• Odredite $\alpha \in \mathbb{R}$ t.d. niz (a_n) zadan s

$$a_n = \sqrt[n]{5 + (2\alpha)^{n \sin \frac{(2n-1)\pi}{2}}}, \quad n \in \mathbb{N}$$

bude konvergentan.

Za $\alpha = 0$ $a_n = \sqrt[n]{5}$
$\lim a_n = 1$

Rj: $\sin \frac{(2n-1)\pi}{2} = \sin\left(-\frac{\pi}{2} + n\pi\right)$

Za $n = 2k, k \in \mathbb{N}$ $\sin\left(-\frac{\pi}{2} + 2k\pi\right) = \sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

Za $n = 2k-1, k \in \mathbb{N}$ $\sin\left(-\frac{\pi}{2} + (2k-1)\pi\right) = \sin\left(-\frac{3\pi}{2}\right)$
$= -\sin\left(2\pi - \frac{\pi}{2}\right) =$
$= \sin\left(\frac{\pi}{2}\right) = 1$

Za $\alpha \neq 0$:

$a_{2k-1} = \left(5 + (2\alpha)^{2k-1}\right)^{\frac{1}{2k-1}}$ $\qquad a_{2k} = \left(5 + \left(\frac{1}{2\alpha}\right)^{2k}\right)^{\frac{1}{2k}}$

• Za $\left|\frac{1}{2\alpha}\right| < 1$ je $\lim \left(\frac{1}{2\alpha}\right)^{2k} = 0 \Rightarrow \lim\limits_{k\to+\infty} a_{2k} = 5^0 = 1$

• Za $\left|\frac{1}{2\alpha}\right| = 1$ $a_{2k} = 6^{\frac{1}{2k}} \to 1$ $\lim\limits_{k\to+\infty} a_{2k} = 1$

• Za $\left|\frac{1}{2\alpha}\right| > 1$ $2 \cdot \left(\frac{1}{2\alpha}\right)^{2k} > 5 + \left(\frac{1}{2\alpha}\right)^{2k} > \left(\frac{1}{2\alpha}\right)^{2k}$

↓ za dovoljno velike k

$\Rightarrow \sqrt[2k]{2} \left|\frac{1}{2\alpha}\right| > a_{2k} > \left|\frac{1}{2\alpha}\right|$ (za dovoljno velike k)

($\sqrt[2k]{2} \to 1$)

teorem o sendviču $\lim\limits_{k\to\infty} a_{2k} = \left|\frac{1}{2\alpha}\right|$

• Za $|2\alpha| < 1$ $\qquad \lim_k (2\alpha)^{2k-1} = 0$

$$\lim a_{2k-1} = 5^0 = 1$$

• Za $|2\alpha| = 1$ $\qquad a_{2k-1} = \begin{cases} 6^{\frac{1}{2k-1}}, & \text{za } \alpha = \frac{1}{2} \\ 4^{\frac{1}{2k-1}}, & \text{za } \alpha = -\frac{1}{2} \end{cases}$

$$\lim_k a_{2k-1} = 1$$

• Za $|2\alpha| > 1 \Rightarrow$

• Za $\alpha > 0$ $\qquad 2 \cdot (2\alpha)^{2k-1} > 5 + (2\alpha)^{2k-1} > (2\alpha)^{2k-1}$

↓ za dovoljno velike k

$$\Rightarrow \sqrt[2k]{2}\, 2\alpha > a_{2k-1} > 2\alpha \quad (\sqrt[2k]{2} \to 1)$$

$$\Rightarrow \lim_k a_{2k-1} = 2\alpha$$

• Za $\alpha < 0$ $\qquad a_{2k-1} = \left(5 - (|2\alpha|)^{2k-1}\right)^{\frac{1}{2k-1}}$

$$-(|2\alpha|)^{2k-1} < 5 - (|2\alpha|)^{2k-1} < \frac{-1}{2} \cdot (|2\alpha|)^{2k-1}$$

↳ za dovoljno velike k

$$\Rightarrow -|2\alpha| < a_{2k-1} < \left(-\frac{1}{2}\right)^{\frac{1}{2k-1}} |2\alpha| \quad \left(\left(-\tfrac{1}{2}\right)^{\frac{1}{2k-1}} \to -1\right) \Rightarrow \lim a_{2k-1} = -|2\alpha| = 2\alpha$$

Zaključak

- Za $\alpha = 0$ niz je konvergentan.

- Za $|2\alpha| \leq 1$ i $\alpha \neq 0$ $\lim a_{2k-1} = 1$, $\lim a_{2k} = \left|\frac{1}{2\alpha}\right|$,

  pa niz konvergira $\iff 1 = |2\alpha| \iff \alpha = \pm \frac{1}{2}$

- Za $|2\alpha| > 1$ $\quad \lim a_{2k-1} = 2\alpha$
  $(|\alpha| > \frac{1}{2})$ $\quad \lim a_{2k} = 1$

  niz ne konvergira jer je za $|\alpha| > \frac{1}{2}$ $2\alpha \neq 1$

$(a_n)_n$ konv za $\alpha = 0$, $\alpha = \pm\frac{1}{2}$.