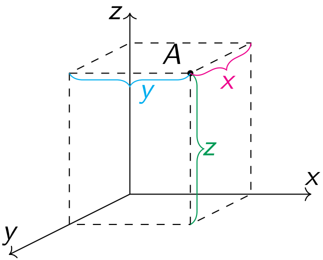
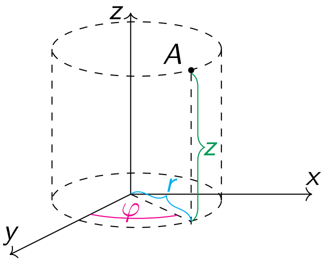
Cilindrične i sferne koordinate

Vježbe 12 - 3.6.2025.

Cilindrične koordinate



Kartezijeve (euklidske) koordinate
(x, y, z)



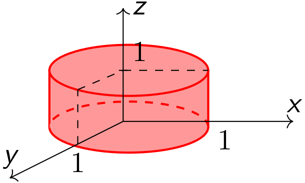
Cilindrične koordinate
(r, φ, z)

Veza Kartezijevih i cilindričnih koordinata

$$x = r\cos\varphi \qquad y = r\sin\varphi \qquad r = \sqrt{x^2 + y^2}$$

Primjer. Promotrimo skup

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}.$$



U Kartezijevim koordinatama:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\ 0 \leq z \leq 1\}.$$

U cilindričnim koordinatama

$$A = \{(r, \varphi, z) \mid 0 \leq r \leq 1,\ 0 \leq \varphi \leq 2\pi,\ 0 \leq z \leq 1\}.$$

Računanje integrala u cilindričnim koordinatama

Teorem

$$\int_A f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_A f(r\cos\varphi, r\sin\varphi, z)\cdot r\,\mathrm{d}r\mathrm{d}\varphi\mathrm{d}z$$

Zadatak 1. Izračunajte:

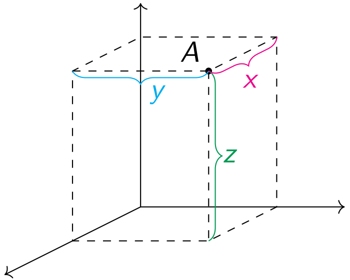
(a) $$\int_V z\sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \text{gdje je} \quad V\ldots \begin{cases} x^2+y^2\leq 1 \\ 0\leq z\leq 2. \end{cases}$$ Rj: $\frac{4\pi}{3}$

(b) $$\int_{C_1} 1\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \text{gdje je} \quad C_1\ldots \begin{cases} x^2+y^2\leq 1 \\ 0\leq z\leq 1 \\ x\geq 0. \end{cases}$$ Rj: $\frac{\pi}{2}$

(c) $$\int_{C_2} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \text{gdje je} \quad C_2\ldots \begin{cases} x^2+y^2\leq 1 \\ 0\leq z\leq 1 \\ y\leq 0. \end{cases}$$ Rj: $\frac{\pi}{3}$

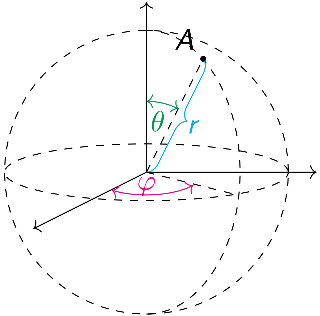
(d) $\int_{C_3} 1\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, gdje je C_3 presjek unutrašnjosti cilindra $x^2+y^2=1$ i područja ispod paraboloida $z=x^2+y^2$, a iznad xy–ravnine. Rj: $\frac{\pi}{2}$

Sferne koordinate



Kartezijeve (euklidske) koordinate

(x, y, z)



Sferne koordinate

(r, φ, θ)

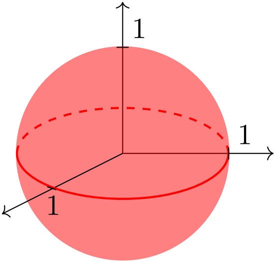
Veza Kartezijevih i sfernih koordinata

$$x = r \sin\theta \cos\varphi \qquad y = r \sin\theta \sin\varphi \qquad z = r\cos\theta$$

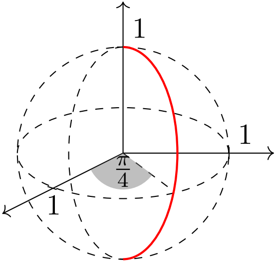
$$r = \sqrt{x^2 + y^2 + z^2}$$

Primjer.

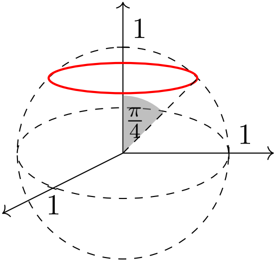
$A_1 \ldots \; r = 1$



$A_2 \ldots \begin{cases} r = 1 \\ \varphi = \frac{\pi}{4} \end{cases}$



$A_3 \ldots \begin{cases} r = 1 \\ \theta = \frac{\pi}{4} \end{cases}$



Računanje integrala u sfernim koordinatama

Teorem

$$\int_A f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$
$$\int_A f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\cdot r^2\sin\theta\,\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$$

Zadatak 2. Izračunajte:

(a) $\int_K 1\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$ gdje je $K \ldots\ x^2 + y^2 + z^2 \leq 1$ Rj: $\frac{4\pi}{3}$

(b) $\int_S (x^2 + y^2 + z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$ gdje je

$$S \ldots \begin{cases} x^2 + y^2 + z^2 \leq 4 \\ y \geq 0 \\ z \geq 0 \end{cases}$$

Rj: $\frac{32}{5}\pi$

(c) $\int_P (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$ gdje je $P \ldots \begin{cases} x^2 + y^2 + z^2 \leq 1 \\ z \geq 0 \end{cases}$ Rj: $\frac{4\pi}{15}$