Quantum Completeness

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Motivated by the idea of eliminating singularities abundantly present in general theory of relativity, we have attempted to analyse them when probed with quantum test particles. Some of the considered examples demonstrated the desired trend of singularity "smearing". Intuitively, this is due to an effective repulsive barrier term, as was noted when final differential equations were obtained. Other examples underwent no significant change and a short discussion on this fact in relation to the theory in general was given. Finally, we have announced future research regarding this approach with the addition of manifold non-commutativity.

I. INTRODUCTION

In the October of 1939, Albert Einstein published a seminal paper[1] elucidating his dissatisfaction with the emergence of the so-called "Schwarzschild singularities", providing calculations and reasoning behind the fact that, as he suspected, such phenomena do not occur. Although a quite reasonable one, this conjecture was proven to be startlingly false. Not only do singularities form, but for certain spacetimes and distributions of matter, this is almost guaranteed, as proved by Hawking and Penrose in their famous treatment of the subject[2]. Ubiquitous as they are, their prevalence is enough of a reason for extensive studies. However, we can provide further motivation.

Consider a spacetime in general theory of relativity (GTR in the following). We consider it to be timelike singular if it is geodesically incomplete[3]. Formally, geodesic completeness is a statement of every maximal geodesic being defined on an entire real line[4]. This definition certainly renders the previous one acceptable since geodesics in GTR represent motion of free-falling test particles so incompleteness would imply that their time evolution is not well defined after a finite amount of elapsed proper time, exactly what we would expect if said particles were to head towards singularity. For completeness, we also note that this definition is quite useful in establishing large classes of singular solutions to Einstein's equation[3] and that, although we shall only consider timelike singularities, spacelike and null can be taken into account as well[5–7].

Probing singularities with point particles is obviously not consistent with the uncertainty principle of quantum mechanics, so we seek novel way of approaching them. Exhaustive research has been and is in progress to determine whether quantum gravitational theories could eliminate them and the current state of affairs suggests that this could be the case, i.e. that the quantum approach appears to "smear" singularities out[8–10].

This begs the question of behaviour of GTR systems when quantum properties are introduced. To answer it, we will consider the motion of quantum test parti-

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cles in singular classical relativistic spacetimes. This will demonstrate that there are static spacetimes with timelike singularities in which a quantum particle is well behaved for all the values of its proper time. Even more significantly, these singularities do not introduce any new ambiguities nor do they require additional boundary conditions when defining test particles. Thus, even though appearing singular when probed with classical particles, said spacetimes are quantum-mechanically non-singular. Intuitively, the origin of this difference lies within an effective repulsive barrier produced in observed spacetime, shielding its classical singularity, as will be seen in the subsequent sections. Consequently, quantum wave packets representing particles simply bounce off the barrier. From this viewpoint, geodesics correspond to the geometric optics limit of infinite frequency waves and only in such unphysical instance is one able to reach the singularity.

How does one go about formulating a condition to determine whether a certain spacetime is singular in quantum theory or not? Quantum physicist would probably suggest analysing the expectation values of quantum observables and their potential divergences, but it is not the procedure we choose to pursue. Approaching this problem more as a relativist would, our choice is obtaining the conditions under which the evolution of any quantum state is uniquely defined for all times.

We denote such a spacetime as non-singular or quantum-mechanically complete. If this is not the case, one lacks a consistent way of evolving the system at hand in time and the spacetime is said to be singular. Note the correspondence with geodesic incompleteness in GTR where maximal geodesics and their extent are of interest.

As a follow-up to previous paragraphs, we could conjecture that the precise criterion for a quantum system to evolve uniquely in time is intimately related to self-adjointness of its Hamiltonian functional. This is easily understood on an intuitive basis as well when one considers the role of Hamiltonian as generator of time evolution in ordinary quantum mechanics. However, an important departure from a standard textbook approach lies in emphasizing the more rigorous definition of self-adjoint (SA in the following) operators, revolving around the fact that an operator is defined not only by its acting prescription, but also its domain of action. This important remark poses little trouble in ordinary quantum mechanics, but becomes significant when considering non-trivial systems, including those evolving on non-trivial manifolds. These ideas will be placed on a concrete footing in the subsequent sections.

II. FORMALISM AND QUANTUM SINGULARITIES

In order to generalise the concept of GTR singularity so as to include quantum mechanical test particles, i.e. probes, we adopt the approach due to Horowitz and Marolf[3] (H&M in the following). As was announced in the Introduction, they proposed that a spacetime be regarded as non-singular if the evolution of quantum states is uniquely determined for all times by their initial conditions.

To formalise this idea, consider a static spacetime $(\mathcal{M}, g)$ with a timelike Killing vector field ξ^{μ} (consult Appendix A). Let t denote the Killing parameter and Σ a static spacetime slice. This notation for a generic slice will be adopted throughout the paper. Next, consider a quantum test particle with mass m and obeying the Klein-Gordon equation for it describes the simplest, scalar example:

$$\left(\nabla^{\mu}\nabla_{\mu} - m^2\right)\psi = 0 \ , \tag{1}$$

which can also be reformulated in terms of the aforementioned Killing vector field with

$$f \equiv -\xi^{\mu}\xi_{\mu} \tag{2}$$

into the following:

$$\frac{\partial^2\psi}{\partial t^2} = f^{1/2}D^i\left(f^{1/2}D_i\right) - fm^2\psi \equiv -A\psi \ , \tag{3}$$

with D_i being the spatial covariant derivative on Σ. We have denoted an operator of the Klein-Gordon equation as A for convenience and separated the spatial and temporal parts of the covariant derivative. This particular separation will soon be exploited directly to explicitly obtain time evolution equation for wave function describing the test particle in question.

Keeping equation (3) in mind, we now turn to the important question of operator self-adjointness. As previously stated, to verify whether an operator is SA, it is not sufficent for its acting prescription to be known, but its domain of action as well. There is a plethora of examples demonstrating this point[11], with operators exhibiting self-adjointness and not, depending on their domain. Motivated by this, we wish to obtain a consistent way of characterising them. Recall from the previous paragraph that such a method applied to operator A in Eq. (3) would conclusively determine whether the investigated spacetime is singular or not. A wonderfully simple procedure of this kind exists. We now present its outline.

Let us examine Hilbert space $L^2(\Sigma)$, i.e. the space of square-integrable functions on Σ. Domain of the operator A, $D(A)$ is chosen in such a way that it does not enclose any of the spacetime singularities. An appropriate set to accomplish this is $C_0^{\infty}(\Sigma)$, the set of smooth functions with compact support on Σ. It is easily seen that operator A is real, positive and symmetric. As an announced departure from standard quantum mechanics, we point out that this does not imply that the operator is SA as well, as can also be seen by considering examples in[11]. However, it is known that this type of operator admits at least one SA extension[12], signifying that it is possible for it to have its domain extended in a way which renders it SA. Should this extension, namely A_E be unique, A will be called essentially self-adjoint[13]. Notice that we do not pay attention to obtaining the extended operator, but rather the questions of its existence and uniqueness (consult Appendix B). We now insert proposed unique SA extension into Eq. (3) for it to become

$$\imath\frac{\partial\psi}{\partial t} = \left(A_E\right)^{1/2}\psi \ , \tag{4}$$

with solution

$$\psi(t) = \exp\left[-\imath t\left(A_E\right)^{1/2}\right]\psi(0) \ . \tag{5}$$

It is now obvious that the quantum test particle admits a unique time evolution on the ground of A_E being unique.

If A fails to be essentially SA, time evolution of the wave function in the previous equation will be ambiguous. In this case, as already mentioned, the spacetime is denoted as quantum-mechanically singular.

Natural follow-up would be devising a way of determining the number of possible SA extensions of an operator of interest. For this purpose, the concept of deficiency indices, due to von Neumann, is used as follows[14,15]. We begin with definitions of deficiency subspaces $N_{\pm}$ as

$$N_{+} = \{\psi \in D(A^{*})\,|A^{*}\psi = Z_{+}\psi,\ \mathrm{Im}Z_{+} > 0\};$$
$$N_{-} = \{\psi \in D(A^{*})\,|A^{*}\psi = Z_{-}\psi,\ \mathrm{Im}Z_{-} > 0\}. \quad (6)$$

These subspaces represent parts of the operator domain which are spanned by eigenvectors having imaginary eigenvalues. The deficiency indices of the operator A are then simply chosen to be the dimensions n_{+} and n_{-} of these subspaces, respectively. It can be shown that a particular index $n_{+}(n_{-})$ depends only on whether $Z_{+}(Z-)$ lies in the upper (lower) half of the complex plane and not on its explicit value. Exploiting this freedom, one commonly sets $Z_{\pm} = \pm\imath\lambda$ in applications, where λ is an arbitrary positive constant necessary due to dimensional consistency. Notice also an intuitive appeal of the previous condition on imaginary eigenvalues. Since SA operators admit only real ones, the deficiency indices in a sense convey information on the subspace spanned by eigenvectors marking operator's departure from self-adjointness. Additional convenience is also seen in the fact that determination of deficiency indices now reduces to counting the number of solutions of the equation

$$A^{*}\psi = Z\psi\ , \quad (7)$$

which are square-integrable (recall our choice of operator domain) or, setting λ equal to 1 for clarity:

$$A^{*}\psi \pm \imath\psi = 0\ . \quad (8)$$

If there exist no such solutions, i.e. $n_{\pm} = 0$, the operator A possesses a unique SA extension, i.e. it is essentially SA. Equivalently, it is sufficient to investigate the functions satisfying Eq. (8) that are not elements of the Hilbert space $L^{2}(\Sigma)$.

III. EXAMPLES

Wishing to put the outlaid formalism into perspective, we will consult several examples of spacetimes and operators which are being extensively investigated at the moment[16–18]. Before this, we briefly mention two of the more elementary, motivational ones.

First example is the classical motion on a real half-line. This motion is classically complete at the end point if there are no initial conditions such that the trajectory runs off to the end point in a finite time. Analogous statement is that the potential grows unbounded from above near the end point. On the other hand, when considering a quantum-mechanical motion on a half-line, a time-independent potential is complete when less restrictive condition is satisfied, i.e. when the associated Hamiltonian is essentially SA on the space of $C_{0}^{\infty}(\langle 0,\ \infty\rangle)$ functions with compact support[13].

Another relevant example is that of a hydrogen atom system which exhibits singularity when analysed classically, but not quantum-mechanically. Since the Coulomb potential is bounded from above near the origin, the electron can reach the origin in a finite time, implying that the classical motion of electron is incomplete. However, it turns complete when probed by the quantum electron (via its bound state). This distinction is also evident in the other measure of completeness, proposed in the Introduction section, i.e. the classical singularity is not reflected in any of the observables related to the bound-state quantum electron.

III.1. Spherically symmetric spacetime and Laplacian operator

Let us consider the motion of a free particle on a $n+1$ dimensional Riemannian manifold $(\mathcal{M}, g)$. In this case, the Hilbert space consists of square-integrable functions on $\mathcal{M}$ with a measure determined by the proper volume element $d^{n+1}x\,(-g)^{1/2}$ and the Hamiltonian operator obviously being Laplacian. Before deriving the result for geodesically incomplete manifold, it is worth noting that for Riemannian manifolds which are geodesically complete, the corresponding Laplacian is essentially SA, i.e. it admits a unique SA extension[19]. Nontrivial implication of this statement is that our general procedure can only amend the situation when applied to Laplacian and class of spacetimes at hand, never worsen it.

We now turn to the case of geodesically incomplete spacetime. One can demonstrate that a unique SA Laplacian exists nonetheless[3] by examination of a spherically symmetric metric of the form

$$ds^2 = dr^2 + h^2(r)d\Omega_n \ , \tag{9}$$

with $d\Omega_n$ denoting the standard metric on the n-sphere. The domain of Laplacian naturally consists of smooth functions with compact support away from the origin since in usual spacetime examples of this kind, origin is the potential coordinate value for the singularity to form. We have stated that in order to see the self-adjointness of the Laplacian it is sufficient to consider its eigenvalue equation with purely imaginary eigenvalues, i.e.

$$\Delta_n \psi = \pm \imath \psi \tag{10}$$

and demonstrate that it admits no square-integrable solutions, consistent with our discussion on the significance of imaginary eigenvalues. Separating it in a familiar manner into $\psi(r, \Omega) = R(r)Y(\Omega)$ yields the following radial equation:

$$\frac{\partial^2 R}{\partial r^2} + \frac{n}{h}\frac{\partial h}{\partial r}\frac{\partial R}{\partial r} - \frac{c}{r^2}R = \pm \imath R \ , \tag{11}$$

with $c \geq 0$ being an eigenvalue of the Laplacian on the n-sphere. To investigate its solutions, we consider the previous equation with c set to zero since $c > 0$ serves only to increase divergence of the solution at the origin, i.e. $r = 0$. This term is an instance of a potential barrier shielding singularity, as announced in the Introduction. For a specific example, we choose $h(r) = r^k$ since it is familiar in GTR. Continuing, notice that in the limit of vanishing r, the term $\pm \imath R(r)$ also turns negligible. It is now straightforward to see that this choice requires the solution to Eq. (11) to be $R(r) = r^\alpha$ with $\alpha = 1 - nk$. However, this in turn implies that there are no square-integrable solutions to the equation if

$$k \geq \frac{3}{n} \ . \tag{12}$$

Notice that this condition is obtained with respect to the aforementioned volume element (measure) $(-g)^{1/2} \propto r^{kn}$. Therefore, one is led to conclude that for any metric of the form (9) with $h(r) = r^k$ $\left(k \geq \frac{3}{n}\right)$ the corresponding spacetime is quantum-mechanically non-singular near the origin. Contrasting it with its classical counterpart which is incomplete for all k but $k = 1$, we see a large class of singular spacetimes amended by the introduction of quantum test particles, thus demonstrating the desired singularity "smearing".

III.2. Spherically symmetric spacetime and Klein-Gordon particle

We now turn to a generalisation of the previous example and examine a static, spherically symmetric spacetime in $n + 2$ dimensions, established by the following metric:

$$ds^2 = -f^2(r)dt^2 + f^{-2}(r)dr^2 + h^2(r)d\Omega_n \ . \tag{13}$$

As before, we consider the equation for imaginary eigenvalues, but the operator is now describing a scalar particle of mass m, given by Klein-Gordon Eq. (3). Upon using the same separation of variables $\psi(r,\Omega) = R(r)Y(\Omega)$, we obtain the radial equation for $R(r)$:

$$\frac{\partial^2 R}{\partial r^2} + \frac{\partial}{\partial r}\ln\left(f^2 h^n\right)\frac{\partial R}{\partial r} - \frac{c}{f^2 h^2}R - \\ -\frac{m^2}{f^2}R = \pm\frac{\imath}{f^4}R \ . \tag{14}$$

Analogously to the earlier example, we again discard the repulsive barrier term $-\frac{m^2}{f^2}R$ since its only potential action is increasing the rate of divergence of the solution near the origin. Recalling Eq. (3), we see that this corresponds to massless test particle.

III.2.1. Reissner-Nordström spacetime

This spacetime is a well-known solution to the Einstein's equation for a system of charged, spherically symmetric black hole. Its metric is given by adopting the choice[20]

$$f^2(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \quad \text{and } h^2(r) = r^2 \ , \tag{15}$$

with M and Q being mass and charge of the black hole, respectively. As before, we consider the system near the origin, which is a classical singularity. Upon inserting $n = 2$, Eq. (14) becomes, to the lowest significant order in r,

$$\frac{\partial^2 R}{\partial r^2} - \frac{2M}{Q^2}\frac{\partial R}{\partial r} = 0 \ . \tag{16}$$

Notice that the reasoning behind discarding the imaginary term ties to the asymptotic form of $f^2(r)$ as r tends to zero, namely $f^2(r)$ tends to GQ^2/r^2. Introducing the abbreviation $b \equiv 2M/(Q^2)$ and solving the equation yields the following two linearly independent solutions:

$$\begin{aligned} R_1(r) =& \text{const. and} \\ R_2(r) =& e^{br} \ . \end{aligned} \tag{17}$$

It is immediately evident that both of the solutions are square integrable near the origin with respect to the measure $f^{-1}(r)h^n(r) \propto r^{n+1} = r^3$, thus demonstrating that the Reissner-Nordström spacetime remains singular even when probed with scalar quantum test particle. It is interesting to note that this behaviour is independent of the specific value of the ratio Q/M[3].

III.2.2. Negative-mass Schwarzschild spacetime

We now focus on the Schwarzschild metric, determined by setting[22]

$$f^2(r) = 1 - \frac{2GM}{r} \text{ and } h^2(r) = r^2 \ , \tag{18}$$

with M being the mass term which is now negative. As before, we consider the system near the origin, which is a classical singularity. To the lowest significant order in r, upon inserting $n = 2$, Eq. (14) turns into

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} = 0 \ . \tag{19}$$

Notice that the reasoning behind discarding the imaginary term once again ties to the asymptotic form of $f^2(r)$ as r tends to zero, namely $f^2(r)$ tends to $-2GM/r$. Solving this equation yields the following two linearly independent solutions:

$$\begin{aligned} R_1(r) =& \text{const. and} \\ R_2(r) =& \ln(r) \ . \end{aligned} \tag{20}$$

The first solution is manifestly well-behaved and square-integrable near the origin with respect to the measure $f^{-1}(r)h^n(r) \propto r^{n+1/2} = r^{5/2}$ and the second one as well since natural logarithm powers tend to zero slower than positive exponentials.

Finally, we conclude that the negative mass Schwarzschild spacetime is singular when probed with a scalar quantum test particle. This inference is fortunate since, according to research[24], a theory exhibiting non-singular negative mass Schwarzschild solution probably does not admit a stable ground state. We also briefly emphasise another peculiarity as a consequence of choosing negative mass term and this is the emergence of naked singularities. These are the ones which can be reached by classical GTR test particles since they are not shielded by the event horizon as ordinary singularities are. This aspect of negative-mass black holes will be briefly discussed in the following example.

III.2.3. Bañados-Teitelboim-Zanelli spacetime

BTZ spacetime is the last one in our analysis. It is three-dimensional, i.e. of one dimension less than the previous ones and in the metric (13) we shall consider r to be a cylindrical coordinate. We choose the other functions to be[25]

$$f^2(r) = -M + \frac{r^2}{L^2} \text{ and } h^2(r) = r^2 \ , \quad (21)$$

where M once again denotes the negative mass term and L represents some characteristic length magnitude of the system. The negative mass term will be allowed to assume values in the interval $[-1, 0]$. This is worth explaining a tad further. Setting $M > 0$ implies that the $r = 0$ singularity is hidden by the event horizon at finite radius. Recall this in previous black hole solutions, except negative-mass Schwarzschild, whose singularity was naked. On the other hand, if we were to set M to equal zero, the spacetime would obviously converge to a vacuum state with vanishing event horizon. However, lowering its value below zero causes a continuous sequence of naked singularities (point particle sources) to appear at the origin. These do not arise as a consequence of some curvature scalar divergence as they did in the hidden singularity examples, but rather because of a topological obstruction of the attempt to continue spacetime since the Ricci tensor contains (due to cosmological constant) a term proportional to the Dirac distribution[26] in addition to the constant curvature ones. It is this particular choice of classical spacetime that we will probe with quantum test particles in order to see whether it remains singular or not since it is qualitatively different than the other and has important implications to stability of our theory in general.

Furthermore, we note that for the mass parameter $M < -1$, the spacetime exhibits point sources with negative mass devoid of physical meaning and is thus excluded from further discussion.

Metric is of the form[25]

$$ds^2 = -f^2(r)dt^2 + f^{-2}(r)dr^2 + r^2 d\theta^2 \ , \quad (22)$$

with θ being the angular coordinate in the polar plane. We have chosen to write this instance of Eq. (13) explicitly since it is particularly simple and the separation of variables will consequently be somewhat more specific than those previously encountered. Precisely, we use the form $\psi(r, \theta) = R(r)e^{\imath l\theta}$ to obtain, similar to Eq. (14) (observe that we could have simply inserted $n = 1$ into

Eq. (14) since $n+2=3$)

$$\frac{\partial^2 R_l}{\partial r^2}+\frac{\partial}{\partial r}\ln\left(f^2 h\right)\frac{\partial R_l}{\partial r}-\frac{l^2}{f^2h^2}R_l- \\ -\frac{m^2}{f^2}R_l=\pm\frac{\imath}{f^4}R_l \ . \tag{23}$$

Notice also the appearance of index l in radial functions since the angular part of the solution is known exactly.

After arguing for the mass term to be discarded as previously, we further conclude that, since $f^2(r)$ tends to $-M$ as r tends to zero, the imaginary term on the RHS remains. This transforms the radial equation into

$$\frac{\partial^2 R_l}{\partial r^2}+\frac{1}{r}\frac{\partial R_l}{\partial r}+\left[\pm\imath-\frac{l^2}{\alpha^2 r^2}\right]R_l=0 \ , \tag{24}$$

where we have introduced a convenient replacement $\alpha^2=-M$ and exploited freedom in setting the constant in the imaginary term to 1 (recall this point from Introduction). General solutions of Eq. (24) are given by

$$R_{1,l}(r)=J_{|l/\alpha|}(kr) \text{ and} \\ R_{2,l}(r)=N_{|l/\alpha|}(kr) \ , \tag{25}$$

where $k=(\imath)^{1/2}$. $J_\nu(kr)$ and $N_\nu(kr)$ are the ν^{th} order Bessel and Neumann functions, respectively. We wish to investigate their behaviour around $r=0$. Considering their asymptotic forms[21], it is obvious that $J_\nu(kr)$ is integrable $\forall\nu$ and consequently $\forall l$. Mesaure used in deciding this is given by $f^{-1}(r)h^n(r)\propto r$. Turning to $N_{|l/\alpha|}(kr)$, we consider its specific asymptotic form for small values of the argument

$$N_{\nu=0}(kr\to 0)=\frac{2}{\pi}\left[\ln\frac{1}{2}kr+\text{const.}\right] \text{ and} \\ N_{\nu\neq 0}(kr\to 0)=-\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{kr}\right)^{\nu} \ . \tag{26}$$

As before, since natural logarithm powers tend to zero slower than positive exponentials, $N_{\nu\neq 0}(kr)$ are square-integrable near the origin. However, the second equation in (26) implies that for values of $\nu>1$ they fail to be as such. If we recall that ν in our example is given by $|l/\alpha|$, it must be that $|l|\leq\alpha$, but we have focused on values of M and consequently α less than one. This leads to the fact that the only acceptable value for $l\in\mathbb{N}_0$ is zero, i.e. $l=0$.

We thus conclude that for $l=0$, there exist two linearly independent square-integrable solutions near the origin and the spacetime is considered to be quantum-mechanically singular, i.e. introducing a quantum probe did not remove the singularity.

In the end, it is worth mentioning the fact that is outside the scope of this paper, but important nevertheless. When mass term M takes the value -1, the spacetime does not exhibit an event horizon, but there is no singularity to hide either, so the solution is permissible. More significantly, it represents the ground state of the theory (recall the non-integrability implication in negative-mass Schwarzschild case), which is important for characterising it as a whole. Reader is advised to consult discussions on this important application of our formalism[24].

IV. CONCLUSION

Wishing to analyse and potentially eliminate singularities present in general theory of relativity, we have extended its formalism to include quantum probes, i.e. quantum test particles. Their behaviour, being fundamentally different from classical ones, as determined by the uncertainty principle, was enough to conjecture the possibility of singularity "smearing". To this end, we have discussed quantum operators and the importance of specifying their action domain when defining them; specifically the ability of extending their domain so as to render them self-adjoint. These were relevant since our definition of a quantum-mechanically non-singular spacetime corresponded to the existence of a unique SA operator meant to evolve quantum systems, its uniqueness allowing us to pose a well-defined idea of this evolution. Remembering that geodesics represent motion of free-falling test particles, it is easy to notice resemblance to the GTR definition of singular, i.e. incomplete spacetime as one with partially undefined maximal geodesics.

Through several examples, we have demonstrated that this approach can, although not always, be fruitful. This is an encouraging aspect of quantum completeness since it seems to imply that the classical situation is somewhat amended when approached quantum-mechanically. We have also mentioned that an intuitive explanation for singularity disappearance lies within emergence of a repulsive potential barrier resulting in an effective singularity "shielding".

Extending on this, we should have considered other types of test particles, e.g. those obeying Dirac equation for a full treatment. This would have provided us with a more complete understanding of the precise effect of quantum probing in GTR spacetimes and is planned within collaboration between the author of the current paper and mentor T. Jurić.

Another significant part of our current and future research is the non-commutative approach. It encompasses modeling the spacetime with a non-commutative manifold, in accordance with the expectation of spacetime quantization[16,18]on Planck's scales of length, time, mass etc. Intuitively, grainy rather than continuous structure of manifold on such scales is also expected to have a singularity "smearing" effect. Specifically, calculations are being carried out for Reissner-Nordström and neutral and charged BTZ black holes at the moment to investigate effects of both quantum probing and non-commutativity simultaneously.

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Appendix A: Killing vectors

Vector field ξ^μ is called Killing if it satisfies the following relation[27]:

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)} = 0 \ , \qquad \text{(A1)}$$

with $\mathcal{L}_\xi$ and $g_{\mu\nu}$ denoting the Lie derivative in direction of vector ξ^μ and the metric tensor, respectively.

Among many of their useful properties, we point out that Kiling vectors are generators of symmetries in spacetime. This formally means that for every vector u^μ tangent to some geodesic γ, quantity $\xi^\mu u_\mu$ is constant along γ. The Killing vector associated with conserved quantity being energy is, as in classical mechanics, generator of time translations, i.e. $\xi = \partial/\partial t$.

Appendix B: Self-adjoint extensions

We present a brief overview of the basic results in self-adjoint extensions theory.

Suppose that the deficiency indices of the subspaces in (6) were obtained and also recall that the operator A is positive, real and symmetric. Then, due to von Neumann's method[15], the following three distinct cases are possible:

a. Operator A is essentially SA if and only if $n_\pm = 0$.

b. Operator A is not SA, but it admits SA extensions if and only if $n_+ = n_- \neq 0$

c. Operator A is not SA, nor does it admit SA extensions if $n_+ \neq n_-$.

In this paper, we have exclusively focused on operators of case a. However, for completeness, we mention that in the b case, with $n \equiv n_\pm$, domains of SA extensions of operator A are given by

$$D(A_E) = \{\psi + \psi_+ + \mathcal{U}\psi_- | \psi \in D(A) \quad \text{and} \quad \mathcal{U} \text{ being a unitary } n \times n \text{ matrix}\} \ . \tag{B1}$$

Here $\psi_\pm$ represent two solutions to equations in (6) with positive and negative imaginary part.

From this construction method, it is also seen that there may exist several SA extensions depending on the unitary matrix $\mathcal{U}$ dimension or, equivalently, deficiency indices. Recall the relation between multitude of SA extensions and uniqueness of time evolution of test particles.

In practice, however, one imposes additional boundary conditions so as to choose a particular SA extension.

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