(4.) $g(x,y)=\begin{cases} \dfrac{xy^4}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

$$0\le |g(x,y)| = \left|\frac{xy^4}{x^2+y^2}\right| = \underbrace{\frac{y^2}{x^2+y^2}}_{\le 1}\cdot|x|\cdot y^2 \le |x|\cdot y^2 \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\lim_{(x,y)\to(0,0)} |g(x,y)| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$$

$\Rightarrow$ g je neprekidna u 0.

Jasno, g je neprekidna i u točkama $(x,y)\neq(0,0)$ jer je kvocijent neprekidnih funkcija.

$$\frac{\partial g}{\partial x}(0,0) = \lim_{t\to 0}\frac{g(t,0)-g(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0 \quad (1)$$

$$\frac{\partial g}{\partial y}(0,0) = \lim_{t\to 0}\frac{g(0,t)-g(0,0)}{t} =$$

$$= \lim_{t\to 0}\frac{0-0}{t} = 0 \quad (2)$$

(1)&(2) $\Rightarrow$ Jedini kandidat za diferencijal je nul-operator

$$0\le h(x,y) = \frac{\|g(x,y)-g(0,0)\|}{\|(x,y)\|} = \frac{\left|\frac{xy^4}{x^2+y^2}\right|}{\sqrt{x^2+y^2}} =$$

$$= \underbrace{\frac{|x|}{\sqrt{x^2+y^2}}}_{\le 1}\cdot\underbrace{\frac{y^2}{x^2+y^2}}_{\le 1}\cdot y^2 \le y^2 \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\| g(x,y) - g(0,0) - \mathbf{0}(x,y)\|}{\|(x,y)\|} = 0$$

$\Rightarrow$ f je diferencijabilna u $(0,0)$.
Jasno, f je diferencijabilna i u ostalim točkama
jer je kvocijent derivabilnih funkcija.

$$f_1(x,y) = xy^4 \qquad \frac{\partial f_1}{\partial x}(x,y) = y^4 \qquad \frac{\partial f_1}{\partial y}(x,y) = 4xy^3$$

$$f_2(x,y) = x^2+y^2 \qquad \frac{\partial f_2}{\partial x}(x,y) = 2x \qquad \frac{\partial f_2}{\partial y}(x,y) = 2y$$

f_1, f_2 imaju sve parcijalne derivacije i te su derivacije neprekidne $\Rightarrow$ f_1 i f_2 su diferencijabilne.

Još trebamo provjeriti je li g klase C^1

Za $(x,y) \neq (0,0)$
$$\frac{\partial g}{\partial x}(x,y) = \frac{y^4(x^2+y^2) - 2x \cdot xy^4}{(x^2+y^2)^2} = \frac{y^4(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial g}{\partial y}(x,y) = \frac{4xy^3(x^2+y^2) - 2y \cdot xy^4}{(x^2+y^2)^2} = \frac{2xy^3(2x^2+2y^2-y^2)}{(x^2+y^2)^2} = \frac{2xy^3(2x^2+y^2)}{(x^2+y^2)^2}$$

$$0 \leq \left|\frac{\partial g}{\partial x}(x,y)\right| \leq \frac{y^4}{(x^2+y^2)^2}\cdot|y^2-x^2| \leq \frac{y^4}{(x^2+y^2)^2}\cdot(x^2+y^2) = \frac{y^4}{x^2+y^2}$$

$$= \underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1} \cdot y^2 \leq y^2 \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial g}{\partial x}(x,y) = 0 \overset{(1)}{=} \frac{\partial g}{\partial x}(0,0)$$

$\Rightarrow \frac{\partial g}{\partial x}$ je neprekidna u $(0,0)$,
a u ostalim točkama je također neprekidna
kao kvocijent nepr. funkcija.

$$0 \le \left|\frac{\partial g}{\partial y}(x,y)\right| = 2|xy^3| \frac{2x^2+y^2}{(x^2+y^2)^2} = 2|x||y| \cdot \frac{y^2}{x^2+y^2} \cdot \frac{2x^2+y^2}{x^2+y^2}$$

$$= 2|xy| \cdot \underbrace{\frac{y^2}{x^2+y^2}}_{\le 1} \cdot \underbrace{\left(1+\underbrace{\frac{x^2}{x^2+y^2}}_{\le 1}\right)}_{\le 2} \le 4|xy| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial g}{\partial y}(x,y) = 0 \overset{(2)}{=} \frac{\partial g}{\partial y}(0,0)$$

$\Rightarrow \frac{\partial g}{\partial x}$ je neprekidna u $(0,0)$,
a u ostalim točkama je također neprekidna kao
kvocijent nepr. funkcija.

Pošto su $\frac{\partial g}{\partial x}$ i $\frac{\partial g}{\partial y}$ neprekidne u svakoj točki $\Rightarrow g$ je
klase C^1.