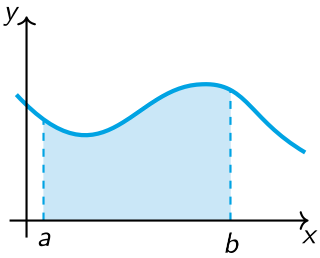
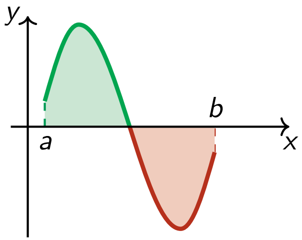
Višestruki integrali

Vježbe 11 - 27.5.2025.

Ponavljanje: integrali

Prisjetimo se: za $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ i $[a, b] \subseteq \mathcal{I}$ vrijedi





Ako je $f \geq 0$, onda je

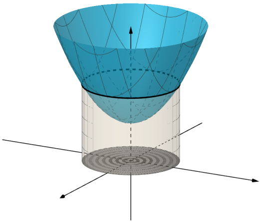
$$\int_a^b f(t)\mathrm{d}t = \begin{array}{c}\text{površina iznad } [a, b] \\ \text{i ispod } \Gamma_f\end{array}$$

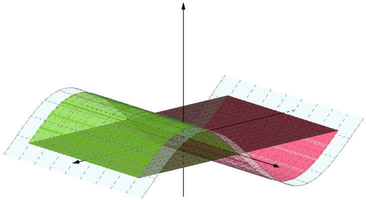
Općenito,

$$\int_a^b f(t)\mathrm{d}t = \begin{array}{c}\text{površina iznad } [a, b] \text{ i ispod } \Gamma_f \\ - \text{ površina ispod } [a, b] \text{ i iznad } \Gamma_f.\end{array}$$

Dvostruki integrali

Za $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ i $A \subseteq \Omega$ vrijedi





Ako je $f \geq 0$, onda je

$$\int_A f \,\mathrm{d}A = \begin{array}{c}\text{volumen iznad } A \\ \text{i ispod } \Gamma_f.\end{array}$$

Općenito,

$$\int_A f \,\mathrm{d}A = \begin{array}{c}\text{volumen iznad } A \text{ i ispod } \Gamma_f \\ - \text{ volumen ispod } A \text{ i iznad } \Gamma_f.\end{array}$$

Svojstva dvostrukih integrala

Za dvostruke (i općenito višestruke) integrale vrijedi:

① za $f, g : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ i $\alpha, \beta \in \mathbb{R}$ je

$$\int_A (\alpha f + \beta g)\,\mathrm{d}A = \alpha \int_A f\,\mathrm{d}A + \beta \int_A g\,\mathrm{d}A;$$

② za disjunktne skupove $A_1, A_2 \subseteq \mathbb{R}^2$ je

$$\int_{A_1 \cup A_2} f\,\mathrm{d}A = \int_{A_1} f\,\mathrm{d}A + \int_{A_2} f\,\mathrm{d}A.$$

Integriranje funkcije $f(x, y) = 1$

Vrijedi

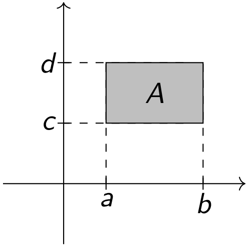
$$\int_a^b 1 \mathrm{d}t = t \Big|_a^b = b - a = \text{duljina intervala } [a, b].$$

Analogno, za skup $A \subseteq \mathbb{R}^2$ vrijedi

$$\int_A 1 \mathrm{d}A = \text{površina skupa } A.$$

Integriranje po pravokutniku

- Za $A = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \wedge c \leq y \leq d\}$



Fubinijev teorem

$$\int_A f \mathrm{d}A = \int_a^b \left(\int_c^d f(x, y) \mathrm{d}y \right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x, y) \mathrm{d}x \right) \mathrm{d}y$$

Zadatak 1. Izračunajte

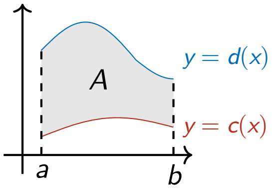
(a) $\int_A (xy + x)\,\mathrm{d}A$, ako je $A = [0,1] \times [1,2]$. Rj: $\frac{5}{4}$

(b) $\int_0^1 \left(\int_{-1}^1 \frac{2x}{y^2+1}\,\mathrm{d}x \right) \mathrm{d}y$ Rj: 0

Integriranje po području između krivulja

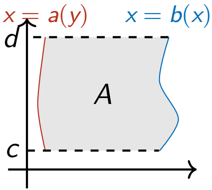
$$A = \{(x, y) \mid a \leq x \leq b \wedge c(x) \leq y \leq d(x)\}$$

$$\int_A f \,\mathrm{d}A = \int_a^b \int_{c(x)}^{d(x)} f(x, y)\mathrm{d}y\mathrm{d}x.$$



$$A = \{(x, y) \mid c \leq y \leq d \wedge a(y) \leq x \leq b(y)\}$$

$$\int_A f \,\mathrm{d}A = \int_c^d \int_{a(y)}^{b(y)} f(x, y)\mathrm{d}x\mathrm{d}y.$$



Zadatak 2. Izračunajte

$$\int_A xy \, \mathrm{d}x\mathrm{d}y$$

ako je $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, e^{-x} \leq y \leq e^x\}$.

Rj: $\frac{1}{8}e^2 + \frac{3}{8}e^{-2}$

Zadatak 3. Zamijenite poredak integracije u dvostrukom integralu

(a) $\int_0^2 \int_{2y}^4 f(x, y)\,\mathrm{d}x\mathrm{d}y$

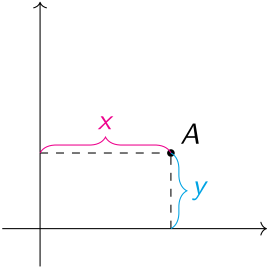
Rj: $\int_0^4 \int_0^{\frac{x}{2}} f(x, y)\, dydx$

(a) $\int_0^1 \int_{x^2-2}^{\sqrt{x}} f(x, y)\,\mathrm{d}y\mathrm{d}x$

Rj: $\int_{-2}^{-1} \int_0^{\sqrt{y+2}} f(x, y)\, dxdy + \int_{-1}^{0} \int_0^1 f(x, y)\, dxdy + \int_0^1 \int_{y^2}^1 f(x, y)\, dxdy$

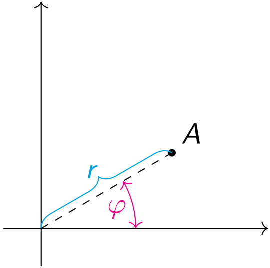
Zamjena varijabli u višestrukim integralima

Polarne koordinate



Kartezijeve (euklidske) koordinate

(x, y)



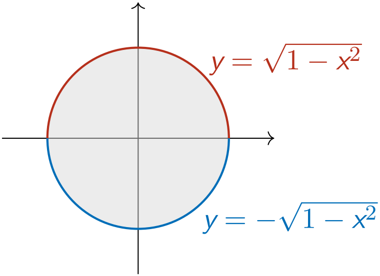
Polarne koordinate

(r, φ)

Veza Kartezijevih i polarnih koordinata

$$x = r\cos\varphi \qquad y = r\sin\varphi \qquad r = \sqrt{x^2 + y^2}$$

Primjer. Promotrimo skup $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.



U Kartezijevim koordinatama:

$$A = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, \, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\}.$$

U polarnim koordinatama

$$A = \{(r, \varphi) \mid 0 \leq r \leq 1, \, 0 \leq \varphi \leq 2\pi\}.$$

Računanje integrala u polarnim koordinatama

Teorem

$$\int_A f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_A f(r\cos\varphi, r\sin\varphi)\cdot r\,\mathrm{d}r\mathrm{d}\varphi$$

Faktor r je **apsolutna vrijednost determinante pripadne Jacobijeve matrice**:

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = |r(\cos^2\varphi + \sin^2\varphi)| = r.$$

Zadatak 4. Prijelazom na polarne koordinate izračunajte:

(a) $\int_A \sqrt{9 - x^2 - y^2}\,\mathrm{d}x\mathrm{d}y$, gdje je $A \ldots \begin{cases} x^2 + y^2 \leq 9 \\ y \leq 0. \end{cases}$ Rj: 9π

(b) $\int_K 3\sqrt{x^2 + y^2}\,\mathrm{d}x\mathrm{d}y$, gdje je $K \ldots \begin{cases} x^2 + y^2 \leq 4 \\ x \leq 0. \end{cases}$ Rj: 8π

(c) $\int_B 3\sqrt{x^2 + y^2}\,\mathrm{d}x\mathrm{d}y$, gdje je $B \ldots x^2 + y^2 \leq 6x.$ Rj: 96

Zadatak 5. Izračunajte:

(a) $\int_{-1}^{1}\int_{x^2-1}^{0}\int_{0}^{2} 1\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$ Rj: $\frac{8}{3}$

(b) $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y} x\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$ Rj: $\frac{1}{24}$