FUNKCIJE

Def. Funkcija $f: A \to B$ je relacija $f \subseteq A \times B$ takva da za svaki $x \in A$ postoji jedinstveni $y \in B$ takav da je $(x,y) \in f$.

(druga Def.)

Definicija: Funkcija je uredena trojka (A, B, f): A – domena, B – kodomena, f – pravilo koje svakom elementu skupa A pridružuje točno jedan element skupa B

Primjeri:
- $|\,|: \mathbb{R} \to \mathbb{R}$ (apsolutna vrijednost)
- $|\,|: \mathbb{C} \to \mathbb{R}$ (modul)
- $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 - 2x + 7$ (kvadratna funkcija)
- $g: \mathbb{N} \to \mathbb{N}$ $g(n) = n+1$ (sljedbenik)
- $h: \mathbb{N} \to \mathbb{N}$ $h(x) = \begin{cases} \frac{x}{2}, & x \text{ paran} \\ 3x+1, & x \text{ neparan} \end{cases}$

Je nužno za svaki $y \in B$ postojati $x \in A$ t.d. je $f(x) = y$.

Def. slika funkcije f $\mathrm{Im}(f) = \{f(x) : x \in A\}$ (druga oznaka R_f)

f je surjekcija na B ako je $\mathrm{Im}(f) = B$

f je injekcija ako $f(x_1) = f(x_2)$ povlači $x_1 = x_2$.

f je bijekcija ako je surjekcija i injekcija.

Neka je $A_1 \subseteq A$ $f(A_1) = \{f(x) : x \in A_1\}$ slika skupa A_1

Neka je $B_1 \subseteq B$ $f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}$ praslika skupa B_1,

Zad. Ispitajte injektivnost, surjektivnost i bijektivnost funkcija $f(x) = x^2$ i $g(x) = |x|$, $f: \mathbb{R} \to \mathbb{R}^+$, $g: \mathbb{R} \to \mathbb{R}$.

Rj:

$x^2 = y$

$x = \sqrt{y} \Rightarrow (\forall y \in \mathbb{R}^+)(\exists x \in \mathbb{R})\ (x^2 = y) \Rightarrow f$ je surjekcija

$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \not\Rightarrow x_1 = x_2$ (npr. $(-3)^2 = 3^2$) $\Rightarrow f$ nije injekcija

$\Rightarrow f$ nije bijekcija

$(\forall y < 0)(\nexists x \in \mathbb{R})\ (|x| = y) \Rightarrow g$ nije surjekcija

$g(x_1) = g(x_2) \Rightarrow |x_1| = |x_2| \not\Rightarrow x_1 = x_2$ (npr. $|-2| = |2|$) $\Rightarrow g$ nije injekcija

$\Rightarrow g$ nije bijekcija

Nap. $f|_{\mathbb{R}^+}$ je injekcija pa i bijekcija

$g: \mathbb{R} \to \mathbb{R}^+$ je surjekcija, ali ne i injekcija

$g|_{\mathbb{R}^+}$ je surjekcija i injekcija pa i bijekcija

Zad. Zadana je funkcija $f: \langle -1, 1\rangle \to \langle -2, 2\rangle$ formulom

$$f(x) = x - x^2.$$

Odredite $B \subseteq \langle -2, 2\rangle$ takav da $g: \langle -1, 1\rangle \to B$, zadana istim pravilom, bude surjekcija. Zatim odredite $A \subseteq \langle -1, 1\rangle$ takav da $h: A \to B$, zadana istim pravilom, bude bijekcija.

Rj.

$f(x) = x - x^2$ je kvadratna funkcija

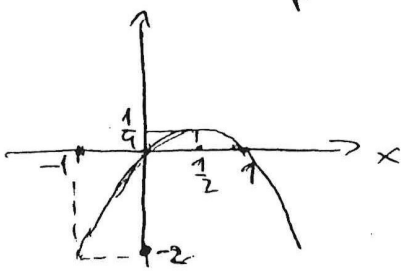
$x_0 = -\frac{b}{2a} = \frac{1}{2}$

$y_0 = \frac{4ac - b^2}{4a} = \frac{1}{4}$

$(\frac{1}{2}, \frac{1}{4})$ je tjeme, a otvor parabole je prema dolje

nultočke: $x_1 = 0$, $x_2 = 1$

graf



Iz grafa i gornje navedenih svojstava iščitamo:

$g: \langle -1, 1\rangle \to \langle -2, \frac{1}{4}]$ je surjekcija $\Rightarrow B = \langle -2, \frac{1}{4}]$

$h: \langle -1, \frac{1}{2}] \to \langle -2, \frac{1}{4}]$ je bijekcija $\Rightarrow A = \langle -1, \frac{1}{2}]$

↓

("komad" $\langle \frac{1}{2}, 1\rangle$ služi za injektivnost)

Zad. Odredite maksimalne skupove $A, B \subseteq \mathbb{R}$ takve da je s

$$f(x) = \frac{x+1}{x-2}$$

zadana bijekcija s A na B.

Odredite inverznu funkciju $f^{-1}: B \to A$.

Rj.

Odmah vidimo: $x \neq 2$ $\quad f(x_1) = f(x_2)$

$$\frac{x_1+1}{x_1-2} = \frac{x_2+1}{x_2-2}$$

$$x_1x_2 - 2x_1 + x_2 - 2 = x_1x_2 + x_1 - 2x_2 - 2$$

$3x_2 = 3x_1$

$x_1 = x_2$

$\Rightarrow f$ je injekcija na $\mathbb{R}\setminus\{2\}$

x stavimo: $\frac{x+1}{x-2} = y$

$x+1 = xy-2y$

$x-xy = -2y-1$

$x(1-y) = -2y-1$

$x = \frac{2y+1}{y-1}$, otito $y \neq 1$

$\Rightarrow (\forall y \in \mathbb{R}\setminus\{1\})(\exists x \in \mathbb{R}\setminus\{2\})\ (f(x)=y)$

↑ tj $x = \frac{2y+1}{y-1}$ $\Rightarrow$ f je surjekcija na $\mathbb{R}$

Traženi maksimalni skupovi su $A = \mathbb{R}\setminus\{2\}$, $B = \mathbb{R}\setminus\{1\}$.

$x = \frac{2y+1}{y-1}$ uz $y \leftarrow x$ i $x \leftarrow f^{-1}(x)$ odmah dobivamo:

$f^{-1}(x) = \frac{2x+1}{x-1}$, $f^{-1}: \mathbb{R}\setminus\{1\} \to \mathbb{R}\setminus\{2\}$.

Neka je $f: A \to B$, $g: D$ i $B \subseteq C$. Tada je kompozicija f i g $g \circ f: A$ definirana s $(g \circ f)(x) = g(f(x))$.

Dokaži:

a) $g \circ f$ injekcija, tj je f injekcija.

b) $g \circ f$ surjekcija, tj g surjekcija.

a) $f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 =$

dom b) Pokažimo. Pretp. da f nije injekcija. Dakle,

$\exists x_1 \neq x_2 + d.\ j.\ f(x_1) = f(x_2) \mid g$

$g(f(x_1)) = g(f(x_2))$

$(g \circ f)(x_1) = (g \circ f)(x_2)$

tj $g \circ f$ injekcija, $x_1 = x_2$. $\not\in$

b) Neka je $g \circ f$ kopa i neka je $d \in D$ proizvoljan. Tada

$\exists a \in A + d.\ j.\ (g \circ f)(a) = g(f(a)) = d$. Ako označimo

$f(a) = b \in B \subseteq C$, da postoji $c \in C$ + d. je $g(c) = d$ pa

g surjekcija.

Def. Neka je $f: A \to B$, $A, B \subseteq \mathbb{R}$. Kažemo da je

f (strogo) rastuća ako $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) \overset{(<)}{\leq} f(x_2)$

f (strogo) padajuća ako $x_1, x_2 \in A$, $x_1 < x_2 \Rightarrow f(x_1) \overset{(>)}{\geq} f(x_2)$

Zad. Neka je $f: A \to B$ strogo rastuća funkcija, $A, B \subseteq \mathbb{R}$. Dokaži

$$x_1 < x_2 \iff f(x_1) < f(x_2).$$

Rj.

$\Rightarrow$ iz definicije

$\Leftarrow$ Neka su $x_1, x_2 \in A$ t.d. je $f(x_1) < f(x_2)$. Pretp. da nije

vrijedi $x_1 < x_2$, tj. da je $x_1 \geq x_2$. Tada je:

1° $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$

2° $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ (er je f strogo rastuća)

Dakle, $f(x_1) \geq f(x_2)$, što je $\not\downarrow$.

Zad. Pokažite da je svaka strogo rastuća funkcija injekcija.

Rj.

Neka je $f: A \to B$ koja je strogo rast

Neka su $x_1, x_2 \in A$, $x_1 \neq x_2$. Tada imamo slučaja:

1° $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x_1) \neq f(x_2)$

2° $x_2 < x_1 \Rightarrow f(x_2) < f(x_1) \Rightarrow f(x_2) \neq f(x_1)$

$\Rightarrow f(x_1) \neq f(x_2)$

Pokazali smo: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, a to je ekvivalentno

definiciji injektivnosti (obrat po kontrapoziciji).

Zad. Neka je $f: A \to B$ strogo rastuća surjekcija. Pokažite

da je $f^{-1}: B \to A$ također strogo rastuća.

Rj.

Neka su $y_1, y_2 \in B$, $y_1 < y_2$. Treba $f^{-1}(y_1) < f^{-1}(y_2)$.

$f(f^{-1}(y_1)) < f(f^{-1}(y_2))$ (tu koristimo $f \circ f^{-1} = id$)

$\Rightarrow y_1 < y_2$ $\lightning$

$f^{-1}(B)=$.

1. način (grafički):

$f^{-1}(B)=\langle -1,0]$

2. način (računski):

$f^{-1}(B)=\{x\in\mathbb{R}: f(x)\in B\}=\{x\in\mathbb{R}: x^3\in\langle -1,0]\}$

$\Rightarrow -1<x^3 \;\text{ i }\; x^3\le 0$

$x^3+1>0 \qquad x\le 0$

$(x+1)\underbrace{(x^2-x+1)}_{>0}>0$

$[D<0, a>0]$

$\Rightarrow x>-1 \;\text{ i }\; x\le 0$

$\Rightarrow x\in\langle -1,0]$

$\Rightarrow f^{-1}(B)=\langle -1,0]$

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Zad. Ako je $f(x)=\frac{x^2+1}{x^2-1}$, odredite D_f i $f^{-1}(\langle -\infty,-2\rangle)$.

Rj.:

$D_f=\mathbb{R}\setminus\{-1,1\}$

$f^{-1}(\langle -\infty,-2\rangle)=\{x\in\mathbb{R}: f(x)\in\langle -\infty,-2\rangle\}=$

$=\{x\in\mathbb{R}: f(x)<-2\}=$

$=\{x\in\mathbb{R}: \frac{x^2+1}{x^2-1}<-2\}$

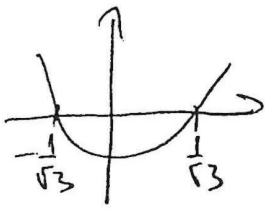
$\frac{x^2+1}{x^2-1}<-2$

$\frac{x^2+1}{x^2-1}+2<0$

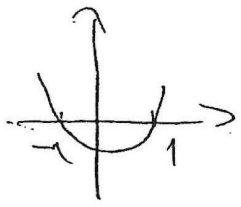
$\frac{x^2+1+2x^2-2}{x^2-1}<0$

$\frac{3x^2-1}{x^2-1}<0$

$3x^2-1$



x^2-1

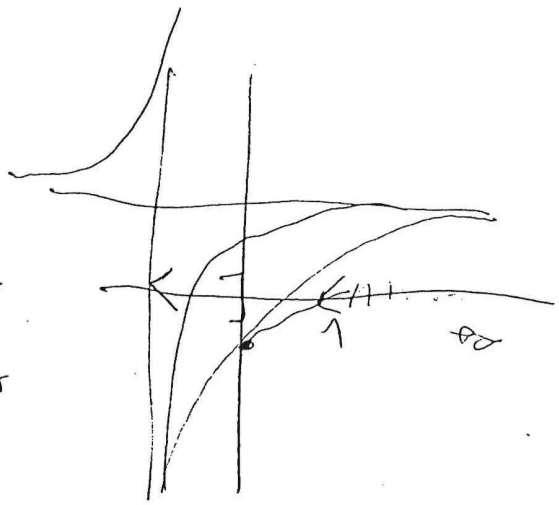


	$-\infty$	-1	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1	$+\infty$
$3x^2-1$	+	+	−	+	+	
x^2-1	+	−	−	−	+	
	+	⊖	+	⊖	+	

$\Rightarrow x\in\langle -1,-\frac{1}{\sqrt{3}}\rangle\cup\langle\frac{1}{\sqrt{3}},1\rangle$

$\Rightarrow f^{-1}(\langle -\infty,-2\rangle)=\langle -1,-\frac{1}{\sqrt{3}}\rangle\cup\langle\frac{1}{\sqrt{3}},1\rangle$

Zad. Odredite prirodnu domenu i sliku funkcije $f(x)=\frac{x-1}{x+1}$. Je li f injekcija na prirodnoj domeni? Odredite $f(\langle 1,+\infty\rangle)$ i $f(\langle -1,0]\rangle)$.



Rj.

$D_f = \mathbb{R}\setminus\{-1\}$

$$f(x)=\frac{x-1}{x+1}=\frac{x+1-2}{x+1}=1-\frac{2}{x+1}=y$$

$$\frac{2}{x+1}=1-y$$

$$x+1=\frac{2}{1-y}$$

$$x=\frac{2}{1-y}-1$$

$\Rightarrow (\forall y\in\mathbb{R}\setminus\{1\})(\exists x\in\mathbb{R}\setminus\{-1\})$ t.d. $f(x)=y$

$\Rightarrow R_f=\mathbb{R}\setminus\{1\}$

$f(x_1)=f(x_2)$

$$1-\frac{2}{x_1+1}=1-\frac{2}{x_2+1}$$

$$\frac{1}{x_1+1}=\frac{1}{x_2+1}$$

$$x_1+1=x_2+1$$

$$x_1=x_2 \Rightarrow f \text{ je injekcija na } D_f$$

$f(\langle 1,+\infty\rangle)=?$

$$\frac{2}{1-y}-1>1$$

$$\frac{2}{1-y}-2>0$$

$$\frac{2(-2+2y)}{1-y}>0$$

$$\frac{y}{1-y}>0$$

	$-\infty$		0		1		$+\infty$
y	$-$	ϕ	$+$		$+$		
$1-y$	$+$		$+$	$\oplus$	$-$		
	$-$		$\oplus$		$-$		

$y\in\langle 0,1\rangle$

$\Rightarrow f(\langle 1,+\infty\rangle)=\langle 0,1\rangle$

$f(\langle -1,0]\rangle=?$

$$\frac{2}{1-y}-1>-1 \quad \& \quad \frac{2}{1-y}-1\le 0$$

$$\frac{2}{1-y}>0 \qquad \frac{2-1+y}{1-y}\le 0$$

$$\Rightarrow 1-y>0 \qquad \frac{y+1}{1-y}\le 0$$

$$y<1$$

$$y\in\langle -\infty,1\rangle$$

	$-\infty$		-1		1		$+\infty$
$y+1$	$-$	ϕ	$+$		$+$		
$1-y$	$+$		$+$	ϕ	$-$		
	$(-)$		$+$		$\to$		

$y\in\langle -\infty,-1]\cup\langle 1,$

$\Rightarrow y\in\langle -\infty,-1]$

$\Rightarrow f(\langle -1,0]\rangle=\langle -\infty,-1]$

Zad. Neka su $f: X \to Y$ i $g: Z \to W$ dvije funkcije takve da je $f(X) \subseteq Z$. Neka je $A \subseteq X$ i $B \subseteq W$. Pokažite da je:

a) $(g \circ f)(A) = g(f(A))$

b) $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$

Rj.

a) $z \in (g \circ f)(A) \Leftrightarrow \exists x \in A$ t.d. $(g \circ f)(x) = z$

$\Leftrightarrow \exists x \in A$ t.d. $g(f(x)) = z$

$\Leftrightarrow \exists y \in f(A)$ t.d. $g(y) = z$

$\Leftrightarrow z \in g(f(A))$

b) $x \in (g \circ f)^{-1}(B) \Leftrightarrow (g \circ f)(x) \in B$

$\Leftrightarrow g(f(x)) \in B$

$\Leftrightarrow f(x) \in g^{-1}(B)$

$\Leftrightarrow x \in f^{-1}(g^{-1}(B))$

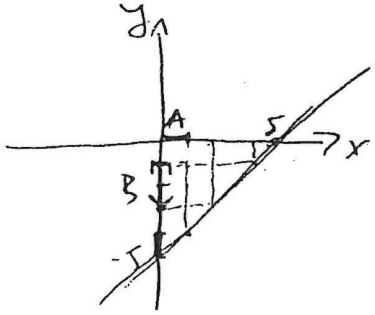
Zad. Skicirajte grafove funkcija f te odredite R_f, $f(A)$ i $f^{-1}(B)$ ako je:

a) $f(x) = x - 5$ $\quad A = [0, 1]$ $\quad B = \langle -3, -1]$

b) $f(x) = x^2 + 1$ $\quad A = \langle -1, 1 \rangle$ $\quad B = [4, 5 \rangle$

c) $f(x) = x^3$ $\quad A = \langle -1, 1 \rangle$ $\quad B = \langle -1, 0]$.

Rj. a)



$R_f = f(\mathbb{R}) = \mathbb{R}$

$f(A) = [-5, -4]$

1. način (pomoću grafa):

$f^{-1}(B) = ?$

$f^{-1}(B) = \langle 2, 4]$

2. način (računski):

$f^{-1}(B) = \{x \in \mathbb{R}: f(x) \in B\} = \{x \in \mathbb{R}: x-5 \in \langle -3, -1]\} = \{x \in \mathbb{R}: -3 < x-5 \leq -1\}$ (6)

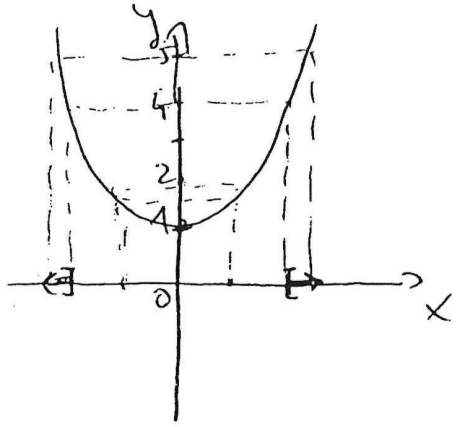
$x-5 > -3 \quad \wedge \quad x-5 \leq -1$

$x > 2 \qquad\qquad x \leq 4$

$\Rightarrow x \in \langle 2, 4]$

$\Rightarrow f^{-1}(B) = \langle 2, 4]$

b)



$R_f = [1, +\infty)$

$f(A) = [1, 2\rangle$

$f^{-1}(B) = ?$

1. način (grafično):

$x^2 + 1 = 4$

$x = \pm 3$

$f^{-1}(B) = \langle -2, -\sqrt{3}] \cup [\sqrt{3}, 2\rangle$

2. način (računski):

$f^{-1}(B) = \{x \in \Omega: f(x) \in B\} =$

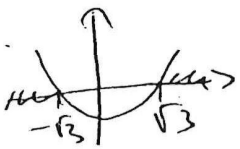
$= \{x \in \Omega: x^2+1 \in [4, 7\rangle\} =$

$= \{x \in \mathbb{R}: 4 \leq x^2+1 < 5\} =$

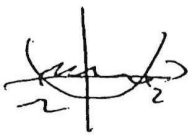
$= \{x \in \mathbb{R}: 3 \leq x^2 < 4\}$

$x^2 \geq 3 \quad \& \quad x^2 < 4$

$x^2 - 3 \geq 0 \quad \& \quad x^2 - 4 < 0$



$x \in \langle -\infty, -\sqrt{3}] \cup [\sqrt{3}, +\infty\rangle$

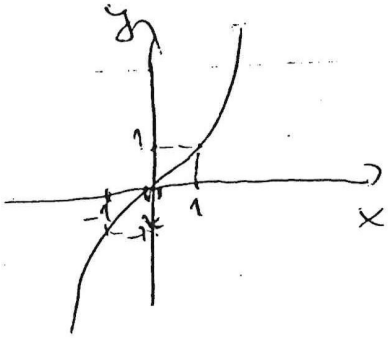


$x \in \langle -2, 2\rangle$

$\Rightarrow x \in \langle -2, \sqrt{3}] \cup [\sqrt{3}, 2\rangle$

$\Rightarrow f^{-1}(B) = \langle -2, \sqrt{3}] \cup [\sqrt{3}, 2\rangle$

c)



$R_f = f(\mathbb{R}) = \mathbb{R}$

$f(A) = \langle -1, 1\rangle$

MM 26.

Zad. Zadanoj funkciji odredite sliku tako da je prikažete kao kompoziciju funkcija:

a) $f:[0,1\rangle \to \mathbb{R}$, $f(x)=x^4+5x^2+6$

b) $f:\langle -1,2] \to \mathbb{R}$, $f(x)=2x^2+3|x|+1$

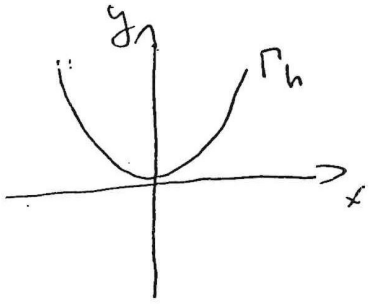
Rj:

a) $f=g\circ h$

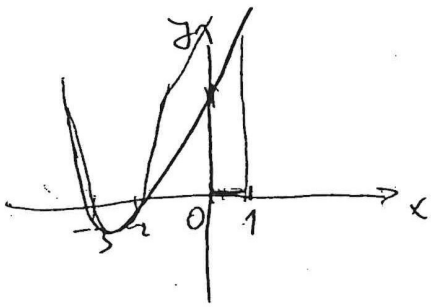
$h:[0,1\rangle \to \mathbb{R}$, $h(x)=x^2$

$g:\mathbb{R}\to\mathbb{R}$, $g(x)=x^2+5x+6=(x+2)(x+3)$

Tada je $R_f=f([0,1\rangle)=(g\circ h)([0,1\rangle)=g(h([0,1\rangle))=(*)=[6,12\rangle$



$\Rightarrow h([0,1\rangle)=[0,1\rangle$



(*) $g([0,1\rangle)=[6,12\rangle$

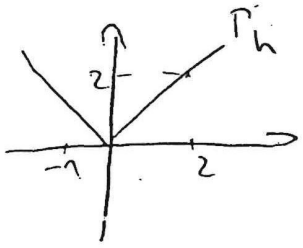
b) $f=g\circ h$

$h:\langle -1,2]\to\mathbb{R}$, $h(x)=|x|$

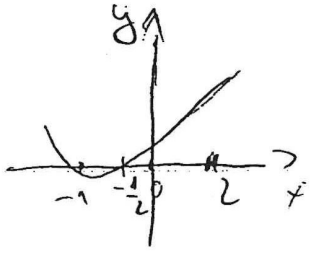
$g:\mathbb{R}\to\mathbb{R}$, $g(x)=2x^2+3x+1=2(x+1)(x+\frac{1}{2})$

Tada je $R_f=f(\langle -1,2])=(g\circ h)(\langle -1,2])=g(h(\langle -1,2]))=g([0,2))=$

$=[1,15]$



$h(\langle -1,2])=[0,2]$



$g([0,2])=[1,15]$

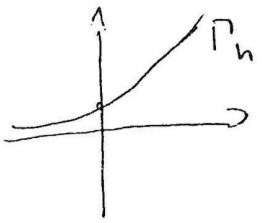
Zad. Neka je $f(x)=4^x-2^{x+1}-3$. Odredite $f([0,2])$ i $f^{-1}([5,45])$ (10)

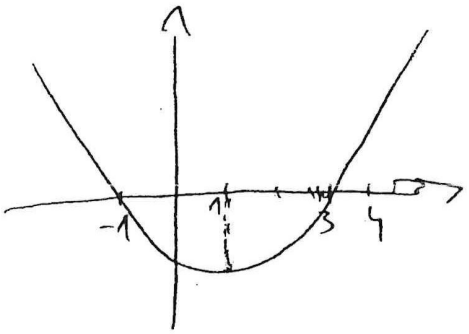
Rj:

$f=g\circ h$

$h(x)=2^x$, $g(x)=x^2-2x-3=(x+1)(x-3)$

$\Rightarrow f([0,2])=(g\circ h)([0,2])=g(h([0,2]))=g([1,4])=[-4,5]$





$x_0=\frac{-b}{2a}=\frac{2}{2}=1 \quad g(1)=-4$

$g(4)=5$

$f^{-1}([5,45])=(g\circ h)^{-1}([5,45])=h^{-1}(g^{-1}([5,45]))=h^{-1}([-6,-2]\cup[4,8]$

$=h^{-1}([-6,-2])\cup h^{-1}([4,$

$=\emptyset$

$2^x>4 \;\&\; 2^x$

$x>2 \;\&\; x<$

$x\in\langle 2,3\rangle$

$x^2-2x-3>5 \;\&\; x^2-2x-3\leq 45$

$x^2-2x-8>0 \;\&\; x^2-2x-48\leq 0$

$x_{1,2}=\frac{2\pm\sqrt{4+32}}{2}$

$=\frac{2\pm 6}{2}\begin{matrix}\nearrow 4\\ \searrow -2\end{matrix}$

$x_{1,2}=\frac{2\pm\sqrt{4+192}}{2}$

$x_{1,2}=\frac{2\pm 14}{2}\begin{matrix}\nearrow 8\\ \searrow -6\end{matrix}$

$x\in\langle -\infty,-2\rangle\cup\langle 4,+\infty\rangle \qquad x\in\langle -6,8\rangle$

$\Rightarrow x\in\langle -6,-2\rangle\cup\langle 4,8\rangle$

$\Rightarrow f^{-1}([5,45])=\langle 2,3\rangle$

$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

ad. Neka je $f(x)=(2+\sqrt{3})^{\quad}+(2-\sqrt{3})^{\quad}$.

Odredite $f^{-1}\left(\left\{\frac{4}{2-\sqrt{3}}\right\}\right)$. JOBE

$f^{-1}\left(\left\{\frac{4}{2-\sqrt{3}}\right\}\right)=\left\{x\in D_f : f(x)=\frac{4}{2-\sqrt{3}}\right\}=\left\{x\in\mathbb{R}: (2+\sqrt{3})^{x^2-2x+1}+(2-\sqrt{3})^{x^2-2x-1}=\frac{4}{2-\sqrt{3}}\right\}$

$(2+\sqrt{3})^{x^2-2x+1}+(2-\sqrt{3})^{x^2-2x+1}=\frac{4}{2-\sqrt{3}} \mid \cdot(2-\sqrt{3})^{x^2-2x+1}$

$1+(2-\sqrt{3})^{2(x^2-2x)}=4(2-\sqrt{3})^{x^2-2x}$

$t:=(2-\sqrt{3})^{x^2-2x} \Rightarrow 1+t^2=4t$

$t^2-4t+1=0$

$t_{1,2}=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$

$t_1=(2+\sqrt{3})=(2-\sqrt{3})^{x^2-2x} \mid \cdot(2-\sqrt{3})$

$1=(2-\sqrt{3})^{x^2-2x+1}$

$(2-\sqrt{3})^0=(2-\sqrt{3})^{x^2-2x+1}$

$x^2-2x+1=0$

$(x-1)^2=0$

$\boxed{x=1}$

$t_2=(2-\sqrt{3})^1=(2-\sqrt{3})^{x^2-2x}$

$1=x^2-2x$

$x^2-2x-1=0$

$x_{1,2}=\frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}$

$\boxed{x_1=1+\sqrt{2}}$ $\boxed{x_2=1-\sqrt{2}}$

$\Rightarrow f^{-1}\left(\left\{\frac{4}{2-\sqrt{3}}\right\}\right)=\{1-\sqrt{2},1,1+\sqrt{2}\}$

Zad. Neka je $f(x)=5^x-5^{3-x}$. Odredite $f^{-1}(\langle-\infty,20])$.

Rj. DZ

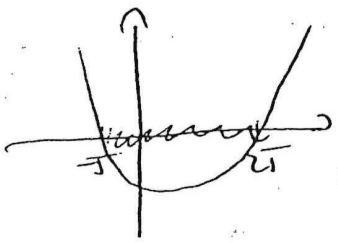
$f^{-1}(\langle-\infty,20])=\{x\in D_f : f(x)\in\langle-\infty,20]\}=\{x\in\mathbb{R}: 5^x-5^{3-x}\le 20\}$

$5^x-5^{3-x}\le 20 \mid \cdot 5^x \ (5^x>0)$

$5^{2x}-20\cdot 5^x-125\le 0$

$t:=5^x \quad t^2-20t-125\le 0$

$t_{1,2}=\frac{20\pm\sqrt{400+500}}{2}=10\pm 15$



$t_1=-5 \quad t_2=25$

$\Rightarrow t\in[-5,25] \Rightarrow -5\le 5^x\le 25=5^2$

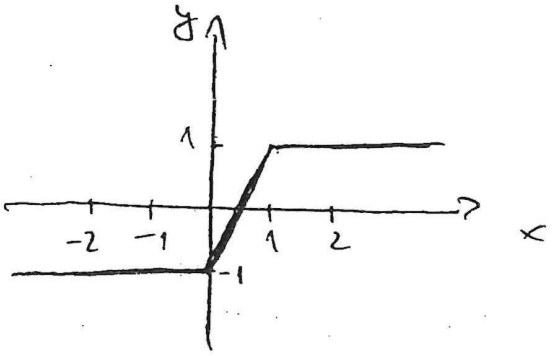
$\Rightarrow x\in\langle-\infty,2]$

$\Rightarrow f^{-1}(\langle-\infty,20])=\langle-\infty,2]$

Zad. Skicirajte graf funkcije f i odredite $f(\langle -2,2\rangle)$; $f^{-1}(\langle 1,2]$ ako je $f(x)=|x|-|x-1|$.

Rj.

$$f(x)=\begin{cases} -x-(-(x-1)) & , x\le 0 \\ x-(-(x-1)) & , 0<x\le 1 \\ x-(x-1) & , x>1 \end{cases} = \begin{cases} -1 & , x\le 0 \\ 2x-1 & , 0<x\le 1 \\ 1 & , x>1 \end{cases}$$



$\Rightarrow f(\langle -2,2\rangle)=[-1,1]$

$f^{-1}(\langle 1,2])=?$

grafički: $f^{-1}(\langle 1,2])=\emptyset$

računski: $f^{-1}(\langle 1,2])=\{x\in D_f : f(x)\in\langle 1,2]\}=$

$=\{x\in\mathbb{R} : 1<f(x)\le 2\}=\emptyset$

↓ jer je $f(x)\le 1,\ \forall x\in\mathbb{R}$

Zad. Odredite prirodnu domenu i sliku funkcije.

a) $f(x)=\sqrt{1-x}$ $\quad h:\langle -\infty,1]\to[0,+\infty\rangle,\ h(x)=1-x$

b) $f(x)=\sqrt[7]{2x^4-1}$. DZ $\quad g:[0,+\infty\rangle\to[0,+\infty\rangle,\ g(x)=\sqrt{x}$

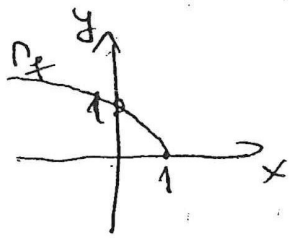
Rj.

a) $D_f=?$

$1-x\ge 0$

$x\le 1$

$D_f=\langle -\infty,1]$



→ graf funkcije f tako izgleda

$\Rightarrow R_f=[0,+\infty\rangle$ (može se obrazložiti i logički)

Skup $y\in[0,+\infty\rangle$ može se zapisati kao $y=\sqrt{1-x}$ za neki $x\in\langle -\infty,1]$.

b) $D_f = ?$

neparni korijen $\Rightarrow D_f = \mathbb{R}$

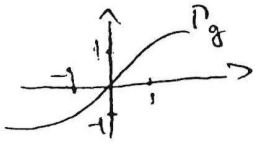
$R_f = ?$

$f = g \circ h$, $h: \mathbb{R} \to \mathbb{R}$, $h(x) = 2x^4 - 1$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = \sqrt[3]{x}$

$$\Rightarrow R_f = R_{g \circ h} = (g \circ h)(\mathbb{R}) = g(h(\mathbb{R})) \overset{(1)}{=} g([-1, +\infty>) \overset{(2)}{=} [-1, +\infty> \checkmark$$

(1) $2x^4 \geq 0 \Rightarrow 2x^4 - 1 \geq -1 \Rightarrow h(\mathbb{R}) \in [-1, +\infty>$

(2) 

(tako izgledaju grafovi neparnih korijena)

Zad. Neka je $f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$.

a) Odredite D_f.

b) Odredite $f^{-1}(\{4, 6\})$.

c) Odredite $f^{-1}([2, 8>)$.

Rj.

a) $x - 1 \geq 0$

$x + 2\sqrt{x-1} \geq 0$

$x - 2\sqrt{x-1} \geq 0$

$x - 1 \geq 0$

$x - 2\sqrt{x-1} \geq 0$

$x \geq 1$

$x \geq 2\sqrt{x-1}$

$x \geq 1$

$x^2 \geq 4(x-1)$

$x \geq 1$

$x^2 - 4x + 4 \geq 0$

$x \geq 1$

$(x-2)^2 \geq 0$

$\Rightarrow D_f = [1, +\infty>$

$(x + 2\sqrt{x-1})(x - 2\sqrt{x-1}) \geq 0$

$x^2 - 4(x-1) \geq 0$

$(x-2)^2 \geq 0$



Dalje pojedinačno:

$$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} = \sqrt{\left(\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}\right)^2} =$$

$$= \sqrt{x + 2\sqrt{x-1} + 2\sqrt{x + 2\sqrt{x-1}} \cdot \sqrt{x - 2\sqrt{x-1}} + x - 2\sqrt{x-1}} =$$

$$= \sqrt{2x + 2\sqrt{x^2 - 4(x-1)}} = \sqrt{2x + 2\sqrt{x^2 - 4x + 4}} = \overset{x \in D_f = [1, +\infty>}{\longrightarrow}$$

$$= \sqrt{2x + 2\sqrt{x^2 - 4(x-1)}} = \sqrt{2x + 2\sqrt{(x-2)^2}} = \sqrt{2x + 2|x-2|} = \ldots$$

$$\ldots = \begin{cases} \sqrt{2x^2-2x+4}, & 1 \le x \le 2 \\ \sqrt{x^2+2x-4}, & x > 2 \end{cases} = \begin{cases} 2, & 1 \le x \le 2 \\ \sqrt{4x-4}, & x > 2 \end{cases} = \begin{cases} 2, & 1 \le x \le 2 \\ 2\sqrt{x-1}, & x > 2 \end{cases}$$

$\Rightarrow f^{-1}([4,64\rangle) = \{5, 10\}$

$2\sqrt{x-1} = 4$ $\quad$ $2\sqrt{x-1} = 6$

$\sqrt{x-1} = 2$ $\quad$ $\sqrt{x-1} = 3$

$x = 5$ $\quad$ $x = 10$

$f^{-1}([2,8>) = ?$

$2\sqrt{x-1} < 8$

$\sqrt{x-1} < 4 \mid^2$

$\Rightarrow x < 17$

$\Rightarrow f^{-1}([2,8>) = [1,17>$

Zad. Ispitajte je li funkcija $f(x) = x^4 - 4x^2 - 5$ injekcija na zadanom intervalu i ako jest, odredite joj sliku i inverz.

a) $[2,3>$ $\quad$ b) $[1,2]$.

Rj.

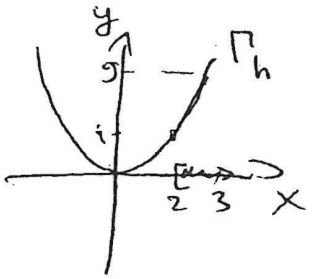
$f = g \circ h$

$h: \mathbb{R} \to \mathbb{R}, \; h(x) = x^2$

$g: \mathbb{R} \to \mathbb{R}, \; g(x) = x^2 - 4x - 5$

a) $h|_{[2,3>} : [2,3> \to \mathbb{R}$ je injekcija

(jer je h na $[2,3>$ strogo rastuća)

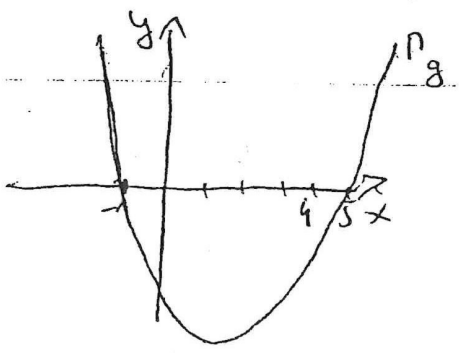


$\Rightarrow h([2,3>) = [4,9>$

$x^2 - 4x - 5 = 0$

$x_{1,2} = \frac{4 \pm \sqrt{16+20}}{2} = \frac{4 \pm 6}{2} \searrow 5 \quad \searrow -1$

$g|_{[4,9>} : [4,9> \to \mathbb{R}$ je injekcija, jer je g na $[4,9>$ strogo ras



$\Rightarrow g([4,9>) = [-5, 40>$

$g(4) = -5$

$g(9) = 40$

$\Rightarrow f|_{[2,3>} = g|_{[4,9>} \circ h|_{[2,3>}$ je injekcija kao kompozicija injekcija (4)

Tada $h|_{[2,3>} : [2,3> \to h([2,3>) = [4,9>$

$g|_{[4,9>} : [4,9> \to g([4,9>) = [-3,40>$

su surjekcije pa su i bijekcije. Stoga je i

$f|_{[2,3>} : [2,3> \to f([2,3>) = (g \circ h)([2,3>) = g(h([2,3>))$

$= g([4,9>) = [-3,40>$ bijekcija.

Nadimo joj inverz: (usput malo objasniti restrikcije funkcije!)

$$(f|_{[2,3>})^{-1} = (g|_{[4,9>} \circ h|_{[2,3>})^{-1} =$$

$$= (h|_{[2,3>})^{-1} \circ (g|_{[4,9>})^{-1}$$

wwww

$(g|_{[4,9>})^{-1} = ?$ $\quad g|_{[4,9>} : [4,9> \to [-3,40>$

Za $z \in [-5,40>$ tražimo $y \in [4,9>$ t.d. je

$$g|_{[4,9>}(y) = z$$

$$y^2 - 4y - 5 = z$$

$$y^2 - 4y - (5+z) = 0$$

$$y_{1,2} = \frac{4 \pm \sqrt{16 + 4(5+z)}}{2} = \frac{4 \pm \sqrt{36+4z}}{2} = 2 \pm \sqrt{9+z}$$

$y \in [4,9>$ => uzimamo $y = 2 + \sqrt{9+z}$

$$\Rightarrow (g|_{[4,9>})^{-1}(z) = 2 + \sqrt{9+z}$$

$(h|_{[2,3>})^{-1} = ?$ $\quad h|_{[2,3>} : [2,3> \to [4,9>$

Za $y \in [4,9>$ tražimo $x \in [2,3>$ t.d. je

$$h|_{[2,3>}(x) = y$$

$x = y$

$x_1 = \sqrt{y}$

$x_{1,2} = \pm\sqrt{y}$

$x \in [2,3\rangle \Rightarrow x = \sqrt{y} \Rightarrow (h|_{[2,3\rangle})^{-1}(y) = \sqrt{y}$

$(f|_{[2,3\rangle})^{-1} : [-5, 40\rangle \to [2,3\rangle$

$(f|_{[2,3\rangle})^{-1}(z) = (h|_{[2,3\rangle})^{-1}\left((g|_{[1,9\rangle})^{-1}(z)\right) =$

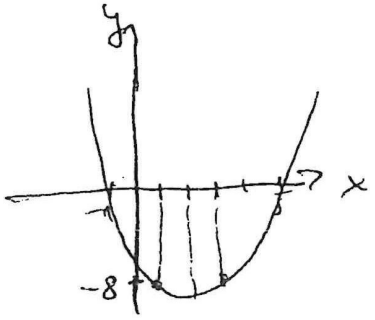
$= (h|_{[2,3\rangle})^{-1}(2+\sqrt{9+z}) = \sqrt{2+\sqrt{9+z}}$

b) $h|_{[1,2]} : [1,2] \to \mathbb{R}$ je injekcija, jer je strogo rastuća na $[1,2]$

$\Rightarrow h|_{[1,2]}([1,2]) = [1,4]$

Međutim, $g|_{[1,4]} : [1,4] \to \mathbb{R}$ <u>nije</u> injekcija, jer je

$g(1) = g(3) = -8$



$\Rightarrow f|_{[1,2]}$ nije injekcija

Zad. Odredite prirodnu domenu funkcija:

a) $f(x) = \dfrac{\sqrt{\ln(3-|x|)}}{\ln|x|}$

b) $f(x) = \log_{\pi}\left(\sqrt{x^2+4x+3} + 5x + 3\right)$

c) $f(x) = \log_{16-x^2} \dfrac{x^2+4x-5}{\sqrt{9^x - 8\cdot 3^x - 9}}$

$\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\}$ $\eta^{\alpha} = y$

$\Rightarrow \eta > 1,$

Rj. $\ln|x| \neq 0 \Rightarrow |x| \neq 1 \Rightarrow x \neq \pm 1$

a) $\left.\begin{array}{l} \cdot\ |x| > 0 \\ \cdot\ 3 - |x| > 0 \end{array}\right\} \Rightarrow 0 < |x| < 3 \Rightarrow x \in \langle -3, 0 \rangle \cup \langle 0, 3 \rangle$

$\cdot \ln(3-|x|) \geq 0$

$\ln(3-|x|) \geq \ln 1$

$3 - |x| \geq 1$

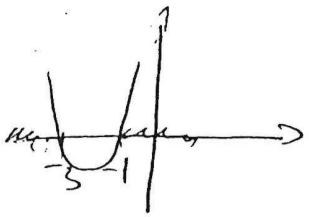
$|x| \leq 2$

$x \in [-2, 2]$

Sve zajedno: $D_f = [-2, -1\rangle \cup \langle -1, 0\rangle \cup \langle 0, 1\rangle \cup \langle 1, 2]$

b) $\cdot\ \sqrt{x^2+4x+3} + \sqrt{x+3} > 0$

$\cdot\ x^2+4x+3 \geq 0 \Rightarrow x_{1,2} = \frac{-4 \pm \sqrt{16-12}}{2} = -2 \pm 1 \begin{array}{l} -3 \\ -1 \end{array}$



$x \in \langle -\infty, -3] \cup [-1, +\infty \rangle$

$\sqrt{x^2+4x+3} > -5x-3$

$1^0\ -5x-3 < 0 \Rightarrow x > -\frac{3}{5}$

$\sqrt{x^2+4x+3} \geq 0 > -5x-3 \Rightarrow$ svijek vrijedi

$\Rightarrow \boxed{x > -\frac{3}{5}}$

$2^0\ -5x-3 \geq 0 \Rightarrow x \leq -\frac{3}{5}$

$\underbrace{\sqrt{x^2+4x+3}}_{\geq 0} > \underbrace{-5x-3}_{\geq 0} \quad /^2$

$x^2+4x+3 > (-5x-3)^2$

$x^2+4x+3 > 25x^2+30x+9$

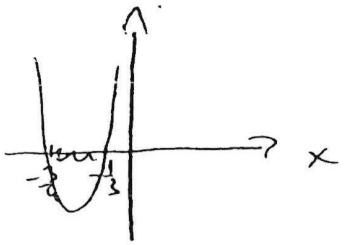
$24x^2+26x+6 < 0$

$12x^2+13x+3 < 0$

$x_{1,2} = \frac{-13 \pm \sqrt{169-144}}{24} = \frac{-13 \pm 5}{24}$

$x_1 = -\frac{18}{24} = -\frac{3}{4}$

$x_2 = -\frac{8}{24} = -\frac{1}{3}$



$x \in \langle -\frac{3}{4}, -\frac{1}{3} \rangle$

Rj. nejednadžbe = Rj $1^0 \cup$ Rj 2^0

$= \langle -\frac{3}{4}, +\infty \rangle$

$D_f = \langle -\frac{3}{4}, +\infty \rangle \cap (\langle -\infty, -3] \cup [-1, +\infty \rangle)$

$= \langle -\frac{3}{4}, +\infty \rangle$

c) • $16-x^2>0$, $16-x^2\neq 1$

$x^2<16$ $\quad x^2\neq 17$

$\Rightarrow |x|<4$ $\quad x\neq\pm\sqrt{17}$

$\Rightarrow \boxed{x\in\langle -4,4\rangle}$

- $9^x-8\cdot 3^x-9\geq 0$
- $\sqrt{9^x-8\cdot 3^x-9}\neq 0$

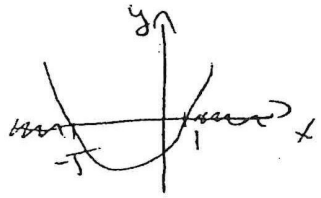
$\Leftrightarrow \quad 9^x-8\cdot 3^x-9>0$

- $\dfrac{x^2+4x-5}{\sqrt{9^x-8\cdot 3^x-9}}>0$

>0

$\Leftrightarrow x^2+4x-5>0$

$x_1=1 \quad x_2=-5$



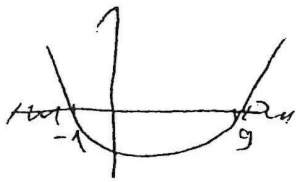
$\boxed{x\in\langle -\infty,-5\rangle\cup\langle 1,+\infty\rangle}$

$t := 3^x$

$t^2-8t-9>0$

$t_{1,2}=\dfrac{8\pm\sqrt{64+36}}{2}=\dfrac{8\pm 10}{2}$

$t_1=-1 \quad t_2=9$



$\Rightarrow t\in\langle -\infty,-1\rangle\cup\langle 9,+\infty\rangle$

tj. $t<-1$ ili $t>9$

$3^x<-1$ $\quad$ $3^x>9=3^2$

$\not\in$ $\quad$ $\boxed{x>2}$

$\Rightarrow D_f=\langle 2,\sqrt{17}\rangle\cup\langle\sqrt{17},4\rangle=\langle 2,4\rangle\setminus\{\sqrt{17}\}$

3. FUNKCIJE

Def Funkcija $f: A \to B$ je relacija $f \subseteq A \times B$ takva da za svaki $x \in A$ postoji jedinstveni $y \in B$ takav da je $(x,y) \in f$.

simbolima. $(\forall x \in A)(\exists! y \in B)((x,y) \in f)$

$(x,y) \in f$ pišemo i $y = f(x)$, $x \overset{f}{\mapsto} y$

A - domena funkcije f

B - kodomena funkcije f

PRIMJERI: $|\cdot| : \mathbb{R} \to \mathbb{R}$ (apsolutna vrijednost)

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 5x - 2$

$g: \mathbb{N} \to \mathbb{N}$, $g(n) = n+1$

$h: \mathbb{N} \to \mathbb{N}$, $h(x) = \begin{cases} \frac{x}{2}, & x \text{ paran} \\ 3x+1 & x \text{ neparan} \end{cases}$

Def Slika funkcije f je skup $\mathrm{Im}(f) = \{ f(x) : x \in A \}$

Def Funkcija $f: A \to B$ je

- injekcija, ako $f(x_1) = f(x_2)$ povlači $x_1 = x_2$
  ⚠ OPK: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- surjekcija, ako je $\mathrm{Im}(f) = B$
- bijekcija, ako je injekcija i surjekcija

Slika skupa $A' \subseteq A$ je $f(A') = \{ f(x) : x \in A' \}$ ($f(A) = \mathrm{Im}(f)$)

Praslika skupa $B' \subseteq B$ je $f^{-1}(B) = \{ x \in A : f(x) \in B' \}$

zd 1 Ispitajte injektivnost, surjektivnost i bijektivnost funkcija

$f: \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2 \qquad g: \mathbb{R} \to \mathbb{R}, \quad g(x) = |x|$

INJ. $f(x_1) = f(x_2) \overset{?}{\Rightarrow} x_1 = x_2$

$x_1^2 = x_2^2 \overset{?}{\Rightarrow} x_1 = x_2$ NE, npr. $1^2 = (-1)^2$ i $1 \neq -1$

$g(x_1) = g(x_2) \overset{?}{\Rightarrow} x_1 = x_2$

$|x_1| = |x_2| \overset{?}{\Rightarrow} x_1 = x_2$ NE!, npr. $|2| = |-2|$ i $2 \neq -2$

SURJ $y \in \mathbb{R}^+ \Rightarrow x = \sqrt{y} \in \mathbb{R}$

$f(x) = x^2 = (\sqrt{y})^2 = y$ DA

postoji li za svaki $y \in \mathbb{R}$ neki $x \in \mathbb{R}$ t.d. je $|x| = y$?

NE! za $y < 0$ ne postoji

	I	S	B
f	NE	DA	NE
g	NE	NE	NE

Nap $f|_{\mathbb{R}^+}$ je injekcija pa i bijekcija

$g: \mathbb{R} \to \mathbb{R}^+$ je surjekcija, ali ne i injekcija

$g|_{\mathbb{R}^+}: \mathbb{R}^+ \to \mathbb{R}^+$ je bijekcija

Zad 2 Zadana je funkcija $f: \langle -1, 1\rangle \to \langle -2, 2\rangle$ formulom

$$f(x) = x - x^2$$

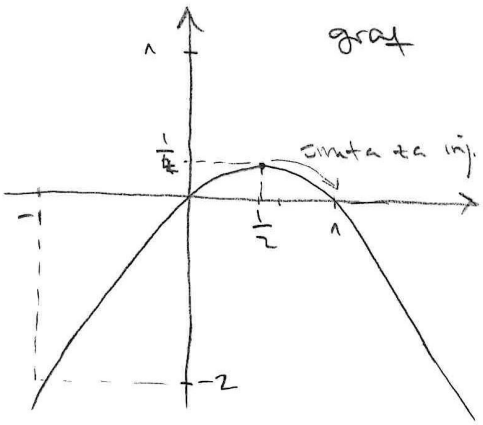
Odredite $B \subseteq \langle -2, 2\rangle$ takav da $g: \langle -1, 1\rangle \to B$ zadana istim pravilom, bude surjekcija. Zatim odredite $A \subseteq \langle -1, 1\rangle$ t.d $h: A \to B$ zadana istim pravilom, bude bijekcija.

Rj $f(x) = x - x^2 = -x^2 + x + 0$ kvadratna funkcija

$x_0 = -\frac{b}{2a} = \frac{1}{2}$ $\quad y_0 = \frac{4ac - b^2}{4a} = \frac{1}{4}$ tjeme: $(\frac{1}{2}, \frac{1}{4})$

$a < 0 \Rightarrow$ otvor prema dolje

nultočke: $x_1 = 0, \; x_2 = 1$



Iz grafa i gore navedenog zitamo:

$g: \langle -1, 1\rangle \to \langle -2, \frac{1}{4}]$ je surjekcija

$h: \langle -1, \frac{1}{2}] \to \langle -2, \frac{1}{4}]$ je bijekcija

Zad 3 Odredite maksimalne skupove $A, B \subseteq \mathbb{R}$ takve da je s

$$f(x) = \frac{x+1}{x-2}$$

zadana bijekcija $f: A \to B$. Odredite inverznu funkciju $f^{-1}: B \to A$

Rj: Očito $x \neq 2$, tj. $2 \notin A$

$$f(x_1) = f(x_2)$$

$$\frac{x_1+1}{x_1-2} = \frac{x_2+1}{x_2-2}$$

$$x_1x_2 - 2x_1 + x_2 - 2 = x_1x_2 - 2x_2 + x_1 - 2$$

$$3x_2 = 3x_1 \Rightarrow x_1 = x_2$$

f je injekcija na $\mathbb{R} \setminus \{2\}$, tj. $A = \mathbb{R} \setminus \{2\}$

$y = f(x) \iff y = \frac{x+1}{x-2} \iff yx - 2y = x + 1$

$\iff x(y-1) = 2y+1$

$\iff x = \frac{2y+1}{y-1}$ osim to $y \neq 1$ $\quad B = \mathbb{R}\setminus\{1\}$

Pokazali smo: $(\forall y \in \mathbb{R}\setminus\{1\})(\exists! x \in \mathbb{R}\setminus\{2\})(y = f(x))$

↓

taj x je $\frac{2y+1}{y-1}$

$f : \mathbb{R}\setminus\{2\} \to \mathbb{R}\setminus\{1\}$ je bijekcija

Inverzna funkcija $f^{-1} = g : \mathbb{R}\setminus\{1\} \to \mathbb{R}\setminus\{2\}$, $g(y) = \frac{2y+1}{y-1}$

Def Prirodna domena funkcije f je najveći skup na kojem možemo odrediti vrijednost $f(x)$. Označavamo je sa D_f.

Zad 2 Odredite prirodne domene sljedećih funkcija.

(a) $f(x) = \frac{1}{2x+5}$

(b) $g(x) = \sqrt{2x^2 - x - 1}$

(c) $h(x) = \sqrt[3]{2x^2 - x - 1}$

(d) $p(x) = \sqrt[4]{\frac{1-t}{t^2-4}}$

(e) $k(x) = 9 \cdot e^{3x-1}$

(f) $f(u) = \frac{u}{\ln(5-3u)}$

(g) $g(x) = \left(\log_{\frac{1}{2}}\left(\frac{x-1}{3-2x}\right) + 1\right)^{\frac{1}{2}}$

(h) $r(t) = \log_7(\sin(7t))$

(i) $h(x) = \operatorname{tg}\left(2x - \frac{4\pi}{3}\right) + \cos\left(\frac{3x}{2} + \pi\right)$

(j) $f(x) = (x^2 + 2x + 1)^{-\frac{1}{2}} + (\sin x + \sqrt{3}\cos x)^{-1}$

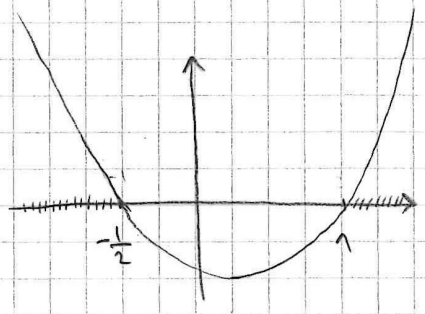
R/ (a) $2x + 5 \neq 0 \Rightarrow x \neq -\frac{5}{2}$

$D_f = \mathbb{R} \setminus \left\{-\frac{5}{2}\right\}$

(b) $2x^2 - x - 1 \geq 0$

$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4}$

$x_1 = 1 \qquad x_2 = -\frac{1}{2}$



$\Rightarrow x \in \langle -\infty, -\frac{1}{2}] \cup [1, +\infty\rangle$

$D_f = \langle -\infty, -\frac{1}{2}] \cup [1, +\infty\rangle$

(c) $D_f = \mathbb{R}$

(d) $t^2 - 4 \neq 0 \Rightarrow t \neq \pm 2$

$$\frac{1-t}{t^2-4} \geq 0 \iff \frac{1-t}{(t+2)(t-2)} \geq 0$$

	$-\infty$		-2		1		2		∞
$1-t$		$+$		$+$	0	$-$		$-$	
$t-2$		$-$		$-$		$-$	0	$+$	
$t+2$		$-$	0	$+$		$+$		$+$	
$\frac{1-t}{(t-2)(t+2)}$		$+$	$\times$	$-$		$+$	$\times$	$-$	

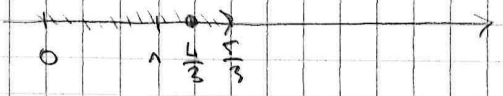
$\Rightarrow D_f = \langle -\infty, -2 \rangle \cup [1, 2\rangle$

(e) $D_f = \mathbb{R}$

(f) $5 - 3u > 0 \iff 3u < 5 \iff u < \frac{5}{3}$

$\ln(5-3u) \neq 0 \iff 5 - 3u \neq e^0 \iff -3u \neq 1 - 5$

$\iff u \neq \frac{4}{3}$



$D_f = \langle -\infty, \frac{4}{3} \rangle \cup \langle \frac{4}{3}, \frac{5}{3} \rangle$

(g) $3 - 2x \neq 0 \Rightarrow x \neq \frac{3}{2}$

$\frac{x-1}{3-2x} > 0$

	$-\infty$		1		$\frac{3}{2}$		∞
$x-1$		$-$	0	$+$		$+$	
$3-2x$		$+$		$+$	0	$-$	
$\frac{x-1}{3-2x}$	0	$-$		$+$	$\times$	$-$	

$\Rightarrow x \in \langle 1, \frac{3}{2} \rangle$

$$\log_{\frac{1}{2}}\left(\frac{x-1}{3-2x}\right)+1 \geq 0$$

$$\log_{\frac{1}{2}}\left(\frac{x-1}{3-2x}\right) \geq -1 \quad \text{(baza je manja od 1!)}$$

$$\frac{x-1}{3-2x} \leq \left(\frac{1}{2}\right)^{-1}$$

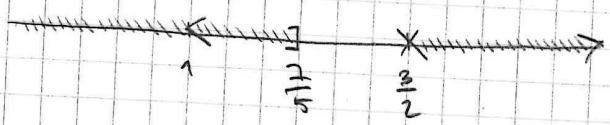
$$\frac{x-1}{3-2x} - 2 \leq 0$$

$$\frac{x-1-6+4x}{3-2x} \leq 0$$

$$\frac{5x-7}{3-2x} \leq 0$$

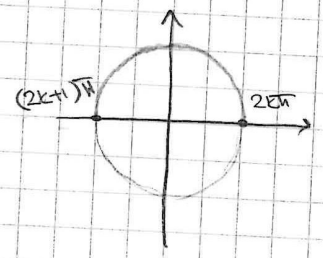
	$-\infty$		$\frac{7}{5}$		$\frac{3}{2}$		∞
$5x-7$		$-$	0	$+$		$+$	
$3-2x$		$+$		$+$	0	$-$	
$\frac{5x-7}{3-2x}$		$-$	0	$+$	$\times$	$-$	

$\Rightarrow x \in \langle -\infty, \frac{7}{5}] \cup \langle \frac{3}{2}, +\infty \rangle$



$D_f = \langle 1, \frac{7}{5}]$

(h) $\sin(7t) > 0$



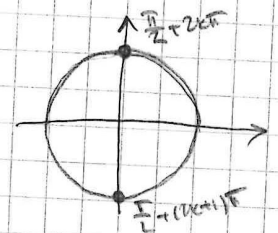
$7t \in \langle 2k\pi, (2k+1)\pi \rangle$ za bilo koji $k \in \mathbb{Z}$

$$2k\pi < 7t < (2k+1)\pi$$

$$\frac{2k\pi}{7} < t < \frac{(2k+1)\pi}{7}$$

$$D_f = \bigcup_{k\in\mathbb{Z}} \left\langle \frac{2k\pi}{7}, \frac{(2k+1)\pi}{7} \right\rangle$$

(i) $2x - \frac{4\pi}{3} \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$



$$2x \neq \frac{11\pi}{6} + k\pi, \quad k \in \mathbb{Z}$$

$$x \neq \frac{11\pi}{12} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}$$

$$D_f = \mathbb{R} \setminus \left\{ \frac{11\pi}{12} + \frac{k\pi}{2}, \; k \in \mathbb{Z} \right\}$$

) $f(x) = \frac{1}{\sqrt{x^2+2x+1}} + \frac{1}{\sin x + \sqrt{3}\cos x}$

$x^2+2x+1 > 0 \iff (x+1)^2 \geq 0 \iff x \neq -1$

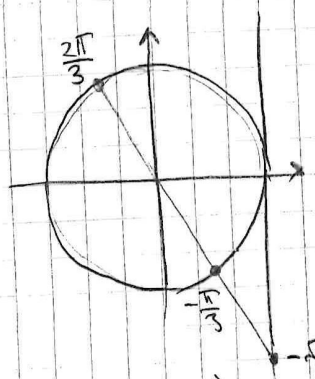
$\sin x + \sqrt{3}\cos x \neq 0$

$\sin x \neq -\sqrt{3}\cos x$

$\frac{\sin x}{\cos x} \neq -\sqrt{3}$

$\operatorname{tg} x \neq -\sqrt{3}$

$x \neq -\frac{\pi}{3} + k\pi, \; k \in \mathbb{Z}$



$D_f = \mathbb{R} \setminus \left(\{-1\} \cup \{-\frac{\pi}{3} + k\pi \mid k \in \mathbb{Z}\} \right)$

$\log a^b = b \log a$

ad 3 Jesu li funkcije $f(x) = \log(x+5)^2$ i $g(x) = 2\log(x+5)$ jednake?

Rješenje domene:

f $(x+5)^2 > 0 \iff x \neq -5$ $\quad D_f = \mathbb{R} \setminus \{-5\}$

g $x+5 > 0 \iff x > -5$ $\quad D_g = \langle -5, +\infty \rangle$

$D_f \neq D_g$, pa f i g nisu jednake (ali podudarajuće se na $D_f \cap D_g = \langle -5, +\infty \rangle$)

Zad 4 Odredite slike sljedećih funkcija:

(a) $f_1(x) = x^2 + 2x - 3$

(b) $f_2(x) = -3\sin(x - 2\pi)$

(c) $f_3(x) = \frac{5x+2}{x-3}$

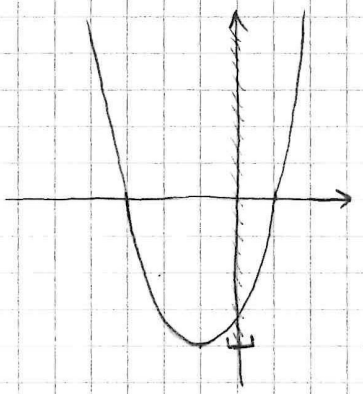
(d) $f_4(x) = \frac{2^x - 2}{2^x + 1}$

Ř. (a) $f_1(x) = x^2 + 2x - 3 = (x-1)(x+3)$

TJEME:

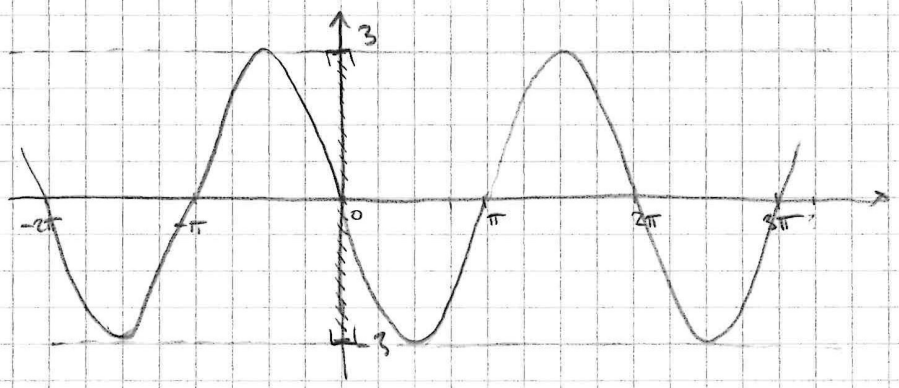
$$x_0 = -\frac{b}{2a} = -\frac{2}{2\cdot 1} = -1$$

$$y_0 = \frac{4ac - b^2}{4a} = \frac{-12-4}{4} = -4 \quad (= f_1(x_0))$$



$$\mathrm{Im}(f_1) = [-4, \infty)$$

(b) $f_2(x) = -3\sin(x - 2\pi) = -3\sin(x)$



$$-1 \leq \sin x \leq 1$$

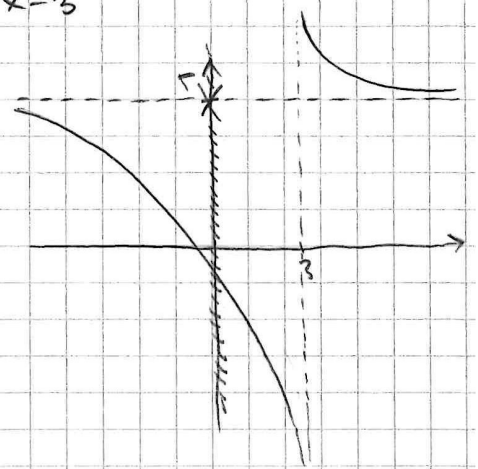
$$3 \geq -3\sin x \geq -3$$

$$\mathrm{Im}(f_2) = [-3, 3]$$

(c) $$f(x) = \frac{5x+2}{x-3} = \frac{5x-15+17}{x-3} = 5 + \frac{17}{x-3}$$

$x \neq 3!$

$$\frac{17}{x-3} \neq 0 \Rightarrow f(x) = 5 + \underbrace{\frac{17}{x-3}} \neq 5$$



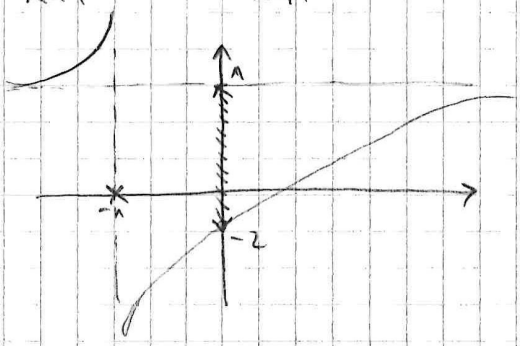
$$\mathrm{Im}(f) = \mathbb{R} \setminus \{5\}$$

(a) $g_1(x) = 2^x$ $\qquad g_2(x) = \frac{x-2}{x+1} = 1 - \frac{3}{x+1}$

$\mathrm{Im}(g_1) = \langle 0, +\infty \rangle$

$g_2(0) = 1 - 3 = -2$

$\mathrm{Im}(g_2) = \mathrm{Im}(f) = \langle -2, 1 \rangle$



Def Neka su $f: A \to B$, $g: B \to C$ funkcije. Kompozicija funkcija f i g je funkcija $g \circ f: A \to C$ definirana s

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in A$$

Zad 5 Odredite kompozicije $f \circ g$, $g \circ f$ te prirodne domene i slike tih funkcija ako je

(a) $f(x) = x - 2$ $\qquad g(x) = 3x + 7$

(b) $f(x) = \log_3(9x)$ $\qquad g(x) = \left(\frac{1}{27}\right)^{2x+1}$

Rj

(a) $(f \circ g)(x) = f(g(x)) = f(3x+7) = (3x+7) - 2 = 3x + 5$

$D_{f \circ g} = \mathbb{R}$, $\quad \mathrm{Im}(f \circ g) = \mathbb{R}$

$(g \circ f)(x) = g(f(x)) = g(x-2) = 3(x-2) + 7 = 3x + 1$

$D_{g \circ f} = \mathbb{R}$, $\quad \mathrm{Im}(g \circ f) = \mathbb{R}$

(b) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{27}\right)^{2x+1} = \log_3\left(9 \cdot \left(\frac{1}{27}\right)^{2x+1}\right)$

$= \log_3\left(3^2 \cdot 3^{-3(2x+1)}\right) = \log_3\left(3^{-6x-1}\right) = -6x - 1$

$D_{f \circ g} = \mathbb{R}$, $\quad \mathrm{Im}(f \circ g) = \mathbb{R}$

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$$(g\circ f)(x) = g(f(x)) = g(\log_3(9x)) = \left(\frac{1}{27}\right)^{2\log_3(9x)+1}$$

$$= 3^{-3\cdot(2\log_3(9x)+1)}$$

$$= 3^{-3}\cdot\left(3^{\log_3(9x)}\right)^{-6}$$

$$= 3^{-3}\cdot(9x)^{-6} = 3^{-3}\cdot 3^{-12}\cdot x^{-6}$$

$$= 3^{-15}\cdot x^{-6}$$

$D_{g\circ f} = \langle 0, +\infty\rangle$ $\quad$ $Im(g\circ f) = \langle 0, +\infty\rangle$

(jer $D_{g\circ f} \subseteq D_f$!)

Zad 6 Zadane su funkcije (1. kol 2016)

$$f(x) = \sqrt{x^2-5x+4} \quad ; \quad g(x) = x^2$$

(a) Odredite domenu i sliku funkcija f i g

(b) Za koji x je moguće definirati $f\circ g$ i $g\circ f$?

(c) Odredite $f\circ g$ i $g\circ f$

Rj

$$x^2-5x+4 \geq 0$$

Zad 6 Neka su $f: A \to B$, $g: B \to C$ injekcije. Dokažite da je $g \circ f$ također injekcija.

Rj

$$x_1, x_2 \in A$$

$$(g \circ f)(x_1) = (g \circ f)(x_2)$$

$$g(f(x_1)) = g(f(x_2))$$

$(g$ inj.$) \Rightarrow f(x_1) = f(x_2)$

$(f$ inj.$) \Rightarrow x_1 = x_2$

Zad 7 Ispitajte jesu li sljedeće f-je bijekcije

(a) $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$

(b) $f: \mathbb{R} \to [0, \infty\rangle, \quad f(x) = ||x| - 1|$

(c) $f: \langle 0, \infty\rangle \to \mathbb{R}, \quad f(x) = \log(x^2) - 10$

Rj (a) $f(x_1) = f(x_2) \to \frac{1}{x_1} = \frac{1}{x_2} \quad / \cdot x_1 x_2 \Rightarrow x_2 = x_1$

f je injekcija

$(\forall x \in \mathbb{R} \setminus \{0\})\left(\frac{1}{x} \neq 0\right)$

$\Rightarrow 0 \notin \mathrm{Im}(f) \Rightarrow f$ nije surjekcija

Def Identiteta na skupu A je funkcija $id_A : A \to A$, $id_A(x) = x$

Def Neka je $f : A \to B$ funkcija. Funkcija $f^{-1} : B \to A$ t.d.

$$f \circ f^{-1} = id_B \quad i \quad f^{-1} \circ f = id_A$$

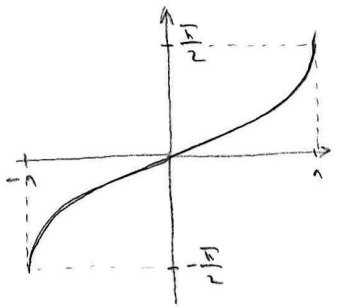
zovemo inverzna funkcija od f

Tm $f : A \to B$ ima inverznu funkciju akko je bijekcija

Nap Graf funkcije i inverzne funkcije su simetrični s obzirom na pravac $y = x$

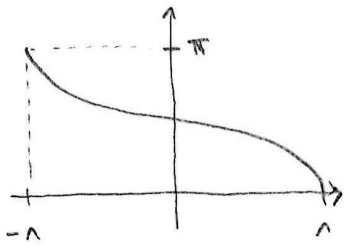
PRIMJER: Ciklometrijske (arkus) funkcije

$\sin : \mathbb{R} \to [-1,1]$ nije bijekcija, ali $\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]} : [-\frac{\pi}{2},\frac{\pi}{2}] \to [-1,1]$ je bijekcija, pa po tm ∧ ima inverznu funkciju $\arcsin : [-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}]$ (arkus sinus)

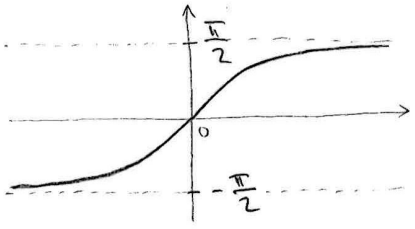


Analogno se definiraju funkcije

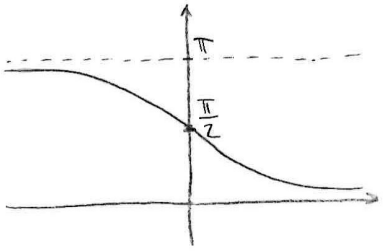
arkus kosinus
$\arccos : [-1,1] \to [0,\pi]$



arkus tangens
$\operatorname{arctg} : \mathbb{R} \to \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$



arkus kotangens
$\operatorname{arcctg} : \mathbb{R} \to \langle 0, \pi \rangle$



Zad 8 Odredite inverzne funkcije sljedećih funkcija

(a) $f(x) = \frac{2x}{x-4}$

(b) $f(x) = (4^{2x+1} + 5)^3$

(c) $f(x) = 2\sin x \cos x$

(d) $f(x) = 2\log_3 x - \log_9 x^2$

(e) $f(x) = \sqrt{\operatorname{arctg}(2^x) - 1}$

Rj (a) $y = \frac{2x}{x-4}$

$xy - 4y = 2x$

$x(y-2) = 4y$

$x = \frac{4y}{y-2}$

$\Rightarrow \underline{f^{-1}(x) = \frac{4x}{x-2}}$

(b) $y = (4^{2x+1} + 5)^3$

$4^{2x+1} + 5 = \sqrt[3]{y}$

$4^{2x+1} = \sqrt[3]{y} - 5$

$2x+1 = \log_4(\sqrt[3]{y} - 5)$

$x = \frac{1}{2}(\log_4(\sqrt[3]{y} - 5) - 1)$

$\Rightarrow \underline{f^{-1}(x) = \frac{1}{2}\log_4(\sqrt[3]{y} - 5) - \frac{1}{2}}$

(c) $y = 2\sin x \cos x$

$y = \sin(2x)$

$2x = \arcsin y$

$x = \frac{1}{2}\arcsin y$

$\Rightarrow \underline{f^{-1}(x) = \frac{1}{2}\arcsin x}$

(d) $y = 2\log_3 x - \log_9 x^2$

$y = 2\log_3 x - \frac{\log_3 x^2}{\log_3 9} = 2\log_3 x - \frac{2\log_3 x}{2} = \log_3 x$

$x = 3^y \quad \Rightarrow \underline{f^{-1}(x) = 3^x}$

(e) $y = \sqrt{\operatorname{arctg}(2^x) - 1}$

$y^2 = \operatorname{arctg}(2^x) - 1$

$y^2 + 1 = \operatorname{arctg}(2^x)$

$2^x = \operatorname{tg}(y^2+1)$

$x = \log_2(\operatorname{tg}(y^2+1))$

$\underline{f^{-1}(x) = \log_2(\operatorname{tg}(x^2+1))}$

Svojstva funkcija

- ① Parnost

Def Neka je f funkcija takva da za svaki $x \in D_f$ vrijedi $-x \in D_f$. Kažemo da je f

1) <u>parna</u>, ako za svaki $x \in D_f$ vrijedi $f(-x) = f(x)$

2) <u>neparna</u>, ako za svaki $x \in D_f$ vrijedi $f(-x) = -f(x)$

Zad 9 Neka je $f : \mathbb{R} \to \mathbb{R}$ parna, $g : \mathbb{R} \to \mathbb{R}$ neparna f-ja. Dokažite da su $f \circ g$, $g \circ f$ parne funkcije.

Rj $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x)$

$(g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x)$

Zad 10 Ispitajte parnost sljedećih funkcija:

(a) $f(x) = |x-4| + |x+4|$

(b) $f(x) = \frac{\cos x^5}{x} + \frac{\sin x^{10}}{x}$

(c) $f(x) = \frac{2 \operatorname{tg} x - 1}{x}$

(d) $f(x) = \ln\left(\frac{1+\sin(2x)}{1-\sin(2x)}\right)$

Rj (a) $f(-x) = |-x-4| + |-x+4| = |-(x+4)| + |-(x-4)|$

$= |x+4| + |x-4| = f(x)$ $\Rightarrow f$ je parna

(b) $f(-x) = \frac{\cos(-x)^5}{-x} + \frac{\sin(-x)^{10}}{-x} = \frac{\cos(-x^5)}{-x} + \frac{\sin x^{10}}{-x}$

$= -\frac{\cos x^5}{x} - \frac{\sin x^{10}}{x} = -f(x)$ $\Rightarrow f$ je neparna

(c) $f(-x) = \frac{2 \operatorname{tg}(-x) - 1}{-x} = \frac{-2 \operatorname{tg} x - 1}{-x} = \frac{2 \operatorname{tg} x + 1}{x}$

$f(-x) \neq f(x)$, $f(-x) \neq -f(x)$ $\Rightarrow f$ nije ni parna, ni neparna

(d) $f(-x) = \ln\left(\frac{1+\sin(2(-x))}{1-\sin(2(-x))}\right) = \ln\left(\frac{1+\sin(-2x)}{1-\sin(-2x)}\right)$

$= \ln\left(\frac{1-\sin(2x)}{1+\sin(2x)}\right) = \ln\left(\frac{1+\sin(2x)}{1-\sin(2x)}\right)^{-1}$

$= -\ln\left(\frac{1+\sin(2x)}{1-\sin(2x)}\right) = -f(x)$ $\Rightarrow f$ je neparna

② Omeđenost

Def Funkcija $f: D_f \to \mathbb{R}$ je

- <u>omeđena odozdo</u> ako $(\exists m \in \mathbb{R})(\forall x \in D_f)\ m \leq f(x)$
- <u>omeđena odozgo</u> ako $(\exists M \in \mathbb{R})(\forall x \in D_f)\ f(x) \leq M$

Funkcija je <u>omeđena</u> ako je omeđena odozdo i odozgo.

<u>Zad 11</u> Ispitajte omeđenost sljedećih funkcija:

(a) $f(x) = 5\sin(\pi x + \sqrt{2})$

(b) $f(x) = \frac{1}{\sqrt{-x^2+4x-3}}$

(c) $f(x) = -3^{-x+2}$

(d) $f(x) = \frac{3x^2}{4+x^2}$

(e) $f(x) = \arcsin\frac{x+2}{x-1} + \sqrt{1-x^2}$

(f) $f(x) = \log\frac{3x+2}{2-3x}$

Rj (a) $-1 \leq \sin(\pi x + \sqrt{2}) \leq 1 \Rightarrow -5 \leq 5\sin(\pi x + \sqrt{2}) \leq 5$

$\Rightarrow f$ je omeđena ($m=-5$, $M=5$)

(b) $\sqrt{-x^2+4x-3} = \sqrt{-x^2+4x-4+1} = \sqrt{\underbrace{1-\underbrace{(x-2)^2}_{\geq 0}}_{\leq 1}} \leq 1$

$\Rightarrow f(x) = \frac{1}{\sqrt{-x^2+4x-3}} \geq 1$

$\Rightarrow f$ je omeđena odozdo ($m=1$)

Nadalje, $\sqrt{-x^2+4x-3} = \sqrt{(x-1)(3-x)}$; Ako je x „blizu" 1 ili 3, onda je nazivnik „blizu" 0 pa će $f(x)$ biti jako velik

$\Rightarrow f$ nije omeđena odozgo

(c) $3^{-x+2} > 0 \Rightarrow -3^{-x+2} < 0 \Rightarrow f$ je omeđena odozgo ($M=0$)

$-3^{-x+2} = -9 \cdot 3^{-x} = -9 \cdot \left(\frac{1}{3}\right)^x \Rightarrow f$ nije omeđena odozdo
$\downarrow \infty$ (jer $x \mapsto a^x$ nije omeđena odozgo)

(d) $0 \leq \underbrace{\frac{3x^2}{4+x^2}}_{\geq 0} \leq \underbrace{\frac{3x^2}{x^2}}_{\geq 4} = 3 \Rightarrow f$ je omeđena ($m=0, M=3$)

II. NAČIN $\frac{3x^2}{4+x^2} = \frac{3x^2+12-12}{x^2+4} = 3 - \underbrace{\frac{12}{x^2+4}}_{\geq 0} \leq 3$

(e) $-\frac{\pi}{2} \leq \arcsin \frac{x+2}{x-1} \leq \frac{\pi}{2}$
$0 \leq \sqrt{1-x^2} \leq 1$ $\Big\}+ \Rightarrow -\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}+1$

$\Rightarrow f$ je omeđena ($m=-\frac{\pi}{2}$, $M=\frac{\pi}{2}+1$)

(f) $\frac{3x+2}{2-3x} = \frac{3x-2+4}{2-3x} = -1 + \underbrace{\frac{4}{\underbrace{2-3x}_{\neq 0}}}_{\neq -1}$

slika ove funkcije je $\mathbb{R} \setminus \{-1\}$

funkcija $x \mapsto \log x$ preslikava $\langle 0, +\infty \rangle$ u $\mathbb{R}$

$\Rightarrow$ slika od f je $\mathbb{R} \Rightarrow f$ nije omeđena odozdo ni odozgo

③ Monotonost

Def Funkcija f **raste** na intervalu $I \subseteq D_f$ ako

$$(\forall x_1, x_2 \in I)(x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2))$$

Funkcija f **pada** na intervalu $I \subseteq D_f$ ako

$$(\forall x_1, x_2 \in I)(x_1 \leq x_2 \Rightarrow f(x_1) \geqslant f(x_2))$$

Zad 12 Odredite intervale monotonosti sljedećih funkcija

(a) $f(x) = 2^{2x+4}$ (c) $f(x) = \frac{3x+7}{x+2}$

(b) $f(x) = \log_{\frac{1}{5}}(x^2+1)$ (d) $f(x) = |2^x - 4|$

Rj (a) $D_f = \mathbb{R}$

$x_1 \leq x_2 \Rightarrow 2x_1 \leq 2x_2 \Rightarrow 2x_1+4 \leq 2x_2+4$

$\Rightarrow 2^{2x_1+4} \leq 2^{2x_2+4} \Rightarrow f(x_1) \leq f(x_2)$

$x \mapsto a^x$ je rastuća za $a>1$

f je rastuća na $\mathbb{R}$

(b) $D_f = \mathbb{R}$

1) $0 \leq x_1 \leq x_2 \Rightarrow x_1^2 \leq x_2^2 \Rightarrow x_1^2+1 \leq x_2^2+1$

$\Rightarrow \log_{\frac{1}{5}}(x_1^2+1) \geqslant \log_{\frac{1}{5}}(x_2^2+1)$

$x \mapsto \log_a x$ je padajuća za $a<1$

2) $x_1 \leq x_2 \leq 0 \Rightarrow x_1^2 \geqslant x_2^2 \Rightarrow x_1^2+1 \geqslant x_2^2+1$

$\Rightarrow \log_{\frac{1}{5}}(x_1^2+1) \leq \log_{\frac{1}{5}}(x_2^2+1)$

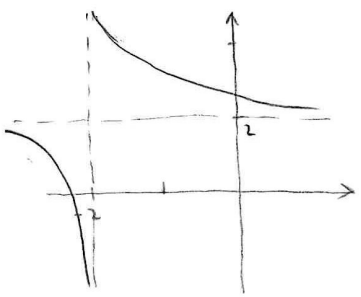
f je padajuća na $\langle -\infty, 0]$ i rastuća na $[0, +\infty\rangle$

(c) $D_f = \mathbb{R} \setminus \{-2\}$

$f(x) = \frac{3x+6+1}{x+2} = 2 + \frac{1}{x+2}$

$x_1 \leq x_2 \Rightarrow x_1+2 \leq x_2+2 \Rightarrow \frac{1}{x_1+2} \geqslant \frac{1}{x_2+2}$

$\Rightarrow 2 + \frac{1}{x_1+2} \geqslant 2 + \frac{1}{x_2+2} \Rightarrow f(x_1) \geqslant f(x_2)$



f je padajuća na $D_f = \langle -\infty, -2\rangle \cup \langle -2, +\infty\rangle$

(a) $f(x) = |2^x - 4| = \begin{cases} 2^x - 4, & 2^x - 4 \geq 0 \Leftrightarrow 2^x \geq 4 \Leftrightarrow x \geq 2 \\ -2^x + 4, & 2^x - 4 \leq 0 \Leftrightarrow 2^x < 4 \Leftrightarrow x < 2 \end{cases}$

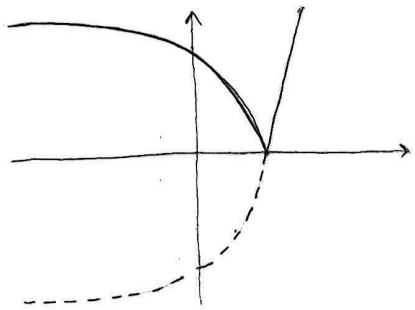
1) $2 \leq x_1 \leq x_2 \Rightarrow 2^{x_1} \leq 2^{x_2} \Rightarrow 2^{x_1} - 4 \leq 2^{x_2} - 4$

$\Rightarrow f(x_1) \leq f(x_2)$

2) $x_1 \leq x_2 \leq 2 \Rightarrow 2^{x_1} \leq 2^{x_2} \Rightarrow -2^{x_1} \geq -2^{x_2}$

$\Rightarrow -2^{x_1} + 4 \geq -2^{x_2} + 4 \Rightarrow f(x_1) \geq f(x_2)$

f je rastuća na $[2, +\infty\rangle$ i padajuća na $\langle -\infty, 2]$



1. REALNE FJE REALNE VARIJABLE

Def. Neka su A i B neprazni skupovi. Funkcija sa skupa A u skup B je pridruživanje f koje svakom elementu $a \in A$ pridružuje točno jedan element $b \in B$ (pišemo $b = f(a)$). Skup A zovemo domena, a skup B kodomena fje f.

Oznaka: $f: A \to B$ („f sa A u B").

Ako je $A, B \subseteq \mathbb{R}$, kažemo da je f realna fja realne varijable.

Prirodna domena i slika fje

Prirodna domena fje f je „najveći" skup na kojemu možemo odrediti vrijednosti $f(x)$ fje f. Označavamo ju $\mathcal{D}_f$.

Zad. 1.1. Odredite prirodne domene sljedećih fja:

a) $f(x) = \frac{1}{2x+5}$

b) $g(x) = \sqrt{2x^2 - x - 1}$

c) $h(x) = \sqrt[3]{2x^2 - x - 1}$

d) $p(t) = \sqrt[4]{\frac{1-t}{t^2-4}}$

e) $k(x) = 9x\, e^{3x-1}$

f) $f(u) = \frac{u}{\ln(5-3u)}$

g) $g(x) = \left(\log_{\frac{1}{2}}\left(\frac{x-1}{3-2x}\right) + 1\right)^{\frac{1}{2}}$

h) $r(t) = \log_7(\sin(7t))$

i) $h(x) = \operatorname{tg}\left(2x - \frac{4\pi}{3}\right) + \cos\left(\frac{3x}{2} + \pi\right)$

j) $f(x) = (x^2+2x+1)^{-\frac{1}{2}} + (\sin x + \sqrt{3}\cos x)^{-1}$

Rj. a) $2x + 5 \neq 0 \to x \neq -\frac{5}{2}$

$\mathcal{D}_f = \mathbb{R} \setminus \{-\frac{5}{2}\}$

b) $2x^2 - x - 1 \geq 0$

$2x^2 - 2x + x - 1 \geq 0$

$2x(x-1) + (x-1) \geq 0$

$(2x+1)(x-1) \geq 0$



$\Rightarrow x \in \langle -\infty, -\frac{1}{2}] \cup [1, \infty\rangle$

$\mathcal{D}_g = \langle -\infty, -\frac{1}{2}] \cup [1, \infty\rangle$

c) $\mathcal{D}_h = \mathbb{R}$ (eksponencijalna fja je definirana za svaki realni broj)

d) $1^\circ\ t^2 - 4 \neq 0 \Rightarrow t \neq \pm 2$

$2^\circ\ \frac{1-t}{t^2-4} \geq 0$

$\frac{1-t}{(t-2)(t+2)} \geq 0$

	$-\infty$		-2		1		2		∞
$1-t$		+		+	0	−		−	
$t-2$		−		−		−	0	+	
$t+2$		−	0	+		+		+	
$\frac{1-t}{(t-2)(t+2)}$		+	X	−	0	+	X	−	

$\Rightarrow t \in \langle -\infty, -2\rangle \cup [1, 2\rangle$

$\mathcal{D}_f = (\mathbb{R} \setminus \{-2, 2\}) \cap (\langle -\infty, -2\rangle \cup [1, 2\rangle) = \langle -\infty, -2\rangle \cup [1, 2\rangle$

e) $\mathcal{D}_k = \mathbb{R}$ (eksponencijalna fja je definirana za svaki realni broj)

f) $1^\circ\ 5 - 3u > 0 \Rightarrow u < \frac{5}{3}$

$2^\circ\ \ln(5-3u) \neq 0 \Rightarrow 5 - 3u \neq e^0 \Rightarrow -3u \neq -4 \Rightarrow u \neq \frac{4}{3}$

$\mathcal{D}_f = (\mathbb{R} \setminus \{\frac{4}{3}\}) \cap \langle -\infty, \frac{5}{3}\rangle = \langle -\infty, \frac{4}{3}\rangle \cup \langle \frac{4}{3}, \frac{5}{3}\rangle$

g) 1° $3-2x \neq 0 \Rightarrow x \neq \frac{3}{2}$

2° $\frac{x-1}{3-2x} > 0$

	$-\infty$		1		$\frac{3}{2}$		∞
$x-1$		$-$	0	$+$		$+$	
$3-2x$		$+$		$+$	0	$-$	
$\frac{x-1}{3-2x}$		$-$	0	$+$	$\times$	$-$	

$\Rightarrow x \in \langle 1, \frac{3}{2} \rangle$

3° $\log_{\frac{1}{2}}\left(\frac{x-1}{3-2x}\right) + 1 \geq 0$

$\log_{\frac{1}{2}}\left(\frac{x-1}{3-2x}\right) \geq -1$

$\frac{x-1}{3-2x} \leq \left(\frac{1}{2}\right)^{-1}$ (baza logaritma je manja od 1)

$\frac{x-1}{3-2x} - 2 \leq 0$

$\frac{x-1-6+4x}{3-2x} \leq 0$

$\frac{5x-7}{3-2x} \leq 0$

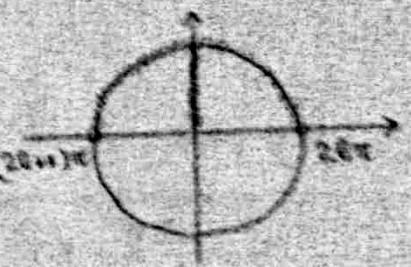
	$-\infty$		$\frac{7}{5}$		$\frac{3}{2}$		∞
$5x-7$		$-$	0	$+$		$+$	
$3-2x$		$+$		$+$	0	$-$	
$\frac{5x-7}{3-2x}$		$-$	0	$+$	$\times$	$-$	

$\Rightarrow x \in (-\infty, \frac{7}{5}] \cup \langle \frac{3}{2}, \infty \rangle$



$\mathcal{D}_g = \langle 1, \frac{7}{5}]$

h) $\sin(7t) > 0$



$7t \in \bigcup_{k \in \mathbb{Z}} \langle 2k\pi, (2k+1)\pi \rangle$

$2k\pi < 7t < (2k+1)\pi \Rightarrow \frac{2k\pi}{7} < t < \frac{(2k+1)\pi}{7}$

$\mathcal{D}_f = \bigcup_{k \in \mathbb{Z}} \langle \frac{2k\pi}{7}, \frac{(2k+1)\pi}{7} \rangle$

c) $2x - \frac{4\pi}{3} \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$ (Kosinus je definiran na čitavom $\mathbb{R}$.)

$2x \neq \frac{11\pi}{6} + k\pi, \quad k \in \mathbb{Z}$

$x \neq \frac{11\pi}{12} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}$

$\mathcal{D}_h = \mathbb{R} \setminus \left\{ \frac{11\pi}{12} + \frac{k\pi}{2} ; k \in \mathbb{Z} \right\}$

d) $f(x) = \frac{1}{\sqrt{x^2+2x+1}} + \frac{1}{\sin x + \sqrt{3}\cos x}$

1° $x^2+2x+1 \geq 0 \Rightarrow (x+1)^2 \geq 0$ ✓

2° $\sqrt{x^2+2x+1} \neq 0 \Rightarrow x^2+2x+1 \neq 0 \Rightarrow (x+1)^2 \neq 0 \Rightarrow x \neq -1$

3° $\sin x + \sqrt{3}\cos x \neq 0$

Rješimo jednadžbu $\sin x + \sqrt{3}\cos x = 0$.

<u>1. način</u>

$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = 0$

$\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x = 0$

$\sin\left(x + \frac{\pi}{3}\right) = 0$

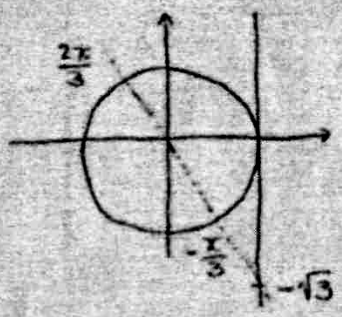
$x + \frac{\pi}{3} = k\pi, \quad k \in \mathbb{Z}$

$x = -\frac{\pi}{3} + k\pi, \quad k \in \mathbb{Z}$

<u>2. način</u>

$\sin x = -\sqrt{3}\cos x \quad / : \cos x \neq 0$

$\operatorname{tg} x = -\sqrt{3}$



$x = -\frac{\pi}{3} + k\pi, \quad k \in \mathbb{Z}$

$\Rightarrow x \neq -\frac{\pi}{3} + k\pi, \quad k \in \mathbb{Z}$

$\mathcal{D}_f = \mathbb{R} \setminus \left(\{-1\} \cup \left\{ -\frac{\pi}{3} + k\pi ; k \in \mathbb{Z} \right\} \right)$

Zad. 1.2. Jesu li fje f i g jednake ako je

$$f(x) = \log(x+5)^2, \quad g(x) = 2\log(x+5)\ ?$$

Rj. Za prirodnu domenu od f imamo

$$(x+5)^2 > 0 \Rightarrow x \neq -5 \Rightarrow \mathcal{D}_f = \mathbb{R}\setminus\{-5\},$$

dok za prirodnu domenu od g vrijedi:

$$x+5 > 0 \Rightarrow x > -5 \Rightarrow \mathcal{D}_g = \langle -5, \infty\rangle.$$

Dalje, $\mathcal{D}_f \neq \mathcal{D}_g$ pa f i g nisu jednake (ali se podudaraju na zajedničkom području definicije, $\langle -5, \infty\rangle$).

Def. Neka je $f: A \to B$ fja. Skup

$$\mathrm{Im} f := \{ f(x) \mid x \in A \}$$

zovemo slika fje f.

Zad. 1.3. Odredite slike sljedećih fja:

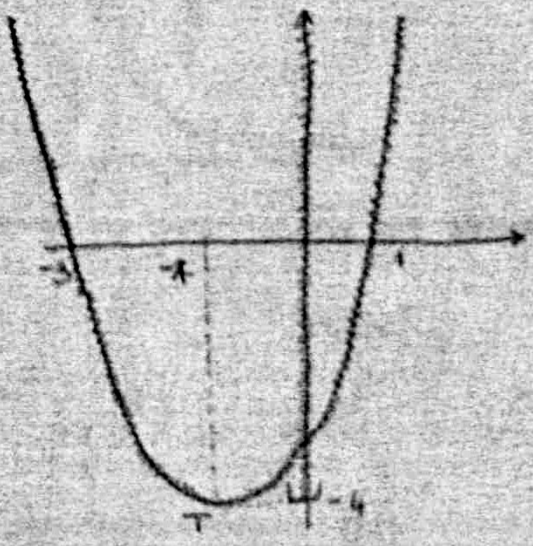
a) $f_1(x) = x^2 + 2x - 3$

b) $f_2(x) = -3\sin(x - 2\pi)$

c) $f_3(x) = \dfrac{5x+2}{x-3}$

d) $f_4(x) = \dfrac{2^x - 2}{2^x + 1}$

Rj. a) $f_1(x) = x^2 + 2x - 3 = (x-1)(x+3)$



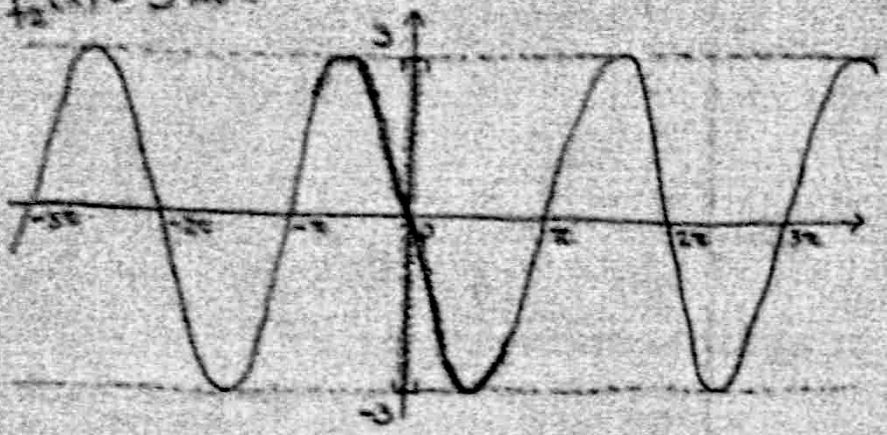
Fja f_1 svoju minimalnu vrijednost postiže u točki tjemenu T:

$$x_T = -\frac{2}{2\cdot 1} = -1,$$

$$y_T = f_1(x_T) = 1 - 2 - 3 = -4.$$

Dalje, $\mathrm{Im} f_1 = [-4, \infty)$.

b) $f_2(x) = -3\sin x$



$-1 \le \sin x \le 1 \rightarrow -3 \le -3\sin x \le 3$

$\mathrm{Im} f_2 = [-3, 3]$

c) $f_3(x) = \frac{5x+2}{x-3} = \frac{5x-15+17}{x-3} = 5 + \frac{17}{x-3}$



Budući da je za sve $x \in \mathbb{R}\setminus\{3\}$

$\frac{17}{x-3} \neq 0,$

slijedi

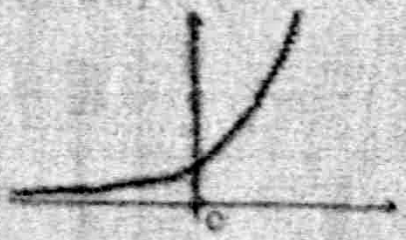
$5 + \frac{17}{x-3} \neq 5.$

Dakle, $\mathrm{Im} f_3 = \mathbb{R}\setminus\{5\}.$

d) Promatrajmo fje

$g_1(x) = 2^x, \quad g_2(x) = \frac{x-2}{x+1}.$

Vrijedi $\mathrm{Im} g_1 = \langle 0, \infty \rangle.$



Fja g_2 „poslikava" interval $\langle 0, \infty \rangle$ u $\langle -2, 1 \rangle$.

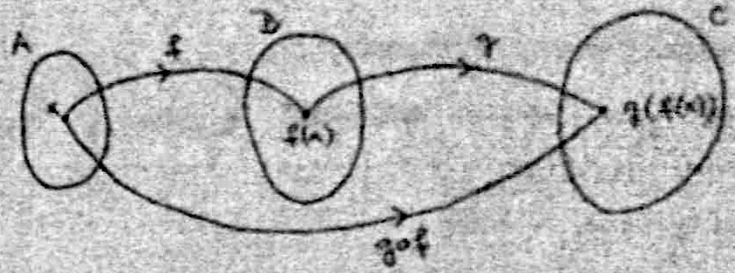


Dakle,

$\mathrm{Im} f_4 = \langle -2, 1 \rangle.$

Kompozicija fja, inverzna fja

Def. Neka su $f: A \to B$ i $g: B \to C$ fje. Kompozicija fja f i g je fja $g \circ f: A \to C$ definirana s $(g \circ f)(x) = g(f(x))$.



Zad. 1.4. Odredite kompozicije $f \circ g$, $g \circ f$ te prirodne domene tih fja ako su fje f i g zadane na svojim prirodnim domenama:

a) $f(x) = x - 2$, $g(x) = 3x + 7$

b) $f(x) = \log_3(9x)$, $g(x) = \left(\frac{1}{27}\right)^{2x+1}$

Rj: a) $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = (3x+7) - 2 = 3x + 5$

$\mathcal{D}_{f \circ g} = \mathbb{R}$, $\mathrm{Im}(f \circ g) = \mathbb{R}$

$(g \circ f)(x) = g(f(x)) = 3(x-2) + 7 = 3x + 1$

$\mathcal{D}_{g \circ f} = \mathbb{R}$, $\mathrm{Im}(g \circ f) = \mathbb{R}$

b) $f: \langle 0, \infty \rangle \to \mathbb{R}$, $g: \mathbb{R} \to \langle 0, \infty \rangle$

$(f \circ g)(x) = f(g(x)) = \log_3\left(9 \cdot \left(\frac{1}{27}\right)^{2x+1}\right) = \log_3\left(3^2 \cdot 3^{-3(2x+1)}\right) = -6x - 1$

$\mathcal{D}_{f \circ g} = \mathbb{R}$, $\mathrm{Im}(f \circ g) = \mathbb{R}$

$(g \circ f)(x) = g(f(x)) = \left(\frac{1}{27}\right)^{2\log_3(9x)+1} = 3^{-6\log_3(9x)-3}$

$= 3^{-6\log_3(9x)} \cdot 3^{-3} = 3^{\log_3(9x)^{-6}} \cdot 3^{-3} = (9x)^{-6} \cdot 3^{-3}$

$\mathcal{D}_{g \circ f} = \langle 0, \infty \rangle$ $(= \mathcal{D}_f)$, $\mathrm{Im}(g \circ f) = \langle 0, \infty \rangle$

Zad. 1.5 (1. kolokvij 2016., grupa B, rad. 1)

b) Zadane su fje

$$f(x) = \sqrt{x^2-5x+4} \quad \text{i} \quad g(x) = x^2.$$

Odredite domene i kodomene fja f i g.

c) Za koje je x moguće definirati kompoziciju $f \circ g$, odnosno $g \circ f$?

d) Odredite $f \circ g$ i $g \circ f$.

Rj. b) Za f:

$x^2-5x+4 \geq 0 \Rightarrow (x-1)(x-4) \geq 0$

$\Rightarrow \mathcal{D}_f = \langle -\infty, 1] \cup [4, \infty \rangle$



Fja $x \mapsto \sqrt{x}$ prestiče svoju prirodnu domenu $\langle 0, \infty \rangle$ ponovno u interval $\langle 0, \infty \rangle$ pa je $\text{Im} f = \langle 0, \infty \rangle$.

Za g: $\mathcal{D}_g = \mathbb{R}$, $\text{Im} g = [0, \infty \rangle$.

c) Kompoziciju $f \circ g$ je moguće definirati za one x iz domene od g za koje je $g(x)$ u domeni od f

$$\mathcal{D}_{f \circ g} = \{ x \in \mathcal{D}_g \mid g(x) \in \mathcal{D}_f \}.$$

$g(x) \in \mathcal{D}_f \Rightarrow x^2 \leq 1$ ili $x^2 \geq 4 \Rightarrow x \in [-1,1]$ ili $x \in \langle -\infty, -2] \cup [2, \infty \rangle$

$\Rightarrow x \in \langle -\infty, -2] \cup [-1, 1] \cup [2, \infty \rangle$

$g \circ f$ je moguće definirati za $x \in \mathbb{R} \setminus (\langle -2, -1 \rangle \cup \langle 1, 2 \rangle)$.

Analogno,

$$\mathcal{D}_{g \circ f} = \{ x \in \mathcal{D}_f \mid f(x) \in \mathcal{D}_g \}.$$

Budući da je $\mathcal{D}_g = \mathbb{R}$, $g \circ f$ je moguće definirati za sve x iz $\mathcal{D}_f$.

d) $(f \circ g)(x) = f(g(x)) = \sqrt{x^4 - 5x^2 + 4}$

$(g \circ f)(x) = g(f(x)) = \left(\sqrt{x^2-5x+4}\right)^2 = x^2 - 5x + 4$

Def. Kažemo da je fja $f: A \to B$:

1) injekcija ako
$$(\forall x_1, x_2 \in A)\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

2) surjekcija ako
$$(\forall y \in B)(\exists x \in A)\ y = f(x),$$

3) bijekcija ako je injekcija i surjekcija.

Nap. Fja $f: A \to B$ je

1) injekcija ako $(\forall x_1, x_2 \in A)\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (često po kontrapoziciji),

2) surjekcija ako $\operatorname{Im} f = B$ (po definiciji slike),

3) bijekcija ako $(\forall y \in B)(\exists! x \in A)\ y = f(x)$ (po definiciji injekcije i surjekcije).

Zad. 1.6. Neka su $f: A \to B$ i $g: B \to C$ injekcije. Dokažite da je $g \circ f$ također injekcija.

Rj Neka su $x_1, x_2 \in A$ t.d. $(g \circ f)(x_1) = (g \circ f)(x_2)$. Imamo

$$(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$$
$$\Rightarrow f(x_1) = f(x_2) \quad (g \text{ injekcija})$$
$$\Rightarrow x_1 = x_2 \quad (f \text{ injekcija})$$

pa je $g \circ f$ injekcija.

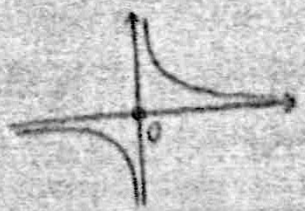
Zad. 1.7. Ispitajte jesu li sljedeće fje bijekcije:

a) $f: \mathbb{R}\setminus\{0\} \to \mathbb{R},\ f(x) = \frac{1}{x}$

b) $f: \mathbb{R} \to [0, \infty),\ f(x) = |x(x-1)|$

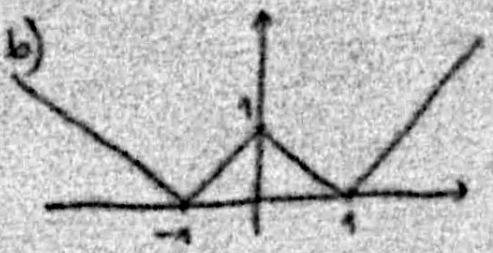
c) $f: \langle 0, \infty \rangle \to \mathbb{R},\ f(x) = \log(x^2) - 10$

Rj a)



f je injekcija: za $x_1, x_2 \in \mathbb{R}\setminus\{0\}$ t.d. $f(x_1) = f(x_2)$ imamo $\frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$. f nije surjekcija jer $\operatorname{Im} f = \mathbb{R}\setminus\{0\}$. Dakle, f nije bijekcija.

b)

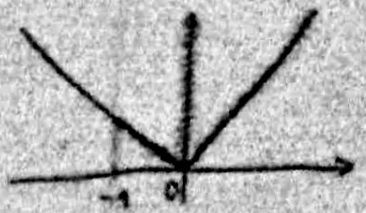


f nije injekcija, npr:

$f(-2) = |1-2|-1| = 1$
$f(0) = |1 0|-1| = 1$
$f(2) = |1 2|-1| = 1$

$\Big\} \; f(-2) = f(0) = f(2).$

No, f je surjekcija. Naime, slika fje $x \mapsto |x|-1$ je $[-1, \infty)$, a taj interval fje $x \mapsto |x|$ preslikava u interval $[0, \infty)$. Dakle, $\operatorname{Im} f = [0, \infty)$ (ali f nije bijekcija).



c) f je injekcija: neka su $x_1, x_2 \in \langle 0, \infty \rangle$ t.d. $f(x_1) = f(x_2)$. Imamo

$$\log(x_1^2) - 10 = \log(x_2^2) - 10$$
$$\Rightarrow \log(x_1^2) = \log(x_2^2)$$
$$\Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1 = x_2 \qquad (\text{jer } x_1, x_2 > 0)$$

f je i surjekcija: za proizvoljan $y \in \mathbb{R}$ tražimo $x \in \langle 0, \infty \rangle$ t.d. $y = f(x)$, tj. rješavamo jednadžbu

$$y = \log(x^2) - 10$$
$$\Rightarrow y + 10 = \log(x^2)$$
$$\Rightarrow 10^{y+10} = x^2$$
$$\Rightarrow x = \sqrt{10^{y+10}}$$

Dakle, f je bijekcija.

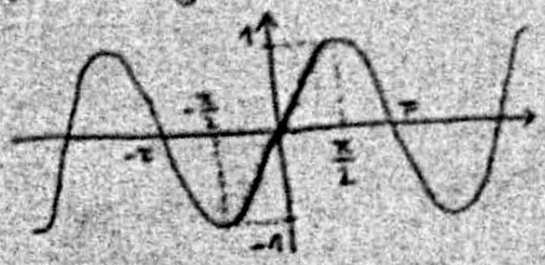
Def. Identiteta na skupu A je fja
$$id_A : A \to A, \quad id_A(x) = x.$$

Def. Neka je $f : A \to B$ fja. Fja $f^{-1} : B \to A$ t.d.
$$f \circ f^{-1} = id_B, \quad f^{-1} \circ f = id_A$$
zovemo inverzna fja fje f.

Tm 1. Fja $f : A \to B$ ima inverznu fju ako i samo ako je f bijekcija.

Nap. Graf fje i inverzne fje su međusobno simetrični obzirom na pravac $y = x$.

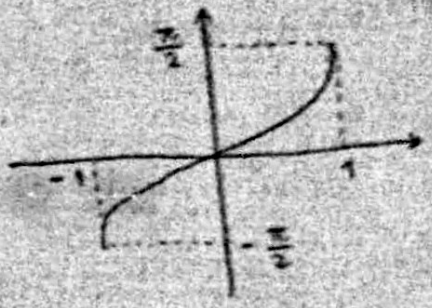
1. (Ciklometrijske fje)



Fja $f: \mathbb{R} \to [-1,1]$, $f(x) = \sin x$ nije bijekcija jer nije injekcija (npr. $f(\pi) = f(-\pi) = 0$).

No, 'sužavanjem' domene na segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ dobivamo bijekciju $f|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ koja prema tom ima inverznu fju

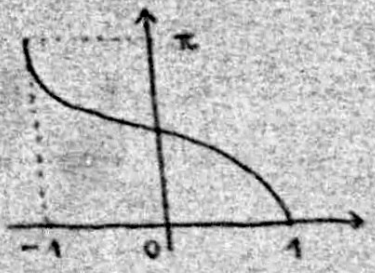
$\arcsin: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (arkus sinus).



Analogno se definiraju fje

arkus kosinus

$\arccos: [-1,1] \to [0,\pi]$



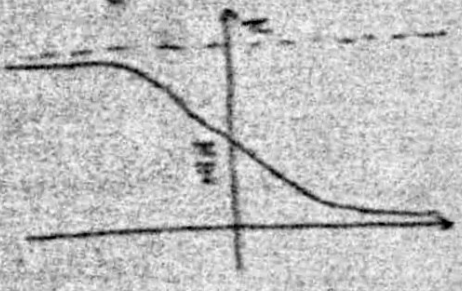
arkus tangens

$\operatorname{arctg}: \mathbb{R} \to \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$



arkus kotangens

$\operatorname{arcctg}: \mathbb{R} \to \langle 0, \pi \rangle$



Zad. 1.8. Odredite inverzne fje sljedećih fja:

a) $f(x) = \frac{2x}{x-4}$

b) $f(x) = (4^{2x+1} + 5)^3$

c) $f(x) = 2 \sin x \cos x$

d) $f(x) = 2 \log_3 x - \log_3 x^2$

e) $f(x) = \sqrt{\operatorname{arctg}(2^x) - 1}$

Rj. a) $y = \frac{2x}{x-4}$

$xy - 4y = 2x$

$x(y-2) = 4y$

$x = \frac{4y}{y-2}$ $\Rightarrow$ $f^{-1}(x) = \frac{4x}{x-2}$

b) $y = (4^{2x+1} + 5)^3$

$4^{2x+1} + 5 = \sqrt[3]{y}$

$2x+1 = \log_4(\sqrt[3]{y} - 5)$

$x = \frac{1}{2}\log_4(\sqrt[3]{y} - 5) - \frac{1}{2} \Rightarrow f^{-1}(x) = \frac{1}{2}\log_4(\sqrt[3]{x} - 5) - \frac{1}{2}$

c) $y = 2\sin x \cos x$

$y = \sin(2x)$

$2x = \arcsin y$

$x = \frac{1}{2}\arcsin y \Rightarrow f^{-1}(x) = \frac{1}{2}\arcsin x$

d) $y = 2\log_3 x - \log_3 x^2$

$y = 2\log_3 x - 2\log_3 x$

$y = 2\log_3 x - 2 \cdot \frac{1}{2}\log_3 x$

$y = \log_3 x$

$x = 3^y \Rightarrow f^{-1}(x) = 3^x$

e) $y = \sqrt{\operatorname{arctg}(2^x) - 1}$

$y^2 = \operatorname{arctg}(2^x) - 1$

$y^2 + 1 = \operatorname{arctg}(2^x)$

$2^x = \operatorname{tg}(y^2+1)$

$x = \log_2(\operatorname{tg}(y^2+1)) \Rightarrow f^{-1}(x) = \log_2(\operatorname{tg}(y^2+1))$

...stva fja

Parnost

Def.: Neka je f fja tad za svaki $x \in \mathcal{D}_f$ vrijedi i $-x \in \mathcal{D}_f$. Kažemo da je f:

1) parna ako za svaki $x \in \mathcal{D}_f$ vrijedi $f(-x) = f(x)$,

2) neparna ako za svaki $x \in \mathcal{D}_f$ vrijedi $f(-x) = -f(x)$.

Zad. 1.9. Neka je $f: \mathbb{R} \to \mathbb{R}$ parna, a $g: \mathbb{R} \to \mathbb{R}$ neparna fja. Dokažite da su $f \circ g$ i $g \circ f$ parne fje.

Rj. $(f \circ g)(-x) = f(g(-x)) = [g \text{ neparna}] = f(-g(x)) = [f \text{ parna}] = f(g(x)) = (f \circ g)(x)$

$(g \circ f)(-x) = g(f(-x)) = [f \text{ parna}] = g(f(x)) = (g \circ f)(x)$

Zad. 1.10. Ispitajte parnost sljedećih fja:

a) $f(x) = |x-4| + |x+4|$

b) $f(x) = \frac{\cos x^5}{x} + \frac{\sin x^{10}}{x}$

c) $f(x) = \frac{2\operatorname{tg} x - 1}{x}$

d) $f(x) = \ln \frac{1+\sin(2x)}{1-\sin(2x)}$

Rj. a) $f(-x) = |-x-4| + |-x+4| = |-(x+4)| + |-(x-4)| = |x+4| + |x-4| = f(x)$

$\Rightarrow f$ je parna

b) $f(-x) = \frac{\cos(-x)^5}{-x} + \frac{\sin(-x)^{10}}{-x} = \frac{\cos(-x^5)}{-x} + \frac{\sin x^{10}}{-x}$

$= -\frac{\cos x^5}{x} - \frac{\sin x^{10}}{x} = -f(x)$

$\Rightarrow f$ je neparna

c) $f(-x) = \frac{2\operatorname{tg}(-x) - 1}{-x} = \frac{-2\operatorname{tg} x - 1}{-x} = \frac{2\operatorname{tg} x + 1}{x}$

$f(-x) \neq f(x)$, $f(-x) \neq -f(x) \Rightarrow f$ nije ni parna, ni neparna

d) $f(-x) = \ln \frac{1+\sin(2(-x))}{1-\sin(2(-x))} = \ln \frac{1+\sin(-2x)}{1-\sin(-2x)} = \ln \frac{1-\sin(2x)}{1+\sin(2x)}$

$= \ln \left(\frac{1+\sin(2x)}{1-\sin(2x)}\right)^{-1} = -\ln \frac{1-\sin(2x)}{1+\sin(2x)} = -f(x)$

$\Rightarrow f$ je neparna

2) Omeđenost

Def. Fja f je:

a) omeđena odozdo ako

$$(\exists m \in \mathbb{R})(\forall x \in D_f)\ m \le f(x).$$

b) omeđena odozgo ako

$$(\exists M \in \mathbb{R})(\forall x \in D_f)\ f(x) \le M.$$

Fja f je omeđena ako je omeđena odozdo i odozgo.

Zad 1.11. Ispitajte omeđenost sljedećih fja:

a) $f(x) = 5\sin(\pi x + \sqrt{2})$

b) $f(x) = \frac{1}{\sqrt{-x^2+4x-3}}$

c) $f(x) = -3^{-x+2}$

d) $f(x) = \frac{3x^2}{4+x^2}$

e) $f(x) = \arcsin\frac{x+2}{x-1} + \sqrt{1-x^2}$

f) $f(x) = \log\frac{3x+2}{2-3x}$

R/ a) $-1 \le \sin(\pi x + \sqrt{2}) \le 1 \Rightarrow -5 \le 5\sin(\pi x + \sqrt{2}) \le 5$

$\Rightarrow f$ je omeđena $(m=-5,\ M=5)$

b) $\sqrt{-x^2+4x-3} = \sqrt{-x^2+4x-4+1} = \sqrt{1-(x-2)^2} \le 1$

$\Rightarrow \frac{1}{\sqrt{-x^2+4x-3}} \ge 1$

$\Rightarrow f$ je omeđena odozdo $(m=1)$

Nadalje,

$$f(x) = \frac{1}{\sqrt{(x-1)(3-x)}}.$$

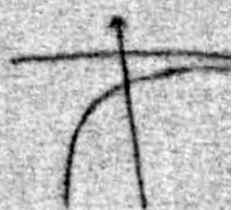
Vidimo je x „blizu" 1 ili 3, nazivnik će biti „blizu" 0 pa
$f(x)$ bit „jako velik". Dakle, f nije odozgo omeđena.

c) $3^{-x+2} > 0 \Rightarrow -3^{-x+2} < 0$

$\Rightarrow f$ je omeđena odozgo $(M=0)$

Nadalje,

$$f(x) = -9\cdot 3^{-x} = -9\cdot\left(\frac{1}{3}\right)^x$$



pa f nije omeđena odozdo (eksponencijalna fja nije omeđ

d) 1. način

$0 \le \frac{3x^2}{4+x^2} \le \frac{3x^2}{x^2} = 3 \Rightarrow f$ je omeđena $(m=0, M=3)$

2. način

$0 \le \frac{3x^2}{4+x^2} = \frac{3x^2+12-12}{x^2+4} = 3 - \frac{12}{x^2+4} < 3 \Rightarrow f$ je omeđena

e) $\left.\begin{array}{l} -\frac{\pi}{2} \le \arcsin\frac{x+2}{x-1} \le \frac{\pi}{2} \\ 0 \le \sqrt{1-x^2} \le 1 \end{array}\right\} + \Rightarrow -\frac{\pi}{2} \le \arcsin\frac{x+2}{x-1} + \sqrt{1-x^2} \le 1 + \frac{\pi}{2}$

$\Rightarrow f$ je omeđena $(m=-\frac{\pi}{2}, M=1+\frac{\pi}{2})$

f) Odredimo $\mathrm{Im} f$. Vrijedi

$$\frac{3x+2}{2-3x} = \frac{3x-2+4}{-3x+2} = -1 + \frac{4}{-3x+2}$$

pa je slika fje $x \mapsto \frac{3x+2}{2-3x}$ skup $\mathbb{R}\setminus\{-1\}$. Fja $x \mapsto \log x$ taj skup preslikava u $\mathbb{R}$ pa je $\mathrm{Im} f = \mathbb{R}$. Dakle, f nije omeđena odozdo niti odozgo.

3) Monotonost

Def. Fja f raste na intervalu $I \subseteq \mathcal{D}_f$ ako

$$(\forall x_1, x_2 \in I)\ x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2).$$

Fja f pada na intervalu $I \subseteq \mathcal{D}_f$ ako

$$(\forall x_1, x_2 \in I)\ x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2).$$

Zad. 1.12. Odredite intervale monotonosti sljedećih fja:

a) $f(x) = 2^{2x+4}$ c) $f(x) = \frac{3x+4}{x+2}$

b) $f(x) = \log_{\frac{1}{2}}(x^2+1)$ d) $f(x) = |2^x - 4|$

Rj. a) $\mathcal{D}_f = \mathbb{R}$

$x_1 < x_2 \Rightarrow 2x_1 \le 2x_2 \Rightarrow 2x_1+4 \le 2x_2+4 \Rightarrow 2^{2x_1+4} \le 2^{2x_2+4} \Rightarrow f(x_1) \le f(x_2)$

(eksponencijalna fja s bazom >1 je rastuća)

$\Rightarrow f$ raste na čitavoj svojoj domeni