Noncommutative Geometry

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The focus of this paper is to motivate and explain the duality between spectral triples and Riemannian manifolds, state the spectral action formalism and present an application of said formalism to modern particle physics models. The first portion of the paper focuses on motivating the idea of noncommutative geometry in an understandable manner, through examples and applications that appeared in physics in recent times. The second portion is focused on providing the mathematical background, which might be unfamiliar to a graduate physics student, while assuming some knowledge of topology and differential geometry. We proceed by laying out the spectral triple formalism, which provides a language to speak of Riemannian manifolds using purely algebraic terms, formulated as operators on a Hilbert space. Furthermore, a result by Connes is cited, which proves the complete equivalence between Riemannian manifolds and the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}$. Next, a principle for constructing physical actions from said spectral triples is outlined and finally, an application of said principle is given on a particle physics model with curved background which reproduces all of modern particle physics, as well as the gravitational action and the couplings between the two. Certain discrepancies between the obtained model and standard particle physics + gravity models exist, which can be explained by imposing some physical constraints and through the use of renormalization theory.

I. The Motivation and Applications for Noncommutative Geometry in Physics

A. What is Noncommutative geometry?

In essence, the program of noncommutative geometry is to impose a commutation relation of the form:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{1}$$

where $\theta^{\mu\nu}$ is an antisymmetric (2,0)-tensor, the modulus of which determines the "scale of noncommutativity" and x^μ are the coordinate functions. As was the case in quantum mechanics, it can be shown that this commutation relation implies:

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}\|\theta^{\mu\nu}\| \tag{2}$$

This implies that two position coordinates cannot be diagonalized simultaneously and therefore that they cannot be measured at the same time. Volume is therefore meaningless on scales smaller than $\|\theta^{\mu\nu}\|^2$ and the idea of describing space as a smooth manifold fails, needing to be replaced by something new [1].

B. Quantum Physics and Quantum Field Theory

The disappearance of the coordinate space is in direct analogy to the disappearance of phase space in quantum mechanics upon imposing the canonical commutation relations: $[x, p] = i\hbar$.

One of the key problems in the early days of Quantum Field Theory (the correct relativistic extension of quantum mechanics) was to remove the divergencies plaguing many fundamental calculations. The historically (and modernly) accepted solution to this problem is the renormalization group (RG) approach. It consists of placing a UV cut-off, meaning the theory only considers energies up to Λ and not any higher. This makes the many divergent integrals finite and the theory well defined. Once the whole thing works with the cut-off, the Λ is made to go to infinity in some special way so that it retains the regular behaviour of important physical quantities and the theory functions without a cut-off all the way up to infinite energy.

While the renormalization group is the historically accepted and predictively successful point of view when it comes to calculations, there were other approaches at the time that seem to be good candidates as well. One such approach to solving this divergent integral problem was to do QFT on a lattice, which yielded finite results but broke Lorentz invariance - a fundamental principle of physics. This led naturally to attempts to quantize the spacetime itself, imposing a minimal length scale and therefore a maximum momentum scale (that is to say energy scale $[E \sim p{\cdot}c]$ for high-energy, nearly free particles), naturally introducing a cut-off volume / energy scale beyond which the theory stops making sense - therefore automatically dealing with the UV divergence problems arising in QFT, all while preserving Lorentz invariance. If the noncommutativity is "small" in an appropriate sense (simply put: that $\|\theta^{\mu\nu}\|$ in equation (1) is small compared to length scales that we are concerned with) the theory could reproduce all the results of the standard model at the energies ($\sim$ length scales) we make measurements at today as the noncommutative behaviour would be "too small" to see. Furthermore it should in principle give predictions for extremely high-energy behaviour where the noncommutativity does comes into play but, as it turns out, such high energy scales are well out of reach for any

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experiments to be done in the near (and far) future. The problem with this approach is however that it mixes IR and UV divergencies and creates problems of it's own, which haven't generally been solved (except in certain lower-dimensional models) [1],[2].

Modern theories of gravity tend to embraces the QFT divergencies, interpreting the high-energy behaviour of QFT as a failure of an incomplete theory that needs to be replaced by an appropriate generalization, taking into account gravitational effects. A simple line of reasoning to this effect is given in the following section.

1. *The Doplicher, Fredenhagen, Roberts argument*

The argument [3-4] goes as follows: Imagine we're interested in looking at some very small regions of spacetime. To "look" at something we need to scatter particles of it that are sensitive to the dimension of the object we're looking at. We therefore need the de Broglie wavelength of the scattering particles to be smaller than or equal to the dimension of the region of interest. It is then clear that if we're interested in a region of dimension L, we have the condition on the de Broglie wavelength given as:

$$\lambda_{dB} = \frac{h}{p} \leq L$$

Using the relativistic equation for a free particle:

$$E = \sqrt{m^2c^4 + p^2c^2} \approx pc$$

where $\approx$ holds when the energy is much larger than the mass of the particle, an assumption that will be justified very shortly. We therefore have the condition on the energy of the particle:

$$E \geq \frac{hc}{L}$$

Now we see that while the numerator is a fairly small number in SI units, $\sim 200MeVfm$, the energy condition can be made as large as need be, if we're interested in small enough regions of spacetime L. No problem in sight yet, but now we employ a ubiquitous feature of any theory of gravity - energy (density) bends spacetime and when it surpasses a certain threshold, a black hole forms, preventing any information from escaping. Combining this with the energy condition from earlier, we arrive at an impasse: as we make the dimension of spacetime we want to investigate smaller, we require larger and larger energies to be able to resolve it. On the other hand, as we shoot higher and higher energy particles at our region of interest, at some critical point, call it L_c, the energy (density) of the probe becomes so high that it creates a black hole and prevents any information about the region of interest from reaching us.

Thus using no assumptions whatsoever and only a few very elementary considerations from quantum mechanics and general relativity, we've arrived at the following conclusion: It is an intrinsic feature of our Universe that there are length scales smaller than some L_c which remain inaccessible to any kind of probe and can therefore be regarded to "not exist".

There is a minimal physical length scale L_c of our spacetime. Any region smaller than L_c^4 will remain inaccessible to any kind of experiment trying to observe what happens inside of it.

This concept can be compactly expressed through equation (1). Spacetime coordinates do not commute, leading to an uncertainty relation which states that measuring distance for scales smaller than some $\|\theta^{\mu\nu}\|$ (roughly speaking) doesn't make much sense.

C. Non-Commutative Field Theory and the Star Product

One way to introduce noncommutativity of coordinates, or in a sense a "minimal length scale", is through the use of a special product between fields (in essence between functions) appearing in the Lagrangian. This kind of approach[1] consists of writing the Lagrangian (for example for a ϕ^4 Klein-Gordon theory) as:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \tag{3}$$

where the $\star$ product between functions f and g is defined as an associative, noncommutative operator of the form:

$$f \star g = fg + \sum_1^\infty \theta^n C_n(f,g) \tag{4}$$

The fg in this equation denotes the ordinary pointwise product between functions, C_n is an order n (noncommutative) differential operator and θ is the module of the tensor from (1) - essentially determining the scale of noncommutativity. Obviously, as "noncommutativity goes to 0", the theory reduces to the usual commutative Lagrangian. These higher order differential terms add additional structure to the theory (namely non-locality), which can be controlled by varying the noncommutativity parameter. Namely, if $(\theta \to 0)$ this implies $(f \star g = fg))$, or in words that the theory reduces to the usual commutative product theory.

D. Effective models - Landau's 2D model

A practical example of noncommuting coordinates, in the form (1), appears when considering the 2-dimensional Landau problem. The problem consists of a single free particle of unit charge, constrained to move in a 2-dimensional plane (x_1, x_2) with a constant magnetic field B imposed in the direction perpendicular to the plane. The Lagrangian of this theory is:

$$\mathcal{L}_m = \frac{m}{2}\dot{\vec{x}}^2 - \dot{\vec{x}} \cdot \vec{A(x)} \tag{5}$$

The canonical momentum comes out to be:

$$\vec{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} - \vec{A(x)} = \vec{p} - \vec{A(x)} \tag{6}$$

which gives the Hamiltonian:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}}\dot{\vec{x}} - \mathcal{L} = \frac{1}{2m}(\vec{\pi} + \vec{A})^2 \tag{7}$$

$\vec{A}$ is given so as to reproduce the magnetic field B in the z direction: $A_i = -\frac{B}{2}\epsilon_{ij}x^j$. Notice that A depends on the position coordinates, and therefore doesn't commute with the position operator. This choice, along with the canonical commutation relation $[x_i, \pi_j] = i\delta_{ij}$, gives that the canonical momenta $\pi_i = p_i - A_j$ do not commute with each other:

$$[\pi_i, \pi_j] = [p_i - A_i, p_j - A_j] \tag{8}$$

$$= [p_i, -\frac{B}{2}\epsilon_{jk}x^k] - [-\frac{B}{2}\epsilon_{ik}x^k, p_j] \tag{9}$$

$$= -\frac{B}{2}i(\epsilon_{ij} + \epsilon_{ij}) = -iB\epsilon_{ij} \tag{10}$$

The momentum space is thus quantized into cells of volume B, as per the relation:

$$\Delta\pi_i\Delta\pi_j \geq \|B\|$$

Another interesting situation comes about if we now consider the limit where the interaction part $\dot{x}^i(-\frac{B}{2}\epsilon_{ij}x^j)$ of (5) dominates over the kinetic part $\frac{m}{2}\dot{x}^2$, we get the condition: $B >> m$, which mathematically reduces to $m \to 0$, or $B \to \infty$.

The canonical momentum is now:

$$\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \frac{\partial(-\frac{B}{2}\dot{x}^i\epsilon_{ij}x^j)}{\partial \dot{x}^k} = -\frac{B}{2}\epsilon_{kj}x^k \sim x^k \tag{11}$$

The upshot is that if we now impose the canonical commutation relation:

$$[x_i, \pi_j] = [x_i, -\frac{B}{2}\epsilon_{jk}x^k] = -\frac{B}{2}\epsilon_{jk}[x_i, x_j] = i\delta_{ij}$$

$$[x_i, x_j] = -i\epsilon^{ij}\frac{1}{B}$$

where we use: $\epsilon_{ij}\epsilon^{ij} = \epsilon_{01}\epsilon^{01} + \epsilon_{10}\epsilon^{10} = 2$ up to an overall sign due to convention. We've arrived at a noncommutativity of position coordinates, simply from the limiting behaviour of a model in standard quantum physics. From this follows the uncertainty relation:

$$\Delta x_i\Delta x_j \geq \frac{1}{B}$$

which effectively quantizes the volume of physical space [1].

This is an example of a noncommutative theory appearing as a specific limit (effective theory) of a commutative theory, a direct analogy to what happens in string theory [1].

II. Mathematical Background

So far I've mentioned a very simple effective model and overviewed some places where noncommutative geometry shows up in physics, without providing any real mathematical detail. The purpose of the following section will be to explain the contemporary (and somewhat mathematical) approach to noncommutative geometry, originally developed by Connes in the 1980s [5].

A. Introduction

The spacetime - a (pseudo) Riemannian manifold - is the stage on which all of our experiments take place on, mathematically represented as the pair: a smooth manifold and a symmetric (0,2)-metric tensor: $(\mathcal{M}, g)$. It turns out that we never actually directly measure the manifold we live on or the metric tensor it is equipped with. What we do measure are smooth functions on the manifold and following this it seems to be assumed, without further justification, that we can reconstruct information about the underlying spacetime. If the functions are really what's key to the spacetime structure, then the algebra of said functions must carry all the information contained in the spacetime - i.e. the topological, differential and geometrical structure of the Riemannian manifold.

It turns out that there exists a correspondence between the space of functions on a manifold $\mathcal{M}$ and some abstract algebra A, which when equipped with some additional structure - **the spectral triple** (A, H, D) - can carry the same information as the spacetime and all the calculations can be done in an analogous way.

As the functions on manifolds are commutative, they always correspond to commutative abstract algebras A in the spectral triple; but surely A can in general be noncommutative! Taking A to be noncommutative and doing spectral triple calculations must, by following the above logic in reverse, correspond to some manifold $(\mathcal{M}_{\text{nonc}}, g_{\text{nonc}})$ on which the algebra of functions themselves is noncommutative - and thus on which coordinates do not commute. It should be noted that this doesn't actually give us a spacetime in the standard sense - it's completely unclear what $(\mathcal{M}_{\text{nonc}}, g_{\text{nonc}})$ is from a differential geometry standpoint, but it doesn't really matter, since what we **do** know is the spectral triple side of things, and that's enough to speak meaningfully about noncommutative spaces and to perform calculations on them as we wish.

The process of justifying these claims will roughly go as follows:

- To start, we only concern ourselves with the algebraic structure of functions on a manifold (a topological space in fact) and state the Gelfand-Naimark theorems, which show that this algebra can be related to the representations of some abstract algebra on a Hilbert space H. At this point, we have the first two ingredients of the spectral triple - the (A, H).
- The theorems only work for topological spaces and we want them to work for full blown spacetimes, so we're pressed to introduce differential structure on the topological space and try to extend it somehow to the algebraic side of things in the "correct" way

so the correspondence survives. Doing so, we introduce the differential calculi and space of derivations of an abstract algebra and relate it to the Dirac operator D. At this point, we have the entire (A, H, D).

- The spectrum of D can be used to reconstruct the manifold structure it corresponds to (roughly speaking, up to some technical equivalence)

- It can be shown that D - the Dirac operator - is capable of reproducing the notion of "distance" between two "points" and thus is equivalent also to the metric on a Riemannian manifold

To be able to talk about these things we need to introduce the obligatory terminology and a few basic results. I will spend a little more time on abstract algebra and a little bit less on differential geometry since the latter is usually more familiar to physicists.

B. Mathematical Preliminaries: Topology, Geometry & Abstract Algebra

I'll skip the long and arduous construction of Riemannian manifolds, the details of which can be found in most of the introductory mathematical literature on the subject [7]. The important bits of information are the following: a manifold is a locally Euclidean topological space, which when equipped with a positive definite metric g (among other things) becomes a Riemannian manifold. A result then follows, which will become important much later:

Definition 1. *We say two Riemannian manifolds (M, g) and (N, h) are **isometric** if there exists a map f (isometry) between these manifolds $f : (M, g) \to (N, h)$ such that the pullback $f^*h = g$ holds. This comes down to the condition that said map preserves distance between the manifolds.*

Now I shall move on to the **algebraic definitions** underlying the theory. While I obviously can't overview every piece of algebraic technology necessary to understand the article, I will try to provide at least the details unfamiliar to most readers with a physics background.

Definition 2. *A Ring $(\mathbf{R}, +, \cdot)$ is a set equipped with two operations, which satisfy the following conditions:*

$+ : \mathbf{R} \cdot \mathbf{R} \to \mathbf{R}$	$\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$
Commutativity:	**Commutativity:*
$\forall x, y \in R$	$\forall x, y \in R$
$x + y = y + x$	$x \cdot y = y \cdot x$
Associativity:	*Associativity:*
$\forall x, y, z \in R$	$\forall x, y, z \in R$
$x + (y + z) = (x + y) + z$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Neutral element:	**Neutral element:*
$\exists 0 \in R : \forall x \in R$	$\exists 1 \in R : \forall x \in R$
$x + 0 = x$	$x \cdot 1 = x$
Inverse element:	**Inverse element:*
$\forall x \in R, \exists(-x) \in R$	$\forall x \in R \backslash \{0\}, \exists(1/x) \in R$
$x + (-x) = 0$	$x \cdot (1/x) = 1$

**Ring multiplication need not generally satisfy the starred conditions. If it does, we call the ring a commutative (commutativity), unital (neutral element) and division ring (inverse element). A field is a special case of a ring, all three at once: a commutative, unital, division ring.*

Definition 3. *A module $(\mathbf{V}, \oplus, \odot)$ over a ring $(\mathbf{R}, +, \cdot$ is a set equipped with two operations, which satisfy the following conditions:*

$\oplus : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$	$\odot : \mathbf{R} \times \mathbf{V} \to \mathbf{V}$
Commutativity:	*Associativity:*
$\forall x, y \in R$	$\forall r, s \in R, \forall v \in V$
$x \oplus y = y \oplus x$	$(r \cdot s) \odot v = r \odot (s \odot v)$
Associativity:	*Distributivity of vectors:*
$\forall x, y, z \in R$	$\forall r \in R, \forall v, w \in V$
$x \oplus (y \oplus z) = (x \oplus y) \oplus z$	$r \odot (v \oplus w) = r \odot v + r \odot w$
Neutral element:	*Distributivity of scalars:*
$\exists 0 \in R : \forall x \in R$	$\forall r, s \in R, \forall v \in V$
$x \oplus 0 = x$	$(r + s) \odot v = r \odot v \oplus s \odot v$
Inverse element:	*Scalar identity:*
$\forall x \in R, \exists(-x) \in R$	$\exists 1 \in R, \forall v \in V$
$x \oplus (-x) = 0$	$1 \odot v = v$

If the ring is in fact a field - that is a commutative, unital, division ring - then this is a definition of a vector space over the field R.

Note: usually $\oplus$ and $\odot$ are written as $+, \cdot$ and it is to be understood from context whether they are the ring or the module operations.

Following are some important properties of modules.

Definition 4. *A generating system S of a D-module V*

is a subset $S \subset V$ such that:

$$(\forall v \in V)\,(\exists\{v^1, ...v^n\} \in D)\,(\exists\{e_1, ..., e_n\} \subset S):\;(v = v^i e_i)$$

The cardinality of S need not be finite, but every v can be written as a finite linear combination of elements from S. A basis is a generating system that is additionally linearly independent:

$$(\forall s \in S)(s \notin span(S\backslash\{s\})$$

If a generating system is called finite if it is of finite cardinality.

Thus - modules are different from vector spaces only in terms of the difference in the ring vs field structure. The key difference comes from the possibility that the elements of the ring don't have a multiplicative inverse, i.e. it not being a division ring.

Theorem 1. *Let D be a division ring. Then a D-module V has a basis.*

Corollary 1.1. *Every vector space has a basis, since any field is a division ring.*

The corollary is obvious since every field is necessarily a division ring (and also a commutative and unital ring). Note that a module over a non-division ring might still have a basis, but it isn't guaranteed as in the case of a division ring.

To see that this is nothing all that abstract, consider the familiar notion of smooth functions $C^\infty(\mathcal{M})$; well behaved in most ways with respect to addition and multiplication, but alas there is a problem - a generic function doesn't have a multiplicative inverse.

Example 1. *Let $\mathcal{M}$ be a smooth differentiable manifold, $C^\infty(\mathcal{M})$ be the space of smooth functions.*

Take for example the smooth function $f \in C^\infty(\mathcal{M})$ and say it vanishes at a point $p \in \mathcal{M}$:

$$f(p) = 0$$

(a completely valid property of a smooth function).

A multiplicative inverse f^{-1} is by definition:

$$\forall x \in \mathcal{M} : f(x) \cdot f^{-1}(x) = 1$$

Evaluating this at point p, we conclude that $f^{-1}(p)$ needs to be divergent, or in other words not a smooth function.

We see that smooth functions $C^\infty(M)$ do not form a division ring and thus, referring to the earlier theorem, any C^∞-module is not guaranteed to have a basis.

Example 2. *Let $\mathcal{M}$ be a smooth manifold, $C^\infty(\mathcal{M})$ be the smooth functions. The vector fields over $\mathcal{M}$, $\Gamma(TM)$, do not constitute a vector space, but are rather a C^∞-module.*

An example of such a C^∞-module not having a basis is the "You can't comb a sphere" theorem [6], which states that it is impossible to construct a nowhere vanishing vector field on a sphere; much less two nowhere vanishing vector fields that are linearly independent, which would thus form a basis.

Definition 5. *A module over a ring is called **free** if it has a basis.*

This doesn't necessarily mean that the ring has to be a division ring. There exist non-division ring modules which do have a basis (e.g. the vector fields on $\mathbb{R}^2$ do have a basis over the ring $C^\infty(\mathbb{R}^2)$; constant fields $\hat{x}$ and $\hat{y}$ form such a basis).

Remark 1. *If a finitely generated D-module F is free, then it is isomorphic to the direct sum of N copies of the ring D:*

$$F \cong \overbrace{D \oplus ... \oplus D}^{N\ times}$$

where N is the cardinality of the generating system.

This is in direct analogy with any finite-dimensional vector spaces being isomorphic to R^n.

Definition 6. *An R-module P is called **projective** if:*

$$(\exists\ R\text{-}module\ Q),\ (\exists\ free\ R\text{-}module\ F) : P \oplus Q = F$$

holds.

Remark 2. *A module being free implies it is projective, as Q from the definition can be picked to be the empty module $\{\emptyset\}$.*

Theorem 2. *(Serre, Swan, and others) The set of all smooth sections of a vector fibre bundle ($E \xrightarrow{\pi} M$) on a smooth manifold $\mathcal{M}$ is a finitely generated projective $C^\infty(\mathcal{M})$-module $P(E)$.*

The slew of definitions continues:

Definition 7. *An associative R-algebra A over a ring R is an R-module $(A, +, \cdot)$ with an additional multiplication operation: $\times : A \times A \to A$ such that $(A, +, \times)$ also satisfies the (unital) ring axioms.*

From this we see that an associative algebra is also a ring, giving meaning to a module structure over an "associative algebra" ring (plus some additional structure).

Definition 8. *A normed ring R is a ring with map $|.| : R \to \mathbb{R}$ such that: $(\forall r, w \in R) : |rs| \leq |r||s|$ and $|1| = 1$*

Definition 9. *A Banach algebra B is an associative R-algebra $(B, +, \cdot, \times)$ over a normed ring, equipped with a norm $\|.\| : B \to \mathbb{R}$ that satisfies the properties:*

- $(\forall x \in B\backslash\{0\}) : \|x\| > 0 \quad (\|x\| = 0) \iff (x = 0)$
- $(\forall x \in B), (\forall r \in R) : \|rx\| = \|r\|\|x\|$
- $(\forall x, y \in B) : \|x + y\| \leq \|x\| + \|y\|$

thus $(B, +, \cdot)$ a normed vector space; and additionally:

- $(\forall x, y \in B) : \|xy\| \leq \|x\|\|y\|$

completing the Banach algebra structure. It is also required that B be complete in the norm, meaning that every Cauchy sequence has a limit in B.

Remark 3. *A vector space with norm and completeness in said norm is what is usually referred to as a "Banach space". A Banach algebra is then the extension of this requirement that said vector space also behave like a ring when equipped with a vector multiplication operation.*

Definition 10. *A Hilbert space is a vector space* $(H, +, \cdot)$ *over a field F, equipped with an inner product map:* $\langle ., . \rangle : H \times H \to \mathbb{R}$ *which satisfies:*

- $(\forall x, y \in H)$:
  $\langle y, x \rangle = \overline{\langle x, y \rangle}$
- $(\forall x, y, z \in H)(\forall a, b \in F)$:
  $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
- $(\forall x \in H)$:
  $\langle x, x \rangle \geq 0$

The Hilbert space inner product automatically induces a norm on H, making it also a Banach space.

A ring can be equipped with further structure that we will need. The definition follows:

Definition 11. *A *-ring R is a ring with a map*

$$(^*: R \times R \to R)$$

that is involutive (it's own inverse) and an antiisomorphism.
Equivalently, the conditions on * *are as follows:*
$(\forall x, y \in R)$:

- $(x + y)^* = x^* + y^*$
- $(xy)^* = y^* x^*$ *(antiisomorphism)*
- $1^* = 1$
- $(x^*)^* = x$ *(involutivity)*

Definition 12. *A *-algebra A is a *-ring with invluotion* $*$ *that is an associative algebra over a *-ring R with involution ', such that:*
$(\forall r \in R)(\forall x \in A) : (rx)^* = r' x^*$

A familiar example of a *-algebra structure is for example the algebra of complex functions over a complex *-field, with complex conjugation as both of the involutions.

Definition 13. *A C*-algebra is a *-algebra that is also a Banach algebra; i.e. has a norm and is complete.*

Remark 4. *A norm of a linear operator on a Hilbert space H is given by:*

$$\|T\| = \sup_{v \in \mathcal{H}, \|v\|=1} \|Tv\|$$

i.e. by it's "largest action on a unit vector in H".
It then turns out that the algebra of all bounded operators on H, call it B(H), is a C-algebra and so is any of it's norm closed subalgebras (that is: restrictions of the algebra to norm* $\leq$ *some number).*

A Hilbert space is chosen to provide the space with an appropriate norm.

Definition 14. *A bounded linear map* $\pi : A \to B$, *between* C^**-algebras A and B is called a *-homomorphism if:*

- $(\forall x, y \in A) : \pi(xy) = \pi(x)\pi(y)$
- $(\forall x \in A) : \pi(x^*) = \pi(x)^*$

*Note: A bijective *-homomorphism is called a *-isomorphism.*

Remark 5. *If X is a (locally) compact Hausdorff space and C(X) is the algebra of continuous complex functions on X, then C(X) is a commutative (non) unital* C^**-algebra over the field of complex numbers.*

Definition 15. *A representation of a finite-dimensional *-algebra A is the pair* (H, π), *where H is (finite-dimensional, complex) inner product space and* π *is a *-algebra map:*

$$\pi : A \to L(H)$$

where L(H) is the space of linear (finite-dimensional, complex) operators acting on H: $L : H \to H$ *such that they preserve the vector space structure.*

Remark 6. *A representation* (H, π) *is called irreducible if there is no subspace* $W \in H$, *other than* $\emptyset$ *and H, such that it is invariant under the action of the entire image of L(H).*

$$(\nexists W \in H)(\forall a \in A) : \pi(a)W \in W$$

as then the choice $(W, \pi|_W)$ *would equally validly represent the algebra.*

Definition 16. *Two representations* (H_1, π_1) *and* (H_2, π_2) *of an algebra A are said to be unitarily equivalent if:*

$$(\exists U : H_1 \to H_2)(\forall a \in A) : \pi_1(a) = U^* \pi_2(a) U$$

Definition 17. *A* ***structure space*** $\widehat{A}$ *of the* C^**-algebra A is the set of all unitary equivalence classes of irreducible representations of A.*

C. Algebrizing the Geometry

The goal is now to build up the different levels of manifold structure:

- The topological - points, open sets, etc.
- The differential - vector fields and differential forms
- And finally the geometrical - the notion of distance on a manifold given by the metric

in algebraic terms.

1. The Gelfand-Naimark theorems

Having introduced a plethora of definitions, we're now terminologically equipped to talk about the first few results; namely the Gelfand-Naimark theorems, which work towards the first level - the topological, giving the prototype of a connection between points on a manifold and states on an abstract algebra.

Remark 7. *For a topological, (locally) compact, Hausdorff space X; the continuous complex-valued functions $C(X)$ form a commutative (non) unital C^*-algebra.*

This is the natural C^*-algebra on the manifold that we're hoping we can relate to an abstract algebra, as is done by the following two theorems [8][9].:

Theorem 3. ***(Gelfand-Naimark-Segal; The Gelfand duality)*** *Every abstract C^*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a concrete C^*-algebra of operators on a Hilbert space H. If the algebra $\mathcal{A}$ is separable then we can take H to be separable.*

Given an abstract algebra A, the theorem guarantees that we can find a Hilbert space H on which there exists a $*-$structure preserving representation of A.

Theorem 4. ***(Gelfand-Naimark)*** *If a C^*-algebra is commutative then it is an algebra of continuous functions on some (locally compact, Hausdorff) topological space.*

An alternative reading of the second theorem is: given a *commutative* algebra A, represented on a specific Hilbert space H, by means of the first theorem, we are guaranteed that this exactly corresponds to the algebra of functions on some topological space, thus providing an equivalence between the pair (A, H) and a topological space M. If, however, the algebra is noncommutative (and those certainly exist), then it seems unclear what this corresponds to, but the appeal of the formalism that is to follow (namely the spectral triple and spectral action principle) is that it works equally well and can be used to represent the 'algebraic side' of a noncommutative space without having to develop any of the technology of differential geometry on geometric noncommutative spaces.

2. Points vs Algebras

The theorems (3) and (4) seem to relate topological spaces and algebras. This begs the question - what does the most basic building block of topological spaces, *the point* correspond to on the algebra side of things? While the formal mathematical construction is known [13], it would turn out to be too big of a digression to be worth introducing. Instead, I'll just cite the result:

Remark 8. *Let A be a commutative C^*-algebra, which by (3) and (4) corresponds to $C(X)$ for some topological space X. Then the following are equivalent:*

- *$x \in X$ is a point.*
- *$\chi_x : A \to \mathbb{C}, \chi_x(f) = f(x)$ is a character of the algebra A.*
- *$\chi_x : A \to \mathbb{C}, \chi_x(f) = f(x)$ is a pure state on A.*
- *$\chi_x : A \to \mathbb{C}, \chi_x(f) = f(x)$ is an irreducible representation of A.*
- *$I_x \subset A, I_x = \{f \in A : f(x) = 0\}$ is a maximal ideal.*

Additionally we need the informally stated theorem:

Theorem 5. *Every irreducible representation of a C^*-algebra is unitarily equivalent to the point-evaluated representation: $\chi_x(f) = f(x)$.*

The important point here is that given an algebra A and classifying, for example, all the irreducible representations (up to unitary equivalence), i.e. the structure space of A, directly corresponds to talking about the points on the manifold X.

This can be expanded on further, providing the algebraic equivalents of all the different things one can define on a topological space; for example: homeomorphisms $\cong$ automorphisms, open subsets $\cong$ ideals of A, Cartesian products $\cong$ tensor products, etc., but none of this detail will be relevant for further developments so I won't spend much time on it.

This entire thing relies on the algebra being commutative. If A is noncommutative the equivalences from the remark above don't hold and while we can indeed still find characters, pure states and irreducible representations of noncommutative algebras, they won't correspond to a notion of a point on some "noncommutative topological space".

A result of this topological equivalence of algebra representations and topological manifolds, which will come up later follows:

Remark 9. *A homeomorphism (and this extends well to diffeomorphism once we introduce differentiable structure) of a topological manifold corresponds to an automorphism of the corresponding algebra A.*

Again, none of these results follow from the text up until here, but the point is that *all of this exists* - it is formally true that all the information in a topological space is somehow encoded in the algebraic structure, as provided by (3) and (4), etc..

3. Differential Calculus

Having A, H, we've dealt with the topological equivalence. Now we want to extend this further to a differentiable and ultimately geometric equivalence between manifolds and abstract algebras. For this we have to acquire a notion of differential calculus and ultimately "distance", or in other words, to promote the "locally compact Hausdorff topological space" in the above theorems to a full-fledged differentiable manifold with metric.

The idea ([8] [10] [11]) is to formulate the definitions of vector fields and forms from differential geometry in algebraic terms. This should then allow us to construct analogous definitions for the abstract algebraic sector.

Starting from the idea of vector fields on manifolds:

Definition 18. *A vector field is a left C^∞-module and a derivation $d : C^\infty \to C^\infty$ on the associative algebra (over $\mathbb{R}$ of C^∞ functions.*

Using the already established analogy: C^∞ function algebra $\sim A$ (in fact the representation of A on H, but whatever) we have the following definition:

Definition 19. *The vector space of derivations of A is defined by the set of K-linear maps:*

$$DerA = \left\{\xi : A \xrightarrow{\sim} \epsilon \mid \xi(a \cdot b) = \xi(a) \cdot b + a \cdot \xi(b)\right\} \ (a, b \in A)$$

where $\xrightarrow{\sim}$ denotes K-linearity in the underlying field and ϵ is an A-bimodule.

The space of derivations is also a Lie algebra with commutator defined by composition of K-linear maps, which will be relevant for the definition of the graded differential algebra, later.

The dual construction then proceeds exactly as in differential geometry:

Definition 20. *The R-vector space of 1-forms on the algebra are then defined as the space of A-linear maps $Lin_A(DerA, A)$ over.*

The graded algebra of forms is then constructed by:

$$\Lambda(DerA, A) := \oplus_p \Lambda^p(DerA, A)$$

where Λ^p denotes p copies of $Lin_A(DerA, A)$ Cartesian producted together, and multilinear and skew-symmetric in A.

Definition 21. *The exterior derivative d defined by:*

$$d : \Lambda^p(DerA, A) \to \Lambda^{p+1}(DerA, A)$$

$$d\omega(X_1, ..., X_n) = \sum_k (-)^{k+1} X_k \omega(X_1, ..., \not{X}_k, ..., X_{p+1}) + \sum_{k<l} (-)^{k+l} \omega([X_k, X_l], X_1, ...\not{X}_k, ...\not{X}_l, ...X_p)$$

for $X \in DerA$. The exterior derivative is a graded derivation of degree +1 if we make the identification

Let's start of with a remark on modules:

Remark 10. *A **left** A-module L of an associative algebra A, whose operations $(A, +, \times)$ satisfy the ring axioms, is defined the same as for a ring, but with ring multiplication only defined when acting from the left.*

$$(\forall a, b \in A)(\forall l \in L) : a \cdot (b \cdot l) = (a \cdot b) \cdot l$$

A right A-module is defined analogously as:

$$(\forall a, b \in A)(\forall r \in R) : (r \cdot a) \cdot b = r \cdot (a \cdot b)$$

A bimodule over two algebras A, B - call it K - is then defined as having multiplication from both sides by the algebra and the algebra being associative:

$$(\forall a \in A)(\forall b \in B)(\forall k \in K) : (a \cdot k) \cdot b = a \cdot (k \cdot b)$$

This technology was introduced so we could talk, algebraically about forms on a manifold:

Remark 11. *Differential one-forms $\Omega^1(M)$ over a manifold M are a bimodule over the associative algebra of smooth functions $C^\infty(M)$. The external derivative d: $C^\infty(M) \to \Omega^1(M)$ is a bimodule-valued derivation on the algebra $C^\infty(M)$:*

$$d(fg) = (df) \cdot g + f \cdot (dg), \forall f, g \in C^\infty(M)$$

Having stated what forms on a manifold are in algebraic language, we can **define** an analogue on an abstract algebra:

Definition 22. *Let A be an algebra. The first-order differential calculus over A is the pair $(\Omega^1(A), d)$ where $\Omega^1(A)$ is a bimodule over A and d is an $\Omega^1(A)$-valued derivation of A:*

$$d(ab) = (da) \cdot b + a \cdot (db), \forall a, b \in A$$

While this seems like a valid definition, the space of first-order differential calculi over A is hardly unique, while the one-forms are uniquely determined on manifolds as the duals of vectors. Due to this, we might ask what happens to vector fields - that is to objects acting as derivations on the continuous functions on a manifold. We proceed through thus developed analogy:

Example 3. *The continuous functions correspond to elements of the algebra, which can be represented on a Hilbert space H by the map π.*

We choose as the algebra A the full graded differential algebra over $C^\infty(M)$ and as the Hilbert space H the (left C^∞-)module of one-forms. Taking $f \in \Omega^0$, i.e. a C^∞ function and $\psi \in \Omega^1$, i.e. a one-form, we have by virtue of d satisfying the Leibniz rule:

$$(df)\psi = d(f \cdot \psi) - f \cdot (d\psi) = [d, f]\psi$$

Thus constructed, $[d, f]$ is an operator on the space of one-forms ψ which satisfies the Leibniz rule and is linear

Generalizing this, we have to figure out what to replace the operator d with when considering general noncommutative algebras and thus noncommutative spaces. This will in fact appear later in the form of the Dirac operator.

4. Spin Manifolds

When talking about a Riemannian manifold with metric (M, g), it can be useful to lift the structure to a spin manifold and work with that instead. This is in a way analogous to the interplay between the Klein-Gordon equation and the Dirac equation; where KG appears as the "square" of the Dirac, but Dirac has additional structure - the spin structure.

Having motivated spin-structures, let's get to the definitions and results, following the developments of [12]:

Definition 23. *Let V be a vector space over a field $\mathbb{F}$, equipped with a quadratic form $Q : V \to \mathbb{F}$, such that:*

$$Q(\lambda v) = \lambda^2 Q(v); (\lambda \in \mathbb{F}, v \in V)$$
$$Q(v + w) + Q(v - w) = Q(v) + Q(w); (v, w \in V)$$

Such a Q on V gives rise to a Clifford (associative, unital) algebra Cl(V,Q) subject to the relation

$$v^2 = Q(v)1$$

where 1 is the multiplicative identity of the algebra.

The Clifford algebra structure is in essence the promotion of a vector space to an associative algebra structure, *but* with the additional condition that the "squares" of the vectors are "normalized" in accordance with some quadratic map Q.

Remark 12. *As the form defining the Clifford algebra is quadratic, it doesn't distinguish between $v \in V$ and $-v \in V$. This implies there exists an algebra automorphism (isomorphism onto itself) and therefore a grading (decomposition given by the automorphism):*

$$\alpha : Cl(V, Q) \to Cl(V, Q)$$
$$\implies Cl(V, Q) = Cl^{[0]}(V, Q) \otimes Cl^{[1]}(V, Q)$$

where

$$Cl^{[l]}(V, Q) = \{x \in Cl(V, Q) | \alpha(x) = (-1)^l x\}$$

Note: It then follows that any product of k vectors (of negative parity, as all true vectors are of negative parity) in the algebra is even if k is even and conversely is odd if k is odd.

$$\alpha(x_1 \cdots x_k) = (-1)^k x_1 \cdots x_k$$

Remark 13. *It is easily seen that:*

$$vw + wv = 2g_Q(v, w)$$

provided we define

$$\begin{aligned} g_Q(v, w) &:= \frac{1}{2}(Q(v + w) - Q(v) - Q(w)) \\ &= \frac{1}{2}((v + w)^2 + v^2 + w^2) \\ &= \frac{1}{2}(vw + wv) \end{aligned} \tag{12}$$

which is the familiar notion of a Clifford algebra, as usually seen in particle physics with γ-matrices. One should note that a Clifford algebra may well be defined over a vector space with complex coefficients; then denoted: $\mathbb{C}l_n$.

The notation for the standard ("Euclidean") Clifford algebras with a particularly simple quadratic map $Q_n(x_1, ..., x_n) = x_1^2 + ... + x_n^2$ are the following:

$$Cl_n^+ := Cl(R^n, Q_n)$$
$$Cl_n^- := Cl(R^n, -Q_n)$$
$$\mathbb{C}l_n := Cl(\mathbb{C}^n, Q_n)$$

where we should note that on $\mathbb{C}$-vector spaces, an overall sign in the definition of the Clifford map Q doesn't matter since vector components can have complex coefficients and thus can change the signature of Q. Thus every signature on a complex vector space is equivalent to the all-pluses signature.

A complexificaton of a Clifford algebra $Cl(V, Q)$ is defined as:

$$\mathbb{C}l(V, Q) = Cl(V, Q) \otimes \mathbb{C}$$

Definition 24. *The chirality operator γ_{n+1} on $\mathbb{C}l_n$, equipped with a vector space basis $\{e_1, ..., e_n\}$, is defined as:*

$$\gamma_{n+1} = (-i)^m e_1 \cdots e_n$$

where n = 2m if n is even and n = 2m+1 if n is odd.

As we know, a manifold has a vector space - the tanget space - at each point, and we will want to define a Clifford algebra at each point in the manifold, take a disjoint union and thus get what is called a Clifford bundle! As Riemannian manifolds (M, g) come equipped with the familiar bilinear form g, the metric, we will be using it to define the required quadratic Clifford map from (12).

Definition 25. *A Riemannian metric on a manifold M is a symmetric bilinear form on vector fields $\Gamma(TM)$*

$$g : \Gamma(TM) \times \Gamma(TM) \to C(M)$$

such that

- *g(X,Y) is a real function if X and Y are real vector fields;*
- *g is $C(M)$-bilinear:*
  $$g(fX, Y) = g(X, fY) = fg(X, Y); \quad (f \in C(M))$$
- *$g(X, X) \geq 0$ for all real vector fields X and $g(X, X) = 0 \iff X = 0$.*

$C(M)$ is the set of continuous functions on the manifold M.

Intuitively, we can now construct what is called a "Clifford algebra bundle".

Definition 26. *The idea with bundles is as usual to construct a Clifford algebra at every point and then take the union.*

A metric g on M is usually represented as a (0,2)-form dependant on some local coordinates (chart) - $g(x)$. Given this, we can take the evaluation of the metric at a point p - $g(p)$ - as the symmetric bilinear form on the vector space at that point - T_pM.

We now have all the ingredients to define the Clifford algebra at p:

$$Cl(T_pM, Q_p)$$

as according to equation (12) we can reconstruct the necessary quadratic form map Q_p from having $g(p)$.

The disjoint union of all such Clifford algebras with transition maps inherited from TM is the definition of a Clifford bundle $Cl(TM)$.

Remark 14. *As the tangent space over a real* ***Riemannian*** *manifold is a real vector space isomorphic to R^n, the Clifford algebras at a point look like $Cl^+(R^n, Q)$ and thus the whole bundle deserved the name $Cl^+(TM)$.*

Note: The sections of a Clifford algebra bundle are usually denoted $\text{Cliff}(M)^+ = \Gamma(Cl^+(TM))$ and the complexified version as

$$\mathbb{C}\text{liff}(M) := \text{Cliff}^+(M) \otimes_{\mathbb{R}} \mathbb{C}$$

We are now ready to define what is meant by a *spin manifold*.

Definition 27. *A Riemannian manifold is called spin^c if there exists a vector bundle $S \to M$ such that there is an algebra bundle isomorphism (meaning it preserves both the bundle and the algebra structure):*

$$\mathbb{C}l(TM) \cong End(S) \quad (M \text{ even-dimensional})$$
$$\mathbb{C}l(TM)^0 \cong End(S) \quad (M \text{ odd-dimensional})$$

If such a pair (M, S) exists, we call S the spinor bundle, and the sections $\Gamma(S)$ spinors.

This definition comes down to providing a representation of the Clifford algebra $\mathbb{C}l(TM)$ on S and $S \to M$ being a vector bundle.
As this is an *algebra* morphism, a multiplication map on the vector fields needs to be established; of the form $\Gamma(S) \times \Gamma(S) \to \Gamma(S)$.

Definition 28. *Let (M, S) be a spin^c structure on M. Clifford multiplication is defined by the linear map:*

$$\begin{aligned} c : \Omega^1 \times \Gamma(S) &\to \Gamma(S) \\ (\omega, X) &\mapsto \omega^\# X \end{aligned} \tag{13}$$

where $\omega^\#$ is the vector field corresponding to the form ω as that vector field whose dual is induced by the metric: $\omega = g(\omega^\#, .)$.

Note that the vector field is defined through ω which takes a vector field and gives a real number, to be understood as a 1-form.

Definition 29. *A Riemannian spin^c manifold is called spin if there exists an antiunitary operator $J_M : \Gamma(S) \to \Gamma(S)$ such that:*

- *J_M commutes with the action of real-valued functions on $\Gamma(S)$*
- *J_M commutes with Cliff$^-$*

We call the pair (S, J_M) a spin structure on M and refer to the operator J_M as the charge conjugation.

Remark 15. *The representation theory of Clifford algebras and spinor bundles gives us the following results:*

- *If the manifold M is even dimensional, we can define the grading on the sections of the vector bundle:*
  $$(\gamma_M \psi)(x) = \gamma_{n+1}(\psi(x)); \quad (\psi = \Gamma(S)).$$
  where γ_{n+1} is defined above.
- *The dimension (modulo 8) of the manifold M determines the behaviour of J_M; specifically*
  $$J_M^2 = \epsilon, \quad J_M x = \epsilon' x J_M;$$
  $$J_M \gamma_M = \epsilon'' \gamma_M J_M$$
  such that $(x \in (\textit{Cliff}^-(M)))$ and $(\epsilon, \epsilon'$ and $\epsilon')' = \{-1, +1\}$ depend on dimension. [12]

5. The Spin Connection and Dirac Operator

Having a spin structure on the manifold now allows us to construct an operator which, when squared, gives the usual Laplace operator - in differential geometry terms equivalent to the metric g.

Intuitively, we're looking for the square root of the usual Laplace, or in other words some operator D which squares to give the Laplace:

$$D^2 \sim g_{\mu\nu} \partial^\mu \partial^\nu$$

once again, similarly to how Dirac "guessed" the generalization of the Klein-Gordon equation, but now in a more general setting [5] [12].

Definition 30. *A connection on a vector bundle $E \to M$ is given by a $\mathbb{C}$-linear map on the space of smooth sections:*

$$\nabla : \Gamma^\infty(E) \to \Omega^1_{dR}(M) \otimes_{C^\infty(M)} \Gamma^\infty(E)$$

satisfying the Leibniz rule:

$$\nabla(fX) = f\nabla(X) + df \otimes X; \quad (f \in C^\infty(M), X \in \Gamma^\infty(E))$$

Definition 31. *A connection is considered compatible with the metric if it holds that for a given inner product $< .,. >: \Gamma^\infty(E) \times \Gamma^\infty(E) \to C^\infty$ it holds:*

$$< \nabla\rho, \phi > + < \rho, \nabla\phi >= d < \rho, \phi >$$

for smooth sections $\rho, \phi \in \Gamma^\infty(E)$.

This is often written in more familiar form, where the inner product is given by the metric, as:

$$\nabla(g_{\mu\nu} V^\mu V^\nu) = g_{\mu\nu} \nabla(V^\mu) V^\nu + g_{\mu\nu} V^\mu \nabla(V^\nu)$$

usually read as: "the covariant derivative doesn't see / is compatible with the metric".

Remark 16. *It can be shown that there exists a unique connection on a Riemannian manifold (M, g) that is compatible with the metric g.*

Definition 32. *A spin connection ∇^S is defined on a spinor bundle $(S \to M)$ as the lift of the Levi-Civita connection.*

6. The Distance on Abstract Algebras and the Dirac Operator

At this point we'd like to give a notion of distance that works on algebraic structures and reproduces the manifold distance function between points.

Remark 17. *The distance function on M as induced by the metric g, is given by the infimum:*

$$d_g(x_i, x_f) = \inf_\gamma \left\{ \int_0^1 g(\dot\gamma(t), \dot\gamma(t))dt : \gamma(0) = x_i, \gamma(1) = x_f \right\} \tag{14}$$

Intuitively - "the shortest path on M, that connects the points x_i and x_f".

Now we get to defining a distance function on the algebra ; but for this we need some additional technology; the Dirac operator [12]:

Definition 33. *Let M be a spin manifold (M, S, J_M). **The Dirac operator** D_M is the composition of the spin connection on S with Clifford multiplication (13):*

$$D_M : \Gamma^\infty(S) \xrightarrow{\nabla^S} \Omega^1(M) \otimes_{C^\infty} \Gamma^\infty(S) \to \Gamma^\infty(S)$$

Theorem 6 (17)**.** *The Dirac operator D_M is self-adjoint on $L^2(S)$ with compact resolvent $(i+D)^{-1}$, and a relation holds for the commutators with elements in $C^\infty(M)$:*

$$[D_M, f] = -ic(df)$$

so that $\|[D_M, f]\| = \|f\|_{Lip}$ is the Lipschitz (semi)-norm of f:

$$\|f\|_{Lip} = \sup_{x \neq y} \left\{ \frac{f(x) - f(y)}{d_g(x, y)} \right\}$$

Here we see the remnants of the old relation $[d, f]\psi$ functioning as a vector field, but this time with a self adjoint operator D_M.

In operator theory [14], a compact resolvent implies that the operator D_M has a discrete set of eigenvalues, subset of $\mathbb{C}$, while the fact that D_M is self-adjoint gives us an orthonormal basis $\{v_1, ...\}$ with eigenvalues $\{\lambda_1, ...\}$.

Theorem 7. *Given a C^∞ algebra (as taken from a Clifford bundle $S \to M$) and a Dirac operator D_M, we can take as the notion of a distance between two characters (i.e. points on a manifold) of said algebra the formula:*

$$d(x, y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \|[D_M, f\| \leq 1\}$$

where this notion turns out to be the same as the distance between the points x and y on the manifold, as given by (14).

This can be formally proven / derived in the theory of Kantorovich optimal transport [15][16].
To see that this indeed does work, consider the example [12]:

Example 4. *Take as the manifold $M = \mathbb{R}$ with the algebra of all functions $C^\infty : \mathbb{R} \to \mathbb{R}$. The Dirac operator is simply the derivative operator ∂_x (up to a complex factor, usually -i to make it self-adjoint). Now imagine we're looking for the distance between two points x and y on the real line.*

Taken as a manifold, there isn't much thinking; the obvious shortest path connecting x and y is the only path connecting x and y, the straight line between them, and the integral from (14) obviously gives $d_g(x, y) = |x - y|$.

The algebraic approach is a little less straight forward (pun not intended) but is still fairly simple to see if we translate it into standard-analysis terms. We consider all elements of the algebra (functions); specifically the difference of their characters $f(x)$ and $f(y)$ (i.e. we look at the difference in value of said algebra elements), such that their commutator

$$[D_M, f] = \partial_x(f(x)) + f(x)\partial_x - f(x)\partial_x = \partial_x(f(x))$$

be less than or equal to 1 (i.e. functions whose slope never becomes greater than 1). So, analytically speaking, we're looking for the function whose slope is always less than or equal to 1, which maximizes it's growth between x and y. After a little thinking (or drawing) it becomes pretty quickly apparent that the function maximizing it's growth is in fact the linear function $f(x) = x$; in which case the distance becomes

$$d(x, y) = |f(x) - f(y)| = |x - y|$$

which exactly reproduces the manifold distance.

Having seen how to effectively reproduce the metric, first by defining a Dirac operator D_M and then by formulating the distance function using said operator, we're effectively done. We know how to reproduce the manifold and all of it's differentiable structure and we additionally know even how to reproduce the Riemannian metric. Along the way we had a lot of things and it's now convenient to package all of this into one big structure - **the Spectral Triple**.

Definition 34. *A **spectral triple** (A, H, D) is given by a unital $*$-algebra A represented as bounded operators on a Hilbert space H and self-adjoint operator D in H such that the resolvent $(i + D)^{-1}$ is a compact operator and $[D, a]$ is bounded for each $a \in A$.*

- *A spectral triple is **even** if the Hilbert space H is endowed with a $\mathbb{Z}_2$ grading γ such that $\gamma a = a\gamma$ and $\gamma D = -D\gamma$.*
- *A real spectral triple (or a spectral triple with real structure) is a spectral triple equipped with the isometry $J : H \to H$ such that:*
  $$J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon''\gamma J$$
  where $\epsilon, \epsilon', \epsilon'' \in \{-1, 1\}$ depend on the dimension of the manifold n (modulo 8).
  It's also required that defining $b^0 = Jb^*J^{-1}$, we demand that:
  $$[a, b^0] = 0, \quad [[D, a], b^0] = 0; \quad (a, b \in A)$$

Given a Riemannian manifold, there's a direct way to assign (A, H, D) such that it reproduces all the information the manifold held.

Remark 18. *The **canonical triple** associated to a (locally) compact Riemannian spin manifold:*

- *$A = C^\infty(M)$, the algebra of smooth functions on M;*
- *$H = L^2(S)$, the Hilbert space of square integrable sections of a spinor bundle $S \to M$;*
- *$D = D_M$, the Dirac operator associated to the Levi-Civita connection lifted to the spinor bundle.*

An application of a *canonical triple* can quickly be seen on the circle $\mathbb{S}^1$

Example 5. *We start by using the prescription of the canonical triple:*

- *Obviously we put $A = C^\infty(\mathbb{S}^1)$.*
- *Then by definition we also automatically have: $H = L^2(\mathbb{S}^1)$, where the fibers are simply one dimensional $\mathbb{C}$-lines.*
- *As the fibre is trivial and the total space is a cylinder (intrinsically flat), the connection is trivial and the Dirac operator becomes: $D_{\mathbb{S}^1} = -i\partial_\varphi$.*

Thus we have: $\left(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), -i\partial_\varphi\right)$ as the spectral triple.

One might ask if this process works in reverse? If we're given a Spectral Triple, can we *reconstruct* the manifold from just this information? This would make sense, all things considered, as it would be a direct generalization of Theorem 4. to fully differentiable / geometric spaces! There in fact exist reconstruction theorems which state that from the information of a real spectral triple; a structure which can be assigned only to special manifolds, one can reconstruct the Riemannian manifold information directly, one to one. This is a fairly recent result and simply stating it is somewhat involved, so we'll satisfy ourselves with the knowledge that it does exist [20].

Remark 19. *Interestingly, even if one doesn't provide the algebra A, just by providing (H, D, γ, J) one can look at the spectrum Σ of the operator D and from it reconstruct quite a lot of information about the Riemannian manifold.*

*An interesting result is that the set of isometric manifolds is a subset of **isospectral** manifolds (those with the same spectrum of their Dirac operators).*

This is an important result which will appear in the forthcoming construction of the action principle for Spectral Triples [17].

The final set of results that we'll need are those concerning almost-commutative spaces and specifically their gauge symmetries.

Definition 35. *Let M be a Riemannian spin manifold with canonical triple $(C^\infty(M), L^2(S), D_M; J_M, \gamma_M)$ and let $(A_F, H_F, D_F; J_F, \gamma_F)$ be a finite spectral triple (i.e. the algebra has finitely many characters $\cong$ the underlying space is 0 dimensional).*

*The **almost-commutative** manifold $M \times F$ is then defined by the real spectral triple:*

$$\begin{aligned} M \times F = (C^\infty(M, A_F), L^2(S \otimes (M \times H_F), \\ D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F) \end{aligned} \tag{15}$$

(AC Spectral Triple or ACST for short)

The intuition here is effectively: "function valued matrices" [5]; the finite space being a finite dimensional matrix space and the functions are the C^∞ ones on the manifold M.

Remark 20 (12)**.** *The full symmetry group - that is the group of automorphism $\alpha : A \to A$ on an almost-commutative manifold $M \times F$ - is given by:*

$$\mathcal{G} = G(M \times F) \rtimes \mathit{Diff}(M) \tag{16}$$

where $G(M \times F)$ is the "gauge group" of the Spectral Triple, defined as the inner unitary operator:

$$G(A, H; J) := \left\{ U = uJuJ^{-1} | u \in \mathcal{U}(A) \right\}$$

where $\mathcal{U}(A)$ is the set of all unitary elements of the $*$-algebra A. When "applied" to the spectral triple, it gives: $U(A, H, D; J, \gamma)U^* = (A, H, UDU^*; J, \gamma)$ meaning that U commutes with both J and γ. This induces the "inner fluctuations" (inner meaning expressed purely in terms of the algebra elements J and u as opposed to some additional "external" operator):

$$UDU^* = D + u[D, u^*] + \epsilon' Ju[D, u^*]J^{-1}$$

With this, we've arrived at the end of the mathematical background and it's time to move on to physics, specifically the reconstruction of the modern particle physics models (similar to the Standard Model).

III. Physics: The Spectral Action and the Particle Physics Models [17]

As the title suggests, the Spectral Action will be an alternative approach to constructing the action of a physical system [12] [17][18], such that it reproduces the results one would arrive at by integrating a Lagrangian over a manifold.

Lagrangians and therefore actions are (at least in physics) obtained by the principle of "guess the one that give the correct equations of motion" and I should start off immediately by saying that this situation will be no different; the correct spectral triple (A, H, D) needs to be guessed and then a generic formula is applied to construct the action from it. While this guessing process is by no means more trivial than in the Lagrangian approach, it turns out that if we do get a spectral action that reproduces particle physics models, the likes of the Standard model for example, it is also automatically coupled to Einstein and Weyl gravity as well as showing features of Grand Unification Theories such as a relationship between the coupling constants. Also, the Higgs mechanism is implemented into the action automatically, in a sense having become part of the noncommutative geometry!

A. The Standard Action

The laws of physics at low enough energies are completely contained in the **scalar** action:

$$S = S_{\text{Einstein}} + S_{\text{Particle Physics}} \tag{17}$$

consisting of the Einstein action for gravity

$$S_{\text{Einstein}} = \frac{1}{16\pi G}\int R\sqrt{g}d^4x$$

and the particle physics action :

$$S_{\text{Particle Physics}} = S_{\text{G}} + S_{\text{f}} + S_{\text{GH}} + S_{\text{H}} + S_{\text{Gf}} + S_{\text{Hf}}$$

containing all the kinetic and interaction terms for the gauge bosons G ($\gamma, W^{\pm}, Z$ and gluons) of spin 1, the Higgs field H and fermions f of spin 1/2 (quarks and leptons).

When constructiong actions, we tend to consider symmetries of the Lagrangian as the key defining feature of the underlying physics, since they imply conserved quantities and thus decide many of the physical laws that follow from said Lagrangian without any prior calculation. Considering this, it's useful to state the symmetry group of both the Einstein Lagrangian and the particle physics Lagrangian.
The total symmetry group is:

$$G = \mathcal{U} \rtimes \text{Diff}(M) \tag{18}$$

where $\mathcal{U} = U(1) \times SU(2) \times SU(3)$ is the symmetry group of the standard model; containing the U(1) charge of Electrodynamics, the SU(2) weak-isospin and the SU(3) color charge. Diff(M) is the diffeomorphism group of the manifold M; locally speaking the group of differentiable coordinate transformations, which is the symmetry group of general relativity.

After identifying the symmetry groups, we usually try to construct all the physical objects that are invariant under said symmetries and put them into the Lagrangian. Then we use the Euler-Lagrange equations and throw away those terms that produce unphysical results (for example: a photon mass term of the form $\frac{1}{2}m^2 A_\mu A^\mu$ isn't present in QED despite satisfying the required symmetries).

B. The Spectral Action

We now want to translate the above symmetries into the language of (real) Spectral Triples $(A, H, D; J, \gamma)$ and following that consider as the action all those quantities that can be made invariant to those symmetries (i.e. "scalar").

We immediately notice that the form of the symmetry group (18) corresponds exactly to the form one gets for an almost-commutative manifold from Remark 16 and thus the particle physics model will be formulated on such a structure.

It should be noted that the stage we're working with is the Hilbert space H, with the algebra representation on H and $D; J, \gamma$ all being operators on it. One needs to know what is meant by "scalar" in this context. Usually, "scalars" are complex-valued functions of algebra elements on the underlying manifold. In the simplest case, the algebra elements are themselves complex-valued and therefore one can just take algebra products and immediately get the necessary scalars to be integrated in the action. On the other hand, when dealing with spinors represented on some Hilbert space, one needs to take functions from the Hilbert space to the complex numbers, say the Hilbert space product $< .,. >: H \to \mathbb{C}$, and use those to make "scalars"; for example $< \psi, D\psi >$ would be a valid candidate.

Having defined what scalars mean, we're now ready to look for all maps from the Hilbert space to the complex numbers, such that they respect the symmetries of the curved particle physics model (18) translated to Spectral Triple language.

Starting with the gravity sector, since we want an action with diffeomorphism invariance we'd want to concern ourselves with the families of isometric manifolds. Translating this into the algebraic language of spectral triples [17], we'd obviously want to call upon the correspondence between the metric of (M, g) and the Dirac operator of (A, H, D). It was touched upon earlier (Remark 19) that isospectrality of the Dirac operator is the corresponding notion to isometry of the metric and as such we have the statement which will underlie the rest of the construction; we require from our action the following :

The physical action depends only on the spectrum Σ of the Dirac operator D.

in direct analogy to the requirement that the Einstein action be isometry invariant, and in fact as we've seen from Remark 19 it is a more general condition.

This all falls well into the almost-commutative Spectral Triple context - identifying the gravity sector with the M part of the ACST gives us the second part of (18).

The standard model symmetry group is obviously more phenomenologically motivated and doesn't follow from a principle as simple as "general diffeomorphism invariance" and thus it's implementation will have to be done much more so "by hand" compared to the elegant "action must depend exclusively on Σ". Considering almost-commutative Spectral Triples, we need to pick the finite space F such that it's gauge group corresponds to $U(1) \times SU(2) \times SU(3)$ as required by the particle physics model. This turns out to be [17-19] exactly the choice:

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{19}$$

where $\mathbb{H}$ are the quaternions, behaving similarly to SU(2), and M_3 is the space of 3x3 complex matrices.

As our "function space" A_F is finite dimensional - the space of 3x3 complex matrices (finite basis of dim 18) times the space of quaternions $\cong$ 2x2 complex unitary matrices (finite basis of dim 4) times a single complex number (finite basis of dim 1), accounting for all the particles (including colour, chirality and antiparticles). The basis elements of this finite space are usually labeled by the elementary particles they represent.

The rest of the spectral geometry then follows [17] from the definition of the almost-commutative Spectral Triple:

$$A = C^{\infty}(M) \otimes A_F$$
$$H = L^2(M,S) \otimes H_F, \quad D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$$
$$J = J_M \otimes J_F, \quad \gamma = \gamma_M \otimes \gamma_F$$

The concept of "internal fluctuations of a Dirac operator "; defined earlier as:

$$D = D_0 + K + JKJ^{-1} \quad K = \sum a_i[D_0, b_i]$$
$$a_i, b_i \in K; \quad K = K^*$$

then essentially corresponds to gauge transformations from particle physics. We're exploiting the freedom we have from requiring that our spectral triple be unitarily equivalent to transformations of it's gauge group.

Taking the almost-commutative approach, we've now prepared the Spectral Triple one should use to construct scalars which will constitute an action with the correct symmetry group. The particle physics model gauge freedom is encoded in the finite space while the gravity background is in the requirement that the action only depend on the spectrum of the total Dirac operator D.

What's left is to "guess" the correct scalar terms which give the action and here Connes provides the correct choice in his original paper [17]; the action is postulated to be:

$$S_{\text{Spectral}} = <\psi, D\psi> + \text{Tr}\left(\chi\left(\frac{D}{\lambda}\right)\right) \tag{20}$$

where λ is some cutoff scale and χ is a smooth cutoff function that goes to 0 as it's argument becomes larger than one. Usually, one also considers the square of the Dirac. A few remarks are in order:

- The pure Dirac term $<\psi, D\psi>$ makes use of the Hilbert space inner product to create a scalar, to be interpreted as the propagator term of all the particles, while the second term takes the trace (in the sense of the sum of eigenvalues) of an operator; again a reasonable way to create a scalar, to be interpreted as the interaction and background gravity terms.
- The χ function essentially allows us to consider general "operator-Laurent series" of the Dirac with the requirement that their behaviour go to 0 as the eigenvalues we consider become large.
- The proposed action obviously depends exclusively on the spectrum of the Dirac operator as it's the only operator on the Hilbert space that appears in the action.

C. Heat-Kernel

The final point is now to provide the expansion of the trace-part of the action [12] and show that it truly does reproduce all that was claimed earlier - gravity and all the couplings between elementary particles on a curved background. To do this, some non-trivial mathematics is required; namely the Heat-Kernel formalism.

I'll first introduce the results in a general setting and then rapid-fire apply them to the particle physics model. We'll also want to restrict to χ being positive definite meaning that it should be a function of the square of the Dirac operator D^2 as this has consequences on the gravity portion of the action expansion.

Firstly, we'll want to expand $\chi(P)$ as a series in integer powers of P, which is guaranteed by χ being a smooth (analytic) function. Next we use the linearity of the trace and for each term we apply the expansion:

$$\text{Tr}(P^{-s}) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \text{Tr} e^{-tP} dt$$

s.t. Re(s) ≥ 0;
where we again have the problematic term $\text{Tr}e^{-tP}$ which can luckily be expanded further (asymptotically, as $t \to 0$) as the titular *heat-kernel expansion*:

$$\text{Tr}e^{-tP} \simeq \sum_{n\geq 0} t^{\frac{n-m}{d}} \int_M a_n(x,P) dv(x)$$

where $m = dim(M)$, d is the order of P in D and $dv(x) = \sqrt{g}\, d^m x$ is the volume element. In the usual case, we have $m = 4$ and $d = 2$.

Combining these two equations, we get by complex integration:

$$\text{Tr}(P^{-s}) = \text{Res}\Gamma(s)|_{s=\frac{n-m}{d}} a_n$$

where a_n are the Seeley-de Witt coefficients, which are known and can be expressed using the curvature scalars of the underlying space, i.e. corresponding metric and it's derivatives.

Plugging this into the trace from the action, we get:

$$\text{Tr}\chi(P) \simeq \sum_{n\geq 0} f_n a_n(P) \tag{21}$$

where f_n are given by the moments of the function χ:

$$f_0 = \int_0^{\infty} \chi(u) u du$$
$$f_2 = \int_0^{\infty} \chi(u) du$$
$$f_{2(n+2)} = (-1)^n \chi^{(n)}(0)$$

s.t. $n \geq 0$, which are all just numerical integrals of some function.

The general Seeley-de Witt coefficients for a Dirac operator:

$$D = \begin{bmatrix} \gamma^{\mu}(D_\mu \otimes 1_N + A_\mu) & \gamma_5 S \\ \gamma_5 S & \gamma^{\mu}(D_\mu \otimes 1_N + A_\mu) \end{bmatrix}$$

are known in the literature [17] and are as follows:

$$a_0 = \frac{\lambda^4}{4\pi^2}\int \sqrt{g}d^4x\text{Tr}(1)$$

$$a_2 = \frac{\lambda^2}{4\pi^2}\int \sqrt{g}d^4x\left[\frac{R}{12}\text{Tr}(1) - 2\text{Tr}(S^2)\right]$$

$$a_4 = \frac{1}{4\pi^2}\int \sqrt{g}d^4x\Bigg[\frac{\text{Tr}(1)}{360}\bigg(3R^{\mu}_{;\mu} - \frac{9}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{4}R^*R^*\bigg) + \text{Tr}\bigg((D_\mu S + [A_\mu, S])^2 - \frac{R}{6}S^2\bigg) - \frac{1}{6}\text{Tr}F_{\mu\nu}F^{\mu\nu} + \text{Tr}S^4 - \frac{1}{3}\text{Tr}(S^2)^{\mu}_{;\mu}\Bigg]$$

where $R^*R^* \equiv \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}_{\rho\sigma}$. In short, as integrals of curvature and connection scalars. The traces over unity depend on the dimension of the space one works with; 1_N being the dimension of the finite dimensional space A_F in the particle physics model case.

D. Back To the Particle Physics Model

The correct action functional for the particle physics model is given by Connes [17] then as:

$$S_{\text{Spectral}} = <\psi, D\psi> + \text{Tr}\left(\chi\left(\frac{D^2}{\lambda^2}\right)\right)$$

What's left is to provide details on how $(H_F, D_F; J_F, \gamma_F)$ for the finite dimensional particle physics sector of the almost-commutative Spectral Triple $(M \times F)$ look. To start, recall that $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and a basis is chosen on the Hilbert space H_F such that the basis elements correspond to the elementary particles. The quarks Q and leptons L are denoted on H as:

$$\psi_Q = \begin{pmatrix} u_L \\ d_L \\ d_R \\ u_R \end{pmatrix} \quad \psi_L = \begin{pmatrix} \nu_L \\ e_L \\ e_R \end{pmatrix}$$

The Dirac and J are taken to satisfy the real spectral triple axioms and $\gamma_{N+1} = \gamma_5$.

Taking the general internal fluctuations of the Dirac in the context of the gauge group $U(1) \times SU(2) \times SU(3)$ imposed on the almost-commutative space then exactly reproduces connection terms which can be identified with the particle physics gauge bosons, but even more so, satisfying the ACST axioms makes room naturally for an additional scalar field H which wasn't manually included in any way. This corresponds to the Higgs with the correct couplings to the rest of the particles.

After some calculations, one arrives at the kinetic terms for quarks and leptons: $\langle\psi_Q, D_Q\psi_Q\rangle, \langle\psi_L, D_L\psi_L\rangle$, where D_q and D_l are equivalent to the particle physics model's covariant derivatives, including the interactions of the different gauge bosons and Higgs with the quarks and leptons. The explicit form of these operators is fairly complicated but in essence the diagonals contain the gauge boson terms of the form:

$$D_\mu \otimes 1_2 - \frac{i}{2}g_{02}A^{\alpha}_{\mu}\sigma^{\alpha} - \frac{i}{6}g_{01}B_\mu \otimes 1_2$$

where A^{α}_{μ} are the weak-isospin gauge bosons, σ are the SU(2) generators and B_μ is the QED photon boson. Off the diagonal, there are either zeroes or the 3x3 family mixing matrices k^d, k^u and k^e and Higgs terms. Obviously all of the ingredients of the particle physics model are here, following solely from the gauge freedom imposed on the Dirac by requiring that the finite space have the same gauge group as the Standard Model (and in fact the Higgs is reproduced for free). For a more complete account of the details of the calculation, I again refer to Connes' original paper [17].

E. The Bosonic Action

As we've seen, $<\psi, D\psi>$ takes care of the quark and lepton kinetic terms and what's left is to evaluate the Trace $\text{Tr}\left(\chi\left(\frac{D^2}{\lambda^2}\right)\right)$. Following through on the Heat kernel formalism for a general operator $P = \frac{D^2}{\lambda^2}$, we get:

$$S_{\text{Spectral}} = \frac{45\lambda^4}{4\pi^2}f_0\int d^4x\sqrt{g} + \frac{3\lambda^2}{4\pi^2}f_2\int d^4x\sqrt{g}\left[\frac{5}{4}R - 2y^2H^*H\right] + \frac{f_4}{4\pi^2}\int d^4x\sqrt{g}\Bigg[\frac{5}{160}\bigg(12R^{\mu}_{;\mu} + 11R^*R^* - 18C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\bigg) + 3y^2\bigg(D_\mu H^* D^\mu H - \frac{1}{6}RH^*H\bigg) + g^2_{03}G^i_{\mu\nu}G^{\mu\nu i} + g^2_{02}F^{\alpha}_{\mu\nu}F^{\mu\nu\alpha} + \frac{5}{3}g^2_{01}B_{\mu\nu}B^{\mu\nu} + 3z^2(H^*H)^2 - y^2(H^*H)^{\mu}_{;\mu}\Bigg] + O\left(\frac{1}{\lambda^2}\right)$$

where x, y, z are some combinations of the traces of the family mixing matrices.

From the action, we can recognize the familiar physical terms as:

- the Einstein gravity term $\int d^4x\sqrt{g}R$ with the factor $\frac{15\lambda^2}{16\pi^2}f_2$
- the Cosmological constant term $\int d^4x\sqrt{g}$ with the factor $\frac{45\lambda^4}{4\pi^2}f_0$
- the Yang Mills terms $G^i_{\mu\nu}G^{\mu\nu i}, F^{\alpha}_{\mu\nu}F^{\mu\nu\alpha}, B_{\mu\nu}B^{\mu\nu}$ with factors $\frac{f_4}{4\pi^2}g^2_{02}$, $\frac{f_4}{4\pi^2}g^2_{03}$ and $\frac{f_4}{4\pi^2}\frac{5}{3}g^2_{01}$

- the Weyl conformal gravity term $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ with the factor $-\frac{f_4}{4\pi^2}\frac{9}{16}$

- and very importantly the Higgs terms $-\frac{3\lambda^2 f_2}{2\pi^2}y^2|H|^2 + \frac{f_4}{4\pi^2}3z^2|H|^4 + \frac{f_4}{4\pi^2}3y^2|D_\mu H|^2$, exactly reproducing the Mexican hat potential.

As we see, all the key gravity + particle physics model terms appear in the action along with correction terms (expected only to be observable at higher order energies). Unlike the usual particle physics models, there seem to exist certain relations between the coupling constants obtained by this theory; for example all the couplings constants of the Yang Mills terms should be equal in any standard particle physics model and thus we should have: $g_{03}^2 = g_{02}^2 = \frac{5}{3}g_{01}^2$, $\frac{15\lambda^2}{4\pi^2}f_2 = \frac{1}{\kappa_0^2}$ and $\frac{g_{03}^2 f_4}{\pi^2} = 1$. This issue is solved through renormalization, by saying the obtained quantities as they appear in the action are correct at some high cutoff energy where unification is in progress and if we want the coupling constants as observed, we should renormalize them to their physical, low-energy, mutually independent values. This will be elaborated on in the conclusion.

Having completed this, we have a Standard Model-like particle physics model on a curved background with higher order couplings and unification predictions. To proceed from here, one would vary the action and reproduce all the equations of motion and proceed with the theory as usual (canonical quantization, etc.).

IV. Conclusion

In this paper, the goal was to collect the information understandable to a student at a predoctoral level and get them started on the road to understanding the vast subject of noncommutative geometry of Connes as well as to demonstrate one of many interesting applications of the spectral triple / spectral action approach, namely the spectral formulation of a particle physics + gravity model.

The first order of business was to motivate and explain the equivalence between the topological/differential/geometrical side on which gravity is formulated and the algebraic side on which particle physics is formulated. This was done through a rather long argument, the results of which can be summarized by the following table:

Points on Topological Space	(pure) States on the Algebra
$C^\infty(\mathcal{M})$	Noncommutative Algebra
Vector Fields	Derivations of Algebra
deRham Complex	Derivation Based Calculus
Vector Bundles of Spaces	Modules over Algebras
de Rham Cohomology	Cyclic Homology
Riemann Manifold	Spectral Triple (A,H,D)
Riemann Metric	Spectral Distance (Dirac op.)
Atiyah-Singer theorem	Connes-Moscovici Index Thm.

The remaining developments then include: the spectral action, the heat kernel expansion and the application of both to a particle physics + gravity model.

This approach is of course one of many and it's important to know where it succeeds and where it fails; namely what it can and cannot do and where there's work still left to be done. Without further ado, some of the boons are the following:

- This approach presents gravity and a particle physics gauge theory in a unified framework, in the algebraic language of spectral triples, without referencing "background" physics or making any explicit distinctions between the way the two are presented.
- The Higgs field, necessary in standard particle physics models to provide a gauge invariant mechanism for giving mass to the gauge bosons, is usually implemented by hand by means of spontaneous symmetry breaking. In this approach however, it comes about in an automatic way; purely by writing the correct gauge group, the spectral action approach homes in on the (more or less) unique theory presented here, which contains the Mexican hat potential *in its symmetry broken form.* It comes about as one of the curvature invariants of the finite space.
- There's also the fact that the Lagrangian obtained from the spectral action principle contains fewer free parameters than the usual particle physics Lagrangian does (albeit at accelerator energies of $\sim$ 14 TeV, no one knows what happens after that) meaning that they cannot be equal (again, at 14 TeV). If they're expected to be equal (at some high energies), this would mean that the free parameters - the coupling constants - in the particle physics Lagrangian would have to exhibit some form of running coupling that would tend to unify some of the coupling constants effectively reducing the number of free parameters, in line with the spectral action result. Taking this seriously, this result can serve as a prediction of high-energy physics behaviour of our modern particle physics / gravity models.
- Not every gauge theory fits into this framework; in fact the Standard Model-like particle physics models discussed here are among the simplest theories that can be described by this formalism [21]. Any

considerably simpler setup tends to fail one of the many conditions required for the whole thing to work (real spectral triple conditions, etc.) This can be used as a mechanism for selecting what theories to look at, which helps in looking for new physics as well as giving us a satisfactory statement that the Standard Model is in some sense "minimal complexity"!

On the other hand, some of the main drawbacks are:

- The particle content of the particle physics model still has to be put in by hand; there is no mechanism to tell us why there are exactly 3 fermionic families, etc., although some people have speculated on this subject, but no definitive / accepted results are known.
- The manifold which we're trying to describe has to be Riemannian for the whole spectral action approach to (completely) work. On Lorentzian manifolds, a Dirac operator can still be found , and in fact work is being done [22-24] (and is more or less complete) to extend the reconstruction theorems of Connes to Lorentzian manifolds. Where the approach nevertheless fails is when one tries to use the heat-kernel expansion on a Lorentzian Dirac operator, as the result only holds for elliptic operators and the rest of the calculation cannot be carried out. The solution is to do everything for a Riemannian manifold and then Wick-rotate once the action / equations of motion are obtained, but this also runs into problems of its own [25].
- The spectral triple yields a classical, non-quantized action and canonical quantization has to be done as it's usually presented in textbooks that just start from the action. Implementing the spectral action principle directly into a path-integral framework is an open problem and work is being done in this area. The basic idea is that the different "paths" are different spectral triples with different Dirac operators and one has to sum over all Dirac operators to obtain the physical quantized action [26]. This would provide quantization at an elementary formalism level rather than after we already have the completely familiar action and physical theory, but its far from completion.

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