Nekomutativni prostori u fizici

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Pregled

Uvod

Povijesni pregled

- divergencije u QFT
  - Heisenberg (1938.): fundamentalna duljina
  - Snyder (1947.): prvi nekomutativni (NC) prostor
  - pad u zaborav
- kvantna gravitacija
  - Seiberg, Witten (1999.): NC kao limes efektivnog polja u teoriji otvorenih struna s B poljem
  - Doplicher, Fredenhagen, Roberts (1995.): NC kao rješenje problema mjerenja na Planckovim skalama
- matematika
  - Alain Connes, Drinfel'd (80-te nadalje)

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Glavna ideja

- želimo "razmazanost" prostora

$$\Delta\hat{x}^{\mu}\Delta\hat{x}^{\nu} \geq l_p^2, \quad l_p = \sqrt{\hbar G/c^3} \approx 10^{-33} \text{ cm}$$

- realizacija: NC algebra

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = l_p^2 \theta^{\mu\nu}\left(\frac{\hat{x}}{l_p}\right)$$

$$\overset{l_p=1}{=} \theta^{\mu\nu}_{(0)} + \theta^{\mu\nu}_{(1)\,\alpha}\hat{x}^{\alpha} + \theta^{\mu\nu}_{(2)\,\alpha\beta}\hat{x}^{\alpha}\hat{x}^{\beta} + \ldots$$

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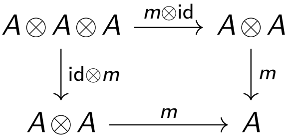
Hopfove algebre

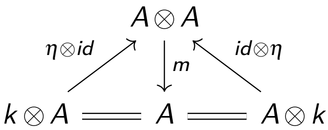
(Ko)algebre

Definicija

Algebra je trojka $(\mathscr{A}, m, \eta)$ gdje su $m : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ i $\eta : k \to \mathscr{A}$ linearne mape koje zovemo multiplikacijom (ili produktom) odnosno jedinicom. One zadovoljavaju relacije

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \quad \text{(asocijativnost)}$$
$$m \circ (\eta \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes \eta) = \mathrm{id} \quad \text{(jedinica)}$$

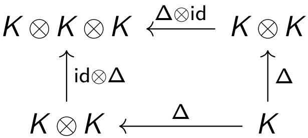


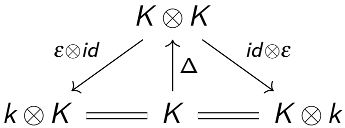


Definicija

Koalgebra je trojka $(\mathscr{K}, \Delta, \varepsilon)$ gdje su $\Delta : \mathscr{K} \to \mathscr{K} \otimes \mathscr{K}$ i $\varepsilon : \mathscr{K} \to k$ linearne mape koje zovemo komultiplikacijom (ili koproduktom) odnosno kojedinicom. One zadovoljavaju relacije

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \quad \text{(koasocijativnost)}$$
$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} \quad \text{(kojedinica)}$$





- primjer: $\Delta(f)(g_1, g_2) = f(g_1 g_2)$, $\varepsilon(f) = f(1)$
- Sweedlerova notacija: $\Delta(a) = \sum a_{(i)} \otimes a_{(j)}$
- svaka Lie algebra! Ako $\Delta(a) = a \otimes 1 + 1 \otimes a$, onda $\Delta([a,b]) = [a,b] \otimes 1 + 1 \otimes [a,b] \Rightarrow$ primjer Hopfove algebre

Definicija

Bialgebra $\mathscr{B}$ je istodobno algebra i koalgebra. Kompatibilnost traži

$$\Delta(ab) = \Delta(a)\Delta(b), \Delta(1) = 1 \otimes 1$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \varepsilon(1) = 1$$

za svaki $a, b \in \mathscr{B}$. **Hopfova algebra** $\mathscr{H}$ je bialgebra s *antipodom*, linearnom mapom $S : \mathscr{H} \to \mathscr{H}$ koja zadovoljava

$$m \circ (S \otimes \mathrm{id}) \otimes \Delta = m \circ (\mathrm{id} \otimes S) \otimes \Delta = \eta \circ \varepsilon.$$

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Opis "deformirane" strukture

- koalgebarska struktura određuje deformirano Leibnitzovo pravilo
- definiramo djelovanje $h \triangleright a$ t.d. $h_1 \triangleright (h_2 \triangleright a) = (h_1 h_2) \triangleright a$
- Hopfova algebra mora djelovati sukladno s koalgebarskom strukturom:

$$\begin{aligned} h \triangleright (ab) &= m \circ (\Delta(h) \triangleright (a \otimes b)) \\ &= \sum \left(h_{(1)} \triangleright a\right) \left(h_{(2)} \triangleright b\right) \end{aligned}$$

- za $\Delta(P^\mu) = P^\mu \otimes 1 + 1 \otimes P^\mu$ i $P_\mu \Leftrightarrow i\partial_\mu$ dobivamo Leibnitzovo pravilo

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Moyalov prostor

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$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$$

- želimo NC algebru $\hat{A}_{\hat{x}}$ izraziti pomoću obične A_x
- treba naći NC množenje t.d.

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) \stackrel{\cong}{\rightarrow} f(x) \star g(x)$$

- hint iz QM - uvodimo Weylov transformat

$$\mathscr{W}(f)(\hat{x}) = \int \frac{\mathrm{d}^n k}{(2\pi)^{\frac{n}{2}}} e^{ik_\mu \hat{x}^\mu} \tilde{f}(k)$$

- inverz je Wignerovov transformat

$$\mathscr{W}^{-1}(\hat{F})(x) = \int \frac{\mathrm{d}^n k}{(2\pi)^{\frac{n}{2}}} e^{-ik_\mu x^\mu} \mathrm{Tr}_{\hat{x}} \hat{F}(\hat{x}) e^{ik_\mu \hat{x}^\mu}$$

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Novi produkt

- definiramo $\star$-produkt

$$f \star g = \mathscr{W}^{-1}(\mathscr{W}(f)\mathscr{W}(g))$$

$$= \int \frac{\mathrm{d}^n k}{(2\pi)^{\frac{n}{2}}} e^{-ik_\mu x^\mu} \mathrm{Tr}_{\hat{x}} \int \frac{\mathrm{d}^n p}{(2\pi)^{\frac{n}{2}}} \frac{\mathrm{d}^n q}{(2\pi)^{\frac{n}{2}}} e^{i(p_\mu + q_\mu)\hat{x}^\mu} e^{\frac{i}{2} p_\mu q_\nu \theta^{\mu\nu}} \tilde{f}(p)\tilde{g}(q) e^{ik_\mu \hat{x}^\mu}$$

$$= \int \frac{\mathrm{d}^n p}{(2\pi)^{\frac{n}{2}}} \frac{\mathrm{d}^n q}{(2\pi)^{\frac{n}{2}}} e^{i(p_\mu + q_\mu)x^\mu} e^{\frac{i}{2} p_\mu q_\nu \theta^{\mu\nu}} \tilde{f}(p)\tilde{g}(q)$$

$$= e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y \to x}$$

gdje smo koristili $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$

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$$f \star g = \mathscr{W}^{-1}(\mathscr{W}(f)\mathscr{W}(g))$$

$$= \int \frac{\mathrm{d}^n k}{(2\pi)^{\frac{n}{2}}} e^{-ik_\mu x^\mu} \mathrm{Tr}_{\hat{x}} \int \frac{\mathrm{d}^n p}{(2\pi)^{\frac{n}{2}}} \frac{\mathrm{d}^n q}{(2\pi)^{\frac{n}{2}}} e^{i(p_\mu + q_\mu)\hat{x}^\mu} e^{\frac{i}{2} p_\mu q_\nu \theta^{\mu\nu}} \tilde{f}(p)\tilde{g}(q) e^{ik_\mu \hat{x}^\mu}$$

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$$= e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y \to x}$$

gdje smo koristili $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$

Novi produkt

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- za $f(x) = x^\mu$, $g(x) = x^\nu$,

$$x^\mu \star x^\nu = x^\mu x^\nu + \frac{i}{2}\theta^{\alpha\beta}\eta_{\alpha\mu}\eta_{\beta\nu}$$
$$\Rightarrow [x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$$

- identificiramo li $P_\mu = i\partial_\mu$, imamo

$$f \star g = m \circ \mathscr{F}^{-1} \triangleright (f(x) \otimes g(y))$$

gdje je $\mathscr{F} = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}$ zakretanje

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- imali smo $\Delta(P^\mu) = P^\mu \otimes 1 + 1 \otimes P^\mu$ i $P_\mu \Leftrightarrow i\partial_\mu \Rightarrow$ Leibnitzovo pravilo
- sad Poincaréova algebra mora djelovati na $A_x^\star$
- jedini konzistentan način: deformacija koalgebarskog sektora

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- Poincaréova algebra ostaje ista, ali imamo koalgebru

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\Delta_\star P_\mu &= P_\mu \otimes 1 + 1 \otimes P_\mu \\
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Posljedice

- može se pokazati da je integral cikličan,

$$\int d^n x\, f_1(x) \star \cdots \star f_k(x) = \mathrm{Tr}\, W(f_1) \cdots W(f_k)$$

stoga slobodna teorija polja ostaje ista

- u ϕ^4 teoriji, imamo $\mathscr{L}_{\text{int}} = \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi$
- vrh dobiva oscilatorni faktor $\prod_{i<j} e^{-\frac{i}{2}\theta_{\mu\nu} k_i^\mu k_j^\nu}$
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- mijenja se statistika: za bozone, umjesto

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Ostali prostori

κ-Minkowski

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}(\hat{x}^\mu a^\nu - \hat{x}^\nu a^\mu)$$

- prostor originalno dobiven iz deformacije Poincaréove algebre
- problemi s konstrukcijom teorije polja, zakretanjem, $\star$-podukta
- modifikacije disperzijskih relacija:

$$E = \frac{\hbar\omega}{1 - \frac{\omega}{c\kappa}}$$

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$$[\hat{x}^\mu, \hat{x}^\nu] = iM^{\mu\nu}$$

- deformacija ovisna o impulsu
- netrivijalne Lorentzove transformacije

$$\Lambda(\omega, k \oplus q) = \Lambda(\omega_1(k,q), k) \oplus \Lambda(\omega_2(k,q), q)$$

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Hvala na pažnji!