbb{R}$ vrijedi

$$\int_{a}^{a+\tau} f(x)\,dx = \int_{0}^{\tau} f(x)\,dx.$$

Rješenje. Neka je $m\in\mathbb{Z}$ takav da je $a < m\tau \le a+\tau$. Zapravo, $m = \left\lfloor \frac{a}{\tau} \right\rfloor$, jer je

$$m\tau = \left\lfloor \frac{a}{\tau} \right\rfloor \tau + \tau \le \frac{a}{\tau}\tau + \tau = a+\tau$$
$$m\tau = \left\lfloor \frac{a}{\tau} \right\rfloor \tau + \tau > (\frac{a}{\tau} - 1)\tau + \tau = a$$

pa je $a < m\tau \le a+\tau$. Sada računamo

$$\int_{a}^{a+\tau} f(x)\,dx = \int_{a}^{m\tau} f(x)\,dx + \int_{m\tau}^{a+\tau} f(x)\,dx = \begin{bmatrix} t=x-\tau & m\tau\mapsto (m-1)\tau \\ dt=dx & a+\tau\mapsto a \end{bmatrix} =$$
$$= \int_{a}^{m\tau} f(x)\,dx + \int_{(m-1)\tau}^{a} f(t+\tau)\,dt = \int_{(m-1)\tau}^{m\tau} f(t)\,dt =$$
$$= \begin{bmatrix} s=t-(m-1)\tau & (m-1)\tau\mapsto 0 \\ ds=dt & m\tau\mapsto\tau \end{bmatrix} = \int_{0}^{\tau} f(s+(m-1)\tau)\,ds = \int_{0}^{\tau} f(x)\,dx.$$

△

Zadatak 2.15 ⋆ Izračunajte integral:

$$\int_{1}^{1+10\pi} \max\{|\sin x|, |\cos x|\}\,dx.$$

Rješenje. Funkcija $x \mapsto \max\{|\sin x|, |\cos x|\}$ je periodična s periodom $\pi/2$ pa je

$$\int_1^{1+10\pi} \max\{|\sin x|, |\cos x|\}\, dx = \int_0^{10\pi} \max\{|\sin x|, |\cos x|\}\, dx =$$
$$= 20\int_0^{\frac{\pi}{2}} \max\{|\sin x|, |\cos x|\}\, dx = 20\left(\int_0^{\frac{\pi}{4}} \cos x\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\, dx\right) =$$
$$= 20(\sin x\Big|_0^{\frac{\pi}{4}} - \cos x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}) = 20(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) = 20\sqrt{2}$$

△

Vrijede formule parcijalne integracije:

$$\int u(x)\, v'(x)\, dx = u(x)v(x) - \int u'(x)v(x)\, dx$$

i

$$\int_a^b u(x)\, v'(x)\, dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\, dx.$$

Ponekad se gornje formule zapisuju kao

$$\int u\, dv = uv - \int v\, du$$

i

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du.$$

Zadatak 2.16 Izračunajte integrale koristeći metodu parcijalne integracije:

$$\text{(a)} \int x\, e^x\, dx \qquad \text{(b)} \int_1^2 \frac{\ln x}{x^2}\, dx.$$

Rješenje.

$$\text{(a)} \int x\, e^x\, dx = \begin{bmatrix} u = x & du = dx \\ dv = e^x\, dx & v = e^x \end{bmatrix} = xe^x - \int e^x\, dx = e^x(x-1) + C$$

$$\text{(b)} \int_1^2 \frac{\ln x}{x^2}\, dx = \begin{bmatrix} u = \ln x & du = \frac{dx}{x} \\ dv = \frac{dx}{x^2} & v = -\frac{1}{x} \end{bmatrix} = -\frac{\ln x}{x}\Big|_1^2 + \int_1^2 \frac{dx}{x^2} = \frac{\ln 2}{2} - \frac{1}{x}\Big|_1^2 = \frac{1+\ln 2}{2}$$

△

Zadatak 2.17 Izračunajte integrale:

$$\text{(a)} \int_0^1 \ln(1+x^2)\,dx \qquad \text{(b)} \int_0^{\frac{\pi}{2}} e^x \sin x\,dx \qquad \text{(c)} \int \ln x\,dx \qquad \text{(d)} \int \operatorname{arctg} x\,dx.$$

Rješenje.

(a)
$$\int_0^1 \ln(1+x^2)\,dx = \begin{bmatrix} u = \ln(1+x^2) & du = \frac{2xdx}{1+x^2} \\ dv = dx & v = x \end{bmatrix} = x\ln(1+x^2)\Big|_0^1 - \int_0^1 \frac{2x^2dx}{1+x^2} =$$
$$= \ln 2 - 2\int_0^1 dx + 2\int_0^1 \frac{dx}{1+x^2} = \ln 2 - 2 + 2\operatorname{arctg} x\Big|_0^1 = \ln 2 - 2 + \frac{\pi}{2}$$

(b)
$$\int_0^{\frac{\pi}{2}} e^x \sin x\,dx = \begin{bmatrix} u = \sin x & du = \cos x\,dx \\ dv = e^x dx & v = e^x \end{bmatrix} = e^x \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\,dx =$$
$$= \begin{bmatrix} u = \cos x & du = -\sin x\,dx \\ dv = e^x dx & v = e^x \end{bmatrix} = e^{\pi/2} - e^x \cos x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x\,dx =$$
$$= e^{\pi/2} + 1 - \int_0^{\frac{\pi}{2}} e^x \sin x\,dx \implies \int_0^{\frac{\pi}{2}} e^x \sin x\,dx = \frac{e^{\pi/2}+1}{2}$$

(c)
$$\int \ln x\,dx = \begin{bmatrix} u = \ln x & du = \frac{dx}{x} \\ dv = dx\,dx & v = x \end{bmatrix} = x\ln x - \int x\frac{dx}{x} = x\ln x - x + C$$

(d)
$$\int \operatorname{arctg} x\,dx = \begin{bmatrix} u = \operatorname{arctg} x & du = \frac{dx}{x^2+1} \\ dv = dx\,dx & v = x \end{bmatrix} = x\operatorname{arctg} x - \int \frac{xdx}{x^2+1} =$$
$$= x\operatorname{arctg} x - \frac{1}{2}\int \frac{d(x^2)}{x^2+1} = x\operatorname{arctg} x - \frac{1}{2}\ln(x^2+1) + C$$

△

Zadaci za vježbu

2.18 Izračunajte integrale:

$$\text{(a)} \int (3x^2 - 2x + 1)(x^3 - x^2 + x - 9)^7 \, dx \qquad \text{(b)} \int \frac{e^x + 1}{e^x + x} \, dx$$

$$\text{(c)} \int_e^{e^e} x^3 e^{x^4} \, dx \qquad \text{(d)} \int_0^4 x\sqrt{x^2 + 9}$$

2.19 Izračunajte integrale:

$$\text{(a)} \int \frac{\ln x}{x^3} \, dx \qquad \text{(b)} \int \frac{x}{\cos^2 x} \, dx \qquad \text{(c)} \int e^x \sin^2 x \, dx \qquad \text{(d)} \int \frac{x e^{\operatorname{arctg} x}}{(1 + x^2)^{3/2}} \, dx$$

2.20 Izračunajte integrale:

$$\text{(a)} \int_0^2 x^2 \sqrt{4 - x^2} \, dx \qquad \text{(b)} \int_0^1 \frac{x^3}{x^6 + 2x^3 + 1} \, dx \qquad \text{(c)} \int_1^e \frac{\sin \ln x}{x} \, dx$$

$$\text{(d)} \int \frac{dx}{\sin x} \qquad \text{(d)} \int \frac{dx}{\cos x}$$

2.21 Izračunajte integrale:

$$\text{(a)} \int \sin 3x \sin 5x \, dx \qquad \text{(b)} \int \left(\frac{\ln x}{x} \right)^2 dx \qquad \text{(c)} \int \sin \ln x \, dx \qquad \text{(d)} \int (x^4 + 3x) \cos x \, dx$$

2.22 Izračunajte integrale:

$$\text{(a)} \int \arcsin^2 x \, dx \qquad \text{(b)} \int x^2 \sqrt{1 + x^2} \, dx \qquad \text{(c)} \int \operatorname{arctg} \sqrt{x} \, dx \qquad \text{(d)} \int x^2 \arccos x \, dx$$

2.23 Izračunajte integrale:

$$\text{(a)} \int \frac{\arcsin x}{x^2} \frac{1 + x^2}{\sqrt{1 - x^2}} \, dx \qquad \text{(b)} \int_1^{16} \operatorname{arctg} \sqrt{\sqrt{x} - 1} \, dx \qquad \text{(c)} \int e^x \sin x \sin 3x \, dx$$

2.24 Izračunajte integrale:

$$\text{(a)} \int \sqrt{\frac{x + \sqrt{1 + x^2}}{1 + x^2}} \, dx \qquad \text{(b)} \int \frac{dx}{x \ln x \ln \ln x} \qquad \text{(c)} \int \frac{\operatorname{arctg} \sqrt{x}}{\sqrt{x}} \frac{1}{1 + x} \, dx$$

$$\text{(d)} \int \frac{1}{1 - x^2} \ln \frac{1 + x}{1 - x} \, dx \qquad \text{(e)} \int \frac{dx}{e^{x/2} + e^x} \qquad \text{(f)} \int e^{x + \ln x} \, dx$$

2.25 Neka je $f\colon \mathbb{R} \to \mathbb{R}$ neprekidna funkcija. Dokažite sljedeće jednakosti:

(a) $\displaystyle\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$

(b) $\displaystyle\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

(c) $\displaystyle\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

2.26 Izračunajte

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$$

2.27 Izračunajte

$$\lim_{t\to+\infty} \frac{\int_1^t \left(\operatorname{arctg}\frac{x+1}{x-1} - \frac{\pi}{4}\right)\,dx}{\ln t}.$$

2.28 Dokažite

$$\lim_{t\to+\infty} \frac{\int_1^{x^2} \frac{e^t}{t}\,dt}{x^{-2}e^{x^2}} = 1.$$

2.29 Odredite sve neprekidne funkcije $f\colon [a,b] \to [0,+\infty\rangle$ takve da je $\displaystyle\int_a^b f(x)\,dx = 0$.