Large solutions for subordinate spectral Laplacian

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Also:

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- Example: $\phi(\lambda) = \lambda^s$, $s \in (0,1)$, $\phi(-\Delta|_D) = (-\Delta_{|D})^s$ is the spectral fractional Laplacian.

Probabilistic background

Underlying process and connection to $\phi(-\Delta|_D)$

Let $W = (W_t)_t$ be a Brownian motion in $\mathbb{R}^d$ with the char. exp. $\xi \mapsto |\xi|^2$.

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Let $W = (W_t)_t$ be a Brownian motion in $\mathbb{R}^d$ with the char. exp. $\xi \mapsto |\xi|^2$. Let $S = (S_t)_t$ be a subordinator with the Laplace exponent ϕ. The killed Brownian motion W^D upon exiting the set D is defined by

$$W_t^D := \begin{cases} W_t, & t < \tau_D := \inf\{t > 0 : W_t \notin D\}, \\ \partial, & t \geq \tau_D, \end{cases}$$

where ∂ is the additional point added to $\mathbb{R}^d$ called the *cemetery*.

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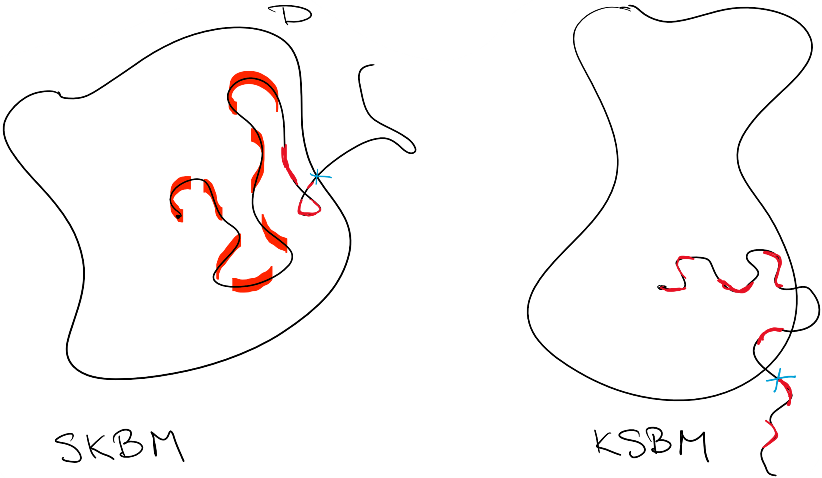
The process

$$X_t = (W^D)_{S_t}, t \geq 0,$$

is called the subordinate killed Brownian motion.

Subordination and killing do not commute!



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- ϕ is a Bernstein function

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- ϕ satisfies the weak scaling condition at infinity: there exists $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0,1)$ s.t.

$$a_1 \left(\frac{t}{s}\right)^{\delta_1} \leq \frac{\phi(t)}{\phi(s)} \leq a_2 \left(\frac{t}{s}\right)^{\delta_2}, \quad t, s \geq 1. \tag{WSC}$$

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The assumption (WSC) drives small space-time behaviour of X.

Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an ONB of $L^2(D)$ s.t. $-\Delta|_D\, \varphi_j = \lambda_j \varphi_j$ in D. We define

$$\phi(-\Delta|_D)u = \sum_{j=1}^{\infty} \phi(\lambda_j)\widehat{u}_j\varphi_j,$$

for $u \in \mathcal{D}(\phi(-\Delta|_D)) := \{v = \sum_{j=1}^{\infty} \widehat{v}_j\varphi_j \in L^2(D) : \sum_{j=0}^{\infty} \phi(\lambda_j)^2|\widehat{v}_j|^2 < \infty\}$.

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$\phi(-\Delta|_D)$ is an unbounded operator, $C_c^\infty(D) \subset \mathcal{D}(\phi(-\Delta|_D))$, and

Operator $\phi(-\Delta|_D)$

Definition in $L^2(D)$

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$\phi(-\Delta|_D)$ is an unbounded operator, $C_c^\infty(D) \subset \mathcal{D}(\phi(-\Delta|_D))$, and

Lemma (B., 2023)

The operator $-\phi(-\Delta|_D)$ is the infinitesimal generator of $L^2(D)$ semigroup generated by $X_t = W^D_{S_t}$, i.e. of the subordinate killed Brownian motion X.

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Properties of $\phi(-\Delta|_D)$

$\phi(-\Delta|_D)$ is a non-local operator with a pointwise representation

Proposition (B., 2023)

For $u \in C^{1,1}(D) \cap \mathcal{D}(\phi(-\Delta|_D))$ *and a.e.* $x \in D$

$$\phi(-\Delta|_D)u(x) = P.V. \int_D [u(x) - u(y)] J_D(x,y)dy + \kappa(x)u(x).$$

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Here

$$J_D(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1 \right) \frac{\phi(|x-y|^{-2})}{|x-y|^d}, \quad x, y \in D.$$

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Theorem (Kim, Song, Vondraček, 2016, B., 2023)

$$G_D^\phi(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1\right) \frac{1}{|x-y|^d \phi(|x-y|^{-2})},\ x,y \in D.$$

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Proposition (B., 2023)

The function

$$P_D^\phi(x,z) := -\frac{\partial}{\partial \mathbf{n}} G_D^\phi(x,z), \quad x \in D, z \in \partial D.$$

is well defined and $(x,z) \mapsto P_D^\phi(x,z) \in C(D \times \partial D)$. *Moreover,*

$$P_D^\phi(x,z) \asymp \frac{\delta_D(x)}{|x-z|^{d+2}\phi(|x-z|^{-2})}, \quad x \in D, z \in \partial D.$$

Nonnegative harmonic functions

Definition

$h \in L^1(D, \delta_D(x)dx)$ is harmonic in D if $\phi(-\Delta|_D)h = 0$ in D in distributional sense.

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A function $h \geq 0$ is harmonic in D if and only if it satisfies the mean-value property, i.e. $h(x) = \mathbb{E}_x[h(X^D_{\tau_U})]$, for every $x \in U \subset\subset D$.

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Theorem (B., 2023)

If $h \geq 0$ is harmonic in D, then there exists a finite measure $\zeta \in \mathcal{M}(\partial D)$ such that

$$h(x) = \int_{\partial D} P^\phi_D(x, z)\zeta(dz), \quad \text{for a.e. } x \in D.$$

Moderate solutions

Theorem (B., 2023)

Let $f \in L^1(D, \delta_D(x)dx)$ *and* $g \in L^1(\partial D)$*, then the problem*

$$\begin{aligned} -\phi(-\Delta|_D)u &= f && \text{in } D, \\ \frac{u}{P_D^\phi \sigma} &= g && \text{on } \partial D, \end{aligned}$$

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has a so-called weak-dual solution $u = G_D^\phi f + P_D^\phi g$. Additionally, if f and g are "regular enough", u is a pointwise solution.

Large solutions

A solution $u : D \to \mathbb{R}$ to the problem

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Lemma (B., Wagner, 2024+)

If $u : D \to \mathbb{R}$ satisfies

$$\lim_{D \ni x \to z} \frac{|u(x)|}{P_D^{\phi}\sigma(x)} = \infty, \quad z \in \partial D,$$

then u is not uniformly bounded in D by any nonnegative harmonic function with respect to $\phi(-\Delta|_D)$.

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Higher Hölder regularity of distributional solutions to $\phi(-\Delta|_D)u = f$ in D

Theorem (B., Wagner, 2024+)

Let $d \geq 3$, $\alpha \in (0,1)$ *and* $k \in \mathbb{N}_0$ *such that* $k + \alpha + 2\delta_1 \notin \mathbb{N}$, *and let* $f \in C^{k+\alpha}(D)$.
If $u \in L^1(D, \delta_D(x)dx)$ *solves* $\phi(-\Delta|_D)u = f$ *in* D *in distributional sense, then* $u \in C^{k+\alpha+2\delta_1}(D)$ *and for any* $K \subset\subset K' \subset\subset D$, *there exists* $C > 0$ *such that*

$$||u||_{C^{k+\alpha+2\delta_1}(K)} \leq C\left(||f||_{C^{k+\alpha}(K')} + ||u||_{L^1(D,\delta_D(x)dx)}\right).$$

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Moreover, if $f \in L^\infty_{loc}(D)$ and $\beta \in (0, 2\delta_1)$, then

$$||u||_{C^\beta(K)} \leq C \left(||f||_{L^\infty(K')} + ||u||_{L^1(D,\delta_D(x)dx)} \right).$$

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$$||u||_{C^\beta(K)} \leq C\left(||f||_{L^\infty(K')} + ||u||_{L^1(D,\delta_D(x)dx)}\right).$$

In particular, if u *is* $\phi(-\Delta|_D)$*-harmonic, then* $u \in C^\infty(D)$, *and* $P^\phi_D\zeta \in C^\infty(D)$ *for all finite measures* ζ *on* ∂D.

Higher Hölder regularity of distributional solutions to $\phi(-\Delta|_D)u = f$ in D: Remarks

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- The goal is to connect $\phi(-\Delta|_D)u = f$ in D to $\phi(-\Delta))\overline{u} = \overline{f}$ in $\mathbb{R}^d$, and to use the parabolic theory of $\partial_t - \Delta_{|D}$.
- At this point, we cannot remove $d \geq 3$ even in the fractional case. In the essential part of the proof we use function

$$\overline{v}(x) := G_{\mathbb{R}^d}\overline{f}(x) = \mathbb{E}_x\left[\int_0^\infty \overline{f}(W_t)dt\right],$$

but $G_{\mathbb{R}^d}$ is the Green function of the Brownian motion and in $d = 2$ the Brownian motion is not transient so $G_{R^d}|\overline{f}| \equiv \infty$ for $f \not\equiv 0$.

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Introduction

We solve

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for $f : D \to [0, \infty)$ such that $f \in C^1(\mathbb{R})$ and

$$(1+m)f(t) \le tf'(t) \le (1+M)f(t), \quad t \in \mathbb{R}, \tag{F}$$

for some $0 < m \le M < \infty$, e.g. $f(t) = t^p$ for $p > 1$.

Approximating sequence

Let $(u_j)_j$ be a sequence of solutions to the problems

$$\begin{aligned} -\phi(-\Delta|_D)u_j &= f(u_j) && \text{in } D, \\ \frac{u_j}{P_D^{\phi}\sigma} &= j && \text{on } \partial D, \end{aligned} \qquad \text{(AP)}$$

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Lemma (B., Wagner, 2024+)

The sequence $(u_j)_j$ increases as $j \to \infty$, and if $f \in C^\alpha(\mathbb{R})$ for $\alpha > 2(\delta_2 - \delta_1)$, then u_j is a pointwise solution to (AP).

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The goal now is to find a Keller-Osserman-type condition that will guarantee that $\lim_j u_j =: u$ is finite and that it is a large solution. This will be obtained by using the method of supersolution.

Construction of a supersolution

Let

$$F(t) = \int_0^t f(s)ds, \quad t > 0,$$

and set $\varphi : (0, \infty) \to (0, \infty)$ as

$$\varphi(t) = \int_t^\infty \frac{ds}{\sqrt{F(s)}}, \quad t > 0.$$

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Denote by ψ the inverse of φ.
A supersolution will be obtained from $U(x) := \psi(V(\delta_D(x)))$, where $V(t)$ is the renewal function of the subordinate Brownian motion with char. exp. $\phi(|\xi|^2)$.

Construction of a supersolution, part 2

Lemma (B., Wagner, 2024+)

The function $U = \psi(V(\delta_D(x))$ satisfies $U \in L^1(D, \delta_D(x)dx)$ if and only if

$$\int_1^\infty \frac{dt}{\phi^{-1}(\varphi(t)^{-2})} < \infty.$$

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If in addition

$$\int_r^\infty \frac{dt}{\phi^{-1}(\varphi(t)^{-2})} \lesssim \frac{r}{\phi^{-1}(\varphi(r)^{-2})}, \qquad r \geq 1, \qquad \text{(KO)}$$

then there exist constants $C > 0$ *and* $\eta > 0$ *such that*

$$\phi(-\Delta|_D)U(x) \geq -Cf(U(x)), \quad x \in D_\eta,$$

where $D_\eta = \{x \in D : \delta_D(x) < \eta\}$.

By modifying U to $\overline{U} := \lambda U + \mu G_D^{\phi}\mathbf{1}$ for some $\mu, \lambda > 0$, we get

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Corollary (B., Wagner, 2024+)

Let f satisfy (F) *and* (KO). *Then there is a function* $\overline{U} \in L^1(D, \delta_D(x)dx) \cap C^{1,1}(D)$ *such that*

$$\phi(-\Delta|_D)\overline{U} \geq -f(\overline{U}), \quad \text{in } D,$$

both in the distributional and pointwise sense. Furthermore, assume that

$$\lim_{s \to 0+} \frac{\psi(s)}{s^2 \phi^{-1}(s^{-2})} = \infty, \tag{B}$$

$$\text{then} \quad \lim_{x \to \partial D} \frac{\overline{U}(x)}{P_D^{\phi}\sigma(x)} = \infty.$$

Large solution under (F), (KO), and (B)

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Theorem (B. Wagner, 2024+)

The function u is in $L^1(D, \delta_D(x)dx)$ and is a distributional and a pointwise solution to the semilinear problem

$$\begin{aligned} -\phi(-\Delta|_D)u &= f(u) && \text{in } D, \\ \frac{u}{P_D^{\phi}\sigma} &= \infty && \text{on } \partial D. \end{aligned}$$

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