SJENE CRNIH RUPA

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Mentor: izv. prof. Ivica Smolić

Samostalni seminar iz istraživanja u fizici

1

Što je sjena crne rupe?

Preuzeto: Event Horizon Telescope kolaboracija (2019.)

SCHWARZSCHILD

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2$$

prirodne jedinice	$c = G = 1$
sferna simetrija	nema naboja, nema rotacije
statičko rješenje	"konstantna u vremenu"
vakuumsko rješenje	$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$
crna rupa	kauzalno odvojeni dio prostorvremena

SCHWARZSCHILD

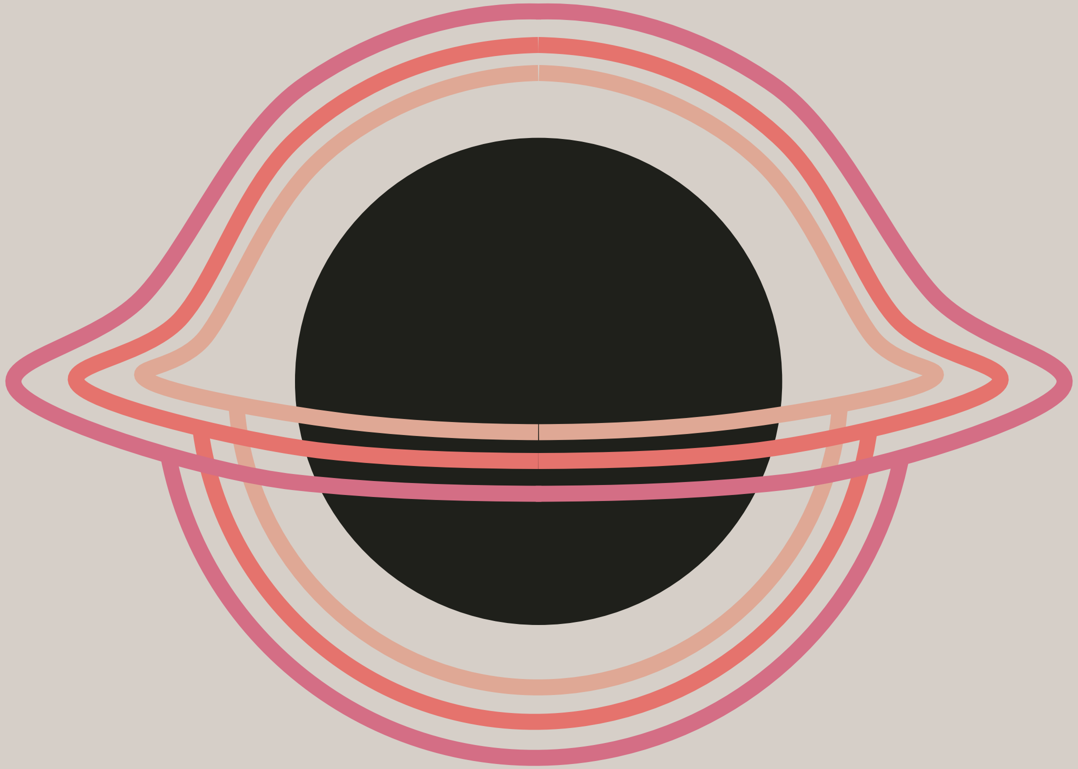
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2$$

$r \to 0$

$r = 2M$

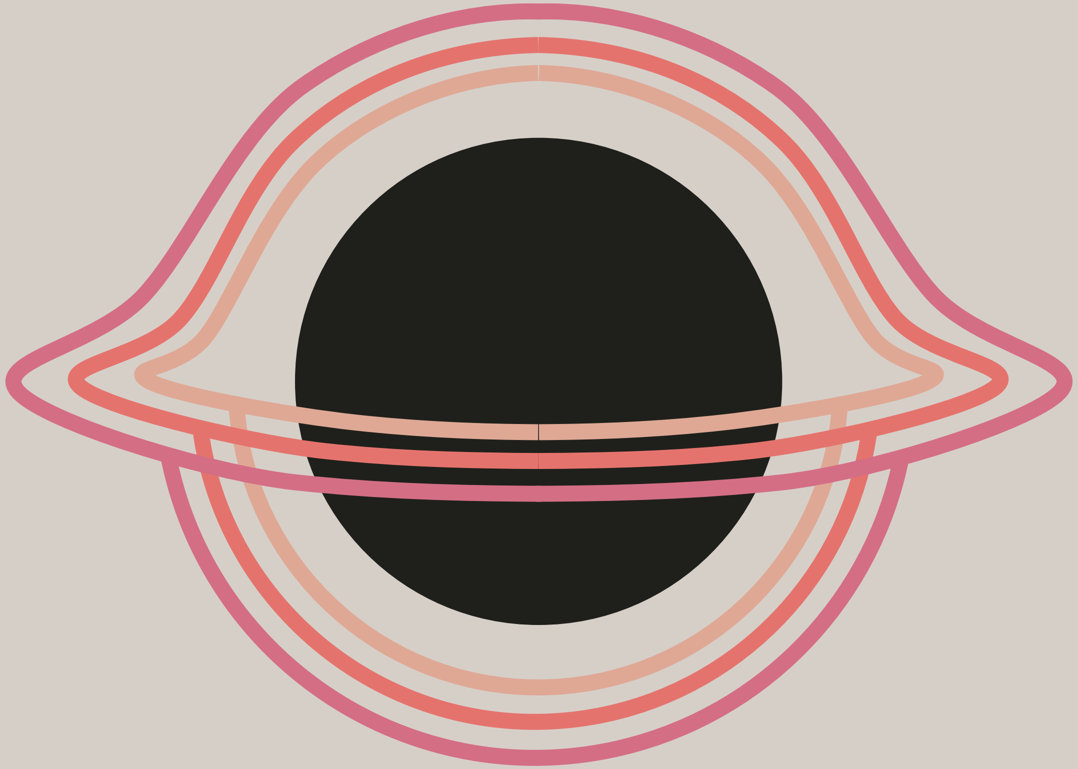
$r \to \infty$

"VIDJETI" CRNU RUPU



"REALNA" CRNA RUPA

"VIDJETI" CRNU RUPU



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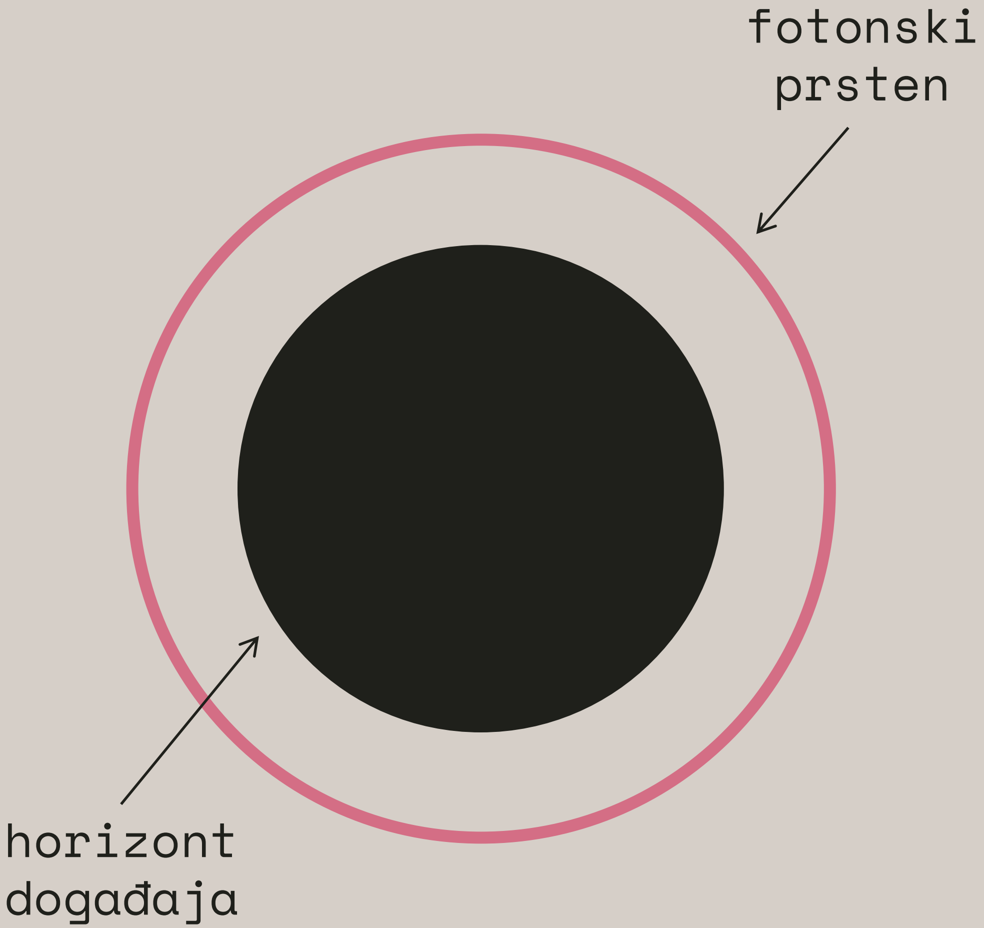
IDEALIZACIJA

"VIDJETI" CRNU RUPU

"REALNA" CRNA RUPA

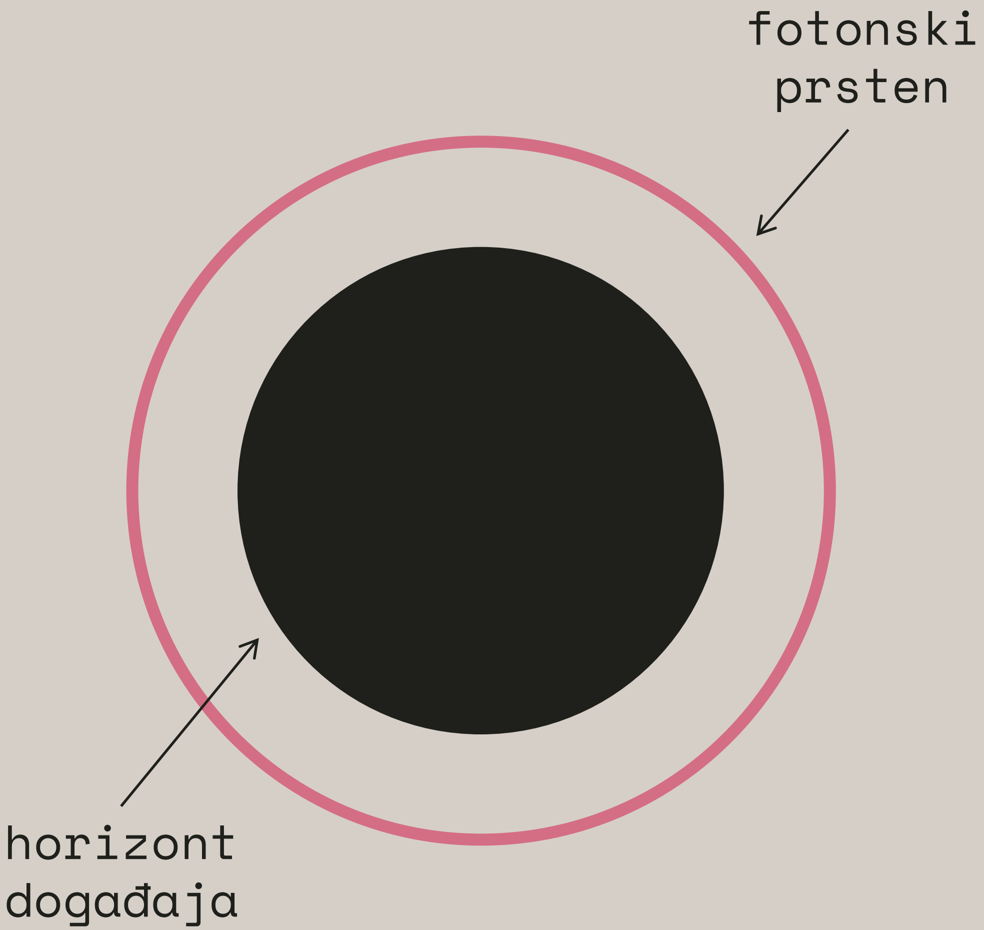


IDEALIZACIJA

"VIDJETI" CRNU RUPU

PRETPOSTAVKE U DALJNJIM IZVODIMA:

- uniformna pozadina
- statički opažač
- nema izvora između opažača i crne rupe



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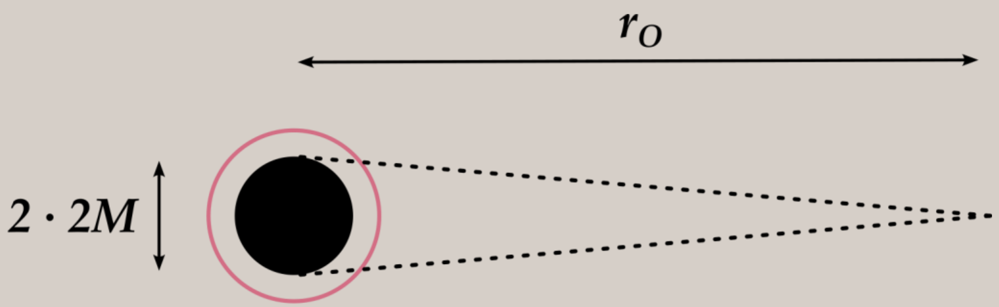
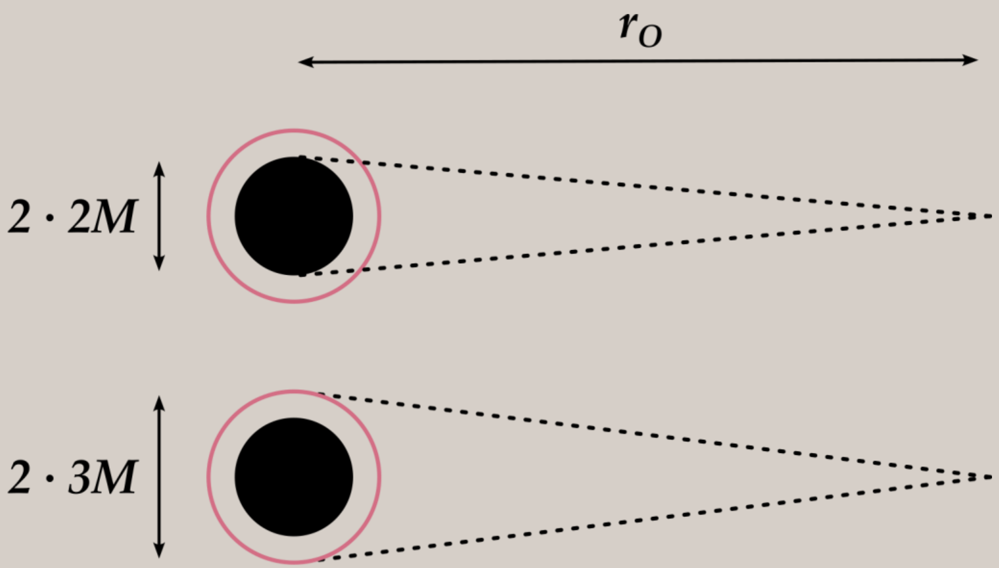


promjer horizonta događaja

Prilagođeno: Perlick, Tsupko (2022.)

"VIDJETI" CRNU RUPU



promjer horizonta događaja

promjer fotonskog prstena

Prilagođeno: Perlick, Tsupko (2022.)

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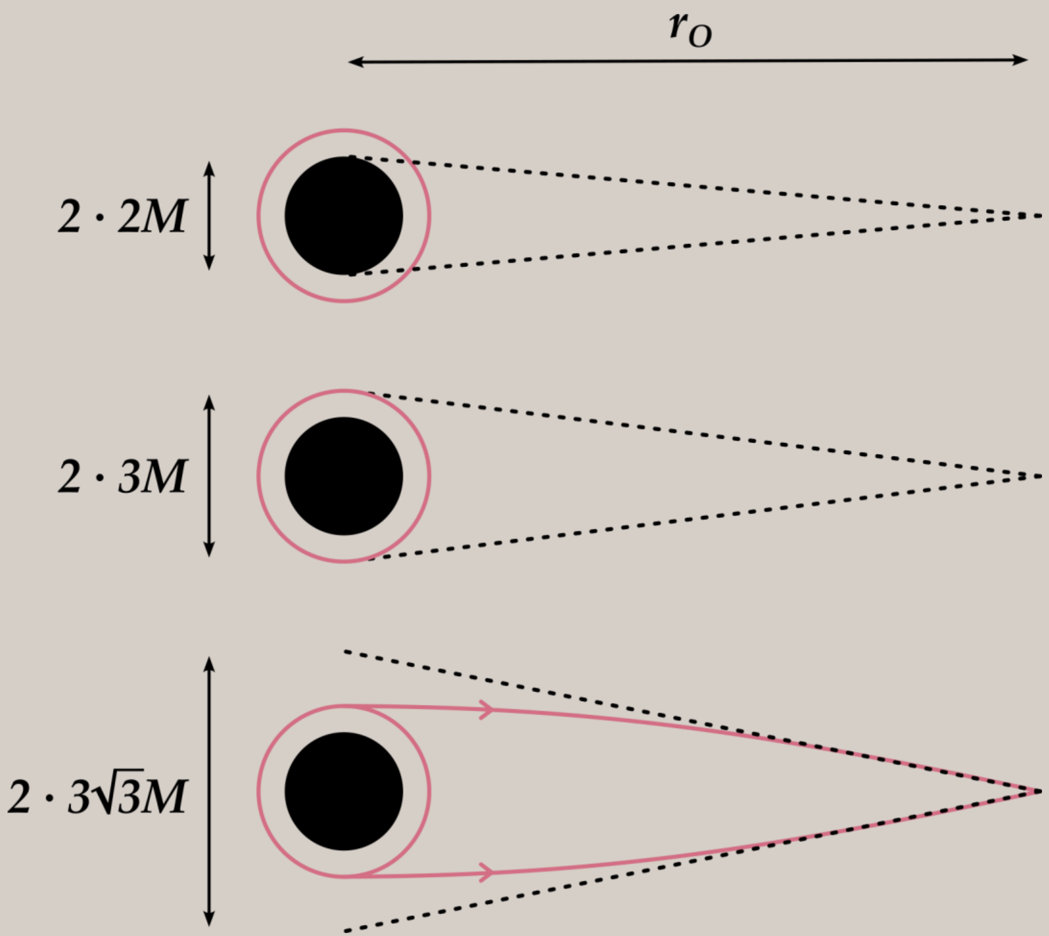


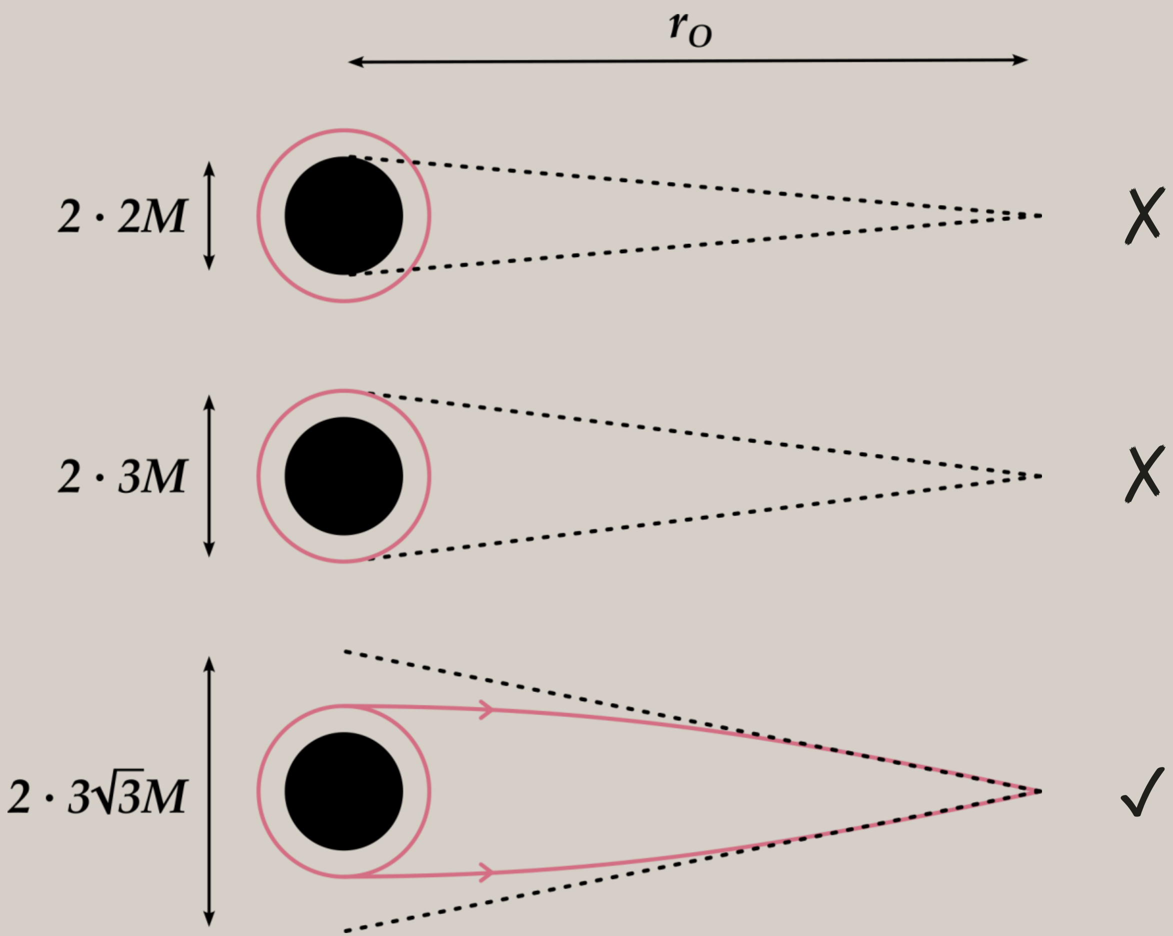
promjer horizonta događaja

promjer fotonskog prstena

promjer fotonskog prstena
+ zakrivljenost prostorvremena

Prilagođeno: Perlick, Tsupko (2022.)

"VIDJETI" CRNU RUPU



promjer horizonta događaja

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promjer fotonskog prstena
+ zakrivljenost prostorvremena

Prilagođeno: Perlick, Tsupko (2022.)

2

Određivanje sjene

SFERNA SIMETRIJA

OPĆENITI POSTUPAK:

1. Izvod izraza za trajektoriju općenite svjetlosne zrake (svjetlosnog geodezika) emitirane od opažača "u prošlost".

   → kut otklona

2. Izdvajanje onih zraka koje asimptotski ulaze u nestabilne kružne orbite.

SFERNA SIMETRIJA

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + D(r)(d\theta^2 + \sin^2\theta d\phi^2)$$

$A(r),\ B(r),\ D(r)$ svugdje pozitivne

SFERNA SIMETRIJA

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + D(r)(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}\left(-A(r)\dot{t}^2 + B(r)\dot{r}^2 + D(r)\dot{\phi}^2\right)$$

E-L jednadžbe → konstante gibanja

$$E \equiv A(r)\dot{t}$$

$$L \equiv D(r)\dot{\phi}.$$

SFERNA SIMETRIJA

E-L jednadžbe → konstante gibanja*

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*po jedinici mase

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udarni parametar

$$b \equiv \frac{L/\mu}{E/\mu} = \frac{L}{E}$$

SFERNA SIMETRIJA

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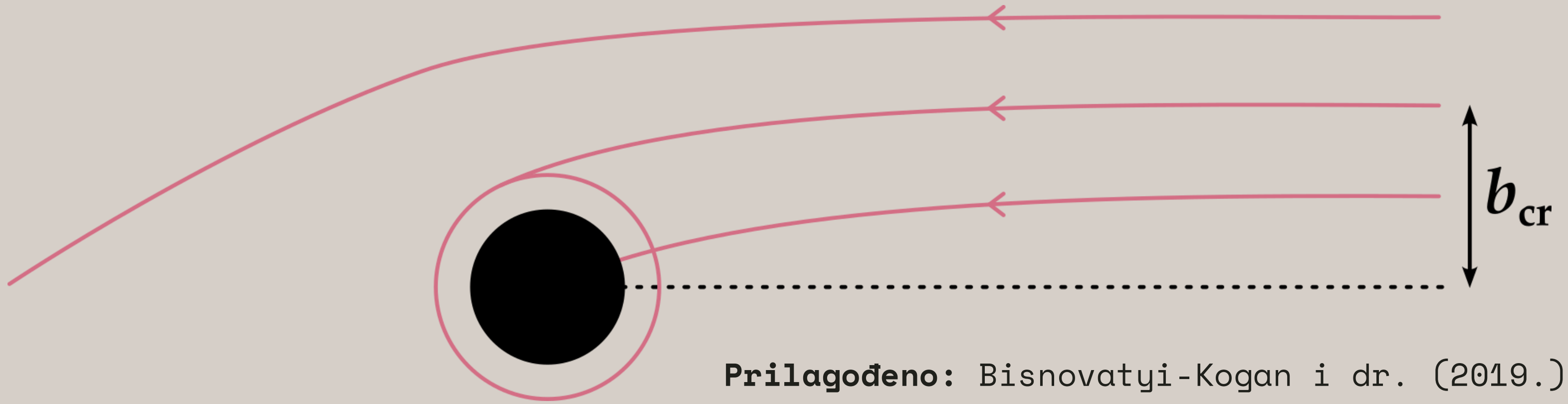


Prilagođeno: Bisnovatyi-Kogan i dr. (2019.)

SFERNA SIMETRIJA

E-L jednadžbe → konstante gibanja*

$$E \equiv A(r)\dot{t}$$

$$L \equiv D(r)\dot{\phi}$$

*po jedinici mase

udarni parametar

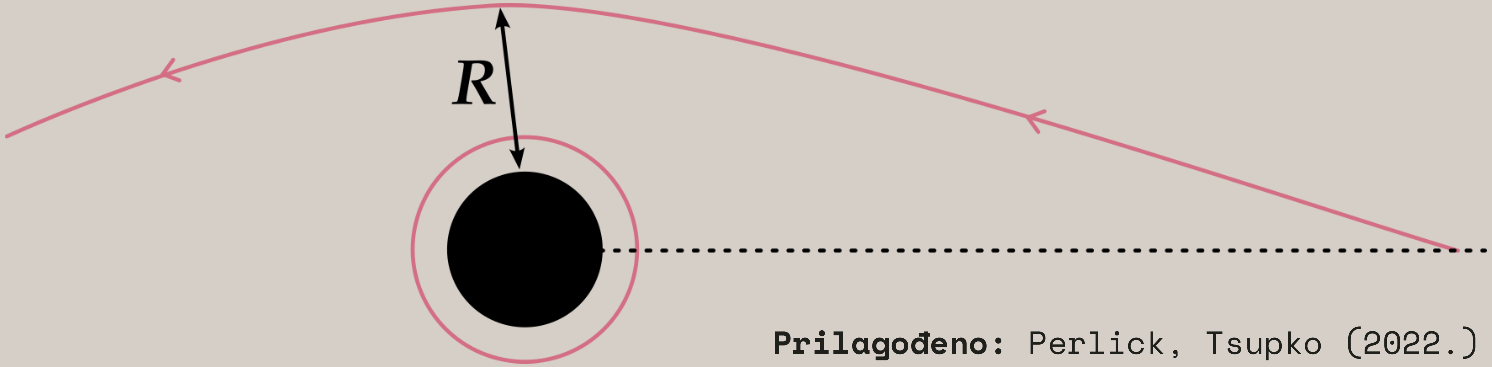
$$b \equiv \frac{L/\mu}{E/\mu} = \frac{L}{E}$$

$$ds^2 = 0$$

svjetlosni geodezici → jednadžba orbite/prolaska

SFERNA SIMETRIJA

$$\frac{1}{b^2} = \frac{E^2}{L^2} = \frac{A(R)}{D(R)}$$



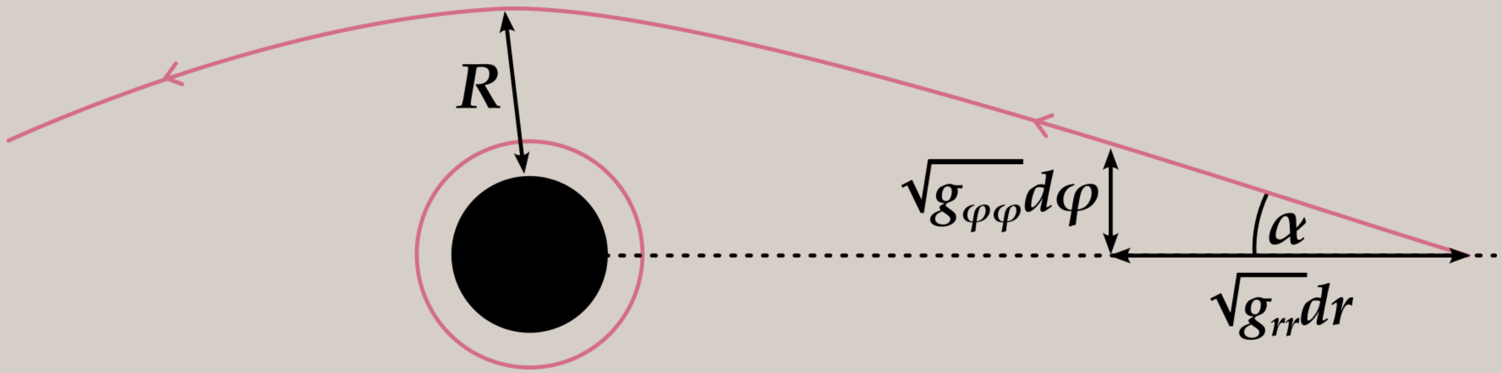
Prilagođeno: Perlick, Tsupko (2022.)

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{D(r)}{B(r)}\left(\frac{h^{-2}(r)}{h^{-2}(R)} - 1\right),\ h^{-2}(r) = \frac{D(r)}{A(r)}$$

orbita = ekstrem

SFERNA SIMETRIJA

$$\sin^2\alpha = \frac{h^{-2}(R)}{h^{-2}(r_O)}$$



Prilagođeno: Perlick, Tsupko (2022.)

asimptotski do orbite → kritični uvjet, tj. **rub sjene**

$$\sin^2\alpha_{sh} = \frac{h^{-2}(r_{ph})}{h^{-2}(r_O)} = \frac{b_{cr}^2}{h^{-2}(r_O)}$$

SFERNA SIMETRIJA

uvjet sjene

$$\sin^2 \alpha_{\mathrm{sh}} = \frac{h^{-2}(r_{\mathrm{ph}})}{h^{-2}(r_O)} = \frac{b_{\mathrm{cr}}^2}{h^{-2}(r_O)}$$

npr. Schwarzschild

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2M}{r}, \quad D(r) = r^2$$

$$h^{-2}(r) = \frac{r^2}{1 - 2M/r}$$

SFERNA SIMETRIJA

uvjet sjene

$$\sin^2 \alpha_{\rm sh} = \frac{h^{-2}(r_{\rm ph})}{h^{-2}(r_O)} = \frac{b_{\rm cr}^2}{h^{-2}(r_O)}$$

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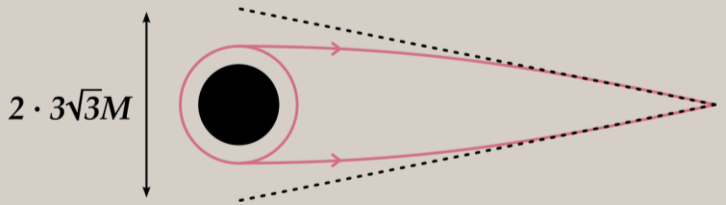


$$\sin^2 \alpha_{\rm sh} = \frac{27M^2(1 - 2M/r_O)}{r_O^2}$$

SJENA S VELIKE UDALJENOSTI

$$\sin^2 \alpha_{\text{sh}} = \frac{27M^2(1 - 2M/r_O)}{r_O^2} \quad \Rightarrow \quad \sin \alpha_{\text{sh}} \approx \alpha_{\text{sh}} \approx \frac{3\sqrt{3}M}{r_O}$$

SJENA S VELIKE UDALJENOSTI

$$\sin^2 \alpha_{\text{sh}} = \frac{27M^2(1 - 2M/r_O)}{r_O^2} \Rightarrow \sin \alpha_{\text{sh}} \approx \alpha_{\text{sh}} \approx \frac{3\sqrt{3}M}{r_O}$$

asimptotska ravnost $\lim_{r \to \infty} A(r),\ B(r),\ D(r)/r^2 = 1$

SJENA S VELIKE UDALJENOSTI

$$\sin^2 \alpha_{\text{sh}} = \frac{27M^2(1-2M/r_O)}{r_O^2} \quad \Rightarrow \quad \sin \alpha_{\text{sh}} \approx \alpha_{\text{sh}} \approx \frac{3\sqrt{3}M}{r_O}$$

asimptotska ravnost

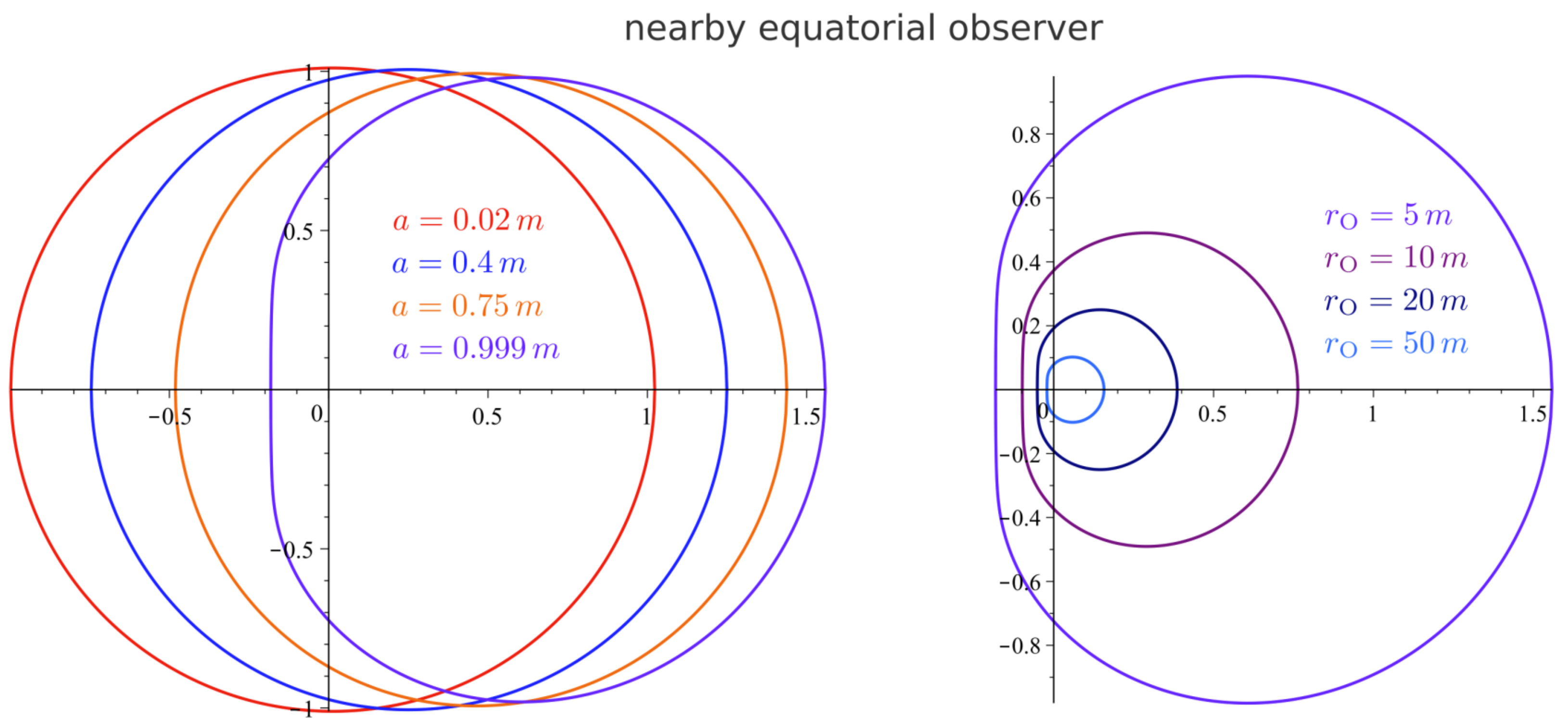
$$\lim_{r\to\infty} A(r),\ B(r),\ D(r)/r^2 = 1$$

$$\sin \alpha_{\text{sh}} \approx \alpha_{\text{sh}}$$

$$= \frac{b_{\text{cr}}\sqrt{A(r_O)}}{\sqrt{D(r_O)}} \to \frac{b_{\text{cr}}}{r_O}$$

✓

KERROVA CRNA RUPA



Preuzeto: Perlick, Tsupko (2022.)

3

Jedinstvenost sjene

$^*ds^2 = -A(r)dt^2 + B(r)dr^2 + D(r)(d\theta^2 + \sin^2\theta d\phi^2)$

DODATNE SPECIJALIZACIJE

1

Specijaliziramo izbor metričkih funkcija*.

$$A(r) = V(r)\tilde{A}(r),\, B(r) = \frac{1}{V(r)\tilde{B}(r)}, \quad V(r) = 1 - 2m/r$$

$$D(r) = r^2$$

m *nije* nužno ADM masa, već fiksira položaj horizonta.

2

Pretpostavljamo asimptotsku ravnost.

3

Funkcije $\tilde{A}(r)$ i $\tilde{B}(r)$ svugdje su pozitivne izvan horizonta i barem su C^1.

SFERNA SIMETRIJA

$$h(r) = \frac{\sqrt{\tilde{A}(r)V(r)}}{r}, \; b_{\mathrm{cr}} = \frac{r_{\mathrm{ph}}}{\sqrt{\tilde{A}(r_{\mathrm{ph}})V(r_{\mathrm{ph}})}}$$

SFERNA SIMETRIJA

$$h(r) = \frac{\sqrt{\tilde{A}(r)V(r)}}{r}, \; b_{\text{cr}} = \frac{r_{\text{ph}}}{\sqrt{\tilde{A}(r_{\text{ph}})V(r_{\text{ph}})}}$$

uvjet degeneracije $\sin\alpha_{\text{sh}} \overset{!}{=} \sin\alpha_{\text{sh}}^{(\text{Schw})}$

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uvjet degeneracije $\sin\alpha_{\mathrm{sh}} \stackrel{!}{=} \sin\alpha_{\mathrm{sh}}^{(\mathrm{Schw})}$

$$b_{\mathrm{cr}} = \frac{3\sqrt{3}M}{\sqrt{\tilde{A}(r_O)}}$$

SFERNA SIMETRIJA

$$h(r) = \frac{\sqrt{\tilde{A}(r)V(r)}}{r}, \; b_{\text{cr}} = \frac{r_{\text{ph}}}{\sqrt{\tilde{A}(r_{\text{ph}})V(r_{\text{ph}})}}$$

uvjet degeneracije

$$\left.\begin{aligned} \sin\alpha_{\text{sh}} &\overset{!}{=} \sin\alpha_{\text{sh}}^{(\text{Schw})} \\ b_{\text{cr}} &= \frac{3\sqrt{3}M}{\sqrt{\tilde{A}(r_O)}} \end{aligned}\right\} \tilde{A}(r) = 1$$

DEGENERACIJA KLASE I

$\tilde{A}(r) = 1$

I

Vrijedi $\tilde{A}(r) = 1$. Izbor funkcije $\tilde{B}(r)$ *nije* dodatno ograničen.

Dovoljan uvjet.

DEGENERACIJA KLASE I

$\tilde{A}(r) = 1$

I

Vrijedi $\tilde{A}(r) = 1$. Izbor funkcije $\tilde{B}(r)$ *nije* dodatno ograničen.

npr. Simpson-Visser

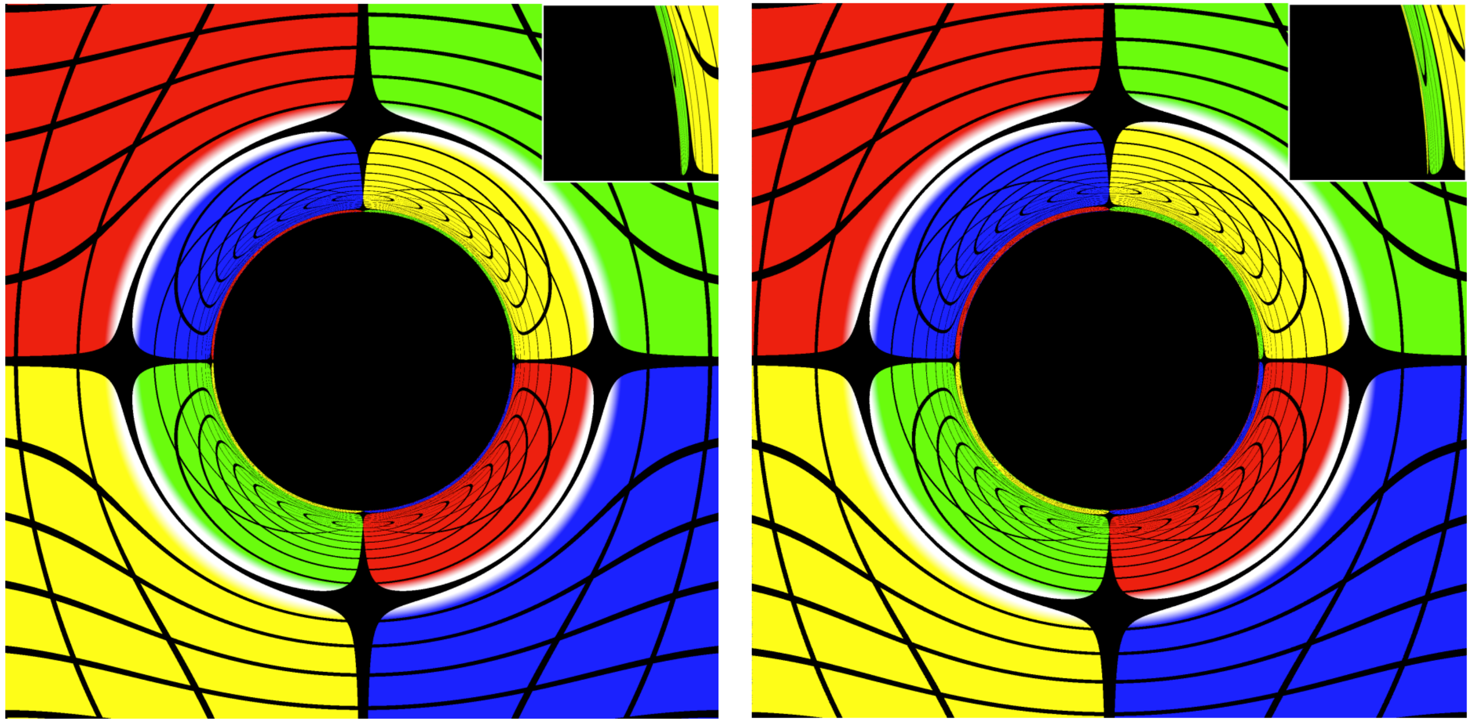
$$ds^2 = -V(r)dt^2 + \frac{1}{V(r)\tilde{B}_{\mathrm{SV}}(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\tilde{B}_{\mathrm{SV}}(r) = 1 - \frac{b^2}{r^2}$$

= regularna crna rupa ako $0 < b < 2M$

DEGENERACIJA KLASE I

$\tilde{A}(r) = 1$



(a) Schwarzschild ($b = 0$) (b) SV BH ($b = 1.99M$)

Preuzeto: Lima Junior i dr. (2021.)

DEGENERACIJA KLASE II

$\tilde{A}(r) \neq 1$

II

Općenito ne vrijedi $\tilde{A}(r) = 1$, ali zadovoljen je uvjet degeneracije za *jedan* od više postojećih fotonskih prstena, koji nazivamo dominantnim fotonskim prstenom.

Nužan i dovoljan uvjet.

DEGENERACIJA KLASE II

$\tilde{A}(r) \neq 1$

ODREĐIVANJE DOMINANTNOG PRSTENA:

$1/b \geqslant h(r)$ mora vrijediti duž geodezika

na točki obrata vrijedi $1/b = h(r)$

funkcija $h(r)$ iščezava na horizontu

+ nenegativna je svugdje izvan

$$h(r) = \frac{\sqrt{\tilde{A}(r)V(r)}}{r}$$

$$b_{\text{cr}} = \frac{3\sqrt{3}M}{\sqrt{\tilde{A}(r_O)}}$$

DEGENERACIJA KLASE II

$\tilde{A}(r) \neq 1$

ODREĐIVANJE DOMINANTNOG PRSTENA:

- $1/b \geqslant h(r)$ mora vrijediti duž geodezika
- na točki obrata vrijedi $1/b = h(r)$
- funkcija $h(r)$ iščezava na horizontu

  \+ nenegativna je svugdje izvan

$$h(r) = \frac{\sqrt{\tilde{A}(r)V(r)}}{r}$$

$$b_{\text{cr}} = \frac{3\sqrt{3}M}{\sqrt{\tilde{A}(r_O)}}$$

nejednakost + izrazi za $h(r)$, $b(r)$ →

$$\tilde{A}(r) \leqslant \frac{r^3}{r-2m}\left(\frac{\tilde{A}(r_O)}{27M^2}\right)$$

DEGENERACIJA SJENE

≠ degeneracija leće

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + D(r)(d\theta^2 + \sin^2\theta d\phi^2) \qquad A(r) = V(r)\tilde{A}(r),\ B(r) = \frac{1}{V(r)\tilde{B}(r)}, \quad V(r) = 1 - 2m/r$$

$$D(r) = r^2$$

I

Vrijedi $\tilde{A}(r) = 1$. *Poseban slučaj klase II.*

II

Općenito ne vrijedi $\tilde{A}(r) = 1$, ali zadovoljen je uvjet:

$$\tilde{A}(r) \leqslant \frac{r^3}{r - 2m}\left(\frac{\tilde{A}(r_O)}{27M^2}\right)$$

i jednakost mora vrijediti bar na jednom $r_{\mathrm{ph}}^{\mathrm{dom}} < r_O$ izvan horizonta.

DEGENERACIJA S KERROM?

*

Isti postupak.

DODATNE SPECIJALIZACIJE:
* asimptotska ravnost
* Hamilton-Jacobi separabilnost

4

Zaključak

1

Odabiremo **položaj opažača**, izvan horizonta događaja i fotonskih orbita u relevantnim koordinatama.

2

Promatramo **geodezike svjetlosnog tipa**. Pritom zamišljamo zrake emitirane od opažača prema crnoj rupi, "u prošlost", i rješavamo jednadžbu geodezika očekujući njihovo skretanje pod utjecajem gravitacije crne rupe.

3

Namećemo **kritični uvjet**, odnosno da minimalni radijus prolaska odgovara (dominantnoj) fotonskoj orbiti. Odgovarajući kutovi zraka-doglednica (do središta crne rupe) opisuju rub sjene.

HVALA NA POZORNOSTI

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