$\boxed{2007}$ (2.) Na vektorskom prostoru $\mathcal{P}_n$ (polinoma stupnja $\leq n$) zadan je linearni operator D_a s

$$(D_a(p))(t) = \frac{p(t+a)-p(t)}{a},$$

pri čemu je a konstanta iz skupa $\mathbb{R}\setminus\{0\}$. Naćite jezgru i sliku te defekt i rang operatora D_a.

rj: $D_a(p) = 0 \iff (D_a(p))(t) = 0 \quad \forall t \in \mathbb{R}$

$$\iff \frac{p(t+a)-p(t)}{a} = 0 \quad \forall t \in \mathbb{R}$$

$$\iff p(t+a) = p(t) \quad \forall t \in \mathbb{R}$$

Stavimo $c := p(a) \in \mathbb{R}$ i $q(t) = p(t) - c$

Ako je $p \in \operatorname{Ker} D_a \Rightarrow p(t+a) = p(t) \quad \forall t \in \mathbb{R}$

$$\Rightarrow p(2a) = p(a+a) = p(a) = c$$

$$p(3a) = p(2a+a) = p(2a) = c$$

$$\vdots$$

$$p(na) = c \quad \forall n \in \mathbb{N}$$

$\Rightarrow q(na) = p(na) - c = 0 \quad \forall n \in \mathbb{N}$

Svi elementi skupa $\{a, 2a, 3a, \dots\}$ su međusobno različiti jer je $a \neq 0 \Rightarrow$ polinom $q \in \mathcal{P}_n$ ima beskonačno mnogo nultočaka $\Rightarrow q \equiv 0 \Rightarrow p(t) = c \quad \forall t \in \mathbb{R}$

$\operatorname{Ker} D_a = \mathcal{P}_0 \Rightarrow d(D_a) = 1 \Rightarrow r(D_a) = n \quad (1)$

$P_0(t)=1\ ,\ P_1(t)=t,\ \ldots,\ P_n(t)=t^n$

$$(D_a(P_k))(t)=\frac{P_k(t+a)-P_k(t)}{a}=\frac{(t+a)^k-t^k}{a}=$$

$$=\frac{((t+a)-t)\left((t+a)^{k-1}+(t+a)^{k-2}\cdot t+\ldots+(t+a)t^{k-2}+t^{k-1}\right)}{a}$$

$$=(t+a)^{k-1}+(t+a)^{k-2}\cdot t+\ldots+(t+a)t^{k-2}+t^{k-1}$$

$\Rightarrow \operatorname{st} D_a(P_k)=k-1 \quad \forall\ k\in\{1,2,3,\ldots,n\}$

$\Rightarrow \{D_a(P_1), D_a(P_2),\ldots,D_a(P_n)\}$ je lin. nez. u $\operatorname{Im} D_a$ (2)

$\overset{(1)\&(2)}{\Longrightarrow} \{D_a(P_1), D_a(P_2),\ldots,D_a(P_n)\}$ je baza za $\operatorname{Im} D_a$.

$$\left.\begin{array}{l}\operatorname{Im} D_a \leq \mathcal{P}_{n-1}\\ r(D_a)=\dim(\mathcal{P}_{n-1})\end{array}\right\} \Rightarrow \boxed{\operatorname{Im} D_a=\mathcal{P}_{n-1}}$$