

Trigonometrijske funkcije

na predavanju su definirali funkcije, proučavali njihova osnovna svojstva, crtali grafove, izveli formule i identitete, rješavali nejednakosti u obliku jednačbi i nejednakosti

Zad. Dokazite da vrijedi:

$$a) \frac{\sin(2k+y) + \sin(2k-y)}{\cos(k+2y) - \cos(k-2y)} = -\frac{\cos x}{\sin y}$$

$$b) \operatorname{tg} 2x + \frac{1}{\cos 2x} = \frac{\cos x + \sin x}{\cos x - \sin x}, \text{ za } x \neq \frac{\pi}{4}(2k+1), k \in \mathbb{Z} \quad \frac{\pi}{2}k + \frac{\pi}{4}$$

$$c) \frac{\sin x - \sin 2x}{\cos x + \cos 2x} = -\operatorname{tg} \frac{x}{2}$$

Rje.

$$a) \text{ Adicijske formule daju: } \frac{\sin 2x \cos y + \cos 2x \sin y + \sin 2x \cos y - \cos 2x \sin y}{\cos x \cos y - \sin x \sin y - \cos x \cos y - \sin x \sin y} =$$

$$= \frac{2 \sin 2x \cos y}{-2 \sin x \sin y} = \frac{2 \sin x \cos x \cos y}{-2 \sin x \sin y \cos y} = -\frac{\cos x}{\sin y}$$

$$b) \text{ Krcimo s desne strane i pomnožimo i brojnik i nazivnik s } \cos x + \sin x$$

$$\frac{\cos x + \sin x}{\cos x - \sin x} \cdot \frac{\cos x + \sin x}{\cos x + \sin x} = \frac{\cos^2 x + 2 \sin x \cos x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1 + \sin 2x}{\cos 2x} = \operatorname{tg} 2x + \frac{1}{\cos 2x}$$

$$c) \text{ Konstruimo zapise od } \sin x \text{ i } \cos x \text{ pomoću } \operatorname{tg} \frac{x}{2} = t, \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$\frac{\sin x - \sin 2x}{\cos x + \cos 2x} = \frac{\sin x - 2 \sin x \cos x}{\cos x + \cos^2 x - \sin^2 x} = \frac{\frac{2t}{1+t^2} - 2 \cdot \frac{2t}{1+t^2} \cdot \frac{1-t^2}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{(1-t^2)^2}{(1+t^2)^2} - \frac{4t^2}{(1+t^2)^2}} =$$

$$= \frac{\frac{2t+t^3-4t+4t^3}{(1+t^2)^2}}{\frac{1-t^4+1+t^2-2t^2-4t^2}{(1+t^2)^2}} = \frac{6t^3-2t}{-6t^2+2} = -\frac{t(6t^2-2)}{6t^2-2} = -t = -\operatorname{tg} \frac{x}{2}$$

Zad. Skicirajte grafove sljedećih funkcija:

$$a) f(x) = \frac{1}{2} \sin(2x + \pi) + 1 = \frac{1}{2} \sin\left(2\left(x + \frac{\pi}{2}\right)\right) + 1$$

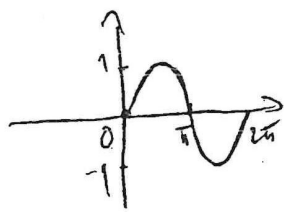
$$b) f(x) = -3 \cos\left(\frac{1}{2}x - \left(\frac{\pi}{3}\right)\right) \quad \frac{\pi}{2}$$

$$c) f(x) = \operatorname{tg}(2x) - 1$$

Z.

Zadatke rješavamo korak po korak, transformirajući jednačbu u poznate grafove u složenije.

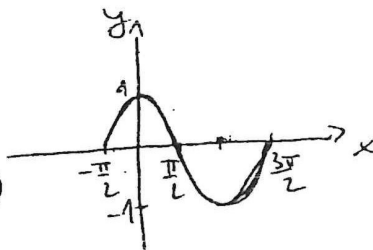
a)



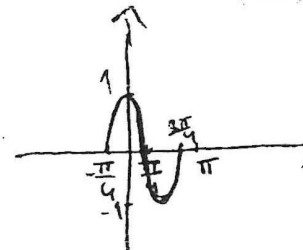
$\sin x$

(crtamo "uzorak")

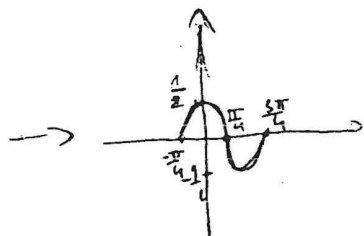
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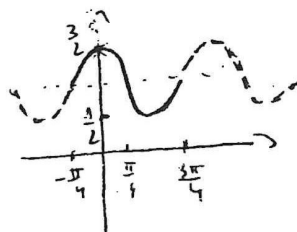
$\sin(x + \frac{\pi}{2})$



$\sin 2(x + \frac{\pi}{2})$



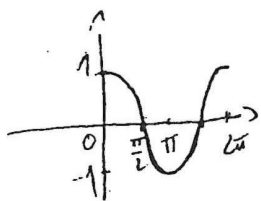
$\frac{1}{2} \sin 2(x + \frac{\pi}{2})$



$\frac{1}{2} \sin 2(x + \frac{\pi}{2}) + 1$

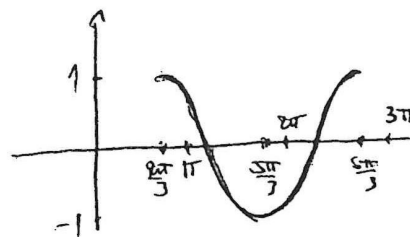
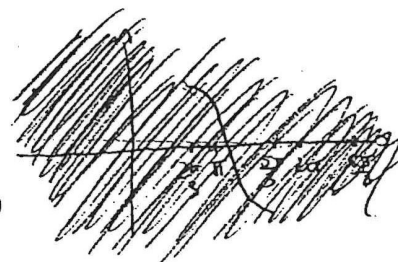
i "proširimo s -----"

b) $f(x) = -3 \cos(\frac{1}{2}x - \frac{\pi}{3}) = -3 \cos \frac{1}{2}(x - \frac{2\pi}{3})$

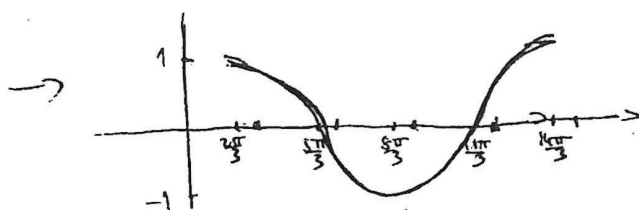


$\cos x$

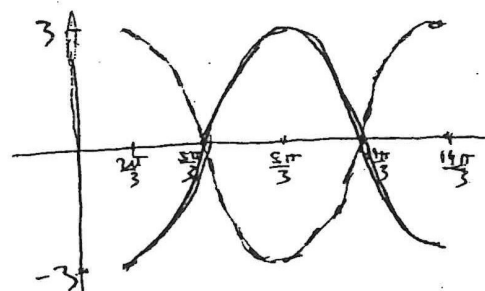
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$\cos(x - \frac{2\pi}{3})$

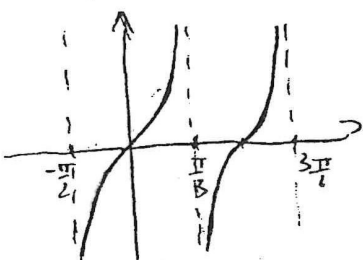


$\cos \frac{1}{2}(x - \frac{2\pi}{3})$

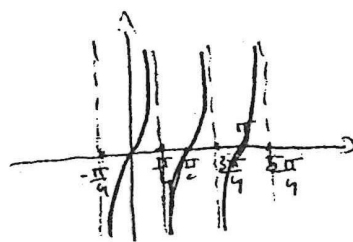


$3 \cos \frac{1}{2}(x - \frac{2\pi}{3})$ -----
 $-3 \cos \frac{1}{2}(x - \frac{2\pi}{3})$ ————

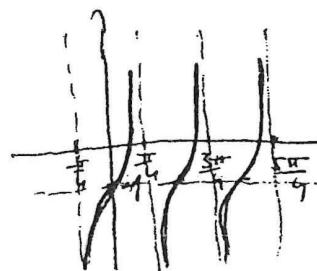
c) $f(x) = \lg 2x - 1$



$\lg x$



$\lg 2x$



$\lg 2x - 1$

Zad. Riješite jednačinu:

- a) $\operatorname{ctg}^2 2x - 4 \operatorname{ctg} 2x + 3 = 0$
 b) $8 \sin^2 x - 3 \sin x \cos x + \cos^2 x = 3$
 c) $3 \sin x - 4 \cos x = 2$
 d) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$
 e) $2 \sin x = 5x^2 + 2x + 3$

Rj.

a) $t = \operatorname{ctg} 2x$

$$t^2 - 4t + 3 = 0$$

$$t_{1,2} = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2}$$

$$t_1 = 3 \quad t_2 = 1$$

$$\operatorname{ctg} 2x = 3$$

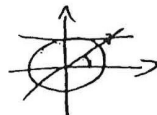
$$2x = \operatorname{arccotg} 3 + k\pi$$

$$x_1 = \operatorname{arccotg} 3 + \frac{k\pi}{2}, k \in \mathbb{Z}$$

$$\operatorname{ctg} 2x = 1$$

$$2x = \frac{\pi}{4} + k\pi$$

$$x_2 = \frac{\pi}{8} + \frac{k\pi}{2}, k \in \mathbb{Z}$$



b) Transformiramo jednačinu redom:

$$8 \sin^2 x - 3 \sin x \cos x + \cos^2 x = 3(\sin^2 x + \cos^2 x)$$

$$5 \sin^2 x - 3 \sin x \cos x - 2 \cos^2 x = 0 \quad |: \cos^2 x !!!$$

!!! Želimo podijeliti jednačinu s $\cos^2 x$. Smijemo li? Hoćemo li tako izgubiti neka rješenja?

$$\cos^2 x = 0 \Leftrightarrow \cos x = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi, \text{ a u tome } x \text{ je } \sin x = \pm 1$$

$$5 \cdot 1 \mp 3 \cdot 1 \cdot 0 - 2 \cdot 0 = 0$$

$$5 \neq 0 \Rightarrow \text{možemo izdvojiti } \cos^2 x$$

Dijeljenjem sad dobivamo:

$$5 \operatorname{tg}^2 x - 3 \operatorname{tg} x - 2 = 0$$

$$t = \operatorname{tg} x$$

$$5t^2 - 3t - 2 = 0$$

$$t_{1,2} = \frac{3 \pm \sqrt{9+40}}{10} = \frac{3 \pm 7}{10}$$

$$t_1 = 1 \quad t_2 = -\frac{2}{5}$$



$$\operatorname{tg} x = -\frac{2}{5}$$

$$x_3 = \operatorname{arctg}(-\frac{2}{5}) + k\pi, k \in \mathbb{Z}$$

$$\operatorname{tg} x = 1$$

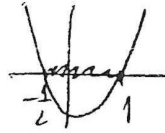
$$x_4 = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}$$

12) $u = \sin x$

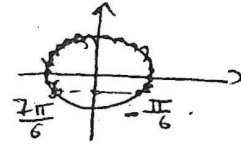
$$2t^2 - t - 1 \leq 0$$

$$t_{1,2} = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4}$$

$$t_1 = 1 \quad t_2 = -\frac{1}{2}$$



$$t \in [-\frac{1}{2}, 1] \Rightarrow -\frac{1}{2} \leq \sin x \leq 1 \Rightarrow \sin x > -\frac{1}{2}$$



$$x \in \bigcup_{k \in \mathbb{Z}} [-\frac{\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi]$$

c) Dargesten die beiden Transformationsregeln:

$$(\frac{1}{5})^{2\cos^2 x - 1} - (\frac{1}{5})^{2\cos^2 x} < 4 \cdot (\frac{1}{5})^{\frac{3}{2}}$$

$$5 \cdot (\frac{1}{5})^{2\cos^2 x} - (\frac{1}{5})^{2\cos^2 x} < 4 \cdot (\frac{1}{5})^{\frac{3}{2}}$$

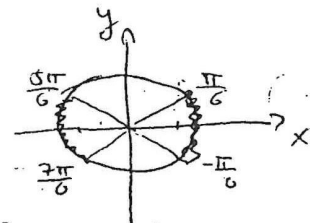
$$4 \cdot (\frac{1}{5})^{2\cos^2 x} < 4 \cdot (\frac{1}{5})^{\frac{3}{2}}$$

$$(\frac{1}{5})^{2\cos^2 x} < (\frac{1}{5})^{\frac{3}{2}}$$

$$2\cos^2 x > \frac{3}{2}$$

$$\cos^2 x > \frac{3}{4}$$

$$\Leftrightarrow |\cos x| > \frac{\sqrt{3}}{2} \Leftrightarrow \cos x > \frac{\sqrt{3}}{2} \text{ \& } \cos x < -\frac{\sqrt{3}}{2}$$



$$x \in \bigcup_{k \in \mathbb{Z}} (-\frac{\pi}{6} + 2k\pi, \frac{\pi}{6} + 2k\pi) \cup (\frac{5\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi)$$

$$\Rightarrow x \in \bigcup_{k \in \mathbb{Z}} (-\frac{\pi}{6} + k\pi, \frac{\pi}{6} + k\pi)$$

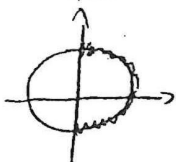
d) $\cos x \cos 2x < 0$

$$1^\circ \cos x > 0 \quad \cos 2x < 0 \quad \text{system}$$

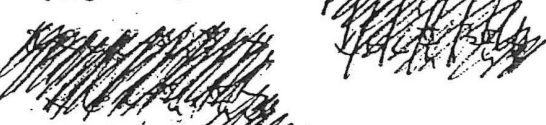
$$2\cos^2 x - 1 < 0$$

$$\cos^2 x < \frac{1}{2}$$

$$-\frac{\sqrt{2}}{2} < \cos x < \frac{\sqrt{2}}{2}$$



$$x \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

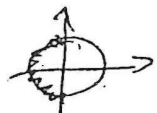


projekt:

$$x \in (\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{5\pi}{4}, \frac{7\pi}{4})$$

$$2^\circ \cos x < 0 \quad \cos 2x > 0 \quad \text{system}$$

$$x \in (\frac{\pi}{2}, \frac{3\pi}{2})$$

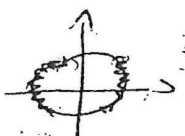


$$2\cos^2 x - 1 > 0$$

$$\cos^2 x > \frac{1}{2}$$

$$\cos x > \frac{\sqrt{2}}{2} \text{ \& } \cos x < -\frac{\sqrt{2}}{2}$$

$$\text{projekt: } x \in (\frac{3\pi}{4}, \frac{5\pi}{4})$$



$$\text{Kombi: } x \in \bigcup_{k \in \mathbb{Z}} (\frac{\pi}{4} + k\pi, \frac{3\pi}{4} + k\pi)$$

$$\bigcup_{k \in \mathbb{Z}} (\frac{5\pi}{4} + k\pi, \frac{7\pi}{4} + k\pi)$$

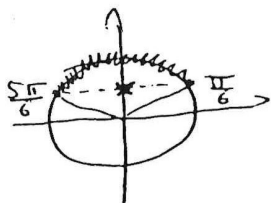
Zad. Riješite nejednačice:

- a) $\sin x \geq \frac{1}{2}$
 b) $|\cos x| < 0.6$
 c) $\operatorname{ctg} 2x < \sqrt{3}$

$$\sin x = \frac{1}{2} \rightarrow$$

Ri.

a)



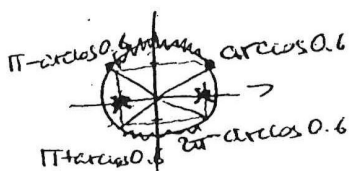
Rješenja jednadžbe $\sin x = \frac{1}{2}$ su $\left(\frac{\pi}{6}\right)$; $\left(\frac{5\pi}{6}\right) + 2k\pi$.

$$\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$$

$$\Rightarrow \left\{ \frac{\pi}{6} + 2k\pi \leq x \leq \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z} \right\}$$

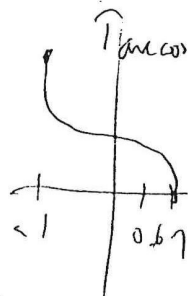
$$i) x \in \bigcup_{k \in \mathbb{Z}} \left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \right] \text{ (dvačiji zapis)}$$

b) $|\cos x| < 0.6 \Leftrightarrow -0.6 < \cos x < 0.6$

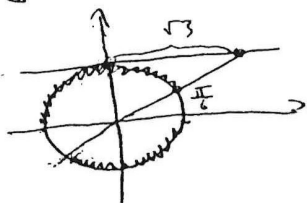


$$x \in \left(\arccos 0.6 + 2k\pi, \pi - \arccos 0.6 + 2k\pi \right) \cup \left(\pi + \arccos 0.6 + 2k\pi, \frac{3\pi}{2} - \arccos 0.6 + 2k\pi \right)$$

$$\Rightarrow x \in \bigcup_{k \in \mathbb{Z}} \left(\arccos 0.6 + k\pi, \pi - \arccos 0.6 + k\pi \right)$$



c) $\operatorname{ctg} 2x < \sqrt{3}$



$$2x \in \left(\frac{\pi}{6} + k\pi, \pi + k\pi \right), k \in \mathbb{Z}$$

$$x \in \bigcup_{k \in \mathbb{Z}} \left(\frac{\pi}{12} + \frac{k\pi}{2}, \frac{\pi}{2} + \frac{k\pi}{2} \right)$$

Zad. Riješite nejednačice:

a) $\operatorname{tg} \left(2x - \frac{\pi}{3} \right) \leq -1$

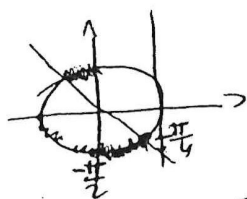
b) $2\sin^2 x - \sin x \leq 1$

c) $0.2^{\cos 2x} - \frac{1}{25^{\cos^2 x}} < 4 \cdot 125^{-\frac{1}{2}}$

d) $\cos x \cos 2x < 0$

Ri.

a)



$$t = 2x - \frac{\pi}{3}$$

$$\operatorname{tg} t \leq -1$$

$$-\frac{\pi}{2} \leq t \leq -\frac{\pi}{4}$$

$$-\frac{\pi}{2} + k\pi \leq t \leq -\frac{\pi}{4} + k\pi$$

$$-\frac{\pi}{2} + k\pi \leq 2x - \frac{\pi}{3} \leq -\frac{\pi}{4} + k\pi$$

$$-\frac{\pi}{6} + k\pi \leq 2x \leq \frac{\pi}{12} + k\pi$$

$$-\frac{\pi}{12} + \frac{k\pi}{2} \leq x \leq \frac{\pi}{24} + \frac{k\pi}{2}, k \in \mathbb{Z}$$

$$\Rightarrow x \in \bigcup_{k \in \mathbb{Z}} \left(-\frac{\pi}{12} + \frac{k\pi}{2}, \frac{\pi}{24} + \frac{k\pi}{2} \right]$$

Stroga <, jer tg nije def u $\frac{\pi}{2}$

c) Konstinuo univerzalnu supstituciju $t = \tan \frac{x}{2}$

$$3 \cdot \frac{2t}{1+t^2} - 4 \frac{1-t^2}{1+t^2} = 2 / -(1+t^2)$$

$$6t - 4(1-t^2) = 2(1+t^2)$$

$$6t - 4 + 4t^2 = 2 + 2t^2$$

$$2t^2 + 6t - 6 = 0 / :2$$

$$t^2 + 3t - 3 = 0$$

$$t_{1,2} = \frac{-3 \pm \sqrt{21}}{2}$$

$$\tan x = \frac{-3 + \sqrt{21}}{2}$$

$$x_1 = \arctan\left(\frac{-3 + \sqrt{21}}{2}\right) + k\pi, k \in \mathbb{Z}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{-3 - \sqrt{21}}{2}$$

$$x_2 = \arctan\left(\frac{-3 - \sqrt{21}}{2}\right) + k\pi, k \in \mathbb{Z}$$

d) Konstinuo formule za pretvaranje zbroja u umnožak:

$$\sin x + \sin 2x + \sin 3x + \sin 4x = 0$$

$$2 \sin \frac{5x}{2} \cos \frac{3x}{2} + 2 \sin \frac{7x}{2} \cos \frac{x}{2} = 0$$

$$2 \sin \frac{5x}{2} (\cos \frac{3x}{2} + \cos \frac{x}{2}) = 0$$

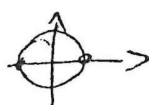
$$\sin a + \sin b$$

$$= 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$1^0 \sin \frac{5x}{2} = 0$$

$$\frac{5x}{2} = k\pi$$

$$x_1 = \frac{2k\pi}{5}, k \in \mathbb{Z}$$



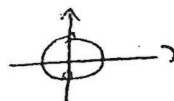
$$2^0 \cos \frac{3x}{2} + \cos \frac{x}{2} = 0$$

$$2 \cos x \cos \frac{x}{2} = 0$$

formule pretvorbe

$$\cos x = 0$$

$$\cos \frac{x}{2} = 0$$



$$x_2 = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$

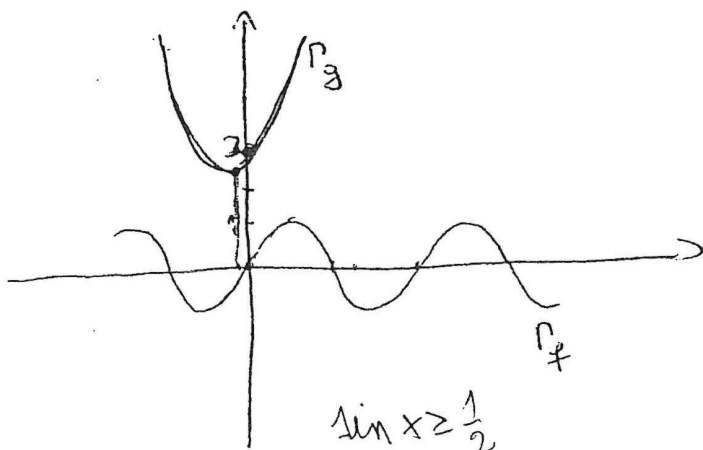


$$\frac{x}{2} = \frac{\pi}{2} + k\pi$$

$$x_3 = \pi + 2k\pi, k \in \mathbb{Z}$$

e) Jednadžbu rješavamo grafički. Skiciramo grafove funkcija

$$f(x) = \sin x \text{ i } g(x) = 5x^2 + 2x + 3$$



$$\sin x \geq \frac{1}{2}$$

$$| \cos x | \leq 0.6$$

$$g(x) = 0$$

$$D = 2^2 - 4 \cdot 5 \cdot 3 = 4 - 60 < 0 \Rightarrow g(x) = \text{nema rješenja}$$

$$x_0 = -\frac{b}{2a} = -\frac{2}{10} = -\frac{1}{5}$$

$$y_0 = \frac{4ac - b^2}{4a} = \frac{60 - 4}{20} = \frac{56}{20} = \frac{14}{5} > 2$$

$$\text{tjeme } \left(-\frac{1}{5}, \frac{14}{5}\right)$$

$$\Rightarrow g(x) > f(x), \forall x \in \mathbb{R}$$

$\Rightarrow$ jednadžbu nema rješenja (s)