3.5 Uvjetne razdiobe

Neka je $(\Omega, \mathcal{F}, \mathbb{P})$ vjer. prostor i X diskretna sluč. var t.d. $D_X = \{a_i : i \in I\}$. Distr. od X (s obzirom na $\underline{\underline{\mathbb{P}}}$) je $\begin{pmatrix} a_1 & a_2 & \cdots \\ p_1 & p_2 & \cdots \end{pmatrix}$ za $p_i := \underline{\underline{\mathbb{P}}}(X = a_i),\ i \in I$.

Za $B \in \mathcal{F}$ t.d. $\mathbb{P}(B) > 0$, <u>uvjetna distribucija</u> od X uz dano B je $\begin{pmatrix} a_1 & a_2 & \cdots \\ p_1^B & p_2^B & \cdots \end{pmatrix}$ za

$$p_i^B := \mathbb{P}(X = a_i \mid B) = \mathbb{P}_B(X = a_i),\ i \in I$$

distr. od X s obzirom na $\mathbb{P}_B$!

$\hookrightarrow$ uvjetno očekivanje od X uz dano B (oznaka $\mathbb{E}[X \mid B]$) defin. kao očekivanje od X s obzirom na $\underline{\underline{\mathbb{P}_B}}$.

Tm. 3.42 Neka je $(H_j)_{j \in J}$ PSD (uz $\mathbb{P}(H_j) > 0, \forall j$) ($J = \{1, \dots, n\}$ ili $J = \mathbb{N}$) te X diskretna sluč. var. t.d. (a) $X \geq 0$, ili (b) $\mathbb{E}[|X|] < \infty$.

$$\Rightarrow \quad \mathbb{E}[X] = \sum_{j \in J} \mathbb{P}(H_j)\, \mathbb{E}[X \mid H_j]. \quad (3.26)$$

Dokaz Neka je $X \sim \begin{pmatrix} a_1 & a_2 & \cdots \\ p_1 & p_2 & \cdots \end{pmatrix}$.

1° Ako je $\mathbb{E}[|X|] < +\infty$,

$$\mathbb{E}[\,|X| \mid H_j] = \sum_{i \in I} |a_i| \cdot \overbrace{\mathbb{P}(X = a_i \mid H_j)}^{= \frac{\mathbb{P}(\{X = a_i\} \cap H_j)}{\mathbb{P}(H_j)}}$$

$$\leq \frac{1}{\mathbb{P}(H_j)} \underbrace{\sum_{i \in I} |a_i| \cdot \mathbb{P}(X = a_i)}_{= \mathbb{E}[|X|]} < +\infty, \quad \forall j.$$

$\overset{(b)}{\Rightarrow} \boxed{\exists\, \mathbb{E}[X \mid H_j], \forall j!}$

ako je $X \geq 0$, $\mathbb{E}[X \mid H_j]$ postoji $\forall j \in J$ jer se D_X ne mijenja.

(2°) $\mathbb{E}[X] = \sum_{i\in I} a_i p_i \overset{FPV}{=} \sum_{i\in I} a_i \cdot \sum_{j\in J} \mathbb{P}(X=a_i \mid H_j)\,\mathbb{P}(H_j)$

$$= \sum_{j\in J} \mathbb{P}(H_j) \underbrace{\sum_{i\in I} a_i\, \mathbb{P}(X=a_i \mid H_j)}_{=\mathbb{E}[X \mid H_j]} . \blacksquare$$

Pr. 3.43 Bacamo novčić t.d. je vjerojatnost za glavu $p \in \langle 0,1\rangle$. Neka je R duljina prvog niza (to može biti niz glava ili niz pisama). $\Rightarrow$ $\mathbb{E}[R] = ?$

Rj. $q := 1-p$, $H_1 := \{$u prvom bacanju $G\}$, $H_2 = H_1^c$

$$\underset{R\geq 1 \geq 0}{\Rightarrow} \mathbb{E}[R] = \underbrace{\mathbb{P}(H_1)}_{=p} \mathbb{E}[R \mid H_1] + \underbrace{\mathbb{P}(H_2)}_{=q} \mathbb{E}[R \mid H_2]$$

- uvjetno na H_1, prvi niz je niz $\underline{\underline{G}}$, te R s obzirom na $\mathbb{P}_{H_1}$ ima istu razdiobu kao sl. var. $\underline{\underline{1+X}}$ gdje je $X \sim G_0(q)$ $\Rightarrow$ $\mathbb{E}[R \mid H_1] = \mathbb{E}[1+X] = 1 + \frac{p}{q} = \frac{1}{q}$ //

  očekujemo prvo P!

- analogno, $\mathbb{E}[R \mid H_2] = 1 + \frac{q}{p} = \frac{1}{p}$ //

$$\Rightarrow \mathbb{E}[R] = p \cdot \frac{1}{q} + q \cdot \frac{1}{p} = \frac{p}{q} + \frac{q}{p} . \blacksquare$$

DZ Ako je R_n duljina n-tog niza, pokažite da je
$(R = R_1)$

$\mathbb{E}[R_{2k-1}] = \mathbb{E}[R_1]$, $\mathbb{E}[R_{2k}] = 2$, $\underline{\underline{\forall k \geq 1}}$ te

(?) $\mathbb{E}[R_1] \geq 2$, uz jednakost $\Leftrightarrow$ $p = q = \frac{1}{2}$!