

2010/2011 (4.)  $T: \mathcal{P}_n \rightarrow \mathcal{P}_n \quad (Tp)(t) = (tp(t+1))''$

Ali je  $p \in \mathcal{P}_n \Rightarrow \text{st } p \leq n$ .

• Za  $p \neq 0$   $\text{st } p = k$ , gdje je  $k \in \{0, 1, \dots, n\}$

•  $\text{st } p(t+1) = k$

•  $\text{st } tp(t+1) = k+1$

•  $\text{st } (tp(t+1))'' = k-1$  za  $k \geq 1$  (\*)

$p \in \text{Ker } T \Leftrightarrow Tp = 0$

• Za  $k \geq 2$  je  $k-1 \geq 1$  pa  $p \notin \text{Ker } T$

• Za  $k \in \{0, 1\}$  je  $p(t)$  oblika  $p(t) = at + b$

$$\begin{aligned} (Tp)(t) &= (t(a(t+1)+b))'' = (at^2 + (a+b)t)'' \\ &= (2at + a+b)' = 2a \end{aligned}$$

$$\Rightarrow Tp = 0 \Leftrightarrow a = 0$$

Dakle,  $\text{Ker } T = \{p(t) = b \mid b \in \mathbb{R}\}$  (konstantni polinomi)

$$\Rightarrow \boxed{d(T) = 1}$$

$$\Rightarrow \boxed{r(T) = \dim \mathcal{P}_n - 1 = (n+1) - 1 = n}$$

$$\dim \text{Im } T = n = \dim \mathcal{P}_{n-1} \quad (1)$$

$$\begin{aligned}
 \text{Iz } (*) \Rightarrow \text{Za } p \in \mathcal{P}_n \quad k = \deg p \leq n &\Rightarrow \text{St } (tp(t+1))^k \leq n-1 \\
 &\Rightarrow \text{St } T_p \leq n-1 \\
 &\Rightarrow T_p \in \mathcal{P}_{n-1}
 \end{aligned}$$

Dakle,  $\text{Im } T \leq \mathcal{P}_{n-1}$  (2)

$$\text{Iz } (1) \& (2) \Rightarrow \boxed{\text{Im } T = \mathcal{P}_{n-1}}$$

Jedna baza za jezgru je polinom  $P_1(t) = 1 \quad \forall t \in \mathbb{R}$

Jedna baza za sliku  $\{1, t, t^2, \dots, t^{n-1}\}$ .