1. Write the PDE explicitly

Since

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2},$$

the equation becomes

$$-\frac{\hbar^2}{2m}(\psi_{xx} + \psi_{yy} + \psi_{zz}) = E\psi.$$

Multiply by $-2m/\hbar^2$:

$$\psi_{xx} + \psi_{yy} + \psi_{zz} + \frac{2mE}{\hbar^2}\psi = 0.$$

Let

$$k^2 = \frac{2mE}{\hbar^2}.$$

Then

$$\psi_{xx} + \psi_{yy} + \psi_{zz} + k^2\psi = 0.$$

2. Separate variables

Assume

$$\psi(x, y, z) = X(x)Y(y)Z(z).$$

Substitute:

$$X''YZ + XY''Z + XYZ'' + k^2XYZ = 0.$$

Divide by XYZ:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0.$$

Each term depends on only one variable, so each must equal a constant.

Write

$$\frac{X''}{X} = -k_x^2, \qquad \frac{Y''}{Y} = -k_y^2, \qquad \frac{Z''}{Z} = -k_z^2,$$

with

$$k_x^2 + k_y^2 + k_z^2 = k^2.$$

So we get three ODEs:

$$X'' + k_x^2X = 0,$$

$$Y'' + k_y^2Y = 0,$$

$$Z'' + k_z^2Z = 0.$$

3. Apply boundary conditions

x-direction

General solution:

$$X(x) = A\sin(k_x x) + B\cos(k_x x).$$

Boundary conditions:

$$X(0) = 0 \implies B = 0.$$

$$X(a) = 0 \implies \sin(k_x a) = 0.$$

Thus

$$k_x a = n_x \pi, \qquad n_x = 1, 2, 3, \ldots$$

so

$$k_x = \frac{n_x \pi}{a}.$$

Similarly,

$$k_y = \frac{n_y \pi}{b}, \qquad k_z = \frac{n_z \pi}{c},$$

with positive integers n_y, n_z.

4. Energy formula

Since

$$k^2 = k_x^2 + k_y^2 + k_z^2,$$

we have

$$\frac{2mE}{\hbar^2} = \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right).$$

Therefore

$$E = \frac{\pi^2 \hbar^2}{2m} \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right).$$

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5. Smallest possible energy

The smallest allowed integers are

$$n_x = n_y = n_z = 1.$$

Hence

$$\boxed{E_{\min} = \frac{\pi^2 \hbar^2}{2m} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)}.$$

This matches the answer in the image.