HD JDŽBE ATMOSFERE U RAZ. KOORD. SUSTAVIMA

- možemo ih zvati i OSNOVNE JDŽBE
- izbor prikaza ovih jdžbi ovisi o svrsi u koju se koriste
- najopćenitiji je vektorski oblik koji se kasnije razlaže na komponente ovisno o koord. sustavu

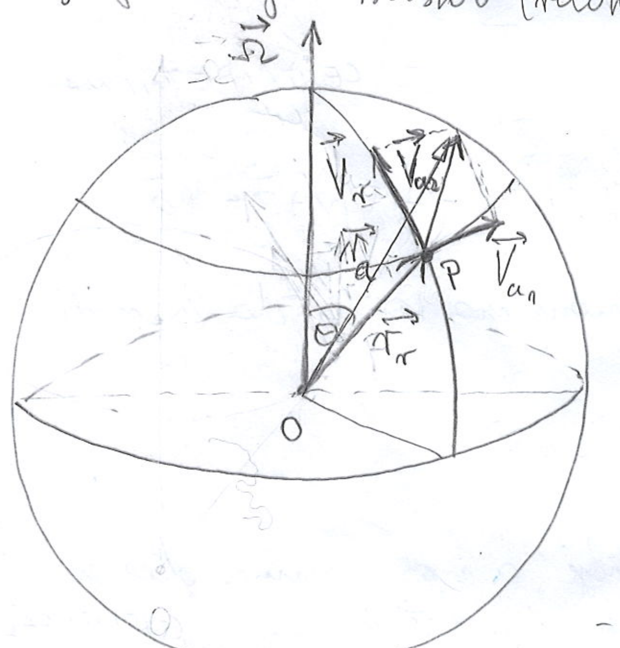
(I) VEKTORSKI OBLIK

Ia JDŽBA GIBANJA

- gledamo li iz sustava Zemlje koja rotira kutnom brzinom $\vec{\Omega}$ (RELATIVNI, NEINERCIJALNI SUSTAV), ona je različita nego gledano iz sustava "Svemira" koji miruje (APSOLUTNI, INERCIJALNI SUSTAV)
- ta jdžba je u biti II. Newtonov zakon ⇒ gledano iz apsolutnog sustava, uzrok gibanja jednak je netto sili ⇒ koje sve sile djeluju na čest zraka? ⇒ desna strana jdžbe:
- gledano iz apsolutnog sustava, brzina čest će biti V_a, a jdžba gibanja (II NZ):

$$\frac{d\vec{V}_a}{dt} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{sila grad. tlaka}} + \underbrace{\vec{g}_*}_{\text{gravitacijska sila}} + \underbrace{\vec{a}_{tr}}_{\text{sila trenja}}$$

- Zemlja je rotirajući sustav (relativan) pa će sve veličine nositi indeks r



- ako točka P miruje u rel. sustavu, u apsolutnom sustavu ona ima brzinu: $\vec{V}_{a1} = \vec{\Omega} \times \vec{r}_r$

$$|\vec{V}_a| = \Omega r_r \sin\theta$$

- ako se točka P giba u rel. sustavu, ona u njemu ima brzinu:

$$\vec{V}_r = \frac{d\vec{r}_r}{dt}$$

- ukupna brzina u apsolutnom sustavu $\vec{V}_{a2}$ je zbroj te dvije brzine:

$$\vec{V}_{a2} = \frac{d\vec{r}_a}{dt} = \vec{V}_r + \vec{V}_{a1} = \frac{d\vec{r}_r}{dt} + \vec{\Omega}\times\vec{r}_r \Rightarrow \boxed{\frac{d\vec{r}_a}{dt} = \left(\frac{d}{dt} + \vec{\Omega}\times\right)\vec{r}_r}$$

- VAŽNO ⇒ iz ove jdžbe se vidi da operator $\frac{d}{dt}$ u apsolutnom sustavu postoje operator $\frac{d}{dt}+\vec{\Omega}\times$ u rel. sustavu:

$$\boxed{\frac{d}{dt}(\vec{aps.}) = \left(\frac{d}{dt}+\vec{\Omega}\times\right)(\vec{rel})}$$

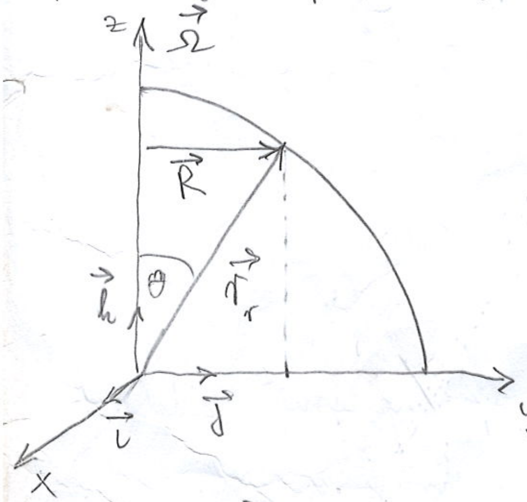
- primjenimo sada taj operator na akceleraciju u apsolutnom sustavu ⇒ sada umjesto brzine $\vec{V}_{a_r}$ pišem samo $\vec{V}_a$:

$$\frac{d\vec{V}_a}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}_a}{dt}\right) = \left(\frac{d}{dt}+\vec{\Omega}\times\right)\left(\frac{d\vec{r}_r}{dt}+\vec{\Omega}\times\vec{r}_r\right) =$$

$$= \frac{d^2\vec{r}_r}{dt^2} + \frac{d}{dt}(\vec{\Omega}\times\vec{r}_r) + \vec{\Omega}\times\frac{d\vec{r}_r}{dt} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}_r) =$$

$$= \frac{d^2\vec{r}_r}{dt^2} + \cancelto{0}{\frac{d\vec{\Omega}}{dt}}\times\vec{r}_r + \vec{\Omega}\times\frac{d\vec{r}_r}{dt} + \vec{\Omega}\times\frac{d\vec{r}_r}{dt} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}_r)$$

- pogledajmo posljednji član u pomoćnom koord. sustavu:



$$\vec{\Omega}\times\vec{r}_r = \Omega\vec{k}\times(r_r\sin\theta\,\vec{j} + r_r\cos\theta\,\vec{k}) =$$

$$= -\Omega r_r\sin\theta\,\vec{i}$$

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}_r) = \Omega\vec{k}\times(-\Omega r_r\sin\theta\,\vec{i}) =$$

$$= -\Omega^2\underbrace{r_r\sin\theta\,\vec{j}}_{\vec{R}} = -\Omega^2\vec{R}$$

CENTRIPETALNA SILA

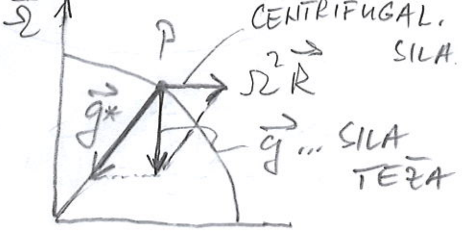
- sada: $$\frac{d\vec{V}_a}{dt} = \frac{d^2\vec{r}_a}{dt^2} = \frac{d^2\vec{r}_r}{dt^2} + 2\vec{\Omega}\times\frac{d\vec{r}_r}{dt} - \Omega^2\vec{R} = -\frac{1}{\rho}\nabla p + \vec{g}^* + \vec{a}_{tr}$$

- sada ću izostaviti indeks r s time da imam na umu da se radi o sustavu Zemlje (rel. sustavu):

$$\frac{d\vec{V}_a}{dt} = \frac{d^2\vec{r}}{dt^2} + 2\vec{\Omega}\times\frac{d\vec{r}}{dt} - \Omega^2\vec{R} = -\frac{1}{\rho}\nabla p + \vec{g}^* + \vec{a}_{tr}$$

$\frac{d\vec{r}}{dt} = \vec{V}$ ... brzina točke P u sustavu Zemlje, a nas zanima akceleracija u sustavu Zemlje ⇒ $\frac{d^2\vec{r}}{dt^2} = ?$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \underbrace{\vec{g}^* + \Omega^2\vec{R}}_{\vec{g}} + \vec{a}_{tr}$$



$$\Rightarrow \boxed{\frac{d\vec{V}}{dt} = \underset{(A)}{-\frac{1}{\rho}\nabla p} \underset{(B)}{- 2\vec{\Omega}\times\vec{V}} + \underset{(C)}{\vec{g}} + \underset{(D)}{\vec{a}_{tr}}}$$

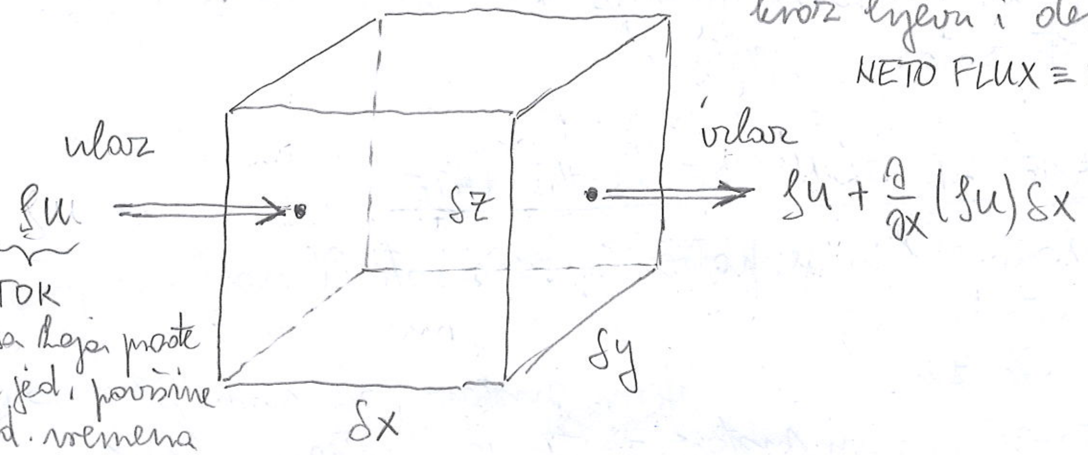
A... sila grad. tlaka C... sila teža
B... Coriolisova sila D... sila trenja

- Coriolisova sila i centrifugalna sila su posljedice rotacije Zemlje, to su fiktivne sile!

IIa) JDŽBA OČUVANJA MASE (JDŽBA KONTINUITETA)

- promatramo tok mase kroz fiksni volumen ⇒ gledamo tok mase kroz lijevu i desnu stranu

NETO FLUX ≡ ulaz − izlaz



- masa koja prođe kroz jed. površine u jed. vremena

$$\Rightarrow \rho u\,\delta y\,\delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\delta x\right]\delta y\,\delta z = \rho u\,\delta y\,\delta z - \rho u\,\delta y\,\delta z -$$

$$- \frac{\partial}{\partial x}(\rho u)\,\delta x\,\delta y\,\delta z = -\frac{\partial}{\partial x}(\rho u)\,\delta V \quad /:\delta V$$

- analogno za y smjer: $\quad -\frac{\partial}{\partial y}(\rho v)\,\delta V \quad /:\delta V$

— || — z smjer: $\quad -\frac{\partial}{\partial z}(\rho w)\,\delta V \quad /:\delta V$ $\quad +$

$$\Rightarrow -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) = -\nabla\cdot(\rho\vec{V})$$

- čemu je to jednako, tj. što mi u biti promatramo? ⇒ promatramo tok mase u jedinici vremena po jedinici volumena:

$$\frac{\delta m}{\delta V\,\delta t} = \frac{\delta\rho}{\delta t} = \left\{\lim_{\delta t\to 0}\right\} = \frac{\partial\rho}{\partial t} \quad \Rightarrow \boxed{\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{V})}$$

III a IZTD-a

$$Q = \frac{dU}{dt} + \frac{dW}{dt}$$

Q... toplina u jedinici vremena
U ... unutrašnja energija
W... rad plina

$dU = c_v dT$; $dW = p\, d\alpha$ $\Rightarrow$ $\boxed{Q = c_v \frac{dT}{dt} + p \frac{d\alpha}{dt}}$

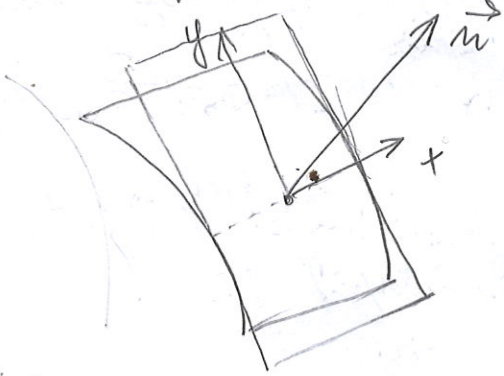
IV a JDŽBA STANJA IP-a

$$p\alpha = RT \; ; \quad \alpha = \frac{1}{\rho}$$

- pogledamo li ove jdžbe imamo 6 nepoznanica: u, v, w, p, ρ, T ali imamo i 6 jdžbi $\Rightarrow$ 3 jdžbe gibanja i 3 ostale pa je sustav u prvoj aproksimaciji zatvoren i rješiv

II GENERALIZIRANE KRIVOLINIJSKE KOORDINATE

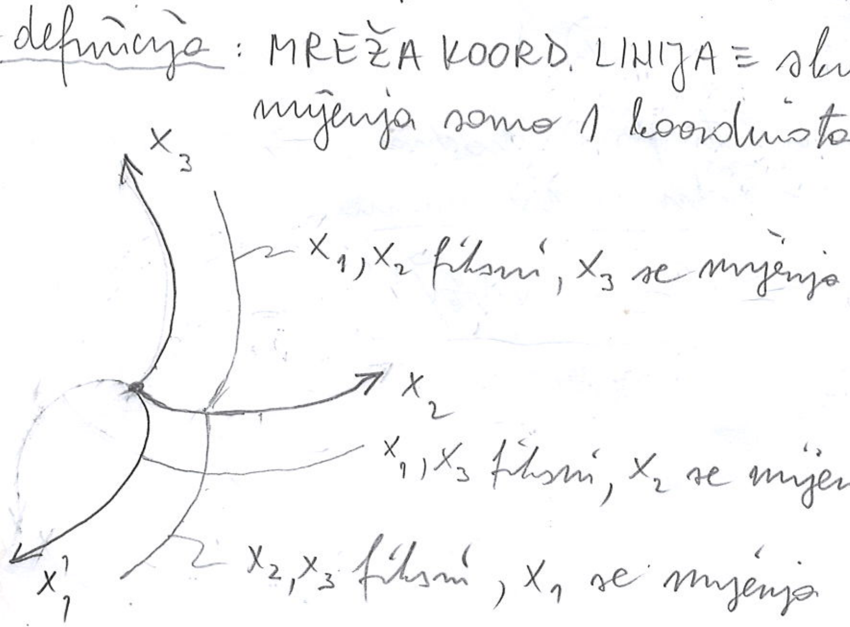
- jdžbe u vektorskom obliku je potrebno raspisati u najpovoljnijem koordinatnom sustavu
- npr. za model cijele Zemlje je to sferni sustav, za model manjeg dijela Zemlje tangencijalni sustav $\Rightarrow$ to je Kartezijev sustav u tangencijalnoj ravnini



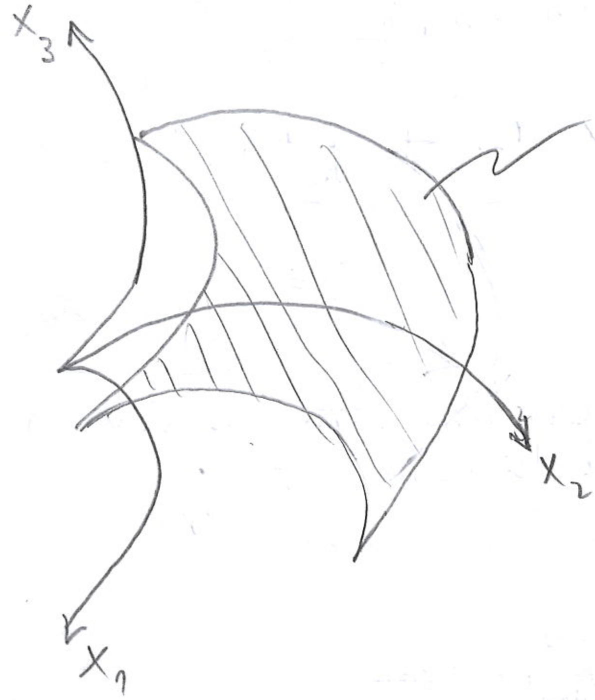
$\Rightarrow \exists$ razl. vrste projekcija i projekcija koordinata

- općenito možemo definirati KRIVOLINIJSKI SUSTAV koji se može primijeniti na specijalne slučajeve

- definicija: MREŽA KOORD. LINIJA $\equiv$ skup krivulja duž kojih se mijenja samo 1 koordinata, a ostale 2 su fiksne



- definicija: KOORDINATNE PLOHE ⇒ 1 koordinatu držim konstantnom, a druge 2 su promjenjive



KOORD. PLOHA ⇒ x_3 je const, a x_1 i x_2 se mijenjaju

- u krivolinijskom sustavu se odnos jedinica koordinata poklapa s dužinskim jedinicama pa se uvodi METRIČKI KOEFICIJENT

$\boxed{\text{II}a}$ METRIČKI KOEFICIJENTI

- govore kako se dužinske jedinice mijenjaju s promjenom krivolinijskih koordinata ⇒ VEZA između pravokutnih i krivolinijskih koordinata ⇒ kako duljina od 1 m u pravokutnim koord. izgleda u krivolinijskim koord.
- neka su (x_1, x_2, x_3) krivolinijske, a (y_1, y_2, y_3) pravokutne koord.
- promatrajmo element krivulje u pravokutnim koord:

$$(ds)^2 = (dy_1)^2 + (dy_2)^2 + (dy_3)^2$$

- budući da je $y_j = y_j(x_1, x_2, x_3)$; $j = 1, 2, 3$

$$\Rightarrow dy_j = \frac{\partial y_j}{\partial x_1} dx_1 + \frac{\partial y_j}{\partial x_2} dx_2 + \frac{\partial y_j}{\partial x_3} dx_3 = \sum_{i=1}^{3} \frac{\partial y_j}{\partial x_i} dx_i$$

$$\Rightarrow (ds)^2 = \underbrace{\left(\sum_{i=1}^{3} \frac{\partial y_1}{\partial x_i} dx_i\right)^2}_{\text{I}} + \underbrace{\left(\sum_{i=1}^{3} \frac{\partial y_2}{\partial x_i} dx_i\right)^2}_{\text{II}} + \underbrace{\left(\sum_{i=1}^{3} \frac{\partial y_3}{\partial x_i} dx_i\right)^2}_{\text{III}} = \sum_{k=1}^{3} \left(\sum_{i=1}^{3} \frac{\partial y_k}{\partial x_i} dx_i\right)^2$$

- sada raspišem sume u svakom članu, kvadriram, zbrojim i grupiram

$$\Rightarrow (ds)^2 = \left[\left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_2}{\partial x_1}\right)^2 + \left(\frac{\partial y_3}{\partial x_1}\right)^2\right](dx_1)^2 + \left[\left(\frac{\partial y_1}{\partial x_2}\right)^2 + \left(\frac{\partial y_2}{\partial x_2}\right)^2 + \left(\frac{\partial y_3}{\partial x_2}\right)^2\right](dx_2)^2 +$$
$$\left[\left(\frac{\partial y_1}{\partial x_3}\right)^2 + \left(\frac{\partial y_2}{\partial x_3}\right)^2 + \left(\frac{\partial y_2}{\partial x_3}\right)^2\right](dx_3)^2 + 2\left[\frac{\partial y_1}{\partial x_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial y_2}{\partial x_1}\frac{\partial y_2}{\partial x_2} + \frac{\partial y_3}{\partial x_1}\frac{\partial y_3}{\partial x_2}\right]dx_1\,dx_2 +$$
$$+ 2\left[\frac{\partial y_1}{\partial x_1}\frac{\partial y_1}{\partial x_3} + \frac{\partial y_2}{\partial x_1}\frac{\partial y_2}{\partial x_3} + \frac{\partial y_3}{\partial x_1}\frac{\partial y_3}{\partial x_3}\right]dx_1\,dx_3 + 2\left[\frac{\partial y_1}{\partial x_2}\frac{\partial y_1}{\partial x_3} + \cdots\right]dx_2\,dx_3$$

- skraćeni zapis: $\boxed{(ds)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\,dx_i\,dx_j}$; $\boxed{g_{ij} = \sum_{k=1}^{3} \frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}}$

g_{ij} .. METRIČKI KOEF $\Rightarrow$ dovode u vezu metričku duljinu ds i krivolinijske koord

simetrični koef. $(g_{ij} = g_{ji}) \Rightarrow$ od njih 9, 6 je različitih $\Rightarrow$ SIMETR

$\Rightarrow$ SIMETRIČNI TENZOR: $\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{bmatrix}$

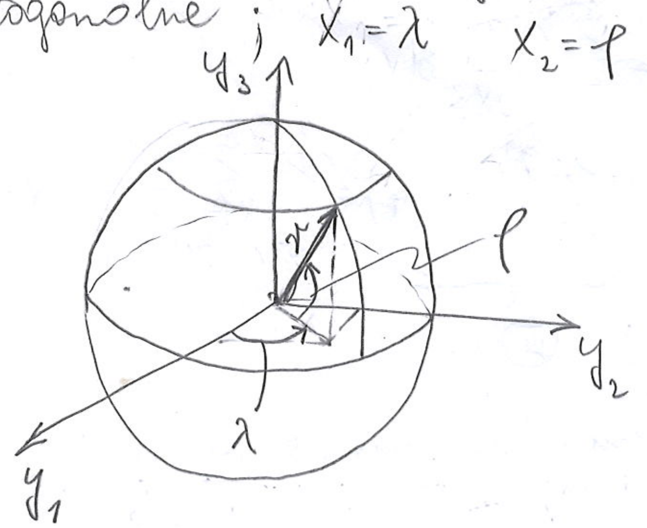
- spec. slučaj: ORTOGONALNE krivolinijske koordinate (najčešće u praksi), linearno nezavisne, čine bazu prostora) $\Rightarrow$ za njih vrijedi:

$$g_{ij} = \begin{cases} 0, & i \neq j \\ \neq 0, & i = j \end{cases} \quad ; \quad i, j = 1, 2, 3$$

$$\Rightarrow \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \Rightarrow (ds)^2 = g_{11}(dx_1)^2 + g_{22}(dx_2)^2 + g_{33}(dx_3)^2$$

- uvodimo supstituciju: $\boxed{h_i = \sqrt{g_{ii}}}$ — LAMEOVI KOEFICIJENTI $i = 1, 2, 3$

PRIMJER: neka krivolinijske koord. budu sferne! $\Rightarrow$ one su ortogonalne ; $x_1 = \lambda$ $\quad x_2 = \varphi \quad x_3 = r$



- napišimo vezu:

$y_1 = r\cos\varphi\cos\lambda$

$y_2 = r\cos\varphi\sin\lambda$

$y_3 = r\sin\varphi$

- sada: $g_{11} = \left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_2}{\partial x_1}\right)^2 + \left(\frac{\partial y_3}{\partial x_1}\right)^2 = \left(\frac{\partial y_1}{\partial \lambda}\right)^2 + \left(\frac{\partial y_2}{\partial \lambda}\right)^2 + \left(\frac{\partial y_3}{\partial \lambda}\right)^2 =$

$$= (-r\cos\varphi \sin\lambda)^2 + (r\cos\varphi\cos\lambda)^2 = r^2\cos^2\varphi = x_3^{\,2}\cos^2 x_2$$

$$g_{22} = \dots = r^2 = x_3^{\,2}$$

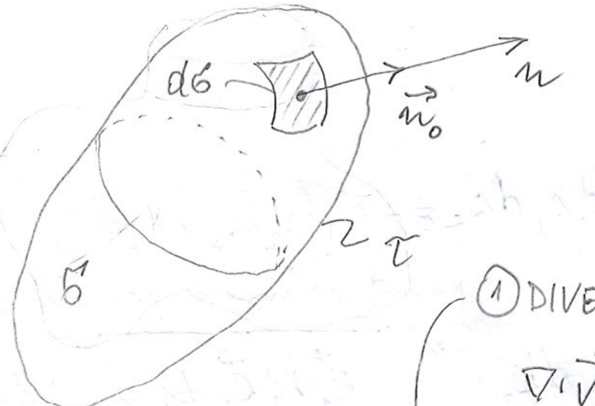
$$g_{33} = \cos^2\varphi + \sin^2\varphi = 1$$

- Lameovi koef: $\boxed{h_1 = h_\lambda = r\cos\varphi}$ $\boxed{h_2 = h_\varphi = r}$ $\boxed{h_3 = h_r = 1}$

- Lameove koeficijente ćemo koristiti za prikaz vektorskih operatora

II.G VEKTORSKI OPERATORI U GENERALIZIRANIM KOORDINATAMA

- to su slijedeći operatori: gradijent, divergencija, rotacija i laplasijan
  ⇒ oni se javljaju u HD jdžbama
- njihove se komponente koriste u prikazu jdžbi u raz. koord. sustavima
- promatrajmo volumen τ u vektorskom polju $\vec{V}$ (npr. brzina) ili skalarnom polju α (npr. tlak)



- volumen τ ima oplošje σ i jedinični normalni vektor na tu površinu ⇒ $\vec{n}_0$
- općenite definicije operatora su:

① DIVERGENCIJA $(\nabla\cdot\vec{V})$ ⇒ skalar

$$\nabla\cdot\vec{V} = \lim_{\tau\to 0}\frac{1}{\tau}\int_\sigma \vec{n}_0\cdot\vec{V}\,d\sigma$$

② ROTACIJA $(\nabla\times\vec{V})$ ⇒ vektor

$$\nabla\times\vec{V} = \lim_{\tau\to 0}\frac{1}{\tau}\int_\sigma \vec{n}_0\times\vec{V}\,d\sigma$$

③ GRADIJENT $(\nabla\alpha)$ ⇒ vektor

$$\nabla\alpha = \lim_{\tau\to 0}\frac{1}{\tau}\int_\sigma \vec{n}_0\,\alpha\,d\sigma$$

- ove definicije ne ovise o izboru koord. sustava, svugdje su iste, ali da bismo ih mogli računati potreban nam je koord. sustav

- želimo raspisati ove operatore u krivolinijskim koord. (x_1, x_2, x_3) čiji su odgovarajući jed. vektori $\vec{e}_i$, $i = 1,2,3$ (ortogonalna baza)

- neka su v_i $(i=1,2,3)$ komponente vektora $\vec{V}$:

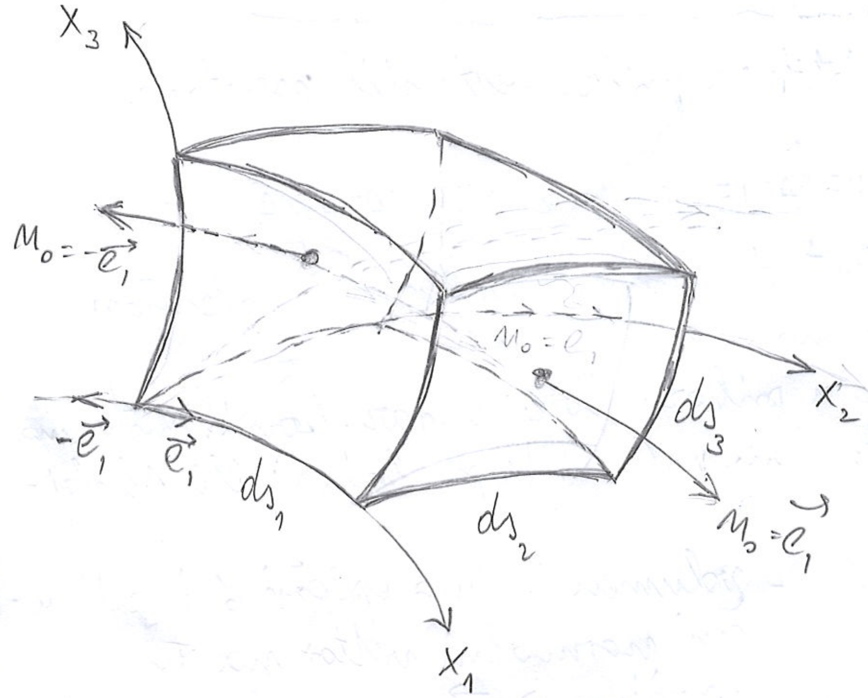
$$\vec{V} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$$

- komponente pomaka u krivolinijskom sustavu su:

$$ds_1 = h_1\,dx_1 \quad ; \quad ds_2 = h_2\,dx_2 \quad ; \quad ds_3 = h_3\,dx_3$$

DIVERGENCIJA $(\nabla\cdot\vec{V})$

- djelić volumena u krivolinijskim koord. zamislimo kao nekakvu iskrivljenu kocku



- koncentrirajmo se na same integrale preko ploha na pojedinoj koordinati
- za ilustraciju promatrajmo plohe $\perp$ na x_1

- stražnja ploha: $\int_\sigma \vec{n}_0\cdot\vec{V}\,d\sigma \approx -\vec{e}_1 v_1\vec{e}_1\,ds_2\,ds_3 = -v_1 h_2 h_3\,dx_2\,dx_3$

- prednja ploha: $\int_\sigma \vec{n}_0\cdot\vec{V}\,d\sigma \approx \vec{e}_1 v_1\vec{e}_1\,ds_2\,ds_3 + \overbrace{\frac{\partial}{\partial x_1}\left(\vec{e}_1 v_1\vec{e}_1\,ds_2\,ds_3\right)dx_1}^{\text{prirast jer smo se pomakli po } x_1} =$

$$= v_1 h_2 h_3\,dx_2\,dx_3 + \frac{\partial}{\partial x_1}\left(v_1 h_2 h_3\,dx_2\,dx_3\right)dx_1$$

$\Rightarrow$ ukupno: stražnja + prednja ploha:

$$\Rightarrow = \frac{\partial}{\partial x_1}\left(v_1 h_2 h_3\,dx_2\,dx_3\right)dx_1$$

- analogno napravimo i za plohe $\perp$ na x_2 i x_3 i zbrojimo
- volumen: $d\tau \approx ds_1\,ds_2\,ds_3 = h_1 h_2 h_3\,dx_1\,dx_2\,dx_3$

$$\nabla\cdot\vec{V} = \frac{1}{h_1 h_2 h_3 \cancel{dx_1 dx_2 dx_3}}\left[\frac{\partial}{\partial x_1}\left(v_1 h_2 h_3\right)\cancel{dx_1 dx_2 dx_3} + \frac{\partial}{\partial x_2}\left(v_2 h_1 h_3\right)\cancel{dx_1 dx_2 dx_3} + \frac{\partial}{\partial x_3}\left(v_3 h_1 h_2\right)\cancel{dx_1 dx_2 dx_3}\right]$$

$$\Rightarrow \boxed{\nabla\cdot\vec{V} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1) + \frac{\partial}{\partial x_2}(h_1h_3v_2) + \frac{\partial}{\partial x_3}(h_1h_2v_3)\right]}$$

ROTACIJA ($\nabla\times\vec{V}$)

- dobije se na sličan način, a skraćeno možemo pisati pomoću determinante:

$$\nabla\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1v_1 & h_2v_2 & h_3v_3 \end{vmatrix}$$

GRADIJENT ($\nabla\alpha$)

$$\nabla\alpha = \frac{\vec{e}_1}{h_1}\frac{\partial\alpha}{\partial x_1} + \frac{\vec{e}_2}{h_2}\frac{\partial\alpha}{\partial x_2} + \frac{\vec{e}_3}{h_3}\frac{\partial\alpha}{\partial x_3}$$

LAPLASIJAN ($\nabla^2\alpha$)

$$\nabla^2\alpha = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\alpha}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\alpha}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\alpha}{\partial x_3}\right)\right]$$

TOTALNA PROMJENA

$$\frac{d\alpha}{dt} = \frac{\partial\alpha}{\partial t} + \vec{V}\cdot\nabla\alpha = \frac{\partial\alpha}{\partial t} + \frac{v_1}{h_1}\frac{\partial\alpha}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial\alpha}{\partial x_2} + \frac{v_3}{h_3}\frac{\partial\alpha}{\partial x_3}$$

Pr Ilustracija na sfernom koord. sustavu:

$x_1 = \lambda$; $x_2 = \varphi$; $x_3 = r$

- jed. vektori: $\vec{e}_\lambda$; $\vec{e}_\varphi$; $\vec{e}_r$

$\vec{V} = u\vec{e}_\lambda + v\vec{e}_\varphi + w\vec{e}_r$

- Lameovi koef.: $h_1 = h_\lambda = r\cos\varphi$

$h_2 = h_\varphi = r$

$h_3 = h_r = 1$

DZ