KOMPLEKSNA ANALIZA
Zadaci za vježbu br. 4
2020./2021.

1. Izračunajte:

(a) $i^{\sin i}$

(b) $(4-3i)^{1+i}$

(c) $\operatorname{Ln}(3-2i)$

(d) $\cos(2+i)$.

Rješenje. (a) Po definiciji je

$$i^{\sin i} = e^{\sin i \cdot \operatorname{Ln} i} = e^{i \operatorname{sh} 1 \cdot \left(\frac{\pi}{2}+2k\pi\right) i} = e^{-\operatorname{sh} 1 \cdot \left(\frac{\pi}{2}+2k\pi\right)}, \qquad k \in \mathbb{Z},$$

pri čemu smo u drugoj jednakosti iskoristili sljedeće:

- $\sin i = \sin(i \cdot 1) = i \operatorname{sh} 1$, s obzirom da je
$$\sin(iz) = i \operatorname{sh} z, \qquad z \in \mathbb{C}$$
(alternativno, $\sin i$ se može izračunati direktno po definiciji funkcije sin).
- $\operatorname{Ln} i = \ln|i| + i(\arg i + 2k\pi) = \ln 1 + i\left(\frac{\pi}{2}+2k\pi\right) = \left(\frac{\pi}{2}+2k\pi\right) i, \qquad k \in \mathbb{Z}.$

(Posebno, glavna vrijednost od $i^{\sin i}$ je $e^{-\frac{\pi}{2}\operatorname{sh} 1}$.)

(b) Imamo

$$(4-3i)^{1+i} = e^{(1+i)\operatorname{Ln}(4-3i)}. \tag{1}$$

Kako je

$$\operatorname{Ln}(4-3i) = \ln|4-3i| + i(\arg(4-3i)+2k\pi) = \ln 5 + i\left(\operatorname{arctg}\left(-\frac{3}{4}\right)+2k\pi\right), \quad k \in \mathbb{Z},$$

imamo

$$\begin{aligned}(1+i)\operatorname{Ln}(4-3i) &= (1+i)\left(\ln 5 + i\left(\operatorname{arctg}\left(-\frac{3}{4}\right)+2k\pi\right)\right)\\ &= \ln 5 - \operatorname{arctg}\left(-\frac{3}{4}\right) - 2k\pi + i\left(\ln 5 + \operatorname{arctg}\left(-\frac{3}{4}\right)+2k\pi\right), \quad k \in \mathbb{Z},\end{aligned}$$

dakle

$$\begin{aligned}(4-3i)^{1+i} &\overset{(1)}{=} e^{\ln 5 - \operatorname{arctg}\left(-\frac{3}{4}\right) - 2k\pi + i\left(\ln 5 + \operatorname{arctg}\left(-\frac{3}{4}\right)+2k\pi\right)}\\ &= e^{\ln 5 - \operatorname{arctg}\left(-\frac{3}{4}\right) - 2k\pi}\left(\cos\left(\ln 5 + \operatorname{arctg}\left(-\frac{3}{4}\right)+2k\pi\right) + i\sin\left(\ln 5 + \operatorname{arctg}\left(-\frac{3}{4}\right)+2k\pi\right)\right)\\ &= 5e^{-\operatorname{arctg}\left(-\frac{3}{4}\right)-2k\pi}\left(\cos\left(\ln 5 + \operatorname{arctg}\left(-\frac{3}{4}\right)\right) + i\sin\left(\ln 5 + \operatorname{arctg}\left(-\frac{3}{4}\right)\right)\right), \qquad k \in \mathbb{Z}.\end{aligned}$$

Posebno, glavna vrijednost od $(4-3i)^{1+i}$ je

$$5e^{-\operatorname{arctg}\left(-\frac{3}{4}\right)}\left(\cos\left(\ln 5+\operatorname{arctg}\left(-\tfrac{3}{4}\right)\right)+i\sin\left(\ln 5+\operatorname{arctg}\left(-\tfrac{3}{4}\right)\right)\right).$$

(c) Imamo

$$\begin{aligned}\operatorname{Ln}(3-2i)&=\ln|3-2i|+i(\arg(3-2i)+2k\pi)\\&=\ln\sqrt{3^2+(-2)^2}+i\left(\operatorname{arctg}\left(-\frac{2}{3}\right)+2k\pi\right)\\&=\frac{1}{2}\ln 13+i\left(\operatorname{arctg}\left(-\frac{2}{3}\right)+2k\pi\right),\qquad k\in\mathbb{Z}.\end{aligned}$$

(d) Imamo

$$\begin{aligned}\cos(2+i)&=\cos 2\cdot\cos i-\sin 2\cdot\sin i\\&=\cos 2\cdot\cos(i\cdot 1)-\sin 2\cdot\sin(i\cdot 1)\\&=\cos 2\cdot\operatorname{ch} 1-\sin 2\cdot i\operatorname{sh} 1\\&=\cos 2\cdot\operatorname{ch} 1-i\sin 2\cdot\operatorname{sh} 1,\end{aligned}$$

pri čemu smo u prvoj jednakosti primijenili adicijsku formulu

$$\cos(z_1+z_2)=\cos z_1\cdot\cos z_2-\sin z_1\cdot\sin z_2,\qquad z_1,z_2\in\mathbb{C},$$

a u trećoj formule

$$\cos(iz)=\operatorname{ch} z\qquad\text{i}\qquad\sin(iz)=i\operatorname{sh} z,$$

koje vrijede za sve $z\in\mathbb{C}$.

2. način: po definiciji funkcije cos imamo

$$\begin{aligned}\cos(2+i)&=\frac{e^{i(2+i)}+e^{-i(2+i)}}{2}\\&=\frac{e^{-1+2i}+e^{1-2i}}{2}\\&=\frac{1}{2}\left(e^{-1}(\cos 2+i\sin 2)+e\left(\cos(-2)+i\sin(-2)\right)\right)\\&=\frac{1}{2}\left(e^{-1}(\cos 2+i\sin 2)+e\left(\cos 2-i\sin 2\right)\right)\\&=\frac{e+e^{-1}}{2}\cos 2-i\frac{e-e^{-1}}{2}\sin 2\\&=\operatorname{ch} 1\cdot\cos 2-i\operatorname{sh} 1\cdot\sin 2.\end{aligned}$$

2. Riješite jednadžbe:

(a) $\operatorname{ch} z - i = 0$

(b) $4\cos z + 5 = 0$

(c) $\ln(z + i) = 0$

(d) $\cos z = \operatorname{ch} z$.

Rješenje. (a) Imamo

$$
\begin{aligned}
\operatorname{ch} z - i = 0 \quad & \Leftrightarrow \quad \frac{e^z + e^{-z}}{2} - i = 0 \\
& \Leftrightarrow \quad (\text{množenjem sa } 2e^z) \quad \Leftrightarrow \quad (e^z)^2 - 2ie^z + 1 = 0 \\
& \Leftrightarrow \quad e^z = \frac{2i \pm \sqrt{(-2i)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{2i \pm 2\sqrt{2}i}{2} = \left(1 \pm \sqrt{2}\right) i \\
& \Leftrightarrow \quad z \in \operatorname{Ln}\left(\left(1 \pm \sqrt{2}\right) i\right) \\
& \Leftrightarrow \quad z = \ln\left|\left(1 \pm \sqrt{2}\right) i\right| + i\left(\arg\left(\left(1 \pm \sqrt{2}\right) i\right) + 2k\pi\right) \text{ za neki } k \in \mathbb{Z} \\
& \Leftrightarrow \quad z = \ln\left(\sqrt{2} \pm 1\right) + i\left(\pm\frac{\pi}{2} + 2k\pi\right) \text{ za neki } k \in \mathbb{Z}.
\end{aligned}
$$

(b) Imamo

$$
\begin{aligned}
4\cos z + 5 = 0 \quad & \Leftrightarrow \quad 4 \cdot \frac{e^{iz} + e^{-iz}}{2} + 5 = 0 \\
& \Leftrightarrow \quad 2e^{iz} + 2e^{-iz} + 5 = 0 \\
& \Leftrightarrow \quad (\text{množenjem sa } e^{iz}) \quad \Leftrightarrow \quad 2\left(e^{iz}\right)^2 + 5e^{iz} + 2 = 0 \\
& \Leftrightarrow \quad e^{iz} = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{-5 \pm 3}{4} \\
& \Leftrightarrow \quad e^{iz} = -2 \quad \text{ili} \quad e^{iz} = -\frac{1}{2} \\
& \Leftrightarrow \quad iz \in \operatorname{Ln}(-2) \quad \text{ili} \quad iz \in \operatorname{Ln}\left(-\tfrac{1}{2}\right) \\
& \Leftrightarrow \quad iz = \ln|-2| + i(\arg(-2) + 2k\pi) \text{ za neki } k \in \mathbb{Z} \\
& \qquad \text{ili} \quad iz = \ln\left|-\tfrac{1}{2}\right| + i\left(\arg\left(-\tfrac{1}{2}\right) + 2k\pi\right) \text{ za neki } k \in \mathbb{Z} \\
& \Leftrightarrow \quad iz = \ln 2 + i(\pi + 2k\pi) \text{ za neki } k \in \mathbb{Z} \\
& \qquad \text{ili} \quad iz = -\ln 2 + i\left(\pi + 2k\pi\right) \text{ za neki } k \in \mathbb{Z} \\
& \Leftrightarrow \quad iz = \pm\ln 2 + i(2k+1)\pi \text{ za neki } k \in \mathbb{Z} \\
& \Leftrightarrow \quad (\text{množenjem sa } -i) \quad \Leftrightarrow \quad z = (2k+1)\pi \pm i\ln 2 \text{ za neki } k \in \mathbb{Z}.
\end{aligned}
$$

(c) Imamo

$$\begin{aligned}
\ln(z+i)=0 \quad &\Leftrightarrow \quad \ln|z+i| + i\arg(z+i) = 0\\
&\Leftrightarrow \quad \begin{cases} \ln|z+i| = 0 \\ \arg(z+i) = 0 \end{cases}\\
&\Leftrightarrow \quad \begin{cases} |z+i| = 1 \\ z+i \in \mathbb{R}_{>0} \end{cases}\\
&\Leftrightarrow \quad z+i = 1\\
&\Leftrightarrow \quad z = 1-i.
\end{aligned}$$

(d) Kako je

$$\operatorname{ch} z = \cos(iz), \qquad z \in \mathbb{C},$$

imamo

$$\begin{aligned}
\cos z = \operatorname{ch} z \quad &\Leftrightarrow \quad \cos z = \cos(iz)\\
&\Leftrightarrow \quad \cos z - \cos(iz) = 0\\
&\Leftrightarrow \quad -2\sin\frac{z+iz}{2}\cdot\sin\frac{z-iz}{2} = 0\\
&\Leftrightarrow \quad \sin\left(\frac{1+i}{2}z\right) = 0 \quad \text{ili} \quad \sin\left(\frac{1-i}{2}z\right) = 0\\
&\Leftrightarrow \quad \frac{1+i}{2}z = k\pi \text{ za neki } k\in\mathbb{Z} \quad \text{ili} \quad \frac{1-i}{2}z = k\pi \text{ za neki } k \in \mathbb{Z} \qquad (2)\\
&\Leftrightarrow \quad z = \frac{2k\pi}{1+i} = \frac{2k\pi}{1+i}\cdot\frac{1-i}{1-i} = k\pi(1-i) \text{ za neki } k\in\mathbb{Z}\\
&\qquad\quad \text{ili} \quad z = \frac{2k\pi}{1-i} = \frac{2k\pi}{1-i}\cdot\frac{1+i}{1+i} = k\pi(1+i) \text{ za neki } k\in\mathbb{Z}\\
&\Leftrightarrow \quad z = k\pi(1\pm i) \text{ za neki } k\in\mathbb{Z}.
\end{aligned}$$

Pritom treća ekvivalencija vrijedi jer je

$$\cos z_1 - \cos z_2 = -2\sin\frac{z_1+z_2}{2}\cdot\sin\frac{z_1-z_2}{2}, \qquad z_1, z_2 \in \mathbb{C}. \tag{3}$$

Jednakost (3) dokaže se pomoću adicijskih formula za $\cos(z_1 \pm z_2)$ točno kao u realnom slučaju: imamo

$$\begin{aligned}
\cos z_1 - \cos z_2 &= \cos\left(\frac{z_1+z_2}{2} + \frac{z_1-z_2}{2}\right) - \cos\left(\frac{z_1+z_2}{2} - \frac{z_1-z_2}{2}\right)\\
&= \cos\frac{z_1+z_2}{2}\cdot\cos\frac{z_1-z_2}{2} - \sin\frac{z_1+z_2}{2}\cdot\sin\frac{z_1-z_2}{2}\\
&\quad - \left(\cos\frac{z_1+z_2}{2}\cdot\cos\frac{z_1-z_2}{2} + \sin\frac{z_1+z_2}{2}\cdot\sin\frac{z_1-z_2}{2}\right)\\
&= -2\sin\frac{z_1+z_2}{2}\cdot\sin\frac{z_1-z_2}{2}, \qquad z_1, z_2 \in \mathbb{C}.
\end{aligned}$$

Nadalje, peta ekvivalencija u (2) vrijedi jer za $w \in \mathbb{C}$ imamo

$$\begin{aligned}
\sin w = 0 \quad &\Leftrightarrow \quad \frac{e^{iw} - e^{-iw}}{2i} = 0 \\
&\Leftrightarrow \quad e^{iw} = e^{-iw} \\
&\Leftrightarrow \quad (\text{množenjem sa } e^{iw}) \quad \Leftrightarrow \quad \left(e^{iw}\right)^2 = 1 \\
&\Leftrightarrow \quad e^{2iw} = 1 \\
&\Leftrightarrow \quad 2iw \in \operatorname{Ln} 1 \\
&\Leftrightarrow \quad 2iw = \ln|1| + i(\arg 1 + 2k\pi) \text{ za neki } k \in \mathbb{Z} \\
&\Leftrightarrow \quad 2iw = 2k\pi i \text{ za neki } k \in \mathbb{Z} \\
&\Leftrightarrow \quad w = k\pi \text{ za neki } k \in \mathbb{Z}.
\end{aligned}$$

3. Funkcijom $f(z) = e^z$ preslikajte skup

$$G := \{x + iy \in \mathbb{C} : 0 < x < 1,\ 0 < y < \pi\}.$$

Rješenje. Sjetimo se da za $z = x + iy \in \mathbb{C}$ vrijedi:

- $|e^z| = e^x$
- $\operatorname{Arg}(e^z) = y$.

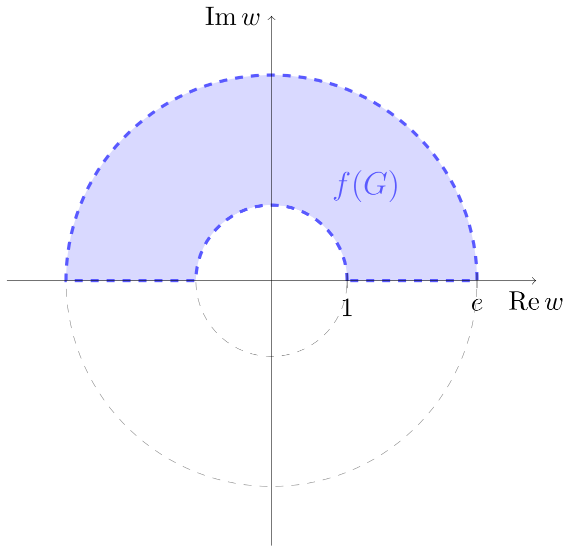
Prema tome, imamo:

- $0 < x < 1 \quad \Leftrightarrow \quad e^0 < e^x < e^1 \quad \Leftrightarrow \quad 1 < |e^z| < e.$
  Ovdje prva ekvivalencija vrijedi jer je funkcija $\mathbb{R} \to \mathbb{R}$, $x \to e^x$, strogo rastuća.
- $0 < y < \pi \quad \Leftrightarrow \quad 0 < \operatorname{Arg}(e^z) < \pi.$

Dakle,

$$f(G) = \{w \in \mathbb{C} : 1 < |w| < e \quad \text{i} \quad 0 < \operatorname{Arg}(w) < \pi\}.$$

Skup $f(G)$ skiciran je na sljedećoj slici:



4. Odredite i skicirajte domenu funkcije

$$f(z) := \ln\left(z^2 + 2z\right).$$

Rješenje. Sjetimo se da je domena funkcije ln skup $\mathbb{C}_\pi := \mathbb{C}\backslash\mathbb{R}_{\leq 0}$. Prema tome, vrijednost $\ln(z^2+z)$ nije definirana za one $z = x + iy \in \mathbb{C}$ koji zadovoljavaju uvjet

$$\begin{aligned}
z^2 + z \in \mathbb{R}_{\leq 0} \quad &\Leftrightarrow \quad (x+iy)^2 + x + iy \in \mathbb{R}_{\leq 0} \\
&\Leftrightarrow \quad x^2 - y^2 + x + (2xy + y)i \in \mathbb{R}_{\leq 0} \\
&\Leftrightarrow \quad \begin{cases} x^2 - y^2 + x \leq 0 \\ 2xy + y = 0 \end{cases} \\
&\Leftrightarrow \quad \begin{cases} x^2 + x \leq y^2 \\ (2x+1)y = 0 \end{cases} \\
&\Leftrightarrow \quad \begin{cases} x^2 + x \leq y^2 & (1) \\ x = -\frac{1}{2} \quad \text{ili} \quad y = 0 & (2) \end{cases}
\end{aligned}$$

Prema (2), imamo dva slučaja:

- Ako je $y = 0$, tada je

$$(1) \quad \Leftrightarrow \quad x^2 + x \leq 0 \quad \Leftrightarrow \quad x(x+1) \leq 0 \quad \Leftrightarrow \quad x \in [-1, 0],$$

  dakle $z = x + iy \in [-1, 0]$.

- Ako je $x = -\frac{1}{2}$, tada je

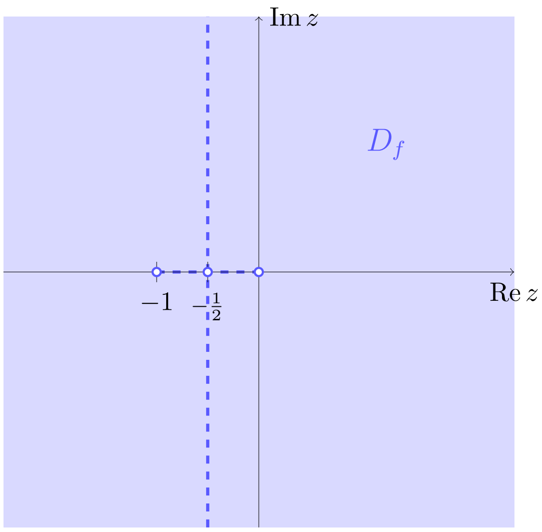
$$(1) \quad \Leftrightarrow \quad \left(-\frac{1}{2}\right)^2 - \frac{1}{2} \leq y^2 \quad \Leftrightarrow \quad -\frac{1}{4} \leq y^2,$$

  a ova je zadnja nejednakost očito zadovoljena za sve $y \in \mathbb{R}$. Dakle, u ovom je slučaju $z = x + iy \in -\frac{1}{2} + i\mathbb{R} := \left\{-\frac{1}{2} + it : t \in \mathbb{R}\right\}$.

Dakle, domena funkcije f je skup

$$\mathbb{C} \setminus \left([-1, 0] \cup \left(-\frac{1}{2} + i\mathbb{R}\right)\right),$$

koji je skiciran na sljedećoj slici:



5.* Dokažite ili opovrgnite sljedeću tvrdnju:

Neka su $z_1, z_2, \frac{z_1}{z_2} \in \mathbb{C}_\pi$. Tada je

$$\ln \frac{z_1}{z_2} = \ln z_1 - \ln z_2.$$

Rješenje. Imamo

$$\begin{aligned}
\ln \frac{z_1}{z_2} = \ln z_1 - \ln z_2 \quad &\Leftrightarrow \quad \ln \left|\frac{z_1}{z_2}\right| + i \arg \frac{z_1}{z_2} = \ln |z_1| + i \arg z_1 - (\ln |z_2| + i \arg z_2) \\
&\Leftrightarrow \quad \ln \left|\frac{z_1}{z_2}\right| + i \arg \frac{z_1}{z_2} = \ln |z_1| - \ln |z_2| + i (\arg z_1 - \arg z_2) \\
&\Leftrightarrow \quad \begin{cases} \ln \left|\frac{z_1}{z_2}\right| = \ln |z_1| - \ln |z_2| & (1) \\ \arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2. & (2) \end{cases}
\end{aligned}$$

Jednakost (1) je očito zadovoljena za sve $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Međutim, jednakost (2) nije zadovoljena kad god $\arg z_1 - \arg z_2 \notin \langle -\pi, \pi]$. Primjerice, za $z_1 := e^{-\frac{3\pi}{4} i}$ i $z_2 := e^{\frac{3\pi}{4} i}$ imamo

$$\arg \frac{z_1}{z_2} = \arg \left(\frac{e^{-\frac{3\pi}{4} i}}{e^{\frac{3\pi}{4} i}} \right) = \arg \left(e^{-\frac{3\pi}{2} i} \right) = \arg i = \frac{\pi}{2},$$

dok je

$$\arg z_1 - \arg z_2 = \arg \left(e^{-\frac{3\pi}{4} i} \right) - \arg \left(e^{\frac{3\pi}{4} i} \right) = -\frac{3\pi}{4} - \frac{3\pi}{4} = -\frac{3\pi}{2}.$$

Prema tome, tvrdnja iz zadatka ne vrijedi.