

1. Polinom je funkcija  $p: \mathbb{R} \rightarrow \mathbb{R}$  oblika

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$$

za neke  $a_0, \dots, a_m \in \mathbb{R}$ ,  $a_m \neq 0$ .

$a_0, \dots, a_m$  zovemo koeficijentima

$a_m$  zovemo vodećim koeficijentom

$m = \text{st } p (\deg p)$  zovemo stepnjem polinoma  $p$

$\mathbb{R}[x]$  prsten polinoma s realnim koeficijentima

$\mathbb{Z}[x]$  prsten polinoma s cjelobrojnim koeficijentima

primjeri

Nulpolinom  $f(x) = 0$ ,  $\forall x \in \mathbb{R}$  po definiciji nije polinom (vodeći koeficijent nije 0), ali ga smatramo polinomom. Mi njegov stepanj ne definiramo (prelovajće!), ali neli ga definiraju kao -1 ili - $\infty$ .

Stepanj konstantnih polinoma je 0.

1. zad. Neka je  $(3x^2 - 2x - 1)^4 = a_8 x^8 + a_7 x^7 + \dots + a_1 x + a_0$ .

Odredite:

a)  $a_2 + a_6 + a_4 + a_2 + a_0$

b)  $a_6 + a_4 + a_2$

I način:

$$(3x^2 - 2x - 1)^4 = 81x^8 - 216x^7 + 108x^6 + 120x^5 - 74x^4 - 40x^3 + 12x^2 + 8x + 1$$

$$a_0 = 1, a_2 = 12, a_4 = -74, a_6 = 108, a_8 = 81$$

II način:

$$\begin{array}{lcl} x=1 & \dots & (3 \cdot 1 - 2 \cdot 1 - 1)^4 = a_8 + a_7 + \dots + a_1 + a_0 = 0 \\ x=-1 & \dots & (3(-1)^2 - 2(-1) - 1)^4 = a_8 - a_7 + \dots - a_1 + a_0 = 256 \end{array} \quad \Bigg| +$$

$$\Rightarrow 2(a_8 + a_6 + a_4 + a_2 + a_0) = 256$$

$$\Rightarrow a_8 + a_6 + a_4 + a_2 + a_0 = 128$$

$$r) x=0 \dots (3 \cdot 0 - 2 \cdot 0 - 1)^4 = a_0 = 1$$

$$a_8 \text{ je vodeći koeficijent: } (3x^2)^4 = a_8 x^8 = 81x^8 \\ \Rightarrow a_8 = 81$$

$$\text{Iz a) imamo: } a_6 + a_4 + a_2 = 46$$

od. Uredite zbroj koeficijenata polinoma koji se  
nalazi u oblasti pozitivnog izraza  $(2-3x+x^2)^{1999} (2+3x+x^2)^{2000}$

$$\therefore x=1 \dots (2-3+1)^{1999} (2+3+1)^{2000} = 0$$

$$f, g \in \mathbb{R}[x]$$

$$(f \pm g)(x) = f(x) \pm g(x), \quad \forall x \in \mathbb{R}$$

$$(f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in \mathbb{R}$$

$$(f \circ g)(x) = f(g(x)), \quad \forall x \in \mathbb{R}$$

$$\Rightarrow f \pm g, f \cdot g, f \circ g \in \mathbb{R}[x]$$

d. Weźmijmy  $f, g \in \mathbb{R}[x]$ . Ocienić stopniowe  $\text{st}(f+g)$ ,  $\text{st}(f-g)$ ,  $\text{st}(f \cdot g)$ ,  $\text{st}(f \circ g)$  pomocą  $\text{st}f$  i  $\text{st}g$ .

$$f(x) = a_n x^n + \dots + a_0, \quad a_n \neq 0 \quad \Rightarrow \text{st}f = n$$

$$g(x) = b_m x^m + \dots + b_0, \quad b_m \neq 0 \quad \Rightarrow \text{st}g = m$$

$$\text{st}(f+g) \leq \max\{\text{st}f, \text{st}g\}$$

$$\text{st}(f-g) \leq \max\{\text{st}f, \text{st}g\}$$

$$\text{st}(f \cdot g) = \text{st}f + \text{st}g$$

$$\text{st}(f \circ g) = \text{st}f \cdot \text{st}g$$

$$(f \circ g)(x) = f(g(x)) = f(b_m x^m + \dots + b_0) =$$

$$= a_n (b_m x^m + \dots + b_0)^n + \dots =$$

$$= a_n (b_m)^n x^{m \cdot n} + \dots$$

tem (o jednakosti polinoma)

#4

Uzeto su  $p(x) = a_n x^n + \dots + a_0$  i  $q(x) = b_m x^m + \dots + b_0$   
dva polinoma. Uvijek:

$$p(x) = q(x), \forall x \in \mathbb{R} \iff \begin{matrix} m = n \\ a_i = b_i, \forall i \in \{0, \dots, n\} \end{matrix}$$

■

2d. Odredite sve polinome  $p \in \mathbb{R}[x]$  koji zadovoljavaju

$$(x+1)p(x) = x^3 + 1, \forall x \in \mathbb{R}$$

$$\deg((x+1)p(x)) = \deg(x^3 + 1)$$

$$1 + \deg p = 3 \Rightarrow \deg p = 2$$

$$\Rightarrow p(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R}$$

$$(x+1)(ax^2 + bx + c) = x^3 + 1$$

$$ax^3 + (a+b)x^2 + (b+c)x + c = x^3 + 1$$

$$\Rightarrow a = 1$$

$$c = 1$$

$$a+b=0 \Rightarrow b = -a = -1$$

$$b+c=0 \Rightarrow b = -c = -1$$

$$\Rightarrow p(x) = x^2 - x + 1$$

2d. Odredite sve polinome  $f \in \mathbb{R}[x]$  tak. uvijek:

$$f(x^2 - 3) = x^2 f(x-1), \quad \forall x \in \mathbb{R}$$

$$n = \deg f$$

Iz jednakosti polinoma imamo:  $2n = 2 + n \Rightarrow n = 2$

Dakle,  $f(x) = ax^2 + bx + c$ ,  $a, b, c \in \mathbb{R}$

$$f(x^2 - 3) = x^2 f(x - 1)$$

$$a(x^2 - 3)^2 + b(x^2 - 3) + c = x^2(x-1)^2 a + x^2(x-1)b + cx^2$$

$$a(x^4 - 6x^2 + 9) + bx^2 - 3b + c = x^2a(x^2 - 2x + 1) + b(x^3 - x^2) + cx^2$$

$$ax^4 + x^2(-6a + b) + (9a - 3b + c) = x^4a + x^3(-2a + b) + x^2(a - b + c)$$

Imamo:  $a = a$  ✓

$$x^3 \dots -2a + b = 0 \Rightarrow \underline{b = 2a}$$

$$x^0 \dots 9a - 3b + c = 0 \Rightarrow \underline{c = 3b - 9a = 6a - 9a = -3a}$$

$$x^1 \dots -6a + b = a - b + c$$

$$-6a + 2a = a - 2a - 3a$$

$$-4a = -4a \quad \checkmark$$

Dakle,  $a \in \mathbb{R}$  parameter.

$$\Rightarrow f(x) = ax^2 + 2ax - 3a = a(x^2 + 2x - 3), \quad a \in \mathbb{R} \setminus \{0\}$$

od.

zadaca)

Odredite  $f \in \mathbb{R}[x]$  ako vrijedi:

$$f(-2x + 1) = 2x^2 - x + 3.$$

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I način

$$t := -2x + 1 \Rightarrow x = \frac{-t+1}{2}$$

$$f(t) = 2 \left( \frac{-t+1}{2} \right)^2 - \frac{-t+1}{2} + 3 = \dots = \frac{1}{2}t^2 - \frac{1}{2}t + 3$$

II način

$$\Delta_x f = 2 \Rightarrow \Delta_x^2 f = 2 \Rightarrow f(x) = ax^2 + bx + c$$

$$\Rightarrow f(-2x+1) = 2x^2 - x + 3$$

$$a(-2x+1)^2 + b(-2x+1) + c = 2x^2 - x + 3$$

$$a = \frac{1}{2}, b = -\frac{1}{2}, c = 3$$

od. Vokreolite polinomom drugog stupnja ako je

$$f(1) = 6, f(2) = 11, f(-1) = 8$$

$$f(x) = ax^2 + bx + c$$

$$f(1) = 6 = a + b + c$$

$$f(2) = 11 = 4a + 2b + c$$

$$f(-1) = 8 = a - b + c$$

Dakle ,  $a + b + c = 6$

$$4a + 2b + c = 11$$

$$a - b + c = 8$$

$$\dots a = 2, b = -1, c = 5$$

2d.

Volredite  $a, b \in \mathbb{R}$  t.d.  $f(x) = x^4 + 2x^3 + ax^2 + 2x + b$  bude kvadratni polinom.

$\therefore$

$$x^4 + 2x^3 + ax^2 + 2x + b = (x^2 + \beta x + \gamma)^2, \quad \forall x \in \mathbb{R}$$

$$x^4 + 2x^3 + ax^2 + 2x + b = x^4 + \beta^2 x^2 + \gamma^2 + 2\beta x^3 + 2\gamma x^2 + 2\beta\gamma x$$

porovnajmo koeficiente:

$$x^4 \dots \quad \beta^2 = 1 \quad \Rightarrow \quad \beta = \pm 1$$

$$x^3 \dots \quad 2\beta = 2 \quad \Rightarrow \quad \beta = 1 \quad \Rightarrow \quad \beta = \alpha = \pm 1$$

$$x^2 \dots \quad 2\gamma + \beta^2 = a$$

$$x^1 \dots \quad 2\beta\gamma = 2 \quad \Rightarrow \quad \gamma = \frac{1}{\beta} = \pm 1 \quad \Rightarrow \quad \underline{a = 3}$$

$$x^0 \dots \quad \gamma^2 = b \quad \Rightarrow \quad \underline{b = 1}$$

2d.

ica) Volredite sve polinome  $p \in \mathbb{R}[x]$  t.d.

$$p(1) + p(x) + p(x^2) = (1+x+x^2)p(x), \quad \forall x \in \mathbb{R}$$

$\therefore$

$$n: \deg p \quad \Rightarrow \quad 2n = n+2 \quad \Rightarrow \quad n = 2$$

Dakle,  $p(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R}$

$$a + b + c + ax^2 + bx + c + ax^4 + bx^2 + c = (1+x+x^2)(ax^2 + bx + c)$$

$$ax^4 + x^2(a+b) + bx + (a+b+c) = ax^4 + x^3(a+b) + x^2(a+b+c) + x(b+c) + c$$

porovnajmo koeficiente:

$$x^4 \dots \quad a = a$$

$$b = -a$$

$$x^2 \dots \quad 0 = 0 + c \quad \Rightarrow \quad c = 0$$

$$\dots \quad 1 = 0 + c \quad \Rightarrow \quad c = 1$$

orem (o dijeljenju polinoma s ostatkom)

Neka su  $f, g \in \mathbb{R}[x]$  polinomi i neka  $g$  nije nulpol.  
Tada postoje jedinstveni polinomi  $q, r \in \mathbb{R}[x]$  td.

$$f = q \cdot g + r \quad \text{i} \quad \text{st} r < \text{st} g.$$

□

$q$  zovemo kvocijent

$r$  zovemo ostatak

ed. Podijelite polinom  $f(x) = x^3 - 2x^2 + x + 3$  s polinomom  
 $g(x) = x^2 + 2x - 1$ .

$$\begin{array}{r} (x^3 - 2x^2 + x + 3) : (x^2 + 2x - 1) = x - 4 \\ \underline{-x^3 - 2x^2 + x} \phantom{+ 3} \\ -4x^2 + 2x + 3 \\ \underline{+4x^2 + 8x - 4} \\ 10x - 1 \end{array}$$

$$\Rightarrow q(x) = x - 4, \quad r(x) = 10x - 1$$

d. Podijelite  $f(x) = x^7 + 5x^6 - 3x^5 + 2x^4 - x^3 + x^2 - 7x + 3$  s  
 $g(x) = x^3 - 2x^2 + 3x + 1$ .

$$(x^7 + 5x^6 - 3x^5 + 2x^4 - x^3 + x^2 - 7x + 3) : (x^3 - 2x^2 + 3x + 1) = x^4 + 7x^3 + 8x^2 - 4x - 1$$

$$\begin{array}{r} -x^7 + 2x^6 - 3x^5 - x^4 \\ \hline \end{array}$$

$$\begin{array}{r} 7x^6 - 6x^5 + x^4 - x^3 \\ -7x^6 + 14x^5 - 21x^4 - 7x^3 \\ \hline \end{array}$$

$$\begin{array}{r} 8x^5 - 20x^4 - 8x^3 + x^2 \\ -8x^5 + 16x^4 - 24x^3 - 8x^2 \\ \hline -4x^4 - 32x^3 - 7x^2 - 7x \\ 4x^4 - 8x^3 + 12x^2 + 4x \\ \hline \end{array}$$

$$-40x^3 + 5x - 3x + 3$$

$$\begin{array}{r} 40x^3 - 80x^2 + 120x + 40 \\ \hline \end{array}$$

$$-75x^2 + 117x + 43$$

$$\Rightarrow \begin{aligned} q(x) &= x^4 + 7x^3 + 8x^2 - 4x - 40 \\ r(x) &= -75x^2 + 117x + 43 \end{aligned}$$

2d. Vobredite ostatak pri dijeljenju polinoma

$$f(x) = x^{100} + 3x^{99} + x^2 - 3x + 9 \text{ sa } g(x) = x^2 + 2x - 3.$$

$$\because \deg r < \deg g = 2 \Rightarrow \deg r \leq 1 \Rightarrow r(x) = ax + b, \quad a, b \in \mathbb{R}$$

$$\Rightarrow f(x) = q(x)g(x) + r(x), \quad \forall x \in \mathbb{R}$$

$$x^{100} + 3x^{99} + x^2 - 3x + 9 = q(x) \cdot (x^2 + 2x - 3) + ax + b$$

$$x^{100} + 3x^{99} + x^2 - 3x + 9 = q(x) \cdot (x+3)(x-1) + ax + b$$

$$= -3 \dots$$

$$(-3)^{100} + 3(-3)^{99} + 9 + 9 + 9 = 0 + a(-3) + b$$

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$$\cancel{3^{100}} - \cancel{3^{100}} + 27 = -3a + b$$

$$\Rightarrow -3a + b = 27$$

$$1 \dots$$

$$1 + \cancel{3} + 1 - \cancel{3} + 9 = 0 + a + b$$

$$\Rightarrow a + b = 11$$

$$\begin{cases} -3a + b = 27 \\ a + b = 11 \end{cases} \Rightarrow \begin{cases} a = -4 \\ b = 15 \end{cases}$$

$$\Rightarrow r(x) = -4x + 15$$

sol.  
ica) Vyhodíte ostatak pri dielení polynómu  
 $f(x) = x^{256} - 8x^{64} + 11x^8 - 7$  a  $g(x) = x^2 - 1$ .

$$\deg f < \deg g = 2 \Rightarrow \deg f \leq 1 \Rightarrow r(x) = ax + b, a, b \in \mathbb{R}$$

$$f(x) = q(x)g(x) + r(x), \quad \forall x \in \mathbb{R}$$

$$x^{256} - 8x^{64} + 11x^8 - 7 = q(x)(x^2 - 1) + ax + b$$

$$x = 1 \dots$$

$$1 - 8 + 11 - 7 = 0 + a + b$$

$$\Rightarrow a + b = -3$$

$$x = -1 \dots$$

$$1 - 8 + 11 - 7 = 0 - a + b$$

$$\Rightarrow -a + b = -3$$

$$\begin{cases} a + b = -3 \\ a - b = 3 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = -3 \end{cases}$$

$$\Rightarrow r(x) = -3$$

(11)

11  
 14. Neka su  $f, g \in R[x]$  dva nekakva polinoma. Kažemo da  $g$  dijeli  $f$  ako postoji polinom  $h \in R[x]$  t.d.

$$f = h \cdot g$$

i pišemo  $g \mid f$ .

ed.  
 (a) Djeljivost polinoma je binarna <sup>relacija</sup> operacija na  $R[x]$ .  
 Ispitajte joj svojstva.

• refleksivnost: DA

• simetričnost: NE

• tranzitivnost: DA

$$f = 1 \cdot f \Rightarrow f \mid f$$

$$x^2 - 1 \mid x^3 - x \text{ ali } x^3 - x \nmid x^2 - 1$$

$$f, g, h \in R[x], f \mid g, g \mid h \stackrel{?}{\Rightarrow} f \mid h$$

$$\exists h_1, h_2 \in R[x] \text{ t.d. } g = f h_1 \text{ i } h = g h_2$$

$$\Rightarrow h = f h_1 h_2, \quad h_1, h_2 \in R[x]$$

$$\Rightarrow f \mid h$$

• antisimetričnost: NE

$$x^2 + 1 \mid 2x + 2 \wedge 2x + 2 \mid x + 1, \text{ ali } x + 1 \nmid 2x + 2$$

$$\forall x \in R$$

4. Za polinom  $p \in R[x]$  kažemo da je normiran ako  $m$  je vodeći koeficijent jednak 1.

5. Neka su  $f, g \in R[x]$ ,  $f, g \neq 0$ . Za polinom  $p \in R[x]$  kažemo da je najveća zajednička mjera polinoma  $f$  i  $g$  ako:

1)  $p \mid f$  i  $p \mid g$

2) ako za  $q \in R[x]$  vrijedi  $q \mid f$  i  $q \mid g$  tada  $q \mid p$ .

6. Za svako dva polinoma  $f, g \neq 0$  postoji jedinstvena normirana najveća zajednička mjera koju označavamo s  $M(f, g)$ .



ŽANJA:

Uvode odnose od definicije s predavateljice (kao uvrsto u  $A[x]$ ) jer smo dobili normiranost.

uključujući odgovarajuće za polinome:

$$M(f, g) = ?$$

$$\text{st } f \geq \text{st } g$$

$$f = q_1 \cdot g + r_1, \quad \text{st } r_1 < \text{st } g$$

$$g = q_2 \cdot r_1 + r_2, \quad \text{st } r_2 < \text{st } r_1$$

$$r_1 = q_3 \cdot r_2 + r_3, \quad \text{st } r_3 < \text{st } r_2$$

$\vdots$

$$r_{m-2} = q_m \cdot r_{m-1} + r_m, \quad \text{st } r_m < \text{st } r_{m-1}$$

$$r_{m-1} = q_{m+1} \cdot r_m \quad \leftarrow \text{ostatak je } 0$$

$\Rightarrow M(f, g)$  je najmanji  $r_m$

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201. Napišite najveću zajedničku mjeru polinoma

$$f(x) = x^4 + x^3 + 2x^2 + x + 1$$

$$g(x) = x^3 - 2x^2 + x - 2$$

$$(x^4 + x^3 + 2x^2 + x + 1) : (x^3 - 2x^2 + x - 2) = x + 3$$

$$-x^4 + 2x^3 - x^2 + 2x$$

$$3x^3 + x^2 + 3x + 1$$

$$-3x^3 + 6x^2 - 3x + 6$$

$$7x^2 + 7$$

$\Rightarrow$

$$q_1(x) = x + 3$$

$$r_1(x) = 7x^2 + 7$$

$$(x^3 - 2x^2 + x - 2) : (7x^2 + 7) = \frac{1}{7}x - \frac{2}{7}$$

$$\begin{array}{r} -x^3 \qquad -x \\ \hline -2x^2 \qquad -2 \\ +2x^2 \qquad +2 \\ \hline 0 \end{array}$$

$$\Rightarrow H(f, g) = \frac{1}{7} r_1(x) = \frac{1}{7} (7x^2 + 7) = x^2 + 1$$

13. Dokazite da je ostatak pri dijeljenju polinoma  $f$  polinomom  $g(x) = x - a$  jednak  $f(a)$ .

$$\because \deg f < \deg g = 1 \quad \Rightarrow \quad r = \text{const} \in \mathbb{R}$$

$$f(x) = q(x)g(x) + r, \quad \forall x \in \mathbb{R}$$

$$f(x) = q(x)(x - a) + r$$

$$x = a \quad \dots \quad f(a) = 0 + r$$

$$\Rightarrow r = f(a)$$

~  
Za  $a \in \mathbb{R}$  možemo da je nultočka polinoma  $f$  ako je  
 $f(a) = 0$ .

zm (Bezout)

Broj  $a \in \mathbb{R}$  je nultočka polinoma  $f$  ako  $x-a$   
dijeli polinom  $f$ . ▣

d. Dokažite da je polinom

$$f(x) = (x^2 + x - 1)^{2m} + (x^2 - x + 1)^{2m} - 2, \quad m \in \mathbb{N}$$

djeljom polinomom  $g(x) = x^2 - x$ .

∴  $g(x) = x(x-1)$

$L$  je djeljiv  $\Delta$   $g$  ako je djeljiv  $\Delta$   $x$  i  $x-1$

$$x \mid f \Leftrightarrow f(0) = 0$$

$$x-1 \mid f \Leftrightarrow f(1) = 0$$

provjerimo:

$$f(0) = (-1)^{2m} + 1^{2m} - 2 = 1 + 1 - 2 = 0 \in$$

$$f(1) = (1+1-1)^{2m} + (1-1+1)^{2m} - 2 = 1 + 1 - 2 = 0 \in$$

$$\Rightarrow g \mid f$$

NE  
 100% Polinom  $f \in \mathbb{R}[x]$  pri dijeljenju s  $x-a$  daje ostatak A  
 a pri dijeljenju s  $x-b$  daje ostatak B,  $a \neq b$ . Odredite  
 ostatak pri dijeljenju s  $(x-a)(x-b)$ .

Prima jednemu od prethodnih zadatka:

$$f(a) = A, \quad f(b) = B$$

$$\text{Nj } r < \text{Nj } (x-a)(x-b) = 2 \quad \Rightarrow \quad \text{Nj } r \leq 1 \quad \Rightarrow \quad r(x) = ux + v$$

$$f(x) = q(x)(x-a)(x-b) + ux + v$$

$$\begin{array}{lcl} x=a & \dots & A = ua + v \\ x=b & \dots & B = ub + v \end{array} \quad \Bigg| - \quad \Rightarrow \quad u(a-b) = A-B$$

$$\Rightarrow u = \frac{A-B}{a-b}$$

$$r(x) = A - ua = A - \frac{(A-B)}{a-b} a = \frac{Ba - Ab}{a-b}$$

$$\Rightarrow r(x) = \frac{A-B}{a-b} x + \frac{Ba - Ab}{a-b}$$

d. Ukoliko je  $f(x) = x^3 + ax^2 + bx + c$ . Odredite  $a, b, c \in \mathbb{R}$  tak. da pri dijeljenju s  $x-1, x-2, x-3$  daje redom ostatke -8, -9, -4

$$\left. \begin{array}{lcl} f(1) = -8 & \Rightarrow & 1+a+b+c = -8 \\ f(2) = -9 & \Rightarrow & 8+4a+2b+c = -9 \\ f(3) = -4 & \Rightarrow & 27+9a+3b+c = -4 \end{array} \right\} \Rightarrow \begin{array}{l} a+b+c = -9 \\ 4a+2b+c = -17 \\ 9a+3b+c = -31 \end{array}$$

$$\dots \Rightarrow a = -3, \quad b = 1, \quad c = -7$$

ad.  
 Pokažite da je polinom  $f(x) = (x+1)^{2m} - x^{2m} - 2x - 1$ ,  $m \in \mathbb{N}$   
 djeljiv polinomom  $g(x) = 2x^3 + 3x^2 + x$ .

$$g(x) = x(x+1)(2x+1)$$

$$g \mid f \Leftrightarrow x \mid f \wedge x+1 \mid f \wedge x+\frac{1}{2} \mid f$$

$$f(0) = 1^{2m} - 0 - 0 - 1 = 1 - 1 = 0 \quad \checkmark$$

$$f(-1) = 0 - 1 + 2 - 1 = 0 \quad \checkmark$$

$$f(-\frac{1}{2}) = (\frac{1}{2})^{2m} - (\frac{1}{2})^{2m} + 1 - 1 = 0 \quad \checkmark$$

2.  
 Neka su  $f, g \in \mathbb{R}[x]$  i  $g(x) = x^3 - 9x^2 - 9x - 1$ .  
 Pokažite da  $M(f, g)$  nije djeljiv polinomom  $x+1$ .

Imamo:  $M(f, g) \mid f$  i  $M(f, g) \mid g$

$$\text{Kao } x+1 \mid M(f, g) \Rightarrow x+1 \mid g \Rightarrow$$

$$(-1) + 1 = 0, \text{ ali}$$

$$\cancel{g(-1) = 0, \text{ ali}}$$

$$g(-1) = -2$$

$$\Rightarrow x+1 \nmid M(f, g)$$

$\Rightarrow \Leftarrow$

$$f(-1) = 1 - 1 + 1 - 1 = 0 \quad \times$$

$$f(i) = 1 + i - 1 - i = 0 \quad \times$$

$$f(-i) = 1 - i - 1 + i = 0 \quad \times$$

$\Rightarrow g \nmid f$

d. Polinom  $f \in \mathbb{R}[x]$  pri dijeljenju s  $x+1$  daje ostatak 4,  
 pri dijeljenju s  $x^2+1$ , ostatak  $2x+3$ . Odredite ostatak  
 pri dijeljenju s  $(x+1)(x^2+1)$ .

$$\deg f < \deg (x+1)(x^2+1) = 3 \Rightarrow r(x) = ax^2 + bx + c$$

$$f(x) = q(x)(x+1)(x^2+1) + ax^2 + bx + c$$

= 1 ...

$$4 = 0 + a - b + c$$

= i ...

$$2i + 3 = 0 - a + ib + c$$

= -i ...

$$-2i + 3 = 0 - a - ib + c$$

$$\Rightarrow \begin{cases} -2a + 2c = 6 \\ c = a + 3 \end{cases}$$

$$-a + ib + a + 3 = 2i + 3 \Rightarrow$$

$$b = 2$$

$$4 = a - b + c$$

$$a = \frac{3}{2}$$

$$6 = a + c$$

$$c = \frac{9}{2}$$

$$6 = 2a + 3$$

$$2a = 3$$

$$a = \frac{3}{2}$$

$$c = \frac{9}{2}$$

$$\Rightarrow r(x) = \frac{3}{2}x^2 + 2x + \frac{9}{2}$$

ad. Dokazite da je polinom

$$f(x) = x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3},$$

$a, b, c, d \in \mathbb{N}$

djeljiv s polinomom  $g(x) = x^3 + x^2 + x + 1$ .

$$\therefore g(x) = (x+1)(x^2+1) = (x+1)(x-i)(x+i)$$

d. Vodećite se polinome  $f \in \mathbb{R}[x]$  koji zadovoljavaju:

$$x f(x-1) = (x-3) f(x), \quad \forall x \in \mathbb{R}$$

$$n = \deg f \Rightarrow n+1 = n+1 \quad \text{u}$$

namamo informaciju o stepenu polinoma  $f$

Prostimo brojeve koji će poništiti ili lijevu ili desnu stranu

$$\begin{aligned} = 0 \dots & \quad 0 f(-1) = -3 \cdot f(0) \Rightarrow f(0) = 0 \\ = 3 \dots & \quad 3 f(2) = 0 \cdot f(3) \Rightarrow f(2) = 0 \\ = 2 \dots & \quad 2 f(1) = -1 \cdot \underbrace{f(2)}_0 = 0 \Rightarrow f(1) = 0 \end{aligned}$$

$$\Rightarrow x, x-1, x-2 \mid f \Rightarrow x(x-1)(x-2) \mid f$$

$$\Rightarrow f(x) = x(x-1)(x-2) g(x), \text{ za neki } g \in \mathbb{R}[x]$$

$$x f(x-1) = (x-3) f(x)$$

$$x(x-1)(x-2)(x-3) g(x-1) = x(x-1)(x-2)(x-3) g(x), \quad \forall x \in \mathbb{R}$$

$$\text{Za } x \neq 0, 1, 2, 3 \text{ imamo } g(x) = g(x-1)$$

$$\text{Dakle, } g(4) = g(3) = g(5) = g(6) = \dots$$

$$\text{Definirajmo } h \in \mathbb{R}[x] : h(x) = g(x) - g(3)$$

$$h(3) = 0 = h(4) = h(5) = h(6) = \dots$$

$\Rightarrow h$  ima beskonačno mnogo nultočaka

$$\Rightarrow h(x) = g(x) - g(3) = 0, \quad \forall x \in \mathbb{R}$$

Horner algorithm

OW 14DA 15.1

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_n \neq 0$$

podijelimo  $f$  s  $x - \alpha$ :  $f(x) = q(x)(x - \alpha) + r$ ,  $\deg q = n-1$   
 $f(\alpha)$

$$q(x) = b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0$$

Izjednačimo koeficijente:

$$a_n = b_{n-1}$$

$$a_{n-1} = b_{n-2} - \alpha b_{n-1}$$

$$a_{n-2} = b_{n-3} - \alpha b_{n-2}$$

$\vdots$

$$a_2 = b_1 - \alpha b_2$$

$$a_1 = b_0 - \alpha b_1$$

$$a_0 = r - \alpha b_0$$

Dakle,

$$b_{n-1} = a_n$$

$$b_{n-2} = \alpha b_{n-1} + a_{n-1}$$

$$b_{n-3} = \alpha b_{n-2} + a_{n-2}$$

$\vdots$

$$b_1 = \alpha b_2 + a_2$$

$$b_0 = \alpha b_1 + a_1$$

$$r = \alpha b_0 + a_0$$

Učimo:

|           |           |           |     |       |       |            |
|-----------|-----------|-----------|-----|-------|-------|------------|
| $a_m$     | $a_{m-1}$ | $a_{m-2}$ |     | $a_2$ | $a_1$ | $a_0$      |
| $b_{m-1}$ | $b_{m-2}$ | $b_{m-3}$ | ... | $b_1$ | $b_0$ | $r = f(2)$ |

d. Podijelite  $f(x) = 2x^5 - x^3 + x + 9$  sa  $g(x) = x - 2$ .

$$\begin{array}{r|rrrrrr} & 2 & 0 & -1 & 0 & 1 & 9 \\ x-2 & 2 & 4 & 7 & 14 & 29 & 66 = f(2) \end{array}$$

$$Q(x) = 2x^4 + 4x^3 + 7x^2 + 14x + 29$$

$$r(x) = 66$$

d. Dokažite da polinom stupnja  $n$  ne može imati više od  $n$  različitih nultočeka.

Pretp. suprotno, tj. da su  $a_1, a_2, \dots, a_{n+1}$  različite nultocke od  $f$ . Po Bezoutovu teoremu  $f$  je djeljiv sa

$$(x - a_1)(x - a_2) \dots (x - a_{n+1})$$

pa se djeljiv i sa njihovim produktom

$$(x - a_1)(x - a_2) \dots (x - a_{n+1})$$

što je polinom  $(n+1)$ -og stupnja. Tada je  $\deg f \geq n+1$

$$\Rightarrow \Leftarrow$$

$$\deg f = n.$$

Primjeri: Jedini polinom s beskonačno mnogo nultočeka  
je  $f(x) = 0$  za  $\forall x \in \mathbb{R}$

$$\Rightarrow g(x) = g(3) = \text{const} \in \mathbb{K}$$

Ockle,  $\checkmark(x) = c \cdot x(x-1)(x-2), \quad c \in \mathbb{R}$

rem

Neka je  $f \in \mathbb{H}[x]$  i  $L = \frac{m}{n} \in \mathbb{Q}$  njegov racionalni multiplikator. To je najviši zajednički djelitelj od  $m$  i  $n$  jednak 1. Jedan brojnik  $m$  dijeli svaki član polinoma  $f$ , a nazivnik  $n$  svaki koeficijent.

□

d. Nađite multiplikatore polinoma:

$$f(x) = 2x^3 - x^2 - 6x + 3$$

djelitelji  $a_0: \pm 1, \pm 3$

djelitelji  $a_n: \pm 1, \pm 2$

multiplikatori racionalne multiplikacije:  $\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$

$$f(1) = -2, \quad f(3) = 30, \quad f(-1) = 6, \quad f(-3) = -42, \quad f\left(\frac{1}{2}\right) = 0$$

$$\Rightarrow x_1 = \frac{1}{2} \text{ multiplikator}$$

$$\Rightarrow x - \frac{1}{2} \text{ (2x-1) dijeli } f$$

$$(2x^3 - x^2 - 6x + 3) : (2x - 1) = x^2 - 3 \quad (\text{ili Hornerom})$$

$$\Rightarrow 2x^3 - x^2 - 6x + 3 = (2x - 1)(x^2 - 3)$$

$$\Rightarrow x_{2,3} = \pm \sqrt{3} \text{ multiplikatori}$$

Mulitčka  $\alpha$  polinoma  $f$  je kretnosti  $k \in \mathbb{N}$  ako  $(x-\alpha)^k$  dijeli  $f$  i  $(x-\alpha)^{k+1}$  ne dijeli  $f$ .

em (Osnovni teorem algebre)

Velika je  $f \in \mathbb{C}[x]$  stupnja  $n \geq 1$ . Onda  $f$  ima točno  $n$  multočaka računajući i njihove kretnosti

$$f(x) = a_n (x-\alpha_1)^{k_1} (x-\alpha_2)^{k_2} \dots (x-\alpha_s)^{k_s}$$

$a_n \dots$  vodeći koeficijent od  $f$

$$k_1, \dots, k_s \in \mathbb{N} \text{ tak. } k_1 + k_2 + \dots + k_s = n = \deg f$$

$\alpha_1, \dots, \alpha_s$  međusobno različite multočke

$k_1, \dots, k_s$  su kretnosti multočaka  $\alpha_1, \dots, \alpha_s$ .

1. Velika je  $f \in \mathbb{R}[x]$  realni polinom koji ima kompleksnu multočku  $z = a + ib$ ,  $a, b \in \mathbb{R}$ . Pokažite da je i  $\bar{z} = a - ib$  multočka od  $f$ .

$$f(x) = a_n x^n + \dots + a_1 x + a_0, \quad a_n, \dots, a_0 \in \mathbb{R}$$

$$f(z) = 0 = \overline{0} = \overline{f(z)}$$

$$\overline{f(z)} = \overline{a_n z^n + \dots + a_1 z + a_0} = \overline{a_n z^n} + \dots + \overline{a_1 z} + \overline{a_0} =$$

$$= \overline{a_n} \overline{z^n} + \dots + \overline{a_1} \overline{z} + \overline{a_0} =$$

$$= a_n \overline{z}^n + \dots + a_1 \overline{z} + a_0 = f(\bar{z})$$

$$\Rightarrow f(\bar{z}) = 0$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \Rightarrow \overline{z^n} = \bar{z}^n$$

$$= -$$

$$\sim$$

$$\sim$$

$$\sim$$

$$\sim$$

$$\sim$$

$$\sim$$

$$\sim$$

$$\sim$$

ad. Vredite sve nultočke polinoma

$$p(x) = x^5 - 2x^4 - 5x^3 + 8x^2 - 16x - 40$$

ako znate da je jedna od njih  $x_0 = 1 - i\sqrt{3}$ .

∴

$$\left. \begin{array}{l} x_0 = 1 - i\sqrt{3} \\ x_1 = 1 + i\sqrt{3} \end{array} \right\} \Rightarrow p \text{ djeljiv } \Delta (x - 1 + i\sqrt{3})(x - 1 - i\sqrt{3}) = x^2 - 2x + 4$$

$$(x^5 - 2x^4 - 5x^3 + 8x^2 - 16x - 40) : (x^2 - 2x + 4) = x^3 - 9x - 10$$

Treba naći nultočke od  $x^3 - 9x - 10$ .

kandidati:  $\pm 1, \pm 2, \pm 5, \pm 10$

$$x_3 = -2 \quad \dots \quad (-2)^3 - 9(-2) - 10 = 0 \quad \text{nultočka}$$

Hornerom odredimo kvocijent od  $(x^3 - 9x - 10) : (x + 2)$

|          |             |   |    |    |     |
|----------|-------------|---|----|----|-----|
| $\alpha$ | $\parallel$ | 1 | 0  | -9 | -10 |
| -2       | $\parallel$ | 1 | -2 | -5 | 0   |

$$\Rightarrow (x^3 - 9x - 10) : (x + 2) = x^2 - 2x - 5$$

$$x^2 - 2x - 5 = 0 \quad \Rightarrow \quad x_{4,5} = 1 \pm \sqrt{6}$$

rek.

Weka je  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  polinom  
 // stupnja  $n$ . Formalna derivacija polinoma  $f$  je polinom  
 ° stupnja  $n-1$

$$f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + 2 a_2 x + a_1.$$

2f.

$$f^{(n)} = (f^{(n-1)})', \quad f^{(0)} = f$$

$n$ -ta formalna derivacija

gjtwa:

$$(\alpha f + \beta g)' = \alpha f' + \beta g',$$

$$\alpha, \beta \in \mathbb{R}$$

$$(fg)' = f'g + fg'$$

$$(f \circ g)' = (f' \circ g) g'$$

rek. Ako je  $\alpha$  multička polinoma kretnosti  $\alpha \geq 2$ ,  
 uola je  $\alpha$  multička polinoma  $f'$  kretnosti  $\alpha-1$ .

□

1. Odredite kretnost multičke  $x=2$  polinoma

$$f(x) = x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$$

ovo!!

$$f(2) = 0$$

$$f'(x) = 5x^4 - 20x^3 + 21x^2 - 4x + 4$$

$$f'(2) = 0$$

$$f''(x) = 20x^3 - 60x^2 + 42x - 4$$

$$f''(2) = 0$$

$$f'''(x) = 60x^2 - 120x + 42$$

$$f'''(2) = 42 \neq 0$$

$\Rightarrow x=2$  je násobná hodnota  $f$  hodnoty 3

d. Dokažte, že polynom  $f(x) = x^m - 1$  nemá násobných hodnot.

OVČ MČ

Albo by  $f$  měl násobné hodnoty, ovšem by  $f$  a  $f'$  měl mák jednoduchých hodnot.

$$f'(x) = mx^{m-1} \quad \dots \quad x=0 \text{ jediná hodnota od } f'$$

$$f(0) = 0^m - 1 = -1 \neq 0$$

$\Rightarrow f$  a  $f'$  nemají jednoduchých hodnot pro  $x$  nemá násobné hodnoty.

Odredite koeficijente  $a$  i  $b$  polinoma.

$$f(x) = x^4 + ax^3 - 2x^2 + b$$

td. bude djeljiv s  $(x-2)^2$ .

$\therefore (x-2)^2 \mid f \Leftrightarrow x=2$  je multipla od  $f$  kratnosti barem 2

$$\begin{aligned} f(2) &= 0 & \Rightarrow & \begin{aligned} 8a+b &= -8 \\ 16+8a-8+b &= 0 \\ 12a &= -24 \end{aligned} \\ f'(2) &= 0 & \Rightarrow & \begin{aligned} 32+12a-8 &= 0 \end{aligned} \end{aligned} \quad \left. \vphantom{\begin{aligned} f(2) &= 0 \\ f'(2) &= 0 \end{aligned}} \right\} \Rightarrow \begin{aligned} a &= -2 \\ b &= 8 \end{aligned}$$

$$f(x) = 4x^3 + 3ax^2 - 4x$$

d.

Odredite ostatak pri dijeljenju polinoma

$$f(x) = x^{100} - x^{50} + 1 \quad \wedge \quad g(x) = x^2 - 2x + 1 \quad \text{ovo!!}$$

$$g(x) = (x-1)^2$$

$$f(x) = q(x)g(x) + r(x), \quad r(x) = ax + b$$

$$x^{100} - x^{50} + 1 = q(x)(x-1)^2 + ax + b \quad (*)$$

$$x=1 \dots$$

$$1 - 1 + 1 = 0 + a + b \quad \Rightarrow \quad a + b = 1$$

$$\text{derivirajmo } (*): \quad 100x^{99} - 50x^{49} = q'(x)(x-1)^2 + q(x)2(x-1) + a$$

$$x=1 \dots$$

$$100 - 50 = 0 + 0 + a \quad \Rightarrow \quad a = 50$$
$$b = -49$$

$$\Rightarrow r(x) = 50x - 49$$

Velik je  $f \in \mathbb{Z}[x]$  polinom s cjelobrojnim koeficijentima koji ima cjelobrojnu nultocku  $a \in \mathbb{Z}, a \neq 0$ . Tada  $a$  dijeli slobodni član polinoma  $f$ .

20. Za) Vediti nultocke polinoma

SKIP

$$f(x) = x^4 - 2x^3 - 5x^2 + 4x + 6$$

Kandidati za cjelobrojne nultocke:  $\pm 1, \pm 2, \pm 3, \pm 6$

$$f(1) = 4, f(-1) = 0, f(2) = -6, f(-2) = 10, f(3) = 0, f(-3) = -84, \\ f(6) = 714, f(-6) = 1530.$$

$$\Rightarrow f \text{ je djeljiv s } (x+1)(x-3)$$

Hornerom (2 puta: jednom za  $-1$ , drugi put za  $3$ ) imamo:

$$(x^4 - 2x^3 - 5x^2 + 4x + 6) : (x^2 - 2x - 3) = x^2 - 2$$

Dakle, nultocke od  $f$  su:  $-1, 3, \pm\sqrt{2}$

SKIP

21.

Dokažite da ne postoji trokut čijene su dužine stranice jednake nultockama polinoma

$$f(x) = 2x^3 - 13x^2 + 25x - 14.$$

$$a_n \text{ djeljivo s } : \pm 1, \pm 2$$

$$a_0 \text{ djeljivo s } : \pm 1, \pm 2, \pm 7, \pm 14$$

$$\text{moguće nultocke: } \pm 1, \pm 7, \pm 14, \pm \frac{1}{2}, \pm \frac{7}{2}$$

$$f(1) = 2 - 13 + 25 - 14 = 0$$

$$\Rightarrow \lambda_1 = 1 \text{ multiplika}$$

Korner:

$$\begin{array}{c|ccc|c} & 2 & -13 & 25 & -14 \\ \hline 1 & 2 & -11 & 14 & 0 \end{array}$$

$$\Rightarrow f(x) = (x-1)(2x^2 - 11x + 14)$$

$$\frac{11 \pm \sqrt{121 - 112}}{4}$$

$$\Rightarrow x_2 = 2, x_3 = \frac{7}{2}$$

Wzajemność boków nie istnieje:  $1+2=3 < \frac{7}{2}$ .

$\Rightarrow$  nie są stronice trójkąta

DO TUD E.I.

zol.  $f(x) = x^3 - (a+4)x^2 + (4a+3)x - 3a$

$$g(x) = x^3 - 9x^2 + 26x - 24$$

Wskaz: we  $a \in \mathbb{R}$  t.d. je  $M(f, g) =$  polinom stopnia 2

Wyznaczmy multożke od  $g$ :

kandydaci:  $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

$$g(1) = -6, g(-1) = -60, g(2) = 0$$

Korner:

$$\begin{array}{c|ccc|c} & 1 & -9 & 26 & -24 \\ \hline 2 & 1 & -7 & 12 & 0 \end{array}$$

$$\Rightarrow g(x) = (x-2)(x^2 - 7x + 12) = (x-2)(x-3)(x-4)$$

multożke od  $g$  su: 2, 3, 4

$$f(2) = a - 2 \Rightarrow a = 2$$

$$f(3) = 0$$

$$f(4) = -3a + 12 \Rightarrow a = 4$$

Wzajemność boków nie istnieje:  $2+3=5 < 4$

ex. Nađite polinom s cjelobrojnim koeficijentima kojemu je  
multočka  $x = \sqrt{2} + \sqrt[3]{3}$ .

g.

$$x - \sqrt{2} = \sqrt[3]{3} \quad |^3$$

$$x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} = 3$$

$$-\sqrt{2}(3x^2 + 2) = 3 - 6x - x^3 \quad |^2$$

$$2(9x^4 + 12x^2 + 4) = 9 + 36x^2 + x^6 - 36x - 6x^3 + 12x^4$$

$$\Rightarrow f(x) = x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1$$

ex. Neka je  $f \in \mathbb{Z}[x]$ . Dokazite: ako su  $f(0)$  i  $f(1)$  neparni brojevi, tada  $f$  nema cjelobrojnih multočka.

Pretp. da  $f$  ima cjelobrojnu multočku  $k \in \mathbb{Z}$ .

$$f(k) = 0 \Rightarrow x - k \mid f$$

$$\Rightarrow \exists g \in \mathbb{Z}[x] \text{ tak. } f(x) = (x - k)g(x)$$

$$\left. \begin{array}{l} x=0 \dots f(0) = -k g(0) \\ x=1 \dots f(1) = (1-k) g(1) \end{array} \right\} \text{ svi su tu iz } \mathbb{Z}$$

$$\Rightarrow k \text{ neparan}$$

$$1-k \text{ neparan} \Rightarrow k \text{ paran}$$

$\Rightarrow \Leftarrow$

primjena u gornjem zadatku  $g$  će imati cjelobrojne koeficijente jer to slijedi iz Hornerovog algoritma.

yp. ako  $\exists a \in \mathbb{H}$  tak  $f(a) = 0$

$$\Rightarrow \quad 1 = \underbrace{(d-a)(d-b)(d-c)h(d)}_{\text{tri čimbenika su iz } \mathbb{H}}$$

$$\Rightarrow \quad d-a, d-b, d-c, h(d) \in \{-1, 1\}$$

$$\Rightarrow \quad d-a = d-b \text{ ili } d-a = d-c \text{ ili } d-b = d-c$$

$$\text{tj. } a=b \text{ ili } a=c \text{ ili } b=c$$

sol.  $\Rightarrow \Leftarrow$   
Dokažite da ne postoji  $p \in \mathbb{H}[x]$  tak.  $p(11) - p(7)$  prost broj.

$$\therefore p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_n, \dots, a_0 \in \mathbb{H}$$

$$p(11) = a_n 11^n + a_{n-1} 11^{n-1} + \dots + a_1 11 + a_0$$

$$p(7) = a_n 7^n + a_{n-1} 7^{n-1} + \dots + a_1 7 + a_0 \quad | -$$

$$p(11) - p(7) = a_n (11^n - 7^n) + a_{n-1} (11^{n-1} - 7^{n-1}) + \dots + a_1 (11 - 7)$$

$$\text{TV: } 4 \mid 11^m - 7^m, \quad \forall m \in \mathbb{N}.$$

Indukcijom:

$$i) \text{ Za } m=1 \text{ imamo: } 11-7=4 \text{ u}$$

$$ii) \text{ Pretp. da za neki } m \in \mathbb{N} \quad 4 \mid 11^m - 7^m, \text{ tj. da postoji } k \in \mathbb{H} \text{ t}$$

$$11^m - 7^m = 4 \cdot k$$

$$\begin{aligned} iii) \quad 11^{m+1} - 7^{m+1} &= 11 \cdot 11^m - 7 \cdot 7^m = 4 \cdot 11^m + 7 \cdot 11^m - 7 \cdot 7^m = \\ &= 4 \cdot 11^m + 7 \underbrace{(11^m - 7^m)}_{4k \text{ pretpostavka}} = 4 \cdot 11^m + 7 \cdot 4 \cdot k = \\ &= 4(11^m + 7k) \\ &\quad \underbrace{\hspace{1cm}}_{\in \mathbb{H}} \end{aligned}$$

1.1 1.1.1 1.1.1.1

Dokažite da ne postoji polinom s cjelobrojnim koeficijentima  $f$  tak.  $f(1)=2$ ,  $f(3)=5$ .

Pretp. suprotno, tj. postoji takav polinom  $f$ .

Definirajmo  $g \in \mathbb{Z}[x]$  i  $g(x) = f(x+1)$

Tadaemo:  $g(0)=2$ ,  $g(2)=5$

$$g(x) = a_n x^n + \dots + a_1 x + a_0$$

$$g(0) = 2 = a_0$$

$$g(2) = \underbrace{a_n 2^n + \dots + a_1 \cdot 2 + 2}_{\text{paran broj}} = \underbrace{5}_{\text{neparan broj}}$$

$\Rightarrow \Leftarrow$

d.

Velika je  $f \in \mathbb{Z}[x]$  i  $a, b, c \in \mathbb{Z}$  međusobno različiti tak.  $f(a)=f(b)=f(c)=2$ .

Dokažite da ne postoji  $d \in \mathbb{Z}$  tak.  $f(d)=3$ .

Definirajmo:  $g(x) = f(x) - 2$ ,  $g \in \mathbb{Z}[x]$ .

$$g(a) = g(b) = g(c) = 0 \Rightarrow (x-a), (x-b), (x-c) \mid g$$

$$\Rightarrow (x-a)(x-b)(x-c) \mid g \Rightarrow \exists h \in \mathbb{Z}[x] \text{ tak.}$$

$$g(x) = (x-a)(x-b)(x-c)h(x)$$

$$f(x) = (x-a)(x-b)(x-c)h(x) + 2$$

|               |   |    |    |     |
|---------------|---|----|----|-----|
|               | 2 | -1 | -6 | 3   |
| 1             | 2 | 1  | -5 | -2  |
| -1            | 2 | -3 | -3 | 6   |
| 3             | 2 | 5  | 9  | 30  |
| -3            | 2 | -7 | 15 | -42 |
| $\frac{1}{2}$ | 2 | 0  | -6 | 0   |
|               |   |    |    |     |

u. Rastavte na parcijalne razlomke  $\frac{2x-1}{(x+1)^2}$

$$\frac{2x-1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad | \cdot (x+1)^2$$

$$2x-1 = A(x+1) + B$$

$$2x-1 = Ax + (A+B)$$

$$\Rightarrow A=2$$

$$A+B=-1 \Rightarrow B=-3$$

$$\Rightarrow \frac{2x-1}{(x+1)^2} = \frac{2}{x+1} - \frac{3}{(x+1)^2}$$

Uvode g ima višestruku reálnu nultčku.

1) ako g ima jedinstvenu komplexnu nultčku?

u razlomku se pojavjuje  $x^2+ax+b$  t.d.  $a^2-4b < 0$ , tj. nema nultčka u  $\mathbb{R}$ .

Tada u restavu pišemo član  $\frac{Cx+D}{x^2+ax+b}$

2) ako g ima višestruke komplexne nultčke?

u razlomku se pojavjuje  $(x^2+ax+b)^n$  t.d.  $a^2-4b < 0$ , tj. nema reálnih nultčka.

Tada u restavu pišemo članove:

$$\frac{C_1x+D_1}{x^2+ax+b} + \frac{C_2x+D_2}{(x^2+ax+b)^2} + \dots + \frac{C_nx+D_n}{(x^2+ax+b)^n}$$



K1 1a) Uvrstimo  $x=1$ :

(7)

$$(3 \cdot 1^2 - 2 \cdot 1 - 1)^4 = a_8 \cdot 1^8 + a_7 \cdot 1^7 + \dots + a_1 \cdot 1 + a_0$$

$$0 = a_8 + a_7 + \dots + a_1 + a_0 \quad (*)$$

Uvrstimo  $x=-1$ :

$$(3 \cdot (-1)^2 - 2 \cdot (-1) - 1)^4 = a_8 \cdot (-1)^8 + \dots + a_1 \cdot (-1) + a_0$$

$$256 = a_8 - a_7 + a_6 - \dots - a_1 + a_0 \quad (**)$$

$$(*) + (**): \quad 256 = 2a_8 + 2a_6 + 2a_4 + 2a_2 + 2a_0$$

$$128 = a_8 + a_6 + a_4 + a_2 + a_0$$

(b)  $x=0$ :

$$(3 \cdot 0^2 - 2 \cdot 0 - 1)^4 = a_8 \cdot 0 + a_7 \cdot 0 + \dots + a_1 \cdot 0 + a_0$$

$$\underline{\underline{1 = a_0}}$$

$$(3x^2 - 2x - 1)^4 = 3^4 x^8 + (\text{niže potencije od } x)$$

$$\Rightarrow \text{vodedi koef je } 3^4 = \underline{\underline{81}}$$

$$a_6 + a_4 + a_2 = 128 - a_8 - a_0 =$$

$$= 128 - 81 - 1 = \underline{\underline{46}}$$

Zad 2 Odredite zbroj koeficijenata polinoma koji se dobije sređivanjem izraza

$$(2 - 3x + x^2)^{2023} (2 + 3x + x^2)^{2024}$$

Y ( $x=1$ ) zbroj koef. je  $(2 \cdot 3 \cdot 1 + 1^2)^{2023} (2 + 3 \cdot 1 + 1^2)^{2024}$   
 $= \underline{\underline{0}}$

Nap Za  $f, g \in \mathbb{R}[x]$  funkcije  $f+g, f-g, f \cdot g$  i  $f \circ g$  (8)  
 $(f \pm g)(x) = f(x) \pm g(x), \quad \forall x \in \mathbb{R}$   
 $(f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in \mathbb{R}$   
 $(f \circ g)(x) = f(g(x)), \quad \forall x \in \mathbb{R}$   
 su također polinomi, tj. elementi  $\mathbb{R}[x]$ .

Zad 3 Neka su  $f, g \in \mathbb{R}[x]$ . Izrazite stupnjere od  $f+g, f \cdot g$  i  $f \circ g$  pomoću  $\text{st } f$  i  $\text{st } g$

Rj Označimo  $\text{st } f = n$  i  $\text{st } g = m$

$$f(x) = a_n x^n + \dots + a_1 x + a_0, \quad a_n \neq 0$$

$$g(x) = b_m x^m + \dots + b_1 x + b_0, \quad b_m \neq 0$$

$$(f+g)(x) = a_n x^n + \dots + a_1 x + a_0 + b_m x^m + \dots + b_1 x + b_0$$

1°)  $n > m$

$$(f+g)(x) = \underbrace{a_n}_{\neq 0} x^n + \dots + (a_m + b_m) x^m + \dots + (a_1 + b_1) x + (a_0 + b_0)$$

$$\Rightarrow \text{st}(f+g) = n$$

2°)  $n < m$

$$(f+g)(x) = \underbrace{b_m}_{\neq 0} x^m + \dots + (a_n + b_n) x^n + \dots + (a_1 + b_1) x + (a_0 + b_0)$$

$$\Rightarrow \text{st}(f+g) = m$$

3°)  $n = m$

$$(f+g)(x) = \underbrace{(a_n + b_n)}_{\text{može biti } 0!} x^n + \dots + (a_1 + b_1) x + (a_0 + b_0)$$

(9)

$(f+g)(x)$  može biti i nultpolinom, npr. za

$$f(x) = 2x - 1 \quad i \quad g(x) = -2x + 1$$

$\Rightarrow$  Jedino što možemo zaključiti je  $\deg(f+g) \leq n$

Iz dva tri slučaja zaključujemo da je

$$\deg(f+g) \leq \max \{ \deg f, \deg g \}$$

$$\begin{aligned} (f \cdot g)(x) &= (a_n x^n + \dots + a_1 x + a_0)(b_m x^m + \dots + b_1 x + b_0) \\ &= \underbrace{a_n b_m}_{\neq 0} x^{m+n} + (a_n b_{m-1} + a_{n-1} b_m) x^{m+n-1} \\ &\quad + \dots \text{ (niže potencije) } \end{aligned}$$

$$\Rightarrow \deg(fg) = m+n = \deg f + \deg g$$

$$\begin{aligned} (f \circ g)(x) &= a_n (g(x))^n + \dots + a_1 (g(x)) + a_0 \\ &= a_n (b_m x^m + \dots + b_1 x + b_0)^n + \dots + a_0 \\ &= \underbrace{a_n b_m^n}_{\neq 0} x^{m \cdot n} + \text{ (niže potencije) } \end{aligned}$$

$$\Rightarrow \deg(f \circ g) = m \cdot n = \deg f \cdot \deg g$$

11 (o jednakosti polinoma) Neka su  $p(x) = a_n x^n + \dots + a_1 x + a_0$  i  $q(x) = b_m x^m + \dots + b_1 x + b_0$  dva polinoma. Vrijedi

$$p(x) = q(x), \forall x \in \mathbb{R} \Leftrightarrow m=n \text{ i } a_i = b_i, \forall i \in \{0, \dots, n\}$$

Zad 4 Odredite sve polinome  $p \in \mathbb{R}[x]$  koji zadovoljavaju

$$(x+1)p(x) = x^3 + 1, \quad \forall x \in \mathbb{R}$$

Rj  $\deg((x+1)p(x)) = \deg(x^3 + 1)$

$$1 + \deg p = 3 \Rightarrow \deg p = 2$$

$$p(x) = ax^2 + bx + c, \quad a \neq 0$$

$$(x+1)(ax^2 + bx + c) = x^3 + 1$$

$$ax^3 + bx^2 + cx + ax^2 + bx + c = x^3 + 1$$

$$ax^3 + (b+a)x^2 + (c+b)x + c = x^3 + 1$$

$$a = 1$$

$$b+a = 0 \Rightarrow b = -a = -1$$

$$c+b = 0 \Rightarrow c = -b = 1$$

$$c = 1 \quad \checkmark$$

Rješenje:  $p(x) = x^2 - x + 1$

II. NAČIN: Za  $x = -1$  imamo

$$(1-1)p(-1) = (-1)^3 + 1$$

$$0 \cdot p(-1) = 0 \rightarrow \text{UVJEK VRIJEDI!}$$

za  $x \neq -1$  mora vrijediti

$$p(x) = \frac{x^3 + 1}{x+1} = \frac{(x+1)(x^2 - x + 1)}{(x+1)} = x^2 - x + 1$$

$$\Rightarrow \text{Rješenje: } p(x) = x^2 - x + 1$$

Zad 5 Odredite sve polinome  $f \in \mathbb{R}[x]$  t.d. vrijedi

(11)

$$f(x^2-3) = x^2 f(x-1), \quad \forall x \in \mathbb{R}$$

Rj.

$$n := \deg f$$

$$\deg(\underbrace{f(x^2-3)}_{\text{kompozicija}}) = \deg(x^2 \cdot f(x-1))$$

$f \circ g(x) = x^2 - 3$

$$2n = 2 + n \Rightarrow \underline{n=2}$$

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

$$a(x^2-3)^2 + b(x^2-3) + c = x^2(a(x-1)^2 + b(x-1) + c)$$

$$\underbrace{ax^4 - 6ax^2 + 9a} + \underbrace{bx^2 - 3b} + c = \underbrace{ax^4 - 2ax^3 + ax^2} + \underbrace{bx^3 - bx^2 + cx^2}$$

$$ax^4 + (b-6a)x^2 + (9a-3b+c) = ax^4 + (b-2a)x^3 + (a-b+c)x^2$$

Imamo:  $a=a \quad \checkmark$

$$0 = b-2a \Rightarrow 2a = b$$

$$b-6a = a-b+c \Rightarrow 2a-6a = a-2a+c$$

$$0 = 0$$

$$-3a = c$$

$$9a-3b+c = 0$$

$$9a-3(2a)+(-3a) = 0 \quad \checkmark$$

Dakle,  $a \in \mathbb{R} \setminus \{0\}$  je slobodan parametar.

Riješenja su svi polinomi oblika

$$f(x) = ax^2 + 2ax - 3a, \quad a \in \mathbb{R} \setminus \{0\}$$

Zad 6 Odredite sve  $f \in \mathbb{R}[x]$  t.d.

$$f(-2x+1) = 2x^2 - x + 3$$

Ry  $n = \deg f \quad n \cdot 1 = 2 \Rightarrow \deg f = 2$

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

$$a(-2x+1)^2 + b(-2x+1) + c = 2x^2 - x + 3$$

$$4ax^2 - 4ax + a - 2bx + b + c = 2x^2 - x + 3$$

$$4a = 2 \Rightarrow a = \frac{1}{2}$$

$$-4a - 2b = -1 \Rightarrow b = \frac{1}{2}(1 - 4a) = -\frac{1}{2}$$

$$a + b + c = 3 \Rightarrow c = 3 - a - b = 3$$

Rješenje:  $f(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 3$

2. NAČIN:

$$t := -2x+1 \Rightarrow x = -\frac{1}{2}(t-1)$$

$$\underline{\underline{f(t)}} = 2\left(-\frac{1}{2}(t-1)\right)^2 + \frac{1}{2}(t-1) + 3$$

$$= \frac{1}{2}(t^2 - 2t + 1) + \frac{1}{2}(t-1) + 3$$

$$= \underline{\underline{\frac{1}{2}t^2 - \frac{1}{2}t + 3}}$$

$$f(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 3$$

13

Zad 7 Odredite polinom  $f \in \mathbb{R}[x]$  t.d.

$$f(0) = 2, \quad f(1) = 6, \quad f(2) = 11, \quad f(-1) = 8$$

Ry  $f(0) = 2 \Rightarrow f(x) = ax^2 + bx + c, \quad a \neq 0$

$$6 = f(1) = a \cdot 1^2 + b \cdot 1 + c$$

$$11 = f(2) = a \cdot 2^2 + b \cdot 2 + c$$

$$8 = f(-1) = a \cdot (-1)^2 + b \cdot (-1) + c$$

$$\begin{cases} a + b + c = 6 \\ 4a + 2b + c = 11 \\ a - b + c = 8 \end{cases}$$

$$\begin{cases} a + b + c = 6 & c = 6 - a - b = 5 \\ 3a + b = 5 & a = \frac{1}{3}(5 - b) = 2 \\ -2b = 2 & \Rightarrow b = -1 \end{cases}$$

$$f(x) = 2x^2 - x + 5$$

Zad 8 Odredite  $a, b \in \mathbb{R}$  t.d.

$$f(x) = x^4 + 2x^3 + ax^2 + 2x + b$$

bude kvadrat nekog polinoma

Ry  $f(x) = (g(x))^2$

$$4 = \deg(f) = 2 \cdot \deg g \Rightarrow \deg g = 2$$

$$g(x) = \alpha x^2 + \beta x + \gamma$$

$$x^4 + 2x^3 + ax^2 + 2x + b = (\alpha x^2 + \beta x + \gamma)^2$$

$$\underline{x^4 + 2x^3 + ax^2 + 2x + b} = \underline{\alpha^2 x^4} + \beta^2 x^2 + \gamma^2 + 2\alpha\beta x^3 + 2\alpha\gamma x^2 + 2\beta\gamma x$$

$$\alpha^2 = 1 \Rightarrow \alpha = \pm 1$$

$$2\alpha\beta = 2 \Rightarrow \beta = \frac{1}{\alpha}$$

$$\beta^2 + 2\alpha\gamma = a$$

$$2\beta\gamma = 2 \Rightarrow \gamma = \frac{1}{\beta} = \pm 1$$

$$\gamma^2 = b$$

$$1^\circ) \quad \alpha = \beta = \gamma = 1$$

$$a = \beta^2 + 2\alpha\gamma = 3$$

$$b = \gamma^2 = 1$$

$$2^\circ) \quad \alpha = \beta = \gamma = -1$$

$$a = 3, \quad b = 1$$

Rješenje:

$$a = 3,$$

$$b = 1$$

Zad 9 Odredite sve polinome  $p \in \mathbb{R}[x]$  t.d.

$$p(1) + p(x) + p(x^2) = (1+x+x^2)p(x), \quad \forall x \in \mathbb{R}$$

R<sub>j</sub>  $\forall p = n \Rightarrow 2n = 2+n \Rightarrow \underline{n=2}$

$$p(x) = ax^2 + bx + c, \quad a \neq 0$$

$$a+b+c + ax^2 + bx + c + ax^4 + bx^2 + c$$

$$= (1+x+x^2)(ax^2 + bx + c)$$

$$ax^4 + (a+b)x^2 + bx + (a+b+3c)$$

$$= ax^4 + (a+b)x^3 + (a+b+c)x^2$$

$$+ (b+c)x + c$$

$$a=a$$

$$a+b=0 \Rightarrow b=-a$$

$$a+b = a+b+c \Rightarrow c=0$$

$$b = b+c \Rightarrow c=0$$

$$a+b+3c = c \quad \checkmark$$

$a$  je  
slobodni  
parametar

Rješenja su svi polinomi

$$p(x) = ax^2 - ax, \quad a \in \mathbb{R} \setminus \{0\}$$

Nulpolinom? DA! i  $p(x)=0$  je tj.

Im (0 dělení s ostátkem) Neka su  $f, g \in \mathbb{R}[x]$  polynomi,  $g \neq 0$ . Tada postoje jedinstveni polinomi  $q, r \in \mathbb{R}[x]$  t.d.

$$f(x) = q(x) \cdot g(x) + r(x), \quad \forall x \in \mathbb{R}$$

!  $\text{st } r < \text{st } g$

$q \dots$  kvocient       $r \dots$  ostatak

Zad 10 Podívejte polinom  $f(x) = x^3 - 2x^2 + x + 3$   
s polynomem  $g(x) = x^2 + 2x - 1$

Rj

$$\begin{array}{r} (x^3 - 2x^2 + x + 3) : (x^2 + 2x - 1) = x - 4 \\ \underline{-x^3 + 2x^2 + x} \phantom{+ 3} \\ -4x^2 + 2x + 3 \\ \underline{+4x^2 + 8x + 4} \\ 10x - 1 \end{array}$$

$$q(x) = x - 4, \quad r(x) = 10x - 1$$

D2

$$f(x) = x^7 + 5x^6 - 3x^5 + 2x^4 - x^3 + x^2 - 7x + 3$$

$$g(x) = x^3 - 2x^2 + 3x + 1$$

Zad 25 Odredite sve  $a \in \mathbb{R}$  t.d najveća zajed. mjera polinoma

$$f(x) = x^3 - (a+4)x^2 + (4a+3)x - 3a$$

$$g(x) = x^3 - 9x^2 + 26x - 24$$

bude stupnja 2

Rj  $\deg(H(f,g)) = 2 \Leftrightarrow f, g$  imaju točno 2 zajed. nultčke

Kandidati za ras. nultčke od  $g$ :

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$$

|    | 1 | -9  | 26 | -24 |
|----|---|-----|----|-----|
| 1  | 1 | -8  | 18 | -6  |
| -1 | 1 | -10 | 36 | -60 |
| 2  | 1 | -7  | 12 | 0   |

$x_1 = 2$

$$g(x) = (x-2)(x^2 - 7x + 12)$$

$$x_{3,4} = \frac{7 \pm \sqrt{49 - 48}}{2}$$

Nultčke od  $g$ : 2, 4, 3

$$\begin{aligned} f(2) &= 8 - (a+4) \cdot 4 + (4a+3) \cdot 2 - 3a \\ &= 8 - 4a - 16 + 8a + 6 - 3a \\ &= a - 2 \end{aligned}$$

$$f(3) = 27 - (a+4) \cdot 9 + (4a+3) \cdot 3 - 3a$$

$$= 27 - 9a - 36 + 12a + 9 - 3a = 0$$

$$f(4) = 64 - (a+4) \cdot 16 + (4a+3) \cdot 4 - 3a$$

$$= 64 - 16a - 64 + 16a + 12 - 3a$$

$$= 12 - 3a$$

Za  $a=2$  ili  $a=4$  zaji. nultočke su 2 i 3, tj. 3 i 4. Inače je 3 jedina zaji. nultočka

Rješenje:  $a \in \{2, 4\}$

Zad 26 Odredite  $f \in \mathbb{Z}[x]$  t.d.  $f(\sqrt{2} + \sqrt[3]{3}) = 0$

Rj

$$\left[ \begin{array}{l} f(2) = 0? \quad f(x) = x - 2 \\ f\left(\frac{\sqrt{5}}{2}\right) = 0? \\ x - \frac{\sqrt{5}}{2} = 0 \Rightarrow 2x = \sqrt{5} \Rightarrow 4x^2 = 5 \\ \Rightarrow \underbrace{4x^2 - 5}_{f(x)} = 0 \end{array} \right]$$

$$x = \sqrt{2} + \sqrt[3]{3} \Rightarrow x - \sqrt{2} = \sqrt[3]{3}$$

$$\Rightarrow x^3 - 3x^2\sqrt{2} + 3x \cdot 2 - 2\sqrt{2} = 3$$

$$\Rightarrow x^3 + 6x - 3 = \sqrt{2}(3x^2 + 2)$$

$$\Rightarrow x^6 + 36x^2 + 9 + 12x^4 - 6x^3 - 36x = 2(9x^4 + 12x^2 + 4)$$

$$f(x) = x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1$$

Zad 27 Neka je  $f \in \mathbb{Z}[x]$ .

Dokažite: ako su  $f(0)$  i  $f(1)$  neparni, onda  $f$  nema cjelobrojnih nultočaka

Rj Pretp. da je  $k \in \mathbb{Z}$  nultoka od  $f$ .

Tada je  $f(x) = (x-k)g(x)$ , pri čemu je

$g \in \mathbb{Z}[x]$  (lako se vidi iz Hornerovog alg.)

$$f(0) = k \cdot g(0) \text{ neparan} \Rightarrow k \text{ neparan}$$

$$f(1) = -(k-1)g(1) \text{ neparan} \Rightarrow k-1 \text{ neparan}$$



Zad 28 Dokažite da ne postoji  $f \in \mathbb{Z}[x]$  t.d.  $f(1) = 2$   
i  $f(3) = 5$

Rj Pretp. da postoji takav  $f \in \mathbb{Z}[x]$ .

$$\text{Imamo } f(x) = (x-1)q(x) + f(1),$$

$$q \in \mathbb{Z}[x]$$

$$\in \mathbb{Z}$$

$$5 = f(3) = (3-1)q(3) + 2 = 2(q(3) + 1)$$

$$\Rightarrow 2 \mid 5 \quad \downarrow$$

Zad 29 Neka je  $f \in \mathbb{Z}[x]$  i  $a, b, c \in \mathbb{Z}$  međusobno različiti t.d.  $f(a) = f(b) = f(c) = 2$ . Dokažite da ne postoji  $d \in \mathbb{Z}$  t.d.  $f(d) = 3$ .

Rj  $f(x) = (x-a)g_1(x) + 2$

$$= (x-b)g_2(x) + 2$$

$$= (x-c)g_3(x) + 2$$

$$g_1, g_2, g_3$$

$$\in \mathbb{Z}[x]$$

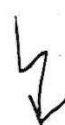
Pretp. da postoji  $d \in \mathbb{Z}$  t.d.  $f(d) = 3$

$$3 = f(d) = (d-a)g_1(d) + 2$$

$$\Rightarrow \underbrace{(d-a)}_{\in \mathbb{Z}} \underbrace{g_1(d)}_{\in \mathbb{Z}} = 1 \Rightarrow d-a = \pm 1$$

Analogno  $d-b = \pm 1, d-c = \pm 1$

$$d-a, d-b, d-c \in \{1, -1\}$$

ali,  $a, b, c$  su različiti  $\Rightarrow d-a, d-b$  i  $d-c$  su različiti 

Zad 30 Dokazite da ne postoji  $f \in \mathbb{Z}[x]$  t.d. je  $f(11) - f(7)$  prost broj

Rj Neka je  $f \in \mathbb{Z}[x]$

$$f(x) = (x-7)g(x) + f(7), \quad g \in \mathbb{Z}[x]$$

$$f(11) = (11-7)g(11) + f(7)$$

$$f(11) - f(7) = 4 \underbrace{g(11)}_{\in \mathbb{Z}}$$

$$\Rightarrow 4 \mid f(11) - f(7) \Rightarrow f(11) - f(7) \text{ nije prost}$$