2. zadaća

1. zad

a) $A = \left\{ (-1)^{n+1} \frac{2n^2-1}{n^2+2} : n \in \mathbb{N} \right\}$

$$\frac{2n^2-1}{n^2+2} = \frac{2(n^2+2)-5}{n^2+2} =$$

$$= 2 - \frac{5}{n^2+2}$$

$$A_1 = \left\{ (-1)^{2k+1} \left(2 - \frac{5}{(2k)^2+2}\right) : k \in \mathbb{N} \right\}$$

$$= \left\{ -2 + \frac{5}{(2k)^2+2} : k \in \mathbb{N} \right\}$$

$$A_2 = \left\{ (-1)^{2k-1+1} \left(2 - \frac{5}{(2k-1)^2+2}\right) : k \in \mathbb{N} \right\}$$

$$= \left\{ 2 - \frac{5}{(2k-1)^2+2} : k \in \mathbb{N} \right\}$$

Odredimo $\inf A_1$ i $\sup A_1$

$$-2+\frac{5}{(2k)^2+2} \geq -2 \quad \forall k \in \mathbb{N}$$

Dokažimo da je $\inf A_1 = -2$. Već smo dokazali da je -2 donja međa tog skupa. Trebamo još dokazati

$$(\forall \varepsilon>0)\ (\exists k\in \mathbb{N})\ \text{t.d.}\ -2+\frac{5}{(2k)^2+2} < -2+\varepsilon,$$

tj. $\quad 5 < ((2k)^2+2)\,\varepsilon$

Arhimedov aksiom primijenjen na ε i 5

$\Longrightarrow \exists\ k_0 \in \mathbb{N}$ t.d.

$$5 < k_0 \cdot \varepsilon$$

Sada je $5 < k_0\cdot\varepsilon \leq k_0^2\cdot\varepsilon < ((2k_0)^2+2)\,\varepsilon$

$$\Longrightarrow 5 < ((2k_0)^2+2)\,\varepsilon$$

$$\Longrightarrow -2+\frac{5}{(2k_0)^2+2} < -2+\varepsilon$$

$$\Rightarrow \boxed{\inf A_1 = -2}$$

$$a_k = -2 + \frac{5}{(2k)^2+2}$$

$$(2k)^2 < (2(k+1))^2 \quad \forall k \in \mathbb{N}$$

$$(2k)^2+2 < (2(k+1))^2+2 \quad \forall k \in \mathbb{N}$$

$$\frac{5}{(2k)^2+2} > \frac{5}{(2(k+1))^2+2} \quad \forall k \in \mathbb{N}$$

$$\Rightarrow \quad -2+\frac{5}{(2k)^2+2} > -2+\frac{5}{(2(k+1))^2+2}$$

$$\Rightarrow \quad a_k > a_{k+1}$$

$\Rightarrow$ Niz (a_k) je padajući

$$\Rightarrow \max A_1 = a_1 = -2+\frac{5}{2^2+2} = -2+\frac{5}{6} = -\frac{7}{6}$$

$$\Rightarrow \boxed{\sup A_1 = -\frac{7}{6}}$$

Analogno se dobije $\boxed{\sup A_2 = 2}$,

$$\boxed{\inf A_2 = \min A_2 = \frac{1}{3}}$$

$$A = A_1 \cup A_2$$

$$\sup A = \max\{\sup A_1, \sup A_2\} = \max\{-\frac{7}{6}, 2\} = 2$$

$$\inf A = \min\{\inf A_1, \inf A_2\} = \min\{-2, \frac{1}{3}\} = -2$$

e) $$\frac{2nm^2 + 4nm - 2n - 3m^2 - 6m + 3}{nm^2 + 2mn} =$$

$$= \frac{2n(m^2+2m) - 3(m^2+2m) - (2n-3)}{n(m^2+2m)} =$$

$$= \frac{(2n-3)(m^2+2m-1)}{n(m^2+2m)} = \frac{2n-3}{n} \cdot \frac{m^2+2m-1}{m^2+2m}$$

$E = E_1 \cdot E_2$, gdje je

$$E_1 = \left\{ \frac{2n-3}{n} : n \in \mathbb{N} \right\}, \quad E_2 = \left\{ \frac{m^2+2m-1}{m^2+2m} : m \in \mathbb{N} \right\}$$

$$a_n = \frac{2n-3}{n} = 2 - \frac{3}{n}$$

$$a_n < a_{n+1} \iff$$

$$\cancel{2} - \frac{3}{n} < \cancel{2} - \frac{3}{n+1} \iff$$

$$\frac{3}{n} > \frac{3}{n+1} \iff$$

$$n+1 > n \checkmark$$

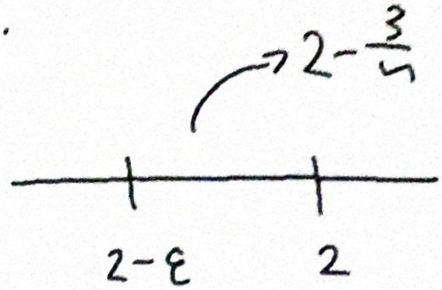
(a_n) je rastući niz $\Rightarrow$

$$\boxed{\inf E_1 = \min E_1 = a_1 = -1}$$

$2 - \frac{3}{n} < 2 \ \forall n \in \mathbb{N} \Rightarrow 2$ je gornja međa skupa E_1. Dokažimo $\sup E_1 = 2$

T.D. $(\forall \varepsilon>0)\ (\exists n\in\mathbb{N})$ t.d.

$$2-\frac{3}{n}>2-\varepsilon,$$



$$\iff \quad \frac{3}{n}<\varepsilon$$

$$\iff \quad 3<n\varepsilon$$

A.A. $\Rightarrow \exists\, n_0\in\mathbb{N}$ t.d. $3<n_0\varepsilon$

$$\Rightarrow \quad 2-\frac{3}{n_0}>2-\varepsilon$$

Dakle 2 je najmanja gornja međa skupa E_1, tj. $\boxed{\sup E_1 = 2}$

$\boxed{E_2}$ $$a_n = \frac{n^2+2n-1}{n^2+2n} = 1 - \frac{1}{n^2+2n}$$

$$a_n < a_{n+1}$$

$$\Longleftrightarrow \quad 1 - \frac{1}{n^2+2n} < 1 - \frac{1}{(n+1)^2+2(n+1)}$$

$$\Longleftrightarrow \quad \frac{1}{n^2+2n} > \frac{1}{n^2+4n+3}$$

$$\Longleftrightarrow \quad \cancel{n^2}+4n+3 > \cancel{n^2}+2n$$

$$\Longleftrightarrow \quad 2n+3 > 0 \quad \checkmark$$

(a_n) je rastući niz

$$\Rightarrow \quad \boxed{\inf E_2 = \min E_2 = a_1 = \frac{2}{3}}$$

$1 - \frac{1}{n^2+2n} < 1 \quad \forall\, n \in \mathbb{N} \Rightarrow 1$ je gornja međa skupa E_2. Dokažimo $\sup E_2 = 1$

T.D. $(\forall \varepsilon > 0)$ $(\exists m \in N)$ t.d.

$$1 - \varepsilon < 1 - \frac{1}{m^2 + 2m}$$

$$\iff \varepsilon (m^2 + 2m) > 1$$

A.A. 1 i ε $\Rightarrow$ $\exists m_0 \in N$ t.d.

$$\varepsilon \cdot m_0 > 1$$

$$\varepsilon \cdot (m_0^2 + 2m_0) > \varepsilon \cdot (2m_0) > \varepsilon \cdot m_0 > 1,$$

tj. $$1 - \varepsilon < 1 - \frac{1}{m_0^2 + 2m_0}$$

$\Rightarrow$ $\boxed{\sup E_2 = 1}$

$\{\sup E_1 \cdot \sup E_2, \sup E_1 \cdot \inf E_2, \inf E_1 \cdot \sup E_2, \inf E_1 \cdot \inf E_2\}$

$= \{2 \cdot 1, 2 \cdot \frac{2}{3}, -1 \cdot 1, -1 \cdot \frac{2}{3}\} = \{2, \frac{4}{3}, -1, -\frac{2}{3}\}$

$\sup E = \max \{2, \frac{4}{3}, -1, -\frac{2}{3}\} = 2$

$\inf E = \min \{2, \frac{4}{3}, -1, -\frac{2}{3}\} = -1$

f) $\frac{nx^2-4nx+2}{n} = \frac{n(x^2-4x)+2}{n} =$

$= x^2-4x+\frac{2}{n}$

$F = F_1 + F_2$, gdje je

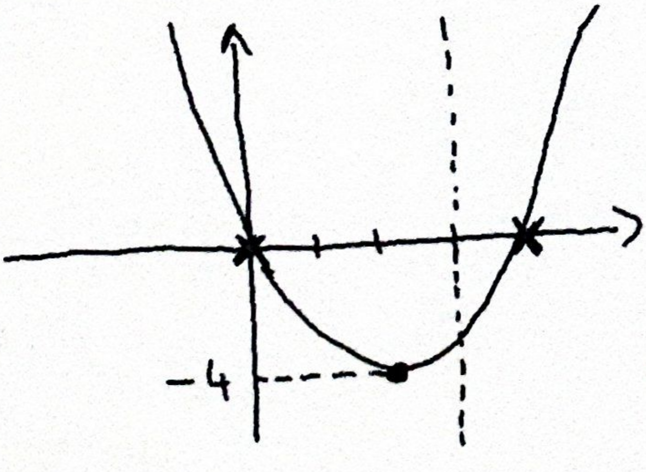
$F_1 = \{ x^2-4x : x \in \langle 0,3] \}$

$F_2 = \{ \frac{2}{n} : n \in \mathbb{N} \}$

$x^2-4x = x^2-4x+4-4 = (x-2)^2-4 \geq -4$

(-4 se dostiže za $x = 2$) $\in \langle 0,3]$

$\min F_1 = -4 \implies \boxed{\inf F_1 = -4}$



$x^2-4x < 0$ za $x \in \langle 0,3]$

$\implies 0$ je gornja međa skupa F_1.

Dokažimo da je

$F_1 = [-4, 0\rangle$

Već smo dokazali $F_1 \subseteq [-4, 0\rangle$

Dokažimo da $\forall y \in [-4, 0\rangle \ \exists x \in \langle 0, 3]$

t.d. $y = x^2 - 4x$.

$\iff x^2 - 4x - y = 0$

$\iff (x-2)^2 - 4 - y = 0$ $\qquad y \in [-4, 0\rangle$

$\iff (x-2)^2 = \underbrace{4+y}_{\geq 0}$ $\qquad 4 + y \in [0, 4\rangle$

$\iff x - 2 = \pm\sqrt{4+y}$ $\qquad \sqrt{4+y} \in [0, 2\rangle$

$\iff x = 2 \pm \sqrt{4+y}$

$$0 < 2 - \sqrt{4+y} \leq 2 - 0 = 2 < 3$$

Za $x = 2 - \sqrt{4+y}$ je $x^2 - 4x = y$

$\Rightarrow F_1 = [-4, 0\rangle \Rightarrow \boxed{\sup F_1 = 0}$

Za DZ $\boxed{\inf F_2 = 0}$, $\boxed{\sup F_2 = 2}$

$\sup F = \sup F_1 + \sup F_2 = 0 + 2 = 2$, $\inf F = \inf F_1 + \inf F_2$
$= -4 + 0 = -4$

g) $\cos(2k\pi) = \cos 0 = 1 \quad \forall k \in \mathbb{N}$

$\cos((2k-1)\pi) = \cos(-\pi) = \cos\pi = -1 \quad \forall k \in \mathbb{N}$

$G_1 = \{2 + \cos(m\pi) : m \in \mathbb{N}\}$

$= \{2+1, 2-1\} = \{3, 1\}$

$\boxed{\sup G_1 = 3},\quad \boxed{\inf G_1 = 1}$

$$G_2 = \left\{\frac{n^2+1}{3n^2+n} : n \in \mathbb{N}\right\}$$

$$a_n \leq a_{n+1} \iff \frac{n^2+1}{3n^2+n} \leq \frac{(n+1)^2+1}{3(n+1)^2+(n+1)}$$

$$\iff (n^2+1)(3n^2+7n+4) \leq (3n^2+n)(n^2+2n+2)$$

$$\iff 3n^4 + 7n^3 + 7n^2 + 7n + 4$$

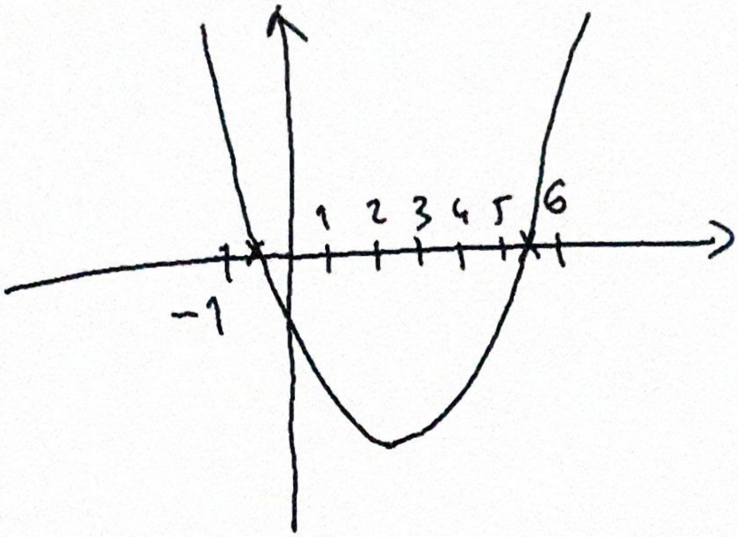
$$\leq 3n^4 + 7n^3 + 8n^2 + 2n$$

$$\iff n^2 - 5n - 4 \geq 0$$

$$n_{1,2} = \frac{5 \pm \sqrt{25+16}}{2} = \frac{5 \pm \sqrt{41}}{2}$$

$\frac{5+\sqrt{41}}{2} \in \langle 5, 6 \rangle$

$\frac{5-\sqrt{41}}{2} \in \langle -1, 0 \rangle$



$$a_n \leq a_{n+1} \iff n \geq \overset{\nearrow \in \mathbb{N}}{\frac{5+\sqrt{41}}{2}}$$

$$\iff n \geq 6$$

za $n < 6$ je $a_n > a_{n+1}$

$$a_1 > a_2 > a_3 > a_4 > a_5 > \underline{\underline{a_6}} \leq a_7 \leq a_8 \leq \dots$$

$$\Rightarrow \min G_2 = a_6 = \frac{6^2+1}{3\cdot 6^2+6} = \frac{37}{114}$$

$$\Rightarrow \boxed{\inf G_2 = \frac{37}{114}}$$

$$a_1 = \frac{1^2+1}{3\cdot 1^2+1} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{n^2+1}{3n^2+n} \leq \frac{1}{2} \iff$$

$$2n^2+2 \leq 3n^2+n \iff$$

$$2 \leq n^2+n \iff$$

$$2 \leq n(n+1) \quad \checkmark \quad (\text{jer je } n \in \mathbb{N})$$

$$\Rightarrow \quad a_n \leq \frac{1}{2} = a_1 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad \max G_2 = a_1 = \frac{1}{2}$$

$$\Rightarrow \quad \boxed{\sup G_2 = \frac{1}{2}}$$

$$G = G_1 \cdot G_2 \qquad G_1, G_2 \subseteq [0, +\infty)$$

$$\sup G = \sup G_1 \cdot \sup G_2 = 3 \cdot \frac{1}{2} = \frac{3}{2}$$

$$\inf G = \inf G_1 \cdot \inf G_2 = 1 \cdot \frac{37}{114} = \frac{37}{114}$$

(3.) Treba dokazati (za zbrajanje u $\mathbb{Z}$)

Ako je $[(m_1, n_1)] = [(m_1', n_1')]$

i $[(m_2, n_2)] = [(m_2', n_2')]$

da je $[(m_1, n_1)] + [(m_2, n_2)]$

$= [(m_1', n_1')] + [(m_2', n_2')]$, tj.

ako je $(m_1, n_1) \sim (m_1', n_1')$

$(m_2, n_2) \sim (m_2', n_2')$

da je $(m_1+m_2, n_1+n_2) \sim (m_1'+m_2', n_1'+n_2')$

$$(m_1, n_1) \sim (m_1', n_1') \iff \boxed{m_1 + n_1' = n_1 + m_1'}$$

$$(m_2, n_2) \sim (m_2', n_2') \iff \boxed{m_2 + n_2' = n_2 + m_2'}$$

$$\left.\begin{array}{l} m_1 + n_1' = n_1 + m_1' \\ m_2 + n_2' = n_2 + m_2' \end{array}\right\} \oplus \Rightarrow m_1 + m_2 + n_1' + n_2' = n_1 + n_2 + m_1' + m_2'$$

$$\Rightarrow [(m_1+m_2, n_1+n_2)] = [(m_1'+m_2', n_1'+n_2')]$$

Slično za ostale operacije (DZ)

4. $([(a,b)]+[(c,d)])+[(e,f)]$

$= [(ad+bc, bd)]+[(e,f)]$

$= [(ad+bc)f+(bd)e, (bd)f]$

$= [(adf+bcf+bde, bdf)]$

$[(a,b)]+([(c,d)]+[(e,f)])$

$= [(a,b)]+[(cf+de, df)] =$

$= [(adf+b(cf+de), b(df)]$

$= [(adf+bcf+bde, bdf)]$

1. $S = \{ z \in \mathbb{N} : (x+y)\cdot z = x\cdot z + y\cdot z \}$

$$\left.\begin{array}{l}(x+y)\cdot 1 = x+y \\ x\cdot 1 + y\cdot 1 = x+y\end{array}\right\} \Rightarrow 1 \in S$$

Pp. $z \in S$ za neki $z \in \mathbb{N}$

$$\begin{aligned}(x+y)\cdot s(z) &= (x+y)\cdot z + (x+y) \\ &\underset{z\in S}{=} (x\cdot z + y\cdot z) + x + y \\ &= x\cdot z + x + y\cdot z + y \\ &= x\cdot s(z) + y\cdot s(z)\end{aligned}$$

$\Rightarrow s(z) \in S$

M.I. $\Rightarrow$ $S = \mathbb{N}$