Zad 2: Neka su $A, B \subseteq \mathbb{R}^+$ odozdo ograničeni skupovi. Dokažite da je $\inf(AB) = \inf A \cdot \inf B$

rj: $C := \{ab : a \in A, b \in B\} = AB$

- $\forall a \in A$ vrijedi $a \geq \inf A$

  $\forall b \in B$ vrijedi $b \geq \inf B$

$a \geq \inf A \ /\cdot b$ $\qquad a \cdot b \geq \inf A \cdot b \geq \inf A \cdot \inf B$

$b \in B \Rightarrow b \geq 0$

$\Rightarrow \forall c = ab \in C$ je $c \geq \inf A \cdot \inf B$

$\Rightarrow \inf A \cdot \inf B$ je donja međa skupa C.

- Kako je C odozdo ograničen $\Rightarrow C$ ima infimum

$\boxed{\inf A \cdot \inf B \leq \inf C}$ (1)

Za $c = ab$ vrijedi $ab \geq \inf C \ /: b \Rightarrow a \geq \frac{\inf C}{b} \quad \forall a \in A$

$\Rightarrow \frac{\inf C}{b}$ je donja međa skupa $A \Rightarrow \frac{\inf C}{b} \leq \inf A$

$\Rightarrow \inf C \leq b \cdot \inf A \Rightarrow$

1° Ako je $\inf A \neq 0$, onda je $\frac{\inf C}{\inf A} \leq b \quad \forall b \in B$

$\Rightarrow \frac{\inf C}{\inf A}$ je donja međa skupa B

$\Rightarrow \frac{\inf C}{\inf A} \leq \inf B \Rightarrow \boxed{\inf C \leq \inf A \inf B}$ (2)

2° Ako je $\inf A = 0 \Rightarrow \frac{\inf C}{b} \leq 0 \Rightarrow \inf C \leq 0$

Kako je $C \subseteq \mathbb{R}^+$, onda je ujedno i $\inf C \geq 0$ $\Big\} \Rightarrow \inf C =$

$\Rightarrow \overset{0}{\widetilde{\inf C}} = \overset{0}{\widetilde{\inf A}} \cdot \inf B$

Also if $\deg A \neq 0$ (1) & (2) $\Rightarrow$ $\deg C = \deg A - \deg B$

(3.) a) $A=\{1\}$, $B=\{0,1\}$

$A \subsetneq B$ $\quad \sup A = 1 = \sup B$

Dakle, ne vrijedi nužno $A \subsetneq B \Rightarrow \sup A < \sup B$

b) $A=\{0\}$, $B=\{0,1\}$

$A \subsetneq B$ $\quad \inf A = \inf B = 0$

Dakle, ne vrijedi nužno $A \subsetneq B \Rightarrow \inf A > \inf B$

c) Ne vrijedi nužno! Stavimo npr. $A=B=\{-1,0\}$.

Tada je $AB = \{-1,0\}\cdot\{-1,0\} = \{1,0\}$

$\sup(AB) = 1$ $\quad \sup A = \sup B = 0$

$\sup(AB) \neq \sup A \cdot \sup B$

<u>Napomena:</u> Ako su $A, B \subseteq [0,+\infty\rangle$, oba ograničena,

onda je $\sup(AB) = \sup A \cdot \sup B$

(vidjeti zad 2.13. u vježbama)

d) Ne vrijedi nužno! Stavimo npr. $A=B=\{-1,0\}$.

Tada je $AB = \{0,1\}$ $\quad \inf AB = 0$, a $\inf A \cdot \inf B = (-1)\cdot(-1) = 1$

$\inf(AB) \neq \inf A \cdot \inf B$.

<u>Napomena:</u> Ako su $A, B \subseteq [0,+\infty\rangle$, oba ograničena,

onda je $\inf(AB) = \inf A \cdot \inf B$ (vidi zad. 2. u zadaći)

e) Ne vrijedi nužno. Stavimo npr. $A = B = \{-1, 0\}$.

Tada je $A - B = \{-1, 0\} - \{-1, 0\} = \{0, -1, 1\} \Rightarrow \sup(A-B) = 1$

$\sup A - \sup B = 0 - 0 = 0$ pa je $\sup(A-B) \neq \sup A - \sup B$.

f) $\sup(A-B) = \sup(A + (-B)) = \sup A + \sup(-B) =$

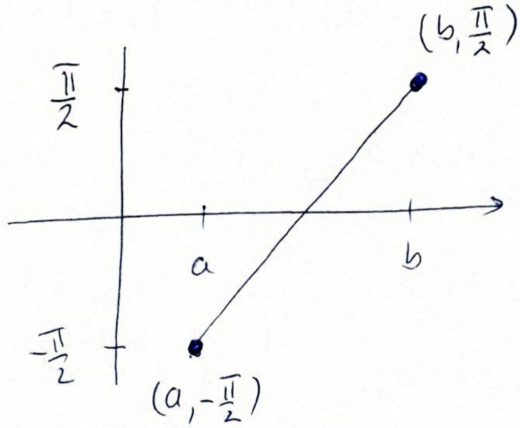
$= \sup A - \inf B$ za sve

neprazne ograničene skupove $A, B \subseteq \mathbb{R}$.

4. a) $tg : \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \to \mathbb{R}$ je bijekcija $\Rightarrow \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \sim \mathbb{R}$

Dokažimo sada da je $\langle a, b \rangle \sim \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

$f(x) = \dfrac{\frac{\pi}{2} - (-\frac{\pi}{2})}{b-a}(x-a) - \frac{\pi}{2}$ je bijekcija iz $\langle a, b \rangle$ u $\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$



$\Rightarrow \boxed{\langle a, b \rangle \sim \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \sim \mathbb{R}}$ (1)

- $\langle a, b \rangle \sim [a, b]$ (2) (dokazano na predavanjima)

- $id : [a, b\rangle \to [a, b]$ je injekcija

  $[c, d] \subseteq [a, b\rangle$ $\exists\, g : [a, b] \to [c, d]$ bijekcija

  $g : [a, b] \to [a, b\rangle$ je injekcija

  Cantor – Bernstein $\Rightarrow [a, b\rangle \sim [a, b]$ (3)

- Analogno se pokaže $[a, b] \sim \langle a, b]$ (4)

(1), (2), (3) & (4) $\Rightarrow \langle a, b \rangle \sim [a, b\rangle \sim \langle a, b] \sim [a, b] \sim \mathbb{R}$

(5.) Pretpostavimo suprotno, tj. da je $\mathbb{R}\setminus\mathbb{Q}$ prebrojiv skup.
Tada postoji bijekcija $f: \mathbb{N} \to \mathbb{R}\setminus\mathbb{Q}$.

$\mathbb{R}\setminus\mathbb{Q} = \{f(1), f(2), \dots\}$ Stavimo $a_i = f(i)$, $i \in \mathbb{N}$
$= \{a_1, a_2, \dots\}$

$g: \mathbb{N} \to \mathbb{Q}$ bijekcija (takva postoji jer je $\mathbb{N} \sim \mathbb{Q}$).

$\mathbb{Q} = \{g(1), g(2), \dots\}$ Stavimo $b_i = g(i)$, $i \in \mathbb{N}$
$= \{b_1, b_2, \dots\}$

$\Rightarrow \mathbb{R} = (\mathbb{R}\setminus\mathbb{Q}) \cup \mathbb{Q} = \{a_1, b_1, a_2, b_2, \dots\}$

$h: \mathbb{N} \to \mathbb{R}$ $h(2n) = a_n$
$h(2n-1) = b_n$ h je bijekcija
$\Rightarrow \mathbb{N} \sim \mathbb{R}$ $\Rightarrow\Leftarrow$

Dakle, $\mathbb{R}\setminus\mathbb{Q}$ je neprebrojiv skup.

(6.) Tvrdnja: Ako je $a \in \mathbb{Q}\setminus\{0\}$, onda je $a\sqrt{2} \in \mathbb{R}\setminus\mathbb{Q}$

dokaz: Ako je $a\sqrt{2} = q \in \mathbb{Q}$, onda je i $\sqrt{2} = \frac{q}{a} \in \mathbb{Q}$ $\Rightarrow\Leftarrow$

IDEJA: Dokažimo da u svakom intervalu $\langle x-\varepsilon, x+\varepsilon\rangle$ postoji element oblika $a\sqrt{2}$, gdje je $a \in \mathbb{Q}\setminus\{0\}$.

$a\sqrt{2} \in \langle x-\varepsilon, x+\varepsilon\rangle \iff x-\varepsilon < a\sqrt{2} < x+\varepsilon \quad /:\sqrt{2}$

$\iff \frac{x}{\sqrt{2}} - \frac{\varepsilon}{\sqrt{2}} < a < \frac{x}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}}$

$\iff a \in \left\langle \frac{x}{\sqrt{2}} - \frac{\varepsilon}{\sqrt{2}}, \frac{x}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \right\rangle$

Uzmemo $a \in \mathbb{Q}\setminus\{0\} \cap \left\langle \frac{x}{\sqrt{2}} - \frac{\varepsilon}{\sqrt{2}}, \frac{x}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \right\rangle$ (takav postoji jer je $\mathbb{Q}$ gust u $\mathbb{R}$).

Tada je $a\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ i $a\sqrt{2} \in \langle x-\varepsilon, x+\varepsilon \rangle$.

$\Rightarrow \langle x-\varepsilon, x+\varepsilon \rangle \cap \mathbb{R} \setminus \mathbb{Q} \neq \emptyset$, tj. $\mathbb{R} \setminus \mathbb{Q}$ je gust u $\mathbb{R}$.