3. poglavlje, zadaci

(1.) a) $X = (x_1, x_2)$

$y = (y_1, y_2)$

- $d(x, y) = \underbrace{\frac{|x_1 - y_1|}{2}}_{\geq 0} + \underbrace{\frac{|x_2 - y_2|}{5}}_{\geq 0} \geq 0$

$\forall x, y \in \mathbb{R}^2$

- $d(x, y) = 0 \iff$

$$\frac{|x_1 - y_1|}{2} + \frac{|x_2 - y_2|}{5} = 0$$

$\iff$ (jer su oba pribrojnika nenegativna)

$$\frac{|x_1 - y_1|}{2} = 0 \ \& \ \frac{|x_2 - y_2|}{5} = 0$$

$\iff \quad x_1 - y_1 = 0 \ \& \ x_2 - y_2 = 0$

$\iff \quad x_1 = y_1 \ \& \ x_2 = y_2$

$\iff \quad x = y$

- $$d(y,x) = \frac{|y_1 - x_1|}{2} + \frac{|y_2 - x_2|}{5}$$

$$= \frac{|x_1 - y_1|}{2} + \frac{|x_2 - y_2|}{5}$$

$$= d(x,y) \qquad \forall x, y \in \mathbb{R}^2$$

- $z = (z_1, z_2) \in \mathbb{R}^2$

$$d(x,z) = \frac{|x_1 - z_1|}{2} + \frac{|x_2 - z_2|}{5}$$

$$= \frac{|(x_1 - y_1) + (y_1 - z_1)|}{2} + \frac{|(x_2 - y_2) + (y_2 - z_2)|}{5}$$

$$\leq \frac{|x_1 - y_1|}{2} + \frac{|y_1 - z_1|}{2} + \frac{|x_2 - y_2|}{5} + \frac{|y_2 - z_2|}{5}$$

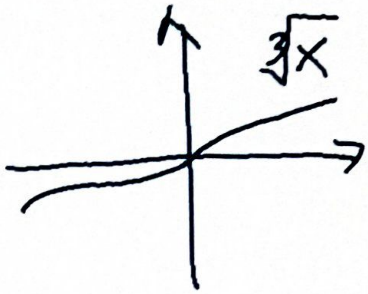
$$= \left(\frac{|x_1 - y_1|}{2} + \frac{|x_2 - y_2|}{5} \right) + \left(\frac{|y_1 - z_1|}{2} + \frac{|y_2 - z_2|}{5} \right)$$

$$= d(x,y) + d(y,z)$$

$$\Rightarrow d(x,z) \leq d(x,y) + d(y,z) \qquad \forall x, y, z \in \mathbb{R}^2$$

$\Rightarrow$ d je metrika na $\mathbb{R}^2$.

b) $$d(x,y) = \sqrt[3]{|y^3 - x^3|}$$



- $d(x,y) = \sqrt[3]{\underbrace{|y^3 - x^3|}_{\geq 0}} \geq 0$

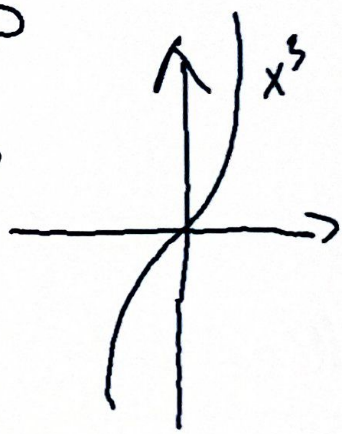
- $d(x,y) = 0 \iff \sqrt[3]{|y^3 - x^3|} = 0$

$$\iff |y^3 - x^3| = 0$$

$$\iff y^3 - x^3 = 0$$

$$\iff y^3 = x^3$$

$$\iff y = x$$



- $d(y,x) = \sqrt[3]{|x^3 - y^3|} = \sqrt[3]{|y^3 - x^3|} = d(x,y)$

- $d(x,z) = \sqrt[3]{|z^3 - x^3|} =$

$$= \sqrt[3]{|z^3 - y^3 + y^3 - x^3|}$$

$$|z^3-y^3+y^3-x^3| \le |z^3-y^3| + |y^3-x^3| \quad (*)$$

Kako je funkcija $x \longmapsto \sqrt[3]{x}$ rastuća funkcija, iz $(*)$ slijedi

$$\sqrt[3]{|z^3-y^3+y^3-x^3|} \le \sqrt[3]{|z^3-y^3| + |y^3-x^3|}$$

Općenito vrijedi $\sqrt[3]{a+b} \le \sqrt[3]{a} + \sqrt[3]{b}$ za $\underline{a, b \ge 0}$

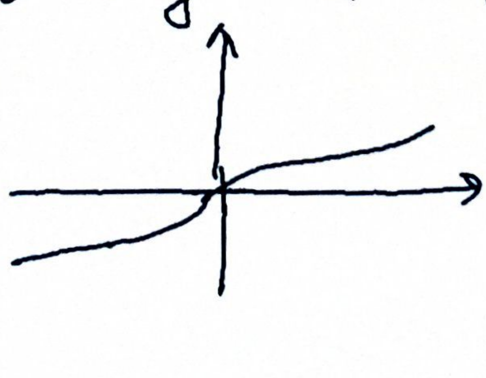
<u>dokaz:</u>

$$\sqrt[3]{a+b} \le \sqrt[3]{a} + \sqrt[3]{b} \quad /^3 \iff$$

$$\cancel{a}+\cancel{b} \le \cancel{a} + 3\sqrt[3]{a^2 b} + 3\sqrt[3]{ab^2} + \cancel{b} \iff$$

$$0 \le 3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right), \quad a$$

ovo je istina jer su $a, b \ge 0$



Stoga je

$$\sqrt[3]{|z^3-y^3+y^3-x^3|} \leq \sqrt[3]{\underbrace{|z^3-y^3|}_{\geq 0}+\underbrace{|y^3-x^3|}_{\geq 0}}$$

$$\leq \sqrt[3]{|z^3-y^3|} + \sqrt[3]{|y^3-x^3|}$$

$$= d(y,z)+d(x,y),$$

odnosno $d(x,z) \leq d(x,y)+d(y,z)$

$\forall x,y,z \in \mathbb{R}$

$\Longrightarrow d$ je metrika na $\mathbb{R}$

c) $d_1(x,y)=\sum_{i=1}^{n} |x_i-y_i|$

- $|x_i-y_i| \geq 0 \ \forall i \in \{1,\dots,n\}$

  $\Longrightarrow d_1(x,y) \geq 0$

- $d_1(x,y)=0 \iff |x_i-y_i|=0 \ \forall i \in \{1,\dots,n\}$

  $\iff x_i-y_i=0$

  $\iff x_i=y_i \ \forall i \in \{1,\dots,n\}$

  $\iff x=y$

- $d_1(y,x) = \sum_{i=1}^{n} |y_i - x_i| = \sum_{i=1}^{n} |x_i - y_i|$

$$= d_1(x,y) \qquad \forall\, x,y \in \mathbb{R}^n$$

- $d_1(x,z) = \sum_{i=1}^{n} |x_i - z_i|$

$$= \sum_{i=1}^{n} |x_i - y_i + y_i - z_i|$$

$$\leq \sum_{i=1}^{n} \left(|x_i - y_i| + |y_i - z_i| \right)$$

$$= \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|$$

$$= d_1(x,y) + d_1(y,z)$$

$$\Longrightarrow d_1(x,z) \leq d_1(x,y) + d_1(y,z) \qquad \forall\, x,y,z \in \mathbb{R}^n$$

$\Longrightarrow$ d_1 je metrika na $\mathbb{R}^n$

d) • $d(x,y) = \max\{\underbrace{\underbrace{2d_1(x,y)}_{\geq 0}}_{\geq 0}, \underbrace{d_2(x,y)}_{\geq 0}\} \geq 0$

$\forall x,y \in \mathbb{R}^n$

• $d(x,y) = 0 \iff$

$\max\{\underbrace{2d_1(x,y)}_{\geq 0}, \underbrace{d_2(x,y)}_{\geq 0}\} = 0$

$\iff 2d_1(x,y) = 0$ & $d_2(x,y) = 0$

$\iff x = y$ (jer su d_1 i d_2 metrike)

• $d(y,x) = \max\{2d_1(y,x), d_2(y,x)\}$

$= \max\{2d_1(x,y), d_2(x,y)\}$

$= d(x,y) \quad \forall x,y \in \mathbb{R}^n$

- $d(x,z) = \max\{2d_1(x,z), d_2(x,z)\}$

Ako je $\max\{2d_1(x,z), d_2(x,z)\} =$
$= 2d_1(x,z)$, onda je

$$
\begin{aligned}
d(x,z) &= 2d_1(x,z) \le 2(d_1(x,y) + d_1(y,z)) \\
&= 2d_1(x,y) + 2d_1(y,z) \\
&\le \max\{2d_1(x,y), d_2(x,y)\} \\
&+ \max\{2d_1(y,z), d_2(y,z)\} \\
&= d(x,y) + d(y,z)
\end{aligned}
$$

Ako je pak $\max\{2d_1(x,z), d_2(x,z)\}$
$= d_2(x,z)$, onda je

$$d(x,z) = d_2(x,z) \le d_2(x,y) + d_2(y,z)$$

$$\leq \max\{2d_1(x,y), d_2(x,y)\}$$

$$+ \max\{2d_1(y,z), d_2(y,z)\}$$

$$= d(x,y) + d(y,z)$$

Dakle, $d(x,z) \leq d(x,y) + d(y,z)$

$\forall x,y,z \in \mathbb{R}^n$

$\Rightarrow$ d je metrika na $\mathbb{R}^n$

(2.) Ako je $x \in A'$, to po definiciji znači da za svaki otvoreni skup $U \subseteq X$ t.d. $x \in U$ je $U \cap A \setminus \{x\} \neq \emptyset$.

$$A \subseteq B \Rightarrow A \setminus \{x\} \subseteq B \setminus \{x\}$$

$$\Rightarrow \underbrace{U \cap (A \setminus \{x\})}_{\neq \emptyset} \subseteq U \cap (B \setminus \{x\}) \Rightarrow U \cap (B \setminus \{x\}) \neq \emptyset$$

Dakle, $U \cap B \setminus \{x\} \neq \emptyset$ za svaki otvoreni skup U koji sadrži x.

$\Rightarrow \quad x \in B'$.

Dokazali smo $A' \subseteq B'$.

Dokažimo sada $\mathrm{Cl}A \subseteq \mathrm{Cl}B$. S predavanja znamo $\mathrm{Cl}A = A \cup A'$

$\mathrm{Cl}B = B \cup B'$

$$\underbrace{A \subseteq B \quad \Rightarrow \quad A' \subseteq B'}$$

$$\Rightarrow \quad A \cup A' \subseteq B \cup B'$$

$$\Rightarrow \quad \mathrm{Cl}A \subseteq \mathrm{Cl}B$$

③ a) $\boxed{\supseteq}$ $A \subseteq A \cup B$

$\underset{\text{prethodni zadatak}}{\Longrightarrow}$ $Cl\,A \subseteq Cl(A \cup B)$ (1)

$B \subseteq A \cup B$

$\underset{\text{prethodni zadatak}}{\Longrightarrow}$ $Cl\,B \subseteq Cl(A \cup B)$ (2)

(1) & (2) $\Rightarrow$ $\boxed{Cl\,A \cup Cl\,B \subseteq Cl(A \cup B)}$ (*)

$\boxed{\subseteq}$ $A \subseteq Cl\,A, \; B \subseteq Cl\,B$

$\Rightarrow A \cup B \subseteq \underbrace{Cl\,A \cup Cl\,B}_{\text{zatvoren skup koji sadrži } A\cup B}$

$\Rightarrow$ $\boxed{Cl(A \cup B) \subseteq Cl\,A \cup Cl\,B}$ (**)

Iz (*) & (**) $\Rightarrow$ $Cl(A \cup B) = Cl A \cup Cl B$

b) $A \cap B \subseteq A \underset{\text{prethodni zadatak}}{\Longrightarrow} Cl(A \cap B) \subseteq Cl A$ (3)

$A \cap B \subseteq B \Rightarrow Cl(A \cap B) \subseteq Cl B$ (4)

(3) & (4) $\Rightarrow$ $Cl(A \cap B) \subseteq Cl A \cap Cl B$

Obratna inkluzija ne vrijedi.

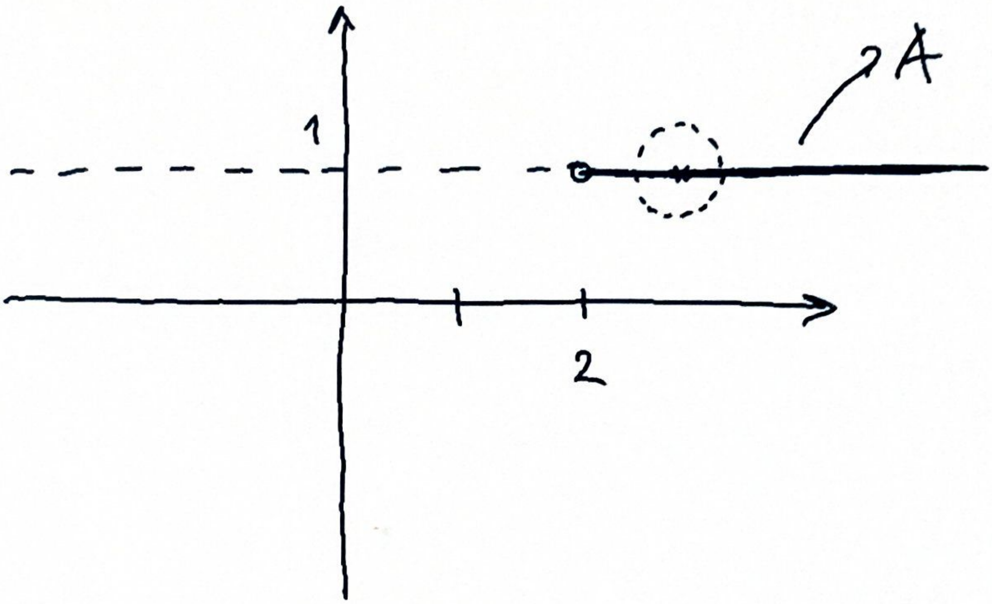
Stavimo npr. $A = \langle 0,1 \rangle$, $B = \langle 1,2 \rangle$

$Cl(A \cap B) = Cl(\emptyset) = \emptyset$

$Cl A \cap Cl B = [0,1] \cap [1,2] = \{1\}$

$Cl A \cap Cl B \nsubseteq Cl(A \cap B)$

(5.) a) $A = \{(x,1) : x > 2\}$ u $\mathbb{R}^2$



$\operatorname{Int} A = \emptyset$

dokaz: Ako je za neki $x_0 > 2$ $(x_0, 1) \in \operatorname{Int} A$, onda $\exists\, r > 0$ t.d. $K((x_0,1), r) \subseteq A$.

Međutim, u $K((x_0,1), r)$ sigurno postoje točke s ordinatom $\neq 1$, npr. $(x_0, 1+\frac{r}{2}) \in K((x_0,1), r)$ $\left(d((x_0, 1+\frac{r}{2}), (x_0,1)) = \frac{r}{2} < r \right)$, dok $(x_0, \underbrace{1+\frac{r}{2}}_{\neq 1}) \notin A$.

$$\mathrm{Cl}\,A = \{(x,1) : x \geq 2\}$$

dokaz: Jasno, $A \subseteq \mathrm{Cl}\,A$ (po def. zatvarača)

Tvrdimo $(2,1) \in \mathrm{Cl}\,A$.

Promatrajmo niz $\left(\left(\underbrace{2+\frac{1}{n}}_{>2},1\right)\right)_n$ u A

$$\lim_n \left(2+\frac{1}{n},1\right) = (2,1)$$

$$\Rightarrow (2,1) \in \mathrm{Cl}\,A$$

$$\Rightarrow \{(x,1) : x \geq 2\} \subseteq \mathrm{Cl}\,A \quad (1)$$

Da bismo dokazali obratnu inkluziju, dovoljno je dokazati da je $\{(x,1) : x \geq 2\}$ zatvoren skup.

$$\{(x,1) : x \geq 2\}^C = \{(x,y) : y \neq 1\} \cup \{(x,y) : x < 2\}$$

$$= (\mathbb{R} \times \langle -\infty, 1\rangle) \cup (\mathbb{R} \times \langle 1, +\infty\rangle) \cup (\langle -\infty, 2\rangle \times \mathbb{R})$$

$\Rightarrow \{(x,1) : x \geq 2\}^c$ je otvoren skup u $\mathbb{R}^2 \Rightarrow \{(x,1) : x \geq 2\}$ je zatvoren u $\mathbb{R}^2$. Kako taj skup sadrži A, onda je $\quad Cl A \subseteq \{(x,1) : x \geq 2\}$ (2)

Iz (1) & (2) $\Rightarrow Cl A = \{(x,1) : x \geq 2\}$

$$\partial A = Cl A \setminus \underbrace{Int A}_{\emptyset} = \{(x,1) : x \geq 2\}$$

c) $Int A = \{(x,y,z) \in \mathbb{R}^3 : x \in \langle -2,3 \rangle, y > 0\}$

<u>dokaz:</u>

$\{(x,y,z) \in \mathbb{R}^3 : x \in \langle -2,3 \rangle, y > 0\} =$

$= \langle -2,3 \rangle \times \langle 0,+\infty \rangle \times \mathbb{R}$ je otvoren u $\mathbb{R}^3$.

$\Rightarrow \{(x,y,z) \in \mathbb{R}^3 : x \in \langle -2,3 \rangle, y > 0\} \subseteq Int A$

Znamo da je $\operatorname{Int} A \subseteq A$ i

$$\{(x,y,z)\in\mathbb{R}^3 : x\in\langle -2,3\rangle, y>0\} \subseteq \operatorname{Int} A.$$

Da bismo dokazali

$$\{(x,y,z)\in\mathbb{R}^3 : x\in\langle -2,3\rangle, y>0\} = \operatorname{Int} A,$$

dovoljno je dokazati da točke iz

$$A \setminus \{(x,y,z)\in\mathbb{R}^3 : x\in\langle -2,3\rangle, y>0\}$$

nisu u $\operatorname{Int} A$, tj. da točke

$(3,y,z)$, $y>0$, $z\in\mathbb{R}$ nisu u $\operatorname{Int} A$.

Ako je $(3,y_0,z_0)\in\operatorname{Int} A$, onda $\exists\, r>0$ t.d. $K((3,y_0,z_0),r)\subseteq \operatorname{Int} A\subseteq A$.

Međutim, $(3+\frac{r}{2},y_0,z_0)\in K((3,y_0,z_0),r)$, ali $(3+\frac{r}{2},y_0,z_0)\notin A$ jer je $3+\frac{r}{2}>3$

$\Rightarrow\Leftarrow$

$$\Rightarrow \{(x,y,z)\in\mathbb{R}^3 : x\in\langle -2,3\rangle, y>0\} = \operatorname{Int} A$$

$$Cl\,A = \{(x,y,z) \in \mathbb{R}^3 : x \in [-2,3],\ y \geq 0\}$$

<u>dokaz:</u> Dokažimo da je

$$\{(x,y,z) \in \mathbb{R}^3 : x \in [-2,3],\ y \geq 0\}$$

zatvoren skup.

$$\{(x,y,z) \in \mathbb{R}^3 : x \in [-2,3],\ y \geq 0\}^c$$

$$= \{(x,y,z) \in \mathbb{R}^3 : x < -2\}$$

$$\cup \{(x,y,z) \in \mathbb{R}^3 : x > 3\}$$

$$\cup \{(x,y,z) \in \mathbb{R}^3 : y < 0\}$$

$$= (\langle -\infty, -2\rangle \times \mathbb{R} \times \mathbb{R}) \cup (\langle 3, +\infty\rangle \times \mathbb{R} \times \mathbb{R})$$

$$\cup (\mathbb{R} \times \langle -\infty, 0\rangle \times \mathbb{R}),$$

a to je unija tri otvorena skupa u $\mathbb{R}^3$ pa je to otvoren skup

$\Rightarrow \{(x,y,z) \in \mathbb{R}^3 : x \in [-2,3],\ y \geq 0\}$ je zatvoren skup koji sadrži A

$$\Rightarrow \mathrm{Cl}A \subseteq \{(x,y,z) \in \mathbb{R}^3 : x \in [-2,3], y \geq 0\}$$

Da bismo dokazali obratnu inkluziju, pošto je $A \subseteq \mathrm{Cl}A$, dovoljno je dokazati da su točke oblika

$$(-2, y_0, z_0), \quad y_0 \geq 0$$

i $(x_0, 0, z_0)$, $x_0 \in [-2, 3]$

u zatvaraču.

Niz $a_n = (\underbrace{-2+\frac{1}{n}}_{\in \langle -2,3]}, \underbrace{y_0+\frac{1}{n}}_{>0}, z_0)$ je niz

u A i $\lim\limits_n a_n = (-2, y_0, z_0)$

$\Rightarrow (-2, y_0, z_0) \in \mathrm{Cl}A$ za $y_0 \geq 0$

Niz $b_n = (x_0, \underbrace{\frac{1}{n}}_{>0}, z_0)$ je niz u A

za $x_0 \in \langle -2, 3]$ i $\lim\limits_n b_n = (x_0, 0, z_0)$

Ako je $x_0 = -2$, onda stavimo

$$b_n = (\underbrace{-2+\tfrac{1}{n}}_{\in\langle -2,3]}, \underbrace{\tfrac{1}{n}}_{>0}, z_0) \longrightarrow (-2, 0, z_0)$$

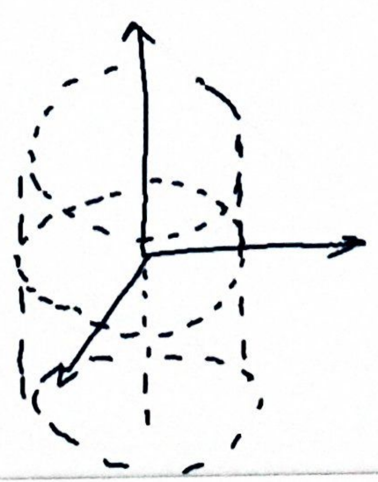
$\Rightarrow (x_0, 0, z_0) \in \mathrm{Cl}\,A$ za $x_0 \in [-2,3]$

$$\Rightarrow \mathrm{Cl}\,A = \{ (x,y,z) \in \mathbb{R}^3 : x \in [-2,3], y \geq 0 \}$$

$$\partial A = \mathrm{Cl}\,A \setminus \mathrm{Int}\,A = \{ (-2,y,z) : y \geq 0 \}$$
$$\cup \{ (3,y,z) : y \geq 0 \}$$
$$\cup \{ (x,0,z) : x \in [-2,3] \}$$

d) $\mathrm{Int}\,A = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 < 1, z \in \langle -1,1\rangle \}$

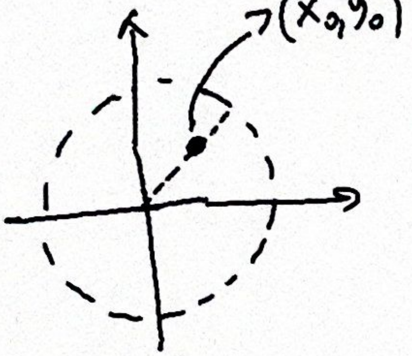
dokaz:



Dokažimo najprije da je

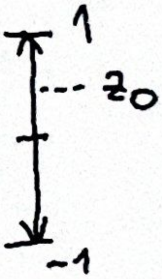
$$\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1,\ z \in \langle -1,1\rangle\} \text{ otvoren u } \mathbb{R}^3.$$

$(x_0,y_0,z_0) \in \mathbb{R}^3$ t.d. $x_0^2+y_0^2<1$, $z_0 \in \langle -1,1\rangle$



$$r_1 = 1-\sqrt{x_0^2+y_0^2}$$

$$r_2 = 1-|z_0|$$



$$r = \min\{r_1, r_2\}$$

$$K((x_0,y_0,z_0), r) \subseteq \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1, z\in\langle -1,1\rangle\}$$

$(x,y,z) \in K((x_0,y_0,z_0), r)$

$$\Rightarrow \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < r \leq r_1$$

$$\Rightarrow \sqrt{(x-x_0)^2+(y-y_0)^2} \leq \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < r_1$$

$$\Rightarrow d((x,y),(x_0,y_0)) < 1 - d((0,0),(x_0,y_0))$$

$$\Rightarrow d((x,y),(0,0)) \leq d((x,y),(x_0,y_0)) + d((x_0,y_0),(0,0))$$

$$< 1 \Rightarrow x^2+y^2<1 \qquad (1)$$

S druge strane

$$(x,y,z) \in K((x_0,y_0,z_0),r)$$

$$\Rightarrow \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < r \leq r_2$$

$$\Rightarrow |z-z_0| \leq \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < r_2$$

$$\Rightarrow |z-z_0| < 1-|z_0|$$

$$\Rightarrow |z| = |z-z_0+z_0| \leq |z-z_0|+|z_0| < 1 \quad (2)$$

$$(1) \,\&\, (2) \Rightarrow K((x_0,y_0,z_0),r)$$

$$\subseteq \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1,\ z \in \langle -1,1 \rangle\}$$

Dokazali smo

$\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1,\ z \in \langle -1,1 \rangle\}$ je otvoren skup (sadržan u A).

$$\Rightarrow \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1,\ z \in \langle -1,1 \rangle\} \subseteq \mathrm{Int} A$$

Da bismo dokazali jednakost, dovoljno je dokazati da preostale točke skupa A (one koje nisu u skupu

$\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1, z \in \langle -1,1 \rangle\}$) nisu

u IntA, tj. da točke oblika

$(x,y,-1)$, $x^2+y^2<1$ nisu u IntA.

Ako $(x_0,y_0,-1)$, $x_0^2+y_0^2<1$ i ako $(x_0,y_0,-1) \in \text{Int}A$, onda $\exists\, r>0$ t.d.

$K((x_0,y_0,-1),r) \subseteq \text{Int}A \subseteq A$

Ali npr. točka $\underbrace{(x_0,y_0,\underbrace{-1-\tfrac{r}{2}}_{<-1})}_{\notin A} \in K((x_0,y_0,-1),r)$

Dakle, $(x_0,y_0,-1) \notin \text{Int}A$

$\Rightarrow \text{Int}A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1, z \in \langle -1,1 \rangle\}$

$$\mathrm{Cl}\,A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 1,\ z \in [-1,1]\}$$

$\boxed{\supseteq}$ Za (x_0, y_0, z_0) t.d. $x_0^2+y_0^2 \le 1$, $z_0 \in [-1,1]$

je $a_n = \left(x_0\left(1-\frac{1}{n}\right), y_0\left(1-\frac{1}{n}\right), z_0\left(1-\frac{1}{n}\right)\right)$

niz u A jer je

$$\left(x_0\left(1-\frac{1}{n}\right)\right)^2 + \left(y_0\left(1-\frac{1}{n}\right)\right)^2$$

$$= \underbrace{(x_0^2+y_0^2)}_{\le 1} \underbrace{\frac{(n-1)^2}{n^2}}_{< 1} < 1$$

$$\left|z_0\left(1-\frac{1}{n}\right)\right| = \underbrace{|z_0|}_{\le 1} \cdot \underbrace{\frac{|n-1|}{|n|}}_{< 1} < 1$$

$$\lim_n a_n = (x_0, y_0, z_0)$$

$\Rightarrow$ Točke oblika (x_0, y_0, z_0) t.d.

$x_0^2+y_0^2 \le 1$, $z_0 \in [-1,1]$ su u $\mathrm{Cl}\,A$

$\boxed{\subseteq}$ Skup

$$\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1,\ z \in [-1,1]\}$$

sadrži A pa da bismo dokazali da sadrži ClA, dovoljno je dokazati da je skup

$$\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1,\ z \in [-1,1]\}$$

zatvoren.

$$f(x,y,z) = x^2+y^2$$

$$\left.\begin{array}{l} p_1(x,y,z) = x \\ p_2(x,y,z) = y \\ p_3(x,y,z) = z \end{array}\right\} \text{neprekidne}$$

$f = p_1^2 + p_2^2$ je neprekidna

$$\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 1,\ z\in[-1,1]\right\}$$

$$= \underbrace{\underbrace{\underbrace{f^{-1}(\underbrace{\langle-\infty,1]}_{\text{zatvoren}})}_{\text{zatvoren}} \cap \underbrace{p_3^{-1}(\underbrace{[-1,1]}_{\text{zatvoren}})}_{\text{zatvoren}}}_{\text{zatvoren}}}_{\text{i sadrži } A}$$

$$\Longrightarrow \operatorname{Cl} A \subseteq \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 1,\ z\in[-1,1]\right\}$$

$$\operatorname{Cl} A = \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 1,\ z\in[-1,1]\right\}$$

$$\partial A = \operatorname{Cl} A \setminus \operatorname{Int} A =$$

$$= \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1,\ z\in[-1,1]\right\}$$
$$\cup \left\{(x,y,1)\in\mathbb{R}^3 : x^2+y^2\le 1\right\} \cup \left\{(x,y,-1)\in\mathbb{R}^3 : x^2+y^2\le 1\right\}$$

6. $A = \{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, \ i = 1, \dots, n \}$

$(x_1, x_2, \dots, x_n) \in A^c \iff \exists \ i \in \{1, \dots, n\}$

t.d. $x_i < 0$

$$A^c = \bigcup_{i=1}^{n} \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_i < 0 \}$$

$A_i = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_i < 0 \}$ je otvoren $\forall i \in \{1, \dots, n\}$

$x_0 = (x_1^{(0)}, \dots, x_n^{(0)}) \in A_i \Rightarrow x_i^{(0)} < 0$

$K(x_0, -x_i^{(0)}) \subseteq A_i$

$x \in K(x_0, -x_i^{0}) \Rightarrow d(x, x_0) < -x_i^{(0)}$

$$\Rightarrow |x_i - x_i^{(0)}| \leq \sqrt{\sum_{j=1}^{n} (x_j - x_j^{(0)})^2} < -x_i^{(0)}$$

$\Rightarrow x_i - x_i^{(0)} < -x_i^{(0)} \Rightarrow x_i < 0$

$\Rightarrow x \in A_i$

$A^c = \bigcup A_i$ je otvoren $\Rightarrow A$ je zatvoren u $\mathbb{R}^n$.

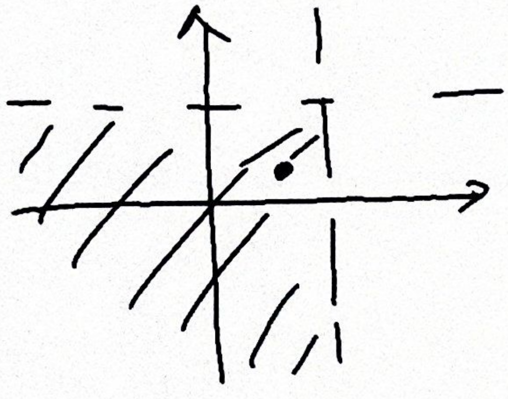
7. $A = \{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i < 1,\ i = 1, \dots, n \}$

$$= \langle -\infty, 1\rangle \times \langle -\infty, 1\rangle \times \dots \times \langle -\infty, 1\rangle$$

za $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}) \in A \Rightarrow x_i^{(0)} < 1$
$\forall i \in \{1, \dots, n\}$

$$K(x_0, r) \subseteq A$$

za $r = \min\{1 - x_1^{(0)}, 1 - x_2^{(0)}, \dots, 1 - x_n^{(0)}\}$



$$d(x, x_0) < r \Rightarrow \sqrt{\sum_{i=1}^{n} |x_i - x_i^{(0)}|^2} < r$$

$$\Rightarrow |x_j - x_j^{(0)}| \le \sqrt{\sum_{i=1}^{n} |x_i - x_i^{(0)}|^2} < r \le 1 - x_j^{(0)}$$

$$\Rightarrow |x_j - x_j^{(0)}| < 1 - x_j^{(0)}$$

$$\Rightarrow x_j - x_j^{(0)} < 1 - x_j^{(0)} \Rightarrow x_j < 1 \quad \forall j \in \{1, \dots, n\}$$

$$\Rightarrow x \in A$$