Gauge invarijantnost i modulacijska nestabilnost u nelinearnim fotoničkim sustavima s umjetnim magnetskim poljima

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Uvod

- Kvantni simulatori
  - Ultrahladni plinovi i fotonički sustavi
- Fotonički sustavi opisani nelinearnom Schrödingerovom jedn.

$$i\hbar\,\partial_t\Psi = \frac{1}{2m}\left(-i\hbar\,\nabla - \mathbf{A}\right)^2\,\Psi + V\left(\mathbf{r}\right)\Psi + \eta\left|\Psi\right|^2\,\Psi$$

- Ψ nije valna funkcija
- Nelinearnost kao promjena indeksa loma

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- Nelinearnost kao promjena indeksa loma
- Modulacijska nestabilnost
  - test stabilnosti svojstvenog rješenja

$$\Psi_0 \;\rightarrow\; \Psi_0(1+\delta(\mathbf{r},t))$$

Uvod - baždarna transformacija

$$i\hbar\,\partial_t \Psi = \frac{1}{2m}\left(-i\hbar\,\nabla - \mathbf{A}\right)^2\,\Psi + V(\mathbf{r})\,\Psi + \eta\,|\Psi|^2\,\Psi$$

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$$\begin{aligned}
\left(-i\hbar\,\nabla - \mathbf{A}'\right)\Psi' &= \left\{\mathbf{A}' = \mathbf{A} + \nabla\chi\right\} \\
&= \left(-i\hbar\,\nabla - \mathbf{A} - \nabla\chi\right) e^{i\alpha}\,\Psi \\
&= e^{i\alpha}\left(-i\hbar\,\nabla - \mathbf{A} - \nabla\chi + \hbar\,\nabla\alpha\right)\Psi \\
&= e^{i\chi/\hbar}\left(-i\hbar\,\nabla - \mathbf{A}\right)\Psi
\end{aligned}$$

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$$\Psi' = \exp\left(i\,\frac{\chi}{\hbar}\right)\,\Psi$$

Rješenje u simetričnom baždarenju

$$\mathbf{A}_S = \frac{B}{2}\left(x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}}\right) \quad \leftarrow \quad \mathbf{B} = \nabla \times \mathbf{A}_S$$

- Uz pretpostavku vremenske faze $\Psi = \exp(-i\frac{E}{\hbar}t)\,\psi$ dobivamo

$$E\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + i\frac{\hbar B}{2m}(x\partial_y - y\partial_x) + \frac{B^2}{8m}\left(x^2 + y^2\right) + V + \eta|\psi|^2\right]\psi$$

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- Izbor potencijala $V = -\frac{B^2}{8m}\left(x^2+y^2\right)$:

$$\Psi = \sqrt{I_0}\,\exp\left(-i\frac{\eta I_0}{\hbar}t\right)$$

Modulacijska nestabilnost hom. rješenja

$$\Psi = \sqrt{I_0}\,\exp\left(-i\frac{\eta I_0}{\hbar}t\right)\,(1+\delta(\mathbf{r},t))$$

$$i\hbar\,\partial_t\delta = -\frac{\hbar^2}{2m}\nabla^2\delta + i\frac{\hbar B}{2m}\,\partial_\phi\delta + \eta I_0(\delta+\delta^*)$$

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$$i\hbar\,\partial_t b = -\frac{\hbar^2}{2m}\nabla^2 a + i\frac{\hbar B}{2m}\,\partial_\phi b + 2\eta I_0 a$$

$$i\hbar\,\partial_t a = -\frac{\hbar^2}{2m}\nabla^2 b + i\frac{\hbar B}{2m}\,\partial_\phi a$$

$$a = \delta + \delta^*$$

$$b = \delta - \delta^*$$

Modulacijska nestabilnost hom. riješenja

- Fourierov transformat u polarnom sustavu:

$$f(\mathbf{r},t) = \frac{1}{2\pi}\int \tilde{f}(\mathbf{k},t)\, e^{-i\,\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} = \sum_{n=-\infty}^{\infty} i^{-n} e^{-in\phi} \int_0^{\infty} \tilde{f}_n(\rho,t) J_n(\rho r)\rho d\rho$$

$$\nabla^2 \left(e^{-in\phi} J_n(\rho r)\right) = \frac{1}{r^2}\left[r^2\,\partial_r^2 + r\,\partial_r + \partial_\phi^2\right] e^{-in\phi} J_n(\rho r) = -\rho^2 e^{-in\phi} J_n(\rho r)$$

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$$i\hbar\,\partial_t\tilde{b}_n = \frac{\hbar^2\rho^2}{2m}\tilde{a}_n + \frac{\hbar B}{2m}n\tilde{b}_n + 2\eta I_0\tilde{a}_n$$

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Modulacijska nestabilnost hom. rješenja

- Općenito rješenje šuma $\delta = 1/2(a+b)$:

$$\delta = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-in(\phi+\pi/2)} e^{-i\frac{B}{2m}nt} \int_0^{\infty} J_n(\rho r)\rho d\rho \left[\tilde{a}_{n1}(1+S)e^{-i\tilde{\omega}t} + \tilde{a}_{n2}(1-S)e^{+i\tilde{\omega}t}\right]$$

- uz pokrate i uvjete:

$$S = \sqrt{1+\frac{4\eta I_0 m}{\hbar^2\rho^2}}$$

$$\tilde{\omega} = \frac{\hbar\rho^2}{2m}S$$

$$\rho_c = \sqrt{-\frac{4\eta I_0 m}{\hbar^2}}$$

$$\begin{cases} \tilde{a}^*_{-n1} = (-1)^n\,\tilde{a}_{n1} \\ \tilde{a}^*_{-n2} = (-1)^n\,\tilde{a}_{n2} \end{cases} \quad \text{za } i\tilde{\omega} \in \mathbb{R}$$

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Prikaz rješenja

- Namećemo početni uvjet ravnog vala $\delta(t=0) = \epsilon \exp(-i\,\mathbf{k}_0 \cdot \mathbf{r})$:

$$\delta(\mathbf{r},t) = \frac{\epsilon}{4S_0}\left\{e^{-i\tilde{\omega}_0 t}\left[(S_0+1)^2 e^{-i\,\mathbf{k}_0\cdot\tilde{\mathbf{r}}} + (S_0^2-1)e^{+i\,\mathbf{k}_0\cdot\tilde{\mathbf{r}}}\right]\right.$$
$$\left. - e^{+i\tilde{\omega}_0 t}\left[(S_0-1)^2 e^{-i\,\mathbf{k}_0\cdot\tilde{\mathbf{r}}} + (S_0^2-1)e^{+i\,\mathbf{k}_0\cdot\tilde{\mathbf{r}}}\right]\right\}$$

- pokrate:

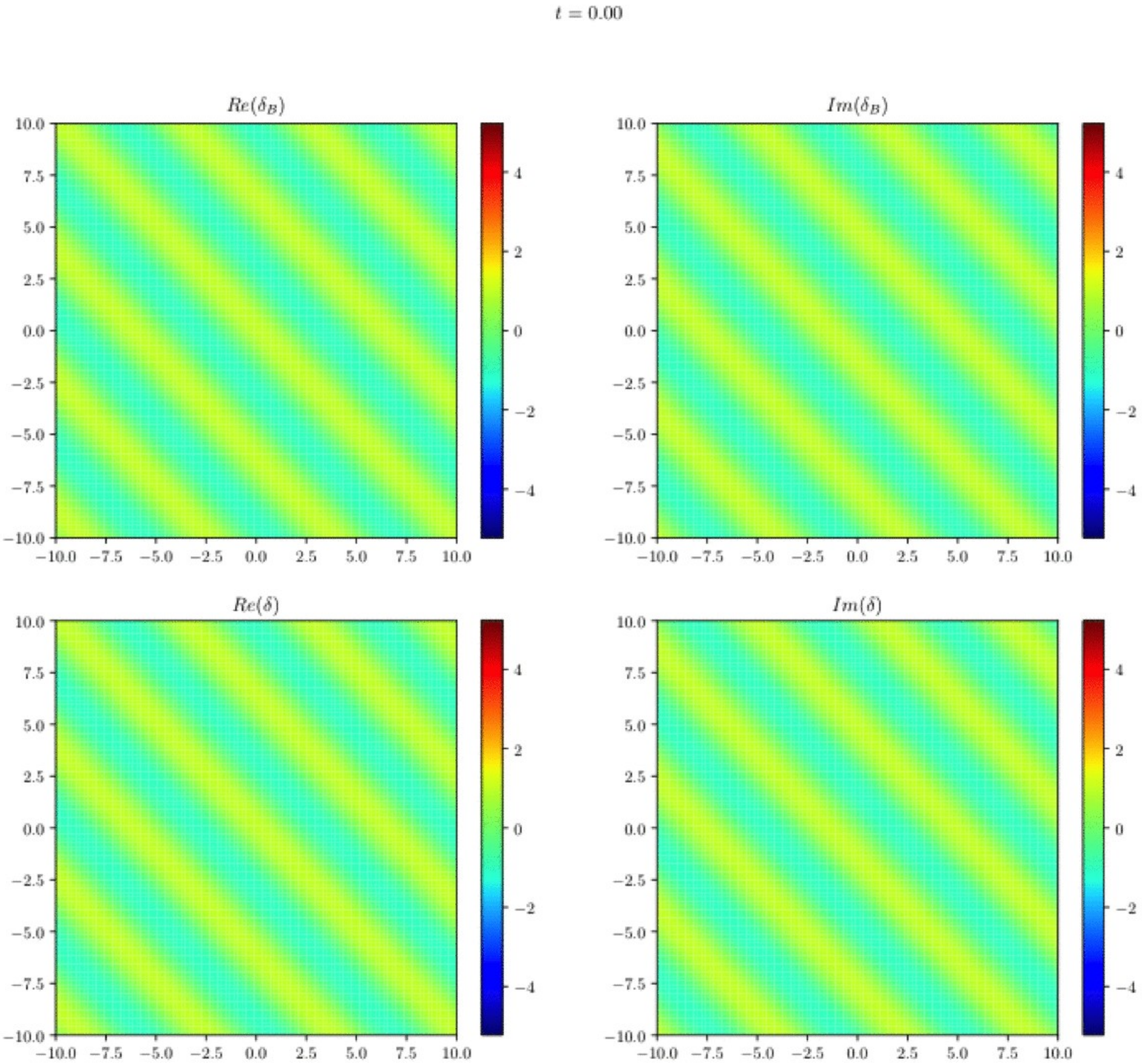
$$S_0 = \sqrt{1 + \frac{4\eta I_0 m}{\hbar^2\,\mathbf{k}_0^2}}\,, \quad \tilde{\omega}_0 = \frac{\hbar\,\mathbf{k}_0^2}{2m}S_0\,, \quad \tilde{\mathbf{r}} = \mathbf{r}\,\cos\frac{B}{2m}t + \hat{\mathbf{z}}\times\mathbf{r}\,\sin\frac{B}{2m}t$$

Prikaz rješenja: divergentni p.u.

$$m = \hbar = -\eta = I_0 = \epsilon = 1$$

$$B = \sqrt{7}$$

$$(k_{0x}, k_{0y}) = (1, 1)$$



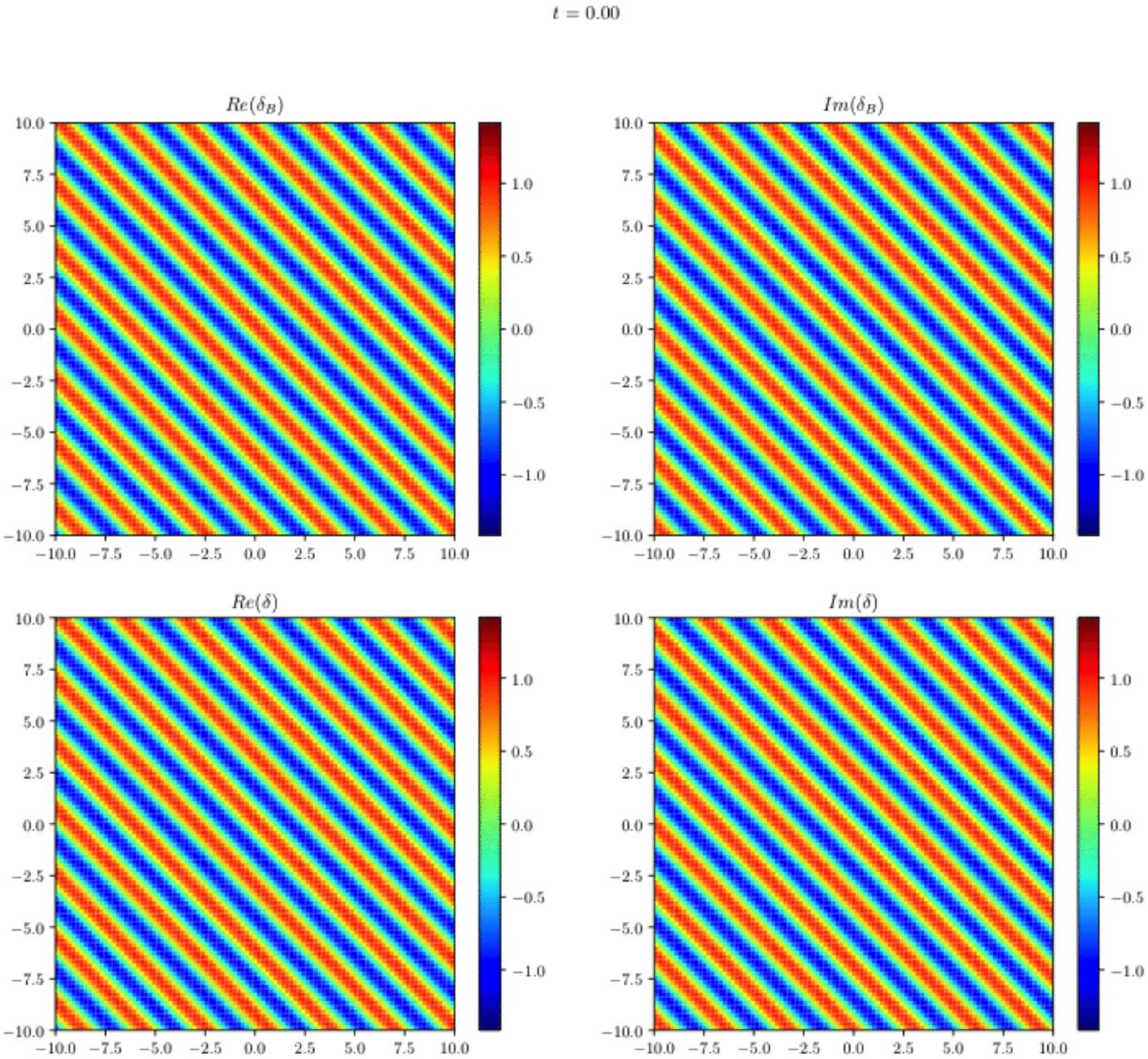
$t = 0.00$
$Re(\delta_B)$
$Im(\delta_B)$
$Re(\delta)$
$Im(\delta)$

Prikaz rješenja: oscilatorni p.u.

$$m = \hbar = -\eta = I_0 = \epsilon = 1$$

$$B = \sqrt{7}$$

$$(k_{0x}, k_{0y}) = (2, 2)$$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Rješenje u Landauovom baždarenju

$$\mathbf{A}_L = Bx\,\hat{\mathbf{y}}\,, \quad V = -\frac{B^2}{8m}\left(x^2 + y^2\right)$$

$$E\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + i\frac{\hbar B}{m}x\partial_y + \frac{B^2}{8m}\left(3x^2 - y^2\right) + \eta|\psi|^2\right]\psi$$

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$$\mathbf{A}_L = \mathbf{A}_S + \nabla\left(\frac{B}{2}xy\right) \quad \rightarrow \quad \psi = \sqrt{I_0}\,\exp\left(i\frac{B}{2\hbar}xy\right)$$

Rješenje u Landauovom baždarenju

$$\Psi = \sqrt{I_0}\exp\left(-i\frac{\eta I_0}{\hbar}t\right)\exp\left(i\frac{B}{2\hbar}xy\right)(1+\delta)$$

- Ubacivanjem u početnu nelinearnu Schrödingerovu jedn.

$$i\hbar\,\partial_t\Psi = \frac{1}{2m}\left(-i\hbar\,\nabla - \mathbf{A}\right)^2\,\Psi + V(\mathbf{r})\,\Psi + \eta\left|\Psi\right|^2\,\Psi$$

$$\boxed{i\hbar\,\partial_t\delta = -\frac{\hbar^2}{2m}\nabla^2\delta + i\frac{\hbar B}{2m}\,\partial_\phi\delta + \eta I_0(\delta+\delta^*)}$$

Generalizacija invarijatnosti šuma

- Na svojstveno stanje ψ_0 početne NLSE nametnemo šum:

$$\Psi = \exp(-i\frac{E}{\hbar}t)\,\psi_0\,, \quad \Psi \to \Psi(1+\delta)$$

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- Generalna (linearizirana) jednadžba šuma:

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- Invarijantna na transformaciju

$$\mathbf{A} \to \mathbf{A} + \nabla\chi\,, \qquad \psi_0 \to \psi_0\,e^{i\chi/\hbar}$$

Zaključak

- Analitičko rješenje za šum homogene funkcije
  - Jednostavan primjer utjecaja magnetskog polja
  - Rotacija faze trebala bi biti jasno opservabilan efekt
  - Test numeričkom pristupu NLSE
- Modulacijska nestabilnost homogene funkcije
  - Jednaka slučaju bez magnetskog polja i potencijala
  - Zanimljiviji efekti na solitonskim rješenjima
- Invarijantnost šuma na baždarnu transformaciju
  - Konceptualna važnost i olakšani pristup problemu

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