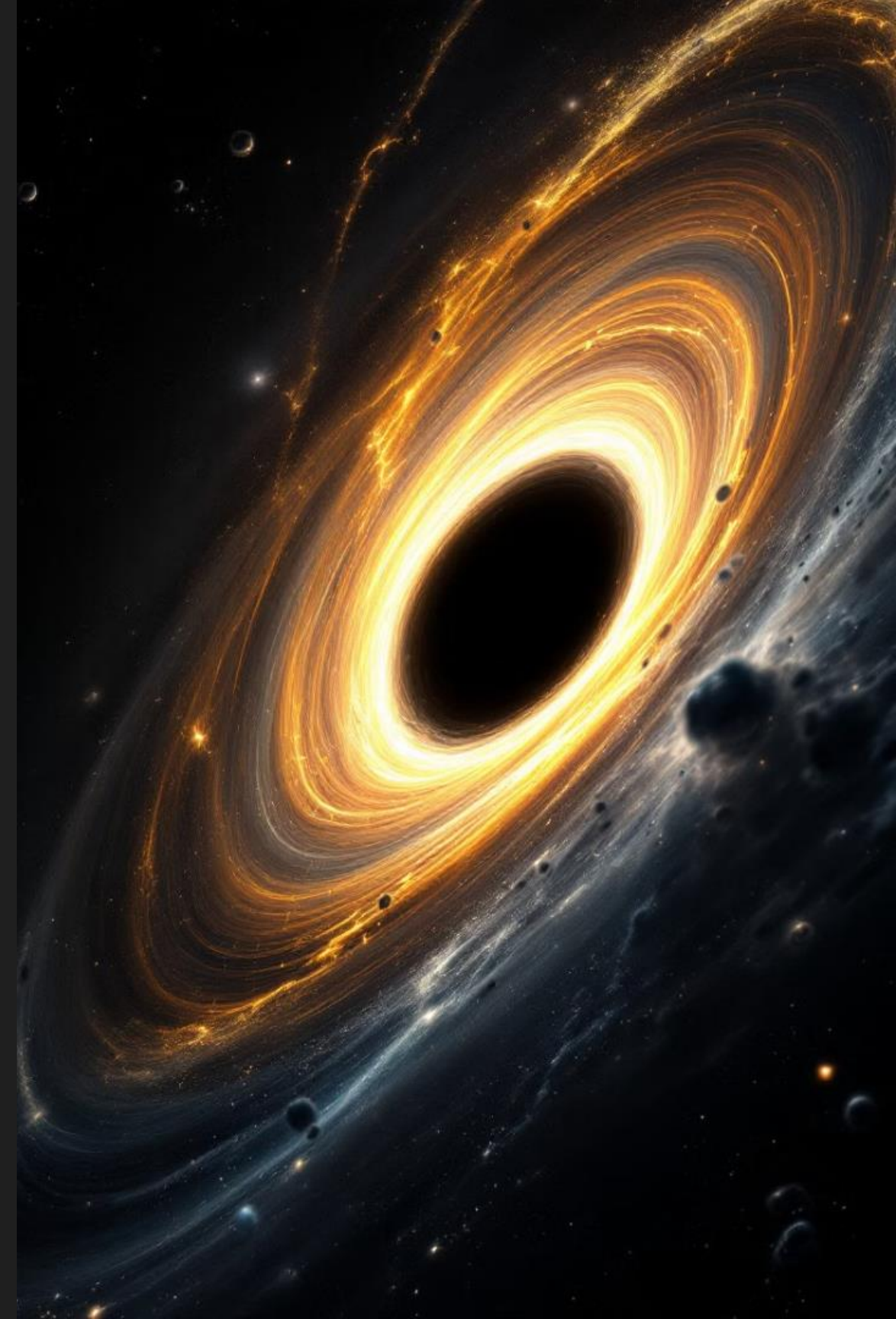


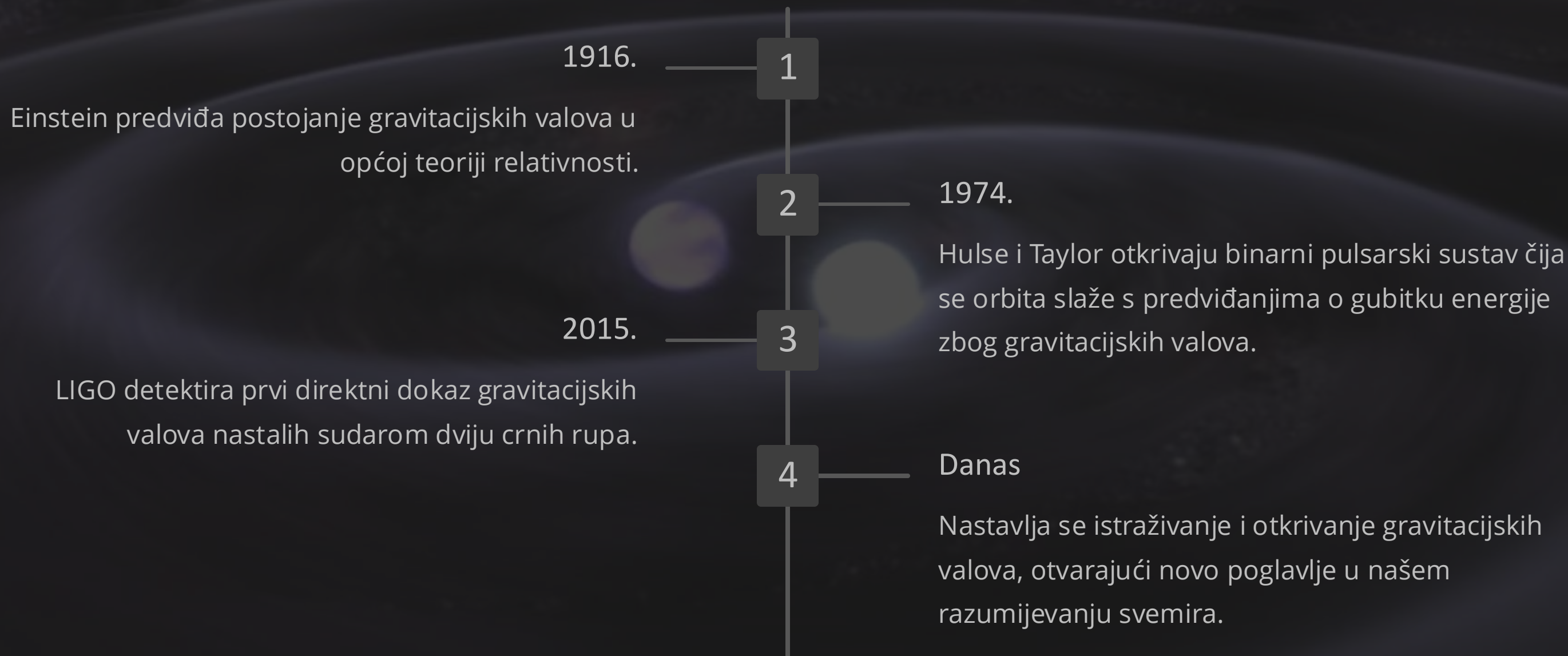
Uvod u Perturbacije Schwarzschildove Metrike

Prezentacija će se baviti analizom perturbacija Schwarzschildove metrike, ključnog rješenja Einsteinovih jednažbi koje opisuje nerotirajuće crne rupe. Fokusirat ćemo se na utjecaj ovih perturbacija na stabilnost crnih rupa te njihovu povezanost s gravitacijskim valovima. Posebnu pažnju posvetit ćemo kvazinormalnim modovima, istražujući njihovu ulogu u dosadašnjim istraživanjima i njihov potencijal za buduće primjene.

Karlo Novak



Povijest Gravitacijskih Valova



LIGO



Einsteinova jednađžba i Schwarzschildova Metrika

Metrika

Einsteinova jednađžba

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Ova jednađžba opisuje vezu između geometrije prostor-vremena (lijeva strana) i raspodjele energije i impulsa (desna strana).

Schwarzschildova Metrika

$$ds^2 = - \left(1 - \frac{2Gm}{c^2} r \right) dt^2 + \left(1 - \frac{2Gm}{c^2} r \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Ova metrika je rješenje Einsteinovih jednađžbi za nerotirajuću crnu rupu. Horizont događaja ($r = 2M$) je granica iza koje ništa, uključujući svjetlost, ne može pobjeći. Singularitet ($r = 0$) je točka u središtu gdje zakrivljenost prostor-vremena postaje beskonačna.

$$g_{\mu\nu} \rightarrow R_{\mu\nu}$$

$$g_{\mu\nu} + h_{\mu\nu} \rightarrow R_{\mu\nu} + \delta R_{\mu\nu}$$

$$\delta R_{\mu\nu} = -\delta\Gamma^\beta_{\mu\nu;\beta} + \delta\Gamma^\beta_{\mu\beta;\nu}$$

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\nu} (h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu})$$

$$\phi_L^M = \mathrm{const} Y_L^M(x_2,x_3) = \mathrm{const} Y_L^M(\theta,\phi) \; (-)^L$$

$$\psi_{L\, ,\mu}^M = \mathrm{const} \frac{\partial}{\partial x^\mu} Y_L^M(x_2x_3) \; (-)^L$$

$$\psi_{L\, \mu\nu}^M = \mathrm{const} Y_{L\, ;\mu\nu}^M \; (-)^L$$

$$\phi_{L\, \mu\nu}^M = \mathrm{const} \gamma_{\mu\nu} Y_L^M \; (-)^L$$

$$\phi_{L\, ,\mu}^M = \mathrm{const} \epsilon_\mu{}^\nu \frac{\partial}{\partial x_\nu} Y_L^M(x_2x_3) \; (-)^{L+1}$$

$$\chi_{L\, \mu\nu}^M = \frac{1}{2} \mathrm{const} [\epsilon_\mu{}^\lambda \psi_{L\, \lambda\nu}^M + \epsilon_\nu{}^\lambda \psi_{L\, \lambda\mu}^M] \; (-)^{L+1}$$

$$\epsilon_2^2 = \epsilon_3^3 = 0, \; \epsilon_2^3 = -\frac{1}{\sin x^2}, \; \epsilon_3^2 = \sin x^2$$

$$\gamma_{\mu\nu} = \frac{g_{\mu\nu}}{r^2}$$

$$\gamma_{22} = 1, \; \gamma_{23} = 0 = \gamma_{32}, \; \gamma_{33} = \sin^2 x^2$$

$$x^2 = \theta, \; x^3 = \phi$$

Podjela Perturbacija

Aksijalne Perturbacije

Neparni paritet. Opisane Regge-Wheelerovom jednačbom.

Polarne Perturbacije

Parni paritet. Opisane Zerillijevom jednačbom.

Aksijalni Valovi

$$h_{\mu\nu}^A = \begin{bmatrix} 0 & 0 & -h_0(t, r) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_L^M & h_0(t, r) \sin \theta \frac{\partial}{\partial \theta} Y_L^M \\ 0 & 0 & -h_1(t, r) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_L^M & h_1(t, r) \sin \theta \frac{\partial}{\partial \theta} Y_L^M \\ \text{Sym} & \text{Sym} & h_2(t, r) \left(\frac{1}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} - \frac{\cot \theta}{\sin \theta} \frac{\partial}{\partial \phi} Y_L^M \right) & \text{Sym} \\ \text{Sym} & \text{Sym} & \frac{1}{2} h_2(t, r) \left(\frac{1}{\sin \theta} \frac{\partial^2}{\partial \phi^2} + \cos \theta \frac{\partial}{\partial \theta} - \sin \theta \frac{\partial^2}{\partial \theta^2} \right) Y_L^M & -h_2(t, r) \sin \theta \frac{\partial^2}{\partial \phi \partial \theta} + \cos \theta \frac{\partial}{\partial \phi} Y_L^M \end{bmatrix}$$

Polarni Valovi

$$h_{\mu\nu}^P = \begin{bmatrix} (1 - \frac{2m^*}{r}) H_0 Y_L^M & H_1 Y_L^M & h_0 \frac{\partial}{\partial \theta} Y_L^M & h_0 \frac{\partial}{\partial \phi} Y_L^M \\ H_1 Y_L^M & (1 - \frac{2m^*}{r}) H_2 Y_L^M & h_1 \frac{\partial}{\partial \theta} Y_L^M & h_1 \frac{\partial}{\partial \phi} Y_L^M \\ \text{Sym} & \text{Sym} & r^2 (K + G \frac{\partial^2}{\partial \theta^2}) Y_L^M & \text{Sym} \\ \text{Sym} & \text{Sym} & r^2 G \left(\frac{\partial^2}{\partial \phi \partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) Y_L^M & r^2 (K \sin^2 \theta + G \frac{\partial^2}{\partial \phi^2} + \sin \theta \cos \theta \frac{\partial}{\partial \theta}) Y_L^M \end{bmatrix}$$

Baždarne ili koordinatne transformacije, kanonska forma perturbacija i jednačbe modova

Različiti valovi mogu predstavljati iste fizičke pojave gledane u različitim sustavima koordinata. Razmotrimo infinitezimalnu koordinatnu transformaciju:

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha} \quad (\xi^{\alpha} \ll x^{\alpha})$$

Infinitezimalni pomaci ξ^{α} transformiraju se poput vektora. U novom sustavu koordinata imat ćemo:

$$g'_{\mu\nu} + h'_{\mu\nu} = g_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} + h_{\mu\nu}$$

Sada je $h_{\mu\nu}$ definiran kao razlika između perturbirane metrike i Schwarzschildove metrike, napisane u sfernim koordinatama. Prema ovoj definiciji, razlika u novom sustavu koordinata imat će vrijednost:

$$h_{\mu\nu}^{\text{new}} = h_{\mu\nu}^{\text{old}} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

Baždarni vektor ξ^α koji pojednostavljuje aksijalne valove mora imati formu:

$$\xi^0 = 0, \quad \xi^1 = 0, \quad \xi^\mu = \Lambda(T, r) \epsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} Y_L^M(\theta, \phi), \quad (\mu, \nu = 2, 3)$$

$$h_{\mu\nu}^A = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ Sym & Sym & 0 & 0 \end{bmatrix} \cdot e^{-ikT} (\sin \theta \frac{\partial}{\partial \theta}) P_L(\cos \theta)$$

$$h_{\mu\nu}^P = \begin{bmatrix} H_0(1 - \frac{2m^*}{r}) & H_1 & 0 & 0 \\ H_1 & H_2(1 - \frac{2m^*}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 K \sin^2 \theta \end{bmatrix} \cdot e^{-ikT} (\sin \theta \frac{\partial}{\partial \theta}) P_L(\cos \theta)$$

Aksijalni valovi

$$\delta\Gamma_{\mu\nu;\beta}^{\beta} - \delta\Gamma_{\mu\beta;\nu}^{\beta} = 0$$

$$(1 - \frac{2m^*}{r})^{-1}kh_0 + \frac{d}{dr}(1 - \frac{2m^*}{r})h_1 = 0, \quad \text{iz} \quad \delta R_{23} = 0;$$

$$(1 - \frac{2m^*}{r})^{-1}k(\frac{dh_0}{dr} - kh_1 - \frac{2h_0}{r}) + (L - 1)(L + 2)\frac{h_1}{r^2} = 0, \quad \text{iz} \quad \delta R_{13} = 0;$$

$$\frac{d}{dr}(kh_1 - \frac{dh_0}{dr}) + 2k\frac{dh_1}{dr} = r^{-2}(1 - \frac{2m^*}{r})^{-1} \cdot (4m^*\frac{dh_0}{dr} - L(L + 1)h_0), \quad \text{iz} \quad \delta R_{03} = 0.$$

Aksijalni valovi

$$\Psi = \left(1 - \frac{2m^*}{r}\right) \frac{h_1}{r}$$

za aksijalne valove (Ψ) dobijemo valnu jednadžbu drugog reda:

$$\frac{d^2\Psi}{dr^{*2}} + [\omega^2 - V_{\text{eff}}(r)] \Psi = 0$$

Ovdje koristimo sljedeće skraćenice:

$$dr^* = \exp\left(\frac{1}{2}\lambda - \frac{1}{2}\nu\right) dr,$$

gdje su λ i ν definirane izrazom za metriku:

$$ds^2 = -e^\nu dT^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

pri čemu vrijedi:

$$e^\nu = e^{-\lambda} = 1 - \frac{2m^*}{r}.$$

$$V_{\text{RW}}(r) = L(L+1) \frac{e^\nu}{r^2} + 6m^* \frac{e^\nu}{r^3}$$

Polarni valovi

$$\frac{dK_{LM}}{dr} + \frac{r-3m}{r(r-2m)}K_{LM} - \frac{1}{r}H_{LM} + \frac{1}{2}\frac{L(L+1)}{i\omega r^2}H_{1LM} = 0,$$

$$\frac{dH_{LM}}{dr} + \frac{r-3m}{r(r-2m)}K_{LM} - \frac{r-4m}{r(r-2m)}H_{LM} + \left[\frac{i\omega r}{r-2m} + \frac{1}{2}\frac{L(L+1)}{i\omega r^2} \right] H_{1LM} = 0,$$

$$\frac{dH_{1LM}}{dr} + \frac{i\omega r}{r-2m}K_{LM} + \frac{i\omega r}{r-2m}H_{LM} + \frac{2m}{r(r-2m)}H_{1LM} = 0,$$

$$-\left[\frac{6m}{r} + (L-1)(L+2) \right] H_{LM} + \left[(L-1)(L+2) - \frac{2\omega^2 r^3}{r-2m} + \frac{2m(r-3m)}{r(r-2m)} \right] K_{LM} + \left[2i\omega r + \frac{L(L+1)m}{i\omega r^2} \right] H_{1LM} = 0.$$

Polarni valovi

$$R_{LM} = (1/\omega) H_{LM}$$

$$\frac{dK_{LM}}{dr} = [\alpha_0(r) + \alpha_2(r)\omega^2]K_{LM} + [\beta_0(r) + \beta_2(r)\omega^2]R_{LM},$$

$$\frac{dR_{LM}}{dr} = [\gamma_0(r) + \gamma_2(r)\omega^2]K_{LM} + [\delta_0(r) + \delta_2(r)\omega^2]R_{LM},$$

$$K_{LM} = f(r)\hat{K}_{LM} + g(r)\hat{R}_{LM}, \quad R_{LM} = h(r)\hat{K}_{LM} + k(r)\hat{R}_{LM}, \quad \frac{dr}{dr^*} = n(r).$$

Polarni valovi

$$\frac{dK_{LM}}{dr^*} = R_{LM}$$

$$\frac{dR_{LM}}{dr^*} = [V_{\text{eff}}(r) - \omega^2] K_{LM}.$$

Ako izaberemo $n(r) = 1 - \frac{2m}{r}$. gdje je m masa Schwarzschildovog polja:

$$f(r) = [\lambda(\lambda + 1)r^2 + 3\lambda mr + 6m^2] \cdot \frac{(\lambda r + 3m)}{r^2}, \quad g(r) = 1,$$

$$h(r) = i [-\lambda r^2 + 3\lambda mr + 3m^2] \cdot \frac{(\lambda r + 3m)}{(r - 2m)}, \quad k(r) = -\frac{ir^2}{(r - 2m)}, \quad \lambda = \frac{1}{2} (L - 1) (L + 2),$$

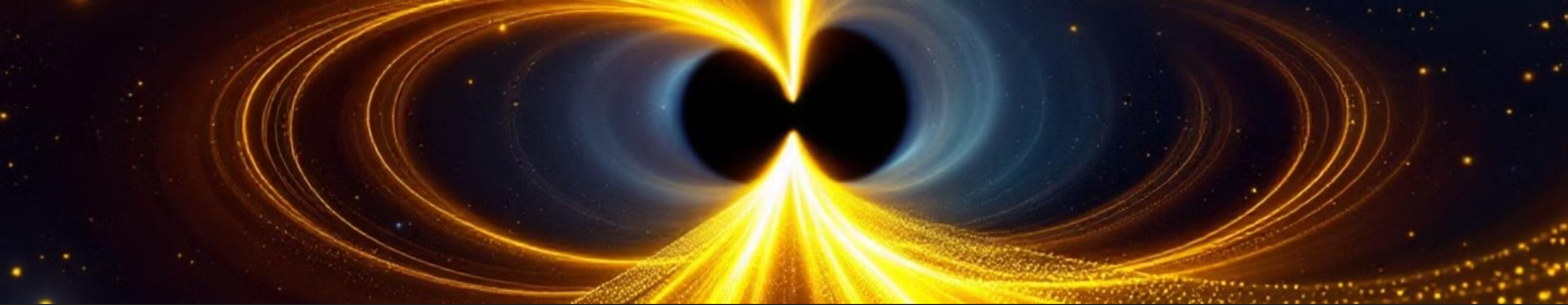
$$\text{gdje je } r^* = r + 2m \ln \left(\frac{r}{2m-1} \right)$$

Polarni valovi

$$\frac{d\hat{K}_{LM}}{dr^*} = \hat{R}_{LM}, \quad \frac{d\hat{R}_{LM}}{dr^*} = [V_Z(r^*) - \omega^2] \hat{K}_{LM},$$

$$\frac{d^2 \hat{K}_{LM}}{dr^{*2}} + [\omega^2 - V_Z(r)] \hat{K}_{LM} = 0.$$

$$V_Z(r) = \left(\frac{1 - 2m}{r} \right) \frac{2\lambda^2 (\lambda + 1) r^3 + 6\lambda^2 m r^2 + 18\lambda m^2 r + 18m^3}{r^3 (\lambda r + 3m)^2}$$



Efektivni Potencijali

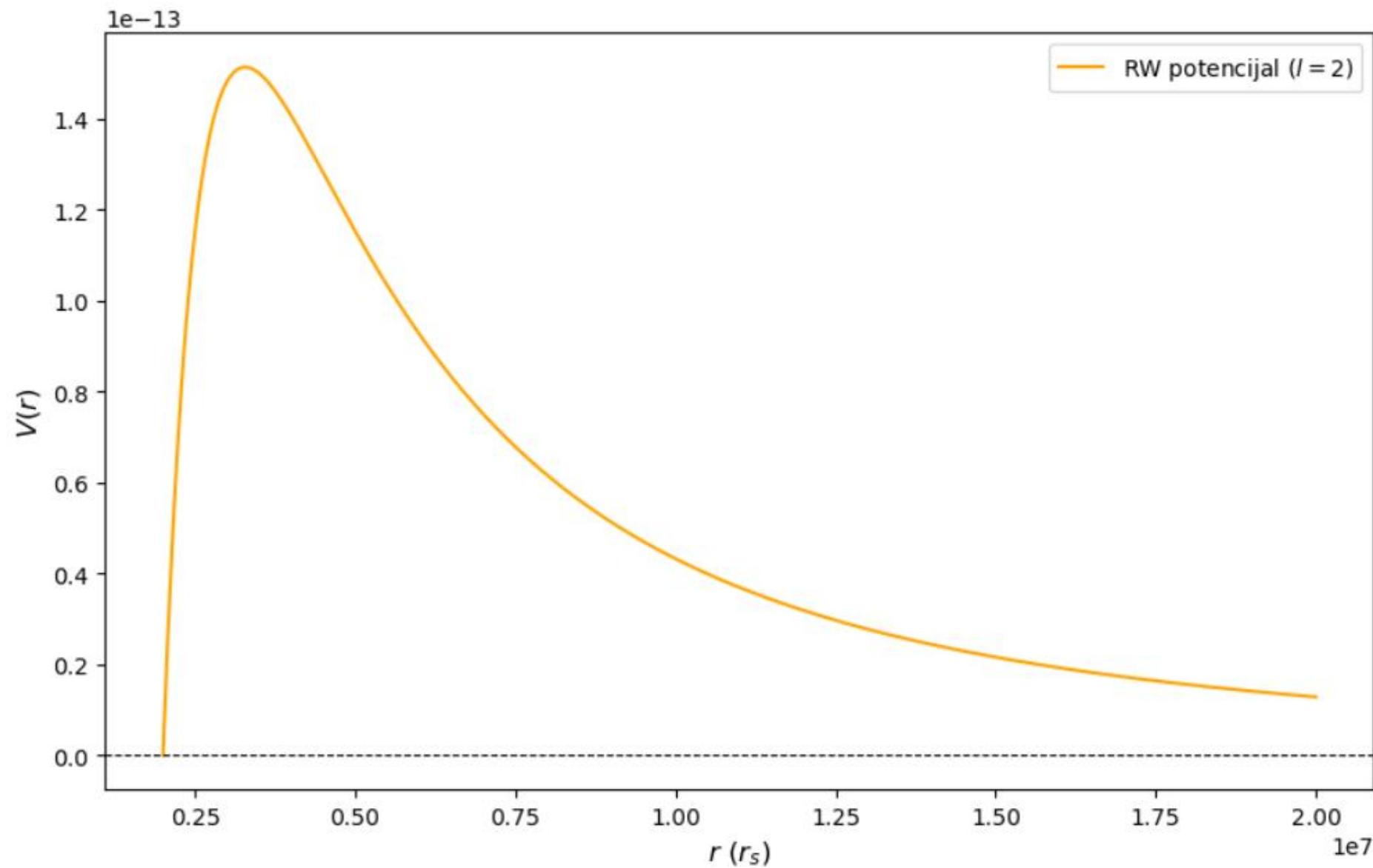
Uloga

Efektivni potencijali opisuju širenje gravitacijskih valova i koriste se za proučavanje stabilnosti i raspršenja gravitacijskog zračenja.

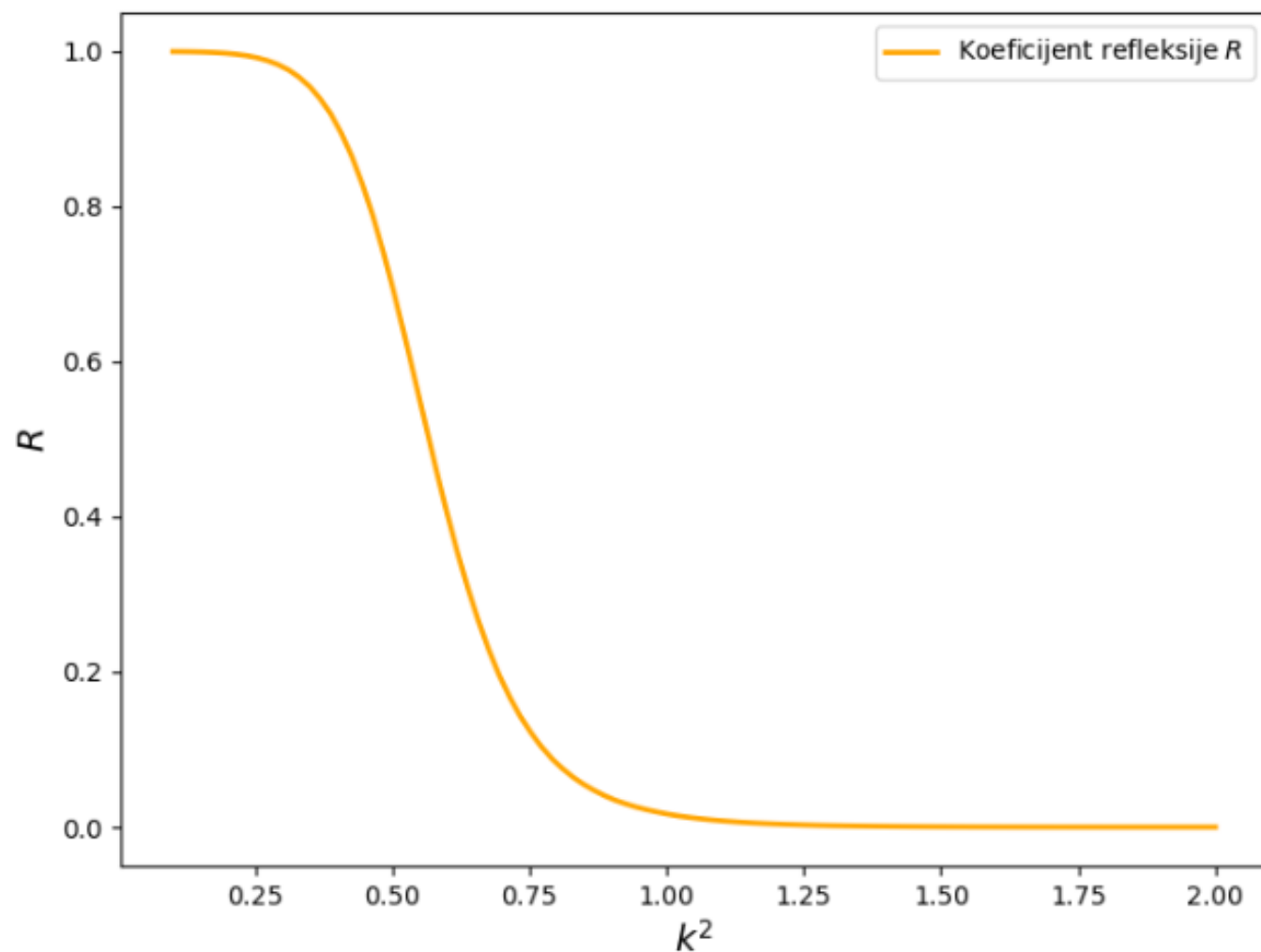
$$\frac{d^2\Psi}{dr^{*2}} + [\omega^2 - V_{\text{eff}}(r)] \Psi = 0$$

$$V_{\text{RW}}(r) = \left(1 - \frac{2m}{r}\right) \left[\frac{\ell(\ell+1)}{r^2} - \frac{6m}{r^3} \right]$$

$$V_Z(r) = \left(\frac{1 - 2m}{r} \right) \frac{2\lambda^2 (\lambda + 1) r^3 + 6\lambda^2 m r^2 + 18\lambda m^2 r + 18m^3}{r^3 (\lambda r + 3m)^2}$$



Slika 1: Efektivni potencijal V_{eff} za neparne gravitacijske valove u najnižem modu $l = 2$ za crnu rupu mase ($M = 10^6 M_\odot$)



Slika 2: Koeficijent refleksije u ovisnosti o kvadratu frekvencije k^2 , za najniži mod $l = 2$. Za niske frekvencije ($k \rightarrow 0$) dolazi do potpune refleksije ($|R| \rightarrow 1$), dok za više frekvencije vidimo brzi pad u R .



Kvazinormalni Modovi (QNM)

Oscilacije

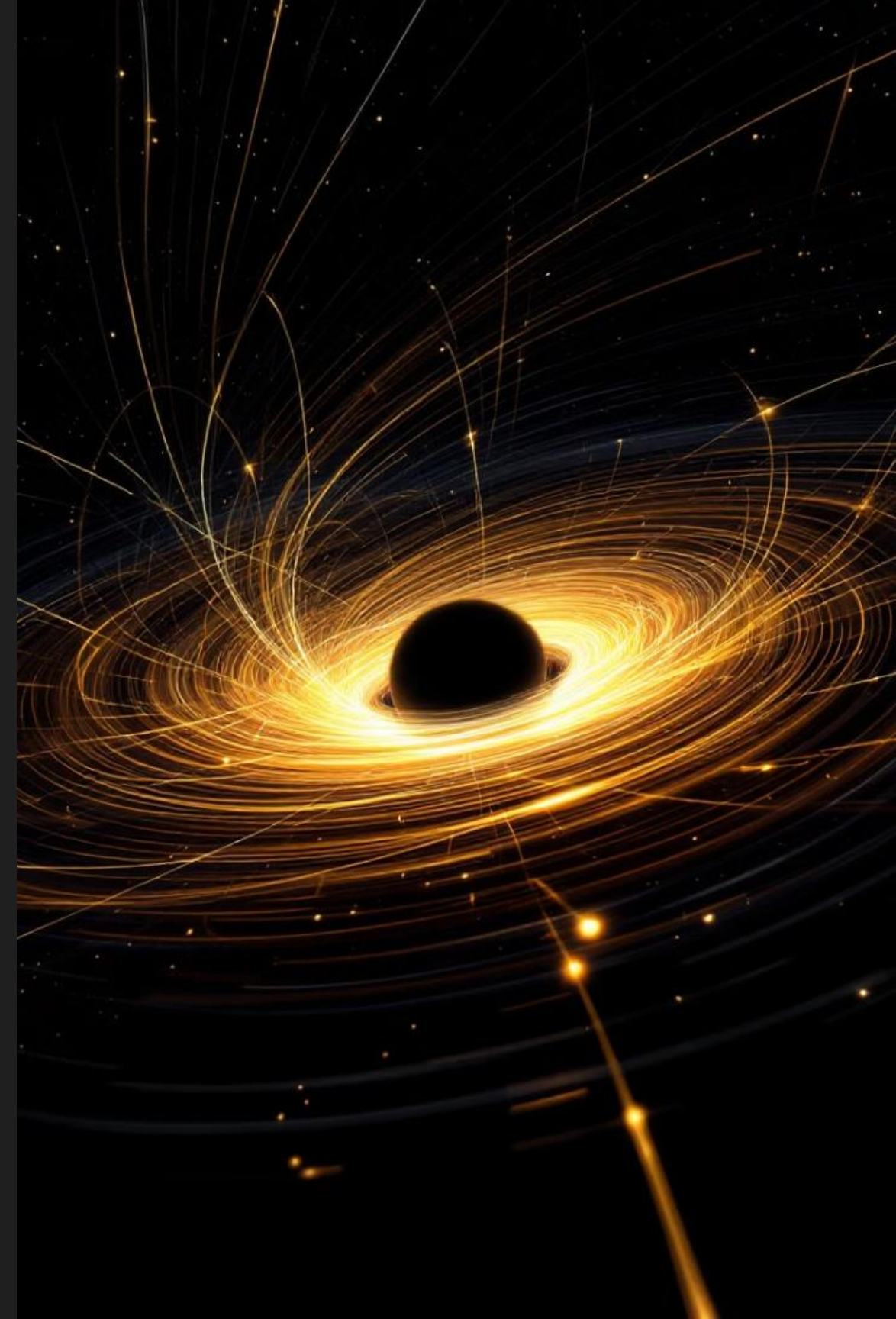
Realni dio kompleksne frekvencije QNM-a.

Prigušenje

Imaginarni dio kompleksne frekvencije QNM-a.

Gravitacijski Valovi

Kvadrupolni mod $L = 2$ predstavlja najniži mod koji generira gravitacijske valove. Viši modovi $L \geq 3$ također pridonose, ali s manjom relativnom snagom. Oni postaju značajni pri analizi detaljnije strukture valnih oblika i energije zračenja u specifičnim frekvencijskim opsezima.



Izvod kvazinormalnih modovi za Zerillijev potencijal

Substitucija:

$$\psi = \exp \left(i \int^x \phi dx \right)$$

Riccatijeva jednačba:

$$i \frac{d\phi}{dx} + \sigma^2 - \phi^2 - V(x) = 0$$

Kvazinormalni mod odgovara rješenju ove jednačbe koja zadovoljava rubne uvjete:

$$\phi \rightarrow -\sigma \quad \text{kad} \quad x \rightarrow +\infty$$

$$\phi \rightarrow +\sigma \quad \text{kad} \quad x \rightarrow -\infty.$$

Izvod kvazinormalnih modovi za Zerillijev potencijal

$$-2i\sigma + \int_{-\infty}^{+\infty} (\sigma^2 - \phi^2) dx = \int_{-\infty}^{+\infty} V(x) dx$$

$$\sigma = \sigma_1 + i\sigma_2 \quad \text{i} \quad \phi = \phi_1 + i\phi_2 \quad (\sigma_1 \geq 0)$$

$$\frac{d\phi_1}{dx} = -2\sigma_1\sigma_2 + 2\phi_1\phi_2$$

$$\frac{d\phi_2}{dx} = \sigma_1^2 - \sigma_2^2 - \phi_1^2 + \phi_2^2 - V$$

$$\phi_1 \rightarrow -\sigma_1 \quad \text{kad} \quad x \rightarrow +\infty, \quad \phi_1 \rightarrow +\sigma_1 \quad \text{kad} \quad x \rightarrow -\infty$$

$$\phi_2 \rightarrow -\sigma_2 \quad \text{kad} \quad x \rightarrow +\infty, \quad \phi_2 \rightarrow +\sigma_2 \quad \text{kad} \quad x \rightarrow -\infty$$

Izvod kvazinormalnih modovi za Zerillijev potencijal

$$\frac{d\phi_2}{dx} = -\sigma_2^2 + \phi_2^2 - V$$

$$\frac{d^2 Z}{dr^{*2}} + (\sigma^2 - V_Z) Z = 0$$

$$V_Z(r) = \left(1 - \frac{2m}{r}\right) \frac{2n^2(n+1)r^3 + 6n^2mr^2}{r^3(nr+3m)^2} + \frac{18nm^2r + 18m^3}{r^3(nr+3m)^2}.$$

$$n = \frac{1}{2}(l-1)(l+2).$$

Zerilli-jev potencijal očito ispunjava uvjet kratkog dosega te imamo:

$$2m \int_{-\infty}^{+\infty} V_Z dr^* = 2n + \frac{1}{2} = (l-1)(l+2) + \frac{1}{2}.$$

Izvod kvazinormalnih modovi za Zerillijev potencijal

$$Z_{\pm} \rightarrow e^{\pm i\omega r^*}$$

$$Z = e^{-i\sigma r^*} \sum_{j=0}^{\infty} \alpha_j r^{-j}, \quad (r^* \rightarrow +\infty)$$

$$Z = e^{+i\sigma r^*} \sum_{j=0}^{\infty} \beta_j (r - 2m)^j, \quad (r^* \rightarrow -\infty)$$

$$\phi = \frac{1}{Z} \frac{dZ}{dr^*}$$

$$M(\sigma) = \phi_{-}(r^*) - \phi_{+}(r^*)$$

Izvod kvazinormalnih modovi za Zerillijev potencijal

l	Chandrasekar	Leaver	3rd-order WKB
2	$0.74734 + 0.17792i$	$0.747343 - 0.177925i$	$0.746324 - 0.178348i$
	$0.69687 + 0.54938i$	$0.693422 - 0.54783i$	$0.692034 - 0.54983i$
		$0.602107 - 0.956554i$	$0.60587 - 0.942128i$
		$0.50301 - 1.410296i$	$0.494924 - 1.345796i$
3	$1.19889 + 0.18541i$	$1.198887 - 0.185406i$	$1.19853 - 0.185456i$
	$1.16402 + 0.56231i$	$1.165288 - 0.562596i$	$1.16471 - 0.56281i$
	$0.85257 + 0.74546i$	$1.10337 - 0.958186i$	$1.1064 - 0.953368i$
		$1.023924 - 1.380674i$	$1.031494 - 1.354858i$
4	$1.61835 + 0.18832i$	$1.61836 - 0.18832i$	$1.618196 - 0.188322i$
	$1.59313 + 0.56877i$	$1.59326 - 0.56886i$	$1.592999 - 0.568732i$
	$1.12019 + 0.84658i$	$1.54542 - 0.957981i$	$1.547272 - 0.957948i$
		$1.47968 - 1.36784i$	$1.486624 - 1.35661i$

Tablica 1: Karakteristične vrijednosti $2m\sigma$ za Zerillijev potencijal koristeći opisanu metodu, Leaverovu metodu i WKB metodu do 3. reda

Izospektralnost

$$Z \rightarrow \begin{cases} e^{+i\sigma r^*} + Ae^{-i\sigma r^*} & (r^* \rightarrow +\infty), \\ Be^{+i\sigma r^*} & (r^* \rightarrow -\infty), \end{cases}$$

$$X \rightarrow \begin{cases} e^{+i\sigma r^*} + \frac{\frac{2}{4}n(n+1)+2mi\sigma}{\frac{2}{3}n(n+1)-2mi\sigma} Ae^{-i\sigma r^*} & (r^* \rightarrow +\infty), \\ Be^{+i\sigma r^*} & (r^* \rightarrow -\infty) \end{cases}$$

Neki od novih teorijskih pristupa

$f(R)$ gravitacija, Einstein-Gauss-Bonnet teorija, nekomutativnost prostora...

Zaključak

