4.5 Zbroj slučajnih varijabli

Pr. 4.33 Ako je $X \sim B(n, \underline{p})$ i $Y \sim B(m, \underline{p})$, te su X i Y nezavisne, $\boxed{X+Y \sim B(m+n, p)}$.

Dokaz Neka su $X_1, X_2, \dots, X_{n+m}$ nezavisne i.d.

$X_i \sim \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$, $\forall i$. Za $X := \sum_{i=1}^{n} X_i$, $Y := \sum_{i=n+1}^{m+n} X_i$,

imamo • $X \sim B(n,p)$, $Y \sim B(m,p)$

npr.

$x_i \in \{0,1\}$, te $(X_i)_i$ nj.d.

$$\mathbb{P}(X=k) = \mathbb{P}\left(\sum_{i=1}^{n} X_i = k\right) = \binom{n}{k} \mathbb{P}(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0)$$

$$\underset{\text{nezavisnost}}{=} \binom{n}{k} \mathbb{P}(X_1 = 1) \cdots \mathbb{P}(X_k = 1)\, \mathbb{P}(X_{k+1} = 0) \cdots \mathbb{P}(X_n = 0)$$

$$= \binom{n}{k} p^k q^{n-k}, \quad \forall k \in \{0, 1, \dots, n\}.$$

• X i Y nezavisne jer $X = g(X_1, \dots, X_n)$, a $Y = h(X_{n+1}, \dots, X_{m+n})$.

Nadalje, $X + Y = \sum_{i=1}^{m+n} X_i \sim B(m+n, p)$. ∎

Pr. 4.34 Ako je $\boxed{X \sim P(\lambda)}$ i $\boxed{Y \sim P(\mu)}$, te su X i Y nezavisne, vrijedi $X+Y \sim \underline{P}(\lambda+\mu)$.

Intuicija: Zakon rijetkih događaja + Pr. 4.33

Dokaz $X, Y \in \mathbb{N}_0 \Rightarrow X+Y \in \mathbb{N}_0$

$\forall k \in \mathbb{N}_0$,

$$\mathbb{P}(X+Y=k) = \sum_{i=0}^{\infty} \mathbb{P}(X=i, \overset{i}{\cancel{X}}+Y=k)$$

$$= \sum_{i=0}^{\infty} \mathbb{P}(X=i, Y=k-i) \underset{Y \in \mathbb{N}_0}{=} \sum_{i=0}^{k} \mathbb{P}(X=i, Y=k-i)$$

"konvolucijska formula"

$$= \sum_{i=0}^{k} \mathbb{P}(X=i)\,\mathbb{P}(Y=k-i)$$

nez.

$$= \sum_{i=0}^{k} \frac{\lambda^i}{i!} e^{-\lambda} \cdot \frac{\mu^{k-i}}{(k-i)!} e^{-\mu} \cdot \frac{k!}{k!}$$

$$= e^{-(\lambda+\mu)} \cdot \frac{1}{k!} \underbrace{\sum_{i=0}^{k} \binom{k}{i} \lambda^i \mu^{k-i}}_{=(\lambda+\mu)^k}$$

$$= e^{-(\lambda+\mu)} \cdot \frac{(\lambda+\mu)^k}{k!} . \quad \boxtimes$$

[Kasnije ćemo vidjeti bolji način preko f-je izvodnica.]