Alpöge's proof of the Riemann hypothesis for Dirichlet L-functions

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November 27, 2026

Abstract

On 20 November 2026, L. Alpöge announced a proof of the Riemann hypothesis for Dirichlet L-functions, found with the help of Claude and released as a formal development in the proof assistant Lean. On 25 November 2026, L. Sernau and colleagues at Anthropic, with Claude, released a shorter formal proof of the same theorem. We give an account of the proof in conventional mathematics, along the architecture of the shorter one.

The proof follows Stepanov's method for curves over finite fields, in which one object is shown to be both small and large: a lattice of automorphic forms on unitary groups over a cyclotomic extension of an imaginary quadratic field, built from a long chain of theta lifts. Its size is measured in two ways. From above, Poincaré series of forms that gain divisibility at the inert primes are short, and a second-moment argument shows that few of their translates already span almost all of the space. From below, the lattice cannot be too dense, since its covolume and index are governed by heights of ratios of Hecke L-values, which the Rallis inner product formula computes. When the two estimates are compared the main terms cancel, and what is left is a one-sided bound for sums over primes in an arithmetic progression, twisted by a quadratic character. A criterion of Landau type turns such bounds into the absence of zeros of Dirichlet L-functions to the right of the critical line.

This work was done with the help of Aleph, a mathematical AI research assistant. The Lean sources were translated into structured definitions and results in conventional language. Results already in the literature were identified and are cited rather than reproved, and we shortened the argument where a known theorem or a more direct route was available.

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1 Introduction

1.1 The theorem

Let $N \geq 1$ be an integer and χ a Dirichlet character modulo N, extended to $\mathbb{Z}$ by $\chi(n) = 0$ for $\gcd(n, N) > 1$, and let $L(s, \chi)$ be the meromorphic continuation to $\mathbb{C}$ of $\sum_{n \geq 1} \chi(n) n^{-s}$; it is entire unless χ is principal, and then its only pole is at $s = 1$. We follow the notation of [214]. The main result of this paper is that

$$L(s, \chi) \neq 0 \qquad \text{for every } s \in \mathbb{C} \text{ with } \operatorname{Re} s > \tfrac{1}{2},$$

except at the pole $s = 1$ when χ is principal (Theorem 2.2.1). By the functional equation, every zero of $L(s, \chi)$ with $0 < \operatorname{Re} s < 1$ then has real part $\frac{1}{2}$, for every Dirichlet character, primitive or not (Corollary 2.2.3). This is the generalised Riemann hypothesis for Dirichlet L-functions; for $N = 1$ it is the Riemann hypothesis [242]: every zero of $\zeta(s)$ other than the trivial zeros $s = -2n$, $n \geq 1$, has real part $\frac{1}{2}$ (Theorem 2.2.2). For the place of these conjectures in the theory of primes see [64, 146, 308]; for instance, by a theorem of von Koch [311], the Riemann hypothesis gives the prime number theorem with an error of order $\sqrt{x} \log x$. Since the Dedekind zeta function of an abelian number field is the product of the L-functions of its characters [317], the theorem is equivalent to the same zero-free half-plane for the Dedekind zeta functions of all abelian number fields.

The proof was found by L. Alpöge with the help of Claude and announced on 20 November 2026 as a formal development in the proof assistant Lean 4 [4], together with a four-page note describing its ideas [5]. On 25 November 2026, L. Sernau and colleagues at Anthropic, with Claude, released a shorter formal proof of the same theorems [259]. This paper gives an account of the proof in conventional mathematics, along the architecture of the shorter one. We have not verified either formal development, and no step of this paper appeals to one.

1.2 Outline of the proof

The idea. The proof follows Stepanov's method for the Riemann hypothesis for curves over finite fields [321, 288, 30] (see [289]), in which an auxiliary object is shown to be both small and large. Here the object is a lattice of automorphic forms on unitary groups over a CM field, built from a long chain of theta lifts. In Alpöge's description, multiplicity is hard to find over the integers, each lift of the chain mimics it, and for a von Mangoldt sum of length X the chain has length about $\sqrt{X}$ [5]. The architecture of [259] isolates this mechanism in three theorems about one arithmetic datum, which we call Theorems A, B and C, and one inequality that combines them, the main inequality. We keep its names: the datum, short forms (Theorem A), enough forms (Theorem B), integral forms are long (Theorem C). Where we relate a term to the description in [5] and no result states the relation, we say that it is our reading.

The reduction (Section 2). Let d be an odd prime, $q \geq 1$ an integer prime to d, and $\lambda_{d,q}(n) = \Lambda(n)\big(1 - (\frac{n}{d})\big)$ for $n \equiv 1 \pmod q$ and 0 otherwise, a non-negative weight. The prime remainder is

$$R_{d,q}(y) = \sum_{n \leq y} \lambda_{d,q}(n)\Big(\frac{1}{n} - \frac{1}{y}\Big) - \frac{\log y}{\varphi(q)} - c_{d,q} \qquad (y \geq 1),$$

with $c_{d,q}$ the constant for which $R_{d,q}(y) \to 0$ (Definition 2.1.1, Theorem 2.1.10). For $d \equiv 7 \pmod 8$ the Laplace transform of $v \mapsto R_{d,q}(e^v)$ is meromorphic, with at most simple poles in $\operatorname{Re} s > 0$, and its residue at $\rho \neq 1$ is

$$\frac{1}{\varphi(q)\,\rho(1-\rho)} \sum_{\xi \bmod q} \Big(\operatorname{ord}_\rho L(\cdot, \xi) - \operatorname{ord}_\rho L\big(\cdot, \xi(\tfrac{\cdot}{d})\big)\Big)$$

(Proposition 2.1.9): the zeros of $L(s,\xi)$ count positively, those of the twists by $(\frac{\cdot}{d})$ negatively. Only a sign is needed. A zero-density estimate for the twists of a character by Legendre symbols (Lemma 2.1.13), in the tradition of [213, 155, 126], gives in every suitable residue class a prime d whose twists have no zero in a prescribed box $\operatorname{Re} s \geq \beta$, $|\operatorname{Im} s| \leq H$ (Proposition 2.1.12). For such a prime the residues on $(\beta, 1)$ are non-negative, and a window form of Landau's theorem on Laplace integrals of functions of one sign [179, 115] gives the one-sided criterion (Proposition 2.1.3): if q_0 is a multiple of 8, $d_0 \equiv 7 \pmod 8$ is prime to q_0, $\beta > \frac{1}{2}$, and

$$R_{d,q_0}(x) \leq x^{\beta-1} \qquad \text{for all large } x$$

for every prime $d \equiv d_0 \pmod{q_0}$, then no $L(s,\psi)$ with ψ modulo q_0 vanishes at a point $\rho \neq 1$ with $\operatorname{Re}\rho > \beta$. Every prime of the class is needed because the prime is chosen after the zero. The arithmetic supplies the hypothesis as the one-sided bound (Proposition 2.4.3): if $18480 \mid q_0$ and $d \equiv q_0/2 - 1 \pmod{q_0}$ is prime, then $R_{d,q_0}(y) \leq y^{-1/2+\varepsilon}$ for every $\varepsilon > 0$ and all large y. Taking $q_0 = 18480N$ gives the theorem (Theorem 2.2.1). Sections 2.1 and 2.2 contain this deduction; the rest of Section 2 reduces the one-sided bound to the main inequality.

The datum and the main inequality. Fix such a pair $P = (q_0, d)$ and put $K = \mathbb{Q}(\sqrt{-d})$. The bound at a large integer x comes from the datum (admissible data over P, Definition 2.3.4). It consists of numbers, places and primes only: scales $x < x_1 < b$, each larger than a high power of the one before, and the long order $a = xb$; the number of lifts m, a power of 2 with $\sqrt{x} \leq m \leq 2\sqrt{x}$, a target rank r of size x^5 and the source rank $k = r + 1 - m$; the CM field $E = K(\zeta)$ with ζ a root of unity of order $q_1 = 3l_0m^{22}q_0$, whose f infinite places form a group Φ; a large sparse set $\Phi_b \subseteq \Phi$ of b-places; and primes p_0, p_1, N of prescribed sizes and congruences. Data exist at every large x (Proposition 2.5.1), by the Siegel–Walfisz theorem [314, 279], a construction after Behrend [18] and a pigeonhole argument in the class group of K. The total remainder rem of the datum measures a sum over the primes inert in K against its expected value (Definition 2.4.1, Definition 2.4.2), and it is a combination of the prime remainders R_{d,q_0} at x and at x_1 (Lemma 2.4.5); the terms at x_1 are negligible by the prime number theorem in progressions of Thorner and Zaman [307]. The main inequality (Theorem 5.9.1) says that for every datum

$$m \cdot \mathrm{rem} \leq f\, a\, (\log b)^5,$$

and it gives $R_{d,q_0}(x) \leq \varphi(q_0)(\log b)^6/m$ up to the negligible terms at x_1 (Lemma 2.4.4). Since $m \geq \sqrt{x}$, this is the one-sided bound: the exponent $\frac{1}{2}$ is that of the number of lifts. That m plays the part of the order of vanishing in Stepanov's method is our reading.

Forms attached to the datum (Section 3). The main inequality compares two bounds for the size of automorphic forms on unitary groups of hermitian spaces over E [180, 147, 249]. The source forms are algebraic modular forms [110] on a definite unitary group of rank k, and the target forms $\mathcal{T}$ live on a unitary group of rank r and signature $(k+\varrho, \varrho+1)$ at every infinite place, $m = 2\varrho + 2$, with a discrete series type at infinity and a principal level cut out by an idempotent of a finite group $\bar{G}$ built from Green's characters [106, 194] (Definition 3.2.31); its dimension is $d_{\mathcal{T}}$. A chain of m theta lifts [322, 134, 136], formed with the powers of a twist ϑ, a character of U(1), carries $\mathcal{T}$ to algebraic modular forms on a definite group, where integrality at a prime and the size of conjugates have a direct meaning. Pulled back, they give the target lattice: integrality at a prime identification $\iota \mid \ell$, and the index and the covolume of a family of target forms (Definition 3.1.3, Definition 3.1.4). In our reading, index and covolume are the finite and the archimedean parts of an arithmetic degree, as in the theory of slopes of Stuhler, Grayson and Bost [291, 105, 40]. Source and target are linked by the normalised Poincaré map $\mathfrak{p}$ (Definition 3.2.37): a map $\mathcal{P}$ from source to target forms, scaled by a constant c_{ss} and divided at each non-split prime ℓ by ℓ to the integral part of the gain $\mathrm{gain}(\ell)$ less a small error. The

gain is the contribution of ℓ to the prime sum, so that the main amount h_{main} of the datum satisfies an identity in which $h_{\mathrm{main}} - m \cdot \mathrm{rem}$ is a term without primes less $2\sum_p \mathrm{gain}(p)\log p$ (Definition 2.4.2). Theorems A and B are stated for arbitrary parameters $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$, with the properties they need as hypotheses. The map used in the proof represents Bessel periods [96] at a suitable point and is fixed in Section 5. Below, $\eta = f\,a\,(\log b)^4$ and $u_{\mathrm{err}} = f\,a\,(\log b)^3$.

Short forms (Section 4). Theorem A has two parts. The first (Theorem 4.3.1) says that the ramified source forms $\mathcal{S}^{\mathrm{ram}}$ are spanned by forms φ, integral at every prime identification, with

$$\|s\mathfrak{p}\,\varphi\|_\sigma^2 \le \exp\bigl(h_{\mathrm{main}} - m\cdot\mathrm{rem} + \eta\bigr)$$

for every $s \in \bar{G}$ and every automorphism σ of $\mathbb{C}$. The second (Theorem 4.4.1) bounds the index of a family $s\mathfrak{p}\,\varphi$ of such images, s in a subset $t \subseteq \bar{G}$, by $\#t \cdot \dim \mathcal{S}^{\mathrm{ram}} \cdot 2u_{\mathrm{err}}$. This is where multiplicity enters: by the identity above, the exponent is a term without primes less twice the gains. Both parts assume a seesaw identity, in the sense of Kudla [171], which writes the chain of lifts composed with $\mathcal{P}$ as c_{ss} times a composite lift; the index form also assumes the divisibility of the composite lifts by $\ell^{\mathrm{gain}(\ell)-\mathrm{err}_\ell}$. Section 4 proves the first for every map represented by Bessel periods at one point (Proposition 4.2.25) and the second for a family of errors that it constructs (Proposition 4.2.54); it opens with the Weil representation (Section 4.1) and the seesaw identity and its divisibility (Section 4.2), whose notions the hypotheses use. In our reading the gain is the order of vanishing: in [5], a derivative of order a of a theta function on an elliptic curve with complex multiplication by K is divisible at an inert prime p by p to the power $a/p + a/p^3 + \dots$ up to a small error.

Enough forms (Section 5). Theorem B (Theorem 5.1.2) assumes that $\mathcal{P}$, scaled and followed by the idempotent e_0, is nearly an isometry and that the second moment over $\bar{G}$ of the Gram operators of its translates is at most $(1 + 9b^{-4})$ times an explicit main term. It says that $d_{\mathcal{T}} \ge 1$ and that a subset $t \subseteq \bar{G}$ satisfies

$$\#t \cdot \dim \mathcal{S}^{\mathrm{ram}} \le 5\,d_{\mathcal{T}} \log(1 + 2b^2) \qquad \text{and} \qquad \dim \sum_{s\in t} s\mathfrak{p}(\mathcal{S}^{\mathrm{ram}}) \ge \Bigl(1 - \frac{4}{b^2}\Bigr) d_{\mathcal{T}} :$$

few translates of the short forms span almost all target forms. Given the two moments, the proof is a greedy choice of translates (Lemma 5.1.3, Proposition 5.1.4). Most of Section 5 proves the second moment for combinations of class Poincaré series (Proposition 5.6.48), bounding the main term with a sieve and the remaining terms with local bounds and the unfolding of Poincaré series near the Bessel subgroup, and bounds $d_{\mathcal{T}}$ by a mass (Proposition 5.8.54) through an equality of traces on unitary groups of the same rank and different signatures (Proposition 5.8.4), in the setting of Labesse [177]; see [235, 97] for masses and volumes. The section ends with the proof of the main inequality (Section 5.9).

Integral forms are long (Section 6). Theorem C (Theorem 6.1.4) is the lower bound: if $y_1, \dots, y_n$ are linearly independent target forms with $n \ge (1 - 4/b^2)\,d_{\mathcal{T}}$, of index at most δ and covolume at most c, then

$$d_{\mathcal{T}}\bigl(h_{\mathrm{main}} - 3\eta\bigr) \le c + 2\delta.$$

Its frame is the product formula [31]: a non-zero algebraic number is not small at all places (Theorem 3.1.5). For an almost full family of target forms it bounds $c + 2\delta$ below by minus the height of the ratios $Z(1)/Z(\vartheta_\iota)$ of the determinants $Z(\vartheta) = \det(\Theta_\vartheta^*\Theta_\vartheta)$ of the chain, with a twist ϑ_ι chosen separately at each prime, up to the cost of completing the family by short auxiliary forms (Lemma 6.4.5). The ratio for the chain is a product over the m lifts (Proposition 6.5.1), and the ratio for one lift is governed by critical values of Hecke L-functions [128], which are squared norms of theta lifts by the Rallis inner product formula [237, 238, 187] in the doubling

form [231] and the Siegel–Weil formula [280, 323, 139]. For a regular family of twists the cost of the m lifts adds up to $d_{\mathcal{T}}(\eta - h_{\mathrm{main}})$ (Theorem 6.6.6), and this is the source of the main amount in Theorem C. The bound for one lift uses Theorem B and the equality of multiplicities of Section 5; no statement used for Theorem B rests on Theorem C. We read the product formula as the count of zeros against poles, the counterpart of the slope inequality of Bost [41] listed among the tools in [5].

Regular families (Section 7). Regular families of twists exist for every datum (Theorem 7.3.1). The condition is on denominators of projections to the space of lifts, that is, on congruences between lifts and forms that are not lifts. Following Ribet [241], Mazur and Wiles [197, 327] and, for unitary groups, Bellaïche and Chenevier [20, 19], such congruences produce Galois cohomology classes with coefficients in Tate twists, and the denominators are bounded by bounding these classes (Theorem 7.6.3, Theorem 7.6.6). There are three inputs: the defect of the crystalline local conditions of Bloch and Kato [28] at the places above ℓ, measured with Breuil–Kisin modules [85, 160, 25] (Theorem 7.7.12); a bound for the odd classes at all primes at once by one explicit integer (Theorem 7.2.8), in the tradition of Stickelberger's theorem and Euler systems [317, 244]; and triangles of auxiliary primes in place of a bound for the even classes (Theorem 7.6.15). The note [5] asks here for "a significant weakening of Vandiver"; that the bound for the odd classes plays this part is our reading.

The comparison. The main inequality follows from the three theorems in a few lines, and the main amount cancels (Theorem 5.9.1). Choose t by Theorem B and, by the first part of Theorem A, a basis $y_1, \dots, y_n$ of $\sum_{s \in t} s\mathfrak{p}(\mathcal{S}^{\mathrm{ram}})$ among the images of short integral forms, so that $(1 - 4/b^2)\, d_{\mathcal{T}} \le n \le d_{\mathcal{T}}$. By Hadamard's inequality [113, 133] its covolume is at most $n(h_{\mathrm{main}} - m \cdot \mathrm{rem} + \eta)$, and by the second part of Theorem A its index is at most $10\, d_{\mathcal{T}} \log(1 + 2b^2)\, u_{\mathrm{err}}$. Theorem C gives

$$d_{\mathcal{T}}(h_{\mathrm{main}} - 3\eta) \le n(h_{\mathrm{main}} - m \cdot \mathrm{rem} + \eta) + 20\, d_{\mathcal{T}} \log(1 + 2b^2)\, u_{\mathrm{err}}.$$

Since $h_{\mathrm{main}} > 0$ (Lemma 2.4.8) and $n \le d_{\mathcal{T}}$, the terms $d_{\mathcal{T}} h_{\mathrm{main}}$ cancel, and $n \ge d_{\mathcal{T}}/2$ gives the main inequality. The cancelled quantities are large, $h_{\mathrm{main}} \ge x^4 f\, a$ against $\eta \le x^{1/25} f\, a$, so the three theorems allow errors smaller than the main amount by a factor x^{-3}.

How the sections depend on each other. Sections 4, 5 and 6 can each be read after Section 3 with its theorem as the goal; of the later sections, Section 2 uses only the main inequality. Below the three theorems the links are few: the seesaw identity of Section 4 uses an unfolding of class integrals from Section 6 (Lemma 6.4.74), Section 6 uses Theorem B and the equality of multiplicities from Section 5, Theorem C uses the regular families of Section 7, and Section 7 uses the ramification and the sizes of the datum and the integral theta jets of Section 4 and the norms of lifts of Section 6. Every definition precedes its first use. A few are stated for an arbitrary value of a parameter, which is specified later, right after the result that provides it; this is how Theorems A and B can be proved for parameters that are fixed only in Section 5.

1.3 How this account was obtained

This account is self-contained down to published results. These are quoted with their sources in the Appendix, arranged by field. Where a published theorem is needed in a form that is not in print, the step from the printed form is proved next to its first use; examples are the twisted spherical blocks behind the Galois representations of depth-zero types (Proposition 7.1.12) and the graph terms of the doubled theta series behind the Rallis inner product formula (Lemma 6.5.14). The proofs written here have not been verified independently.

This work was done with the help of Aleph, a mathematical AI research assistant that runs on an ensemble of Claude, ChatGPT, Gemini and bespoke models. Claude Code and Codex

agents translated the Lean code of the formal developments [4, 259] into the Aleph database, as structured definitions and results in conventional language, each with the statements it rests on. At this step the results that are already in the literature were identified, and they are quoted with citations to their sources rather than reproved; the paper quotes 124 results. We also shortened the argument where a known theorem or a more direct route was available. For example, the prime d of the criterion is found by a zero-density estimate proved directly from a mean value of theta series twisted by Jacobi symbols (Lemma 2.1.13), the decay of the prime remainders is deduced from the prime number theorem in progressions of Thorner and Zaman [307] rather than from a Tauberian argument (Theorem 2.1.10), and the primes of the datum are found by the Siegel–Walfisz theorem (Theorem A.1.9).

Several other accounts of the argument appeared within days of the first formal development. Adriazola [1] wrote the argument as one continuous proof in conventional mathematics, in two parts, with its open steps stated. Zhang [333] wrote a preliminary account of the proof and its principal ingredients, with an emphasis on their sources and history. Rubinstein [245, 246] maintains a guide to the proof and an exposition of the argument specialised to the Riemann zeta function. Naskręcki [222] built a map of the formal proof and a record of checks, with an atlas of the route. Epstein [75] wrote a human-readable blueprint of the formal development, Brox [45] leads a programme to distil the formal proof into shorter proofs, and Jaswal [152] wrote an account of the Riemann hypothesis for Dirichlet L-functions after Alpöge. We used the map of [222] as a reading aid for [259]; no statement or proof of this paper rests on it.

1.4 Conventions

Throughout, $P = (q_0, d)$ is a parameter pair and D a datum over P, that is, admissible data in the sense of Definition 2.3.4; x, x_1 and b are its scales, $a = xb$ is the long order, m is the number of lifts and f is the number of infinite places of E. These letters keep their meaning throughout, and log is the natural logarithm. Theorems A, B and C and the main inequality hold for every datum, Theorems A and B for all parameters with the properties they assume, and their constants are absolute; the small parameter ε enters only through the existence of data and the proof of the one-sided bound. Theorem A is the pair Theorem 4.3.1, Theorem 4.4.1, Theorem B is Theorem 5.1.2 and Theorem C is Theorem 6.1.4. Item labels such as (AD1)–(AD6) refer to the items of the definition in which they occur. The names datum, short forms, enough forms, integral forms are long and main inequality are used in the Introduction, in headings and between results; the statements use the terms of the definitions.

Several letters have more than one meaning, always separated by context. The letter N is the modulus of the character in the main theorem and its proof, the modulus of the class in the choice of the prime, the level of the cyclotomic field in the bound for the odd classes, and the level prime of the datum everywhere else. The letter t is the integer $5r$ of the datum, a set of translates in Theorems A and B, and the number of Tate twists in the bound for the odd classes. The letter T is the scale of the zero count in the choice of the prime and, in the main term of Theorem B, the finite set of places of the sieve. The letter s is a complex variable in Sections 2.1 and 2.2, the exponent with $2^s = m^{22}$ in the definition of the datum, and an element of $\bar{G}$ in the translates $s\mathfrak{p}$. A twist is a character of U(1) as in Definition 3.1.2; the quadratic twist of a Dirichlet character in the criterion is unrelated.

The explicit constants, such as the factor 18480 of the modulus, the exponents 10^6 and 10^{12} in the conditions on the scales, the exponent 22 that ties the field to the number of lifts and the powers of $\log b$ in the errors, are not optimised, and we do not know which of them are forced by the method.

2 The reduction to the main inequality

This section reduces Theorem 2.2.1 to the main inequality Theorem 5.9.1, following [259]. The first two subsections are analytic and need only a sign: they define the prime remainder, choose the prime d, prove the one-sided criterion, and deduce the theorem and its two consequences from the one-sided bound. The last three are arithmetic: they define the datum and its sizes, define the prime sums and the main amount and prove the one-sided bound from the main inequality, and prove that data exist at every large integer. All proofs rest on quoted results of analytic and algebraic number theory, with one exception: Lemma 2.4.4 uses the main inequality, which is proved at the end of Section 5. The definitions of the datum and of the prime sums are used in all later sections, and the two results that close Section 2.4, Lemma 2.4.8 and Lemma 2.4.9, serve the main inequality rather than the one-sided bound.

2.1 Prime remainders and the one-sided criterion

This section proves the one-sided criterion Proposition 2.1.3. The prime is chosen after the zero, so that no twist $L(s, \xi(\frac{\cdot}{d}))$ vanishes in the box $\operatorname{Re} s \geq \beta$, $|\operatorname{Im} s| \leq |\operatorname{Im} \rho|$; the residues of the Laplace transform of the prime remainder are then non-negative on the real segment $(\beta, 1)$, and a window form of Landau's theorem excludes every pole with real part above β. The rest of the section proves the three inputs.

The window form of Landau's theorem (Proposition 2.1.4) replaces the sign of the function in the classical theorem [179, 325, 115, 214] by the sign of the residues at the real poles, and it asks for the upper bound only on windows $[U, \lambda U]$ of the variable. Its proof compares single Taylor coefficients on a vertical line: an isolated nearest singularity forces large coefficients, and the bound on a window caps them. The four lemmas behind it are elementary.

The pole data (Proposition 2.1.9) follow from the orthogonality of characters. Holomorphy at $s = 1$ needs the prime number theorem for the prime remainders (Theorem 2.1.10), which we derive from the prime number theorem in progressions of Thorner and Zaman [307] for the fixed modulus qd.

The choice of the prime (Proposition 2.1.12) is a count: few primes d have a twist with a zero in the box (Lemma 2.1.13), while the prescribed class contains many primes. The zero-density estimate is proved with theta series, as in Riemann's and Hecke's proofs of the functional equation [242, 128, 64]. A zero forces a large tail integral of a theta series (Lemma 2.1.14), and the mean of these integrals over the twists by Jacobi symbols is small by the Pólya–Vinogradov inequality (Lemma 2.1.16). Mean values over quadratic characters are the subject of [155, 126], and zero-density estimates from mean values go back to [213]; the power saving for one box needed here follows from the elementary mean value.

Notation for Dirichlet characters and their L-functions is that of [214]. For an integer $N \geq 1$, a **Dirichlet character modulo** N is a character χ of $(\mathbb{Z}/N\mathbb{Z})^{\times}$, extended to $\mathbb{Z}$ by $\chi(n) = 0$ for $\gcd(n, N) > 1$; the **principal character** modulo N is denoted $\chi_{0,N}$, and χ is **primitive** if its conductor equals N. The complex variable is s. The **Dirichlet L-function** of χ is the meromorphic continuation to $\mathbb{C}$ of

$$L(s, \chi) = \sum_{n \geq 1} \chi(n)\, n^{-s} = \prod_{p} (1 - \chi(p)\, p^{-s})^{-1}, \qquad \operatorname{Re} s > 1,$$

where χ is taken as a character modulo N, so that the Euler factors at the primes $p \mid N$ are equal to 1 also when χ is imprimitive. It is entire unless $\chi = \chi_{0,N}$, in which case its only pole is a simple pole at $s = 1$. The Riemann zeta function is $\zeta(s) = L(s, \chi_{0,1})$; its **trivial zeros** are the points $s = -2n$, $n \geq 1$. The von Mangoldt function is $\Lambda(n)$.

Definition 2.1.1. Let d be a prime and $q \geq 1$ an integer. The **weight** $\lambda_{d,q}\colon \mathbb{Z}_{\geq 1} \to [0,\infty)$ is

$$\lambda_{d,q}(n) = \begin{cases} \Lambda(n)(1 - (\frac{n}{d})) & \text{if } n \equiv 1 \pmod q, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

so that $0 \leq \lambda_{d,q}(n) \leq 2\Lambda(n)$. The **count error** is $\mathcal{E}_{d,q}(t) = \sum_{n \leq t} \lambda_{d,q}(n) - t/\varphi(q)$ for real $t \geq 0$. The **remainder constant** is

$$c_{d,q} = \int_1^\infty \frac{\mathcal{E}_{d,q}(t)}{t^2}\, dt$$

if this improper integral converges, and $c_{d,q} = 0$ otherwise. The **prime remainder** is the function of real $y \geq 1$

$$R_{d,q}(y) = \sum_{n \leq y} \lambda_{d,q}(n)\Big(\frac{1}{n} - \frac{1}{y}\Big) - \frac{\log y}{\varphi(q)} - c_{d,q} = \int_1^y \frac{\mathcal{E}_{d,q}(t)}{t^2}\, dt - c_{d,q}. \tag{2.2}$$

When d is fixed we write λ_q, $\mathcal{E}_q$, c_q and R_q.

Definition 2.1.2. Let $f\colon [0,\infty) \to \mathbb{R}$ be a measurable function, $\widehat{f}\colon \mathbb{C} \to \mathbb{C}$ a function and $\mathcal{Z} \subset \mathbb{C}$ a set. The triple $(f, \widehat{f}, \mathcal{Z})$ is **Laplace pole data** if:

(LP1) there is $C_0 \geq 0$ with $|f(u)| \leq C_0 e^{2u}$ for all $u \geq 0$;

(LP2) $\widehat{f}(s) = \int_0^\infty f(u) e^{-su}\, du$ for every s with $\operatorname{Re} s > 2$;

(LP3) every $\rho \in \mathcal{Z}$ has $\operatorname{Re}\rho > 0$, and $\widehat{f}$ is holomorphic on $\{s : \operatorname{Re} s > 0\} \setminus \mathcal{Z}$;

(LP4) $\mathcal{Z}$ has no accumulation point in $\{s : \operatorname{Re} s > 0\}$;

(LP5) every $\rho \in \mathcal{Z}$ is a simple pole of $\widehat{f}$ with nonzero residue c_ρ, that is, $\widehat{f}(s) - c_\rho/(s-\rho)$ agrees on a punctured neighbourhood of ρ with a function holomorphic at ρ; and if ρ is real, then c_ρ is real and $c_\rho > 0$.

The values of $\widehat{f}$ at the points of $\mathcal{Z}$ and in $\operatorname{Re} s \leq 0$ play no role.

Let $\theta \in \mathbb{R}$ and $\lambda > 1$. A measurable function $f\colon [0,\infty) \to \mathbb{R}$ has **integrated window bounds of exponent θ and ratio λ** if for every $\eta > 0$ and every $U_0 \geq 0$ there are $U \geq U_0$ and a function $R\colon [U, \lambda U] \to [0,\infty)$ such that $u \mapsto R(u)e^{-\theta u}$ is integrable on $[U, \lambda U]$, $f(u) \leq R(u)$ for all $u \in [U, \lambda U]$, and

$$\int_U^{\lambda U} R(u)\, e^{-\theta u}\, du \leq e^{\eta U}.$$

Proposition 2.1.3. *Let $q_0 \geq 1$ be a multiple of 8, let $d_0 \equiv 7 \pmod 8$ be an integer prime to q_0, and let $\beta > \frac{1}{2}$. Suppose that for every prime $d \equiv d_0 \pmod{q_0}$ there is x_d with*

$$R_{d,q_0}(x) \leq x^{\beta - 1} \qquad \text{for all real } x \geq x_d.$$

Then $L(\rho, \psi) \neq 0$ for every Dirichlet character ψ modulo q_0 and every $\rho \neq 1$ with $\operatorname{Re}\rho > \beta$.

Proof. Fix ψ and ρ as in the statement and put $H = |\operatorname{Im}\rho|$.

Choice of d. The class a of d_0 modulo q_0 is a unit, and since $8 \mid q_0$ every prime $d \equiv d_0 \pmod{q_0}$ satisfies $d \equiv d_0 \equiv 7 \pmod 8$. By Proposition 2.1.12 with $N = q_0$, β and H, there is a prime $d \equiv d_0 \pmod{q_0}$ with

$$L\big(s, \xi(\tfrac{\cdot}{d})\big) \neq 0 \qquad \text{for all } \xi \bmod q_0 \text{ and all } s \text{ with } \operatorname{Re} s \geq \beta,\ |\operatorname{Im} s| \leq H. \tag{2.3}$$

Then $\gcd(d, q_0) = \gcd(d_0, q_0) = 1$, so Proposition 2.1.9 applies to d and $q = q_0$. Let $F = F_{q_0}$ be the meromorphic function given there and $r = r_{q_0}$ its residues (2.10). For $s \neq 1$ put $n(s) = \sum_{\xi \bmod q_0} \operatorname{ord}_s L(\cdot, \xi) \geq 0$. By (2.3), every twisted order in (2.10) vanishes in the box, so

$$r(s) = \frac{n(s)}{\varphi(q_0)\, s(1-s)} \qquad \text{for } s \neq 1,\ \operatorname{Re} s > \beta,\ |\operatorname{Im} s| \leq H. \tag{2.4}$$

Sign at real points. Let $x > \beta$ be real, $x \neq 1$. By (2.4), $r(x)$ is real. If $x < 1$ then $x(1-x) > 0$ and $r(x) \geq 0$. If $x > 1$ then $L(x, \xi) \neq 0$ for every ξ by the Euler product, so $n(x) = 0$ and $r(x) = 0$. At $x = 1$ the function F is holomorphic. So F has no pole of negative or non-real residue at a real point $x > \beta$.

Shifted pole data. Let $f(u) = R_{d,q_0}(e^u)\, e^{(1-\beta)u}$ for $u \geq 0$; it is measurable by Proposition 2.1.9(1). Let $\mathcal{Z}$ be the set of z with $\operatorname{Re} z > 0$ at which $F(\cdot + \beta)$ has a pole, and let $\widehat{f}(z) = F(z + \beta)$ for $z \notin \mathcal{Z}$ and $\widehat{f}(z) = 0$ for $z \in \mathcal{Z}$. We check (LP1)–(LP5). (LP1): $|f(u)| \leq Ce^u e^{(1-\beta)u} \leq Ce^{2u}$, as $\beta > 0$. (LP2): for $\operatorname{Re} z > 2$ we have $\operatorname{Re}(z + \beta) > 2$, so $\widehat{f}(z) = \int_0^\infty R_{d,q_0}(e^u) e^{(1-z-\beta)u}\, du = \int_0^\infty f(u) e^{-zu}\, du$. (LP3), (LP4): the poles of the meromorphic function F form a discrete closed subset of $\mathbb{C}$, so $\mathcal{Z}$ has no accumulation point, and $\widehat{f}$ is holomorphic on $\{\operatorname{Re} z > 0\} \setminus \mathcal{Z}$. (LP5): every pole of F in $\operatorname{Re} s > \beta$ is simple by Proposition 2.1.9(2), with nonzero residue $r(z + \beta)$ (a simple pole with residue 0 is removable); if $z \in \mathcal{Z}$ is real then $z + \beta > \beta$ is real, and by the previous paragraph $r(z + \beta)$ is real and ≥ 0, hence > 0. So $(f, \widehat{f}, \mathcal{Z})$ is Laplace pole data.

Window bounds. By hypothesis there is u_1 with $R_{d,q_0}(e^u) \leq e^{(\beta-1)u}$, that is $f(u) \leq 1$, for $u \geq u_1$. Let $0 < \theta < 1$ and $\lambda = 2$. Given $\eta > 0$ and $U_0 \geq 0$, choose $U \geq \max(U_0, u_1)$ with $\theta^{-1} \leq e^{\eta U}$ and put $R \equiv 1$ on $[U, 2U]$. Then $f \leq R$ on $[U, 2U]$ and $\int_U^{2U} e^{-\theta u}\, du \leq \theta^{-1} \leq e^{\eta U}$. So f has integrated window bounds of exponent θ and ratio 2, for every $\theta \in (0, 1)$.

Landau. By Proposition 2.1.4, $\operatorname{Re} z \leq \theta$ for every $z \in \mathcal{Z}$ and every $\theta \in (0, 1)$, so $\operatorname{Re} z \leq 0$; as $\operatorname{Re} z > 0$ on $\mathcal{Z}$, the set $\mathcal{Z}$ is empty. Thus F is holomorphic on $\operatorname{Re} s > \beta$, and in particular $r(\rho) = 0$.

Conclusion. The point ρ satisfies $\rho \neq 1$, $\operatorname{Re} \rho > \beta$, $|\operatorname{Im} \rho| \leq H$, and $\rho \neq 0$. By (2.4), $n(\rho) = \varphi(q_0)\rho(1-\rho)r(\rho) = 0$, so $\operatorname{ord}_\rho L(\cdot, \psi) = 0$, that is $L(\rho, \psi) \neq 0$. □

Proposition 2.1.4. *Let $(f, \widehat{f}, \mathcal{Z})$ be Laplace pole data, let $0 < \theta < 1$ and $\lambda > 1$, and suppose that f has integrated window bounds of exponent θ and ratio λ. Then $\operatorname{Re} \rho \leq \theta$ for every $\rho \in \mathcal{Z}$.*

Proof. Suppose, to the contrary, that some $\rho_0 \in \mathcal{Z}$ has $\operatorname{Re} \rho_0 > \theta$. For an integer $k \geq 0$ write $A_k(s) = (-1)^k \widehat{f}^{(k)}(s)/k!$ wherever $\widehat{f}$ is holomorphic near s. Let $\sigma > 2$ and C_2 be as in Lemma 2.1.7; since $\widehat{f}$ coincides with the Laplace integral of f on the open half-plane $\operatorname{Re} s > 2$ by (LP2), that lemma applies to the derivatives of $\widehat{f}$ at σ and $\sigma + i\gamma$. By Lemma 2.1.5, $\operatorname{Re} \rho \leq 2 < \sigma$ for every $\rho \in \mathcal{Z}$. We use three elementary facts.

(C) Cauchy's estimate. If $\widehat{f}$ is holomorphic on $B(z, r)$ and $0 < r' < r$, then $|A_k(z)| \leq Mr'^{-k}$ for all k, with $M = \max_{|s-z|=r'} |\widehat{f}(s)|$.

(P) Coefficients near a simple pole. Let $\rho \in B(z, r)$, $\rho \neq z$, let $\widehat{f}$ be holomorphic on $B(z, r) \setminus \{\rho\}$ and let $\widehat{f}(s) - c/(s - \rho)$ agree near ρ with a function holomorphic at ρ. Then $\widehat{f}(s) - c/(s - \rho)$ extends holomorphically to $B(z, r)$, and the k-th Taylor coefficient of $c/(s - \rho)$ at z is $-c/(\rho - z)^{k+1}$. By (C), for every $r' \in (0, r)$ there is M with

$$\Big| A_k(z) - \frac{c}{(z - \rho)^{k+1}} \Big| \leq Mr'^{-k} \qquad \text{for all } k.$$

(G) Growth. Let $0 < d < e \le E$, $c > 0$ and $M, C \in \mathbb{R}$. It is impossible that for every $\eta > 0$ and every U_0 there are $U \ge U_0$ and an integer $k \ge U$ with

$$c\, d^{-k} \le M e^{-k} + C\, e^{\eta U} E^{-k}.$$

Indeed, take $\eta = \frac{1}{2}\log(e/d) > 0$ and $U_0 \ge 0$. Then $e^{\eta U} \le e^{\eta k} = (e/d)^{k/2}$ and $E^{-k} \le e^{-k}$, so the inequality gives $c \le (|M| + |C|)(e/d)^{k/2}(d/e)^k = (|M| + |C|)(d/e)^{k/2}$ with $k \ge U_0$ arbitrarily large, whence $c \le 0$.

The set $\mathcal{Z}$ has no accumulation point in $\operatorname{Re} s > 0$ by (LP4), so Lemma 2.1.6 applies with $\theta < \sigma$ and ρ_0. All discs below have radius at most $\sigma - \theta < \sigma$ and centre on the line $\operatorname{Re} s = \sigma$, so they lie in $\operatorname{Re} s > 0$, where $\widehat{f}$ is holomorphic off $\mathcal{Z}$ by (LP3).

Case 2 of Lemma 2.1.6: the nearest singularity is real. There are a real $\beta \in \mathcal{Z}$ with $\theta < \beta < \sigma$ and r with $\sigma - \beta < r \le \sigma - \theta$ such that β is the only point of $\mathcal{Z}$ in $B(\sigma, r)$. By (LP5) the residue c_β is real and positive. Apply (P) with $z = \sigma$, $\rho = \beta$ and $r' = (\sigma - \beta + r)/2$: since $c_\beta(\sigma - \beta)^{-k-1}$ is real,

$$A_k(\sigma) \ge \frac{c_\beta}{\sigma - \beta}(\sigma - \beta)^{-k} - M r'^{-k} \qquad \text{for all } k.$$

Given $\eta > 0$ and U_0, Lemma 2.1.7 with $\gamma = 0$ gives $U \ge U_0$ and $k \ge U$ with $A_k(\sigma) \le |A_k(\sigma)| + A_k(\sigma) \le C_2 e^{\eta U}(\sigma - \theta)^{-k}$. Hence

$$\frac{c_\beta}{\sigma - \beta}(\sigma - \beta)^{-k} \le M r'^{-k} + C_2 e^{\eta U}(\sigma - \theta)^{-k},$$

which contradicts (G) with $d = \sigma - \beta < e = r' < E = \sigma - \theta$.

Case 1 of Lemma 2.1.6: the nearest singularity is not real. There is $D \le \sigma - \theta$ such that $B(\sigma, D)$ contains no point of $\mathcal{Z}$, and there are $\gamma \in \mathbb{R}$, $\rho \in \mathcal{Z}$ and $r \le D$ such that, with $z = \sigma + i\gamma$ and $d = |z - \rho|$, we have $d < r$ and ρ is the only point of $\mathcal{Z}$ in $B(z, r)$. Moreover, $d > 0$ because $\operatorname{Re}\rho < \sigma$. Put $r' = (d + r)/2$, so $d < r' < r \le D$. By (P) at z there is M with $|A_k(z)| \ge |c_\rho| d^{-k-1} - M r'^{-k}$, and by (C) at σ, applied on $B(\sigma, D)$, there is M' with $|A_k(\sigma)| \le M' r'^{-k}$, for all k. Given $\eta > 0$ and U_0, Lemma 2.1.7 gives $U \ge U_0$ and $k \ge U$ with

$$|A_k(z)| \le C_2 e^{\eta U}(\sigma - \theta)^{-k} - A_k(\sigma) \le C_2 e^{\eta U}(\sigma - \theta)^{-k} + M' r'^{-k}.$$

Hence $\frac{|c_\rho|}{d} d^{-k} \le (M + M') r'^{-k} + C_2 e^{\eta U}(\sigma - \theta)^{-k}$, with $|c_\rho| > 0$ by (LP5). This contradicts (G) with $d < e = r' < E = \sigma - \theta$.

Both cases are impossible, so $\operatorname{Re}\rho \le \theta$ for all $\rho \in \mathcal{Z}$.

Relation to Landau's theorem. Landau's theorem [179] (see [144], [214]) states that a Laplace integral of an eventually non-negative function has a singularity at the real point of its abscissa of convergence; its proof expands at a real point σ and uses $f \ge 0$ to continue the Taylor series to the left. The argument above never continues the Laplace integral to the left of $\operatorname{Re} s = 2$: the hypothesis holds only on the windows $[U, \lambda U]$, so it compares single Taylor coefficients instead. A singularity at distance d from a point of the line $\operatorname{Re} s = \sigma$ forces coefficients of size d^{-k} there, while Lemma 2.1.7, for $k \approx \frac{1+\lambda}{2}\sigma U$, bounds them by $e^{\eta U}(\sigma - \theta)^{-k}$. If the singularity nearest to σ is real, the positivity of its residue gives the real coefficient $A_k(\sigma)$ a definite sign (case 2); otherwise the coefficient at σ is negligible and the coefficient at a point $\sigma + i\gamma$ near the singularity is large (case 1). Positivity of residues at real poles thus replaces the non-negativity of f. □

Lemma 2.1.5. *Let $f\colon [0, \infty) \to \mathbb{R}$ be measurable with $|f(u)| \le C_0 e^{2u}$ for all $u \ge 0$. Then $F(s) = \int_0^\infty f(u) e^{-su}\, du$ converges absolutely and is holomorphic in $\operatorname{Re} s > 2$, $|F(s)| \le C_0/(\sigma - 2)$ whenever $\operatorname{Re} s \ge \sigma > 2$, and for every integer $k \ge 0$*

$$\frac{(-1)^k F^{(k)}(s)}{k!} = \int_0^\infty f(u)\, \frac{u^k}{k!}\, e^{-su}\, du \qquad (\operatorname{Re} s > 2).$$

If $(f, \widehat{f}, \mathcal{Z})$ is Laplace pole data, then $\widehat{f} = F$ on $\operatorname{Re} s > 2$, so $\widehat{f}$ satisfies the same identity there, and every $\rho \in \mathcal{Z}$ has $\operatorname{Re}\rho \le 2$.

Proof. Let $b > 0$. For $\operatorname{Re} z \ge 2 + b$ and $u > 0$,

$$\left|f(u)(-u)^k e^{-zu}\right| \le C_0\, u^k e^{-bu},$$

which is integrable on $(0,\infty)$. Hence all the integrals $\int_0^\infty f(u)(-u)^k e^{-zu}\,du$ converge absolutely in $\operatorname{Re} z > 2$, and differentiation under the integral sign (dominated convergence, with the dominating function $C_0 u^{k+1} e^{-bu}$ on the half-plane $\operatorname{Re} z \ge 2+b$, $b = (\operatorname{Re} s - 2)/2$) shows that the k-th integral is holomorphic with derivative the $(k+1)$-st. By induction $F^{(k)}(s) = \int_0^\infty f(u)(-u)^k e^{-su}\,du$, which is the displayed identity. For $\operatorname{Re} s \ge \sigma > 2$,

$$|F(s)| \le \int_0^\infty C_0 e^{2u} e^{-\sigma u}\,du = \frac{C_0}{\sigma - 2}.$$

Now let $(f, \widehat{f}, \mathcal{Z})$ be Laplace pole data. By (LP2) $\widehat{f} = F$ on the open half-plane $\operatorname{Re} s > 2$, so the derivatives agree there. Suppose $\rho \in \mathcal{Z}$ had $\operatorname{Re}\rho > 2$, and put $\sigma' = (2 + \operatorname{Re}\rho)/2$. On a neighbourhood of ρ contained in $\operatorname{Re} s > \sigma'$ we have $|\widehat{f}(s)| \le C_0/(\sigma' - 2)$, so $(s-\rho)\widehat{f}(s) \to 0$ as $s \to \rho$. By (LP5), however, $(s-\rho)\widehat{f}(s) = c_\rho + (s-\rho)h(s)$ near ρ with h holomorphic at ρ, which tends to $c_\rho \ne 0$. Hence $\operatorname{Re}\rho \le 2$. □

Lemma 2.1.6. *Let $\mathcal{Z} \subset \mathbb{C}$ be a set without accumulation points in $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, and let $0 < \theta < \sigma$ be real numbers such that $\operatorname{Re}\rho < \sigma$ for every $\rho \in \mathcal{Z}$. Suppose that some $\rho_0 \in \mathcal{Z}$ has $\operatorname{Re}\rho_0 > \theta$. Write $B(z,r)$ for the open disc of centre z and radius r. Then one of the following holds.*

1. *There are $D \le \sigma - \theta$ with $B(\sigma, D) \cap \mathcal{Z} = \emptyset$, and $\gamma \in \mathbb{R}$, $\rho \in \mathcal{Z}$ and $r \le D$ such that $|\sigma + i\gamma - \rho| < r$ and $B(\sigma + i\gamma, r) \cap \mathcal{Z} = \{\rho\}$.*
2. *There are a real number $\beta \in \mathcal{Z}$ with $\beta > \theta$ and a real r with $\sigma - \beta < r \le \sigma - \theta$ such that $B(\sigma, r) \cap \mathcal{Z} = \{\beta\}$.*

Proof. Finiteness. If $z \in \mathbb{C}$ and $0 \le t < \operatorname{Re} z$, the closed disc $\overline{B}(z,t)$ is a compact subset of $\{\operatorname{Re} s > 0\}$; covering it by neighbourhoods each containing at most one point of $\mathcal{Z}$ shows that $\mathcal{Z} \cap \overline{B}(z,t)$ is finite.

The dichotomy at σ. The set $F = \mathcal{Z} \cap \overline{B}(\sigma, \sigma - \theta)$ is finite, since $\sigma - \theta < \sigma$. We first show that one of the following holds:

(i) there are $D \le \sigma - \theta$ with $B(\sigma, D) \cap \mathcal{Z} = \emptyset$ and $\rho_1 \in \mathcal{Z}$ with $\sigma - \operatorname{Re}\rho_1 < D$;

(ii) conclusion 2 holds.

If $F = \emptyset$, then (i) holds with $D = \sigma - \theta$ and $\rho_1 = \rho_0$. Otherwise let $d = \min_{w \in F} |w - \sigma| \le \sigma - \theta$, attained at $\rho_s \in F$; then $B(\sigma, d) \cap \mathcal{Z} = \emptyset$. If some $\rho_1 \in \mathcal{Z}$ has $\sigma - \operatorname{Re}\rho_1 < d$, then (i) holds with $D = d$. Otherwise $\sigma - \operatorname{Re} w \ge d$ for all $w \in \mathcal{Z}$. A point w with $|w - \sigma| \le d$ and $\sigma - \operatorname{Re} w \ge d$ satisfies $(\operatorname{Re} w - \sigma)^2 + (\operatorname{Im} w)^2 \le d^2 \le (\operatorname{Re} w - \sigma)^2$, hence $\operatorname{Im} w = 0$ and $w = \sigma - d$. So $\beta = \sigma - d$ is the only point of F at distance d from σ, and it is real. Moreover $d < \sigma - \theta$: otherwise $\sigma - \operatorname{Re}\rho_0 \ge d = \sigma - \theta$, contradicting $\operatorname{Re}\rho_0 > \theta$. Hence $\beta > \theta$. Let $r_1 > d$ be the least distance from σ of a point of $F \setminus \{\beta\}$ (or $r_1 = d + 1$ if there is none) and $r = \min(r_1, \sigma - \theta)$. Then $\sigma - \beta = d < r \le \sigma - \theta$, and every point of $\mathcal{Z}$ in $B(\sigma, r)$ lies in F and hence equals β: this is conclusion 2.

From (i) to conclusion 1. Assume (i). Choose $\varepsilon > 0$ with $\varepsilon \le (D - (\sigma - \operatorname{Re}\rho_1))/2$ and $\varepsilon \le (\sigma - D)/2$ (note $D < \sigma$), and put $c_0 = \sigma + i \operatorname{Im}\rho_1$. The set $F' = \mathcal{Z} \cap \overline{B}(c_0, D+\varepsilon)$ is finite, since $D + \varepsilon < \sigma = \operatorname{Re} c_0$, and it contains ρ_1 because $|c_0 - \rho_1| = \sigma - \operatorname{Re}\rho_1$. For distinct $a, b \in F'$, the points $\sigma + i\gamma$ equidistant from a and b satisfy

$$2\gamma(\operatorname{Im} a - \operatorname{Im} b) = (\sigma - \operatorname{Re} a)^2 + (\operatorname{Im} a)^2 - (\sigma - \operatorname{Re} b)^2 - (\operatorname{Im} b)^2.$$

If $\operatorname{Im} a = \operatorname{Im} b$ this would force $(\operatorname{Re} b - \operatorname{Re} a)((\sigma - \operatorname{Re} a) + (\sigma - \operatorname{Re} b)) = 0$, impossible since $a \ne b$ and both real parts are below σ; so each pair excludes at most one value of γ. Choose $\gamma \in (\operatorname{Im}\rho_1 - \varepsilon, \operatorname{Im}\rho_1 + \varepsilon)$ outside this finite set of values, and put $z = \sigma + i\gamma$. Then the distances $|z - w|$, $w \in F'$, are pairwise distinct; let $\rho \in F'$ be the nearest point to z. Since $|z - \rho| \le |z - \rho_1| \le |\gamma - \operatorname{Im}\rho_1| + (\sigma - \operatorname{Re}\rho_1) < \varepsilon + (\sigma - \operatorname{Re}\rho_1) < D$, there is $r \le D$ with $|z - \rho| < r$ and $|z - w| \ge r$ for all $w \in F' \setminus \{\rho\}$. A point $w \in \mathcal{Z} \cap B(z, r)$ satisfies $|w - c_0| < r + \varepsilon \le D + \varepsilon$, so $w \in F'$ and hence $w = \rho$. Together with $B(\sigma, D) \cap \mathcal{Z} = \emptyset$ from (i), this is conclusion 1. □

Lemma 2.1.7. *Let $f\colon [0,\infty) \to \mathbb{R}$ be measurable with $|f(u)| \le C_0 e^{2u}$ for all $u \ge 0$, and let $F(s) = \int_0^\infty f(u) e^{-su}\,du$ for $\operatorname{Re} s > 2$. Let $0 < \theta < 1$ and $\lambda > 1$, and suppose that f has integrated window bounds of exponent θ and ratio λ. Put $l = 2 + 6/(\lambda - 1)$ and $\sigma = l\lambda(l+2)$. Then $\sigma > 2$, and there is $C_2 \ge 0$ such that for every $\eta > 0$ and every $U_0 \in \mathbb{R}$ there are $U \ge U_0$ and an integer $k \ge U$ with*

$$\Big|\frac{F^{(k)}(\sigma + i\gamma)}{k!}\Big| + \frac{(-1)^k F^{(k)}(\sigma)}{k!} \le C_2 \frac{e^{\eta U}}{(\sigma - \theta)^k} \qquad \text{for every } \gamma \in \mathbb{R};$$

here $(-1)^k F^{(k)}(\sigma)/k!$ is real.

Proof. Let C_1 be the constant attached to λ in Lemma 2.1.8; we show the claim with $C_2 = 2 + 2C_0C_1$. We have $\sigma > 2 > \theta > 0$. Let $\eta > 0$ and $U_0 \in \mathbb{R}$. The window bound for η and $\max(U_0, 1)$ gives $U \ge \max(U_0, 1)$ and $R\colon [U, \lambda U] \to [0,\infty)$ with $f \le R$ on $[U, \lambda U]$ and $\int_U^{\lambda U} R(u) e^{-\theta u}\,du \le e^{\eta U}$. Since $U \ge 1$, part 2 of Lemma 2.1.8 gives an integer $k \ge U$ with

$$\int_{(0,\infty)\setminus[U,\lambda U]} K_{k,\sigma-2}(u)\,du \le C_1 \sigma^{-k}. \tag{2.5}$$

Reduction to the positive part of f. Put $w = K_{k,\sigma}$ and $A(s) = (-1)^k F^{(k)}(s)/k!$. By Lemma 2.1.5, for every $\gamma \in \mathbb{R}$,

$$A(\sigma + i\gamma) = \int_0^\infty f(u)\, w(u)\, e^{-i\gamma u}\,du, \qquad A(\sigma) = \int_0^\infty f(u)\, w(u)\,du \in \mathbb{R},$$

the integrals converging absolutely because $|f| w \le C_0 K_{k,\sigma-2}$. As $|A(\sigma + i\gamma)| \le \int_0^\infty |f| w$ and $|f| + f = 2\max(f, 0)$,

$$|A(\sigma + i\gamma)| + A(\sigma) \le \int_0^\infty (|f(u)| + f(u)) w(u)\,du = 2\int_0^\infty \max(f(u), 0)\, w(u)\,du. \tag{2.6}$$

This is the only place where the sign of f enters. In Landau's classical argument [179] one has $f \ge 0$, and (2.6) reads $|A(\sigma + i\gamma)| \le A(\sigma)$. Here only the positive part of f has to be controlled, and only where w has its mass.

The window. For $u \in [U, \lambda U]$ we have $0 \le \max(f(u), 0) \le R(u)$, since $f \le R$ and $R \ge 0$, and $w(u) = K_{k,\sigma-\theta}(u)\, e^{-\theta u} \le (\sigma - \theta)^{-k} e^{-\theta u}$ by part 1 of Lemma 2.1.8. Hence

$$2\int_U^{\lambda U} \max(f, 0)\, w\,du \le 2(\sigma - \theta)^{-k} \int_U^{\lambda U} R(u) e^{-\theta u}\,du \le 2(\sigma - \theta)^{-k} e^{\eta U}.$$

Outside the window. There $\max(f,0) \le |f| \le C_0 e^{2u}$ and $e^{2u} w(u) = K_{k,\sigma-2}(u)$, so by (2.5)

$$2\int_{(0,\infty)\setminus[U,\lambda U]} \max(f,0)\, w\, du \le 2C_0C_1\sigma^{-k} \le 2C_0C_1\, e^{\eta U}(\sigma-\theta)^{-k},$$

using $0 < \sigma - \theta < \sigma$ and $e^{\eta U} \ge 1$. Adding the two parts in (2.6) gives the claim with $C_2 = 2 + 2C_0C_1$. □

Lemma 2.1.8. *For an integer $k \ge 0$ and real $a > 0$ put $K_{k,a}(u) = u^k e^{-au}/k!$ for $u \ge 0$.*

1. *$\int_0^\infty K_{k,a}(u)\, du = a^{-k-1}$, and $0 \le K_{k,a}(u) \le a^{-k}$ for all $u \ge 0$.*
2. *Let $\lambda > 1$ and put*

$$l = 2 + \frac{6}{\lambda - 1}, \qquad \sigma = l\lambda(l+2), \qquad C_1 = \frac{e}{\sigma - 2 - l} + \frac{1}{\sigma - 2 + l}.$$

Then $\sigma > 2 + l$, and for every real $U \ge 1$ the integer $k = \lceil (1+\lambda)\sigma U/2 \rceil$ satisfies $k \ge U$ and

$$\int_{(0,\infty)\setminus[U,\lambda U]} K_{k,\sigma-2}(u)\, du \le C_1\, \sigma^{-k}.$$

Proof. Part 1. The substitution $t = au$ gives $\int_0^\infty u^k e^{-au}\, du = a^{-k-1}\Gamma(k+1) = a^{-k-1}k!$. For $u \ge 0$ the number $(au)^k/k!$ is a term of the series of e^{au}, so $a^k K_{k,a}(u) = (au)^k e^{-au}/k! \le 1$.

Part 2. The kernel $K_{k,\sigma-2}$ has mass $(\sigma-2)^{-k-1}$ and is concentrated near $u = k/(\sigma-2) \approx \frac{1+\lambda}{2} \cdot \frac{\sigma}{\sigma-2} U$, a point inside $[U, \lambda U]$ when σ is large. The claim bounds its mass outside this window explicitly. Write $s = \sigma - 2$ and $V = \lambda U$. Since $l > 2$ and $\lambda > 1$ we have $\sigma > l(l+2)$, so $s - l > l^2 + l - 2 > 0$; in particular $\sigma > 2 + l$. We use the two identities

$$\frac{(1+\lambda)(l+2)}{2} = l\lambda - 1, \qquad \frac{(1+\lambda)(l-2)}{2} = l + 1. \tag{2.7}$$

A Chernoff bound. Let $u > 0$ with $u \notin [U, V]$. Then $l(U-u) \ge 0$ or $l(u-V) \ge 0$, hence

$$e^{-su} \le e^{-su+l(u-V)} + e^{-su+l(U-u)} = e^{-lV} e^{-(s-l)u} + e^{lU} e^{-(s+l)u}.$$

Multiplying by $u^k/k!$, integrating over $(0,\infty)$ and using part 1 (note $s - l > 0$),

$$\int_{(0,\infty)\setminus[U,V]} K_{k,s}(u)\, du \le e^{-lV}(s-l)^{-k-1} + e^{lU}(s+l)^{-k-1}. \tag{2.8}$$

Conversion to powers of σ. For $a > 0$ and real x, the inequality $1 + t \le e^t$ with $t = (\sigma - a)/a$ gives $\sigma/a \le e^{(\sigma-a)/a}$, hence

$$e^x a^{-k-1} = e^x \Big(\frac{\sigma}{a}\Big)^k a^{-1}\sigma^{-k} \le e^{x + k(\sigma-a)/a}\, a^{-1}\sigma^{-k}. \tag{2.9}$$

First term of (2.8). Here $x = -lV = -l\lambda U$ and $a = s - l = \sigma - 2 - l = (l\lambda - 1)(l+2)$, so $(\sigma - a)/a = (l+2)/a = 1/(l\lambda - 1)$. By the first identity of (2.7), $(1+\lambda)\sigma/2 = (l\lambda - 1)\, l\lambda$, so $k \le (1+\lambda)\sigma U/2 + 1 = (l\lambda - 1)l\lambda U + 1$ and $k/(l\lambda - 1) \le l\lambda U + 1$ (as $l\lambda - 1 \ge 1$). The exponent in (2.9) is therefore at most 1, and the first term is at most $e\,(\sigma - 2 - l)^{-1}\sigma^{-k}$.

Second term of (2.8). Here $x = lU$ and $a = s + l = \sigma - 2 + l$, so $(\sigma - a)/a = -(l-2)/(\sigma - 2 + l)$. By the second identity of (2.7), $k(l-2) \ge (1+\lambda)\sigma U(l-2)/2 = (l+1)\sigma U$; and $l(\sigma - 2 + l) \le (l+1)\sigma$, since this is equivalent to $l^2 - 2l \le \sigma$, which holds because $\sigma > l(l+2)$. Hence $k(l-2)/(\sigma - 2 + l) \ge lU$, the exponent in (2.9) is at most 0, and the second term is at most $(\sigma - 2 + l)^{-1}\sigma^{-k}$.

Adding the two bounds gives the claimed inequality with $s = \sigma - 2$. Finally $k \ge (1+\lambda)\sigma U/2 \ge U$ because $\sigma > 2$ and $\lambda > 1$. □

Proposition 2.1.9. *Let $d \equiv 7 \pmod 8$ be a prime and $q \geq 1$ an integer prime to d. Then:*

1. *there is $C \geq 0$ with $|R_q(e^v)| \leq Ce^v$ for all $v \geq 0$, and $v \mapsto R_q(e^v)$ is measurable;*
2. *the function $F_q(s) = \int_0^\infty R_q(e^v)\, e^{(1-s)v}\, dv$, holomorphic on $\operatorname{Re} s > 2$, extends to a meromorphic function F_q on $\mathbb{C}$ which is holomorphic at $s = 1$ and has at most simple poles in $\operatorname{Re} s > 0$;*
3. *for every $\rho \neq 1$ with $\operatorname{Re}\rho > 0$ the residue of F_q at ρ (zero if F_q is holomorphic at ρ) is*

$$r_q(\rho) = \frac{1}{\varphi(q)\,\rho(1-\rho)} \sum_{\xi \bmod q} \Big(\operatorname{ord}_\rho L(\cdot,\xi) - \operatorname{ord}_\rho L(\cdot,\xi(\tfrac{\cdot}{d}))\Big), \tag{2.10}$$

the sum running over the Dirichlet characters ξ modulo q.

Proof. Step 0: the L-functions involved. Let ξ be a character modulo q. The character $\xi(\frac{\cdot}{d})$ modulo qd is not principal: by the Chinese remainder theorem there is $n \equiv 1 \pmod q$ which is a quadratic non-residue modulo d, and then $\gcd(n, qd) = 1$ and $\xi(n)(\frac{n}{d}) = -1$. Hence, by Theorem A.1.2 and Theorem A.1.4, $L(s, \xi(\frac{\cdot}{d}))$ is entire and $L(1, \xi(\frac{\cdot}{d})) \neq 0$; likewise $L(s,\xi)$ is entire with $L(1,\xi) \neq 0$ if $\xi \neq \chi_{0,q}$, while $L(s, \chi_{0,q})$ is holomorphic on $\mathbb{C} \setminus \{1\}$ with a simple pole at 1. All these functions are non-zero on $\operatorname{Re} s > 1$ by the Euler product, so none vanishes identically. For a meromorphic function $f \not\equiv 0$ and a point ρ, writing $f(s) = (s-\rho)^m g(s)$ with g holomorphic and non-zero near ρ gives $f'/f = m/(s-\rho) + g'/g$ near ρ: so f'/f is meromorphic with at most simple poles, and its residue at ρ is $\operatorname{ord}_\rho f = m$.

Step 1: growth and measurability. By (2.1) and Lemma 2.1.11(2),

$$0 \leq \sum_{n \leq t} \lambda_q(n) \leq 2 \sum_{n \leq t} \Lambda(n) \leq 6t,$$

so $|\mathcal{E}_q(t)| \leq 7t$ for $t \geq 0$. The function $\mathcal{E}_q$ is a step function minus a linear function, hence measurable and locally bounded, so by (2.2) the function

$$R_q(y) = \int_1^y \mathcal{E}_q(t) t^{-2}\, dt - c_q$$

is continuous on $[1,\infty)$ and $|R_q(y)| \leq 7\log y + |c_q|$. Thus $v \mapsto R_q(e^v)$ is continuous, hence measurable, and $|R_q(e^v)| \leq 7v + |c_q| \leq (7 + |c_q|)e^v$ for $v \geq 0$. This is (1), with $C = 7 + |c_q|$.

Step 2: the transform as a Mellin transform. Let $\sigma = \operatorname{Re} s > 1$. Substituting $y = e^v$, the integral $F_q(s) = \int_1^\infty R_q(y)\, y^{-s}\, dy$ converges absolutely by Step 1. Since $\int_1^\infty \int_t^\infty 7t^{-1} y^{-\sigma}\, dy\, dt = 7/(\sigma-1)^2 < \infty$, Fubini's theorem gives

$$\int_1^\infty \Big(\int_1^y \frac{\mathcal{E}_q(t)}{t^2}\, dt\Big) y^{-s}\, dy = \int_1^\infty \frac{\mathcal{E}_q(t)}{t^2} \cdot \frac{t^{1-s}}{s-1}\, dt, \qquad \text{so} \qquad F_q(s) = \frac{1}{s-1}\Big(\int_1^\infty \mathcal{E}_q(t)\, t^{-s-1}\, dt - c_q\Big).$$

Put $D_q(s) = \sum_{n \geq 1} \lambda_q(n) n^{-s}$, absolutely convergent as $0 \leq \lambda_q \leq 2\Lambda$. Since $\lambda_q(1) = 0$, Fubini's theorem gives

$$\int_1^\infty (\sum_{n \leq t} \lambda_q(n)) t^{-s-1}\, dt = \sum_n \lambda_q(n) \int_n^\infty t^{-s-1}\, dt = D_q(s)/s,$$

and $\int_1^\infty (t/\varphi(q))\, t^{-s-1}\, dt = 1/(\varphi(q)(s-1))$. With

$$G_q(s) = D_q(s) - \frac{s}{\varphi(q)(s-1)} \qquad (\operatorname{Re} s > 1)$$

this reads $\int_1^\infty \mathcal{E}_q(t)t^{-s-1}\,dt = G_q(s)/s$, hence

$$\int_1^\infty \mathcal{E}_q(t)\,t^{-s-1}\,dt = \frac{G_q(s)}{s}, \qquad F_q(s) = \frac{1}{s-1}\Big(\frac{G_q(s)}{s} - c_q\Big) \qquad (\operatorname{Re} s > 1). \tag{2.11}$$

Step 3: G_q through logarithmic derivatives. For $\gcd(n,q) = 1$, orthogonality gives

$$\frac{1}{\varphi(q)} \sum_{\xi \bmod q} \xi(n) = 1 \text{ if } n \equiv 1 \pmod q, \quad \text{and 0 otherwise;}$$

for $\gcd(n,q) > 1$ both sides vanish. With $(\frac{n}{d}) = 0$ for $d \mid n$, (2.1) becomes, for all $n \geq 1$,

$$\lambda_q(n) = \frac{1}{\varphi(q)} \sum_{\xi \bmod q} \Big(\Lambda(n)\xi(n) - \Lambda(n)\,\xi(n)\big(\tfrac{n}{d}\big)\Big).$$

For every Dirichlet character χ and $\operatorname{Re} s > 1$, logarithmic differentiation of the Euler product gives $\sum_n \Lambda(n)\chi(n)n^{-s} = -\frac{L'}{L}(s,\chi)$. Hence, for $\operatorname{Re} s > 1$,

$$G_q(s) = \frac{1}{\varphi(q)} \sum_{\xi \bmod q} \Big(-\frac{L'}{L}(s,\xi) + \frac{L'}{L}\big(s,\xi(\tfrac{\cdot}{d})\big)\Big) - \frac{s}{\varphi(q)(s-1)}. \tag{2.12}$$

By Step 0 the right side is meromorphic on $\mathbb{C}$ with at most simple poles; we denote this continuation again by G_q. It is holomorphic on $\operatorname{Re} s > 1$, where all the L-functions are holomorphic and non-zero. At a point $\rho \neq 1$ the last term is holomorphic, so by Step 0

$$\operatorname{res}_\rho G_q = \frac{1}{\varphi(q)} \sum_{\xi \bmod q} \Big(\operatorname{ord}_\rho L\big(\cdot,\xi(\tfrac{\cdot}{d})\big) - \operatorname{ord}_\rho L(\cdot,\xi)\Big) \qquad (\rho \neq 1). \tag{2.13}$$

At $s = 1$: for $\xi \neq \chi_{0,q}$ and for every twist the L-function is holomorphic and non-zero at 1, so its logarithmic derivative is holomorphic there; $\operatorname{ord}_1 L(\cdot,\chi_{0,q}) = -1$, so $-\frac{L'}{L}(s,\chi_{0,q}) = \frac{1}{s-1} + (\text{holomorphic})$; and $\frac{s}{s-1} = 1 + \frac{1}{s-1}$. The two polar parts $\frac{1}{\varphi(q)(s-1)}$ cancel, and G_q is holomorphic at 1.

Step 4: the constant. By Theorem 2.1.10, which applies since d is an odd prime and q is prime to d, the integral $\int_1^\infty |\mathcal{E}_q(t)|t^{-2}\,dt$ converges, so $c_q = \int_1^\infty \mathcal{E}_q(t)t^{-2}\,dt$ by Definition 2.1.1. For real $s > 1$ and $t \geq 1$, $|\mathcal{E}_q(t)t^{-s-1}| \leq |\mathcal{E}_q(t)|t^{-2}$; by dominated convergence the left side of (2.11) tends to c_q as $s \to 1^+$, while the right side $G_q(s)/s$ tends to $G_q(1)$ by Step 3. Hence $c_q = G_q(1)$.

Step 5: continuation and residues. Let $h(s) = G_q(s)/s$, meromorphic on $\mathbb{C}$, holomorphic near 1 with $h(1) = G_q(1) = c_q$. By (2.11), $F_q(s) = (h(s) - h(1))/(s-1)$ on $\operatorname{Re} s > 1$. The right side is meromorphic on $\mathbb{C}$, holomorphic on $\operatorname{Re} s > 1$ (so in particular on $\operatorname{Re} s > 2$, where it is the integral) and holomorphic at $s = 1$, where the singularity is removable; it is the meromorphic continuation of F_q, unique by the identity theorem. Let $\rho \neq 1$ with $\operatorname{Re}\rho > 0$. Near ρ the functions $1/(s(s-1))$ and $h(1)/(s-1)$ are holomorphic, and G_q has at most a simple pole at ρ; hence $F_q(s) = G_q(s)/(s(s-1)) - h(1)/(s-1)$ has at most a simple pole at ρ, with residue (zero if F_q is holomorphic at ρ)

$$\frac{\operatorname{res}_\rho G_q}{\rho(\rho-1)} = \frac{1}{\varphi(q)\,\rho(1-\rho)} \sum_{\xi \bmod q} \Big(\operatorname{ord}_\rho L(\cdot,\xi) - \operatorname{ord}_\rho L\big(\cdot,\xi(\tfrac{\cdot}{d})\big)\Big) = r_q(\rho)$$

by (2.13). Together with the holomorphy at 1 this proves (2), and the display is (3). □

Theorem 2.1.10. *Let d be an odd prime and $q \geq 1$ an integer prime to d. Then the integral $\int_1^\infty \mathcal{E}_q(t)\, t^{-2}\, dt$ converges absolutely, so that $R_q(y) = -\int_y^\infty \mathcal{E}_q(t)\, t^{-2}\, dt$ for $y \geq 1$, and for every $B \geq 0$ there is y_B with*

$$|R_q(y)| \leq (\log y)^{-B} \qquad \text{for all } y \geq y_B.$$

In particular $R_q(y) \to 0$ as $y \to \infty$.

Proof. Put $M = qd \geq 3$ and, for b prime to M, $S(t;b) = \sum_{n \leq t,\ n \equiv b\,(M)} \Lambda(n)$.

Step 1: primes in the classes modulo M. Fix $A > 0$. We show $S(t;b) = t/\varphi(M) + O(t(\log t)^{-A})$ for $t \geq 2$, uniformly in b. By Theorem A.1.8(1) applied to the modulus M, an exceptional zero β_1, if it exists, satisfies $\frac{1}{2} < 1 - 1/(13\log(3M)) \leq \beta_1 < 1$; it depends only on M. By Theorem A.1.8(2), for $X \geq M^{12}$,

$$\Big|\vartheta(X;M,b) - \frac{X}{\varphi(M)}\Big| \leq \frac{X^{\beta_1}}{\beta_1 \varphi(M)} + A_4\Big(X^{-c_4/\log M} + e^{-c_4(\log X)^{3/5}(\log\log X)^{-1/5}}\Big)\frac{3X}{\varphi(M)},$$

the first term omitted if there is no exceptional zero; here $X^{\beta_1}/\beta_1 \leq 2X$ was used in the factor of the error term. Since $\beta_1 < 1$ and M are fixed, each term on the right is at most $C_1 X(\log X)^{-A}$ for $X \geq M^{12}$; for $2 \leq X < M^{12}$ the same holds after enlarging C_1, as both sides are bounded there. The prime powers $p^k \leq t$ with $k \geq 2$ contribute to $S(t;b)$ at most $\sum_{2 \leq k \leq \log_2 t} \vartheta(t^{1/k}) \leq 3\sqrt{t}\, \log_2 t$, by $\vartheta(x) \leq \sum_{n \leq x} \Lambda(n) \leq 3x$ (Lemma 2.1.11(2)). Hence $|S(t;b) - t/\varphi(M)| \leq C_2 t(\log t)^{-A}$ for $t \geq 2$.

Step 2: the count error. By the Chinese remainder theorem, as $\gcd(q,d) = 1$, the integers n with $n \equiv 1 \pmod q$ and $d \nmid n$ are those in the classes $b \in \mathcal{B}$ modulo M, where $\mathcal{B}$ consists of the $d-1$ classes $b \equiv 1 \pmod q$ prime to d, and $b \mapsto b \bmod d$ is a bijection from $\mathcal{B}$ onto $(\mathbb{Z}/d\mathbb{Z})^\times$. The prime powers $n \leq t$ with $d \mid n$ are the powers of d; they contribute $\Lambda(n)(1 - (\frac{n}{d})) = \log d$ each, at most $\log t$ in total. Therefore, by (2.1),

$$\sum_{n \leq t} \lambda_q(n) = \sum_{b \in \mathcal{B}} \Big(1 - \big(\tfrac{b}{d}\big)\Big) S(t;b) + O(\log t).$$

By Step 1, $\varphi(M) = \varphi(q)(d-1)$ and $\sum_{b \in \mathcal{B}} (\frac{b}{d}) = \sum_{u \in (\mathbb{Z}/d)^\times} (\frac{u}{d}) = 0$, the right side is $(d-1)t/\varphi(M) + O(t(\log t)^{-A}) = t/\varphi(q) + O(t(\log t)^{-A})$. So

$$|\mathcal{E}_q(t)| \leq C_3\, t(\log t)^{-A} \qquad (t \geq 2). \tag{2.14}$$

Step 3: the remainder. Take $A = B + 2$. By (2.14), for $y \geq 2$,

$$\int_y^\infty \frac{|\mathcal{E}_q(t)|}{t^2}\, dt \leq C_3 \int_y^\infty \frac{dt}{t(\log t)^{B+2}} = \frac{C_3}{B+1}(\log y)^{-B-1},$$

and $\mathcal{E}_q(t)t^{-2}$ is bounded and piecewise continuous on $[1,2]$. So the integral defining c_q in Definition 2.1.1 converges absolutely, $c_q = \int_1^\infty \mathcal{E}_q(t)t^{-2}\, dt$, and (2.2) gives $R_q(y) = -\int_y^\infty \mathcal{E}_q(t)t^{-2}\, dt$. Hence $|R_q(y)| \leq \frac{C_3}{B+1}(\log y)^{-B-1} \leq (\log y)^{-B}$ as soon as $\log y \geq C_3$. □

Lemma 2.1.11. *1. For every real $x \geq 1$, $\sum_{n \leq x} \Lambda(n) \leq (\log 4)\, x + 2\sqrt{x}\, \log x$.*

2. For every real $x \geq 0$, $\sum_{n \leq x} \Lambda(n) \leq 3x$.

3. For every real $x \geq 1$,

$$\Big|\sum_{n \leq x} \frac{\Lambda(n)}{n} - \log x\Big| \leq 3 + \log 2.$$

Proof. Write $\vartheta(y) = \sum_{p \le y} \log p$, so that $\vartheta(y) \le y \log 4$ for $y \ge 0$ by Chebyshev's bound [214], and $\vartheta(y) = 0$ for $y < 2$.

Item 1. Since $\Lambda(p^k) = \log p$ and Λ vanishes off the prime powers,

$$\sum_{n \le x} \Lambda(n) = \sum_{k \ge 1} \vartheta(x^{1/k}).$$

The term with $k = 1$ is at most $x \log 4$. A term with $k \ge 2$ is non-zero only if $x^{1/k} \ge 2$, that is, $k \le \log x / \log 2$, and it is at most $x^{1/k} \log 4 \le \sqrt{x} \log 4$. There are fewer than $\log x / \log 2$ such k, so these terms total at most $(\log x / \log 2)\sqrt{x} \log 4 = 2\sqrt{x} \log x$.

Item 2. For $x < 2$ the sum is 0. For $x \ge 1$, with $u = \sqrt{x}$ and $\log u \le u/e$ (valid for $u > 0$), $2\sqrt{x} \log x = 4u \log u \le \frac{4}{e} x$; as $\log 4 + \frac{4}{e} < 2.9$, item 1 gives the bound $3x$.

Item 3. Let $k = \lfloor x \rfloor \ge 1$ and $S = \sum_{n \le k} \Lambda(n)/n$, which equals $\sum_{n \le x} \Lambda(n)/n$.

- Since $\log m = \sum_{d \mid m} \Lambda(d)$ for $m \ge 1$, Legendre's identity $\log k! = \sum_{n \le k} \Lambda(n) \lfloor k/n \rfloor$ holds.
- $k \log k - k \le \log k! \le k \log k$; the lower bound holds because $\log k! = \sum_{m=2}^{k} \log m \ge \int_1^k \log t \, dt = k \log k - k + 1$.
- From $k/n - 1 < \lfloor k/n \rfloor \le k/n$ and item 2,

$$kS - 3k \le kS - \sum_{n \le k} \Lambda(n) \le \log k! \le kS.$$

Hence $\log k - 1 \le S \le \log k + 3$. Since $\log k \le \log x < \log(k+1) \le \log k + \log 2$, we get $-1 - \log 2 < S - \log x \le 3$, which gives item 3. □

Proposition 2.1.12. *Let $N \ge 1$ be an integer, let $\beta > \frac{1}{2}$ and $H \ge 0$ be real, and let $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ be a class such that every prime $d \equiv a \pmod N$ satisfies $d \equiv 7 \pmod 8$. Then there is a prime $d \equiv a \pmod N$ such that*

$$L(s, \xi(\tfrac{\cdot}{d})) \ne 0$$

for every Dirichlet character ξ modulo N and every s with $\operatorname{Re} s \ge \beta$, $|\operatorname{Im} s| \le H$.

Proof. Step 0: reductions. If $N = 1$, the prime 2 lies in the class a and $2 \not\equiv 7 \pmod 8$, against the hypothesis; so $N \ge 2$. Every prime $d \equiv a \pmod N$ is odd, as $d \equiv 7 \pmod 8$, and prime to N, as a is a unit. Put $\sigma = \min(\beta, 1) \in (\frac{1}{2}, 1]$; the box $\operatorname{Re} s \ge \beta$, $|\operatorname{Im} s| \le H$ lies in the box $\operatorname{Re} s \ge \sigma$, $|\operatorname{Im} s| \le H$.

Step 1: passage to primitive characters. For each character ξ modulo N let ψ_ξ be the primitive character inducing it, of modulus $M_\xi \mid N$. Let $d \equiv a \pmod N$ be prime; then $d \nmid 2M_\xi$, and $\psi_\xi(\frac{\cdot}{d})$ is not principal (it takes the value -1 at an $n \equiv 1 \pmod{M_\xi}$ which is a non-residue modulo d). The characters $\xi(\frac{\cdot}{d})$ modulo Nd and $\psi_\xi(\frac{\cdot}{d})$ modulo $M_\xi d$ agree at every n prime to Nd, and the first vanishes at the other n. Comparing Euler products on $\operatorname{Re} s > 1$,

$$L(s, \xi(\tfrac{\cdot}{d})) = L(s, \psi_\xi(\tfrac{\cdot}{d})) \prod_{p \mid N,\ p \nmid M_\xi} \Big(1 - \psi_\xi(p)(\tfrac{p}{d}) p^{-s}\Big),$$

and both sides are entire by Theorem A.1.2, so the identity holds on $\mathbb{C}$. For $\operatorname{Re} s > 0$ each factor of the product is non-zero, as $|\psi_\xi(p)(\frac{p}{d})p^{-s}| \le p^{-\operatorname{Re} s} < 1$. Hence a zero of $L(s, \xi(\frac{\cdot}{d}))$ with $\operatorname{Re} s \ge \sigma$, $|\operatorname{Im} s| \le H$ is a zero of $L(s, \psi_\xi(\frac{\cdot}{d}))$.

Step 2: few bad primes. Call a prime $d \equiv a \pmod N$ bad if some $L(s, \xi(\frac{\cdot}{d}))$, ξ modulo N, vanishes at a point with $\operatorname{Re} s \ge \sigma$, $|\operatorname{Im} s| \le H$. By Lemma 2.1.13 applied to ψ_ξ, σ and H there

are $\delta_\xi > 0$ and $C_\xi \geq 0$ such that for $X \geq 2$ at most $C_\xi X^{1-\delta_\xi}$ primes $d \leq X$ with $d \equiv 7 \pmod 8$ and $d \nmid M_\xi$ give $L(s, \psi_\xi(\frac{\cdot}{d}))$ a zero in that box. Every bad prime satisfies $d \equiv 7 \pmod 8$ and $d \nmid M_\xi$ (Step 0, as $M_\xi \mid N$). By Step 1, with $\delta = \min_\xi \delta_\xi > 0$ and $C' = \sum_\xi C_\xi$, at most $C' X^{1-\delta}$ bad primes are $\leq X$, for every $X \geq 2$.

Step 3: many primes in the class. Apply Theorem A.1.8(2) with $M = N \geq 2$ and $b = a$, prime to N. Put $m(X) = (X - \chi_1(a) X^{\beta_1}/\beta_1)/\varphi(N)$ (with $m(X) = X/\varphi(N)$ if there is no exceptional zero) and $\varepsilon(X) = A_4\big(e^{-c_4 \log X/\log N} + e^{-c_4 (\log X)^{3/5} (\log\log X)^{-1/5}}\big)$. The exceptional zero, if any, depends only on N and satisfies $\frac{1}{2} < \beta_1 < 1$ by Theorem A.1.8(1), so $X^{\beta_1}/\beta_1 \leq X/4$ and $m(X) \geq \frac{3}{4} X/\varphi(N)$ for large X; also $\varepsilon(X) \to 0$. Hence for large X, $\vartheta(X; N, a) \geq (1 - \varepsilon(X))\, m(X) \geq X/(2\varphi(N))$, and the number of primes $p \leq X$ with $p \equiv a \pmod N$ is at least $\vartheta(X; N, a)/\log X \geq X/(2\varphi(N) \log X)$.

Step 4: a good prime. Since $X^\delta/\log X \to \infty$, for large X we have $X/(2\varphi(N)\log X) > C' X^{1-\delta}$, so by Steps 2 and 3 some prime $d \leq X$ with $d \equiv a \pmod N$ is not bad. For this d, $L(s, \xi(\frac{\cdot}{d})) \neq 0$ for every ξ modulo N and every s with $\operatorname{Re} s \geq \sigma$, $|\operatorname{Im} s| \leq H$, in particular for every s with $\operatorname{Re} s \geq \beta$, $|\operatorname{Im} s| \leq H$ (Step 0). □

Lemma 2.1.13. *Let $M \geq 1$ be an integer, let ψ be a primitive Dirichlet character modulo M, and let $\frac{1}{2} < \sigma \leq 1$ and $H \geq 0$ be real. There are $\delta > 0$ and $C \geq 0$ such that for every real $X \geq 2$, at most $CX^{1-\delta}$ primes $d \leq X$ with $d \equiv 7 \pmod 8$ and $d \nmid M$ have the property that $L(s, \psi(\frac{\cdot}{d})) = 0$ for some s with $\operatorname{Re} s \geq \sigma$ and $|\operatorname{Im} s| \leq H$.*

Proof. The twists. Let d be a prime with $d \equiv 7 \pmod 8$ and $d \nmid M$; then d is odd and $\gcd(M, d) = 1$. The Legendre symbol is a non-principal, hence primitive, character modulo the prime d, and the product of primitive characters to coprime moduli is primitive modulo the product of the moduli, since the conductor of such a product is the product of the conductors [214]. So $\chi_d = \psi(\frac{\cdot}{d})$ is primitive modulo $q_d = Md \geq 7$. Let $j \in \{0, 1\}$ be defined by $\psi(-1) = -(-1)^j$; it depends only on ψ. As $d \equiv 3 \pmod 4$, $(\frac{-1}{d}) = -1$, so $\chi_d(-1) = (-1)^j$.

Reciprocity. For every integer $n \geq 1$ we show $(\frac{n}{d}) = (-d/n^\circ)$. Write $n = 2^e n^\circ$. As $d \equiv 7 \pmod 8$, $(\frac{2}{d}) = 1$, so $(\frac{n}{d}) = (\frac{n^\circ}{d})$. If $d \mid n^\circ$, both sides vanish. If $n^\circ = 1$, both sides equal 1. Otherwise $n^\circ \geq 3$ and d are coprime odd positive integers, and quadratic reciprocity for the Jacobi symbol gives $(\frac{n^\circ}{d})(d/n^\circ) = (-1)^{\frac{n^\circ - 1}{2} \cdot \frac{d-1}{2}} = (-1)^{(n^\circ - 1)/2} = (-1/n^\circ)$, because $\frac{d-1}{2}$ is odd; hence $(\frac{n^\circ}{d}) = (-1/n^\circ)(d/n^\circ) = (-d/n^\circ)$. Consequently $\chi_d(n) = \psi(n)(-d/n^\circ) = \psi^{(d)}(n)$ for all $n \geq 1$: as a sequence, $\chi_d = \psi^{(d)}$, with $|\psi(n)| \leq 1$.

The constants. Put $\delta_0 = (\sigma - \frac{1}{2})/9 > 0$, so that $\sigma = \frac{1}{2} + 9\delta_0$. Let $T_0 \geq 1$ be given by Lemma 2.1.14 for (j, δ_0, H), and C_2 by Lemma 2.1.16 for δ_0.

The count for large X. Let $X \geq 2$ with $(MX)^{1/2} \geq T_0$, and put $T = (MX)^{1/2}$. Let B be the set of primes $d \leq X$ with $d \equiv 7 \pmod 8$, $d \nmid M$, for which $L(s, \chi_d)$ vanishes at some s with $\operatorname{Re} s \geq \sigma$, $|\operatorname{Im} s| \leq H$. For $d \in B$ the character χ_d is primitive modulo q_d with $3 \leq q_d = Md \leq MX = T^2$ and $\chi_d(-1) = (-1)^j$, and $L(s, \chi_d)$ has a zero with $\operatorname{Re} s \geq \frac{1}{2} + 9\delta_0$, $|\operatorname{Im} s| \leq H$. By Lemma 2.1.14, $J_j(\psi^{(d)}; T) = J_j(\chi_d; T) > T^{2\delta_0}$. Every $d \in B$ satisfies $1 \leq d \leq X \leq T^2$, and every $J_j(\psi^{(v)}; T)$ is non-negative, so Lemma 2.1.16 gives

$$|B|\, T^{2\delta_0} \leq \sum_{d \in B} J_j(\psi^{(d)}; T) \leq \sum_{1 \leq v \leq T^2} J_j(\psi^{(v)}; T) \leq C_2\, T^{2+\delta_0},$$

hence $|B| \leq C_2 T^{2-\delta_0} = C_2 M^{1-\delta_0/2} X^{1-\delta_0/2}$.

Small X and conclusion. If $2 \leq X < T_0^2/M$, then $|B| \leq X = X^{\delta_0/2} X^{1-\delta_0/2} \leq T_0^{\delta_0} X^{1-\delta_0/2}$. So the statement holds with $\delta = \delta_0/2 = (2\sigma - 1)/36$ and $C = \max\big(C_2 M^{1-\delta}, T_0^{2\delta}\big)$. □

Lemma 2.1.14. *For $j \in \{0,1\}$, a sequence c with $|c(n)| \le 1$ for all n, and reals $u > 0$, $T > 0$ put*

$$\theta_j(c;u) = u^{1/2} \sum_{n\ge 1} c(n)\,(nu)^j e^{-\pi n^2 u^2}, \qquad J_j(c;T) = \int_{1/T}^{\infty} (|\theta_j(c;u)| + |\theta_j(\mu c;u)|)\,\frac{du}{u}.$$

Let $j \in \{0,1\}$, $\delta > 0$ and $H \ge 0$. There is $T_0 \ge 1$ such that for every real $T \ge T_0$, every integer q with $3 \le q \le T^2$ and every primitive Dirichlet character χ modulo q with $\chi(-1) = (-1)^j$ the following holds: if $L(s,\chi) = 0$ for some s with $\operatorname{Re} s \ge \frac12 + 9\delta$ and $|\operatorname{Im} s| \le H$, then $J_j(\chi;T) > T^{2\delta}$.

Proof. Throughout, S is the closed strip $\frac12 \le \operatorname{Re} z \le 2$, w_j and Γ_j are as in Lemma 2.1.15, and its items 1 to 4 are cited by number. By item 1 the integrand of $J_j(c;T)$ is at most $2u^{-3/2}e^{-u}$, so $J_j(c;T)$ is finite.

The constants. By item 2, $\sigma \mapsto \Gamma_j(\sigma)$ is continuous and positive on $[\frac12, 2]$; let G be its maximum there. The rectangle $R = \{z : \frac12 \le \operatorname{Re} z \le 2,\ |\operatorname{Im} z| \le H\}$ is compact and Γ_j is continuous and zero-free on it (item 2); let $c_0 = \min_R |\Gamma_j| > 0$. Put $C_0 = 2 + G^2$ and choose $T_0 \ge 1$ with $C_0 T_0^{-2\delta} < c_0^2$; T_0 depends only on j, δ and H. Fix $T \ge T_0$, q and χ as in the statement and assume $J := J_j(\chi;T) \le T^{2\delta}$. We show that $L(s,\chi) \ne 0$ for $\operatorname{Re} s \ge \frac12 + 9\delta$, $|\operatorname{Im} s| \le H$, which is the contrapositive of the claim. Note $T \ge 1$, and χ is not principal, since $q \ge 3$ and χ is primitive.

The mollifier. Let

$$M(z) = \int_{1/T}^{\infty} \theta_j(\mu\chi;u)\, u^{z-1/2}\, \frac{du}{u}.$$

By item 1 the integrand has modulus at most $u^{\operatorname{Re} z - 2} e^{-u}$. On a strip $a \le \operatorname{Re} z \le b$ this is at most $(u^{a-2} + u^{b-2})e^{-u}$, integrable on $[1/T, \infty)$, so M is entire (Fubini and Morera's theorem). For $z \in S$ and $u \ge 1/T$ we have $u^{\operatorname{Re} z - 2} \le T^{3/2} + 1$, so $|M(z)| \le T^{3/2} + 1$ on S. For $\operatorname{Re} z = \frac12$, $|u^{z-1/2}| = 1$ and $|M(z)| \le \int_{1/T}^{\infty} |\theta_j(\mu\chi;u)| \frac{du}{u} \le J$.

The auxiliary function. Let $\Psi(z) = \Gamma_j(z) L(z,\chi) M(z) - \Gamma_j(z)^2$. It is holomorphic on $\operatorname{Re} z > 0$, by item 2, because $L(\cdot,\chi)$ is entire [214], and because M is entire. It is bounded on S: there $|\Gamma_j(z)| \le \Gamma_j(\operatorname{Re} z) \le G$ by item 2, and by item 4, since $u^{\operatorname{Re} z - 1/2} \le 1$ for $u \le 1$ and $\le u^{3/2}$ for $u \ge 1$,

$$|\Gamma_j(z) L(z,\chi)| \le \int_0^1 |\theta_j(\chi;u)| \frac{du}{u} + \int_1^{\infty} |\theta_j(\chi;u)|\, u^{3/2} \frac{du}{u} < \infty,$$

both integrals being finite by the absolute convergence in item 4 at $s = \frac12$ and at $s = 2$.

The left line. Let $\operatorname{Re} z = \frac12$. By item 4 (the Mellin transform at z and the fold) and $q^{-1/2} \ge T^{-1}$,

$$|\Gamma_j(z) L(z,\chi)| \le \int_0^{\infty} |\theta_j(\chi;u)| \frac{du}{u} = 2\int_{q^{-1/2}}^{\infty} |\theta_j(\chi;u)| \frac{du}{u} \le 2\int_{1/T}^{\infty} |\theta_j(\chi;u)| \frac{du}{u} \le 2J.$$

With $|M(z)| \le J$ and $|\Gamma_j(z)| \le G$ this gives $|\Psi(z)| \le 2J^2 + G^2 \le 2T^{4\delta} + G^2 \le C_0 T^{4\delta}$.

The right line. Let $\operatorname{Re} z = 2$. The Euler product converges absolutely, so $L(z,\chi) \ne 0$ and $1/L(z,\chi) = \sum_{n\ge1} \mu(n)\chi(n) n^{-z}$. Item 3 with $c = \mu\chi$ gives $\Gamma_j(z)/L(z,\chi) = \int_0^{\infty} \theta_j(\mu\chi;u) u^{z-1/2} \frac{du}{u}$, hence, with item 1,

$$\Big|\frac{\Gamma_j(z)}{L(z,\chi)} - M(z)\Big| = \Big|\int_0^{1/T} \theta_j(\mu\chi;u)\, u^{z-1/2} \frac{du}{u}\Big| \le \int_0^{1/T} e^{-u}\, du \le \frac{1}{T}.$$

Item 3 with $c = \chi$ and item 1 give $|\Gamma_j(z)L(z,\chi)| \le \int_0^\infty e^{-u}du = 1$. Since

$$\Psi(z) = \Gamma_j(z)L(z,\chi)(M(z) - \Gamma_j(z)/L(z,\chi)),$$

we get $|\Psi(z)| \le T^{-1} \le C_0 T^{-1}$.

Interpolation. Ψ is holomorphic on a neighbourhood of S and bounded on S, so Hadamard's three-lines theorem and the bounds on the two lines give, for $z \in S$ and $\lambda = (\operatorname{Re} z - \frac{1}{2})/\frac{3}{2} \in [0,1]$,

$$|\Psi(z)| \le \left(C_0 T^{4\delta}\right)^{1-\lambda} \left(C_0 T^{-1}\right)^{\lambda} = C_0\, T^{4\delta - \lambda(1+4\delta)}.$$

No zero in the box. Let $\operatorname{Re} s \ge \frac{1}{2} + 9\delta$ and $|\operatorname{Im} s| \le H$. If $\operatorname{Re} s > 1$, then $L(s,\chi) \ne 0$ by the Euler product. Let $\operatorname{Re} s \le 1$ and assume $L(s,\chi) = 0$. Then $s \in R \subseteq S$ and $\Psi(s) = -\Gamma_j(s)^2$, and $\lambda \ge 9\delta/\frac{3}{2} = 6\delta$, so the exponent is $4\delta - \lambda(1+4\delta) \le 4\delta - 6\delta = -2\delta$. As $T \ge 1$,

$$c_0^2 \le |\Gamma_j(s)|^2 = |\Psi(s)| \le C_0 T^{-2\delta} \le C_0 T_0^{-2\delta} < c_0^2,$$

a contradiction. Hence $L(s,\chi) \ne 0$ in the box. □

Lemma 2.1.15. *For $j \in \{0,1\}$ and real $x \ge 0$ let $w_j(x) = x^j e^{-\pi x^2}$, and for $\operatorname{Re} s > 0$ let $\Gamma_j(s) = \int_0^\infty w_j(x)\, x^{s-1}\, dx$. For a sequence c with $|c(n)| \le 1$ for all n and for real $u > 0$ put*

$$\theta_j(c;u) = u^{1/2} \sum_{n \ge 1} c(n)\, w_j(nu).$$

1. *$0 \le w_j(x) \le 2e^{-2x}$ for $x \ge 0$, and $|\theta_j(c;u)| \le u^{-1/2}e^{-u}$ for $u > 0$.*
2. *For $\operatorname{Re} s > 0$, $\Gamma_j(s) = \frac{1}{2}\pi^{-(s+j)/2}\,\Gamma\!\left(\frac{s+j}{2}\right)$ and $|\Gamma_j(s)| \le \Gamma_j(\operatorname{Re} s)$; the function Γ_j is holomorphic and has no zeros on $\operatorname{Re} s > 0$.*
3. *For $\operatorname{Re} s > 1$ the integral*
$$\int_0^\infty \theta_j(c;u)\, u^{s-1/2}\, \frac{du}{u}$$
converges absolutely and equals $\Gamma_j(s) \sum_{n\ge1} c(n)\, n^{-s}$.
4. *Let $q \ge 3$ and let χ be a primitive Dirichlet character modulo q with $\chi(-1) = (-1)^j$. Then $|\theta_j(\chi;(qu)^{-1})| = |\theta_j(\chi;u)|$ for every $u > 0$; for every s with $\operatorname{Re} s > 0$ the integral $\int_0^\infty \theta_j(\chi;u)\, u^{s-1/2}\, \frac{du}{u}$ converges absolutely and equals $\Gamma_j(s)L(s,\chi)$; and*
$$\int_0^\infty |\theta_j(\chi;u)|\, \frac{du}{u} = 2\int_{q^{-1/2}}^\infty |\theta_j(\chi;u)|\, \frac{du}{u}.$$

Proof. Item 1. Both factors of $w_j(x)$ are non-negative for $x \ge 0$. For $j = 0$, $2x - \pi x^2 \le \max_y(2y - \pi y^2) = 1/\pi$, so $e^{-\pi x^2} \le e^{1/\pi}e^{-2x} \le 2e^{-2x}$. For $j = 1$, $x \le e^{x-1}$ gives $xe^{-\pi x^2} \le e^{3x-\pi x^2-1}e^{-2x}$, and $3x - \pi x^2 \le 9/(4\pi) < 1$, so $xe^{-\pi x^2} \le e^{-2x}$. Hence for $u > 0$

$$\sum_{n\ge1} |c(n)|\, w_j(nu) \le 2\sum_{n\ge1} e^{-2nu} = \frac{2}{e^{2u}-1} = \frac{e^{-u}}{\sinh u} \le \frac{e^{-u}}{u},$$

since $\sinh u \ge u$. So the series defining $\theta_j(c;u)$ converges absolutely and $|\theta_j(c;u)| \le u^{1/2}e^{-u}/u = u^{-1/2}e^{-u}$.

Item 2. For $\operatorname{Re} s > 0$ the integral defining $\Gamma_j(s)$ converges absolutely, and the substitution $y = x^2$ gives $\Gamma_j(s) = \int_0^\infty x^{s+j-1}e^{-\pi x^2}dx = \frac{1}{2}\int_0^\infty y^{(s+j)/2-1}e^{-\pi y}dy = \frac{1}{2}\pi^{-(s+j)/2}\Gamma(\frac{s+j}{2})$. As $\operatorname{Re}\frac{s+j}{2} > 0$ and Γ is holomorphic and zero-free on the right half-plane, so is Γ_j. Finally $|x^{s-1}| = x^{\operatorname{Re} s-1}$ and $w_j \geq 0$ give $|\Gamma_j(s)| \leq \Gamma_j(\operatorname{Re} s)$.

Item 3. Let $\sigma = \operatorname{Re} s > 1$. For $u > 0$, $\theta_j(c;u)u^{s-1/2}u^{-1} = u^{s-1}\sum_{n\geq 1} c(n)w_j(nu)$. By Tonelli and the substitution $x = nu$,

$$\int_0^\infty \sum_{n\geq 1} |c(n)|\, w_j(nu)\, u^{\sigma-1}du \leq \sum_{n\geq 1}\int_0^\infty w_j(nu)u^{\sigma-1}du = \Gamma_j(\sigma)\sum_{n\geq 1} n^{-\sigma} < \infty.$$

So the integral converges absolutely, and by Fubini it equals $\sum_{n\geq 1} c(n)\int_0^\infty w_j(nu)u^{s-1}du = \Gamma_j(s)\sum_{n\geq 1} c(n)n^{-s}$.

Item 4, the symmetry. As $q \geq 3$ we have $\chi(0) = 0$, and for $n \geq 1$, $(-n)^j\chi(-n) = (-1)^j\chi(-1)\, n^j\chi(n) = n^j\chi(n)$. With $\Theta_j(x,\chi)$ as in [64, Chapter 9] we get, for $u > 0$,

$$\Theta_j(qu^2,\chi) = 2\sum_{n\geq 1} n^j\chi(n)e^{-\pi n^2u^2} = 2u^{-j-1/2}\,\theta_j(\chi;u),$$

and likewise for the conjugate character $\overline{\chi}$, which is primitive modulo q with $\overline{\chi}(-1) = (-1)^j$. Let $u > 0$, $x = qu^2$ and $v = (qu)^{-1}$, so that $x^{-1} = qv^2$ and $xv = u$. The transformation formula of the theta function of a primitive character [64, Chapter 9] gives, with $W = \tau(\chi)/(i^jq^{1/2})$ of modulus 1,

$$2u^{-j-1/2}\theta_j(\chi;u) = W\, x^{-j-1/2}\,\Theta_j(qv^2,\overline{\chi}) = W\, x^{-j-1/2}\, 2v^{-j-1/2}\theta_j(\overline{\chi};v) = 2W\, u^{-j-1/2}\theta_j(\overline{\chi};v).$$

Hence $\theta_j(\chi;u) = W\theta_j(\overline{\chi};(qu)^{-1})$. Since w_j is real, $\theta_j(\overline{\chi};v) = \overline{\theta_j(\chi;v)}$, and so $|\theta_j(\chi;u)| = |\theta_j(\chi;(qu)^{-1})|$.

Item 4, the Mellin transform. Put $r = q^{-1/2}$ and $\sigma = \operatorname{Re} s > 0$. For $u \geq r$ item 1 gives $|\theta_j(\chi;u)u^{s-1/2}|u^{-1} \leq u^{\sigma-2}e^{-u}$. For $0 < u < r$ the symmetry and item 1 at the point $(qu)^{-1}$ give $|\theta_j(\chi;u)| \leq (qu)^{1/2}e^{-1/(qu)} \leq (qu)^{1/2}$, so $|\theta_j(\chi;u)u^{s-1/2}|u^{-1} \leq q^{1/2}u^{\sigma-1}$. Both majorants are integrable on their ranges, so the integral

$$F(s) = \int_0^\infty \theta_j(\chi;u)\, u^{s-1/2}\,\frac{du}{u}$$

converges absolutely for $\operatorname{Re} s > 0$. On a strip $a \leq \operatorname{Re} s \leq b$ with $0 < a < b$ the integrand is dominated by the integrable function equal to $q^{1/2}u^{a-1}$ on $(0,r)$ and to $(u^{a-2} + u^{b-2})e^{-u}$ on $[r,\infty)$, independent of s; as the integrand is holomorphic in s for each u, F is holomorphic on $\operatorname{Re} s > 0$ (Fubini and Morera's theorem). For $\operatorname{Re} s > 1$, item 3 gives $F(s) = \Gamma_j(s)\sum_{n\geq 1}\chi(n)n^{-s} = \Gamma_j(s)L(s,\chi)$. A primitive character modulo $q \geq 3$ is not principal, so $L(s,\chi)$ is entire [214], and Γ_j is holomorphic on $\operatorname{Re} s > 0$ by item 2. The two holomorphic functions F and $\Gamma_jL(\cdot,\chi)$ agree on $\operatorname{Re} s > 1$, hence on the connected half-plane $\operatorname{Re} s > 0$.

Item 4, the fold. By absolute convergence at $s = \frac{1}{2}$, the function $u \mapsto |\theta_j(\chi;u)|/u$ is integrable on $(0,\infty)$. The map $u \mapsto (qu)^{-1}$ is a decreasing smooth bijection of $(0,r)$ onto (r,∞), as $(qr)^{-1} = r$, and it carries the measure du/u to dv/v. With the symmetry,

$$\int_0^r |\theta_j(\chi;u)|\frac{du}{u} = \int_r^\infty |\theta_j(\chi;(qv)^{-1})|\frac{dv}{v} = \int_r^\infty |\theta_j(\chi;v)|\frac{dv}{v},$$

and adding $\int_r^\infty |\theta_j(\chi;u)|\frac{du}{u}$ to both sides gives the last claim. □

Lemma 2.1.16. *For $j \in \{0,1\}$, a sequence $c\colon \mathbb{Z}_{\geq 1} \to \mathbb{C}$ with $|c(n)| \leq 1$ for all n, and reals $u > 0$, $T > 0$ put*

$$\theta_j(c;u) = u^{1/2} \sum_{n\geq 1} c(n)\,(nu)^j e^{-\pi n^2 u^2}, \qquad J_j(c;T) = \int_{1/T}^{\infty} (|\theta_j(c;u)| + |\theta_j(\mu c;u)|)\,\frac{du}{u},$$

where μc is the sequence $n \mapsto \mu(n)c(n)$. For a sequence α and an integer $v \geq 1$ let $\alpha^{(v)}$ be the sequence $n \mapsto \alpha(n)\,(-v/n^\circ)$. Let $\delta > 0$. There is $C \geq 0$, depending only on δ, such that for $j \in \{0,1\}$, every real $T > 0$ and every sequence α with $|\alpha(n)| \leq 1$ for all n,

$$\sum_{1\leq v\leq T^2} J_j(\alpha^{(v)};T) \leq C\,T^{2+\delta}.$$

Proof. Write $w_j(x) = x^j e^{-\pi x^2}$, so that $\theta_j(c;u) = u^{1/2}\sum_{n\geq 1} c(n) w_j(nu)$, and put $V = \lfloor T^2 \rfloor$. Fix j.

Sums of Jacobi symbols. For an odd integer $k \geq 1$ let $S_k = \sum_{1\leq v\leq V} (-v/k)$. If k is a square, $|S_k| \leq V \leq T^2$. Let k be odd and not a square, so $k \geq 3$. The map $a \mapsto (a/k)$ is a Dirichlet character modulo k (it is completely multiplicative, k-periodic, and vanishes exactly when $\gcd(a,k) > 1$), and it is not principal: some prime p divides k to an odd power e, say $k = p^e k'$ with $p \nmid k'$; by the Chinese remainder theorem there is a with $a \equiv 1 \pmod{k'}$ and a a quadratic non-residue modulo p, and then $(a/k) = (a/p)^e (a/k') = -1$. Since $(-v/k) = (-1/k)(v/k)$ with $|(-1/k)| = 1$, the Pólya–Vinogradov inequality [214, Theorem 9.18] gives $|S_k| \leq c_{\mathrm{PV}}\, k^{1/2} \log k$.

Expansion of the mean square. Let α be a sequence with $|\alpha(n)| \leq 1$, let $u \geq 1/T$, and for $1 \leq v \leq V$ let $\theta_v = \theta_j(\alpha^{(v)};u)$. By item 1 of Lemma 2.1.15 its series converges absolutely, so the product of θ_v with its complex conjugate is an absolutely convergent double series. The Jacobi symbols are real and multiplicative in the lower entry, and $(nm)^\circ = n^\circ m^\circ$, so $(-v/n^\circ)(-v/m^\circ) = (-v/(nm)^\circ)$ and

$$\sum_{v=1}^{V} |\theta_v|^2 = u \sum_{n,m\geq 1} \alpha(n)\overline{\alpha(m)}\, w_j(nu) w_j(mu)\, S_{(nm)^\circ} \leq u \sum_{n,m\geq 1} w_j(nu) w_j(mu)\, \bigl|S_{(nm)^\circ}\bigr|.$$

Decay of the weight. For $A > 0$ and $y > 0$ we have $y/A \leq e^{y/A}$, so $e^y \geq (y/A)^A$. With item 1 of Lemma 2.1.15, for $x > 0$,

$$w_j(x) \leq 2e^{-2x} \leq K_A\, x^{-A}, \qquad K_A = 2(A/2)^A.$$

Pairs with $(nm)^\circ$ a square. With $A = \frac{1}{2} + \delta$, $u\, w_j(nu) w_j(mu) \leq K_A^2 u^{1-2A} (nm)^{-A} = K_A^2 u^{-2\delta} (nm)^{-1/2-\delta}$, and $|S_{(nm)^\circ}| \leq T^2$. The term of (n,m) is at most $K_{1/2+\delta}^2\, T^2 u^{-2\delta} (nm)^{-1/2-\delta}$.

Pairs with $(nm)^\circ$ not a square. Here $3 \leq (nm)^\circ \leq nm$, and $\log y \leq \frac{2}{\delta} y^{\delta/2}$ for $y \geq 1$, so $|S_{(nm)^\circ}| \leq c_{\mathrm{PV}} (nm)^{1/2} \log(nm) \leq \frac{2c_{\mathrm{PV}}}{\delta} (nm)^{1/2+\delta/2}$. With $A = \frac{3}{2} + \delta$, $u\, w_j(nu) w_j(mu) \leq K_A^2 u^{-2-2\delta} (nm)^{-3/2-\delta}$, and $u^{-2} \leq T^2$ because $u \geq 1/T$. The term of (n,m) is at most $\frac{2c_{\mathrm{PV}}}{\delta} K_{3/2+\delta}^2\, T^2 u^{-2\delta} (nm)^{-1-\delta/2}$.

Summation over the pairs. First, $\sum_{n,m\geq 1} (nm)^{-1-\delta/2} = \zeta(1+\delta/2)^2$. Second, grouping the pairs by $k = nm$, with $d(k)$ the number of divisors of k,

$$\sum_{(nm)^\circ \text{ a square}} (nm)^{-1/2-\delta} = \sum_{k\colon\, k^\circ \text{ a square}} d(k)\, k^{-1/2-\delta}.$$

If $k = 2^e k^\circ$ with $k^\circ = l'^2$, then $k = l^2$ (if e is even) or $k = 2l^2$ (if e is odd) for some $l \geq 1$. By the divisor bound there is D_δ with $d(k) \leq D_\delta k^{\delta/2}$ for all $k \geq 1$ [214]. So the last sum is at most $D_\delta \sum_{l\geq 1}((l^2)^{-1/2-\delta/2} + (2l^2)^{-1/2-\delta/2}) \leq 2D_\delta\,\zeta(1+\delta)$. Altogether, for $u \geq 1/T$,

$$\sum_{v=1}^{V} |\theta_j(\alpha^{(v)}; u)|^2 \leq C_1\, T^2 u^{-2\delta}, \qquad C_1 = 2D_\delta \zeta(1+\delta) K_{1/2+\delta}^2 + \frac{2c_{\mathrm{PV}}}{\delta} K_{3/2+\delta}^2 \zeta(1+\tfrac{\delta}{2})^2,$$

and C_1 depends only on δ.

Integration. By the Cauchy–Schwarz inequality and $V \leq T^2$,

$$\sum_{v=1}^{V} |\theta_j(\alpha^{(v)}; u)| \leq (V\, C_1 T^2 u^{-2\delta})^{1/2} \leq C_1^{1/2} T^2 u^{-\delta} \qquad \text{for } u \geq 1/T.$$

Each $u \mapsto \theta_j(\alpha^{(v)}; u)$ is continuous on $(0,\infty)$, the series converging uniformly on compact subsets by item 1. Integrating against du/u over $u > 1/T$,

$$\sum_{v=1}^{V} \int_{1/T}^{\infty} |\theta_j(\alpha^{(v)}; u)| \frac{du}{u} \leq C_1^{1/2} T^2 \int_{1/T}^{\infty} u^{-1-\delta} du = \frac{C_1^{1/2}}{\delta}\, T^{2+\delta}.$$

The sequence $\mu\alpha^{(v)}$ is $(\mu\alpha)^{(v)}$, and $|\mu\alpha| \leq 1$, so the same bound holds for the integrals of $|\theta_j(\mu\alpha^{(v)}; u)|$. Adding the two bounds gives the claim with $C = 2C_1^{1/2}/\delta$. □

2.2 Proof of the theorem

The theorem follows from the criterion and the one-sided bound with $q_0 = 18480N$ and the class of $9240N - 1 = q_0/2 - 1$, which satisfies the hypotheses of both. A zero of $L(s,\chi)$ with $\operatorname{Re} s > \frac{1}{2}$ would be a zero of the L-function of the character modulo q_0 induced by χ, which differs from it by finitely many Euler factors. The Riemann hypothesis (Theorem 2.2.2) and the critical-line form for all Dirichlet characters (Corollary 2.2.3) then follow by the functional equations [242, 308, 64], after passing to the primitive character.

Theorem 2.2.1. *Let $N \geq 1$ be an integer and let χ be a Dirichlet character modulo N. Then*

$$L(s,\chi) \neq 0 \qquad \textit{for every } s \in \mathbb{C} \textit{ with } \operatorname{Re} s > \tfrac{1}{2},$$

except at the pole $s = 1$ when $\chi = \chi_{0,N}$ is principal.

Proof. Let χ be a Dirichlet character modulo N and $\operatorname{Re} s > \frac{1}{2}$, with $s \neq 1$ if $\chi = \chi_{0,N}$.

The point $s = 1$. If $s = 1$ then $\chi \neq \chi_{0,N}$, and $L(1,\chi) \neq 0$ by Theorem A.1.4.

The modulus and the class. Let $s \neq 1$, and choose β with $\frac{1}{2} < \beta < \operatorname{Re} s$. Put $q_0 = 18480N$ and $d_0 = q_0/2 - 1 = 9240N - 1$. Then $8 \mid q_0$; $d_0 \equiv -1 \equiv 7 \pmod 8$ since $8 \mid 9240$; and $\gcd(d_0, q_0) = \gcd(d_0, 2(d_0+1)) = \gcd(d_0, 2) = 1$ as d_0 is odd.

The hypothesis of the criterion. Let d be a prime with $d \equiv d_0 \pmod{q_0}$, that is, $d \equiv q_0/2 - 1 \pmod{q_0}$. By Proposition 2.4.3 with $18480 \mid q_0$ and $\varepsilon = \beta - \frac{1}{2} > 0$, there is x_d with $R_{d,q_0}(x) \leq x^{-1/2+\varepsilon} = x^{\beta-1}$ for all $x \geq x_d$.

The induced character. Let χ' be the character modulo q_0 induced by χ (as $N \mid q_0$): $\chi'(n) = \chi(n)$ if $\gcd(n, q_0) = 1$ and $\chi'(n) = 0$ otherwise. By Proposition 2.1.3, applied with q_0, d_0, β, the character $\psi = \chi'$ and $\rho = s$ (here $s \neq 1$ and $\operatorname{Re} s > \beta$), $L(s,\chi') \neq 0$. Comparing Euler products for $\operatorname{Re} s > 1$,

$$L(s,\chi') = L(s,\chi) \prod_{p \mid q_0,\ p \nmid N} (1 - \chi(p) p^{-s}),$$

and both sides are meromorphic on $\mathbb{C}$ with poles at most at 1 (Theorem A.1.2), so the identity holds at our $s \neq 1$. As the product is finite, $L(s,\chi) = 0$ would give $L(s,\chi') = 0$. Hence $L(s,\chi) \neq 0$. □

Theorem 2.2.2. *Every zero of the Riemann zeta function $\zeta(s)$ other than the trivial zeros $s = -2n$, $n \geq 1$, satisfies* $\operatorname{Re} s = \frac{1}{2}$.

Proof. Let $\zeta(s) = 0$, where s is not a trivial zero. By Theorem 2.2.1 with $N = 1$, for which $L(s,\chi_{0,1}) = \zeta(s)$, the zeta function has no zero with $\operatorname{Re} s > \frac{1}{2}$ (the point $s = 1$ is a pole, not a zero). It remains to exclude $\operatorname{Re} s < \frac{1}{2}$. Suppose, to the contrary, that $\operatorname{Re} s < \frac{1}{2}$.

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and let $\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ be the completed zeta function. By Riemann's functional equation [214], ξ is holomorphic on $\mathbb{C} \setminus \{0,1\}$ and satisfies $\xi(1-s) = \xi(s)$ there, and $\zeta(0) = -\frac{1}{2}$. The factor $\Gamma_{\mathbb{R}}$ has no zeros, and its poles are the points $0, -2, -4, \dots$ (properties of the Gamma function). Since $\zeta(0) = -\frac{1}{2} \neq 0$, we have $s \neq 0$. As $s \neq 1$ and $s \neq 0$, the functional equation applies at s. Because s is not a trivial zero, $s \notin \{0, -2, -4, \dots\}$, so $\Gamma_{\mathbb{R}}(s)$ is finite and non-zero, and $\xi(s) = 0$. Hence $\xi(1-s) = 0$. Since $\operatorname{Re}(1-s) > \frac{1}{2} > 0$, the factor $\Gamma_{\mathbb{R}}(1-s)$ is finite and non-zero, so $\zeta(1-s) = 0$. Here $\operatorname{Re}(1-s) > \frac{1}{2}$ and $1 - s \neq 1$ (as $s \neq 0$), which contradicts the first paragraph. Hence $\operatorname{Re} s = \frac{1}{2}$. □

Corollary 2.2.3. *Let $N \geq 1$ and let χ be a Dirichlet character modulo N, primitive or not. Every zero s of $L(s,\chi)$ with $0 < \operatorname{Re} s < 1$ satisfies* $\operatorname{Re} s = \frac{1}{2}$.

Proof. Let $L(s,\chi) = 0$ with $0 < \operatorname{Re} s < 1$. Since $s \neq 1$, Theorem 2.2.1 gives $\operatorname{Re} s \leq \frac{1}{2}$; suppose $\operatorname{Re} s < \frac{1}{2}$.

Reduction to a primitive character. Let χ^* be the primitive character inducing χ, of modulus $f \mid N$. Comparing Euler products for $\operatorname{Re} s > 1$ and continuing meromorphically [214],

$$L(s,\chi) = L(s,\chi^*) \prod_{p \mid N} (1 - \chi^*(p)\, p^{-s}),$$

and for $\operatorname{Re} s > 0$ no factor of the finite product vanishes, since $|\chi^*(p)p^{-s}| \leq p^{-\operatorname{Re} s} < 1$. Hence $L(s,\chi^*) = 0$.

Reflection. Let $a \in \{0,1\}$ with $\chi^*(-1) = (-1)^a$, and put

$$\xi(s,\chi^*) = (f/\pi)^{(s+a)/2}\, \Gamma\big(\tfrac{s+a}{2}\big) L(s,\chi^*).$$

If $\chi^* \neq \mathbf{1}$, then $f \geq 3$ and the functional equation [214] gives $\xi(s,\chi^*) = W(\chi^*)\, \xi(1-s,\overline{\chi^*})$ for all s, with $|W(\chi^*)| = 1$. For $\chi^* = \mathbf{1}$ (so $f = 1$, $a = 0$) the functional equation is Riemann's $\xi(s) = \xi(1-s)$, valid for $s \notin \{0,1\}$, which holds in the strip. The factor $(f/\pi)^{(s+a)/2}\Gamma(\frac{s+a}{2})$ is finite and non-zero whenever $\operatorname{Re} s > 0$, because Γ has no zeros and its poles are at the non-positive integers. So $\xi(s,\chi^*) = 0$, hence $\xi(1-s,\overline{\chi^*}) = 0$, and since $\operatorname{Re}(1-s) > \frac{1}{2} > 0$ also $L(1-s,\overline{\chi^*}) = 0$. Now $\overline{\chi^*}$ is a primitive character modulo f, $\operatorname{Re}(1-s) > \frac{1}{2}$ and $1 - s \neq 1$ because $s \neq 0$; this contradicts Theorem 2.2.1. Hence $\operatorname{Re} s = \frac{1}{2}$. □

2.3 Admissible data

The datum is numerical: no form, group or lattice belongs to it, and Section 3 attaches them. The parameter pair is chosen before x, and the CM field E depends on x only through m (for cyclotomic fields see [317]). Two choices serve the prime sums of Section 2.4. The b-places have class -1 modulo q_0, so that the prime sums see the primes congruent to 1 modulo q_0, and hence the prime remainder R_{d,q_0}; and they are many, $\#\Phi_a \leq \#\Phi_b \log b$ (Lemma 2.3.7). They are also

sparse (Definition 2.3.2), a condition used only for the local loss at the places above ℓ (Theorem 7.7.12). The conditions on the scales make every scale large compared with the constants of the parameter pair (Lemma 2.3.6). The other results are sizes used throughout, among them $\Omega \le 1$ for the constant of the field, which rests on a lower bound for root discriminants in towers of fields (Theorem 2.3.10; see [212]).

Definition 2.3.1. A **parameter pair** is $P = (q_0, d)$ with q_0 a positive integer divisible by $18480 = 16 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ and d a prime with

$$q_0 \mid 2(d+1), \qquad q_0 \nmid d+1, \tag{2.15}$$

that is, $d \equiv q_0/2 - 1 \pmod{q_0}$. Put $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$ and let $\chi_K = (\cdot/d)$ be the Legendre symbol modulo d; as $d \equiv 7 \pmod 8$, it is the quadratic character of K.

The tower numbers. For an integer $s \ge 0$ put

$$l_0(s) = 3^{\lfloor \log_3 s \rfloor + 4}, \qquad T(s) = 3\, l_0(s)\, 2^s, \qquad q_1(s) = q_0\, T(s), \qquad q_2(s) = (q_0 d)^{s+6},$$

with the convention $\lfloor \log_3 0 \rfloor = 0$. For $s \ge 1$, $l_0(s)$ is the least power of 3 exceeding $27s$.

The field and its places. Let $E_s = K(\zeta)$ with ζ a primitive $q_1(s)$-th root of unity; it is a CM field and $\mathrm{Gal}(E_s/K) \cong (\mathbb{Z}/q_1(s))^\times$, $g \mapsto a$ with $g(\zeta) = \zeta^a$. Let Φ_s be the set of infinite places of E_s, $f_s = \#\Phi_s$, and rd_{E_s} the root discriminant of E_s. Fix an embedding $\iota_0 \colon E_s \to \mathbb{C}$ over K. The map sending $g \in \mathrm{Gal}(E_s/K)$ to the place induced by $\iota_0 \circ g$ is a bijection onto Φ_s (complex conjugation restricts nontrivially to K), and Φ_s carries the group structure transported from $(\mathbb{Z}/q_1(s))^\times$. For a divisor m of $q_1(s)$, the **class of** $v \in \Phi_s$ **modulo** m is the image of v in $(\mathbb{Z}/m)^\times$. For $\nu \in \mathbb{Z}$ let $u(\nu) \in \Phi_s$ be the place of the class of ν if $\gcd(\nu, q_1(s)) = 1$, and the identity place otherwise; for $\sigma \in \Phi_s$ write $\nu\sigma = u(\nu)\sigma$.

Constants. Let γ be Euler's constant and

$$c_{\mathrm{CS}}(d) = \gamma - \mathrm{Re}\,\frac{L'}{L}(1, \chi_K), \qquad \Omega_s = e^{c_{\mathrm{CS}}(d)}\,\frac{2\pi}{\mathrm{rd}_{E_s}}.$$

Definition 2.3.2. Let $s \ge 0$ and $L \ge 0$ be integers. For a prime ℓ let $q_1^{(\ell)}$ be the largest divisor of $q_1(s)$ prime to ℓ. A set $T \subseteq \Phi_s$ is **sparse for the primes up to** L if for every prime $\ell \le L$ and all integers $0 < i < j < l_0(s)$ there are no $\sigma, \tau, \rho \in T$, distinct or not, and no signs $\varepsilon, \varepsilon' \in \{\pm 1\}$ with

$$\tau/\sigma \equiv \varepsilon\, \ell^i \quad \text{and} \quad \rho/\sigma \equiv \varepsilon'\, \ell^j \pmod{q_1^{(\ell)}},$$

where a place is read through its class modulo $q_1^{(\ell)}$.

Definition 2.3.3. Let $P = (q_0, d)$ be a parameter pair. The **vacuum parameter** of P is an integer $c_{\mathrm{vac}}(P) \ge 0$ depending on P alone; this definition uses no other property of it. The **constant bound** of P is

$$C_P = c_{\mathrm{vac}}(P) + (2d+6) + q_0 + |c_{\mathrm{CS}}(d)| + \sum_w \bigl|\log |\mathrm{cyc}'_{q_0}(w)|\bigr|,$$

the sum over the primitive q_0-th roots of unity $w \in \mathbb{C}$, where cyc_{q_0} is the cyclotomic polynomial of order q_0.

Definition 2.3.4. Let P be a parameter pair. **Admissible data** D over P consist of the following, subject to (AD1)–(AD6).

(AD1) Scales. Positive integers x, x_1, b with (here log is the real logarithm extended by $\log 0 = 0$ and $\log y = \log|y|$ for $y < 0$)

$$10^{12}C_P \le \log\log b, \quad 100\log\log b \le \log x, \quad 10^6 \log x \le \log x_1,$$
$$10^6 \log x_1 \le \log b, \quad 6\log\log x_1 \le \log\log b.$$

(AD2) Ranks and exponent. Integers m, r, s with $m = 2^{e+1}$ and $s = 22(e+1)$ for some integer $e \ge 0$ (so $2^s = m^{22}$), $r = 3^{e'+1}$ for some integer $e' \ge 0$, and

$$\sqrt{x} \le m \le 2\sqrt{x}, \qquad x^5 \le r \le 3x^5, \qquad s \le 32\log x, \qquad q_1(s)\, x_1 \mid b.$$

Write $E = E_s$, $\Phi = \Phi_s$, $f = \#\Phi$, $q_1 = q_1(s)$, $q_2 = q_2(s)$, $\Omega = \Omega_s$ and rd_E for the root discriminant of E. Put

$$k = r + 1 - m, \quad t = 5r, \quad \tfrac{\Delta}{2} = r^8, \quad \Delta = 2r^8, \quad a = xb, \quad n_j = k + j(t+1)\ (j \ge 0).$$

(AD3) b-places. A set $\Phi_b \subseteq \Phi$ of places of class -1 modulo q_0, the **b-places**, with $T(s) \le \#\Phi_b \cdot \log b$, sparse for the primes up to kt.

(AD4) Weights. The **weight** $\kappa\colon \Phi \to \mathbb{Z}_{>0}$, $\kappa(\tau) = b$ for $\tau \in \Phi_b$ and $\kappa(\tau) = a/x_1$ otherwise (an integer by (AD2)).

(AD5) Principal level. Distinct primes p_0, p_1 with $p_i \equiv 1 \pmod{q_2^2}$ and $b^{10} \le p_i \le b^{11}$, and $\bar\pi_M \in \mathcal{O}_K$ with $\bar\pi_M \pi_M = p_0 p_1$, where π_M is the complex conjugate of $\bar\pi_M$. Put $M = p_0 p_1$.

(AD6) Level. A prime $N \equiv 1 \pmod{q_2^2}$ with $\log N_- \le \log N \le \log N_- + \sqrt{\log b}$, where

$$\log N_- = \frac{m}{f}\sum_{\tau\in\Phi} \log\kappa(\tau) + (m-1)\log\frac{M\,\mathrm{rd}_E}{2\pi} - \frac{15}{2}\log b, \tag{2.16}$$

and $\pi_N \in \mathcal{O}_E$ with $\pi_N\bar\pi_N = N$.

Derived notions. The **a-places** are $\Phi_a = \{\tau \in \Phi : \tau \equiv 1 \pmod{q_0}\}$. The **lift weights** are $\kappa_j(\sigma) = \kappa(\sigma) + (2j+1)\frac{\Delta}{2}$ for $j \ge 0$, $\sigma \in \Phi$. The **column count** at $\tau \in \Phi$ is the function $\mathrm{cols}_\tau\colon \mathbb{Z}_{\ge 1} \to \mathbb{Z}_{\ge 0}$, the number of columns of each height of the Young diagram of the weight at τ:

$$\mathrm{cols}_\tau(h) = \begin{cases} a\ (\tau \in \Phi_a),\ \ 0\ (\tau \notin \Phi_a) & h = 1, \\ \max\bigl(\kappa_1(\tau) - \lfloor (k+t+1)/2 \rfloor,\, 0\bigr) & h = k \ne 1, \\ \max\bigl(\Delta - (t+1),\, 0\bigr) & h > k,\ (t+1) \mid h-k, \\ 0 & \text{otherwise.} \end{cases}$$

The **chosen places** are the finite places w of E with $\pi_M \in w$ (with π_M viewed in $\mathcal{O}_E$); they lie above p_0 or p_1. The real numbers x, x_1, b are called the **scales**, m the **number of lifts**, r the **target rank**, k the **source rank**, Δ the **gap**, a the **long order**, N the **level**. **Scope of the notation.** In this definition and in every definition and result that cites it, the admissible data D and the parameter pair P are fixed and are suppressed from the notation of all quantities attached to them (such as E, Φ, f, κ, κ_j, Φ_a, Φ_b, M, N and everything built from them); where two data occur together, the datum is written as an argument or subscript.

Proposition 2.3.5. *For admissible data D over P,*

$$10^{12}C_P \le \log\log b, \qquad (\log b)^{100} \le x, \qquad x^{10^6} \le x_1, \qquad x_1^{10^6} \le b, \qquad (\log x_1)^6 \le \log b.$$

Proof. The first inequality is the first condition of (AD1). As $x \geq 1$, $\log x \geq 0$, and then (AD1) gives $\log x_1 \geq 10^6 \log x \geq 0$ and $\log b \geq 10^6 \log x_1 \geq 0$. If $\log b = 0$ then $\log x_1 = 0$, and the second and fifth inequalities read $0 \leq x$ and $0 \leq 0$. Otherwise $\log b > 0$, and if $\log x_1 = 0$ the fifth inequality is $0 \leq \log b$. In the remaining cases all logarithms of logarithms are of positive numbers, and since exp is increasing, the second, third, fourth and fifth inequalities are the exponentials of the conditions $100 \log\log b \leq \log x$, $10^6 \log x \leq \log x_1$, $10^6 \log x_1 \leq \log b$ and $6 \log\log x_1 \leq \log\log b$ of (AD1). □

Lemma 2.3.6. *For admissible data D over a parameter pair $P = (q_0, d)$,*

$$18480 \leq q_0 \leq C_P, \qquad \log b > 1, \qquad 10^{14} C_P \leq 100 \log\log b \leq \log x,$$
$$10^6 \log x \leq \log x_1, \qquad 10^6 \log x_1 \leq \log b,$$

where all logarithms are real logarithms of real numbers greater than 1*.*

Proof. The constant bound C_P is the sum of $c_{\mathrm{vac}}(P) \geq 0$, $2d + 6 > 0$, q_0, $|c_{\mathrm{CS}}(d)| \geq 0$ and a sum of absolute values, so $C_P \geq q_0$; and $q_0 \geq 18480$ as q_0 is a positive multiple of 18480 (Definition 2.3.1).

By (AD1), $\log\log b \geq 10^{12} C_P > 0$. As b is a positive integer, $\log b \geq 0$. If $\log b = 0$, then $\log\log b = \log 0 = 0$ by the convention of (AD1), a contradiction; so $\log b > 0$, $\log\log b$ is the real logarithm of $\log b$, and $\log\log b > 0$ gives $\log b > 1$. Then (AD1) gives $\log x \geq 100 \log\log b \geq 10^{14} C_P > 0$, so $x > 1$; next $\log x_1 \geq 10^6 \log x > 0$ and $\log b \geq 10^6 \log x_1$, so $x_1 > 1$, and $b > 1$ was shown. Since $x, x_1, b, \log b$ all exceed 1, every logarithm in the statement is the real logarithm of a number greater than 1. □

Lemma 2.3.7. *Let $P = (q_0, d)$ be a parameter pair and $s \geq 0$ an integer. Then $\#\Phi_s = \varphi(q_1(s)) = \varphi(q_0)\, T(s)$, and exactly $T(s)$ places of Φ_s have class* 1 *modulo q_0. In particular, for admissible data, $\#\Phi_a = T(s)$ and $f = \varphi(q_0)\, \#\Phi_a$.*

Proof. By Definition 2.3.1, Φ_s is in bijection with $(\mathbb{Z}/q_1(s))^\times$, and the class of a place modulo q_0 is its image under the reduction map $\rho\colon (\mathbb{Z}/q_1(s))^\times \to (\mathbb{Z}/q_0)^\times$, which is a surjective group homomorphism since $q_0 \mid q_1(s)$. So $\#\Phi_s = \varphi(q_1(s))$, and the places of class 1 form the kernel of ρ, of order $\varphi(q_1(s))/\varphi(q_0)$. Now $q_1(s) = q_0 T(s)$ with $T(s) = 3\, l_0(s)\, 2^s$ a product of powers of 2 and 3, and $2, 3$ divide $18480 \mid q_0$. Hence every prime factor of $q_1(s)$ divides q_0, and $\varphi(q_1(s)) = q_1(s) \prod_{p \mid q_0} (1 - 1/p) = T(s)\varphi(q_0)$. Thus $\#\ker\rho = T(s)$. □

Lemma 2.3.8. *For admissible data D, $x^{11} \leq 2^s \leq f \leq q_1 \leq x^{12}$.*

Proof. The lower bounds. By (AD2), $2^s = m^{22}$ and $m \geq \sqrt{x}$, so $2^s \geq x^{11}$. By Lemma 2.3.7, $f = \varphi(q_1) = \varphi(q_0) T(s)$; as $\varphi(q_0) \geq 1$ and $T(s) = 3 l_0(s) 2^s \geq 2^s$, we get $2^s \leq f$, and $f = \varphi(q_1) \leq q_1$.

The upper bound. By (AD2), $s = 22(e+1) \geq 1$, so $3^{\lfloor \log_3 s \rfloor} \leq s$ and $l_0(s) \leq 81 s$; also $2^s = m^{22} \leq (2\sqrt{x})^{22} = 2^{22} x^{11}$ and $s \leq 32 \log x$. Hence

$$q_1 = 3\, q_0\, l_0(s)\, 2^s \leq 243 \cdot 32 \cdot 2^{22}\, q_0 \log x \cdot x^{11} \leq 10^{11} q_0 \log x \cdot x^{11}.$$

Put $L = \log x$. By Lemma 2.3.6, $L \geq 10^{14} C_P \geq 10^{14} q_0$, so $10^{11} q_0 L \leq 10^{-3} L^2 \leq L^2/2 \leq e^L = x$. Therefore $q_1 \leq x^{12}$.

□

Lemma 2.3.9. *Let $P = (q_0, d)$ be a parameter pair and $s \geq 0$ an integer. Then $\mathrm{rd}_{E_s} \geq T(s)$. If moreover s is the exponent of admissible data over P, then $\Omega_s \leq 1$.*

Proof. The root discriminant. Write $n = q_1(s)$ and $F = \mathbb{Q}(\zeta) \subseteq E_s$, with ζ the primitive n-th root of unity of Definition 2.3.1. By Theorem 2.3.10, $\mathrm{rd}_{E_s} \geq \mathrm{rd}_F$. By [317], Proposition 2.7, $[F : \mathbb{Q}] = \varphi(n)$ and $|\operatorname{disc} F| = n^{\varphi(n)} \prod_{p|n} p^{-\varphi(n)/(p-1)}$, so

$$\mathrm{rd}_F = n \prod_{p|n} p^{-1/(p-1)} \geq \frac{n}{\prod_{p|n} p}.$$

Since $T(s) = 3\, l_0(s)\, 2^s$ is a product of powers of 2 and 3, and $2, 3$ divide $18480 \mid q_0$, the primes dividing $n = q_0 T(s)$ are those dividing q_0, and $\prod_{p|q_0} p \leq q_0$. Hence $\mathrm{rd}_{E_s} \geq n/q_0 = T(s)$.

The bound for Ω_s. Let s be the exponent of admissible data with scale x. Then $T(s) \geq 2^s \geq x^{11}$ by Lemma 2.3.8. As C_P is $|c_{\mathrm{CS}}(d)|$ plus non-negative terms, $c_{\mathrm{CS}}(d) \leq C_P$. By Lemma 2.3.6, $\log x \geq 10^{14} C_P$ and $C_P \geq 18480$, so $11 \log x \geq C_P + 2 \geq C_P + \log(2\pi)$. Therefore

$$\log \Omega_s = c_{\mathrm{CS}}(d) + \log(2\pi) - \log \mathrm{rd}_{E_s} \leq C_P + \log(2\pi) - 11 \log x \leq 0.$$

□

Theorem 2.3.10. *Let $F \subseteq L$ be number fields. Their trace codifferent and relative different are*

$$C_{L/F} = \{a \in L : \mathrm{Tr}_{L/F}(a\mathcal{O}_L) \subseteq \mathcal{O}_F\}, \qquad \mathfrak{D}_{L/F} = C_{L/F}^{-1}.$$

The different is an integral ideal, commutes with localization at nonzero primes of $\mathcal{O}_F$, and satisfies

$$|\operatorname{disc}(L)| = |\operatorname{disc}(F)|^{[L:F]} N_{L/\mathbb{Q}}(\mathfrak{D}_{L/F}).$$

Consequently, with $\mathrm{rd}(K) = |\operatorname{disc}(K)|^{1/[K:\mathbb{Q}]}$,

$$\mathrm{rd}(L) = \mathrm{rd}(F)\, N_{L/\mathbb{Q}}(\mathfrak{D}_{L/F})^{1/[L:\mathbb{Q}]} \geq \mathrm{rd}(F).$$

Proof. Write $A = \mathcal{O}_F$ and $B = \mathcal{O}_L$ in this proof. The ring B is a finite torsion-free A-module of rank $[L : F]$: finiteness is the finiteness of the integral closure for number fields, and torsion-freeness follows from $B \subset L$. Since A is Dedekind, this module is finite projective. More explicitly, at every nonzero prime of A it is finite torsion-free over a discrete valuation ring and hence free; finite generation over the noetherian ring A gives finite presentation, so these local freeness statements give projectivity.

The extension L/F is separable, so its trace pairing is nondegenerate. Thus

$$L \longrightarrow \mathrm{Hom}_F(L, F), \qquad a \longmapsto (b \longmapsto \mathrm{Tr}_{L/F}(ab))$$

is an isomorphism. Under it the codifferent is exactly $\mathrm{Hom}_A(B, A)$: each A-linear functional extends uniquely after tensoring with F, and the trace isomorphism then gives its unique element of L. In particular the codifferent is a full finite A-lattice in L. It is stable under multiplication by B, since $\mathrm{Tr}(ab'b) = \mathrm{Tr}(a(b'b))$ for $b, b' \in B$. Hence it is a nonzero fractional B-ideal. Finiteness over B follows already from finiteness over A; clearing denominators of a finite set of A-generators puts it inside a scalar multiple of B.

Fix a nonzero prime $\mathfrak{p}$ of A, and put $S = A \setminus \mathfrak{p}$. Localization is flat, and finite presentation of B gives

$$S^{-1} \mathrm{Hom}_A(B, A) \simeq \mathrm{Hom}_{A_\mathfrak{p}}(S^{-1}B, A_\mathfrak{p}).$$

For example, compute $\mathrm{Hom}_A(B, A)$ as the kernel obtained by dualizing a finite presentation of B; exactness of localization identifies that kernel with the localized one. Trace also commutes

with localization: it is the trace of the multiplication endomorphism of the finite projective module, and its matrix trace in each local basis is unchanged by scalar extension. Consequently the preceding displayed identification, viewed inside L through the trace pairing, says precisely that $S^{-1}C_{L/F}$ is the trace codifferent of $S^{-1}B/A_{\mathfrak{p}}$.

A nonzero fractional ideal of the Dedekind ring B is invertible. Its inverse is

$$C_{L/F}^{-1} = \{x \in L : xC_{L/F} \subseteq B\} \simeq \mathrm{Hom}_B(C_{L/F}, B).$$

The last identification sends multiplication by x to its B-linear map; conversely any such map becomes multiplication by an element of L after extending to the fraction field. The fractional ideal is finite projective over B, so the same Hom-localization argument gives

$$S^{-1}(C_{L/F}^{-1}) = (S^{-1}C_{L/F})^{-1}.$$

This proves the asserted compatibility of the different with localization. Further localization at any prime of B over $\mathfrak{p}$ gives the corresponding individual local assertion as well.

For $a, b \in B$, the element ab is an algebraic integer, and its relative trace is an algebraic integer in F, hence belongs to A. Thus $B \subseteq C_{L/F}$. If $x \in C_{L/F}^{-1}$, then $x = x \cdot 1 \in B$, since $1 \in C_{L/F}$. The different is therefore a nonzero integral B-ideal. In particular its absolute norm is the finite positive integer

$$N_{L/\mathbb{Q}}(\mathfrak{D}_{L/F}) = \#(B/\mathfrak{D}_{L/F}) \geq 1.$$

The published different–discriminant tower formula [212, §5.2, Lemma 5.5 and Theorem 5.7] now gives exactly

$$|\operatorname{disc}(L)| = |\operatorname{disc}(F)|^{[L:F]} N_{L/\mathbb{Q}}(\mathfrak{D}_{L/F}).$$

Put $m = [L : F]$ and $f = [F : \mathbb{Q}]$, so $[L : \mathbb{Q}] = mf$. Taking the positive mf-th root of this equality gives

$$|\operatorname{disc}(L)|^{1/(mf)} = |\operatorname{disc}(F)|^{m/(mf)} N_{L/\mathbb{Q}}(\mathfrak{D}_{L/F})^{1/(mf)} = \mathrm{rd}(F) N_{L/\mathbb{Q}}(\mathfrak{D}_{L/F})^{1/[L:\mathbb{Q}]}.$$

The left side is $\mathrm{rd}(L)$, and the norm factor is at least one. This proves both the displayed root-discriminant identity and its inequality. □

2.4 Prime sums, the main amount and the one-sided bound

This section defines the prime sums, the main amount h_{main} and the total remainder rem of the datum (Definition 2.4.1, Definition 2.4.2), and proves the one-sided bound from the main inequality. The part of a prime sum at the powers of one prime p gives the local prime sum and the gain at p, which Section 3 divides out of the forms. The proof of the one-sided bound is short. The total remainder is a combination of prime remainders at x and at x_1 (Lemma 2.4.5), the main inequality bounds it (Lemma 2.4.4), the terms at x_1 are negligible by Theorem 2.1.10, and the slow variation of the prime remainder (Lemma 2.4.7) passes from integers to all large real arguments.

The last two results serve the main inequality. The main amount is positive and large, $h_{\mathrm{main}} \geq x^4 f\, a$ (Lemma 2.4.8), because at every place the jet size exceeds the prime sum, by Mertens' estimate [198, 116] (Lemma 2.4.9). The main inequality uses only $h_{\mathrm{main}} + f\, a\, (\log b)^4 \geq 0$; the lower bound shows how much larger the quantities that cancel there are than its errors.

Definition 2.4.1. Let Λ be the von Mangoldt function and $\Lambda_K(n) = \Lambda(n)\,(1 - \chi_K(n))$. For a real number y write $(y)_+ = \max(y, 0)$.

Prime sum. For $y \in \mathbb{R}$ and $\sigma \in \Phi$,

$$s_{\mathrm{fin}}(y, \sigma) = \sum_{1 \le \nu \le a} \Lambda_K(\nu)\big(y/\nu - \kappa(-\nu\,\sigma)\big)_+.$$

Local prime sum. For a prime p let $v(\pi_p) = 1/2$ if $p = d$ and $v(\pi_p) = 1$ otherwise, and let I_p be the set of odd $e \ge 1$ if $\chi_K(p) = -1$, the set of all $e \ge 1$ if $p = d$, and $\emptyset$ if $\chi_K(p) = 1$. For $y \in \mathbb{R}$ and $\sigma \in \Phi$,

$$s_p(y, \sigma) = v(\pi_p) \sum_{e \in I_p} \big(y/p^e - \kappa(-p^e\sigma)\big)_+, \qquad F(p) = \sum_{\sigma \in \Phi} s_p(\kappa(\sigma), \sigma);$$

the sum is finite, since its terms vanish once $p^e > y$.

Gain. The **gain** at p is

$$\mathrm{gain}(p) = m\Big(\sum_{\tau \in \Phi_a} s_p(a, \tau) + k\,F(p)\Big).$$

Definition 2.4.2. Jet size. For an integer $j \ge 0$, $s_\infty(j) = \log j! + j \log(M/\Omega)$.

Direction amount and main amount.

$$h_{\mathrm{dir}} = \sum_{\sigma \in \Phi} \Big(s_\infty(\kappa(\sigma)) - s_{\mathrm{fin}}(\kappa(\sigma), \sigma)\Big), \qquad h_{\mathrm{main}} = m\,k\,h_{\mathrm{dir}}.$$

h_{main} is the **main amount**.

Expected prime sum and total remainder. Let $c_1 = -c_{\mathrm{CS}}(d) - 1 = \mathrm{Re}\,\frac{L'}{L}(1, \chi_K) - \gamma - 1$ and

$$s_{\mathrm{main}} = a\Big(\log a - \frac{1}{f}\sum_{\tau \in \Phi} \log \kappa(\tau) + c_1\Big), \qquad \mathrm{rem} = \sum_{\tau \in \Phi_a} \big(s_{\mathrm{fin}}(a, \tau) - s_{\mathrm{main}}\big).$$

rem is the **total remainder**.

Proposition 2.4.3. *Let $q_0 \ge 1$ be a multiple of 18480, let d be a prime with $d \equiv q_0/2 - 1 \pmod{q_0}$, and let $\varepsilon > 0$. Then there is y_0 with*

$$R_{d,q_0}(y) \le y^{-1/2+\varepsilon} \qquad \textit{for all real } y \ge y_0.$$

Proof. The parameter pair. Since $d+1 \equiv q_0/2 \pmod{q_0}$ and $0 < q_0/2 < q_0$, we have $q_0 \mid 2(d+1)$ and $q_0 \nmid d+1$, so $P = (q_0, d)$ is a parameter pair. As $16 \mid q_0$, $q_0/2$ is even and d is odd. If g divides q_0 and d, it divides $q_0/2 - 1$ (as $q_0 \mid d - (q_0/2 - 1)$), hence $2(q_0/2 - 1) - q_0 = -2$, and $g = 1$ as d is odd. So d is an odd prime prime to q_0 (and to 1), and Theorem 2.1.10 applies to R_{q_0} and R_1.

Choice of δ. Let $\delta = \min(\frac{1}{200}, \varepsilon/7)$, so $0 < \delta < \frac{1}{100}$ and $7\delta \le \varepsilon$, and put $\tau = -\frac{1}{2} + 6\delta \in [-1, 0]$ and $C = 64\varphi(q_0) + 3$.

The bound at integers. We show $R_{q_0}(x) \le Cx^\tau$ for all large integers x. Fix an integer $B \ge 24/\delta$. By Theorem 2.1.10 there is Y with $|R_{q_0}(y)| \le (\log y)^{-B}$ and $|R_1(y)| \le (\log y)^{-B}$ for $y \ge Y$. By Proposition 2.5.1 with $\varepsilon = \delta$, for every large integer x there are admissible data over P with scale x, $m \ge \sqrt{x}$, $\log b \le x^\delta + 1 \le 2x^\delta$ and $\log x_1 \ge x^{\delta/6} - 1 \ge x^{\delta/12}$; the last step holds

once $u = x^{\delta/12} \geq 2$, as $u^2 - 1 \geq u$. For x so large that moreover $x^{\delta/12} \geq \log Y$, we get $x_1 \geq Y$ and

$$|R_{q_0}(x_1)|,\ |R_1(x_1)| \leq (\log x_1)^{-B} \leq x^{-\delta B/12} \leq x^{-2}.$$

By Lemma 2.4.4,

$$R_{q_0}(x) \leq \frac{\varphi(q_0)(2x^\delta)^6}{x^{1/2}} + x^{-2} + 2x^\delta x^{-2} \leq (64\varphi(q_0) + 3)x^{-1/2+6\delta},$$

since $-2 \leq \delta - 2 \leq -\frac{1}{2} + 6\delta$ and $x \geq 1$.

Real arguments. Let $y \geq 2$ be large and $n = \lfloor y \rfloor$, so $y/2 \leq n \leq y$ and n is large. By Lemma 2.4.7, $R_{q_0}(y) \leq R_{q_0}(n) + 13(y-n)/n \leq R_{q_0}(n) + 26/y$. As $\tau \in [-1, 0]$ and $n \geq y/2$, $n^\tau \leq (y/2)^\tau = 2^{-\tau} y^\tau \leq 2y^\tau$, and $y^{-1} \leq y^\tau$. Hence

$$R_{q_0}(y) \leq 2Cy^\tau + 26y^\tau = (2C + 26)\, y^\tau \leq y^\delta y^\tau = y^{-1/2+7\delta} \leq y^{-1/2+\varepsilon}$$

as soon as $y^\delta \geq 2C + 26$. □

Lemma 2.4.4. *For admissible data D over a parameter pair $P = (q_0, d)$,*

$$R_{q_0}(x) \ \leq\ \frac{\varphi(q_0)\,(\log b)^6}{m} + |R_{q_0}(x_1)| + \log b \cdot |R_1(x_1)|.$$

Proof. By Lemma 2.3.7, $\#\Phi_a = T(s) \geq 1$ and $f = \varphi(q_0)\#\Phi_a$; by (AD3), $\#\Phi_a = T(s) \leq \#\Phi_b \log b$, so $\#\Phi_b > 0$ and $\#\Phi_a/\#\Phi_b \leq \log b$. Also $a, m > 0$. By Theorem 5.9.1, rem $\leq fa(\log b)^5/m$. Solving Lemma 2.4.5 for $R_{q_0}(x)$,

$$R_{q_0}(x) = \frac{\text{rem}}{a\,\#\Phi_b} + R_{q_0}(x_1) - \frac{\#\Phi_a}{\#\Phi_b} R_1(x_1) \leq \frac{f(\log b)^5}{m\,\#\Phi_b} + |R_{q_0}(x_1)| + \frac{\#\Phi_a}{\#\Phi_b}|R_1(x_1)|.$$

The first term is $\varphi(q_0)\frac{\#\Phi_a}{\#\Phi_b}\frac{(\log b)^5}{m} \leq \frac{\varphi(q_0)(\log b)^6}{m}$, and the last is at most $\log b\,|R_1(x_1)|$. □

Lemma 2.4.5. *For admissible data D over a parameter pair $P = (q_0, d)$,*

$$\text{rem} = a\Big(\#\Phi_b(R_{q_0}(x) - R_{q_0}(x_1)) + \#\Phi_a\, R_1(x_1)\Big).$$

Proof. Write $S_q(y) = \sum_{n \leq y} \lambda_q(n)(\frac{1}{n} - \frac{1}{y})$ for real $y \geq 1$, so that $S_q(y) = R_q(y) + \frac{\log y}{\varphi(q)} + c_q$ by (2.2). Recall $a = xb$, and by (AD2) $x_1 \mid b$, so $1 \leq x \leq a$ and $1 \leq x_1 \leq b \leq a$; the weight is $\kappa = a/x$ on Φ_b and $\kappa = a/x_1$ off Φ_b (AD4).

Step 1: a cut-off sum. For a real y with $1 \leq y \leq a$ and $q \in \{1, q_0\}$,

$$\sum_{1 \leq \nu \leq a} \lambda_q(\nu)\Big(\frac{a}{\nu} - \frac{a}{y}\Big)_+ = a\, S_q(y), \tag{2.17}$$

since the term is $a\lambda_q(\nu)(\frac{1}{\nu} - \frac{1}{y})$ for $\nu \leq y$ and 0 for $\nu > y$, and $y \leq a$.

Step 2: which places are b-places. Let $\nu \geq 1$ and $\tau \in \Phi_a$. The class map modulo q_0 is a group homomorphism on Φ, so the class of $\nu \cdot (-1)\tau = u(-\nu)\tau$ is the class of $u(-\nu)$. Every prime factor of q_1 divides q_0 (proof of Lemma 2.3.7). If $\nu \equiv 1 \pmod{q_0}$ then $\gcd(\nu, q_1) = 1$ and $u(-\nu)$ has class -1. If $\nu \not\equiv 1$, then either $\gcd(\nu, q_1) = 1$ and $u(-\nu)$ has class $-\nu \neq -1$, or $u(-\nu)$

is the identity place, of class $1 \neq -1$ as $q_0 > 2$. As every b-place has class -1 (AD3), we get: if $\nu \not\equiv 1 \pmod{q_0}$, no $u(-\nu)\tau$ with $\tau \in \Phi_a$ is a b-place; if $\nu \equiv 1 \pmod{q_0}$, multiplication by $u(-\nu)$ is a bijection from Φ_a (class 1) onto the places of class -1, so exactly $\#\Phi_b$ of the $\tau \in \Phi_a$ have $u(-\nu)\tau \in \Phi_b$.

Step 3: the sum of the prime sums. For every place σ, $(\frac{a}{\nu} - \kappa(\sigma))_+ = (\frac{a}{\nu} - \frac{a}{x_1})_+ + \mathbf{1}_{\sigma\in\Phi_b}((\frac{a}{\nu} - \frac{a}{x})_+ - (\frac{a}{\nu} - \frac{a}{x_1})_+)$. Summing over $\tau \in \Phi_a$ with $\sigma = u(-\nu)\tau$, Step 2 gives

$$\sum_{\tau\in\Phi_a} \Lambda_K(\nu)\Big(\frac{a}{\nu} - \kappa(-\nu\tau)\Big)_+ = \#\Phi_a\,\lambda_1(\nu)\Big(\frac{a}{\nu} - \frac{a}{x_1}\Big)_+ + \#\Phi_b\,\lambda_{q_0}(\nu)\Big(\Big(\frac{a}{\nu} - \frac{a}{x}\Big)_+ - \Big(\frac{a}{\nu} - \frac{a}{x_1}\Big)_+\Big).$$

Summing over $1 \le \nu \le a$ and applying (2.17) with $(q, y) = (1, x_1), (q_0, x), (q_0, x_1)$, and $\varphi(1) = 1$,

$$\begin{aligned}\sum_{\tau\in\Phi_a} s_{\mathrm{fin}}(a,\tau) &= a\Big(\#\Phi_b(R_{q_0}(x) - R_{q_0}(x_1)) + \#\Phi_a R_1(x_1)\Big)\\ &\quad + a\Big(\#\Phi_a(\log x_1 + c_{d,1}) + \#\Phi_b\,\frac{\log x - \log x_1}{\varphi(q_0)}\Big);\end{aligned}$$

the constant c_{q_0} cancels between the cut-offs x and x_1.

Step 4: the expected value. Since $\log\kappa = \log a - \log x$ on Φ_b and $\log a - \log x_1$ elsewhere, $\sum_{\tau\in\Phi}\log\kappa(\tau) = f(\log a - \log x_1) - \#\Phi_b(\log x - \log x_1)$. By Lemma 2.3.7, $f = \varphi(q_0)\#\Phi_a > 0$, hence

$$\#\Phi_a\, s_{\mathrm{main}} = a\,\#\Phi_a\Big(\log x_1 + \frac{\#\Phi_b(\log x - \log x_1)}{f} + c_1\Big) = a\Big(\#\Phi_a(\log x_1 + c_1) + \#\Phi_b\,\frac{\log x - \log x_1}{\varphi(q_0)}\Big).$$

Step 5: conclusion. d is an odd prime ($d \equiv 7 \pmod 8$, Definition 2.3.1), so Lemma 2.4.6 applies: $\frac{L'}{L}(1,\chi_K)$ is real and $c_{d,1} = \frac{L'}{L}(1,\chi_K) - \gamma - 1 = c_1$. Subtracting Step 4 from Step 3, $\mathrm{rem} = \sum_{\tau\in\Phi_a} s_{\mathrm{fin}}(a,\tau) - \#\Phi_a\, s_{\mathrm{main}}$ is the claimed expression. □

Lemma 2.4.6. *Let d be an odd prime and $\chi_K = (\frac{\cdot}{d})$ the Legendre symbol, a real non-principal Dirichlet character modulo d. Then $\frac{L'}{L}(1,\chi_K)$ is real and*

$$c_{d,1} = \frac{L'}{L}(1,\chi_K) - \gamma - 1.$$

Proof. A Dirichlet series. For real $s > 1$ the Euler products give $\sum_n \Lambda(n)n^{-s} = -\frac{\zeta'}{\zeta}(s)$ and $\sum_n \Lambda(n)\chi_K(n)n^{-s} = -\frac{L'}{L}(s,\chi_K)$, so

$$\sum_{n\ge1}\lambda(n)n^{-s} = -\frac{\zeta'}{\zeta}(s) + \frac{L'}{L}(s,\chi_K).$$

By Theorem A.1.2, $(s-1)\zeta(s) = 1 + \gamma(s-1) + O((s-1)^2)$ near $s = 1$, so $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ extends holomorphically to a neighbourhood of 1 with value $-\gamma$ at 1. Also $L(s,\chi_K)$ is entire (Theorem A.1.2) and $L(1,\chi_K) \neq 0$ (Theorem A.1.4), so $\frac{L'}{L}(s,\chi_K)$ is holomorphic near 1; it is real for real $s > 1$, as χ_K is real, hence real at $s = 1$. Therefore

$$G(s) = \sum_{n\ge1}\lambda(n)n^{-s} - \frac{s}{s-1} \tag{2.18}$$

extends continuously to $s = 1$ from the right, with $G(1) = -\gamma + \frac{L'}{L}(1,\chi_K) - 1$ (note $\frac{s}{s-1} = 1 + \frac{1}{s-1}$).

A Mellin transform. For real $s > 1$, since $\lambda \geq 0$ and $\lambda(1) = 0$, Tonelli's theorem gives $\int_1^\infty (\sum_{n \leq t} \lambda(n)) t^{-s-1}\, dt = \sum_n \lambda(n) \int_n^\infty t^{-s-1}\, dt = \frac{1}{s} \sum_n \lambda(n) n^{-s}$, and $\int_1^\infty t \cdot t^{-s-1}\, dt = \frac{1}{s-1}$. Subtracting,

$$\int_1^\infty \mathcal{E}(t)\, t^{-s-1}\, dt = \frac{1}{s} \sum_n \lambda(n) n^{-s} - \frac{1}{s-1} = \frac{G(s)}{s} \qquad (s > 1).$$

The limit $s \to 1^+$. By Theorem 2.1.10 (with $q = 1$, which is prime to d), $\int_1^\infty |\mathcal{E}(t)| t^{-2}\, dt < \infty$. For $s \geq 1$ and $t \geq 1$, $|\mathcal{E}(t) t^{-s-1}| \leq |\mathcal{E}(t)| t^{-2}$, so by dominated convergence the left side tends to $\int_1^\infty \mathcal{E}(t) t^{-2}\, dt = c_{d,1}$ (Definition 2.1.1) as $s \to 1^+$, while the right side tends to $G(1)$. Hence $c_{d,1} = G(1) = \frac{L'}{L}(1, \chi_K) - \gamma - 1$. □

Lemma 2.4.7. *Let d be a prime and $q \geq 1$ an integer. For all real $1 \leq y \leq z$,*

$$|R_q(z) - R_q(y)| \leq 13\, \frac{z - y}{y}.$$

Proof. By (2.2), $R_q(z) - R_q(y) = \int_y^z \mathcal{E}_q(t) t^{-2}\, dt$; the constant c_q cancels. For $t \geq 0$ both $\sum_{n \leq t} \lambda_q(n)$ and $t/\varphi(q)$ lie in $[0, 6t]$: the first because $0 \leq \lambda_q \leq 2\Lambda$ and $\sum_{n \leq t} \Lambda(n) \leq 3t$ (Lemma 2.1.11(2)), the second because $\varphi(q) \geq 1$. Hence $|\mathcal{E}_q(t)| \leq 6t$, and

$$|R_q(z) - R_q(y)| \leq \int_y^z \frac{6}{t}\, dt \leq 6\, \frac{z - y}{y} \leq 13\, \frac{z - y}{y}.$$

□

Lemma 2.4.8. *For admissible data D, $x^4 f\, a \leq h_{\mathrm{main}}$; in particular $h_{\mathrm{main}} > 0$.*

Proof. Sizes. Put $L = \log x$. By Lemma 2.3.6, $L \geq 10^{14} C_P \geq 10^{14} q_0$; in particular $x \geq e^{10^{14}}$, so $\sqrt{x}/2 \geq L \geq q_0 \geq \varphi(q_0)$ and $x^5/2 \geq 2\sqrt{x}$. By (AD2), $\sqrt{x} \leq m \leq 2\sqrt{x}$ and $r \geq x^5$, so

$$k = r + 1 - m \geq x^5 - 2\sqrt{x} \geq \tfrac{1}{2} x^5 > 0, \qquad m\, k \geq \sqrt{x} \cdot \tfrac{1}{2} x^5 \geq \varphi(q_0)\, x^5.$$

Conclusion. By Lemma 2.4.9, $h_{\mathrm{dir}} \geq b\, T(s) \geq 0$, and $mk > 0$, so $h_{\mathrm{main}} = mk\, h_{\mathrm{dir}} \geq mk\, b\, T(s) \geq \varphi(q_0)\, x^5\, b\, T(s)$. By Lemma 2.3.7, $f = \varphi(q_0) T(s)$, and $a = xb$, so $\varphi(q_0) x^5 b T(s) = x^4 f a$. Hence $x^4 f a \leq h_{\mathrm{main}}$, and $x^4 f a > 0$ gives $h_{\mathrm{main}} > 0$. □

Lemma 2.4.9. *For admissible data D and every place $\sigma \in \Phi$,*

$$0 \leq s_{\mathrm{fin}}(\kappa(\sigma), \sigma) \leq \kappa(\sigma) \log b, \qquad s_\infty(\kappa(\sigma)) \geq 20\, \kappa(\sigma) \log b,$$

hence $s_\infty(\kappa(\sigma)) - s_{\mathrm{fin}}(\kappa(\sigma), \sigma) \geq \kappa(\sigma) \log b \geq 0$; and

$$h_{\mathrm{dir}} \geq b\, T(s) = b\, \#\Phi_a.$$

Proof. By Lemma 2.3.6, $x, x_1, b > 1$, $\log x \leq \log x_1$ and $\log b \geq 10^6 \log x_1$, $\log x_1 \geq 10^6 \log x \geq 10^6$. By (AD4), κ takes the values $b = a/x$ and a/x_1, and $a/x_1 = xb/x_1 \leq b$ since $x \leq x_1$. So $a/x_1 \leq \kappa(\tau) \leq b$ for every place τ.

The prime sum. Fix $\sigma \in \Phi$ and put $y = \kappa(\sigma) \leq b$. In $s_{\mathrm{fin}}(y, \sigma) = \sum_{1 \leq \nu \leq a} \Lambda_K(\nu)(y/\nu - \kappa(-\nu\sigma))_+$ every term is non-negative. For $\nu > x_1$, $y/\nu < b/x_1 \leq xb/x_1 = a/x_1 \leq \kappa(-\nu\sigma)$, so

the term vanishes. For $\nu \le x_1$, $(y/\nu - \kappa(-\nu\sigma))_+ \le y/\nu$ as $\kappa > 0$, and $\Lambda_K(\nu) \le 2\Lambda(\nu)$. By Lemma 2.1.11, item 3, with $x_1 \ge 1$,

$$0 \le s_{\mathrm{fin}}(y,\sigma) \le 2y \sum_{\nu \le x_1} \frac{\Lambda(\nu)}{\nu} \le 2y\,(\log x_1 + 3 + \log 2) \le 3y \log x_1 \le y \log b,$$

using $\log x_1 \ge 10^6$ and $\log b \ge 10^6 \log x_1$.

The jet size. y is a positive integer, so $\log y! \ge 0$. By (AD5), $M = p_0 p_1 \ge b^{20}$, and $0 < \Omega_s \le 1$ by Lemma 2.3.9, so $\log(M/\Omega_s) \ge \log M \ge 20 \log b$. Hence $s_\infty(y) = \log y! + y \log(M/\Omega_s) \ge 20y \log b$.

The difference. Combining, $s_\infty(y) - s_{\mathrm{fin}}(y,\sigma) \ge 19y \log b \ge y \log b \ge 0$, since $\log b > 1$.

The direction amount. Every summand of h_{dir} is non-negative, and on Φ_b the weight is b. Dropping the places outside Φ_b and using (AD3), $T(s) \le \#\Phi_b \log b$,

$$h_{\mathrm{dir}} \ge \sum_{\sigma \in \Phi_b} b \log b = b\,\#\Phi_b \log b \ge b\,T(s).$$

Finally $T(s) = \#\Phi_a$ by Lemma 2.3.7. □

2.5 Existence of admissible data

This section proves that data exist at every large integer x, for every parameter pair and every $\varepsilon \in (0, \frac{1}{100})$ (Proposition 2.5.1). The construction is explicit except for the choice of primes. The b-places are given by exponents from Behrend's sphere construction [18], chosen so that no relation that would violate sparseness holds between distinct exponents (Lemma 2.5.4). The primes come from windows of length one for the logarithm, by the Siegel–Walfisz theorem (Lemma 2.5.3; [314, 64, 214]). A pigeonhole argument in the class group of K makes $p_0 p_1$ a norm from K (Lemma 2.5.5), and $N \equiv 1 \pmod 4$ is a sum of two squares [116], hence a norm from E (Lemma 2.5.6). Since Siegel's theorem [279] is ineffective, so is the threshold for x; the proof needs only its existence.

Proposition 2.5.1. *Let P be a parameter pair and $0 < \varepsilon < 1/100$. There is x_0 such that for every integer $x \ge x_0$ there are admissible data D over P with scale x and*

- $x_1 = \lfloor \exp(x^{\varepsilon/6}) \rfloor$;
- *b the least integer with $q_1(s)\,x_1 \mid b$ and $b > \exp(x^\varepsilon)$;*
- *m the least power of 2 with $m \ge \sqrt{x}$, and r the least power of 3 with $r \ge x^5$;*
- $x^{\varepsilon/6} - 1 \le \log x_1$ *and* $x^\varepsilon \le \log b \le x^\varepsilon + 1$.

Proof. Let P and ε be given, let h be the class number of K, $A_0 = \lceil 70 \log(q_0 d)/\varepsilon \rceil$, $U_0(A_0)$ the number of Lemma 2.5.3 for $A = A_0$, and $Y = \max(7,\, 2h+2,\, U_0(A_0))$. Let n and x_0 be as in Lemma 2.5.2 for P, ε and Y. Let $x \ge x_0$ be an integer and let W, m, r, s, x_1, b, k, t and $L = \log b$ be as in Lemma 2.5.2; x_1, b, m, r are the numbers prescribed in the statement. We complete them to admissible data.

(AD1), (AD2) and the last item. These are (SC1) and (SC2) of Lemma 2.5.2.

Windows. For every real $U \ge L$ there is a prime $p \equiv 1 \pmod{q_2(s)^2}$ with $U < \log p \le U + 1$. Indeed $U \ge L \ge Y \ge U_0(A_0)$, and $1 \le q_2(s)^2 \le L^{A_0} \le U^{A_0}$ by (SC4); so Lemma 2.5.3 applies with $q = q_2(s)^2$.

(AD3). As $r \geq x^5 \geq 2\sqrt{x} \geq m$, $k \geq 1$ and kt is a positive integer. By (SC1) and (SC3), Lemma 2.5.4 applies to s, the integer kt in the role of L, n and W. It gives a set $\Phi_b \subseteq \Phi_s$ of places of class -1 modulo q_0, sparse for the primes up to kt, with $T(s) \leq \#\Phi_b\, W^6 \leq \#\Phi_b \log b$, as $W^6 \leq L$ by (SC3).

(AD4). Let $\kappa(\tau) = b$ for $\tau \in \Phi_b$ and $\kappa(\tau) = xb/x_1$ otherwise. As $x_1 \mid b$ by (AD2), xb/x_1 is an integer; as $1 \leq x \leq x_1$ (SC1) and $x_1 \leq b$, $1 \leq xb/x_1 \leq b$. So κ is the weight of (AD4) and $1 \leq \kappa(\tau) \leq b$ for all $\tau \in \Phi_s$.

(AD5). As $L \geq Y \geq 2h+2$ and every $U \in [10L, 11L]$ is $\geq L$, Lemma 2.5.5 applies. It gives distinct primes $p_0, p_1 \equiv 1 \pmod{q_2(s)^2}$ with $b^{10} = e^{10L} \leq p_i \leq e^{11L} = b^{11}$ and $\bar{\pi}_M \in \mathcal{O}_K$ with $\bar{\pi}_M \pi_M = p_0 p_1$. Put $M = p_0 p_1$; then $b^{20} \leq M \leq b^{22}$.

(AD6). We apply Lemma 2.5.6 to s, m ($m \geq 2$ by (SC1)), b ($L \geq Y \geq 7$, and $q_1(s)d \leq b$ by (SC4)), the weight κ and M. Its number λ is $\log N_-$ of (AD6), formed with $\Phi = \Phi_s$, $f = \#\Phi_s$ and $E = E_s$. As $m \geq 2$, every $U \in [mL, 24mL]$ is $\geq L$, so the window hypothesis holds. Hence there are a prime $N \equiv 1 \pmod{q_2(s)^2}$ with $\log N_- \leq \log N \leq \log N_- + \sqrt{\log b}$ and $\pi_N \in \mathcal{O}_{E_s}$ with $\pi_N \bar{\pi}_N = N$.

So $(x, x_1, b;\, m, r, s;\, \Phi_b, \kappa;\, p_0, p_1, \bar{\pi}_M;\, N, \pi_N)$ satisfies (AD1)–(AD6): these are admissible data over P with scale x and the prescribed x_1, b, m, r, and the last item holds by (SC2). This holds for every integer $x \geq x_0$. □

Lemma 2.5.2. *Let P be a parameter pair, $0 < \varepsilon < 1/100$ and Y a real number. Put $n = \lceil 72/\varepsilon \rceil$ and $A_0 = \lceil 70 \log(q_0 d)/\varepsilon \rceil$. There is x_0 such that for every integer $x \geq x_0$ the following hold, where $W = x^{\varepsilon/6}$, m is the least power of 2 with $m \geq \sqrt{x}$, r the least power of 3 with $r \geq x^5$, $s = 22 \log_2 m$, $x_1 = \lfloor \exp W \rfloor$, b the least integer with $q_1(s)\, x_1 \mid b$ and $b > \exp(x^\varepsilon)$, $k = r+1-m$, $t = 5r$ and $L = \log b$:*

(SC1) *x, x_1, b, m, r, s satisfy (AD1) and (AD2); moreover $m \geq 2$, $s \geq 1$ and $x \leq x_1$;*

(SC2) *$x^{\varepsilon/6} - 1 \leq \log x_1$ and $x^\varepsilon \leq L \leq x^\varepsilon + 1$;*

(SC3) *$1 \leq W \leq 2^s \leq W^n$, $(2l_0(s))^n \leq W$, $3\, l_0(s)\, n\, 2^n \leq W^3$, $2\, l_0(s)\, (kt+1) < 2^s$ and $W^6 \leq L$;*

(SC4) *$L \geq Y$, $q_1(s)\, d \leq b$ and $q_2(s)^2 \leq L^{A_0}$.*

Proof. Choice of x_0. Write $y = \log x$. Each of the following conditions holds for all large y, since ε, n, P (hence q_0, d, C_P) and Y are fixed and exponentials in y outgrow powers of y:

(a) $y \geq 16$;

(b) $(1 - 100\varepsilon)\, y \geq 100 \log 2$;

(c) $e^{\varepsilon y/6} \geq 1 + 10^6 y$;

(d) $\varepsilon y \geq 10^{12} C_P$ and $e^{\varepsilon y} \geq Y$;

(e) $\log(7776 \cdot 2^{22} q_0 d\, y) + 11y + e^{\varepsilon y/6} \leq e^{\varepsilon y}$;

(f) $(5184\, y)^n \leq e^{\varepsilon y/6}$ and $7776\, n\, 2^n y \leq e^{\varepsilon y/2}$;

(g) $5184\, y\, (45 e^{10y} + 1) < e^{11y}$.

Take x_0 such that (a)–(g) hold for $y \ge \log x_0$, and let $x \ge x_0$ be an integer. By (a), $x \ge e^{16} > 2^{22}$; by (c), $W = e^{\varepsilon y/6} \ge 16$.

Ranks. As $\sqrt{x} > 1$, $m = 2^{e+1}$ for an integer $e \ge 0$ with $2^e < \sqrt{x}$, so $\sqrt{x} \le m < 2\sqrt{x}$ and $m \ge 2$; and $s = 22(e+1) \ge 22$, $2^s = m^{22}$. Likewise $r = 3^{e'+1}$ with $e' \ge 0$ and $x^5 \le r < 3x^5$. From $m < 2\sqrt{x}$, $s = 22\log_2 m < 22 + 11y/\log 2 < 22 + 15.9y \le 32y$ by (a). As $3^{\lfloor \log_3 s \rfloor} \le s$, $l_0(s) \le 81s \le 2592y$. Hence

$$2^s = m^{22} < 2^{22}x^{11} \le x^{12}, \qquad q_1(s) = 3q_0 l_0(s) 2^s \le 7776 \cdot 2^{22} q_0\, y\, x^{11}.$$

Scales. As $W \ge 0$, $1 \le x_1 \le e^W < x_1 + 1 \le 2x_1$, so $W - \log 2 < \log x_1 \le W$; this gives the first part of (SC2), and $\log x_1 \ge W - 1 \ge 10^6 y$ by (c), in particular $x \le x_1$ and $\log x_1 \ge 15$. Put $Q = q_1(s)x_1$ and $Z = e^{x^\varepsilon}$. By the bound on $q_1(s)$ and (e), $\log(Q\,d) \le \log(7776 \cdot 2^{22} q_0 dy) + 11y + W \le x^\varepsilon$, so $Q \le Qd \le Z$. As b is the least multiple of Q exceeding Z, $Z < b \le Z + Q \le 2Z$, whence $x^\varepsilon < L \le x^\varepsilon + \log 2$; this is the second part of (SC2), and $q_1(s)d \le Qd \le Z < b$.

(AD1). By (SC2), $\log\log b \ge \log x^\varepsilon = \varepsilon y$ and $\log\log b \le \log(2x^\varepsilon) = \varepsilon y + \log 2$. So $10^{12} C_P \le \log\log b$ by (d), and $100\log\log b \le 100\varepsilon y + 100\log 2 \le y = \log x$ by (b). Next $10^6 \log x \le \log x_1$ was shown above. As $W \ge 16$, $W^5 \ge 16^5 > 10^6$, so $10^6 \log x_1 \le 10^6 W \le W^6 = x^\varepsilon \le L$. Finally $0 < \log x_1 \le W$ gives $6\log\log x_1 \le 6\log W = \log x^\varepsilon \le \log\log b$. All of x, x_1, b are positive integers.

(AD2). $m = 2^{e+1}$, $s = 22(e+1)$, $r = 3^{e'+1}$, $\sqrt{x} \le m \le 2\sqrt{x}$, $x^5 \le r \le 3x^5$, $s \le 32\log x$ and $q_1(s)x_1 \mid b$ were all shown above. This completes (SC1).

(SC3). $1 \le 16 \le W$. As $\varepsilon/6 < 1$, $W \le x \le x^{11} \le m^{22} = 2^s$. As $n\varepsilon/6 \ge 12$ and $x \ge 1$, $2^s \le x^{12} \le x^{n\varepsilon/6} = W^n$. By $l_0(s) \le 2592y$ and (f), $(2l_0(s))^n \le (5184y)^n \le W$ and $3l_0(s)n2^n \le 7776\, n 2^n y \le W^3$. As $m \ge 1$, $k \le r$, so $kt \le 5r^2 < 45x^{10}$, and by (g), $2l_0(s)(kt+1) \le 5184y(45x^{10}+1) < x^{11} \le 2^s$. Finally $W^6 = x^\varepsilon \le L$ by (SC2).

(SC4). $L \ge x^\varepsilon = e^{\varepsilon y} \ge Y$ by (d), and $q_1(s)d \le b$ was shown. As $s \le 32y$ and $y \ge 2$, $2s + 12 \le 70y$, so, with $q_0 d > 1$ and $\varepsilon A_0 \ge 70\log(q_0 d)$,

$$q_2(s)^2 = (q_0 d)^{2s+12} \le e^{70y\log(q_0 d)} \le e^{\varepsilon y A_0} = (x^\varepsilon)^{A_0} \le L^{A_0}.$$

□

Lemma 2.5.3. *For every real $A \ge 1$ there is a real number $U_0(A) \ge 2$ such that for every real $U \ge U_0(A)$ and every integer q with $1 \le q \le U^A$ there is a prime $p \equiv 1 \pmod q$ with $U < \log p \le U + 1$.*

Proof. Let $c = c_A$ and $C = C_A$ be the constants of the Siegel–Walfisz theorem [64] for this A. Since $U^A(4Ce^{-c\sqrt{U}} + 3(U+1)^2 e^{-U/2}) \to 0$ as $U \to \infty$, there is $U_0(A) \ge 2$ such that

$$(e-1)\,U^{-A} > 4C\,e^{-c\sqrt{U}} + 3(U+1)^2 e^{-U/2} \qquad (U \ge U_0(A)). \tag{2.19}$$

Let $U \ge U_0(A)$ and $1 \le q \le U^A$, and put $X_0 = e^U$, $X_1 = e^{U+1}$. Both are at least $e^2 > 2$, and $q \le U^A = (\log X_0)^A < (\log X_1)^A$, so the theorem applies at X_0 and at X_1 with $a = 1$. As $\varphi(q) \le q \le U^A$ and $e + 1 < 4$,

$$\psi(X_1; q, 1) - \psi(X_0; q, 1) \ge \frac{X_1 - X_0}{\varphi(q)} - CX_1 e^{-c\sqrt{U+1}} - CX_0 e^{-c\sqrt{U}} \ge e^U\big((e-1)U^{-A} - 4Ce^{-c\sqrt{U}}\big).$$

The left side is the sum of $\Lambda(n)$ over the $n \equiv 1 \pmod q$ in $(X_0, X_1]$. Those n that are not primes are prime powers p^k with $k \ge 2$; then $p \le X_1^{1/2}$ and $2 \le k \le \log X_1/\log 2$, so there are

at most $X_1^{1/2}(U+1)/\log 2$ of them, each with $\Lambda(n) = \log p \le U+1$. Their total contribution is at most $(U+1)^2 e^{(U+1)/2}/\log 2 \le 3(U+1)^2 e^{U/2}$. Hence the sum of $\log p$ over the primes $p \equiv 1 \pmod q$ with $X_0 < p \le X_1$ is at least

$$e^U\big((e-1)U^{-A} - 4Ce^{-c\sqrt{U}} - 3(U+1)^2 e^{-U/2}\big) > 0$$

by (2.19). So there is such a prime p, and $U < \log p \le U+1$. □

Lemma 2.5.4. *Let P be a parameter pair, let $s \ge 1$, $L \ge 0$ and $n \ge 1$ be integers and $W \ge 1$ a real number with*

$$W \le 2^s \le W^n, \qquad (2l_0(s))^n \le W, \qquad 3\,l_0(s)\,n\,2^n \le W^3, \qquad 2\,l_0(s)\,(L+1) < 2^s.$$

Then there is a set $T \subseteq \Phi_s$ of places of class -1 modulo q_0, sparse for the primes up to L, with $T(s) \le \#T \cdot W^6$.

Proof. Write $l_0 = l_0(s)$, $q_1 = q_1(s)$, and $q_0 = 2^v u$ with u odd; as $16 \mid q_0$, $v \ge 4$. Then $3l_0 = 3^c$ with $c = \lfloor \log_3 s \rfloor + 5 \ge 5$, and $q_1 = 2^{s+v} \cdot 3l_0 u$. The proof constructs T as the places of the classes $-g^e$, with $g = 1 + 3l_0 q_0$ and e in a Behrend-type set.

Step 1: four facts on orders. Every prime factor of q_1 divides $3l_0q_0$, so $\gcd(g, q_1) = 1$; let $\gamma = u(g) \in \Phi_s$.

(i) The class of g modulo 2^{s+v} has order 2^s. Indeed $g = 1 + 2^v a$ with $a = 3l_0 u$ odd. For $w \ge 2$ and odd c', $(1+2^w c')^2 = 1 + 2^{w+1}(c' + 2^{w-1}c'^2)$ with $c' + 2^{w-1}c'^2$ odd; by induction $g^{2^k} = 1 + 2^{v+k}a_k$ with a_k odd for all $k \ge 0$. So $g^{2^s} \equiv 1$ and $g^{2^{s-1}} \not\equiv 1 \pmod{2^{s+v}}$.

(ii) $3^c \mid q_1$ and $g \equiv 1 \pmod{3^c}$, as $3^c = 3l_0$ divides $g-1$.

(iii) The class of 4 modulo 3^c has order $3^{c-1} = l_0$. For $w \ge 1$ and $3 \nmid b'$, $(1+3^w b')^3 = 1 + 3^{w+1}(b' + 3^w b'^2 + 3^{2w-1}b'^3)$, and the bracket is $\equiv b' \pmod 3$; by induction $4^{3^k} = 1 + 3^{k+1}b_k$ with $3 \nmid b_k$, so $4^{3^{c-1}} \equiv 1$ and $4^{3^{c-2}} \not\equiv 1 \pmod{3^c}$.

(iv) If ℓ is an odd prime, $o \ge 1$ and $\ell^{2o} \equiv 1 \pmod{2^{s+1}}$, then $2^{s+1} \le (\ell+1)\,2o$. Write $2o = 2^w o'$ with o' odd and $w \ge 1$. Then $\ell^{2o} - 1 = (\ell^{2^w} - 1)\sum_{t=0}^{o'-1} \ell^{2^w t}$, the sum of o' odd numbers is odd, and $\ell^{2^w} - 1 = (\ell-1)(\ell+1)\prod_{k=1}^{w-1}(\ell^{2^k}+1)$ with each $\ell^{2^k}+1 \equiv 2 \pmod 4$. One of $\ell \pm 1$ is $\equiv 2 \pmod 4$, so the 2-adic valuation of $(\ell-1)(\ell+1)$ is at most $1 + \log_2(\ell+1)$. Hence $s+1 \le \log_2(\ell+1) + w$, that is, $2^{s+1} \le (\ell+1)2^w \le (\ell+1)\,2o$.

Step 2: a Behrend-type set of exponents. As $2^s/W \ge 1$, there is a largest integer $\kappa \ge 0$ with $H = 2^\kappa$ satisfying $H^n \le 2^s/W$; then $2^s/W < (2H)^n$, and $H \le W$ since $H^n \le 2^s/W \le W^{n-1} \le W^n$. Put $B = 2l_0 H$; then $B^n = (2l_0)^n H^n \le W \cdot 2^s/W = 2^s$. Following Behrend's sphere construction [18], let $V = \{0, \dots, H-1\}^n$ with $\|v\|^2 = \sum_\nu v_\nu^2$. The values of $\|v\|^2$ on V are integers in $[0, n(H-1)^2]$, at most $n(H-1)^2 + 1 \le nH^2$ of them; as $\#V = H^n$, some ρ has $\Sigma = \{v \in V : \|v\|^2 = \rho\}$ with $\#\Sigma \ge H^n/(nH^2)$. Let $\phi(v) = \sum_{\nu=1}^n v_\nu B^{\nu-1}$. Base-$B$ expansions are unique, so ϕ is injective on vectors with entries in $\{0, \dots, B-1\}$, and $\phi(v) < B^n \le 2^s$ for such v. Put $S = \phi(\Sigma) \subseteq \{0, \dots, 2^s - 1\}$; $\#S = \#\Sigma$.

(B) If $e_1, e_2, e_3 \in S$ and α, α' are positive integers with $\alpha + \alpha' \le 2l_0$ and $(\alpha+\alpha')e_2 \equiv \alpha e_1 + \alpha' e_3 \pmod{2^s}$, then $e_1 = e_2 = e_3$. Write $e_k = \phi(v_k)$ with $v_k \in \Sigma$. The vectors $(\alpha+\alpha')v_2$ and $\alpha v_1 + \alpha' v_3$ have entries in $[0, 2l_0(H-1)] \subseteq [0, B-1]$, so $(\alpha+\alpha')e_2 = \phi((\alpha+\alpha')v_2)$ and $\alpha e_1 + \alpha' e_3 = \phi(\alpha v_1 + \alpha' v_3)$ lie in $[0, 2^s)$; being congruent modulo 2^s, they are equal, and by injectivity $(\alpha+\alpha')v_2 = \alpha v_1 + \alpha' v_3$. Then

$$\alpha\alpha'\|v_1 - v_3\|^2 = (\alpha+\alpha')(\alpha\|v_1\|^2 + \alpha'\|v_3\|^2) - \|\alpha v_1 + \alpha' v_3\|^2 = (\alpha+\alpha')^2\rho - (\alpha+\alpha')^2\|v_2\|^2 = 0,$$

so $v_1 = v_3$, and then $(\alpha+\alpha')v_2 = (\alpha+\alpha')v_1$ gives $v_2 = v_1$.

Step 3: the set T and its size. For $e \in S$ let $\tau_e = u(-1)\gamma^e \in \Phi_s$, the place of the class of $-g^e$, and $T = \{\tau_e : e \in S\}$. As $q_0 \mid g-1$, every τ_e has class -1 modulo q_0. If $\tau_e = \tau_{e'}$ then $g^e \equiv g^{e'} \pmod{2^{s+v}}$, so $e \equiv e' \pmod{2^s}$ by (i), and $e = e'$ as $0 \le e, e' < 2^s$. So $\#T = \#S \ge H^n/(nH^2)$ and, using $2^s < W(2H)^n$, $3l_0 n 2^n \le W^3$ and $H \le W$,

$$T(s) = 3l_0\, 2^s < 3l_0\, W\, 2^n H^n = \frac{H^n}{nH^2} \cdot 3l_0 n 2^n \cdot W \cdot H^2 \le \#T \cdot W^3 \cdot W \cdot W^2 = \#T \cdot W^6.$$

Step 4: T is sparse for the primes up to L. Suppose $\ell \le L$ is prime, $0 < i < j < l_0$, $\sigma = \tau_{e_1}$, $\tau = \tau_{e_2}$, $\rho = \tau_{e_3} \in T$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ with $\tau/\sigma \equiv \varepsilon \ell^i$ and $\rho/\sigma \equiv \varepsilon' \ell^j \pmod{q_1^{(\ell)}}$. Here $\tau/\sigma = \gamma^{e_2}\gamma^{-e_1}$ and $\rho/\sigma = \gamma^{e_3}\gamma^{-e_1}$, the signs cancelling.

If $\ell = 2$: 3^c is odd and divides q_1, hence $q_1^{(2)}$. Reducing modulo 3^c and using (ii), $1 \equiv \varepsilon 2^i \pmod{3^c}$, so $4^i \equiv 1 \pmod{3^c}$, which contradicts (iii) as $0 < i < l_0$.

If ℓ is odd: $2^{s+v} \mid q_1^{(\ell)}$. Let $\bar g$ be the class of g in $(\mathbb{Z}/2^{s+v})^\times$. Squaring the two congruences modulo 2^{s+v} gives $\bar g^{\,2(e_2-e_1)} = \ell^{2i}$ and $\bar g^{\,2(e_3-e_1)} = \ell^{2j}$, so $\bar g^{\,2j(e_2-e_1)} = \ell^{2ij} = \bar g^{\,2i(e_3-e_1)}$ and, by (i), $2j(e_2-e_1) \equiv 2i(e_3-e_1) \pmod{2^s}$. With $\alpha = 2(j-i) > 0$ and $\alpha' = 2i > 0$ this reads $(\alpha+\alpha')e_2 \equiv \alpha e_1 + \alpha' e_3 \pmod{2^s}$, and $\alpha+\alpha' = 2j < 2l_0$. By (B), $e_1 = e_2$, so $\tau = \sigma$ and $1 \equiv \varepsilon\ell^i \pmod{2^{s+v}}$; hence $\ell^{2i} \equiv 1 \pmod{2^{s+1}}$, and (iv) with $o = i$ gives $2^{s+1} \le (\ell+1)\,2i \le (L+1)\,2l_0 < 2^s$, a contradiction.

So no such $\ell, i, j, \sigma, \tau, \rho, \varepsilon, \varepsilon'$ exist, and T has all the stated properties. □

Lemma 2.5.5. *Let P be a parameter pair, $s \ge 0$ an integer, h the class number of K, and $L \ge 2h+2$ a real number. Suppose that for every real U with $10L \le U \le 11L$ there is a prime $p \equiv 1 \pmod{q_2(s)^2}$ with $U < \log p \le U+1$. Then there are distinct primes p_0, p_1 with $p_i \equiv 1 \pmod{q_2(s)^2}$ and $e^{10L} \le p_i \le e^{11L}$ $(i = 0, 1)$, and an element $\bar\pi \in \mathcal{O}_K$ with $\bar\pi\,\pi = p_0 p_1$, where π is the complex conjugate of $\bar\pi$.*

Proof. Step 1: $h+1$ primes. For $i = 0, \dots, h$ put $U_i = 10L + 2i$. Since $L \ge 2h+2$, $10L \le U_i$ and $U_i + 1 \le 10L + 2h + 1 \le 11L$, so by hypothesis there is a prime $\wp_i \equiv 1 \pmod{q_2(s)^2}$ with $U_i < \log \wp_i \le U_i + 1$; in particular $e^{10L} \le \wp_i \le e^{11L}$. The intervals $(U_i, U_i+1]$ are pairwise disjoint, so the $\wp_i$ are pairwise distinct.

Step 2: each $\wp_i$ splits in K. As $d \mid q_2(s)$, $\wp_i \equiv 1 \pmod d$, so $\wp_i \ne d$ and $\chi_K(\wp_i) = (1/d) = 1$. As χ_K is the quadratic character of K and $\wp_i$ does not divide the discriminant $-d$ of K, $\wp_i$ splits: $\wp_i \mathcal{O}_K = \mathfrak{p}_i \bar{\mathfrak{p}}_i$ with prime ideals $\mathfrak{p}_i \ne \bar{\mathfrak{p}}_i$, the bar denoting complex conjugation [224], Chapter I.

Step 3: pigeonhole in the class group. The class group of K has h elements and there are $h+1$ ideals $\mathfrak{p}_0, \dots, \mathfrak{p}_h$, so $[\mathfrak{p}_i] = [\mathfrak{p}_j]$ for some $i \ne j$. Then $[\mathfrak{p}_i\bar{\mathfrak{p}}_j] = [\mathfrak{p}_j\bar{\mathfrak{p}}_j] = [\wp_j\mathcal{O}_K] = 1$, so $\mathfrak{p}_i\bar{\mathfrak{p}}_j = \alpha\mathcal{O}_K$ for some $\alpha \in \mathcal{O}_K$. Conjugating, $\bar\alpha\mathcal{O}_K = \bar{\mathfrak{p}}_i\mathfrak{p}_j$, hence $\alpha\bar\alpha\,\mathcal{O}_K = \mathfrak{p}_i\bar{\mathfrak{p}}_i\mathfrak{p}_j\bar{\mathfrak{p}}_j = \wp_i\wp_j\mathcal{O}_K$. So $\alpha\bar\alpha = \eta\,\wp_i\wp_j$ with $\eta \in \mathcal{O}_K^\times$. As $\alpha\bar\alpha = |\alpha|^2 > 0$, η is a positive real unit of the imaginary quadratic field K; its units are roots of unity, so $\eta = 1$.

Put $p_0 = \wp_i$, $p_1 = \wp_j$ and $\bar\pi = \alpha$. Then $p_0 \ne p_1$, both are $\equiv 1 \pmod{q_2(s)^2}$ and lie in $[e^{10L}, e^{11L}]$, and $\bar\pi\pi = \alpha\bar\alpha = p_0p_1$. □

Lemma 2.5.6. *Let P be a parameter pair, $s \ge 0$ and $m \ge 2$ integers, b a positive integer with $L = \log b \ge 7$ and $q_1(s)\,d \le b$, $\kappa\colon \Phi_s \to \mathbb{Z}$ with $1 \le \kappa(\tau) \le b$ for all τ, and M an integer with $b^{20} \le M \le b^{22}$. Put $f = \#\Phi_s$ and*

$$\lambda = \frac{m}{f}\sum_{\tau\in\Phi_s} \log\kappa(\tau) + (m-1)\log\frac{M\,\mathrm{rd}_{E_s}}{2\pi} - \frac{15}{2}L.$$

Then $mL \le \lambda \le 24mL$. If moreover for every real U with $mL \le U \le 24mL$ there is a prime $p \equiv 1 \pmod{q_2(s)^2}$ with $U < \log p \le U + 1$, then there are a prime $N \equiv 1 \pmod{q_2(s)^2}$ with $\lambda \le \log N \le \lambda + \sqrt{L}$ and an element $\pi_N \in \mathcal{O}_{E_s}$ with $\pi_N \bar{\pi}_N = N$, where $\bar{\pi}_N$ is the complex conjugate of π_N.

Proof. Write $q_1 = q_1(s)$, $E = E_s$.

Step 1: $1 \le \mathrm{rd}_E \le q_1 d$. As d is odd and $\gcd(d, 2(d+1)) = 1$, $d \nmid q_0$; as $3 \mid q_0 \mid 2(d+1)$, $d \ne 3$; so $d \nmid q_1$. The discriminant of $\mathbb{Q}(\zeta)$ divides a power of q_1, hence is prime to the discriminant $-d$ of K. So $[E : \mathbb{Q}] = 2\varphi(q_1)$ and $|d_E| = |d_{\mathbb{Q}(\zeta)}|^2 \, d^{\varphi(q_1)}$, and $|d_{\mathbb{Q}(\zeta)}| \le q_1^{\varphi(q_1)}$ [224], Chapter I. Hence $1 \le |d_E| \le (q_1 d)^{2\varphi(q_1)}$ and $1 \le \mathrm{rd}_E = |d_E|^{1/(2\varphi(q_1))} \le q_1 d$.

Step 2: $mL \le \lambda \le 24mL$. Each $\log \kappa(\tau)$ lies in $[0, L]$, so the mean $\frac{1}{f} \sum_\tau \log \kappa(\tau)$ lies in $[0, L]$. As $b = e^L \ge e^7 > 2\pi$ and $1 \le \mathrm{rd}_E \le q_1 d \le b$,

$$b^{19} \le \frac{b^{20}}{2\pi} \le \frac{M \, \mathrm{rd}_E}{2\pi} \le \frac{b^{22} \cdot b}{2\pi} \le b^{23},$$

so $\log(M \mathrm{rd}_E / 2\pi) \in [19L, 23L]$. Hence, as $m \ge 2$ and $L > 0$,

$$\lambda \ge 19(m-1)L - \tfrac{15}{2}L = mL + (18m - \tfrac{53}{2})L \ge mL, \qquad \lambda \le mL + 23(m-1)L - \tfrac{15}{2}L \le 24mL.$$

Step 3: the prime N. By Step 2 the hypothesis applies with $U = \lambda$: there is a prime $N \equiv 1 \pmod{q_2(s)^2}$ with $\lambda < \log N \le \lambda + 1 \le \lambda + \sqrt{L}$, as $L \ge 7$.

Step 4: N is a norm from E. As $4 \mid q_0 \mid q_2(s)$, $N \equiv 1 \pmod 4$, so $N = a^2 + c^2$ with integers a, c by Fermat's two-squares theorem [116], Chapter XX. As $4 \mid q_1$, $\iota = \zeta^{q_1/4} \in \mathcal{O}_E$ satisfies $\iota^2 = -1$, and complex conjugation maps roots of unity to their inverses, so $\bar{\iota} = -\iota$. With $\pi_N = a + c\iota \in \mathcal{O}_E$, $\pi_N \bar{\pi}_N = (a + c\iota)(a - c\iota) = a^2 + c^2 = N$. □

3 Forms attached to the datum

This section defines the objects of Theorems A, B and C. Section 3.1 states the conventions on number fields, Hecke characters and unitary dual pairs, and sets up the arithmetic of the comparison for an abstract set of twists: twisted lattices with their index and covolume, the product formula Theorem 3.1.5, from which Section 6 derives Theorem C, and Hadamard's bound Lemma 3.1.7 [113, 133]; it can be read on its own. Section 3.2 defines the forms: hermitian spaces with a point, the source and target forms, the chain of m theta lifts that makes the target forms a twisted lattice, and the normalised Poincaré map for arbitrary parameters $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$. The parameters used in the proof are fixed in Section 5.6, and the main inequality is proved in Section 5.9.

3.1 Conventions, twists, heights and twisted lattices

The section opens with the conventions used throughout, stated once: number fields and adeles [324, 224], class field theory [53, 9], Hecke characters and their L-functions [128, 294], unitary dual pairs and global theta lifts [172, 121], p-adic Hodge theory [85], Galois cohomology [265, 201, 225] and ℓ-adic Galois representations over CM fields [60]. They come first because the heights below already use the last of them.

The rest of the section sets up the frame in which forms are measured and proves its theorem, the product formula Theorem 3.1.5, the arithmetic core of Theorem C. A twisted lattice (Definition 3.1.4) is a complex vector space with one linear map for each twist. Integrality at a prime identification $\iota \mid \ell$ is read through every twist allowed at ℓ, while conjugate norms and covolume are read at the trivial twist alone. This asymmetry is the point of the construction: in the product formula each ι may be measured with its own twist ξ_ι, at the price of the height of the volume ratios $Z(1)/Z(\xi_\iota)$. The proof bounds the Gram determinant of a basis, a non-zero algebraic number, at each ι by the index and the heights and at the infinite places by the covolume, through the inequalities of Fischer and Hadamard [133], and concludes by the product formula for algebraic numbers [31] (Lemma 3.1.6). A height in the sense of Definition 3.1.3 uses one multiplier for all prime identifications at once, which is what lets a family indexed by ι have a height. The twists of the datum are the m-fold twists of Definition 3.1.2, and Hadamard's bound Lemma 3.1.7 is the upper bound for the covolume that the main inequality uses.

Notation for number fields, local fields and adeles is that of [224] and [53]. Let F be a number field with ring of integers $\mathcal{O}_F$. Places of F are written v, and finite places are identified with the nonzero primes of $\mathcal{O}_F$. We write F_v for the completion and $|\cdot|_v$ for its normalised absolute value. For finite v, ϖ_v is a uniformiser of $\mathcal{O}_{F_v}$ and $q_v = \mathrm{N}v$ is the cardinality of the residue field. For a finite extension E/F, $w \mid v$ means that the place w of E lies above v, and $\mathrm{N}_{E/F}$ is the norm (on elements, ideles and idele classes). A **non-archimedean local field** is a field complete for a discrete valuation with finite residue field. The adele ring is $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$, where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{A}_{F,f}$ is the ring of finite adeles. On the ideles $\mathbb{A}_F^\times$ we set $|x|_{\mathbb{A}} = \prod_v |x_v|_v$. The idele class group is $C_F = \mathbb{A}_F^\times / F^\times$; C_F^1 is its subgroup of norm one and C_F^0 the connected component of the identity. We write $G_F = \mathrm{Gal}(\overline{F}/F)$; $G_{F_v} \subset G_F$ is a decomposition group, I_v its inertia subgroup and Frob_v an arithmetic Frobenius. For a finite set S of places, $G_{F,S}$ is the Galois group of the maximal extension of F unramified outside S.

Notation follows Tate's thesis [296] and [224], Ch. VII. A **Hecke character** of a number field F is a continuous character $\chi = \prod_v \chi_v \colon C_F \to \mathbb{C}^\times$. Hecke characters are unitary unless stated otherwise. Fix a subgroup of C_F that $|\cdot|_{\mathbb{A}}$ maps isomorphically onto $\mathbb{R}_{>0}$. Every unitary Hecke character can be written $\chi = \chi_0 |\cdot|_{\mathbb{A}}^{i\tau}$ with a unique $\tau \in \mathbb{R}$ and χ_0 trivial on this subgroup. The conductor of χ is $\mathfrak{f}(\chi)$. We write $\mathcal{S}(\mathbb{A}_F)$ for the Schwartz–Bruhat space, $\psi_{\mathrm{add}} = \prod_v \psi_{\mathrm{add},v}$

for the standard additive character of $\mathbb{A}_F/F$ (trivial on F), and $\hat{f}$ for the Fourier transform with respect to the self-dual measure. The **Tate zeta integral** $Z(f,\chi,s)$ is $\int_{\mathbb{A}_F^\times} f(x)\chi(x)|x|_{\mathbb{A}}^s\, d^\times x$ for $\operatorname{Re} s > 1$, together with its meromorphic continuation; it satisfies $Z(f,\chi,s) = Z(\hat{f},\chi^{-1},1-s)$. The **Hecke L-function** is $L(s,\chi) = \prod_{v\nmid\infty,\ \chi_v \text{ unramified}} (1-\chi_v(\varpi_v)q_v^{-s})^{-1}$ for $\operatorname{Re} s > 1$. For a Dirichlet character, viewed as a Hecke character of $\mathbb{Q}$ through class field theory, this is the Dirichlet L-function of the inducing primitive character. The **completed L-function** is $\xi(s,\chi) = (|d_F|\,\mathrm{N}\mathfrak{f}(\chi))^{s/2} L_\infty(s,\chi)L(s,\chi)$, with the archimedean gamma factors L_∞. The **root number** $W(\chi)$ is defined by $\xi(s,\chi) = W(\chi)\,\xi(1-s,\overline{\chi})$; it satisfies $|W(\chi)| = 1$. The Dirichlet density of a set of primes is meant in the sense of [224], VII.13.

Let E/F be a CM extension of number fields with complex conjugation c, and write $x \mapsto \bar{x}$ for c. Let Y be a finite-dimensional E-vector space with a nondegenerate form $\langle\ ,\ \rangle$ that is hermitian ($\langle y,x\rangle = \overline{\langle x,y\rangle}$) or skew-hermitian ($\langle y,x\rangle = -\overline{\langle x,y\rangle}$). Then $\mathrm{U}(Y)$ denotes its unitary group over F, and $[\mathrm{U}(Y)] = \mathrm{U}(Y)(F)\backslash\mathrm{U}(Y)(\mathbb{A}_F)$. The signature of Y at a real place w of F is that of the hermitian form $\langle\ ,\ \rangle$, resp. $i\langle\ ,\ \rangle$, at the complex place of E above w.

Let V be a hermitian space and W a skew-hermitian space. Then $V \otimes_E W$ with the form $\mathrm{tr}_{E/F}(\langle\ ,\ \rangle_V \otimes \overline{\langle\ ,\ \rangle_W})$ is a symplectic F-space, and $(\mathrm{U}(V),\mathrm{U}(W))$ is a dual pair in its symplectic group. Let η be the quadratic character of E/F. Fix a nontrivial additive character ψ_{add} of $\mathbb{A}_F/F$ and Hecke characters χ_V,χ_W of $\mathbb{A}_E^\times/E^\times$ whose restrictions to $\mathbb{A}_F^\times$ are $\eta^{\dim V}$, resp. $\eta^{\dim W}$. Then $\omega = \omega_{\psi_{\mathrm{add}},\chi_V,\chi_W}$ denotes the Weil representation of $\mathrm{U}(V)(\mathbb{A}_F)\times\mathrm{U}(W)(\mathbb{A}_F)$ on a model of the Schwartz space $\mathcal{S}(V^{\dim W}(\mathbb{A}_F))$ (Schrödinger or mixed model) [172], [121]. For φ in this space, set

$$\theta(g,h;\varphi) = \sum_{x\in V^{\dim W}(F)} (\omega(g,h)\varphi)(x), \qquad \Theta(\varphi)(h) = \int_{[\mathrm{U}(V)]} \theta(g,h;\varphi)\,dg,$$

when the integral converges. The function $\Theta(\varphi)$ on $[\mathrm{U}(W)]$ is the **theta integral** of φ (the theta lift of the trivial representation). Let P be a parabolic subgroup of $\mathrm{U}(W)$ stabilising an isotropic line $Ex \subset W$. Its Levi factor is $\mathrm{Res}_{E/F}\mathbb{G}_m \times \mathrm{U}(W_1)$, where $W_1 = \langle x,y\rangle^\perp$ for an isotropic y with $\langle x,y\rangle = 1$. The constant term of a function f on $[\mathrm{U}(W)]$ along P is $f_P(h) = \int_{N(F)\backslash N(\mathbb{A}_F)} f(nh)\,dn$, where N is the unipotent radical and the measure has total mass 1. This notation holds throughout the parts of this paper on theta lifting.

Definition 3.1.1. Let P be a parameter pair, $s \geq 0$, $E = E_s$ and $\Phi = \Phi_s$. Let $z \mapsto \bar{z}$ be complex conjugation of E, extended to the adeles $\mathbb{A}_E$, and let $\mathrm{U}(1)(\mathbb{A}) = \{u \in \mathbb{A}_E : u\bar{u} = 1\}$, containing $\mathrm{U}(1)(E) = \{z \in E : z\bar{z} = 1\}$. An **automorphic character** of $\mathrm{U}(1)$ is a continuous homomorphism $\mu\colon \mathrm{U}(1)(\mathbb{A}) \to \{z \in \mathbb{C} : |z| = 1\}$ trivial on $\mathrm{U}(1)(E)$. For a finite place w of E let k_w be its residue field and $\mathrm{N}w = \#k_w$. Let $e\colon \Phi \to \mathbb{Z}$ and let $\psi = (\psi_w)_w$ be a family of characters $\psi_w\colon k_w^\times \to \mathbb{C}^\times$, one for each finite place w, almost all trivial. The character μ has **weight e and tame part ψ** if

$$\mu(u) = \prod_{v\in\Phi} u_v^{e(v)} \prod_w \psi_w(u \bmod w)$$

for every $u \in \mathrm{U}(1)(\mathbb{A})$ whose finite components are integral; here $u_v \in \mathbb{C}$ is the component of u at v read through the embedding of E chosen at v (the CM type), and $u \bmod w \in k_w^\times$ is the reduction of the component at w.

Definition 3.1.2. Automorphic characters of $\mathrm{U}(1)$, $\mathrm{U}(1)(\mathbb{A})$, $\mathrm{U}(1)(E)$, the norms $\mathrm{N}w$ of finite places and characters of weight e and tame part ψ are those of Definition 3.1.1 for the CM field E of D.

Tame characters. A **tame character** is an automorphic character ϑ of weight 0 with tame part ψ, together with a finite set Q of finite places of E (its **modulus**) such that $\psi_w = 1$ for $w \notin Q$, $\bar{w} \notin Q$ for $w \in Q$, and $\mathrm{N}Q = \prod_{w\in Q} \mathrm{N}w$ is prime to MN. Its n-th power ϑ^n is the tame character with the same modulus Q and tame part ψ^n.

Twists. A **twist** is a tame character with $\psi_w \neq 1$ for every $w \in Q$. An m**-fold twist** is a twist with $\psi_w^n \neq 1$ for all $w \in Q$ and all $1 \leq n \leq m$; then ϑ^n is a twist for $1 \leq n \leq m$. The **trivial twist** 1 has $Q = \emptyset$ and trivial tame part; it is an m-fold twist. A twist ϑ is **allowed at** a prime ℓ if $\ell \nmid \mathrm{N}Q$.

Definition 3.1.3. A **prime identification** is a pair $\iota = (\ell, \iota)$ of a prime $\ell = \ell(\iota)$ and a field isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$; write $\iota \mid \ell$. On $\overline{\mathbb{Q}}_\ell$ let $\|\cdot\|$ be the absolute value extending the ℓ-adic one with $\|\ell\| = \ell^{-1}$. A complex number z is **integral at** ι if $\|\iota(z)\| \leq 1$; these numbers form the valuation ring $\mathcal{O}_\iota \subset \mathbb{C}$. The **conjugate** of ι is $\bar{\iota}\colon z \mapsto \iota(\bar{z})$, with the same prime. Wherever an isomorphism $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$ is used together with a prime identification ι, in particular for the coefficients of ℓ-adic representations, it is ι^{-1}, and $\|\cdot\|$ is the ℓ-adic absolute value $|\cdot|_\ell$ of $\overline{\mathbb{Q}}_\ell$.

Let $y = (y_\iota)$ be a family of complex numbers, one for each prime identification, and $c \in \mathbb{R}$. The family y has **height at most** c if there are $H \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{R}$ and real numbers $h_\ell \geq 0$, one for each prime ℓ, with

(HG1) $|\sigma(H)| \leq e^\alpha$ for every field automorphism σ of $\mathbb{C}$;

(HG2) $\|\iota(H\, y_\iota)\| \leq \ell^{h_\ell}$ for every prime identification $\iota \mid \ell$;

(HG3) $\alpha + \sum_{\ell\in S} h_\ell \log \ell \leq c$ for every finite set S of primes.

Definition 3.1.4. Let V be a complex vector space and $\mathcal{T}_{\mathrm{tw}}$ a set with a distinguished element 1 (the **twists**). A **twisted lattice** L on V with twists $\mathcal{T}_{\mathrm{tw}}$ consists of

(TW1) for each $\xi \in \mathcal{T}_{\mathrm{tw}}$ a complex inner product space S_ξ and a $\mathbb{C}$-linear map $\Theta_\xi\colon V \to S_\xi$;

(TW2) a relation ξ **is allowed at** ℓ between twists and primes;

(TW3) for each ξ and each prime identification ι an $\mathcal{O}_\iota$-submodule $\Lambda_\xi(\iota) \subseteq S_\xi$, the vectors **integral at** ι, such that $\|\iota\langle\varphi, \varphi'\rangle\| \leq 1$ for $\varphi \in \Lambda_\xi(\bar{\iota})$, $\varphi' \in \Lambda_\xi(\iota)$ whenever ξ is allowed at $\ell(\iota)$;

(TW4) a finite set C, a $\mathbb{C}$-linear coordinate map $\mathrm{co}\colon S_1 \to \mathbb{C}^C$ and a matrix $G \in M_C(\mathbb{Q})$, positive definite over $\mathbb{C}$, with $\langle\varphi, \varphi'\rangle = \mathrm{co}(\varphi)^*\, G\, \mathrm{co}(\varphi')$ for $\varphi, \varphi' \in S_1$.

The **integral vectors** of V at $\iota \mid \ell$ are

$$V(\iota) = \{v \in V : \Theta_\xi v \in \Lambda_\xi(\iota) \text{ for every } \xi \text{ allowed at } \ell\}.$$

Let $y = (y_i)_{i\in I}$ be a finite family in V. A square matrix $u \in M_I(\mathcal{O}_\iota)$ **carries** y **into** $V(\iota)$ **with loss** δ if $\sum_k u_{ik} y_k \in V(\iota)$ for all i and $\|\iota(\det u)\| \geq \ell^{-\delta}$.

(TI) Index. y has **index at most** δ if there are reals $\delta_\ell \geq 0$, one for each prime, with $\sum_{\ell\in U} \delta_\ell \log \ell \leq \delta$ for every finite set U of primes, and for every $\iota \mid \ell$ a matrix $u \in M_I(\mathcal{O}_\iota)$ that carries y into $V(\iota)$ with loss δ_ℓ. The matrix may depend on ι but not on the twist.

(TC) Covolume. For an automorphism σ of $\mathbb{C}$ and $v \in V$ let $\sigma\Theta_1 v = \sigma(\mathrm{co}(\Theta_1 v)) \in \mathbb{C}^C$, coordinatewise, and $\|v\|_\sigma^2 = \mathrm{Re}\big((\sigma\Theta_1 v)^* G\, (\sigma\Theta_1 v)\big)$, the **conjugate norm**. The family y has **covolume at most** c if for every automorphism σ of $\mathbb{C}$

$$\big|\det\big((\sigma\Theta_1 y_i)^*\, G\, (\sigma\Theta_1 y_k)\big)_{i,k\in I}\big| \leq e^c.$$

(Vo) Volume. If V is a finite-dimensional inner product space, the **volume** at ξ is $Z(\xi) = \det(\Theta_\xi^*\Theta_\xi)$, the Gram determinant of the images under Θ_ξ of an orthonormal basis of V.

Theorem 3.1.5. *Let L be a twisted lattice on a finite-dimensional complex inner product space V, with twists $\mathcal{T}_{\mathrm{tw}}$ and maps Θ_ξ, and assume that Θ_ξ is injective for every $\xi \in \mathcal{T}_{\mathrm{tw}}$. Let $y_1, \dots, y_n, z_1, \dots, z_{n'} \in V$ be linearly independent with $n + n' = \dim_{\mathbb{C}} V$. For every prime identification $\iota \mid \ell$ let $\xi_\iota \in \mathcal{T}_{\mathrm{tw}}$ be a twist allowed at ℓ. Let $c, \delta, b_{\mathrm{sh}}, b_{\mathrm{den}}, b_{\mathrm{vol}} \in \mathbb{R}$ and $\gamma\colon \mathcal{T}_{\mathrm{tw}} \to \mathbb{C} \setminus \{0\}$, and assume:*

1. *the family $y = (y_1, \dots, y_n)$ has covolume at most c and index at most δ;*
2. *$\gamma(\xi)\,\Theta_\xi z_j \in \Lambda_\xi(\iota)$ for every j, every prime identification $\iota \mid \ell$ and every twist ξ allowed at ℓ;*
3. *$\|z_j\|_\sigma^2 \le e^{b_{\mathrm{sh}}}$ for every j and every automorphism σ of $\mathbb{C}$;*
4. *the family $\iota \mapsto |\gamma(\xi_\iota)|^{-2}$ has height at most b_{den}, and the family $\iota \mapsto Z(1)/Z(\xi_\iota)$ has height at most b_{vol}.*

Then

$$c + 2\delta + n'\,(b_{\mathrm{sh}} + b_{\mathrm{den}}) + b_{\mathrm{vol}} \;\ge\; 0.$$

Proof. Write $w = (w_p)_{p \in I \sqcup J}$ for the family $y_1, \dots, y_n, z_1, \dots, z_{n'}$, with $I = \{1, \dots, n\}$ and $J = \{1, \dots, n'\}$; it is a basis of V. For $\xi \in \mathcal{T}_{\mathrm{tw}}$ put

$$\mathrm{Gr}(\xi) = \det\big(\langle \Theta_\xi w_p, \Theta_\xi w_q\rangle\big)_{p,q \in I \sqcup J}, \qquad \gamma_\iota = \gamma(\xi_\iota).$$

The proof writes $\mathrm{Gr}(1)$ as a product of three families, bounds the height of each, bounds the conjugates of $\mathrm{Gr}(1)$, and compares the two by the product formula.

Step 1: factorisation. Let $e_1, \dots, e_N$ be an orthonormal basis of V and $W \in \mathrm{GL}_N(\mathbb{C})$ the matrix with $w_q = \sum_p W_{pq} e_p$. Then $(\langle \Theta_\xi w_p, \Theta_\xi w_q\rangle) = W^* A_\xi W$ with $A_\xi = (\langle \Theta_\xi e_p, \Theta_\xi e_q\rangle)$, so $\mathrm{Gr}(\xi) = |\det W|^2 Z(\xi)$. Since Θ_ξ is injective, the vectors $\Theta_\xi e_p$ are linearly independent and $Z(\xi) = \det A_\xi \neq 0$. Hence, for every ι,

$$\mathrm{Gr}(1) = \Big(|\gamma_\iota|^{2n'} \mathrm{Gr}(\xi_\iota)\Big) \cdot \Big(|\gamma_\iota|^{-2}\Big)^{n'} \cdot \frac{Z(1)}{Z(\xi_\iota)}.$$

Moreover $\mathrm{Gr}(1) \neq 0$, since $\Theta_1 w$ is linearly independent.

Step 2: the family $x_\iota = |\gamma_\iota|^{2n'}\mathrm{Gr}(\xi_\iota)$ has height at most 2δ. Let (δ_ℓ) be the numbers of (TI) for y. Fix $\iota \mid \ell$ and put $\xi = \xi_\iota$, allowed at ℓ. The conjugate $\bar\iota$ has the same prime, so (TI) gives $u \in M_I(\mathcal{O}_\iota)$ carrying y into $V(\iota)$ and $u' \in M_I(\mathcal{O}_{\bar\iota})$ carrying y into $V(\bar\iota)$, both with loss δ_ℓ. Let $v = \mathrm{diag}(u, \gamma_\iota \mathbf{1}_J)$ and $v' = \mathrm{diag}(u', \gamma_\iota \mathbf{1}_J)$, and $a_p = \sum_q v_{pq}\Theta_\xi w_q$, $a'_p = \sum_q v'_{pq}\Theta_\xi w_q$. For $p \in I$, $a_p = \Theta_\xi(\sum_k u_{pk} y_k) \in \Lambda_\xi(\iota)$ because $\sum_k u_{pk} y_k \in V(\iota)$ and ξ is allowed at ℓ; for $p \in J$, $a_p = \gamma_\iota \Theta_\xi z_p \in \Lambda_\xi(\iota)$ by hypothesis (2). Likewise $a'_p \in \Lambda_\xi(\bar\iota)$. By (TW3) every entry of $(\langle a'_p, a_q\rangle) = \overline{v'}\,(\langle \Theta_\xi w_p, \Theta_\xi w_q\rangle)\,v^{\mathsf T}$ lies in $\mathcal{O}_\iota$, hence so does its determinant $\overline{\det v'}\,\mathrm{Gr}(\xi)\det v$:

$$\|\iota(\overline{\det v'})\| \cdot \|\iota(\mathrm{Gr}(\xi))\| \cdot \|\iota(\det v)\| \le 1.$$

Now $\|\iota(\det v)\| = \|\iota(\det u)\|\,\|\iota(\gamma_\iota)\|^{n'} \ge \ell^{-\delta_\ell}\|\iota(\gamma_\iota)\|^{n'}$, and since $\iota(\bar z) = \bar\iota(z)$, $\|\iota(\overline{\det v'})\| = \|\bar\iota(\det u')\|\,\|\iota(\bar\gamma_\iota)\|^{n'} \ge \ell^{-\delta_\ell}\|\iota(\bar\gamma_\iota)\|^{n'}$. As $|\gamma_\iota|^2 = \gamma_\iota\bar\gamma_\iota$,

$$\|\iota(x_\iota)\| = \|\iota(\gamma_\iota)\|^{n'}\|\iota(\bar\gamma_\iota)\|^{n'}\|\iota(\mathrm{Gr}(\xi_\iota))\| \le \ell^{2\delta_\ell}.$$

So $H = 1$, $\alpha = 0$, $h_\ell = 2\delta_\ell$ satisfy (HG1)–(HG3) with bound 2δ.

Step 3: the constant family $\mathrm{Gr}(1)$ has height at most $2\delta + n' b_{\mathrm{den}} + b_{\mathrm{vol}}$. By Lemma 3.1.8(e), a product of finitely many families of heights at most $c_1, \dots, c_k$ has height at most $c_1 + \cdots + c_k$. Apply this to the three factors of the factorisation in Step 1, using Step 2 and hypothesis (4), the middle factor n' times.

Step 4: $|\sigma(\mathrm{Gr}(1))| \le e^{c + n' b_{\mathrm{sh}}}$ for every automorphism σ of $\mathbb{C}$. Let co and $G \in M_C(\mathbb{Q})$ be as in (TW4), and for an automorphism τ write τw_p for $\tau(\mathrm{co}(\Theta_1 w_p)) \in \mathbb{C}^C$ and $M_\tau = ((\tau w_p)^* G (\tau w_q))_{p,q}$. Let $\sigma'(z) = \overline{\sigma(\bar{z})}$. Since G is rational and $\sigma(\bar{a}) = \overline{\sigma'(a)}$,

$$\sigma\langle \Theta_1 w_p, \Theta_1 w_q \rangle = (\sigma' w_p)^* G\, (\sigma w_q), \qquad \text{so} \qquad \sigma(\mathrm{Gr}(1)) = \det\big((\sigma' w_p)^* G (\sigma w_q)\big).$$

The matrix G is positive definite hermitian, so the Cauchy–Schwarz inequality for Gram determinants, Lemma A.6.5(c), applied to the families $(\sigma' w_p)$ and (σw_q), gives

$$|\sigma(\mathrm{Gr}(1))|^2 \le \det M_{\sigma'} \det M_\sigma.$$

Each M_τ is positive semidefinite hermitian; by Fischer's inequality, Lemma A.6.5(a), $\det M_\tau$ is at most the product of the determinants of its $I \times I$ and $J \times J$ blocks. The first is at most e^c in absolute value by (TC) for y; the second is at most the product $\prod_j \|z_j\|_\tau^2 \le e^{n' b_{\mathrm{sh}}}$ of its diagonal entries by Hadamard's inequality, Lemma A.6.5(b), and hypothesis (3). Hence $\det M_\tau \le e^{c + n' b_{\mathrm{sh}}}$ for $\tau = \sigma, \sigma'$, and the claim follows.

Step 5: product formula. Let $z \in \mathbb{C} \setminus \{0\}$, let the constant family z have height at most c_1, and let $|\sigma(z)| \le e^{c_2}$ for all automorphisms σ. Then $c_1 + c_2 \ge 0$: this is Lemma 3.1.6(b), with $c = c_1$ and $c' = c_2$.

Conclusion. Apply Step 5 to $z = \mathrm{Gr}(1) \ne 0$ (Step 1), with $c_1 = 2\delta + n' b_{\mathrm{den}} + b_{\mathrm{vol}}$ (Step 3) and $c_2 = c + n' b_{\mathrm{sh}}$ (Step 4): $c + 2\delta + n'(b_{\mathrm{sh}} + b_{\mathrm{den}}) + b_{\mathrm{vol}} \ge 0$. □

Lemma 3.1.6. *Let $c, c' \in \mathbb{R}$.*

(a) If the constant family 1 has height at most c, then $0 \le c$.

(b) If $z \in \mathbb{C} \setminus \{0\}$, the constant family z has height at most c, and $|\sigma(z)| \le e^{c'}$ for every field automorphism σ of $\mathbb{C}$, then $-c \le c'$.

Proof. **The extension principle used twice.** Let Ω be an algebraically closed field of characteristic 0 with $\#\Omega = \#\mathbb{C}$, let $F \subset \mathbb{C}$ be a subfield of countable transcendence degree over $\mathbb{Q}$, and $\tau\colon F \to \Omega$ a field embedding. Then τ extends to an isomorphism $\mathbb{C} \to \Omega$: both $\mathbb{C}$ and Ω are algebraic closures of purely transcendental extensions of F, respectively $\tau(F)$, of transcendence degree $\#\mathbb{C}$, and an isomorphism of such fields extends to their algebraic closures (Steinitz; see [183], Chapter VIII, §1 and Chapter V, §2). We apply this with $\Omega = \mathbb{C}$ and with $\Omega = \overline{\mathbb{Q}}_\ell$, which is algebraically closed of characteristic 0 and of cardinality $\#\mathbb{C}$.

(a) Let (H, α, h) witness the height bound c for the constant family 1: $H \ne 0$, $|\sigma(H)| \le e^\alpha$ for all automorphisms σ, $\|\iota(H)\| \le \ell^{h_\ell}$ for all $\iota \mid \ell$, and $\alpha + \sum_{\ell \in S} h_\ell \log \ell \le c$ for all finite sets S of primes.

First, H is algebraic. Otherwise, choose an integer n with $10^n \pi > e^\alpha$; the number $w = 10^n \pi$ is transcendental, so $H \mapsto w$ defines an isomorphism $\mathbb{Q}(H) \to \mathbb{Q}(w) \subset \mathbb{C}$, which by the extension principle extends to an automorphism σ of $\mathbb{C}$ with $|\sigma(H)| = 10^n \pi > e^\alpha$, a contradiction.

Let $F = \mathbb{Q}(H)$, $n_F = [F : \mathbb{Q}] \ge 1$ and $q = \mathrm{N}_{F/\mathbb{Q}}(H) \in \mathbb{Q}^\times$. For $\Omega = \mathbb{C}$ or $\Omega = \overline{\mathbb{Q}}_\ell$, $q = \prod_\tau \tau(H)$, the product over the n_F embeddings $\tau\colon F \to \Omega$, and every such τ is the restriction of

an automorphism σ of $\mathbb{C}$, respectively of an isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, i.e. of a prime identification $\iota \mid \ell$. Hence

$$\log|q| \le n_F\,\alpha, \qquad \log|q|_\ell = \log\|q\| \le n_F\,h_\ell \log\ell \quad \text{for every prime } \ell.$$

Let S be the finite set of primes dividing the numerator or the denominator of q; $\log|q|_\ell = 0$ for $\ell \notin S$. The product formula for $q \in \mathbb{Q}^\times$ gives

$$0 = \log|q| + \sum_{\ell\in S} \log|q|_\ell \le n_F\Big(\alpha + \sum_{\ell\in S} h_\ell \log\ell\Big) \le n_F\,c,$$

so $c \ge 0$.

(b) By Lemma 3.1.8(d) the constant family z^{-1} has height at most c'; by Lemma 3.1.8(e) the constant family $z^{-1}z = 1$ has height at most $c + c'$; by (a), $c + c' \ge 0$. □

Lemma 3.1.7. *Let L be a twisted lattice on V, let $y_1, \dots, y_n \in V$ and $h \in \mathbb{R}$. If $\|y_i\|_\sigma^2 \le e^h$ for all i and all automorphisms σ of $\mathbb{C}$, then the family (y_i) has covolume at most nh.*

Proof. Fix σ and put $w_i = \sigma\Theta_1 y_i \in \mathbb{C}^C$. Since $G \in M_C(\mathbb{Q})$ is positive definite over $\mathbb{C}$, it is real symmetric and $\langle w, w'\rangle_G = w^*Gw'$ is a positive definite hermitian form on $\mathbb{C}^C$. The matrix $A = (w_i^*Gw_k)_{i,k}$ is the Gram matrix of $w_1, \dots, w_n$ for this form, hence hermitian and positive semidefinite, with real diagonal entries $A_{ii} = \|y_i\|_\sigma^2 \ge 0$. By Hadamard's inequality for positive semidefinite hermitian matrices ([133], Theorem 7.8.1), $0 \le \det A \le \prod_i A_{ii} \le e^{nh}$. So $|\det A| \le e^{nh}$ for every σ, which is (TC) with $c = nh$. □

Lemma 3.1.8. *Let $c, c' \in \mathbb{R}$ and let $y = (y_\iota)$, $y' = (y'_\iota)$ be families of complex numbers.*

(a) ***Monotonicity.*** *If y has height at most c, $c \le c'$, and $\|\iota(y'_\iota)\| \le \|\iota(y_\iota)\|$ for every ι, then y' has height at most c'.*

(b) ***Units.*** *If y has height at most c, and N_ι is a positive integer with $\ell \nmid N_\iota$ and $e_\iota \in \mathbb{Q}$ for every $\iota \mid \ell$, then $\iota \mapsto N_\iota^{e_\iota} y_\iota$ has height at most c (here $N^e > 0$ is the real power).*

(c) ***Direct bounds.*** *If (e_ℓ) are reals, one for each prime, with $\sum_{\ell\in T} e_\ell \le c$ for every finite set T of primes, and $\|\iota(y_\iota)\| \le \exp(e_\ell)$ for every $\iota \mid \ell$, then y has height at most c. In particular the constant family 1 has height at most 0.*

(d) ***Inverse constants.*** *If $H \in \mathbb{C} \setminus \{0\}$ and $|\sigma(H)| \le e^c$ for every field automorphism σ of $\mathbb{C}$, then the constant family H^{-1} has height at most c.*

(e) ***Products.*** *If y has height at most c and y' has height at most c', then yy' has height at most $c + c'$. Hence, if $y^{(i)}$ has height at most c_i for i in a finite set I, then $\prod_{i\in I} y^{(i)}$ has height at most $\sum_{i\in I} c_i$; and y^n has height at most nc for every integer $n \ge 0$.*

Proof. **(a)** Let (H, α, h) witness (HG1)–(HG3) for y with bound c. The same triple works for y' with bound c': (HG1) is unchanged, $\|\iota(Hy'_\iota)\| = \|\iota(H)\|\,\|\iota(y'_\iota)\| \le \|\iota(H)\|\,\|\iota(y_\iota)\| = \|\iota(Hy_\iota)\| \le \ell^{h_\ell}$ gives (HG2), and (HG3) holds since $c \le c'$.

(b) Fix $\iota \mid \ell$, write $e_\iota = a/q$ with $a \in \mathbb{Z}$, $q \ge 1$, and put $u = N_\iota^{e_\iota} \in \mathbb{R}_{>0}$. Then $u^q = N_\iota^a$, so $\|\iota(u)\|^q = \|\iota(N_\iota)\|^a = |N_\iota|_\ell^a = 1$, as N_ι is an integer prime to ℓ. Hence $\|\iota(uy_\iota)\| = \|\iota(y_\iota)\|$ and (a) with $c' = c$ applies.

(c) Take $H = 1$, $\alpha = 0$ and $h_\ell = \max(e_\ell, 0)/\log\ell \ge 0$. (HG1) holds as $|\sigma(1)| = 1$. (HG2): $\ell^{h_\ell} = \exp(\max(e_\ell, 0)) \ge \exp(e_\ell) \ge \|\iota(y_\iota)\|$. (HG3): for a finite set T of primes let

$T' = \{\ell \in T : e_\ell \geq 0\}$; then $\alpha + \sum_{\ell \in T} h_\ell \log \ell = \sum_{\ell \in T'} e_\ell \leq c$. For the constant family 1 take $e_\ell = 0$ and $c = 0$.

(d) Take the multiplier H itself, $\alpha = c$ and $h_\ell = 0$: (HG1) is the hypothesis, $\|\iota(H \cdot H^{-1})\| = 1 = \ell^0$, and $\alpha + \sum_{\ell \in T} h_\ell \log \ell = c$.

(e) If (H, α, h) and (H', α', h') witness the bounds for y and y', then $(HH', \alpha+\alpha', h+h')$ witnesses the bound $c+c'$ for yy': $HH' \neq 0$; $|\sigma(HH')| \leq e^{\alpha+\alpha'}$; $\|\iota(HH' y_\iota y'_\iota)\| = \|\iota(Hy_\iota)\| \, \|\iota(H' y'_\iota)\| \leq \ell^{h_\ell + h'_\ell}$; and the sums in (HG3) add. The finite products follow by induction on $\#I$, the empty product being the constant family 1, of height at most 0 by (c). The power y^n is the product of n copies of y. □

3.2 Hermitian spaces, Weil data and forms

This section defines the forms of the three theorems and the objects that Sections 4, 5 and 6 share.

Every hermitian space of the paper is read in one framework (Definition 3.2.2): spaces with a point, framed hermitian spaces, the lattice at a point of the group with its first minimum $\lambda_V(g)$ and its height, the standard definite spaces, orthogonal sums and $\mathrm{Hom}(S, T)$ (for hermitian lattices see [270], and for first minima [52]). With it come the Cayley measure, Flensted-Jensen's functions, forms of a type with discrete-series kernels, the masses of definite hermitian lattices [97], and the facts about E that the Weil data need. With the Weil data (Definition 3.2.12) the Weil representation of a unitary dual pair is split as in Kudla [172]; for the theta correspondence see [322, 134, 136].

All groups are unitary groups of hermitian spaces over E [180, 147, 249] (Definition 3.2.13): the standard definite spaces X_j; the space Y_0 of rank r and signature $(k + \varrho, \varrho + 1)$ at every infinite place; the spaces Y_j of the chain, each with a fixed frame (Definition 3.2.15); and the diagonal spaces W_j. The target forms are the good forms on Y_0, of principal level at the chosen places and of a discrete series type at every infinite place [119], cut out by an idempotent built from Green's characters [106, 194] (Definition 3.2.16). The source forms are algebraic modular forms [110] on $\mathrm{U}(X_0)$ (Definition 3.2.17).

The theta model, Weyl modules and the mixed model (Definition 3.2.18, Definition 3.2.19, Definition 3.2.21) carry the theta correspondence. The exchange data (Definition 3.2.22) are formed with a family of bounded linear functionals, which is a parameter of the definition; it is specified in Section 6, after the results that construct it. On the exchange data rest the descent pairs, the lift characters, the grids and the triangle of lifts, and the chain of m theta lifts that makes the target forms a twisted lattice, the target lattice (Definition 3.2.31, Lemma 3.2.32). The error unit $u_{\mathrm{err}} = f\, a\, (\log b)^3$ (Definition 3.2.34) is the size in which the errors of the three theorems are measured.

The section closes with the Bessel periods of source forms (Definition 3.2.36; see [96]) and the normalised Poincaré map (Definition 3.2.37). The definition is parametric. Its inputs are a linear map $\mathcal{P}$ from source forms to forms of principal level on Y_0, a non-zero scalar c_{ss} and divisibility errors $\mathrm{err}_\ell \geq 0$ within the error unit, and a result that needs a further property of them states it as a hypothesis. The parameters are fixed in Section 5.6, with $\mathcal{P}$ a map that represents a Bessel period. In our reading c_{ss} is the constant of a seesaw identity in the sense of Kudla [171].

Definition 3.2.1. Let $N \geq 1$, let $J \in \mathrm{M}_N(\mathbb{C})$ be a **hermitian involution** ($J^* = J$, $J^2 = 1$), and $\mathrm{U}(J) = \{g \in \mathrm{GL}_N(\mathbb{C}) : g^* J g = J\}$. For $y \in \mathrm{M}_N(\mathbb{R})$ let $S(y)$ be the skew-hermitian matrix

with entries

$$S(y)_{aa} = \mathrm{i}\, y_{aa}, \qquad S(y)_{ab} = \frac{(1+\mathrm{i})\, y_{ab} - (1-\mathrm{i})\, y_{ba}}{\sqrt{2}} \quad (a \neq b),$$

and $D(y) = 1 - \frac{1}{2}JS(y)$. Put

$$\operatorname{vol} \mathrm{U}(j) = \prod_{i=0}^{j-1} \frac{2\pi^{i+1}}{i!} \qquad (j \geq 0).$$

The **Cayley measure** of $\mathrm{U}(J)$ is the image of the measure $|\det D(y)|^{-2N} \operatorname{vol} \mathrm{U}(N)^{-1}\, dy$ on $\{y \in \mathrm{M}_N(\mathbb{R}) : \det D(y) \neq 0\}$, dy the Lebesgue measure of $\mathrm{M}_N(\mathbb{R}) = \mathbb{R}^{N\times N}$, under the **Cayley chart** $y \mapsto 2D(y)^{-1} - 1 = (1 + \frac{1}{2}JS(y))(1 - \frac{1}{2}JS(y))^{-1} \in \mathrm{U}(J)$.

Definition 3.2.2. Let E be a CM field with maximal totally real subfield E^+, let Φ be the set of infinite places of E^+, and fix for every $v \in \Phi$ an embedding $E \to \mathbb{C}$ above v; through these, $E \otimes_{E^+} E_v^+ = \mathbb{C}$ and $\mathbb{A}_E^n = (\mathbb{C}^n)^\Phi \times \mathbb{A}_{E,f}^n$. Index sets are finite and vectors indexed by an index set n are columns. For $z \in \mathbb{A}_E^n$, $z_\infty \in \mathbb{C}^{\Phi\times n}$ and $z_f \in \mathbb{A}_{E,f}^n$ are its infinite and finite components, and $\|z_\infty\| = \left(\sum_{v\in\Phi}\sum_{i\in n} |z_{v,i}|^2\right)^{1/2}$ is the euclidean norm. Hermitian forms are linear in the first argument; a hermitian form h' that is conjugate-linear in the first argument is read as the form $h(u,u') = h'(u',u)$.

(VP1) Signs, groups and measures. For signs $s\colon n \to \{\pm 1\}$ let $\varepsilon_s \in \mathrm{GL}_n(\mathbb{A}_E)$ be $\mathrm{diag}(s)$ at every infinite place and 1 at every finite place, $\mathrm{U}(\operatorname{diag} s) = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^*\mathrm{diag}(s)g = \mathrm{diag}(s)\}$,

$$\begin{aligned} G_\infty(s) &= \prod_{v\in\Phi} \mathrm{U}(\operatorname{diag} s), \\ G_f(n) &= \{g \in \mathrm{GL}_n(\mathbb{A}_{E,f}) : g^*g = 1\}, \\ \mathrm{G}(s) &= \{g \in \mathrm{GL}_n(\mathbb{A}_E) : g^*\varepsilon_s g = \varepsilon_s\} = G_\infty(s) \times G_f(n), \end{aligned}$$

and let the **standard level** be $\mathbf{K}_0(n) = G_f(n) \cap \mathrm{GL}_n(\widehat{\mathcal{O}}_E)$. $G_\infty(s)$ and $G_f(n)$ are embedded in $\mathrm{G}(s)$ by $z \mapsto (z,1)$ and $k \mapsto (1,k)$, and $\det\colon \mathrm{G}(s) \to \mathrm{U}(1)(\mathbb{A}_{E^+})$ is the determinant. The **measure** of $\mathrm{U}(\operatorname{diag} s)$ is its Cayley measure (Definition 3.2.1 with $J = \mathrm{diag}(s)$) if $n \neq \emptyset$ and the unit point mass if $n = \emptyset$; the measure of $G_\infty(s)$ is the product over $v \in \Phi$ of these, that of $G_f(n)$ is its Haar measure giving $\mathbf{K}_0(n)$ mass 1, and that of $\mathrm{G}(s)$ is their product.

(VP2) Spaces with a point. A **space with a point** V with signs s on n is a pair (J_V, x_V) of a hermitian matrix $J_V \in \mathrm{GL}_n(E)$, the form being $h_V(u,u') = u'^* J_V u$ on E^n, and a **point** $x_V \in \mathrm{GL}_n(\mathbb{A}_E)$ with

$$x_V^* J_V x_V = \varepsilon_s.$$

Its **real frame** is the infinite component of x_V and its **finite frame** is the finite component $x_{V,f}$. Its **coordinate map** is $c_V(u) = x_V^{-1}u \in \mathbb{A}_E^n$ for $u \in E^n$; the vectors $c_V(u)$ are the **rational vectors** of V in the point, and the **lattice** of V is $\{u \in E^n : c_V(u)_f \in \widehat{\mathcal{O}}_E^n\}$. For an E^+-algebra R put $\mathrm{U}(V)(R) = \{a \in \mathrm{GL}_n(E \otimes_{E^+} R) : a^* J_V a = J_V\}$. The map $g \mapsto x_V g x_V^{-1}$ identifies $\mathrm{G}(s)$ with $\mathrm{U}(V)(\mathbb{A}_{E^+})$; through it $\mathrm{U}(V)(\mathbb{A}_{E^+})$ is written as $G_\infty(s) \times G_f(n)$, the **rational points** are $\Gamma_V = x_V^{-1}\mathrm{U}(V)(E^+)x_V \subseteq \mathrm{G}(s)$, and $[\mathrm{U}(V)] = \Gamma_V\backslash\mathrm{G}(s)$ carries the **quotient measure** of the measure of $\mathrm{G}(s)$ by counting measure on Γ_V.

(VP3) Framed hermitian spaces. A **framed hermitian space** U with signs s on n is a hermitian space (U,h) over E of dimension $\#n$ together with **frames**: for every $v \in \Phi$

a basis $(b_{v,i})_{i\in n}$ of $U\otimes_{E,v}\mathbb{C}$ with $h(b_{v,i},b_{v,j})=s(i)\delta_{ij}$, and an isometry x_U of $\mathbb{A}_{E,f}^n$ with the form $\sum_i z_i\bar{z}_i'$ onto $U\otimes_E\mathbb{A}_{E,f}$. Its **coordinate map** is the $\mathbb{A}_E$-linear isomorphism $c_U\colon U\otimes_E\mathbb{A}_E\to\mathbb{A}_E^n$ whose component at $v\in\Phi$ sends a vector to its coordinates in the frame $(b_{v,i})_{i\in n}$ and whose finite component is x_U^{-1}; for $u\in U$, $c_U(u)$ is the **coordinate vector** of u, and the **lattice** of U is $\{u\in U : c_U(u)_f\in\widehat{\mathcal{O}}_E^n\}$. Let $\mathrm{U}(U)$ be the unitary group of (U,h) over E^+. Through the frames, $\mathrm{U}(U)(\mathbb{A}_{E^+})=\{c_U^{-1}gc_U : g\in\mathrm{G}(s)\}$ is written as $\mathrm{G}(s)=G_\infty(s)\times G_f(n)$; the **rational points** are $\Gamma_U=c_U\,\mathrm{U}(U)(E^+)\,c_U^{-1}\subseteq\mathrm{G}(s)$, the quotient $\Gamma_U\backslash\mathrm{G}(s)$ carries the **quotient measure** of the measure of $\mathrm{G}(s)$ by counting measure on Γ_U, and ρ_U is the action of $G_f(n)$ on functions ϕ on $\mathrm{G}(s)$ by right translation, $(\rho_U(k)\phi)(g)=\phi(g(1,k))$. *Reading through a basis.* For an E-basis $b=(b_i)_{i\in n}$ of U let $J_b\in\mathrm{GL}_n(E)$ have entries $(J_b)_{ij}=h(b_j,b_i)$, and let $x_b\in\mathrm{GL}_n(\mathbb{A}_E)$ be the matrix whose column i consists of the coordinates in b of $b_{v,i}$ at $v\in\Phi$ and of $x_U(e_i)$ at the finite places (e_i the standard basis). The **space with a point of U in the basis** b is $V_b=(J_b,x_b)$. Conversely, a space with a point V is the framed hermitian space (E^n,h_V) whose frame at v consists of the columns of the component of x_V at v and whose isometry is $z\mapsto x_{V,f}z$; then V is its own space with a point in the standard basis.

(VP4) Moved point, first minimum and height. Let W be a space with a point or a framed hermitian space, with signs s on n and coordinate map c_W, and let its vectors be those of E^n, respectively of U. For $g\in\mathrm{G}(s)$ the **lattice at** g and the **length at** g are

$$L_g(W)=\{u : (g^{-1}c_W(u))_f\in\widehat{\mathcal{O}}_E^n\},\qquad |u|_g=\|(g^{-1}c_W(u))_\infty\|,$$

and the **first minimum** and the **height** of g for W are

$$\lambda_W(g)=\min(1,\ \inf\{|u|_g : 0\neq u\in L_g(W)\}),\qquad \mathrm{ht}_W(g)=\lambda_W(g)^{-1},$$

with $\inf\emptyset=+\infty$ (so $\lambda_W(g)=1$ if $n=\emptyset$). $L_1(W)$ is the lattice of W. For a space with a point V and $g\in\mathrm{G}(s)$ the **moved space** is $V_g=(J_V,x_Vg)$, a space with a point with the same signs.

(VP5) Standard spaces, orthogonal sums and $\mathrm{Hom}(S,T)$. The **standard definite space** I_n is the space with a point $(1,1)$ with signs $+1$ on n. The **orthogonal sum** $V\oplus V'$ of spaces with a point with signs s on n and s' on n' is the space with a point on $n\sqcup n'$ with block diagonal matrix and block diagonal point and the concatenated signs. For spaces with a point S (signs s on n) and T (signs s' on n'), $\mathrm{Hom}(S,T)$ is the space with a point on $n'\times n$ with signs $s'\otimes s\colon (i,j)\mapsto s'(i)s(j)$, matrix $J_T\otimes(J_S^{-1})^{\mathsf{T}}$ and point $x_T\otimes(x_S^{-1})^{\mathsf{T}}$, its vectors being read as $n'\times n$ matrices X, on which the matrix and the point act by $X\mapsto J_TXJ_S^{-1}$ and $X\mapsto x_TXx_S^{-1}$. The **pair map** sends $(g,h)\in\mathrm{G}(s)\times\mathrm{G}(s')$ to $h\otimes(g^{-1})^{\mathsf{T}}\in\mathrm{G}(s'\otimes s)$, the map $X\mapsto hXg^{-1}$; its determinant is $\det(h)^{\#n}\det(g)^{-\#n'}$.

Definition 3.2.3. Let n be a nonempty finite set and $\varepsilon\in\{\pm1\}^n$. The **unitary group** $\mathrm{U}(\mathrm{diag}\,\varepsilon)$ is the group of invertible complex $n\times n$ matrices g with $g^*\mathrm{diag}(\varepsilon)g=\mathrm{diag}(\varepsilon)$, and K_ε is its subgroup of unitary matrices, a compact group with Haar probability measure dk. For a complex $n\times n$ matrix X the **block-diagonal part** $\mathrm{bd}_\varepsilon(X)$ keeps the entries X_{ij} with $\varepsilon_i=\varepsilon_j$ and replaces the others by 0. A matrix z is **accretive** if $z+z^*$ is positive definite. For $P\subseteq n$ and a matrix y, $\Delta_P(y)$ is the determinant of the submatrix of y with rows and columns in P ($\Delta_\emptyset(y)=1$).

Parameters. An integer vector $l\in\mathbb{Z}^n$ is read as the doubled Harish-Chandra parameter of $\lambda=\frac{1}{2}l$. Put

$$2\rho_l(i)=\#\{j : l_j<l_i\}-\#\{j : l_i<l_j\},\qquad 2\rho_l^{\mathrm{c}}(i)=\#\{j : \varepsilon_j=\varepsilon_i,\ l_j<l_i\}-\#\{j : \varepsilon_j=\varepsilon_i,\ l_i<l_j\},$$

$l_i^\rho = \lfloor \frac{1}{2}(l_i + 2\rho_l(i)) \rfloor$ (the vector $\lambda + \rho$) and

$$\mu_l(i) = l_i^\rho - 2\rho_l^{\mathrm{c}}(i) \qquad (i \in n),$$

the **lowest K-type** of l (for the signs ε; the vector $\lambda + \rho - 2\rho_{\mathrm{c}}$).

Products of principal minors. For $w \in \mathbb{Z}^n$ and an invertible $n \times n$ matrix g,

$$m_w(g) = (\det g)^{\min_i w_i} \prod_{\min_i w_i < t \le \max_i w_i} \Delta_{\{i:\, w_i \ge t\}}(g). \tag{3.1}$$

Flensted-Jensen's function. Let $l \in \mathbb{Z}^n$ and $\|m_{\mu_l}\|^2 = \int_{K_\varepsilon} |m_{\mu_l}(k)|^2\, dk$, a positive number since m_{μ_l} is continuous on K_ε and $m_{\mu_l}(1) = 1$. For an invertible matrix a and an accretive matrix z put

$$\mathrm{FJ}_l(a, z) = \frac{1}{\|m_{\mu_l}\|^2} \int_{K_\varepsilon} \frac{m_{\mu_l}(a\,k)\,\overline{m_{\mu_l}(k)}}{m_{-l^\rho}(k^* z\, k)}\, dk; \tag{3.2}$$

the integrand is continuous on K_ε, because $k^* z k$ is accretive and an accretive matrix has nonzero principal minors. For $g \in \mathrm{U}(\operatorname{diag}\varepsilon)$ such that $\mathrm{bd}_\varepsilon(g)$ is invertible and $\mathrm{bd}_\varepsilon(g)^{-1} g$ is accretive, **Flensted-Jensen's function** of $\lambda = \frac{1}{2} l$ is

$$c_\lambda(g) = \mathrm{FJ}_l(\mathrm{bd}_\varepsilon(g),\ \mathrm{bd}_\varepsilon(g)^{-1} g).$$

Definition 3.2.4. Let E be a CM field with maximal totally real subfield E^+, and let Φ be the set of infinite places of E^+, each $v \in \Phi$ with a chosen embedding $E \to \mathbb{C}$ above it. Let $N \ge 1$.

(FT1) Framed spaces. A **framed space** of rank N is a framed hermitian space U with signs $s\colon \{1, \dots, N\} \to \{\pm 1\}$ in the sense of (VP3) of Definition 3.2.2 (signs, one for all $v \in \Phi$, frames at $v \in \Phi$ and the isometry x_U as there), its hermitian form being written conjugate-linear in the first argument: the standard form on $\mathbb{A}_{E,f}^N$ is then written $\sum_i \bar{z}_i z_i'$, and a form h' so written is read as the form $h(u, u') = h'(u', u)$ of the preamble of Definition 3.2.2. Put $G_\infty = G_\infty(s)$, $G_f = G_f(\{1, \dots, N\})$ and $\mathrm{K}_f = \mathbf{K}_0(\{1, \dots, N\})$, the groups of (VP1). Through the frames $\mathrm{U}(U)(\mathbb{A}_{E^+}) = G_\infty \times G_f$; $\Gamma_U \subseteq G_\infty \times G_f$ and ρ_U (right translation in the second factor) are those of (VP3).

(FT2) Forms of type e. The measure dx of G_∞ and the measures of G_f and of $G_\infty \times G_f$ are those of (VP1) of Definition 3.2.2, and integrals over $\Gamma_U \backslash (G_\infty \times G_f)$ are for its quotient measure of (VP3); integrals of functions that are not integrable are 0 (Bochner integrals). For a function $e\colon G_\infty \to \mathbb{C}$, a **form of type** e on U is a continuous bounded $F\colon G_\infty \times G_f \to \mathbb{C}$ that is left invariant under Γ_U, right invariant under an open subgroup of G_f, and, if e is integrable,

$$\int_{G_\infty} e(x)\, F(g\,(x, 1))\, dx = F(g) \qquad (g \in G_\infty \times G_f).$$

For an open subgroup $K \subseteq G_f$, $\mathcal{A}_K(U, e)$ is the space of forms of type e that are right invariant under K, with the **Petersson product** $\langle F, F' \rangle = \int_{\Gamma_U \backslash (G_\infty \times G_f)} \overline{F}\, F'$.

(FT3) Discrete series type kernels. A **parameter** for the signs s is $\lambda \in (\frac{1}{2}\mathbb{Z})^N$, and c_λ is Flensted-Jensen's function of λ for the signs s. Its **formal degree** is $d_\lambda = \left(\int_{\mathrm{U}(\operatorname{diag} s)} |c_\lambda|^2\right)^{-1}$, for the Cayley measure of $\mathrm{U}(\operatorname{diag} s)$ ($d_\lambda = 0$ if c_λ is not square integrable). For a family $\lambda = (\lambda(v))_{v \in \Phi}$ of parameters the **type kernel** is

$$e_\lambda(x) = \prod_{v \in \Phi} d_{\lambda(v)}\, \overline{c_{\lambda(v)}(x_v)} \qquad (x \in G_\infty),$$

a **form of parameter** λ is a form of type e_λ, and $\mathcal{A}_K(U, \lambda) = \mathcal{A}_K(U, e_\lambda)$.

(FT4) Functions cuspidal at a split place. For a compact open subgroup $K \subseteq G_f$ and a finitely supported $a = \sum_h a_h\, h \in \mathbb{C}[G_f]$ put $\phi_a^K(g) = \sum_h a_h \mathbf{1}_K(h^{-1}g)$. Let w be a finite place of E with $w \neq \bar{w}$ and $\iota_w\colon \mathrm{GL}_N(E_w) \to G_f$ the embedding at the place of E^+ below w (component 1 elsewhere). A locally constant compactly supported $\phi\colon G_f \to \mathbb{C}$ is **cuspidal at** w if for every $\emptyset \neq T \subsetneq \{1,\dots,N\}$, with ϵ the coordinate projector on T and $U_T = \{1 + \epsilon Z(1-\epsilon) : Z \in \mathrm{M}_N(k_w)\} \subseteq \mathrm{GL}_N(k_w)$, for every $t \in \mathrm{GL}_N(k_w)$, every $g \in G_f$ and every choice of lifts $\tilde{u} \in \mathrm{GL}_N(\mathcal{O}_w)$ of the elements ut ($u \in U_T$),

$$\sum_{u \in U_T} \phi(g\, \iota_w(\tilde{u})) = 0.$$

(FT5) Lattices at a point, principal levels, cusp frames. Let U be a framed space of rank N with hermitian form h, and $G^U = G_\infty \times G_f = \mathrm{G}(s)$. The coordinate vector $c(u) = c_U(u) \in \mathbb{A}_E^N$ of $u \in U$ is that of (VP3) of Definition 3.2.2, $\|\cdot\|$ is the euclidean norm on $\prod_{v\in\Phi} \mathbb{C}^N$, and for $g \in G^U$ the lattice at g, $L_g = L_g(U)$, and the length at g, $|u|_g$, are those of (VP4). For an integer $M \geq 1$ the **principal level** $\mathrm{K}_f(M) = \{k \in \mathrm{K}_f : k - 1 \in M\,\mathrm{M}_N(\widehat{\mathcal{O}}_E)\}$ (so $\mathrm{K}_f(1) = \mathrm{K}_f$). For a nonzero isotropic subspace $A \subseteq U$, N_A is the unipotent radical of the stabiliser of A in $\mathrm{U}(U)$ (a unipotent group over E^+), with $N_A(\mathbb{A}_{E^+}) \subseteq G^U$ through the frames and $N_A(E^+) \subseteq \Gamma_U$. For $\delta \geq 0$, $Z_\delta = \{z \in G_\infty : \max_{v\in\Phi} \|z_v - 1\| \leq \delta\}$ (operator norm), and G_∞, G_f are embedded in G^U as in (VP1). For $\delta \geq 0$, $g \in G^U$ and a subgroup $J \subseteq G_f$, U has a **cusp frame of size** δ **at** g **covered by** J if there are a nonzero isotropic subspace $A \subseteq U$ and reals $0 \leq a \leq b$ with $ab \leq \delta$ such that A is spanned by vectors $u \in L_g$ with $|u|_g \leq a$, $A^\perp$ is spanned by vectors $w \in L_g$ with $|w|_g \leq b$, and

$$N_A(\mathbb{A}_{E^+})\, g \subseteq N_A(E^+)\, g\, Z_\delta\, J.$$

Definition 3.2.5. All symbols introduced here are local to this definition and to the results that cite it. Let E be a CM field with complex conjugation $a \mapsto \bar{a}$ (so $\tau(\bar{a}) = \overline{\tau(a)}$ for every embedding $\tau\colon E \to \mathbb{C}$) such that

(Um) Unramified conjugation for every maximal ideal $\mathfrak{P}$ of $\mathcal{O}_E$ there is $\theta \in \mathcal{O}_E$ with $\theta - \bar{\theta} \notin \mathfrak{P}$.

Put $\ell_E = \log_2[E:\mathbb{Q}]$. A **definite hermitian space** is a finite-dimensional E-vector space V with a map $h\colon V \times V \to E$, conjugate-linear in the first and linear in the second variable, with $h(y,x) = \overline{h(x,y)}$ and $\tau(h(x,x)) > 0$ for all $x \neq 0$ and all embeddings τ; $\mathrm{U}(V)$ is the group of E-linear bijections g of V with $h(gx,gy) = h(x,y)$. For a subspace U, $U^\perp = \{x : h(u,x) = 0\ \forall u \in U\}$; then $V = U \oplus U^\perp$ and $(U, h|_U)$, $(U^\perp, h|_{U^\perp})$ are definite hermitian spaces.

A **lattice** in V is a finitely generated $\mathcal{O}_E$-submodule M with $EM = V$; it is **integral** if $h(M,M) \subseteq \mathcal{O}_E$; its dual is $M^\vee = \{x \in V : h(y,x) \in \mathcal{O}_E\ \forall y \in M\}$, and for integral M the **discriminant index** is $\mathrm{dn}(M) = [M^\vee : M]$ (index of abelian groups). M has **cyclic discriminant** if $M^\vee = M + \mathcal{O}_E t$ for some $t \in V$. $\mathrm{U}(V)$ acts on lattices by $M \mapsto gM$; $\mathrm{Aut}(M)$ is the stabiliser of M, a finite group. Let F be the fixed field of $a \mapsto \bar{a}$. A lattice M' lies in the **genus** of M if for every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_F$ there is an isometry g of $V_\mathfrak{p} = V \otimes_F F_\mathfrak{p}$ (an $E \otimes_F F_\mathfrak{p}$-linear bijection preserving the extension of h, which is sesquilinear over $E \otimes_F F_\mathfrak{p}$) with $g\widehat{M}_\mathfrak{p} = \widehat{M'}_\mathfrak{p}$, where $\widehat{M}_\mathfrak{p} = M \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\mathfrak{p}}$. The **mass** of M is $\mathrm{mass}(M) = \sum_{M'} |\mathrm{Aut}(M')|^{-1}$, M' over representatives of the $\mathrm{U}(V)$-orbits in the genus of M. For $j \geq 1$, $\mathrm{mass}(I_j)$ is the mass of the lattice $\mathcal{O}_E^j$ in E^j with $h(x,y) = \sum_i \bar{x}_i y_i$.

For a subgroup $\Gamma \subseteq \mathrm{U}(V)$, a Γ-stable set $\mathcal{G}$ of lattices and a function $N\colon \mathcal{G} \to [0,\infty]$ constant on Γ-orbits, the **mass sum**

$$\mathrm{ms}_\Gamma(\mathcal{G}; N) = \sum_{[M]\in\Gamma\backslash\mathcal{G}} \frac{N(M)}{|\mathrm{Stab}_\Gamma(M)|} \in [0,\infty],$$

the sum over the Γ-orbits with any representative M; $\mathrm{ms}_\Gamma(\mathcal{G}) = \mathrm{ms}_\Gamma(\mathcal{G}; 1)$. For $n \geq 1$, $\mathcal{L}_n(V)$ is the set of integral lattices M in V with $\mathrm{dn}(M) = n$, and $m_n(V) = \mathrm{ms}_{\mathrm{U}(V)}(\mathcal{L}_n(V))$. For a lattice L and integers $j \geq 0$, $n \geq 1$, $\mathrm{Pr}_{j,n}(L)$ is the set of j-dimensional subspaces $U \subseteq V$ such that the **primitive sublattice** $L \cap U$, a lattice in $(U, h|_U)$, has $\mathrm{dn}(L \cap U) = n$.

Lemma 3.2.6. *Let E be a CM number field containing a primitive 5040th root of unity, and let $R = \widehat{\mathcal{O}}_E = \prod_{w\nmid\infty} \mathcal{O}_{E_w}$ with complex conjugation. For a matrix over R, write $A^* = \bar{A}^t$. Let n, k be nonnegative integers.*

1. *For every $H \in \mathrm{GL}_n(R)$ satisfying $H^* = H$, there are $T \in \mathrm{GL}_n(R)$ and $W \in M_n(R)$ such that*
$$T^*HT = I_n, \qquad T = I_n + W(H - I_n).$$
In particular, $TX = X$ for every matrix X over R satisfying $HX = X$.

2. *If $X, Y \in M_{n,k}(R)$ have left inverses over R and satisfy $X^*X = Y^*Y$, there is $g \in \mathrm{GL}_n(R)$ satisfying*
$$g^*g = I_n, \qquad gX = Y.$$

Both assertions hold at every finite place, including those of residue characteristic two; the common Gram matrix in the second assertion need not be invertible.

Proof. The coefficient involution. Put $F = E^+$. The given root of unity supplies $i \in \mathcal{O}_E$ with $i^2 = -1$, $\bar{i} = -i$, and a primitive seventh root ξ. Set $\omega = -(\xi + \xi^2 + \xi^4)$. The identity $1+\xi+\cdots+\xi^6 = 0$ gives $\omega+\bar{\omega} = 1$. Hence $c = i\omega$ satisfies $c-\bar{c} = i \in \mathcal{O}_E^\times$. In particular, over the integers S_v of each completion of F, the product R_v of the integers of E above v is a quadratic etale S_v-algebra, split or unramified. Here is a direct check that includes residue characteristic two: for $x \in R_v$, the elements

$$d_x = (x - \bar{x})/(c - \bar{c}), \qquad x - d_x c$$

are fixed by conjugation and therefore belong to S_v. Thus $(1, c)$ is an S_v-basis of R_v; the discriminant of its quadratic relation is $(c - \bar{c})^2$, a unit. Also $e = c/(c - \bar{c})$ is integral and $e + \bar{e} = 1$.

A norm calculation with a congruence. We will use the following elementary consequence. Let S be a complete discrete valuation ring with finite residue field, let R/S be a split or unramified quadratic etale algebra, and let B_0 be a commutative finite free S-algebra. Write $B = B_0 \otimes_S R$, with conjugation on the second factor. If I is an ideal of B_0 and $u \in B_0^\times$ satisfies $u - 1 \in I$, there is $z \in B^\times$ such that

$$z - 1 \in IB, \qquad z\bar{z} = u.$$

A trace-one element $e \in R$ exists: in the split case take $(1, 0)$; in the unramified case lift a residue element of trace 1, whose trace is a unit, and divide by that trace. To prove the norm assertion, first work modulo a power of a uniformizer of S. The resulting B_0 is a finite Artinian ring with nilpotent Jacobson radical J. Modulo J, it is a product of finite fields. On each factor,

B is a quadratic finite-field extension or a split quadratic algebra; its norm on units is onto, respectively by the cyclic multiplicative group of a finite field or by multiplication in the split case. The image of I on each factor is either zero or the whole field. Choose the norm preimage to be 1 on the former factors, where $u = 1$. Lift these choices to a unit $z \in 1 + IB$. Initially $u/(z\bar{z}) - 1 \in I \cap J$. If this error is $d \in I \cap J^\ell$, choose $y = ed \in (I \cap J^\ell)B$, where $e + \bar{e} = 1$. Then

$$(1 + y)(1 + \bar{y}) = 1 + d + y\bar{y} \equiv 1 + d \pmod{J^{\ell+1}}.$$

Replacing z by $z(1 + y)$ removes the error modulo $J^{\ell+1}$, retains $z \equiv 1 \pmod{IB}$, and retains invertibility. Nilpotence finishes the finite-ring calculation. Finally B is compact, IB is closed, and the norm equations modulo successive powers define nested nonempty closed subsets of $1 + IB$. Their intersection gives the stated z. The congruence modulo the uniformizer already ensures that its norm, and hence z, is a unit. This argument uses trace one rather than division by two.

Orthonormal correction preserving fixed vectors. For a hermitian $H \in \mathrm{GL}_n(R_v)$, its characteristic polynomial Q has coefficients in S_v: conjugate transpose preserves the characteristic polynomial, and $H^* = H$. Form

$$B_0 = S_v[t]/(Q(t)), \qquad B = B_0 \otimes_{S_v} R_v.$$

The image of t is a unit since the constant coefficient of Q is a unit. Apply the norm calculation to $I = (t - 1)B_0$ and $u = t^{-1}$. We obtain $z \equiv 1 \pmod{t - 1}$ and $\bar{z}z = t^{-1}$. Evaluation $t \mapsto H$ is a ring homomorphism from B to the commutative subalgebra generated by H and scalar matrices, by Cayley–Hamilton, and intertwines conjugation with adjoint. Its value $T_v = z(H)$ is invertible, satisfies

$$T_v^* H T_v = 1, \qquad T_v = 1 + W_v(H - 1)$$

for some integral matrix W_v. Taking the product over finite places of F gives these statements over $R = \widehat{\mathcal{O}}_E$, with no restricted-product condition to check because every component is integral. This proves the orthonormal-frame assertion and the stronger displayed congruence.

An invertible map compatible with adjoints. Choose integral left inverses $LX = 1$ and $MY = 1$. Put

$$A_0 = YL + M^*X^*(1 - XL).$$

Direct multiplication using $X^*X = Y^*Y$ gives $A_0X = Y$ and $A_0^*Y = X$. We need to make A_0 invertible while retaining both equalities. Over R, for arbitrary square matrices A, P, Q, there exists C for which

$$A + (1 - AP)C(1 - QA)$$

is invertible. Indeed this may be checked and chosen separately at each residue field. Over a field write $A = UDV$ with U, V invertible and $D = \mathrm{diag}(D_1, 0)$, D_1 invertible. Take $C = U\,\mathrm{diag}(0, 1)V$. After multiplication by U^{-1} and V^{-1}, the displayed matrix has block form

$$\begin{pmatrix} D_1 + D_1P'_{12}Q'_{21}D_1 & -D_1P'_{12} \\ -Q'_{21}D_1 & 1 \end{pmatrix}, \qquad P' = VPU, \quad Q' = VQU.$$

Its Schur complement is D_1, so it is invertible, including when either block has size zero. Lift each chosen C to the local integer ring. The resulting matrix has unit determinant at every finite place, so it and its inverse lie in $M_n(R)$.

Apply this observation with $A = A_0$, $P = L^*Y^*$ and $Q = XM$, and denote the corrected matrix by A. The correction annihilates X on the right because $(1 - XMA_0)X = 0$. Its adjoint annihilates Y on the right because $(1 - A_0L^*Y^*)^*Y = 0$. Thus

$$A \in \mathrm{GL}_n(R), \qquad AX = Y, \qquad A^*Y = X.$$

Extension. Set $H = A^*A$. It is hermitian invertible, and $HX = X$. Apply the orthonormal correction just proved to choose $T = 1 + W(H-1)$ with $T^*HT = 1$. Then $TX = X$. Therefore $g = AT$ is integral and invertible, $g^*g = 1$, and $gX = Y$. No nondegeneracy of X^*X was used. □

Lemma 3.2.7. *Let E be a CM number field containing a primitive 5040th root of unity. Put*

$$R = \widehat{\mathcal{O}}_E = \prod_{w \nmid \infty} \mathcal{O}_{E_w},$$

with the involution induced by complex conjugation. There exists $c \in R^\times$ such that $\bar{c}\,c = -1$.

Proof. Apply Lemma 3.2.6(1) with $n = 1$ and $H = (-1) \in \mathrm{GL}_1(R)$. This matrix is hermitian, since complex conjugation fixes -1, and it is invertible with inverse itself. The field hypothesis is exactly that of the cited lemma. We obtain $T \in \mathrm{GL}_1(R)$ satisfying $T^*HT = (1)$. Writing $T = (c)$ gives $c \in R^\times$ and

$$-\bar{c}\,c = 1,$$

hence $\bar{c}\,c = -1$, as required. □

Lemma 3.2.8. *Let $P = (q_0, d)$ be a parameter pair and $s \geq 0$ an integer; write $q_1 = q_1(s)$ and $E = E_s$.*

1. *Every prime ℓ dividing q_1 divides q_0, differs from d, and splits in K: $\ell\mathcal{O}_K = \mathfrak{p}_\ell\, c(\mathfrak{p}_\ell)$ with $\mathfrak{p}_\ell \neq c(\mathfrak{p}_\ell)$. Moreover $d \equiv 7 \pmod 8$.*
2. *$[E : K] = \varphi(q_1)$, $\mathcal{O}_E = \mathcal{O}_K[\zeta]$, and $\mathfrak{D}_{E/K} = \mathrm{cyc}'_{q_1}(\zeta)\,\mathcal{O}_E$; the element $\mathrm{cyc}'_{q_1}(\zeta)$ divides q_1 in $\mathcal{O}_E$.*
3. *There is an integral ideal $\mathfrak{b}$ of $\mathcal{O}_E$ such that*

$$\mathfrak{D}_{E/K} = \mathfrak{b}\,c(\mathfrak{b}), \qquad \mathfrak{b} + c(\mathfrak{b}) = \mathcal{O}_E, \qquad \sigma(\mathfrak{b}) = \mathfrak{b} \quad \text{for every } \sigma \in \mathrm{Gal}(E/K).$$

Proof. Item 1. By Definition 2.3.1, $16 \mid q_0$ and $q_0 \mid 2(d+1)$, so $8 \mid d+1$, that is $d \equiv 7 \pmod 8$ and $-d \equiv 1 \pmod 8$. As $T(s) = 3\,l_0(s)\,2^s$ is a product of powers of 2 and 3, and $6 \mid 18480 \mid q_0$, every prime $\ell \mid q_1 = q_0 T(s)$ divides q_0. If $\ell = 2$: since $-d \equiv 1 \pmod 8$, 2 splits in K, and $2 \neq d$. If ℓ is odd: $\ell \mid q_0 \mid 2(d+1)$ gives $\ell \mid d+1$, so $\ell \nmid d$ (in particular $\ell \neq d$) and $-d \equiv 1 \pmod \ell$ is a nonzero square modulo ℓ, so ℓ splits in K. Both cases use the decomposition law in the quadratic field K, whose discriminant is $-d$ as $-d \equiv 1 \pmod 4$ [224, Chapter I]. If $\ell\mathcal{O}_K = \mathfrak{p}_\ell\mathfrak{p}'_\ell$ with $\mathfrak{p}_\ell \neq \mathfrak{p}'_\ell$, then $\mathfrak{p}'_\ell = c(\mathfrak{p}_\ell)$, because c restricts to the non-trivial automorphism of K ($c(\sqrt{-d}) = -\sqrt{-d}$, as every embedding sends $\sqrt{-d}$ to a purely imaginary number) and $\mathrm{Gal}(K/\mathbb{Q})$ permutes the primes above ℓ transitively.

Item 2. Put $F = \mathbb{Q}(\zeta)$, so $E = KF$. The ring of integers of F is $\mathbb{Z}[\zeta]$, and $F/\mathbb{Q}$ is ramified only at the primes dividing q_1 [317, Chapter 2]; $K/\mathbb{Q}$ is ramified only at d, which does not divide q_1 by item 1. Hence $K \cap F = \mathbb{Q}$ (a subfield of K different from $\mathbb{Q}$ is K, which is ramified at d, while F is not), the discriminants of K and F are coprime, and therefore $[E : K] = [F : \mathbb{Q}] = \varphi(q_1)$ and $\mathcal{O}_E = \mathcal{O}_K\,\mathbb{Z}[\zeta] = \mathcal{O}_K[\zeta]$ [224, Chapter I, §2]. The monic polynomial $\mathrm{cyc}_{q_1} \in \mathbb{Z}[X]$ of degree $\varphi(q_1) = [E : K]$ vanishes at ζ, so it is the minimal polynomial of ζ over K; as $\mathcal{O}_E = \mathcal{O}_K[\zeta]$, the different is $\mathfrak{D}_{E/K} = \mathrm{cyc}'_{q_1}(\zeta)\mathcal{O}_E$ [224, Chapter III, §2]. Finally write $X^{q_1} - 1 = \mathrm{cyc}_{q_1}(X)\,Q(X)$ with $Q \in \mathbb{Z}[X]$; differentiating and evaluating at ζ gives $q_1\zeta^{q_1-1} = \mathrm{cyc}'_{q_1}(\zeta)\,Q(\zeta)$, and ζ is a unit, so $\mathrm{cyc}'_{q_1}(\zeta) \mid q_1$ in $\mathcal{O}_E$.

The different is stable under $\mathrm{Gal}(E/\mathbb{Q})$. Let $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ ($E/\mathbb{Q}$ is Galois, as the compositum of two Galois extensions). Then $\zeta' = \sigma(\zeta)$ is again a primitive q_1-th root of unity, ζ and ζ' are powers of each other, so $\mathcal{O}_K[\zeta'] = \mathcal{O}_K[\zeta] = \mathcal{O}_E$, and the argument of item 2 with ζ' in place of ζ gives $\mathfrak{D}_{E/K} = \mathrm{cyc}'_{q_1}(\zeta')\mathcal{O}_E = \sigma\big(\mathrm{cyc}'_{q_1}(\zeta)\mathcal{O}_E\big) = \sigma(\mathfrak{D}_{E/K})$, since cyc_{q_1} has rational coefficients. Writing $\mathfrak{D}_{E/K} = \prod_{\mathfrak{P}} \mathfrak{P}^{n_{\mathfrak{P}}}$ over the maximal ideals of $\mathcal{O}_E$, it follows that $n_{\sigma(\mathfrak{P})} = n_{\mathfrak{P}}$ for all σ and $\mathfrak{P}$.

Item 3. If $n_{\mathfrak{P}} > 0$ then $\mathfrak{P} \supseteq \mathfrak{D}_{E/K} \ni q_1$, so $\mathfrak{P}$ lies above a prime $\ell \mid q_1$, hence above $\mathfrak{p}_\ell$ or $c(\mathfrak{p}_\ell)$ (item 1). Put

$$\mathfrak{b} = \prod_{\ell \mid q_1} \prod_{\mathfrak{P} \mid \mathfrak{p}_\ell} \mathfrak{P}^{n_{\mathfrak{P}}},$$

an integral ideal. The map $\mathfrak{P} \mapsto c(\mathfrak{P})$ is a bijection from the primes above $\mathfrak{p}_\ell$ onto those above $c(\mathfrak{p}_\ell)$, and $n_{c(\mathfrak{P})} = n_{\mathfrak{P}}$; hence $c(\mathfrak{b}) = \prod_{\ell \mid q_1} \prod_{\mathfrak{P} \mid c(\mathfrak{p}_\ell)} \mathfrak{P}^{n_{\mathfrak{P}}}$ and $\mathfrak{b}\, c(\mathfrak{b}) = \mathfrak{D}_{E/K}$. The primes dividing $\mathfrak{b}$ lie above some $\mathfrak{p}_\ell$, those dividing $c(\mathfrak{b})$ above some $c(\mathfrak{p}_{\ell'})$; as the $2 \cdot \#\{\ell \mid q_1\}$ primes $\mathfrak{p}_\ell, c(\mathfrak{p}_\ell)$ of K are pairwise distinct, $\mathfrak{b}$ and $c(\mathfrak{b})$ have no common prime factor, so $\mathfrak{b} + c(\mathfrak{b}) = \mathcal{O}_E$. Finally $\sigma \in \mathrm{Gal}(E/K)$ fixes each $\mathfrak{p}_\ell$, so it permutes the primes above $\mathfrak{p}_\ell$, preserving the exponents $n_{\mathfrak{P}}$; hence $\sigma(\mathfrak{b}) = \mathfrak{b}$. □

Lemma 3.2.9. *1. Let E be a CM field that contains a primitive 28-th root of unity. Then E satisfies (Um): there is $y \in \mathcal{O}_E$ such that $y - \bar{y}$ is a root of unity of order 4. Consequently, if E^+ is the fixed field of $a \mapsto \bar{a}$, no maximal ideal of $\mathcal{O}_{E^+}$ ramifies in E.*

2. Let $P = (q_0, d)$ be a parameter pair and $s \geq 0$ an integer. Then $5040 \mid q_1(s)$, so E_s contains a primitive 5040-th root of unity; in particular E_s satisfies (Um), and E_s is unramified over its maximal totally real subfield E_s^+ at every finite place.

Proof. Item 1. Let $\eta \in E$ be a primitive 28-th root of unity, and put $i_E = \eta^7$, a root of unity of order 4, and $\zeta_7 = \eta^4$, a primitive 7-th root of unity. Let

$$g = \sum_{a=1}^{6} \Big(\frac{a}{7}\Big)\zeta_7^a \in \mathcal{O}_E.$$

As $7 \equiv 3 \pmod 4$, the evaluation of the quadratic Gauss sum gives $g^2 = (\frac{-1}{7})7 = -7$ [317, Chapter 4]. Put $\omega = (1+g)/2$. Then

$$\omega^2 - \omega + 2 = \frac{(1+g)^2 - 2(1+g) + 8}{4} = \frac{g^2 + 7}{4} = 0,$$

so ω is an algebraic integer, $\omega \in \mathcal{O}_E$.

Conjugates. Let $\tau\colon E \to \mathbb{C}$ be an embedding. By Definition 3.2.5, $\tau(\bar{a}) = \overline{\tau(a)}$ for all $a \in E$. Since $\tau(i_E)^2 = -1$ and $\tau(g)^2 = -7$, both $\tau(i_E)$ and $\tau(g)$ are purely imaginary, so $\tau(\bar{i}_E) = -\tau(i_E)$ and $\tau(\bar{g}) = -\tau(g)$; as τ is injective, $\bar{i}_E = -i_E$ and $\bar{g} = -g$. Hence $\bar{\omega} = (1-g)/2 = 1 - \omega$.

The element y. Put $y = i_E\,\omega \in \mathcal{O}_E$. Then

$$y - \bar{y} = i_E\,\omega - (-i_E)(1-\omega) = i_E.$$

As $i_E^4 = 1$, i_E is a unit of $\mathcal{O}_E$, so $y - \bar{y} \notin \mathfrak{P}$ for every maximal ideal $\mathfrak{P}$ of $\mathcal{O}_E$: this is (Um) with $\theta = y$ for every $\mathfrak{P}$. The last assertion of item 1 is Lemma 3.2.10 applied to E.

Item 2. We have $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$. By Definition 2.3.1, $18480 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ divides q_0, so $2^4 \cdot 5 \cdot 7 \mid q_0$; and $3\,l_0(s) = 3^{\lfloor \log_3 s \rfloor + 5}$ is divisible by 3^5. Hence $5040 \mid q_0 \cdot 3\,l_0(s) \mid q_0 T(s) = q_1(s)$. With ζ the primitive $q_1(s)$-th root of unity of Definition 2.3.1, $\zeta^{q_1(s)/5040} \in E_s$ is a

primitive 5040-th root of unity, and $\zeta^{q_1(s)/28}$ a primitive 28-th root of unity. As E_s is a CM field (Definition 2.3.1), item 1 applies; its maximal totally real subfield is the fixed field E_s^+ of complex conjugation. □

Lemma 3.2.10. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field with (Um) and F the fixed field of $a \mapsto \bar{a}$. Then no maximal ideal of $\mathcal{O}_F$ ramifies in E: every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_F$ is either split ($\mathfrak{p}\mathcal{O}_E = \mathfrak{P}\bar{\mathfrak{P}}$ with $\mathfrak{P} \neq \bar{\mathfrak{P}}$) or inert ($\mathfrak{p}\mathcal{O}_E$ is a maximal ideal).*

Proof. $[E : F] = 2$, so for a maximal ideal $\mathfrak{p}$ of $\mathcal{O}_F$ either $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}\bar{\mathfrak{P}}$ with $\mathfrak{P} \neq \bar{\mathfrak{P}}$, or $\mathfrak{p}\mathcal{O}_E$ is maximal, or $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}^2$. In the last case $\bar{\mathfrak{P}} = \mathfrak{P}$ (it is the only prime above $\mathfrak{p}$) and the residue degree is 1, so $\mathcal{O}_F/\mathfrak{p} \to \mathcal{O}_E/\mathfrak{P}$ is bijective: for $\theta \in \mathcal{O}_E$ there is $a \in \mathcal{O}_F$ with $\theta - a \in \mathfrak{P}$, and conjugating, $\bar{\theta} - a \in \bar{\mathfrak{P}} = \mathfrak{P}$. Hence $\theta - \bar{\theta} \in \mathfrak{P}$ for every $\theta \in \mathcal{O}_E$, which contradicts (Um) for this $\mathfrak{P}$. □

Lemma 3.2.11. *Let P be a parameter pair, $s \geq 0$ and $E = E_s$. There is a continuous homomorphism $\upsilon\colon \mathbb{A}_E^\times \to \{z \in \mathbb{C} : |z| = 1\}$, trivial on $E^\times$, such that*

(i) $\upsilon(\bar{x}) = \upsilon(x)^{-1}$ *for every* $x \in \mathbb{A}_E^\times$;

(ii) $\upsilon(x) = \prod_{v \in \Phi_s} x_v/|x_v|$ *for every* $x \in \mathbb{A}_E^\times$ *whose components at all finite places are units, $x_v \in \mathbb{C}^\times$ being the component at v read through the embedding chosen at v;*

(iii) $\upsilon(\varpi) = -1$ *for every finite place w with $\bar{w} = w$ and every idele ϖ whose component at w is a uniformiser of E_w and whose other components are* 1.

Proof. Write $q = q_1(s)$, $F = E^+$, and $G = \mathbb{A}_E^\times$. Let

$$U = \mathbb{A}_{E,\infty}^\times \times \widehat{\mathcal{O}}_E^\times, \qquad p(x) = \prod_{v \in \Phi_s} x_v/|x_v|.$$

The subgroup U is open in G, and p is a continuous circle-valued character on it.

The exact unit compatibility. Every root of unity of E is a power of the chosen ζ. To check this, put $q = q_1(s)$. We have $\gcd(q, d) = 1$, and $d > 64$: indeed every prime divisor of q divides q_0, whereas $q_0 \mid 2(d+1)$ excludes $d \mid q_0$, and $18480 \leq q_0 \leq 2(d+1)$. The quadratic Gauss sum of conductor d has square $-d$, so $K \subset \mathbb{Q}(\zeta_d)$ and $E \subset \mathbb{Q}(\zeta_{qd})$. Since qd is even, the roots of unity of this last field have order dividing qd. One can see the last assertion directly from the degree formula: if it contains a root of order N and $L = \operatorname{lcm}(N, qd)$, then $\varphi(L) = \varphi(qd)$; for an even divisor qd of L, the prime factor formula for φ forces $L = qd$. If a root of unity of E had order not dividing q, then E would contain primitive roots of orders q and d, hence all of $\mathbb{Q}(\zeta_{qd})$. This would imply

$$\varphi(q)(d-1) \leq [E : \mathbb{Q}] = 2\varphi(q),$$

contrary to $d > 64$. Thus it is ζ^j for some integer $j \geq 0$.

Since $4 \mid q$, put $i_0 = \zeta^{q/4}$ and $\epsilon = i_0(\zeta - 1)$. The minimal polynomial of ζ over K is cyc_q, by Lemma 3.2.8(2). Its degree is even. The cyclotomic constant-term and value-at-one identities give

$$\mathrm{Norm}_{E/K}(\zeta) = 1, \qquad \mathrm{Norm}_{E/K}(\zeta - 1) = \mathrm{cyc}_q(1) = 1;$$

here $12 \mid q$, so q is not a prime power. Hence ϵ is an integral unit of norm 1 to K. Since $\bar{i}_0 = -i_0$ and $\bar{\zeta} = \zeta^{-1}$,

$$\zeta\bar{\epsilon} = \zeta(-i_0)(\zeta^{-1} - 1) = i_0(\zeta - 1) = \epsilon.$$

Consequently, for every root of unity $\rho = \zeta^j$, the actual unit $\delta = \epsilon^j$ satisfies

$$\rho\bar{\delta} = \delta, \qquad \prod_{v\in\Phi_s} \delta_v = \mathrm{Norm}_{E/K}(\delta) = 1. \tag{1}$$

The product here is the norm because the chosen embeddings are exactly the embeddings over K.

Triviality on the overlap. Let

$$B = E^\times\{b\bar{b} : b \in G\} \subset G.$$

This is a subgroup. Suppose $x = yb\bar{b} \in U$, with $y \in E^\times$ and $b \in G$. At every finite place w, integrality and invertibility of x_w give

$$\mathrm{ord}_w(y) + \mathrm{ord}_w(b) + \mathrm{ord}_{\bar{w}}(b) = 0.$$

The same equation at $\bar{w}$ shows $\mathrm{ord}_w(y) = \mathrm{ord}_w(\bar{y})$. Thus $\rho = \bar{y}/y$ is an algebraic unit, and every complex conjugate of ρ has modulus one. By Lemma A.2.1 it is a root of unity, since all embeddings of its field extend to embeddings of the CM field E. Choose the unit δ of (1) for this ρ, and put $a = y\delta$. Then

$$\bar{a} = \bar{y}\bar{\delta} = y\rho\bar{\delta} = y\delta = a,$$

so $a \in F^\times$. The finite idele $u = ab_f\bar{b}_f$ is fixed by conjugation and is a unit at every finite place, since $u = \delta x_f$.

Apply Lemma 3.2.6(1) in rank one to the hermitian unit u. It gives an integral finite unit t with $\bar{t}\,u\,t = 1$, hence $u = t^{-1}\bar{t}^{-1}$. Thus u is a norm from the finite ideles of E. Let η be the quadratic idele-class character of E/F supplied by Theorem A.2.8. Both u and $b_f\bar{b}_f$ are norms, so $\eta(a_f) = 1$, where a finite idele is given infinite components 1. Reciprocity kills the principal idele a, and its real local character is the sign. Therefore

$$\prod_{v\in\Phi_s} \mathrm{sign}(a_v) = 1. \tag{2}$$

At infinity $b_v\bar{b}_v = |b_v|^2 > 0$, so $p(x) = p(y)$. Equation (1) gives $p(\delta) = 1$, whence $p(y) = p(a)$; equation (2) gives $p(a) = 1$. We have proved

$$p|_{U\cap B} = 1. \tag{3}$$

This verifies compatibility on all principal elements and norms in the overlap, not just on the roots of unity themselves.

Extension and continuity. Equation (3) makes

$$\chi(ub) = p(u) \qquad (u \in U,\ b \in B)$$

a well-defined homomorphism on UB, trivial on B. It extends algebraically to a circle-valued homomorphism v on G. For completeness, take a maximal extension to a subgroup of G by Zorn's lemma. If that subgroup omits $z \in G$ and no positive power of z belongs to it, assign z any value on the circle. Otherwise let n be the least positive exponent with z^n in the subgroup, and assign z an nth root of the prescribed value of z^n. In either case the formula on products of old elements and powers of z is well-defined and extends the character, contradicting maximality. Such roots exist on the circle. Thus the maximal subgroup is G.

The restriction of v to the open subgroup U is the continuous character p. A homomorphism continuous on an open neighborhood of the identity is continuous everywhere, so v is continuous. It kills $E^\times \subset B$, and $v(x\bar{x}) = 1$ for every x, giving (i). Its restriction to U is p, giving (ii).

The quadratic restriction and the inert sign. Restrict υ to the embedded ideles of F. It kills $F^\times$ and $N_{E/F}\mathbb{A}_E^\times$. By Theorem A.2.8,

$$\mathbb{A}_F^\times/\big(F^\times N_{E/F}\mathbb{A}_E^\times\big) \simeq \mathrm{Gal}(E/F)$$

has order two. This restriction is nontrivial: its value on the idele that is -1 at exactly one real place and 1 elsewhere is -1, by (ii). It is therefore exactly η.

Now let $w = \bar{w}$ be finite, and let v be its place in F. By Lemma 3.2.9(2), E_w/F_v is unramified; as w is fixed by conjugation, it is the nonsplit extension of degree two. A uniformizer π_v of F_v is also a uniformizer of E_w. Its local Artin symbol is the nontrivial Frobenius of this quadratic extension, so $\upsilon(\pi_v) = \eta(\pi_v) = -1$. Every other uniformizer of E_w differs from π_v by a local unit. An idele supported on such a unit belongs to U and has phase 1, so υ is 1 on it. Thus (iii) holds for every indicated uniformizer, including at dyadic places. □

Definition 3.2.12. Let P be a parameter pair, $s \geq 0$, $E = E_s$, F its maximal totally real subfield, and $\sqrt{-d} \in K$ the element whose image in $\mathbb{C}$ is $i\sqrt{d}$. The **Weil data** of E are the following.

(WD1) The imaginary element. $\jmath = -1/\sqrt{-d} \in E$, and $c_0 = (1-\jmath^{-1})/2 = (1+\sqrt{-d})/2$.

(WD2) The Weil idele. $\alpha \in \mathbb{A}_{E,f}^\times$ is the finite idele with $\alpha_w = \mathrm{cyc}'_{q_1}(\zeta)^{-1}$ at the finite places w with $c_0 \in w$ and $\alpha_w = 1$ at the other finite places, cyc_{q_1} being the cyclotomic polynomial of order q_1.

(WD3) The splitting character. A **splitting character** of E is a continuous homomorphism $\upsilon\colon \mathbb{A}_E^\times \to \{z \in \mathbb{C} : |z| = 1\}$, trivial on $E^\times$, such that $\upsilon(\bar{x}) = \upsilon(x)^{-1}$ for every idele x; $\upsilon(x) = \prod_{v\in\Phi_s} x_v/|x_v|$ for every idele x whose finite components are units; and $\upsilon(\varpi) = -1$ for every finite place w with $\bar{w} = w$ and every idele ϖ whose component at w is a uniformiser of E_w and whose other components are 1. The splitting character υ of the Weil data is a fixed one, which exists by Lemma 3.2.11. We write υ also for its restriction to $\mathrm{U}(1)(\mathbb{A}) = \{u \in \mathbb{A}_E : u\bar{u} = 1\}$; as an automorphic character of $\mathrm{U}(1)$ it has weight 1 and trivial tame part.

(WD4) The additive character. This item is stated for any totally complex number field E (for instance a CM field) with a set Φ of complex embeddings containing one of each pair of complex conjugate embeddings; for $E = E_s$ it is $\Phi = \Phi_s$ with the embeddings chosen above. The adeles are $\mathbb{A}_E = \mathbb{C}^\Phi \times \mathbb{A}_{E,f}$, with components x_v ($v \in \Phi$) and x_f, and E embedded diagonally by $a \mapsto ((v(a))_{v\in\Phi}, a)$. The **standard additive character** of E is $\psi_E\colon \mathbb{A}_E/E \to \{z \in \mathbb{C} : |z| = 1\}$, $\psi_E(x) = \exp\big(-2\pi i \sum_{v\in\Phi} 2\,\mathrm{Re}\, x_v\big)\,\psi_{E,f}(x_f)$, where its **finite part** $\psi_{E,f}\colon \mathbb{A}_{E,f} \to \{z \in \mathbb{C} : |z| = 1\}$ is $\psi_{E,f}(y) = \exp\big(2\pi i\,\mathrm{Tr}_{E/\mathbb{Q}}(a)\big)$ for any $a \in E$ with $y - a \in \widehat{\mathcal{O}}_E$. Where the field E is fixed by the context, $\psi_{E,f}$ is also written ψ_f.

(WD5) The vacuum. For $n \geq 0$ the **vacuum** of rank n is the function on $\mathbb{A}_E^n$

$$e_n(x) = \exp\Big(-\pi \sum_{i=1}^{n} \sum_{v\in\Phi_s} \mathrm{Im}(\jmath_v)\,|x_{i,v}|^2\Big) \cdot \begin{cases} \psi_{E,f}\big(\frac{c_0\jmath}{2} \sum_{i=1}^n \bar{x}_{i,f} x_{i,f}\big) & \text{if } x_{i,f} \in \alpha\widehat{\mathcal{O}}_E \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

(WD6) Heisenberg data in rank n. Let $s = (s_1, \ldots, s_n) \in \{\pm 1\}^n$ and $W_n = \mathbb{A}_E^n$. The **bicharacter** of sign s is

$$\beta_s(x, y) = \psi_E\Big(\tfrac{\jmath}{2}\, \bar{y}^{\mathsf{T}} S\, x\Big) \qquad (x, y \in W_n),$$

where S is the diagonal matrix with entries s_i at the infinite places and 1 at the finite places. Let $\kappa_v = (2\pi \operatorname{Im} \jmath_v)^{-1/2} = (2\pi/\sqrt{d})^{-1/2}$ for $v \in \Phi_s$. The **vacuum measure** μ_n on W_n is the image of $\pi^{-fn}\,du \otimes dz$ under $(u,z) \mapsto \big((\kappa_v u_{v,i})_{v,i}, \alpha z\big)$, where $f = \#\Phi_s$, du is Lebesgue measure on $\mathbb{C}^{\Phi_s \times n} = \mathbb{R}^{2fn}$ and dz is the Haar measure on $\mathbb{A}_{E,f}^n$ giving $\widehat{\mathcal{O}}_E^n$ mass 1.

(WD7) The theta model. The **theta model** $\mathcal{M}_s$ is the space of μ_n-integrable $f\colon W_n \to \mathbb{C}$ with $f(x) = \int_{W_n} f(y)\,\beta_s(y,x)\,e_n(x-y)\,d\mu_n(y)$ for every x, with the norm of $L^1(\mu_n)$, the inner product $\langle f, g\rangle = \int \bar{f} g\, d\mu_n$, the **Heisenberg operators** $(\rho(z)f)(x) = \beta_s(z,x) f(x-z)$ ($z \in W_n$) and the **vacuum vector** $v^0 = e_n$.

(WD8) Finite coordinates. For a finite adelic vector $x \in \mathbb{A}_{E,f}^n$ the **finite coordinate conversion** is $\mathrm{fv}(x) = (0, \alpha x) \in W_n$ (archimedean part 0), and the **finite Heisenberg operator** is

$$\rho_f(x) = \psi_{E,f}\Big(\jmath\alpha\bar{\alpha}\,\tfrac{c_0}{2}\sum_{i=1}^{n} \bar{x}_i x_i\Big)\,\rho(\mathrm{fv}(x)).$$

(WD9) Theta functionals. Let V be a hermitian space over E of rank n, given with an element $x_V \in \mathrm{GL}_n(\mathbb{A}_E)$ with $\bar{x}_V^{\mathsf{T}} J_V x_V = S$, J_V the Gram matrix of V in a basis over E; let $V(E) = \{x_V^{-1} u : u \in E^n\} \subset W_n$ be its rational vectors read through x_V. A **theta functional** of V is a bounded linear functional θ on $\mathcal{M}_s$ with $\theta(\rho(\gamma)f) = \theta(f)$ for $\gamma \in V(E)$, $f \in \mathcal{M}_s$, and

$$\int_{W_n/V(E)} \big|\theta(\rho(x)v^0)\big|^2\,dx = 1,$$

the quotient carrying its invariant probability measure. Fix a unit c of $\widehat{\mathcal{O}}_E$ with $\bar{c}c = -1$; one exists by Lemma 3.2.7, which applies since E contains a primitive 5040th root of unity by (2) of Lemma 3.2.9. For signs s on $\{1,\dots,n\}$ the **standard space of signs** s has $J_V = \mathrm{diag}(s_1,\dots,s_n)$ and the diagonal point x_V whose entry i is 1 if $s_i = +1$ and, if $s_i = -1$, the idele with components 1 at the infinite places and finite part c; the **standard lines** of sign ± 1 are the standard spaces of rank 1.

Definition 3.2.13. All spaces are hermitian spaces over E (relative to complex conjugation) with a basis and an $\mathcal{O}_E$-lattice; a space is **standard definite** of rank n if it is E^n with the form $\sum_i z_i \bar{z}_i'$ and the lattice $\mathcal{O}_E^n$. Put $\varrho = m/2 - 1$, so that $m = 2\varrho + 2$ and $r = k + 2\varrho + 1$.

(HP1) For $j \geq 0$, X_j is the standard definite space of rank n_j; $Y_m \subset X_m$ is the span of the first $n_m - 1$ basis vectors, a standard definite space of rank $n_m - 1 = r + mt$.

(HP2) Y_0 has rank r, with basis $e_1,\dots,e_\varrho$, the basis of X_0, a vector z, and $e_1^*,\dots,e_\varrho^*$; its Gram matrix is that of X_0 on X_0, $\langle e_i, e_{i'}^*\rangle = \delta_{ii'}$, $\langle z, z\rangle = -N$, and all other products of basis vectors vanish. So Y_0 is X_0 plus ϱ hyperbolic planes and a line of norm $-N$, of signature $(k+\varrho, \varrho+1)$ at every infinite place. The **numbering** of the basis of Y_0 is a bijection num from the set of these r basis vectors onto $\{0,\dots,r-1\}$, fixed once and for all (any bijection may be taken); $\mathrm{num}(c)$ is the **number** of the basis vector c. The **sign** of a basis vector c of Y_0 is $(-1)^{e(c)}$ with $e(c) = \max(\mathrm{num}(c) + 1 - k, 0)$: it is -1 exactly when $\mathrm{num}(c) - k$ is non-negative and even, that is for the $\varrho + 1$ numbers $k, k+2, \dots, k + 2\varrho = r - 1$, and $+1$ for the other $k + \varrho$ basis vectors, which matches the signature. Lists indexed by the basis of Y_0 (signs, parameters) are written in this numbering.

(HP3) For $0 \le j < m$, Y_j has rank $n_j + (2\varrho + 1 - j) = r + jt$: its Gram matrix is the identity on X_j and antidiagonal on a string of $2\varrho + 1 - j$ further vectors $v_1, \dots, v_{2\varrho+1-j}$, with $\langle v_i, v_{2\varrho+2-j-i}\rangle = 1$ except that the middle vector, present when $2\varrho+1-j$ is odd, has norm $-N$; for $j = 0$ this is (HP2). For $j \ge m$, Y_j is standard definite.

(HP4) For $1 \le j \le m$, W_j is the diagonal space of rank r on the basis of Y_0 in which the basis vector c has norm $+1$ if $\mathrm{num}(c) < k+j-1$ and -1 if $\mathrm{num}(c) \ge k+j-1$ (the **signs of** W_j: $+1$ on the first $k + j - 1$ basis vectors in the numbering of (HP2), -1 on the remaining $m - j$); it has the lattice $\mathcal{O}_E^r$.

(HP5) Lattices and frames. Let $\varpi \in \mathbb{A}_{E,f}^\times$ be the finite idele with $\varpi_w \mathcal{O}_{E,w} = \pi_M \mathcal{O}_{E,w}$ at the chosen places and $\varpi_w = 1$ at the other finite places, let $\alpha \in \mathbb{A}_{E,f}^\times$ be the Weil idele of (WD2) of Definition 3.2.12, and put $\beta = \alpha\,\bar{\varpi}^{-1}$ (a finite idele; the letter a is the long order of the data) and, for $0 \le j < m$, $u_j = (\beta/\bar{\beta})^{\lceil (m-j)/2 \rceil}$. For $0 \le j < m$ let $s_j = 2\varrho + 1 - j$, let y_j be the last basis vector of X_j and, for odd s_j, $z_j = v_{(s_j+1)/2}$ the middle string vector of (HP3). The **graded basis** L_j of $Y_j \otimes_E \mathbb{A}_{E,f}$ keeps the basis vectors of X_j other than y_j and replaces $y_j, v_1, \dots, v_{s_j}$ by

$$\bar{\pi}_N^{-1} y_j + u_j N^{-1} z_j, \quad u_j v_1, \dots, u_j v_{s_j} \qquad (s_j \text{ odd}),$$

$$y_j, \quad u_j \beta^{-1} v_i \ (i \le s_j/2), \quad u_j \bar{\beta}\, v_i \ (i > s_j/2) \qquad (s_j \text{ even}).$$

The **lattice** M_{Y_j} of Y_j is the $\widehat{\mathcal{O}}_E$-span of L_j. A **frame** of Y_j consists of an $\widehat{\mathcal{O}}_E$-basis of M_{Y_j}, orthonormal for the form, of the shape $L_j g$ with $g \in \mathrm{GL}_{r+jt}(\widehat{\mathcal{O}}_E)$ reducing to 1 at every chosen place, and, at each $v \in \Phi$, a basis of $Y_j \otimes_{E,v} \mathbb{C}$ in which the form is diagonal with the signs of Y_j (-1 exactly $\lceil s_j/2 \rceil$ times); for $j = 0$ the vectors of this basis are indexed by the basis of Y_0, the vector of index c having norm equal to the sign of c of (HP2). For $j \ge m$ the lattice M_{Y_j} is $\widehat{\mathcal{O}}_E^{r+jt}$, and the **standard frame** of Y_j consists of the standard basis of $\widehat{\mathcal{O}}_E^{r+jt}$ and, at each $v \in \Phi$, the standard basis of $Y_j \otimes_{E,v} \mathbb{C} = \mathbb{C}^{r+jt}$. With a frame (for $j \ge m$, the standard frame), Y_j is a framed hermitian space with the signs of Y_j in the sense of (VP3) of Definition 3.2.2: its frames at $v \in \Phi$ are the bases at v of the frame, and $x_U(e_i)$ is the i-th vector of $L_j g$ (of the standard basis for $j \ge m$); in the basis b of Y_j its space with a point is $V_b = (J_b, x_b)$ of (VP3).

Lemma 3.2.14. *Let $0 \le j < m$, let $J_j \in M_{r+jt}(E)$ be the Gram matrix of Y_j in its basis (HP3), and regard the graded basis L_j as a matrix in $\mathrm{GL}_{r+jt}(\mathbb{A}_{E,f})$ whose columns are the vectors of L_j. Then:*

1. *there is $k \in \mathrm{GL}_{r+jt}(\widehat{\mathcal{O}}_E)$ whose reduction at every chosen place is 1 and with $(L_j k)^* J_j (L_j k) = 1$;*
2. *for every $v \in \Phi$ and every list $\varepsilon \in \{\pm 1\}^{r+jt}$ with exactly $\lceil s_j/2 \rceil$ entries -1 there is $y \in \mathrm{GL}_{r+jt}(\mathbb{C})$ with $y^* J_j^v y = \mathrm{diag}(\varepsilon)$, where J_j^v is the image of J_j under v.*

In particular Y_j has a frame as in (HP5).

Proof. Write $n = r + jt$, $s = s_j$, $u = u_j$, $\pi = \pi_N$, and $h(x, x') = x^* J_j x'$ (conjugate linear in the first variable). Since $u = (\beta/\bar{\beta})^c$ for an integer c, $\bar{u}u = 1$; and $\pi\bar{\pi} = N$.

Step 1: the Gram matrix $H = L_j^* J_j L_j$ lies in $\mathrm{GL}_n(\widehat{\mathcal{O}}_E)$. The vectors of X_j other than y_j are orthonormal and orthogonal to all other vectors of L_j. For even s there is no middle vector;

v_i pairs only with v_{s+1-i}, which lies in the other half of the string, and $h(u\beta^{-1}v_i, u\bar{\beta}\,v_{s+1-i}) = \bar{u}\,\bar{\beta}^{-1}u\bar{\beta} = 1$, so $H = J_j$. For odd s put $y' = \bar{\pi}^{-1}y_j + uN^{-1}z_j$, where z_j pairs only with itself, $h(z_j, z_j) = -N$. Then

$$h(y', y') = (\pi\bar{\pi})^{-1} + \bar{u}uN^{-2}(-N) = 0, \qquad h(y', uz_j) = \bar{u}uN^{-1}(-N) = -1, \qquad h(uz_j, uz_j) = -N,$$

$h(uv_i, uv_{s+1-i}) = 1$ for $v_i \neq z_j$, and y' is orthogonal to the other string vectors. So H has integer entries, and up to a permutation it is block diagonal with blocks 1, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$, each of determinant ± 1. Hence H is hermitian and $H \in \mathrm{GL}_n(\widehat{\mathcal{O}}_E)$.

Step 2: a hermitian $H \in \mathrm{GL}_n(\widehat{\mathcal{O}}_E)$ satisfies $l^*Hl = 1$ for some $l \in \mathrm{GL}_n(\widehat{\mathcal{O}}_E)$. As $\widehat{\mathcal{O}}_E = \prod_v R_v$ over the finite places v of E^+, $R_v = \mathcal{O}_E \otimes \mathcal{O}_{E^+,v}$, it suffices to find $l_v \in \mathrm{GL}_n(R_v)$ for each v. Since $18480 \mid q_0 \mid q_1(s)$, E contains i and ζ_3, neither in the totally real E^+; so $E = E^+(i) = E^+(\zeta_3)$. For $v \nmid 2$ put $\omega = i$, for $v \mid 2$ put $\omega = \zeta_3$; then $\omega \in R_v$ and $\omega - \bar{\omega}$ ($= 2i$, resp. $\pm\sqrt{-3}$) is a unit of R_v, so $E \otimes E_v^+$ is an unramified quadratic field extension of E_v^+ or $E_v^+ \times E_v^+$, and R_v is its maximal order. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_{E^+,v}$; $\mathfrak{p}R_v$ is the Jacobson radical of R_v. Induct on n, with $M = R_v^n$ and $h_H(x, x') = x^*Hx'$. If $h_H(x, x) \in \mathfrak{p}$ for all $x \in M$, then for $x, x' \in M$ and $t = h_H(x, x')$, polarisation applied to $x + x'$ and to $\omega x + x'$ gives $t + \bar{t} \in \mathfrak{p}R_v$ and $\bar{\omega}t + \omega\bar{t} \in \mathfrak{p}R_v$, so $(\bar{\omega} - \omega)t \in \mathfrak{p}R_v$ and $t \in \mathfrak{p}R_v$; then $H \equiv 0$ modulo $\mathfrak{p}R_v$, contradicting $\det H \in R_v^\times$. So some x has $c = h_H(x, x) \in \mathcal{O}_{E^+,v}^\times$. The norm $R_v^\times \to \mathcal{O}_{E^+,v}^\times$ is surjective (for the unramified field extension this is the classical surjectivity of the norm on units; in the split case $(c', 1) \mapsto c'$), so $c^{-1} = \bar{\lambda}\lambda$ with $\lambda \in R_v^\times$, and $e_1 = \lambda x$ has $h_H(e_1, e_1) = 1$. Then $M = R_v e_1 \oplus e_1^\perp$ via $x' \mapsto (h_H(e_1, x')e_1,\ x' - h_H(e_1, x')e_1)$; $e_1^\perp$ is a direct summand of M, hence free of rank $n - 1$ (R_v is local or a product of two local rings and the rank is constant), and the Gram matrix of h_H on a basis of it is hermitian with unit determinant. By induction $e_1^\perp$ has an orthonormal basis; together with e_1 it is the columns of the required l_v.

Step 3: correction at the chosen places. Take l from Step 2 for H of Step 1, so $(L_j l)^* J_j (L_j l) = 1$. So each chosen w lies over a place v of E^+ split in E, distinct chosen places lie over distinct v, and $R_v = \mathcal{O}_w \times \mathcal{O}_{\bar{w}}$ with conjugation exchanging the factors. In these coordinates $g = (A, B)$ satisfies $g^*g = 1$ if and only if $B = (A^\mathsf{T})^{-1}$ (identifying $\mathcal{O}_{\bar{w}}$ with $\mathcal{O}_w$ by conjugation), so $A \mapsto (A, (A^\mathsf{T})^{-1})$ is an isomorphism from $\mathrm{GL}_n(\mathcal{O}_w)$ onto the integral unitary group of the identity form at v. As $\mathrm{GL}_n(\mathcal{O}_w) \to \mathrm{GL}_n(k_w)$ is surjective, choose $A_w \in \mathrm{GL}_n(\mathcal{O}_w)$ reducing to the inverse of the reduction of l at w, and let $g \in \mathrm{GL}_n(\widehat{\mathcal{O}}_E)$ be $(A_w, (A_w^\mathsf{T})^{-1})$ at these v and 1 elsewhere. Then $g^*g = 1$, and $k = lg$ satisfies $(L_j k)^* J_j (L_j k) = g^*g = 1$ and reduces to 1 at every chosen place. This is (1).

Step 4: the infinite places. J_j has rational entries, so $J_j^v = J_j$. In $Y_j \otimes_{E,v} \mathbb{C}$ take the basis of X_j, the vectors $(v_i \pm v_{s+1-i})/\sqrt{2}$ for $i < (s+1)/2$, and, for odd s, $z_j/\sqrt{N}$. They are orthogonal, of norms 1, ± 1 and -1 respectively, since $h(v_i + v_{s+1-i}, v_i - v_{s+1-i}) = -1 + 1 = 0$. The number of norms -1 is $\lfloor s/2 \rfloor + [s \text{ odd}] = \lceil s/2 \rceil$. Reordering this basis to match ε gives y. This is (2), and (1) with (2) is a frame as in (HP5). □

Definition 3.2.15. (XF1) Fixed frames. For $0 \leq j < m$ the **fixed frame** of Y_j is one frame of Y_j in the sense of (HP5), fixed once and for all; frames of Y_j exist by Lemma 3.2.14. Its **finite part** is the $\widehat{\mathcal{O}}_E$-basis $L_j g$ of M_{Y_j} ($g \in \mathrm{GL}_{r+jt}(\widehat{\mathcal{O}}_E)$ reducing to 1 at every chosen place), and its **part at** $v \in \Phi$ is the basis of $Y_j \otimes_{E,v} \mathbb{C}$ in which the form is diagonal with the signs of Y_j. For $j \geq m$ the fixed frame of Y_j is the standard frame of (HP5). With its fixed frame, Y_j is the framed hermitian space of (HP5) (in the sense of (VP3) of Definition 3.2.2); the point of Y_j is the point x_b of its space with a point $V_b = (J_b, x_b)$ in the basis b of Y_j ((VP3)), the space with a point given by its fixed frame.

Definition 3.2.16. Labels. Let k_0 be a finite field and $r \geq 1$. A **label** of degree r over k_0 is a

monic squarefree $c \in k_0[X]$ of degree r with $c(0) \neq 0$. It is **primitive** if it is irreducible and has a root of multiplicative order $|k_0|^r - 1$ in an algebraic closure. For an integer M_0 it is **allowed for** M_0 if no root of c in k_0 is an M_0-th root of unity. Let Ξ_c be Green's character of $\mathrm{GL}_r(k_0)$ attached to the label c, formed with a fixed faithful character of the multiplicative group of an algebraic closure of k_0, and

$$e_c = \frac{\Xi_c(1)}{|\mathrm{GL}_r(k_0)|} \sum_{g \in \mathrm{GL}_r(k_0)} \overline{\Xi_c(g)}\, g \ \in \mathbb{C}[\mathrm{GL}_r(k_0)].$$

A tuple $(c_w)_{w \in S}$ of labels of degree r over finite fields k_w is **good for** M_0 if every c_w is allowed for M_0 and at least one is primitive.

The group $\bar{G}$ and e_0. For $i = 0, 1$ let S_i be the set of chosen places above p_i, and for a chosen place w let k_w be its residue field. Put $\bar{G} = \prod_{w \in S_0 \cup S_1} \mathrm{GL}_r(k_w)$. A family $c = (c_w)_{w \in S_0 \cup S_1}$ is a **good family** if for $i = 0$ and for $i = 1$ the tuple $(c_w)_{w \in S_i}$ is good for q_2. The **good idempotent** is

$$e_0 = \sum_{c \text{ good family}} \ \bigotimes_{w \in S_0 \cup S_1} e_{c_w} \ \in \mathbb{C}[\bar{G}],$$

a central idempotent.

Target parameter. At $v \in \Phi$ the **target parameter** is the sequence of r integers

$$\varsigma(v) = \Big(\tfrac{k}{2} - 1,\ \tfrac{k}{2} - 2,\ \ldots,\ 1 - \tfrac{k}{2};\ \tfrac{\Delta}{2} - \kappa_1(v),\ \ldots,\ \tfrac{\Delta}{2} - \kappa_m(v)\Big),$$

with $k - 1$ entries before the semicolon (k is even).

Forms of principal level and good forms. Let V be a hermitian space of rank r on the basis of Y_0 with given signs, the same at every infinite place (for example Y_0 or W_j). Let $\mathrm{K}_M \subseteq \mathrm{U}(V)(\mathbb{A}_f)$ be the group of integral elements that reduce to 1 at every chosen place. The **forms of principal level** $\mathcal{A}_M(V)$ are the automorphic forms on $\mathrm{U}(V)(\mathbb{A})$ that are right invariant under K_M and whose archimedean type at each $v \in \Phi$ is the discrete series with Harish-Chandra parameter $\varsigma(v)$ (for the signs of V at v). The group $\bar{G}$ acts on $\mathcal{A}_M(V)$: $g \in \bar{G}$ acts by right translation by any integral element of $\mathrm{U}(V)(\mathbb{A}_f)$ whose reductions at the chosen places are the components of g. The **good forms** on V are $\mathcal{A}^{\mathrm{good}}(V) = e_0\, \mathcal{A}_M(V)$. **Archimedean type.** Precisely, V with its frames is a framed space of rank r with signs s ((FT1) of Definition 3.2.4), so $\mathrm{U}(V)(\mathbb{A}) = G_\infty \times G_f$ with $G_\infty = \prod_{v \in \Phi} \mathrm{U}(\mathrm{diag}\, s)$, and the entry of $\varsigma(v)$ of number i is attached to the basis vector c with $\mathrm{num}(c) = i$, num the numbering of the basis of Y_0 of (HP2) of Definition 3.2.13, in which the signs s of V are listed. A function on $\mathrm{U}(V)(\mathbb{A})$ **has archimedean type** ς if it is a form of parameter $\varsigma = (\varsigma(v))_{v \in \Phi}$, that is, of type e_ς, built from Flensted-Jensen's functions $c_{\varsigma(v)}$ of Definition 3.2.3 for the signs s (normalised by $c_{\varsigma(v)}(1) = 1$) and their formal degrees $d_{\varsigma(v)}$ ((FT3) of Definition 3.2.4). Thus $\mathcal{A}_M(V) = \mathcal{A}_{\mathrm{K}_M}(V, \varsigma)$, with the Petersson product.

Definition 3.2.17. Weight. At $v \in \Phi$ let $\lambda_v = (a, 0, \ldots, 0)$ if $v \in \Phi_a$ and $\lambda_v = 0$ otherwise (highest weights of GL_k), and let $W_\lambda = \bigotimes_{v \in \Phi} W_{\lambda_v}$, where W_{λ_v} is the irreducible algebraic representation of $\mathrm{GL}_k(\mathbb{C})$ of highest weight λ_v (determinant exponent 0); For each $v \in \Phi$ let $F_v \colon E \hookrightarrow \mathbb{C}$ be the chosen embedding over K, and let ρ_{λ_v} denote the standard integral Weyl representation in its bideterminant-coordinate basis. Write $\rho_\lambda((A_v)_v) = \bigotimes_v \rho_{\lambda_v}(A_v)$. The rational action of $\gamma \in \mathrm{U}(X_0)(E^+)$ on W_λ is $\rho_\lambda((\overline{F_v(\gamma)})_v)$, where conjugation and F_v are applied entrywise.

Level. Let $\mathrm{K}_0 \subseteq \mathrm{U}(X_0)(\mathbb{A}_f)$ be the group of integral elements that reduce to 1 at every chosen place and whose last row is $\equiv (0, \ldots, 0, 1)$ modulo π_N.

Source forms. A **source form** is a function $\varphi\colon \mathrm{U}(X_0)(\mathbb{A}_f) \to W_\lambda$ with $\varphi(\gamma g \kappa) = \gamma \cdot \varphi(g)$ for all $\gamma \in \mathrm{U}(X_0)(E^+)$, $g \in \mathrm{U}(X_0)(\mathbb{A}_f)$, $\kappa \in \mathrm{K}_0$. They form a finite-dimensional complex vector space $\mathcal{S}$. Put $d_v = a$ for $v \in \Phi_a$ and $d_v = 0$ otherwise. For these one-row weights the Weyl-coordinate indices are $\alpha = (\alpha_v)_v$, with $\alpha_v \in \mathbb{Z}^k_{\geq 0}$ and $\sum_i \alpha_{v,i} = d_v$. In the monomial evaluation coordinates, fix the positive definite diagonal Gram matrix

$$G_{\alpha,\alpha} = \prod_{v \in \Phi} \frac{d_v!}{\prod_{i=1}^k \alpha_{v,i}!}, \qquad G_{\alpha,\beta} = 0 \quad (\alpha \neq \beta).$$

The source Weyl norm is $\|\xi\|^2 = \xi^* G \xi$. Thus an evaluation vector $(y^\alpha)_{|\alpha|=d_v}$ has squared norm $(\sum_i |y_i|^2)^{d_v}$.

Ramified source forms. For each finite place w of E dividing π_N fix a **scaling** $z_w \in \mathrm{U}(1)(\mathbb{A}_f)$: integral, equal to 1 away from w and $\bar{w}$, with component at w reducing to a generator of $k_w^\times$ (of order $N-1$). A source form φ is **ramified** if for every such w

$$\sum_{i=0}^{N-2} \varphi(\,\cdot\, z_w^i) = 0,$$

where z_w acts as a scalar matrix. They form the subspace $\mathcal{S}^{\mathrm{ram}} \subseteq \mathcal{S}$.

Integral source forms. For an embedding $j\colon \mathbb{C} \hookrightarrow \mathbb{C}_\ell$ put $\iota_{j,v}(z) = j(F_v(\bar{z}))$ for $z \in E$. Denote by $\iota^f_{j,v}$ the resulting map from the finite adeles of E to $\mathbb{C}_\ell$, obtained by projecting to the induced finite place and extending this field embedding continuously. For $g \in \mathrm{U}(X_0)(\mathbb{A}_f)$ and $\xi \in W_\lambda$, define

$$\mathrm{gauge}_j(g,\xi) = \max_I \left| \left[\rho_\lambda((\iota^f_{j,v}(g^{-1}))_v)\, j(\xi) \right]_I \right|_\ell,$$

where I runs through the same integral Weyl-coordinate basis and $j(\xi)$ is applied coordinatewise. A source form φ is **integral at** j if $\mathrm{gauge}_j(g, \varphi(g)) \leq 1$ for every g. In the normalisation $|\ell|_\ell = \ell^{-1}$, the coefficient ring is $\{z \in \mathbb{C} : |j(z)|_\ell \leq 1\}$. **Independence of the scaling.** If z'_w is another such scaling, its residue generator is the c-th power of that of z_w for an integer c coprime to $N-1$. Thus $z'_w z_w^{-c} \in \mathrm{K}_0$ and its right-translation operator is T_w^c, where $T_w \varphi(g) = \varphi(g z_w)$ and $T_w^{N-1} = 1$. Multiplication by c permutes the residues modulo $N-1$, so $\sum_{i=0}^{N-2} (T_w^c)^i = \sum_{i=0}^{N-2} T_w^i$. Hence the defining ramified subspace is independent of the scaling. The operator $(N-1)^{-1} \sum_i T_w^i$ is a projector; $\prod_w (1 - T_w)$ has range $\mathcal{S}^{\mathrm{ram}}$ and is not in general a projector.

Definition 3.2.18. Index sets are finite.

(WR1) Spaces with points. For the CM field $E = E_s$ with $F = E^+$: signs s on n, the element ε_s, the groups $\mathrm{U}(\mathrm{diag}\, s)$, $G_\infty(s)$, $G_f(n)$ and $\mathrm{G}(s) = G_\infty(s) \times G_f(n)$, the standard level $\mathbf{K}_0(n)$, the determinant and the measures are those of (VP1) of Definition 3.2.2; spaces with a point $V = (J_V, x_V)$, their form $h_V(u,u') = u'^* J_V u$, real and finite frames, lattice, rational vectors, the unitary group $\mathrm{U}(V)$, the identification $g \mapsto x_V g x_V^{-1}$ of $\mathrm{G}(s)$ with $\mathrm{U}(V)(\mathbb{A}_F)$, the rational points Γ_V and $[\mathrm{U}(V)] = \Gamma_V \backslash \mathrm{G}(s)$ with its quotient measure are those of (VP2); for $g \in \mathrm{G}(s)$ the moved space V_g, the first minimum $\lambda_V(g)$ and the height $\mathrm{ht}_V(g) = \lambda_V(g)^{-1}$ are those of (VP4); the standard definite space I_n and the orthogonal sum $V \oplus V'$ are those of (VP5). Let c be the unit of $\widehat{\mathcal{O}}_E$ with $\bar{c}c = -1$ fixed in (WD9) of Definition 3.2.12. For an index set p the **hyperbolic space** $\mathbb{H}_p$ has index set $p \sqcup p$, signs $+1$ on the first and -1 on the second copy, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the point equal to $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ at every infinite place and to $\begin{pmatrix} 1 & -c \\ c_0 & c\,\bar{c}_0 \end{pmatrix}$ at the finite places (blocks of size p, scalars times the identity).

(WR2) The space $\mathrm{Hom}(S,T)$**.** For spaces with points S (signs s on n) and T (signs s' on n'), the space with a point $\mathrm{Hom}(S,T)$ on $n'\times n$ with signs $s'\otimes s$, its vectors read as $n'\times n$ matrices, and the pair map $(g,h)\mapsto h\otimes(g^{-1})^{\mathsf T}$ from $\mathrm G(s)\times\mathrm G(s')$ to $\mathrm G(s'\otimes s)$, the map $X\mapsto hXg^{-1}$, with determinant $\det(h)^{\#n}\det(g)^{-\#n'}$, are those of (VP5) of Definition 3.2.2.

(WR3) The Weil representation on the theta model. For signs s on n and $g\in\mathrm G(s)$ the **multiplier** is

$$\Delta_s(g)=\prod_{v\in\Phi}\overline{\det(g_v)_-}\cdot\upsilon(\det u)\,|\det u|^{-1/2},$$

where $(g_v)_-$ is the block of g_v on the indices i with $s_i=-1$ (its determinant is 1 if there are none), $u\in\mathrm{GL}_n(\mathbb A_{E,f})$ is any element with $u\widehat{\mathcal O}_E^n=\widehat{\mathcal O}_E^n\cap g_f^{-1}\widehat{\mathcal O}_E^n$, and $\det u$ is read as an idele with components 1 at the infinite places. The **Weil representation** of $\mathrm G(s)$ on $\mathcal M_s$ is

$$(\omega_s(g)f)(x)=\Delta_s(g)\int_{\mathbb A_E^n}f(y)\,\beta_s(gy,x)\,e_n(x-gy)\,d\mu_n(y)\qquad(f\in\mathcal M_s,\ x\in\mathbb A_E^n).$$

Transporting elements. For signs s on n let V_s be the standard space of signs s of (WD9) of Definition 3.2.12, a space with a point with signs s. Let V be a space with a point with signs s. A **transporting element** of V is a $k\in\mathrm G(s)$ that carries the rational vectors of V into those of V_s: for every $u\in E^n$ there is $u'\in E^n$ with $k\,x_V^{-1}u=x_{V_s}^{-1}u'$.

(WR4) The Weil representation of a pair. For spaces with points S,T as in (WR2), $\mathcal M_{S,T}=\mathcal M_{s'\otimes s}$ is the theta model of $\mathrm{Hom}(S,T)$. For automorphic characters α,β of $\mathrm U(1)$ the **Weil representation of the pair** is

$$\omega_{\alpha,\beta}(g,h)=\alpha(\det h)\,\beta(\det g)^{-1}\,\omega_{s'\otimes s}(h\otimes(g^{-1})^{\mathsf T})\qquad(g\in\mathrm G(s),\ h\in\mathrm G(s')),$$

so that $\omega_{\alpha,\beta}(g,h)=\alpha(\det h)\beta(\det g)^{-1}\omega_{1,1}(g,h)$.

(WR5) The Fock model. For signs ε on an index set ι let ν_ι be $\pi^{-\#\iota}$ times Lebesgue measure on $\mathbb C^\iota=\mathbb R^{2\#\iota}$,

$$\beta_\varepsilon^\infty(x,y)=\exp\bigl((y^*\mathrm{diag}(\varepsilon)x-x^*\mathrm{diag}(\varepsilon)y)/2\bigr)\quad\text{and}\quad e^\infty(x)=\exp(-x^*x/2).$$

The **Fock model** $\mathcal F_\varepsilon$ is the space of ν_ι-integrable $c\colon\mathbb C^\iota\to\mathbb C$ with

$$c(x)=\int c(y)\beta_\varepsilon^\infty(y,x)e^\infty(x-y)\,d\nu_\iota(y)\quad\text{for all }x,$$

with the norm of $L^1(\nu_\iota)$, the inner product $\int\bar c c'\,d\nu_\iota$ and the **real Heisenberg operators** $(\rho^\infty(y)c)(x)=\beta_\varepsilon^\infty(y,x)\,c(x-y)$ $(y\in\mathbb C^\iota)$. The **Fock vector** of a polynomial $P\in\mathbb C[X_i:i\in\iota]$ is $v_P(x)=P(\zeta(x))e^\infty(x)$, where $\zeta_i(x)=\bar x_i$ if $\varepsilon_i=1$ and $\zeta_i(x)=x_i$ if $\varepsilon_i=-1$; $v_1=e^\infty$ is the **real vacuum**. For $u\in\mathrm U(\mathrm{diag}\,\varepsilon)$,

$$(\omega_\varepsilon^\infty(u)c)(x)=\overline{\det u_-}\int c(y)\,\beta_\varepsilon^\infty(uy,x)\,e^\infty(x-uy)\,d\nu_\iota(y),$$

u_- the block of u on the indices of sign -1. For S,T as in (WR2) and $\gamma,\gamma'\colon\Phi\to\mathbb Z$, the Fock model of $\mathrm{Hom}(S,T)$ is $\mathcal F_{S,T}=\mathcal F_\varepsilon$ for $\iota=\Phi\times n'\times n$ and $\varepsilon(v,i,j)=s'(i)s(j)$, and the **real Weil representation of the pair with weights** (γ,γ') is

$$\Omega_{\gamma,\gamma'}(x,y)=\prod_{v\in\Phi}\det(y_v)^{\gamma(v)}\det(x_v)^{-\gamma'(v)}\,\omega_\varepsilon^\infty\Bigl(\bigoplus_{v\in\Phi}y_v\otimes(x_v^{-1})^{\mathsf T}\Bigr)$$
$$(x\in G_\infty(s),\ y\in G_\infty(s')).$$

(WR6) Data. For signs s on n, $v \in \mathcal{F}_\varepsilon$ with $\iota = \Phi \times n$, $\varepsilon(w,i) = s_i$, and $f \in \mathcal{M}_s$, the datum $v \otimes f \in \mathcal{M}_s$ is $\int v(y)\,\rho(\mathrm{rv}(y))f\,d\nu_\iota(y)$, where $\mathrm{rv}(y) \in \mathbb{A}_E^n$ has component $\kappa_w y_{w,i}$ at $w \in \Phi$ and 0 at the finite places. Let A be a finite set of finite places of E containing the conjugate of none of its members. For $z \in \mathbb{A}_{E,f}^n$ let z_A keep the components at A and set the others to 0. P^A is the mean, over $y \in \widehat{\mathcal{O}}_E^n$ with the Haar probability measure, of $\rho_f(y_{A\cup\bar{A}})$. For $y \in \mathbb{A}_{E,f}^n$ supported at $A \cup \bar{A}$, τ_y is the function $x \mapsto \psi_{E,f}\big(\jmath\alpha\bar{\alpha}\,\frac{1}{2}\,\bar{y}^{\mathsf{T}}(2x+y)_A\big)\mathbf{1}_A(x+y)$ of finite adelic vectors x, $\mathbf{1}_A$ the indicator of the vectors integral at A; $\mathcal{T}_A$ is the span of the τ_y. Fix a linear map $\Psi \mapsto (c_y(\Psi))_y$ from $\mathcal{T}_A$ to finitely supported families with $\sum_y c_y(\Psi)\tau_y = \Psi$. For $\Psi \in \mathcal{T}_A$ the **Schrödinger datum** is $\Psi \otimes_A f = \sum_y c_y(\Psi)\,\rho(\mathrm{fv}(y))P^A f$. The **datum** of (v,Ψ) at A is $v \otimes (\Psi \otimes_A v^0)$; at $A = \emptyset$ and $\Psi = \mathbf{1}_\emptyset = 1$ it is $v \otimes v^0$, and v^0 is the **vacuum**.
Products of data. Let s_1, s_2, s be signs on index sets n_1, n_2, n and $e\colon n_1 \sqcup n_2 \to n$ a bijection with $s \circ e = s_1 \sqcup s_2$. For $f_1 \in \mathcal{M}_{s_1}$ and $f_2 \in \mathcal{M}_{s_2}$ the **product** $f_1 \boxtimes_e f_2 \in \mathcal{M}_s$ is the function $x \mapsto f_1\big((x\circ e)|_{n_1}\big)\,f_2\big((x\circ e)|_{n_2}\big)$ on $\mathbb{A}_E^n$. For spaces with points S_1, S_2 (signs on n_1, n_2) and T (signs on n'), the **source product** of $f \in \mathcal{M}_{S_1,T}$ and $f' \in \mathcal{M}_{S_2,T}$ is $f \boxtimes^{\mathrm{sc}} f' = f \boxtimes_e f' \in \mathcal{M}_{S_1\oplus S_2,T}$ for the bijection e of $(n'\times n_1)\sqcup(n'\times n_2)$ onto $n'\times(n_1\sqcup n_2)$ that is the identity on each part; for T_1, T_2 (signs on n_1', n_2') and S (signs on n), the **target product** of $f \in \mathcal{M}_{S,T_1}$ and $f' \in \mathcal{M}_{S,T_2}$ is $f \boxtimes^{\mathrm{tg}} f' = f \boxtimes_e f' \in \mathcal{M}_{S,T_1\oplus T_2}$ for the bijection e of $(n_1'\times n)\sqcup(n_2'\times n)$ onto $(n_1'\sqcup n_2')\times n$ that is the identity on each part.

(WR7) Conjugation and doubled data. For an index set ι let $\sigma_c\colon \mathbb{A}_E^\iota \to \mathbb{A}_E^\iota$ be the map $x \mapsto (x_\infty, c^{-1}x_f)$, where c is the unit of (WD9) of Definition 3.2.12 (as in (WR1)) acting on the finite components. The **conjugate** of a function f on $\mathbb{A}_E^\iota$ is the function

$$(\mathrm{cj}\,f)(x) = \overline{f(\sigma_c x)} \qquad (x \in \mathbb{A}_E^\iota).$$

Let n, n' be index sets and $n \sqcup n$ the disjoint union of two copies of n; for $x \in \mathbb{A}_E^{n'\times(n\sqcup n)}$ let $x_1, x_2 \in \mathbb{A}_E^{n'\times n}$ be its restrictions to n' times the first and the second copy. For a complex number c_{d} with $|c_{\mathrm{d}}| = 1$ and functions f, f' on $\mathbb{A}_E^{n'\times n}$, the **doubled datum** of (f,f') with constant c_{d} is the function on $\mathbb{A}_E^{n'\times(n\sqcup n)}$

$$\mathrm{dbl}_{c_{\mathrm{d}}}(f,f')(x) = f(x_1)\,c_{\mathrm{d}}\,\overline{f'(\sigma_c x_2)} = f(x_1)\,c_{\mathrm{d}}\,(\mathrm{cj}\,f')(x_2).$$

Definition 3.2.19. Let $N = (N_\tau)_{\tau\in\Phi}$ and let κ be a **weight**: natural numbers $\kappa(\tau,h)$ for $\tau \in \Phi$ and $1 \le h < N_\tau$, the number of columns of height h of a Young diagram at τ. Its **highest weight** at τ is $\lambda_\tau(i) = \sum_{i<h<N_\tau}\kappa(\tau,h)$ for $0 \le i < N_\tau$.

(WM1) Products of minors. Let $y_{\tau,a,c}$ be variables ($\tau \in \Phi$, $1 \le a \le N_\tau$ a row, $c \ge 0$ a column). For a set A of h rows, the **minor** $\Delta_{\tau,A}(y)$ is the determinant of $(y_{\tau,a,c})_{a\in A,\,0\le c<h}$. A **column filling** T of κ assigns to every τ, every $1 \le h < N_\tau$ and every $1 \le j \le \kappa(\tau,h)$ a set $T(\tau,h,j)$ of h rows; its **product of minors** is $\Delta_T = \prod_{\tau,h,j}\Delta_{\tau,T(\tau,h,j)}$. Let ∇_κ be the $\mathbb{Z}$-module spanned by the Δ_T. The **standard basis** is a fixed $\mathbb{Z}$-basis $(b_i)_{i\in I_\kappa}$ of ∇_κ consisting of products of minors [3].

(WM2) The Weyl module. The **Weyl module** of weight κ is $\mathbb{V}_\kappa = \mathbb{C}^{I_\kappa}$, the coordinate vectors in the basis dual to (b_i); $\mathbb{V}_\kappa(R) = R^{I_\kappa}$ for a ring R. For matrices $z = (z_\tau)$, $z_\tau \in \mathrm{M}_{N_\tau}(R)$, the **Schur matrix** $S_\kappa(z) \in \mathrm{M}_{I_\kappa}(R)$ is defined by $b_i(zy) = \sum_j S_\kappa(z)_{ij}\,b_j(y)$, where $(zy)_{\tau,a,c} = \sum_{a'}(z_\tau)_{aa'}y_{\tau,a',c}$. More generally, let $N'_\tau \ge N_\tau$ be ranks and κ a weight for the ranks N' with $\kappa(\tau,h) = 0$ for $N_\tau < h < N'_\tau$; in the ranks N it is read through its

values at $h < N_\tau$. Put $d_\tau = \kappa(\tau, N_\tau)$ if $N_\tau < N'_\tau$ and $d_\tau = 0$ otherwise, and let $\Delta_\tau(y)$ be the minor of y on all N_τ rows. For $z_\tau \in \mathrm{M}_{N'_\tau \times N_\tau}(R)$, $S_\kappa(z) \in \mathrm{M}_{I'_\kappa \times I_\kappa}(R)$ is defined by

$$b'_i(zy) = \sum_j S_\kappa(z)_{ij}\, b_j(y) \prod_\tau \Delta_\tau(y)^{d_\tau},$$

with $(b'_i)_{i \in I'_\kappa}$ the standard basis for the ranks N'. For integers $e = (e_\tau)$ and $h = (h_\tau) \in \prod_\tau \mathrm{U}(N_\tau)$ (or a product of unitary groups of signed diagonal forms),

$$\rho_{\kappa,e}(h) = \prod_\tau \det(\bar{h}_\tau)^{e_\tau}\, S_\kappa(\bar{h})$$

acts on $\mathbb{V}_\kappa$ by matrix multiplication, $\bar{h}$ the entrywise complex conjugate.

(WM3) Tensors and the Gram matrix. Let $\mathcal{C}_\kappa$ be the set of column fillings in which every $T(\tau, h, j)$ is listed as an increasing h-tuple of rows. The **filling matrix** $F_\kappa \in \mathrm{M}_{\mathcal{C}_\kappa \times I_\kappa}(\mathbb{Z})$ is defined by $\Delta_T = \sum_i (F_\kappa)_{Ti}\, b_i$; it embeds $\mathbb{V}_\kappa$ into the tensor product $\bigotimes_{\tau,h,j} \Lambda^h \mathbb{C}^{N_\tau}$ of exterior powers, written in its standard basis indexed by $\mathcal{C}_\kappa$. The **Gram matrix** is $G_\kappa = F_\kappa^{\mathsf{T}} F_\kappa \in \mathrm{M}_{I_\kappa}(\mathbb{Z})$, the matrix of the form for which the basis tensors are orthonormal, and

$$\langle \xi, \xi' \rangle_\kappa = \bar{\xi}^{\mathsf{T}} G_\kappa\, \xi', \qquad \|\xi\|_\kappa^2 = \langle \xi, \xi \rangle_\kappa \qquad (\xi, \xi' \in \mathbb{V}_\kappa).$$

(WM4) Polynomials and evaluations. The **polynomial** of $\xi \in \mathbb{V}_\kappa$ is $\mathrm{Pol}_\kappa(\xi) = \sum_i (G_\kappa \xi)_i\, b_i$; the **evaluation vector** at an array y over R is $\mathrm{ev}_\kappa(y) = (b_i(y))_{i \in I_\kappa} \in \mathbb{V}_\kappa(R)$.

(WM5) Hook number. The **Fock pairing** of polynomials is $\langle p, q \rangle_{\mathrm{F}} = p(\partial) q\,(0)$, $p(\partial)$ the operator obtained by substituting $\partial/\partial y_{\tau,a,c}$ for $y_{\tau,a,c}$. The **highest polynomial** is $f_\kappa = \prod_\tau \prod_h \Delta_{\tau,\{1,\dots,h\}}^{\kappa(\tau,h)}$, and the **hook number** is the positive integer $h_\kappa = \langle f_\kappa, f_\kappa \rangle_{\mathrm{F}}$.

(WM6) Doubled Harish-Chandra parameters. Let n be a nonempty finite index set, $N = \#n$, and take the ranks $N_\tau = N$ ($\tau \in \Phi$), the rows of the arrays of (WM1) being indexed by n. For a weight κ and $e\colon \Phi \to \mathbb{Z}$, a family $l = (l_\tau)_{\tau \in \Phi}$ of maps $l_\tau\colon n \to \mathbb{Z}$ is a **doubled Harish-Chandra parameter** of (κ, e) if every l_τ is injective and every value $l_\tau(a)$ equals $2(e_\tau + \lambda_\tau(p)) + N - 1 - 2p$ for some $0 \le p < N$. Its **highest vector** is $v_l^+ = \mathrm{ev}_\kappa(y^l)$, where $y^l_{\tau,a,c} = 1$ if $c = \#\{a' \in n : l_\tau(a') > l_\tau(a)\}$ and $y^l_{\tau,a,c} = 0$ otherwise.

(WM7) Algebraic forms. Let Φ be the set of infinite places of $E = E_s$, and let n, N be as in (WM6); $G_f(n)$, $G_\infty(s)$ for signs s on n, the standard definite space I_n and its rational points are those of (VP1), (VP5), (VP2) of Definition 3.2.2, a rational point γ of I_n having the components $\gamma_\infty \in G_\infty(1)$ and $\gamma_f \in G_f(n)$. An **algebraic form** of weight (κ, e) on n is a function $\eta\colon G_f(n) \to \mathbb{V}_\kappa$, right invariant under an open subgroup of $G_f(n)$, with $\eta(\gamma_f g) = \rho_{\kappa,e}(\gamma_\infty)\, \eta(g)$ for every rational point γ of I_n and every $g \in G_f(n)$; they form the space $\mathcal{A}_n(\kappa, e)$. For a doubled Harish-Chandra parameter l of (κ, e) and signs s on n, an algebraic form η defines the function

$$F_\eta^l(x, g) = \left\langle v_l^+, \rho_{\kappa,e}(x^{-1})\, \eta(g) \right\rangle_\kappa \qquad (x \in G_\infty(s),\ g \in G_f(n)).$$

For a function F on $G_\infty(1) \times G_f(n)$, continuous in the first variable, its **algebraic form** for l is the function

$$F_l^\flat(g) = \#I_\kappa \int_{G_\infty(1)} F(x, g)\, \rho_{\kappa,e}(x)\, v_l^+\, dx \qquad (g \in G_f(n)),$$

dx the Haar probability measure of the compact group $G_\infty(1)$.

(WM8) Gaps and the block rule. Let n, n' be finite index sets, $N = \#n$, $N' = \#n'$, with signs $s\colon n \to \{\pm 1\}$ and $t\colon n' \to \{\pm 1\}$, and let $h, A \in \mathbb{N}$. A map $l\colon n \to \mathbb{Z}$ has **gap** h **for** s if l is injective, $l(a) \equiv N - 1 \pmod 2$ for every $a \in n$, and $|l(a) - l(a')| \geq 2h$ whenever $s(a) \neq s(a')$; it has **large gap** A **for** s if it has gap $500N^3 + A$ for s. For $d \in \mathbb{N}$ the **doubled block** is $[d] = \{d - 1 - 2b :\ 0 \leq b < d\} \subset \mathbb{Z}$. For maps $l\colon n \to \mathbb{Z}$, $l'\colon n' \to \mathbb{Z}$ and integers γ, γ', the **shifted entries** are

$$\tilde{l}(a) = l(a) - 2\gamma' + N' \quad (a \in n), \qquad \tilde{l}'(b) = l'(b) - 2\gamma + N \quad (b \in n'),$$

and the pair (l, l') satisfies the **block rule** for the weights (γ, γ') and the signs (s, t) if

(a) l has gap $500(N + N')^3$ for s, and l' has gap $500(N + N')^3$ for t;

(b) if $N \leq N'$, then $\tilde{l}'(n') = \tilde{l}(n) \cup [N' - N]$, and if $N' \leq N$, then $\tilde{l}(n) = \tilde{l}'(n') \cup [N - N']$, as sets of integers;

(c) if $a \in n$ and $b \in n'$ satisfy $\tilde{l}(a) = \tilde{l}'(b)$, then $s(a) = t(b)$ when $\tilde{l}(a) > 0$, and $s(a) = -t(b)$ when $\tilde{l}(a) < 0$.

For families $l = (l_\tau)_{\tau \in \Phi}$ of maps $n \to \mathbb{Z}$ and $l' = (l'_\tau)_{\tau \in \Phi}$ of maps $n' \to \mathbb{Z}$ and maps $\gamma, \gamma'\colon \Phi \to \mathbb{Z}$, the parameters l and l' **correspond** for the weights (γ, γ') and the signs (s, t) if for every $\tau \in \Phi$ the pair (l_τ, l'_τ) satisfies the block rule for the weights $(\gamma(\tau), \gamma'(\tau))$ and the signs (s, t).

(WM9) The gauge. Let Φ, E, n, N be as in (WM7) and (κ, e) a weight with $e\colon \Phi \to \mathbb{Z}$. For a prime identification $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ and $\tau \in \Phi$, the embedding $x \mapsto \iota(\tau(\bar{x}))$ of E into $\overline{\mathbb{Q}}_\ell$ induces a place w of E above ℓ; let $f_{\iota,\tau}\colon \mathbb{A}_{E,f} \to \overline{\mathbb{Q}}_\ell$ be the projection to E_w followed by the continuous extension of this embedding, applied entrywise to matrices, and $f_\iota(g) = (f_{\iota,\tau}(g))_{\tau \in \Phi}$. For $g \in G_f(n)$ and $\xi \in \mathbb{V}_\kappa$, the **gauge** of ξ at g is

$$\mathrm{gau}_\iota(g, \xi) = \max_{i \in I_\kappa} \Big| \Big(\prod_{\tau \in \Phi} \det f_{\iota,\tau}(g^{-1})^{e_\tau}\ S_\kappa\big(f_\iota(g^{-1})\big)\, \iota(\xi) \Big)_i \Big|_\ell,$$

with the Schur matrix of (WM2) over $\overline{\mathbb{Q}}_\ell$, $\iota(\xi)$ taken coordinatewise and $|\cdot|_\ell$ the absolute value of $\overline{\mathbb{Q}}_\ell$ with $|\ell|_\ell = \ell^{-1}$.

Lemma 3.2.20. *Let $n \geq 1$, and let κ^+, κ^- be weights of rank n over Φ; put $\kappa(\tau, i) = \kappa^+(\tau, i) + \kappa^-(\tau, n - i)$ and $e^-(\tau) = \sum_{1 \leq i < n} \kappa^-(\tau, i)$. There is an integer matrix $C \in \mathrm{M}_{(I_{\kappa^+} \times I_{\kappa^-}) \times I_\kappa}(\mathbb{Z})$ such that*

(a) C has an integral left inverse;

(b) $C^{\mathsf{T}}(G_{\kappa^+} \otimes G_{\kappa^-})C = G_\kappa$;

(c) over every field R of characteristic 0, $C\,\mathrm{ev}_\kappa(y_{\mathrm{id}}) = \mathrm{ev}_{\kappa^+}(y_{\mathrm{id}}) \otimes \mathrm{ev}_{\kappa^-}(y_{w_0})$, where y_{id} (resp. y_{w_0}) is the array with entry 1 at row a, column $a - 1$ (resp. column $n - a$) at every τ and 0 elsewhere;

(d) over every field R of characteristic 0 and for all $g = (g_\tau) \in \prod_\tau \mathrm{GL}_n(R)$,

$$C \prod_\tau \det(g_\tau)^{-e^-(\tau)} S_\kappa(g) = \big(S_{\kappa^+}(g) \otimes S_{\kappa^-}((g^{-1})^{\mathsf{T}})\big)\, C.$$

Proof. If $n = 1$ all weights are empty ($1 \le h < n$ is void), every I_λ is a point, all Gram matrices, Schur matrices and evaluation vectors are 1, $e^- = 0$, and $C = (1)$ works. Let $n \ge 2$. Put $\kappa^c(\tau, h) = \kappa^-(\tau, n-h)$ for $1 \le h < n$; then $\kappa = \kappa^+ + \kappa^c$, and $\sum_{1 \le h < n} \kappa^c(\tau, h) = e^-(\tau)$. The matrix C will be $(1 \otimes J)\, C'$, where C' splits κ into κ^+ and κ^c (Step 2) and J identifies the weight κ^c with the weight κ^- composed with $g \mapsto (g^{-1})^{\mathsf{T}}$ (Step 3).

Step 1: general facts. Let λ be a weight of rank n and R a field of characteristic 0.

(i) The b_i^λ are linearly independent over R: they are polynomials with integer coefficients, linearly independent over $\mathbb{Z}$, hence over $\mathbb{Q}$, and a family of vectors with rational coordinates that is independent over $\mathbb{Q}$ stays independent over every field containing $\mathbb{Q}$. So $S_\lambda(z)$ is uniquely determined by $b_i(zy) = \sum_j S_\lambda(z)_{ij} b_j(y)$, and $S_\lambda(zz') = S_\lambda(z) S_\lambda(z')$, since $b_i(zz'y) = \sum_j S_\lambda(z)_{ij} b_j(z'y) = \sum_{j,k} S_\lambda(z)_{ij} S_\lambda(z')_{jk} b_k(y)$. Evaluating the defining identity at an array y over R gives $S_\lambda(z)\, \mathrm{ev}_\lambda(y) = \mathrm{ev}_\lambda(zy)$.

(ii) *Spanning.* The vectors $\mathrm{ev}_\lambda(g\, y_{\mathrm{id}}) = S_\lambda(g)\, \mathrm{ev}_\lambda(y_{\mathrm{id}})$, $g \in \prod_\tau \mathrm{GL}_n(R)$, span $\mathbb{V}_\lambda(R)$. Indeed $(g\, y_{\mathrm{id}})_{\tau,a,c} = (g_\tau)_{a,c+1}$ for $0 \le c < n$ and 0 for $c \ge n$. Let (c_i) be a linear form vanishing on these vectors, and $p = \sum_i c_i b_i$. Every minor in p has size $h \le n-1$ and uses only the columns $c \le n-2$, so $p(y) = q(g)$ under the substitution $y_{\tau,a,c} = (g_\tau)_{a,c+1}$ ($c \le n-2$), a bijection between the variables of p and some of the entries of g; q is a polynomial vanishing on $\prod_\tau \mathrm{GL}_n(R)$. Then $q \cdot \prod_\tau \det g_\tau$ vanishes on all of $\prod_\tau \mathrm{M}_n(R)$, hence is the zero polynomial (R is infinite), so $q = 0$, $p = 0$, and $c = 0$ by (i).

(iii) *Integral left inverse of F_λ.* For a filling $T \in \mathcal{C}_\lambda$ the column T of $F_\lambda\, \mathrm{ev}_\lambda(y)$ is $\sum_i (F_\lambda)_{Ti} b_i(y) = \Delta_T(y)$. The Δ_T span ∇_λ over $\mathbb{Z}$ (WM1), so the map $\mathbb{Z}^{\mathcal{C}_\lambda} \to \mathbb{Z}^{I_\lambda}$ with matrix F_λ^{T} is surjective; $\mathbb{Z}^{I_\lambda}$ being free, it has a section, i.e. an integer matrix Q_λ^{T} with $F_\lambda^{\mathsf{T}} Q_\lambda^{\mathsf{T}} = 1$, so $Q_\lambda F_\lambda = 1$. In particular F_λ is injective over every ring, and $F_\lambda Q_\lambda$ is the identity on the image of F_λ.

Step 2: the multiplication matrix C'. For $T^+ \in \mathcal{C}_{\kappa^+}$ and $T^c \in \mathcal{C}_{\kappa^c}$ let $T^+ \sqcup T^c \in \mathcal{C}_\kappa$ have, at τ and height h, the columns $T^+(\tau, h, j)$ for $j \le \kappa^+(\tau, h)$ followed by the columns $T^c(\tau, h, j)$; this is a bijection $\mathcal{C}_{\kappa^+} \times \mathcal{C}_{\kappa^c} \to \mathcal{C}_\kappa$, and $\Delta_{T^+} \Delta_{T^c} = \Delta_{T^+ \sqcup T^c}$. Hence $\nabla_{\kappa^+} \nabla_{\kappa^c} \subseteq \nabla_\kappa$, and there is a unique integer matrix $M \in \mathrm{M}_{I_\kappa \times (I_{\kappa^+} \times I_{\kappa^c})}(\mathbb{Z})$ with

$$b_j^{\kappa^+} b_k^{\kappa^c} = \sum_i M_{i,(j,k)}\, b_i^\kappa .$$

Put $C' = M^{\mathsf{T}}$. Its properties:

(a′) The products $b_j^{\kappa^+} b_k^{\kappa^c}$ span ∇_κ over $\mathbb{Z}$, since every Δ_T with $T \in \mathcal{C}_\kappa$ is a product $\Delta_{T^+} \Delta_{T^c}$ and each factor is an integer combination of its basis. So M is surjective over $\mathbb{Z}$ and, as in Step 1 (iii), $C' = M^{\mathsf{T}}$ has an integral left inverse L'.

(b′) Expanding $\Delta_{T^+} \Delta_{T^c}$ once through (F_{κ^+}), (F_{κ^c}) and M and once through F_κ and comparing coefficients in the basis b^κ gives $(F_{\kappa^+} \otimes F_{\kappa^c})\, C' = \Pi F_\kappa$, where Π is the permutation matrix of the bijection $(T^+, T^c) \mapsto T^+ \sqcup T^c$. As $\Pi^{\mathsf{T}} \Pi = 1$ and $(A \otimes B)^{\mathsf{T}} (A \otimes B) = A^{\mathsf{T}} A \otimes B^{\mathsf{T}} B$,

$$C'^{\mathsf{T}} (G_{\kappa^+} \otimes G_{\kappa^c}) C' = F_\kappa^{\mathsf{T}} \Pi^{\mathsf{T}} \Pi F_\kappa = G_\kappa .$$

(c′) Evaluating the defining identity of M at an array y over R: $C'\, \mathrm{ev}_\kappa(y) = \mathrm{ev}_{\kappa^+}(y) \otimes \mathrm{ev}_{\kappa^c}(y)$.

(d′) Over a field R of characteristic 0 and for $z \in \prod_\tau \mathrm{M}_n(R)$: substituting zy in the defining identity of M and expanding both sides in the basis $b^\kappa(y)$ by the definition of the Schur matrices gives, by Step 1 (i), $C' S_\kappa(z) = (S_{\kappa^+}(z) \otimes S_{\kappa^c}(z))\, C'$.

Step 3: the duality matrix J. *Jacobi's identity.* For $A \in \mathrm{GL}_n(R)$ over a commutative ring and subsets $I, K \subseteq \{1,\dots,n\}$ of equal size, $\det((A^{-1})_{K,I}) = \pm \det(A_{I^c,K^c})/\det A$, with a sign depending only on I, K (complementary minors of the inverse, [133], 0.8.4). Let $g \in \prod_\tau \mathrm{GL}_n(R)$, $y = g\, y_{\mathrm{id}}$ and $z = (g^{-1})^{\mathsf{T}} y_{w_0}$, so that $y_{\tau,a,c} = (g_\tau)_{a,c+1}$ and $z_{\tau,a,c} = ((g_\tau^{-1})^{\mathsf{T}})_{a,n-c}$ for $0 \le c < n$. For a set I of h' rows ($1 \le h' < n$), $\Delta_{\tau,I}(z)$ is the minor of $(g_\tau^{-1})^{\mathsf{T}}$ on the rows I and the columns $n, n-1, \dots, n-h'+1$, and $\Delta_{\tau,I^c}(y)$ is the minor of g_τ on the rows I^c and the columns $1,\dots,n-h'$. Jacobi's identity with $K = \{n-h'+1,\dots,n\}$ (and the reversal of the order of h' columns) gives a sign $u(\tau, I) \in \{\pm 1\}$, depending only on n, τ, I, with

$$u(\tau, I)\, \Delta_{\tau,I^c}(y) = \det(g_\tau)\, \Delta_{\tau,I}(z) \qquad \text{over every commutative ring.}$$

The star. For $T \in \mathcal{C}_{\kappa^c}$ let $T^* \in \mathcal{C}_{\kappa^-}$ have the columns $T^*(\tau, h', j) = T(\tau, n-h', j)^c$ ($1 \le h' < n$, $j \le \kappa^-(\tau, h') = \kappa^c(\tau, n-h')$); $T \mapsto T^*$ is a bijection (its inverse takes complements again). Put $\mathrm{sg}(T) = \prod_{\tau,h,j} u(\tau, T(\tau,h,j)^c)$ and let $P \in \mathrm{M}_{\mathcal{C}_{\kappa^-} \times \mathcal{C}_{\kappa^c}}(\mathbb{Z})$ have $P_{T^*,T} = \mathrm{sg}(T)$ and all other entries 0; it is a signed permutation matrix, $P^{\mathsf{T}}P = 1$. Taking the product of the displayed identity over the columns of T (with $I = T(\tau,h,j)^c$, so $I^c = T(\tau,h,j)$) gives $\mathrm{sg}(T)\Delta_T(y) = \prod_\tau \det(g_\tau)^{e^-(\tau)} \Delta_{T^*}(z)$, that is, by Step 1 (iii),

$$P\, F_{\kappa^c}\, \mathrm{ev}_{\kappa^c}(g\, y_{\mathrm{id}}) = \prod_\tau \det(g_\tau)^{e^-(\tau)}\ F_{\kappa^-}\, \mathrm{ev}_{\kappa^-}\big((g^{-1})^{\mathsf{T}} y_{w_0}\big). \tag{$*$}$$

The matrix J. Put $J = Q_{\kappa^-} P F_{\kappa^c} \in \mathrm{M}_{I_{\kappa^-} \times I_{\kappa^c}}(\mathbb{Z})$. Over $R = \mathbb{C}$, $(*)$ shows that $PF_{\kappa^c}v$ lies in the image of F_{κ^-} for v in the spanning set of Step 1 (ii), so $F_{\kappa^-}Jv = F_{\kappa^-}Q_{\kappa^-}PF_{\kappa^c}v = PF_{\kappa^c}v$ there, hence for all v:

$$F_{\kappa^-} J = P F_{\kappa^c} \qquad \text{(integer matrices).}$$

Consequences: (1) $K = Q_{\kappa^c} P^{\mathsf{T}} F_{\kappa^-}$ is an integral left inverse of J, since $KJ = Q_{\kappa^c} P^{\mathsf{T}} P F_{\kappa^c} = 1$. (2) $J^{\mathsf{T}} G_{\kappa^-} J = (F_{\kappa^-}J)^{\mathsf{T}}(F_{\kappa^-}J) = F_{\kappa^c}^{\mathsf{T}} P^{\mathsf{T}} P F_{\kappa^c} = G_{\kappa^c}$. (3) Over a field R of characteristic 0, $(*)$, the injectivity of F_{κ^-} and Step 1 (i) give

$$J\, S_{\kappa^c}(g)\, \mathrm{ev}_{\kappa^c}(y_{\mathrm{id}}) = \prod_\tau \det(g_\tau)^{e^-(\tau)} S_{\kappa^-}((g^{-1})^{\mathsf{T}})\, \mathrm{ev}_{\kappa^-}(y_{w_0});$$

at $g = 1$, $J\, \mathrm{ev}_{\kappa^c}(y_{\mathrm{id}}) = \mathrm{ev}_{\kappa^-}(y_{w_0})$. (4) For $g, h \in \prod_\tau \mathrm{GL}_n(R)$, by (3) and the multiplicativity of Step 1 (i),

$$\begin{aligned} S_{\kappa^-}((g^{-1})^{\mathsf{T}})\, J\, S_{\kappa^c}(h) \mathrm{ev}_{\kappa^c}(y_{\mathrm{id}}) &= \prod_\tau \det(h_\tau)^{e^-(\tau)} S_{\kappa^-}(((gh)^{-1})^{\mathsf{T}}) \mathrm{ev}_{\kappa^-}(y_{w_0}) \\ &= \prod_\tau \det(g_\tau)^{-e^-(\tau)}\, J\, S_{\kappa^c}(g)\, S_{\kappa^c}(h) \mathrm{ev}_{\kappa^c}(y_{\mathrm{id}}), \end{aligned}$$

so by Step 1 (ii) $S_{\kappa^-}((g^{-1})^{\mathsf{T}})\, J = \prod_\tau \det(g_\tau)^{-e^-(\tau)}\, J\, S_{\kappa^c}(g)$.

Step 4: the matrix $C = (1 \otimes J)\, C'$. It is an integer matrix indexed by $(I_{\kappa^+} \times I_{\kappa^-}) \times I_\kappa$.

(a) $L'(1 \otimes K)$ is an integral left inverse: $L'(1 \otimes K)(1 \otimes J)C' = L'C' = 1$ by Step 2 (a′) and Step 3 (1).

(b) $C^{\mathsf{T}}(G_{\kappa^+} \otimes G_{\kappa^-})C = C'^{\mathsf{T}}(G_{\kappa^+} \otimes J^{\mathsf{T}} G_{\kappa^-} J)C' = C'^{\mathsf{T}}(G_{\kappa^+} \otimes G_{\kappa^c})C' = G_\kappa$ by Step 3 (2) and Step 2 (b′).

(c) $C\, \mathrm{ev}_\kappa(y_{\mathrm{id}}) = (1 \otimes J)(\mathrm{ev}_{\kappa^+}(y_{\mathrm{id}}) \otimes \mathrm{ev}_{\kappa^c}(y_{\mathrm{id}})) = \mathrm{ev}_{\kappa^+}(y_{\mathrm{id}}) \otimes \mathrm{ev}_{\kappa^-}(y_{w_0})$ by Step 2 (c′) and Step 3 (3).

(d) By Step 2 (d′) and Step 3 (4),

$$C\prod_\tau \det(g_\tau)^{-e^-(\tau)} S_\kappa(g) = \Big(S_{\kappa^+}(g)\otimes \prod_\tau \det(g_\tau)^{-e^-(\tau)} J S_{\kappa^c}(g)\Big) C' \\ = (S_{\kappa^+}(g)\otimes S_{\kappa^-}((g^{-1})^{\mathsf{T}}))C.$$

□

Definition 3.2.21. Let V, V_0 be spaces with points with signs s on n, s_0 on n_0, and p an index set.

(MX1) Splittings. A **splitting** of V over V_0 along p is a triple (k,l,i) of matrices over E, k,l of size $n\times p$ and i of size $n\times n_0$, with $k^*J_Vk=0$, $l^*J_Vl=0$, $l^*J_Vk=1$, $i^*J_Vi=J_{V_0}$, $i^*J_Vk=0$, $i^*J_Vl=0$ and $\#n=\#n_0+2\#p$: $V=(lE^p\oplus kE^p)\perp iV_0$. Its **adelic frames** are $k_{\mathbb{A}}=x_V^{-1}k$, $l_{\mathbb{A}}=x_V^{-1}l$ and $i_{\mathbb{A}}=x_V^{-1}ix_{V_0}$, and its **coordinates** are $\mathrm{pr}_k=l_{\mathbb{A}}^*\varepsilon_s$, $\mathrm{pr}_l=k_{\mathbb{A}}^*\varepsilon_s$ and $\mathrm{pr}_i=\varepsilon_{s_0}i_{\mathbb{A}}^*\varepsilon_s$, which send $l_{\mathbb{A}}a+i_{\mathbb{A}}w+k_{\mathbb{A}}b$ to b, a and w. The **opposite splitting** is (l,k,i). The splitting is **base-adapted** if the adelic matrix with column blocks $l_{\mathbb{A}}$, $k_{\mathbb{A}}$, $i_{\mathbb{A}}$ has orthonormal columns at every infinite place and entries in $\widehat{\mathcal{O}}_E$ at the finite places. The **tautological splitting** of $\mathbb{H}_p\oplus V_0$ over V_0 has l, k, i equal to the column blocks of the identity matrix for the first copy of p, the second copy of p and n_0. For a space with a point T with signs t on n', the splitting of V induces the splitting of $\mathrm{Hom}(V,T)$ over $\mathrm{Hom}(V_0,T)$ along $n'\times p$ with frames $J_T^{-1}\otimes(k^*J_V)^{\mathsf{T}}$, $1\otimes(l^*J_V)^{\mathsf{T}}$ and $1\otimes(J_{V_0}^{-1}i^*J_V)^{\mathsf{T}}$.

(MX2) Levi and radical elements. For $m\in \mathrm{GL}_p(\mathbb{A}_E)$ and $g_0\in \mathrm{G}(s_0)$, $\mathrm{lev}(m,g_0)=k_{\mathbb{A}}m\mathrm{pr}_k+i_{\mathbb{A}}g_0\mathrm{pr}_i+l_{\mathbb{A}}(m^{-1})^*\mathrm{pr}_l\in\mathrm{G}(s)$ (it sends k to km, l to $l(m^{-1})^*$ and acts as g_0 on iV_0). For an adelic $p\times p$ matrix M the **radical centre element** is $z(M)=1+k_{\mathbb{A}}(M-M^*)\mathrm{pr}_l\in\mathrm{G}(s)$. For an adelic $n_0\times p$ matrix c and an adelic $p\times p$ matrix b the **radical element** is the element of $\mathrm{G}(s)$

$$\mathrm{rad}(c,b)=1-i_{\mathbb{A}}c\,\mathrm{pr}_l+k_{\mathbb{A}}c^*\varepsilon_{s_0}\mathrm{pr}_i-k_{\mathbb{A}}\Big(\tfrac{1}{2}\big(\jmath^{-1}(b+b^*)+c^*\varepsilon_{s_0}c\big)\Big)\mathrm{pr}_l.$$

Let A be a splitting of a space with a point W' over W (signs s on n) along a one-element set, so that rad_A and lev_A (the elements above for A) take $n\times 1$ and 1×1 matrices, the latter read as adeles. For an adelic $n\times 1$ matrix y, an adele b, $u\in\mathbb{A}_E^\times$ and $g\in\mathrm{G}(s)$ the **line radical** and the **line Levi element** are $\mathrm{rad}_A^{\mathrm{ln}}(y,b)=\mathrm{rad}_A(x_W^{-1}y,2b)$ and $\mathrm{lev}_A^{\mathrm{ln}}(u,g)=\mathrm{lev}_A(u^{-1},g)$.

(MX3) Centre coefficients. For a function D on $\mathrm{G}(s)$, $T\in\mathrm{M}_p(E)$ and $g\in\mathrm{G}(s)$, the **centre coefficient** is

$$\mathrm{coef}(T;D)(g)=\int_{\mathrm{M}_p(\mathbb{A}_E)/\mathrm{M}_p(E)}\psi_E\big(\mathrm{tr}(TM)\big)\,D\big(z(M)g\big)\,dM,$$

whenever $M\mapsto D(z(M)g)$ is integrable on the quotient (it is invariant under $\mathrm{M}_p(E)$ when D is left invariant under Γ_V).

(MX5) Composites, shifts and adapted points. Let $D=(k,l,i)$ be a splitting of V over V_0 along p and $D'=(k',l',i')$ a splitting of V_0 over a space with a point V_1 along q. The **composite** $D\circ D'$ is the splitting of V over V_1 along $p\sqcup q$ with frames $(k\mid ik')$, $(l\mid il')$ (column blocks indexed by p and q) and embedding ii'. Let A be a splitting of V over V_0 along a one-element set o, $\bar{A}$ its opposite splitting, B a splitting of V_0 over V_1 along p, and for a splitting C write z_C for the radical centre element (MX2) of C. For an adelic $1\times p$

matrix a let M_a be the $(o \sqcup p) \times (o \sqcup p)$ matrix with the block a in the position (o,p) and 0 elsewhere. The **shifts** are

$$z^{+}_{A,B}(a) = z_{\bar{A}\circ B}(M_a), \qquad z^{-}_{A,B}(a) = z_{A\circ B}(M_a) \qquad \text{in } \mathrm{G}(s).$$

For $g \in \mathrm{G}(s)$ and $m \in \mathrm{GL}_p(\mathbb{A}_E)$, D is **adapted at g in the basis m** if the adelic matrix $g^{-1}(l_{\mathbb{A}}(m^{-1})^* \mid k_{\mathbb{A}} m \mid i_{\mathbb{A}})$ has orthonormal columns at every infinite place and entries in $\widehat{\mathcal{O}}_E$ at the finite places; base-adapted means adapted at 1 in the basis 1.

(MX7) Real zero terms and real slices. Let D be a splitting of V over V_0 along p and $m \in \mathrm{GL}_p(\mathbb{A}_E)$ (in the applications D is adapted at 1 in the basis m, (MX5); the formulas below do not use it). Let $\mathcal{F}_s$ be the Fock model $\mathcal{F}_\varepsilon$ of (WR5) for $\iota = \Phi \times n$, $\varepsilon(w', i) = s_i$, and $\mathcal{F}_{s_0}$ likewise; v_1 is the real vacuum of $\mathcal{F}_{s_0}$ and rv is as in (WR6). For an adelic matrix M let M_∞ be the complex matrix, block diagonal over Φ, whose block at v is M_v. For $y \in \mathbb{C}^{\Phi \times n}$ put

$$a(y) = (m^* \mathrm{pr}_l)_\infty\, y, \qquad w(y) = (\mathrm{pr}_i)_\infty\, y, \qquad \chi(y) = \psi_E\Big(\tfrac{\jmath}{2}\,(\mathrm{pr}_k \mathrm{rv}(y))^*\, \mathrm{pr}_l \mathrm{rv}(y)\Big).$$

The **real zero term** of $v \in \mathcal{F}_s$ is the Bochner integral in $\mathcal{F}_{s_0}$

$$Z^{\infty}_{D,m}(v) = \int_{\mathbb{C}^{\Phi\times n}} v(y)\,\chi(y)\,e^{-a(y)^* a(y)}\,\rho^{\infty}(w(y))v_1\,d\nu_{\Phi\times n}(y),$$

and the **real slice** of v at $b \in \mathbb{C}^{\Phi\times p}$ is

$$\mathrm{rsl}_{D,m}(b; v) = Z^{\infty}_{D,m}\Big(\rho^{\infty}\big((l_{\mathbb{A}}(m^{-1})^*)_\infty\, b\big)v\Big) \in \mathcal{F}_{s_0}.$$

Along a space of homomorphisms. For a space with a point U with signs on an index set r, let D_U be the splitting of $\mathrm{Hom}(V,U)$ over $\mathrm{Hom}(V_0,U)$ along $r \times p$ induced by D (MX1), and let the **adapted basis** be $m_U = (x_U^{-1})^* \otimes 1_p \in \mathrm{GL}_{r\times p}(\mathbb{A}_E)$, the Kronecker product with the identity of size p. The **real zero term** and the **real slices** of D along U are $Z^{\infty}_{D,U} = Z^{\infty}_{D_U,m_U}$ and $\mathrm{rsl}_{D,U}(b;v) = \mathrm{rsl}_{D_U,m_U}(b;v)$, for v in the Fock model of $\mathrm{Hom}(V,U)$ and $b \in \mathbb{C}^{\Phi\times(r\times p)}$ (the Fock models of (WR5) for the signs of $\mathrm{Hom}(V,U)$ and $\mathrm{Hom}(V_0,U)$).

(MX9) Representability and the Siegel bracket. An adelic $p \times p$ matrix T' is **represented by the signs** $t\colon p \to \{\pm 1\}$ if for every $v \in \Phi$ there is $c \in \mathrm{M}_p(\mathbb{C})$ with $c^* \mathrm{diag}(t)\, c = T'_v$. Let $\delta_\jmath \in \mathbb{A}_E^\times$ be the idele with components $\jmath_v \kappa_v^2$ at $v \in \Phi$ and finite part $\jmath\alpha\bar{\alpha}$ (κ_v of (WD6), α of (WD2) of Definition 3.2.12). For $T \in \mathrm{M}_p(E)$ and an adelic $p\times p$ matrix γ the **Siegel bracket** is $T\{\gamma\} = \delta_\jmath^{-1}\gamma^* T\gamma$. For a set S of finite places of E, with $\bar{S}$ the set of complex conjugate places, an adele, or an adelic matrix, is **integral on** $S \cup \bar{S}$ if its components at the places of $S \cup \bar{S}$ are integral, and an adelic square matrix X is **unimodular on** $S \cup \bar{S}$ if X is invertible and X and X^{-1} are integral on $S \cup \bar{S}$.

(MX11) Partial means, shears and frame stabilisers. For a set R of finite places of E, $\mathrm{e}_R \in \mathbb{A}_E$ is the adele with component 1 at the places of R and 0 at all other places, finite and infinite; an adele e acts on an adelic matrix X entrywise, giving eX. Let A be a splitting of V over V_0 along a one-element set, with adelic frames $k_{A,\mathbb{A}}$, $l_{A,\mathbb{A}}$, and $\bar{A}$ its opposite splitting; let B be a splitting of V_0 over a space with a point V_1 along p, and D a splitting of V over a space with a point V_2 along p, with adelic frame $k_{D,\mathbb{A}}$ (MX1). *Frame stabilisers.* For a set S of finite places of E, the **frame stabiliser of D off S** is the stabiliser of the adelic $n \times p$ matrix $(1 - \mathrm{e}_{S\cup\bar{S}})k_{D,\mathbb{A}}$ in $\mathrm{G}(s)$, acting by left multiplication:

$$\mathrm{Stab}_{\mathrm{G}(s)}((1 - \mathrm{e}_{S\cup\bar{S}})k_{D,\mathbb{A}}) = \{g \in \mathrm{G}(s) : g\,(1 - \mathrm{e}_{S\cup\bar{S}})k_{D,\mathbb{A}} = (1 - \mathrm{e}_{S\cup\bar{S}})k_{D,\mathbb{A}}\}.$$

Partial means. Let f be a function on $\mathrm{G}(s)$, $T \in \mathrm{M}_p(E)$, and $\mathrm{coef}(T;f)$ the centre coefficient (MX3) of f along D, that is, formed with the radical centre elements $z_D(M)$. For $x \in \mathrm{G}(s)$ the **partial means** of f at x are

$$\mathrm{coef}^{+}_{A,B,D}(T;f)(x) = \int \mathrm{coef}(T;f)(z^{-}_{A,B}(b)\,x)\,db,$$
$$\mathrm{coef}^{-}_{A,B,D}(T;f)(x) = \int \mathrm{coef}(T;f)(z^{+}_{A,B}(b)\,x)\,db,$$

the integrals over $\mathrm{M}_{1\times p}(\mathbb{A}_E)/\mathrm{M}_{1\times p}(E)$, with the shifts $z^{\pm}_{A,B}$ of (MX5); each is defined at x when its integrand is defined for every $b \in \mathrm{M}_{1\times p}(\mathbb{A}_E)$, invariant under $\mathrm{M}_{1\times p}(E)$ and integrable on the quotient. The **shifted means** of f at x are the functions

$$a \mapsto \mathrm{coef}^{+}_{A,B,D}(T;f)(z^{+}_{A,B}(a)\,x), \qquad b \mapsto \mathrm{coef}^{-}_{A,B,D}(T;f)(z^{-}_{A,B}(b)\,x)$$

of $a, b \in \mathrm{M}_{1\times p}(\mathbb{A}_E)$, where defined.

Shears. For a set U of finite places of E, an **A-shear for U** is an element $\nu \in \mathrm{G}(s)$ of the form

$$\nu = 1 + \mathrm{e}_{\bar{U}}\, k_{A,\mathbb{A}}\, c - \mathrm{e}_U\, a \qquad \text{with } c \in \mathrm{M}_{1\times n}(\mathbb{A}_E),\ c\,k_{A,\mathbb{A}} = 0,\ a \in \mathrm{M}_n(\mathbb{A}_E).$$

A function F on $\mathrm{G}(s)$ is **invariant under the A-shears for U at** $x \in \mathrm{G}(s)$ if F is defined at x and at νx, and $F(\nu x) = F(x)$, for every A-shear ν for U. The $\bar{A}$-shears are formed in the same way with $k_{\bar{A},\mathbb{A}} = l_{A,\mathbb{A}}$.

(MX12) Straight points and dual indices. Let A, B, D be as in (MX11), o the one-element index set of A, and S a set of finite places of E; for a splitting C write $k_{C,\mathbb{A}}$ for its adelic frame (MX1). A **straight point** of (A,B,D) off S is a tuple (X,u,t,t',γ) with $X,u \in \mathrm{G}(s)$, $t,t' \in \mathbb{A}_E$ and $\gamma \in \mathrm{M}_p(\mathbb{A}_E)$ such that

- X and u have the same components at the places of $S \cup \bar{S}$, and X lies in the frame stabiliser of D off S (MX11);
- with block matrices for the index set $o \sqcup p$ of the composites (MX5),

$$u\,k_{A\circ B,\mathbb{A}} = k_{A\circ B,\mathbb{A}} \begin{pmatrix} t & 0 \\ 0 & \gamma \end{pmatrix}, \qquad u\,k_{\bar{A}\circ B,\mathbb{A}} = k_{\bar{A}\circ B,\mathbb{A}} \begin{pmatrix} t' & 0 \\ 0 & \gamma \end{pmatrix};$$

- $\bar{t}'\,t = 1$.

Its **opposite** is (X,u,t',t,γ), a straight point of $(\bar{A},B,D)$ off S.

The **denominator index** of a finite adele a is the index $[\widehat{\mathcal{O}}_E : \{y \in \widehat{\mathcal{O}}_E : ay \in \widehat{\mathcal{O}}_E\}]$ of additive groups.

For $T \in \mathrm{M}_p(E)$ and $\gamma \in \mathrm{M}_p(\mathbb{A}_E)$ with $T\{\gamma\}$ (MX9) invertible, the **dual index of γ on $S \cup \bar{S}$ for T** is the denominator index of the finite part of $\mathrm{e}_{S\cup\bar{S}}\,\det(T\{\gamma\})^{-1}$.

Definition 3.2.22. Throughout, $E = E_s$ for a parameter pair P and $s \geq 0$ (a CM field containing a primitive 5040-th root of unity), with totally real subfield F, infinite places Φ, and its Weil data of Definition 3.2.12: the **splitting character** υ of (WD3), also written for its restriction to $\mathrm{U}(1)(\mathbb{A})$, and the additive character ψ_E of (WD4). For an automorphic character α of $\mathrm{U}(1)$ let $\tilde{\alpha}(x) = \alpha(x/\bar{x})$ on $\mathbb{A}_E^{\times}$ be its **base change**, and for $p \in \mathbb{Z}$ let $\alpha[p] = \tilde{\alpha}\,\upsilon^{-p}$ be its **space character** in rank p.

(XD1) Spaces. A **space** V is a space with a point with **signs** $s\colon n_V \to \{\pm 1\}$ in the sense of (VP2) of Definition 3.2.2 (for $E = E_s$, as in (WR1) of Definition 3.2.18), whose lattice, frames and rational points are those of (VP2), standard definite spaces and orthogonal sums those of (VP5), and hyperbolic spaces those of (WR1); $\operatorname{rk} V = \#n_V$ and $\operatorname{neg}(V) = \#\{i : s_i = -1\}$; the zero space has rank 0; $\mathbb{H} = \mathbb{H}_p$ with $\#p = 1$ is the hyperbolic plane. $\mathrm{U}(V)$ is its unitary group, $\mathrm{U}(V)(\mathbb{A}_F) = \mathrm{G}(s)$ through the point, and $[\mathrm{U}(V)] = \mathrm{U}(V)(F)\backslash\mathrm{U}(V)(\mathbb{A}_F) = \Gamma_V\backslash\mathrm{G}(s)$ with the quotient measure of (VP2) (the measures of (VP1): Cayley measures at the infinite places, mass 1 on $\mathbf{K}_0(n)$).

(XD2) Functionals, theta kernels and integrals. The exchange data are formed with a family of functionals, which is a parameter of this definition: for every space with a point V with signs s, a bounded linear functional θ_V on the theta model $\mathcal{M}_s$ is given, and nothing else is assumed of it. For spaces S, T with signs s, s' and automorphic characters α, β of $\mathrm{U}(1)$, $\omega_{\alpha,\beta}$ is the Weil representation of the pair on $\mathcal{M}_{S,T}$ ((WR4) of Definition 3.2.18), and

$$\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h) = \theta_{\mathrm{Hom}(S,T)}\big(\omega_{\alpha,\beta}(g,h)\varphi\big) \qquad (\varphi \in \mathcal{M}_{S,T},\ g \in \mathrm{G}(s),\ h \in \mathrm{G}(s'))$$

is its **theta kernel**, $\theta_{\mathrm{Hom}(S,T)}$ being the functional of the family at the space with a point $\mathrm{Hom}(S,T)$ ((VP5) of Definition 3.2.2); (g,h) acts by $\alpha(\det h)\beta(\det g)^{-1}$ times the Weil operator of $X \mapsto hXg^{-1}$. A **datum** is a vector $\varphi \in \mathcal{M}_{S,T}$. For a Fock vector $v \in \mathcal{F}_{S,T}$, a finite set A of finite places containing the conjugate of none of its members and $\Phi \in \mathcal{T}_A$, the datum $v \otimes \Phi \otimes \Phi^\circ$ is $v \otimes (\Phi \otimes_A v^0)$ of (WR6), and $\Phi^\circ = v^0$ is the **vacuum**. The **theta integral** over $[\mathrm{U}(T)]$ is $I^{S,T}_{\alpha,\beta}(\varphi)(g) = \int_{[\mathrm{U}(T)]} \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)\,dh$, a function on $\mathrm{U}(S)(\mathbb{A}_F)$. A datum $\varphi \in \mathcal{M}_{S,T}$ is **target-bounded** if there are $B \in \mathbb{N}$ and $C \geq 0$ with $|\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)| \leq C\,\mathrm{ht}_S(g)^B$ for all automorphic characters α, β of $\mathrm{U}(1)$, all $g \in \mathrm{G}(s)$ and all $h \in \mathrm{U}(T)(\mathbb{A}_F)$, ht_S the height of (VP4) of Definition 3.2.2.

(XD3) Schwartz vectors. A Fock vector $v \in \mathcal{F}_\varepsilon$ for signs ε on an index set ι ((WR5) of Definition 3.2.18; in particular $v \in \mathcal{F}_{S,T}$, or v in the Fock models $\mathcal{F}_s$, $\mathcal{F}_{s_0}$ of (MX7) of Definition 3.2.21) has the **moments** $\mathrm{mom}_N(v) = \int_{\mathbb{C}^\iota} (1 + \|z\|)^N |v(z)|\,d\nu_\iota(z) \in [0,\infty]$ ($N \in \mathbb{N}$, $\|z\| = \max_{i\in\iota} |z_i|$), and is a **Schwartz vector** if $\mathrm{mom}_N(v) < \infty$ for every N. The Fock vectors of polynomials are Schwartz vectors.

(XD4) Moduli and tame characters. A **modulus** is an ideal $\mathfrak{m} = \prod_{w\in\Sigma(\mathfrak{m})} w^{e_w}$ ($e_w \geq 1$) of $\mathcal{O}_E$ whose nonempty set of places $\Sigma(\mathfrak{m})$ contains the conjugate of none of its members; $R_\mathfrak{m} = \mathcal{O}_E/\mathfrak{m}$, and $\mathcal{O}_\mathfrak{m} = \prod_{w\in\Sigma(\mathfrak{m})} \mathcal{O}_{E,w}$. Where a set of places is expected (as in $L_*(\mathfrak{m}, \nu; e)$ below), $\mathfrak{m}$ stands for $\Sigma(\mathfrak{m})$. Let $\mu\colon \Phi \to \mathbb{Z}$. An automorphic character ν of $\mathrm{U}(1)$ is **tame of weight μ for** $(\mathfrak{m}, \chi)$ if it has weight μ and a tame part ψ with $\psi_w \neq 1$ exactly for $w \in \Sigma(\mathfrak{m})$, and $\chi\colon R_\mathfrak{m} \to \mathbb{C}$ is the multiplicative character $u \mapsto \prod_{w\in\Sigma(\mathfrak{m})} \psi_w(u \bmod w)$ on $R_\mathfrak{m}^\times$, extended by 0 to the nonunits. A **split tame character** of weight μ is a pair $\mathfrak{c} = (\nu, \Sigma)$ of an automorphic character ν of $\mathrm{U}(1)$ and a nonempty finite set Σ of finite places of E containing the conjugate of none of its members, such that $\mu(v) > 0$ for every $v \in \Phi$ and ν has weight μ and a tame part ψ with $\psi_w \neq 1$ exactly for $w \in \Sigma$. Put $E_\Sigma = \prod_{w\in\Sigma} E_w$ and $\mathcal{O}_\Sigma = \prod_{w\in\Sigma} \mathcal{O}_{E,w}$, and read $t \in E_\Sigma$ as the finite adele with components t_w at $w \in \Sigma$ and 1 elsewhere. For a character ξ of $\mathbb{A}_E^\times/E^\times$ and $k \in \mathbb{Z}$ the **local character** $\xi_{\Sigma,k}$ of E_Σ is $\xi_{\Sigma,k}(t) = \xi(t)\prod_{w\in\Sigma} |t_w|_w^{k/2}$ for t invertible and 0 otherwise, $|\cdot|_w$ the normalised absolute value; the **semilocal character** of $\mathfrak{c}$ is $\chi_\mathfrak{c} = \tilde{\nu}_{\Sigma,0}$, $t \mapsto \nu(t/\bar{t})$.

(XD5) Data modulo $\mathfrak{m}$. For a function Φ on matrices over $R_\mathfrak{m}$ let $\Phi^\natural(x) = \Phi(x_{\Sigma(\mathfrak{m})} \bmod \mathfrak{m})$ if the components of x at $\Sigma(\mathfrak{m})$ are integral and $\Phi^\natural(x) = 0$ otherwise. If $\Phi^\natural \in \mathcal{T}_{\Sigma(\mathfrak{m})}$ ((WR6)

of Definition 3.2.18), the **datum modulo** $\mathfrak{m}$ of Φ and a Fock vector v is $\mathrm{mod}_{\mathfrak{m}}(\Phi, v) = v \otimes (\Phi^{\natural} \otimes_{\Sigma(\mathfrak{m})} v^0)$.

(XD6) Matched functions. Let W have index set $n = b \sqcup a$ (a fixed bijection σ). Functions Φ on $a \times n$ matrices and Ψ on $n \times b$ matrices over $R_{\mathfrak{m}}$ are **matched** for (σ, χ) if Φ vanishes off the matrices $(k^{-1})_a$ formed by the rows a of k^{-1}, Ψ vanishes off the matrices k_b formed by the columns b of k, as k runs over $\mathrm{GL}_n(R_{\mathfrak{m}})$, and $\Phi((k^{-1})_a) = \chi(\det k)\, \Psi(k_b)$ for every such k. The same words over the ring $\mathcal{O}_{\mathfrak{m}}$, with χ read on $\mathcal{O}_{\mathfrak{m}}^{\times}$ through the residue maps, define matched functions on matrices over $\prod_{w \in \Sigma(\mathfrak{m})} E_w$ supported on integral matrices.

(XD7) The L-value product. For a finite set S of finite places, an automorphic character ν of $\mathrm{U}(1)$ and $e \in \mathbb{N}$,

$$L_*(S, \nu; e) = \Big(\prod_{w \in S} \mathrm{N}w \Big)^{e^2} \prod_{z=1}^{e} |L^S(z, \tilde{\nu})|^2, \qquad L_*(S, \nu; -e) = L_*(S, \nu; e)^{-1},$$

where $L^S(z, \tilde{\nu})$ is the partial L-function of $\tilde{\nu}$ with the Euler factors at the places of $S \cup \bar{S}$ removed (at $z = 1$ its value by analytic continuation). For a split tame character $\mathfrak{c} = (\nu, \Sigma)$ of (XD4) and real $z \geq 1$, the **sharp** L**-value** is

$$L^{\sharp}_{\mathfrak{c}}(z) = \big|L^{\Sigma}(z, \tilde{\nu})\big| \prod_{w \in \Sigma} (\mathrm{N}w)^{z - 1/2}.$$

(XD8) The equal-rank ratio. Let X be a space of even rank, ν tame of weight μ for $(\mathfrak{m}, \chi)$, $\gamma = \nu^{-1} \upsilon^{\mathrm{rk}\, X/2}$, and z a Schwartz vector for (X, X). Let Φ_χ be the function $x \mapsto \chi(\det x)^{-1}$ on invertible integral $X \times X$ matrices over $\mathcal{O}_{\mathfrak{m}}$ and 0 elsewhere. Put $I_X(z) = I^{X,X}_{\gamma,\gamma}(z \otimes \Phi_\chi \otimes \Phi^{\circ})$. The **equal-rank ratio** is

$$r_X(\nu, z) = \frac{|I_X(z)(1)|^2}{L_*(\mathfrak{m}, \nu; \mathrm{rk}\, X/2)}.$$

(XD9) Local data and the two sides. Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, S, T spaces with index sets n_S, n_T, and $v \in \mathcal{F}_{S,T}$. For a function Φ on $n_T \times n_S$ matrices over E_Σ let $\Phi^{\sharp}(y) = \Phi(y_\Sigma)$ and $\Phi^{\flat}(y) = \Phi((\bar{y})_{\bar{\Sigma}})$ for finite adelic $n_T \times n_S$ matrices y, y_Σ the matrix of the components at Σ (read as functions of the vectors of $\mathrm{Hom}(S, T)$). If $\Phi^{\sharp} \in \mathcal{T}_\Sigma$, the **local datum** is $v \otimes [\Phi]_\Sigma = v \otimes (\Phi^{\sharp} \otimes_\Sigma v^0)$; if $\Phi^{\flat} \in \mathcal{T}_{\bar{\Sigma}}$, the **conjugate local datum** is $v \otimes [\Phi]_{\bar{\Sigma}} = v \otimes (\Phi^{\flat} \otimes_{\bar{\Sigma}} v^0)$ ((WR6) of Definition 3.2.18). The **large side** is

$$\theta^{\mathrm{lg}}_{\mathfrak{c}}(S, T; v, \Phi) = I^{S,T}_{\alpha,\alpha}(v \otimes [\Phi]_\Sigma), \qquad \alpha = \nu^{-1} \upsilon^{\lfloor \mathrm{rk}\, S/2 \rfloor},$$

and, for a function Ψ on $n_S \times n_T$ matrices over E_Σ and $\Psi^{\mathsf{T}}(x) = \Psi(x^{\mathsf{T}})$, the **small side** is

$$\theta^{\mathrm{sm}}_{\mathfrak{c}}(S, T; v, \Psi) = I^{S,T}_{\alpha',\beta'}(v \otimes [\Psi^{\mathsf{T}}]_{\bar{\Sigma}}), \qquad \alpha' = \nu\, \upsilon^{\lfloor \mathrm{rk}\, S/2 \rfloor},\ \beta' = \upsilon^{\mathrm{rk}\, T},$$

functions on $\mathrm{U}(S)(\mathbb{A}_F)$ (in the exchange identity the small side is taken for $T = V'$).

(XD10) Standard and plane data. Let $\mathfrak{c}$ be a split tame character with semilocal character $\chi = \chi_{\mathfrak{c}}$, χ^{-1} meaning $u \mapsto \chi(u)^{-1}$ on $\mathcal{O}_\Sigma^{\times}$ and 0 elsewhere. For index sets b, n and $s \in \mathrm{M}_{b \times n}(E_\Sigma)$ the **standard column datum** is the function on $n \times b$ matrices over E_Σ

$$\Psi^{\mathrm{st}}_s(y) = \chi^{-1}(\det(sy)) \ \text{ if } y \in \mathrm{M}_{n \times b}(\mathcal{O}_\Sigma), \qquad \Psi^{\mathrm{st}}_s(y) = 0 \ \text{ otherwise;}$$

for a bijection $\sigma\colon b \sqcup a \to n$ the **standard row datum** is the function on $a \times n$ matrices x over E_Σ with $\Phi^{\mathrm{st}}_{\sigma,s}(x) = \chi^{-1}(\det M)$ if $M \in \mathrm{M}_n(\mathcal{O}_\Sigma)$ and 0 otherwise, M the $n \times n$ matrix whose row $\sigma(j)$ is row j of s ($j \in b$) and whose row $\sigma(i)$ is row i of x ($i \in a$). Let A be a base-adapted splitting of a space W' (index set n') over a space W (index set n) along a one-element set ((MX1) of Definition 3.2.21), $(\mathrm{pr}_i)_\Sigma \in \mathrm{M}_{n\times n'}(E_\Sigma)$ the components at Σ of the finite part of its coordinate pr_i, V' a space with index set r', $\sigma'\colon b \sqcup r' \to n'$ a bijection and $v \in \mathcal{F}_{W',V'}$. For $y \in \mathrm{M}_{b\times n}(E_\Sigma)$ the **plane datum** is the local datum

$$\varphi^{\mathrm{pl}}_{A,\sigma'}(v, y) = v \otimes \left[\Phi^{\mathrm{st}}_{\sigma',\, y(\mathrm{pr}_i)_\Sigma}\right]_\Sigma \in \mathcal{M}_{W',V'},$$

defined when $(\Phi^{\mathrm{st}}_{\sigma',y(\mathrm{pr}_i)_\Sigma})^\sharp \in \mathcal{T}_\Sigma$. With $\mathrm{rad}^{\mathrm{ln}}_A$ and lev_A of (MX2) of Definition 3.2.21, the **plane constant term** at $u \in \mathbb{A}_E^\times$ and $g \in \mathrm{U}(W)(\mathbb{A}_F)$ is

$$\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v, y)(u, g) = \int_{\mathrm{M}_{n\times 1}(\mathbb{A}_E)/\mathrm{M}_{n\times 1}(E)} \int_{\mathbb{A}_E/E} \theta^{\mathrm{lg}}_{\mathfrak{c}}\big(W', V'; v, \Phi^{\mathrm{st}}_{\sigma',y(\mathrm{pr}_i)_\Sigma}\big)\big(\mathrm{rad}^{\mathrm{ln}}_A(q_1, q_2)\,\mathrm{lev}_A(u, g)\big)\, dq_2\, dq_1,$$

the iterated integral, q_2 inner, both quotients with their invariant probability measures.

(XD11) Equivariant lifts. Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, X a space with signs r on m, and b an index set. Let $\mathcal{L}$ be a linear space of functions on $m \times b$ matrices over E_Σ and $\mathcal{D}$ a linear map from the functions on $m \times b$ matrices over E_Σ to the functions on $\mathrm{G}(r)$. The pair $(\mathcal{L}, \mathcal{D})$ is an **equivariant lift** for $(\mathfrak{c}, X, b)$ if:

- (a) every $\Psi \in \mathcal{L}$ is locally constant with compact support and vanishes at every matrix without a left inverse;
- (b) for $\Psi \in \mathcal{L}$, $\mathcal{D}\Psi$ is continuous and left invariant under Γ_X;
- (c) for $\Psi \in \mathcal{L}$ and $g \in G_f(m)$ with component 1 at every finite place w with $w \notin \Sigma$ and $\bar{w} \notin \Sigma$, the function $\Phi(y) = \Psi(g_\Sigma y)$ lies in $\mathcal{L}$ and $\mathcal{D}\Phi(x(1, g)) = \upsilon_{\Sigma,1}(\det g_\Sigma)^{-\#b}\,\mathcal{D}\Psi(x)$ for all $x \in \mathrm{G}(r)$, where $g_\Sigma \in \mathrm{GL}_m(E_\Sigma)$ is the matrix of the components of g at Σ and $\upsilon_{\Sigma,1}$ is the local character (XD4) of the splitting character;
- (d) for $\Psi \in \mathcal{L}$, $x \in \mathrm{G}(r)$ and $k \in \mathbf{K}_0(m)$ with component 1 at every $w \in \Sigma$, $\mathcal{D}\Psi(x(1, k)) = \mathcal{D}\Psi(x)$.

(XD12) The row-column exchange identity. Let $\mu\colon \Phi \to \mathbb{Z}$, let X, V, V' be spaces with index sets m, a, b, let $\sigma\colon b \sqcup a \to m$ be a bijection, $v \in \mathcal{F}_{X,V}$, $y \in \mathcal{F}_{X,V'}$, and let ℓ be a real function on the split tame characters of weight μ. The data (σ, v, y, ℓ) satisfy the **row-column exchange identity** for (X, V, V') and μ if v and y are Schwartz vectors (XD3) and there is $C \in \mathbb{R}$ such that for every split tame character $\mathfrak{c} = (\nu, \Sigma)$ of weight μ there is $c \in \mathbb{C}$ with

- (a) $C\,|c|^2 = L_*(\Sigma, \nu; \#a - \lfloor \#m/2 \rfloor)\,\ell(\mathfrak{c})$ (XD7);
- (b) $\theta^{\mathrm{lg}}_{\mathfrak{c}}(X, V; v, \Phi^{\mathrm{st}}_{\sigma,z}) = c\,\theta^{\mathrm{sm}}_{\mathfrak{c}}(X, V'; y, \Psi^{\mathrm{st}}_z)$ as functions on $\mathrm{U}(X)(\mathbb{A}_F)$, for every $z \in \mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$, with the two sides of (XD9) and the standard row and column data of (XD10) for the semilocal character $\chi_{\mathfrak{c}}$.

Definition 3.2.23. For $n \geq 1$, I_n, $G_f(n)$ and $\mathbf{K}_0(n)$ are as in (VP5), (VP1) of Definition 3.2.2, and for a finite place $w \neq \bar{w}$, $\mathrm{red}_w\colon \mathbf{K}_0(n) \to \mathrm{GL}_n(k_w)$ reduces the component at w. For a subgroup $K \subseteq G_f(n)$ and a function f on $G_f(n)$, the **class sum** is $\sum_{[h]} \frac{1}{\#\Gamma_h} f(h)$ over the double cosets $\mathrm{U}(\mathrm{I}_n)(E)\,h\,K$, with $\Gamma_h = \mathrm{U}(\mathrm{I}_n)(E) \cap hKh^{-1}$, whenever the summand is constant on double cosets.

(DP1) Level systems. A **level system** Σ consists of two finite families of finite places of E, the **chosen places** M_Σ and the **twist places** Q_Σ, all distinct and none equal to the conjugate of itself or of another.

(DP2) Descent pairs. A **descent pair** O over E, for ranks $r < n$, consists of a level system Σ_O; signs ε_O on $\{1,\dots,r\}$ and a space $W(O)$ of rank r with these signs; an injective map $\mathrm{inc}_O\colon \{1,\dots,r\} \to \{1,\dots,n\}$; a map $\mathrm{ord}_O\colon \{1,\dots,r\} \to \mathbb{N}$; natural numbers $\kappa_O^+(\tau,i)$, $\kappa_O^-(\tau,i)$ for $\tau \in \Phi$, $1 \le i < n$; and an automorphic character μ_O of $\mathrm{U}(1)$ of some weight wt_O and tame part $(\psi_{O,w})_w$ with $\psi_{O,w} = 1$ for $w \notin M_O \cup Q_O$; each $\psi_{O,w}$, a character of $k_w^\times$, is extended to k_w by $\psi_{O,w}(0) = 0$ (also when it is trivial). Put $r_\pm(O) = \#\{c : \varepsilon_O(c) = \pm 1\}$. Over any ring, Inc_O is the $n \times r$ matrix with entry 1 at $(\mathrm{inc}_O(c), c)$ and 0 elsewhere; the **inc-rows** of an $n \times r$ matrix form the $r \times r$ matrix of its rows in the image of inc_O, and the **inc-block** of an $n \times n$ matrix is its $r \times r$ submatrix on that image.

(DP3) Weight. $\kappa_O(\tau,i) = \kappa_O^+(\tau,i) + \kappa_O^-(\tau,n-i)$ and $e_O(\tau) = -\sum_{1\le i<n} \kappa_O^-(\tau,i)$; the forms take values in $\mathbb{V}_O = \mathbb{V}_{\kappa_O}$ (rank n at every τ), on which a rational point γ of $\mathrm{U}(\mathrm{I}_n)$ acts by $\rho_O(\gamma) = \rho_{\kappa_O, e_O}((\gamma_\tau)_\tau)$.

(DP4) Level. $\mathbf{L}(O)$ is the group of families $(j_w) \in \prod_{w \in M_O \cup Q_O} \mathrm{GL}_n(k_w)$ with $j_w \mathrm{Inc}_O = \mathrm{Inc}_O$ for $w \in M_O$ and, for $w \in Q_O$, vanishing entries of j_w in the rows of the image of inc_O and the columns outside it; its character is $\eta_O(j) = \prod_{w \in Q_O} \psi_{O,w}(\det(\text{inc-block of } j_w))^{-1}$. The **level** is $\mathbf{K}(O) = \{g \in \mathbf{K}_0(n) : (\mathrm{red}_w g)_w \in \mathbf{L}(O)\}$, with the character $\tilde\eta_O(g) = \eta_O((\mathrm{red}_w g)_w)$.

(DP5) Forms. $\mathrm{Form}(O)$ is the space of $\Xi\colon G_f(n) \to \mathbb{V}_O$ with $\Xi(\gamma h g) = \tilde\eta_O(g)\,\rho_O(\gamma)\,\Xi(h)$ for rational points γ of $\mathrm{U}(\mathrm{I}_n)$, $h \in G_f(n)$, $g \in \mathbf{K}(O)$, with the inner product $\langle \Xi, \Xi' \rangle$ given by the class sum for $\mathbf{K}(O)$ of $h \mapsto \langle \Xi(h), \Xi'(h) \rangle_{\kappa_O}$. The group $\mathrm{GL}_r^{\mathrm{ch}}(O) = \prod_{w \in M_O} \mathrm{GL}_r(k_w)$ acts on $\mathrm{Form}(O)$: (g_w) acts by $\prod_{w \in M_O} \psi_{O,w}(\det g_w)$ times right translation by any $k \in \mathbf{K}_0(n)$ with $\mathrm{red}_w k = 1$ for $w \in Q_O$ and, for $w \in M_O$, $\mathrm{red}_w k$ acting as g_w on the span of the basis vectors in the image of inc_O and as the identity on the others.

(DP6) Characters and theta model. $\chi_O = \upsilon^{r_-(O)}$ and $\chi'_O = \mu_O \upsilon^{\lfloor n/2 \rfloor}$. Then $\omega_O = \omega_{\chi_O, \chi'_O}$ is the Weil representation of the pair $(S,T) = (W(O), \mathrm{I}_n)$ of (WR4) of Definition 3.2.18, (u,h) acting by $\chi_O(\det h)\chi'_O(\det u)^{-1}$ times the Weil representation at $X \mapsto hXu^{-1}$, on $\mathcal{M}_O = \mathcal{M}_{W(O),\mathrm{I}_n}$, and $\theta_O = \theta_{\mathrm{Hom}(W(O),\mathrm{I}_n)}$ is the functional of the exchange data at the space with a point $\mathrm{Hom}(W(O), \mathrm{I}_n)$ ((XD2) of Definition 3.2.22).

(DP7) The block datum. For families $X = (X_w)_{w \in M_O \cup Q_O}$ of $n \times r$ matrices over k_w,

$$\Psi_O(X) = \prod_{w \in Q_O} \psi_{O,w}(\det(\text{inc-rows of } X_w)) \ \text{ if } X_w = \mathrm{Inc}_O \text{ for all } w \in M_O,$$
$$\Psi_O(X) = 0 \text{ otherwise,}$$

with $\psi_{O,w}(0) = 0$ as in (DP2), so that $\Psi_O(X) = 0$ when the inc-rows of some X_w, $w \in Q_O$, are singular. Let $S_O = M_O \cup Q_O$ and $\Phi_O(x) = \Psi_O((x_w \bmod w)_w)$ if the components of x at S_O are integral, $\Phi_O(x) = 0$ otherwise, a function of the components at S_O of finite adelic $n \times r$ matrices, read as vectors of $\mathbb{A}_{E,f}^{nr}$; it lies in $\mathcal{T}_{S_O}$. The **block datum** is the Schrödinger datum $\Upsilon(O) = \Phi_O \otimes_{S_O} v^0 \in \mathcal{M}_O$ of (WR6) of Definition 3.2.18.

(DP8) The theta datum. Let $X_{\tau,i,c}$ ($\tau \in \Phi$, $1 \le i \le n$, $1 \le c \le r$) be the Fock variables of $\mathrm{Hom}(W(O), \mathrm{I}_n)$. For a sign σ, the **letter substitution** Let_σ sends the variable $y_{\tau,i,a}$ of (WM1) to $\sum_c X_{\tau,i,c}$ over c with $\varepsilon_O(c) = \sigma$, $\mathrm{ord}_O(c) = a$. Let C_O be a fixed integer

matrix as in Lemma 3.2.20 for (κ_O^+, κ_O^-), and $\mathrm{Har}_O(\xi)$ the image of $C_O\xi$ under the linear map $\alpha \otimes \beta \mapsto \mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}\alpha)\,\mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}\beta)$. The **theta datum** of $\xi \in \mathbb{V}_O$ is the datum $\mathrm{J}_O(\xi) = v_{\mathrm{Har}_O(\xi)} \otimes \Upsilon(O) \in \mathcal{M}_O$ of (WR6), v_P the Fock vector of the polynomial P in the Fock model of $\mathrm{Hom}(W(O), \mathrm{I}_n)$ of (WR5).

(DP9) The normalised descent. For $u \in \mathrm{U}(W(O))(\mathbb{A})$ and $h \in G_f(n)$ let $\mathcal{C}_O(u,h)$ be the linear form $\xi \mapsto \overline{\theta_O(\omega_O(u,(1,h))\,\mathrm{J}_O(\bar{\xi}))}$ on $\mathbb{V}_O$. The **normalised descent** of $\Xi \in \mathrm{Form}(O)$ is the function on $\mathrm{U}(W(O))(\mathbb{A})$

$$\mathrm{Th}_O(\Xi)(u) = \sum_{[h]} \frac{1}{\#\Gamma_h}\,\mathcal{C}_O(u,h)(\Xi(h)),$$

the class sum for $\mathbf{K}(O)$.

(DP10) Hecke polynomials. For a finite place w with $w \neq \bar{w}$ and $w, \bar{w} \notin S_O$, $q = \mathrm{N}w$, and $0 \le i \le n$, $T_{w,i}$ is the operator of the double coset of $\mathrm{diag}(\varpi_w 1_i, 1_{n-i})$ in

$$\mathrm{GL}_n(\mathcal{O}_{E,w})\backslash\mathrm{GL}_n(E_w)/\mathrm{GL}_n(\mathcal{O}_{E,w}),$$

acting by right translation through the embedding of $\mathrm{GL}_n(E_w)$ in $G_f(n)$ at w. The **Hecke polynomial operator** is

$$H_w(X) = \sum_{i=0}^{n} (-X)^i q^{i(i-1)/2}\,T_{w,i} \qquad (X \in \mathbb{C})$$

on $\mathrm{Form}(O)$, and for $t = (t_i)_{0\le i\le r}$ in $\mathbb{C}$, $H_t^{(r)}(X) = \sum_{i=0}^{r}(-X)^i q^{i(i-1)/2} t_i$. For an automorphic character a of $\mathrm{U}(1)$ and $p \ge 0$ put $a_w\{p\} = a[p](\varpi_w)\,q^{p/2}$, ϖ_w the idele with a uniformiser at w and 1 elsewhere. The **lift Hecke root** at w is $X_*(w) = \big(\chi_{O,w}\{r\}\,q^{\,n-r-1}\big)^{-1}$. A form $y \in \mathrm{Form}(O)$ **has the Hecke polynomials of a lift** of a family $z = (z(w))_w$, $z(w) \in \mathbb{C}^{r+1}$, if for every such w, y is fixed by the elements of $\mathbf{K}_0(n)$ with components 1 away from $w, \bar{w}$, and

$$H_w(X)\,y = \prod_{j=0}^{n-r-1}(1 - q^j\chi_{O,w}\{r\}X)\cdot H_{z(w)}^{(r)}\Big(\frac{q^{n-r}\chi_{O,w}\{r\}}{\chi'_{O,w}\{n\}}X\Big)\,y \qquad (X \in \mathbb{C}).$$

(DP11) The parameter and fitting. With the rows of the arrays of (WM1) for $\mathbb{V}_O$ numbered $1,\dots,n$, the **parameter** of O is the doubled Harish-Chandra parameter $l_O = (l_{O,\tau})_{\tau\in\Phi}$ of (κ_O, e_O) in the sense of (WM6) given by

$$l_{O,\tau}(a) = 2\big(e_O(\tau) + \lambda_{O,\tau}(a-1)\big) + n - 1 - 2(a-1) \qquad (\tau \in \Phi,\ 1 \le a \le n),$$

$\lambda_{O,\tau}$ the highest weight of κ_O at τ. A family $l = (l_\tau)_{\tau\in\Phi}$ of maps $l_\tau\colon \{1,\dots,r\} \to \mathbb{Z}$ **fits** O (and O **fits the parameter** l) if

(i) l and l_O correspond, in the sense of (WM8), for the weights $\tau \mapsto r_-(O)$ and $\tau \mapsto \lfloor n/2\rfloor + \mathrm{wt}_O(\tau)$ on Φ and the signs $(\varepsilon_O, \mathbf{1}_n)$, where $\mathbf{1}_n$ is the constant sign $+1$ on $\{1,\dots,n\}$;

(ii) for every $\tau \in \Phi$ and $1 \le c \le r$, the basis vectors of each sign are numbered from 0 by decreasing $\varepsilon_O\, l_\tau$:

$$\mathrm{ord}_O(c) = \#\{c' \in \{1,\dots,r\} :\ \varepsilon_O(c') = \varepsilon_O(c),\ \varepsilon_O(c)\,l_\tau(c) < \varepsilon_O(c)\,l_\tau(c')\};$$

(iii) for every $\tau \in \Phi$, $\kappa_O^+(\tau, i) = 0$ for $r_+(O) < i < n$ and $\kappa_O^-(\tau, i) = 0$ for $r_-(O) < i < n$.

Lemma 3.2.24. *Let $e\colon \Phi \to \mathbb{Z}$, let S be a finite set of finite places of E, and let $\psi = (\psi_w)_w$ be characters $\psi_w\colon k_w^\times \to \mathbb{C}^\times$ with $\psi_w = 1$ for $w \notin S$. The elements of $\mathrm{U}(1)(E)$ that are integral at every finite place are exactly the roots of unity of E, and the following are equivalent:*

1. *there is an automorphic character of $\mathrm{U}(1)$ of weight e and tame part ψ;*
2. *for every root of unity $\zeta \in E$,*

$$\prod_{v\in\Phi} \zeta_v^{e(v)} \prod_{w\in S} \psi_w(\zeta \bmod w) = 1.$$

In particular (2), and hence (1), holds if $\prod_{v\in\Phi} \zeta_v^{e(v)} = 1$ and $\psi_w(\zeta \bmod w) = 1$ for every root of unity $\zeta \in E$ and every $w \in S$.

Proof. **Step 1: integral norm-one elements.** A root of unity ζ is integral and $\zeta\bar{\zeta} = |\zeta|^2 = 1$ in every complex embedding, so $\zeta\bar{\zeta} = 1$. Conversely, let $\gamma \in E$ be integral with $\gamma\bar{\gamma} = 1$. For every embedding $\sigma\colon E \to \mathbb{C}$, $\sigma(\bar{\gamma}) = \overline{\sigma(\gamma)}$ (E is CM), so $|\sigma(\gamma)| = 1$; an algebraic integer all of whose conjugates have absolute value 1 is a root of unity (Kronecker).

Step 2: (1) implies (2). If μ has weight e and tame part ψ and ζ is a root of unity, then $\zeta \in \mathrm{U}(1)(E) \cap H$, so $1 = \mu(\zeta)$ equals the formula at ζ; the factors with $w \notin S$ are 1.

Step 3: (2) implies (1). $H = \mathrm{U}(1)(E \otimes \mathbb{R}) \times \prod_w \mathrm{U}(1)(\mathcal{O}_{E,w})$ is an open subgroup. Define $\theta\colon H \to \mathbb{C}^\times$ by $\theta(u) = \prod_{v\in\Phi} u_v^{e(v)} \prod_{w\in S} \psi_w(u \bmod w)$; it is a continuous homomorphism with $|\theta| = 1$ (each $|u_v| = 1$; the second product depends only on the components at the finitely many $w \in S$ and is locally constant there). By Step 1, $\mathrm{U}(1)(E) \cap H$ consists of the roots of unity, on which $\theta = 1$ by (2). Hence $\tilde{\theta}(\gamma h) = \theta(h)$ ($\gamma \in \mathrm{U}(1)(E)$, $h \in H$) is a well-defined homomorphism on the subgroup $H_1 = \mathrm{U}(1)(E)H$, trivial on $\mathrm{U}(1)(E)$. The group $\mathrm{U}(1)(\mathbb{A})$ is abelian and $\mathbb{C}^\times$ is divisible, so $\tilde{\theta}$ extends to a homomorphism $\chi\colon \mathrm{U}(1)(\mathbb{A}) \to \mathbb{C}^\times$. The quotient $\mathrm{U}(1)(\mathbb{A})/H_1$ is finite: $\mathrm{U}(1)(E)\backslash\mathrm{U}(1)(\mathbb{A})$ is compact, the torus $\mathrm{U}(1)$ being anisotropic over F, and H_1 is open. The homomorphism $|\chi|$ to the positive reals is trivial on H_1 (as $|\theta| = 1$), so it factors through a finite group; a positive real of finite order is 1, so $|\chi| = 1$. On the open subgroup H, $\chi = \theta$ is continuous, so χ is continuous. Thus χ is an automorphic character of $\mathrm{U}(1)$, and for $u \in H$, $\chi(u) = \theta(u) = \prod_v u_v^{e(v)} \prod_w \psi_w(u \bmod w)$ since $\psi_w = 1$ off S: χ has weight e and tame part ψ.

Step 4: the last assertion. Under its hypotheses every factor in (2) is 1. □

Lemma 3.2.25. *Let D be admissible data.*

(a) For every chosen place w, q_2^2 divides $\#k_w - 1$.

(b) Let $c \in \mathbb{Z}$ and let $\psi = (\psi_w)_w$ be characters $\psi_w\colon k_w^\times \to \mathbb{C}^\times$ such that $\psi_w = 1$ for every finite place w that is not chosen, and $\psi_w(x) = 1$ for every chosen w and every $x \in k_w^\times$ with $x^{q_1} = 1$. Then there is an automorphic character of $\mathrm{U}(1)$ of weight $-c\,\kappa_0$ and tame part ψ.

Proof. **(a).** A chosen place w contains π_M, hence $\pi_M\bar{\pi}_M = p_0p_1$, hence p_0 or p_1; so $\#k_w = p_i^f$ for some $i \in \{0, 1\}$ and $f \geq 1$. By (AD5), $p_i \equiv 1 \pmod{q_2^2}$, so $p_i^f \equiv 1 \pmod{q_2^2}$.

Step 1: $d \nmid q_1$. The primes dividing $q_1 = q_0 \cdot 3\, l_0(s)\, 2^s$ are those dividing q_0, since $6 \mid q_0$. If $d \mid q_0$, then $q_0 \mid 2(d+1)$ gives $d \mid 2(d+1)$, so $d \mid 2$, contradicting $d \equiv 7 \pmod 8$.

Step 2: every root of unity of E is a power of ζ. Since $\mathrm{Gal}(E/K) \cong (\mathbb{Z}/q_1)^\times$ and $[K:\mathbb{Q}] = 2$, $[E:\mathbb{Q}] = 2\varphi(q_1)$. Let $y \in E$ be a root of unity of order N and $L = \mathrm{lcm}(N, q_1)$. Then $\mathbb{Q}(\zeta_L) = \mathbb{Q}(y,\zeta) \subseteq E$, so $\varphi(q_1) \mid \varphi(L) \mid 2\varphi(q_1)$. Suppose $L \neq q_1$ and let p be a prime dividing L/q_1. If $p \mid q_1$ then $p\,\varphi(q_1) \mid \varphi(L)$, so $p = 2$; if $p \nmid q_1$ then $(p-1)\varphi(q_1) \mid \varphi(L)$, so $p \leq 3$, which is impossible as $6 \mid q_1$. Hence L/q_1 is a power of 2, $\varphi(L) = 2\varphi(q_1)$ and $E = \mathbb{Q}(\zeta_L)$. The prime d ramifies in K (its discriminant is $-d$, as $-d \equiv 1 \pmod 4$), hence in $\mathbb{Q}(\zeta_L)$, so $d \mid L$; as the primes dividing L divide q_1, $d \mid q_1$, contrary to Step 1. Therefore $L = q_1$, $y^{q_1} = 1$, and y is a power of ζ, the q_1-th roots of unity of E being the q_1 powers of ζ.

Step 3: the infinite factor. By (AD2), $q_1 x_1 \mid b$; so $q_1 \mid b$ and $q_1 \mid a/x_1 = x \cdot (b/x_1)$, and by (AD4) $q_1 \mid \kappa(v)$ for every $v \in \Phi$. Hence $\kappa_0(v) = \kappa(v) + \Delta/2 \equiv \Delta/2 \pmod{q_1}$. For a root of unity $y = \zeta^e$ of E (Step 2), $y^{q_1} = 1$, so $\prod_{v\in\Phi} y_v^{-c\kappa_0(v)} = \left(\prod_{v\in\Phi} y_v\right)^{-c\Delta/2}$. The embedding chosen at the place of $g \in \mathrm{Gal}(E/K)$ is $\iota_0 \circ g$, so $\prod_{v\in\Phi} y_v = \iota_0(\mathrm{N}_{E/K}(y))$, and

$$\mathrm{N}_{E/K}(\zeta) = \prod_{u\in(\mathbb{Z}/q_1)^\times} \zeta^u = \zeta^{\sum_u u} = 1,$$

because $u \mapsto -u$ is an involution of $(\mathbb{Z}/q_1)^\times$ without fixed points ($q_1 > 2$), so $\sum_u u \equiv 0 \pmod{q_1}$. Thus the infinite factor is 1.

Step 4: the finite factor and conclusion. Let S be the set of chosen places, finite since its members contain $\pi_M \neq 0$. For $w \in S$ and a root of unity y of E, $x = y \bmod w$ satisfies $x^{q_1} = 1$ by Step 2, so $\psi_w(x) = 1$. Hence for every root of unity y of E,

$$\prod_{v\in\Phi} y_v^{-c\kappa_0(v)} \prod_{w\in S} \psi_w(y \bmod w) = 1,$$

and $\psi_w = 1$ for $w \notin S$. By Lemma 3.2.24 (applied with $e = -c\kappa_0$ and this S) there is an automorphic character of U(1) of weight $-c\kappa_0$ and tame part ψ. □

Definition 3.2.26. Lift characters. At every chosen place w fix a character η_w of $k_w^\times$ of order q_2^2 (it exists by (a) of Lemma 3.2.25, $k_w^\times$ being cyclic), and let $\omega = (\omega_w)_w$ with $\omega_w = \eta_w^{q_2}$ at the chosen places and $\omega_w = 1$ at the other finite places. The data D carry three automorphic characters of U(1):

(LF1) $\mu^\flat$, a fixed automorphic character of weight $-\kappa_0$ and trivial tame part;

(LF2) υ, the splitting character of the Weil data restricted to $\mathrm{U}(1)(\mathbb{A})$, of weight 1 and trivial tame part;

(LF3) χ_M, a fixed automorphic character of weight 0 and tame part ω.

$\mu^\flat$ and χ_M exist by (b) of Lemma 3.2.25, with $c = 1$, $\psi = 1$ and with $c = 0$, $\psi = \omega$. For $j \geq 0$ the **untwisted lift character** and, for a tame character ϑ, the **lift character** are

$$\mu_j^0 = \mu^\flat\,(\upsilon^{-\Delta}\chi_M)^j, \qquad \mu_j^0\vartheta;$$

μ_j^0 has weight $-\kappa_j$ and tame part ω^j.

Single lifts.

(LF4) The descent pair of lift j. For $1 \leq j \leq m$ and a tame character ϑ with modulus Q, $O_{j,\vartheta}$ is the descent pair for ranks $r < r + t$ with: the level system with the chosen places of D and the places of Q; $W(O_{j,\vartheta}) = W_j$ with its signs; inc the inclusion of the r basis indices of W_j as the first r of $r + t$; $\mathrm{ord}(c) = \mathrm{num}(c)$ if the sign of c in W_j is $+1$ and

$r-1-\mathrm{num}(c)$ if it is -1, $\mathrm{num}(c)\in\{0,\dots,r-1\}$ being the number of the basis vector c in the numbering of (HP2) of Definition 3.2.13; for $\tau\in\Phi$ and $1\le h<r+t$,

$$\kappa^+(\tau,h)=\big(\varsigma_\tau(h-1)-\max(\varsigma_\tau(h),\tfrac{t-1}{2}-\kappa_j(\tau))-1\big)^+,$$
$$\kappa^-(\tau,h)=\big(\min(\varsigma_\tau(r-h-1),-\kappa_j(\tau)-\tfrac{t-1}{2})-\varsigma_\tau(r-h)-1\big)^+,$$

where $y^+=\max(y,0)$ and $\varsigma_\tau(i)$ is the entry i of the target parameter for $0\le i\le r-1$, extended by $\varsigma_\tau(i)=\frac{\Delta}{2}-\kappa_{i+2-k}(\tau)$ for $i\ge r$ (and the arguments $h-1$, $r-h-1$, $r-h$ read as 0 when negative); and the character $\mu_j^0\vartheta$.

(LF5) The single lift. Let O be any descent pair, for ranks $r_O<n$ ((DP2) of Definition 3.2.23; for $O=O_{j,\vartheta}$, $n=r+t$), the rows of the arrays of (WM1) of Definition 3.2.19 numbered $1,\dots,n$. With l_O the parameter of O ((DP11) of Definition 3.2.23), a doubled Harish-Chandra parameter of (κ_O,e_O), the **highest vector** of O is $\xi_O=v_{l_O}^+$, which is $\mathrm{ev}_{\kappa_O}(y_{\mathrm{id}})$ for the array y_{id} with entry 1 at row a and column $a-1$ and 0 elsewhere. For a function F on $\mathrm{U}(\mathrm{I}_n)(\mathbb{A})=G_\infty(1)\times G_f(n)$, continuous in the first variable, the **algebraic form attached to** F is its algebraic form $F^\flat=F^\flat_{l_O}\colon G_f(n)\to\mathbb{V}_O$ for l_O, of weight (κ_O,e_O), as in (WM7) of Definition 3.2.19. For $O=O_{j,\vartheta}$ and $y\in\mathcal{A}_M(W_j)$, $\Theta_{j,\vartheta}(y)\in\mathrm{Form}(O)$ is the algebraic form attached to the function

$$h\longmapsto\int_{[\mathrm{U}(W_j)]}\theta_O(\omega_O(u,h)\,\mathrm{J}_O(\xi_O))\,y(u)\,du\qquad(h\in\mathrm{U}(\mathrm{I}_n)(\mathbb{A})).$$

The **single lift** $\Theta_{j,\vartheta}\colon\mathcal{A}^{\mathrm{good}}(W_j)\to\mathcal{A}_{j,\vartheta}(V^{\mathrm{def}})$ is its restriction to the good forms, with $V^{\mathrm{def}}=\mathrm{I}_{r+t}$ and $\mathcal{A}_{j,\vartheta}(V^{\mathrm{def}})=\mathrm{Form}(O_{j,\vartheta})$ with its inner product. Its image $\mathcal{L}_{j,\vartheta}=\Theta_{j,\vartheta}(\mathcal{A}^{\mathrm{good}}(W_j))$ is the **space of lifts**.

Definition 3.2.27. Let n be a nonempty finite index set and $s\colon n\to\{\pm1\}$ signs; matrices are indexed by n, $n_\pm=\{i\in n:s(i)=\pm1\}$, and a **weight** is a map $w\colon n\to\mathbb{Z}$.

(HV1) Orders, dominance and the holomorphic gap. An **order** is an injective map $o\colon n\to\mathbb{Z}$; it is **positive-first** for s if $o(i)<o(j)$ whenever $s(i)=+1$ and $s(j)=-1$. A weight w is K**-dominant** for (s,o) if $w(j)\le w(i)$ whenever $s(i)=s(j)$ and $o(i)<o(j)$. For $\gamma\in\mathbb{N}$, a weight w has **holomorphic gap** γ for s if $w(i)+\gamma\le w(j)$ whenever $s(i)=+1$ and $s(j)=-1$.

(HV2) Holomorphic model functions. For a weight w let $w^\sharp(i)=2\sum_{j\in n}|w(j)|$ for $i\in n_+$ and $w^\sharp(i)=0$ for $i\in n_-$. For $g\in\mathrm{U}(\mathrm{diag}\,s)$ put

$$m^{\mathrm{hol}}_{s,w}(g)=\frac{m_{w+w^\sharp}(g)}{m_{w^\sharp}(g)},$$

and for a family $u=(u_v)_{v\in\Phi}$ of weights the **holomorphic model function** of weight u is

$$m^{\mathrm{hol}}_{s,u}(x)=\prod_{v\in\Phi}m^{\mathrm{hol}}_{s,u_v}(x_v)\qquad(x=(x_v)_{v\in\Phi}\in G_\infty(s)).$$

(HV3) Functions holomorphic of a weight. Let o be an order. A complex matrix b is o**-triangular** if it is invertible and $b_{ij}=0$ whenever $o(j)<o(i)$; for a weight w put $b^w=\prod_{i\in n}b_{ii}^{w(i)}$ (the diagonal entries of an o-triangular matrix are nonzero). Let $u=(u_v)_{v\in\Phi}$ be a family of weights. A function $F\colon G_\infty(s)\to\mathbb{C}$ is **holomorphic of weight** u **for** o

if there is a function $\tilde{F}\colon \mathrm{M}_n(\mathbb{C})^\Phi \to \mathbb{C}$ such that for every $x \in G_\infty(s)$ and every family $b = (b_v)_{v\in\Phi}$ of o-triangular matrices, $\tilde{F}$ is complex analytic at $xb = (x_v b_v)_{v\in\Phi}$ and

$$\tilde{F}(xb) = \Big(\prod_{v\in\Phi} b_v^{u_v}\Big) F(x).$$

A function F on $\mathrm{G}(s)$ is **adelically holomorphic of weight** u **for** o if for every $g \in \mathrm{G}(s)$ the function $x \mapsto F\big(g\,(x,1)\big)$ on $G_\infty(s)$ is holomorphic of weight u for o.

(HV4) Peaks. Let X be a set and $F\colon X \to \mathbb{C}$ a function with $\sup_{y\in X} |F(y)| < \infty$. A **peak** of F is a point $x \in X$ with $|F(x)| > \frac{1}{2}\sup_{y\in X} |F(y)|$.

(HV5) Oscillator operators. Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma, \gamma' \in \mathbb{Z}$, and let $\mathbb{C}[X] = \mathbb{C}[X_{ac} : a \in I',\, c \in I]$, the polynomials in the entries of an $I' \times I$ matrix; the variable X_{ac} has the **sign** $t(a)s(c)$. If X_{ac} has sign $+1$, the **raising operator** ra_{ac} is the multiplication by X_{ac} and the **lowering operator** lo_{ac} is $\partial/\partial X_{ac}$; if X_{ac} has sign -1, $\mathrm{ra}_{ac} = -\partial/\partial X_{ac}$ and lo_{ac} is the multiplication by X_{ac}. The **row units** and the **column units** of (s,t) with the **shifts** (γ,γ') are the operators on $\mathbb{C}[X]$

$$E^{\mathrm{row}}_{aa'} = \sum_{c\in I} \mathrm{ra}_{ac}\,\mathrm{lo}_{a'c} + \gamma\,\delta_{aa'} \quad (a, a' \in I'),$$
$$E^{\mathrm{col}}_{cc'} = -\sum_{a\in I'} \mathrm{lo}_{ac}\,\mathrm{ra}_{ac'} + (\gamma' + \#I')\,\delta_{cc'} \quad (c, c' \in I),$$

δ being Kronecker's symbol.

(HV6) K-highest vectors and harmonic polynomials. Let J be a finite set with signs ε, $o\colon J \to \mathbb{Z}$ a map, $(E_{jj'})_{j,j'\in J}$ linear operators on a complex vector space M, and $\nu\colon J \to \mathbb{Z}$. A vector $x \in M$ is **K-highest of weight ν for o** if $E_{jj}x = \nu(j)\,x$ for every $j \in J$ and $E_{jj'}x = 0$ whenever $\varepsilon(j) = \varepsilon(j')$ and $o(j') < o(j)$. In the setting of (HV5), a polynomial $H \in \mathbb{C}[X]$ is **termwise harmonic** if

$$\frac{\partial^2 H}{\partial X_{ac}\,\partial X_{a'c}} = 0 \ \text{ whenever } t(a) \neq t(a'), \qquad \frac{\partial^2 H}{\partial X_{ac}\,\partial X_{ac'}} = 0 \ \text{ whenever } s(c) \neq s(c')$$

$(a, a' \in I',\ c, c' \in I)$: no monomial of H contains two variables of one column from rows of opposite signs, or two variables of one row from columns of opposite signs. With $\partial_{ac} = \partial/\partial X_{ac}$, the **row Laplacians** and the **column Laplacians** of (s,t) are the operators on $\mathbb{C}[X]$

$$\Delta^{\mathrm{row}}_{aa'} = \sum_{c\in I:\ t(a)s(c)=-1} \partial_{ac}\,\partial_{a'c} \quad (a, a' \in I',\ t(a) \neq t(a')),$$
$$\Delta^{\mathrm{col}}_{cc'} = \sum_{a\in I':\ t(a)s(c)=1} \partial_{ac}\,\partial_{ac'} \quad (c, c' \in I,\ s(c) \neq s(c')),$$

and H is **Laplace harmonic** if $\Delta^{\mathrm{row}}_{aa'}H = 0$ for all $a, a' \in I'$ with $t(a) \neq t(a')$ and $\Delta^{\mathrm{col}}_{cc'}H = 0$ for all $c, c' \in I$ with $s(c) \neq s(c')$.

(HV7) Joint harmonics. In the setting of (HV5), let $l\colon I' \to \mathbb{Z}$, $l'\colon I \to \mathbb{Z}$, $\nu\colon I' \to \mathbb{Z}$ and $\nu'\colon I \to \mathbb{Z}$. A polynomial $H \in \mathbb{C}[X]$ is **jointly highest of weights (ν,ν') for** (l,l'), for the row and column units of (s,t) with the shifts (γ,γ'), if $H \neq 0$, H is termwise harmonic, H is K-highest of weight ν for l for the row units $(E^{\mathrm{row}}_{aa'})$ (signs t), and H is K-highest of weight ν' for l' for the column units $(E^{\mathrm{col}}_{cc'})$ (signs s). Let $\lambda\colon I \to \mathbb{Z}$, $\lambda'\colon I' \to \mathbb{Z}$. A **joint harmonic** for the signs (s,t), the weights (γ,γ') and the parameters (λ,λ') is a polynomial $H \in \mathbb{C}[X]$ such that

(a) the pair (λ, λ') satisfies the block rule for the weights (γ, γ') and the signs (s, t) ((WM8));

(b) H is jointly highest of weights $(\mu_{\lambda'}, \mu_{-\lambda})$ for $(\lambda', -\lambda)$, for the row and column units of (s, t) with the shifts $(\gamma, -\gamma')$, where $\mu_{\lambda'}$ is the lowest K-type of λ' for the signs t and $\mu_{-\lambda}$ that of $-\lambda$ for the signs s.

Definition 3.2.28. Put $\mathfrak{b} = \pi_M^b \in \mathcal{O}_E$, the **deep level**, where b is the scale of (AD1), a positive integer. For an m-fold twist ϑ the **grid modulus** is $\mathfrak{m}_\vartheta = \mathfrak{b} \cdot \prod_{w \in Q} w$; $\mathfrak{m}_1 = \mathfrak{b}$. Fix for each ϑ a finite idele t_ϑ, integral, with component 1 at the finite places not dividing $\mathfrak{m}_\vartheta$ and generating $\mathfrak{m}_\vartheta \widehat{\mathcal{O}}_E$; with the Weil idele α of (WD2) of Definition 3.2.12, the **flip idele** is $\mathrm{flp}_\vartheta = \overline{\alpha t_\vartheta}/(\alpha t_\vartheta)$, a finite idele of norm one, read as the scalar matrix $\mathrm{flp}_\vartheta \cdot 1$ in $G_f(I)$ for any index set I.

The two grids. A **node** is a triple (z, i, j) with $z \in \{\mathrm{I}, \mathrm{II}\}$, the grid I or the grid II (for $z \in \{\mathrm{I}, \mathrm{II}\}$ we write $\neg z$ for the other grid), and $0 \le i, j \le m$ (the **row index** i and the **column index** j). The node (I, i, j) carries a space Y_j^i of rank $r + (i + j)t$ and the node (II, i, j) a space $\tilde{Y}_i^j$ of rank $r + 1 + i(t + 1) + j(t - 1)$, each with explicit signs and an explicit doubled Harish-Chandra parameter at every infinite place, such that

(GP1) $Y_j^0 = Y_j$ for $0 \le j \le m$ (the spaces $Y_0, \dots, Y_m$ form the first column of the grid Y);

(GP2) $\tilde{Y}_0^0 = Y_0 \oplus \langle 1 \rangle$, and $\tilde{Y}_j^j = Y_j^j \oplus \langle (-1)^j \rangle$ for $j \le m$;

(GP3) $\tilde{Y}_j^{j+1}$ and Y_j^{j+1} are the same space up to a sign-preserving renumbering of the basis;

(GP4) the node (z, i, j) has $\lfloor (m - j + [z])/2 \rfloor$ negative signs ($[\mathrm{I}] = 0$, $[\mathrm{II}] = 1$); in particular the spaces Y_m^i and $\tilde{Y}_i^m$ at the nodes (z, i, m) of the row m are positive definite;

(GP5) (signs and parameters) put $\varsigma_n(v) = \frac{k}{2} - 1 - n$ for $0 \le n < k - 1$ and $\varsigma_n(v) = \frac{\Delta}{2} - \kappa_{n+2-k}(v)$ for $n \ge k - 1$ ($v \in \Phi$), so that $(\varsigma_0(v), \dots, \varsigma_{r-1}(v))$ is the target parameter $\varsigma(v)$. The basis of the space at the node (z, i, j) is indexed by the **index set** $n_{z,i,j}$ of the node, the disjoint union of the **base** $\{0, \dots, r - 1 + [z]\}$, the **up blocks** $\{1, \dots, i\} \times \{0, \dots, t + [z] - 1\}$ and the **down blocks** $\{1, \dots, j\} \times \{0, \dots, t - [z] - 1\}$; the base index $n \in \{0, \dots, r - 1\}$ is the basis vector c of Y_0 with $\mathrm{num}(c) = n$ (the numbering of (HP2) of Definition 3.2.13, in which $\varsigma(v)$ is listed), and for $z = \mathrm{II}$ the base index r is the added line of (GP2). Every block index has the sign $+1$; the base index n has the sign $(-1)^{e + \min(e,j)}$ with $e = \max(n + 1 - k, 0)$. With $\delta = 1$ if $z = \mathrm{I}$ and $i + j$ is odd, $\delta = 0$ otherwise, the **grid parameter** $p_{z,i,j}(v)$ is the integer vector with the entries

$$\begin{aligned} p_{z,i,j}(v)(n) &= 2\varsigma_n(v) + [z] - \delta, \\ p_{z,i,j}(v)(l, c) &= 2\kappa_l(v) + t + [z] - 1 - 2c - \delta, \\ p_{z,i,j}(v)(l, c) &= -2\kappa_l(v) + t - [z] - 1 - 2c - \delta \end{aligned}$$

at a base index n, an up index (l, c) and a down index (l, c) respectively. The doubled Harish-Chandra parameter of Y_j^i at v is $p_{\mathrm{I},i,j}(v)$, and that of $\tilde{Y}_i^j$ is $-p_{\mathrm{II},i,j}(v)$. *Offsets.* For an integer $o \ge 0$, the **signs with offset** o and the **grid parameter with offset** o, $p^o_{z,i,j}(v)$, of the node (z, i, j) are given by the same formulas, with the sign $(-1)^{e + \min(e,j)}$ of the base index n replaced by $(-1)^{o + e + \min(e \dot{-} o, j)}$ ($x \dot{-} y = \max(x - y, 0)$) and, at a down index (l, c), $\kappa_l(v)$ replaced by $\kappa_{l+o}(v)$; offset 0 gives the grids, $p^0_{z,i,j} = p_{z,i,j}$.

$\mathcal{F}(z, i, j)$ denotes the **grid forms** at a node: the forms of the type of the negated parameter, that is conjugate forms on Y_j^i and forms on $\tilde{Y}_i^j$. An **edge** is a pair of nodes $((z, i, j), (z, i + 1, j))$

or $((z,i,j),(z,i,j+1))$ in one grid; it is **vertical** if its character carries the twist (in the grid Y: the second index changes; in the grid $\tilde{Y}$: the first). For an m-fold twist ϑ, an **edge tuple** S is a matrix $S(e)$ over $\mathcal{O}_E/\mathfrak{m}_\vartheta$ of the size rk(target) $\times$ rk(source) for every edge e. The **lift move** along $((z,i,j),(z,i+1,j))$ and the **transposed move** along $((z,i,j+1),(z,i,j))$ with matrix $S(e)$ are the theta lift, respectively the transposed theta lift, between the two nodes with finite datum the indicator of $S(e)$ modulo $\mathfrak{m}_\vartheta$, the joint harmonic of the edge at the infinite places, and the theta kernel $\theta_{\alpha,\beta}^{Y',Y''}$ of (XD2) in which Y' is the space at the node (z,i,j), Y'' the space at the other node of the edge, α the edge character (GP6) at the rank $N = \operatorname{rk} Y'$ and β the edge character at the rank $N = \operatorname{rk} Y''$. Thus, for both kinds of move, the move twists the determinant of the unitary group of each of the two nodes by the edge character taken at the rank of the other node ($\alpha(\det h)$ for h in the group of Y'', $\beta(\det g)^{-1}$ for g in the group of Y').

(GP6) Edge characters. For $l \geq 0$, a flag $c \in \{\mathrm{II}, \mathrm{I}\}$ and a rank N, the **edge character** is $\mu_{l+1}^0 \vartheta^{l+1} v^{\lfloor N/2 \rfloor}$ if $c = \mathrm{II}$ and $\mu_{l+1}^0 v^{\lceil N/2 \rceil}$ if $c = \mathrm{I}$; it has weight $\tau \mapsto \gamma_{c,l}(N)(\tau) = \lfloor (N + [\neg c])/2 \rfloor - \kappa_{l+1}(\tau)$. The lift move along $((z,i,j),(z,i+1,j))$ uses $c = z$, $l = i$; the transposed move along $((z,i,j+1),(z,i,j))$ uses $c = \neg z$, $l = j$; the rank N at which it is taken is fixed by the rule of the moves above. The character modulo $\mathfrak{m}_\vartheta$ of an edge, χ_e in (PD2), is $u \mapsto \prod_{w|\mathfrak{m}_\vartheta} \psi_w(u \bmod w)$ for the tame part ψ of $\mu_{l+1}^0 \vartheta^{l+1}$ (vertical edges) or of μ_{l+1}^0 (other edges), l the index that changes along the edge.

(GP7) Joint harmonics. Let s be signs on a nonempty finite set I and t signs on a nonempty finite set I', and for $v \in \Phi$ let $\gamma_v, \gamma_v' \in \mathbb{Z}$, $\lambda_v \colon I \to \mathbb{Z}$ and $\lambda_v' \colon I' \to \mathbb{Z}$. The **joint harmonic at** v, H_v, is a fixed joint harmonic for the signs (s,t), the weights (γ_v, γ_v') and the parameters (λ_v, λ_v') ((HV7) of Definition 3.2.27), a polynomial in the variables X_{ac} ($a \in I'$, $c \in I$), if one exists, and $H_v = 0$ if none exists. The **joint harmonic** is $H = \prod_{v \in \Phi} H_v$, with H_v written in the variables $X_{v,a,c}$: a polynomial in the variables of the Fock model of (WR5) of Definition 3.2.18 on the index set $\Phi \times I' \times I$ with the signs $(v,a,c) \mapsto t(a)s(c)$; v_H is its Fock vector. The joint harmonic of the lift move from $p = (z,i,j)$ to $q = (z,i+1,j)$ is the one for the signs of p and q, $\gamma = \gamma_{z,i}(\operatorname{rk} p)$, $\gamma' = \gamma_{z,i}(\operatorname{rk} q)$, $\lambda = -p_{z,i,j}$, $\lambda' = -p_{z,i+1,j}$; that of the transposed move between $p = (z,i,j)$ and $q = (z,i,j+1)$ is the one for the signs of p and q, $\gamma = \gamma_{\neg z,j}(\operatorname{rk} p)$, $\gamma' = \gamma_{\neg z,j}(\operatorname{rk} q)$, $\lambda = p_{z,i,j}$, $\lambda' = p_{z,i,j+1}$, read on $q \times p$ matrices.

(GP8) Points. The spaces of the grids are spaces with points ((VP2) of Definition 3.2.2, as in (WR1) of Definition 3.2.18): $Y_j^0 = Y_j$ for $0 \leq j \leq m$ is the space with a point of Y_j with its fixed frame ((XF1) of Definition 3.2.15) in the basis b of (HP1), (HP3) of Definition 3.2.13, that is, $V_b = (J_b, x_b)$ of (VP3) of Definition 3.2.2: J its Gram matrix in that basis and the point whose finite component is the matrix of the finite part and whose component at $v \in \Phi$ is the matrix of the part at v of the fixed frame, the basis at v numbered as in (GP5), $\tilde{Y}_0^0 = Y_0 \oplus \langle 1 \rangle$ with $\langle 1 \rangle$ the standard line of sign $+1$, and at every other node the space is the standard space of its signs ((WD9) of Definition 3.2.12), its basis numbered as in (GP5). In particular the space at a node (z,i,m) of the row m, whose signs are all $+1$ by (GP4), is the standard definite space I_N of its rank N; for the node $(\mathrm{I},0,m)$ this is (HP1) of Definition 3.2.13, Y_m being the standard definite space of rank $r + mt$.

Paths and turns. A **monotone path** from (i_0,j_0) to (i,j) is a sequence of steps $(i',j'+1) \to (i',j')$ (**down**) or $(i',j') \to (i'+1,j')$ (**right**). A **path through the two grids** Γ consists of an index $l = l(\Gamma) \leq m$ and two monotone paths Γ_{I}, Γ_{II} from $(0,m)$ to (l,l), the first in the grid Y (read from $Y_m^0 = Y_m$), the second in the grid $\tilde{Y}$ (read from $\tilde{Y}_0^m$). The **first path** has $l = 0$ and both halves going down the first column; the **last path** has $l = m$ and both halves

going right along the row m. A **turn** of Γ at (i,j) $(0 \le i,j < m)$ replaces a corner at which one half comes down to (i,j) and leaves to the right by the corner through $(i+1,j+1)$ (for the half in $\tilde{Y}$ the node is named (j,i)), or, if $i = j = l(\Gamma)$ and both halves end with a step down, replaces l by $l+1$ and both final steps down by steps right. The edges of Γ are the edges of its two halves; along a half, the composite of the moves by the matrices of S is the linear map $\mathrm{fm}_{\Gamma_z}(S)\colon \mathcal{F}(z,0,m) \to \mathcal{F}(z,l,l)$.

Path data. For a finite set w of edges and $N \in \mathbb{N}$, a function Ψ of edge tuples is a **path datum** of bound N on w if

(PD1) $\Psi(S)$ depends only on the matrices $S(e)$, $e \in w$;

(PD2) $\epsilon_w(g)\,\Psi(g\cdot S) = \Psi(S)$ for every admissible family $g = (g_p)$ of invertible matrices modulo $\mathfrak{m}_\vartheta$ at the nodes, where $(g\cdot S)(p,q) = g_q^{-1} S(p,q) g_p$ and $\epsilon_w(g) = \prod_{(p,q)\in w} \chi_{(p,q)}(\det g_q / \det g_p)$ with χ_e the character modulo $\mathfrak{m}_\vartheta$ of the edge (admissible: in the level of the node at Y_m^0 and $\tilde{Y}_0^m$, of the form $g \oplus 1$ at $\tilde{Y}_i^i$ for the element g at Y_i^i, free elsewhere);

(PD3) $\Psi(S) \neq 0$ only if $S(e)$ has a left inverse for every vertical $e \in w$;

(PD4) every value $\Psi(S)$ is a sum of at most N roots of unity.

Periods. A **period triple** $x = (\vartheta, y', \varphi)$ is an m-fold twist, a conjugate form y' on Y_m and a form φ on $\tilde{Y}_0^m$. For a family $\Psi = (\Psi_\vartheta)$ and a real point g of $\mathrm{U}(\tilde{Y}_l^l)$, $l = l(\Gamma)$, the **period** of Ψ on Γ is

$$\mathrm{per}_\Gamma(\Psi; g)(x) = \sum_S \Psi_\vartheta(S)\,\langle \mathrm{fm}_{\Gamma_\mathrm{I}}(S)\,y',\ \mathrm{fm}_{\Gamma_\mathrm{II}}(S)\,\varphi\rangle_{l,g}, \tag{3.3}$$

the sum over the edge tuples supported on the edges of Γ, where $\langle u, v\rangle_{l,g}$ is the period over $[\mathrm{U}(Y_l^l)]$ of u against the restriction along $\mathrm{U}(Y_l^l) \to \mathrm{U}(\tilde{Y}_l^l)$ of v translated by g and, on the added line, by the l-th power flp_ϑ^l of the flip idele. For a function Ξ of period triples, Ψ is **proportional** to Ξ on Γ, written $\Psi \propto_\Gamma \Xi$, if for every real point g there is $\beta \in \mathbb{C}$ (possibly 0) with $\mathrm{per}_\Gamma(\Psi; g)(x) = \beta\,\Xi(x)$ for every period triple x.

Transfer of periods. For paths Γ', Γ, a function k from the m-fold twists to $\mathbb{N}$ and a function c from the m-fold twists to $\mathbb{C}^\times$, Γ' **transfers periods to** Γ with bound k and constant c if for every function N and every family Ψ' of path data Ψ'_ϑ of bound $N(\vartheta)$ on the edges of Γ' there is a family Ψ of path data Ψ_ϑ of bound $k(\vartheta)N(\vartheta)$ on the edges of Γ such that, for every Ξ,

$$\Psi \propto_\Gamma \Xi \implies \Psi' \propto_{\Gamma'} \big(x \mapsto \Xi(x)/c(\vartheta_x)\big).$$

Definition 3.2.29. For signs s on n, dx is the measure of $G_\infty(s)$, the product over $v \in \Phi$ of the Cayley measures of $\mathrm{U}(\mathrm{diag}\,s)$ of (VP1) of Definition 3.2.2; for $x \in G_\infty(s)$ put $\|x\| = \prod_{v\in\Phi} \|x_v\|$ (operator norms). For a function k on $G_\infty(s)$ and F on $\mathrm{G}(s)$, $R(k)F(g) = \int_{G_\infty(s)} k(x) F(g(x,1))\,dx$.

(ED1) Kernels. A function e on $G_\infty(s)$ is an **idempotent kernel** if it is continuous and integrable, $e(x^{-1}) = \overline{e(x)}$, $e(1) \neq 0$, and $\int e(z)\,e((zx)^{-1}y)\,dz = e(x^{-1})e(y)/e(1)$ for all x, y. For $B \in \mathbb{N}$ it is a **cusp kernel of order** B if moreover $x \mapsto |e(x)|\,\|x\|^{B+1}$ is integrable, $\sup_x |e(x)|\,\|x\|^B < \infty$, and $\int a(x) e(x^{-1}y)\,dx = e(y)$ for all y, for some continuous compactly supported a on $G_\infty(s)$.

(ED2) Forms of a type. A **type** on a space with a point V with signs s is a function e on $G_\infty(s)$. A **form of type** e on V is a function F on $\mathrm{G}(s)$ that is continuous, bounded, left invariant under Γ_V, right invariant under an open subgroup of $G_f(n)$ (it **has a level**), and satisfies $R(e)F = F$ if e is integrable. The **Petersson product** of bounded measurable functions on $\Gamma_V\backslash\mathrm{G}(s)$ is $\langle F, F'\rangle_V = \int_{[\mathrm{U}(V)]} \overline{F} F'$.

(ED3) Theta kernels with a real part. Let S, T, V be spaces with points with signs s on n, s' on n', η on ι, let $\varepsilon(v,i) = \eta_i$ on $\Phi \times \iota$, and let Ω map $G_\infty(s) \times G_\infty(s')$ to bounded operators of $\mathcal{F}_\varepsilon$. A **theta kernel** for (S,T) on V is a pair (ω, θ) of a homomorphism ω from $\mathrm{G}(s) \times \mathrm{G}(s')$ to the bounded operators of $\mathcal{M}_\eta$ with $(p,\varphi) \mapsto \omega(p)\varphi$ continuous and a bounded linear functional θ on $\mathcal{M}_\eta$ with $\theta(\omega(\gamma g, \gamma' h)\varphi) = \theta(\omega(g,h)\varphi)$ for $\gamma \in \Gamma_S$, $\gamma' \in \Gamma_T$. It **has real part** Ω if $\omega((x,1),(y,1))(v \otimes f) = (\Omega(x,y)v) \otimes f$ for $x \in G_\infty(s)$, $y \in G_\infty(s')$, $v \in \mathcal{F}_\varepsilon$, $f \in \mathcal{M}_\eta$. The **lift** of a function F on $\mathrm{G}(s)$ with datum $\varphi \in \mathcal{M}_\eta$ is the function on $\mathrm{G}(s')$

$$L_\varphi F(h) = \int_{[\mathrm{U}(S)]} \theta(\omega(g,h)\varphi) F(g)\, dg.$$

(ED4) Half edges and edges. Let s, s', η, ε and Ω be as in (ED3), e a type on the spaces with signs s, e' one on those with signs s', and $\Phi \in \mathcal{F}_\varepsilon$. Then Φ is a **half edge for** Ω **from** e **to** e' if:

(a) e is a cusp kernel of order 0;

(b) the **zeta integral** $\int_{G_\infty(s)} \langle \Phi, \Omega(x,1)\Phi\rangle\, \overline{e(x)}/e(1)\, dx$ is not 0;

(c) for all spaces with points S, T, V with signs s, s', η, every theta kernel for (S,T) on V with real part Ω, every $f \in \mathcal{M}_\eta$ and every form F of type e on S: if $L_{\Phi\otimes f}F$ has a level, it is a form of type e' on T;

(d) for every polynomial P there is a continuous compactly supported a on $G_\infty(s')$ with $L_{v_P \otimes f}F = R(a)\, L_{\Phi\otimes f}F$ for all S, T, V, theta kernels, f and F as in (c).

Φ is an **edge for** Ω **from** e **to** e' if it is a half edge for Ω from e to e' and a half edge for $(y,x) \mapsto \Omega(x,y)$ from $\bar{e}'$ to $\bar{e}$. An edge **for the weights** (γ,γ') is an edge for $\Omega_{\gamma,\gamma'}$ ($\eta = s' \otimes s$, $\iota = n' \times n$).

Definition 3.2.30. Let D be admissible data and $z \in \{\mathrm{I}, \mathrm{II}\}$. The **triangle** z has the **positions** (i,n), $i, n \geq 0$: the row i is the offset, the column n the node $(z,0,n)$. Write $n_n = n_{z,0,n}$ and $N_n = \#n_n$; the index set n_{n+1} is n_n together with the down block $n+1$ (GP5).

(TV1) Spaces. For signs s on n, the **signed space** is the space with a point $(\mathrm{diag}(s), x_s)$, x_s diagonal with entry 1 where $s = 1$ and the idele with infinite components 1 and finite component c where $s = -1$. The **opposite space** of $V = (J_V, x_V)$ with signs s is $V^- = (-J_V, c\, x_V)$, with signs $-s$, c read as the idele with infinite components 1; $\mathrm{G}(-s)$ and $\mathrm{G}(s)$ are the same group of matrices. The space $V^i_{z,n}$ at the position (i,n) has the index set n_n and the signs with offset i of the node $(z,0,n)$: $V^0_{\mathrm{I},n} = Y_n$ ($n \geq 0$), $V^0_{\mathrm{II},0} = \tilde{Y}^0_0$, $V^{i+1}_{z,0} = (V^i_{z,0})^-$, and in all other cases ($n \geq 1$, and $i \geq 1$ or $z = \mathrm{II}$) the signed space of these signs.

(TV2) Triangle forms and the triangle modulus. The **triangle forms** $\mathcal{F}^i_\triangle(z,n)$ at (i,n) are the forms of type e_λ on $V^i_{z,n}$ ((ED2)), $\lambda = \frac{1}{2}p^i_{z,0,n}$. The **triangle modulus** of an m-fold twist ϑ is $\mathfrak{t}_\vartheta(\mathrm{I}) = \pi_M \prod_{w\in Q} w$ and $\mathfrak{t}_\vartheta(\mathrm{II}) = \mathfrak{b}$; write $R_\vartheta = R_{\mathfrak{t}_\vartheta(z)}$.

(TV3) Triangle lifts and weighted chains. The **point datum** Φ^{pt}_n is the indicator function of the $n_{n+1} \times n_n$ matrix over R_ϑ with entry 1 at (a,a) ($a \in n_n$) and 0 elsewhere. Put $S = V^i_{z,n}$, $T = V^i_{z,n+1}$, let ν_1, ν_2 be the edge characters (GP6) with $c = \neg z$ and $l = n+i$ at the ranks N_n and N_{n+1}, and H the joint harmonic (GP7) for the signs of S and T, the weights $\gamma = \gamma_{\neg z, n+i}(N_n)$, $\gamma' = \gamma_{\neg z, n+i}(N_{n+1})$ and the doubled parameters $\lambda = p^i_{z,0,n}$, $\lambda' = p^i_{z,0,n+1}$. The **triangle lift** $\mathrm{Tl}^i_{\vartheta,z,n}$ sends $F \in \mathcal{F}^i_\triangle(z,n)$ to the lift $L_\varphi F$ ((ED3)) for the

theta kernel $\theta^{S,T}_{\nu_1,\nu_2}$ and the datum $\varphi = \mathrm{mod}_{\mathfrak{t}_\vartheta(z)}(\Phi^{\mathrm{pt}}_n, v_H)$, a function on the adelic group of T. The **weighted chain** of row i is $\mathrm{Wc}^i_{\vartheta,z,0} = \mathrm{id}$ and

$$\mathrm{Wc}^i_{\vartheta,z,n+1} = \#\mathrm{GL}_{n_n}(\mathcal{O}_E/\mathfrak{t}_\vartheta(z)) \cdot \mathrm{Tl}^i_{\vartheta,z,n} \circ \mathrm{Wc}^i_{\vartheta,z,n}.$$

(TV4) Base level and base forms. The **base level** $\mathrm{K}^{\mathrm{bs}}_z \subseteq \mathbf{K}_0(n_0)$ is the group of the elements whose reduction modulo every chosen place is 1 if $z = \mathrm{I}$, and modulo $\mathfrak{b}$ is 1 if $z = \mathrm{II}$. The **base forms** $\mathcal{F}^{\mathrm{bs}}_z$ are the $F \in \mathcal{F}^0_\triangle(z, 0)$ with $F(gk) = F(g)$ for all g and all $k \in \mathrm{K}^{\mathrm{bs}}_z$.

(TV5) The opposite space and the first lifts. The **copy** at $(i, 0)$ of a function F on the adelic group of $V^0_{z,0}$ is the same function on the adelic group of $V^i_{z,0}$, the same group of matrices by (TV1). The **first lift** of row i is $F \mapsto \mathrm{Tl}^i_{\vartheta,z,0}(\text{copy of } F \text{ at } (i, 0))$.

(TV6) Left and reversed pairs. For an m-fold twist ϑ, the **left-pair character** of row i at the rank N is $\chi^{\mathrm{lp}}_{\vartheta,z,i}(N) = \chi^{-1}\upsilon^N$, χ the edge character (GP6) with $c = \neg z$, $l = i$ at the rank N; it has the weight $\delta_i(N) = N - \gamma_{\neg z,i}(N)$. Let $\mu = \frac{1}{2}p^{i+1}_{z,0,n}$ and $\mu' = \frac{1}{2}p^i_{z,0,n+1}$. A polynomial Φ in the variables of the $n_{n+1} \times n_n$ matrices is a **left pair** at (i, n) if e_μ, $e_{-\mu}$ are cusp kernels of order 0 ((ED1)) with $e_{-\mu} = \overline{e_\mu}$, the same holds for μ', and v_Φ is an edge for the weights $(\delta_i(N_n), \delta_i(N_{n+1}))$ from $e_{-\mu}$ to $e_{-\mu'}$ ((ED4)). A polynomial Φ in the variables of the $n_n \times n_{n+1}$ matrices is a **reversed pair** at (i, n) if the same kernel conditions hold and v_Φ is an edge for the weights $(\delta_i(N_{n+1}), \delta_i(N_n))$ from $e_{-\mu'}$ to $e_{-\mu}$.

(TV7) Descents, ascents and the Fourier datum. For a left pair Φ at (i, n) and a function Ψ on the $n_{n+1} \times n_n$ matrices over R_ϑ, let $S = V^{i+1}_{z,n}$, $T = V^i_{z,n+1}$, $\varphi = \mathrm{mod}_{\mathfrak{t}_\vartheta(z)}(\Psi, v_\Phi)$ and $(\nu_1, \nu_2) = (\chi^{\mathrm{lp}}_{\vartheta,z,i}(N_n), \chi^{\mathrm{lp}}_{\vartheta,z,i}(N_{n+1}))$. The **descent** $\mathrm{Ds}^i_{\vartheta,z,n}(\Phi, \Psi)$ sends $F \in \mathcal{F}^i_\triangle(z, n+1)$ to the function

$$g \longmapsto \int_{[\mathrm{U}(T)]} \theta^{S,T}_{\nu_1,\nu_2}(\varphi)(g, h)\, F(h)\, dh$$

on the adelic group of S. For a reversed pair Φ at (i, n) and a function Ψ on the $n_n \times n_{n+1}$ matrices over R_ϑ, the **ascent** $\mathrm{As}^i_{\vartheta,z,n}(\Phi, \Psi)$ is defined by the same formula with $S = V^i_{z,n+1}$, $T = V^{i+1}_{z,n}$ and $(\nu_1, \nu_2) = (\chi^{\mathrm{lp}}_{\vartheta,z,i}(N_{n+1}), \chi^{\mathrm{lp}}_{\vartheta,z,i}(N_n))$; it sends $\mathcal{F}^{i+1}_\triangle(z, n)$ to functions on the adelic group of $V^i_{z,n+1}$. Fix an integral finite idele t generating $\mathfrak{t}_\vartheta(z)\widehat{\mathcal{O}}_E$ with components 1 at the places not dividing it, and let ψ_ϑ be the additive character of R_ϑ with $\psi_\vartheta(x \bmod \mathfrak{t}_\vartheta(z)) = \psi_{E,f}(\jmath\alpha\bar{\alpha}\, t^{-1}x)$ for $x \in \widehat{\mathcal{O}}_E$, α the Weil idele. The **Fourier datum** of a function Ψ on the $N' \times N$ matrices over R_ϑ is the function on the $N \times N'$ matrices $\widehat{\Psi}(\rho) = \sum_x \overline{\Psi(x)}\, \psi_\vartheta(\mathrm{tr}(\rho x))$.

Definition 3.2.31. Target forms. The **target forms** are the good forms on Y_0, $\mathcal{T} = \mathcal{A}^{\mathrm{good}}(Y_0)$, and $d_\mathcal{T} = \dim_\mathbb{C} \mathcal{T}$ is the **target dimension**.

The chain. Y_m is the standard definite space I_n of rank $n = r + mt$ ((HP1) of Definition 3.2.13), so that $\mathrm{U}(Y_m)(\mathbb{A}_F) = G_\infty(1) \times G_f(n)$ ((VP1), (VP2) of Definition 3.2.2). For an m-fold twist ϑ with modulus Q, let $\mathcal{A}_\vartheta(Y_m)$ be the space of algebraic forms $G_f(n) \to W_{\lambda^\circ}$ of the weight (κ_Y, e_Y) of Y_m ((WM7) of Definition 3.2.19; the weight and $W_{\lambda^\circ} = \mathbb{V}_{\kappa_Y}$ are defined below), of the level K_ϑ of integral elements that reduce to 1 at the chosen places and to block upper triangular matrices at the places of Q, with the character of ϑ on K_ϑ defined below. The **chain of lifts** Θ_ϑ sends $y \in \mathcal{T}$ to the algebraic form $F^\flat_l$ attached to F ((WM7) of Definition 3.2.19, for the doubled Harish-Chandra parameter $l = (2\lambda(Y_m)_v)_{v\in\Phi}$ of (κ_Y, e_Y) defined below), where F is the function on $\mathrm{U}(Y_m)(\mathbb{A}_F) = G_\infty(1) \times G_f(n)$ obtained from y by applying lift 1, lift 2, ..., lift m (defined below) in turn.

The target lattice. The **target lattice** consists of the following data on $\mathcal{T}$, in the roles of (TW1)–(TW4) of a twisted lattice, with the m-fold twists as twists and the trivial twist 1:

(TL1) $S_\vartheta = \mathcal{A}_\vartheta(Y_m)$ with the **Petersson inner product** $\langle\varphi,\varphi'\rangle = \sum_c (\#\Gamma_c)^{-1}\varphi(g_c)^* G_W\,\varphi'(g_c)$, where g_c are fixed representatives of the classes $c \in \mathrm{U}(Y_m)(E^+)\backslash\mathrm{U}(Y_m)(\mathbb{A}_f)/\mathrm{K}_\vartheta$, $\Gamma_c = \mathrm{U}(Y_m)(E^+)\cap g_c\mathrm{K}_\vartheta g_c^{-1}$ and values are written in the standard basis of W_{λ°; and the map Θ_ϑ;

(TL2) ϑ allowed at ℓ if and only if $\ell \nmid \mathrm{N}Q$;

(TL3) $\Lambda_\vartheta(\iota)$ the forms **integral at** ι: the φ such that for every $g \in \mathrm{U}(Y_m)(\mathbb{A}_f)$ all coordinates of $g_\ell^{-1}\iota(\varphi(g))$ have absolute value at most 1, where ι is applied coordinatewise and g_ℓ acts on the factor of $v \in \Phi$ of $W_{\lambda^\circ}\otimes\mathbb{C}_\ell$ through its component at the place of E induced by $x \mapsto \iota(v(\bar{x}))$;

(TL4) coordinates on S_1: the values of a form at fixed representatives of the classes

$$\mathrm{U}(Y_m)(E^+)\backslash\mathrm{U}(Y_m)(\mathbb{A}_f)/\mathrm{K}_1,$$

in the standard basis of the Weyl module, with the rational Gram matrix of the Weyl module on each class.

So a target form y is integral at $\iota \mid \ell$ if and only if $\Theta_\vartheta y$ is integral at ι for every m-fold twist ϑ with $\ell \nmid \mathrm{N}Q$, and the volume at ϑ is the **chain determinant** $Z(\vartheta) = \det(\Theta_\vartheta^*\Theta_\vartheta)$. **The lifts, the level and the character.** Here μ_j^0 is the untwisted lift character of Definition 3.2.26. The **level places** of ϑ are the chosen places and the places of Q; the product of their prime ideals is the triangle modulus $\mathfrak{t}_\vartheta(\mathrm{I}) = \pi_M\prod_{w\in Q} w$ ((TV2) of Definition 3.2.30), a modulus in the sense of (XD4). For $1 \le j \le m$ put $\nu_j = \mu_j^0\vartheta^j$, and let H_j be the joint harmonic (GP7) for the signs of Y_{j-1} and of Y_j ((GP1), (GP5)), the weights $\gamma_{\mathrm{II},j-1}(\mathrm{rk}\,Y_{j-1})$ and $\gamma_{\mathrm{II},j-1}(\mathrm{rk}\,Y_j)$ of (GP6), and the doubled parameters $p_{\mathrm{I},0,j-1}$ and $p_{\mathrm{I},0,j}$ of (GP5). Let $\mathbf{1}_j^{\mathrm{inc}}$ be the indicator function of the $\mathrm{rk}\,Y_j \times \mathrm{rk}\,Y_{j-1}$ matrix over $R_{\mathfrak{t}_\vartheta(\mathrm{I})}$ with entry 1 at (a,a) for every basis index a of Y_{j-1} and 0 elsewhere, the basis indices of Y_{j-1} being the first $\mathrm{rk}\,Y_{j-1}$ basis indices of Y_j ((GP5)). **Lift** j sends a function F on $\mathrm{U}(Y_{j-1})(\mathbb{A}_F)$ for which the integral below converges absolutely to the function

$$h \longmapsto \int_{[\mathrm{U}(Y_{j-1})]} \theta^{Y_{j-1},Y_j}_{\nu_j\upsilon^{\lfloor \mathrm{rk}\,Y_{j-1}/2\rfloor},\,\nu_j\upsilon^{\lfloor \mathrm{rk}\,Y_j/2\rfloor}}\big(\mathrm{mod}_{\mathfrak{t}_\vartheta(\mathrm{I})}(\mathbf{1}_j^{\mathrm{inc}}, v_{H_j})\big)(g,h)\,F(g)\,dg$$

on $\mathrm{U}(Y_j)(\mathbb{A}_F)$ (the theta kernel of (XD2) with the datum of (XD5), the quotient measure of (XD1)), multiplied by the index, in the group of integral elements of $\mathrm{U}(Y_{j-1})(\mathbb{A}_f)$, of its subgroup of the elements that reduce to 1 at every level place. Write the basis of Y_m as a block of size r (the basis of Y_0) followed by m blocks of size t, numbered $0,1,\dots,m$. Then K_ϑ is the group of integral elements of $\mathrm{U}(Y_m)(\mathbb{A}_f)$ whose reductions are 1 at the chosen places and block upper triangular at the places of Q, and its character is

$$g \longmapsto \prod_w\prod_{l=1}^{m}\psi_{l,w}(\det g_w^{(l)}),$$

w over the level places, $g_w^{(l)}$ the diagonal block l of the reduction of g at w, $\psi_l = (\omega_w^l\psi_w^l)_w$ the tame part of $\mu_l^0\vartheta^l$ formed from the tame part ω^l of μ_l^0 ((LF3) of Definition 3.2.26) and the tame part ψ^l of ϑ^l (Definition 3.1.2); a tame part is in general not determined by the character alone, and the character of K_ϑ is taken with this family. **Parameters and the weight.** Let

$0 \le j \le m$, $v \in \Phi$, and $\epsilon_j = \frac{1}{2}$ for odd j, $\epsilon_j = 0$ for even j. List the part at v of the fixed frame of Y_j ((XF1) of Definition 3.2.15) as r vectors numbered $0, \dots, r-1$ (for $j = 0$ the number of a vector is the number num of its index, a basis vector of Y_0, (HP2) and (HP5) of Definition 3.2.13) followed by j blocks of t vectors, with the signs: $(-1)^{c-j}$ at the vector of number $k-1+c$ for $j < c \le m-1$, and $+1$ at all other vectors . The **parameter** $\lambda(Y_j)_v$ of Y_j at v attaches the entry $\varsigma(v)_i - \epsilon_j$ to the vector of number i and the entry $\frac{t-1}{2} - i - \kappa_l(v) - \epsilon_j$ to vector i $(0 \le i < t)$ of block l $(1 \le l \le j)$; it is taken as the Harish-Chandra parameter at v for the group $\mathrm{U}(Y_j)(E_v^+)$, written in the frame as the unitary group of the diagonal matrix of these signs. The **weight** $\lambda^\circ = (\lambda_v^\circ)_{v\in\Phi}$ of Y_m: with $n = r + mt$ and the entries of $\lambda(Y_m)_v$ listed in decreasing order $\lambda_1 \ge \dots \ge \lambda_n$,

$$\lambda_v^\circ = \Big(\lambda_p - \frac{n+1}{2} + p\Big)_{1\le p\le n},$$

the parameter minus half the sum of the positive roots; the **weight of** Y_m is (κ_Y, e_Y) on the index set of Y_m ((WM1), (WM6) of Definition 3.2.19) with $\kappa_Y(v,h) = \lambda^\circ_{v,h} - \lambda^\circ_{v,h+1}$ $(1 \le h < n)$ and $e_Y(v) = \lambda^\circ_{v,n}$, $\lambda^\circ_{v,p}$ the entry p of λ°_v; and $W_{\lambda^\circ} = \mathbb{V}_{\kappa_Y}$ is its Weyl module (WM2), acted on through ρ_{κ_Y,e_Y}, which realises the tensor product over $v \in \Phi$ of the Weyl modules of GL_n of the highest weights λ_v°, with its standard basis of products of minors and the integer Gram matrix $G_W = G_{\kappa_Y}$ (WM3) of the $\mathrm{U}(n)$-invariant form for which the basis tensors are orthonormal; so the summands of (TL1) are the Gram pairings $\langle \varphi(g_c), \varphi'(g_c)\rangle_{\kappa_Y}$.

Lemma 3.2.32. *Let ϑ be an m-fold twist. Then:*

1. $\mathrm{U}(Y_m)(E^+) \cap gK_\vartheta g^{-1} = \{1\}$ *for every $g \in \mathrm{U}(Y_m)(\mathbb{A}_f)$; in particular every Γ_c is trivial and $\langle \varphi, \varphi'\rangle = \sum_c \varphi(g_c)^* G_W\, \varphi'(g_c)$ on $\mathcal{A}_\vartheta(Y_m)$;*

2. *for every prime identification $\iota \mid \ell$ with $\ell \nmid \mathrm{N}Q$, and all $\varphi \in \Lambda_\vartheta(\bar\iota)$, $\varphi' \in \Lambda_\vartheta(\iota)$, $\|\iota\langle \varphi, \varphi'\rangle\| \le 1$;*

3. *for $\vartheta = 1$, the coordinates* co *of (TL4) and the block diagonal matrix G with one block G_W for each class satisfy (TW4): $G \in M_C(\mathbb{Q})$ is positive definite and $\langle \varphi, \varphi'\rangle = \mathrm{co}(\varphi)^* G\, \mathrm{co}(\varphi')$ on S_1.*

Hence (TL1)–(TL4) satisfy (TW1)–(TW4), and the target lattice is a twisted lattice.

Proof. Write $n = r + mt$ for the rank of Y_m, $\Gamma = \mathrm{U}(Y_m)(E^+)$ and $K = K_\vartheta$.

Step 1: elements of $\Gamma \cap gKg^{-1}$ have finite order and unipotent reduction. The group $\mathrm{U}(Y_m)(E^+ \otimes \mathbb{R})$ is compact (Y_m is definite), so Γ is discrete in $\mathrm{U}(Y_m)(\mathbb{A}_f)$, and $\Gamma \cap gKg^{-1}$ is discrete and compact, hence finite. Let $\gamma = gkg^{-1}$ in it, $k \in K$. Its characteristic polynomial $P \in E[X]$ is that of k, so its coefficients are integral at every finite place: $P \in \mathcal{O}_E[X]$. At a chosen place w the reduction of k is 1, so $P \equiv (X-1)^n$ modulo w.

Step 2: $\gamma = 1$. Let ζ be an eigenvalue of γ, a root of unity of order d, $L = E(\zeta)$ and $\mathfrak{P}$ a prime of L above a chosen place w, of residue characteristic $p \in \{p_0, p_1\}$. From $P(\zeta) = 0$ and $P \equiv (X-1)^n$ we get $(\zeta - 1)^n \in \mathfrak{P}$, so $\zeta \equiv 1$ modulo $\mathfrak{P}$. If d has two distinct prime factors, $1 - \zeta$ is a unit; if $d = q^a$ with a prime $q \ne p$, $1 - \zeta$ divides q, a unit at $\mathfrak{P}$; both contradict $\zeta \equiv 1$ unless $d = 1$. If $d = p^a$ with $a \ge 1$, then $\mathbb{Q}(\zeta_p) \subseteq L$ and $p - 1 \le [L : \mathbb{Q}] \le n[E : \mathbb{Q}]$. But $[E : \mathbb{Q}] = 2\varphi(q_1(s)) \le 2b$ by $q_1(s) \mid b$ (AD2), $n = r(1 + 5m) \le 3x^5(1 + 10\sqrt{x}) \le b$ by (AD1), (AD2), and $p \ge b^{10}$ (AD5), so $p - 1 > 2b^2 \ge n[E : \mathbb{Q}]$, a contradiction. Hence all eigenvalues of γ are 1; as γ has finite order it is semisimple, so $\gamma = 1$. With the definition of the Petersson product in (TL1) this is (1).

Step 3: integrality of the Weyl pairing. Let $\iota \mid \ell$, $\varphi \in \Lambda_\vartheta(\bar\iota)$, $\varphi' \in \Lambda_\vartheta(\iota)$ and $g \in \mathrm{U}(Y_m)(\mathbb{A}_f)$; let $\xi = \varphi(g)$, $\xi' = \varphi'(g)$ in the standard basis. As G_W has integer entries,

$$\iota(\xi^* G_W \xi') = (\bar\iota\xi)^{\mathsf T} G_W\, (\iota\xi'),$$

$\bar\iota\xi$ and $\iota\xi'$ taken coordinatewise. We show that the two translates in (TL3) leave this value unchanged. For $v \in \Phi$ let $j_v \colon E \to \overline{\mathbb{Q}}_\ell$, $j_v(x) = \iota(v(x))$. As E is a CM field, $v(\bar x) = \overline{v(x)}$, so the embedding attached in (TL3) to $\bar\iota$ at v is $x \mapsto \bar\iota(v(\bar x)) = \iota(v(x)) = j_v(x)$, and the one attached to ι is $x \mapsto j_v(\bar x)$. Extend j_v to $E \otimes \mathbb{Q}_\ell$ through the completion at the place it induces and put $A_v = j_v(g_\ell) \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, $A = (A_v)_v$. Since Y_m is standard definite, $\bar g^{\mathsf T} g = 1$, so $\bar g = (g^{\mathsf T})^{-1}$ and the component of g_ℓ for ι at v is $j_v(\bar g_\ell) = (A_v^{\mathsf T})^{-1}$. Let ρ be the action of $\prod_v \mathrm{GL}_n$ on W_{λ° in its standard basis. By the definition of W_{λ° in Definition 3.2.31, each $W_{\lambda_v^\circ}$ is spanned, inside a tensor product of exterior powers of the standard representation with its orthonormal basis of wedge tensors, by the standard basis vectors, with integer coordinate matrix F_v and a GL_n-stable column space; the action is a power of the determinant times the restriction of the tensor action, and $G_W = \bigotimes_v F_v^{\mathsf T} F_v$. So Lemma 3.2.33 applies over $\overline{\mathbb{Q}}_\ell$: $G_W \rho(a) = \rho(a^{\mathsf T})^{\mathsf T} G_W$. With $x = g_\ell^{-1}\bar\iota\xi = \rho(A)^{-1}\bar\iota\xi$ and $x' = g_\ell^{-1}\iota\xi' = \rho((A^{\mathsf T})^{-1})^{-1}\iota\xi' = \rho(A^{\mathsf T})\iota\xi'$,

$$x^{\mathsf T} G_W x' = (\bar\iota\xi)^{\mathsf T}\rho(A^{-1})^{\mathsf T} G_W \rho(A^{\mathsf T})\iota\xi' = (\bar\iota\xi)^{\mathsf T}\rho(A^{-1})^{\mathsf T}\rho(A)^{\mathsf T} G_W\, \iota\xi' = (\bar\iota\xi)^{\mathsf T} G_W\, \iota\xi' = \iota(\xi^* G_W \xi').$$

By (TL3) both x and x' have coordinates of absolute value at most 1, and G_W has integer entries, so $\|\iota(\xi^* G_W \xi')\| = \|x^{\mathsf T} G_W x'\| \le 1$ by the ultrametric inequality.

Step 4: (2) and (3). By (1), $\iota\langle\varphi,\varphi'\rangle = \sum_c \iota(\varphi(g_c)^* G_W \varphi'(g_c))$, a finite sum of terms of absolute value at most 1 by Step 3; the ultrametric inequality gives (2). For (3), (1) gives $\langle\varphi,\varphi'\rangle = \mathrm{co}(\varphi)^* G\, \mathrm{co}(\varphi')$ with $G = \bigoplus_c G_W$. The matrix $G_W = F^{\mathsf T}F$ for the integer matrix F expressing the standard basis of the Weyl module in the orthonormal basis of tensors, and F is injective, so G_W, and hence G, is rational and positive definite. (TW1) and (TW2) are (TL1), (TL2); (TW3) is (2), as ϑ is allowed at ℓ exactly when $\ell \nmid \mathrm{N}Q$; (TW4) is (3). □

Lemma 3.2.33. *Let $n \ge 1$, let $h_1, \dots, h_N \in \{1, \dots, n\}$, let $T = \bigotimes_{c=1}^N \wedge^{h_c}\mathbb{Q}^n$ with its basis of wedge tensors $e_{I_1} \otimes \cdots \otimes e_{I_N}$ (I_c increasing h_c-tuples), and for a commutative ring K and $a \in \mathrm{M}_n(K)$ let $T(a)$ be the matrix of $\bigotimes_c \wedge^{h_c} a$ in this basis (its entries are products of minors of a). Let F be an integer matrix with linearly independent columns (the coordinates of vectors $w_1, \dots, w_d \in T$) whose column space is stable under $T(a)$ for every $a \in \mathrm{GL}_n(\mathbb{C})$, and put $G = F^{\mathsf T}F$. Then G is a positive definite rational matrix, and for every field K of characteristic 0 and every $a \in \mathrm{GL}_n(K)$ there is a unique matrix $P(a) \in \mathrm{M}_d(K)$ with $T(a)F = FP(a)$, namely $P(a) = G^{-1}F^{\mathsf T}T(a)F$; and for every $e \in \mathbb{Z}$, with $\rho(a) = \det(a)^e P(a)$,*

$$G\,\rho(a) = \rho(a^{\mathsf T})^{\mathsf T}\, G.$$

The same holds for families: if $(F_v, G_v, \rho_v)_{v\in\Phi}$ are as above, $G = \bigotimes_v G_v$ and $\rho(a) = \bigotimes_v \rho_v(a_v)$ for $a = (a_v) \in \prod_v \mathrm{GL}_n(K)$, then $G\rho(a) = \rho(a^{\mathsf T})^{\mathsf T} G$ with $a^{\mathsf T} = (a_v^{\mathsf T})$.

Proof. G **and** P. $G = F^{\mathsf T}F$ is rational, and positive definite because F has independent columns ($x^{\mathsf T}Gx = \|Fx\|^2 > 0$ for real $x \ne 0$); so G is invertible over $\mathbb{Q}$. If $T(a)F = FX$ then $X = G^{-1}F^{\mathsf T}T(a)F$ (multiply by $G^{-1}F^{\mathsf T}$), which gives uniqueness.

Existence over every field of characteristic 0**.** The matrix

$$\Delta(a) = T(a)F - FG^{-1}F^{\mathsf T}T(a)F$$

has entries that are polynomials with rational coefficients in the entries of a. For $a \in \mathrm{GL}_n(\mathbb{C})$ the column space of F is $T(a)$-stable, so $T(a)F = FX$ for some X, and then $X = G^{-1}F^{\mathsf T}T(a)F$

and $\Delta(a) = 0$. As $\mathrm{GL}_n(\mathbb{C})$ is the complement of the hypersurface $\det = 0$ in $\mathrm{M}_n(\mathbb{C})$, it is Zariski dense, so the entries of Δ are the zero polynomials. A field K of characteristic 0 contains $\mathbb{Q}$, hence $\Delta(a) = 0$ for every $a \in \mathrm{M}_n(K)$, which is $T(a)F = FP(a)$.

Transposes. In the wedge basis the entry of $\wedge^h a$ at (I, J) is the minor $\det a_{I,J}$, and $\det(a^{\mathsf{T}})_{I,J} = \det a_{J,I}$; so $\wedge^h(a^{\mathsf{T}}) = (\wedge^h a)^{\mathsf{T}}$, and taking tensor products, $T(a^{\mathsf{T}}) = T(a)^{\mathsf{T}}$.

The identity. For $a \in \mathrm{GL}_n(K)$,

$$G\,P(a) = F^{\mathsf{T}}FP(a) = F^{\mathsf{T}}T(a)F = \big(T(a)^{\mathsf{T}}F\big)^{\mathsf{T}}F = \big(T(a^{\mathsf{T}})F\big)^{\mathsf{T}}F = \big(FP(a^{\mathsf{T}})\big)^{\mathsf{T}}F = P(a^{\mathsf{T}})^{\mathsf{T}}G,$$

and multiplying by $\det(a)^e = \det(a^{\mathsf{T}})^e$ gives $G\rho(a) = \rho(a^{\mathsf{T}})^{\mathsf{T}}G$. For families,

$$G\rho(a) = \bigotimes_v G_v\rho_v(a_v) = \bigotimes_v \rho_v(a_v^{\mathsf{T}})^{\mathsf{T}}G_v = \rho(a^{\mathsf{T}})^{\mathsf{T}}G,$$

since transposition and products commute with Kronecker products. □

Definition 3.2.34. The **error unit** and the **denominator budget** of D are

$$u_{\mathrm{err}} = f\,a\,(\log b)^3, \qquad \beta_{\mathrm{den}} = \frac{u_{\mathrm{err}}}{m\,k} = \frac{f\,a\,(\log b)^3}{m\,k}.$$

Lemma 3.2.35. *Let $P = (q_0, d)$ be a parameter pair and $s \geq 0$ an integer. There is $\varpi \in \mathcal{O}_E$ such that*

(i) $\bar{\varpi} = -\varpi$ and $\varpi\mathcal{O}_E$ is the different of $E/\mathbb{Q}$;

(ii) $|\varpi|_v = \sqrt{d}\,|\mathrm{cyc}'_{q_1}(\zeta)|_v$ for every $v \in \Phi_s$;

(iii) $\prod_{v\in\Phi_s} |\varpi|_v = \mathrm{rd}_E^f$.

Proof. The different of $E/\mathbb{Q}$. By Lemma 3.2.8 (1), $d \equiv 7 \pmod 8$, so $-d \equiv 1 \pmod 4$ and $\mathcal{O}_K = \mathbb{Z}[\alpha]$ with $\alpha = (1+\sqrt{-d})/2$, a root of $g(X) = X^2 - X + (1+d)/4$. For a monogenic ring of integers the different is generated by the derivative of the minimal polynomial at the generator [224] (Ch. III, Prop. 2.4), so the different of $K/\mathbb{Q}$ is $g'(\alpha)\mathcal{O}_K = (2\alpha - 1)\mathcal{O}_K = \sqrt{-d}\,\mathcal{O}_K$. By Lemma 3.2.8 (2) the different of E/K is $\mathrm{cyc}'_{q_1}(\zeta)\mathcal{O}_E$. The different is multiplicative in towers (loc. cit., Prop. 2.2), hence

$$\mathfrak{D}_{E/\mathbb{Q}} = \mathrm{cyc}'_{q_1}(\zeta)\sqrt{-d}\,\mathcal{O}_E.$$

A phase making $\mathrm{cyc}'_{q_1}(\zeta)$ real. Complex conjugation of the CM field E sends ζ to ζ^{-1} and $\sqrt{-d}$ to $-\sqrt{-d}$, since under every embedding into $\mathbb{C}$ it is induced by complex conjugation. As $16 \mid q_0 \mid q_1$, the element $i = \zeta^{q_1/4}$ is a square root of -1 in E with $\bar{i} = -i$. Let J be the set of integers $1 < j < q_1$ prime to q_1; each $j \in J$ is odd because q_1 is even. Since $\mathrm{cyc}_{q_1}(X) = \prod_{j\in J\cup\{1\}}(X - \zeta^j)$,

$$\mathrm{cyc}'_{q_1}(\zeta) = \prod_{j\in J}(\zeta - \zeta^j).$$

For $j \in J$ put $a_j = (1+j)/2$ and $e_j = (1-j)/2$, both integers, so that $\zeta - \zeta^j = \zeta^{a_j}(\zeta^{e_j} - \zeta^{-e_j})$, and put $t_j = -i(\zeta^{e_j} - \zeta^{-e_j})$. Then $\bar{t}_j = i(\zeta^{-e_j} - \zeta^{e_j}) = t_j$ and $\zeta - \zeta^j = \zeta^{a_j}\,i\,t_j$. The element $\mu = \prod_{j\in J}\zeta^{a_j} i$ is a root of unity, so $c = \mu^{-1} \in \mathcal{O}_E^\times$ satisfies $c\bar{c} = 1$, and $c\,\mathrm{cyc}'_{q_1}(\zeta) = \prod_{j\in J} t_j$ is fixed by complex conjugation.

The element ϖ. Put $\varpi = c\,\mathrm{cyc}'_{q_1}(\zeta)\sqrt{-d} \in \mathcal{O}_E$. Then $\bar{\varpi} = c\,\mathrm{cyc}'_{q_1}(\zeta)\cdot(-\sqrt{-d}) = -\varpi$, and $\varpi\mathcal{O}_E = \mathrm{cyc}'_{q_1}(\zeta)\sqrt{-d}\,\mathcal{O}_E = \mathfrak{D}_{E/\mathbb{Q}}$ because c is a unit; this is (i). For $v \in \Phi_s$ and an embedding ι inducing v, $\iota(\bar{c}) = \overline{\iota(c)}$, so $|c|_v^2 = |c\bar{c}|_v = 1$, and $|\sqrt{-d}|_v = \sqrt{d}$; this gives (ii).

The product. Each $v \in \Phi_s$ is induced by exactly two embeddings, ι and $\bar{\iota}$, with $|\bar{\iota}(\varpi)| = |\iota(\varpi)|$, and there are $[E:\mathbb{Q}] = 2f$ embeddings. So

$$\prod_{v\in\Phi_s} |\varpi|_v^2 = \prod_{\iota} |\iota(\varpi)| = |N_{E/\mathbb{Q}}(\varpi)| = N(\mathfrak{D}_{E/\mathbb{Q}}) = |\operatorname{disc} E| = \mathrm{rd}_E^{2f},$$

by (i), the norm of the different being the absolute discriminant (loc. cit., Thm. 2.9), and the definition $\mathrm{rd}_E = |\operatorname{disc} E|^{1/[E:\mathbb{Q}]}$. Taking positive square roots gives (iii). □

Definition 3.2.36. Let D be admissible data. Write $G = \mathrm{U}(Y_0)$ and regard $G(\mathbb{A}) = \mathrm{U}(Y_0)(\mathbb{A}_{E^+})$ as a group of operators on $Y_0 \otimes_E \mathbb{A}_E$, matrices (g_{ik}) being taken in the basis of Y_0 of (HP3) (g_{ik} the coefficient of the basis vector i in g applied to the basis vector k); $G(E) = \mathrm{U}(Y_0)(E^+)$, and $G(F_\infty) = \prod_{v\in\Phi} \mathrm{U}(Y_0 \otimes_{E,v} \mathbb{C})$ is embedded as $g \mapsto (g,1)$. Let ϖ be the generator of the different of $E/\mathbb{Q}$ of Lemma 3.2.35.

Grades. The **grade** of a basis vector of Y_0 is $\mathrm{gr}(x) = 0$ for the basis vectors x of X_0 and $\mathrm{gr}(v_i) = 2(\varrho + 1 - i)$ for $1 \le i \le 2\varrho+1$ (so $\mathrm{gr}(v_{\varrho+1}) = 0$).

Bessel group. The **Bessel group** $R(\mathbb{A}) \subseteq G(\mathbb{A})$ consists of the g with $g_{ik} = \delta_{ik}$ for every string vector k with $\mathrm{gr}(k) \ge 0$ and every basis vector i with $\mathrm{gr}(i) \le \mathrm{gr}(k)$; its **unipotent radical** $\mathsf{U}(\mathbb{A})$ consists of the g with $g_{ik} = \delta_{ik}$ whenever $\mathrm{gr}(i) \le \mathrm{gr}(k)$; $R(E) = R(\mathbb{A}) \cap G(E)$. Let $\iota\colon \mathrm{U}(X_0)(\mathbb{A}_{E^+}) \to G(\mathbb{A})$ act on X_0 and fix every v_i. For $r \in R(\mathbb{A})$ its **Levi part** $a(r)$ is the block of the matrix of r on the basis of X_0, and its **unipotent part** is $u(r) = r\,\iota(a(r))^{-1}$.

Unipotent character. Let $\mathfrak{e}$ be the E-linear endomorphism of Y_0 that is 0 on X_0 and on $v_{2\varrho+1}$, and sends $v_i \mapsto v_{i+1}$ for $1 \le i \le 2\varrho$, $i \ne \varrho+1$, and $v_{\varrho+1} \mapsto -Nv_{\varrho+2}$. For $g \in G(\mathbb{A})$ put

$$\psi_{\mathsf{U}}(g) = \psi_E\Big(-\frac{\mathrm{tr}(\mathfrak{e}\,g)}{2M\varpi}\Big), \tag{3.4}$$

$\mathfrak{e}$ and g being read as matrices over $\mathbb{A}_E$ in the basis of Y_0; on $\mathsf{U}(\mathbb{A})$ it is the **unipotent character**.

Bessel source function. For $h \in \mathrm{U}(X_0)(\mathbb{A}_{E^+})$ write $h_v \in \mathrm{U}(k)$ ($v \in \Phi$) for its components at the infinite places (matrices in the standard basis of X_0, through F_v) and $h_f \in \mathrm{U}(X_0)(\mathbb{A}_{E^+,f})$ for its finite part. For a function $\varphi\colon \mathrm{U}(X_0)(\mathbb{A}_{E^+,f}) \to W_\lambda$ (for instance a source form) and a linear form t on W_λ, the **Bessel source function** is the function on $R(\mathbb{A})$

$$\mathcal{W}_{t,\varphi}(r) = \psi_{\mathsf{U}}(u(r))\; t\Big(\rho_\lambda\big((\overline{a(r)_v})_{v\in\Phi}\big)^{-1}\varphi(a(r)_f)\Big), \tag{3.5}$$

the bar denoting the entrywise complex conjugate.

Bessel period. Fix a left Haar measure dr of $R(\mathbb{A})$; integrals over $[R] = R(E)\backslash R(\mathbb{A})$ are for the quotient of dr by counting measure on $R(E)$, and an integral of a function that is not integrable is 0. Forms $y \in \mathcal{A}_M(Y_0)$ are read on $G(\mathbb{A})$ through the frames of Y_0. For $g \in G(F_\infty)$, a linear form t on W_λ, $\varphi \in \mathcal{S}$ and $y \in \mathcal{A}_M(Y_0)$, the **Bessel period** of φ at (g,t) against y is

$$\mathrm{BP}_{g,t}(\varphi, y) = \int_{[R]} \mathcal{W}_{t,\varphi}(r)\,\overline{y(r\,(g,1))}\,dr. \tag{3.6}$$

Definition 3.2.37. Let D be admissible data. The objects below are defined for three **parameters**: a $\mathbb{C}$-linear map $\mathcal{P}\colon \mathcal{S} \to \mathcal{A}_M(Y_0)$, a constant $c_{\mathrm{ss}} \in \mathbb{C}^\times$, and a family $\mathrm{err} = (\mathrm{err}_\ell)_\ell$ of

nonnegative real numbers indexed by the rational primes ℓ which satisfies the **error conditions**

$$\begin{aligned}&\mathrm{err}_\ell = 0 \quad \text{if } \ell \neq N \text{ and } \big(\chi_K(\ell) = 1 \text{ or } \ell \geq x_1\big),\\ &\sum_{\ell \in U} \mathrm{err}_\ell \log \ell \leq u_{\mathrm{err}} \quad \text{for every finite set } U \text{ of rational primes.}\end{aligned}$$

Nothing else is assumed of the parameters here; a result that needs a further property of them states it as a hypothesis. The objects below depend on $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$; once the parameters are fixed, this dependence is not written.

Gain exponents. For a rational prime ℓ the **gain exponent** is $g_\ell = \lfloor \mathrm{gain}(\ell) - \mathrm{err}_\ell \rfloor \in \mathbb{Z}$.

Inverse gain factor. Let S_{ns} be the set of primes $p < x_1$ with $\chi_K(p) \neq 1$. The **inverse gain factor** is

$$c_B = \prod_{\ell \in S_{\mathrm{ns}} \cup \{N\}} \ell^{-g_\ell} \in \mathbb{Q}_{>0}.$$

Normalised Poincaré map. The **normalised Poincaré map** is the linear map $\mathfrak{p}\colon \mathcal{S}^{\mathrm{ram}} \to \mathcal{T}$,

$$\mathfrak{p}\varphi = c_B\, c_{\mathrm{ss}}^{-1}\, \#\bar{G} \cdot e_0\, \mathcal{P}\varphi;$$

it takes values in $\mathcal{T} = e_0 \mathcal{A}_M(Y_0)$, as $\mathcal{P}\varphi \in \mathcal{A}_M(Y_0)$. For $s \in \bar{G}$ its **translate** is $s\mathfrak{p} = s \circ \mathfrak{p}$, with s acting on $\mathcal{A}_M(Y_0)$ as in Definition 3.2.16.

4 Short forms: the upper bound (Theorem A)

This section proves Theorem A, the upper half of the comparison: short integral source forms span the ramified source forms (Theorem 4.3.1), and the index of their images under the translates of the normalised Poincaré map is small (Theorem 4.4.1). Both are stated for arbitrary parameters $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$, with the seesaw identity as a hypothesis and, for the index form, the uniform local property of the errors.

The key idea is to move from the indefinite group of Y_0, where the target forms live, to definite groups, where integrality and the size of conjugates can be read off coordinates. The seesaw identity [171] makes this move: the chain of lifts applied to $\mathcal{P}\varphi$ is c_{ss} times a composite of m definite theta lifts of φ, whose kernels are Schur jets of theta functions with complex multiplication by K at a fixed CM period. Two estimates for these jets give the two forms of the theorem. At the infinite places their conjugates are Gaussian moments of bounded theta values, and this bounds the size (Proposition 4.3.2). At a prime p the composite lift of the forms of a submodule of small colength is divisible by $p^{\mathrm{gain}(p)}$ up to a loss, and the losses, weighted by $\log p$, sum to at most u_{err} (Proposition 4.4.2). In our reading this divisibility is the multiplicity that [5] attributes to each lift.

The subsections follow the hypotheses. Section 4.1 proves the laws of the Weil representation on the theta model and fixes the theta functionals. Section 4.2 fixes the integral vacuum and the CM period and shows that the hypotheses of Theorem A can be met: the seesaw identity holds for every map represented by Bessel periods at one point (Proposition 4.2.25), and a family of divisibility errors exists (Proposition 4.2.54). Theorem A follows, in its spanning form (Section 4.3) and its index form (Section 4.4). The last two subsections treat the good projection, which carries bounds and integrality from the chain of lifts to the target lattice, and construct the short integral source forms. For other readings of this part of the argument see [223, 75].

4.1 The Weil representation: basic lemmas

This section proves the laws of the Weil representation on an arithmetic model of the theta model, and it makes the fixed choices with which every theta kernel of the paper is formed (Definition 4.1.15). The representation and the correspondence are classical [322, 134, 136, 236, 206]; what is proved here is their form with explicit normalisations, integral data and theta functionals that do not depend on the choices.

The laws (Lemma 4.1.1) rest on the comparison with the classical representation (Lemma 4.1.2), through the Stone–von Neumann theorem in Mackey's form [195], the splittings of Kudla and of Harris, Kudla and Sweet [172, 121] and the Siegel–Levi formula [191]; on the Fock model [14, 157] (Lemma 4.1.5); and on the normalised integral action (Proposition 4.1.14), whose multipliers are Gauss sums, as in Weil. The theta functionals are unique and join coherently (Lemma 4.1.7), exist on every framed space (Lemma 4.1.9) and are invariant under rational points (Lemma 4.1.11); spaces with a point of the same signs differ by a rational isometry and an adelic unitary point, by Hasse's norm theorem and Landherr's classification [123, 180] (Lemma 4.1.8). For the Heisenberg group and its models see [83]. The fixed choices give the canonical theta functional of every space with a point, normalised, independent of the choices and multiplicative (Lemma 4.1.17), and with it the theta kernel of a pair. Section 4.2 forms the vacuum with these choices, and Section 6 forms the exchange data with the canonical functionals.

Lemma 4.1.1. *Let $E = E_s$. Let S, T be spaces with points with signs s on n and s' on n', and α, β automorphic characters of* $\mathrm{U}(1)$.

1. *There is a unit c of $\widehat{\mathcal{O}}_E$ with $\bar{c}c = -1$.*

2. *For signs s, each $\omega_s(g)$ is a bounded operator of $\mathcal{M}_s$ preserving the inner product, $\omega_s(gg') = \omega_s(g)\omega_s(g')$, $\omega_s(1) = 1$, $(g,f) \mapsto \omega_s(g)f$ is continuous, and $\omega_s(g)\rho(z)\omega_s(g)^{-1} = \rho(gz)$ for $z \in \mathbb{A}_E^n$. The same holds for $\omega_\varepsilon^\infty$ on $\mathcal{F}_\varepsilon$, and $\omega_{\alpha,\beta}$, $\Omega_{\gamma,\gamma'}$ are representations.*

3. *(Comparison.) Let $\langle X, Y\rangle = \mathrm{Tr}_{E/F}(\jmath\, h(X,Y))$, h the form of $\mathrm{Hom}(S,T)$, and ψ_F the character of $\mathbb{A}_F/F$ with $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$. There is a linear isomorphism from the space of vectors of $\mathcal{M}_{S,T}$ that are smooth for ρ onto the Schrödinger model of the Heisenberg representation of the symplectic F-space $(\mathrm{Hom}(S,T)(\mathbb{A}_E), \langle\,,\rangle)$ with central character ψ_F, intertwining ρ with that representation, $\omega_{\alpha,\beta}$ with the Weil representation $\omega_{\psi_F,\chi}$ of Theorem A.5.4 for the dual pair $(\mathrm{U}(S),\mathrm{U}(T))$ with the splitting characters $\beta[\#n']$ on $\mathrm{U}(S)$ and $\alpha[\#n]$ on $\mathrm{U}(T)$.*

4. *(Split places.) Let A be a finite set of finite places of E containing the conjugate of none of its members, $\Psi \in \mathcal{T}_A$, $f \in \mathcal{M}_{S,T}$, and $g \in G_f(n)$, $h \in G_f(n')$ with components 1 at the places outside $A \cup \bar{A}$. For a matrix k over $\mathbb{A}_{E,f}$ let $\det_A k$ be the idele with component $\det k_w$ at $w \in A$ and 1 elsewhere. Then $\Psi'(x) = \Psi(h^{-1}xg)$ lies in $\mathcal{T}_A$ and*

$$\omega_{\alpha,\beta}((1,g),(1,h))(\Psi \otimes_A f) = \frac{\alpha[\#n](\det_A h)}{\beta[\#n'](\det_A g)}\, |\det_A g|^{\#n'/2}\, |\det_A h|^{-\#n/2}\ \Psi' \otimes_A f.$$

5. *(Real points.) Let α, β have weights γ, γ', i.e. $\alpha(z) = \prod_{v\in\Phi} z_v^{\gamma(v)}$ for $z \in \mathrm{U}(1)(\mathbb{A})$ with finite components 1, and likewise β with γ'. For $x \in G_\infty(s)$, $y \in G_\infty(s')$, $k \in G_f(n)$, $k' \in G_f(n')$, $v \in \mathcal{F}_{S,T}$ and $f \in \mathcal{M}_{S,T}$,*

$$\omega_{\alpha,\beta}((x,k),(y,k'))(v \otimes f) = \big(\Omega_{\gamma,\gamma'}(x,y)v\big) \otimes \big(\omega_{\alpha,\beta}((1,k),(1,k'))f\big).$$

If moreover x_v, y_v are block diagonal for the partitions of the indices by sign at every $v \in \Phi$, x_v^σ, y_v^σ denoting the blocks of sign $\sigma \in \{\pm 1\}$, then for every polynomial P in the variables $X_{v,a,b}$ ($v \in \Phi$, $a \in n'$, $b \in n$)

$$\Omega_{\gamma,\gamma'}(x,y)v_P = \prod_{v\in\Phi}\prod_{\sigma=\pm1} \det(y_v^\sigma)^{\gamma(v)-\#\{i\in n: s_i=-\sigma\}} \det(x_v^\sigma)^{\#\{i\in n': s_i'=-\sigma\}-\gamma'(v)}\ v_{P'},$$

where P' is P after the substitution

$$X_{v,a,b} \mapsto \sum_{i\in n', j\in n} c_{v,a,b,i,j} X_{v,i,j}$$

with $c_{v,a,b,i,j} = (y_v)_{ia}(x_v^{-1})_{bj}$ if $s'(a) = s(b)$ and $c_{v,a,b,i,j} = (y_v^{-1})_{ai}(x_v)_{jb}$ otherwise.

6. *(The vacuum.) Let $\mathfrak{m}$ be a modulus with $A = \Sigma(\mathfrak{m})$, Φ a function on $n' \times n$ matrices over $R_\mathfrak{m}$, and $\Phi^\natural(x) = \Phi(x_A \bmod \mathfrak{m})$ if the components of x at A are integral, $\Phi^\natural(x) = 0$ otherwise. Then $\Phi^\natural \in \mathcal{T}_A$, and for $k \in \mathbf{K}_0(n)$, $k' \in \mathbf{K}_0(n')$ (with infinite components 1) and $v \in \mathcal{F}_{S,T}$*

$$\omega_{\alpha,\beta}((1,k),(1,k'))(v \otimes (\Phi^\natural \otimes_A v^0)) = \alpha(\det k')\,\beta(\det k)^{-1}\ v \otimes (\Phi^\natural_{k,k'} \otimes_A v^0),$$

with $\Phi_{k,k'}(X) = \Phi(\bar{k}'^{-1}X\bar{k})$, $\bar{k}, \bar{k}'$ the reductions modulo $\mathfrak{m}$. In particular

$$\omega_{\alpha,\beta}((1,k),(1,k'))(v \otimes v^0) = \alpha(\det k')\beta(\det k)^{-1}\, v \otimes v^0.$$

7. *(Zero terms.) For every splitting (k,l,i) along p of a space with a point V (signs s on n) over a space with a point V_0 (signs s_0 on n_0) fix theta functionals θ_V, θ_{V_0} in the sense of (WD9) locally. For every $f \in \mathcal{M}_s$ there is exactly one vector $Z(f) \in \mathcal{M}_{s_0}$ with*

$$\theta_{V_0}\big(\rho(y)Z(f)\big) = \int_{\mathbb{A}_E^p/E^p} \theta_V\big(\rho(i_{\mathbb{A}}y + k_{\mathbb{A}}b)f\big)\,db \qquad (y \in \mathbb{A}_E^{n_0}),$$

and $f \mapsto Z(f)$ is a bounded linear map.

Proof. The parameter relations give $5040 \mid q_1$: indeed $18480 = 16 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ divides q_0, and $q_1 = q_0\, 3l_0(s)2^s$. Thus $E = K(\zeta_{q_1})$ contains a primitive 5040th root of unity. Apply Lemma 3.2.7.

The boundedness, pairing invariance, genuine product and identity laws, joint continuity and Heisenberg conjugation in (2) are those of Proposition 4.1.14, applied to the same (WR3) kernel and, at infinity, the same (WR5) kernel. The pair maps in (VP5) of Definition 3.2.2 are group homomorphisms; their determinant factors in (WR4)–(WR5) are characters. Hence the pair operators are representations as well, as stated in that proposition.

For (3), apply the action-comparison assertion of Lemma 4.1.2 with $r = \#n$, $t = \#n'$ and the same spaces and characters. Its form

$$B(X,Y) = \mathrm{Tr}_{E/F}\left(\jmath\,\mathrm{tr}(J_S^{-1}Y^*J_TX)\right)$$

is $\mathrm{Tr}_{E/F}(\jmath h(X,Y))$ for the Hom form in (VP5) of Definition 3.2.2. Its additive character satisfies the required trace identity, and its original-factor splitting characters are exactly $\beta[\#n']$ and $\alpha[\#n]$. Thus its single Heisenberg identification intertwines precisely the two prescribed pair representations. That identification extends to a unitary map of Hilbert completions and restricts to a topological isomorphism on the polarized Schwartz–Bruhat spaces. In particular it supplies the asserted linear isomorphism on smooth Heisenberg vectors, with the topology needed to transport continuous functionals. No separate scalar or character is chosen in this application.

Apply Lemma 4.1.16(1) with $r_S = \#n$, $r_T = \#n'$ and the given conjugate-disjoint set A. Its inverse position transform is $X \mapsto h^{-1}Xg$, and its determinant factors are exactly $\det_A h$ and $\det_A g$. This gives the membership and full identity in (4), including an arbitrary complementary vector f. The modulus-data identity and the vacuum assertion in (6) are Lemma 4.1.16(2) for the same modulus and integral unitary points. Integral invertibility ensures that the reduced matrices act on precisely the specified residue matrix space.

The real/finite separation and the polynomial determinant and substitution formula in (5) are Lemma 4.1.5. Its signs, character weights, compact sign blocks and Fock variables are precisely those in (WR5); its first identity applies to every integrable Fock vector and every vector of the theta model.

For (7), apply Lemma 4.1.6 to the given splitting (k,l,i) and its fixed source and residual spaces. Its formula is precisely the theta-function mean in (7), with probability Haar measure on $\mathbb{A}_E^p/E^p$. It applies to every vector of the full L^1 carrier and gives a bounded linear operator. The injectivity of the residual theta-function map supplies uniqueness, including empty isotropic or residual index sets. □

Lemma 4.1.2. *Let S,T have ranks r,t, and let α,β be automorphic characters of* U(1). *Put*

$$B(X,Y) = \mathrm{Tr}_{E/F}\Big(\jmath\,\mathrm{tr}(J_S^{-1}Y^*J_TX)\Big), \qquad \chi_{\mathrm{in}} = \beta[t], \quad \chi_{\mathrm{out}} = \alpha[r].$$

Let ψ_F be the standard character with $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$. Fix a theta functional $\theta_{\mathrm{Hom}(S,T)}$ of $\mathrm{Hom}(S,T)$ in the sense of (WD9), locally for this statement. There is a linear Heisenberg isomorphism from the ρ-smooth vectors of the SAME canonical carrier $\mathcal{M}_{S,T}$ onto a polarized Schwartz–Bruhat model for (B,ψ_F), intertwining the literal $\omega_{\alpha,\beta}$ with Kudla's representation whose characters on the original factors $\mathrm{U}(S),\mathrm{U}(T)$ are respectively $\chi_{\mathrm{in}},\chi_{\mathrm{out}}$. The identification extends to a unitary isomorphism of the Hilbert completions, and its restriction and inverse are continuous for the Schwartz–Bruhat topology (real Schwartz seminorms and finite compact-open inductive limits). It carries the fixed $\theta_{\mathrm{Hom}(S,T)}$ to a nonzero multiple of the rational theta distribution.

The character convention is made explicit as follows. In the both-linear-first ordinary tensor convention put

$$h_V(v,v') = v^{\mathsf{T}} J_T \bar{v}', \qquad h_W(z,z') = -\jmath z^{\mathsf{T}} \overline{J_S^{-1}} \bar{z}', \qquad \mathcal{A}(v\otimes z) = \bar{v}\,\bar{z}^{\mathsf{T}}.$$

Use the full-trace tensor pairing. The ordinary tensor splitting parameters are $\chi_V = \chi_{\mathrm{in}}^{-1}$ and $\chi_W = \chi_{\mathrm{out}}^{-1}$; the original pair maps to $(\bar{h}, \overline{g^{-\mathsf{T}}})$. Thus the displayed assertion specifies the original-factor characters, not their unreversed names on the transported tensor factors. No scalar character is left unspecified. Rank zero uses the one-dimensional Heisenberg model and retains the original determinant twists.

Proof. **The Heisenberg model, without choosing the unitary-group phase.** The commutator of (WD7) is the character of B, since its product bicharacter is $\psi_F(B/2)$. The form B is nondegenerate. The prescribed measure is the self-dual one in the given scaled coordinates: its finite assertion is the inverse-different identity in Lemma 4.1.3, and its real assertion follows by inserting the displayed κ_v. By Lemma 4.1.10 the vacuum is cyclic, has norm one, and its Weyl integral is the rank-one operator

$$U \longmapsto U(0)v^0.$$

Pass to the L^2 completion. Apply Theorem A.5.9 to complementary rational Lagrangians. The multiplicity space is one-dimensional: the Weyl integral just displayed would otherwise be the vacuum rank-one projector tensored with the identity of that multiplicity space. Thus a unitary Heisenberg identification with the polarized model exists.

We spell out the smooth domain and its topology. Choose rational complementary Lagrangians and identify the polarized position group with $X = \mathbb{A}_F^d$. At infinity write its underlying real vector space as $\mathbb{R}^D$. In this realization the infinitesimal Heisenberg operators are nonzero constant multiples of ∂_{x_j} and x_j, with constants fixed by the given additive character. From the reproducing formula and $|e_n| \le 1$,

$$\|F\|_\infty \le \|F\|_1, \qquad \|F\|_2^2 \le \|F\|_\infty \|F\|_1 \le \|F\|_1^2.$$

Thus a vector smooth for the coherent norm is smooth in the Hilbert completion. Its polarized representative u satisfies $x^\alpha \partial^\beta u \in L^2(\mathbb{R}^D)$ for every pair of multi-indices, initially in the distributional sense.

Here is the required estimate, including the global decay. For an integer M with $4M > D$, Fourier inversion and Cauchy–Schwarz give

$$\|h\|_\infty \le C_{D,M} \|(1-\Delta)^M h\|_2 \le C'_{D,M} \sum_{|\eta|\le 2M} \|\partial^\eta h\|_2.$$

Indeed the first bound follows by inserting $(1+|\xi|^2)^{-M}$ in the integral of $\widehat{h}(\xi)$; its square is integrable by $4M > D$. Apply it to $h = x^\alpha \partial^\beta u$. The Leibniz rule expresses its right side as

finitely many norms $\|x^{\alpha'}\partial^{\beta'}u\|_2$. The estimate applies to these distributional derivatives directly: Plancherel identifies their weak derivatives with multiplication of their Fourier transforms, so $(1+|\xi|^2)^M\widehat{h} \in L^2$ and hence $\widehat{h} \in L^1$. Fourier inversion supplies a bounded continuous representative for each h. Doing this for every derivative of u gives one smooth representative with the stated weighted derivative bounds. Hence every Schwartz seminorm $\sup_x |x^\alpha\partial^\beta u(x)|$ is finite and is bounded by finitely many coherent smooth-vector seminorms. This proves one continuous inclusion, rather than only set-theoretic regularity.

The vacuum itself is smooth for the coherent norm. Its real translates and their derivatives are polynomials times a Gaussian, uniformly dominated on compact parameter sets by an integrable Gaussian; its finite translates have an open stabilizer. The preceding inclusion therefore makes its polarized representative ϕ_0 Schwartz at infinity. At finite adeles choose compact open position and modulation groups U, V fixing a vector, shrinking so that $U \subset V^\perp$. Modulation invariance forces support in the compact open set $V^\perp$, and translation invariance makes the function constant on cosets of U. Thus this finite fixed space is finite-dimensional, consisting of functions on $V^\perp/U$. Conversely any finite Schwartz–Bruhat function is fixed by such a pair. Outside finitely many places the pair can and does equal the integral position and dual lattices. These observations apply in particular to the finite part of ϕ_0.

For the reverse inclusion, take a Schwartz–Bruhat function u in the polarized model. Its coherent coefficient is

$$F_u(z) = \langle \rho(z)\phi_0, u\rangle.$$

In real position coordinates this is, up to the prescribed unit central phase, $\int \overline{\phi_0(t-x)}u(t)e^{-i\langle\xi,t\rangle}dt$. Differentiating in x, ξ, and integrating by parts in t, shows that for every integer m,

$$|F_u(x,\xi)| \le C_m(u)(1+|x|+|\xi|)^{-m},$$

where $C_m(u)$ is bounded by finitely many Schwartz seminorms of u, with ϕ_0 fixed. To see both decay variables at once use $1+|x| \le (1+|t|)(1+|t-x|)$ before the integration by parts in ξ; all remaining integrals are bounded by integrable polynomial weights times the two Schwartz functions. The same calculation applies after every infinitesimal Heisenberg operator on u. At finite places the coefficient is locally constant and compactly supported. Choosing $m > 2D$ therefore bounds each coherent L^1 smooth-vector seminorm by finitely many Schwartz seminorms, on each finite fixed space.

The coefficient just constructed is the original vector, not merely a formal function. With the self-dual measures, Fubini and Plancherel in the modulation variable give the polarized orthogonality identity

$$\int \overline{\langle \rho(z)\phi_0, u_1\rangle}\langle \rho(z)\phi_0, u_2\rangle \, d\mu(z) = \langle u_1, u_2\rangle \|\phi_0\|_2^2.$$

For Schwartz functions all integrals here are absolutely controlled by the preceding decay estimates; the identity extends by density to L^2. Since $\|\phi_0\|_2 = 1$, the weak resolution $u = \int F_u(z)\rho(z)\phi_0 \, d\mu(z)$ follows by pairing with an arbitrary Schwartz function. The same integral converges in the coherent norm when $F_u \in L^1$, by the Bochner reproducing assertion of Lemma 4.1.10. Its Hilbert image is the displayed u, so the polarized Schwartz function lies in the original coherent carrier and is smooth there. This also proves continuity of the inverse inclusion.

Finally the adelic Schwartz–Bruhat topology is the inductive limit over finite position/modulation fixed spaces of their tensor products with the real Schwartz space. The finite spaces just described are finite-dimensional, and the estimates in both directions hold on every such

space. We have therefore proved the asserted equality of smooth domains and both continuous maps for precisely this topology.

The genuine action and Schwartz preservation required here are the conclusions of Proposition 4.1.14. After the fixed Heisenberg identification, its ratio with any classical implementer at a group element is scalar. For two genuine splittings this ratio is a continuous character.

The exact local scalar of a hyperbolic Levi. We record the calculation in a polarized hyperbolic block of the canonical kernel. If the action on its second isotropic half is an invertible matrix Q, the zero-position functional after that action has scalar

$$z(Q) = \upsilon(\det Q)^{-1} |\det Q|_E^{1/2}. \tag{Kuc1}$$

This assertion is local and includes real and finite places. Here the absolute value is the normalized local absolute value on E; at a real place of F, $|u|_E = |u|_{\mathbb{C}}^2$.

Here are its modulus and phase directly from the kernel. Use the adapted hyperbolic frame of (WR1); the finite matrix and its inverse are integral, and the real matrix is orthonormal for the auxiliary positive metric. In its polarized Heisenberg realization the adapted vacuum is a nonnegative Gaussian at infinity and the integral-position indicator at finite places, up to ONE constant. At finite places the adapted integral lattice splits into dual isotropic lattices. The vacuum semicharacter restricts to one on each, because the hermitian form vanishes on each. Its Schrödinger vector is therefore fixed by translations of the first lattice and modulations of the dual lattice. Modulation invariance confines support to the integral position lattice, and translation invariance makes it constant there; hence it is precisely a scalar times that indicator. The trace-one semicharacter gives the original mixed phase between these halves, so no extra quadratic multiplier is inserted. At a real place the corresponding one-variable Gaussian integral gives the stated nonnegative Gaussian. The constant cancels below. The position action is a scalar times the inverse linear substitution, by Heisenberg covariance. Comparing its norm fixes the modulus $|z(Q)| = |\det Q|_E^{1/2}$.

Write its nonnegative vacuum function as ϕ_0, and set

$$I = \int \phi_0(x)^2 dx > 0, \qquad J = \int \phi_0(Q^* x)\phi_0(x) dx \geq 0.$$

The actual kernel and unitarity give $\langle v^0, \omega(U)v^0\rangle = 1/\overline{\Delta(U)}$. Here is a rank-one verification retaining the phase. Write $P_U = \int e(y)\rho(Uy)\, d\mu(y)$. The vacuum projector identity and covariance make $P_U = \omega(U)P_1\omega(U)^{-1}$, the orthogonal projector onto $\mathbb{C}\omega(U)v^0$. The literal kernel gives $\omega(U)v^0 = \Delta(U)P_U v^0$. As $\|\omega(U)v^0\| = 1$, cancellation yields $\langle \omega(U)v^0, v^0\rangle = \Delta(U)^{-1}$, proving the asserted pairing. Taking the same pairing in position coordinates gives

$$z(Q) J \,\overline{\Delta(U)} = I. \tag{Kuc2}$$

Thus $J \neq 0$, and its positive sign determines the phase.

For the finite adapted frame C, Lemma 4.1.13 gives exactly

$$\Delta(C^{-1}\operatorname{diag}(Q^{-*}, Q)C) = |\Delta(Q)|^2\, \overline{\upsilon(\det Q)|\det Q|_E^{-1/2}}.$$

Consequently its product with $\upsilon(\det Q)$ is strictly positive. At infinity the negative block is $(Q^{-*} + Q)/2$; multiplying its adjoint by Q gives $(1 + Q^*Q)/2 > 0$. Hence the same product is strictly positive there too. Equation (Kuc2), together with the norm, proves (Kuc1), including its phase. The finite integral frame exists by Lemma 3.2.6; this includes places over two.

Apply the scalar to the actual Hom blocks. Double the TARGET space first, with the diagonal and antidiagonal isotropic halves. A target Levi is $m(A) = \operatorname{diag}(A, A^{-*})$. Use the

second target half for the zero-position half of Hom. Its block on that half is A^{-*}, repeated r times. Formula (Kuc1) therefore contributes

$$\upsilon(\det A)^{-r}|\det A|_E^{-r/2}.$$

The target determinant twist is $\tilde{\alpha}(\det A)$, since $\det_E m(A) = \det A/\overline{\det A}$. The resulting polarized operator is consequently

$$\alpha[r](\det A)|\det A|_E^{-r/2}\,\varphi(A^{-1}Z). \tag{Kuc3}$$

Double the SOURCE space instead. For $m(D) = \mathrm{diag}(D, D^{-*})$, the original action $X \mapsto Xm(D)^{-1}$ on Hom has block D^* on the half coming from the second source summand, repeated t times. Formula (Kuc1) contributes $\upsilon(\det D)^t|\det D|_E^{t/2}$. The inverse source twist is $\tilde{\beta}(\det D)^{-1}$. Hence

$$\beta[t](\det D)^{-1}|\det D|_E^{t/2}\,\varphi(ZD). \tag{Kuc4}$$

The substitutions are also forced directly by the translation/modulation covariance on these position halves.

The space characters have the required restrictions. At a split place υ restricts trivially to $F^\times$. At a fixed finite place E/F is unramified quadratic: a place over two is split by the trace-one, norm-two seventh-root element; at an odd fixed place the images of $i, -i$ are distinct, so conjugation acts nontrivially on the residue field. Thus the fixed-place value -1 on a uniformizer and value one on units in (WD3) are exactly the quadratic character. At infinity $\upsilon(x) = x/|x|$ restricts to sign. Therefore

$$\upsilon|_{\mathbb{A}_F^\times} = \eta_{E/F}, \qquad \alpha[r]|_{\mathbb{A}_F^\times} = \eta_{E/F}^r, \qquad \beta[t]|_{\mathbb{A}_F^\times} = \eta_{E/F}^t.$$

The base-change factors are trivial on $\mathbb{A}_F^\times$, and inversion does not change a quadratic restriction.

Compare the same classical forms and phases. The matrix-to-tensor calculation in the statement is valid over every quadratic field or split algebra. On pure tensors it gives

$$\jmath\,\mathrm{tr}(J_S^{-1}Y^*J_TX) = \overline{h_V(v,v')h_W(z,z')}.$$

Taking the full trace proves that the tensor transport has the same B, hence the same additive character. It also sends the original pair to $(\bar{h}, \overline{g^{-\mathsf{T}}})$, with determinants $(\det h)^{-1}, \det g$. These are algebraic identities; the calculation in Lemma 4.5.4 makes both inverses explicit and does not use split idempotents for these identities.

Use this dictionary for the classical comparison. At a nonarchimedean place the doubled skew-factor formula is Theorem A.5.7. Passing to the other factor interchanges the hermitian and skew-hermitian factors by the fixed imaginary scalar and conjugate dual. The full-trace both-linear-first convention is converted to that quotation's half-trace convention by multiplying the hermitian form by two and conjugating the skew-form value. This leaves the unitary groups and the determinant characters unchanged and gives precisely the original trace pairing. At a real place use Theorem A.5.8, with the same conjugate-dual substitution; its real-part pairing becomes the full-trace pairing by that same factor two. The ordinary complex absolute value in that quotation is converted by $|u|_E = |u|_{\mathbb{C}}^2$. Its character restrictions are just the restrictions checked above. These substitutions give (Kuc3)–(Kuc4) for the classical splitting as well. In the split algebra this is additionally the exact formula of Lemma 4.5.4, including its choice of position half.

For clarity, the sign change on the ordinary tensor parameters is necessary: the prescribed pair twist becomes the standard tensor characters $\chi_V = \beta[t]^{-1}$, $\chi_W = \alpha[r]^{-1}$. Conjugation of

the target group and conjugate inverse transpose of the source give exactly the original-factor characters in (Kuc3)–(Kuc4). An assignment of these characters to the unreversed tensor factors would be a different splitting.

Equality on the doubled group. On a doubled hyperbolic acting space the ratio of the two genuine actions is a continuous character ν. Equations (Kuc3)–(Kuc4) make it one on the Siegel Levi. In a hyperbolic basis with Gram matrix $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, write

$$n_+(Z) = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}, \quad n_-(Z) = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix}, \quad m(Q) = \operatorname{diag}(Q, Q^{-*}), \qquad Z + Z^* = 0.$$

Choose q in the local base field with $q \neq 0$ and $q^2 \neq 1$. Conjugation by $m(q1)$ multiplies the upper parameter by q^2 and the lower parameter by q^{-2}. Thus

$$n_+(Z) = [m(q1), n_+(Z/(q^2 - 1))],$$

and the corresponding formula with $q^{-2} - 1$ holds for n_-. Characters kill commutators, so $\nu = 1$ on both radicals.

We verify that this determines the character on the whole local group. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and a is invertible, its defining unitary equations give

$$ca^{-1} + (ca^{-1})^* = 0, \quad a^{-1}b + (a^{-1}b)^* = 0, \quad d = ca^{-1}b + a^{-*}.$$

The first and third follow from $g^* H g = H$, the second from $g H g^* = H$. Consequently

$$g = n_-(ca^{-1})\, m(a)\, n_+(a^{-1}b).$$

So ν is one on the open cell $\det a \neq 0$.

In fact every group element is a quotient of two points of this cell, over any infinite base field of characteristic different from two. To prove this without a density assumption, regard the unitary group as an algebraic group over the base field. Over its algebraic closure the quadratic étale algebra is the product of two copies of that closure. A point becomes a pair (g_1, g_2) satisfying $g_2^{\mathsf{T}} H g_1 = H$, so $g_2 = H(g_1^{-1})^{\mathsf{T}} H$. Thus the algebraic group becomes GL_{2r} and is geometrically integral.

Let $\mathfrak{u} = \{X : X^* H + H X = 0\}$, a vector space over the base field. The rational Cayley map

$$X \longmapsto (1 - X)(1 + X)^{-1}$$

identifies its nonempty open set $\det(1 + X) \neq 0$ with the group open set $\det(1 + g) \neq 0$, its inverse being $(1 - g)(1 + g)^{-1}$. Indeed $X^\dagger = -X$ for $\dagger = H^{-1}(\cdot)^* H$, so the commuting factors $1 \pm X$ give $\mathcal{C}(X)^\dagger = \mathcal{C}(X)^{-1}$, and the inverse formula gives the converse. Two is invertible, so these opens contain 0 and 1 respectively.

Write $\mathcal{U}$ for the cell where a is invertible; over a split quadratic algebra this means nonvanishing of the norm of $\det a$. Fix a group point g. The three open subsets $\mathcal{U}$, $g\mathcal{U}$ and the Cayley image have nonempty intersection over the algebraic closure, by geometric integrality. Pullback gives a nonempty base-field-defined open subset of the affine vector space $\mathfrak{u}$. It contains a base-field point: choose a basis, and a nonzero polynomial defining a principal open contained in it; over an infinite field such a polynomial cannot vanish on every coordinate tuple, by induction on the number of variables. Hence there is $x \in \mathcal{U}$ with $g^{-1}x \in \mathcal{U}$. The displayed block factorization applies to both, and

$$g = x(g^{-1}x)^{-1}.$$

Consequently the opposite parabolics generate the whole group, not just a dense subgroup. Since ν is one on their Levi and radicals it is one everywhere. This argument includes real and nonarchimedean local fields, split quadratic algebras and hyperbolic rank one; rank zero is trivial.

The standard Kudla splitting on an arbitrary acting space is, by its defining doubling construction, the restriction to the first summand of the doubled splitting. The real statement is explicitly in Theorem A.5.8; for the nonarchimedean field case it is the first-factor clause of Theorem A.5.10, and in the split case it is the doubled restriction used in Lemma 4.5.4. The actual kernel restricts to the original action tensored with the identity, by the orthogonal product assertion of Proposition 4.1.14; the extra external character is also the identity on the second component. Equality on the doubled group therefore implies equality on each original factor, by evaluating against any nonzero vector in the unused factor. The two factor actions commute, and so their product gives the desired pair comparison.

We describe the adelic assembly explicitly. Enlarge a finite set of places to contain the ramification of the two characters, the nonintegral entries of the chosen frames and their inverses, the different, the denominators of the fixed Weil data, and the places above two. At every remaining finite place, the scaled vacuum is the integral-lattice vacuum, its polarized representative is a scalar times the characteristic function of the integral position lattice, and both have norm one. Choose the local Heisenberg identification to send the first to that characteristic function with scalar exactly one. At the finitely many remaining places choose the local unitary identification used in the preceding comparison.

The coherent vacuum and its phase-space measure are products. Fubini therefore gives

$$\left\langle \bigotimes_v F_v, \bigotimes_v G_v \right\rangle = \prod_v \langle F_v, G_v \rangle$$

when all but finitely many factors are the normalized vacua. Such tensors have dense span in the coherent Hilbert completion: the reproducing integral of Lemma 4.1.10 reduces this to approximating its L^1 coefficient by compactly supported functions; compact finite-adelic pieces admit finite locally constant partitions, and real compact pieces admit ordinary smooth approximations. Each approximant is a finite sum of products of local coefficients and hence of local vacuum translates. The polarized $L^2(\mathbb{A}_F^d)$ has the same elementary-tensor density, by the same product-measure approximation. Thus the tensor product of the chosen local unitary maps extends to a unitary map of these two Hilbert completions.

For an adelic group element, at almost every finite place its matrix and inverse are integral and its two character components are unramified. The actual operator fixes the local vacuum exactly: the integral kernel changes variables through the integral unitary matrix, which preserves its bicharacter, lattice, measure and vacuum, and its multiplier and external characters are one. The compared classical operator is the same local operator and fixes the same characteristic function. Therefore both operators on an elementary tensor are tensor products with only finitely many nonvacuum factors. The local equalities already proved give equality on every elementary tensor, with no infinite product of phases. Unitarity extends that equality to the completions. On the smooth domains the map and its inverse are the continuous maps established above, on each finite fixed space; the inductive-limit conclusion gives the adelic Schwartz–Bruhat statement.

The theta distribution and zero ranks. The theta functional fixed locally in the statement is bounded, rational-Heisenberg invariant and normalized by (WD9). The estimates above make its restriction a continuous distribution T on the polarized Schwartz–Bruhat space $\mathcal{S}(X)$, $X = \mathbb{A}_F^d$. The rational position and momentum lattices are $L = F^d$. For $b \in L$ let M_b be

multiplication by $\psi_F(b\cdot x)$. Rational-Heisenberg invariance gives

$$T((\psi_F(b\cdot x)-1)\varphi)=0,\qquad T(\varphi(\cdot+a))=T(\varphi)\quad (a,b\in L).$$

The annihilator of L in X is L itself. Here is the place compatibility explicitly, in one coordinate. For an adele z annihilating F, choose $q\in F$ with $z_f-q\in\widehat{\mathcal{O}}_F$, by clearing its finitely many denominators and the Chinese remainder theorem. Subtract q. For every b in the inverse different $\mathcal{D}_F^{-1}$ the finite character of bz_f is now one, so the archimedean trace pairing makes $\mathrm{Tr}_{F_\infty/\mathbb{R}}(bz_\infty)$ integral. Trace duality of $\mathcal{D}_F^{-1}$ and $\mathcal{O}_F$ implies $z_\infty\in\mathcal{O}_F$. Subtract this algebraic integer as well. Now $z_\infty=0$, and the annihilator condition says $\psi_{F,f}(bz_f)=1$ for all $b\in F$. The diagonal F is dense in $\mathbb{A}_{F,f}$ by the Chinese remainder theorem. Continuity and nondegeneracy of the local trace-character pairings therefore force $z_f=0$. Reversing the two rational subtractions proves $z\in F$. This argument coordinatewise proves $L^\perp=L$ and fixes the global, rather than a merely local, annihilator.

If $x\notin L$, choose $b\in L$ with $\psi_F(b\cdot x)\neq 1$. On a sufficiently small neighborhood this modulation difference is a nonzero smooth function with smooth reciprocal. Dividing a compactly supported test function by it and using the first invariance identity shows that T vanishes there. A finite partition on compact supports gives $\operatorname{supp}T\subset L$.

We eliminate derivative distributions explicitly. Isolate $\lambda\in L$ in a product neighborhood meeting L only there. For a compact test φ in this neighborhood with $\varphi(\lambda)=0$, shrink its finite neighborhood until φ is one smooth archimedean function times the indicator of that neighborhood near λ; the difference is supported away from λ and is killed by the support assertion. Choose $D=[F:\mathbb{Q}]d$ rational momentum vectors $b_1,\dots,b_D$ whose real trace-linear forms ℓ_i span the archimedean cotangent space, using a rational basis and nondegeneracy of the trace pairing. Shrink the finite neighborhood into all their character kernels. Since $\psi_F(b_i\lambda)=1$ globally, on this neighborhood

$$\psi_F(b_i\cdot x)=\exp(2\pi i\,\ell_i(x_\infty-\lambda_\infty)).$$

Thus the finite phases have canceled the archimedean phases at λ. Put $u_i=\ell_i(x_\infty-\lambda_\infty)$ and shrink so that $|u_i|<1/4$. Hadamard's explicit formula $f(u)-f(0)=\sum_i u_i\int_0^1(\partial_i f)(tu)\,dt$ and the smooth extension

$$\frac{u_i}{e^{2\pi i u_i}-1}\Big|_{u_i=0}=\frac{1}{2\pi i}$$

give compactly supported smooth, finite-locally-constant functions G_i such that $\varphi-\sum_i(M_{b_i}-1)G_i$ vanishes near λ. One inserts archimedean cutoffs equal to one on a smaller neighborhood; all introduced differences are supported away from λ. The support assertion and modulation invariance therefore give $T(\varphi)=0$. It follows that T locally is a_λ times evaluation at λ. This proves absence of all archimedean derivatives of a point mass; finite germs are already constant on a sufficiently small compact open set, so they introduce no further terms.

Rational translations now imply $a_\lambda=a_0$ for every $\lambda\in L$. Hence on compactly supported smooth tests, $T(\varphi)=a_0\sum_{\lambda\in L}\varphi(\lambda)$. This extends to all Schwartz–Bruhat functions with the required continuity. On a fixed compact finite-adelic support, the possible rational coordinates lie in a fixed fractional-ideal lattice in F_∞^d (contained in one after bounding the finitely many denominators). Its point count in a ball of radius R is $O((1+R)^D)$. Therefore a Schwartz bound of order greater than D makes the sum absolutely convergent and bounds it by a fixed Schwartz seminorm on that finite level. Ordinary archimedean cutoffs converge in every Schwartz seminorm, proving the claimed extension and equality. The scalar a_0 is nonzero: normalization makes the original theta functional nonzero, and smooth vectors are dense in the coherent

norm by the reproducing integral and the smooth Gaussian vacuum translates. Thus the fixed functional is carried to a nonzero multiple of the rational Dirac comb.

If one rank is zero, Hom has rank zero and its Heisenberg and Schwartz models are $\mathbb{C}$. The actual action is exactly the remaining external determinant character. The defining zero-partner convention of the classical splitting retains that character: the rank-zero space character is the base change and descends on the norm-one determinant to the original character (or its displayed source inverse). If both ranks are zero all operators are the identity. This proves the whole assertion. □

Lemma 4.1.3. *Let A be a finite conjugate-disjoint set of finite places of E, and let n be a finite index set, including the empty set. Write $Q_A = \prod_{w\in A} E_w^n$, $L_A = \prod_{w\in A} \mathcal{O}_{E_w}^n$, with Haar measure giving L_A volume one. Functions depending only on the components at A are identified with functions on Q_A. Put*

$$\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t), \quad b_A(y,z) = \psi_0(\bar{z}^{\mathsf{T}}y/2), \quad a_A(y) = \psi_0(c_0\bar{y}^{\mathsf{T}}y/2)$$

for finite vectors supported on $A \cup \bar{A}$. Define, locally in this statement,

$$(R_A(y)\Psi)(x) = \psi_0(\bar{y}^{\mathsf{T}}(2x+y)_A/2)\Psi(x+y_A).$$

For the same prescribed Weil data,

$$(\jmath\alpha\bar{\alpha})\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1}\widehat{\mathcal{O}}_E.$$

Consequently $\widehat{\mathcal{O}}_E^n$ is its own annihilator under $(u,v) \mapsto \psi_0(\bar{u}^{\mathsf{T}}v)$, with integral-lattice Haar mass one. Moreover:

1. *$\mathcal{T}_A = C_c^\infty(Q_A)$. The operator $\Psi \otimes_A f$ of (WR6) is independent of the chosen linear section. For $\Psi, \Psi' \in \mathcal{T}_A$ and $f, f' \in \mathcal{M}_s$,*

$$\langle \Psi \otimes_A f, \Psi' \otimes_A f' \rangle = \left(\int_{Q_A} \overline{\Psi(x)}\Psi'(x)\,dx \right) \langle P^A f, P^A f' \rangle.$$

   In particular $\mathbf{1}_A \otimes_A f = P^A f$.

2. *For y supported on $A \cup \bar{A}$,*

$$\rho(\mathrm{fv}(y))(\Psi \otimes_A f) = (R_A(y)\Psi) \otimes_A f.$$

   The corresponding (WD8) formula is obtained by replacing $R_A(y)$ with

$$a_A(y)R_A(y)\Psi(x) = \psi_0(\bar{y}^{\mathsf{T}}(x + c_0 y)_A)\Psi(x + y_A).$$

   It has exactly the (WD8) phase, independently of the real signs.

3. *The map $\Psi \mapsto \Psi \otimes_A v^0$ is injective, takes $\mathbf{1}_A$ to v^0, and its image is exactly the span of $\rho(\mathrm{fv}(y))v^0$ with y supported on $A \cup \bar{A}$. It is the fixed vacuum-coefficient transform: at a point whose finite scaled coordinate on these places is z, its selected finite coefficient is*

$$\mathcal{I}_A(\Psi)(z) = \langle R_A(z)\mathbf{1}_A, \Psi \rangle_{L^2(Q_A)}.$$

   The other finite and real coordinates retain their original vacuum factors. This formula concerns coherent coefficients, not theta-orbit evaluation.

4. *If $\mathfrak{m}$ is supported on A and Φ is any function on $(\mathcal{O}_E/\mathfrak{m})^n$, its extension $\Phi^\natural$ by zero from integral position coordinates belongs to $\mathcal{T}_A$, and its (WR6) datum is the image under this same fixed transform. In particular class indicators are transported without any further phase or scalar choice. For every polynomial Fock vector v_P, the actual (WR6) datum $v_P \otimes (\Psi \otimes_A v^0)$ has coherent coefficient equal to the product of v_P in normalized real coordinates, $\mathcal{I}_A(\Psi)$ in the selected finite coordinates and the original finite vacuum elsewhere. Thus finite residue data and polynomial Fock test vectors are transported in the same realization. The selected finite correspondence and its inverse are continuous for the usual Schwartz–Bruhat inductive limit topology.*

Proof. **The actual Weil idele gives the inverse different.** Write $\mathcal{O}_f = \widehat{\mathcal{O}}_E$, $\delta = \mathrm{cyc}'_{q_1}(\zeta)$ and $\gamma = \jmath\alpha\bar{\alpha}$. By Lemma 3.2.8(1)–(2), every prime dividing δ lies above a rational prime $\ell \mid q_1$, every such ℓ divides q_0, and $\mathfrak{D}_{E/K} = \delta\mathcal{O}_E$. Put $v_2(q_0) = h \geq 4$. The parameter congruence gives $d+1 = (q_0/2)(2a+1)$ for an integer a, so $v_2(d+1) = h-1$ and $(d+1)/4$ is divisible by 2. Every odd $\ell \mid q_1$ also divides $(d+1)/4$. Since

$$c_0 + \bar{c}_0 = 1, \qquad c_0\bar{c}_0 = (d+1)/4,$$

exactly one of $c_0, \bar{c}_0$ belongs to each prime of E above $\ell \mid q_1$.

For each finite place w, let $e_w = v_w(\delta)$, with additive valuation normalized by $v_w(\varpi_w) = 1$. The relative different is conjugation invariant. Indeed its inverse is the trace-dual $\{t : \mathrm{Tr}_{E/K}(t\mathcal{O}_E) \subseteq \mathcal{O}_K\}$; conjugating this condition gives the same condition, since conjugation preserves both integer rings and commutes with trace. Thus $v_w(\bar{\delta}) = e_w$. The literal (WD2) definition gives

$$v_w(\alpha) = -e_w \mathbf{1}_{c_0 \in w}, \qquad v_w(\bar{\alpha}) = -e_w \mathbf{1}_{\bar{c}_0 \in w}.$$

If $e_w > 0$, exactly one indicator is one. If $e_w = 0$, both displayed valuations are zero regardless of the indicators. Hence at every finite place $v_w(\alpha\bar{\alpha}) = -e_w$, proving

$$(\alpha\bar{\alpha})\mathcal{O}_f = \mathfrak{D}_{E/K}^{-1}\mathcal{O}_f. \tag{PT1}$$

Here is the absolute trace normalization as well. Write $\tau = \sqrt{-d} = 2c_0 - 1$. The integer ring of K is $\mathbb{Z}[c_0]$, with basis $1, c_0$. Its trace-dual basis is $-\bar{c}_0/\tau, 1/\tau$: their traces against $1, c_0$ are respectively $(1,0)$ and $(0,1)$, since $c_0\bar{c}_0 = (d+1)/4$ and $\bar{\tau} = -\tau$. Therefore its codifferent is $\tau^{-1}\mathcal{O}_K$. The tower identity for trace duals can be seen directly here. Take the $\mathcal{O}_K$-basis $1, \zeta, \ldots, \zeta^{[E:K]-1}$, provided by Lemma 3.2.8(2), and its relative trace-dual basis v_i. If u_i denotes the original relative basis and k_j, l_j the displayed basis and dual basis of K, then

$$\mathrm{Tr}_{E/\mathbb{Q}}(u_i k_j v_a l_b) = \mathrm{Tr}_{K/\mathbb{Q}}(k_j l_b \,\mathrm{Tr}_{E/K}(u_i v_a)) = \delta_{ia}\delta_{jb}.$$

Thus the absolute codifferent is $\mathfrak{D}_{E/K}^{-1}\tau^{-1}\mathcal{O}_E$. As $\jmath = -\tau^{-1}$, (PT1) proves the stated equality for γ.

The local character $t \mapsto \psi_{E,w}(t)$ is the rational local character of conductor $\mathbb{Z}_\ell$ composed with $\mathrm{Tr}_{E_w/\mathbb{Q}_\ell}$. This follows from (WD4), by choosing the rational approximation with integral components at all other finite places. Its annihilator on $\mathcal{O}_{E_w}$ is consequently the local codifferent: the condition is exactly $\mathrm{Tr}(t\mathcal{O}_{E_w}) \subseteq \mathbb{Z}_\ell$. Multiplication by γ_w, whose ideal is that codifferent, makes the annihilator precisely $\mathcal{O}_{E_w}$. Taking products proves self-annihilation, including every rank. Conjugation preserves $\psi_{E,f}$, by its trace definition, and $\bar{\gamma} = -\gamma$; hence $\psi_0(\bar{t}) = \psi_0(t)^{-1}$. These facts discharge every character hypothesis of Lemma 4.1.4.

The selected coefficient and its phases. Put $Q = Q_A$, $L_Q = L_A$ and $B = \mathbf{1}_{L_Q}$. Read a selected scaled vector as (q, p), with $q = z_A$ and $p = \bar{z}_{\bar{A}}$ on A. Denote by χ_A the restriction of ψ_0 to adeles supported at A. Both copies of L_Q have volume one and are character duals. The literal formulas become

$$R(q,p)\Psi(x) = \chi_A(p \cdot (x + q/2))\Psi(x+q), \qquad b((q,p),(a,b')) = \chi_A((b' \cdot q - p \cdot a)/2). \quad \text{(PT2)}$$

Multiplying two such operators leaves the scalar $\chi_A((b' \cdot q - p \cdot a)/2)$ after the exponent of $R(q+a, p+b')$ is subtracted. Thus $R(y)R(z) = b(y,z)R(y+z)$. Translation and the modulus-one phase preserve the L^2 norm, so these operators are unitary.

On the selected integral box the vacuum phase is

$$e_A(q,p) = \chi_A(-p \cdot q/2)\mathbf{1}_{L_Q}(q)\mathbf{1}_{L_Q}(p). \quad \text{(PT3)}$$

Indeed its original phase is $\chi_A((c_0 - \bar{c}_0)p \cdot q/2)$, differing by the integral argument $c_0 p \cdot q$. It is real and even. Direct integration now gives

$$\langle R(q,p)B, B\rangle = \chi_A(-p \cdot q/2)\mathbf{1}_{L_Q}(q) \int_{L_Q} \chi_A(-p \cdot x)\, dx = e_A(q,p). \quad \text{(PT4)}$$

When q is not integral the supports are disjoint; otherwise the last integral is precisely the character-orthogonality indicator in p.

For a test function Ψ the fixed coefficient transform is therefore

$$J\Psi(q,p) = \langle R(q,p)B, \Psi\rangle = \chi_A(-p \cdot q/2) \int B(x+q)\chi_A(-p \cdot x)\Psi(x)\, dx. \quad \text{(PT5)}$$

In particular $JB = e_A$. Moving $R(y)$ across this inner product, using unitarity and the product just calculated, gives

$$J(R(y)\Psi)(z) = b(y,z)J\Psi(z-y). \quad \text{(PT6)}$$

The scalar is $b(-y,z)$ in the first argument, hence is conjugated; this gives exactly $b(y,z)$. Thus (PT6) is the actual coherent action, with the prescribed orientation.

Finite Fourier expansion spans all test functions. All $R(y)B$ are locally constant and compactly supported. Conversely any test function is a finite sum of coset indicators of one common compact open additive subgroup $U \subseteq L_Q$. For $\mathbf{1}_U$, choose a product of local integral ideals $I \subseteq U$. The pairing of L_Q/I with $I^\perp/L_Q$ induced by χ_A is perfect: the two groups have the same order, and an element pairing trivially with all of L_Q belongs to L_Q by the conductor computation. Its subcharacters trivial on U are exactly the character group of L_Q/U. Hence

$$\mathbf{1}_U(x) = \frac{B(x)}{[L_Q : U]} \sum_{\eta \in (L_Q/U)^\vee} \eta(x). \quad \text{(PT7)}$$

Each summand is a pure modulation $R(0,p)B$; pure position translations give all its cosets. This proves $\mathcal{T}_A = C_c^\infty(Q)$.

The full mixed Gram calculation. For fixed q put $F_q(x) = B(x+q)\Psi(x)$ and $G_q(x) = B(x+q)\Psi'(x)$. Choose a common compact open subgroup U of constancy contained in L_Q and a compact open group H containing their supports. Their Fourier transforms are

$$\widehat{F}_q(p) = \mathrm{vol}(U) \sum_{x \in H/U} F_q(x)\chi_A(-p \cdot x)\mathbf{1}_{U^\perp}(p).$$

The corresponding formula holds for G_q. Character orthogonality on $U^\perp$ gives

$$\int_{U^\perp} \chi_A(p\cdot(x-y))\,dp = \mathrm{vol}(U)^{-1}\mathbf{1}_{x-y\in U}.$$

The reciprocal volume follows from the same finite perfect pairing as in (PT7), with the integral lattices of mass one. Multiplying the finite sums, every unequal coset pair vanishes and therefore

$$\int_p \overline{\widehat{F}_q(p)}\widehat{G}_q(p)\,dp = \mathrm{vol}(U)\sum_{x\in H/U}\overline{F_q(x)}G_q(x) = \int_x \overline{\Psi(x)}\Psi'(x)B(x+q)\,dx.$$

In (PT5) the common scalar has modulus one, so it cancels in the mixed product. Integration in q, followed by $\int_q B(x+q)dq = 1$, proves

$$\langle J\Psi, J\Psi'\rangle_{L^2(W_A)} = \langle \Psi, \Psi'\rangle_{L^2(Q)}. \tag{PT8}$$

All expressions here are finite sums or integrals of test functions; their supports make Fubini legitimate.

For arbitrary actual f, Lemma 4.1.4 now gives $P^A f = e_A f_{\mathrm{off}}$ pointwise, with $f_{\mathrm{off}}(w) = f(0,w)$ in the complementary carrier. In particular every selected Heisenberg operator leaves this complementary factor unchanged. If $\Psi = \sum_y c_y R(y)B$, then (PT6) gives, pointwise,

$$\sum_y c_y\rho(\mathrm{fv}(y))P^A f = J\Psi\, f_{\mathrm{off}}. \tag{PT9}$$

This proves independence of the chosen coefficient section. Every model vector is bounded by its L^1 norm, directly from its reproducing equation and $|e_n|\le 1$, and hence belongs to L^2. Product measures, (PT8) and $\|e_A\|_2^2 = 1$ give

$$\langle J\Psi f_{\mathrm{off}}, J\Psi' f'_{\mathrm{off}}\rangle = \langle\Psi,\Psi'\rangle\langle f_{\mathrm{off}}, f'_{\mathrm{off}}\rangle = \langle\Psi,\Psi'\rangle\langle P^A f, P^A f'\rangle.$$

This is clause (1), for all model vectors, without a density assertion. Equation (PT6) proves its raw intertwining in clause (2). For the corrected action, $a_A(y) = \chi_A((c_0-\bar c_0)p\cdot q/2)$, so multiplication by the phase of (PT2) gives

$$\chi_A\big(p\cdot x + (1+c_0-\bar c_0)p\cdot q/2\big) = \chi_A\big(p\cdot(x+c_0q)\big).$$

Here $1+c_0-\bar c_0 = 2c_0$. This is exactly the displayed (WD8) phase and proves the second identity of clause (2).

The vacuum itself factors with selected factor e_A. Its complementary squared norm is one: the finite vacuum has modulus one on its integral box of mass one, and each normalized real factor has squared integral $\pi^{-1}\int_{\mathbb{C}} e^{-|u|^2}du = 1$. The vacuum is in the reproducing carrier. On the finite factor its reproducing integral is supported on $x\in\mathcal{O}_f^n$; for integral x,t, subtracting its constant phase leaves c_0t^*t and a conjugate pair with opposite signs divided by two. The first is integral, and the character of the latter equals that of $\bar c_0 x^*t$, also integral, because the discarded half conjugate sum is fixed by conjugation and has character one. The integral is thus the vacuum value times the unit mass of the box. On a real coordinate the Gaussian calculation in the next paragraph with degree zero is the reproducing identity. Product measures give it globally. Consequently (PT9), (PT8) and (PT7) prove the injectivity, exact range, vacuum value and coefficient formula of clause (3).

Residue functions, real polynomial insertion and topology. An integral residue function extended by zero is a finite sum of compact open coset indicators, so (PT7) and section independence identify its actual datum without an additional scalar or phase.

Here is the real polynomial calculation for the same datum. In one normalized coordinate u of sign $+1$,

$$\beta^{\infty}(y,u)e^{\infty}(u-y) = e^{-|u|^2/2-|y|^2/2+\bar{u}y}.$$

For sign -1 the final term is $\bar{y}u$. With π^{-1} Lebesgue measure, polar integration gives

$$\pi^{-1}\int_{\mathbb{C}} \bar{y}^a y^b e^{-|y|^2}\,dy = \begin{cases} a!, & a=b, \\ 0, & a\neq b. \end{cases}$$

Expanding the exponential in the first kernel therefore reproduces $\bar{u}^a e^{-|u|^2/2}$ from $\bar{y}^a e^{-|y|^2/2}$; the second reproduces $u^a e^{-|u|^2/2}$. The power series and all its polynomial multiples are dominated on bounded u sets by a constant times $e^{-|y|^2/2}$, so integration term by term is valid. Tensoring over coordinates and adding monomials proves the Fock reproducing identity for every polynomial vector, in every sign pattern. Gaussian decay gives its L^1 integrability. Finite and real factors and measures separate in the actual (WR6) integral. It therefore changes only the real vacuum factor in (PT9), to the stated polynomial Fock vector. Absolute Bochner convergence follows from its integrable coefficient and the constant L^1 norm of each vacuum translate. This proves the product assertion of clause (4).

For completeness, (PT5) takes a fixed Schwartz–Bruhat stage into a fixed such stage: the q support is contained in the original support plus L_Q, and a common constancy subgroup $U \subseteq L_Q$ bounds the p support by $U^{\perp}$. Inversion follows from finite Fourier inversion in the very same finite quotient:

$$\Psi(x) = \int_{q,p} \chi_A(p\cdot(x+q/2))B(x+q)J\Psi(q,p)\,dq\,dp.$$

Indeed the inner integral is $B(x+q)\Psi(x)$ by Fourier inversion, and the q integral of that box is one. On any fixed stage of the image, bounded q support bounds the recovered support by its sum with L_Q, and bounded p support gives a common annihilator subgroup of constancy in x. Both maps thus preserve suitable finite-dimensional stages in both directions and are continuous for the inductive limit topologies. Empty A and rank zero give the one-dimensional constant factor in every formula. □

Lemma 4.1.4. *Put $\chi(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$. Assume that $\chi(\bar{t}) = \chi(t)^{-1}$, that χ is trivial on $\widehat{\mathcal{O}}_E$, and that $\widehat{\mathcal{O}}_E$ is its own annihilator for the pairing $(u,v)\mapsto\chi(uv)$. Let A be any finite conjugate-disjoint set of finite places and let n be any finite index set, including the empty set. Split the actual adelic coordinates into the finite coordinates at $A\cup\bar{A}$, scaled by α, and their complement:*

$$F(z,w) = f(\mathrm{fv}(z)+w), \qquad z\in W_A = \prod_{v\in A\cup\bar{A}} E_v^n.$$

Here w has zero components at those finite places. In these coordinates $\mu_n = dz\,d\mu_{\mathrm{off}}(w)$, with each selected integral lattice of mass one. Put

$$L = \prod_{v\in A\cup\bar{A}} \mathcal{O}_{E_v}^n, \qquad e_A(z) = \chi(c_0\bar{z}^{\mathsf{T}}z/2)\mathbf{1}_L(z).$$

Let $\mathcal{M}_{\mathrm{off}}$ be the reproducing carrier defined by the same (WD7) equation on the complementary coordinates, with the complementary vacuum, bicharacter and product measure. Then for every $f\in\mathcal{M}_s$ the function $f_{\mathrm{off}}(w) = F(0,w)$ belongs to $\mathcal{M}_{\mathrm{off}}$, satisfies $\|f_{\mathrm{off}}\|_1 \le \|f\|_1$, and

$$(P^A f)(\mathrm{fv}(z)+w) = e_A(z)f_{\mathrm{off}}(w).$$

The range of P^A consists exactly of the functions $e_A(z)h(w)$ with $h\in\mathcal{M}_{\mathrm{off}}$. In particular the complementary factor is unchanged by any operator acting solely on the selected coordinates.

Proof. All coordinates in the proof are the actual scaled finite coordinates in the statement. Let $Q = \prod_{v\in A} E_v^n$ and $L_Q = \prod_{v\in A} \mathcal{O}_{E_v}^n$. Write a selected vector as (q,p), where $q = z_A$ and $p = \bar{z}_{\bar{A}}$ is read on A. Let χ_A denote χ on adeles supported at A, with dot products summed over the coordinates. The assumed conjugation identity gives

$$b((q,p),(a,b')) = \chi_A((b'\cdot q - p\cdot a)/2).$$

The two copies of L_Q are exact character duals, both with mass one. In particular

$$\int_{L_Q} \chi_A(r\cdot t)\,dr = \mathbf{1}_{L_Q}(t). \tag{PN1}$$

A nontrivial character has integral zero by translation; the assumed annihilator identifies its trivial case. Since $c_0 + \bar{c}_0 = 1$ and c_0 is integral, on L the finite vacuum phase satisfies

$$e_A(q,p) = \chi_A(-p\cdot q/2)\mathbf{1}_{L_Q}(q)\mathbf{1}_{L_Q}(p). \tag{PN2}$$

Indeed the unrestricted phase is $\chi_A((c_0-\bar{c}_0)p\cdot q/2)$, and its difference from $-p\cdot q/2$ is $c_0 p\cdot q$, an integral argument. Thus e_A is real and even, has modulus one on L, and has both squared norm and L^1 norm equal to one.

We calculate the compact mean on every selected coherent kernel, without assuming a tensor decomposition of model vectors. Put $t = (a,b')$, $z = (q,p)$ and $l = (h,k) \in L$. The selected kernel is

$$K_t(z) = b(t,z)e_A(z-t).$$

The mean of the corrected operators on this kernel is

$$\int_L e_A(l)b(l,z)b(t,z-l)e_A(z-l-t)\,dl. \tag{PN3}$$

If $q-a$ or $p-b'$ is not integral, this integral is zero, by its support. Suppose both are integral. Using (PN2), the exponent of its integrand under χ_A, written without the character, is

$$\begin{aligned} &-\frac{k\cdot h}{2} + \frac{p\cdot h - k\cdot q}{2} + \frac{(p-k)\cdot a - b'\cdot(q-h)}{2} - \frac{(p-k-b')\cdot(q-h-a)}{2} \\ &\quad = (p-k)\cdot(h+a) - \frac{p\cdot q + b'\cdot a}{2}. \end{aligned} \tag{PN4}$$

Integrating first in h gives $\mathbf{1}_{L_Q}(p)$ by (PN1), since k is integral. If this is nonzero, b' is integral too. Integrating next in k gives $\mathbf{1}_{L_Q}(a)\chi_A(p\cdot a)$. If nonzero, q is integral as well, and $\chi_A(p\cdot a) = 1$. The remaining scalar is $\chi_A(-p\cdot q/2)\chi_A(-b'\cdot a/2) = e_A(z)e_A(t)$. If the original support condition fails, both $e_A(t)e_A(z)$ and (PN3) vanish. We have proved in all cases that

$$\int_L e_A(l)b(l,z)K_t(z-l)\,dl = e_A(z)e_A(t). \tag{PN5}$$

The Heisenberg operator preserves the reproducing equation. To see this directly, substitute $y = l + t$ in that equation for $\rho(l)f$; the requisite phase equality is

$$\beta_s(l,t)\beta_s(l+t,x) = \beta_s(l,x)\beta_s(t,x-l).$$

It follows by expanding the alternating bilinear exponent. Translation invariance of the measure and the modulus-one factor make each operator an L^1 isometry. Hence their compact mean P^A

is L^1 bounded with norm at most one and preserves the reproducing equation. Fubini here is justified by $\|f\|_1 < \infty$, the compact mean having total variation one, and $|e_n| \le 1$.

Apply this compact mean to the (WD7) reproducing equation for f, whose kernel and measure split as selected and complementary factors. Formula (PN5) gives

$$(P^A f)(\mathrm{fv}(z) + w) = e_A(z) \int f(\mathrm{fv}(t) + u) e_A(t) \beta_{\mathrm{off}}(u, w) e_{\mathrm{off}}(w - u)\, dt\, d\mu_{\mathrm{off}}(u).$$

The integral is exactly $F(0, w)$ by the same original reproducing equation evaluated at selected coordinate zero: $b(t, 0) = 1$ and $e_A(-t) = e_A(t)$. This proves the asserted pointwise factorization. Its product form and $\|e_A\|_1 = 1$ give $\|f_{\mathrm{off}}\|_1 = \|P^A f\|_1 \le \|f\|_1$. Since $P^A f$ satisfies (WD7), substituting its product form into that equation and evaluating at selected coordinate zero gives the complementary reproducing equation; the selected integral is $\int e_A(t)^2 dt = 1$. Thus $f_{\mathrm{off}} \in \mathcal{M}_{\mathrm{off}}$.

Finally, (PN5) with $t = 0$ says precisely that e_A satisfies the selected reproducing equation. If $h \in \mathcal{M}_{\mathrm{off}}$, tensoring its equation with this selected equation proves that $e_A h$ is in $\mathcal{M}_s$; Tonelli gives its L^1 integrability. The pointwise factorization already proved fixes this function, because $e_A(0) = 1$. Hence these and only these functions form the range. Empty A or empty n makes Q a point; every displayed integral and character is then the unit point-mass identity. □

Lemma 4.1.5. *Let S, T have signs s, s' on finite index sets n, n', respectively. Let α, β have weights γ, γ', i.e. $\alpha(z) = \prod_{v \in \Phi} z_v^{\gamma(v)}$ for $z \in \mathrm{U}(1)(\mathbb{A})$ with finite components 1, and likewise β with γ'. For $x \in G_\infty(s)$, $y \in G_\infty(s')$, $k \in G_f(n)$, $k' \in G_f(n')$, $v \in \mathcal{F}_{S,T}$ and $f \in \mathcal{M}_{S,T}$,*

$$\omega_{\alpha,\beta}((x, k), (y, k'))(v \otimes f) = (\Omega_{\gamma,\gamma'}(x, y)v) \otimes (\omega_{\alpha,\beta}((1, k), (1, k'))f).$$

If moreover x_v, y_v are block diagonal for the partitions of the indices by sign at every $v \in \Phi$, x_v^σ, y_v^σ denoting the blocks of sign $\sigma \in \{\pm 1\}$, then for every polynomial P in the variables $X_{v,a,b}$ ($v \in \Phi$, $a \in n'$, $b \in n$)

$$\Omega_{\gamma,\gamma'}(x, y)v_P = \prod_{v \in \Phi} \prod_{\sigma = \pm 1} \det(y_v^\sigma)^{\gamma(v) - \#\{i \in n : s_i = -\sigma\}} \det(x_v^\sigma)^{\#\{i \in n' : s'_i = -\sigma\} - \gamma'(v)}\, v_{P'},$$

where P' is P after the substitution $X_{v,a,b} \mapsto \sum_{i \in n', j \in n} c_{v,a,b,i,j} X_{v,i,j}$ with $c_{v,a,b,i,j} = (y_v)_{ia}(x_v^{-1})_{bj}$ if $s'(a) = s(b)$ and $c_{v,a,b,i,j} = (y_v^{-1})_{ai}(x_v)_{jb}$ otherwise.

Proof. All operators below are the specified integral kernels on the fixed model. We first prove the real and finite separation, including its extension to arbitrary data.

In normalized coordinates the phase space, vacuum, measure and bicharacter are products of their real and finite factors. Let $\mathcal{M}_f$ be the range of the finite reproducing projection and write e_f for its vacuum. The bounded reproducing projection Π of Lemma 4.1.10 factors as $\Pi_\infty \otimes \Pi_f$ by Fubini, where Π_∞ has range $\mathcal{F}_{S,T}$. Thus finite sums of pointwise products $u \boxtimes w$, with $u \in \mathcal{F}_{S,T}$ and $w \in \mathcal{M}_f$, are dense in the actual adelic model: apply Π to finite sums of L^1 simple tensors, which are dense in product L^1.

For the real model the rank-one identity is

$$\int v(z) \rho_\infty(z) u\, d\nu(z) = u(0) v. \tag{1}$$

This follows from the full identity of Lemma 4.1.10 by taking the full vectors $v \boxtimes e_f$ and $u \boxtimes e_f$ and integrating first in the finite variable. The finite vacuum identity gives $\int e_f(t) \rho_f^{\mathrm{raw}}(t) e_f\, d\mu_f(t) =$

e_f, so cancellation of its nonzero finite factor leaves (1). Equivalently it is the signed Gaussian twisted convolution calculation. Consequently the literal (WR6) operation satisfies

$$v \otimes (u \boxtimes w) = u(0)\, v \boxtimes w. \tag{2}$$

Its norm is at most $\|v\|_1 \|f\|_1$ on an arbitrary f, directly from its Bochner definition and Heisenberg isometry.

Put $U_\infty = \bigoplus_v y_v \otimes (x_v^{-1})^{\mathsf{T}}$ and $U_f = k' \otimes (k^{-1})^{\mathsf{T}}$. The multiplier in (WR3) factors exactly as $\Delta_\infty(U_\infty)\Delta_f(U_f)$: its negative real minor uses only the infinite part, and its lattice index uses only the finite part. The product kernel and Fubini therefore give

$$\omega_{\alpha,\beta}((x,k),(y,k'))(u \boxtimes w) = c_\infty c_f \, (\omega_\varepsilon^\infty(U_\infty)u) \boxtimes (K_f(U_f)w), \tag{3}$$

where $c_\infty = \prod_v \det(y_v)^{\gamma(v)} \det(x_v)^{-\gamma'(v)}$, $c_f = \alpha(\det k')\beta(\det k)^{-1}$, and $K_f(U_f)$ is precisely the finite factor of the (WR3) kernel, not an independently chosen action. These integrals are absolutely convergent in L^1: the norm of an integral kernel with matrix U and multiplier Δ is at most $|\Delta|\|e\|_1$, by translation invariance and integration of $|e(\,\cdot - Uz)|$. The same estimate applies to each factor. Its range is in the reproducing model because it is a Bochner superposition of vacuum translates; for the real factor use the same projection calculation. Thus every operator used in (2)–(3) is bounded on the stated spaces.

For $f = u \boxtimes w$, applying (3) to (2) makes the left side of the desired datum identity

$$c_\infty c_f \, u(0)(\omega_\varepsilon^\infty(U_\infty)v) \boxtimes (K_f(U_f)w).$$

On the right side, the finite operator sends f to $c_f \, u \boxtimes K_f(U_f)w$, since its real kernel is the identity reproducing projection. Formula (2) with first argument $\Omega_{\gamma,\gamma'}(x,y)v = c_\infty \omega_\varepsilon^\infty(U_\infty)v$ gives exactly the same expression. The dense product span and the displayed bounds extend the identity to every f and every integrable Fock vector v. This proves the first formula without assuming a splitting comparison or a group law.

We next compute the compact polynomial action. Suppose x_v, y_v preserve their sign blocks. Each block is an ordinary unitary matrix, since its form is the scalar $+1$ or -1 times the Euclidean form. Consequently U_∞ is Euclidean unitary and commutes with the Hom sign matrix. It preserves the real vacuum, measure and bicharacter. Change variables $z \mapsto U_\infty z$ in the defining real kernel; the reproducing equation yields

$$\omega_\varepsilon^\infty(U_\infty)c(u) = \overline{\det(U_\infty)_-}\; c(U_\infty^{-1}u). \tag{4}$$

For precision, the integral after this change of variables is the reproducing projection of $c \circ U_\infty^{-1}$. This function already belongs to the real model because its reproducing equation is carried to that of c by the same measure- and vacuum-preserving change of variables. Thus (4) holds pointwise.

At a place v, the negative Hom coordinates are the blocks with row sign σ and column sign $-\sigma$. On this block U_∞ is $y_v^\sigma \otimes ((x_v^{-\sigma})^{-1})^{\mathsf{T}}$. The determinant of a Kronecker product is the first determinant to the dimension of the second factor times the second determinant to the dimension of the first factor. Since the determinants of all these compact blocks have modulus one,

$$\overline{\det(U_\infty)_-} = \prod_v \prod_{\sigma=\pm 1} \det(y_v^\sigma)^{-\#\{i : s_i = -\sigma\}} \det(x_v^\sigma)^{\#\{i : s_i' = -\sigma\}}. \tag{5}$$

Empty blocks have determinant one, so (5) also covers a missing sign and zero multiplicities. Multiplying by c_∞, and writing the determinant of x_v or y_v as the product of its sign-block determinants, gives exactly the exponents in the stated formula.

It remains to compute the polynomial in (4). In matrix coordinates $U_\infty^{-1}Z = y_v^{-1}Zx_v$. On positive Hom coordinates the Fock variable is $\bar{Z}_{a,b}$. Conjugating this inverse transform and using unitarity gives its coefficient $(y_v)_{ia}(x_v^{-1})_{bj}$ on $\bar{Z}_{i,j}$. On negative Hom coordinates the Fock variable is $Z_{a,b}$ and the inverse transform has coefficient $(y_v^{-1})_{ai}(x_v)_{jb}$ on $Z_{i,j}$. The sign-block hypothesis ensures that a nonzero coefficient has the same Hom sign, so these are exactly coefficients of the corresponding Fock variables and there are no cross terms. Substitution into P, followed by (4)–(5), proves the second formula for every polynomial, including a constant and polynomials using full-height minors. □

Lemma 4.1.6. *Let θ_V and θ_{V_0} be theta functionals in the sense of (WD9), fixed locally for this statement. For every splitting (k, l, i) of a space with a point V over V_0 along a finite index set p, there is a unique bounded complex-linear map*

$$Z : \mathcal{M}_s \longrightarrow \mathcal{M}_{s_0}$$

such that, for every $f \in \mathcal{M}_s$ and every $y \in \mathbb{A}_E^{n_0}$,

$$\theta_{V_0}(\rho(y)Zf) = \int_{\mathbb{A}_E^p/E^p} \theta_V(\rho(i_{\mathbb{A}}y + k_{\mathbb{A}}b)f)\, db.$$

The measure is probability Haar and the formula holds for every vector of the prescribed L^1 Banach carrier, not only its Schwartz test vectors. For each fixed splitting a finite constant C exists with $\|Zf\|_1 \le C\|f\|_1$. The displayed characterization includes the empty removed or residual index sets and makes no prior choice of a zero-term operator.

Proof. We use the actual bounded canonical action and its Heisenberg covariance from Proposition 4.1.14, the Banach reproducing identities and injectivity of theta functions from Lemma 4.1.10, and uniqueness of bounded rationally invariant functionals from Lemma 4.1.7. The argument constructs the map on the whole L^1 carrier by an absolutely convergent vector integral; no extension from point evaluation on a dense Schwartz space is asserted.

A fixed actual coordinate transport. Put $H = \mathbb{H}_p$ with precisely the point of (WR1). Write x_H for that point, and regard the columns of $[l\ k\ i]$ in the stated order. The matrix

$$m = x_V^{-1}[l\ k\ i]\,\mathrm{diag}(x_H, x_{V_0})$$

is an invertible adelic matrix. Invertibility follows from the direct-sum equations in (MX1): pairing a relation among its columns successively with k, l, i makes every coefficient zero. Its Gram matrix for ε_s is $\varepsilon_{H\oplus s_0}$, by those same equations and the point identities. At infinity Sylvester's law of inertia therefore identifies the positive and negative index counts on its two sides. Choose once a permutation matching these signs; at finite places any such permutation is integral and preserves the identity form. If P is its coordinate permutation matrix, then $g = mP^{-1}$ lies in $\mathrm{G}(s)$.

Coefficient pullback under P is the literal isometric identification of the product canonical model of $H \oplus V_0$ with the reordered model: its bicharacter, vacuum and measure are products. This follows either by substitution in (WD7), or by the coordinate assertion of Lemma 4.1.7. Compose it with the actual operator $\omega_s(g)$ from Proposition 4.1.14. Denote this fixed bounded invertible map by

$$A : \mathcal{M}_{H\oplus V_0} \longrightarrow \mathcal{M}_s.$$

There is no freely chosen Heisenberg intertwiner here. The permutation and canonical operator give, by the covariance in that proposition,

$$A\rho_{H\oplus 0}(z)A^{-1} = \rho_V(mz). \tag{1}$$

In particular $m(0,y) = i_{\mathbb{A}} y$. It also sends the hyperbolic coordinate corresponding to the rational vector in the k-copy to $k_{\mathbb{A}} b$ with exactly the point coordinates prescribed by (MX1). We do not require any rational-unitary invariance of theta under A.

A bounded compact mean. For $F \in \mathcal{M}_{H \oplus V_0}$ define

$$\lambda(F) = \int_{\mathbb{A}_E^p / E^p} \theta_V(\rho_V(k_{\mathbb{A}} b) A F)\, db. \tag{2}$$

The integrand descends to the quotient. Indeed adding $\gamma \in E^p$ introduces the rational vector $k_{\mathbb{A}}\gamma$, which fixes θ_V; its bicharacter with $k_{\mathbb{A}} b$ is one because $k^* J_V k = 0$. Strong continuity of ρ makes the descended integrand continuous. Its absolute value is at most $\|\theta_V\| \|A\| \|F\|_1$, since Heisenberg operators are L^1 isometries. Thus (2) is a bounded complex-linear functional, with

$$\|\lambda\| \le \|\theta_V\| \|A\|. \tag{3}$$

For $w \in \mathcal{M}_H$, the bounded functional $v \mapsto \lambda(w \boxtimes v)$ on $\mathcal{M}_{s_0}$ is rationally Heisenberg invariant. To check this directly, take a rational vector γ of V_0. Equation (1) changes its action into $\rho_V(i_{\mathbb{A}}\gamma)$. This vector is rational in the point of V, because i is a rational matrix. It is orthogonal to $k_{\mathbb{A}} b$. Consequently it commutes with that translation with no scalar, and rational invariance of θ_V removes it in every integrand in (2). This proves the asserted invariance, not merely invariance of its average.

By Lemma 4.1.7, this functional is a scalar multiple of the fixed θ_{V_0}. Choose once $u \in \mathcal{M}_{s_0}$ with $\theta_{V_0}(u) = 1$; such a vector exists since the theta functional is nonzero by its normalization. Then

$$c(w) = \lambda(w \boxtimes u), \qquad \lambda(w \boxtimes v) = c(w)\theta_{V_0}(v). \tag{4}$$

The function c is a bounded linear functional and $\|c\| \le \|\lambda\| \|u\|_1$, since the L^1 norm of a coefficient product is the product norm. The scalar in (4) therefore has one fixed bounded realization; it is not chosen separately for each test vector.

A uniformly bounded vector kernel. Let e_H, e_0 be the actual canonical vacua. In product phase coordinates $z = (z_H, z_0)$ put

$$K(z) = c(\rho_H(z_H) e_H)\, \rho_0(z_0) e_0.$$

This is a continuous $\mathcal{M}_{s_0}$-valued function, by strong continuity and boundedness of c, and

$$\|K(z)\|_1 \le B := \|c\| \|e_H\|_1 \|e_0\|_1 \qquad \text{for every } z. \tag{5}$$

The coherent models are separable Banach spaces: they are closed subspaces of the L^1 spaces of the finite-dimensional adelic phase groups, which are second countable with sigma-finite Haar measure. Thus this continuous vector kernel is strongly measurable. For every $F \in \mathcal{M}_{H \oplus V_0}$ the integral

$$TF = \int F(z) K(z)\, d\mu_{H \oplus 0}(z) \tag{6}$$

is an absolutely convergent Bochner integral in $\mathcal{M}_{s_0}$. In particular T is bounded and linear, with $\|TF\|_1 \le B \|F\|_1$. This is the point where the full L^1 domain is established.

For any residual phase vector y, equation (4) gives on a translated product vacuum

$$\theta_{V_0}(\rho_0(y) K(z)) = \lambda(\rho_{H \oplus 0}(0,y) \rho_{H \oplus 0}(z) e_{H \oplus 0}). \tag{7}$$

There is no missing cross factor: the hyperbolic and residual factors are orthogonal, while the possible residual projective factor occurs identically on both sides. Integrate (7) against $F(z)$.

Every exchange is bounded by a constant times $|F(z)|$, by (3), (5) and the isometry of $\rho_0(y)$. The Bochner reproducing identity of Lemma 4.1.10 then yields

$$\theta_{V_0}(\rho_0(y)TF) = \lambda(\rho_{H\oplus 0}(0,y)F). \tag{8}$$

The zero term and uniqueness. Define $Z = TA^{-1}$. It is bounded with $\|Zf\|_1 \leq B\|A^{-1}\|\|f\|_1$. For $F = A^{-1}f$, substitute (2) into (8), use (1), and use orthogonality of $i_{\mathbb{A}}y$ and $k_{\mathbb{A}}b$. The result is precisely

$$\theta_{V_0}(\rho_0(y)Zf) = \int_{\mathbb{A}_E^p/E^p} \theta_V(\rho_V(i_{\mathbb{A}}y + k_{\mathbb{A}}b)f)\, db.$$

The integrand is periodic for the same rational/isotropic reason as in (2); its continuity and boundedness prove its integrability on probability Haar. This checks every quantifier in the required formula for every f and y.

If two residual vectors have this theta function, their difference has identically zero theta function and is zero by Lemma 4.1.10. This proves pointwise uniqueness and hence uniqueness of the bounded linear map. It also shows that the auxiliary sign permutation and the auxiliary vector u do not affect the final map. If p is empty, the compact mean is the mean over the one-point quotient and the same argument applies. If the residual rank is zero, its model is the one-dimensional constant carrier and its normalized theta functional is nonzero; the construction and bounds remain valid. Thus no exceptional rank or domain is omitted. □

Lemma 4.1.7. *For a framed hermitian space V, the vector space of bounded rational-Heisenberg-invariant linear functionals on its coherent model is one-dimensional. Two normalized theta functionals therefore differ by a scalar of absolute value one.*

For framed spaces V_1, V_2, the pointwise coefficient product

$$(f_1 \boxtimes f_2)(z_1, z_2) = f_1(z_1)f_2(z_2)$$

belongs to the canonical model of $V_1 \oplus V_2$. There is one scalar z_{12} of absolute value one, depending only on the three fixed theta functionals, such that

$$\theta_{V_1\oplus V_2}(f_1 \boxtimes f_2) = z_{12}\theta_{V_1}(f_1)\theta_{V_2}(f_2)$$

for every pair of coherent vectors.

If a specified continuous additive coordinate isomorphism between two such phase spaces preserves their bicharacters, vacua, vacuum measures and rational lattices, its literal coefficient pullback intertwines their models and their Heisenberg operators; the two normalized theta functionals under this pullback likewise differ by one scalar of absolute value one.

Proof. Fix V, write W for its adelic phase space and L_V for its rational vectors. We use the Banach, Bochner and rank-one identities of Lemma 4.1.10, and the existence of a nonzero normalized functional from Lemma 4.1.9.

The continuous characters

$$\chi_q(x) = \beta_s(x,q)^2, \qquad q \in L_V,$$

separate the points of W/L_V. Here are the arithmetic details of this assertion. Transporting by the fixed frame reduces the pairing to the trace pairing on E^n with an invertible rational matrix (the rational hermitian matrix times $\jmath$). The annihilator of E in $\mathbb{A}_E$ for the standard trace character is E. Indeed, subtract a rational element to make the finite component integral,

using additive approximation. Testing on $\mathcal{O}_E$ then places the infinite component in its trace-dual lattice, which is the embedded inverse different. Subtract that rational element also. The resulting adele has zero infinite part and its finite character pairs trivially with all of E; density of E in $\mathbb{A}_{E,f}$ and nondegeneracy of the local trace characters force its finite component to vanish. The converse is rational triviality of the global character. Apply this argument in each coordinate and use invertibility of the rational pairing matrix. Thus the common kernel of the χ_q is exactly L_V. Their span contains the constants and is closed under multiplication and complex conjugation. The Stone–Weierstrass theorem therefore makes this span uniformly dense in the continuous functions of the compact quotient.

Let R, S be bounded invariant functionals, without normalizing them. For coherent u, v define

$$\mathcal{B}_{R,S}(u,v) = \int_{W/L_V} \overline{R(\rho(x)u)}S(\rho(x)v)\,dx, \qquad c_{R,S} = \mathcal{B}_{R,S}(e,e).$$

The integral is continuous, periodic and bounded by $\|R\|\|S\|\|u\|_1\|v\|_1$; the projective phases cancel. Translating the quotient proves its Heisenberg invariance. The rank-one operator $\mathcal{P}u = u(0)e = \int e(z)\rho(z)u\,d\mu(z)$ is self-adjoint for this form, since e is real and even. Hence $\mathcal{B}_{R,S}(e,v) = c_{R,S}v(0)$. Bochner reproduction of u and invariance now give, with absolutely justified exchanges,

$$\mathcal{B}_{R,S}(u,v) = c_{R,S}\langle u, v\rangle. \tag{1}$$

In particular no positivity or irreducibility assumption on this cross form is used.

Suppose $c_{R,S} = 0$. The continuous periodic function $H_{u,v}(x) = \overline{R(\rho(x)u)}S(\rho(x)v)$ has zero coefficient against every χ_q. Indeed $\rho(x)\rho(q) = \beta_s(x,q)^2\rho(q)\rho(x)$, rational invariance absorbs $\rho(q)$, and (1) gives

$$\int H_{u,v}(x)\chi_q(x)\,dx = \mathcal{B}_{R,S}(u, \rho(q)v) = 0.$$

Uniform density of the character span forces $H_{u,v} = 0$. At $x = 0$ this says $\overline{R(u)}S(v) = 0$ for all u, v, so at least one of R, S is zero.

Choose a normalized theta functional θ. For any invariant bounded S put $S_0 = S - c_{\theta,S}\theta$. Since $c_{\theta,\theta} = 1$, we have $c_{\theta,S_0} = 0$. The preceding conclusion and $\theta \neq 0$ imply $S_0 = 0$. This proves one-dimensionality. If S is also normalized, its vacuum mean is $|c_{\theta,S}|^2$, so its scalar has absolute value one.

For the orthogonal sum, its vacuum, measure, bicharacter and reproducing projection are literal products. Thus $\boxtimes$ is bounded bilinear, with $\|f_1 \boxtimes f_2\|_1 = \|f_1\|_1\|f_2\|_1$, and its product span is dense in the model: apply the bounded product reproducing projection to the dense simple-tensor span of product L^1. For fixed f_2, the functional $f_1 \mapsto \theta_{V_1\oplus V_2}(f_1 \boxtimes f_2)$ is bounded and rationally invariant. Uniqueness makes it $r(f_2)\theta_{V_1}(f_1)$. Choose a coherent f_1 with $\theta_{V_1}(f_1) = 1$ to see that r is itself bounded and invariant. Uniqueness again gives $r = z_{12}\theta_{V_2}$. This scalar is nonzero, since otherwise the nonzero sum functional would vanish on its dense product span. Apply the equality to independent Heisenberg translates of the two vacua and integrate its absolute square on the product rational quotient. All three vacuum means are one and the quotient probability measure is the product measure; hence $|z_{12}|^2 = 1$.

Finally let the stated coordinate bijection be ϕ from the new phase space to the old one. Changing variables in the reproducing integral proves that $f \mapsto f \circ \phi$ is an isometric model isomorphism. Direct substitution gives $(\rho_{\mathrm{old}}(z)f) \circ \phi = \rho_{\mathrm{new}}(\phi^{-1}z)(f \circ \phi)$. The map on compact rational quotients preserves probability Haar measure, so pulling back a normalized theta functional retains both rational invariance and vacuum mean one. The first assertion gives its scalar of absolute value one. These statements concern the specified pullback and require each listed coordinate identity; they introduce no freely chosen intertwiner. □

Lemma 4.1.8. *Let V, V_1 be spaces with points with the same signs s on n. There are $\gamma \in \mathrm{GL}_n(E)$ with $\gamma^* J_{V_1}\gamma = J_V$ and $k \in \mathrm{G}(s)$ with $x_V = \gamma^{-1}x_{V_1}k$; and for every such pair (γ, k), $\Gamma_V = k^{-1}\Gamma_{V_1}k$.*

Proof. Write $E = F(\sqrt{\delta})$ with $\delta \in F$ totally negative, so that for every place v of F the norms $z\bar{z}$ of $z \in E \otimes_F F_v$ are the elements $a^2 - \delta b^2$, $a, b \in F_v$.

Existence. The determinants $d = \det J_V$ and $d_1 = \det J_{V_1}$ lie in $F^\times$ (J_V, J_{V_1} are hermitian and invertible); put $y = d/d_1$. At a finite place v of F, the v-components of $x_V^* J_V x_V = \varepsilon_s = 1$ and of the same identity for V_1 give $d\,N_v(\det x_{V,v}) = 1 = d_1 N_v(\det x_{V_1,v})$, $N_v(z) = z\bar{z}$; so $y = N_v(\det x_{V_1,v}/\det x_{V,v}) = a^2 - \delta b^2$ with $a, b \in F_v$. At a real place σ of F, i.e. $\sigma = v|_F$ for $v \in \Phi$, the component $x_{V,v} \in \mathrm{GL}_n(\mathbb{C})$ satisfies $x_{V,v}^* J_{V,v} x_{V,v} = \mathrm{diag}(s)$, so $\sigma(d)|\det x_{V,v}|^2 = (-1)^{\#\{s_i=-1\}}$, and likewise for V_1; hence $\sigma(y) > 0$, which is $a^2 - \sigma(\delta)b^2$ with $b = 0$. By the Hasse norm theorem A.2.11, $y = a^2 - \delta b^2 = z\bar{z}$ with $z = a + b\sqrt{\delta} \in E^\times$, i.e. $\det J_V = z\bar{z}\det J_{V_1}$. For an embedding $\tau\colon E \to \mathbb{C}$, τ or $\bar{\tau}$ is the embedding chosen at some $v \in \Phi$, and $\tau(J_V)$ is $J_{V,v}$ or its complex conjugate, congruent to $\mathrm{diag}(s)$ by the last identity; the same holds for J_{V_1}, so $\tau(J_V)$ and $\tau(J_{V_1})$ have the same signature. Landherr's theorem A.2.12 (with $G = J_{V_1}$, $G' = J_V$) gives $\gamma \in \mathrm{GL}_n(E)$ with $\gamma^* J_{V_1}\gamma = J_V$. Put $k = x_{V_1}^{-1}\gamma x_V \in \mathrm{GL}_n(\mathbb{A}_E)$. Since $(x_{V_1}^{-1})^* \varepsilon_s x_{V_1}^{-1} = J_{V_1}$,

$$k^*\varepsilon_s k = x_V^*\gamma^* J_{V_1}\gamma x_V = x_V^* J_V x_V = \varepsilon_s,$$

so $k \in \mathrm{G}(s)$, and $x_V = \gamma^{-1}x_{V_1}k$.

Rational points. Let (γ, k) be such a pair. An element $g \in \mathrm{GL}_n(E)$ preserves $J_V = \gamma^* J_{V_1}\gamma$ if and only if $\gamma g \gamma^{-1}$ preserves J_{V_1}, and then $x_V^{-1} g x_V = k^{-1}x_{V_1}^{-1}(\gamma g\gamma^{-1})x_{V_1}k$. So $\Gamma_V = k^{-1}\Gamma_{V_1}k$. □

Lemma 4.1.9. *Let P be a parameter pair and $s \geq 0$.*

(a) Every hermitian space over E_s with a frame has a theta functional.

(b) Let $s = (s_1, \ldots, s_n)$ be signs and let θ_+, θ_- be theta functionals of the standard lines of sign $+1$ and -1. Then the standard space of signs s has a theta functional θ with $\theta(\rho(x)v^0) = \prod_{i=1}^n \theta_{s_i}(\rho(x_i)v^0)$ for all $x \in \mathbb{A}_E^n$.

Proof. Write $W = W_n$, $e = e_n$ and $L_V = V(E)$ in the fixed frame. We use the reproducing Banach model, strongly continuous Heisenberg operators and vacuum identities of Lemma 4.1.10; its first assertion is independent of the existence of a theta functional.

We first record a uniform sampling bound for this particular Gaussian-lattice vacuum:

$$C_V := \sup_{t \in W} \sum_{\gamma \in L_V} |e(t + \gamma)| < \infty. \tag{2}$$

Here is a direct verification. Choose a compact set of representatives covering W/L_V. On it the finite components of t lie in a fixed compact set, so the condition $(t + \gamma)_f \in \alpha\widehat{\mathcal{O}}_E^n$ restricts γ_f to a fixed compact set. For this restricted set of lattice vectors, the number with archimedean norm at most R is at most $C_1(1 + R)^{2fn}$, with a constant C_1 independent of t in the chosen compact set. To see this, choose a relatively compact open neighbourhood U with the translates $\gamma + U$ disjoint, using discreteness of L_V. The translates corresponding to those vectors fit into the product of an archimedean ball of radius $R + C_2$ and a fixed compact finite set. Comparison of Haar measures gives the asserted polynomial bound. The real factor of $|e(t + \gamma)|$ is at most $C_3 \exp(-c_1\|\gamma_\infty\|^2)$ on that compact set, with $c_1 > 0$. Summing over integral annuli proves (2). The sum is unchanged under $t \mapsto t + \gamma_0$, so this proves its bound on all of W. The same estimate proves uniform convergence of the vacuum lattice sum on compact sets.

Reproduction, Tonelli and (2) imply, for every model vector,

$$\sum_{\gamma\in L_V} |f(t+\gamma)| \le \int |f(z)| \sum_{\gamma\in L_V} |e(t+\gamma-z)|\, d\mu_n(z) \le C_V \|f\|_1. \tag{3}$$

For $a \in W$ put $u_a = \rho(-a)e$ and define

$$R_a(f) = \sum_{\gamma\in L_V} \langle u_a, \rho(\gamma) f\rangle.$$

Indeed, inner-product invariance and vacuum evaluation identify the absolute value of each summand with $|f(-a-\gamma)|$. Thus (3) proves absolute convergence and the bound $|R_a(f)| \le C_V\|f\|_1$. Since $\beta_s(\gamma,\delta) = 1$ for rational vectors, the projective law and reindexing give $R_a(\rho(\delta)f) = R_a(f)$ for every $\delta \in L_V$.

At least one of these bounded invariant functionals is nonzero. For,

$$R_a(u_a) = \sum_{\gamma\in L_V} \beta_s(a,\gamma)^2 \langle e, \rho(\gamma)e\rangle. \tag{4}$$

The coefficients are absolutely summable by (3) and their coefficient at 0 is $\langle e, e\rangle = 1$. For every nonzero $\gamma \in L_V$, $a \mapsto \beta_s(a,\gamma)^2$ is a nontrivial continuous character of W/L_V. To verify nontriviality, transport the hermitian form back to its rational nondegenerate matrix; at a nonzero coordinate of γ the corresponding linear form is nonzero, and nondegeneracy of the local additive character provides an adele on which its character is not one. The character is periodic since the form of two rational vectors is rational. The invariant probability mean of a nontrivial character is zero (translate by a point where its value differs from one). Termwise integration of (4) therefore gives mean 1. In particular some R_a is nonzero.

Fix such a bounded invariant functional R. The function $x \mapsto |R(\rho(x)e)|^2$ is continuous and L_V-periodic by the projective law. Its mean m_R is finite on the compact quotient. It is positive: if its mean were zero, continuity and full support of Haar measure would make it identically zero; applying R to the Bochner reproducing formula would then give $R(f) = 0$ for every f. Thus $\theta = m_R^{-1/2} R$ is a bounded rationally invariant functional with the exact vacuum mean one. This proves (a). It uses no action of a rational unitary group.

We prove (b) on the same product model, without making a tensor completion choice. For the prescribed normalized line functionals put $g_i(z_i) = \theta_{s_i}(\rho(z_i)e_1)$. Each $|g_i|$ is a continuous periodic function on the compact line quotient, hence bounded. The formula

$$\Theta(f) = \int_{W_n} f(z) \prod_{i=1}^{n} g_i(z_i)\, d\mu_n(z)$$

defines a bounded linear functional on $\mathcal{M}_s$. For a product $f(z) = \prod_i f_i(z_i)$ with $f_i \in \mathcal{M}_{s_i}$, Fubini and reproduction give $\Theta(f) = \prod_i \theta_{s_i}(f_i)$. Such products have dense linear span in $\mathcal{M}_s$: simple product functions are dense in the product L^1 space; the bounded reproducing projection factors as the product of the line projections, so their projected simple functions are precisely these products and are dense in its range. The product Heisenberg law now proves rational invariance first on that dense span, and then everywhere by boundedness.

The vacuum and its translates factor as products. Hence

$$\Theta(\rho(x)e_n) = \prod_i g_i(x_i).$$

For the standard signed space its rational lattice and compact quotient are the products of the line lattices and quotients; probability Haar measure is their product. Fubini and the prescribed

normalization of each line give $\int |\Theta(\rho(x)e_n)|^2 dx = \prod_i \int |g_i(x_i)|^2 dx_i = 1$. Thus Θ is the required theta functional with the exact tensor phase. □

Lemma 4.1.10. *For every sign vector s, the following assertions hold.*

1. *The space $\mathcal{M}_s$ is a closed reproducing subspace of $L^1(\mu_n)$, with its specified continuous representatives. The operators $\rho(z)$ are strongly continuous L^1-isometries, preserve its inner product, and satisfy*

$$\rho(z)\rho(w) = \beta_s(z,w)\rho(z+w), \qquad \langle \rho(z)e_n, f\rangle = f(z).$$

The vacuum is real and even, $e_n(0) = \langle e_n, e_n\rangle = 1$. For $f, u \in \mathcal{M}_s$ the following are Bochner integrals in $\mathcal{M}_s$:

$$f = \int f(z)\rho(z)e_n \, d\mu_n(z), \qquad \int f(z)\rho(z)u \, d\mu_n(z) = u(0)f.$$

2. *If V is a framed hermitian space with this model, $L_V = V(E)$ its rational vectors in the frame, and θ any theta functional of V, then, for all $u, v \in \mathcal{M}_s$,*

$$\int_{W_n/L_V} \overline{\theta(\rho(x)u)}\theta(\rho(x)v)\, dx = \langle u, v\rangle.$$

Here the integrand is a well-defined continuous function on the compact quotient, and dx is its invariant probability measure. In particular the full adelic theta function determines the vector.

Proof. We use the arithmetic normalization in the preliminary conclusions of Lemma 4.1.3. Its Weil data (WD1)–(WD8) are exactly those of the present statement; the finite index set has rank n, including $n = 0$, and its conjugate-disjoint set can be taken to be empty. Hence, for these same $\jmath, \alpha, \psi_{E,f}$,

$$(\jmath\alpha\bar{\alpha})\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1}\widehat{\mathcal{O}}_E.$$

In particular, for $\chi(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}\, t)$, χ is trivial on $\widehat{\mathcal{O}}_E$, and the integral lattice $\widehat{\mathcal{O}}_E^n$ is its own annihilator under $(t,z) \mapsto \chi(t^*z)$, and its Haar measure is one. These are the normalization and character-orthogonality conclusions stated in that lemma, with the finite scaled coordinates and measure of (WD6), so no coordinate or measure is changed.

In the finite coordinates of (WD6) put $B = \widehat{\mathcal{O}}_E^n$ and $a(z) = \chi(c_0 z^* z/2)$. The finite vacuum is $a\,\mathbf{1}_B$ and B has measure one. We have $\chi(\bar{t}) = \chi(t)^{-1}$, and $\chi(c_0 z^* z) = 1$ on B; hence $a(z)$ is real, even and of modulus one on B. The finite twisted convolution identity is

$$\int e(z)\beta_s(z,y)\beta_s(z+y,x)e(x-y-z)\, dz = e(y)e(x). \tag{1}$$

Indeed its support requires $l = x - y \in B$. With $z \in B$, expansion of its exponent leaves, apart from its constant factor, the character

$$z \longmapsto \chi\big((y^* + \bar{c}_0 l^*)z\big).$$

Its integral is $\mathbf{1}_B(y)$, by the preceding annihilator calculation and $\bar{c}_0 l \in B$. If $x, y \in B$, the remaining constant is $a(x)a(y)$: subtracting the two exponents cancels a conjugate pair under $\chi(\bar{t}) = \chi(t)^{-1}$, using $c_0 + \bar{c}_0 = 1$. This proves (1), including the zero cases.

In normalized real coordinates the vacuum is $\exp(-|z|^2/2)$ and the measure is π^{-fn} times Lebesgue measure. Identity (1) follows coordinate by coordinate from

$$\pi^{-1}\int_{\mathbb{C}} \exp(-|z|^2 + bz + c\bar{z})\,dz = \exp(bc).$$

This formula follows by the two real Gaussian integrals, initially for real linear terms and then for complex terms by dominated entire continuation. Substitution of the defining exponential for β_s gives (1) for either sign. The product of the finite and real calculations proves (1) on W_n. They also give $e_n(0) = 1$, real evenness, $|e_n| \le 1$, integrability and $\int |e_n|^2 d\mu_n = 1$.

For clarity, all subsequent integrals use this fixed measure. The operator

$$(\Pi f)(x) = \int f(z)\beta_s(z,x)e_n(x-z)\,d\mu_n(z)$$

is bounded on L^1 with norm at most $\|e_n\|_1$. Its values are continuous, by dominated convergence and translation continuity of the vacuum. Identity (1) with $y = 0$, and Fubini, give $\Pi^2 = \Pi$. Its range is exactly the specified model; thus that model is Banach. Point evaluation has norm at most one, since $|e_n| \le 1$. In particular every model vector is bounded and belongs to L^2. The bicharacter identities give the displayed projective law. They also give $\Pi\rho(a) = \rho(a)\Pi$: after $z = a + t$ in its defining integral the multiplier is $\beta_s(a,t)\beta_s(a+t,x) = \beta_s(a,x)\beta_s(t,x-a)$. Thus $\rho(a)$ really preserves the stated reproducing range. Translation invariance of μ_n gives both norm and inner-product preservation. Strong continuity follows from L^1 translation continuity and dominated convergence for the multiplier $\beta_s(z,x)$.

The defining reproducing equation is also the first displayed Bochner identity. Indeed $z \mapsto f(z)\rho(z)e_n$ is strongly measurable with integrable norm $\|e_n\|_1|f(z)|$, and evaluation of its integral gives exactly Πf. In particular the linear span of the vacuum translates is dense in $\mathcal{M}_s$. Direct conjugation of the reproducing kernel gives $\langle \rho(z)e_n, f\rangle = f(z)$.

For the second Bochner identity, the integrand has integrable norm $\|u\|_1|f(z)|$. It suffices by bilinear continuity and the preceding density to take $f = \rho(a)e_n$ and $u = \rho(b)e_n$. At x its integral is

$$\int \beta_s(a,z)e_n(z-a)\beta_s(z,x)\beta_s(b,x-z)e_n(x-z-b)\,d\mu_n(z).$$

Put $z = a + t$. The bicharacter factors become $\beta_s(a,x)\beta_s(t,b)\beta_s(t+b,x-a)$. Identity (1), with $y = b$ and x there replaced by $x - a$, therefore evaluates this integral as $e_n(b)\beta_s(a,x)e_n(x-a) = u(0)f(x)$. This proves the claimed rank-one identity on all model vectors.

We now prove the mean identity without any unitary-group action. Put

$$\mathcal{B}_\theta(u,v) = \int_{W_n/L_V} \overline{\theta(\rho(x)u)}\theta(\rho(x)v)\,dx.$$

The rational lattice is isotropic for β_s, since its form is rational and ψ_E is trivial on E. Rational invariance and the projective law show that the product in this integral is periodic; the scalar factors have modulus one and cancel. Strong continuity gives continuity. The quotient is compact because L_V is an invertible adelic-linear image of E^n; the latter is discrete and cocompact by the additive adelic fundamental-domain construction. In particular the integral exists. More explicitly, $|\mathcal{B}_\theta(u,v)| \le \|\theta\|^2\|u\|_1\|v\|_1$. Translation of the invariant probability measure gives $\mathcal{B}_\theta(\rho(z)u, \rho(z)v) = \mathcal{B}_\theta(u,v)$.

Let $\mathcal{P}u = \int e_n(z)\rho(z)u\,d\mu_n(z) = u(0)e_n$. Since e_n is real and even and $\rho(z)^{-1} = \rho(-z)$, invariance and absolute Bochner bounds give $\mathcal{B}_\theta(\mathcal{P}u, v) = \mathcal{B}_\theta(u, \mathcal{P}v)$. Take $u = e_n$. The vacuum

normalization in (WD9) is $\mathcal{B}_\theta(e_n, e_n) = 1$, so $\mathcal{B}_\theta(e_n, v) = v(0)$. Apply the first Bochner identity to an arbitrary u and use invariance once more:

$$\mathcal{B}_\theta(u, v) = \int \overline{u(z)}\mathcal{B}_\theta(\rho(z)e_n, v)\, d\mu_n(z) = \int \overline{u(z)}(\rho(-z)v)(0)\, d\mu_n(z) = \int \overline{u(z)}v(z)\, d\mu_n(z).$$

All exchanges are justified by the displayed bounded-form estimate and $\|u\|_1 < \infty$. If the full theta function vanishes, its mean square is zero and hence the vector is zero. □

Lemma 4.1.11. *As in the parameter data, $\mu_{5040} \subset E$. Consider the following coordinate-kernel oscillator construction simultaneously for finite-dimensional framed hermitian spaces over E, including their negatives, orthogonal sums and residual Witt spaces. In a frame of signs s, its operator at $U \in \mathrm{U}(V)(\mathbb{A}_F)$, read in the given frame, is literally*

$$(\Omega_V(U)F_0)(z) = \Delta_V(U) \int F_0(y)\beta_s(Uy, z)e_n(z - Uy)\, d\mu_n(y).$$

Its finite multiplier is

$$\Delta_{V,f}(U) = \chi_f(\det t_U), \qquad \chi_f(t) = \upsilon_f(t)|t|_f^{-1/2}, \qquad t_U\widehat{\mathcal{O}}_E^n = \widehat{\mathcal{O}}_E^n \cap U_f^{-1}\widehat{\mathcal{O}}_E^n$$

in the finite coordinates $z_f = \alpha x$; its infinite multiplier is the product of the conjugate determinants of the negative principal blocks. Assume that these operators give genuine continuous unitary actions with exact Heisenberg covariance and preserve the polarized Schwartz domain of the coherent models. In rank zero use $\mathbb{C}$ with its trivial action and theta functional the identity. No rational-unitary theta invariance is assumed.

Let S, T be two of these framed spaces and $W = \mathrm{Hom}_E(S, T)$ with its original Hom form. Choose its theta functional θ_W as in (WD9) of Definition 3.2.12. Let $\Omega^{S,T}$ be the pair action obtained from Ω_W by the linear map $Z \mapsto r_{\mathrm{out}}Zr_{\mathrm{in}}^{-1}$ and the prescribed automorphic determinant characters. For every $r_{\mathrm{in}} \in \mathrm{U}(S)(F)$, $r_{\mathrm{out}} \in \mathrm{U}(T)(F)$, and every vector F_0 in the Schwartz domain,

$$\theta_W\Big(\Omega^{S,T}(r_{\mathrm{in}}, r_{\mathrm{out}})F_0\Big) = \theta_W(F_0).$$

The rational pair operator is exactly the canonical rational lift in the polarized Weil representation. In the theta-coefficient realization $c_{F_0}(Z) = \theta_W(\rho_W(Z)F_0)$, one has

$$c_{\Omega^{S,T}(r_{\mathrm{in}}, r_{\mathrm{out}})F_0}(Z) = c_{F_0}(r_{\mathrm{out}}^{-1}Zr_{\mathrm{in}}).$$

The conclusion is unchanged by the single nonzero normalization scalar of the theta functional. It neither identifies a theta-coefficient function with a raw coherent coefficient nor asserts that another oscillator construction has the displayed kernel.

Proof. A split polarized model with its vacuum fixed. We first treat a split hermitian space of rank $2d$. Write a rational hyperbolic basis as (λ, κ), with Gram matrix

$$H = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \qquad z = \lambda a + \kappa b.$$

All notation in the proof is local. Put $t_v = \mathrm{Im}\, \jmath_v > 0$. On $X = \mathbb{A}_E^d$ use the Haar measure whose infinite factor is $\prod_{v\in\Phi}(2t_v)^d\, dx_v$ and whose finite factor gives $B = \alpha\widehat{\mathcal{O}}_E^d$ mass one. It is self-dual for $(b, x) \mapsto \psi_E(\jmath b^{\mathsf{T}}x)$. At the finite places this follows from the inverse-different identity in

Lemma 4.1.3; at infinity it follows from the ordinary real Fourier transform with frequency scale $2t_v$ in each of the $2d$ real coordinates. On $L^2(X)$ put

$$(R(a,b)\varphi)(x) = \psi_E(\jmath\bar{b}^{\mathsf{T}}(x+a/2))\varphi(x+a), \quad v(x) = 2^{d\#\Phi/2}\exp\left(-2\pi\sum_v t_v|x_v|^2\right)\mathbf{1}_B(x_f). \quad (1)$$

The measure just specified gives $\|v\|_2 = 1$: at one complex coordinate

$$2(2t)\int_{\mathbb{C}} e^{-4\pi t|x|^2}\,dx = 1.$$

The identity $\psi_E(\bar{u}) = \psi_E(u)$ and $\bar{\jmath} = -\jmath$ show, by multiplying (1), that R has exactly the Heisenberg cocycle of the hyperbolic form.

We verify the vacuum, including its finite phase. By Lemma 3.2.7 choose $c \in \widehat{\mathcal{O}}_E^\times$ with $c\bar{c} = -1$, and put $\epsilon = c_0$. At finite places use

$$x_f = \begin{pmatrix} 1 & -c \\ \epsilon & c\bar{\epsilon} \end{pmatrix}, \quad x_f^{-1} = \begin{pmatrix} \bar{\epsilon} & 1 \\ \bar{c}\epsilon & -\bar{c} \end{pmatrix}.$$

Multiplication gives $x_f^* H x_f = 1$ and the displayed inverse, so both matrices are integral. At infinity use $x_\infty = 2^{-1/2}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, which carries H to $\mathrm{diag}(1,-1)$ and preserves the Euclidean norm. Take the product of these blocks, with a sign-preserving reindexing if necessary, and call it x_H.

In scaled finite coordinates put $\chi(u) = \psi_{E,f}(\jmath\alpha\bar{\alpha}u)$. The finite coefficient of $\mathbf{1}_{\widehat{\mathcal{O}}_E^d}$ for (1) is

$$\mathbf{1}_{\widehat{\mathcal{O}}_E^d}(a)\mathbf{1}_{\widehat{\mathcal{O}}_E^d}(b)\,\chi(\bar{b}^{\mathsf{T}}a/2). \quad (2)$$

Indeed the overlap of the two supports forces a integral, and integration of $x \mapsto \chi(\bar{b}^{\mathsf{T}}x)$ over the integral lattice forces b integral. In the same coordinates the prescribed coherent vacuum, read through x_f, is

$$\mathbf{1}_{\widehat{\mathcal{O}}_E^d}(a)\mathbf{1}_{\widehat{\mathcal{O}}_E^d}(b)\,\chi(c_0(\bar{a}^{\mathsf{T}}b + \bar{b}^{\mathsf{T}}a)/2).$$

For integral a, b, set $u = \bar{b}^{\mathsf{T}}a$. The quotient of its phase by that in (2) is

$$\chi((c_0\bar{u} - \bar{c}_0 u)/2) = \chi(-\bar{c}_0 u) = 1.$$

Here $\chi(\bar{z}) = \chi(z)^{-1}$, $c_0 + \bar{c}_0 = 1$, and integrality of $\bar{c}_0 u$ justify all equalities, also at places over 2. At infinity, completing the square with $y = x + a/2$ gives

$$\langle v, R(a,b)v\rangle = \exp(-\pi t_v(|a|^2 + |b|^2))$$

per place. More explicitly the remaining integral is $(4t_v)^d \int_{\mathbb{C}^d} e^{-4\pi t_v|y|^2 + 4\pi i t_v \operatorname{Im}(\bar{b}^{\mathsf{T}}y)}\,dy = e^{-\pi t_v|b|^2}$. Thus $\langle v, R(a,b)v\rangle = e_{2d}(x_H^{-1}(a,b))$ exactly.

These coefficient identities give a unitary Heisenberg identification of the coherent Hilbert completion with $L^2(X)$: send $\rho(x_H^{-1}(a,b))e_{2d}$ to $R(a,b)v$. Their Gram matrices agree by the cocycle and the coefficient identity. The coherent span is dense by the reproducing formula of Lemma 4.1.10. The target span is dense as well: a vector orthogonal to all $R(a,b)v$ has, for each a, Fourier transform of $\overline{v(x+a)}\,f(x)$ equal to zero. Fourier inversion makes this product zero almost everywhere; the translates of the nonzero Gaussian times $\mathbf{1}_B$ cover X, so $f = 0$.

This identification and its inverse identify the Schwartz–Bruhat spaces, not just their Hilbert completions. Here are the requisite estimates. The coefficient $\langle R(a,b)v, \varphi\rangle$ is Schwartz when φ

is Schwartz. At infinity, differentiating in a, b inserts derivatives or polynomial multiples of the two Schwartz functions in its defining integral. To gain any power of b, integrate by parts in x; to gain any power of a, use $a_j = (x_j + a_j) - x_j$ and expand that power between the two factors. Cauchy–Schwarz bounds the resulting integrals by finitely many Schwartz seminorms of φ and v. Thus every polynomially weighted derivative of this coefficient is bounded. At a finite place, the support of a lies in the difference of two fixed compact sets, and the support of b lies in the annihilator of a common compact-open period subgroup; both are compact. Local constancy is immediate from the same integral. Conversely the resolution

$$\varphi = \int \langle R(a,b)v, \varphi\rangle \, R(a,b)v \, da \, db \tag{3}$$

follows by integrating b first: Fourier inversion forces the two position variables to agree, and the remaining a integral is $\int |v(x+a)|^2 \, da = 1$. For Schwartz φ the preceding estimates make (3) convergent in every real Schwartz seminorm and at a fixed finite level. They also show continuity of both transforms at that level. Finite restricted tensors, and then their inductive limit, give the adelic assertion. In particular the inverse sends Schwartz functions to the actual L^1 coherent carrier.

The literal kernel is bounded on the coherent L^1 space: Tonelli and translation invariance give $\|\Omega(g)F\|_1 \le |\Delta(g)| \, \|e_n\|_1 \|F\|_1$. Thus the following compositions with bounded theta functionals are legitimate.

For a split space V with its given point x_V, put $q = x_V^{-1}(\lambda, \kappa)x_H$. It is unitary in the prescribed coordinates. Compose the preceding identification with $\Omega_V(q)^{-1}$ and denote it by J. Exact covariance gives

$$J\rho_V(\lambda a + \kappa b)J^{-1} = R(a,b), \qquad Ju = v, \quad u = \Omega_V(q)e_{2d}. \tag{4}$$

Here, as throughout, rational vectors in this display are read through the given point. The assumed Schwartz preservation of $\Omega_V(q)^{\pm 1}$ makes (4) a fixed polarized Schwartz identification. Its maps are continuous for that topology. To justify this point for the operator being transported, fix finite position and modulation subgroups fixing an input finite level. Covariance sends them to compact-open Heisenberg subgroups containing smaller position and modulation subgroups; thus the image lies in one fixed finite level. Such a fixed space is finite-dimensional, since modulation invariance bounds the position support by the compact annihilator and position invariance imposes a fixed period. On its tensor product with the real Schwartz space, the graph of the operator is closed: convergence in Schwartz implies convergence in L^2, where the operator is unitary. The closed graph theorem gives continuity on this Fréchet space. Taking the inductive limit over finite levels proves the assertion.

The theta distribution and the exact slice formulas. We record the distribution argument in sufficient detail to fix its scope. Let T be a continuous functional on $\mathcal{S}(\mathbb{A}_E^d)$ invariant under the rational translations and the modulations $x \mapsto \psi_E(\jmath \bar{b}^{\mathsf{T}} x)$, $b \in E^d$. The annihilator of E^d is E^d. Indeed Theorem A.5.2(a), together with local self-duality, says this for the standard trace pairing; multiplication by the rational $\jmath$ and conjugation preserve E^d. If $x \notin E^d$, choose b for which this modulation is not 1 at x. On a sufficiently small product neighborhood it is nonzero after subtracting 1. Divide a compactly supported test function there by this smooth nonvanishing function. Modulation invariance shows that T vanishes on that neighborhood. Thus its support is contained in the discrete set E^d.

Fix $\xi \in E^d$ and isolate it in a product neighborhood. At a fixed finite level a test function is locally a single smooth function of the infinite variables. Choose $[E : \mathbb{Q}]d$ rational modulation parameters whose real trace-linear forms are a basis of the real cotangent space; a number-field basis and its trace-dual basis provide them. Shrink the finite neighborhood into their

finitely many character kernels. Since each full character has value 1 at ξ, the corresponding characters in this neighborhood have the form $\exp(2\pi i u_j)$, where u_j are real linear coordinates centered at ξ. Hadamard's identity $f(u)-f(0)=\sum_j u_j \int_0^1 (\partial_j f)(tu)\,dt$ and the smooth extension of $u_j/(\exp(2\pi i u_j)-1)$ at zero show that every test vanishing at ξ is a sum of modulation differences near ξ. Cutoffs leave a remainder supported away from ξ, already annihilated by T. Consequently T is a scalar multiple of evaluation at ξ there: no derivatives of point masses remain. Finite-variable germs are locally constant, so introduce no additional distributions. A compact test meets only finitely many rational points. A partition of unity therefore gives $T(\varphi)=\sum_\xi a_\xi \varphi(\xi)$ on compact tests, and rational translation invariance makes all a_ξ the same scalar a. On any fixed finite compact support, the rational points lie in a fractional ideal lattice in the infinite vector space. That lattice has at most $C(1+R)^{[E:\mathbb{Q}]d}$ points in a ball of radius R, by covering with translates of a bounded fundamental parallelepiped. Hence $\sum_\xi |\varphi(\xi)|$ is bounded by a Schwartz seminorm with decay exponent greater than $[E:\mathbb{Q}]d$. The comb is continuous at every finite level. Infinite compact cutoffs converge in these seminorms, so

$$T(\varphi) = a \sum_{\xi \in E^d} \varphi(\xi) \tag{5}$$

on the full Schwartz–Bruhat space. A nonzero T has $a \neq 0$.

Apply this to $T = \theta_V J^{-1}$. It is continuous by (3), rational-Heisenberg invariant by (4), and nonzero: Schwartz vacuum translates are dense in the coherent L^1 carrier, while the normalization of theta makes θ_V nonzero. Thus (5) holds with $a \neq 0$. Define, locally,

$$\operatorname{ev} F = \int_{\mathbb{A}_E^d/E^d} \theta_V(\rho_V(\kappa b)F)\,db, \qquad S_x F = \operatorname{ev}(\rho_V(\lambda x)F).$$

The quotient measure is probability measure. Inserting (5) and (1), orthogonality of characters gives

$$\operatorname{ev} F = a(JF)(0), \qquad S_x F = a(JF)(x). \tag{6}$$

The exchange of sum and quotient integral is justified by $\sum_\xi |(JF)(\xi)| < \infty$; the characters have modulus one. This proves, without an implicit model comparison, both identities needed here:

$$\theta_V(F) = \sum_{\xi \in E^d} S_\xi F, \qquad \int \overline{S_x F}\, S_x G\,dx = |a|^2 \langle F, G\rangle. \tag{7}$$

The first sum is absolutely convergent and the second is the L^2 identity for the explicitly unitary J. For the adapted vacuum in (4), (6) yields exactly

$$S_x u = a\, 2^{d\#\Phi/2} |e_d(x)|^2, \tag{8}$$

so its nonzero scalar and its positive Gaussian and lattice factors have all been determined.

Calibration of the Levi phase by the literal kernel. Let $L(A)$ act on the hyperbolic coordinates by $\operatorname{diag}(A^{-*}, A)$, for $A \in \mathrm{GL}_d(\mathbb{A}_E)$. The unitary operator $D_A\varphi(x) = |\det A|_{\mathbb{A}}^{1/2}\varphi(A^*x)$ has the same Heisenberg covariance as $J\Omega_V(L(A))J^{-1}$. Their quotient is bounded and commutes with every position translation and modulation. The bounded commutant theorem A.5.9(i) applies: $\mathbb{A}_E^d$ is a second countable locally compact abelian group and both operators are unitary on its L^2 space. Therefore, for some scalar z_A,

$$J\Omega_V(L(A))J^{-1}\varphi(x) = z_A \varphi(A^*x), \qquad |z_A| = |\det A|_{\mathbb{A}}^{1/2}. \tag{9}$$

We now determine its phase; no uniqueness argument determines that phase.

For every unitary matrix g in signed coordinates the literal kernel gives

$$\langle e_n, \Omega(g) e_n \rangle = \overline{\Delta(g)}^{-1}. \tag{10}$$

Here is a direct proof retaining its phase. By Lemma 4.1.10,

$$P = \int e_n(y) \rho(y)\, d\mu_n(y), \qquad PF = F(0) e_n$$

is the orthogonal projection onto the unit vector e_n. Conjugating this Bochner integral by the unitary $\Omega(g)$ and using exact covariance gives

$$P_g = \int e_n(y) \rho(gy)\, d\mu_n(y) = \Omega(g) P \Omega(g)^{-1},$$

the orthogonal projection onto $\mathbb{C}\Omega(g)e_n$. The literal kernel, applied to e_n, is exactly $\Omega(g)e_n = \Delta(g) P_g e_n$. Since $\Omega(g)e_n$ is a unit vector, the projection formula gives $1 = \Delta(g)\langle \Omega(g)e_n, e_n\rangle$. Taking conjugates proves (10). The integrals are absolutely Bochner convergent because $e_n \in L^1$ and the Heisenberg action is isometric.

For q in (4), the conjugate $q^{-1} L(A) q$ has signed matrix $x_H^{-1} \operatorname{diag}(A^{-*}, A) x_H$. Write its multiplier as $M(A)$. At infinity its negative principal block is $D = (A^{-*} + A)/2$. The matrix $D^* A = (1 + A^* A)/2$ is positive definite, whence

$$\overline{\det D}\, \frac{\det A}{|\det A|} > 0.$$

At finite places the integral frame x_f and Lemma 4.1.13 give

$$M_f(A) = |\Delta(A_f)|^2 \overline{\chi_f(\det A_f)}, \qquad M_f(A) \upsilon_f(\det A_f) > 0.$$

Multiplying these equalities with the infinite factors gives

$$M(A) \upsilon(\det A) > 0. \tag{11}$$

By genuineness and unitarity, (10) applied after conjugation by q gives

$$\langle u, \Omega_V(L(A)) u \rangle = \overline{M(A)}^{-1}.$$

On the other hand (9) makes this coefficient

$$z_A K_A, \qquad K_A = \int_X \upsilon(x) \upsilon(A^* x)\, dx > 0.$$

The positivity is literal: its real integrand is a product of positive Gaussians, and the finite support is the intersection of two compact-open lattices, of positive Haar measure. The integral is finite by Cauchy–Schwarz. Equations (11) and this equality show $z_A \upsilon(\det A) > 0$. Together with the absolute value in (9), they give the exact formula

$$J \Omega_V(L(A)) J^{-1} \varphi(x) = \upsilon(\det A)^{-1} |\det A|_{\mathbb{A}}^{1/2} \varphi(A^* x). \tag{12}$$

Rational parabolics and generation. In these hyperbolic coordinates the unipotent radical fixing κE^d consists of $n_-(C) = \left(\begin{smallmatrix} 1 & 0 \\ C & 1 \end{smallmatrix}\right)$, $C^* = -C$. The multiplication operator

$$Q_C \varphi(x) = \psi_E(\jmath x^* C x/2) \varphi(x)$$

is unitary, preserves Schwartz functions and has exactly the covariance of $n_-(C)$: multiplying the phases in (1) gives $Q_C R(a,b) Q_C^{-1} = R(a, b + Ca)$. The bounded commutant theorem again makes the actual action a scalar multiple of Q_C. Evaluation at zero is invariant under this radical by Lemma 4.1.12, applied to $P = \kappa \mathbb{A}_E^d$: it is modulation invariant, and $n_-(C) - 1$ strictly raises the two-step isotropic grading. The comparison scalar is therefore one, since $Q_C \varphi(0) = \varphi(0)$ and some Schwartz function has nonzero value at zero.

If A and C are rational, the scalar in (12) is 1 by the product formula and the automorphy of υ. The map $\xi \mapsto A^* \xi$ permutes E^d. Also $\psi_E(\jmath \xi^* C \xi/2) = 1$ for $\xi \in E^d$. Formula (5) now proves invariance of theta under the rational Levi and radical, hence under this rational Siegel parabolic. Repeating (1)–(12) with λ and κ interchanged proves invariance under the opposite parabolic: their Gram matrix is still H, the same finite self-duality and positive vacuum calculation apply, and rationality is unchanged.

Here is an algebraic proof that these two parabolics generate all rational points. For $g = \left(\begin{smallmatrix} a & b \\ c & d_0 \end{smallmatrix}\right)$ with a invertible, the equations $g^* H g = H$ and $g H g^* = H$ give

$$ca^{-1} + (ca^{-1})^* = 0, \quad a^{-1}b + (a^{-1}b)^* = 0, \quad d_0 = ca^{-1}b + a^{-*}.$$

Thus

$$g = n_-(ca^{-1}) \operatorname{diag}(a, a^{-*}) n_+(a^{-1}b). \tag{13}$$

Each factor is in one of the two parabolics. Let U be this nonempty open cell; invertibility of a is the nonvanishing of the F-polynomial $\mathrm{N}_{E/F}(\det a)$. The algebraic unitary group G is geometrically integral: over $\bar{F}$ the algebra E becomes $\bar{F} \times \bar{F}$, and a unitary point is a pair (g_1, g_2) with $g_2 = H(g_1^{-1})^{\mathsf{T}} H$. Thus $G_{\bar{F}} \simeq \mathrm{GL}_{2d, \bar{F}}$.

The Cayley map

$$X \longmapsto (1 - X)(1 + X)^{-1}$$

is an F-isomorphism from the open subset $1 + X$ invertible in the F-vector space $\{X : X^* H + HX = 0\}$ onto the open subset $1 + g$ invertible in G. Its inverse is $(1 - g)(1 + g)^{-1}$; the identity $X^\dagger = -X$ gives $((1-X)(1+X)^{-1})^\dagger = ((1-X)(1+X)^{-1})^{-1}$ and verifies both directions. For any $g \in G(F)$, the three nonempty opens U, gU and the Cayley open have nonempty intersection by geometric integrality. Its inverse image in that vector space is a nonempty F-open subset. It has an F-point: in affine coordinates a nonempty open contains a principal open for a nonzero polynomial, and a nonzero polynomial cannot vanish on every point over an infinite field, by induction on the number of variables. Choose x in the intersection. Both x and $g^{-1}x$ satisfy (13), so $g = x(g^{-1}x)^{-1}$ belongs to the subgroup generated by the two parabolics. This proves theta invariance on every split space. Rank zero is immediate.

Doubling and the canonical rational lift. For arbitrary V with Gram matrix J_V, the space $V \perp V^-$ is split, with dual isotropic frames

$$\kappa = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} J_V^{-1}/2 \\ -J_V^{-1}/2 \end{pmatrix}.$$

Their pairings are $0, 0, 1$ by direct matrix multiplication. Choose theta on this doubled space by Lemma 4.1.9. Orthogonal sums have product vacua, measures and bicharacters. The finite intersection lattice and the negative infinite block are direct sums, so their multipliers multiply. Fubini in the displayed kernel therefore gives exactly

$$\Omega_{V \perp V^-}(r, 1)(F \boxtimes Y) = \Omega_V(r) F \boxtimes Y. \tag{14}$$

We justify the polarized distribution argument for this possibly nonsplit space as well. For its underlying symplectic F-space choose complementary rational Lagrangians. The Heisenberg

representation on the coherent Hilbert completion has multiplicity one in Theorem A.5.9(ii): any bounded commutant commutes with the projection P in (10), hence acts by a scalar on e_n and then on its dense Heisenberg span. A higher-dimensional multiplicity space would have nonscalar bounded commutants. Thus a unitary polarized identification exists.

Its vacuum image ϕ_0 is Schwartz–Bruhat. At infinity the coherent vacuum and all its Heisenberg derivatives are integrable polynomials times Gaussians; differentiation under the integral is justified by a Gaussian dominating each derivative on a compact parameter set. In the polarized Hilbert space the infinitesimal Heisenberg operators are constant multiples of the operators x_j and ∂_j. Therefore $x^\alpha \partial^\beta \phi_0 \in L^2$ for all multi-indices. For $4M > D$, with D the real position dimension, Fourier inversion and Cauchy–Schwarz give

$$\|h\|_\infty \le C_{D,M} \|(1-\Delta)^M h\|_2 \le C'_{D,M} \sum_{|\eta| \le 2M} \|\partial^\eta h\|_2.$$

Indeed insert $(1+|\xi|^2)^{-M}$ in the Fourier integral; its square is integrable. This also holds for weak derivatives by Plancherel and furnishes their continuous representatives. Apply the inequality to $h = x^\alpha \partial^\beta \phi_0$ and expand derivatives by the Leibniz rule. Every weighted derivative is bounded, so ϕ_0 has a Schwartz representative. At finite adeles the coherent vacuum is fixed by sufficiently small compact-open position and modulation groups; in the polarized model modulation invariance confines support to a compact annihilator and translation invariance gives local constancy. Thus its finite part lies in a finite-dimensional Schwartz–Bruhat fixed space.

For any polarized Schwartz φ, the coefficient $\langle \rho(z)\phi_0, \varphi \rangle$ is Schwartz and belongs to L^1, by the integration-by-parts and finite-support estimates preceding (3), now with the fixed Schwartz window ϕ_0. Fubini and Fourier inversion in the modulation variable give (3) with this unit window. Thus its inverse is the actual coherent vector, by the Bochner reproduction of Lemma 4.1.10, and is continuous into the coherent L^1 norm on every finite level. Consequently every bounded theta functional restricts to a continuous distribution on this polarized Schwartz space. The rational translations and modulations are those of the chosen rational Lagrangians. The proof of (5), with F in place of E, applies verbatim; it uses only the number-field trace pairing and its rational lattice.

For fixed Schwartz Y, the functional $F \mapsto \theta_{V \perp V^-}(F \boxtimes Y)$ is bounded on the coherent L^1 carrier and rational-Heisenberg invariant on V. By that distribution argument it is $a_Y \theta_V(F)$ on Schwartz vectors. Some a_Y is nonzero. Otherwise theta on the doubled space would vanish on every pure Schwartz product, in particular on all Heisenberg translates of its vacuum, since its vacuum and these translates factor into the two summands. They have dense span in its coherent L^1 carrier by Lemma 4.1.10, while theta is bounded and nonzero by its normalization. This is a contradiction. Split-space invariance, (14), and cancellation of such an $a_Y \neq 0$ give

$$\theta_V(\Omega_V(r)F) = \theta_V(F) \qquad (r \in \mathrm{U}(V)(F)). \tag{15}$$

For clarity we also check the character hypothesis needed to compare with the classical rational lift. The field contains i and a primitive seventh root ξ. Set $u_0 = -(\xi + \xi^2 + \xi^4)$, so $u_0 + \bar{u}_0 = 1$. The integral element $z = iu_0$ satisfies $z - \bar{z} = i$, a unit. Moreover $E = F(i)$, and $z \notin F$; hence $F(z) = E$. The quadratic order $\mathcal{O}_F[z]$ has discriminant $(z-\bar{z})^2 = -1$, so it is finite étale at every finite prime. Over a completed discrete valuation ring a finite étale algebra is a product of unramified discrete valuation rings and is integrally closed; thus it is the full local integral closure. Consequently E/F is split or unramified at every finite place. At a split place the two restrictions of υ to $F_v^\times$ cancel by conjugation. At an unramified nonsplit place it is 1 on units and -1 on a uniformizer. At a real place it is the sign character. Thus $\upsilon|_{\mathbb{A}_F^\times} = \eta_{E/F}$.

To compare a single hermitian space V with the canonical rational lift, apply Theorem A.5.4 to the hermitian factor V and the skew-hermitian line with form $h_-(a,b) = -\jmath a\bar{b}$. Use splitting characters $\upsilon^{\dim_E V}$ and υ; their restrictions are the powers of $\eta_{E/F}$ required by that theorem. Write $h_V(x,y) = \bar{y}^{\mathsf{T}} J_V x$, linear in its first variable. In this convention the tensor symplectic form on $V \otimes_E E$ is

$$\mathrm{Tr}_{E/F}\big(h_V(x,y)\overline{h_-(1,1)}\big) = \mathrm{Tr}_{E/F}(\jmath h_V(x,y)).$$

Thus its Weyl bicharacter is precisely the one in the statement. The operator of the classical pair at $(r,1)$ is the canonical rational lift, implements the same rational Heisenberg automorphism, and preserves the rational theta distribution by Theorem A.5.4. Transport it to the same Hilbert Heisenberg model. Its quotient with the literal rational operator is a bounded scalar commutant, by the rank-one argument above. Both preserve the same nonzero theta distribution: for the literal operator this is (15), and for the canonical operator it is Theorem A.5.4(ii), together with (5) in a rational polarization. Their scalar quotient is consequently 1.

Finally apply (15) to $V = W = \mathrm{Hom}_E(S,T)$. A rational pair acts on W by $Z \mapsto r_{\mathrm{out}} Z r_{\mathrm{in}}^{-1}$. Its external automorphic determinant characters are 1 at rational determinants, so its operator has the just-proved rational normalization. Exact covariance gives

$$\rho_W(Z)\Omega^{S,T}(r_{\mathrm{in}}, r_{\mathrm{out}}) = \Omega^{S,T}(r_{\mathrm{in}}, r_{\mathrm{out}})\rho_W(r_{\mathrm{out}}^{-1} Z r_{\mathrm{in}}).$$

Applying theta and (15) proves the asserted inverse-pullback formula. All theta identities are homogeneous in its one nonzero normalization scalar. □

Lemma 4.1.12. *Let E/F be a quadratic extension of number fields, let $\mathbb{W}$ be a nondegenerate skew-Hermitian E-space, and fix a nontrivial adelic central character of its Heisenberg group. Let $\mathscr{S}$ be its adelic Schwartz oscillator model, with a genuine continuous Weil action ω of $U(\mathbb{W})(\mathbb{A}_F)$ preserving $\mathscr{S}$ and satisfying exact Heisenberg covariance. A fixed polarized intertwiner may be used to realize this same space in another model. Let $P \subset \mathbb{W}$ be an E-rational isotropic subspace. Put $Q = \mathrm{Stab}(P)$ and let Q^0 be the kernel of the action of Q on $\mathbb{W}_0 = P^\perp/P$. Suppose $\ell : \mathscr{S} \to \mathbb{C}$ is continuous and $\ell(\rho(p)f) = \ell(f)$ for every $p \in P(\mathbb{A}_F)$. Then $\ell(\omega(g)f) = \ell(f)$ for every g in the commutator subgroup of $Q^0(\mathbb{A}_F)$. In particular this holds for a unitary g such that, in a grading with positive subspace P, zero subspace $\mathbb{W}_0$ and negative subspace dual to P, the operator $g-1$ strictly raises degree. If a continuous rational-Heisenberg-invariant theta functional θ is given, the probability mean*

$$\ell_P(f) = \int_{P(F)\backslash P(\mathbb{A}_F)} \theta(\rho(p)f)\, dp$$

is such an ℓ. Consequently, with

$$C_P(f,w) = \ell_P(\rho(w)f),$$

one has $C_P(\omega(g)f, w) = C_P(f, g^{-1}w)$ for each of the preceding g. No comparison between a coherent coefficient and a theta value is asserted.

Proof. Choose a polarization containing P. In the corresponding Schrödinger coordinates write the model as Schwartz functions $\varphi(x,z)$, where x is the coordinate dual to P and z belongs to a polarization of $\mathbb{W}_0$. The operators $\rho(p)$ multiply by the characters of x. Define $e\varphi = \varphi(0,\cdot)$. This is a continuous surjection onto the residual Schwartz space, with continuous right inverse $\psi(z) \mapsto \eta(x)\psi(z)$ for any Schwartz η with $\eta(0) = 1$. We first verify

$$\ker e = \overline{\mathrm{span}\{(\rho(p)-1)\varphi : p \in P(\mathbb{A}_F),\ \varphi \in \mathscr{S}\}}. \tag{1}$$

The inclusion from right to left is immediate. Here are the details of the converse, including the adelic topology. Locally at an archimedean place, the ideal of smooth functions vanishing at $x = 0$ is generated by the real coordinate functions: for a function vanishing there, integrate its first derivatives along tx, $0 \le t \le 1$. On a sufficiently small coordinate neighborhood each coordinate function is a smooth multiple of $\exp(i\epsilon x_j) - 1$. For compactly supported functions, a partition of unity handles that neighborhood and its complement; at any point of the complement choose a character different from 1 and divide by that character minus 1 on a smaller neighborhood. Cutoffs tending to 1 approximate a Schwartz function in every Schwartz seminorm, so the same conclusion holds with closure for Schwartz functions and with Schwartz-valued parameters z. At a finite place, a locally constant function vanishing at 0 vanishes on a neighborhood of 0; its compact support is covered by finitely many sets on each of which some additive character minus 1 is invertible. Division gives the same conclusion. Apply these arguments to a finite restricted tensor product containing the function and then use density of such tensors in the adelic Schwartz space. The additive pairing with $P(\mathbb{A}_F)$ supplies exactly all these characters. This proves (1).

Exact covariance implies that $\omega(g)$ preserves $\ker e$ for $g \in Q^0(\mathbb{A}_F)$, so it induces a continuous invertible operator A_g on the residual Schwartz space. For $v \in P^\perp$, the vectors gv and v differ by an element of P; that element is orthogonal to v, so no Heisenberg phase remains after passage to the quotient. Thus A_g commutes with every residual translation and modulation.

For completeness the required commutant statement is valid on Schwartz space itself, without a bounded-Hilbert-operator assumption. If A commutes with all modulations, evaluation of $A\psi$ at a point z_0 annihilates $(\chi - \chi(z_0))\psi$ for every character χ. The proof of (1), translated to z_0, says that the closed span of these functions is the kernel of evaluation at z_0. Hence $(A\psi)(z_0) = a(z_0)\psi(z_0)$. Commutation with all translations makes $a(z_0)$ independent of z_0. Therefore A is scalar. Applied to A_g, this gives $A_g = \lambda(g)1$. Since ω is a genuine representation, λ is a character of $Q^0(\mathbb{A}_F)$, and so is 1 on its commutator subgroup. By (1), every continuous $P(\mathbb{A}_F)$-invariant functional factors through e. This proves the first assertion.

We check explicitly the additional assertion about a degree-raising g. Choose a Witt decomposition $P \oplus \mathbb{W}_0 \oplus P^\vee$. The kernel Q^0 is $N \rtimes \mathrm{GL}_E(P)$, where the parabolic radical N is two-step nilpotent. In exponential coordinates its elements have coordinates (C, Z) of weights $1, 2$ and multiplication

$$(C, Z)(C', Z') = (C + C', Z + Z' + \tfrac{1}{2}B(C, C')),$$

with B alternating and bilinear. These coordinates are obtained directly from the upper block-unitriangular unitary equations; equivalently the finite polynomial exponential and logarithm identify the radical with its two-step Lie algebra. Let t act by 2 on P, by 1 on $\mathbb{W}_0$ and by $1/2$ on $P^\vee$. It is unitary and belongs to Q^0, and conjugation by t sends (C, Z) to $(2C, 4Z)$. Thus

$$[t, n(C, Z/3)] = n(C, Z);$$

the bilinear cross term is zero because it involves two multiples of the same C. Every element of N is consequently a commutator in Q^0.

For a strictly degree-raising g, the induced map on $\mathbb{W}_0$ is the identity and its P-block is upper unitriangular in a degree-ordered basis. Write $g = n\, m(A)$. Gaussian elimination expresses A as a product of elementary upper unipotent matrices $1 + ae_{ij}$. Each such matrix is a commutator in $\mathrm{GL}_E(P)$: conjugate $1 + ae_{ij}$ by a diagonal matrix whose ith entry is 2, jth entry is 1, and use $[d, 1 + ae_{ij}] = 1 + ae_{ij}$. If $\dim P \le 1$ there are no such factors. It follows that g belongs to the commutator subgroup of Q^0. The case $P = 0$ gives $g = 1$ and requires no coordinates.

Finally the theta mean is defined on a compact additive quotient. Rational Heisenberg invariance makes its integrand periodic, and Haar translation gives $\ell_P(\rho(p_0)f) = \ell_P(f)$ for

every adelic $p_0 \in P$; isotropy removes all cross phases. Continuity follows from compactness and continuity of the oscillator action on Schwartz space. Exact covariance gives $\rho(w)\omega(g) = \omega(g)\rho(g^{-1}w)$, and the asserted coefficient formula follows. Transport through a fixed polarized intertwiner conjugates all these identities and does not alter their scalar. □

Lemma 4.1.13. *Let E be a CM number field, let $\mathbb{A}_{E,f}$ be its finite adeles, and put $\mathcal{O}_f = \widehat{\mathcal{O}}_E$. Conjugation of E acts on its finite adeles, and $g^* = \bar{g}^{\,t}$. Let $\psi_{E,f}$ be the standard additive character obtained from the trace to the finite adeles of $\mathbb{Q}$. Fix $j \in E^\times$ with $\bar{j} = -j$ and $\alpha \in \mathbb{A}_{E,f}^\times$, and put*

$$c_0 = (1 - j^{-1})/2, \qquad \psi_0(t) = \psi_{E,f}(j\alpha\bar{\alpha}t).$$

Assume $c_0 \in \mathcal{O}_E$ and that $\mathcal{O}_f$ is self-dual for $(z, w) \mapsto \psi_0(\bar{z}w)$. Let $\upsilon_f : \mathbb{A}_{E,f}^\times \to \mathbb{C}^\times$ be a continuous character such that

$$|\upsilon_f(t)| = 1, \qquad \upsilon_f(u) = 1 \ (u \in \mathcal{O}_f^\times), \qquad \upsilon_f(\bar{t}) = \upsilon_f(t)^{-1}.$$

Write $\chi_f(t) = \upsilon_f(t)|t|_f^{-1/2}$.

For each integer $n \geq 0$, put $L = \mathcal{O}_f^n$, and normalize additive Haar measure μ by $\mu(L) = 1$. For $x, z \in \mathbb{A}_{E,f}^n$, use the following local notation:

$$\begin{aligned} \beta_f(x,z) &= \psi_0(z^*x/2), & s_L(x) &= \psi_0(c_0x^*x/2), \\ v_0(z) &= s_L(z)\mathbf{1}_L(z), & (\rho(x)v)(z) &= \beta_f(x,z)v(z-x), \\ H(x) &= s_L(x)\rho(x), & \mathcal{V}_{\mathrm{coh}} &= \mathrm{span}_{\mathbb{C}}\{\rho(x)v_0 : x \in \mathbb{A}_{E,f}^n\} \subset C_c^\infty(\mathbb{A}_{E,f}^n). \end{aligned}$$

For $g \in \mathrm{GL}_n(\mathbb{A}_{E,f})$, set

$$J(g) = L \cap g^{-1}L, \qquad I(g) = [L : J(g)].$$

Choose $u_g \in \mathrm{GL}_n(\mathbb{A}_{E,f})$ with $u_gL = J(g)$ and put $\Delta(g) = \chi_f(\det u_g)$. These choices exist, and $\Delta(g)$ is independent of them. The notation Δ below uses this definition in the rank of its argument.

For $g^*g = 1$, define the operator by the literal integral

$$(\Omega(g)v)(z) = \Delta(g)\int_{\mathbb{A}_{E,f}^n} v(y)\beta_f(gy, z)v_0(z - gy)\, d\mu(y), \qquad v \in \mathcal{V}_{\mathrm{coh}}.$$

This integral is absolutely convergent at every z, defines an endomorphism of $\mathcal{V}_{\mathrm{coh}}$, and satisfies

$$\Omega(g)\rho(x) = \rho(gx)\Omega(g).$$

For every such g,

$$\Delta(g) \neq 0, \qquad I(g) \in \mathbb{Z}_{>0}, \qquad |\Delta(g)|^2 = I(g), \qquad \overline{\Delta(g)} = \upsilon_f(\det g)\Delta(g),$$

and, for any set of representatives of $L/J(g)$,

$$\Omega(g)v_0 = \frac{\Delta(g)}{I(g)} \sum_{y \in L/J(g)} H(gy)v_0.$$

Each summand is independent of its representative. In particular, for every field embedding $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ with $|\sigma(\upsilon_f(\det g))| = 1$,

$$\left|\frac{\sigma(\Delta(g))}{\Delta(g)}\right| = 1.$$

For arbitrary invertible finite adelic matrices, in every rank,

$$|\Delta(g)|^2 = I(g), \qquad \Delta(g^{-1}) = \chi_f(\det g)\Delta(g), \qquad \Delta(g^*) = \overline{\Delta(g)},$$
$$\Delta(k_1 g k_2) = \Delta(g) \quad (k_1, k_2 \in \mathrm{GL}_n(\mathcal{O}_f)), \qquad \Delta(g_1 \oplus g_2) = \Delta(g_1)\Delta(g_2).$$

Consequently, if $A \in \mathrm{GL}_r(\mathbb{A}_{E,f})$, $k \in \mathrm{GL}_{2r}(\mathcal{O}_f)$ *and* $A^{-*} = (A^*)^{-1}$, *then*

$$\Delta\Big(k^{-1}\operatorname{diag}(A^{-*}, A)k\Big) = |\Delta(A)|^2\overline{\chi_f(\det A)},$$
$$\Delta\Big(k^{-1}\operatorname{diag}(A^{-*}, A)k\Big)\, \upsilon_f(\det A) = |\Delta(A)|^2 |\det A|_f^{-1/2} > 0.$$

Proof. We first record the character identities used in the integral. Conjugation invariance of the trace and $\bar{j} = -j$ give

$$\psi_0(\bar{t}) = \psi_0(t)^{-1}, \qquad c_0 + \bar{c}_0 = 1.$$

The character ψ_0 is trivial on $\mathcal{O}_f$: self-duality implies $\psi_0(\bar{z}w) = 1$ for $z, w \in \mathcal{O}_f$, and we take $z = 1$. It is also trivial on every fixed element of conjugation. Indeed, if $t = \bar{t}$, applying the preceding identity to $t/2$ gives $\psi_0(t/2)^2 = 1$, and hence $\psi_0(t) = 1$. In particular $\beta_f(x, x) = 1$. The continuity of an additive character of finite adeles implies its local constancy, so υ_0 and all its translates in the statement are locally constant and compactly supported.

We establish the scalar formulas in arbitrary rank. At each finite place w, write q_w for the residue cardinality, choose a uniformizer ϖ_w, and express the matrix by elementary divisors as

$$g_w = k_w \operatorname{diag}(\varpi_w^{a_{w,1}}, \dots, \varpi_w^{a_{w,n}})k'_w, \qquad k_w, k'_w \in \mathrm{GL}_n(\mathcal{O}_{E,w}), \quad a_{w,i} \in \mathbb{Z}.$$

This is Smith normal form over a discrete valuation ring, applied after multiplying by a large enough power of ϖ_w. The matrix and its inverse are integral at all but finitely many places, so all the exponents vanish there. Since the integral invertible matrices preserve the standard lattice, the local intersection is

$$J(g)_w = (k'_w)^{-1} \operatorname{diag}(\varpi_w^{\max(0,-a_{w,1})}, \dots, \varpi_w^{\max(0,-a_{w,n})})\mathcal{O}_{E,w}^n.$$

Taking these local bases, and the standard bases elsewhere, supplies u_g with $u_g L = J(g)$. Any two such matrices differ on the right by $\mathrm{GL}_n(\mathcal{O}_f)$. Their determinant ratio lies in $\mathcal{O}_f^\times$, on which χ_f is one. Thus $\Delta(g)$ is well defined and nonzero.

Put $t_w(g) = \sum_i \max(0, -a_{w,i})$. Counting the successive residue quotients of the displayed diagonal lattice gives

$$\operatorname{ord}_w \det u_g = t_w(g), \qquad I(g) = \prod_w q_w^{t_w(g)}, \qquad |\det u_g|_f = I(g)^{-1}.$$

The product is finite, and every factor is a positive integer. Since υ_f is circle-valued, these formulas give $|\Delta(g)|^2 = I(g)$.

Two ideles with the same finite valuations differ by an element of $\mathcal{O}_f^\times$, so χ_f depends only on those valuations. The elementary divisors of g_w^{-1} are the negatives of those of g_w, in reverse order. Therefore

$$t_w(g^{-1}) - t_w(g) = \sum_i a_{w,i} = \operatorname{ord}_w \det g.$$

It follows that $\Delta(g^{-1}) = \chi_f(\det g)\Delta(g)$. Transposition preserves elementary divisors, and conjugation carries those at w to those at $\bar{w}$. Moreover

$$\chi_f(\bar{t}) = \overline{\chi_f(t)},$$

because conjugation preserves the finite idele modulus and inverts v_f. Applying this to the valuation description proves $\Delta(g^*) = \overline{\Delta(g)}$.

If $g^*g = 1$, then $g^* = g^{-1}$ and $|\det g|_f = 1$. The last assertion follows by taking the finite modulus of $\overline{\det g}\det g = 1$, since conjugation leaves that modulus unchanged. The inverse and adjoint formulas consequently yield

$$\overline{\Delta(g)} = v_f(\det g)\Delta(g).$$

Multiplying by $\Delta(g)$ yields $v_f(\det g)\Delta(g)^2 = I(g)$. A field embedding of $\mathbb{C}$ fixes the positive integer $I(g)$. Applying σ to this equality and taking absolute values, under the stated hypothesis on $\sigma(v_f(\det g))$, gives $|\sigma(\Delta(g))|^2 = I(g)$. Division by $|\Delta(g)|^2 = I(g)$ proves the embedding assertion.

We next prove the vacuum expansion as an equality of the explicit functions. Direct substitution into the definition of ρ gives

$$\rho(x)\rho(y) = \psi_0(y^*x/2)\rho(x+y).$$

Indeed the extra factor is initially $\psi_0(-x^*y/2)$, which equals the displayed factor by conjugation. Expanding $s_L(x+y)$, and applying the same character identity to its term containing x^*y, gives

$$H(x)H(y) = \psi_0(\bar{c}_0 y^*x)H(x+y).$$

To see the coefficient, divide $s_L(x)s_L(y)\psi_0(y^*x/2)$ by $s_L(x+y)$. The resulting character at $((1-c_0)y^*x - c_0x^*y)/2$ equals the character at $\bar{c}_0y^*x$, since $1-c_0 = \bar{c}_0$.

For $t \in L$ one has $H(t)v_0 = v_0$. Translation by t preserves L, so both sides vanish off L. At $z = t + r \in L$, their ratio is

$$\frac{s_L(t)\psi_0((t+r)^*t/2)s_L(r)}{s_L(t+r)} = \psi_0(t^*t/2)\psi_0(\bar{c}_0r^*t) = 1.$$

The first factor is one because $t^*t/2$ is fixed by conjugation; the second is one because $\bar{c}_0r^*t \in \mathcal{O}_f$. Thus this calculation also applies at places above 2.

Suppose $g^*g = 1$, $x \in L$ and $t \in J(g)$. Then $gt \in L$ and $(gt)^*(gx) = t^*x \in \mathcal{O}_f$. The cocycle formula and the preceding invariance give

$$H(g(x+t))v_0 = H(gx)H(gt)v_0 = H(gx)v_0.$$

Consequently $x \mapsto H(gx)v_0$ is constant on the cosets of $J(g)$ in L. The equality $s_L(gy) = s_L(y)$ and the definition of the integral operator give, pointwise,

$$\begin{aligned}\Omega(g)v_0 &= \Delta(g)\int_L s_L(y)\rho(gy)v_0\,d\mu(y)\\ &= \Delta(g)\int_L H(gy)v_0\,d\mu(y)\\ &= \frac{\Delta(g)}{I(g)}\sum_{y\in L/J(g)} H(gy)v_0.\end{aligned}$$

Here $J(g)$ is an open subgroup of compact L, the quotient is finite, and every coset has measure $I(g)^{-1}$. The integral is thus a finite sum of functions, and the same argument proves independence of representatives.

It remains to check the operator on the stated function space. For any compactly supported locally constant v, the pointwise kernel is bounded on its compact domain of integration, so its

integral is absolutely convergent. For $g^*g = 1$, replace y by $t + x$ in the kernel for $\Omega(g)\rho(x)v$. The two factors that involve x are

$$\beta_f(x, t + x)\beta_f(g(t + x), z) = \beta_f(x, t)\beta_f(gt, z)\beta_f(gx, z).$$

On the other hand, the kernel for $\rho(gx)\Omega(g)v$ has the corresponding product

$$\beta_f(gx, z)\beta_f(gt, z - gx) = \beta_f(gx, z)\beta_f(gt, z)\beta_f(x, t).$$

The second equality uses $\beta_f(gt, -gx) = \psi_0(-x^*t/2) = \beta_f(x, t)$, and $\beta_f(x, x) = 1$ was established above. The remaining functions in both integrands are $v(t)v_0(z-gx-gt)$. Translation invariance of Haar measure proves $\Omega(g)\rho(x)v = \rho(gx)\Omega(g)v$. In particular, applying this to $v = v_0$ expresses the image of each generator $\rho(x)v_0$ as a finite linear combination of Weyl translates of v_0, by the vacuum expansion. Thus the integral defines an endomorphism of $\mathcal{V}_{\mathrm{coh}}$ and proves the claimed covariance there.

Finally, if k_1, k_2 are integral invertible matrices, then

$$L \cap (k_1 g k_2)^{-1} L = k_2^{-1}(L \cap g^{-1}L).$$

The matrix $k_2^{-1}u_g$ is an admissible choice for this lattice, and $\chi_f(\det k_2) = 1$. This proves integral left and right invariance. For a block diagonal matrix the intersection lattice is the direct sum of the two intersection lattices. Choosing the block diagonal matrix $u_{g_1} \oplus u_{g_2}$ proves multiplicativity under direct sums. Using these two assertions and the already proved inverse and adjoint identities gives

$$\begin{aligned}\Delta\Big(k^{-1}\operatorname{diag}(A^{-*}, A)k\Big) &= \Delta(A^{-*})\Delta(A)\\ &= \overline{\Delta(A^{-1})}\Delta(A)\\ &= \overline{\chi_f(\det A)}|\Delta(A)|^2.\end{aligned}$$

Multiplication by $v_f(\det A)$ gives the positive real number $|\Delta(A)|^2|\det A|_f^{-1/2}$, as asserted. In rank zero all determinants, indices and products are one, and the same statements hold. □

Proposition 4.1.14. *For every rank n and sign vector s, the literal integral in (WR3) defines a bounded operator $\omega_s(g)$ of $\mathcal{M}_s$ for every $g \in \mathrm{G}(s)$. These operators satisfy*

$$\begin{gathered}\omega_s(1) = 1, \qquad \omega_s(g)\omega_s(h) = \omega_s(gh), \qquad \omega_s(g)^{-1} = \omega_s(g^{-1}),\\ \langle\omega_s(g)u, \omega_s(g)v\rangle = \langle u, v\rangle, \qquad \omega_s(g)\rho(z)\omega_s(g)^{-1} = \rho(gz),\\ \|\omega_s(g)u\|_1 \le 2^{n\#\Phi}|\Delta_s(g)|\,\|u\|_1.\end{gathered}$$

The map $(g, u) \mapsto \omega_s(g)u$ is jointly continuous for the group topology and the prescribed L^1 norm. The Heisenberg action is strongly continuous and isometric for that norm and the L^2 pairing.

The same assertions hold for the literal real operators of (WR5), with $2^{\#\iota}|\overline{\det g_-}|$ in the norm bound. The real and finite factors and orthogonal sums are compatible with the prescribed product coordinates: on product coherent vectors,

$$\omega_{s\oplus s'}(g \oplus h)(u \otimes v) = \omega_s(g)u \otimes \omega_{s'}(h)v.$$

The finite subgroup acts with the real vacuum fixed, and preserves the span of the finite Heisenberg translates of the full vacuum. Under the prescribed scaled finite coordinates this restricted operator is exactly the normalized finite kernel of Lemma 4.1.13. In particular

$$\begin{aligned}\langle\omega_s(g)v^0, \omega_s(g)v^0\rangle &= 1,\\ \langle\omega_s(g)v^0, \rho(z)\omega_s(g)v^0\rangle &= \langle v^0, \rho(g^{-1}z)v^0\rangle.\end{aligned}$$

The determinant-twisted pair operators $\omega_{\alpha,\beta}$ and $\Omega_{\gamma,\gamma'}$ are genuine representations.

Each $\omega_s(g)$ extends to a unitary operator of the L^2 completion of $\mathcal{M}_s$. Under every fixed unitary identification of this completion with a polarized Schrödinger model intertwining the Heisenberg actions, the transported operator maps the Schwartz–Bruhat space onto itself.

Proof. **The reproducing algebra and the scalar to be calculated.** We use local notation (W, μ, β, e) for one prescribed real factor, finite factor, or their product. The measure and coordinates are exactly those of (WD5)–(WD7), after the given real and finite coordinate conversions. The vacuum is real and even, satisfies $e(0) = 1$, $|e| \le 1$, and obeys

$$\int_W e(t)\beta(t,b)\beta(t+b,a)e(a-b-t)\,d\mu(t) = e(b)e(a). \tag{Oc1}$$

Here and below the local labels refer only to this calculation. To check this identity in a real factor, insert $e(t) = \exp(-t^*t/2)$ and $\beta(t,b) = \exp((b^*Jt - t^*Jb)/2)$ with $J^2 = 1$. Completing the square coordinate by coordinate uses

$$\pi^{-1}\int_{\mathbb{C}} \exp(-|z|^2 + uz + v\bar{z})\,dz = \exp(uv).$$

It gives exactly $\exp(-b^*b/2 - a^*a/2)$. At a finite factor, write $L = \widehat{\mathcal{O}}_E^n$, $\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$ and $e(t) = \psi_0(c_0t^*t/2)\mathbf{1}_L(t)$ in the scaled coordinates. The self-duality is the preliminary conclusion of Lemma 4.1.3. On $t \in L$, the integrand of (Oc1) is

$$\frac{e(a-b)}{\beta(a,b)}\psi_0(a^*t).$$

This follows by expanding $e(a-b-t)$ and using $c_0 + \bar{c}_0 = 1$; the two opposite bicharacters cancel. Its integral is zero unless $a \in L$, by character orthogonality. If $a \in L$, the remaining value equals $e(a)e(b)$, since either both $a-b, b$ belong to L or neither does, and on L the semicharacter satisfies $e(x)e(y)\beta(x,y) = e(x+y)$. The finite vacuum takes values in $\{0, 1, -1\}$ because its square on L is $\psi_0(c_0t^*t) = 1$. Thus it is real and even. Product measures give (Oc1) for the adelic vacuum. The Gaussian calculation also gives $\int |e|^2 = 1$.

Let $Pf(x) = \int f(y)\beta(y,x)e(x-y)\,d\mu(y)$. Twisted convolution and (Oc1) with $b = 0$ give $P^2 = P$. Young's inequality gives $\|P\|_{L^1\to L^1} \le \|e\|_1$. Hence its range is a closed Banach subspace of L^1. Here twisted convolution is $(u*v)(x) = \int u(y)\beta(y,x)v(x-y)\,d\mu(y)$; its associativity follows from the bicharacter identity and Fubini, and (Oc1) at $b = 0$ gives $e * e = e$. Every class in the range has a unique continuous representative given by the displayed integral; point evaluation is bounded by $\|f\|_1$. This is the carrier in (WD7), with its stated norm. The same argument applies to (WR5). The operators $\rho(z)$ preserve it, are isometries for the L^1 norm and for the L^2 pairing, and are strongly continuous, by continuity of translations and unitary modulations on L^1. Moreover

$$\langle \rho(x)e, f\rangle = f(x), \qquad f = \int_W f(y)\rho(y)e\,d\mu(y). \tag{Oc2}$$

The first formula is the reproducing identity, since e is real and even. The second holds as a Banach integral: its norm is dominated by $|f(y)|\|e\|_1$. In particular the span of translates of e is dense in the carrier.

For f, v in this carrier, put

$$P_g(f)v = \int_W f(y)\rho(gy)v\,d\mu(y).$$

The group elements under consideration preserve β and μ. Thus these integrals are bounded by $\|f\|_1\|v\|_1$. For $g = 1$,

$$P_1(f)v = v(0)f. \tag{Oc3}$$

It suffices by (Oc2) and boundedness to verify this for $f = \rho(a)e$, $v = \rho(b)e$. At x, substituting $y = t + a$ into the left side gives $\beta(a, x)$ times the left side of (Oc1) with arguments $b, x - a$, and therefore $e(b)\beta(a, x)e(x - a)$, as required.

These identities give a scalar version of Schur's lemma without selecting an intertwiner. If a bounded operator T commutes with every $\rho(y)$, then

$$Tf = TP_1(f)e = P_1(f)Te = (Te)(0)f. \tag{Oc4}$$

The prescribed normalized operator is $\omega(g)f = \Delta(g)P_g(f)e$. Its literal pointwise formula is (WR3), respectively (WR5). It is bounded by $|\Delta(g)|\|e\|_1$ and carries the carrier into itself. Replacing the integration variable by $y + z$ proves

$$\omega(g)\rho(z) = \rho(gz)\omega(g); \tag{Oc5}$$

the phases cancel because $\beta(gz, gy) = \beta(z, y)$.

For later scalar evaluation, the same substitution proves

$$P_g(P_h(w)v)x = P_{gh}(w)\big(P_g(v)x\big). \tag{Oc6}$$

Indeed the bounded map $v \mapsto P_g(v)x$ intertwines $\rho(z)$ and $\rho(gz)$; apply this to the Banach integral defining $P_h(w)v$. All interchanges are dominated by products of L^1 norms. Let $k = gh$. By (Oc5), $\omega(k^{-1})\omega(g)\omega(h)$ commutes with every $\rho(z)$, so (Oc4) makes it a scalar operator. Applying (Oc6) three times and then (Oc3), its scalar is

$$c(g, h) = \Delta(g)\Delta(h)\Delta(k^{-1}) \iint e(hx)e(ky)\beta(x, y)e(x + y)\, d\mu(x)d\mu(y). \tag{Oc7}$$

More explicitly, the successive nonconstant terms are

$$\begin{aligned} P_{k^{-1}}\big(P_g(P_h(e)e)e\big)e &= P_{k^{-1}}\big(P_k(e)(P_g(e)e)\big)e \\ &= P_1(e)\big(P_{h^{-1}}(e)(P_{k^{-1}}(e)e)\big). \end{aligned}$$

Evaluation at zero and the substitutions $u = -hx$, $v = -ky$ give the double integral in (Oc7), using evenness of e and $\beta(-y, x) = \beta(x, y)$. This establishes the exact scalar rather than merely its existence.

The identity $\Delta(g^{-1}) = \overline{\Delta(g)}$ holds at finite places by Lemma 4.1.13, and at real places because the negative principal block of $g^{-1} = Jg^*J$ is the adjoint of the negative principal block of g. If $c(g, 1) = 1$, (Oc7) gives $\omega(g^{-1})\omega(g) = 1$. The adjoint relation follows directly from (Oc2):

$$\langle u, \omega(g)v\rangle = \Delta(g) \int v(y)\overline{u(gy)}\, d\mu(y) = \langle \omega(g^{-1})u, v\rangle. \tag{Oc8}$$

Thus $\omega(g)$ preserves the L^2 pairing. Applying this to the triple operator in (Oc7) shows $|c(g, h)| = 1$. Once $c(g, 1) = 1$ holds for every g, the inverse operators are two-sided, and

$$\omega(g)\omega(h) = c(g, h)\omega(gh). \tag{Oc9}$$

Associativity therefore gives the ordinary scalar cocycle identity. Consequently

$$R = \{h : c(g, h) = 1 \text{ for every } g\}$$

is a subgroup. The following two calculations prove that it is the whole group in the prescribed real and finite factors.

The finite scalar has phase one. Use the scaled finite coordinates and write $L = \widehat{\mathcal{O}}_E^n$, $\Delta = \Delta_f$, $J(u) = L \cap u^{-1}L$, $I(u) = [L : J(u)]$, and $\chi_f(t) = \upsilon_f(t)|t|_f^{-1/2}$. For g, h in the finite unitary group put

$$M = \{(x, y) : x + y \in L,\ hx \in L,\ ghy \in L\}, \qquad N = M \cap (L \times L) = J(h) \times J(gh).$$

The module M is compact and N is open in it. Thus $Q = M/N$ is a finite $\widehat{\mathcal{O}}_E$-module. Expanding the three semicharacters in (Oc7) shows that its integrand on M is

$$F(x, y) = \psi_0(\bar{c}_0 y^* x),$$

and it is zero off M. Indeed the product of the three semicharacters is the semicharacter cocycle times $s_L(x + y)^2$, which is one when $x + y \in L$.

The value F is constant on cosets of N. To verify this, take $(x, y), (x', y') \in M$ with $x' - x, y' - y \in L$. The difference of the arguments, after removing $\bar{c}_0$, is

$$\begin{aligned} y'^* x' - y^* x = {} & (x' + y')^*(x' - x) - x'^*(x' - x) \\ & + (y' - y)^*(x + y) - (y' - y)^* y. \end{aligned}$$

The first and third terms are integral because the sums and differences lie in L. The second is integral after applying the unitary matrix h to both factors, and the fourth after applying gh. Since $\bar{c}_0$ is integral and ψ_0 is trivial on integral adeles, the two values agree. Let $F_Q(q)$ be the descended value. For a representative (x, y) of q and $t \in \widehat{\mathcal{O}}_E$,

$$F_Q(tq) = \psi_0(\bar{t}t\, \bar{c}_0 y^* x). \tag{Oc10}$$

Furthermore $tq = 0$ whenever the matrix th^{-1} is integral: then $tx = th^{-1}(hx) \in L$ and $t(x+y) \in L$, hence $ty \in L$.

The volume of N is $I(h)^{-1} I(gh)^{-1}$. Using $|\Delta(u)|^2 = I(u)$ and $\Delta(u^{-1}) = \overline{\Delta(u)}$ from Lemma 4.1.13, the exact scalar formula becomes

$$c(g, h)\Delta(h^{-1})\Delta(gh) = \Delta(g) \sum_{q \in Q} F_Q(q). \tag{Oc11}$$

If h is integral invertible, the preceding annihilation assertion with $t = 1$ gives $Q = 0$. Integral right invariance of Δ gives $\Delta(gh) = \Delta(g)$ and $\Delta(h^{-1}) = 1$, so $c(g, h) = 1$. In particular $c(g, 1) = 1$ for every g. The operator argument above therefore proves unitarity, $|c(g, h)| = 1$ and the cocycle identity, without using the desired phase-one assertion.

We prove that every element supported on one orbit of finite places under conjugation belongs to the subgroup R. This suffices: every restricted adelic matrix is a product of an integral invertible unitary matrix and finitely many matrices supported on individual conjugate-place orbits. The scalar integral and its multiplier factor over these orbits. This follows from the product lattice and character, Fubini on the compact module M, and the valuation formula for Δ in Lemma 4.1.13. Thus, when h is supported on one orbit, the components of g off that orbit cancel from (Oc11), and we may take g supported there too.

First suppose the orbit is $\{w, \bar{w}\}$ with $w \neq \bar{w}$. Its unitary group is the general linear group at w, with the other component determined by inverse adjoint. We may reduce to h_w^{-1} integral. Indeed choose an integer power $u = \varpi_w^m I$ such that both u and $h_w^{-1}u$ are integral, and let b

be the supported unitary matrix with $b_w = u^{-1}$. Both b_w^{-1} and $(bh)_w^{-1}$ are integral. Once the assertion is proved for these two matrices, subgroup closure gives $h = b^{-1}(bh) \in R$.

Under this integrality condition, the annihilation assertion following (Oc10), applied to the idempotent which is one off $\bar{w}$ and zero at $\bar{w}$, shows that Q is supported at $\bar{w}$. Let t be the complementary idempotent, supported at $\bar{w}$. Then $tq = q$ and $\bar{t}t = 0$, so (Oc10) gives $F_Q(q) = 1$. The sum in (Oc11) is therefore the positive integer $\#Q$. The multiplier ratio in that equation is positive real as well. To check its phase explicitly, for a supported unitary matrix with component A at w, the inverse-adjoint and direct-factor formulas for the same Δ give

$$\Delta(g) = |\Delta(A^{(w)})|^2 \overline{\chi_f(\det A^{(w)})},$$

where $A^{(w)}$ is the rank-n finite adelic matrix with component A at w and identity elsewhere. Conjugation exchanges its determinant idele with the one at $\bar{w}$, so this follows from the inverse-adjoint formula in Lemma 4.1.13; it is a product over places, not a block sum of doubled rank. In the ratio $\Delta(g)/(\Delta(h^{-1})\Delta(gh))$ these characters cancel by multiplicativity of determinants. Only positive moduli remain. Hence $c(g,h) > 0$, and $|c(g,h)| = 1$ gives $c(g,h) = 1$. This proves the split case including residue characteristic two.

Now suppose $w = \bar{w}$. The field E contains a primitive 5040th root: from the parameter relations $q_1 = q_0\, 3l_0(s)2^s$ and $18480 \mid q_0$, one has $5040 \mid q_1$, so the appropriate power of ζ_{q_1} has the required order. It supplies $i \in \mathcal{O}_E$ with $i^2 = -1$, $\bar{i} = -i$, and a primitive seventh root ξ. Set

$$\omega = -(\xi + \xi^2 + \xi^4).$$

The identity $1 + \xi + \cdots + \xi^6 = 0$ gives $\omega + \bar{\omega} = 1$ and $\omega\bar{\omega} = 2$. A self-conjugate place cannot lie over 2: if it did, $\omega\bar{\omega}$ would imply one of $\omega, \bar{\omega}$ belongs to its prime ideal, then both would, contradicting their sum 1. In its odd residue field the image of i is moved by conjugation. The fixed residue field has some cardinality r, the full residue field has r^2 elements, and conjugation is its nontrivial quadratic automorphism. Thus $i^r = -i$, so $r \equiv 3 \pmod 4$.

The character in (WD3) has value -1 at a uniformizer of w. The valuation formula for Δ therefore makes each multiplier of a supported matrix a nonnegative integral power of $-r$. The module Q is supported at w by the annihilation assertion after (Oc10), so it has no nonzero 2-torsion. Apply Lemma 3.2.6(1) in rank one to $H = -1$. It gives a unit $a \in \widehat{\mathcal{O}}_E^{\times}$ with $\bar{a}a = -1$. Multiplication by a permutes Q and, by (Oc10), conjugates F_Q. Consequently

$$S = \sum_{q \in Q} F_Q(q)$$

is real. Multiplication by i fixes F_Q and has square negation. Its nonzero orbits have exactly four elements: an orbit of size one or two would give $q = -q$, hence $q = 0$. Each $F_Q(q)$ is a root of unity because an additive character of finite adeles obtained from the rational trace has torsion values. Therefore

$$S = 1 + 4A \tag{Oc12}$$

for an algebraic integer A, obtained by summing one value from each nonzero orbit.

Equation (Oc11) now reads

$$c(g,h)(-r)^k = S(-r)^m$$

for some nonnegative integers k, m. Since S is real and $|c(g,h)| = 1$, one has $c(g,h) \in \{1, -1\}$. If it were -1, (Oc12) would make

$$(-r)^m A = -\frac{(-r)^k + (-r)^m}{4}$$

a rational algebraic integer, and therefore an integer. But $-r \equiv 1 \pmod 4$, so the numerator on the right is 2 (mod 4), a contradiction. Thus $c(g,h) = 1$ at the nonsplit places too. The subgroup argument and product factorization prove the finite phase-one identity for all g, h.

The real Gaussian scalar and the completed action. We compute in one normalized real factor with sign matrix J, so $J^* = J$, $J^2 = 1$, and put $P = (1+J)/2$, $Q = 1 - P$. Let $g^*Jg = h^*Jh = J$ and $k = gh$. Write

$$D(u) = \overline{\det u_-}, \qquad T(u) = P + Qu.$$

Then $\det T(u) = \det u_-$ and $D(u) = \overline{\det T(u)}$. The relation $u^{-1} = Ju^*J$ gives $D(u^{-1}) = \overline{D(u)}$. The negative block is invertible: the negative-negative block of $u^*Ju = J$ says $u_-^* u_- = 1 + b^*b$ for the corresponding positive-negative block b. Thus $D(u) \neq 0$.

The integrand of (Oc7) is $\exp(-z^*Az)$ for $z = (x, y)$, where

$$A = \frac{1}{2}\left[\begin{pmatrix} h^*h & 0 \\ 0 & k^*k \end{pmatrix} + \begin{pmatrix} 1 & 1+J \\ 1-J & 1 \end{pmatrix}\right]. \tag{Oc13}$$

Indeed the cross term in its exponent is $-\operatorname{Re}(x^*y) + (y^*Jx - x^*Jy)/2$, which is the cross term of the Gaussian product times $\beta(x, y)$. Moreover

$$A + A^* = \begin{pmatrix} h^*h & 0 \\ 0 & k^*k \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} > 0.$$

Thus the complex Gaussian integral, with the prescribed π^{-2n} normalized Lebesgue measure, is $\det(A)^{-1}$. The formula for an accretive complex matrix follows from the usual positive Gaussian integral by holomorphic continuation on the connected domain $A + A^* > 0$; absolute Gaussian domination on compact subsets justifies that continuation.

Here is the determinant with its phase retained. Direct block multiplication gives

$$A = \begin{pmatrix} T(h) & Q \\ 0 & T(k^{-1})k \end{pmatrix}^* \begin{pmatrix} P & -Q \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -T(h) & -T(h) + T(g^{-1})k \end{pmatrix}. \tag{Oc14}$$

For this check, use $P^2 = P$, $Q^2 = Q$, $PQ = 0$, $J = P - Q$, $k = gh$, $h^*Jh = J$ and $k^*Jk = J$. Expanding the four blocks, the difference from (Oc13) is one half of $\operatorname{diag}(h^*Jh - J, J - k^*Jk)$, which is zero. The middle factor has determinant $\det(P + Q) = 1$. The last factor has determinant $\det(T(g^{-1})k)$, by subtracting its first block column from its second. The first factor has determinant $\overline{\det T(h) \det T(k^{-1}) \det k}$. Since $|\det k| = 1$, multiplication yields

$$\det A = D(g)D(h)D(k^{-1}). \tag{Oc15}$$

Consequently the product of this multiplier and the double Gaussian integral is exactly one. This proves $c(g, h) = 1$ for the real factor, with the very determinant specified in (WR5), and not merely a scalar of modulus one.

In the full adelic coordinates, the vacuum, bicharacter and Haar measure are the prescribed products of real factors and the scaled finite factor. The scalar integral (Oc7) factors by Fubini: at the real places there is Gaussian domination, and at the finite places its support is the compact module M. The multiplier in (WR3) is their product. Both scalar calculations therefore give $c(g, h) = 1$ adelically. Equations (Oc7)–(Oc9) prove the genuine group law and pairing unitarity. The identity operator is given by the reproduction formula and $\Delta(1) = 1$. Inverses are the operators at g^{-1}, and (Oc5) gives the conjugation formula for Heisenberg operators.

The same calculations give the norm bound

$$\|\omega_s(g)f\|_1 \le |\Delta_s(g)|\,\|e_n\|_1\|f\|_1 = 2^{n\#\Phi}|\Delta_s(g)|\|f\|_1.$$

In a real Fock space with N coordinates the corresponding bound is $2^N|D(g)|$. The finite operator has bound $|\Delta_f(g_f)|$, because its vacuum has L^1 norm one. Every scalar multiplier is continuous: the real one is a matrix minor; the finite one is invariant under right multiplication by the open integral unitary subgroup, by Lemma 4.1.13. For fixed f, the Banach-valued integrand $f(y)\rho(gy)e$ is continuous in g and dominated by $|f(y)|\|e\|_1$, independently of g. The adelic and real groups are first countable, so dominated convergence along sequences proves strong continuity. The norm bound and local boundedness of Δ then give joint continuity in (g, f).

Orthogonal sums have product vacua and bicharacters, product Haar measures, and multiplicative multipliers: the finite statement is in Lemma 4.1.13, while the negative real block is block diagonal. Fubini in the literal kernels therefore proves

$$\omega_{s\oplus s'}(g \oplus g')(u \otimes v) = \omega_s(g)u \otimes \omega_{s'}(g')v. \tag{Oc16}$$

Separating the real and finite factors proves the same identity for a real Fock vector tensored with a finite coherent vector. In particular the real vacuum stays fixed under the finite subgroup, and the finite span of its Heisenberg translates is preserved by Lemma 4.1.13, with its exact vacuum expansion.

Pairing unitarity and covariance also give, with no further normalization,

$$\langle\omega_s(g)v^0, \rho(z)\omega_s(g)v^0\rangle = \langle v^0, \rho(g^{-1}z)v^0\rangle, \qquad \langle\omega_s(g)v^0, \omega_s(g)v^0\rangle = 1.$$

Central group elements commute with all $\omega_s(g)$ by the genuine group law. Pullback along the pair homomorphism and multiplication by the given determinant characters prove that the pair operators, including their real weighted versions, are representations.

Finally complete the carrier in its invariant L^2 norm. Each $\omega_s(g)$ extends to a unitary operator with the already proved exact Heisenberg covariance. Under any fixed unitary Heisenberg identification of this completion with a polarized Schrödinger model, that operator is a unitary normalizer of the same Heisenberg representation. Weil's unitary-normalizer theorem [322] makes it an automorphism of the Schwartz–Bruhat space. Thus it preserves the inverse image of that space under the fixed identification. This argument does not select another scalar: the operator transported is exactly the one whose normalized integral and genuine product law were proved above. □

Definition 4.1.15. Let P be a parameter pair, $s \ge 0$ and $E = E_s$. The **fixed choices** of the Weil data of E are the following.

(WC1) Standard theta functionals. Fix theta functionals θ_+, θ_- of the standard lines of sign $+1$, -1. For signs s on $\{1, \dots, n\}$, the **theta functional** θ_{V_s} of the standard space V_s of signs s is the fixed theta functional of V_s with

$$\theta_{V_s}(\rho(x)v^0) = \prod_{i=1}^{n} \theta_{s_i}(\rho(x_i)v^0) \qquad (x \in \mathbb{A}_E^n);$$

these exist by (b) of Lemma 4.1.9.

(WC2) Fixed transporting elements. For a space with a point V with signs s on n, $k_V \in \mathrm{G}(s)$ is a fixed transporting element of V (WR3), with $k_{V_s} = 1$; transporting elements exist by Lemma 4.1.8.

(WC3) Canonical theta functionals and theta functions. The **canonical theta functional** of a space with a point V with signs s is

$$\theta_V(f) = \theta_{V_s}(\omega_s(k_V)f) \qquad (f \in \mathcal{M}_s),$$

with θ_{V_s} of (WC1) and k_V of (WC2); $\omega_s(k_V)$ is a bounded operator of $\mathcal{M}_s$ by Proposition 4.1.14, so θ_V is a bounded linear functional on $\mathcal{M}_s$, and for $V = V_s$ it is the functional of (WC1). For a space with a point V and $f \in \mathcal{M}_s$, the **theta function** of f is $x \mapsto \theta_V(\rho(x)f)$ on $\mathbb{A}_E^n$; a vector evaluated at a point always means its theta function.

(WC4) Theta kernels. For spaces with points S (signs s on n), T (signs s' on n') and automorphic characters α, β of $\mathrm{U}(1)$, the **theta kernel** of $\varphi \in \mathcal{M}_{S,T}$ is

$$\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h) = \theta_{\mathrm{Hom}(S,T)}(\omega_{\alpha,\beta}(g,h)\varphi) \qquad (g \in \mathrm{G}(s),\ h \in \mathrm{G}(s')),$$

with the canonical theta functional $\theta_{\mathrm{Hom}(S,T)}$ (WC3) of the space with a point $\mathrm{Hom}(S,T)$ ((VP5) of Definition 3.2.2).

Lemma 4.1.16. *Let S, T have ranks r_S, r_T and their fixed points. Let α, β be automorphic characters of* $\mathrm{U}(1)$.

1. *Let A be a finite conjugate-disjoint set of finite places, let $\Psi \in \mathcal{T}_A$ on the matrix position space of* $\mathrm{Hom}(S,T)$, *and let f be in its theta model. If finite unitary points g, h of S, T are identity outside $A \cup \bar{A}$, write* $\det_A$ *for the determinant idele supported at A. Then $\Psi'(X) = \Psi(h^{-1}Xg)$ belongs to $\mathcal{T}_A$, and*

$$\omega_{\alpha,\beta}((1,g),(1,h))(\Psi \otimes_A f) = \frac{\alpha[r_S](\det_A h)}{\beta[r_T](\det_A g)} |\det_A g|^{r_T/2} |\det_A h|^{-r_S/2} (\Psi' \otimes_A f).$$

2. *Let $\mathfrak{m}$ be a modulus, $A = \Sigma(\mathfrak{m})$, and let Φ be a function on r_T by r_S matrices over $R_{\mathfrak{m}}$. Write $\Phi^\natural$ for its integral-position extension by zero. For integral finite unitary points k, k' of S, T and every Fock vector v,*

$$\omega_{\alpha,\beta}((1,k),(1,k'))(v \otimes (\Phi^\natural \otimes_A v^0)) = \alpha(\det k')\beta(\det k)^{-1} v \otimes (\Phi^\natural_{k,k'} \otimes_A v^0),$$

where $\Phi_{k,k'}(X) = \Phi(\bar{k}'^{-1} X \bar{k})$ and the bars on k, k' here denote residue reduction. In particular the same operator sends $v \otimes v^0$ to $\alpha(\det k')\beta(\det k)^{-1} v \otimes v^0$.

Proof. We use the fixed (WR6) identification and its section independence from Lemma 4.1.3, and the literal finite kernel and multiplier formulas of Lemma 4.1.13.

First consider a single finite theta model and a unitary matrix U supported at $A \cup \bar{A}$. Let a be its matrix on A, extended by identity elsewhere, and put $\chi_f(t) = \upsilon_f(t)|t|^{-1/2}$. Unitarity specifies its conjugate components, so

$$U = a(a^*)^{-1}.$$

The supports of $a - 1$ and $(a^*)^{-1} - 1$ are disjoint. Write $I(a) = [\mathcal{O}_f^n : \mathcal{O}_f^n \cap a^{-1}\mathcal{O}_f^n]$. Intersections and their frame determinants factor across those disjoint places. Thus the inverse and adjoint multiplier identities of Lemma 4.1.13 give

$$\Delta(U) = \Delta(a)\Delta((a^*)^{-1}) = \Delta(a)\chi_f(\overline{\det a})\overline{\Delta(a)} = I(a)\overline{\chi_f(\det a)}. \tag{1}$$

Moreover $a(\mathcal{O}_f^n \cap a^{-1}\mathcal{O}_f^n) = a\mathcal{O}_f^n \cap \mathcal{O}_f^n$, so Haar scaling gives

$$I(a^{-1}) = I(a)/|\det a|, \qquad I(U) = I(a)^2/|\det a|. \tag{2}$$

Let B be the integral position box, J the fixed coefficient transform of Lemma 4.1.3, and put $T_a\Psi(x) = \Psi(a^{-1}x)$. Direct substitution into the polarized formula there gives

$$T_a R_A(y) = R_A(Uy)T_a. \tag{3}$$

The character pairing is preserved because U acts by a on position and by its inverse adjoint on the conjugate coordinates. Both sides of (3) have translation by $(Uy)_A$ and identical modulation exponents after this substitution.

The vector $\omega(U)v^0$ belongs to the finite coherent span by Lemma 4.1.13 and is fixed by the transformed corrected integral Heisenberg lattice. Indeed clause (2) of Lemma 4.1.3 identifies a corrected integral translation with

$$\Psi(x) \longmapsto \psi_0\big(\overline{l}^{\mathsf{T}}(x + c_0 l)_A\big)\Psi(x + l_A).$$

For integral l this fixes B: translation preserves its support and the phase is integral there, so its character is one. Clause (3) of that lemma identifies JB with the selected vacuum. Heisenberg covariance of the literal finite kernel then gives the stated transformed invariance. In the position realization the fixed space for this transformed lattice is one dimensional, spanned by T_aB: its modulation subgroup forces support in aL_A, by character orthogonality, and its translation subgroup forces constancy there. Hence $\omega(U)v^0 = c\,J(T_aB)$ in the selected finite factor.

The scalar is fixed by coherent evaluation at zero. In the literal finite kernel the integrand at zero is $e(y)e(-Uy)$. On the intersection where it is nonzero, unitarity makes the two semicharacter factors equal. Their product is $\psi_0(c_0\bar{y}^{\mathsf{T}}y) = 1$, since y is integral and c_0 is integral. The integral is therefore $I(U)^{-1}$. On the other hand

$$J(T_aB)(0) = \langle B, T_aB\rangle = \mathrm{vol}(L_A \cap aL_A) = |\det a|/I(a).$$

Using (1)–(2) gives $c = \overline{\chi_f(\det a)}$. Consequently

$$\omega(U)J\Psi = \overline{\chi_f(\det a)}\, J(T_a\Psi) \tag{4}$$

for every Ψ: first apply covariance to $R_A(y)B$, then use their spanning property. For a general vector f in (WR6), use the scaled selected coordinates z_A and denote all other coordinates by z_{off}. The preliminary normalization in Lemma 4.1.3 identifies $(\jmath\alpha\bar{\alpha})\mathcal{O}_f$ with the absolute inverse different. Hence ψ_0 is trivial on $\mathcal{O}_f$ and the integral lattice is its own annihilator; conjugation preserves that lattice, so this also holds for the pairing $(u, v) \mapsto \psi_0(uv)$. Trace invariance, $\bar{\jmath} = -\jmath$ and $\overline{\alpha\bar{\alpha}} = \alpha\bar{\alpha}$ give $\psi_0(\bar{t}) = \psi_0(t)^{-1}$. The character, measure, corrected lattice action and vacuum therefore satisfy every hypothesis of Lemma 4.1.4. Its pointwise projection identity gives

$$(P^A f)(z_A, z_{\mathrm{off}}) = e_A(z_A)f(0, z_{\mathrm{off}}),$$

where e_A is the selected finite vacuum and $z_{\mathrm{off}} \mapsto f(0, z_{\mathrm{off}})$ belongs to the complementary reproducing carrier. Expanding the finite sum in (WR6), every selected translation leaves the spectator factor fixed. Clause (3) of Lemma 4.1.3, applied to the selected vacuum, identifies the sum of its selected coefficients with $J\Psi$. Thus

$$(\Psi \otimes_A f)(z_A, z_{\mathrm{off}}) = (J\Psi)(z_A)f(0, z_{\mathrm{off}}).$$

Here J is the same fixed coefficient transform as above. Since U is identity on the complementary coordinates, the literal (WR3) kernel is a product: the complementary integral reproduces $f(0, z_{\mathrm{off}})$, while (4) applies to $J\Psi$ in the selected factor. Fubini is justified by integrability of both factors and the bounded vacuum kernel. Therefore

$$\omega(U)(\Psi \otimes_A f) = \overline{\chi_f(\det a)}\,((T_a\Psi) \otimes_A f).$$

The spectator factor remains $f(0, z_{\mathrm{off}})$.

Apply (4) to the matrix space with $U(X) = hXg^{-1}$. Its selected determinant is

$$\det_A U = (\det_A h)^{r_S}(\det_A g)^{-r_T}.$$

Its inverse changes the position variable to $h^{-1}Xg$. This preserves Schwartz–Bruhat functions, by invertibility and continuity, so the latter function is in $\mathcal{T}_A$ by Lemma 4.1.3. Multiply (4) by the exact (WR4) twist $\alpha(\det h)\beta(\det g)^{-1}$. For a unitary point supported on the selected conjugate pairs, its full determinant is the selected determinant divided by its conjugate. Hence the twist values are the base-change character values $\tilde{\alpha}(\det_A h)$ and $\tilde{\beta}(\det_A g)^{-1}$. Combining them with $\overline{\chi_f(\det_A U)} = \upsilon(\det_A U)^{-1}|\det_A U|^{-1/2}$ gives precisely the scalar in clause (1).

For clause (2), let $U(X) = k'Xk^{-1}$. This matrix and its inverse are integral at every finite place. Thus its lattice intersection is the integral lattice, its multiplier is one, and $e(Uz) = e(z)$: the support is preserved and the finite Hermitian norm in its phase is preserved. Changing variable in the literal coherent kernel and applying the (WD7) reproducing identity yields the actual coherent identity

$$\omega(U)f(z) = f(U^{-1}z). \tag{5}$$

It fixes the vacuum. By the coefficient transform formula, or directly by applying (5) to a finite sum of named translates, it changes Ψ to $\Psi(k'^{-1}Xk)$. Integral invertible matrices preserve integral positions and commute with reduction; this new function is exactly the extension by zero of $\Phi(\bar{k}'^{-1}X\bar{k})$. Real and finite operators commute because their bicharacters, vacua and measures factor. Thus the actual real integral defining $v \otimes f$ is unchanged; its absolute Bochner convergence permits the finite bounded operator to pass through it. Finally (WR4) supplies the stated determinant twist. Taking the position box instead of $\Phi^\natural$, and using $\mathbf{1}_A \otimes_A v^0 = v^0$, proves the final vacuum assertion. All calculations persist in rank zero and for empty A, with determinant, index and Haar factors equal to one. □

Lemma 4.1.17. *Use any fixed choices (WC1)–(WC3) of Definition 4.1.15; in particular θ_V denotes its canonical theta functional, with the chosen standard line factors, fixed transporting elements and rank-zero convention specified there. Let $V = (J_V, x_V)$ have signs s, and let $k \in \mathrm{G}(s)$ be any transporting element in the sense of (WR3). Then*

$$k\,V(E) = V_s(E), \qquad \theta_V(f) = \theta_{V_s}(\omega_s(k)f) \quad (f \in \mathcal{M}_s),$$

independently of the transporting element. This canonical functional is bounded, rational-Heisenberg invariant and normalized:

$$\theta_V(\rho(q)f) = \theta_V(f) \quad (q \in V(E)), \qquad \int_{W_n/V(E)} |\theta_V(\rho(x)e_n)|^2\,dx = 1.$$

For spaces with points V_1, V_2 and their block orthogonal sum,

$$\theta_{V_1 \oplus V_2}(f_1 \boxtimes f_2) = \theta_{V_1}(f_1)\theta_{V_2}(f_2) \qquad (f_i \in \mathcal{M}_{s_i}).$$

The same exact equality holds when the block sum and the coefficient product are read through a sign-preserving bijection of their coordinate index sets as in (WR6). Empty summands use the identity functional of rank zero.

Proof. Use the fixed choices (WC1)–(WC3) of Definition 4.1.15. Their prescribed standard vacuum-orbit product is supplied by Lemma 4.1.9.

First every transporting element maps the rational vector set onto the standard one, although its definition only requires containment. Indeed

$$B = x_{V_s} k x_V^{-1}$$

maps E^n into E^n. Testing the standard basis shows that all entries of B belong to E. Its adelic matrix is invertible, so its determinant is a nonzero element of E and $B \in \mathrm{GL}_n(E)$. Therefore $B(E^n) = E^n$ and $kV(E) = V_s(E)$. At least one such element exists by Lemma 4.1.8: its rational isometry γ and unitary k satisfy $x_V = \gamma^{-1} x_{V_s} k$, hence $B = \gamma$.

We need exact rational-unitary invariance of the fixed standard functional. It follows from Lemma 4.1.11 for the actual construction, as follows. Realize V_s as $\mathrm{Hom}(\mathrm{I}_1, V_s)$ with trivial external determinant characters. Its Gram matrix, point, signs, kernel, multiplier and rational vectors are literally those of V_s, after the coordinate identification $(i,1) \mapsto i$. Both standard theta functionals agree under this identification: the locally prescribed product gives the same product of the chosen line values on every translated vacuum, and the vacuum translates have dense span by Lemma 4.1.10. Thus this identification adds no scalar.

Here is the test-domain verification for this application. Regard the coherent carrier as a subspace of its Hilbert completion. In a fixed real coordinate system its vacuum is

$$e_\infty(u) = \exp\!\left(-\sum_\ell a_\ell |u_\ell|^2\right), \qquad a_\ell = \pi \operatorname{Im} \jmath_\ell > 0.$$

For a fixed adelic displacement z, its real coherent coefficient is exactly

$$(\rho(z)e)_\infty(u) = \beta_\infty(z,u) \exp\!\left(-\sum_\ell a_\ell |u_\ell - z_\ell|^2\right).$$

The first factor is the exponential of an imaginary real-linear function of u. Every derivative in u, multiplied by any polynomial in u, is therefore a polynomial times this same shifted Gaussian and phase. In particular every Schwartz seminorm is finite. More explicitly, for z in a compact real set and for each finite collection of derivatives in z and u, their absolute values are bounded by

$$C(1 + \|u\|)^M \exp(-a_{\min}\|u\|^2/2), \qquad a_{\min} = \min_\ell a_\ell > 0,$$

after increasing C, M. To obtain this bound, use $\|u - z\|^2 \geq \|u\|^2/2 - \|z\|^2$ and differentiate the displayed exponential. The majorant is integrable, as are its products with every polynomial. Difference quotients and their derivatives converge under this majorant. Consequently the real Heisenberg orbit of each displaced vacuum is C^∞ in the actual coherent L^1 norm, with derivatives given by these polynomial-Gaussian functions; it is also C^∞ in the Hilbert norm. This proves smoothness in the specified carrier, rather than only pointwise smoothness of a formula.

At the finite places put $L = \alpha \widehat{\mathcal{O}}_E^n$. The finite vacuum is

$$\mathbf{1}_L(u)\, \psi_{E,f}\!\left(\frac{c_0 \jmath}{2} \bar{u}^{\mathsf{T}} u\right).$$

It has compact support. The quadratic polynomial in the character is continuous and the character has an open kernel, so this function is locally constant. There is a single compact open subgroup H that works on its whole support: choose $H \subset L$ small enough that both the quadratic difference $Q(u+h) - Q(u)$, for $u \in L, h \in H$, and every needed bilinear character

difference lie in that open kernel. Such an H exists because L is compact and the bilinear maps are continuous; equivalently, cover L by finitely many neighborhoods on which those characters are constant and intersect their translation subgroups. This gives $e_f(u-h)=e_f(u)$ and $\beta_f(h,u)=1$ on L, while $L+h=L$. Hence $\rho(h)e_f=e_f$ for $h\in H$.

For a displaced vacuum the support is the compact open coset z_f+L, and multiplication by $\beta_f(z_f,u)$ is again locally constant there. Shrink H further into the kernel of this last character and its commutation character with z_f. The same calculation then gives $\rho(h)\rho(z_f)e_f=\rho(z_f)e_f$. Thus every displaced vacuum has an open finite Heisenberg stabilizer and compactly supported, locally constant finite coefficient; no condition that a denominator be odd is used. The argument takes place in the local fields themselves and includes the places above 2.

To identify precisely the test domain in a polarized model, apply the first assertion of Lemma 4.1.2 to $\mathrm{Hom}(\mathrm{I}_1,V_s)$ with trivial external characters. Its coordinate map $(i,1)\mapsto i$ identifies the Heisenberg action and Hilbert carrier with the present ones. That assertion identifies the Heisenberg-smooth vectors of this same carrier, with their real Schwartz seminorms and finite compact-open inductive limits, with Schwartz–Bruhat vectors, continuously in both directions. The smoothness and open stabilizers just proved therefore place every displaced vacuum in this precise polarized Schwartz domain. Only the domain assertion of that result is used here, not its assertion about the theta distribution. No theta normalization or rational-unitary invariance has been inferred from a Hilbert-space isomorphism.

The actions on all signs, negatives, sums and residual spaces are the literal kernels of (WR3). They are genuine, continuous and unitary by Proposition 4.1.14; the last assertion of that result says that, in this fixed Heisenberg identification, they preserve the Schwartz–Bruhat domain in both directions. Thus their images of the above test vectors are admissible test vectors as well. Normalized functionals on the auxiliary spaces exist by Lemma 4.1.9. Finally $\mu_{5040}\subset E$ is part of the parameter field hypothesis. These facts verify every action, domain, auxiliary-space and field hypothesis of Lemma 4.1.11 on the present kernels. That result gives

$$\theta_{V_s}(\omega_s(r)f)=\theta_{V_s}(f)\qquad(r\in\Gamma_{V_s}).\tag{1}$$

initially for the Schwartz domain, in particular for every displaced vacuum.

This identity extends to every $f\in\mathcal{M}_s$. Indeed Lemma 4.1.10 gives the Bochner identity $f=\int f(z)\rho(z)e_n\,d\mu_n(z)$ in the coherent L^1 norm. Approximating that integrable Banach-valued function by simple functions shows that the linear span of the displaced vacua is dense in $\mathcal{M}_s$. Both sides of (1) are bounded linear functionals, since the fixed standard theta functional is bounded and $\omega_s(r)$ is bounded by Proposition 4.1.14. Equality on the dense span therefore proves (1) on all of $\mathcal{M}_s$.

Let k,k' be transporting elements and put $r=k'k^{-1}$. By the onto assertion, r preserves $V_s(E)$. More explicitly $r=x_{V_s}^{-1}Dx_{V_s}$ with $D\in\mathrm{GL}_n(E)$. Since $r^*\varepsilon_s r=\varepsilon_s$ and $x_{V_s}^*J_{V_s}x_{V_s}=\varepsilon_s$, it follows that $D^*J_{V_s}D=J_{V_s}$. Hence $r\in\Gamma_{V_s}$, an actual rational unitary point. The genuine group law and (1) now give

$$\theta_{V_s}\omega_s(k')=\theta_{V_s}\omega_s(r)\omega_s(k)=\theta_{V_s}\omega_s(k).$$

Thus the canonical functional is independent of the transporting element. This conclusion is exact; it is not inferred from uniqueness up to a unit scalar.

Its boundedness follows from boundedness of the two factors in its definition. For $q\in V(E)$, exact covariance gives

$$\theta_{V_s}(\omega_s(k)\rho(q)f)=\theta_{V_s}(\rho(kq)\omega_s(k)f)=\theta_{V_s}(\omega_s(k)f),$$

since $kq \in V_s(E)$. Thus it is rational-Heisenberg invariant. Set $v_k = \omega_s(k)e_n$. The map $x \mapsto kx$ induces a topological isomorphism $W_n/V(E) \to W_n/V_s(E)$, so it sends invariant probability measure to invariant probability measure. The all-vector mean identity of Lemma 4.1.10, applied to the standard functional and v_k, gives

$$\begin{aligned}\int_{W_n/V(E)} |\theta_V(\rho(x)e_n)|^2\,dx &= \int_{W_n/V_s(E)} |\theta_{V_s}(\rho(y)v_k)|^2\,dy \\ &= \langle v_k, v_k\rangle = \langle e_n, e_n\rangle = 1.\end{aligned}$$

The penultimate equality uses actual unitarity of Proposition 4.1.14. Vacuum normalization alone would not justify this step; the all-vector mean is essential.

We finally prove the factorization with scalar exactly one. For standard signed spaces, the locally prescribed line product gives the product identity on every pair of translated vacua, because its defining line product is coordinatewise. The product map is bounded bilinear, with L^1 norm equal to the product of the two norms, by the product measure and Tonelli. Each functional is bounded. Extending first in one factor and then in the other along the dense spans of vacuum translates proves

$$\theta_{V_{s_1\sqcup s_2}}(f_1 \boxtimes f_2) = \theta_{V_{s_1}}(f_1)\theta_{V_{s_2}}(f_2). \tag{2}$$

Choose any transporting elements k_1, k_2 for V_1, V_2. Their block sum transports $V_1 \oplus V_2$ onto the standard space of concatenated signs: both the points and the rational vector sets are block sums, and the same finite norm-minus-one unit is fixed for every standard line. Independence just proved lets us use this block sum instead of any previously selected transporter of the sum. The exact tensor action of Proposition 4.1.14 gives

$$\begin{aligned}\theta_{V_1\oplus V_2}(f_1 \boxtimes f_2) &= \theta_{V_{s_1\sqcup s_2}}\big(\omega_{s_1}(k_1)f_1 \boxtimes \omega_{s_2}(k_2)f_2\big) \\ &= \theta_{V_1}(f_1)\theta_{V_2}(f_2),\end{aligned}$$

by (2).

For completeness this argument is unchanged by a sign-preserving coordinate bijection. Reindexing is the explicit permutation of coefficient variables; it carries the coordinate product vacuum, bicharacter and measure to the reindexed ones. Reindexing the integral kernel conjugates each operator by the same permutation: the negative principal block is reindexed on both its rows and columns and has the same determinant, and the finite intersection lattice is reindexed with the same determinant character. The standard theta functionals agree under that reindexing because their prescribed line products agree on every vacuum translate, hence on the whole model by density. Reindex $k_1 \oplus k_2$ in this manner and repeat the displayed calculation. This proves precisely the last asserted product equality. In rank zero all maps and multipliers are the identity and the standard functional is the identity of $\mathbb{C}$, so the same proof includes empty summands. □

Definition 4.1.18. Let V, V_0 be spaces with points with signs s on n, s_0 on n_0, p an index set, and (k, l, i) a splitting of V over V_0 along p ((MX1) of Definition 3.2.21).

(MX4) Zero terms and slices. For $f \in \mathcal{M}_s$ write $f^\theta(x) = \theta_V(\rho(x)f)$ $(x \in \mathbb{A}_E^n)$. A **zero term** of f along the splitting is a vector $Z(f) \in \mathcal{M}_{s_0}$ with

$$\theta_{V_0}(\rho(y)Z(f)) = \int_{\mathbb{A}_E^p/E^p} f^\theta(i_{\mathbb{A}}y + k_{\mathbb{A}}b)\,db \qquad (y \in \mathbb{A}_E^{n_0}).$$

The canonical functionals θ_V, θ_{V_0} are theta functionals in the sense of (WD9) by Lemma 4.1.17. Thus the displayed zero term exists and is unique by Lemma 4.1.6, equivalently (7) of Lemma 4.1.1. The **slice** of f at $x \in \mathbb{A}_E^p$ is $f[x] = Z(\rho(l_{\mathbb{A}}x)f)$. For a space with a point T with signs t on n', the **slice** of $f \in \mathcal{M}_{V,T}$ at an adelic $n' \times p$ matrix x is $f[x] = Z_{\mathrm{Hom}}(\rho(x\mathrm{pr}_k)f) \in \mathcal{M}_{V_0,T}$, where Z_{Hom} is the zero term along the splitting of $\mathrm{Hom}(V,T)$ over $\mathrm{Hom}(V_0,T)$ induced by the splitting ((MX1) of Definition 3.2.21) and $x\mathrm{pr}_k$ is read as a vector of $\mathrm{Hom}(V,T)$.

(MX6) Theta terms and integrable terms. Let D be a splitting of V over V_0 along p, T a space with a point with signs t on n', α, β automorphic characters of $\mathrm{U}(1)$ and $\varphi \in \mathcal{M}_{V,T}$. For $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(t)$ and $x \in \mathrm{M}_{n'\times p}(E)$ the **theta term** of φ at x is

$$\mathrm{term}_x(\varphi)(g,h) = \theta_{\mathrm{Hom}(V_0,T)}\Big((\omega_{\alpha,\beta}(g,h)\varphi)[x_T^{-1}x]\Big),$$

the slice being that of (MX4) for D and T. For $g \in \mathrm{G}(s)$ and a measurable function y on $\mathrm{G}(t)$ that is left invariant under Γ_T, φ has **integrable terms at g with weight y** (for D, T, α, β) if for every measurable set $R \subset \mathrm{G}(t)$ of representatives of $\Gamma_T\backslash\mathrm{G}(t)$

$$\int_R \sum_{x\in \mathrm{M}_{n'\times p}(E)} \big|\mathrm{term}_x(\varphi)(g,h)\, y(h)\big|\, dh < \infty,$$

dh the measure of $\mathrm{G}(t)$; integrable terms at g means weight $y = 1$.

(MX8) The vacuum factor. For a splitting D of V over V_0, with v^0 the vacuum vectors of $\mathcal{M}_s$ and $\mathcal{M}_{s_0}$ ((WD7) of Definition 3.2.12), the **vacuum factor** is

$$\mathrm{vf}(D) = \frac{\langle v^0, Z(v^0)\rangle}{\langle v^0, v^0\rangle} \qquad \text{(inner products of } \mathcal{M}_{s_0}\text{)}.$$

(MX10) Terms of the constant term along a line. Let A be a splitting of a space with a point W' over W (signs s on n) along a one-element set, with the line radicals $\mathrm{rad}_A^{\mathrm{ln}}$ of (MX2) of Definition 3.2.21. Let V be a space with a point with signs t on r and $f \in \mathcal{M}_{W',V}$. For $x \in \mathrm{M}_{r\times 1}(E)$ the **term of x in the constant term along A** is

$$\mathrm{ct}_A(x;f) = \int_{\mathrm{M}_{n\times 1}(\mathbb{A}_E)/\mathrm{M}_{n\times 1}(E)} \mathrm{term}_x(f)\big(\mathrm{rad}_A^{\mathrm{ln}}(y,0),1\big)\, dy \quad \text{if } x^*J_V x = 0,$$
$$\mathrm{ct}_A(x;f) = 0 \text{ otherwise,}$$

with the theta term of (MX6) for the splitting A, the space V and trivial characters $\alpha = \beta = 1$.

(MX13) Rational Lagrangians and their terms. Let W be a space with a point with signs on an index set m_W of even cardinality, p an index set with $2\#p = \#m_W$, $V_\emptyset$ the **zero space** (the space with a point on the empty index set, the standard space of the empty signs), and T a space with a point with signs on n'. For a splitting $D = (k,l,i)$ of W over $V_\emptyset$ along p ((MX1) of Definition 3.2.21; i is the empty $m_W \times \emptyset$ matrix), the **term** of D is the zero term (MX4)

$$Z_D \colon \mathcal{M}_{W,T} \longrightarrow \mathcal{M}_{V_\emptyset,T} = \mathbb{C}$$

along the splitting of $\mathrm{Hom}(W,T)$ over $\mathrm{Hom}(V_\emptyset,T)$ induced by D ((MX1) of Definition 3.2.21). A **rational Lagrangian** of W is an E-subspace $L \subseteq E^{m_W}$ of dimension $\#p$ on which the form h_W vanishes. A rational Lagrangian L is **split along** p if $L = kE^p$ for some splitting (k,l,i) of W over $V_\emptyset$ along p; for each such L one such splitting D_L is fixed, and the **term** of L is $Z_L = Z_{D_L}$.

4.2 The vacuum, conjugate jets, the seesaw identity, the chain gain and the divisibility errors

This section proves the two properties that Theorem A assumes of its parameters: the seesaw identity (Proposition 4.2.25), for every linear map represented by Bessel periods at one point, and the uniform local property of a family of divisibility errors (Proposition 4.2.54), which does not depend on the map. The identity is a seesaw identity in the sense of Kudla [171]. In our reading, the dual pairs of consecutive spaces Y_j and those of the definite spaces X_j sit in common symplectic spaces, and the identity compares the two theta correspondences there [322, 134, 136, 236, 206]; for the splittings see [172, 121, 228]. What the general principle does not give, and the section proves, is the arithmetic: one constant for all twists, integral coefficients, and control of the denominators and supports of the finite data. The section has five parts: the integral vacuum and the CM period, the conjugated jets, the seesaw identity, the divisibility of the chain at every prime, and the family of errors.

The rank-one vacuum has a scalar multiple whose theta values are integral with controlled conjugates (Proposition 4.2.1), by comparison with theta functions on CM elliptic curves [77, 219, 220, 142, 272]; right after it, the vacuum parameter of the datum is specified as the least such constant. The CM period is fixed next (Lemma 4.2.3), and the jet pair and the vacuum scales are specified after it. Between the two come the critical pairs and the lift of a character from U(1) (Definition 4.2.4, Definition 4.2.5), defined for an arbitrary jet pair and used in Section 6. The normalised CM curve has good reduction everywhere [266, 281, 108], and its period carries a factor that is a unit at every prime, by Kronecker's first limit formula averaged over the ideal classes (Lemma 4.2.9; see [329]). In our reading this is the classical route to the formula of Chowla and Selberg [57].

Proposition 4.2.1. *For clarity the rank-one vacuum used here is the vector v_s^0 in the coherent reproducing realization of the Heisenberg representation of the standard positive hermitian line over E_s. Put $\jmath = -1/\sqrt{-d}$, $c_0 = (1+\sqrt{-d})/2$, and use the fixed half-inverse-different idele α_s. Its normalized positive-type matrix coefficient is*

$$e_s(z) = \exp\left(-\frac{\pi}{\sqrt{d}}\sum_{\tau\in\Phi_s}|z_\tau|^2\right)\mathbf{1}_{\alpha_s\widehat{\mathcal{O}}_{E_s}}(z_f)\psi_{E_s,f}\left(\frac{c_0\jmath}{2}\bar{z}_f z_f\right).$$

The Heisenberg bicharacter is, explicitly, $\mathfrak{b}(z,w) = \beta_+(z,w) = \psi_{E_s}((\jmath/2)\bar{w}z)$, as in (WD6), and the carrier is the model (WD7) of integrable functions satisfying

$$f(x) = \int f(y)\mathfrak{b}(y,x)e_s(x-y)\,dy, \qquad (\rho(z)f)(x) = \mathfrak{b}(z,x)f(x-z).$$

The measure is μ_1 of (WD6), the theta functional θ_s is this canonical normalized positive-line functional, and $v_s^0 = e_s$ is the vacuum of (WD5) and (WD7). The finite coordinate conversion is $\mathrm{fv}(x) = (0,\alpha_s x)$ of (WD8). Define the finite theta value

$$\theta_s^\circ(x) = \theta_s(\rho(0,\alpha_s x)v_s^0), \qquad x\in\mathbb{A}_{E_s,f}.$$

In particular this is a theta value, not the value $e_s(0,\alpha_s x)$ of the underlying coefficient function.

There exists an integer $C\geq 0$, depending only on P, such that for every integer $s\geq 0$ there is one scalar $c_s\in\mathbb{C}^\times$ satisfying, simultaneously for every $x\in\mathbb{A}_{E_s,f}$,

$$|\iota(c_s\theta_s^\circ(x))|_p \leq 1 \quad \text{for every prime } p \text{ and every ring homomorphism } \iota:\mathbb{C}\longrightarrow\mathbb{C}_p,$$

and

$$|\sigma(c_s\theta_s^\circ(x))| \leq \exp\bigl(Cq_1(s)^2\bigr) \quad \text{for every } \sigma\in\mathrm{Aut}(\mathbb{C}/K).$$

Thus the set of nonnegative integers satisfying the displayed uniform property is nonempty.

Proof. **1. A fixed normalized CM elliptic lattice.** Write $N = q_1(s)$ and $f = [E_s : K] \le N$. Put $\omega = (1+\sqrt{-d})/2$. Since $d \equiv 7 \pmod 8$, the algebra $\mathcal{O}_K/2$ is $\mathbb{F}_2 \times \mathbb{F}_2$ and the images of $\omega, \bar{\omega}$ are its complementary nonzero idempotents. In $\mathbb{C}/\mathcal{O}_K$ choose the nonzero two-torsion point t with $\omega t \neq 0$ and $\bar{\omega} t \neq 0$; it corresponds to $(1,1)$ in that product. Scale the lattice to $\Lambda_0 = \Omega\mathcal{O}_K$ so that $\wp''(t) = 2$, and keep this choice independently of s. This is possible because, at a nonzero two-torsion point, $\wp''(t) = 2(\wp(t) - e')(\wp(t) - e'') \neq 0$. Choose a with $a^2 = 12\wp(t)$. The elliptic curve with this marked point has equation

$$B: \quad y^2 + axy = x^3 + x, \qquad t = (0,0), \qquad \Delta = a^4 - 64, \qquad j(B) = \frac{(a^4-48)^3}{a^4-64}. \tag{1}$$

These identities follow by translating $X = \wp(z) - \wp(t)$ and completing the square in $\wp'(z)/2$; the coefficient of X is $\wp''(t)/2 = 1$. By Theorem A.2.17, $j(B)$ is algebraic. Equation (1), monic in a, makes a algebraic.

2. The same normalized discriminant is a unit at every prime. We prove

$$|\iota(a)|_p \le 1, \qquad |\iota(\Delta)|_p = 1 \tag{2}$$

for every prime and every $\iota : \mathbb{C} \to \mathbb{C}_p$. Work over a finite local extension over which B, its full CM action and its marked two-torsion are defined and B has good reduction; Theorem A.2.17 supplies this extension. Let L be this finite local extension, put $S = \operatorname{Spec} \mathcal{O}_L$, and let $\mathcal{B}/S$ be the good elliptic scheme with generic fibre B_L. The scheme S is noetherian and normal, and its generic point $U = \operatorname{Spec} L$ is open and dense. Both source and target $\mathcal{B}$ are abelian, hence semiabelian, schemes. The semiabelian homomorphism extension theorem [77, Chapter I, Proposition 2.7], equivalently [178, Proposition 3.3.1.5], therefore extends each CM endomorphism of B_L uniquely to $\mathcal{B}$; this is the precise application of Theorem A.2.19. Uniqueness shows that these extensions preserve addition, composition and the identity, since the corresponding equalities hold on the generic fibre. Thus the entire $\mathcal{O}_K$ action extends to this same good model, including at $p = 2$.

The marked point reduces away from the origin. For odd p this follows because reduction is injective on prime-to-p torsion. For $p = 2$ here are the needed details. The full $\mathcal{O}_K \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ action splits the good curve's 2-divisible group into two factors of height one: on the characteristic-zero Tate module each factor has rank one. Their dimensions add to one. Over the algebraically closed residue field a height-one 2-divisible group of dimension zero is etale and one of dimension one is connected; thus exactly one factor is etale. This also proves ordinary reduction in this case. The point t, being nonzero in both $\mathcal{O}_K/2$ factors, has nonzero image in the etale factor. Therefore its reduction is not the origin.

Choose an integral good Weierstrass equation. The marked point has integral affine coordinates, since its reduction is affine. Translate those coordinates to $(0,0)$. The inverse formula and $2t = 0$ give $a_3 = a_6 = 0$, so the equation is

$$y^2 + a_1xy = x^3 + a_2x^2 + a_4x.$$

Smoothness at the reduction of $(0,0)$ forces a_4 to be a unit: the two partial derivatives there are $-a_4$ and 0. After a finite integral extension choose a root b of the monic equation $b^2 + a_1b = a_2$ and replace y by $y + bx$. The x^2 coefficient disappears; a_4 is unchanged. A unit fourth root of a_4 then changes this good equation to $y^2 + a'xy = x^3 + x$. Its discriminant $(a')^4 - 64$ is a unit and a' is integral.

This is the same normalization as (1), not merely an isomorphic elliptic curve. A characteristic-zero change between the two equations preserving the origin and $(0,0)$ has $x = u^2x'$ and

$y = u^3y' + b'x'$. Comparing the x coefficients gives $u^4 = 1$. Discriminants change by u^{12}, a unit. Hence $\iota(\Delta)$ is a unit as claimed. Finally $a^4 = \Delta + 64$ proves integrality of a. This includes $p = 2$.

3. Reduced theta torsion values: integrality and all conjugate bounds. Let σ_0 be the reduced sigma function of Λ_0, normalized by $\sigma_0'(0) = 1$, and let ϑ be its reduced translate by t, normalized by $\vartheta(0) = 1$. For an entire section G of degree one put

$$G[u] = e^{-\pi|u|^2/(2A_0)}G(u), \qquad A_0 = \mathrm{Area}(\Lambda_0).$$

The usual Weierstrass addition formulas in normalization (1) give

$$\vartheta[2u] = \vartheta[u]^4 - \sigma_0[u]^4,$$

$$\sigma_0[2u]^2 = \vartheta[u]^2\sigma_0[u]^2(4\vartheta[u]^4 + a^2\vartheta[u]^2\sigma_0[u]^2 + 4\sigma_0[u]^4). \tag{3}$$

These are identities of complex numbers, including their Gaussian factors: the factor at $2u$ is the fourth power of the factor at u. One can obtain (3) directly by substituting $x = \sigma_0^2/\vartheta^2$ into the duplication formula on (1); equality off the zeros extends by holomorphy. The normalization $\wp''(t) = 2$ gives the coefficient 1 in the first formula.

Set

$$D_a(v, w) = ((v^2 - w^2)^2,\ vw(4v^2 + a^2vw + 4w^2)).$$

The nonzero pair $W(u) = (\vartheta[u]^2, \sigma_0[u]^2)$ satisfies $W(2u) = D_a(W(u))$. It never vanishes because the zeros of sigma and theta are the disjoint classes 0 and t. If u is torsion and $l \in \Lambda_0$, the entries of $W(u + l)$ differ from those of $W(u)$ by roots of unity. Indeed the reduced translation law has a sign multiplier and the additional phase is the exponential of πi times the alternating lattice pairing of u, l; this pairing is rational for torsion u. The maximum absolute value of the pair, after any complex automorphism or any p-adic embedding, consequently depends only on u mod Λ_0. Its doubling orbit is finite.

By (2), the two reduced quartics $D_{\iota(a)}$ have no nonzero common zero over the residue field. To check this explicitly, the first component forces $w = v$ or $w = -v$. For a nonzero pair the second then forces respectively $8 + a^2 = 0$ or $8 - a^2 = 0$, each contradicting $a^4 - 64 \neq 0$; this argument also applies in characteristic two. Divide a pair by a coordinate of maximum norm and reduce. Homogeneity gives

$$\|D_{\iota(a)}(v, w)\|_{\max} = \|(v, w)\|_{\max}^4.$$

The finite orbit of u now implies $\|\iota W(u)\|_{\max}^{4^i} = \|\iota W(u)\|_{\max}^{4^j}$ for some $i \neq j$. The norm is nonzero, so it equals one. In particular

$$|\iota(\vartheta[u])|_p \leq 1 \quad \text{for all torsion } u,\ p,\ \iota. \tag{4}$$

For the complex bound no CM theta reciprocity is required. There are finitely many algebraic conjugates a' of a. On the compact set $\max(|v|, |w|) = 1$ each $\|D_{a'}(v, w)\|_{\max}$ has a positive minimum, since its two components have no common nonzero zero. Let $0 < \delta \leq 1$ be a common lower bound for these finitely many minima. Then

$$\|D_{a'}(v, w)\|_{\max} \geq \delta\|(v, w)\|_{\max}^4.$$

Apply any $\gamma \in \mathrm{Aut}(\mathbb{C})$ to (3). For $r_i = \|\gamma W(2^i u)\|_{\max}$ we have $r_{i+1} \geq \delta r_i^4$. If $r_0 > \delta^{-1/3}$, iteration gives $r_i \geq \delta^{-1/3}(\delta^{1/3}r_0)^{4^i} \to \infty$, contradicting the finite orbit. Thus, with the fixed number $M = \max(1, \delta^{-1/6})$,

$$|\gamma(\vartheta[u])| \leq M \quad \text{for all torsion } u,\ \gamma. \tag{5}$$

All operations here apply to the actual reduced values, not to their absolute values before conjugation. Equations (4)–(5) therefore avoid any claim that an automorphism commutes with a Gaussian.

4. The precise tight-frame theta product. Fix s. Put $\mathfrak{a} = E_s \cap \alpha_s \widehat{\mathcal{O}}_{E_s}$ and $A = \sqrt{2\pi/\sqrt{d}}$. In positive Fock coordinates the rational lattice is $\Gamma = \{A(\tau\lambda)_\tau : \lambda \in \mathfrak{a}\}$. Here are the different and duality computations. The absolute different is $\mathfrak{D}_{E_s/\mathbb{Q}} = (\sqrt{-d})\mathfrak{D}_{E_s/K}$, and the prescribed half-inverse-different ideal satisfies $\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E_s/K}^{-1}$. Hence $\jmath\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E_s/\mathbb{Q}}^{-1}$ as fractional ideals. On rational vectors the alternating Fock form is, up to sign, $\mathrm{Tr}_{E_s/\mathbb{Q}}(\jmath\lambda\bar{\mu})$: indeed $A^2 = 2\pi/\sqrt{d}$ and the trace is the sum over each complex pair. Its integral dual is therefore $\jmath^{-1}\mathfrak{D}_{E_s/\mathbb{Q}}^{-1}\bar{\mathfrak{a}}^{-1} = \mathfrak{a}$. This proves that Γ is self-dual. The fixed sign semicharacter is the one determined by c_0. Notice also

$$\mathcal{O}_{E_s} \subset \mathfrak{a}, \qquad \mathrm{Tr}_{E_s/K}(\mathfrak{a}\bar{\mathfrak{a}}) \subset \mathcal{O}_K. \tag{6}$$

For (6), $\mathfrak{a}\bar{\mathfrak{a}}$ is the inverse relative different; equivalently this follows place by place from the specified half-inverse-different idele. These statements involve the same α_s as the proposition.

For $0 \le k < N$ define the complex-linear form

$$L_k(z) = \frac{\Omega}{NA} \sum_{\tau \in \Phi_s} \overline{\tau(\zeta^k)} z_\tau, \qquad S(z) = \prod_{k=0}^{N-1} \vartheta(L_k(z))^N. \tag{7}$$

Distinct K-embeddings are distinct on ζ, and character orthogonality gives

$$\sum_{k=0}^{N-1} \tau(\zeta^k)\overline{\tau'(\zeta^k)} = N\delta_{\tau,\tau'}.$$

Since $A_0 = |\Omega|^2\sqrt{d}/2$, the sum of the degree-$N$ hermitian forms of the factors in (7) is exactly the standard Fock form:

$$\sum_k N\,\frac{\pi}{A_0} L_k(z)\overline{L_k(w)} = \sum_\tau z_\tau \bar{w}_\tau. \tag{8}$$

There is no unspecified scale in this identity.

For $\lambda \in \mathfrak{a}$ one has $L_k(NA(\tau\lambda)_\tau) = \Omega\,\mathrm{Tr}_{E_s/K}(\lambda\bar{\zeta}^k) \in \Lambda_0$ by (6). Thus S is a theta function for $N\Gamma$. The semicharacter is the prescribed one, not an arbitrary multiplier. Indeed $t = \Omega/2$ modulo Λ_0 is the unique distinguished point. For $l = \Omega(a+b\omega)$, the sigma sign is $(-1)^{a+b+ab}$ and translation by t adds the factor $(-1)^b$. Thus the theta multiplier is $(-1)^{a+ab} = (-1)^{\mathrm{N}_{K/\mathbb{Q}}(a+b\omega)}$, since $(d+1)/4$ is even. This is the canonical multiplier with trace-one parameter $c_0 = \omega$. More generally, the canonical multiplier of a hermitian form B with trace-one parameter c_0 is $\exp(\pi i\,(\pi^{-1}\,\mathrm{Im}\,B)(\bar{c}_0 u, u))$. Canonical multipliers multiply under sums of forms. The forms in (8) are K-linear and use the same c_0 as the marked-point theta of Λ_0; hence their product multiplier at a period is exactly that of the Fock form. This proves the signed automorphy for $N\Gamma$. Also $S(0) = 1$.

5. Fourier inversion in the fixed model. Use the canonical positive-line functional (WC3) of Definition 4.1.15 throughout. It is bounded, rational-Heisenberg invariant and normalized by Lemma 4.1.17; on this standard line its transporting element is the identity. Thus the normalized theta mean used here applies to this SAME fixed functional, not to an independently selected phase. The all-vector theta mean identity is Lemma 4.1.10(2), and the actual finite-part entire map and its injectivity are Lemma 4.2.2. In its coordinates $\mathrm{rv}(z) = (z/A, 0)$, so its map is exactly $\mathcal{T}v(z) = e^{\|z\|^2/2}\theta_s(\rho(z/A, 0)v)$. Its rational finite-translation formula is precisely the raw

formula in (9); multiplying by the literal (WD8) scalar gives the signed formula. The vacuum's signed lattice invariance follows from (WD5) and the self-duality calculation above, hence its entire image lies in the prescribed principal theta line. Its nonzero image follows from injectivity and $\langle v_s^0, v_s^0\rangle = 1$, the latter supplied by Lemma 4.1.10(1). Thus all transport used below is on the prescribed model. We recall the finite argument to keep track of the carrier and the scalar. Write

$$(U_y F)(z) = \exp\left(-\sum_\tau \bar{y}_\tau z_\tau - \|y\|^2/2\right) F(z+y)$$

for reduced Fock translation, and write R_λ for its signed version for $\lambda \in \Gamma$. On the theta space for $N\Gamma$ the R_λ form a representation of $G = \Gamma/N\Gamma$, of cardinal $D = N^{2f}$. The commutator pairing identifies its character group with $Y = N^{-1}\Gamma/\Gamma$; self-duality of Γ proves perfection. If $\mathcal{A} = \sum_{\lambda\in G} R_\lambda$, then $\mathcal{A}U_y S$ is a theta function for Γ. The latter space is one-dimensional by Theorem A.2.16.

Let $\mathcal{T}$ denote the Fock theta map of the fixed coherent model: $\mathcal{T}v(z)$ is its theta value at the real vector z/A, multiplied by $e^{\|z\|^2/2}$. Thus $\mathcal{T}v(0) = \theta_s(v)$ and

$$\mathcal{T}(\rho(0,\gamma_f)v) = U_{-A(\tau\gamma)_\tau}\mathcal{T}v, \qquad \mathcal{T}(H(\alpha_s^{-1}\lambda_f)v) = R_{-A(\tau\lambda)_\tau}\mathcal{T}v. \tag{9}$$

Here $H(x) = \rho_f(x) = \psi_{E_s,f}(\jmath\alpha_s\bar{\alpha}_s\, c_0\bar{x}x/2)\rho(\mathrm{fv}(x))$, exactly as in (WD8) of Definition 3.2.12; the raw operators are those of (WD7), and the canonical theta functional of (WC3) is normalized by Lemma 4.1.17 in the sense of (WD9). To check the sign and scalar, put $r = A(\tau\gamma)_\tau$ and let $t_v(z) = \theta_s(\rho(z/A, 0)v)$. The archimedean cocycle in these coordinates is $\mathfrak{b}_\infty(r,z) = \exp(-i\,\mathrm{Im}\sum_\tau \bar{r}_\tau z_\tau)$. Rational invariance therefore gives

$$\theta_s(\rho(z/A,\gamma_f)v) = e^{\,i\,\mathrm{Im}\sum_\tau \bar{r}_\tau z_\tau}\, t_v(z-r).$$

Multiplying by $e^{\|z\|^2/2}$ and using $(\|z\|^2 - \|z-r\|^2)/2 = \mathrm{Re}\sum \bar{r}_\tau z_\tau - \|r\|^2/2$ gives exactly $U_{-r}\mathcal{T}v$. For H the additional, unaltered scalar is $\psi_{E_s,f}(\jmath\alpha_s\bar{\alpha}_s\, c_0\bar{x}x/2)$ with $x = \alpha_s^{-1}\lambda_f$, as in (WD8); multiplying U_{-r} by that scalar is precisely R_{-r}. Thus raw ρ and signed H have the distinct transformations in (9). The Gaussian vacuum makes $\mathcal{T}v$ entire on its finite Heisenberg span. The map is injective there: if its entire function vanishes, rational invariance and density give zero finite theta values, and then zero theta values everywhere by continuity; their squared mean is the invariant squared norm.

The function $\mathcal{T}v_s^0$ is a nonzero theta function for Γ. The lattice translations fix the vacuum with the stated sign; nonzero follows from its norm one and the normalized theta mean. Consequently there are scalars b_y such that

$$\mathcal{A}U_y S = b_y \mathcal{T}v_s^0.$$

Character orthogonality, including the identity character, gives the exact inversion equation

$$\sum_{y\in Y} U_{-y}\mathcal{A}U_y S = DS. \tag{10}$$

Indeed expand the left side and commute U_y past R_λ: its coefficient is the sum of the characters of G at λ, hence D for $\lambda = 0$ and zero otherwise. Equation (10) writes S as a linear combination of translates of $\mathcal{T}v_s^0$. Using (9) it constructs a vector F in the actual finite Heisenberg span with $\mathcal{T}F = S$. No independently normalized theta functional has been substituted. Since $S(0) = 1$, $F \neq 0$.

Apply the same Fourier average to this vector. Some $b_y \neq 0$, since otherwise (10) would give $S = 0$. For one such y equations (9) give finite vectors a_g, b and one scalar $c_s = b_y \neq 0$ with

$$c_s v_s^0 = \sum_{g \in G} H(a_g)\rho(0, \alpha_s b)F, \qquad |G| = N^{2f}. \tag{11}$$

There are no denominators in this chosen averaging identity; the factor D^{-1} was used only to construct F in (10).

6. Values of the product vector. At a rational finite translation represented by $\gamma \in E_s$, (7)–(9) and (8) show

$$\theta_s(\rho(0, \gamma_f)F) = \prod_{k=0}^{N-1} \vartheta\Big[-\frac{\Omega}{N} \operatorname{Tr}_{E_s/K}(\gamma\bar{\zeta}^k)\Big]^N. \tag{12}$$

The brackets mean the reduced value of paragraph 3. Each argument is a torsion point of $\mathbb{C}/\Lambda_0$, since its quotient by Ω lies in K. The Gaussian factors in the product agree exactly with the reduced Fock translation factor by (8); this is why (12) involves reduced theta values.

The representation is locally constant at finite places on its finite Heisenberg span. The image $\alpha_s^{-1}E_s$ is dense in the finite adeles. Every finite translate of F therefore equals a rational finite translate, not merely a limit of their theta values. Thus (12), (4) and (5) imply, for every finite x,

$$|\iota(\theta_s(\rho(0, \alpha_s x)F))|_p \leq 1, \qquad |\gamma(\theta_s(\rho(0, \alpha_s x)F))| \leq M^{N^2}. \tag{13}$$

The equality of translates is chosen before applying the possibly discontinuous field map. There is consequently no continuity assumption on ι or γ.

7. The same scalar works everywhere and has the required bound. Evaluate (11) after $\rho(0, \alpha_s x)$ and apply θ_s. The Heisenberg product law gives

$$c_s\theta_s^\circ(x) = \sum_{g \in G} \xi_{g,x}\, \theta_s(\rho(0, \alpha_s x_{g,x})F), \tag{14}$$

where each $\xi_{g,x}$ is a value of a finite adelic additive character, hence a root of unity. The coefficients have p-adic norm one at every embedding, and so do all their complex conjugates in absolute value. The ultrametric triangle inequality and (13) prove the required all-prime integrality of this one c_s. The ordinary triangle inequality gives

$$|\gamma(c_s\theta_s^\circ(x))| \leq N^{2f} M^{N^2} \leq \exp\big((2 + \log M)N^2\big),$$

because $f \leq N$ and $\log N \leq N$. Choose the nonnegative integer $C = \lceil \log M \rceil + 2$. Its construction used only the fixed normalized CM lattice for K, so C depends on P alone. This proves both quantifiers in the statement, in fact the complex bound for every automorphism of $\mathbb{C}$. Thus the set of nonnegative integers satisfying the displayed uniform property is nonempty.

□

From now on, for every parameter pair P, the vacuum parameter $c_{\mathrm{vac}}(P)$ is the **vacuum constant** of P: the least integer $C \geq 0$ such that for every $s \geq 0$ there is $c \in \mathbb{C}^\times$ for which the values $F(x) = c\,\theta_s^\circ(x)$, $x \in \mathbb{A}_{E_s,f}$, satisfy

$$\|\iota(F(x))\|_p \leq 1 \quad \text{and} \quad |\sigma(F(x))| \leq e^{C\, q_1(s)^2}$$

for every prime p, every ring homomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_p$ and every automorphism σ of $\mathbb{C}$ over K. Here $\theta_s^\circ\colon \mathbb{A}_{E_s,f} \to \mathbb{C}$ is the finite theta value of the vacuum of the standard positive hermitian line over E_s of Proposition 4.2.1, which shows that such integers C exist. The constant bound C_P is taken with this value of $c_{\mathrm{vac}}(P)$.

Lemma 4.2.2. *Let V have positive signs, let $\iota = \Phi_s \times n$, and let $\mathcal{M}_{\mathrm{fin},V}$ be the linear span in its canonical model of the vectors $\rho(\mathrm{fv}(t))e_n$, $t \in \mathbb{A}^n_{E,f}$. For $F_0 \in \mathcal{M}_{\mathrm{fin},V}$ put*

$$\mathscr{E}_{V,F_0}(Z) = \exp(\|Z\|^2/2)\theta_V(\rho(\mathrm{rv}(Z))F_0) \qquad (Z \in \mathbb{C}^\iota).$$

Then this function is entire, the map $F_0 \mapsto \mathscr{E}_{V,F_0}$ is injective, and

$$\mathscr{E}_{V,F_0}(Z) = \sum_{\mathbf{m}\in\mathbb{N}^\iota} \frac{\theta_V(v_{X^{\mathbf{m}}} \otimes F_0)}{\mathbf{m}!} Z^{\mathbf{m}}.$$

The series is locally uniformly convergent. For every polynomial Q and every $c \in \mathbb{C}$,

$$Q(\partial_Z)\mathscr{E}_{V,F_0}(cZ)|_{Z=0} = \theta_V(v_{Q(c\cdot)} \otimes F_0).$$

These are ordinary derivatives.

More precisely, let $q \in V(E)$ in the fixed frame, write $q = \mathrm{rv}(a) + \mathrm{fv}(t)$ with $a_{v,i} = \kappa_v^{-1} q_{v,i}$ and $t = \alpha^{-1} q_f$. Then

$$\mathscr{E}_{V,\rho(\mathrm{fv}(t))F_0}(Z) = \exp(\bar{a}^{\mathsf{T}} Z - \|a\|^2/2)\mathscr{E}_{V,F_0}(Z - a).$$

For $\rho_f(t)$ in place of $\rho(\mathrm{fv}(t))$, the right side is multiplied by exactly $\psi_{E,f}(\jmath\alpha\bar{\alpha}\, c_0\, t^* t/2)$. The normalized real coordinate is $\mathrm{rv}(Z)_v = \kappa_v Z_v$; in particular, if $A = \sqrt{2\pi/\sqrt{d}}$, its physical infinite coordinate is Z/A.

Proof. Every vector in the specified finite span has the form

$$F_0(u, z_f) = e^\infty(u) f_0(z_f), \qquad e^\infty(u) = \exp(-\|u\|^2/2),$$

in the normalized real and finite coordinates. Indeed finite Heisenberg translation affects only the finite vacuum factor, and its span has this same real factor. The finite factor is integrable, since it is a finite sum of translates of the compactly supported finite vacuum. The scalar in (WD8) changes neither this span nor this assertion.

For positive signs the explicit real bicharacter gives the identity

$$(\rho(\mathrm{rv}(Z))e^\infty)(u) = \exp(-\|Z\|^2/2)\exp(\bar{u}^{\mathsf{T}} Z) e^\infty(u).$$

Consequently, in the canonical model,

$$\exp(\|Z\|^2/2)\rho(\mathrm{rv}(Z))F_0 = \sum_{\mathbf{m}\in\mathbb{N}^\iota} \frac{Z^{\mathbf{m}}}{\mathbf{m}!} \bar{u}^{\mathbf{m}} e^\infty(u) f_0(z_f). \tag{3}$$

This is a locally uniformly convergent series in the model's L^1 norm: on $\max_i |Z_i| \le R$, the sum of absolute values is bounded by

$$\exp\Big(R\sum_i |u_i| - \|u\|^2/2\Big)|f_0(z_f)|,$$

which is integrable. Tails tend to zero uniformly on any smaller polydisc by the same bound or the power-series majorant with a larger radius. The reproducing closed-range assertion of Lemma 4.1.10 shows that the polynomial coefficients in (3) belong to the model; alternatively they are the norm derivatives of the model-valued entire function in (3).

They are exactly the specified data $v_{X^{\mathbf{m}}} \otimes F_0$, not a new realization. In fact (WR6), applied to the displayed product form of F_0, is the reproducing integral for the real polynomial Fock vector times its unchanged finite factor:

$$(v_Q \otimes F_0)(u, z_f) = Q(\bar{u})e^{\infty}(u)f_0(z_f).$$

One may verify the real reproducing equation directly by the Gaussian integral with polynomial insertions (differentiate the Gaussian identity under its integrable majorant). This also proves that the Bochner integral defining this datum converges in the canonical L^1 norm. Applying the bounded functional θ_V to (3) proves the entire expansion and permits ordinary termwise differentiation. The monomial derivative $\partial^{\mathbf{m}} Z^{\mathbf{m}}|_0 = \mathbf{m}!$ cancels its displayed denominator; linearity gives the formula for Q and c with no further factorial or conjugation.

We verify the finite translation formula directly, including its sign. Set $q_\infty = \mathrm{rv}(a)$ and $q_f = \mathrm{fv}(t)$. Real and finite Heisenberg translations commute. The projective law in Lemma 4.1.10 and rational invariance give

$$\theta_V(\rho(\mathrm{rv}(Z))\rho(q_f)F_0) = \beta_s(\mathrm{rv}(Z), q_\infty)\,\theta_V(\rho(\mathrm{rv}(Z-a))F_0).$$

For example commute $\rho(q)$ to the left, where θ_V absorbs it; the commutator contributes

$$\beta_s(\mathrm{rv}(Z), q_\infty)^2,$$

and combining with $\rho(-q_\infty)$ contributes its inverse once. The surviving bicharacter is

$$\exp((\bar{a}^{\mathsf{T}} Z - \bar{Z}^{\mathsf{T}} a)/2).$$

Multiplying by $\exp(\|Z\|^2/2)$ and dividing by the Gaussian at $Z - a$ gives exactly

$$\exp(\bar{a}^{\mathsf{T}} Z - \|a\|^2/2).$$

Formula (WD8) supplies the additional scalar for $\rho_f(t)$ verbatim. In particular neither its sign nor its factor $1/2$ is discarded.

Finally suppose $\mathscr{E}_{V,F_0}$ vanishes identically. The Gaussian never vanishes, so the theta function of F_0 vanishes at every real adelic vector. Rational invariance and the same projective commutation calculation show that it vanishes at every sum of a real vector and a vector of $V(E)$. This set is dense in W_n: the image of E^n in the finite adeles is dense by additive strong approximation, and multiplication by the fixed finite frame is a homeomorphism. The theta function is continuous by strong Heisenberg continuity and boundedness of θ_V. It therefore vanishes on all of W_n. The all-vector mean identity of Lemma 4.1.10 gives $\langle F_0, F_0\rangle = 0$, and hence $F_0 = 0$ in the model, including its continuous representative. This proves injectivity on exactly the asserted finite-part domain. □

Lemma 4.2.3. *There exist $\Omega_{\mathrm{CM}} \in \mathbb{C}^\times$, $a \in \overline{\mathbb{Q}}$ and $u_{\mathrm{CS}} \in \overline{\mathbb{Q}}^\times$, depending only on P, with the following properties. Put*

$$\Lambda_0 = \Omega_{\mathrm{CM}}\mathcal{O}_K, \qquad t_0 = \Omega_{\mathrm{CM}}/2.$$

Then t_0 is nonzero two-torsion, $\omega t_0, \bar{\omega} t_0 \notin \Lambda_0$ for $\omega = (1+\sqrt{-d})/2$, and

$$\wp''_{\Lambda_0}(t_0) = 2, \qquad g_2(\Lambda_0) = 12\wp_{\Lambda_0}(t_0)^2 - 4, \qquad a^2 = 12\wp_{\Lambda_0}(t_0).$$

The curve $B_a : Y^2 + aXY = X^3 + X$ is a good integral model at every finite place, with invariant differential $dX/(2Y + aX)$ pulling back to dz. The number u_{CS} is a unit under every embedding into every $\mathbb{C}_p$.

For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, let Λ^σ be the actual period lattice of $B_{\sigma a}$ for the differential $dX/(2Y+\sigma(a)X)$. There are a nonzero fractional ideal $\mathfrak{b}_\sigma$ and $\Omega_\sigma \neq 0$ such that

$$\Lambda^\sigma = \Omega_\sigma \mathfrak{b}_\sigma^2/N(\mathfrak{b}_\sigma), \quad A_\sigma := \mathrm{Area}(\Lambda^\sigma) = |\Omega_\sigma|^2\sqrt{d}/2, \quad A_\sigma|\sigma u_{\mathrm{CS}}|^2 = \pi e^{c_{\mathrm{CS}}(d)}\, 2\pi/\sqrt{d}.$$

At the identity these choices can be $\mathfrak{b}_1 = \mathcal{O}_K$ and $\Omega_1 = \Omega_{\mathrm{CM}}$.

For every $s \geq 0$ put $f = f_s$ and define the principal root and the nonzero scale

$$s_0 = \exp\!\left(\frac{1}{f}\,\mathrm{Log}((\upsilon(1,\alpha_s)|\alpha_s|^{-1/2})^{-1})\right), \qquad \eta = (s_0/u_{\mathrm{CS}})\sqrt{2\pi/\sqrt{d}}\,\Omega_{\mathrm{CM}}^{-1}.$$

Some positive power of s_0 belongs to K. For every σ over K, the conjugate-model scale

$$t_\sigma = \sigma(s_0/u_{\mathrm{CS}})\sqrt{2\pi/\sqrt{d}}\,\Omega_\sigma^{-1}$$

satisfies $t_1 = \eta$ and $|t_\sigma|^2 = \Omega_s^{-1}$. If $\pi_M \in K$ has $|\pi_M|^2 = M$, then $|\sigma(\pi_M)t_\sigma|^2 = M/\Omega_s$.

The choices above can be made together with, for every s, a scalar $c_s \neq 0$ satisfying the simultaneous all-prime integral and all-conjugate bounds defining $c_{\mathrm{vac}}(P)$.

Proof. **The principal normalized lattice.** Start with the lattice $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$ and its point $1/2$. Neither $\omega/2$ nor $\bar\omega/2$ belongs to this lattice: otherwise ω or $\bar\omega$ would belong to $2\mathcal{O}_K$, contrary to their coefficients in the displayed integral basis. The point $1/2$ is nonzero two-torsion.

Write $e = \wp_{\mathcal{O}_K}(1/2)$. It is a simple root of the nonsingular Weierstrass cubic. If e', e'' are the other two roots, the differential equation for $\wp$ gives

$$\wp''_{\mathcal{O}_K}(1/2) = 2(e-e')(e-e'') \neq 0.$$

Choose any fourth root Ω_{CM} of $\wp''_{\mathcal{O}_K}(1/2)/2$. Scaling of the Weierstrass function gives

$$\wp''_{\Omega_{\mathrm{CM}}\mathcal{O}_K}(\Omega_{\mathrm{CM}}/2) = 2.$$

Since $\wp'' = 6\wp^2 - g_2/2$, this is exactly the asserted g_2 identity. Choose a square root a of $12\wp_{\Lambda_0}(t_0)$. All hypotheses of Lemma 4.2.6 now hold. That lemma proves algebraicity and integrality of a, good reduction at every finite place, and the exact differential assertion. In particular the initial period lattice of this equation and differential is literally Λ_0, not a further homothety.

Apply Lemma 4.2.8 to this lattice and marked point. It supplies one algebraic u_{CS} with the asserted all-prime unit property and all the conjugate area identities. Its square-ideal conclusion supplies $\mathfrak{b}_\sigma, \Omega_\sigma$. At the identity the original principal lattice already has the required form, so use $\mathcal{O}_K, \Omega_{\mathrm{CM}}$ there. The quadratic Dirichlet character is real; hence $L'(1,\chi_K)/L(1,\chi_K)$ is real, and the constant in that lemma equals $c_{\mathrm{CS}}(d)$.

The fixed finite idele meets the period-root hypotheses. Fix s and put $\delta = \mathrm{cyc}'_{q_1(s)}(\zeta)$ and $\omega = (1+\sqrt{-d})/2$. By Lemma 3.2.8, δ generates the relative different and divides $q_1(s)$. Every rational prime dividing $q_1(s)$ divides q_0 and therefore divides $(d+1)/4$: for odd primes this follows from $q_0 \mid 2(d+1)$; for 2 it follows from $16 \mid q_0$. Thus at every place in the support of δ, exactly one of $\omega, \bar\omega$ is nonunit, since $\omega\bar\omega = (d+1)/4$ and $\omega + \bar\omega = 1$. The relative different is invariant under complex conjugation, so the valuations of δ at conjugate places agree. The definition of α_s selects δ^{-1} exactly where ω is nonunit. Consequently, at each finite place, the sum of the valuations of α_s and $\bar\alpha_s$ is $-v(\delta)$. Outside the support of δ both valuations are zero. This proves the fractional-ideal identity

$$(\alpha_s\bar\alpha_s) = \mathfrak{D}_{E_s/K}^{-1}.$$

The splitting character in the fixed Weil data is unitary and trivial on finite units, with the prescribed infinite part. Thus every hypothesis of Lemma 4.2.11 holds for this actual idele and character.

The argument of the principal logarithm is nonzero. Its displayed exponential has fth power $(\upsilon(1,\alpha_s)|\alpha_s|^{-1/2})^{-1}$, so it is one of the roots allowed in Lemma 4.2.11. That lemma proves the positive-power assertion and $|\sigma(s_0)|^2 = \mathrm{rd}_{E_s}/\sqrt{d}$. Its second conclusion, using the actual area identity just obtained, gives exactly $|t_\sigma|^2 = \Omega_s^{-1}$ and the asserted formula after multiplication by π_M. At the identity the prescribed choice $\Omega_1 = \Omega_{\mathrm{CM}}$ gives $t_1 = \eta$. This calculation uses the periods of the conjugate equations; it makes no identification of Ω_σ with $\sigma(\Omega_{\mathrm{CM}})$.

Finally, Proposition 4.2.1 proves that the set of admissible nonnegative integral bounds in the specification of $c_{\mathrm{vac}}(P)$ is nonempty. Its least element therefore belongs to that set. For this least bound and every s, choose a witnessing nonzero scalar c_s. These are statements about the fixed theta model and functional, and impose no additional condition on the normalized elliptic lattice chosen above. Hence they can be chosen together with the curve, unit and scales already constructed. □

Definition 4.2.4. Convention. For a real number u, u^{-1} means $1/u$ if $u \neq 0$ and 0 if $u = 0$; so $(uu')^{-1} = u^{-1}u'^{-1}$ for all reals u, u'.

Partial L-functions. For an automorphic character ν of U(1) and a finite set S of finite places of E, we write $L^S(z,\nu)$ for the partial L-function $L^S(z,\tilde{\nu})$ of (XD7) of Definition 3.2.22, of the base change $\tilde{\nu}(x) = \nu(x/\bar{x})$, with the Euler factors at the places of S and of $\bar{S}$ removed ($z \in \mathbb{C}$; below it is evaluated at integers). Write S_M for the set of chosen places.

Critical pairs. A **critical pair** $p = (\nu, z)$ consists of an automorphic character ν of U(1) of weight $-c$, for a function $c\colon \Phi \to \mathbb{Z}_{\geq 0}$, with tame part trivial off the chosen places, and an integer z with

$$1 \leq z \leq t, \qquad z \leq c(v) \leq 4b \quad \text{for every } v \in \Phi.$$

For $1 \leq z' \leq z$ the pair (ν, z') is again a critical pair, a **lower point** of p. The critical pair p is **flat** if $c(v) = z$ for every $v \in \Phi$, that is, ν has weight $-z$ at every infinite place.

Ratios. For a critical pair $p = (\nu, z)$, a tame character ϑ with modulus Q and a sign $s \in \{\pm 1\}$, the **ratio** is

$$r_p(\vartheta, s) = \frac{|L^{S_M \cup Q}(z, (\nu\vartheta^s)^{-1})|}{|L^{S_M}(z, \nu^{-1})|} \in \mathbb{R}_{\geq 0}, \qquad r_p(\vartheta) = r_p(\vartheta, 1),$$

and the **ratio product** of an automorphic character ν up to z is $F_{\nu,z}(\vartheta) = \prod_{z'=1}^{z} r_{(\nu,z')}(\vartheta)$ (for a critical pair (ν, z), the product of the ratios of its lower points).

Ramified and main pairs. A critical pair $p = (\nu, z)$ with tame part ψ is **ramified** if $\psi_w \neq 1$ exactly at the chosen places w. For $1 \leq j \leq m$ it is a **main pair for** j if

(RH1) $10 \leq z$ and $m < z \leq r$;

(RH2) $\psi = \omega^j$;

(RH3) $\kappa_j(v) + r - t \leq c(v) \leq \kappa_j(v)$ for every $v \in \Phi$.

Ramified ratios. A function F of the m-fold twists is a **ramified ratio** if there are a ramified critical pair (ν, z) and an integer $1 \leq n \leq m$ with $F(\vartheta) = F_{\nu,z}(\vartheta^n)$ for every m-fold twist ϑ.

Chain and turn ratios. For an m-fold twist ϑ, the **chain ratios** ($0 \le i < j \le m$) and the **turn ratios** ($0 \le i, j < m$) are

$$q_{ij}(\vartheta) = F_{\mu_j^0/\mu_i^0,\,t}(\vartheta^{\,j-i}), \qquad \tau_{ij}(\vartheta) = F_{\mu_{i+1}^0\mu_{j+1}^0,\,t}(\vartheta^{\,j+1}).$$

Absolute value up to a norm power. A function c from the m-fold twists to $\mathbb{C}$ has **absolute value** g **up to a norm power**, for a function g from the m-fold twists to $\mathbb{R}$, if there are $C \in \mathbb{R}$ and $n \in \mathbb{Z}$ with $|c(\vartheta)|^2 = C\,(\mathrm{N}Q)^n g(\vartheta)^2$ for every m-fold twist ϑ with modulus Q ($C = 0$ allowed).

Definition 4.2.5. (OL1) L-value data. For $n \ge 2$, **L-value data** X of rank n consist of a level system Σ_X, natural numbers $\ell_X(\tau)$ ($\tau \in \Phi$), the **row lengths**, and an automorphic character ν_X of $\mathrm{U}(1)$ of weight $\tau \mapsto -(\ell_X(\tau) + \lfloor n/2 \rfloor)$ whose tame part is trivial off $M_X \cup Q_X$. Its **descent pair** O_X has ranks $1 < n$: the level system Σ_X, the sign $+1$ and $W(O_X)$ the standard definite line, $\mathrm{inc}(1)$ a fixed index, $\mathrm{ord} = 0$, $\kappa^+(\tau, 1) = \ell_X(\tau)$, $\kappa^+(\tau, i) = 0$ for $i \ge 2$, $\kappa^- = 0$, and $\mu_{O_X} = \nu_X$. So κ_{O_X} is one row of length $\ell_X(\tau)$ at τ, $e_{O_X} = 0$, $\chi_{O_X} = 1$ and $\chi'_{O_X} = \nu_X \upsilon^{\lfloor n/2 \rfloor}$.

(OL2) The kernel form. For $g \in G_f(1)$ the **kernel form** $k_X(g)\colon G_f(n) \to \mathbb{V}_{O_X}$ is defined by

$$\xi^{\mathsf{T}} G_{\kappa_{O_X}}\, k_X(g)(h) = \theta_{O_X}\big(\omega_{O_X}((1,g),(1,h))\,\mathrm{J}_{O_X}(\xi)\big) \qquad (\xi \in \mathbb{V}_{O_X},\ h \in G_f(n)),$$

$(1, g)$ and $(1, h)$ the adelic points with archimedean component 1; it lies in $\mathrm{Form}(O_X)$.

(OL3) The lift. With $\mathbf{K}_X^{\mathrm{ch}} \subseteq \mathbf{K}_0(1)$ the elements reducing to 1 at every chosen place of Σ_X, the **lift** of X is the class sum $\lambda_X = \sum_{[g]} \frac{1}{\#\Gamma_g}\, k_X(g) \in \mathrm{Form}(O_X)$ over $\mathrm{U}(1)(E)\backslash G_f(1)/\mathbf{K}_X^{\mathrm{ch}}$, $\Gamma_g = \mathrm{U}(1)(E) \cap g\mathbf{K}_X^{\mathrm{ch}} g^{-1}$.

(OL4) Critical pairs. Let D be admissible data, $p = (\nu, z)$ a critical pair with $z \ge 2$ (ν of weight $-c$) and ϑ a tame character with modulus Q. The L-value data $X(p, \vartheta)$ of rank $2z$ have the chosen places of D and the places of Q as level system, row lengths $c(\tau) - z$ and character $\nu\vartheta$; put $j_p = \sum_{\tau\in\Phi}(c(\tau) - z)$. The lift depends on two parameters attached to the parameter pair P alone: a **jet pair** $(\Omega_{\mathrm{CM}}, u_{\mathrm{CS}}) \in \mathbb{C}^\times \times \overline{\mathbb{Q}}^\times$ and, for every integer $s' \ge 0$, a **vacuum scale** $c_{s'} \in \mathbb{C}^\times$; this definition uses of them only that they are nonzero (and u_{CS} algebraic). Let s be the exponent of D ($E = E_s$, $f = \#\Phi_s$). The **lift of** p **at** ϑ is $\varphi_{p,\vartheta} = c_s^{2z}\,(\pi_M \eta_P)^{j_p}\,\lambda_{X(p,\vartheta)}$, where the **jet scale** is

$$\eta_P = \frac{s_0}{u_{\mathrm{CS}}}\sqrt{2\pi/\sqrt{d}}\;\Omega_{\mathrm{CM}}^{-1}, \qquad s_0 = \exp\Big(\tfrac{1}{f}\,\mathrm{Log}\big((\upsilon(\alpha)|\alpha|^{-1/2})^{-1}\big)\Big),$$

with Log the principal branch, α the Weil idele and υ the splitting character of E_s ((WD2), (WD3)), and $(\Omega_{\mathrm{CM}}, u_{\mathrm{CS}})$ the jet pair; its **algebraic part** is $\mathrm{alg}_p(\vartheta) = \|\varphi_{p,\vartheta}\|^2/\mathrm{N}Q^{2z-1}$, and its **denominator** is $n_\nu = M^{4fz}(Md)^{j_p}$. For an integer $N \ge 1$, the N**-integral algebraic part** of p at ϑ is

$$I_p^N(\vartheta) = N\, n_\nu^2\, \mathrm{alg}_p(\vartheta).$$

From now on, for every parameter pair P, the jet pair $(\Omega_{\mathrm{CM}}, u_{\mathrm{CS}})$ and the vacuum scales c_s ($s \ge 0$) are fixed as a choice provided by Lemma 4.2.3: Ω_{CM} and u_{CS} have the properties stated there for its period and its unit, so that the jet scale η_P is the scale η of that lemma, and for every $s \ge 0$ the values $c_s\theta_s^\circ(x)$, $x \in \mathbb{A}_{E_s,f}$, satisfy

$$\|\iota(c_s\theta_s^\circ(x))\|_p \le 1 \quad \text{and} \quad |\sigma(c_s\theta_s^\circ(x))| \le e^{c_{\mathrm{vac}}(P)\,q_1(s)^2}$$

for every prime p, every ring homomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_p$ and every automorphism σ of $\mathbb{C}$ over K.

Lemma 4.2.6. *Let $d \equiv 7 \pmod 8$ be prime, $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$, and $\omega = (1+\sqrt{-d})/2$. Let $M \subset \mathbb{C}$ be a rank-two lattice stable under $\mathcal{O}_K$, and let $t \notin M$ satisfy*

$$2t \in M, \qquad \omega t \notin M, \qquad \bar{\omega} t \notin M.$$

Suppose $g_2(M) = 12\wp_M(t)^2 - 4$ and choose $a \in \mathbb{C}$ with $a^2 = 12\wp_M(t)$. Then a is an algebraic integer and

$$|\iota(a^4 - 64)|_p = 1$$

for every prime p and every field embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$. The equation

$$B_a : \quad Y^2 + aXY = X^3 + X$$

is a good integral model at every finite place after passing to a number field containing a. Its invariant differential

$$\frac{dX}{2Y + aX}$$

pulls back to dz under $X = \wp_M(z) - \wp_M(t)$, $Y = (\wp'_M(z) - aX)/2$.

Proof. The algebraicity, integrality of a, normalized equation, differential comparison and unit discriminant at odd primes follow from Lemma 4.2.7, after translating its uniformization parameter by t. It remains to prove the unit assertion at two.

Fix a two-adic embedding. By Theorem A.2.17, over a finite extension of $\mathbb{Q}_2$ the curve has a smooth integral elliptic model with its full CM action and its marked two-torsion point defined. Enlarge the field finitely as needed below. Every CM endomorphism extends to the good elliptic scheme by Theorem A.2.19. Its 2-divisible group therefore has the action of $\mathcal{O}_K \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, since two splits in K. The two idempotents split this group into factors of height one: on the generic fibre the Tate module of a CM elliptic curve is locally free of rank one over $\mathcal{O}_K \otimes \mathbb{Z}_2$, and the ranks of the finite flat kernels, hence their heights, are unchanged on specialization.

Over an algebraic closure of the residue field the two dimensions sum to one. A height-one divisible group of dimension zero is etale, while one of dimension one is connected; thus one factor is etale. The point t has nonzero component in both factors of the generic two-torsion, because the images of $\omega, \bar{\omega}$ are the two complementary nonzero idempotents modulo two and both $\omega t, \bar{\omega} t$ are nonzero. Its component in the etale factor remains nonzero after reduction: specialization identifies the geometric points of a finite etale group scheme over a strictly henselian valuation ring with those of its special fibre. Hence t reduces away from the identity.

Choose a good integral Weierstrass equation. Because t reduces away from the identity, it has integral affine coordinates. Translate it to $(0,0)$. The equation has $a_6 = 0$, and the inverse formula for a point of order two gives $a_3 = 0$. It therefore has the form

$$y^2 + a_1 xy = x^3 + a_2 x^2 + a_4 x.$$

At the reduction of $(0,0)$ its two partial derivatives are $-a_4$ and zero. Smoothness forces a_4 to be a unit. Choose an integral root b of $b^2 + a_1 b = a_2$ after finite extension, and substitute the corresponding shear in y; this removes the coefficient of x^2 and does not change a_4. A unit fourth root of a_4 then changes the equation to

$$y^2 + a'xy = x^3 + x,$$

with a' integral and discriminant $(a')^4 - 64$ a unit.

This last equation has the identical differential normalization up to a fourth root of unity, rather than an arbitrary isomorphism scale. Indeed a characteristic-zero change between two Weierstrass equations preserving the identity and the marked point $(0,0)$ has the form $X = u^2X'$, $Y = u^3Y' + sX'$; translations in X, Y vanish because they preserve that point. Since both equations have coefficient of X equal to one and $a_3 = 0$, comparison gives $u^4 = 1$. Discriminants differ by u^{12}, a unit. The original discriminant $a^4 - 64$ is therefore a two-adic unit as well. This proves the assertion for the chosen embedding, and the argument applies to every embedding.

Finally the coefficients of B_a are integral and its discriminant is a unit at every finite place of any number field containing a. Its projective Weierstrass model is consequently a smooth elliptic scheme there. The differential identity is the explicit coordinate calculation already cited. □

Lemma 4.2.7. *Let $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$ with $d \equiv 7 \pmod 8$ prime. Let $\Lambda \subset \mathbb{C}$ be a rank-two lattice stable under $\mathcal{O}_K$, let $t \notin \Lambda$ with $2t \in \Lambda$, and choose $a_1 \in \mathbb{C}$ such that*

$$g_2(\Lambda) = 12\wp_\Lambda(t)^2 - 4, \qquad a_1^2 = 12\wp_\Lambda(t).$$

Put

$$x(z) = \wp_\Lambda(z+t) - \wp_\Lambda(t), \qquad y(z) = \tfrac{1}{2}(\wp'_\Lambda(z+t) - a_1 x(z)).$$

Then a_1 is an algebraic integer. For every prime p and every embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, one has $|\iota(a_1)|_p \le 1$. If p is odd, then

$$|\iota(a_1)^4 - 64|_p = 1.$$

The Weierstrass equation

$$Y^2 + a_1XY = X^3 + X$$

defines the same complex elliptic curve as $\mathbb{C}/\Lambda$, and after applying ι it has good reduction at every odd p, with discriminant $\iota(a_1)^4 - 64$.

Fix such an odd p, write $O = \mathcal{O}_{\mathbb{C}_p}$, $a = \iota(a_1)$, and let $x_p, y_p \in \mathbb{C}_p[[z]]$ be the Taylor series of x, y read through ι. There is a unique $X_0(U) \in U^2O[[U]]$ satisfying

$$U^2 + aX_0(U)U = X_0(U)^3 + X_0(U), \qquad X_0(U) = U^2 + \text{terms of degree at least three},$$

and

$$x_p(z) = X_0(y_p(z)).$$

Moreover $y_p(0) = 0$, $y_p'(0) = 1$, and

$$x_p' = 2y_p + ax_p, \qquad y_p' = 1 + 3x_p^2 - ay_p.$$

In particular the subring $O[[y_p]] = \{f(y_p) : f \in O[[U]]\}$ is preserved by d/dz, and it contains x_p.

Proof. **The normalized model and its modulus.** Write $e = \wp_\Lambda(t)$. As t is a nonzero point of order two, $\wp'_\Lambda(t) = 0$. The Weierstrass differential equation gives $g_3(\Lambda) = 4e^3 - g_2(\Lambda)e = -8e^3 + 4e$. With $a_1^2 = 12e$ this becomes

$$g_2 = (a_1^4 - 48)/12, \qquad g_3 = -a_1^2(a_1^4 - 72)/216.$$

Consequently

$$g_2^3 - 27g_3^2 = a_1^4 - 64, \qquad j(\Lambda) = \frac{(a_1^4 - 48)^3}{a_1^4 - 64}. \tag{1}$$

The denominator is nonzero because Λ is a lattice.

For completeness Λ is homothetic to a fractional $\mathcal{O}_K$-ideal. Choose $0 \neq \omega \in \Lambda$. The rational span of Λ is a two-dimensional $\mathbb{Q}$-space stable under K, and contains the two-dimensional space $K\omega$; hence they coincide. Thus $I = \omega^{-1}\Lambda \subset K$ is a nonzero finitely generated $\mathcal{O}_K$-module spanning K, so is a fractional ideal. By the classical integrality of singular moduli, $j(I) = j(\Lambda)$ is an algebraic integer; see J. S. Milne, Modular Functions and Modular Forms (2017), Corollary 12.16. Equation (1) shows that $u = a_1^4$ is a root of the monic polynomial

$$(U - 48)^3 - j(\Lambda)(U - 64).$$

Thus u is integral over the ring of algebraic integers, hence an algebraic integer, and a_1 is one as well, by its monic equation $V^4 - u = 0$. Therefore its image under every p-adic embedding is integral.

Let now p be odd and put $a = \iota(a_1)$ and $\Delta = a^4 - 64$. We already have $|\Delta|_p \leq 1$. If $|\Delta|_p < 1$, then $a^4 - 48 = \Delta + 16$ has norm one, since 16 is a p-adic unit. Equation (1) would imply $|\iota(j(\Lambda))|_p = |\Delta|_p^{-1} > 1$, contradicting its integrality. Hence $|\Delta|_p = 1$.

The change from the standard equation $v^2 = 4u^3 - g_2u - g_3$ is $X = u - e$, $Y = (v - a_1X)/2$. Substitution gives exactly $Y^2 + a_1XY = X^3 + X$, with inverse $u = X + e$, $v = 2Y + a_1X$. Its discriminant is $a_1^4 - 64$, by the usual Weierstrass invariant formulas. The coefficients are algebraic, and at each odd p they lie in the valuation ring of a finite extension of $\mathbb{Q}_p$ with unit discriminant. Its projective Weierstrass model is therefore a smooth elliptic scheme over that ring, proving good reduction; base change gives the same conclusion over O.

The integral local coordinate is the original one. In $O[[U]]$, the equation $U^2 + aXU - X^3 - X = 0$ has derivative -1 with respect to X at $(X, U) = (0, 0)$. Coefficient recursion therefore gives a unique solution with $X(0) = 0$: at each positive degree the new coefficient occurs with coefficient -1, while all remaining terms depend only on earlier coefficients. All coefficients are integral. The linear coefficient is zero and the quadratic coefficient is one, so the solution is the asserted $X_0(U)$.

The original analytic functions satisfy this equation. Also $x(0) = y(0) = 0$, $x'(0) = \wp'_\Lambda(t) = 0$, and

$$y'(0) = \wp''_\Lambda(t)/2 = (6e^2 - g_2/2)/2 = 1.$$

Thus y_p has a formal compositional inverse. Composing the original curve equation with that inverse gives a zero-constant solution in $\mathbb{C}_p[[U]]$, which by the same uniqueness is X_0. Hence $x_p = X_0(y_p)$.

Differentiation preserves the integral parameter ring. The definition gives $x' = 2y + a_1x$. Differentiating the Weierstrass equation yields

$$(2y + a_1x)y' + (a_1y - 3x^2 - 1)x' = 0.$$

Substitute $x' = 2y + a_1x$. This series is not zero, since its linear coefficient is two. The formal series ring in characteristic zero is a domain, so cancellation gives $y' = 1 + 3x^2 - a_1y$. These identities carry over under ι. Since $x_p = X_0(y_p)$ with integral X_0 and $a \in O$, the series y'_p belongs to $O[[y_p]]$. For $f \in O[[U]]$, its formal derivative f' is integral and $(f(y_p))' = f'(y_p)y'_p$. Both factors belong to $O[[y_p]]$. This proves the preservation claim. □

Lemma 4.2.8. *Let $d \equiv 7 \pmod 8$ be prime, let $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$, and let $M \subset \mathbb{C}$ and its distinguished marked two-torsion point satisfy the hypotheses of Lemma 4.2.6. Write $a^2 = 12\wp_M(t)$, and put*

$$c_d = \gamma - \frac{L'(1, \chi_K)}{L(1, \chi_K)},$$

where χ_K is the quadratic Dirichlet character of discriminant $-d$. For $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, let M^σ be the period lattice of

$$Y^2 + \sigma(a)XY = X^3 + X$$

with respect to its invariant differential $dX/(2Y + \sigma(a)X)$. Then there is an algebraic number $u \neq 0$, chosen before σ, such that

$$|\iota(u)|_p = 1 \quad \textit{for every prime } p \textit{ and every } \iota : \mathbb{C} \hookrightarrow \mathbb{C}_p,$$

and

$$\mathrm{Area}(M^\sigma)|\sigma(u)|^2 = \pi e^{c_d}\frac{2\pi}{\sqrt{d}} \quad \textit{for every } \sigma \in \mathrm{Aut}(\mathbb{C}/K).$$

The initial lattice M and every M^σ use the displayed invariant differential, without a further complex rescaling.

Moreover, for every σ there exist a nonzero fractional $\mathcal{O}_K$-ideal $\mathfrak{b}_\sigma$ and $\Omega_\sigma \neq 0$ such that

$$M^\sigma = \Omega_\sigma\, \mathfrak{b}_\sigma^2/N(\mathfrak{b}_\sigma), \qquad \mathrm{Area}(M^\sigma) = |\Omega_\sigma|^2\sqrt{d}/2.$$

Proof. **The ideal classes admit representatives with product one.** Put $\mathcal{C} = \mathrm{Cl}(\mathcal{O}_K)$ and $h = |\mathcal{C}|$. We first check that squaring permutes $\mathcal{C}$. If a fractional ideal $\mathfrak{a}$ has class of order dividing two, then $\mathfrak{a}/\bar{\mathfrak{a}}$ is principal, say $\mathfrak{a} = \alpha\bar{\mathfrak{a}}$. Ideal norms show $\alpha\bar{\alpha} = 1$. Write $\alpha = \beta/\bar{\beta}$: if $\alpha \neq -1$ take $\beta = 1 + \alpha$, and if $\alpha = -1$ take $\beta = \sqrt{-d}$. Hence $\beta^{-1}\mathfrak{a}$ is invariant under conjugation. Factor it into prime ideals. Exponents at split conjugate primes are equal, so their factors are rational principal ideals. Inert primes are already rational principal ideals. The only ramified prime is the principal ideal $(\sqrt{-d})$. Thus the invariant ideal is principal and so is $\mathfrak{a}$. The finite class group has no nontrivial element of order two, and squaring is bijective.

Choose any fractional representative $\mathfrak{b}_c$ of each class c and set

$$\mathfrak{a}_c = \mathfrak{b}_{c^2}\mathfrak{b}_c^{-1}.$$

Its class is c, and bijectivity of squaring gives the exact identity $\prod_c \mathfrak{a}_c = \mathcal{O}_K$.

Choose normalized curves in every homothety class. The lattice M is a complex multiple of a fractional $\mathcal{O}_K$-ideal, as proved in Lemma 4.2.7. For each c, the lattice $\mathfrak{a}_c M$ has a unique nonzero two-torsion point whose two components for $\mathcal{O}_K/2 = \mathbb{F}_2 \times \mathbb{F}_2$ are nonzero. At any nonzero two-torsion point the value of $\wp''$ is nonzero: in the cubic model it is twice the product of the differences from the other two distinct roots. Multiplication of the lattice by a suitable fourth root therefore gives a scalar $\mu_c \neq 0$ and a normalized marked lattice

$$M_c = \mu_c\mathfrak{a}_c M, \qquad \wp''_{M_c}(t_c) = 2. \tag{1}$$

Choose $a_c^2 = 12\wp_{M_c}(t_c)$ and put $\delta_c = a_c^4 - 64$. By Lemma 4.2.6 each curve B_{a_c} has good reduction at every prime and δ_c is an algebraic unit.

The scalars μ_c are algebraic. Indeed, for any nonzero $b \in \mathfrak{a}_c$, multiplication by $\mu_c b$ induces an analytic isogeny from the curve of M to that of M_c. Both curves have algebraic equations. This isogeny is defined over a number field after finite extension: its finite kernel consists of torsion points with algebraic coordinates; quotient by that kernel is algebraic, and the induced complex isomorphism to the target is algebraic, since the change-of-Weierstrass-equation parameters satisfy algebraic equations. Pullback of the fixed invariant differentials has multiplier $\mu_c b$, which is consequently algebraic. Since $b \in K^\times$, μ_c is algebraic too.

The product of the homothety scalars is a unit at every prime. Fix any p and embedding ι. A complex scalar λ inducing an isogeny between two of these normalized curves

is integral under ι. To prove this, take a finite local field containing the algebraic equations and isogeny. Both normalized models are smooth elliptic schemes by Lemma 4.2.6. The isogeny extends by Theorem A.2.19. Their displayed invariant differentials are generators over the valuation ring, so pullback has an integral coefficient. That coefficient is precisely $\iota(\lambda)$ by the complex differential comparison.

For a fractional ideal $\mathfrak{a}$ let $v_\iota(\mathfrak{a})$ be the minimum valuation of the images of any finite generating set; this is the valuation of the extended principal fractional ideal $\mathfrak{a}\mathcal{O}_{\mathbb{C}_p}$ and does not depend on the generators. Every $\mu_c b$ for $b \in \mathfrak{a}_c$ induces an isogeny, so the preceding paragraph gives

$$v_p(\iota(\mu_c)) + v_\iota(\mathfrak{a}_c) \geq 0.$$

Conversely (1) gives $M = \mu_c^{-1}\mathfrak{a}_c^{-1}M_c$. Applying the same argument to its multipliers gives the opposite inequality, because $v_\iota(\mathfrak{a}_c^{-1}) = -v_\iota(\mathfrak{a}_c)$. Hence equality holds. Summing over c and using $\prod_c \mathfrak{a}_c = \mathcal{O}_K$ yields

$$\sum_c v_p(\iota(\mu_c)) = 0. \tag{2}$$

Choose a $12h$th root

$$u^{12h} = \prod_{c\in\mathcal{C}} \mu_c^{12}\delta_c. \tag{3}$$

All factors are nonzero algebraic numbers, so u is nonzero algebraic. Equation (2) and the unit property of every δ_c show that the right side of (3) has valuation zero for every p, ι. Therefore $v_p(\iota(u)) = 0$ for every such choice. This proves the all-prime unit assertion for one fixed u.

Conjugation preserves the entire ideal relation with its exact differential scalar. For two normalized lattices M_1, M_2 related by $M_2 = \mu\mathfrak{a}M_1$, each multiplier μb, $b \in \mathfrak{a}$, is an isogeny. Conjugating its algebraic equation by σ gives an isogeny with differential multiplier $\sigma(\mu)b$, since σ fixes K and the conjugate differential is exactly the displayed generator. Thus $\sigma(\mu)\mathfrak{a}M_1^\sigma \subset M_2^\sigma$. Apply the same argument to $M_1 = \mu^{-1}\mathfrak{a}^{-1}M_2$. Multiplying its inclusion by $\sigma(\mu)\mathfrak{a}$ proves the reverse inclusion. In particular

$$M_c^\sigma = \sigma(\mu_c)\mathfrak{a}_c M^\sigma. \tag{4}$$

For a CM lattice the area of $\mathfrak{a}M$ is $N(\mathfrak{a})$ times the area of M. This follows first for integral ideals by the lattice index and then for fractional ideals by denominator clearing. Consequently

$$\operatorname{Area}(M_c^\sigma) = |\sigma(\mu_c)|^2 N(\mathfrak{a}_c)\operatorname{Area}(M^\sigma). \tag{5}$$

The normalized Weierstrass discriminant of M_c^σ is $\sigma(\delta_c)$, because its equation has coefficient $\sigma(a_c)$ and the same invariant differential.

The class-group mean determines the product for each conjugate separately. The first conclusion of Lemma 4.2.9, applied to K with the positive embedding of $\sqrt{-d}$, says that for fractional ideals $\mathfrak{j}_c$ representing every class,

$$\frac{1}{h}\sum_c \log\Bigl(N(\mathfrak{j}_c)^6|\Delta_{\mathrm{W}}(\mathfrak{j}_c)|\Bigr) = 12\log(2\pi) - 6\log d + 6c_d. \tag{6}$$

Its hypotheses hold because $d > 4$ is prime and $d \equiv 3 \pmod 4$. The covolume identity $\operatorname{Area}(\mathfrak{j}) = N(\mathfrak{j})\sqrt{d}/2$ turns (6) into

$$\prod_c \operatorname{Area}(\mathfrak{j}_c)|\Delta_{\mathrm{W}}(\mathfrak{j}_c)|^{1/6} = \left(\pi e^{c_d}\frac{2\pi}{\sqrt{d}}\right)^h. \tag{7}$$

This identity also holds for arbitrary complex homotheties of the ideal lattices: area scales by the squared absolute value, while the Weierstrass discriminant scales by the inverse twelfth power. For each fixed σ, the lattices in (4) represent every homothety class exactly once, since the $\mathfrak{a}_c$ represent all classes and multiplication by the fixed class of M^σ permutes the class group. Thus (7) applies to the family $(M_c^\sigma)_c$.

On the other hand, multiply (5), use $\prod_c N(\mathfrak{a}_c) = 1$ and take the absolute value of the σ-image of (3). This gives the exact equality

$$\prod_c \operatorname{Area}(M_c^\sigma)|\sigma(\delta_c)|^{1/6} = \Big(\operatorname{Area}(M^\sigma)|\sigma(u)|^2\Big)^h.$$

Comparing with (7) proves equality of the hth powers of two positive real numbers. Taking their positive hth roots gives the asserted area identity for this σ. The construction (3) preceded σ, so it holds simultaneously for all automorphisms over K.

Finally, the CM action transported to the conjugate curve shows that M^σ is an $\mathcal{O}_K$-stable lattice. It is homothetic to a fractional ideal: choose a nonzero period λ; the rational span of the lattice is a one-dimensional K-vector space, so $\lambda^{-1}M^\sigma$ is a finitely generated $\mathcal{O}_K$-submodule of K. Since squaring permutes the ideal classes, choose a fractional ideal $\mathfrak{b}_\sigma$ whose squared class equals this lattice's ideal class. Thus $M^\sigma = \Omega_\sigma \mathfrak{b}_\sigma^2/N(\mathfrak{b}_\sigma)$ for a suitable nonzero complex scalar Ω_σ. The ideal in this expression has norm one; its covolume is $\sqrt{d}/2$. This gives the asserted area formula, using the original period lattice and its fixed differential.

□

Lemma 4.2.9. *Let $d > 4$ be a prime with $d \equiv 3 \pmod 4$, let $K = \mathbb{Q}(s_d)$ be the imaginary quadratic field generated by an element s_d with $s_d^2 = -d$, and let $\phi_K : K \hookrightarrow \mathbb{C}$ be the embedding with $\phi_K(s_d) = i\sqrt{d}$ (Lemma 4.2.10). Write $\mathcal{C} = \mathrm{Cl}(\mathcal{O}_K)$, $h = |\mathcal{C}|$, and χ_d for the quadratic Dirichlet character of discriminant $-d$. For each $c \in \mathcal{C}$ choose a nonzero fractional ideal $\mathfrak{a}_c$ representing c and a period pair Λ_c whose lattice is $\phi_K(\mathfrak{a}_c)$. Put $n_c = N(\mathfrak{a}_c) > 0$ and $\Delta_c = g_2(\Lambda_c)^3 - 27g_3(\Lambda_c)^2$, with $g_2 = 60\sum' \lambda^{-4}$ and $g_3 = 140\sum' \lambda^{-6}$. Then $L(1,\chi_d) > 0$, and, writing $\mathcal{D} = L'(1,\chi_d)/L(1,\chi_d)$,*

$$\frac{1}{h}\sum_{c\in\mathcal{C}} \log(n_c^6|\Delta_c|) = 12\log(2\pi) - 6\log d + 6\gamma - 6\mathcal{D}.$$

Each summand is unchanged by replacing its fractional ideal by another representative of the same class.

Proof. By Lemma 4.2.10, $\operatorname{covol}(\phi_K(\mathcal{O}_K)) = \sqrt{d}/2$. For an integral ideal, its additive index multiplies this covolume by its norm. Multiplying a fractional ideal by a nonzero integer and then cancelling that integer's squared absolute value gives

$$\operatorname{covol}(\phi_K(\mathfrak{a})) = N(\mathfrak{a})\sqrt{d}/2.$$

Choose an oriented basis ω_1, ω_2 for this lattice, and put $\tau = \omega_2/\omega_1$, $y = \operatorname{Im}\tau > 0$ and $k = 2/\sqrt{d}$. Thus $N(\mathfrak{a}) = k|\omega_1|^2 y$.

Let $w = |\mathcal{O}_K^\times|$. A nonzero $x \in \mathfrak{a}$ gives the integral ideal $(x)\mathfrak{a}^{-1}$ in the inverse ideal class, and every integral ideal in that class occurs in this way, exactly w times. Its norm is $|\phi_K(x)|^2/N(\mathfrak{a})$. Consequently, for $\operatorname{Re} s > 1$,

$$\zeta_K([\mathfrak{a}]^{-1}, s) = \frac{N(\mathfrak{a})^s}{w}\sum_{0\neq x\in\mathfrak{a}} |\phi_K(x)|^{-2s} = \frac{k^s}{w}E(\tau,s), \qquad \zeta_K(s) = \frac{k^s}{w}\sum_{c\in\mathcal{C}} E(\tau_c, s).$$

The Euler factors at a split, inert or ramified rational prime give $\zeta_K(s) = \zeta(s)L(s,\chi_d)$: respectively the quotient factors are $(1-p^{-s})^{-1}$, $(1+p^{-s})^{-1}$ and 1. Set

$$D_c = \frac{1}{12}\log\frac{y_c^6|\Delta_W(\tau_c)|}{(2\pi)^{12}}, \qquad \overline{D} = \frac{1}{h}\sum_c D_c.$$

By the first limit formula [329], Theorem 7.19, in its Weierstrass normalisation, with $t = s-1$, the averaged Eisenstein series is $\pi/t + 2\pi(\gamma - \log 2 - \overline{D}) + O(t)$. For completeness, the familiar Laurent constant of ζ follows by writing

$$\zeta(s) - \frac{1}{s-1} = \sum_{n\geq 1}\left(n^{-s} - \int_n^{n+1} u^{-s}\,du\right).$$

The series is locally uniformly convergent near 1 by the derivative bound $O(n^{-3/2})$ on a sufficiently small disc, and its value at 1 is the harmonic-minus-logarithm limit γ. Hence $\zeta(s) = 1/t + \gamma + O(t)$. The above Eisenstein formula continues ζ_K near 1; division by ζ continues L there and gives

$$L(1,\chi_d) = \frac{kh\pi}{w} > 0, \qquad \gamma + \mathcal{D} = 2(\gamma - \log 2 - \overline{D}) + \log k.$$

It follows that $2\overline{D} = \gamma - \log 2 - \frac{1}{2}\log d - \mathcal{D}$.

The absolutely convergent defining sums give $g_2(\Lambda_c) = \omega_{1,c}^{-4} g_2(\tau_c)$ and $g_3(\Lambda_c) = \omega_{1,c}^{-6} g_3(\tau_c)$. Thus $n_c^6|\Delta_c| = k^6 y_c^6 |\Delta_W(\tau_c)|$. Averaging its logarithm and using the last formula gives exactly the asserted class mean. Replacing $\mathfrak{a}$ by $\alpha\mathfrak{a}$ multiplies its norm by $|\phi_K(\alpha)|^2$ and its discriminant by $\phi_K(\alpha)^{-12}$, so the positive product is unchanged. Basis choice changes neither lattice nor its defining sums. □

Lemma 4.2.10. *Let d be a positive squarefree integer with $d \equiv 3 \pmod 4$, let K be a field generated over $\mathbb{Q}$ by an element s with $s^2 = -d$, let $\mathcal{O}_K$ be its ring of integers, and put $\omega = (1+i\sqrt{d})/2 \in \mathbb{C}$. There is a unique field embedding $\phi_K : K \hookrightarrow \mathbb{C}$ with $\phi_K(s) = i\sqrt{d}$. It satisfies*

$$\phi_K(a+bs) = a + b\,i\sqrt{d} \quad (a,b\in\mathbb{Q}), \qquad \phi_K(\mathcal{O}_K) = \mathbb{Z} + \mathbb{Z}\,\omega,$$

and $\phi_K(\mathcal{O}_K)$ is a lattice in $\mathbb{C}$ of covolume $\sqrt{d}/2$.

Proof. **The embedding.** Since $-d < 0$ is not the square of a rational number, $X^2 + d$ is irreducible over $\mathbb{Q}$. Hence $[K:\mathbb{Q}] = 2$, $1, s$ is a $\mathbb{Q}$-basis of K, and $K \cong \mathbb{Q}[X]/(X^2+d)$ with s corresponding to the class of X. A field embedding $K \hookrightarrow \mathbb{C}$ fixes $\mathbb{Q}$ and is determined by the image of s, which must be a root of $X^2 + d$; the root $i\sqrt{d}$ defines one, by the universal property of the quotient. This proves existence, uniqueness and the formula $\phi_K(a+bs) = a + b\,i\sqrt{d}$. In particular $\phi_K(K) = \mathbb{Q} + \mathbb{Q}\,i\sqrt{d}$.

Integrality is detected after embedding. Since ϕ_K is an injective ring homomorphism fixing $\mathbb{Z}$, a monic polynomial $f \in \mathbb{Z}[T]$ satisfies $f(x) = 0$ if and only if $f(\phi_K(x)) = \phi_K(f(x)) = 0$. Hence $x \in K$ is integral over $\mathbb{Z}$ if and only if $\phi_K(x)$ is, and

$$\phi_K(\mathcal{O}_K) = \{z \in \mathbb{Q} + \mathbb{Q}\,i\sqrt{d} : \ z \text{ is integral over } \mathbb{Z}\}.$$

The elements of $\mathbb{Z} + \mathbb{Z}\omega$ are integral. The number ω is a root of

$$T^2 - T + \frac{d+1}{4},$$

which lies in $\mathbb{Z}[T]$ because $d \equiv 3 \pmod 4$. Integral elements of $\mathbb{C}$ form a ring, so $\mathbb{Z}+\mathbb{Z}\omega$ consists of integral elements, and it lies in $\mathbb{Q} + \mathbb{Q}\, i\sqrt{d}$.

Every integral element lies in $\mathbb{Z} + \mathbb{Z}\omega$. Let $z = a + b\, i\sqrt{d}$ with $a, b \in \mathbb{Q}$ be integral over $\mathbb{Z}$. Its complex conjugate $\bar{z}$ is a root of the same monic integer polynomial, hence integral. So $A = z + \bar{z} = 2a$ and $z\bar{z} = a^2 + db^2$ are rational numbers integral over $\mathbb{Z}$, hence rational integers. Put $B = 2b$. Then

$$dB^2 = 4(a^2 + db^2) - A^2 \in \mathbb{Z}.$$

If a prime ℓ divided the reduced denominator of B, then $v_\ell(dB^2) \le 1 - 2 < 0$, because d is squarefree; hence $B \in \mathbb{Z}$. Now $A^2 + dB^2 = 4(a^2 + db^2) \equiv 0 \pmod 4$, and $d \equiv 3 \pmod 4$ gives $A^2 - B^2 \equiv 0 \pmod 4$. Squares are 0 or 1 modulo 4, so $A \equiv B \pmod 2$, and

$$z = \frac{A - B}{2} + B\,\omega \in \mathbb{Z} + \mathbb{Z}\omega.$$

With the previous step this proves $\phi_K(\mathcal{O}_K) = \mathbb{Z} + \mathbb{Z}\omega$.

The covolume. The numbers 1 and ω are linearly independent over $\mathbb{R}$, since $\operatorname{Im}\omega = \sqrt{d}/2 > 0$. So $\mathbb{Z}+\mathbb{Z}\omega$ is a lattice, and the parallelogram spanned by $1, \omega$ has area $|\operatorname{Im}(\bar{1}\cdot\omega)| = \sqrt{d}/2$. □

Lemma 4.2.11. *Let α be the fixed finite idele with $(\alpha\bar{\alpha}) = \mathfrak{D}_{E/K}^{-1}$. Let υ be the unitary Hecke splitting character, trivial on finite units, whose value on $(z_\tau)_\tau$ at infinity is $\prod_{\tau\in\Phi} z_\tau/|z_\tau|$. Write $|\alpha|$ for its finite idele norm, with complex archimedean absolute values squared, and set*

$$\chi = \upsilon(1, \alpha)|\alpha|^{-1/2}.$$

For any $s_0 \in \mathbb{C}^\times$ with $s_0^f = \chi^{-1}$, some positive power of s_0 belongs to K, and

$$|\sigma(s_0)|^2 = \frac{\mathrm{rd}_E}{\sqrt{d}} \qquad (\sigma \in \mathrm{Aut}(\mathbb{C}/K)).$$

Suppose the conjugate normalized CM lattices have areas $A_\sigma = |\Omega_\sigma|^2\sqrt{d}/2$, and one fixed $u \in \mathbb{C}^\times$ satisfies

$$A_\sigma|\sigma(u)|^2 = \pi e^{c_{\mathrm{CS}}(d)}\,\frac{2\pi}{\sqrt{d}}.$$

Then the conjugate scale

$$t_\sigma = \frac{\sigma(s_0/u)}{\Omega_\sigma}\sqrt{\frac{2\pi}{\sqrt{d}}}$$

has $|t_\sigma|^2 = \Omega^{-1}$. If $|\pi_M|^2 = M$ and $\pi_M \in K$, the multiplied scale has $|\sigma(\pi_M)t_\sigma|^2 = M/\Omega$.

Proof. The ideal class group of the number field E is finite. Choose an integer $h > 0$ and $y \in E^\times$ such that the fractional ideal of α^h is (y). Thus α^h/y_f is a finite unit. The character is trivial on these units and on diagonal $E^\times$, so

$$\upsilon(1, \alpha)^h = \upsilon(1, y_f) = \prod_{\tau\in\Phi} \frac{|\tau y|}{\tau y}.$$

The product formula gives

$$|\alpha|^h = |y_f| = \prod_{\tau\in\Phi} |\tau y|^{-2}.$$

Consequently

$$\chi^h = \prod_{\tau\in\Phi} \frac{|\tau y|^2}{\tau y} = \prod_{\tau\in\Phi} \overline{\tau y} = \mathrm{N}_{E/K}(\bar{y}) \in K^\times.$$

It follows that $s_0^{fh} = \chi^{-h} \in K^\times$. For σ fixing K, $\sigma(s_0)^{fh} = s_0^{fh}$; taking ordinary absolute values and the positive (fh)-th root gives $|\sigma(s_0)| = |s_0|$.

For a finite idele a the norm is $\prod_w (\mathrm{N}w)^{-\operatorname{ord}_w(a_w)}$. Complex conjugation preserves this norm. The prescribed ideal equation for α therefore yields

$$|\alpha|^2 = \mathrm{N}_{E/\mathbb{Q}}(\mathfrak{D}_{E/K}).$$

The discriminant tower identity and $|\operatorname{disc}(K)| = d$ give

$$\mathrm{N}_{E/\mathbb{Q}}(\mathfrak{D}_{E/K}) = \frac{|\operatorname{disc}(E)|}{d^f} = \left(\frac{\mathrm{rd}_E}{\sqrt{d}}\right)^{2f}.$$

All the quantities on the right are positive. Since υ is unitary, $|\chi|^2 = |\alpha|^{-1}$, whence

$$(|s_0|^2)^f = |\chi|^{-2} = |\alpha| = (\mathrm{rd}_E/\sqrt{d})^f.$$

Taking the positive f-th root proves the first assertion for every σ.

Finally the assumed area identity gives directly

$$\begin{aligned} |t_\sigma|^2 &= \frac{|\sigma(s_0)|^2}{|\sigma(u)|^2 |\Omega_\sigma|^2} \frac{2\pi}{\sqrt{d}} \\ &= \frac{\pi |\sigma(s_0)|^2}{A_\sigma |\sigma(u)|^2} = \frac{\mathrm{rd}_E}{2\pi e^{c_{\mathrm{CS}}(d)}} = \Omega^{-1}. \end{aligned}$$

An embedding fixing K fixes π_M; multiplying by its squared modulus M proves the last claim. The existence of the unit u and the conjugate lattices with the displayed area identity is not inferred from this calculation. □

The size bound of Theorem A needs every conjugate of a jet of a theta function, and Proposition 4.2.12 supplies them. For an automorphism σ fixing K, the conjugate of every polynomial jet of the theta germ of a step is a Gaussian moment, at a scale of squared modulus M/Ω, of bounded theta values of one vector; these are exactly the hypotheses of the single-kernel bound Proposition 4.3.3. Two ideas meet here: the main theorem of complex multiplication applied to theta functions (Lemma 4.2.13), in our reading a form of Shimura's reciprocity law [271, 272], and the fact that in the Fock model a jet is a Gaussian moment of theta values [14, 157]. They are combined in one CM Galois transport of the theta model (Lemma 4.2.14), whose form on a standard definite space (Lemma 4.2.16) is used again in Section 6.

Proposition 4.2.12. *For $j < m$ put $n = n_j$, $n' = n_{j+1}$, $D_j = fnn'$ and $A_j = C_P(q_1 nn' + 1)^2$. For finite adelic group points $g \in \mathrm{U}(X_j)(\mathbb{A}_f)$, $h \in \mathrm{U}(X_{j+1})(\mathbb{A}_f)$ and finite translation v, set*

$$F_{j,g,h,v} = \omega_j(g,h) H_j(v) v_j^{\mathrm{int}}, \qquad f_{j,g,h,v}(Z) = \theta_{F_{j,g,h,v}}(\pi_M \eta_P Z).$$

For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and $j < m$ there are a permutation $\epsilon_{\sigma,j}$ of the CM-coordinate blocks and $t_{\sigma,j} \in \mathbb{C}^\times$ with

$$|t_{\sigma,j}|^2 = M/\Omega$$

such that, for every g, h, v, there is a vector $F'_{\sigma,j,g,h,v}$ in the SAME coherent theta model with measurable theta-value function

$$\Phi_{\sigma,j,g,h,v}(y) = \theta\Big(\rho(\mathrm{rv}y) F'_{\sigma,j,g,h,v}\Big), \qquad \mathrm{rv}y = ((2\pi/\sqrt{d})^{-1/2} y, 0),$$

satisfying

$$|\Phi_{\sigma,j,g,h,v}(y)| \le e^{A_j} \quad (y \in \mathbb{C}^{D_j}), \tag{J1}$$

and for every complex polynomial P in D_j variables,

$$\sigma\big(P(\partial_Z) f_{j,g,h,v}(Z)|_{Z=0}\big) = \pi^{-D_j} \int_{\mathbb{C}^{D_j}} P^\sigma(\epsilon_{\sigma,j}(t_{\sigma,j}\bar{y}))\, e^{-|y|^2/2} \Phi_{\sigma,j,g,h,v}(y)\, dy. \tag{J2}$$

The same vector and function work for every P. The measure dy is real Lebesgue measure; the Gaussian is part of the integrand. No derivative factorial divisor is inserted.

For arbitrary $\sigma \in \mathrm{Aut}(\mathbb{C})$ the same (J1)–(J2) hold with a measurable function Φ, using the following specified extension when σ does not fix K. Put $\tau = \mathrm{conj} \circ \sigma$, use the witnesses for τ, and set

$$\epsilon_{\sigma,j} = \epsilon_{\tau,j}, \quad t_{\sigma,j} = \overline{t_{\tau,j}}, \quad \Phi_{\sigma,j,g,h,v}(y) = \overline{\Phi_{\tau,j,g,h,v}(\bar{y})}.$$

In particular these are exactly the hypotheses (K3)–(K4) of Proposition 4.3.3 for the actual fixed upper kernels. The assertion includes the CM jet-conjugation and period comparison and the bound at all real theta arguments; it does not replace them by a bound on finite theta values or on the coherent coefficient.

Proof. Fix $j < m$, and put $R = nn'$ and $\delta = fR = D_j$.

The upper Hom space is the specified standard space. By (HP1), X_j and X_{j+1} have identity Gram matrices, their standard bases and standard lattices. In (VP5) their Hom space therefore has Gram matrix $I_{n'} \otimes I_n = I_R$ and point $I_{n'} \otimes I_n = I_R$. Its matrix-coordinate indexing is a sign-preserving bijection with the R coordinates of I_R. Its transporting element is the identity by (WC2); the product formula of (WC1) and (WC3) give exactly the canonical theta functional of this standard space. Thus this identification changes neither the theta normalization nor any finite or real coordinate. The integral vacuum is the same $c_s^R v_R^0$ specified in the common transport theorem Lemma 4.2.14.

If $R = 0$, the vacuum and theta value are 1 and the polynomial ring has no variables. Take the unique empty permutation, $t_{\sigma,j} = \sqrt{M/\Omega}$ for σ fixing K, and $F' = 1$. Both formulas are the integral on a point and the identity $\sigma(P(0)) = P^\sigma(0)$. The stated extension handles the other automorphisms. Assume henceforth $R > 0$.

One common transport map supplies the bounded conjugate theta function. Fix $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Apply (CT1), (CT4), (CT10) and the character assertion of Lemma 4.2.14 to rank R, the same parameter pair, field, fixed jet scale and vacuum scale. Denote its map by $L_{\sigma,R}$, its finite idele by r_σ, its CM-block permutation by e_σ, and its scalar by t_σ^0. These choices precede all input vectors and polynomials.

For the given finite g, h, v, put $F = \omega_j(g,h) H_j(v) c_s^R v_R^0$. It belongs to the finite part: signed finite translations preserve that part, and the finite unitary action preserves it by the normalized finite action in the common theorem. More explicitly (WR4) writes the pair action as $a(g,h)\omega(k)$, where $k = h \otimes (g^{-1})^{\mathsf{T}} \in G_f(R)$ and $a(g,h)$ is the prescribed product of finite determinant-character values. The last assertion of the common theorem gives $|\sigma(a(g,h))| = 1$. The signed operator is a root-of-unity multiple of a raw finite translation. Heisenberg covariance in Lemma 4.1.1(2) moves this translation past $\omega(k)$. Consequently, for every finite x, $\theta_R(R_f(x)F)$ is $a(g,h)$ times a root of unity times a value $\theta_R(R_f(x')\omega(k) c_s^R v_R^0)$ for a finite x'. No integrality restriction on g, h, v, x is needed. Formula (CT10) gives

$$|\sigma\theta_R(R_f(x)F)| \le \exp\big(C_P(q_1 R + 1)^2\big) = e^{A_j}.$$

Set $F' = L_{\sigma,R} F$. Since r_σ is an idele, (CT4) with $x = r_\sigma^{-1} z$ implies

$$|\theta_R(R_f(z)F')| \le e^{A_j} \qquad (z \in \mathbb{A}_{E,f}^R).$$

For a rational physical vector $\gamma \in E^R$, the defining rational Heisenberg invariance of the canonical theta functional identifies the absolute value at its real component with that at the opposite finite component; the Weyl phases have modulus one. The rational vectors are dense in the archimedean space. Strong continuity of the real Heisenberg action and boundedness of the canonical theta functional therefore extend this bound to every real vector. In particular

$$\Phi(y) = \theta_R(\rho(\mathrm{rv}y)F'), \qquad |\Phi(y)| \le e^{A_j}.$$

This function is continuous, hence measurable. The vector F' was fixed before every polynomial. This proves (J1).

Ordinary Gaussian moments recover all polynomial jets. Every vector of the finite part has real factor $e^{-|x|^2/2}$ in normalized coordinates. Directly from the Heisenberg formula,

$$(\rho(\mathrm{rv}z)v^0)(x) = \exp(-|x|^2/2 - |z|^2/2 + \bar{x}\cdot z).$$

Thus $e^{|z|^2/2}\rho(\mathrm{rv}z)U$ is entire with values in the coherent L^1 carrier for a finite-part vector U. On compact sets of z, every derivative is dominated by a polynomial in $|x|$ times $e^{-|x|^2/2+B|x|}$, an integrable function. This justifies differentiation in that carrier.

For every polynomial Q, translation invariance of the carrier norm and Gaussian integrability justify the vector-valued integral and Fubini in

$$\pi^{-\delta}\int Q(\bar{y})e^{-|y|^2/2}\rho(\mathrm{rv}y)U\,dy = Q(\partial_z)\big(e^{|z|^2/2}\rho(\mathrm{rv}z)U\big)\Big|_{z=0}.$$

Indeed its real coefficient at x is

$$e^{-|x|^2/2}\pi^{-\delta}\int Q(\bar{y})e^{-|y|^2+\bar{x}\cdot y}\,dy = Q(\bar{x})e^{-|x|^2/2}.$$

Coordinatewise angular integration gives $\pi^{-1}\int_{\mathbb{C}} \bar{y}^a y^b e^{-|y|^2}\,dy = \mathbf{1}_{a=b}a!$; the exponential Taylor factorial cancels exactly. Applying the bounded theta functional gives the corresponding scalar identity for the entire function $\mathcal{E}_U$. The measure is real Lebesgue measure, and no derivative factorial divisor occurs.

The holomorphic Fock germ in the statement is $f_{j,g,h,v}(Z) = \mathcal{E}_F(\pi_M\eta_P Z)$ in these same normalized coordinates. Apply (CT1) to F. Its coefficientwise identity, followed by replacing Z by $\pi_M Z$, gives

$$\sigma\,\mathrm{Tay}_0\, f_{j,g,h,v}(Z) = \mathrm{Tay}_0\,\mathcal{E}_{F'}(\pi_M t^0_\sigma e^{-1}_\sigma Z).$$

Here $\pi_M \in K$ is fixed by σ. Define $t_{\sigma,j} = \pi_M t^0_\sigma$ and let $\epsilon_{\sigma,j}$ be the permutation on polynomial arguments with $(\epsilon_{\sigma,j}w)_{\tau,c} = w_{e_\sigma(\tau),c}$. This is exactly the dual permutation to the convention $(e^{-1}_\sigma Z)_{\nu,c} = Z_{e^{-1}_\sigma(\nu),c}$ in (CT1): a derivative at (τ,c) selects the coefficient at $(e_\sigma(\tau),c)$. The Gaussian identity with $Q(\bar{y}) = P^\sigma(\epsilon_{\sigma,j}(t_{\sigma,j}\bar{y}))$ now proves (J2). Its integral is absolutely convergent by (J1) and polynomial Gaussian integrability. Furthermore (CT1) and $|\pi_M|^2 = M$ give $|t_{\sigma,j}|^2 = M\Omega^{-1}$, with $\Omega_s = \Omega$ as defined in the statement. No automorphism is passed through the Gaussian integral or applied to a complex period.

Complex conjugation gives the specified remaining automorphisms. If σ does not fix K, put $\tau = \mathrm{conj}\circ\sigma$. Conjugate the identity for τ, and then change variables $y \mapsto \bar{y}$ in its absolutely convergent integral. Lebesgue measure and the real Gaussian are unchanged. The resulting polynomial is $P^\sigma(\epsilon_{\tau,j}(\overline{t_{\tau,j}}\bar{y}))$ and the remaining factor is $\overline{\Phi_{\tau,j,g,h,v}(\bar{y})}$. Its bound remains e^{A_j}. These are precisely the witnesses prescribed in the statement, for the same input vector and every polynomial. Hence (J1)–(J2), and therefore (K3)–(K4), hold in the asserted scope.

□

Lemma 4.2.13. *Let $d \equiv 7 \pmod 8$ be prime, put $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$ and $\omega = (1+\sqrt{-d})/2$, and let $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Let $M, M' \subset \mathbb{C}$ be $\mathcal{O}_K$-stable lattices with marked nonzero two-torsion points t, t', respectively, whose ω- and $\bar\omega$-multiples are nonzero modulo the respective lattices. Suppose*

$$\wp''_M(t) = \wp''_{M'}(t') = 2, \qquad a^2 = 12\wp_M(t), \qquad (a')^2 = 12\wp_{M'}(t'), \qquad a' = \sigma(a).$$

Use the invariant differentials of the normalized equations $Y^2+aXY = X^3+X$ and $Y^2+a'XY = X^3+X$, so that their uniformizing parameter is z. For either marked lattice L put $A_L = \mathrm{Area}(L)$ and $H_L(u,z) = \pi\bar u z/A_L$. Let s_L be its odd reduced sigma function, normalized by $s'_L(0) = 1$, and put

$$\theta_L(z) = e^{-H_L(t_L,z)}\,\frac{s_L(z+t_L)}{s_L(t_L)}, \qquad (T^L_u\theta_L)(z) = e^{-H_L(u,z)-H_L(u,u)/2}\theta_L(z+u).$$

Here $t_M = t$, $t_{M'} = t'$. Write Tay_0 for the ordinary holomorphic Taylor series at zero, including its usual factorial denominators; σ on such a series means coefficientwise application.

Suppose $M = \Omega_{\mathrm{CM}}\mathcal{O}_K$ with $\Omega_{\mathrm{CM}} \neq 0$. Fix $\Omega' \neq 0$ and an integer $e \geq 1$ such that

$$e\Omega' \in M' \subseteq \Omega' e^{-1}\mathcal{O}_K.$$

There is one finite adele $r_\sigma \in \mathbb{A}_{K,f}$ such that, for every $x \in K$, there is an open neighbourhood U_x of $r_\sigma x$ in $\mathbb{A}_{K,f}$ for which

$$\sigma\Big(\mathrm{Tay}_0(T^M_{\Omega_{\mathrm{CM}}x}\theta_M)\Big) = \mathrm{Tay}_0(T^{M'}_{\Omega' x'}\theta_{M'}) \qquad (x' \in K,\ x'_f \in U_x).$$

The periods, marked points, differentials and theta normalizations in this identity are the ones fixed above; no further scalar or quadratic exponential is inserted.

Proof. **Transport first preserves the normalized Taylor series.** In coordinates centered at the marked point set

$$x_L(z) = \wp_L(z+t_L) - \wp_L(t_L), \qquad y_L(z) = \big(\wp'_L(z+t_L) - a_L x_L(z)\big)/2.$$

They satisfy

$$y_L^2 + a_L x_L y_L = x_L^3 + x_L, \quad x'_L = 2y_L + a_L x_L, \quad y'_L = 1 + 3x_L^2 - a_L y_L, \quad x_L(0) = y_L(0) = 0.$$

The identity $\wp''_L(t_L) = 2$ fixes $y'_L(0) = 1$ and the differential dz. Recursion on coefficients in these two differential equations shows

$$\sigma(\mathrm{Tay}_0\, x_M) = \mathrm{Tay}_0\, x_{M'}, \qquad \sigma(\mathrm{Tay}_0\, y_M) = \mathrm{Tay}_0\, y_{M'}. \tag{1}$$

There is no convergence assertion about applying σ to an infinite sum here.

We recall explicitly the section calculation used below. Let $\mathcal{V}_L(r)$ be the entire theta sections with the rth power of the multiplier of θ_L and hermitian form rH_L, for $r \geq 0$. They are spanned by

$$\theta_L^{b_0}(x_L\theta_L^2)^{b_1}(y_L\theta_L^3)^{b_2}, \qquad b_0 + 2b_1 + 3b_2 = r. \tag{2}$$

Indeed division by θ_L^r identifies such sections with elliptic meromorphic functions having no pole except at t_L, of order at most r. The functions x_L, y_L have pole orders $2, 3$ there. The cubic equation reduces every polynomial to a sum of a polynomial in x_L and y_L times a polynomial in x_L. Their leading pole orders give exactly $0, 2, 3, \ldots, r$. Successive subtraction removes every

leading pole of order at least two. A single remaining simple pole is impossible by the residue theorem on a fundamental parallelogram; an elliptic entire remainder is constant. This proves (2), also for $r = 0, 1$. The identity $x_L\theta_L^2 = s_L^2$ follows from their equal zero divisors and their leading term z^2 at zero; the latter uses $\wp_L''(t_L) = 2$. The same reduced translation law and $2t_L \in L$ show that θ_L is even.

Put

$$q(Z) = \mathrm{Tay}_0(\theta_{M'})(Z)/\sigma(\mathrm{Tay}_0(\theta_M))(Z).$$

Both denominator and numerator have constant term one. By (1), coefficient transport of the homogeneous expressions (2) defines a semilinear bijection $\mathcal{V}_M(r) \to \mathcal{V}_{M'}(r)$ whose Taylor series is q^r times the conjugate series. It is well defined: a polynomial relation between the sections, divided by θ^r on its nonzero locus, is a relation in the function field of the displayed cubic; conjugation preserves that equation. The inverse is given by σ^{-1}. This description also shows that zero divisors and evaluation in the corresponding line-bundle fibres are transported by the algebraic conjugation of the marked cubics.

For clarity the multiplication maps used with this description have their exact usual scalars. The analytic map $z \mapsto bz$, $b \in \mathcal{O}_K$, is algebraic on the cubic: its coordinate functions are elliptic meromorphic functions, hence rational functions in a Weierstrass function and its derivative. The invariant differential pulls back to $b\,dz$. Conjugating its rational functions therefore gives the multiplication map with differential $\sigma(b) = b$ on the other cubic. A map between complex elliptic curves fixing the identity is determined by this differential, since its lift to their universal covers is linear. Thus the conjugated multiplication map is the same b-map, with the same degree $b\bar{b}$.

The exact multiplier also verifies the multiplication rule $f(bz) \in \mathcal{V}_L(r\,b\bar{b})$ for $f \in \mathcal{V}_L(r)$ and $b \in \mathcal{O}_K$. Here is its sign check. Put $E_L(u, z) = \operatorname{Im} H_L(u, z)/\pi$; on L this is an integral unimodular alternating form. The reduced sigma sign is 1 on $2L$ and -1 on its other residue classes. Translating it by t_L therefore gives the multiplier

$$\chi_L(l) = \epsilon_L(l)\exp(2\pi i E_L(l, t_L)) = (-1)^{E_L(\bar{\omega}l,l)} \quad (l \in L).$$

For the last equality work modulo 2: the two idempotents $\omega, \bar{\omega}$ split $L/2L$ into two lines. In eigencoordinates (l_1, l_2) their alternating pairing is nonzero, the sigma sign has exponent $l_1 + l_2 + l_1l_2$, and t_L has both half-lattice coordinates nonzero. Its translation contributes $l_1 + l_2$, leaving $l_1l_2 = E_L(\bar{\omega}l, l)$ modulo 2. This proves the identity without an unspecified trivialization. Since $H_L(bu, bz) = b\bar{b}\,H_L(u, z)$ and $E_L(\bar{\omega}bl, bl) = b\bar{b}\,E_L(\bar{\omega}l, l)$, one has $\chi_L(bl) = \chi_L(l)^{b\bar{b}}$. Substitution into the theta automorphy formula proves the multiplication rule exactly. For $b = 0$ the pullback is constant.

Apply this to $b = \omega$, of degree $N = \omega\bar{\omega}$. The pullback of θ_M and its transported section have conjugate zero divisors, because the multiplication maps are conjugate. Their counterparts on M' therefore have the same zero divisor. Their quotient is an elliptic entire function and is constant, and both have value one at zero. Comparing Taylor series gives

$$q(\omega Z) = q(Z)^N. \tag{3}$$

If $q \neq 1$, write $q = 1 + cZ^v + \cdots$ with $v > 0$ and $c \neq 0$. The first coefficient in (3) gives $\omega^v = N$. Since $|\omega| > 1$ and $N = |\omega|^2$, absolute values force $v = 2$. Then $\omega^2 = \omega\bar{\omega}$ implies $\omega = \bar{\omega}$, a contradiction. Consequently $q = 1$. Thus transport in every degree is exactly coefficientwise conjugation of Taylor series, with no extra quadratic term.

A finite collection of torsion translations is transported by one period. The preceding homogeneous coordinates give the following useful fibre statement. For any point w

of the first uniformization there are w' of the second and $\mu \neq 0$ such that for every r and every pair of transported sections f, f' of degree r,

$$f(w) = \mu^r \, \sigma^{-1}(f'(w')). \tag{4}$$

To see this, conjugate the point on the cubic. Where θ is nonzero, the affine coordinates x, y specify the conjugate point and the ratio of the two θ values specifies μ. At its zero the homogeneous coordinates $X = x\theta^2$ and $Y = y\theta^3$ are nonzero and satisfy $Y^2 = X^3$; their ratios have the form μ^2, μ^3, since their quotient determines μ and the relation checks both powers. Formula (2) then proves (4) for all sections at once. A lift $w' \in \mathbb{C}$ exists by elliptic uniformization.

For $b \in \mathcal{O}_K$, apply (4) to $f(bz)$, of degree $r\, b\bar{b}$. Its transported section is $f'(bz)$ by the Taylor identity and uniqueness of an entire function from its Taylor series. Hence

$$f(bw) = \mu^{r\, b\bar{b}} \sigma^{-1}(f'(bw')). \tag{5}$$

In particular corresponding sections vanish at bw, bw' simultaneously.

Fix a finite set $S \subset K$. Choose a positive integer n such that $nx \in \mathcal{O}_K$ for all $x \in S$, and put $w = \Omega_{\mathrm{CM}}/(2n^2)$. Choose w', μ as in (4). Since $2n^2 w$ is a period, apply (5) to the section $x_M \theta_M^2 = s_M^2$, which vanishes exactly at the periods. It follows that

$$v = 2n^2 w' \in M'. \tag{6}$$

For $x \in S$ put $u = 2n^2 xw$, $u' = 2n^2 xw'$. Both

$$z \longmapsto (T_u^M \theta_M)(nz), \qquad z \longmapsto (T_{u'}^{M'} \theta_{M'})(nz)$$

are sections of degree n^2: $nu \in M$ and $nu' \in M'$ give their translation multipliers. Transport the second section backwards. It is a constant multiple of the first, as follows directly from their zero divisors. A zero of the first satisfies $nz + u + t \in M$. Since $2t \in M = 2n^2 w \mathcal{O}_K$, it has the form $z = (b - 2nx)w$ with $b \in \mathcal{O}_K$. At the corresponding point $(b - 2nx)w'$, the argument $nz + u$ becomes nbw'; equation (5) for θ says exactly that this is a zero on the other curve. The section-fibre version of (5), applied with $b - 2nx$, then shows that the backwards transported section vanishes at the original zero. The zeros are simple. Sections of the same degree with these zeros have constant quotient, proving the assertion.

That constant is one, not just a root of unity. Put $p = n^2 xw$, $p' = n^2 xw'$, so $u = 2p, u' = 2p'$, and evaluate at $z = -nxw$ using (5) with $b = -nx$. The section and the unshifted section $\theta(nz)$ have the same degree n^2, and therefore the same fibre factor in (5). The reduced translation identity

$$(T_{2p}^M \theta_M)(-p) = \theta_M(p)$$

and evenness of θ make the constant equal to one if $\theta_M(p) \neq 0$. If it is zero, (5) shows $\theta_{M'}(p') = 0$ too. Then $p + t \in M$, and the reduced automorphy law gives $(T_{2p}^M \theta_M)(0) = -1$; the same holds on M'. Explicitly $2p$ is a period congruent to $2t$ modulo $2M$, whose sigma sign is -1, and the alternating pairing with p vanishes. Comparing constant Taylor coefficients again makes the scalar one. Undo the substitution by the nonzero integer n. We have proved

$$\sigma(\mathrm{Tay}_0(T^M_{\Omega_{\mathrm{CM}} x} \theta_M)) = \mathrm{Tay}_0(T_{vx}^{M'} \theta_{M'}) \quad (x \in S) \tag{7}$$

for one $v \in M'$ depending on S.

Compactness produces one finite adele before all translations. Choose a period v_S as in (7) for every finite subset S of K, and put $b_S = v_S/\Omega' \in e^{-1}\mathcal{O}_K$. Direct the finite subsets

by inclusion. The finite components of b_S lie in the compact set $e^{-1}\widehat{\mathcal{O}}_K \subset \mathbb{A}_{K,f}$. Choose a cluster point r_σ of this net.

Fix $x \in K$ and choose $m \geq 1$ with $mx \in \mathcal{O}_K$. Take a coset neighbourhood U_x of $r_\sigma x$ whose difference set is contained in $2me\,\widehat{\mathcal{O}}_K$. By the cluster-point property there is a finite S containing x for which $(b_S x)_f \in U_x$. The vector mxv_S is a period of M'. If $x'_f \in U_x$ with $x' \in K$, then

$$x' - b_S x = 2me\,c, \qquad c \in \mathcal{O}_K,$$

because $K \cap \widehat{\mathcal{O}}_K = \mathcal{O}_K$. Thus $\Omega' x' = \Omega' b_S x + 2m(e\Omega' c)$, where the parenthesis is a period. For any torsion vector z with $mz \in M'$ and any period $l \in M'$,

$$T^{M'}_{z+2ml}\theta_{M'} = T^{M'}_{z}\theta_{M'}.$$

Indeed the multiplier at $2ml$ is one and the remaining phase is $\exp(-2\pi i\,E(mz,l)) = 1$, where E is the integral alternating Riemann form. Apply this with $z = xv_S$ and $l = e\Omega' c$. Equation (7) now gives the required identity for every $x'_f \in U_x$. The cluster point was chosen once, before x, and every later equality is an equality of actual entire translates before applying a field automorphism. This proves all quantifiers of the assertion. □

Lemma 4.2.14. *For $R \geq 1$, let $\mathcal{M}_R$ be the complex span of the finite Heisenberg translates of the unit vacuum v^0_R in the standard positive space I_R. Write*

$$R_f(x) = \rho(\mathrm{fv}(x)), \qquad \mathcal{E}_F(Z) = e^{|Z|^2/2}\theta_R(\rho(\mathrm{rv}(Z))F), \quad Z \in \mathbb{C}^{\Phi\times\{1,\dots,R\}}.$$

For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ there are one finite idele $r_\sigma \in \mathbb{A}^\times_{E,f}$, one permutation $e_\sigma(\tau) = \sigma\tau$ of Φ, and $t_\sigma \neq 0$, independent of R, and for every $R \geq 1$ one injective σ-semilinear map $L_{\sigma,R} : \mathcal{M}_R \longrightarrow \mathcal{M}_R$, all fixed before vectors, translations, finite group points, Schwartz functions, and polynomials, with the following properties.

1. *The germs $\mathcal{E}_F$ are entire and*

$$\sigma\,\mathrm{Tay}_0\,\mathcal{E}_F(\eta_P Z) = \mathrm{Tay}_0\,\mathcal{E}_{L_{\sigma,R}F}(t_\sigma e_\sigma^{-1}Z), \qquad |t_\sigma|^2 = \Omega_s^{-1}. \tag{CT1}$$

Taylor series are ordinary coefficient series and σ is applied coefficientwise. The convention is $(e_\sigma^{-1}Z)_{\nu,c} = Z_{e_\sigma^{-1}(\nu),c}$. Equivalently, for every polynomial Q with complex coefficients,

$$\sigma\Big(\theta_R(v_{Q(\eta_P\,\cdot)} \otimes F)\Big) = Q^\sigma(\partial_Z)\mathcal{E}_{L_{\sigma,R}F}(t_\sigma e_\sigma^{-1}Z)\Big|_{Z=0}. \tag{CT2}$$

There is no derivative factorial divisor.

2. *Put $\psi_0(w) = \psi_{E,f}(\jmath\alpha_s\bar{\alpha}_s w)$, as in the fixed finite model. If $\chi_{\mathrm{cyc}}(\sigma) \in \widehat{\mathbb{Z}}^\times$ is defined by the action of σ on roots of unity, then*

$$r_\sigma\bar{r}_\sigma = \chi_{\mathrm{cyc}}(\sigma), \qquad \sigma\psi_0(w) = \psi_0(r_\sigma\bar{r}_\sigma w) \quad (w \in \mathbb{A}_{E,f}). \tag{CT3}$$

In particular r_σ has valuation zero at conjugation-fixed places and opposite valuations at paired places.

3. *For every finite x and every $F \in \mathcal{M}_R$,*

$$L_{\sigma,R}R_f(x)F = R_f(r_\sigma x)L_{\sigma,R}F, \qquad \theta_R(R_f(r_\sigma x)L_{\sigma,R}F) = \sigma\theta_R(R_f(x)F). \tag{CT4}$$

The first identity also holds for the signed operators H with their prescribed phases.

4. *Fix any finite set $\mathcal{A}$ of places disjoint from its conjugate, as used for the polarized position coordinates. There is a factorization, fixed before F and every Schwartz function,*

$$r_\sigma = u_{\sigma,\mathcal{A}} a_{\sigma,\mathcal{A}}, \quad u_{\sigma,\mathcal{A}} \bar{u}_{\sigma,\mathcal{A}} = 1, \quad a_{\sigma,\mathcal{A}} \in \widehat{\mathcal{O}}_E^\times, \quad (a_{\sigma,\mathcal{A}})_w = 1 \ (w \in \mathcal{A}).$$

For $L^0_{\sigma,R} = \omega(u_{\sigma,\mathcal{A}})^{-1} L_{\sigma,R}$ there is $C_{\sigma,R,\mathcal{A}} \neq 0$ such that

$$L^0_{\sigma,R} v^0_R = C_{\sigma,R,\mathcal{A}} v^0_R, \qquad L_{\sigma,R}(\Psi \otimes_{\mathcal{A}} v^0_R) = C_{\sigma,R,\mathcal{A}} \omega(u_{\sigma,\mathcal{A}})((\sigma \circ \Psi) \otimes_{\mathcal{A}} v^0_R) \tag{CT5}$$

for every Schwartz–Bruhat function Ψ in those position coordinates. The transform, vacuum, and scalar in both identities are the same.

5. *For every finite unitary g, let $\Delta_R(g)$ be the actual scalar in its fixed normalized finite integral, and $I_R(g) = [\widehat{\mathcal{O}}_E^R : \widehat{\mathcal{O}}_E^R \cap g^{-1}\widehat{\mathcal{O}}_E^R]$. Then*

$$L_{\sigma,R}\omega(g)F = m_{\sigma,R}(g)\omega(g)L_{\sigma,R}F, \qquad m_{\sigma,R}(g) = \frac{\sigma(\Delta_R(g))}{\Delta_R(g)}, \quad |m_{\sigma,R}(g)| = 1, \tag{CT6}$$

and $\Delta_R(g)\overline{\Delta_R(g)} = I_R(g) \in \mathbb{Z}_{>0}$. For $\sigma^\dagger = \mathrm{conj} \circ \sigma \circ \mathrm{conj}$,

$$m_{\sigma,R}(g)\overline{m_{\sigma^\dagger,R}(g)} = 1. \tag{CT7}$$

6. *With the same $C_{\sigma,R,\mathcal{A}}$, put $\lambda_{\sigma,R,\mathcal{A}} = \sigma(c_s^R) C_{\sigma,R,\mathcal{A}}$. Then*

$$|\lambda_{\sigma,R,\mathcal{A}}| \leq \exp(c_{\mathrm{vac}}(P) q^2 R). \tag{CT8}$$

This bounds the image of the fixed integral vacuum, not the unscaled scalar $C_{\sigma,R,\mathcal{A}}$.

7. *Uniformly for every full adelic translation y and finite unitary g,*

$$|\theta_R(\rho(y)\omega(g)v^0_R)| \leq \exp((2d+6)(qR+1)^2). \tag{CT9}$$

Consequently for every field automorphism σ of $\mathbb{C}$, every finite x, g,

$$\left|\sigma\theta_R(R_f(x)\omega(g)c_s^R v^0_R)\right| \leq \exp(C_P(qR+1)^2). \tag{CT10}$$

The last assertion for automorphisms not fixing K is an absolute-value assertion, and does not assert existence of $L_{\sigma,R}$ for them. Rank zero in (CT8)–(CT10) is the empty tensor with value one.

8. *Let ν be a smooth unitary automorphic character of the norm-one torus of E/E^+, with integral archimedean weight $-c(\tau)$. Every finite value of ν is algebraic and all its conjugates have modulus one. For σ over K, the finite character $\sigma \circ \nu_f$, together with archimedean weight $-c'$, where $c'(e_\sigma \tau) = c(\tau)$, defines a smooth unitary automorphic character ν^σ. It has the same unramified places and preserves every specified triviality condition on finite subgroups. The character obtained with $\sigma^\dagger$ is the same ν^σ.*

All assertions use the actual fixed theta normalization. In particular no automorphism is applied to a Gaussian integral, to π, or to an unspecified period.

Proof. **The normalized products and their actual finite-part realizations.** Fix s and $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. All ranks below are standard positive ranks. Write $\mathcal{M}_{\mathrm{fin}} = \mathcal{M}_R$, $\mathcal{H} = \mathbb{A}_{E,f}^R$, $\mathcal{L} = \widehat{\mathcal{O}}_E^R$, $\epsilon = e_\sigma$ and $\eta = \eta_P$. Put $N = q_1(s)$, $A = \sqrt{2\pi/\sqrt{d}}$, and use the original period Ω_{CM},

actual conjugate lattice Λ_σ, with $\Lambda_0 = \Omega_{\mathrm{CM}}\mathcal{O}_K$, and simultaneous unit of Lemma 4.2.3. Write $\beta = s_0/u_{\mathrm{CS}}$. The specifications in the statement identify the actual chosen scale with

$$\eta_P = \beta A/\Omega_{\mathrm{CM}}, \qquad t_\sigma = \sigma(\beta)A/\Omega_\sigma, \qquad |t_\sigma|^2 = \Omega_s^{-1}. \tag{C0}$$

The principal root defining s_0 is taken once in the original data; $\sigma(\beta)$ is its image, not a new choice of principal root.

We check the marked-lattice hypotheses before applying Lemma 4.2.13. On the actual conjugate curve $B_{\sigma a}$, take the point $(0,0)$, and let t_σ^{mark} be its uniformizing class. It is nonzero two-torsion. Completing the square and translating the cubic gives $\wp(t_\sigma^{\mathrm{mark}}) = (\sigma a)^2/12$ and $g_2 = (\sigma a)^4/12 - 4$; hence $\wp''(t_\sigma^{\mathrm{mark}}) = 6\wp(t_\sigma^{\mathrm{mark}})^2 - g_2/2 = 2$. Its ω- and $\bar\omega$-multiples are nonzero. The original CM endomorphisms have algebraic graphs: the kernel of $[\omega]$, respectively $[\bar\omega]$, is killed by the integer $N_{K/\mathbb{Q}}(\omega)$, since their product is that multiplication map. Its points therefore have algebraic coordinates, by the multiplication equations on the algebraic curve; the quotient is constructed over $\overline{\mathbb{Q}}$ by the invariant function field under those algebraic translations. The resulting analytic isomorphism from this quotient back to the original elliptic curve is algebraic. Indeed an origin-preserving isomorphism between Weierstrass equations has the form $(X,Y) \mapsto (u^2X + r, u^3Y + u^2vX + w)$; comparison of the coefficients makes u, r, v, w algebraic: at least one of the two nonsingular Weierstrass invariants is nonzero, so u^4 or u^6 is algebraic, and the remaining coefficients are then solved over the same algebraic closure. Thus the endomorphisms and their graphs are defined over $\overline{\mathbb{Q}}$, with differentials $\omega, \bar\omega$. Conjugating their graphs gives the corresponding endomorphisms of $B_{\sigma a}$, since σ fixes K. They carry the conjugate point to the conjugates of the original nonzero points. This also fixes the parameter $a' = \sigma a$, rather than an arbitrary square root. All uniformizations use exactly the invariant differentials in Lemma 4.2.3.

The ideal shape there is $\Lambda_\sigma = \Omega_\sigma \mathfrak{b}_\sigma^2/N(\mathfrak{b}_\sigma)$. Put $I_\sigma = \mathfrak{b}_\sigma^2/N(\mathfrak{b}_\sigma)$. Since $\mathfrak{b}_\sigma\bar{\mathfrak{b}}_\sigma = N(\mathfrak{b}_\sigma)\mathcal{O}_K$, we have $I_\sigma\bar I_\sigma = \mathcal{O}_K$. Also $\mathrm{Area}(I_\sigma) = \sqrt{d}/2$: multiplication of an ideal by its index multiplies its covolume by that index, so $\mathrm{Area}(\mathfrak{b}_\sigma^2) = N(\mathfrak{b}_\sigma)^2\sqrt{d}/2$, while division by the positive rational $N(\mathfrak{b}_\sigma)$ divides the real area by its square. These formulas extend to fractional ideals by clearing a positive integer denominator. Choose a positive integer e_σ with $e_\sigma \in I_\sigma \subseteq e_\sigma^{-1}\mathcal{O}_K$, possible by commensurability of fractional ideals. At the identity use $I_0 = \mathcal{O}_K$, $e_0 = 1$. In particular $\mathrm{Area}(\Lambda_\sigma) = |\Omega_\sigma|^2\sqrt{d}/2$, the precise normalization needed below.

Apply Lemma 4.2.15(iii)–(v) in rank R, to the original and conjugate marked lattices, retaining the actual ideal $\mathfrak{a} = E \cap \alpha\widehat{\mathcal{O}}_E$. It supplies actual vectors $F_0, F_\sigma \in \mathcal{M}_R$ with

$$\mathcal{E}_{F_j}(Z) = S_j(Z) := \prod_{k=0}^{N-1}\prod_{i=1}^{R} \vartheta_{\Lambda_j}(\ell_{j,k,i}(Z))^N, \qquad \ell_{j,k,i}(Z) = \frac{\Omega_j}{NA}\sum_{\tau\in\Phi} \overline{\tau(\zeta^k)}Z_{\tau,i}.$$

Write $\theta_j = \vartheta_{\Lambda_j}$ for $j = 0, \sigma$. Their theta values at zero are one. The translates of F_0 span $\mathcal{M}_R$. The conjugate product has the exact signed periods $Ne_\sigma\Gamma$, not merely $N\Gamma$ for a nonprincipal conjugate ideal, and is realized by the full finite Fourier formula of that lemma. No sign or scalar is discarded. The area computation and character orthogonality give

$$\sum_{k,i} NH_j(\ell_{j,k,i}(Z), \ell_{j,k,i}(W)) = \bar Z^{\mathsf{T}}W, \qquad H_j(u,w) = \pi\bar u w/\,\mathrm{Area}(\Lambda_j). \tag{C1}$$

Indeed the character sum contributes $N\delta_{\tau,\nu}$, leaving the coefficient $\pi|\Omega_j|^2/(A^2\,\mathrm{Area}(\Lambda_j)) = 1$. This checks the full normalization rather than inferring it from the ideal shape alone. The finite-part germ is entire and injective by Lemma 4.2.2; its ordinary coefficient formula will also be used.

The orbit of every finite-part vector is locally constant. The raw Weyl law obtained by substituting (WD6)–(WD7) is

$$R_f(x)R_f(y) = b(x,y)R_f(x+y), \qquad b(x,y) = \psi_0(\bar{y}^{\mathsf{T}}x/2). \tag{C2}$$

Here the identity $\psi_0(\bar{y}^{\mathsf{T}}x) = \psi_0(\bar{x}^{\mathsf{T}}y)^{-1}$ follows because $\jmath$ is anti-invariant under conjugation and the additive character uses the absolute trace. Each character $b(x,\cdot)$ or $b(\cdot,x)$ has open kernel: a finite additive character is trivial on an open fractional ideal, and multiplication by a fixed finite adele is continuous.

In fact $R_f(u)v^0 = v^0$ for $u \in 2\mathcal{L}$. To see this directly, use the scaled finite coordinate $z = \alpha^{-1}x_f$. Translation by $u \in \mathcal{L}$ preserves the support $z \in \mathcal{L}$. On that support, the ratio of the translated finite vacuum factor to the original is

$$\psi_0\Big(((1-c_0)\bar{z}^{\mathsf{T}}u - c_0\bar{u}^{\mathsf{T}}z + c_0\bar{u}^{\mathsf{T}}u)/2\Big) = 1 \quad (u \in 2\mathcal{L}).$$

Every argument in the parentheses divided by 2 is integral, since c_0 and $1-c_0$ are integral. The real factor is unchanged, since the translation has zero real component. Outside the support both vectors vanish.

Now write $F = \sum_{a=1}^{m} c_a R_f(x_a)v^0$. Choose an open additive subgroup $U \subseteq 2\mathcal{L}$ on which all the finitely many characters $b(u,x_a)$ and $b(x_a,u)$ are 1. Such a subgroup exists by intersecting their open kernels with $2\mathcal{L}$. Formula (C2) shows that $R_f(u)F = F$ for $u \in U$. For any fixed x, shrink U further by the kernel of $b(x,\cdot)$. Then

$$R_f(x+u)F = b(x,u)^{-1}R_f(x)R_f(u)F = R_f(x)F. \tag{C3}$$

This proves local constancy of $x \mapsto R_f(x)F$ as an exact equality of vectors, for every $F \in \mathcal{M}_{\mathrm{fin}}$. It also proves local constancy after any continuous scalar substitution, even if that scalar has not yet been proved to be an idele.

The rational translate formula has a negative trace coordinate. For $\gamma \in E^R$ put

$$x_\alpha(\gamma) = \alpha^{-1}\gamma_f, \quad a(\gamma)_{\tau,i} = A\tau(\gamma_i), \quad q_{k,i}(\gamma) = -\frac{1}{N}\operatorname{Tr}_{E/K}(\gamma_i\bar{\zeta}^k).$$

Then

$$\ell_{j,k,i}(a(\gamma)) = -\Omega_j q_{k,i}(\gamma). \tag{C4}$$

The rational-translation identity of Lemma 4.2.2 gives

$$\mathcal{E}_{R_f(x_\alpha(\gamma))F_j}(Z) = e^{\overline{a(\gamma)}^{\mathsf{T}}Z - \|a(\gamma)\|^2/2}S_j(Z - a(\gamma)).$$

With $T_u^j\theta_j(w) = e^{-H_j(u,w)-H_j(u,u)/2}\theta_j(w+u)$, (C1) and (C4) rewrite this as the exact entire-function identity

$$\mathcal{E}_{R_f(x_\alpha(\gamma))F_j}(Z) = \prod_{k,i}(T^j_{\Omega_j q_{k,i}(\gamma)}\theta_j)(\ell_{j,k,i}(Z))^N. \tag{C5}$$

Indeed the shifted arguments on the two sides agree by (C4). Multiplying the exponential factors on the right gives exactly the single exponential on the preceding display, by (C1). Thus no extra phase, scalar, or quadratic exponential is omitted.

One rational scalar meets all the CM Taylor neighborhoods. Apply Lemma 4.2.13 once to the normalized pair $(\Lambda_0, \Lambda_\sigma)$. It gives one $r_\sigma \in \mathbb{A}_{K,f}$, independent of R, and, for each $q \in K$, a neighborhood U_q of $r_\sigma q$, such that

$$\sigma\operatorname{Tay}_0(T^0_{\Omega_{\mathrm{CM}}q}\theta_0) = \operatorname{Tay}_0(T^\sigma_{\Omega_\sigma q'}\theta_\sigma) \quad (q' \in K,\ q'_f \in U_q). \tag{C6}$$

Fix $\gamma \in E^R$. The map

$$b \longmapsto R_f(b\, x_\alpha(\gamma))F_\sigma \quad (b \in \mathbb{A}_{K,f})$$

is locally constant by (C3) and continuity of multiplication and of the inclusion into $\mathbb{A}_{E,f}$. Therefore it equals its value at r_σ on an open neighborhood of r_σ. Intersect this neighborhood with the preimages of $U_{q_{k,i}(\gamma)}$ under the finitely many continuous maps $b \mapsto bq_{k,i}(\gamma)$. The resulting intersection is still an open neighborhood of r_σ. Density of K in its finite adeles, the usual additive approximation theorem (or the Chinese remainder theorem after clearing a denominator), supplies $b_\gamma \in K$ in this intersection. Consequently

$$R_f(r_\sigma x_\alpha(\gamma))F_\sigma = R_f(x_\alpha(b_\gamma\gamma))F_\sigma, \qquad (b_\gamma q_{k,i}(\gamma))_f \in U_{q_{k,i}(\gamma)}. \tag{C7}$$

This is a finite simultaneous choice, and (C7) is an exact vector equality made before applying σ.

All applications of the automorphism are coefficientwise formal algebra. After substituting the indicated scales, the linear forms become

$$\ell_{0,k,i}(\eta Z) = \frac{\beta}{N}\sum_\tau \overline{\tau(\zeta^k)}Z_{\tau,i}, \qquad \ell_{\sigma,k,i}(t_\sigma\epsilon^{-1}Z) = \frac{\sigma(\beta)}{N}\sum_\tau \overline{\epsilon(\tau)(\zeta^k)}Z_{\tau,i}. \tag{C8}$$

Here $(\epsilon^{-1}Z)_{\nu,i} = Z_{\epsilon^{-1}(\nu),i}$. Since E/K is the stipulated cyclotomic CM extension, the restriction of σ commutes with conjugation on E; hence the second linear form in (C8) is the coefficientwise image under σ of the first. The periods and the real number A have already canceled in (C8); no assertion about $\sigma(A)$, $\sigma(\pi)$, or $\sigma(\Omega_{\mathrm{CM}})$ is involved.

Take ordinary Taylor series in (C5), substitute ηZ on the original side, and use (C6) with $q = q_{k,i}(\gamma)$, $q' = b_\gamma q$. The equality $q_{k,i}(b_\gamma\gamma) = b_\gamma q_{k,i}(\gamma)$ follows from K-linearity of the trace. Equations (C7),(C8) give

$$\sigma\,\mathrm{Tay}_0\,\mathcal{E}_{R_f(x_\alpha(\gamma))F_0}(\eta Z) = \mathrm{Tay}_0\,\mathcal{E}_{R_f(r_\sigma x_\alpha(\gamma))F_\sigma}(t_\sigma\epsilon^{-1}Z). \tag{C9}$$

This step is valid in the ring of formal multivariable power series. Each substituted linear form has zero constant coefficient, so a coefficient of total degree m involves only univariate coefficients of degree at most m, and only finitely many terms. The product has only NR factors. The automorphism is therefore commuted only with finite sums and products for each coefficient, never with an infinite analytic limit.

The extension to all finite translations uses equality, not continuity of σ. For prescribed $x \in \mathcal{H}$, local constancy (C3) shows that

$$y \longmapsto \left(R_f(y)F_0,\ R_f(r_\sigma y)F_\sigma\right)$$

is constant on some open neighborhood of x. The set $x_\alpha(E^R) = \alpha^{-1}E^R$ is dense in $\mathcal{H}$, since E is dense in its finite adeles and multiplication by the fixed idele α^{-1} is a homeomorphism. Choose γ with $x_\alpha(\gamma)$ in that neighborhood. Both vectors in (C9) are then exactly the corresponding vectors for x. Thus

$$\sigma\,\mathrm{Tay}_0\,\mathcal{E}_{R_f(x)F_0}(\eta Z) = \mathrm{Tay}_0\,\mathcal{E}_{R_f(r_\sigma x)F_\sigma}(t_\sigma\epsilon^{-1}Z) \quad (x \in \mathcal{H}). \tag{C10}$$

No sequence, limit, or continuity property of σ has been used.

The proposed semilinear map descends through every linear relation. The scale t_σ is nonzero, by Lemma 4.2.3. Consequently the complex-linear map

$$J_\sigma(F) = \mathrm{Tay}_0\,\mathcal{E}_F(t_\sigma\epsilon^{-1}Z)$$

is injective: undo the nonzero dilation and coordinate permutation formally, and use the injective germ of Lemma 4.2.2. The same holds for $J_0(F) = \mathrm{Tay}_0\,\mathcal{E}_F(\eta Z)$.

For a finite expression $F = \sum_a c_a R_f(x_a)F_0$, set

$$L_\sigma F = \sum_a \sigma(c_a) R_f(r_\sigma x_a) F_\sigma. \tag{C11}$$

Such expressions exist by the cyclicity in Lemma 4.2.15(v). If $\sum_a c_a R_f(x_a)F_0 = 0$, apply J_σ to the proposed right side of (C11). By (C10), the result is

$$\sum_a \sigma(c_a)\sigma J_0(R_f(x_a)F_0) = \sigma J_0\left(\sum_a c_a R_f(x_a)F_0\right) = 0.$$

Injectivity of J_σ proves that the proposed right side is zero. Thus (C11) is independent of its presentation and defines a σ-semilinear map $\mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$. Its full Taylor identity is

$$\sigma J_0(F) = J_\sigma(L_\sigma F) \qquad (F \in \mathcal{M}_{\mathrm{fin}}). \tag{C12}$$

If $L_\sigma F = 0$, (C12), injectivity of the field automorphism on coefficients and injectivity of J_0 give $F = 0$. This proves injectivity. All choices precede F and every polynomial test. The same finite scalar r_σ, included from K into E, works for every positive rank.

The ordinary polynomial consequence. Write a polynomial $Q(Z) = \sum_{\mathbf{m}} q_{\mathbf{m}} Z^{\mathbf{m}}$, a finite sum. If $J_0(F) = \sum_{\mathbf{m}} a_{\mathbf{m}} Z^{\mathbf{m}}$, then Lemma 4.2.2 gives

$$\theta(v_{Q(\eta\,\cdot)} \otimes F) = \sum_{\mathbf{m}} q_{\mathbf{m}} \mathbf{m}!\, a_{\mathbf{m}}.$$

Apply σ, which fixes the integers $\mathbf{m}!$, and use (C12) coefficient by coefficient. The result is exactly

$$\sigma\theta(v_{Q(\eta\,\cdot)} \otimes F) = Q^\sigma(\partial_Z)\mathcal{E}_{L_\sigma F}(t_\sigma \epsilon^{-1} Z)\Big|_{Z=0}.$$

Thus (CT2) has ordinary derivatives and no additional factorial divisor. The direction of the block permutation is fixed by (C8): differentiation with respect to $Z_{\tau,i}$ becomes differentiation of the image entire function in its $(\epsilon(\tau), i)$ coordinate.

The exact finite phase and the idele. We first calculate in rank one, using the same r_σ already chosen for every rank. The orbit maps just proved locally constant give open neighborhoods of zero fixing F_0 and F_σ. Intersect the first with the inverse image of the second under multiplication by r_σ. This remains an open neighborhood of zero, without assuming that r_σ is invertible. It contains a nonzero rational integer b_0: in the restricted-product topology only finitely many local divisibilities have to be imposed, and a sufficiently divisible integer satisfies them. Consequently

$$R_f(b_0)F_0 = F_0, \qquad R_f(r_\sigma b_0)F_\sigma = F_\sigma.$$

The literal raw Weyl product is $R_f(x)R_f(y) = \psi_0(\bar{y}^{\mathsf{T}}x/2)R_f(x+y)$. For every finite x, this gives

$$R_f(2x)F_0 = \psi_0(b_0 x)R_f(2x+b_0)F_0, \qquad R_f(2r_\sigma x)F_\sigma = \psi_0(r_\sigma \bar{r}_\sigma b_0 x)R_f(r_\sigma(2x+b_0))F_\sigma.$$

Apply the already proved translated-germ identity at $2x$ and $2x + b_0$, applying σ only to coefficients. The common germ at $2x + b_0$ is nonzero: its original vector is nonzero because raw translation is invertible and $\mathcal{E}_{F_0}(0) = 1$; the nonzero dilation and the injective germ preserve this, and coefficientwise σ preserves a nonzero coefficient. Cancellation therefore proves

$$\sigma\psi_0(b_0 x) = \psi_0(r_\sigma \bar{r}_\sigma b_0 x).$$

Multiplication by the nonzero rational integer b_0 is a bijection of the finite adeles. Thus this is the claimed phase identity for every argument w.

For each n, write $\sigma(\zeta_n) = \zeta_n^{a_n}$. The compatible classes $a_n \in (\mathbb{Z}/n\mathbb{Z})^\times$ define $u_{\mathrm{cyc}} = \chi_{\mathrm{cyc}}(\sigma) \in \widehat{\mathbb{Z}}^\times$. For each fixed w, choose a positive integer n with $nw \in \widehat{\mathcal{O}}_E$. Since ψ_0 is trivial there, its value at aw, for $a \in \widehat{\mathbb{Z}}$, depends only on $a \bmod n$. Thus evaluation at u_{cyc} raises $\psi_0(w)$ to the power a_n. All these values are roots of unity, so $\sigma\psi_0(w) = \psi_0(u_{\mathrm{cyc}}w)$. Combining the two identities gives

$$\psi_{E,f}(\jmath\alpha\bar{\alpha}(r_\sigma\bar{r}_\sigma - u_{\mathrm{cyc}})w) = 1 \quad (w \in \mathbb{A}_{E,f}).$$

At any finite place choose w supported at that place. The local trace character is nontrivial: the trace to $\mathbb{Q}_p$ is onto, since on rational scalars it is multiplication by the nonzero local degree, and the standard character of $\mathbb{Q}_p$ is nontrivial. It is therefore nondegenerate under multiplication: if a nonzero coefficient c annihilated all w, multiplication by c would make this character trivial on the entire local field, contradicting its nontriviality. Since $\jmath\alpha\bar{\alpha}$ is an idele, every local component of $r_\sigma\bar{r}_\sigma - u_{\mathrm{cyc}}$ is zero. This proves the exact equality in (CT3), without a continuity assumption on σ. It also proves that r_σ is an idele, because $r_\sigma^{-1} = \bar{r}_\sigma/u_{\mathrm{cyc}}$ is a finite adele. Valuations give the stated fixed-place and paired-place conclusions.

Return now to any $R \geq 1$ and its chosen F_0, F_σ. On the spanning vector $R_f(x)F_0$, the phase law and the definition of $L_{\sigma,R}$ give

$$\begin{aligned} L_{\sigma,R}R_f(y)R_f(x)F_0 &= \sigma\psi_0(\bar{x}^{\mathsf{T}}y/2)R_f(r_\sigma(x+y))F_\sigma \\ &= R_f(r_\sigma y)R_f(r_\sigma x)F_\sigma. \end{aligned}$$

Semilinearity extends this to the full finite span, proving raw covariance. The signed operator is $H(y) = \psi_0(c_0y^*y/2)R_f(y)$, with the same $c_0 \in K$. Applying (CT3) to $c_0y^*y/2$ gives $\sigma\psi_0(c_0y^*y/2) = \psi_0(c_0(r_\sigma y)^*(r_\sigma y)/2)$. Hence the signed covariance also holds, at dyadic places as well as elsewhere. Finally the constant Taylor coefficient is $\theta_R(L_{\sigma,R}F) = \sigma\theta_R(F)$. Apply it to $R_f(x)F$ and use raw covariance to obtain the second identity in (CT4).

Integral-weight characters and their conjugates. We first prove the character assertion, which will also control the multiplier moduli. The ideal class group of E is finite by Theorem A.2.3: that bound supplies an integral representative of bounded index in every class, and there are only finitely many integral ideals of bounded index. Indeed an ideal of index n contains $n\mathcal{O}_E$, so for each bounded n it is the inverse image of a subgroup of the finite additive group $\mathcal{O}_E/n\mathcal{O}_E$.

Let ν have the weights in the statement, and put $U = \{v \in \widehat{\mathcal{O}}_E^\times : v\bar{v} = 1\}$. Its restriction to this compact group has finite image: smoothness gives an open kernel, and its quotient is finite. For a finite norm-one idele v, choose $h > 0$ and $\gamma \in E^\times$ with v^h/γ an integral finite unit, by finiteness of the ideal class group. Put $\xi = \gamma/\bar{\gamma}$. Then $\xi\bar{\xi} = 1$ and

$$w = v^{2h}/\xi = (v^h/\gamma)/\overline{(v^h/\gamma)} \in U.$$

The principal-idele relation and the archimedean weight $-c$ give

$$\nu_f(v)^{2h} = \nu_f(w)\prod_{\tau\in\Phi}\tau(\xi)^{c(\tau)}. \tag{C13}$$

The first factor is a root of unity. Every $\tau(\xi)$ is algebraic, and every one of its conjugates has modulus one: embeddings of a CM field commute with complex conjugation, and $\xi\bar{\xi} = 1$. Equation (C13) first proves algebraicity of $\nu_f(v)$, and then, after applying any embedding of its algebraic closure, proves that every conjugate has modulus one.

Set $c'(\epsilon\tau) = c(\tau)$ and $\nu'_f = \sigma \circ \nu_f$. This finite character has exactly the same open kernel as ν_f, so is smooth and continuous even though σ need not be continuous on $\mathbb{C}$. Its values are unitary by (C13). For a rational norm-one ξ,

$$\nu'_f(\xi) = \sigma\left(\prod_\tau \tau(\xi)^{c(\tau)}\right) = \prod_\tau (\epsilon\tau)(\xi)^{c(\tau)}.$$

This is the inverse of the archimedean character of weight $-c'$ at ξ. Thus the product of the finite and archimedean characters is automorphic. Equality of the finite kernels preserves every asserted subgroup triviality and gives the same unramified places. There is no permutation of finite places in this construction.

For $\sigma^\dagger = \mathrm{conj} \circ \sigma \circ \mathrm{conj}$, the CM embedding property gives $\sigma^\dagger \circ \tau = \sigma \circ \tau$ on E, so its permutation is again ϵ. Since $\nu_f(v)$ and its σ-image have modulus one,

$$\sigma^\dagger(\nu_f(v)) = \overline{\sigma(\overline{\nu_f(v)})} = \overline{\sigma(\nu_f(v)^{-1})} = \sigma(\nu_f(v)).$$

Hence the same ν' is obtained with $\sigma^\dagger$, proving the last assertion of the statement.

The actual finite Weil operator and its multiplier. For a vector in $\mathcal{M}_R$, its real coefficient is the unit Gaussian times its finite coefficient. In (WR3), for $(1, g)$, the real multiplier is one and its real integral reproduces this same Gaussian, by the reproducing identity of (WR5). In normalized finite coordinates its remaining factor is literally

$$\varphi \longmapsto \left[z \longmapsto \Delta_R(g) \int \varphi(y)\psi_0(z^*gy/2)v_f(z - gy)\, d\mu_f(y)\right], \qquad \mu_f(\mathcal{L}) = 1.$$

This is the operator in Lemma 4.1.13, not an intertwiner chosen up to scalar. Its hypotheses hold: c_0 is integral by (WD1), the self-dual lattice and measure are those of Lemma 4.1.3, and the splitting character is unitary, trivial on finite integral units and obeys $\upsilon(\bar{z}) = \upsilon(z)^{-1}$, by (WD3). In particular these finite unitary operators preserve $\mathcal{M}_R$, as do their inverses, by the same assertion for g^{-1} and the genuine group law.

The chosen-place factorization and the vacuum line. Put $\mathcal{L} = \widehat{\mathcal{O}}_E^R$. Factor $r_\sigma = ua$ as follows. At $w \in \mathcal{A}$, take $u_w = r_{\sigma,w}$ and $u_{\bar{w}} = \bar{r}_{\sigma,w}^{-1}$. At every remaining pair of places choose an orientation, remove the valuation of r_σ at the first place by a power of a uniformizer, and use its conjugate inverse at the second. At a conjugation-fixed place take $u = 1$. The valuation conclusions of (CT3) show that $a = r_\sigma/u$ is an integral unit at every place. These choices are ideles because there are only finitely many nonzero valuations. They give $u\bar{u} = 1$ and $a_{\mathcal{A}} = 1$ exactly. Fix them before all further vectors.

We verify the fixed line used here rather than invoking abstract irreducibility. In finite coordinates the vacuum is $v_f(z) = \psi_0(c_0 z^* z/2)\mathbf{1}_{\mathcal{L}}(z)$. For $\ell \in \mathcal{L}$, the support of $H(\ell)v_f$ is unchanged, and on that support its quotient by v_f is

$$\psi_0(c_0\ell^*\ell + \bar{c}_0 z^*\ell/2 - c_0\ell^* z/2) = \psi_0(c_0\ell^*\ell + \bar{c}_0 z^*\ell) = 1. \tag{C14}$$

The first equality is multiplication of the three prescribed phase factors. For the second use $\psi_0(\bar{w}) = \psi_0(w)^{-1}$ and $1 - c_0 = \bar{c}_0$. The last equality uses integrality of $c_0, \bar{c}_0$ and triviality of ψ_0 on integral arguments, from Lemma 4.1.3. This proves $H(\ell)v_R^0 = v_R^0$, including at places above two.

The raw Weyl law gives, for every x,

$$H(\ell)H(x)v_R^0 = \psi_0(x^*\ell)H(x)v_R^0 \quad (\ell \in \mathcal{L}). \tag{C15}$$

Indeed it is the ratio of the products in the two orders, whose half-cocycles combine to the displayed full character. The characters in (C15) depend only on $x \bmod \mathcal{L}$, and are distinct for distinct cosets: their equality is exactly annihilation of $x - x'$ by all ℓ, and self-duality in Lemma 4.1.3 says that this happens precisely for $x - x' \in \mathcal{L}$. Vectors from the same coset differ by a nonzero scalar, by the Heisenberg product and (C14). Vectors with finitely many distinct characters are linearly independent as eigenspaces: pass to the finite quotient of $\mathcal{L}$ by their common open kernel and average against each character. Thus the vectors fixed by every $H(\ell)$ in $\mathcal{M}_R$ are precisely $\mathbb{C}v_R^0$.

Let $L^0 = \omega(u)^{-1} L_{\sigma,R}$. Exact unitary Heisenberg covariance, (CT4), and $u\bar{u} = 1$ show that

$$L^0 R_f(x) = R_f(ax) L^0, \qquad L^0 H(x) = H(ax) L^0.$$

For $\ell \in \mathcal{L}$, $a^{-1}\ell \in \mathcal{L}$; hence $H(\ell) L^0 v_R^0 = L^0 H(a^{-1}\ell) v_R^0 = L^0 v_R^0$. The fixed-line calculation gives $L^0 v_R^0 = C v_R^0$. Its scalar is nonzero, by injectivity of $L_{\sigma,R}$ and invertibility of $\omega(u)$. This is the first assertion of (CT5), with $C = C_{\sigma,R,\mathcal{A}}$.

The exact polarized Schwartz law. Let $\mathcal{I}_{\mathcal{A}}(\Psi) = \Psi \otimes_{\mathcal{A}} v_R^0$ be the fixed transform. By Lemma 4.1.3(1)–(3), its domain is the full Schwartz–Bruhat space, it is independent of the chosen section, and

$$\mathcal{I}_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}}) = v_R^0, \qquad \mathcal{I}_{\mathcal{A}}(\tau_y) = R_f(y) v_R^0$$

for y supported on $\mathcal{A} \cup \bar{\mathcal{A}}$. Every Schwartz function is a finite linear combination of these τ_y, by the stated equality $\mathcal{T}_{\mathcal{A}} = C_c^\infty(Q_{\mathcal{A}})$.

We compute their conjugates pointwise. Since $a_{\mathcal{A}} = 1$ and $a\bar{a} = r_\sigma \bar{r}_\sigma = \chi_{\mathrm{cyc}}(\sigma)$, one has $(\bar{a})_{\mathcal{A}} = \chi_{\mathrm{cyc}}(\sigma)$. Using the literal formula in (WR6) gives

$$\begin{aligned} \tau_{ay}(x) &= \psi_0\Big(\bar{a}\bar{y}^{\mathsf{T}}(2x + ay)_{\mathcal{A}}/2\Big)\, \mathbf{1}_{\mathcal{A}}(x + ay) \\ &= \psi_0\Big(\chi_{\mathrm{cyc}}(\sigma)\bar{y}^{\mathsf{T}}(2x + y)_{\mathcal{A}}/2\Big)\, \mathbf{1}_{\mathcal{A}}(x + y) = \sigma(\tau_y(x)). \end{aligned}$$

The indicator is fixed because it takes values zero and one; the character equality is (CT3). Consequently

$$L^0 \mathcal{I}_{\mathcal{A}}(\tau_y) = C R_f(ay) v_R^0 = C \mathcal{I}_{\mathcal{A}}(\sigma \circ \tau_y).$$

For a finite expression $\Psi = \sum_y c_y \tau_y$, semilinearity on the left and linearity of $\mathcal{I}_{\mathcal{A}}$ on the right give $L^0 \mathcal{I}_{\mathcal{A}}(\Psi) = C \mathcal{I}_{\mathcal{A}}(\sigma \circ \Psi)$. All such expressions give the same vector by section independence; no automorphism has been moved through a Haar integral. Restore $\omega(u)$ to obtain the second assertion of (CT5), with the same C.

The finite multiplier law. With $J = \mathcal{L} \cap g^{-1}\mathcal{L}$ and $I = [\mathcal{L} : J]$,

$$\omega(g) v_R^0 = \frac{\Delta_R(g)}{I} \sum_{y \in \mathcal{L}/J} H(gy) v_R^0. \tag{C16}$$

The denominator is an integer. Applying L^0 gives

$$L^0 \omega(g) v_R^0 = C \frac{\sigma(\Delta_R(g))}{I} \sum_{y \in \mathcal{L}/J} H(agy) v_R^0.$$

Multiplication by the scalar unit a permutes $\mathcal{L}/J$: it preserves both lattices defining J and commutes with g. The last sum is therefore exactly the sum in (C16). This proves $L^0 \omega(g) v_R^0 = m_{\sigma,R}(g)\omega(g) L^0 v_R^0$. For $R_f(x) v_R^0$, use first the genuine covariance $\omega(g) R_f(x) = R_f(gx)\omega(g)$, next covariance for L^0, and then $ag = ga$. The same identity follows on each such translate and hence

on their complex span by semilinearity. Since u is a central unitary scalar and the actual action has the genuine group law of Lemma 4.1.1, restoring $\omega(u)$ proves (CT6) on $\mathcal{M}_R$.

Finally Lemma 4.1.13 gives $\overline{\Delta_R(g)} = \upsilon_f(\det g)\Delta_R(g)$ and $\Delta_R(g)\overline{\Delta_R(g)} = I$. The restriction of υ to the norm-one torus has integral weight by (WD3); the character assertion proved above therefore gives $|\sigma\upsilon_f(\det g)| = 1$. Apply σ to $\upsilon_f(\det g)\Delta_R(g)^2 = I$. It follows that $|\sigma\Delta_R(g)|^2 = I = |\Delta_R(g)|^2$, so $|m_{\sigma,R}(g)| = 1$. Moreover

$$\overline{m_{\sigma^\dagger,R}(g)} = \frac{\sigma(\overline{\Delta_R(g)})}{\overline{\Delta_R(g)}}, \qquad m_{\sigma,R}(g)\overline{m_{\sigma^\dagger,R}(g)} = \frac{\sigma(I)}{I} = 1.$$

This proves (CT7) with the exact integer index, without assuming that σ commutes with conjugation on arbitrary complex numbers.

The scalar of the integral vacuum. For the quantitative assertions write $q = N$, $c = c_{\mathrm{vac}}(P)$, $v = v_R^0$, $L = L_{\sigma,R}$, $r = r_\sigma$, and retain the same $u = u_{\sigma,\mathcal{A}}$, $C = C_{\sigma,R,\mathcal{A}}$ for the arbitrary fixed choice $\mathcal{A}$. Fix $R \geq 1$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Write

$$L^0 = \omega(u)^{-1}L, \qquad L^0 v = Cv, \qquad \lambda = \sigma(c_s^R)C, \qquad V = c_s^R v.$$

Thus $LV = \lambda\omega(u)v$, by semilinearity. Exact tensor factorization in Lemma 4.1.17, with the SAME positive-line functional and scale in each coordinate, gives

$$\sigma\theta_R(R_f(x)V) = \prod_{i=1}^{R} \sigma\Big(c_s\theta_+(R_f(x_i)v_+^0)\Big).$$

The rank-one values are precisely those of Proposition 4.2.1; the specified common jet-pair and vacuum-scale choice bounds every factor by $\exp(cq^2)$, so the absolute value of this product is at most $B_0 = \exp(cq^2R)$, for every finite x. By (CT4),

$$|\theta_R(R_f(rx)LV)| \leq B_0.$$

Because r is an idele, multiplication by r is a bijection of the finite coordinate space. Thus this inequality holds with every finite argument, not only a selected orbit.

The function $z \mapsto |\theta_R(\rho(z)LV)|$ is continuous by strong continuity of ρ and boundedness of θ_R. Its modulus descends to the compact quotient of physical adelic vectors by E^R: rational theta invariance and the Weyl product law change it under rational translation only by a scalar of modulus one. The image of finite adelic vectors is dense in this quotient. Indeed, every nonempty product open set $U_\infty \times U_f$ meets

$$E^R + (\{0\} \times \mathbb{A}_{E,f}^R) :$$

choose a rational vector with archimedean component in U_∞, using the density of E^R in its archimedean vector space, and then choose the finite vector to place the finite component in U_f. The normalization fv is an invertible finite-coordinate change and does not change this density statement. Consequently continuity extends the bound B_0 to every full adelic argument.

Now apply identity (2) of Lemma 4.1.10, in its normalization as a probability, to LV. Unitarity in Lemma 4.1.1 and $\|v\|_2 = 1$ give

$$|\lambda|^2 = \|LV\|_2^2 = \int |\theta_R(\rho(z)LV)|^2\, dz \leq B_0^2.$$

This proves (CT8). The conjugated scale cannot be suppressed: the conclusion is

$$|\sigma(c_s^R)C| \leq \exp(cq^2R),$$

or equivalently $|C| \le \exp(cq^2R)/|\sigma(c_s)|^R$, rather than the same bound on C alone. No continuity or modulus-preservation property of σ has been used.

Fourier coefficients of the squared theta orbit. Fix any finite unitary g and any full adelic vector y. Put

$$w = \omega(g)v, \qquad F = \rho(y)w, \qquad h(z) = |\theta_R(\rho(z)F)|^2.$$

The function h is a well-defined continuous function on the same compact quotient Q. For a rational vector $\gamma \in E^R$, put

$$\chi_\gamma(z) = \frac{\beta(z,\gamma)}{\beta(\gamma,z)}.$$

These are all the continuous characters of Q. Indeed the commutator cancels the half in the Weyl cocycle and leaves the full nondegenerate rational trace pairing. Conjugation and multiplication by the fixed nonzero rational $\jmath$ are bijections of the rational dual vector space. Apply Theorem A.5.2(a) in each coordinate. The finite and infinite coordinate normalizations transport this statement to Q.

There is an exact calculation of the coefficients, not an inference from a coefficient bound. The Weyl law and rational invariance give

$$\theta_R(\rho(z)\rho(\gamma)F) = \beta(z,\gamma)\theta_R(\rho(z+\gamma)F),$$
$$\theta_R(\rho(z)F) = \theta_R(\rho(\gamma)\rho(z)F) = \beta(\gamma,z)\theta_R(\rho(z+\gamma)F).$$

Therefore $\theta_R(\rho(z)\rho(\gamma)F) = \chi_\gamma(z)\theta_R(\rho(z)F)$. Using Lemma 4.1.10(2), with $u = F$ and $v = \rho(\gamma)F$, gives

$$\int_Q \chi_\gamma(z)h(z)\,dz = \langle F, \rho(\gamma)F\rangle.$$

If Fourier coefficients are defined using the inverse character, replace γ by $-\gamma$. The indexing is a bijection and every absolute-value assertion is unchanged.

Conjugating $\rho(\gamma)$ by $\rho(y)$ changes it only by the unit-modulus Weyl commutator. Conjugating next by $\omega(g)$ replaces its finite coordinate by $g^{-1}\gamma_f$ and leaves its real coordinate unchanged. Reproduction in Lemma 4.1.10(1) computes the remaining vacuum inner product; real evenness of the vacuum removes the possible sign, and conjugation does not affect its modulus. The literal real Gaussian and finite vacuum support in (WD6) therefore give

$$|\widehat{h}(\gamma)| = \exp\left(-\frac{\pi}{\sqrt{d}}\sum_{\tau\in\Phi,i}|\tau(\gamma_i)|^2\right)\mathbf{1}_{\{g^{-1}\gamma_f\in\alpha_s\widehat{\mathcal{O}}_E^R\}}. \tag{1}$$

Thus a vacuum inner product computes the Fourier coefficient of a squared theta value; no raw coherent coefficient has replaced the theta value.

One-separation and the numerical sum. The support set of (1) is an additive subgroup of E^R. If γ is supported, unitarity of g and $\alpha_s\bar{\alpha}_s\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/K}^{-1}\widehat{\mathcal{O}}_E$ imply

$$q_\gamma = \sum_i \bar{\gamma}_i\gamma_i \in \mathfrak{D}_{E/K}^{-1}.$$

The trace-dual definition of this fractional ideal implies $\mathrm{Tr}_{E/K}(q_\gamma) \in \mathcal{O}_K$. At the fixed embedding this trace is $\sum_{\tau,i}|\tau(\gamma_i)|^2$, which is real and positive when $\gamma \neq 0$. Since $\mathcal{O}_K \cap \mathbb{R} = \mathbb{Z}$, it is then an integer at least one. Differences of supported vectors are supported. Hence their real vectors $(\tau(\gamma_i))_{\tau,i}$ are one-separated in $\mathbb{R}^\nu$, where $\nu = 2fR$.

For completeness the packing estimate needs no unspecified lattice constant. Put $B = \nu + 1$. The radius-1/2 balls about a finite subset of a one-separated set $\mathscr{S}$ are disjoint, and for u in the ball about x, $e^{-B|x|} \le e^{B/2} e^{-B|u|}$. Integration and polar coordinates give

$$\sum_{x \in \mathscr{S}} e^{-B|x|} \le e^{B/2} 2^{\nu} \frac{\nu!}{B^{\nu}} \le e^{\nu} 3^{\nu} \qquad (\nu \ge 1).$$

The last inequality uses $\nu! \le (\nu+1)^{\nu}$, $(\nu+1)/2 \le \nu$ and $2^{\nu} \le 3^{\nu}$. The assertion for the whole set follows by monotone convergence. Completing the square gives

$$e^{-\pi r^2/\sqrt{d}} \le e^{\sqrt{d} B^2} e^{-Br} \quad (r \ge 0),$$

because the optimal completion constant $B^2\sqrt{d}/(4\pi)$ is no larger than the printed constant. Thus (1) is absolutely summable and

$$\sum_{\gamma} |\widehat{h}(\gamma)| \le \exp\Big(\sqrt{d}(\nu+1)^2 + \nu(1 + \log 3) \Big).$$

The absolutely convergent Fourier series is uniformly convergent. Its difference from h has all Fourier coefficients zero and hence is zero: finite linear combinations of characters are dense in $C(Q)$, since they form a self-adjoint algebra containing constants and separating points. In particular its value at the identity bounds $h(0)$ by the displayed sum. This is also the absolutely summable inversion assertion of Theorem A.5.2(b), transported through the rational dual pairing and taken coordinatewise. Absolute summability justifies the finite-coordinate iterations.

Since $f \le q$, $\nu \le 2qR$, $\sqrt{d} \le d$ and $1 + \log 3 \le 3$, one has

$$\sqrt{d}(\nu+1)^2 + \nu(1+\log 3) \le d(2qR+1)^2 + 6qR \le 2(2d+6)(qR+1)^2.$$

Taking square roots proves (CT9), uniformly for all full y and finite g. All Fourier integrals existed before inversion because h was continuous on the compact quotient; absolute convergence was proved before using the Fourier series pointwise.

Conjugate theta values. For σ fixing K, (CT4)–(CT6) give the literal vector equality

$$L(R_f(x)\omega(g)c_s^R v) = \lambda\, m_{\sigma,R}(g) R_f(rx)\omega(gu)v,$$

where $|m_{\sigma,R}(g)| = 1$ and the SAME λ was bounded above. Here u is central in the finite unitary group, and all factors were fixed before x, g. The degree-zero theta identity in (CT4), followed by (CT9), yields

$$|\sigma\theta_R(R_f(x)\omega(g)c_s^R v)| \le \exp\Big(cq^2 R + (2d+6)(qR+1)^2 \Big) \le \exp\Big(C_P (qR+1)^2 \Big).$$

The last step uses $q^2 R \le (qR+1)^2$ and $C_P \ge c + 2d + 6$, directly from Definition 2.3.3. For an arbitrary automorphism σ not fixing the imaginary quadratic field K, $\tau = \mathrm{conj} \circ \sigma$ fixes K, and $|\tau(z)| = |\sigma(z)|$ for every complex number z. Apply the preceding inequality to τ and the SAME original theta value. This does not change the model, vacuum, finite points or scale. Rank zero is the empty tensor with theta value one. This proves (CT10) with no factor 4^{fR}.

□

Lemma 4.2.15. *Put* $\omega = (1 + \sqrt{-d})/2$ *and* $A = \sqrt{2\pi_{\mathbb{R}}/\sqrt{d}}$.

1. *Let $\mathfrak{b}$ be a nonzero fractional $\mathcal{O}_K$-ideal with $\mathfrak{b}\bar{\mathfrak{b}} = \mathcal{O}_K$, let $\Omega \neq 0$, and put $\Lambda = \Omega\mathfrak{b}$. Let $t \notin \Lambda$ satisfy $2t \in \Lambda$ and $\omega t, \bar{\omega} t \notin \Lambda$, and suppose $\wp'_\Lambda(t) = 2$. Set $V_\Lambda = \mathrm{Area}(\Lambda) = |\Omega|^2\sqrt{d}/2$, $H_\Lambda(u,z) = \pi_{\mathbb{R}}\bar{u}z/V_\Lambda$, and let ς_Λ be the reduced sigma function with derivative 1 at zero. Define*

$$\vartheta_\Lambda(z) = e^{-H_\Lambda(t,z)}\frac{\varsigma_\Lambda(z+t)}{\varsigma_\Lambda(t)}, \quad X_\Lambda(z) = \wp_\Lambda(z+t) - \wp_\Lambda(t),$$
$$Y_\Lambda(z) = \frac{\wp'_\Lambda(z+t) - a_\Lambda X_\Lambda(z)}{2}, \qquad a_\Lambda^2 = 12\wp_\Lambda(t).$$

Then $Y^2 + a_\Lambda XY = X^3 + X$, with differential $dX/(2Y + a_\Lambda X) = dz$, a_Λ is an algebraic integer, and $|\iota(a_\Lambda^4 - 64)|_p = 1$ for every prime p and every field embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$. The exact theta multiplier is

$$\chi_\Lambda(\lambda) = \epsilon_\Lambda(\lambda)e^{H_\Lambda(\lambda,t) - H_\Lambda(t,\lambda)} \in \{1,-1\}, \quad \vartheta_\Lambda(z+\lambda) = \chi_\Lambda(\lambda)e^{H_\Lambda(\lambda,z) + H_\Lambda(\lambda,\lambda)/2}\vartheta_\Lambda(z),$$

where ϵ_Λ is the sigma sign.

2. *For a degree-one section G put $G[u] = e^{-\pi_{\mathbb{R}}|u|^2/(2V_\Lambda)}G(u)$, and set*

$$W(u) = (\vartheta_\Lambda[u]^2, \varsigma_\Lambda[u]^2), \qquad D_a(v,w) = ((v^2 - w^2)^2,\ vw(4v^2 + a^2vw + 4w^2)).$$

Then $W(2u) = D_{a_\Lambda}(W(u))$. For every torsion point $u \in \mathbb{Q}\Lambda$ and every p, ι, $\|\iota W(u)\|_{\max} = 1$. There is $M_\Lambda \geq 1$, depending only on the finite set of algebraic conjugates of a_Λ, such that

$$|\sigma(\vartheta_\Lambda[u])|, |\sigma(\varsigma_\Lambda[u])| \leq M_\Lambda \qquad (u \in \mathbb{Q}\Lambda,\ \sigma \in \mathrm{Aut}(\mathbb{C})).$$

3. *Put $\mathfrak{a} = E \cap \alpha\widehat{\mathcal{O}}_E$. Then*

$$\mathcal{O}_E \subset \mathfrak{a}, \quad \mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E/K}^{-1}, \quad \mathfrak{D}_{E/\mathbb{Q}} = (\sqrt{-d})\mathfrak{D}_{E/K}, \quad \mathrm{Tr}_{E/K}(\mathfrak{a}\bar{\mathfrak{a}}) \subset \mathcal{O}_K.$$

For every finite coordinate set C, put $\Gamma = \{A(\tau(\gamma_c))_{\tau,c} : \gamma \in \mathfrak{a}^C\}$. It is self-dual for

$$E_\Gamma(u,v) = \pi_{\mathbb{R}}^{-1}\,\mathrm{Im}\sum_{\tau,c}\bar{u}_{\tau,c}v_{\tau,c}, \qquad E_\Gamma(A\Phi\gamma, A\Phi\delta) = \mathrm{Tr}_{E/\mathbb{Q}}(\jmath\sum_c \gamma_c\bar{\delta}_c).$$

Its prescribed sign is exactly

$$\chi_\Gamma(A\Phi\gamma) = \psi_{E,f}\left(\frac{c_0\jmath}{2}\sum_c \bar{\gamma}_{c,f}\gamma_{c,f}\right) = (-1)^{\mathrm{Tr}_{E/K}(\sum_c \gamma_c\bar{\gamma}_c)}.$$

The exponent is an integer.

4. *Put $r = f|C|$, $h_0 = \mathscr{E}_{v^0}$ and*

$$(U_yS)(Z) = e^{-\bar{y}^{\mathsf{T}}Z - \|y\|^2/2}S(Z+y), \qquad R_\lambda = \chi_\Gamma(\lambda)U_\lambda \quad (\lambda \in \Gamma).$$

For every positive integer m and every entire S satisfying $R_\lambda S = S$ for $\lambda \in m\Gamma$, choose representatives, including zero, of $G = \Gamma/m\Gamma$ and $Y = m^{-1}\Gamma/\Gamma$, put $D = m^{2r}$ and $\mathcal{P} = \sum_{\lambda\in G} R_\lambda$. There are scalars b_y such that

$$\mathcal{P}U_{-y}S = b_y h_0, \qquad \sum_{y\in Y} b_y U_y h_0 = DS.$$

There is a unique F in the actual finite Heisenberg span of v^0 with $\mathscr{E}_F = S$. If $S \neq 0$, this F is cyclic for that finite Heisenberg span: the spans of $\rho(\mathrm{fv}(x))F$ and $\rho(\mathrm{fv}(x))v^0$, over all

finite x, coincide. Writing $\lambda = A\Phi(\gamma_\lambda)$ and $y = A\Phi(\gamma_y)$, if $S \neq 0$ at least one $b_y \neq 0$, and for every y

$$b_y v^0 = \sum_{\lambda \in G} \rho_f(-\alpha^{-1}\gamma_{\lambda,f})\, \rho(\mathrm{fv}(\alpha^{-1}\gamma_{y,f}))F.$$

All $|G| = |Y| = D$ terms and all signs are retained.

5. *Let $\Lambda, \Omega, \mathfrak{b}$ be as in (i), and let $e \geq 1$ be an integer with $e \in \mathfrak{b}$. For $0 \leq k < N$ and $c \in C$, put*

$$L_{k,c}(Z) = \frac{\Omega}{NA} \sum_{\tau \in \Phi} \overline{\tau(\zeta^k)} Z_{\tau,c}, \qquad S_\Lambda(Z) = \prod_{k=0}^{N-1} \prod_{c \in C} \vartheta_\Lambda(L_{k,c}(Z))^N.$$

Then

$$\sum_{k,c} N H_\Lambda(L_{k,c}(U), L_{k,c}(Z)) = \bar{U}^{\mathsf{T}} Z.$$

The function S_Λ has the exact signed periods $Ne\Gamma$ in (iv), and $S_\Lambda(0) = 1$. Consequently it has a unique cyclic realization F_Λ in the same actual finite part, with $\mathscr{E}_{F_\Lambda} = S_\Lambda$ and $\theta(F_\Lambda) = 1$. The finite inversion and vacuum expansion of (iv) have exactly $(Ne)^{2f|C|}$ terms. For $\mathfrak{b} = \mathcal{O}_K$ one may use $e = 1$. All assertions include $C = \varnothing$, with the rank-zero identity functional.

Proof. We prove the assertions with the indicated theta functionals and retain their semicharacters throughout.

First, the norm of a fractional ideal of the imaginary quadratic field satisfies $\mathfrak{b}\bar{\mathfrak{b}} = \mathrm{N}(\mathfrak{b})\mathcal{O}_K$. For an integral ideal its covolume is its index times the covolume of $\mathcal{O}_K$; clearing a positive integer denominator gives the same formula for fractional ideals. Thus $\mathrm{N}(\mathfrak{b}) = 1$ and $\mathrm{Area}(\Omega\mathfrak{b}) = |\Omega|^2\sqrt{d}/2$. The differential equation for $\wp$ gives $\wp'' = 6\wp^2 - g_2/2$. Consequently the hypothesis $\wp''(t) = 2$ is precisely $g_2 = 12\wp(t)^2 - 4$. Apply Lemma 4.2.6 to this lattice, marked point and chosen square root a_Λ. Translation of its uniformizing variable by t gives exactly the displayed equation for X, Y and its invariant differential. This proves the integrality and unit-discriminant assertions as well.

Write $\varsigma = \varsigma_\Lambda$, $\vartheta = \vartheta_\Lambda$, $H = H_\Lambda$, and $V = V_\Lambda$ temporarily. The reduced sigma quasiperiod formula is

$$\varsigma(z + \lambda) = \epsilon_\Lambda(\lambda) e^{H(\lambda,z) + H(\lambda,\lambda)/2} \varsigma(z).$$

Substitution in the definition of ϑ gives the multiplier stated in (i). The imaginary part of $H(\lambda, \mu)/\pi_{\mathbb{R}}$ is integral for periods λ, μ. Since $2t$ is a period,

$$\left(e^{H(\lambda,t) - H(t,\lambda)}\right)^2 = e^{4i\,\mathrm{Im}\,H(\lambda,t)} = 1.$$

Hence the multiplier is a sign, without a change of theta section.

Here are the elliptic identities, including their normalization, needed for (ii). The sigma addition formula is

$$\frac{\varsigma(u+v)\varsigma(u-v)}{\varsigma(u)^2\varsigma(v)^2} = \wp(v) - \wp(u). \tag{1}$$

Indeed, for fixed nonperiodic v, the left side is elliptic in u by the quasiperiod formula. Its only poles are the double poles at periods, with leading term $-u^{-2}$ at zero, because ς is odd and $\varsigma'(0) = 1$. Adding $\wp(u)$ therefore gives a constant elliptic function. At $u = v$ that constant is $\wp(v)$. Continuation proves (1) also at the omitted arguments.

Put $e = \wp(t)$. Since $t \notin \Lambda$ and $2t \in \Lambda$, the sigma sign of $2t$ is -1. Thus $\varsigma(z-t) = -e^{-2H(t,z)}\varsigma(z+t)$. Equation (1) with $v = t$ gives

$$\vartheta(z)^2 = (\wp(z) - e)\varsigma(z)^2. \tag{2}$$

The elliptic function $(\wp(z+t) - e)(\wp(z) - e)$ has no poles: the double zero at a nonzero two-torsion point cancels the possible double pole of the other factor. Its value at zero is $\wp''(t)/2 = 1$. Therefore, with $X(z) = \wp(z+t) - e$,

$$X(z)X(z+t) = 1, \qquad \varsigma(z)^2 = X(z)\vartheta(z)^2. \tag{3}$$

Using (1) with its first argument $z+t$, and then (2)–(3), yields

$$\vartheta(z+w)\vartheta(z-w) = \vartheta(z)^2\vartheta(w)^2 - \varsigma(z)^2\varsigma(w)^2. \tag{4}$$

In particular $\vartheta(2z) = \vartheta(z)^4 - \varsigma(z)^4$.

For the normalization of the other duplication identity put $c = e^{-H(t,t)/2}\varsigma(t)$ and $B(z) = e^{H(t,z)+H(t,t)/2}$. Then

$$\varsigma(z+t) = B(z)c\vartheta(z), \qquad \vartheta(z+t) = -B(z)\varsigma(z)/c.$$

Applying (3) at $z+t$ and then at z proves $c^4 = 1$. Taking the limit $v \to u$ in (1) gives $\varsigma(2u) = -\wp'(u)\varsigma(u)^4$. The quasiperiod formula for $2t$ says $\varsigma(2z+2t) = -B(z)^4\varsigma(2z)$. Use $u = z+t$ in the duplication identity to obtain $\varsigma(2z) = X'(z)\vartheta(z)^4$. The curve equation in (i) gives

$$X'(z)^2 = 4X(z)^3 + a_\Lambda^2 X(z)^2 + 4X(z).$$

Together with (3), this proves

$$\varsigma(2z)^2 = \vartheta(z)^2\varsigma(z)^2(4\vartheta(z)^4 + a_\Lambda^2\vartheta(z)^2\varsigma(z)^2 + 4\varsigma(z)^4). \tag{5}$$

The reducing Gaussian at $2z$ is the fourth power of the reducing Gaussian at z. Squaring (4) at $w = z$ and using (5) proves the exact identity $W(2z) = D_{a_\Lambda}(W(z))$.

The two entries of $W(z)$ never vanish simultaneously, since their zero sets are respectively $-t + \Lambda$ and Λ. Over any field in which $a^4 - 64 \neq 0$, the two homogeneous coordinates of D_a have no common nonzero zero. In fact the vanishing of the first forces $w = v$ or $w = -v$; if the pair is nonzero, the second then forces $a^2 = -8$ or $a^2 = 8$, respectively. This also proves the assertion in residue characteristic two. At a nonarchimedean embedding, the integrality of a_Λ and the unit property of $a_\Lambda^4 - 64$ show that the reduction of this pair has no common projective zero. Scaling a nonzero pair so that its maximum norm is one, at least one reduced coordinate is a unit. Homogeneity gives, for every pair w,

$$\|D_{\iota(a_\Lambda)}(w)\|_{\max} = \|w\|_{\max}^4. \tag{6}$$

For a torsion point u, the doubling orbit modulo Λ is finite. Under a period translation the reduced value of a degree-one section is multiplied by its sign and by $\exp(i \operatorname{Im} H(\lambda, u))$; this last number is a root of unity when $u \in \mathbb{Q}\Lambda$. Consequently the norms of the entries of $W(2^j u)$ have a repetition, under every indicated embedding. By (6), their positive maximum norms are the successive powers $r_0^{4^j}$. A repetition forces $r_0 = 1$, proving the nonarchimedean assertion.

For the complex assertion consider the finitely many conjugates a' of the algebraic integer a_Λ. On the compact set $\|w\|_{\max} = 1$ each $\|D_{a'}(w)\|_{\max}$ has a positive minimum. Choose a common lower bound $0 < \delta \leq 1$. Then $\|D_{a'}(w)\|_{\max} \geq \delta\|w\|_{\max}^4$. For every automorphism σ apply the

exact duplication polynomial to the numbers $\sigma(W(2^j u))$. Their norms are bounded along this orbit, since period multipliers remain roots of unity. If its initial maximum were greater than $\delta^{-1/3}$, the recurrence $r_{j+1} \geq \delta r_j^4$ would give $r_j \geq \delta^{-1/3}(\delta^{1/3} r_0)^{4^j} \to \infty$, a contradiction. Thus each reduced theta or sigma value is bounded by $M_\Lambda = \delta^{-1/6}$. This argument applies to the actual reduced complex numbers before applying σ; it requires no commutation of σ with a real Gaussian.

We next prove (iii). Put $\delta_E = \mathrm{cyc}'_N(\zeta)$. By Lemma 3.2.8, $\mathfrak{D}_{E/K} = (\delta_E)$, its support lies over the primes dividing N, and these primes divide q_0 and split in K. At such a prime $c_0\bar{c}_0 = (d+1)/4$ has positive valuation, whereas $c_0 + \bar{c}_0 = 1$. For odd primes this follows from the parameter divisibility by q_0; at two the same conclusion follows from $d \equiv 7 \pmod 8$. Exactly one of $c_0, \bar{c}_0$ is therefore a nonunit at each relevant place. Complex conjugation preserves the different. The literal selection in (WD2) consequently gives

$$(\alpha\bar{\alpha})\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/K}^{-1}\widehat{\mathcal{O}}_E.$$

Every component of α is either 1 or δ_E^{-1}, so $\mathcal{O}_E \subset \mathfrak{a}$. Taking the global ideals determined by the finite completions gives $\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E/K}^{-1}$. The tower formula for differents, with the different $(\sqrt{-d})$ of $K/\mathbb{Q}$, gives the asserted absolute different; it is also the different computation in Lemma 3.2.35. The defining trace-duality property of the inverse relative different gives the displayed trace inclusion.

Expansion at the pairs of conjugate embeddings, using $\jmath_v = i/\sqrt{d}$ and $A^2 = 2\pi_{\mathbb{R}}/\sqrt{d}$, gives exactly

$$\pi_{\mathbb{R}}^{-1} \operatorname{Im} \sum_{\tau,c} \overline{A\tau\gamma_c} A\tau\delta_c = \operatorname{Tr}_{E/\mathbb{Q}}(\jmath \sum_c \gamma_c \bar{\delta}_c). \tag{7}$$

The trace dual of $\mathfrak{a}$ for the form on the right is $\jmath^{-1}\mathfrak{D}_{E/\mathbb{Q}}^{-1}\bar{\mathfrak{a}}^{-1} = \mathfrak{a}$, since multiplication by -1 does not change an ideal. This is also the full real dual: a trace-dual basis of the rational basis of $\mathfrak{a}$ gives the full dual lattice in its real span. Thus (7) proves self-duality of Γ, in every number of coordinates.

Put $T_\gamma = \operatorname{Tr}_{E/K} \sum_c \gamma_c \bar{\gamma}_c$. It is in $\mathcal{O}_K$ by trace duality, and fixed by conjugation, hence is an integer. For a rational element the finite character is the inverse of its infinite character. Since $c_0\jmath = (\jmath - 1)/2$, (WD4) therefore gives

$$\psi_{E,f}(c_0\jmath \sum_c \bar{\gamma}_c\gamma_c/2) = e^{-\pi i T_\gamma} = (-1)^{T_\gamma}. \tag{8}$$

This proves the exact sign. For later use, it is a semicharacter for E_Γ: if $z = \operatorname{Tr}_{E/K} \sum_c \gamma_c \bar{\delta}_c = m + n\omega$, then $T_{\gamma+\delta} - T_\gamma - T_\delta = 2m + n$, whereas $\operatorname{Tr}_{K/\mathbb{Q}}(\jmath z) = -n$. Thus

$$\chi_\Gamma(\lambda + \mu) = \chi_\Gamma(\lambda)\chi_\Gamma(\mu)e^{\pi i E_\Gamma(\lambda,\mu)}. \tag{9}$$

We give the finite Fourier argument, together with its attachment to the actual vacuum. In the finite normalized coordinates put $\chi(u) = \psi_{E,f}(\jmath\alpha\bar{\alpha}\, u)$ and $L = \widehat{\mathcal{O}}_E^C$. The finite vacuum is $\chi(c_0 z^* z/2)1_L(z)$. For $y \in L$ the quotient of $\rho_f(y)e_n$ by e_n on its support is

$$\chi(c_0 y^* y + \bar{c}_0 z^* y/2 - c_0 y^* z/2) = \chi(c_0 y^* y + \bar{c}_0 z^* y) = 1. \tag{10}$$

Here $\chi(\bar{u}) = \chi(u)^{-1}$, and $\jmath\alpha\bar{\alpha}\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1}\widehat{\mathcal{O}}_E$, so χ is trivial on integral arguments. The support is unchanged by translation by y. Thus (10) proves the actual signed vacuum invariance, including places above two.

For y, z in the complex vector space a direct multiplication gives

$$U_y U_z = e^{\pi i E_\Gamma(y,z)} U_{y+z}, \qquad U_y R_\lambda U_{-y} = e^{2\pi i E_\Gamma(y,\lambda)} R_\lambda. \tag{11}$$

Equation (9) implies $R_\lambda R_\mu = R_{\lambda+\mu}$ for lattice vectors. On the space with the $m\Gamma$ periods this is an action of $\Gamma/m\Gamma$. If $y \in m^{-1}\Gamma$, (11) shows that $U_{-y}S$ has the same $m\Gamma$ periods. Hence $\mathcal{P}U_{-y}S$ is fixed by all R_λ for $\lambda \in \Gamma$.

The precise translation formula of Lemma 4.2.2 reads, with $a = A\Phi\gamma$,

$$\mathscr{E}_{\rho(\mathrm{fv}(\alpha^{-1}\gamma_f))F} = U_{-a}\mathscr{E}_F, \qquad \mathscr{E}_{\rho_f(-\alpha^{-1}\gamma_f)F} = R_a \mathscr{E}_F \quad (\gamma \in \mathfrak{a}^C). \tag{12}$$

The second identity uses (8), also for $-\gamma$. In view of (10), h_0 is fixed by all these R_a. It is nonzero by the injectivity in Lemma 4.2.2, since v^0 has norm one by Lemma 4.1.10. The dimension-one theorem A.2.16 applies to Γ, (9), and the positive Hermitian form $H_0(Z,W) = \pi_{\mathbb{R}}^{-1} Z^{\mathsf{T}} \bar{W}$. Its imaginary part on lattice vectors is $-E_\Gamma$, which gives the same integral self-duality and the same sign in (9). Its theta automorphy formula is exactly $R_\lambda h = h$. Consequently

$$\mathcal{P}U_{-y}S = b_y h_0. \tag{13}$$

Since Γ has rank $2r$, both G and Y have order $D = m^{2r}$. The pairing $(y,\lambda) \mapsto e^{2\pi i E_\Gamma(y,\lambda)}$ between these groups is perfect: its kernel on the first group is zero by self-duality, and their orders agree. The sum of a nontrivial character is zero, because translating the sum by an element where that character is not one multiplies the same sum by a scalar different from one. Therefore (11)–(13) give

$$\sum_{y\in Y} b_y U_y h_0 = \sum_{\lambda\in G} \left(\sum_{y\in Y} e^{2\pi i E_\Gamma(y,\lambda)} \right) R_\lambda S = DS. \tag{14}$$

No representative contributes a suppressed phase: the calculation uses the chosen representatives in each occurrence, and R_λ factors through G on the period space.

Define

$$F = D^{-1} \sum_{y\in Y} b_y \rho(\mathrm{fv}(-\alpha^{-1}\gamma_{y,f}))v^0.$$

Equation (12) and (14) give $\mathscr{E}_F = S$. Its uniqueness follows from the same injectivity. Applying (12) to (13) and using injectivity once more gives precisely

$$b_y v^0 = \sum_{\lambda\in G} \rho_f(-\alpha^{-1}\gamma_{\lambda,f})\rho(\mathrm{fv}(\alpha^{-1}\gamma_{y,f}))F.$$

If $S \neq 0$, some b_y is nonzero by (14), so this identity puts v^0 in the finite Heisenberg span of F. Conversely the definition puts F in the finite Heisenberg span of v^0. The Heisenberg composition law, whose extra factor is a nonzero scalar, makes each such span invariant under every finite Heisenberg operator. The two cyclic spans are therefore equal. This proves all of (iv).

Finally, distinct embeddings over K give distinct images of the primitive root ζ. The finite geometric sum gives

$$\sum_{k=0}^{N-1} \tau(\zeta^k)\overline{\nu(\zeta^k)} = N\,\delta_{\tau,\nu}.$$

Expand the linear forms in (v), use this identity separately in each coordinate, and use the area calculation at the start. The coefficient that remains is $\pi_{\mathbb{R}}|\Omega|^2/(V_\Lambda A^2) = 1$. This proves the full Hermitian identity, not merely its diagonal restriction.

Put $m = Ne$. For $\gamma \in \mathfrak{a}^C$ one has

$$L_{k,c}(mA\Phi\gamma) = \Omega e \ \mathrm{Tr}_{E/K}(\gamma_c \overline{\zeta^k}) \in \Omega e \mathcal{O}_K \subset \Lambda. \tag{15}$$

The trace is integral because $\overline{\zeta^k} \in \mathcal{O}_E \subset \bar{\mathfrak{a}}$ and $\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E/K}^{-1}$. The parameter $N = q_1(s)$ is even. Thus every sign in the multiplier of each factor in (15), raised to its exponent N, is exactly one. The Hermitian identity already proved sums the remaining exponents and gives, for $a \in m\Gamma$,

$$S_\Lambda(Z + a) = e^{\bar{a}^{\mathsf{T}} Z + \|a\|^2/2} S_\Lambda(Z).$$

On the other hand (8) gives $\chi_\Gamma(mA\Phi\gamma) = (-1)^{m^2 T_\gamma} = 1$, since m is even. These are exactly the signed $m\Gamma$ periods of (iv). Every theta factor is one at zero, so $S_\Lambda(0) = 1$. Applying (iv) proves its unique cyclic realization and the stated exact number of terms. Evaluation of its entire function at zero gives $\theta(F_\Lambda) = 1$. For $\mathfrak{b} = \mathcal{O}_K$, the integer $e = 1$ satisfies (15). If C is empty, the lattice is zero, each finite group has one element, the products are one, and every displayed identity is the identity on the rank-zero line. □

Lemma 4.2.16. *Let $R \geq 1$, $\iota_R = \Phi \times \{1, \dots, R\}$, $D = fR$, and let $\mathcal{M}_{\mathrm{fin}}$ be the complex span of the vectors $\rho(\mathrm{fv}(t))v^0$, $t \in \mathbb{A}_{E,f}^R$, in the theta model of the standard definite space I_R. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ there are $t_\sigma \in \mathbb{C}$ with $|t_\sigma|^2 = \Omega_s^{-1}$ and a permutation ϵ_σ of Φ such that for every $F \in \mathcal{M}_{\mathrm{fin}}$ there is a bounded continuous function $\Theta_{\sigma,F}$ on $\mathbb{C}^{\iota_R}$ with:*

(a) $|\Theta_{\sigma,F}(y)| \leq \sup_{x \in \mathbb{A}_{E,f}^R} |\sigma(\theta_R(\rho(\mathrm{fv}(x))F))|$ for every $y \in \mathbb{C}^{\iota_R}$;

(b) for every polynomial Q with complex coefficients in the variables $Z_{\tau,c}$ ($(\tau, c) \in \iota_R$), the integral below converges absolutely and

$$\sigma\Big(\theta_R(v_{Q(\eta_P\,\cdot)} \otimes F)\Big) = \pi^{-D} \int_{\mathbb{C}^{\iota_R}} Q^\sigma\big(t_\sigma\, \bar{y}^{\epsilon_\sigma}\big)\, e^{-|y|^2/2}\, \Theta_{\sigma,F}(y)\, dy,$$

where $Q(\eta_P\,\cdot)$ is the polynomial $Z \mapsto Q(\eta_P Z)$, Q^σ is Q with σ applied to its coefficients, $(\bar{y}^{\epsilon_\sigma})_{\tau,c} = \overline{y_{\epsilon_\sigma(\tau),c}}$, $|y|^2 = \sum_{\tau,c} |y_{\tau,c}|^2$ and dy is Lebesgue measure on $\mathbb{C}^{\iota_R} = \mathbb{R}^{2D}$.

Proof. Fix $R \geq 1$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Apply Lemma 4.2.14 in the SAME standard positive space I_R, with the specified jet pair of the statement. Take its common $L_\sigma, t_\sigma, \epsilon_\sigma, r_\sigma$, fixed before any F or polynomial. Write $\mathsf{R}_f(x) = \rho(\mathrm{fv}(x))$ and

$$E_F(Z) = e^{|Z|^2/2} \theta_R(\rho(\mathrm{rv}(Z))F).$$

We use precisely its normalized coefficientwise Taylor identity and finite translation identity

$$\sigma\, \mathrm{Taylor}_0\, E_F(\eta_P Z) = \mathrm{Taylor}_0\, E_{L_\sigma F}(t_\sigma \epsilon_\sigma^{-1} Z), \qquad \theta_R(\mathsf{R}_f(r_\sigma x) L_\sigma F) = \sigma \theta_R(\mathsf{R}_f(x)F). \tag{1}$$

In particular r_σ is an idele and $|t_\sigma|^2 = \Omega_s^{-1}$. The coordinate convention is $(\epsilon_\sigma^{-1} Z)_{\nu,c} = Z_{\epsilon_\sigma^{-1}(\nu),c}$.

First the supremum in (a) is finite. Choose a finite expression $F = \sum_{j=1}^u a_j \mathsf{R}_f(x_j) v^0$. The Heisenberg product law of Lemma 4.1.10(i) gives $\mathsf{R}_f(x)F = \sum_j a_j \xi_j(x) \mathsf{R}_f(x + x_j) v^0$, where each $\xi_j(x)$ is a root of unity: its finite additive-character argument lies in a compact open subgroup after multiplication by a suitable positive integer. Automorphisms preserve absolute values of roots of unity. Apply (CT10) of Lemma 4.2.14 in rank R, with finite group point 1, to the SAME fixed nonzero scale c_s. Thus, for every x,

$$|\sigma\theta_R(\mathsf{R}_f(x)F)| \leq |\sigma(c_s^R)|^{-1} \left(\sum_j |\sigma(a_j)| \right) e^{C_P (q_1(s)R+1)^2}. \tag{2}$$

Here $\omega(1) = 1$ is the literal identity operator (equivalently its reproducing integral). The nonzero denominator is fixed. Denote the finite supremum by $S_{\sigma,F}$; for $F = 0$ it is zero.

Put $G = L_\sigma F$ and define $\Theta_{\sigma,F}(y) = \theta_R(\rho(\mathrm{rv}(y))G)$. This is continuous by strong continuity in Lemma 4.1.10(i) and boundedness of the canonical functional in Lemma 4.1.17. The second identity in (1) and surjectivity of multiplication by the idele r_σ bound every finite translate of G by $S_{\sigma,F}$.

Here is the exact passage to real translates. For a rational vector q in the original framed space, decompose it as $q = \mathrm{rv}(a) + \mathrm{fv}(x)$. The Heisenberg law and rational invariance of Lemma 4.1.17 imply

$$|\theta_R(\rho(\mathrm{rv}(a))G)| = |\theta_R(\rho(-\mathrm{fv}(x))G)| \le S_{\sigma,F}. \tag{3}$$

Indeed insert $\rho(q)$ to the left of $\rho(-\mathrm{fv}(x))$; its product differs from $\rho(\mathrm{rv}(a))$ only by a phase of modulus one. The real coordinates of rational vectors are dense: a $\mathbb{Q}$-basis of E identifies $E \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^\Phi$, and rational coefficients are dense in real coefficients; the fixed nonzero coordinate scalings preserve density. Apply continuity to (3). This proves $|\Theta_{\sigma,F}(y)| \le S_{\sigma,F}$ for every y, hence (a) and boundedness.

It remains to compute the integral. By (CT1) of Lemma 4.2.14, $H(Z) := E_G(Z) = \sum_\beta h_\beta Z^\beta$ is entire with locally uniformly convergent Taylor series and ordinary-derivative coefficients. Write $Q(Z) = \sum_\alpha q_\alpha Z^\alpha$, a finite sum, and define $(\epsilon_{\sigma *}\alpha)_{\epsilon_\sigma(\tau),c} = \alpha_{\tau,c}$. The ordinary derivative identity (CT2) of Lemma 4.2.14 gives

$$\sigma\theta_R(v_{Q(\eta_P\,\cdot)} \otimes F) = \sum_\alpha \sigma(q_\alpha) t_\sigma^{|\alpha|} \alpha!\, h_{\epsilon_{\sigma *}\alpha}. \tag{4}$$

This also fixes the permutation direction: differentiation in $Z_{\tau,c}$ differentiates the image germ in coordinate $(\epsilon_\sigma(\tau), c)$.

By definition $\Theta_{\sigma,F}(y) = e^{-|y|^2/2} H(y)$. Its established bound proves absolute integrability of the proposed integrand, since the remaining factor is a polynomial times $e^{-|y|^2/2}$. Use polar coordinates and Fubini. At fixed radii the entire Taylor series of H converges uniformly on the compact product of circles. Angular integration of $Q^\sigma(t_\sigma \bar{y}^{\epsilon_\sigma}) H(y) e^{-|y|^2}$ therefore selects, for each of the finitely many α in Q, exactly $h_{\epsilon_{\sigma *}\alpha}$. Radial integration gives

$$\pi^{-1} \int_{\mathbb{C}} |w|^{2a} e^{-|w|^2}\, dw = a!, \qquad a \in \mathbb{N}.$$

The product of these factors is $(\epsilon_{\sigma *}\alpha)! = \alpha!$. Consequently the integral in (b) equals the right side of (4), proving (b). Only compact angular tori were used for Taylor-series interchange; no interchange of an infinite series with an unbounded radial integral is needed. Constant and nonhomogeneous polynomials and $F = 0$ are included. Every application of σ is to a coefficient before integration. The choices of $t_\sigma, \epsilon_\sigma$ precede all F, Q, as required. □

The seesaw identity (Proposition 4.2.25) comes after its one-step transfer (Proposition 4.2.17). Its statement fixes an arithmetic realisation of all the spaces before any twist or source form, takes any map $\mathcal{P}$ represented by Bessel periods at one point [96], and asserts one nonzero constant such that, for every ordinary twist, the chain of lifts of $\mathcal{P}\varphi$ is the constant times a composite definite lift of φ, with integral coefficient functions of controlled size. The constant precedes all twists, and the coefficients precede all source forms. The one-step transfer moves a Bessel period through one theta kernel: the theta coefficients along the isotropic part have an absolutely convergent Fourier expansion, the classes that do not survive are killed by a unitary twist, and the surviving maps form one rational Bessel block; reduction theory controls the growth [33, 208]. The results after the identity are printed in the order in which its proof uses them. Consecutive

steps join into one definite kernel, ordinary Schur jets give the one-edge coefficients uniformly in the twist through the polynomial Cauchy decomposition [135], the outer variables are cut into finite sets, and one unit normalises all the coefficient expansions at once, which is what lets the constant be chosen before the twists.

Proposition 4.2.17. *Let E/F be the same CM extension, let $N \in F^{\times}$, and fix $c \in F^{\times}$. Let $V = X \perp J_s$ and $V' = X' \perp J_{s-1}$, where $s \geq 1$, X, X' are positive definite, and J_l has basis $v_0, \dots, v_{l-1}$ with antidiagonal Gram matrix, the middle entry replaced by $-N$ when l is odd. Give X, X' grade zero and $v_a \in J_l$ grade $l - 1 - 2a$. In rational coordinates write these grades as d, d'. Put $G = \mathrm{U}(V)(\mathbb{A}_F)$, $G' = \mathrm{U}(V')(\mathbb{A}_F)$ and $\Gamma' = \mathrm{U}(V')(F)$.*

Let $R \subset G$ consist of matrices g such that $g_{ia} = \delta_{ia}$ whenever a is a string index with $d(a) \geq 0$ and $d(i) \leq d(a)$; define $R' \subset G'$ in the same way. Let $U^{(2)} \subset R$ consist of u for which $(u-1)_{ia} = 0$ if $d(i) - d(a) < 2$. The inclusion of $\mathrm{U}(X)$ acts trivially on the string. Define the rational matrix S to vanish on X and by

$$Sv_b = \begin{cases} c\epsilon_b v_{b+1}, & b+1 < s, \\ 0, & b+1 = s, \end{cases} \qquad \epsilon_b = \begin{cases} -N, & 2b+1 = s, \\ -Nc, & 2(b+1) = s, \\ 1, & \text{otherwise.} \end{cases}$$

Put $\chi(u) = \psi_E((\jmath/2)\,\mathrm{tr}(Su))$. Let $W : R \to \mathbb{C}$ be continuous, left invariant by $R(F)$, and satisfy $W(ur) = \chi(u)W(r)$ for $u \in U^{(2)}$.

Define $T : V \to V'$ to kill X and to send v_a to $t_a v'_a$ when $a < s-1$, and to zero otherwise, where

$$t_a = \begin{cases} 1, & s \text{ odd},\ 2a+1 < s, \\ -Nc, & s \text{ odd},\ 2a+1 = s, \\ c, & s \text{ odd},\ 2a+1 > s, \\ c, & s \text{ even},\ 2a+1 < s, \\ 1, & s \text{ even},\ 2a+1 > s. \end{cases}$$

Let p_+ retain precisely the entries (b, a) of an adelic matrix for which $d'(b) - d(a) > 0$. For a matrix A in rational coordinates let $[A]$ denote its coordinates in the fixed adelic orthonormal frames: if L, L' send framed coordinates to rational coordinates, then $[A] = (L')^{-1}AL$.

Fix a continuous unitary automorphic character a of $\mathrm{U}(1)$ and put $a_l = a\upsilon^{\lfloor l/2 \rfloor}$. On the coherent model of $\mathrm{Hom}(V, V')$ use

$$\omega_a(g, h) = a_{\dim V}(\det h)\, a_{\dim V'}(\det g)^{-1} \omega_{\mathrm{Hom}}(A \mapsto hAg^{-1}).$$

Let f be a finite sum of tensors of Fock polynomial vectors and finite Heisenberg translates of the vacuum. With additive quotients given probability measure, define

$$C_f(g, h) = \int_{\mathrm{Mat}(E) \backslash \mathrm{Mat}(\mathbb{A}_E)} \theta_{\mathrm{Hom}}(\rho([p_+ z])\, \rho([T])\, \omega_a(g, h) f)\, dz,$$
$$U_f W(h) = \int_{R(F) \backslash R} W(r) C_f(r, h)\, dr.$$

The matrix quotient has the row and column sizes of $\mathrm{Hom}(V, V')$. Measures on reductive and Bessel quotients are induced compatibly by fixed Haar measures.

Then $U_f W$ is left invariant under $R'(F)$ and

$$\int_{R(F) \backslash R} W(r) \theta(\omega_a(r, h) f)\, dr = \sum_{\gamma \in R'(F) \backslash \Gamma'} U_f W(\gamma h). \tag{1}$$

The series is absolutely convergent. Let y be a continuous Γ'-invariant rapidly decreasing function on G': on each Siegel set, for every $A > 0$ it is bounded in absolute value by a constant times H^{-A}, where H is the usual positive reduction-theoretic height. Then the following integrals are absolutely convergent, including the product integral needed to exchange their order:

$$\int_{\Gamma'\backslash G'} \sum_{\gamma} |U_f W(\gamma h) y(h)|\, dh < \infty,$$

$$\int_{R(F)\backslash R} W(r) \int_{\Gamma'\backslash G'} \theta(\omega_a(r,h)f) y(h)\, dh\, dr = \int_{R'(F)\backslash G'} U_f W(h) y(h)\, dh. \tag{2}$$

Proof. Use the theta-orbit expression

$$C_D(\varphi, w) = \int_{[P_D]} \theta(\rho(p)\rho(w)\varphi)\, dp, \qquad P_D = \{z : z_{ba} = 0 \text{ unless } D_b - d_a > 0\},$$

for $D = 0, d'$, with probability measure. The full matrix quotient in the statement pushes forward under its rational positive-entry projection to the probability quotient $[P_{d'}]$. Hence its displayed $C_f(g,h)$ is exactly $C_{d'}(\omega_a(g,h)f, T)$, with the same frame coordinates and operator order. By the theta-function evaluation convention (WC3) of Definition 4.1.15, point evaluation here means $\theta(\rho(x)\varphi)$. Fixed frames conjugate every displayed map and pairing.

Application of the fixed model laws. Use clauses (2),(3) of Lemma 4.1.1 for the actual (WR3) kernel: its bounded continuous genuine action, exact covariance, and comparison with the polarized oscillator model. This is a dependency on those laws, not an inference of the group law from covariance. The exact coherent Banach, projector, continuous-representative and strongly continuous isometric Heisenberg facts are clause (1) of Lemma 4.1.10, for this same (WD7) carrier. The fixed canonical theta functional is the bounded functional of (WC3) of Definition 4.1.15.

The test-domain hypotheses can be checked before any constant-term calculation. Clause (4) of Lemma 4.1.3, with the empty selected set, identifies $v_P \otimes v^0$ with its polynomial-Gaussian real coefficient times the finite vacuum. Finite Heisenberg translations commute with the real datum integral in (WR6), by the product bicharacter and Fubini. Thus each specified tensor has exactly that real polynomial-Gaussian factor and a finite translated-vacuum factor. In normalized real coordinates a polynomial vector is $P(\zeta(x))e^{-|x|^2/2}$. Differentiating its Heisenberg orbit produces polynomial times shifted Gaussian functions, because the translation phase is exponential-linear. For translation parameters in a compact set, derivatives of any fixed order are bounded by $C(1 + \|x\|)^m e^{-\|x\|^2/4}$; this follows from $\|x - z\|^2 \geq \|x\|^2/2 - \|z\|^2$. The bound is integrable, so difference quotients converge in the coherent L^1 norm. Thus the prescribed polynomial vectors are smooth for the full real Heisenberg action.

At finite places the vacuum has compact-open support and locally constant phase by (WD5). On that compact support choose a common compact-open translation subgroup on which the phase differences and the Heisenberg bilinear characters are all trivial. This is possible by continuity of the character and compactness of the support. It fixes the vacuum under the raw Heisenberg action. Conjugating by each of the finitely many finite translations and shrinking into their commutation-character kernels gives a common open stabilizer of the finite test factors. The same vectors are smooth under the finite reductive pair group: by Lemma 4.1.1(6), a sufficiently small subgroup of the standard compact level fixes the vacuum, after intersecting the kernels of the finite determinant characters. Those kernels are open on the compact level: a continuous character of a profinite compact group has compact profinite image in the circle, and such a closed subgroup of the circle is finite. Shrink that subgroup so that its action on each of the finitely many translation parameters differs from that parameter by the preceding

Heisenberg stabilizer, with commutation phase one. Covariance then shows that it fixes every translated vacuum in the test vector. Intersect these finitely many subgroups. This also verifies the finite reductive smoothness required later by Lemma 4.2.22.

Now use the exact same-carrier comparison of Lemma 4.1.2: the Heisenberg-smooth domain of $\mathcal{M}_{V,V'}$ is identified with the polarized Schwartz–Bruhat model, with continuous inverse and with a unitary extension on Hilbert completions. The preceding smoothness checks put f in this domain. This is an application of its stated domain assertion, not a second proof by Sobolev embedding. The genuine action and exact covariance from Lemma 4.1.1(2) extend unitarily to the Hilbert completion. Under the fixed comparison, each such operator is therefore a unitary Heisenberg normalizer for the underlying trace symplectic space over F. The hypotheses of Theorem A.5.3 apply: that space is finite-dimensional and nondegenerate, the standard character is nontrivial, and a rational polarization can be chosen by symplectic linear algebra. It follows that the operator and its inverse preserve the Schwartz–Bruhat domain. The rank-zero domain is $\mathbb{C}$ and needs no comparison.

The bounded coherent theta functional restricts continuously to this Schwartz domain: the zeroth seminorm of the coherent smooth-vector topology is its L^1 norm, and the comparison and inverse are continuous for the stated Schwartz topology. Its compact isotropic means are continuous as well. Indeed the projected image in $\mathcal{M}$ of a compact set of additive quotient representatives is a bounded family of isometries $\rho(p)$, so each mean has norm at most $\|\theta\|$ on the coherent carrier. Hence the functional used in Lemma 4.1.12 satisfies precisely that result's continuity and translation-invariance hypotheses. The latter invariance follows by translation of the probability measure on the isotropic quotient; its bicharacter is trivial on two vectors of that subspace. No fixed phase of the comparison is used.

For rational normalization apply Lemma 4.1.11 to this literal kernel. Its field hypothesis holds: q_0 is divisible by $16 \cdot 3 \cdot 5 \cdot 7$, and $q_1 = q_0\, 3 l_0\, 2^s$ is divisible by 5040, so E_s contains μ_{5040}. Its genuine-action, covariance and Schwartz hypotheses are just the preceding model laws. It gives rational pair theta invariance exactly, independently of the single nonzero theta normalization scalar.

Compactness and source measures. Let the grade filtration be the spans of vectors of grade at least a given value. The defining columns of R fix each nonnegative string vector modulo strictly higher grades. Thus they preserve all positive filtration spaces and act as the identity on their string graded quotients. More precisely, $(V_{\geq r})^{\perp} = V_{>-r}$ for $r > 0$, since the Gram matrix pairs exactly opposite grades. Unitarity therefore gives preservation of the negative filtration too, and duality gives identity on its string graded quotients. On grade zero the only possible nonidentity diagonal block is on X: when the string has a middle vector, its grade-zero image is fixed, and orthogonality to that vector excludes mixing with X on the grade-zero quotient.

The homomorphism taking this X block maps R onto $\mathrm{U}(X)$, with section the block action trivial on the string. Its kernel U consists of unitary matrices $1 + A$ with A strictly grade raising. Such A is nilpotent, so U is an F-unipotent group. This proves $R = U \rtimes \mathrm{U}(X)$ with the stated rational structure. By Theorem A.4.3, $U(F)\backslash U(\mathbb{A}_F)$ is compact. If $X \neq 0$, its positive definiteness satisfies Theorem A.4.5, so $\mathrm{U}(X)(F)\backslash \mathrm{U}(X)(\mathbb{A}_F)$ is compact; if $X = 0$ this factor is trivial.

Choose compact covering sets K_U, K_X for these two quotients. For $r = uh$, first write $h = \delta_X k_X$ with δ_X rational and $k_X \in K_X$. Then $r = \delta_X(\delta_X^{-1} u \delta_X) k_X$. Write the middle unipotent factor as $\delta_U k_U$. Hence $r = (\delta_X \delta_U) k_U k_X$ and

$$R = R(F) K_U K_X.$$

The existence of compact covering sets follows by lifting a finite cover of each compact quotient

by relatively compact coordinate neighborhoods. Thus $[R]$ is compact and W is bounded.

The quotient has finite invariant measure. To check unimodularity, the determinant of the Levi action on $\operatorname{Lie} U$ is an F-rational character of $\mathrm{U}(X)$. This group has no nontrivial such characters: after extension to E a character of $\mathrm{GL}(X)$ is a power of its determinant, while the nontrivial element of $\mathrm{Gal}(E/F)$ sends that power to its inverse; invariance forces the exponent to be zero. The rank-zero case is immediate. Both factors of the semidirect product are unimodular, so R is unimodular as well. The rational subgroup is discrete: its matrix entries lie in the discrete diagonal copy of E in $\mathbb{A}_E$. Compactness now gives finite quotient measure for the fixed compatible Haar measures.

Expansion and transformation laws. The monomial Gram matrices pair opposite grades. Thus P_D is isotropic and $P_D^{\perp}$ is the space of nonnegative-degree maps. By Lemma 4.2.21,

$$\theta(\varphi) = \sum_{[w]\in\mathcal{M}(F)/P_D^{\perp}(F)} C_D(\varphi, w), \qquad \sum_{[w]} |C_D(\varphi, w)| < \infty, \quad \mathcal{M} = \operatorname{Hom}(V, V'). \tag{A}$$

Its quantitative bound and Lemma 4.2.22 are uniform on compact source representatives. Rational theta invariance and a rational probability-measure substitution give

$$C_D(\omega_a(\gamma, \delta)\varphi, w) = C_D(\varphi, \delta^{-1} w\gamma) \tag{B}$$

for $\gamma \in R(F)$ and rational δ preserving the target D-flag. For $D = 0$, every $\delta \in \Gamma'$ is allowed. The same law for strictly grading-raising adelic elements follows from Lemma 4.1.12, with determinant twists 1. The Heisenberg law gives, for $q \in P_D(\mathbb{A})$,

$$C_D(\varphi, w + q) = b(w, q) C_D(\varphi, w). \tag{C}$$

For rational changes in $P_D^{\perp}(F)$ the coefficient is unchanged; rational phases are 1.

Nonsurviving orbit cancellation. Set $F_w(r) = W(r) C_D(\omega_a(r, h)f, w)$. For use in the source character, let h_b, h'_b be the antidiagonal Gram entries on the two strings. Adjointing the diagonal map gives

$$T^{\dagger} T v_b = \frac{h'_b t_{s-2-b} t_b}{h_{b+1}} v_{b+1} \quad (0 \le b < s - 1),$$

and zero on X and the last string vector. For $s = 2k + 1$, the coefficient is c except at $b = k$, where it is $-Nc$; at $b = k - 1$ the numerator contains $-Nc$ and the denominator $-N$, giving c. For $s = 2k$, it is c except at $b = k - 1$, where the middle target Gram entry is $-N$ and both factors are c, giving $-Nc^2$. These are exactly $c\epsilon_b$ in the statement. For $s = 1$ there are no such indices. Therefore $T^{\dagger} T = S$. If $u - 1$ raises degree by at least 2 and $w(u - 1) \in P_D$, (B),(C) and genericity give

$$F_w(ur) = \frac{b(Tu, T)}{b(wu, w)} F_w(r). \tag{D}$$

Survival means that every such quotient is 1. In a nonsurviving class choose a nontrivial unit quotient.

For any rationally stable set of classes, the absolute estimate in (A) permits regrouping by $R(F)$-orbits. For one orbit let $\Lambda_w \subset R(F)$ be its stabilizer. Rational covariance makes F_w left Λ_w-invariant. Discrete unfolding identifies the orbit integral with $\int_{\Lambda_w \backslash R} F_w$: the translates of a fundamental domain for $R(F)$ indexed by $\Lambda_w \backslash R(F)$ partition one for Λ_w. The same absolute estimate proves integrability there. Apply Lemma 4.2.23 with (D). No assumption that u normalizes Λ_w is required. Every stable collection of nonsurviving classes therefore contributes zero.

A grading-independent agreeing Fourier projection. Let $\mathcal{A}_D$ consist of the classes with $w_{ba} = T_{ba}$ where $d_a > \max(0, d'_b)$. For maps $y : V' \to V$ retain exactly these transposed entries. Their adjoints $y^\dagger$ have positive degree for both choices of D: adjoint reverses grades, and $d_a > \max(0, d'_b) \geq D_b$. The Heisenberg commutator and invariance of the P_D mean give

$$C_D(\rho(y^\dagger)\varphi, w) = \psi_E(\jmath \operatorname{tr}(yw)) C_D(\varphi, w).$$

Expand the translated theta by (A), divide by $\psi_E(\jmath \operatorname{tr}(yT))$, and average y modulo rational maps. The expanded absolute sum is the fixed finite sum $\sum |C_D(\varphi, w)|$, so averaging and summation commute. Character orthogonality imposes exactly the selected entry equalities. Consequently

$$\sum_{\mathcal{A}_D} C_D(\varphi, w) = \int_{[y]} \frac{\theta(\rho(y^\dagger)\varphi)}{\psi_E(\jmath \operatorname{tr}(yT))}\, dy. \tag{E}$$

The right side does not depend on D. Hence coarse and fine agreeing sums are equal. This argument does not interchange two noncommuting positive-subspace means.

Orbit geometry and the pointwise identity. By Lemma 4.2.19, each coarse surviving class lies in a rational target translate of $\mathcal{A}_0$. The complement is source-stable and nonsurviving, so its integral is zero. The same lemma's exact stabilizer conclusion says those translates are disjoint, indexed by $R'(F)\backslash\Gamma'$. In the fine quotient every surviving agreeing class is $[T]$. The set $\mathcal{A}_{d'} \setminus \{[T]\}$ is source-stable: agreement is preserved by the right action of $R(F)$, and $Tr - T$ has nonnegative fine degree. It too has integral zero. Applying (E) at $\varphi = \omega_a(r, h)f$ gives

$$\int_{[R]} W(r) \sum_{\mathcal{A}_0} C_0(\omega_a(r, h)f, w)\, dr = \int_{[R]} W(r) C_{d'}(\omega_a(r, h)f, T)\, dr = U_f W(h).$$

By coarse covariance, the block $\delta^{-1}\mathcal{A}_0$ gives $U_f W(\delta h)$. Summing the disjoint blocks proves (1), scalar 1. Moreover,

$$\sum_{\delta \in R'(F)\backslash\Gamma'} |U_f W(\delta h)| \leq \int_{[R]} |W(r)| \sum_{[w] \text{ coarse}} |C_0(\omega_a(r, h)f, w)|\, dr. \tag{F}$$

Here one first bounds the absolute value of each block integral by the integral of its absolute coefficient sum, then uses disjointness. No arbitrary rational target element is assumed to preserve the fine grading.

The target invariance uses the next-string lemma. Apply Lemma 4.2.26 with precisely the present V, V', s, c, N, T, grades and W. Its test-vector hypothesis is the finite sum specified here. Its source domain is R, and the assumed left rational invariance and $U^{(2)}$ character of W are identical. Its positive fine-degree space is $P_{d'}$. The full matrix-quotient probability mean in this statement pushes forward onto that space, as checked at the start of the proof, with the same frames and operator order; therefore its C_f and $U_f W$ are the functions in this proof, not merely proportional functions. That lemma gives continuity and left $R'(F)$-invariance of $U_f W$. It also supplies the next-string covariance when needed, with the same c, N, so no additional adjoint or genericity calculation is required here.

Absolute majorants and final unfolding. The Banach space in Lemma 4.2.22 is the present coherent L^1 carrier, with its strongly continuous isometric Heisenberg action and bounded rational-invariant theta functional. The target unitary group is reductive because its form is nondegenerate; choose its faithful defining matrix realization. The rational action on the trace symplectic Hom space and its exact covariance are those of the pair representation already checked. All three smoothness requirements of that lemma were checked above. The additional

source group can be the reductive ambient group $\mathrm{U}(V)$, while its argument is confined to the single compact covering set $K_U K_X$ of $[R]$. Its permitted compact-source uniformity therefore applies. Use Theorem A.4.2 to choose reduced target representatives with finite components in a fixed compact. The full coarse absolute coefficient sum is invariant under rational target reindexing. By Lemma 4.2.22, (F) and the absolute theta product integral over $[R]$ are bounded by $C\mathcal{H}(h)^b$ there. Rapid decrease with exponent greater than b bounds their products with $|y|$. The reductive quotient has finite volume by Theorem 4.2.18; rank zero is trivial. This proves the absolute product and series integrability. Here take the positive reduction height to be the inverse first lattice minimum cut off at one, in the fixed frame. The comparison Lemma 4.2.24 proves $\mathcal{H}(h) \le CH(h)^b$ on adapted Siegel representatives. Hence rapid decrease for every power of H supplies every matrix-height order required above. Replacing a faithful matrix realization by another does not change this conclusion. Fubini now exchanges the two integrals. Substitute (1), integrate its absolutely convergent series, and use $y(\delta h) = y(h)$. The translates $\delta\mathcal{D}$ of a Γ' fundamental domain, with δ representing $R'(F)\backslash\Gamma'$, partition a fundamental domain for $R'(F)$. Haar substitution gives (2), with no extra scalar. □

Theorem 4.2.18. *Let E/F be a CM quadratic extension and let W be a positive-dimensional finite-dimensional E-vector space with a nondegenerate hermitian or skew-hermitian form. Put $G = \mathrm{U}(W)$. For every nonzero Haar measure dh on $G(\mathbb{A}_F)$, the quotient measure induced by counting measure on the discrete subgroup $G(F)$ satisfies*

$$0 < \mathrm{vol}_{dh}\big(G(F)\backslash G(\mathbb{A}_F)\big) < \infty.$$

No restriction on the signatures or isotropy of W is imposed.

Proof. We apply Theorem A.4.6 with the given number field F and the actual group $G = \mathrm{U}(W)$. This is a connected reductive linear algebraic group: after base change to an algebraic closure of F, the quadratic algebra splits and the unitary equation identifies it with $\mathrm{GL}_{\dim_E W}$, which is connected and reductive.

Here its rational-character condition holds for the actual group G. Its derived group is $\mathrm{SU}(W)$, and the determinant identifies its abelianisation with the norm-one torus

$$T = \ker\big(\mathrm{Res}_{E/F}\,\mathbb{G}_m \xrightarrow{\mathrm{N}_{E/F}} \mathbb{G}_m\big).$$

For dimension one the same assertion simply says $G = T$. After extension to E, the character group of T is $\mathbb{Z}$, on which the nontrivial element of $\mathrm{Gal}(E/F)$ acts by negation. Its invariant subgroup is zero. Thus every F-rational character of G is trivial, and Theorem A.4.6 gives discreteness and finite quotient volume for one Haar measure. All nonzero Haar measures differ by a positive scalar, so finiteness holds for the specified measure. Positivity follows because the rational subgroup is discrete: an identity neighbourhood small enough to be disjoint from its nontrivial rational translates has positive Haar measure and injects into the quotient. Multiplying a skew-hermitian form by any fixed nonzero element of E negated by conjugation makes it hermitian without changing its unitary group, so the same argument covers that case. □

Lemma 4.2.19. *All notation in this statement is local. Let E/F be a CM extension, let $N, c \in F^\times$, and let*

$$V = X \perp J_s, \qquad V' = X' \perp J_{s-1} \qquad (s \ge 1),$$

where X, X' are positive definite. The string space J_l has basis $v_0, \dots, v_{l-1}$, antidiagonal Gram entries 1, and central entry $-N$ when l is odd. Give X, X' grade zero and give v_a grade $l-1-2a$.

Denote the grades by d, d' and use the dual-index involutions of orthogonal bases of X, X' and of the string bases. Let $T : V \to V'$ kill X and v_{s-1} and send v_a to $t_a v'_a$ for $a < s-1$, where

$$t_a = \begin{cases} 1, & s \text{ odd},\ 2a+1 < s, \\ -Nc, & s \text{ odd},\ 2a+1 = s, \\ c, & s \text{ odd},\ 2a+1 > s, \\ c, & s \text{ even},\ 2a+1 < s, \\ 1, & s \text{ even},\ 2a+1 > s. \end{cases}$$

Use the standard trace character ψ_E and a fixed $0 \neq \jmath \in E$ with $\bar{\jmath} = -\jmath$. For $D = 0$ or $D = d'$ and an adelic map $w : V \to V'$, consider the following condition: for every unitary u of V such that $u - 1$ raises source grades by at least 2 and $w(u-1)$ raises source grades to target weights D by at least 1,

$$\psi_E\left(\frac{\jmath}{2}\operatorname{tr}(T^\dagger T u)\right) = \psi_E\left(\frac{\jmath}{2}\operatorname{tr}(w^\dagger w u)\right). \tag{$*$}$$

Here a matrix raises weights by at least b if its entry from weight e to weight e' vanishes whenever $e' - e < b$. Say, within this statement only, that w agrees with T if

$$w_{ba} = T_{ba} \qquad (d(a) > 0,\ d'(b) < d(a)).$$

Let R be the unitary matrices g satisfying $ge_a - e_a \in V_{>d(a)}$ for every string index a with $d(a) \geq 0$; define R' similarly.

The following assertions hold.

1. *If w is rational and satisfies $(*)$ for $D = 0$, then w is injective on $V_{>0}$ and there is $\delta \in \mathrm{U}(V')(F)$ such that δw agrees with T.*
2. *If w agrees with T and satisfies $(*)$ for $D = d'$, then $w - T$ raises grades by at least zero.*
3. *If w agrees with T, $g \in R$ and $\delta \in \mathrm{U}(V')(\mathbb{A}_F)$, then $\delta w g$ agrees with T if and only if $\delta \in R'$.*

Proof. Choose orthogonal rational bases of the definite complements. Write J, J' for the resulting Gram matrices and σ, σ' for their dual-index involutions. The complement Gram matrices are diagonal, and the full Gram matrices pair only dual indices of opposite grades. We may therefore apply Lemma 4.2.20. All its tests use matrices raising grades by at least 2, hence unitary matrices of the kind appearing in $(*)$.

First compute the adjoint of the string map. It gives

$$T^\dagger T(v_a) = \lambda_a v_{a+1} \quad (0 \leq a < s-1), \qquad T^\dagger T(X) = T^\dagger T(v_{s-1}) = 0,$$

where

$$\lambda_a = \begin{cases} -Nc, & s \text{ odd},\ 2a+1 = s, \\ -Nc^2, & s \text{ even},\ 2(a+1) = s, \\ c, & \text{otherwise}. \end{cases} \tag{$**$}$$

Indeed the only entry of $T^* J' T$ in column a is in row $s-2-a$. Its value is $t_a t_{s-2-a}$ times the corresponding antidiagonal entry of J'. Multiplication by J^{-1} moves this row to $a+1$. If s is odd, the central source entry contributes $(-N)^{-1}$ at $a+1 = (s-1)/2$ and cancels the factor $-N$ of t_{s-2-a} there; at $a = (s-1)/2$ no such division occurs and the value is $-Nc$. If s is even,

only $a = (s-2)/2$ encounters the target central entry, and gives $-Nc^2$. All other products give c. Every λ_a is nonzero. These calculations include $s = 1$, when the list is empty.

We prove injectivity in the first assertion. Suppose $v \in V_{>0}$ and $wv = 0$. If $v \neq 0$, choose its nonzero string coordinate with largest index a. Its grade is the smallest grade in the support of v and is positive. Test Lemma 4.2.20 at the next string index $i = a+1$. Its grade is $k = d(a)-2$, so the first support condition holds. The second holds because $wv = 0$. The third holds because every support grade of v is positive and the target weights are all zero. The conclusion says that the i-coordinate of $T^\dagger Tv$ equals that of $w^\dagger wv = 0$. By $(**)$ it is $\lambda_a v_a \neq 0$, a contradiction. Thus $w|_{V_{>0}}$ is injective. The map T is also injective there, directly from its nonzero diagonal string coefficients.

For two positive-grade basis indices a, b, test the same lemma with $v = e_a$ and i the dual of b. Its grade is $-d(b)$. Since $d(a), d(b) \geq 1$, the first support inequality follows from $d(a)+d(b) \geq 2$. The second support condition is vacuous: zero target weight is never at most $\max(-d(b), -1)$. The third is vacuous on the support because $d(a) > 0$. Consequently

$$[(T^\dagger T - w^\dagger w)e_a]_{\sigma(b)} = 0.$$

Multiplication by the nonzero Gram entry $J_{b,\sigma(b)}$ gives

$$h'(Te_b, Te_a) = h'(we_b, we_a).$$

By sesquilinearity, the injective rational maps w, T have the same Gram form on $V_{>0}$. The map $wv \mapsto Tv$ is therefore an isometry between subspaces of the nondegenerate rational Hermitian space V'. Witt's extension theorem [249] extends it to a rational unitary δ. In fact $\delta w = T$ on all positive-grade vectors, which implies the required agreement. If this subspace is zero, take $\delta = 1$.

We next prove the second assertion by ascending induction on the target grade. We must show $w_{ba} = T_{ba}$ whenever $d'(b) < d(a)$. This is already part of agreement when $d(a) > 0$. Assume $d(a) \leq 0$, and that all the required equalities in rows of grade below $d'(b)$ have been proved. The inequality $d'(b) < d(a) \leq 0$ makes b a negative-grade string index. Let b^* be its dual target index. Choose the source string index i with the same string position as b^*. Then

$$d(i) = 1 - d'(b), \qquad Te_i = t_i e_{b^*}, \qquad t_i \neq 0.$$

Because $d(i) > 0$, agreement says that column i of w equals this column of T in all rows of grade less than $d(i)$.

Apply Lemma 4.2.20 with $v = e_i$ and selected source index $\sigma(a)$, whose grade is $k = -d(a) \geq 0$. The first support condition holds since $d(i) = 1 - d'(b) \geq 2 - d(a) = k + 2$. If $d'(p) \leq k$, then $d'(p) < d(i)$, so column agreement applies; its sole possible nonzero row is b^*, whose grade $-d'(b) > k$. Thus $(we_i)_p = 0$, proving the second support condition. For the third, only the support index i matters. If $d'(p) + d(i) \leq 0$, then $d'(p) \leq d'(b) - 1$. Induction gives $w_{pa} = T_{pa}$ there, and this entry of T is zero because its possible nonzero row has grade $d(a) - 1 \geq d'(b)$. Thus the third condition holds as well.

The resulting moment equality, multiplied by the Gram entry in row a, gives

$$[w^* J' w]_{ai} = [T^* J' T]_{ai}.$$

Expand the left side by target rows. Rows p with $d'(p) < d'(b)$ have $w_{pa} = 0$ by the induction just used. For every other row, its dual has grade at most $-d'(b) = d(i)-1$, so column agreement makes $w_{\sigma'(p),i}$ zero unless $p = b$, when it is t_i. Therefore the displayed equality is exactly

$$\overline{w_{ba}} J'_{b,b^*} t_i = \overline{T_{ba}} J'_{b,b^*} t_i.$$

Both factors following the entries are nonzero rational scalars. Hence $w_{ba} = T_{ba}$. This proves every negative-degree entry by induction. There are no such entries at the empty-string endpoint.

Finally we identify the stabilizer of the agreeing block. For a positive source string index a, the defining property of R gives $ge_a = e_a + \sum_{d(p)>d(a)} g_{pa}e_p$. All indices in this sum are positive string indices, and agreement sends each such e_p to Te_p plus target grades at least $d(p)$. Since T lowers string grade by exactly one, it follows that

$$wge_a = Te_a + \text{terms of target grade at least } d(a).$$

Thus wg agrees with T. If $\delta \in R'$, then δ preserves every positive-grade flag and changes a nonnegative target string vector only by vectors of strictly larger grade. Since $Te_a = t_a e'_a$ has grade $d(a) - 1 \geq 0$, applying δ to the last display again gives agreement.

Conversely suppose δwg agrees with T. Descend through the nonnegative target string grades. For a target string index k of grade $d'(k) \geq 0$, choose the source string index a at the same position. Then $d(a) = d'(k) + 1 > 0$ and

$$wge_a = t_a e_k + \sum_{d'(p)>d'(k)} z_p e_p.$$

Both wg and δwg agree, so the entries of $(\delta - 1)wge_a$ in rows b with $d'(b) \leq d'(k)$ vanish. By the descending induction, $(\delta - 1)_{bp} = 0$ in those rows for all string indices p of greater grade. Such indices are all the ones occurring in the sum. Hence $t_a(\delta - 1)_{bk} = 0$ and therefore $\delta_{bk} = (1)_{bk}$ for all these rows. This is precisely the defining condition of R'. The argument is vacuous, and the conclusion still correct, when the target string is empty. □

Lemma 4.2.20. *Let E/F be a quadratic extension of number fields with conjugation, let ψ_E be the standard trace character of $\mathbb{A}_E/E$, and let $0 \neq \jmath \in E$ satisfy $\bar{\jmath} = -\jmath$. All the following notation is local. Let J, J' be nondegenerate Hermitian matrices indexed by finite sets I, I'. Suppose there is an involution σ of I such that $J_{ip} = 0$ unless $p = \sigma(i)$, and $J_{i,\sigma(i)} \neq 0$. Give these indices integer grades d_i with $d_{\sigma(i)} = -d_i$, and give I' arbitrary integer weights D_b. Write $z^\dagger = J^{-1}z^*J'$ for the adjoint of a map between the two spaces. For an endomorphism of the first space use the corresponding adjoint with $J' = J$.*

Fix adelic matrices $T, w : I \to I'$. Suppose that, for every unitary u of the first space satisfying

$$(u-1)_{ip} = 0 \quad (d_i - d_p < 2), \qquad [w(u-1)]_{bp} = 0 \quad (D_b - d_p < 1),$$

one has

$$\psi_E\left(\frac{\jmath}{2}\operatorname{tr}(T^\dagger T u)\right) = \psi_E\left(\frac{\jmath}{2}\operatorname{tr}(w^\dagger w u)\right).$$

Fix $i \in I$, put $k = d_i$, and let $v \in \mathbb{A}_E^I$ satisfy

$$\begin{aligned} v_p &= 0 \quad \text{if } d_p < \max(1, k+2), \\ (wv)_b &= 0 \quad \text{if } D_b \leq \max(k, -1), \\ w_{b,\sigma(i)} = 0 \text{ or } v_p &= 0 \quad \text{if } D_b + d_p \leq 0. \end{aligned}$$

Then

$$[(T^\dagger T - w^\dagger w)v]_i = 0.$$

Proof. We use the linear-second Hermitian convention $h(z,t) = z^* J t$. Extend all forms adelically. For an arbitrary $z \in \mathbb{A}_E$ put $x = z e_{\sigma(i)}$. The support of v lies in strictly positive grades, so $h(v,v) = 0$. Since $h(x,v)$ can involve only v_i, and $k < k+2$, it also vanishes. Put $q = h(x,x)$, which is fixed by conjugation, and define

$$u = 1 + v x^* J - x v^* J - \frac{q}{2}\, v v^* J.$$

This matrix is unitary. Explicitly, if $A = u - 1$, the identities $h(v,v) = h(v,x) = 0$ give

$$A + A^\dagger = -q\, v v^* J, \qquad A^\dagger A = q\, v v^* J.$$

Hence $u^\dagger u = 1$. The formula is a polynomial in finitely many adelic entries, so it defines an adelic unitary matrix.

We verify the two degree conditions in the phase hypothesis. The row $x^* J$ is supported at column i, of grade k. Every nonzero row of $v x^* J$ therefore raises grade by at least 2. The row $v^* J$ has its nonzero columns at the duals of indices in the support of v; these columns have grades at most $-\max(1, k+2)$. Since x has grade $-k$, the term $x v^* J$ also raises grade by at least 2. The same assertion for $v v^* J$ follows by adding two positive support grades. Thus $u - 1$ raises grades by at least 2.

In $w(u-1)$ the first term is $(wv)x^* J$. Its nonzero rows have $D_b > \max(k, -1)$ by the support hypotheses, and its only possible column has grade k, so its degree is at least 1. The second term is $(wx)v^* J$; its entry at the column dual to p contains $w_{b,\sigma(i)} \bar{v}_p$. It vanishes whenever $D_b + d_p \le 0$ by the last condition of the support hypotheses. The last term is a scalar multiple of $(wv)v^* J$. Every nonzero row of wv has $D_b \ge 0$, and every support index of v has $d_p \ge 1$. It too has degree at least 1. Consequently the phase hypothesis applies to the displayed transformation for every z.

We compute its phase, retaining the factor $1/2$. For any map $y : I \to I'$ set $B_y = y^* J' y$, a Hermitian matrix. Cyclicity of trace gives

$$\operatorname{tr}(y^\dagger y u) = \operatorname{tr}(J^{-1} B_y) + x^* B_y v - v^* B_y x - \frac{q}{2}\, v^* B_y v.$$

The first and last terms are fixed by conjugation. The standard trace character satisfies $\psi_E(\jmath r) = 1$ for every conjugation-fixed adele r, since its trace from E to F is zero. If $a = x^* B_y v$, the middle terms are $a - \bar{a}$, and

$$\psi_E\!\left(\frac{\jmath}{2}(a - \bar{a})\right) = \psi_E(\jmath a).$$

Thus the trace identity implies

$$\psi_E\!\left(\frac{\jmath}{2} \operatorname{tr}(y^\dagger y u)\right) = \psi_E(\jmath x^* B_y v).$$

Apply this with $y = T, w$ in the phase hypothesis, divide the two characters, and obtain

$$\psi_E\big(\jmath \bar{z}\, [\, (T^* J' T - w^* J' w) v \,]_{\sigma(i)}\big) = 1 \qquad (z \in \mathbb{A}_E).$$

The trace pairing on the additive adeles is nondegenerate. Since multiplication by $\jmath$ and conjugation are bijections, the bracket is zero. Finally

$$[\, (T^* J' T - w^* J' w) v \,]_{\sigma(i)} = J_{\sigma(i),i} [(T^\dagger T - w^\dagger w) v]_i.$$

Its rational coefficient is nonzero and therefore invertible adelically. The asserted coordinate vanishes. □

Lemma 4.2.21. *Let F be a number field, W a finite-dimensional symplectic F-space with alternating form Ω, and ψ_F the standard additive character of $\mathbb{A}_F/F$. All notation below is local. Put*

$$\beta(z,w)=\psi_F\Big(\tfrac{1}{2}\Omega(z,w)\Big).$$

Let B be a complex Banach space carrying a strongly continuous isometric Heisenberg action in the convention

$$\rho(z)\rho(w)=\beta(z,w)\rho(z+w).$$

Let $\theta\in B^*$ satisfy $\theta(\rho(q)f)=\theta(f)$ for $q\in W(F)$ and $f\in B$. Fix a rational isotropic subspace $P\subset W$. Give $P(F)\backslash P(\mathbb{A}_F)$ probability measure.

Suppose $f\in B$ is a smooth vector for $P(F_\infty)$ and is fixed by a compact open subgroup $K_P\subset P(\mathbb{A}_{F,f})$. For a class $[w]\in W(F)/P^\perp(F)$ set

$$C_P(f,[w])=\int_{P(F)\backslash P(\mathbb{A}_F)}\theta\big(\rho(p)\rho(w)f\big)\,dp.$$

This is independent of the representative. The family is absolutely summable and

$$\theta(f)=\sum_{[w]\in W(F)/P^\perp(F)}C_P(f,[w]).$$

More precisely, fix a real basis $X_1,\dots,X_m$ of $P(F_\infty)$ and an integer $r>m/2$. There is $A_{P,K_P,r}<\infty$, depending only on this basis, P,K_P,r and the additive character, such that

$$\sum_{[w]}|C_P(f,[w])|\le A_{P,K_P,r}\,\|\theta\|\sum_{|\alpha|\le 2r}\|d\rho(X)^\alpha f\|_B.$$

The zero-dimensional case is read with one summand and the identity derivative.

Proof. Write $F_f(p)=\theta(\rho(p)f)$. Isotropy makes ρ an ordinary additive-group representation on $P(\mathbb{A}_F)$. Rational invariance of θ therefore makes F_f periodic under $P(F)$. Its invariance under K_P follows from $\rho(k)f=f$. It is continuous and smooth in the infinite variables.

For rational w, the Heisenberg relation and alternation give

$$\rho(p)\rho(w)=\beta(p,w)^2\rho(w)\rho(p),\qquad \theta(\rho(p)\rho(w)f)=\psi_F(\Omega(p,w))F_f(p).$$

Consequently

$$C_P(f,[w])=\int_{P(F)\backslash P(\mathbb{A}_F)}\psi_F(\Omega(p,w))F_f(p)\,dp. \tag{$*$}$$

The character depends only on $[w]$. Nondegeneracy of Ω identifies $W(F)/P^\perp(F)$ with the rational dual of P. The standard adelic trace pairing identifies this rational dual with the full character group of $P(F)\backslash P(\mathbb{A}_F)$. Thus $(*)$ lists all Fourier coefficients, with their signs reversed if the Fourier coefficient convention uses the inverse character. There is no scalar: the averaging measure has mass one.

We give an absolute bound sufficient to justify inversion. Additive strong approximation gives

$$P(\mathbb{A}_{F,f})=P(F)+K_P.$$

Indeed in an F-basis of P this is the Chinese remainder theorem with denominators at the finitely many prescribed places. The subgroup

$$L_P=P(F)\cap K_P$$

is a full lattice in the real vector space $P(F_\infty)$: in the same basis it is an additive subgroup commensurable with $\mathcal{O}_F^{\dim_F P}$. Therefore

$$P(F)\backslash P(\mathbb{A}_F)/K_P \simeq P(F_\infty)/L_P.$$

The image of the adelic probability measure is the Haar probability measure on this torus. Characters nontrivial on K_P have coefficient zero, by translating the integral in $(*)$. The remaining characters are exactly the dual lattice L_P^*.

Choose the Euclidean structure for which the displayed basis is orthonormal, and let $\Delta = \sum_j \partial_{X_j}^2$ on this torus. Repeated integration by parts gives, for a frequency $\nu \in L_P^*$,

$$|C_P(f,\nu)| \le (1+4\pi^2\|\nu\|^2)^{-r}\|(1-\Delta)^r F_f\|_\infty.$$

The sum of these weights over L_P^* is finite because $2r > m$; for example the number of dual-lattice points in a ball of radius R is at most a constant times $(1+R)^m$, by packing disjoint translates of one bounded fundamental parallelepiped. Expand $(1-\Delta)^r$. Each derivative satisfies

$$\partial_X^\alpha F_f(p) = \theta(\rho(p)d\rho(X)^\alpha f), \qquad |\partial_X^\alpha F_f(p)| \le \|\theta\|\,\|d\rho(X)^\alpha f\|_B,$$

since the action is isometric. Summing the preceding estimates gives the asserted bound with a finite constant depending only on the stated fixed data.

In particular the Fourier series is absolutely and uniformly convergent. It equals F_f: its continuous sum has the same Fourier coefficients, and a continuous function on a real torus with all Fourier coefficients zero is zero, as follows by convolution with the product Fejér kernels, which converge uniformly to that function. Evaluate the series at the identity. Every character is one there, so the value is $\theta(f)$ and gives the displayed sum. If $P = 0$, the quotient and character group are singletons and the assertion is immediate. □

Lemma 4.2.22. *Let $\mathbb{W}$ be a symplectic space over a number field F, let B be a complex Banach space with a strongly continuous isometric Heisenberg action ρ, and let $\theta \in B^*$ be rational-Heisenberg invariant. Let H be a reductive F-group with a faithful matrix realization, a rational symplectic action on $\mathbb{W}$, and a strongly continuous representation ω on B with exact Heisenberg covariance. Assume f is smooth both for $H(\mathbb{A}_{F,f})$ and for the finite Heisenberg group, and is smooth for the full archimedean Heisenberg group. Let $\mathcal{K}_f \subset H(\mathbb{A}_{F,f})$ be compact. For $h_\infty \in H(F_\infty)$ put*

$$\mathcal{H}(h_\infty) = \max(1, \|h_\infty\|, \|h_\infty^{-1}\|).$$

For each rational isotropic $P \subset \mathbb{W}$ there are constants C, b depending only on these fixed data, $f, P, \mathcal{K}_f$, and the chosen norms such that

$$\sum_{[w]\in \mathbb{W}(F)/P^\perp(F)} |C_P(\omega(h_\infty h_f)f,[w])| \le C\mathcal{H}(h_\infty)^b \qquad (h_f \in \mathcal{K}_f). \tag{1}$$

The same assertion holds uniformly after multiplying h by an element of a fixed compact set in any additional reductive group acting with the same covariance. Consequently on a finite-measure quotient with a fixed Siegel covering whose finite components lie in such compact sets, every function y satisfying $|y(h)| \le C_A\mathcal{H}(h_\infty)^{-A}$ for every $A > 0$ makes the bound in (1) integrable. In particular this conclusion requires no numerical value of b.

Proof. Choose a real basis $X_1, \dots, X_m$ of $P(F_\infty)$ and an integer $r > m/2$. We verify uniformly all hypotheses of the absolute bound in Lemma 4.2.21.

The orbit of f under $\mathcal{K}_f$ is finite. Indeed smoothness means that an open subgroup fixes f; its right cosets form a discrete quotient, and the image of a compact set in that quotient is finite. Denote the resulting vectors by $f_1, \dots, f_q$. Each f_i is fixed by some compact open subgroup of the finite Heisenberg group: covariance transports an open stabilizer of f to one of f_i. Intersect the corresponding compact open subgroups of $P(\mathbb{A}_{F,f})$ and call the result K_P. The archimedean action commutes with the finite Heisenberg action. Thus every vector under consideration has this same additive finite level. Every f_i remains archimedean-Heisenberg-smooth, and only finitely many norms of its derivatives of order at most $2r$ will be needed. The case of an additional compact factor is identical: its finite orbit is again finite, and its archimedean part is absorbed into the compact sets below.

We record explicitly polynomial growth of the Banach operator norm. A strongly continuous representation is uniformly bounded in operator norm on each compact set, by the uniform boundedness principle. For the real reductive group, use a Cartan decomposition $k_1 \exp(A) k_2$ with A in a closed positive chamber of a maximal real split subspace. In a faithful matrix realization the weights on that subspace span its dual. The bounds on the matrix and inverse matrix therefore imply

$$\|A\| \le c_0 + c_1 \log \mathcal{H}(h_\infty).$$

To see the last assertion directly, restrict the faithful representation to the split torus: its weight spaces are finite-dimensional, its eigenvalues are $\exp(\lambda(A))$, and the weights have no common kernel. Norm equivalence bounds every $|\lambda(A)|$ by a constant plus $\log \mathcal{H}$, hence bounds $\|A\|$. Compact components and the compact part of the center introduce only a bounded factor. Put $n = \max(1, \lceil \|A\| \rceil)$. The elements $\exp(A/n)$ lie in a fixed compact set, so if their operator norms are at most $M \ge 1$, then

$$\|\omega(h_\infty)\| \le C_0 M^n \le C_1 \mathcal{H}(h_\infty)^{b_0}. \tag{2}$$

The argument applies to the finite product of archimedean factors and to its finitely many components. It does not require a unitary Banach norm for ω.

Differentiating exact covariance gives

$$d\rho(X_1)^{\alpha_1} \cdots d\rho(X_m)^{\alpha_m} \omega(h_\infty) f_i = \omega(h_\infty) d\rho(h_\infty^{-1} X_1)^{\alpha_1} \cdots d\rho(h_\infty^{-1} X_m)^{\alpha_m} f_i.$$

Here the notation on the right means the symplectic action of h_∞^{-1} on the Heisenberg Lie algebra. Its coefficients in a fixed real basis are polynomial functions of the entries of h_∞ and h_∞^{-1}, since the action is rational algebraic. Expanding the ordered products uses finitely many fixed ordered Heisenberg derivatives of f_i of degree at most $2r$; noncommutativity causes no problem because their order is retained. Their norms, multiplied by (2), are bounded by $C_2 \mathcal{H}^{b_1}$. Possible central derivatives are fixed scalar multiples on this central-character representation and obey the same bound.

Apply Lemma 4.2.21 with the common K_P and these derivatives. Its fixed lattice sum is finite and independent of h, and the resulting inequality is exactly (1). On a finite-measure reduced quotient choose $A > b$ in the rapid bound. The product of its right side with $|y|$ is then bounded, hence integrable. There are only finitely many sets in a fixed Siegel covering, so summing their bounds preserves integrability. All the coefficient sums used here are sums of absolute values, before orbit regrouping; Tonelli therefore applies to them. □

Lemma 4.2.23. *Let G be a locally compact second countable Hausdorff group with left Haar measure, let $\Gamma \subset G$ be discrete, and give $\Gamma\backslash G$ its quotient of left Haar measure by counting measure. Let $F : G \to \mathbb{C}$ be continuous and left Γ-invariant, with $\int_{\Gamma\backslash G} |F| < \infty$. If*

$$F(ug) = cF(g) \quad (g \in G), \qquad |c| = 1, \quad c \ne 1$$

for some $u \in G$, then $\int_{\Gamma\backslash G} F = 0$. No normalization of Γ by u is required.

Proof. The issue is that left multiplication by u need not define a map of $\Gamma\backslash G$. We instead use right translation on an intermediate quotient.

Put

$$D(t,x) = |F(tux) - cF(tx)|, \qquad S = \{s \in G : D(st,x) = D(t,x) \text{ for all } t,x \in G\}.$$

The identity belongs to S, the defining identity is preserved by multiplication and inversion, and continuity makes S closed. Left Γ-invariance gives $\Gamma \subset S$. The identity $F(uy) = cF(y)$ and $|c| = 1$ give $u \in S$. For every $s \in S$, substituting $t = 1$ yields

$$F(sux) = cF(sx). \tag{1}$$

Choose a left Haar measure ds on S. We recall the quotient-in-stages formula in a form that does not assume S unimodular. Choose a nonnegative Borel cutoff κ on G with $\int_S \kappa(s^{-1}x)\,ds = 1$ for every x. Such a cutoff is obtained from locally finite, relatively compact coordinate neighborhoods of the quotient by the closed subgroup, a partition of unity, and positive compactly supported functions on the fibers, dividing by their positive fiber integrals. The quotient formula and Tonelli, first on nonnegative compactly supported tests and then by monotone convergence, give for any nonnegative left Γ-invariant a

$$\int_{\Gamma\backslash G} a = \int_G \kappa(x) \int_{\Gamma\backslash S} a(sx)\,d\dot{s}\,dx. \tag{2}$$

Here $d\dot{s}$ is left Haar on S divided by counting measure on Γ. One can verify (2) without a measure on $S\backslash G$: the map $(x,s) \mapsto sx$ carries $\kappa(x)\,dx\,ds$ to Haar measure on G, since changing x to $s^{-1}y$ gives exactly the displayed cutoff integral. Dividing the free left Γ action in this identity gives (2). Absolute integrability permits (2) for complex a as well.

Apply (2) to $|F|$ and F. If the asserted integral were nonzero, there would exist x for which

$$I_x = \int_{\Gamma\backslash S} F(sx)\,d\dot{s}$$

is both absolutely convergent and nonzero. Right multiplication by $u \in S$ is a well-defined map of $\Gamma\backslash S$. It scales the quotient measure by a positive constant d, the same constant by which it scales left Haar on S; this follows by applying the quotient formula to right-translated nonnegative tests. Therefore (1) gives

$$cI_x = \int_{\Gamma\backslash S} F(sux)\,d\dot{s} = dI_x.$$

It follows that $c = d$ is positive real. Together with $|c| = 1$ this gives $c = 1$, a contradiction. Thus the original integral is zero. □

Lemma 4.2.24. *Let E/F be a CM extension, $V = E^n$ a nondegenerate hermitian space, and $x \in \mathrm{GL}_n(\mathbb{A}_E)$ a fixed frame. Write adelic unitary elements in this frame as g. Put*

$$L_g = \{v \in E^n : (x_f g_f)^{-1} v \in \widehat{\mathcal{O}}_E^n\}, \quad \lambda(g) = \min\left(1, \inf_{0 \neq v \in L_g} \|(x_\infty g_\infty)^{-1} v\|\right), \quad H_{\mathrm{lat}}(g) = \lambda(g)^{-1},$$

where the archimedean norm is Euclidean over all CM places; for $n = 0$ put $\lambda = 1$. There is a unitary Siegel set $\mathfrak{S}$ covering the rational quotient, with finite components contained in a compact set, and constants $C, b > 0$ such that

$$\max(1, \|\pi(g_\infty)\|, \|\pi(g_\infty^{-1})\|) \le C H_{\mathrm{lat}}(g)^b \qquad (g \in \mathfrak{S})$$

for any fixed faithful algebraic matrix realization π of the archimedean group. In particular decay faster than every power of H_{lat} implies the reduced matrix-height decay needed in Lemma 4.2.22.

Proof. The finite lattice condition is commensurable with $\mathcal{O}_E^n$. Its image under the Minkowski embeddings and the invertible real frame is a full discrete lattice, so $\lambda(g) > 0$ when $n > 0$.

Use Theorem A.4.2, choosing a minimal parabolic adapted to a rational Witt basis. Write its relative Witt rank as r. Its split torus acts in that basis by

$$a = \mathrm{diag}(a_1, \dots, a_r, 1, \dots, 1, a_r^{-1}, \dots, a_1^{-1}).$$

Here $a_i > 0$ are the real coordinates of the maximal rational split torus of the restriction of scalars; they have the same value at the embeddings over F. The simple roots are a_i/a_{i+1} and either a_r or a_r^2, with the last choice depending on whether the anisotropic part is nonzero. Thus the inequalities defining a fixed Siegel chamber imply

$$a_i \le c_1 a_1, \qquad a_i^{-1} \le c_2, \qquad \max(1, \|a\|, \|a^{-1}\|) \le c_3 \max(1, a_1). \tag{1}$$

For example $a_r \ge \min(t, \sqrt{t}) > 0$, and successive inequalities $a_i \ge t a_{i+1}$ prove all of these bounds with constants depending only on the chamber. This argument does not require $t \ge 1$.

Write $g = pak$, where p ranges over the fixed compact subset of the parabolic and k over the maximal compact group. In rational coordinates this parabolic preserves the extreme rational isotropic line Ev_1, and $av_1 = a_1 v_1$. Hence $p^{-1}v_1 = \eta(p)v_1$, with $\eta(p)$ bounded at every archimedean place on the compact set. The real norms of k^{-1} are bounded as well.

There is one positive rational integer M, independent of p, k, a, such that $Mv_1 \in L_g$ for all these representatives. Indeed $a_f = 1$, while p_f, k_f range over compact sets. Their inverse actions on the fixed vector, with the fixed frame x_f included, form a compact subset of $\mathbb{A}_{E,f}^n$. Every such compact subset is contained in $M^{-1}\widehat{\mathcal{O}}_E^n$ for some positive integer M: it is integral outside finitely many places, and its finitely many negative valuations have a common bound. Thus the nonzero rational vector Mv_1 satisfies the required simultaneous finite lattice condition. At infinity the same vector has length

$$\|(x_\infty g_\infty)^{-1} M v_1\| = \|g_\infty^{-1} x_\infty^{-1} M v_1\| \le c_4 a_1^{-1}; \tag{2}$$

the equality simply rewrites the fixed rational Witt vector in the frame. In rational coordinates the inverse product is $k^{-1}a^{-1}p^{-1}$, so preservation of its extreme line and the preceding compact bounds prove the inequality.

Since λ is the first minimum cut off at one, (2) gives $a_1 \le c_4 H_{\mathrm{lat}}(g)$. Compactness of the real p, k factors and (1) give

$$\max(1, \|g_\infty\|, \|g_\infty^{-1}\|) \le c_5 \max(1, a_1) \le c_6 H_{\mathrm{lat}}(g). \tag{3}$$

The fixed change between Witt and framed coordinates only changes the constants. If $r = 0$, the split torus is trivial and the same Siegel covering is compact, so (3) holds directly using $H_{\mathrm{lat}} \ge 1$. This also covers rank zero. Finally Theorem A.4.1(N) compares any fixed faithful algebraic matrix realization polynomially with the standard one, proving the stated bound with some $b > 0$.

For the last assertion, $\mathcal{H} \le C H_{\mathrm{lat}}^b$ implies $H_{\mathrm{lat}}^{-A} \le C^{A/b} \mathcal{H}^{-A/b}$. Arbitrarily large A therefore give decay of every matrix-height order. □

Proposition 4.2.25. *Let D be admissible. The arithmetic realization and operators in this statement are fixed as follows, before choosing a twist or a source form.*

Use precisely the Weil data (WD1)–(WD9) of Definition 3.2.12 and the canonical spaces and Hom spaces (VP2),(VP5) of Definition 3.2.2, with Weil actions, Fock coordinates and datum maps (WR3)–(WR6) of Definition 3.2.18, with the same splitting character as the lower chain. In particular $\jmath$, c_0, α, ψ_E, the vacuum v^0, its measure and Heisenberg operators are those defined there. The canonical theta functionals and theta functions are (WC3), and the theta kernels are (WC4), of Definition 4.1.15. In framed coordinates the bicharacter is

$$\beta(z,w)=\psi_E\left(\frac{\jmath}{2}\bar{w}^{\mathsf{T}}Sz\right),$$

where S has the prescribed signs at infinity and is the identity at the finite places. A vector evaluated at z means $\theta_V(\rho(z)f)$, not its coherent coefficient $f(z)$. The rank-one phases, tensor products and frame intertwiners are the same fixed choices as in the lower chain; no independent rescaling or additional realization is made.

Let $c_s\neq 0$ be the vacuum scale, whose theta values satisfy the bounds with constant $c_{\mathrm{vac}}(P)$, and use $v^{\mathrm{int}}=c_s^n v^0$ in rank n. Set

$$H(x)=\psi_{E,f}\left(\frac{c_0\jmath\alpha\bar{\alpha}}{2}\sum_i \bar{x}_i x_i\right)\rho(0,\alpha x).$$

Fix the CM curve with lattice $\Omega_{\mathrm{CM}}\mathcal{O}_K$ and its Chowla–Selberg factor u_{CS}, a unit at every prime identification, with $\mathrm{Area}(\Lambda^\sigma)|\sigma u_{\mathrm{CS}}|^2=\pi e^{c_{\mathrm{CS}}(d)}2\pi/\sqrt{d}$ for σ over K. The common derivative scale is

$$s_0=\exp\Big(f^{-1}\operatorname{Log}((\upsilon(1,\alpha)|\alpha|^{-1/2})^{-1})\Big),\qquad \eta_P=(s_0/u_{\mathrm{CS}})\sqrt{2\pi/\sqrt{d}}/\Omega_{\mathrm{CM}}.$$

The normalized model and its intertwiners are the realization of the lower-chain theta kernels, including their additive character, vacuum, measures and theta functional.

Define column numbers and determinant exponents by

$$C_\tau(h)=\begin{cases} a\mathbf{1}_{\Phi_a}(\tau), & h=1,\\ \kappa_1(\tau)-\lfloor n_1/2\rfloor, & h=k,\\ \Delta-(t+1), & h>k,\ t+1\mid h-k,\\ 0, & \text{otherwise,}\end{cases}\qquad e_0(\tau)=0,\quad e_j(\tau)=\lfloor n_{j-1}/2\rfloor-\kappa_j(\tau)\ (j>0).$$

Let $\mathcal{V}_j$ have highest weight $\lambda^X_{j,\tau,i}=e_j(\tau)+\sum_{i<h<n_j}C_\tau(h)$ for $0\le i<n_j$, with its integral Young realization and column-filling Gram form. Put $K(h)=\Delta/2-(t+1)/2$ if $h\ge k$ and $t+1\mid h-k$, and $K(h)=0$ otherwise. The final target Weyl module $\mathcal{V}^Y$ has columns $C^Y_\tau(h)=K(h)+C_\tau(h+1)-K(h+1)$, $1\le h<n_m-1$, determinant exponent $e_m(\tau)$ and highest weight λ^Y. Put $b_\tau=|\lambda^X_{m,\tau}|-|\lambda^Y_\tau|$. Fix the integral branching map $\beta:\mathcal{V}_m\to\mathcal{V}^Y$ spanning the intertwiners for $\mathrm{diag}(g,1)$ and satisfying

$$\beta\rho_X(\mathrm{diag}(g,z))=\left(\prod_\tau z_\tau^{b_\tau}\right)\rho_Y(g)\beta,\qquad \|\beta v\|^2\le e^{x^{70}}\|v\|^2.$$

Its normalized restriction is

$$(\mathscr{R}_\beta A)(h)=\left(\prod_\tau \tau(\bar{\pi}_N)^{-b_\tau}\right)\beta A(\mathrm{diag}(h,1)).$$

Let K_j^X consist of integral elements of $\mathrm{U}(X_j)(\mathbb{A}_f)$ that reduce to 1 at chosen places and have last row $(0,\dots,0,1)$ modulo π_N. For a tame character ϑ set $\nu_j = \mu_j^0\vartheta^j$. In the finite part $\mathcal{F}_j$ of the model on $\mathrm{Hom}(X_j, X_{j+1})$, the pair action has character arguments $A_j = \nu_{j+1}\upsilon^{\lfloor n_j/2\rfloor}$ and $B_j = \nu_{j+1}\upsilon^{\lfloor n_{j+1}/2\rfloor}$:

$$\omega_{\vartheta,j}(g,h) = A_j(\det h)B_j(\det g)^{-1}\,\omega_{\mathrm{Hom}}(u\mapsto hug^{-1}).$$

The finite part is the complex span of the finite Heisenberg translates of v^0. For $F\in\mathcal{F}_j$ let $J_j(F):\mathcal{V}_j\to\mathcal{V}_{j+1}$ be specified by

$$u(J_j(F)v) = P_{u,v}(\partial_Z)\theta_F(\pi_M\eta_P Z)|_{Z=0}\,,\qquad P_{u,v}(Z) = u(S_C(Z)v).$$

Here S_C is the rectangular Schur matrix induced by exterior powers, with all columns $h<n_{j+1}$; in particular height-n_j columns are retained. The germ θ_F is the holomorphic theta germ in the fixed Fock embedding. No factorial divisor is added. Define

$$(T_{\vartheta,j,F}\varphi)(h) = \int_{\mathrm{U}(X_j)(E^+)\backslash\mathrm{U}(X_j)(\mathbb{A}_f)} J_j(\omega_{\vartheta,j}(g,h)F)\varphi(g)\,dg,\qquad \mathrm{vol}(K_j^X)=1.$$

Composition extends linearly in the tensor $F_0\otimes\cdots\otimes F_{m-1}$; write this extension as $\mathscr{C}_{\vartheta,F}$.

Set $\mathcal{X} = \prod_{j<m}\mathrm{Mat}_{n_{j+1},n_j}(\mathbb{A}_{E,f})$. For each $i<m/2$ and $y_i\in\prod_{w|\pi_N}k_w^{n_{2i+1}}$, let $z_{2i}(y_i)$ have only last column $-y_i/N$ at the selected places, and $z_{2i+1}(y_i)$ only the conjugate last row at their conjugates; they vanish elsewhere. Use the integer representatives $0,\dots,N-1$ under $k_w=\mathbb{F}_N$. For a finitely supported $\Psi:\mathcal{X}\to\mathbb{C}$ define

$$F_\Psi = \sum_x \Psi(x)\bigotimes_{i<m/2}\left(\sum_{y_i} H_{2i}(z_{2i}(y_i))H_{2i}(x_{2i})v_{2i}^{\mathrm{int}}\otimes H_{2i+1}(z_{2i+1}(y_i))H_{2i+1}(x_{2i+1})v_{2i+1}^{\mathrm{int}}\right),$$

$$B_{\vartheta,\Psi} = \mathscr{R}_\beta\circ\mathscr{C}_{\vartheta,F_\Psi}.$$

The gluing sum is unnormalized.

The preceding realization and operators are fixed independently of the map and pair in the following hypothesis. Let $\mathcal{P}:\mathcal{S}\to\mathcal{A}_M(Y_0)$ be any $\mathbb{C}$-linear map. Suppose there are

$$g_\infty\in\mathrm{U}(Y_0)((E^+)_\infty),\qquad t_0\in\mathrm{Hom}_{\mathbb{C}}(W_\lambda,\mathbb{C}),$$

fixed before twists and source forms, such that for every $\varphi\in\mathcal{S}$ and $y\in\mathcal{A}_M(Y_0)$ the Bessel integral is absolutely convergent and represents this map:

$$\int_{[R]}\left|\mathcal{W}_{t_0,\varphi}(r)\,\overline{y(r(g_\infty,1))}\right|dr<\infty,\qquad \langle y,\mathcal{P}\varphi\rangle = \mathrm{BP}_{g_\infty,t_0}(\varphi,y).$$

Then there exists one $c_{\mathrm{ss}}\in\mathbb{C}^\times$ such that for every ordinary twist ϑ with modulus Q and $q=\mathrm{N}Q$ there exists such a Ψ_ϑ with the following properties.

1. *Every translation x in its support satisfies $\bar\pi_M x_j\in\mathcal{O}_w^{n_{j+1}\times n_j}$ for $w\nmid q$, and $x_{j,w}=0$ for $w\nmid Mq$.*
2. *Every coefficient is integral at every prime identification above every rational prime $\ell\nmid q$. If $Q=\varnothing$, then $\sum_x|\sigma\Psi_\vartheta(x)|\le M^{3fmn_m^2}$ for every $\sigma\in\mathrm{Aut}(\mathbb{C})$.*
3. *The tensor F_{Ψ_ϑ} is fixed by $\bigotimes_{j<m}\omega_{\vartheta,j}(b_j,b_{j+1})$ for all $b_j\in K_j^X$, $b_m=1$.*

4. *For every* $\varphi \in \mathcal{S}^{\mathrm{ram}}$,

$$\Theta_\vartheta(\mathcal{P}\varphi) = c_{\mathrm{ss}} B_{\vartheta,\Psi_\vartheta}(\varphi).$$

Here Θ_ϑ *is the same normalized lower composite on principal forms, extended to ordinary tame characters, with lower step* j *multiplied by the index of its level in the integral stabilizer. On* m*-fold twists it restricts to the shared chain of Definition 3.2.31; the equality is in the full algebraic form space on* Y_m *of weight* λ^Y.

The scalar precedes all twists, and the datum Ψ_ϑ *precedes every source form. Neither operator is required to be nonzero.*

Proof. All spaces and Hom spaces in the proof are (VP2),(VP5) of Definition 3.2.2; their realizations are the (WD1)–(WD9), (WR3)–(WR6) models, with the canonical theta functionals, theta functions and theta kernels of (WC3)–(WC4) of Definition 4.1.15. The actual action and its theta comparison are Lemma 4.1.1; its zero terms are the operators of (MX4) of Definition 4.1.18, with their laws supplied by Lemma 4.1.1. In particular a vector evaluated at a point means its theta function, as stipulated in (WC3) of Definition 4.1.15; it never means the value of its coherent coefficient. The common curve, differential, Chowla–Selberg unit and principal-root scale can be chosen by Lemma 4.2.3, together with the simultaneous vacuum normalization of Proposition 4.2.1. The matrix and normalized restriction required here are precisely those of Lemma 4.6.6: the column formulas give $b_\tau = a\mathbf{1}_{\Phi_a}(\tau) - m\Delta/2$, including the full-height columns, and the stated norm bound. We retain these choices throughout.

We first prove the reduction starting from the explicit Bessel coefficient

$$W_{t_0,\varphi}(u\iota_0(g)) = \chi_0(u)\, t_0(\rho_0(g_\infty)^{-1}\varphi(g_f)). \tag{A}$$

The initial-period identity is the assumed absolutely convergent pairing identity for the fixed witnesses (g_∞, t_0), as checked explicitly at the end.

Unfolding and the common finite tensors. Use the literal positive theta mean

$$C_{l,f}(r,h) = \int_{[P_l]} \theta_l(\rho(p)\rho([T_l])\omega_{\vartheta,l}(r,h)f)\, dp. \tag{B}$$

Here and below additive rational quotients carry probability Haar. Formula (B) is the point-evaluation convention of (WC3) of Definition 4.1.15, in the order displayed; there is no interchange of $\rho(p)$ and $\rho(T_l)$.

Apply Proposition 4.2.17 successively, beginning with (A). At each stage Lemma 4.2.26 supplies the continuous target-generic function on every fixed right translate, for the same string constant and additive character. Thus the next application uses a function on its own Bessel group and no extension of its domain. The absolute product and series estimates of the step proposition justify the transposed kernel identity against the next good form, and justify each Fubini interchange. The specified lower-chain forms and their successive theta lifts are the good forms to which those transpositions apply. Their real joint harmonics and finite point data are exactly the test data of (B).

Fix a twist for the moment. Choose the cuts O_l of Proposition 4.2.31 to meet all bounded string-lattice orbits, not merely the support of one form. These sets depend only on the datum and the twist: their construction enumerates the finite intermediate lattices between the two fixed conductor bounds and then chooses representatives. Consequently the same cuts work for all ramified source forms, covectors and real outer points. Put $y_m = 1$ and define

$$\mathcal{F}_\vartheta = \sum_{(y_0,\dots,y_{m-1}) \in \prod_{l<m} O_l} \left(\prod_{l<m} \mu_l(y_l)^{-1} \right) \bigotimes_{i<m/2} \mathcal{Y}_{\vartheta,y,i}. \tag{C}$$

Each $\mathcal{Y}$ is the actual finite evaluated tensor (DE3) of Lemma 4.2.27. This is a finite sum, fixed before the source form. The cuts' representatives are integral away from q.

The real cut and the exact quotient integration. Here is the measure argument used at every outer stage $1 \le l < m$. Write $G_l = G_{l,\infty} \times G_{l,f}$, $R_l = R_{l,\infty} \times R_{l,f}$ and $\Gamma_l = R_l(F)$. Give $G_{l,f}$ its Haar measure with A_l of mass one. Give $R_{l,f}$ its Haar measure η_l with $K_l = R_{l,f} \cap A_l$ of mass one, and choose the Haar measure on $R_{l,\infty}$ so that their product is exactly the Bessel Haar measure already used in the step identity. These choices depend on D, not the twist or its level J_l. The groups are unimodular. For the Bessel group this follows directly from its decomposition into its unipotent radical and the unitary group of X_l: the latter acts on the radical by representations whose real or local absolute determinants are powers of $|\det u \, \overline{\det u}| = 1$. Thus inversion preserves each Haar measure.

Choose a continuous nonnegative real cut α_l on $G_{l,\infty}$ with

$$\int_{R_{l,\infty}} \alpha_l(rg)\, dr = 1 \quad (g \in G_{l,\infty}). \tag{K1}$$

We describe its existence and topology. The closed Lie subgroup $R_{l,\infty}$ has a vector-space complement to its Lie algebra in that of $G_{l,\infty}$; choose an inner product and its orthogonal complement. The inverse function theorem applied to multiplication and the exponential map gives local sections of $R_{l,\infty}\backslash G_{l,\infty}$. This quotient is a second-countable manifold and hence has a locally finite partition of unity subordinate to relatively compact section domains. Fix a nonnegative compactly supported continuous function on $R_{l,\infty}$ of Haar integral one. In each local trivialization $g = r\, s(q)$ multiply this function of r by the partition function of q, and sum. Extend each term by zero outside its section domain. The supports of the partition functions are contained in those domains, so the result is continuous and locally bounded. Local finiteness and unimodularity give (K1). In particular the measure $\alpha_l(g)\, dg$ is sigma-finite. All these sections, complements and cuts are chosen once from the fixed real groups.

Put

$$\psi_l(x) = \begin{cases} \mu_l(xJ_l)^{-1}, & xJ_l \in O_l, \\ 0, & xJ_l \notin O_l, \end{cases} \qquad w_l(g) = \alpha_l(g_\infty)\psi_l(g_f).$$

The function ψ_l is nonnegative, locally constant and compactly supported, since O_l is finite and J_l is compact open. Stability of O_l under K_l implies that $r \mapsto \mathbf{1}_{O_l}(r^{-1}xJ_l)$ is constant on the right K_l cosets counted in the definition of μ_l. Furthermore μ_l is constant on an $R_{l,f}$ orbit, by the bijection $rK_l \mapsto srK_l$ for a fixed $s \in R_{l,f}$. Hence

$$\int_{R_{l,f}} \psi_l(r^{-1}x)\, d\eta_l(r) = \begin{cases} 1, & \mu_l(xJ_l) > 0, \\ 0, & \mu_l(xJ_l) = 0. \end{cases} \tag{K2}$$

There are exactly $\mu_l(xJ_l)$ cosets in the first case, each of mass one. Together (K1),(K2) give orbit integral one for w_l on the covered set $U_l = \{g : \mu_l(g_f J_l) > 0\}$, and zero off it.

For every continuous left Γ_l-invariant f with $\int_{\Gamma_l\backslash G_l} |f| < \infty$ and $f = 0$ off U_l, the exact compatible quotient measures consequently give

$$\int_{\Gamma_l\backslash G_l} f = \int_{G_l} w_l(t) \int_{\Gamma_l\backslash R_l} f(rt)\, dr\, dt. \tag{K3}$$

Here is also the absolute-integrability justification. Choose measurable fundamental domains for the discrete rational subgroup in R_l and G_l. Such domains exist for these second-countable locally compact groups, by choosing a countable cover of relatively compact sets with disjoint

rational translates and successively removing earlier translates. Tonelli on nonnegative functions, followed by $t \mapsto r^{-1}t$, unfolds the rational translates of those domains and gives

$$\int_{G_l} w_l(t) \int_{\Gamma_l \backslash R_l} |f(rt)|\, dr\, dt = \int_{\Gamma_l \backslash G_l} |f(g)| \int_{R_l} w_l(r^{-1}g)\, dr\, dg = \int_{\Gamma_l \backslash G_l} |f| < \infty.$$

The last equality is (K1),(K2) and the support hypothesis. Thus the complex integrand in (K3) is absolutely integrable on its actual product domain; Fubini is now applicable. Applying the same unfolded calculation to its real and imaginary parts proves (K3), with no unweighted-class-sum substitution and no interchange of a divergent integral.

We need (K3) only with a separated Bessel period. Write $I_l = [A_l : J_l]$. Suppose that for all t with $t_f J_l \in O_l$,

$$\int_{\Gamma_l \backslash R_l} f(rt)\, dr = I_l\, a(t_\infty) b(t_f J_l), \tag{K4}$$

where a is one fixed function of the real variable, whereas b may vary with the finite data. No measurability of an arbitrary family of chosen real covectors is assumed. Define a scalar depending ONLY on a by

$$\mathcal{I}_l(a) = \begin{cases} \displaystyle\int_{G_{l,\infty}} \alpha_l(g) a(g)\, dg, & a \in L^1(\alpha_l\, dg), \\ 0, & \text{otherwise.} \end{cases} \tag{K5}$$

On its actual domain $L^1(\alpha_l\, dg)$ the integral is a bounded linear functional of norm at most one. We make no assertion that (K5) is linear on all measurable functions.

This scalar always gives the required product integral. The actual inner period in (K3) is strongly measurable almost everywhere for the weighted outer measure, and its absolute value is integrable, by the absolute Fubini argument above. Its finite absolute factor would be

$$\int_{G_{l,f}} \psi_l(x) |b(xJ_l)|\, dx = I_l^{-1} \sum_{u \in O_l} \frac{|b(u)|}{\mu_l(u)} < \infty, \tag{K6}$$

since a J_l coset has mass I_l^{-1}; this finite function is measurable regardless of the function a.

If every $b(u)$ is zero, the actual weighted period is zero on the support of w_l. Its integral and the finite sum below are both zero, irrespective of a. Otherwise choose a coset uJ_l on which $b(u) \neq 0$. It has positive finite Haar measure and ψ_l is a positive constant there. Fubini for the ACTUAL measurable integrable inner period supplies a finite point x in this coset whose real-variable section belongs to $L^1(\alpha_l\, dg)$. Equation (K4) identifies this section with $I_l b(u) a$. Division by its nonzero constant proves $a \in L^1(\alpha_l\, dg)$, including strong measurability up to a null set. We have therefore justified measurability and integrability before using any separated-product formula. Ordinary Fubini now factors the complex product in (K4), and (K6) computes its finite factor. In both cases (K3)–(K6) prove

$$\int_{\Gamma_l \backslash G_l} f = \mathcal{I}_l(a) \sum_{u \in O_l} \mu_l(u)^{-1} b(u). \tag{K7}$$

In particular the factor I_l in the normalized lower lift cancels exactly the finite coset mass I_l^{-1}. The only alternatives in (K5) depend on the fixed function a, not on the twist, b, J_l, source form or test form. This argument deals also with cancellations between separate real summands: it applies to the actual combined period (K4).

The terminal bounded intertwiner on the whole Weyl module. Combining two factors (B) uses Lemma 4.2.27 with its actual positive and plane zero terms and stacked string translation.

Its last endpoint is $\operatorname{diag}(g,1)$. Thus for every tuple ξ of real outer points it supplies a covector $t_\xi \in \mathcal{V}_m^*$, before every twist, finite tuple and source form, for which the last recurrence is the coefficient

$$k \longmapsto t_\xi\Big(\rho_X(\operatorname{diag}(k_\infty,1))^{-1} A(\operatorname{diag}(k_f,1))\Big), \tag{K8}$$

where $A = \mathscr{C}_{\vartheta,\otimes_i \mathcal{Y}_{\vartheta,y,i}}\varphi$. The determinant factors are exactly those already included in that lemma's coefficient and the chosen modules; no independent character is inserted.

Let $K = \prod_\tau \mathrm{U}(n_m - 1)$ with Haar probability measure, $V = \mathcal{V}_m$, $W = \mathcal{V}^Y$, $d_Y = \dim W$, and let v_+ be the highest vector of W in (WM6). All actions below are the same conjugate-embedding actions in (WM2),(WM7) of Definition 3.2.19. For EVERY covector $t \in V^*$ define on ALL of V

$$\mathcal{A}_t(v) = d_Y \int_K \rho_Y(k) v_+ \, t\Big(\rho_X(\operatorname{diag}(k,1))^{-1} v\Big)\, dk. \tag{K9}$$

This is a Bochner integral in the finite-dimensional Banach space W. Its integrand is continuous on a compact group. The Weyl Gram norms make the unitary actions isometries, so

$$\|\mathcal{A}_t(v)\| \le d_Y \|v_+\| \|t\| \|v\|.$$

Consequently $t \mapsto \mathcal{A}_t$ is bounded linear from V^* to $\operatorname{Hom}_{\mathbb{C}}(V,W)$, with their finite-dimensional operator norms. There is no map defined only on an unspecified subspace, and no extension or complement is needed.

For $h \in K$, put $k = hk'$ in (K9). Left invariance of probability Haar and the representation law give

$$\mathcal{A}_t \rho_X(\operatorname{diag}(h,1)) = \rho_Y(h)\mathcal{A}_t.$$

Thus Lemma 4.6.6, with its SAME modules, gives $\mathcal{A}_t = \gamma(t)\beta$. The scalar can be chosen as a linear continuous function of t before the family: if $\beta \neq 0$, fix $v_* \in V$ and $u_* \in W^*$ with $u_*(\beta v_*) \neq 0$, and set $\gamma(t) = u_*(\mathcal{A}_t v_*)/u_*(\beta v_*)$. If $\beta = 0$, the spanning assertion forces every $\mathcal{A}_t = 0$ and we take $\gamma = 0$. Both choices have a finite operator bound from (K9) and are independent of every finite datum and twist.

We verify precisely how this average evaluates the terminal period. In (WM7) the algebraic extraction is $d_Y \int_K F(k,h)\rho_Y(k)v_+\, dk$, so (K8),(K9) give $\gamma(t_\xi)\beta A(\operatorname{diag}(h,1))$. Equivalently this holds after pairing with every automorphic target test form. To see the measures and topology in this equivalent formulation, write its algebraic value as $z(h)$. Its conjugate coefficient is $\langle z(h), \rho_Y(k)v_+\rangle$. The compact integral against (K8) is exactly

$$d_Y^{-1}\langle z(h), \mathcal{A}_t(v)\rangle = d_Y^{-1}\gamma(t)\langle z(h), \beta v\rangle.$$

The vector v_+ has norm one: in the filling-tensor realization it is the tensor of the specified unit exterior basis vectors. The Weyl representation W is irreducible, and averaging the rank-one orthogonal projection onto $\mathbb{C}v_+$ over K gives $d_Y^{-1}\operatorname{Id}_W$. Indeed the average commutes with K, hence is scalar, and its trace is one. Thus the compact integral pairing the two scalar coefficients of algebraic forms z and B is $d_Y^{-1}\langle z(h), B(h)\rangle$.

Apply Lemma 6.4.74(2) to these products on the standard definite space Y_m, with a common open finite level fixing the two forms (intersect their levels and $\mathbf{K}_0$). They are continuous and rationally invariant by (WM7). The quotient is a finite union of compact real fibres, so the integrals are absolutely convergent. That lemma retains exactly the class weights and multiplies both compact averages by the same real Haar mass and the same normalized class integral. Therefore the preceding compact calculation is also the full Petersson calculation; no level index depending on the twist survives in its ratio. With

$$A_N = \prod_\tau \tau(\bar\pi_N)^{-b_\tau} \neq 0$$

it says that the terminal period is $\gamma(t_\xi)/A_N$ times the pairing with $\mathscr{R}_\beta A$. The normalization by A_N is exactly the one in the statement.

Successive outer operations and their uniformity. For every tuple of real outer points choose the covector supplied by Lemma 4.2.27 before all twists, finite data and forms, and fix these choices once. Set $a_{\mathrm{top}}(\xi) = \gamma(t_\xi)/A_N$. Apply (K7) backwards through the stages $m-1, \dots, 1$. At a given stage and fixed remaining real variables, replace the function of that stage's real variable by its scalar (K5). This defines one function of the remaining variables BEFORE all twists, finite data and forms. No continuity or measurability of these arbitrary simultaneous choices is needed: (K7) was proved for an arbitrary fixed function a. Whenever an actual finite coefficient is nonzero, the actual integrable period itself forces the relevant real section into L^1; when every such finite coefficient is zero, both sides are zero. In particular no unproved measurable-selection claim or extension of integration is involved.

For completeness, the induction applies (K7) to the actual function which is the current recurrence times the transposed good test form. It is continuous, rational-Bessel invariant and absolutely integrable by Proposition 4.2.17(2) at precisely that stage; the rapid-decrease hypothesis is satisfied by that good test form. Its finite support is covered by Proposition 4.2.31, and that proposition supplies the stability and positivity hypotheses in (K2),(K6). Its Bessel period is (K4), by the next step of the recurrence and the already established next-stage identity. Thus every use of Fubini is justified by the displayed absolute integral for that actual product, not by integrability of separate real coefficient summands. The induction replaces its finite coefficient by $b \mapsto \sum_{u \in O_l} b(u)/\mu_l(u)$. These are finite linear operations and commute with every tensor and coefficient map. Their combined effect is exactly (C).

At the initial stage use the assumed absolutely convergent pairing identity for the SAME fixed witnesses (g_∞, t_0), as displayed in (J) below. The initial real variable is evaluated at g_∞ rather than integrated. Here is the finite normalization if a transposed form has level J_0 smaller than the fixed chosen level J_M. Since $\mathcal{P}\varphi$ has level J_M, its pairing is unchanged by averaging that test form over J_M. If the test form has level $J_0 \subset J_M$, this average is $[J_M : J_0]^{-1} \sum_{u \in J_M/J_0} z(\cdot\, u)$. The factor from the first normalized lower lift is

$$[A_0 : J_0]\,[J_M : J_0]^{-1} = [A_0 : J_M].$$

The representatives are precisely O_0, and $\mu_0 = 1$. The remaining factor is fixed before the twist; it is not a conductor factor. Put a_{real} equal to this fixed index times the resulting fixed real function evaluated at g_∞. It may be zero.

Combining (K7)–(K9) and the finite linear sum (C) therefore proves

$$L_\vartheta = a_{\mathrm{real}}\, \mathscr{R}_\beta \mathscr{C}_{\vartheta, \mathcal{F}_\vartheta}. \tag{D}$$

Here L_ϑ is the normalized lower composite attached to (A). Both sides have the required target weight: (K9) is the explicit algebraic extraction and the right side uses the given branching map. Equality against all target coefficient tests, or the positive Petersson pairing just computed, proves equality in the full algebraic form space. Every scalar choice above precedes all twists; $\mathcal{F}_\vartheta$ is fixed before the source form. No nonvanishing of either operator, real integral, or intermediate scalar was used.

Invariance before expansion. The tensor (C) is fixed by the internal upper levels. We prove this on (C), rather than try to prove it separately for the translated vacua which will occur later. Let $b_l \in K_l^X$ and $b_m = 1$. The inclusions of these elements into the lower groups preserve the appropriate string maps and the two isotropic splittings. Covariance of the zero term follows by change of variables in its probability means. The determinant calculation in Lemma 4.2.27

identifies the resulting action with the full upper pair action, including its splitting-character powers. Applied to all pairs, it gives

$$\left(\bigotimes_{l<m}\omega_{\vartheta,l}(b_l,b_{l+1})\right)\bigotimes_{i<m/2}\mathcal{Y}_{\vartheta,y,i}=\bigotimes_{i<m/2}\mathcal{Y}_{\vartheta,\iota(b)y,i}. \tag{E}$$

This is the same computation as (DE7), now also for the first and middle blocks: these blocks fix the stacked string map, preserve all removed coordinates, and their residual actions are respectively the first and middle definite blocks. The determinant factors are the three factors already computed there, so no new character scalar occurs.

The point data descend to their prescribed right-level classes: the lower integral principal level fixes the inclusion indicator by Lemma 4.1.16(2), and the character factors are one there. The cut proposition says precisely that left multiplication by these inclusions permutes each O_l and preserves its multiplicity. Reindex (C) using (E). This proves

$$\mathcal{F}_\vartheta\in\mathcal{U}_\vartheta,\qquad \mathcal{U}_\vartheta=\{F:\ (\otimes_l\omega_{\vartheta,l}(b_l,b_{l+1}))F=F \text{ for all such } (b_l)\}. \tag{F}$$

The space $\mathcal{U}_\vartheta$ is a complex linear subspace. This proof precedes, and does not assume any invariance of, a vacuum expansion.

Arithmetic expansion and the single nonzero scalar. For each pair apply Lemma 4.2.32 to (C). Let $c_{\mathrm{ev}}=\prod_{i<m/2}c_i$, fixed before every twist. Taking tensor products of its finite expansions and then the weighted finite sum gives

$$\mathcal{F}_\vartheta=c_{\mathrm{ev}}\sum_x r_{\vartheta,x}G_x,\qquad B_0^{2m}r_{\vartheta,x}\text{ integral at every }\iota\mid\ell,\ \ell\nmid q. \tag{G}$$

Here G_x is the literal product of the unnormalized gluing sums in the statement. There are $m/2$ pairs and each has denominator B_0^4, whence B_0^{2m}. Cut multiplicities are units at the allowed coefficient primes and introduce no other denominator. If $Q=\emptyset$, the pairwise conjugate absolute coefficient sums are at most one, whereas the total absolute weight of the cuts is at most B_0^m. The triangle inequality therefore gives

$$\sum_x|\sigma r_{\vartheta,x}|\le B_0^m\quad(\sigma\in\mathrm{Aut}(\mathbb{C})). \tag{H}$$

The translations have $\bar\pi_M$ denominator away from q. The exact support deletion in the evaluation lemma replaces every G_x by a representative supported above Mq, with equality of vectors. It is a finite pushforward of the coefficients and does not alter (F), (G) or the upper bound (H).

Apply Lemma 4.2.33 with parameter the whole ordinary twist, denominator $d_{\mathrm{clr}}=B_0^{2m}$, invariant subspace $\mathcal{U}_\vartheta$, linear operator $R_\vartheta=\mathscr{R}_\beta\mathscr{C}_{\vartheta,\bullet}$, and the subring which is the intersection of all allowed prime-identification valuation rings. This intersection is a unital ring, so the lemma applies simultaneously to all those identifications. Take its two scalar parameters to be a_{real} and c_{ev}, and use the exact supported representative map just described. The lemma gives one

$$c_{\mathrm{ss}}=\begin{cases}a_{\mathrm{real}}c_{\mathrm{ev}}/B_0^{2m}, & a_{\mathrm{real}}c_{\mathrm{ev}}\neq0,\\ 1, & a_{\mathrm{real}}c_{\mathrm{ev}}=0\end{cases}$$

and integral finitely supported coefficients Ψ_ϑ. They are chosen before the source form, their tensor remains in $\mathcal{U}_\vartheta$, and their empty-conductor bound is

$$B_0^{2m}B_0^m=B_0^{3m}=M^{3fmn_m^2}.$$

If either common intermediate scalar vanishes, the lemma takes the zero tensor, which satisfies every support, integrality and invariance condition, and the displayed nonzero scalar 1. Thus no division by a possibly zero evaluation has occurred.

Combining (D) with this normalization proves all the stated conclusions for L_ϑ, uniformly over ordinary twists. The finite tensor and the final nonzero scalar have exactly the required quantifier order.

Initial period attachment. Use the Bessel source function and period in Definition 3.2.36. For $r = u\,\iota_0(h)$ its formula is

$$\mathcal{W}_{t_0,\varphi}(r) = \psi_{\mathsf{U}}(u)\, t_0\Big(\rho_\lambda((\overline{h_v})_{v\in\Phi})^{-1}\varphi(h_f)\Big). \tag{I}$$

The coefficient representation denoted by ρ_0 in (A) is precisely this conjugate-embedding Weyl action; hence (I) is (A), not a coefficient of a translated source form. The string basis and generic constant are the fixed ones of the Bessel period: writing ϖ for its purely imaginary different generator gives $\jmath c = -(M\varpi)^{-1}$, so its character $\psi_E(-\operatorname{tr}(\mathfrak{e}u)/(2M\varpi))$ is the character χ_0 in (A).

For the SAME fixed witnesses (g_∞, t_0), the hypothesis on the linear map $\mathcal{P}$ gives, for every principal form y,

$$\langle y, \mathcal{P}\varphi\rangle = \int_{[R_0]} \mathcal{W}_{t_0,\varphi}(r)\,\overline{y(r(g_\infty,1))}\,dr. \tag{J}$$

The absolute-convergence hypothesis supplies integrability for this exact product and the same quotient Haar measure. The right translation is $r(g_\infty, 1)$, in precisely the order used by the transposed kernel in the first step above. Thus the normalized lower composite L_ϑ in (D) is $\Theta_\vartheta(\mathcal{P}\varphi)$. No class-coordinate formula or specified Poincaré map is used: (J) is an input for the arbitrary linear map $\mathcal{P}$. In particular, φ in (I) is the original source form.

Substituting this identification into (D)–(H) proves the required operator identity and every support, coefficient and invariance conclusion for the stated actual map, with one nonzero scalar preceding all twists and one finite datum preceding every ramified source form. □

Lemma 4.2.26. *Let P be the positive fine-degree subspace of $\operatorname{Hom}_E(V, V')$, namely the matrices with entries only where $d'(b) - d(a) > 0$. In the fixed frames put*

$$C_f(g,h) = \int_{P(F)\backslash P(\mathbb{A}_F)} \theta\big(\rho(p)\rho([T])\omega_a(g,h)f\big)\,dp, \qquad U_f W(h) = \int_{R(F)\backslash R} W(r)C_f(r,h)\,dr,$$

where the first quotient has probability measure. Thus the evaluation is theta evaluation, not raw coherent coefficient evaluation. Put $\ell = s - 1$. Define S' to vanish on X' and on the last string vector, and for $b + 1 < \ell$ put

$$S'v'_b = c\epsilon'_b v'_{b+1}, \qquad \epsilon'_b = \begin{cases} -N, & 2b+1 = \ell, \\ -Nc, & 2(b+1) = \ell, \\ 1, & \textit{otherwise.} \end{cases}$$

Let $U'^{(2)}$ consist of the unitary matrices u' with $(u'-1)_{ba} = 0$ whenever $d'(b) - d'(a) < 2$, and put $\chi'(u') = \psi_E((\jmath/2)\operatorname{tr}(S'u'))$. Then $U_f W$ is continuous, left invariant under $R'(F)$, and

$$U_f W(u'h) = \chi'(u') U_f W(h) \qquad (u' \in U'^{(2)},\ h \in G').$$

In particular, for every $h_0 \in G'$, the function $r' \mapsto U_f W(r'h_0)$ on R' is continuous, left $R'(F)$-invariant and generic for the target string grading with the same c, N. No extension of the domain of W is required.

Proof. All adjoints and pairings below use the original Hom form and fixed frames. The exact continuous genuine action, covariance and topological Schwartz comparison are Lemma 4.1.1(2),(3). The bounded theta functional and strongly continuous isometric Heisenberg action on the same coherent carrier are supplied by Lemma 4.1.10(1). Polynomial Gaussian vectors and the finite vacuum translates are Schwartz and have a common finite additive level. Hence all compact means below are continuous on the Schwartz domain. The probability mean over P is unchanged by adelic translation in P, since P is isotropic and rational Heisenberg invariance makes the integrand periodic.

For clarity write

$$\Lambda_P(\varphi) = \int_{[P]} \theta(\rho(p)\varphi)\, dp, \qquad C(\varphi, w) = \Lambda_P(\rho([w])\varphi).$$

The monomial Gram matrices pair opposite grades; consequently $P^\perp$ is the nonnegative fine-degree subspace. A target unitary raising degree strictly acts on the Hom space by an operator raising fine degree strictly. By Lemma 4.1.12 its action fixes Λ_P exactly, with scalar one. Its determinant is one, so the external character factors also equal one. Exact covariance therefore gives

$$C(\omega_a(1, u')\varphi, T) = C(\varphi, u'^{-1}T).$$

For $q \in P$ the Heisenberg multiplication law, with the bicharacter of (WD6), gives

$$C(\varphi, T + q) = \beta(T, q)C(\varphi, T).$$

Since T has fine degree -1, whereas $u' - 1$ raises target degree at least 2, both $(u' - 1)T$ and $(u'^{-1} - 1)T$ have positive fine degree. Apply the preceding shift identity with $q = u'^{-1}T - T$. Preservation and alternation of the bicharacter give

$$\beta(T, u'^{-1}T - T) = \beta(u'T, T).$$

It follows, for $\varphi = \omega_a(r, h)f$, that

$$C_f(r, u'h) = \beta(u'T, T)C_f(r, h). \tag{A}$$

We calculate this factor, including the middle coefficients. Let h_a and h'_a denote the antidiagonal Gram entries for J_s and J_{s-1}. The adjoint of the diagonal string map is

$$T^\dagger v'_a = \frac{h'_a\, t_{s-2-a}}{h_{a+1}} v_{a+1}$$

when $a + 1 < s$, with zero contribution where the corresponding string index does not exist. Thus

$$TT^\dagger v'_a = \frac{h'_a\, t_{s-2-a} t_{a+1}}{h_{a+1}} v'_{a+1} \quad (a + 1 < s - 1).$$

The map vanishes on X' and on the last string vector. Inserting the displayed t values gives coefficient c except as follows. For odd $s = 2k + 1$, the exceptional position is $a = k - 1$, where $h'_a = 1$, $h_{a+1} = -N$ and $t_{s-2-a} = t_{a+1} = -Nc$; the coefficient is $-Nc^2$. For even $s = 2k$, the exceptional position is $a = k - 1$, where $h'_a = -N$, $h_{a+1} = 1$, $t_{s-2-a} = c$ and $t_{a+1} = 1$; it is $-Nc$. All other positions have no middle Gram entry and have one factor 1 and one factor c. These are exactly the coefficients defining S' for length $s - 1$. For $s = 1, 2$ the target shift is zero. Therefore $TT^\dagger = S'$ in every case. The defining Hom bicharacter now gives

$$\beta(u'T, T) = \psi_E\big((\jmath/2)\operatorname{tr}(T^\dagger u'T)\big) = \psi_E\big((\jmath/2)\operatorname{tr}(S'u')\big) = \chi'(u').$$

Here the trace identity is cyclic, so also applies to rectangular T. Integrating (A) proves the required covariance.

We check the other assertions without assuming any period-transfer identity. The column equations give $R = U \rtimes U(X)$ with U unipotent: remove the free X diagonal block and the remaining unitary is strictly flag raising. The adelic quotients of these factors are compact by Theorem A.4.3 and Theorem A.4.5. Reducing the Levi component and then the unipotent component gives a compact set of representatives of $R(F)\backslash R$. Therefore W is bounded, the outer integral is finite, and joint continuity of the actions and bounded theta imply continuity after both compact integrations, locally uniformly in h.

For $\delta \in R'(F)$, rational theta invariance from Lemma 4.1.11 and rational change of variables in $[P]$ give

$$C_f(r, \delta h) = C(\omega_a(r, h)f, \delta^{-1}T).$$

Its hypotheses hold for the literal WR action by Lemma 4.1.1; the parameter field contains μ_{5040}. The column condition for δ shows that $\delta^{-1}T - T$ has nonnegative fine degree. Indeed the defining column equations on nonnegative string columns, paired with their opposite columns in the unitary equation, make the action filtration preserving, with identity on each string graded quotient. A changed entry of any string column therefore goes to strictly higher grade. All grades are integers, and the source column of T is exactly one grade higher than its target column; hence each changed entry has fine degree at least zero. There is no input on X or on the last source string vector. Thus this difference lies in $P^{\perp}(F)$. The coefficient is unchanged on adding such a rational map: commute its rational Heisenberg operator to theta; its pairing with P is zero and its remaining rational phase is one. Hence $C_f(r, \delta h) = C_f(r, h)$, proving the left invariance.

Finally fix any h_0. Apply the established identities with $h = r'h_0$. Left invariance, continuity and (A) give precisely the asserted generic function on R'. Throughout, the input W was evaluated only at $r \in R$. □

Lemma 4.2.27. *For $0 \le i < m/2$ put $j = 2i$, and write $T_j : Y_j \to Y_{j+1}$ for the string map. Give*

$$\mathcal{W}_i = \mathrm{Hom}(Y_{j+1}, Y_j^- \perp Y_{j+2})$$

the joined grading, and put $\Sigma_i = (T_j^{\dagger}, T_{j+1})$, with the negative first summand and frames used in the joining map. The following construction uses the fixed theta model, not the values of its coefficient function.

(DE1) Evaluation. *Remove the strictly positive-degree isotropic subspace of $\mathcal{W}_i$. For $j+2 < m$ its degree-zero target has, besides $X_j^- \perp X_{j+2}$, the hyperbolic plane with basis*

$$(z_j - z_{j+2})/(2N), \quad z_j + z_{j+2}.$$

Remove the second isotropic direction of this plane, tensored with X_{j+1}^*. For $j + 2 = m$ there is no such plane: the last basis vector of X_m is identified with z_j/π_N in Y_j^-, and the other positive target vectors are Y_m. These are splittings in the sense of Definition 3.2.21.

Write c_{join} for the fixed finite integral unit denoted c in Proposition 4.2.28, so that $c_{\mathrm{join}}\overline{c_{\mathrm{join}}} = -1$. It is fixed before the twists and finite data. Let $\gamma_\infty = 1$ and $\gamma_f = c_{\mathrm{join}}\alpha/\bar{\alpha}$, and let $\mathscr{J}_i(p, p')(A, B) = p(\gamma A^*)p'(B)$ be the lower joining map with the negative first frame prescribed there. Let Z_i be their successive zero term in the sense of (MX4) of Definition 4.1.18. Its finite-coordinate map $Z_{i,f}$ is the map obtained by separating real and finite

coordinates in this SAME zero term; the real Gaussian factor is fixed once and its nonzero normalization is included in that separation. In finite coordinates set

$$\mathcal{E}_i = \mathscr{A}_f^{-1}\, Z_{i,f}\, \rho_f(\Sigma_{i,f}). \tag{DE2}$$

Here $\Sigma_{i,f}$ means the finite coordinates of Σ_i, divided by the Weil idele α, so that ρ_f is exactly (WD8). The domain is the finite joined Heisenberg span and the codomain is $\mathcal{F}_j \otimes \mathcal{F}_{j+1}$ of Proposition 4.2.25. This defines a linear map on that domain.

For an ordinary twist ϑ and finite points $y_l \in \mathrm{U}(Y_l)(\mathbb{A}_{F,f})$, let $p_{\vartheta,l}$ be the specified finite point datum of the lower chain and put

$$p^{y}_{\vartheta,l} = \omega_{\vartheta,l}((1,y_l),(1,y_{l+1}))p_{\vartheta,l}, \qquad \mathcal{Y}_{\vartheta,y,i} = \mathcal{E}_i \mathscr{J}_i(p^{y}_{\vartheta,j} \otimes p^{y}_{\vartheta,j+1}). \tag{DE3}$$

The lower action in (DE3) is on Y_l, Y_{l+1}, with their ranks in the powers of υ.

These maps have the following properties.

1. *Fix real points ξ_l and an initial covector t_{in}. Define W_0 on $R_0 = U_0 \rtimes \mathrm{U}(X_0)(\mathbb{A}_F)$ by*

$$W_0(u\iota_0(g)) = \chi_0(u) t_{\mathrm{in}}(\rho_0(g_\infty)^{-1}\varphi(g_f)).$$

Put $a_l = (\xi_l, y_l)$. In the following recurrence use the literal theta constant term

$$\mathcal{D}_{l,f}(r,h) = \int_{[P_l]} \theta_l(\rho([p])\rho([T_l])\omega_{\vartheta,l}(r,h)f)\, dp, \tag{DE4}$$

where P_l is the strictly positive-degree matrix space and its additive quotient has probability measure. Put $f^a_l = \omega_{\vartheta,l}(a_l,1)f_l$ for the prescribed full lower datum, and recursively

$$V_{l+1}(h) = \int_{[R_l]} W_l(r)\mathcal{D}_{l,f^a_l}(r,h)\, dr, \qquad W_{l+1}(r) = V_{l+1}(ra_{l+1}). \tag{DE5}$$

For every $2i < m$ there is a covector t_{2i}, chosen before ϑ, φ and all finite y_l, such that

$$W_{2i}(u\iota_{2i}(g)) = \chi_{2i}(u)(\mathscr{C}^{(2i)}_{\vartheta,\mathcal{Y}}\varphi)_{t_{2i}}(g). \tag{DE6}$$

Here $\mathscr{C}^{(2i)}$ is the first i paired upper lifts with the SAME tensors (DE3), and $A_t(g) = t(\rho(g_\infty)^{-1}A(g_f))$. For the last pair the corresponding coefficient of $V_m(\iota_m(g)a_m)$ is evaluated on the upper chain at $\mathrm{diag}(g,1)$, $g \in \mathrm{U}(Y_m)(\mathbb{A}_F)$. The final covector is again chosen before every twist, finite point and source form. The integrals defining these statements are absolutely convergent.

2. *For $j+2 < m$, finite $u \in \mathrm{U}(X_{j+2})(\mathbb{A}_{F,f})$ acts through the inclusion ι_{j+2} on the lower last space and through the second upper factor after evaluation. Thus*

$$\mathcal{E}_i\mathscr{J}_i(p \otimes \omega_{\mathrm{lower}}(1,\iota_{j+2}(u))p') = (\mathrm{Id} \otimes \omega_{\mathrm{upper}}(1,u))\mathcal{E}_i\mathscr{J}_i(p \otimes p'). \tag{DE7}$$

The character on both actions is the same character of this edge, with the rank adjustments prescribed by the lower and upper spaces. In particular, if $y^{-1}_{j+2}\iota_{j+2}(u)y_{j+2} \in J_{j+2}$, then

$$(\mathrm{Id} \otimes \omega_{\vartheta,j+1}(1,u))\mathcal{Y}_{\vartheta,y,i} = \mathcal{Y}_{\vartheta,y,i}. \tag{DE8}$$

No character scalar occurs in (DE8).

Proof. We use the canonical theta functionals, theta functions and theta kernels of (WC3)–(WC4) of Definition 4.1.15, and the actual action and zero-term laws, clauses (2), (3), (5), (6) of Lemma 4.1.1; the zero-term constructor itself is Lemma 4.1.6. The Heisenberg law and tensor identifications are those of Lemma 4.1.10 and Lemma 4.1.13. In particular a zero term throughout this proof is the mean of a theta function, as in (MX4) of Definition 4.1.18; no value of the coherent coefficient function is substituted for it.

The two splittings and the finite map. Opposite grades pair and no other grades pair. In the joined Hom space the strictly positive-degree maps are therefore isotropic, their orthogonal complement is the space of nonnegative-degree maps, and the quotient is the degree-zero Hom space. The source and last target strings have odd length, whereas the middle string has even length. Thus the zero-degree Hom maps have source X_{j+1}. In the middle case the two remaining string vectors have norms $N, -N$ in $Y_j^- \perp Y_{j+2}$. The displayed two plane vectors have norms zero and mutual product one. Their orthogonal complement is exactly $X_j^- \perp X_{j+2}$. In the last case the only remaining string vector has norm $N = \pi_N \bar{\pi}_N$ and division by π_N gives norm one. It supplies the missing last positive coordinate of X_m; the lower target is Y_m, of rank $n_m - 1$. These calculations verify both splittings and the different last endpoint, including all dimensions.

Here is also an explicit description of separation into real and finite parts. In position coordinates for either splitting write a test function as a finite sum of products of a polynomial Gaussian in real variables and a compactly supported locally constant function in finite variables. The zero term keeps the coordinate on the isotropic annihilator at zero and then uses the residual position variables. This assertion follows directly by expanding its theta function and integrating over the rational additive quotient: every nonzero rational frequency has mean zero, and the zero frequency has mean one. Schwartz absolute summability permits this integration term by term. Tensor products of such functions separate the real restriction from the finite restriction. For the real Gaussian with zero polynomial both restrictions are nonzero Gaussians; fix its normalization once. The finite restriction hence gives a well-defined linear $Z_{i,f}$. The same calculation for a Heisenberg translate gives a finite Heisenberg translate of the residual vacuum, or zero: the eliminated position must lie in the support lattice, and the eliminated modulation must annihilate that lattice. Thus $Z_{i,f}$ maps the finite Heisenberg span into the residual finite Heisenberg span. The conclusion is independent of an expression as a sum, since it is the restriction of the unique zero term of Lemma 4.1.6. No extension from polynomial tests to all integrable functions is being asserted here.

At finite places the signed adjoint with $\gamma_f = c_{\mathrm{join}} \alpha / \bar{\alpha}$ preserves the alternating Heisenberg pairing with the required reversed first sign. Pullback by it followed by multiplication in independent blocks sends a product of two translated vacua to the translated joined vacuum, with precisely the Heisenberg multiplication phase. Conversely every translation of the joined vacuum splits into its two blocks. This proves the lower joining assertion on the finite spans without a density argument. On the definite residual space its inverse is the isomorphism in Proposition 4.2.28. These facts prove that (DE2) and (DE3) are actual linear maps on the stated spaces.

Combining the two constant terms. Fix one even source stage. Write the two lower test vectors as f, f'. The lower joining formula and the nonzero fixed theta-product normalization give a fixed nonzero factor z_i relating the product of their theta functions to the theta function of their join. The product formula for the normalized theta functionals is Lemma 4.1.7. Its coordinate-pullback clause applies to the signed adjoint: conjugate transpose reverses the first alternating sign, $c_{\mathrm{join}} \overline{c_{\mathrm{join}}} = -1$ restores that sign, and the factor $\alpha / \bar{\alpha}$ restores the finite vacuum lattice and measure. The negative first frame makes the rational lattices match. The finite first-

block computation was just given. At infinity they are obtained by taking the signed adjoint in the fixed Fock coordinates, with the first real coordinates conjugated exactly when their sign changes. Thus they use the fixed tensor theta functionals, not separately chosen phases.

The two translations by the string maps join into Σ_i. The positive averaging variables map onto the positive subspace of the joined Hom space by taking the positive part of their stacked signed adjoint. This map is onto: in each positive-degree matrix entry either the first or the second original positive block contains that entry, and its coefficient is an invertible scalar. Its kernel accounts for duplicate coordinates. A surjective homomorphism between compact additive adelic quotients pushes probability Haar measure to probability Haar measure. Hence those variables give exactly the first zero term, with no multiplicity factor.

The remaining unipotent variables give the plane zero term in the middle case. To check its character, the two adjacent string maps satisfy

$$T_j^\dagger T_j = S_j, \qquad T_j T_j^\dagger = S_{j+1} = T_{j+1}^\dagger T_{j+1}.$$

These equalities can be checked on successive string vectors: all ordinary coefficients are the common constant, and the middle coefficients are $-Nc$ or $-Nc^2$ according to parity, as in Proposition 4.2.17. Consequently

$$\Sigma_i^\dagger \Sigma_i = -T_j T_j^\dagger + T_{j+1}^\dagger T_{j+1} = 0.$$

On the grade-zero target the only nonzero string column of Σ_i is the grade-one column, and it is the common generic constant times $z_j + z_{j+2}$. The unipotent character in (DE5) cancels exactly the Heisenberg phase of that column. All other plane columns vanish. Positive-degree and plane averaging vectors are mutually orthogonal, so their averages commute without a commutator phase. This proves that the remaining mean is precisely Z_i after translation by Σ_i. For the last pair there is no plane variable, so the same calculation stops after the first zero term.

There is no hidden twist in this contraction. For completeness, let v, b, w be the three lower ranks, let A, B, C be the upper ranks, let q_{pl} be the number of removed planes, and let P_+ be the dimension of the removed positive target part. Then

$$b = B + s', \quad s' \text{ even}, \qquad v + w = A + C + 2q_{\mathrm{pl}} + 2P_+.$$

The determinant on the positive Hom subspace is $(\det g \det h)^{s'/2} (\det r)^{-P_+}$. The zero-term action supplies its inverse splitting-character factor, and the plane mean supplies $\upsilon(\det r)^{q_{\mathrm{pl}}}$. The three exponents before these contractions are

$$b - \lfloor b/2 \rfloor, \quad \lfloor v/2 \rfloor - \lfloor w/2 \rfloor - v, \quad \lfloor b/2 \rfloor.$$

Substitution gives respectively $\lceil B/2 \rceil, -(A+C)/2, \lfloor B/2 \rfloor$. These are exactly the exponents of the definite joined upper action. The arbitrary two edge characters are untouched. Determinants of the embedding at the last endpoint are preserved by $\mathrm{diag}(g, 1)$. Thus the calculation holds for all ordinary twists with the same normalization.

Real coefficients and induction. For fixed real points ξ_l, the preceding zero-term calculation separates the full joined datum as one fixed real vector times its finite vector (DE3). The real vector depends on the prescribed real data and those points, but not on the twist or the finite points: real twisting weights are unchanged by an ordinary tame twist, and (WR5)–(WR6) separate the real and finite actions. The fixed constants from compatible Haar disintegration and theta joining can be included in this real vector. They are chosen before any twist.

The Bessel quotient is compact: its unipotent part has compact quotient by Theorem A.4.3, and its definite Levi has compact quotient by Theorem A.4.5. Its generic source coefficient

is bounded. The translated polynomial-Gaussian tests have absolutely summable theta series uniformly on compact sets; the finite factors have compact support and are locally constant. The two source integrations and all additive means may consequently be interchanged. Disintegrating the two Bessel quotients into their unipotent and definite parts, the calculation above leaves the double definite theta integral with that fixed real vector and finite tensor (DE3). Formula (2) of Proposition 4.2.28 turns it into a coefficient of the two upper lifts, with a single new covector before all twists, finite tensors, forms and group points.

At stage zero take $t_0 = t_{\mathrm{in}}$. Apply this two-step calculation inductively. The required genericity on each next Bessel group is the explicit conclusion of Lemma 4.2.26, for precisely the literal theta mean (DE4): $T_l T_l^{\dagger} = S_{l+1}$, and the same common constant and additive character are retained at every right translate. On an odd-length source string the strictly raising radical raises grade by at least two. Hence its prescribed generic covariance and its restriction to the definite Levi determine the whole source function. The induction therefore proves (DE6) on all of R_{2i}, not just on its Levi. Moving the outer point a_{l+1} into the second argument of the kernel is exactly the representation identity $\omega(r,h)\omega(a_l,1) = \omega(ra_l,h)$; moving its finite component into both adjacent data gives exactly (DE3). This also proves the assertion for the last pair with the endpoint already verified above. No assertion about a Poincare series or its initial unfolding is used.

Finite equivariance and the level consequence. The block action of $\mathrm{U}(X_{j+2})$ fixes Σ_i, preserves both isotropic subspaces, and acts trivially on their removed coordinates. Change variables in their probability means and use Heisenberg covariance. The result is the residual action on X_{j+2} and no determinant on a removed coordinate. The preceding determinant computation identifies its splitting character with the upper last-edge character. Applying the inverse definite joining map proves (DE7).

Finally the point datum is the indicator of the inclusion matrix modulo the level. An element of J_{j+2} is integral and reduces to one there. Clause (6) of Lemma 4.1.1 therefore fixes this indicator; its determinant character is one since its tame character is trivial on the principal level and its unramified part is trivial on units. The group law gives

$$\omega(1,\iota(u))\omega(y_{j+1},y_{j+2}) = \omega(y_{j+1},y_{j+2})\omega(1,y_{j+2}^{-1}\iota(u)y_{j+2}).$$

The last factor fixes the unmoved datum. Insert this equality in (DE7) to obtain (DE8), with scalar one. □

Proposition 4.2.28. *Fix $0 \le j < m-1$, and put $X = X_j$, $X' = X_{j+1}$, $X'' = X_{j+2}$, of ranks n, n', n''. For a vector-valued algebraic form φ on X and $\ell \in \mathcal{V}_j^*$ put*

$$\varphi_\ell(g) = \ell(\rho_j(g_\infty)^{-1}\varphi(g_f)).$$

Write $[H] = \mathrm{U}(H)(F)\backslash\mathrm{U}(H)(\mathbb{A}_F)$ for each definite space H, with the compatible quotient measure.

Choose once a finite integral unit c with $c\bar{c} = -1$, let $\gamma_\infty = 1$, $\gamma_f = c\alpha/\bar{\alpha}$, and give X^- the negative form and the frame obtained from that of X by multiplication by $\bar{\gamma}$. Put $W = \mathrm{Hom}(X', X^- \perp X'')$. In the coherent coefficient realizations define the joining map by

$$\mathscr{A}(f,f')(A,B) = f(\gamma A^*)f'(B), \qquad A : X' \to X^-, \quad B : X' \to X''.$$

Here the adjoint and the factors are read in the specified framed coordinates; at infinity the adjoint exchanges the signed Fock coordinates. On finite parts it induces $\mathscr{A}_f : \mathcal{F}_j \otimes \mathcal{F}_{j+1} \to \mathcal{F}(W)$.

For characters a, b of $\mathrm{U}(1)$ put $a_l = a\upsilon^{\lfloor l/2 \rfloor}$, $b_l = b\upsilon^{\lfloor l/2 \rfloor}$, and define the action on the joined model by

$$\Omega_{a,b}(g,r,h) = \frac{a_n(\det r)}{a_{n'}(\det g)} \frac{b_{n'}(\det h)}{b_{n''}(\det r)} \frac{\upsilon^{n'}(\det g)}{\upsilon^{n}(\det r)} \cdot \omega_W(A \mapsto (g \oplus h)Ar^{-1}).$$

The matrix of g on X^- is the same as on X.

The map $\mathscr{A}_f$ is an isomorphism onto the finite Heisenberg span of the joined vacuum. There is one $z \in \mathbb{C}^\times$, fixed by these models and independent of a, b, such that

$$\theta_W(\mathscr{A}(f,f')) = z\,\theta_j(f)\theta_{j+1}(f'), \qquad \mathscr{A}(\omega_a(g,r)f, \omega_b(r,h)f') = \Omega_{a,b}(g,r,h)\mathscr{A}(f,f'). \tag{1}$$

In the first formula f, f' range over their coherent Banach carriers.

Let u belong to the real coherent carrier of W, and let $\ell \in \mathcal{V}_j^*$. For $Y \in \mathcal{F}_j \otimes \mathcal{F}_{j+1}$, the notation $u \otimes \mathscr{A}_f Y$ means the coherent coefficient function which is the product of the indicated real and finite coefficient functions. There exists $\ell'' \in \mathcal{V}_{j+2}^*$, depending on u, ℓ and the fixed normalizations, such that, for every ordinary twist ϑ, every such Y, every source form φ on X, and $h \in \mathrm{U}(X'')(\mathbb{A}_F)$,

$$\begin{aligned} \int_{[X]} \varphi_\ell(g) \int_{[X']} \theta_W\Big(\Omega_{\nu_{j+1},\nu_{j+2}}(g,r,h)(u \otimes \mathscr{A}_f Y)\Big)\, dr\, dg \\ = \Big((T_{\vartheta,j+1} \circ T_{\vartheta,j})_Y \varphi\Big)_{\ell''}(h). \end{aligned} \tag{2}$$

The subscript Y on the composite denotes its tensor-linear extension in the two finite data. The integrals are absolutely convergent. In particular ℓ'' is chosen before the twist, finite data, source form and group point.

Proof. The spaces and Hom spaces in this statement are (VP2),(VP5) of Definition 3.2.2; their operators use the canonical (WD1)–(WD9), (WR3)–(WR6) realization, as fixed in Proposition 4.2.25. Apply Lemma 4.2.29 to the three consecutive positive definite spaces. Its finite unit is available over the actual parameter field, its negative frame is exactly $\bar{\gamma} x_X$, and its literal scaled-adjoint joining is the displayed $\mathscr{A}$. The result proves the finite-span isomorphism and both formulas (1), with a fixed scalar z of absolute value one. In particular it checks the rational lattices, finite phase, vacuum and multiplier correction, not just the signatures of the spaces. The external characters in that result are exactly the a_l, b_l used here.

We record the convergence needed throughout. The action Proposition 4.1.14 is the same prescribed kernel, with a bounded operator on the coherent Banach carrier, exact covariance and joint continuity. The fixed polarized realization of Lemma 4.1.2 checks the Schwartz-domain hypotheses of Lemma 4.1.11; that result gives rational-unitary theta invariance, including for the joined Hom space. The identity extends from Schwartz vectors to the whole coherent carrier by boundedness: finite vacuum-translate sums are dense by Lemma 4.1.10 and lie in that Schwartz domain. All external determinant characters are automorphic. Hence the integrands here descend to the stated rational quotients.

For a definite unitary group that quotient is compact, with finite class sets at open finite level and compact real fibers. Choose compact sets of representatives; the explicit operator norm bound of Proposition 4.1.14 is bounded on them, since its multiplier is continuous. The coefficient of a fixed algebraic form and the bounded theta functional are bounded there as well. Thus the displayed double integrals are absolutely convergent. These same bounds justify

Fubini and give a bounded linear functional of the real datum for every fixed tuple $(\vartheta, Y, \varphi, h)$. No bound uniform in this tuple is needed below.

We first prove (2) for polynomial real data. In the Fock variables the two blocks of W are independent. The first, negative Hom block has variables given by the entries of A, and the second, positive Hom block by the entries of $\bar{B}$. Every polynomial in these two variable sets is a finite sum

$$R(U, V) = \sum_{a=1}^{d} P_a(U^{\mathsf{T}}) Q_a(V).$$

Indeed each monomial is a product of its first-block and second-block monomials, and transposition merely reindexes the first variables. This includes constant polynomials and all full-height minors.

It suffices by linearity to take one such product. Let F_0, F_1 be finite data on the two edges and $Y = F_0 \otimes F_1$. The specified real flip is conjugate transpose, so on the positive first carrier $P(\overline{A^*}) = P(A^{\mathsf{T}})$. The second carrier is unchanged. The product vacuum is preserved exactly by the joining. The literal (WR6) datum formula, or the real rank-one identity obtained from Lemma 4.1.10, therefore gives the pointwise coefficient identity

$$v_R \otimes \mathscr{A}_f(F_0 \otimes F_1) = \mathscr{A}(v_P \otimes F_0, v_Q \otimes F_1). \tag{3}$$

There is no theta evaluation in (3); it is an identity of coherent coefficient functions.

Apply (1) with $a = \nu_{j+1}$ and $b = \nu_{j+2}$. The joined kernel for (3) is

$$z\, \theta_j(\omega_{\vartheta,j}(g, r)(v_P \otimes F_0))\, \theta_{j+1}(\omega_{\vartheta,j+1}(r, h)(v_Q \otimes F_1))\,. \tag{4}$$

The rank shifts in these two pair characters are precisely those in the upper operators, by their definitions in Proposition 4.2.25.

The one-edge coefficient identity Lemma 4.2.30, for P and the original covector ℓ, supplies one covector ℓ_1 before every twist, finite datum, source form and target point. Integrating (4) first in g therefore replaces the first theta factor, multiplied by $\varphi_\ell(g)$, by $(T_{\vartheta,j,F_0}\varphi)_{\ell_1}(r)$. Apply the same result to edge $j+1$, polynomial zQ and covector ℓ_1. It supplies one covector ℓ_2 with the same uniform order of quantifiers. Integration in r is then

$$\left(T_{\vartheta,j+1,F_1} T_{\vartheta,j,F_0}\varphi\right)_{\ell_2}(h).$$

The scale, ordinary derivatives, source dimension and fixed quotient-volume factors are exactly those of Lemma 4.2.30, so they are already included in ℓ_1, ℓ_2. In particular this calculation does not replace the actual jet map by a renormalized one.

For a sum of product polynomials, sum these covectors. For general Y in the algebraic tensor product, extend the equality from simple tensors by linearity. The chosen covector is unaffected by this extension because the one-edge covectors were chosen before the finite data. We have proved (2) for every polynomial real datum with one covector for the entire indicated family.

We give the Banach density and closure argument to retain this simultaneous quantifier for arbitrary real data. In the signed Fock coordinates $\zeta_i(w) = \bar{w}_i$ on positive coordinates and w_i on negative coordinates, every real model vector has the form

$$f(w) = G(\zeta(w)) \exp(-\|w\|^2/2)$$

with G entire and $f \in L^1$. This follows directly from its reproducing integral: after division by the Gaussian, its kernel is exponential-linear in $\zeta(w)$. On each compact set its derivatives are

dominated by a polynomial times a Gaussian in the integration variable, bounded by a fixed multiple of $|f|$; thus differentiation and the entire power series are justified under the integral.

For $t \in \mathbb{R}/2\pi\mathbb{Z}$ let $\mathcal{R}_t f(w) = G(e^{it}\zeta(w))\exp(-\|w\|^2/2)$. This is the pullback by unitary coordinate rotations, with opposite rotations in the two sign sets. It preserves the bicharacter, Gaussian and measure, so a change of variables in the reproducing equation shows that it preserves the real model. It is an L^1 isometry and strongly continuous, by translation continuity of rotations on L^1 (first for continuous compactly supported functions, then by their density). If G_d is the degree-d homogeneous Taylor polynomial, its circle Fourier coefficient is $G_d(\zeta(w))e^{-\|w\|^2/2}$ for $d \geq 0$ and zero for $d < 0$. This follows on compact sets by the locally uniform Taylor expansion; the averaging operator is bounded in L^1, so these are also the L^1 identities. The Fejer means are therefore polynomial vectors:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\mathcal{R}_t f\,dt = \sum_{d=0}^{N}\left(1 - \frac{d}{N+1}\right) G_d(\zeta(w))e^{-\|w\|^2/2}.$$

They converge to f in L^1: on a small neighbourhood of 0 use strong continuity, and on its complement use the vanishing mass of the nonnegative Fejer approximate identity and the isometry bound. Thus polynomial vectors are dense in exactly the real coherent Banach norm used here.

Let $\mathcal{I}$ be the set of all tuples $(\vartheta, Y, \varphi, h)$ in (2), and give $\mathbb{C}^{\mathcal{I}}$ its product topology. The map

$$\mathcal{L} : \mathcal{V}_{j+2}^{*} \longrightarrow \mathbb{C}^{\mathcal{I}}, \qquad \mathcal{L}(q)_{\vartheta,Y,\varphi,h} = \big((T_{\vartheta,j+1} \circ T_{\vartheta,j})_Y \varphi\big)_q(h)$$

is linear and has finite-dimensional range. Such a subspace of a Hausdorff topological vector space is closed. Concretely, choose finitely many coordinate evaluations that separate that range; an invertible coordinate matrix recovers its finitely many coefficients continuously, proving closedness directly even for this possibly infinite index set.

Sending the real datum u to the family of left sides of (2) is linear and continuous into this product topology: each individual coordinate is bounded by the compact-representative operator estimates above and by the bounded (WR6) datum map. The inverse image of range $\mathcal{L}$ is therefore closed in the real Banach carrier. It contains every polynomial vector by the preceding calculation and hence contains every u by density. Membership supplies a single ℓ'' for the entire family $\mathcal{I}$, chosen before the twist, finite tensor, source form and group point. This proves (2), its absolute convergence, and the stated order of quantifiers without a nonvanishing assumption on any lift. □

Lemma 4.2.29. *Let X, X', X'' be positive definite spaces with points over E_s, of ranks n, n', n''. A finite integral unit c with $c\bar{c} = -1$ exists. Fix one, put $\gamma_\infty = 1$, $\gamma_f = c\alpha/\bar{\alpha}$, and let X^- have form $-h_X$ and point $\bar{\gamma}x_X$. Set $W = \mathrm{Hom}(X', X^- \oplus X'')$. The literal coefficient joining*

$$\mathscr{A}(f, f')(A, B) = f(\gamma A^*)f'(B)$$

is bounded bilinear from the two original coherent models to the model of W. Its restriction induces an isomorphism of the algebraic tensor product of their finite Heisenberg spans onto the finite Heisenberg span of the joined vacuum.

There is one scalar $z \in \mathbb{C}$ with $|z| = 1$, fixed by these three theta functionals and the chosen coordinates, such that

$$\theta_W(\mathscr{A}(f, f')) = z\,\theta_{\mathrm{Hom}(X,X')}(f)\theta_{\mathrm{Hom}(X',X'')}(f')$$

for all coherent f, f'. For characters a, b of $\mathrm{U}(1)$ put $a_l = a\upsilon^{\lfloor l/2 \rfloor}$, $b_l = b\upsilon^{\lfloor l/2 \rfloor}$, and let $\omega_a(g,r)$ and $\omega_b(r,h)$ be their pair actions with character arguments $(a_n, a_{n'})$ and $(b_{n'}, b_{n''})$. Then

$$\mathscr{A}(\omega_a(g,r)f, \omega_b(r,h)f') = \Omega_{a,b}(g,r,h)\mathscr{A}(f,f'),$$

where, with the same matrix of g acting on X^-,

$$\Omega_{a,b}(g,r,h) = \frac{a_n(\det r)}{a_{n'}(\det g)} \frac{b_{n'}(\det h)}{b_{n''}(\det r)} \frac{\upsilon^{n'}(\det g)}{\upsilon^{n}(\det r)} \omega_W(C \mapsto (g \oplus h)Cr^{-1}).$$

The theta scalar z is independent of these external characters.

Proof. The field E_s contains a primitive 5040th root of unity. Indeed $16 \cdot 3 \cdot 5 \cdot 7$ divides q_0 and an additional factor 3 divides $T(s)$, so $5040 = 16 \cdot 9 \cdot 5 \cdot 7$ divides q_1. Apply Lemma 3.2.6 in rank one to the invertible hermitian matrix $H = -1$. Its integral invertible change of basis satisfies $-\bar{c}c = 1$, as required. We use this one fixed c throughout.

Write $x_X, x_{X'}$ for the original points and $J_X, J_{X'}$ for the rational Gram matrices. They satisfy $x_X^* J_X x_X = 1$ and $x_{X'}^* J_{X'} x_{X'} = 1$, because these original spaces are positive at infinity and the finite frame forms are also the identity. The new point really has the required opposite real signs and standard finite form: $\gamma\bar{\gamma} = 1$ at infinity and -1 at the finite places, so $(\bar{\gamma}x_X)^*(-J_X)(\bar{\gamma}x_X)$ is -1 at infinity and 1 at finite places.

We check the actual coordinate change $\phi(A) = \gamma A^*$ from the first reversed Hom space to $\mathrm{Hom}(X, X')$. In normalized real coordinates it is conjugate transpose; the reversed Hom signs are negative, whereas the original ones are positive. Conjugating the trace reverses the skew imaginary part and changing the sign of the form reverses it a second time. Thus its two real bicharacters agree with the same argument order. In normalized finite coordinates it is $A_0 \mapsto cA_0^*$, because $A = \alpha A_0$ implies $\gamma A^* = c\alpha A_0^*$. This is an integral-lattice automorphism. In the finite bicharacter the norm factor $c\bar{c} = -1$ cancels the sign from conjugating $\jmath$; cyclic invariance of the matrix trace gives the same bicharacter identity. These maps preserve the vacuum measures: conjugate transpose is orthogonal at infinity, and the normalized finite integral lattice is preserved and has mass one.

The real Gaussian is unchanged. The finite lattice support is unchanged, and the finite vacuum phase is inverted. It is equal to its inverse on that support: if $A = \alpha A_0$ is supported there, then

$$\psi_{E,f}\big(c_0 \jmath \alpha\bar{\alpha} \operatorname{tr}(A_0^* A_0)\big) = 1.$$

This is the inverse-different assertion and integral-character calculation in Lemma 4.1.10; c_0 and the trace are integral. The displayed scalar is exactly the square of the finite vacuum phase. Hence ϕ preserves the vacuum exactly.

It also identifies the actual rational lattices, not merely their signatures. For a rational matrix $\Xi : X \to X'$ define its original adjoint $\Xi^\dagger = J_X^{-1}\Xi^* J_{X'}$. The reversed rational vector has framed matrix $A = (\bar{\gamma}x_X)^{-1}\Xi^\dagger x_{X'}$. Substitution gives

$$\gamma A^* = x_{X'}^* J_{X'}\, \Xi\, J_X^{-1}(x_X^*)^{-1} = x_{X'}^{-1}\Xi x_X.$$

The adjoint is bijective on rational matrices, so this proves equality of the two rational lattices under ϕ.

Changing variables by this specified map in the reproducing equation gives an isometric model pullback $Lf = f \circ \phi$, with

$$L\rho(z) = \rho(\phi^{-1}z)L, \qquad Le = e_{\mathrm{rev}}.$$

The product of this pullback with the second coefficient function is exactly $\mathscr{A}$. The canonical product measure gives $\|\mathscr{A}(f,f')\|_1 = \|f\|_1\|f'\|_1$. The explicit coordinate checks now meet every hypothesis of Lemma 4.1.7. Its normalized pullback and orthogonal-sum conclusions prove the asserted theta equality and $|z| = 1$. They involve only the fixed theta phases and coordinates, so z has no dependence on a, b.

We next verify the kernel multiplier, which is needed for the action identity. Put

$$U(Z) = rZg^{-1}, \qquad U_{\mathrm{rev}}(A) = gAr^{-1}.$$

Direct adjunction, using the unitary equalities for g, r, gives $\phi U_{\mathrm{rev}} = U\phi$. At infinity the first Hom space is positive and the reversed one negative, while g, r are ordinary compact unitary matrices. Consequently the literal real multipliers are

$$\Delta_\infty(U) = 1, \qquad \Delta_\infty(U_{\mathrm{rev}}) = \overline{\det U_{\mathrm{rev}}} = \det U.$$

For the finite multipliers use Lemma 4.1.13, whose self-duality hypotheses were checked in Lemma 4.1.10; the splitting character of (WD3) is trivial on finite integral units and satisfies the required conjugation law. The reversed operator is the reindexed transpose of U^{-1}. Transpose and reindexing preserve its finite multiplier: over each discrete valuation ring an invertible matrix is reduced by integral invertible left and right matrices to a diagonal matrix of elementary divisors, and the lattice intersection in the multiplier contributes on a divisor ϖ^a the exponent $\max(0, -a)$. Transposition has the same elementary divisors and integral factors have trivial character. The inverse formula of that lemma therefore gives

$$\Delta_f(U_{\mathrm{rev}}) = \chi_f(\det U)\Delta_f(U) = \upsilon_f(\det U)\Delta_f(U).$$

In the last equality $|\det U|_f = 1$: conjugate finite places cancel by the unitary determinant equation, and at a conjugation-fixed place the norm-one determinant has absolute value one. At infinity the splitting character has weight one. Thus

$$\frac{\Delta(U)}{\Delta(U_{\mathrm{rev}})} = \upsilon(\det U)^{-1} = \frac{\upsilon^{n'}(\det g)}{\upsilon^{n}(\det r)}. \tag{1}$$

Every multiplier here is nonzero; at finite places this also follows from its positive lattice index.

Use $\phi U_{\mathrm{rev}} = U\phi$, vacuum and measure preservation in the literal kernel integral. They give pointwise on all coherent vectors

$$L\omega(U) = \frac{\Delta(U)}{\Delta(U_{\mathrm{rev}})}\omega(U_{\mathrm{rev}})L. \tag{2}$$

There is no arbitrary implementing scalar in (2): the scalar is precisely (1). On an orthogonal sum the real negative-block determinant and the finite intersection lattice both factor; the latter factorization is also in Lemma 4.1.13. Fubini in the product kernel therefore intertwines the product of the two actions with the action on W. Absolute Bochner bounds are $\| \int f(y)\rho(Uy)e\, d\mu(y)\|_1 \le \|f\|_1\|e\|_1$, so these substitutions and products hold on the entire Banach carriers. Multiplying (2) by the two external pair characters gives exactly the displayed $\Omega_{a,b}$ and proves the action equality.

Finally consider the finite spans. On normalized finite coordinates ϕ is the bijection $A_0 \mapsto cA_0^*$. The preceding Heisenberg and vacuum identities show that $\mathscr{A}$ sends the tensor of two finite vacuum translates to the joined vacuum translated by the two corresponding blocks. All joined finite translates occur, since that block coordinate map is onto. This proves surjectivity onto the asserted span.

For injectivity write a tensor as $\sum_{i=1}^{d} f_i \otimes f_i'$ with the f_i linearly independent as finite coefficient functions. Evaluations span the dual of their finite-dimensional function span: a function killed by every evaluation is zero. Choose d evaluations with independent restrictions to this span and invert their evaluation matrix. If $\sum_i f_i(\phi A) f_i'(B) = 0$ for all A, B, evaluation and that inverse matrix give $f_i'(B) = 0$ for every i, B. Thus the tensor was zero. This proof uses coefficient functions throughout and never identifies a coefficient value with a theta evaluation. □

Lemma 4.2.30. *Fix $0 \le j < m$. Put*

$$d_j = \sum_{\tau \in \Phi} \sum_{1 \le h < n_{j+1}} h C_\tau(h), \qquad c_j = \pi_M \eta_P \neq 0, \qquad e_{B,j}(\tau) = \lfloor n_{j+1}/2 \rfloor - \kappa_{j+1}(\tau).$$

For $u \in \mathcal{V}_{j+1}^*$ and $v \in \mathcal{V}_j$ write $P_{u,v}(Z) = u(S_C(Z)v)$, retaining every height $h < n_{j+1}$, including $h = n_j$. Then

$$e_j(\tau) = e_{B,j}(\tau) + C_\tau(n_j), \quad \Omega_{\vartheta,j,\infty}(x,y) v_{P_{u,v}} = v_{P_{u\rho_{j+1}(y)^{-1},\, \rho_j(x)v}},$$

where $\Omega_{\vartheta,j,\infty}$ is the actual real pair action for edge j; it is independent of the ordinary twist. For finite data F_0 and any finite points g_f, h_f,

$$\begin{aligned} &\theta(\omega_{\vartheta,j}((x,g_f),(y,h_f))(v_{P_{u,v}} \otimes F_0)) \\ &\qquad = c_j^{-d_j}\, u\rho_{j+1}(y)^{-1} J_j(\omega_{\vartheta,j,f}(g_f,h_f)F_0)\rho_j(x)v. \end{aligned}$$

In particular the ordinary scaled jet and this coefficient use the same vector, model and derivative convention.

For every polynomial R on the edge matrix variables and $\ell \in \mathcal{V}_j^*$ there exists $\ell' \in \mathcal{V}_{j+1}^*$ such that, simultaneously for every ordinary twist ϑ, finite datum F_0, algebraic source form φ of weight $\mathcal{V}_j$ and point h,

$$\int_{[X_j]} \varphi_\ell(g)\, \theta(\omega_{\vartheta,j}(g,h)(v_R \otimes F_0))\, dg = (T_{\vartheta,j,F_0}\varphi)_{\ell'}(h).$$

The integral is absolutely convergent. If $R = P_{u,v}$ one can take

$$\ell' = \frac{b_j\, \ell(v)}{(\dim \mathcal{V}_j) c_j^{d_j}}\, u,$$

where $b_j > 0$ is the fixed ratio of the compatible adelic quotient measure to the measure obtained from probability Haar measure at infinity and the finite class measure used to define T_{ϑ,j,F_0}.

Proof. We first check the weights and the full-height factors. The admissible rank inequalities give $x > 4$, $r \ge x^5$, $m \le 2\sqrt{x}$, hence $k = r+1-m > 1$. Also $r \ge 3$ and $n_1 = 6r+2-m < 6r+2$, while $\kappa_1 \ge 3r^8 + 1$. Thus $C_\tau(k) = \kappa_1(\tau) - \lfloor n_1/2 \rfloor > 0$ and $\Delta - (t+1) = 2r^8 - 5r - 1 > 0$; the displayed column counts are nonnegative without a hidden truncation. There is no nonzero column count strictly between two consecutive ranks, since those ranks differ by $t+1$.

For $j = 0$, $e_0 = 0$ and $C_\tau(k) = \kappa_1(\tau) - \lfloor n_1/2 \rfloor$ give the first identity. For $j > 0$, use $\kappa_{j+1} - \kappa_j = \Delta$ and $\lfloor n_{j+1}/2 \rfloor - \lfloor n_{j-1}/2 \rfloor = t+1$. Subtracting the exponents gives $e_j - e_{B,j} = \Delta - (t+1) = C_\tau(n_j)$. The character ν_{j+1} has weight $-\kappa_{j+1}$ and every ordinary tame twist has weight zero. Therefore the actual two real pair weights are

$$e_{A,j} = \lfloor n_j/2 \rfloor - \kappa_{j+1} = e_{j+1}, \qquad e_{B,j} = \lfloor n_{j+1}/2 \rfloor - \kappa_{j+1},$$

uniformly in the twist.

Here is the representation calculation with its conjugations. Temporarily omit the determinant character from a Weyl representation, leaving the Young realization formed from the columns of heights strictly below its rank. Exterior functoriality of the rectangular Schur matrix gives

$$S_C(ZG) = \det(G)^{C_\tau(n_j)} S_C(Z)\, S_{C,<n_j}(G)$$

at each place. The height-n_j exterior power on the source is its determinant, and the intermediate column counts are zero. The target realization uses the same columns, now all strictly below n_{j+1}; thus no target factor is discarded.

For positive Hom signs, Lemma 4.1.5 says that the polynomial substitution is $Z \mapsto y^{\mathsf{T}} Z (x^{-1})^{\mathsf{T}}$, with scalar $\prod_\tau \det(y_\tau)^{e_{A,j}(\tau)} \det(x_\tau)^{-e_{B,j}(\tau)}$. Since x, y are ordinary unitary matrices, $y^{\mathsf{T}} = \bar{y}^{-1}$ and $(x^{-1})^{\mathsf{T}} = \bar{x}$. The conjugate Weyl action on the source is the Young action at $\bar{x}$ times $\det(\bar{x})^{e_j}$, and the inverse target action is the Young action at $\bar{y}^{-1}$ times $\det(y)^{e_{j+1}}$. The additional full-height factor is $\det(\bar{x})^{C_\tau(n_j)}$. The identity $e_j = e_{B,j} + C_\tau(n_j)$ makes these factors agree exactly. This proves the asserted real polynomial identity for the actual conjugate Weyl modules. In particular dropping full-height columns would give a wrong source determinant exponent.

The scale c_j is nonzero: π_M and the CM period are nonzero, the Chowla–Selberg factor is a unit and the defining exponential factor in η_P never vanishes. The polynomial $P_{u,v}$ is homogeneous of degree d_j, since the Schur matrix has exactly the stated columns. The ordinary generating-series identity of Lemma 4.2.2, on the same finite-part vector, gives

$$u(J_j(F_0)v) = c_j^{d_j} \theta(v_{P_{u,v}} \otimes F_0).$$

Indeed ordinary monomial differentiation cancels its generating-series factorial; no divided derivatives or second scale enter. The finite part is preserved by the actual action in Proposition 4.1.14. Apply the real/finite datum separation of Lemma 4.1.5 and the real identity just proved. This gives the displayed kernel coefficient formula with arbitrary finite points.

We explain the quotient integrals before integrating this formula. The canonical action is genuine and continuous by Proposition 4.1.14. Its fixed polarized Schwartz realization is Lemma 4.1.2; hence the domain and covariance hypotheses of Lemma 4.1.11 apply to this very kernel, in all the auxiliary spaces there. That result gives rational pair theta invariance. On arbitrary coherent vectors the same identity follows by continuity: finite sums of vacuum translates are dense by Lemma 4.1.10, they are Schwartz in the fixed polarized realization, and both sides are bounded functionals. The determinant characters are automorphic, so they are trivial on rational points also. Thus the kernel integrands and the algebraic-form coefficients descend to the stated quotients. The definite adelic quotients are compact. Equivalently their finite class sets at any fixed open level are finite and their real fibers are compact unitary groups. On compact sets of representatives continuity bounds the operator norm and all coefficients, proving absolute convergence and Fubini.

For $R = P_{u,v}$, disintegrate the quotient measure into probability real Haar measure and the finite class measure of the specified lift. The ratio b_j is fixed by uniqueness of invariant quotient measure and is independent of the test vector, twist and form; finite stabilizers are included in the class measure, so this disintegration also holds for levels smaller than the source level. Schur orthogonality on the irreducible compact representation ρ_j is the precise endomorphism identity

$$\int_{G_\infty(X_j)} \rho_j(x) v\, \ell \rho_j(x)^{-1}\, dx = \frac{\ell(v)}{\dim \mathcal{V}_j} \operatorname{id}_{\mathcal{V}_j}.$$

For completeness the integral commutes with every $\rho_j(x)$ by Haar invariance, hence is scalar by irreducibility; its trace is $\ell(v)$, which gives the displayed scalar. Apply this to the coefficient formula and integrate the finite variable. The remaining integral is literally $J_j(\omega_{\vartheta,j,f}(g_f,h_f)F_0)\varphi(g_f)$ with the measure defining T_{ϑ,j,F_0}. This proves the last statement with the stated explicit covector.

Finally consider an arbitrary polynomial R. Its compact orbit span is finite dimensional. Average against the source compact type ρ_j to project to its isotypic component; all other types have zero integral against $\ell\rho_j(x)^{-1}$, by the same Schur orthogonality. The polynomial Cauchy decomposition [135], identifies that isotypic component exactly with the span of the $P_{u,v}$. Here its shape at each place has columns $C_\tau(h)$ for $h \le n_j$. Indeed the source highest weight, after removing the fixed pair determinant weight $e_{B,j}$, has entries $C_\tau(n_j) + \sum_{i<h<n_j} C_\tau(h)$, which uniquely determine that partition. Its length is at most n_j; the target has the same partition padded by zeros because the intermediate columns vanish. The classical decomposition applies to the finite-dimensional complex row and column spaces and is multiplicity free, so no additional source occurrence can contribute. This argument also works for a polynomial with several degrees, by applying it degree by degree.

Choose once a finite decomposition of this projection as $\sum_a P_{u_a,v_a}$. Sum the preceding explicit covectors. Every operation in this choice uses only the polynomial, ℓ, the fixed real weights, the measure ratio and c_j. The ordinary twists have identical real weights, so the resulting covector works simultaneously for all twists and all finite data, forms and target points. This is the required order of quantifiers. □

Proposition 4.2.31. *Fix admissible D and an ordinary twist ϑ of modulus Q, and write $q = \mathrm{N}Q$. Let $\mathfrak{q}$ be a finite idele with valuation one at each place of Q and zero at the other places, chosen compatibly with the fixed generators of the level ideals. For $0 \le j < m$ put $s_j = m-1-j$ and grade $Y_j = X_j \perp J_{s_j}$ as in Proposition 4.2.17. Use the same constant*

$$c = -(\jmath M \delta_E)^{-1}$$

in their string maps and generic characters, where $\delta_E = \epsilon \mathrm{cyc}'_{q_1}(\zeta)\sqrt{-d}$ and $\epsilon \in \mathcal{O}_E^\times$ is the fixed norm-one unit for which $\epsilon\mathrm{cyc}'_{q_1}(\zeta)$ is real. These choices are made before ϑ and agree with the Bessel period defining $\mathcal{P}$.

Write R_j for the graded Bessel subgroup, $G_{j,f} = \mathrm{U}(Y_j)(\mathbb{A}_{F,f})$, A_j for the stabilizer of its framed integral lattice, and $J_j \subset A_j$ for the subgroup reducing to one at the chosen places and at Q. For $x \in G_{j,f}$ let $\Lambda_j(x)$ be the lattice spanned by its translated frame, in rational coordinates; let $\Lambda_j^-(x)$ be its intersection with the string subspace. A lattice is graded if every grade projection of each of its vectors still belongs to it. Say that the orbit of x has bounded string lattice if some $r \in R_j(\mathbb{A}_{F,f})$ has $\Lambda_j(rx)$ graded and

$$\bar{\mathfrak{q}}^{\,j}\Lambda_j^-(1) \subseteq \Lambda_j^-(rx), \qquad \mathfrak{q}^{\,j}\Lambda_j^-(rx) \subseteq \Lambda_j^-(1). \tag{1}$$

Here are the functions to which the assertion applies. Fix a ramified source form φ, its coefficient covector t, real points ξ_j and finite points y_j, with y_0 integral and reducing to one at the chosen places. Write $a_j = (\xi_j, y_j)$ and let $f_{\vartheta,j}$ be the specified lower datum on $\mathrm{Hom}(Y_j, Y_{j+1})$. Let W_0 be the Bessel source function

$$W_0(uh) = \chi_0(u)\, t\big(\rho_0(h_\infty)^{-1}\varphi(h_f)\big), \qquad h \in \mathrm{U}(X_0)(\mathbb{A}_F),$$

using the unipotent–Levi decomposition of R_0. Recursively define, for $j < m-1$,

$$\Phi_{j+1}(h) = \int_{[R_j]} W_j(r) C_{\omega_{\vartheta,j}(a_j,1)f_{\vartheta,j}}(r,h)\, dr, \qquad W_{j+1}(r) = \Phi_{j+1}(ra_{j+1}). \tag{2}$$

The notation C is exactly the positive-degree coefficient constant term, and the character of step j is ν_{j+1}.

If $1 \le j < m$ and $\Phi_j(a,x) \neq 0$, then the orbit of x has bounded string lattice.

For each $0 \le j < m$ there is a finite set $O_j \subset G_{j,f}/J_j$ with the following properties. For $j = 0$ it is the image of the integral chosen level and put $\mu_0 = 1$. For $j > 0$ it is stable under $R_j(\mathbb{A}_{F,f}) \cap A_j$. Define

$$\mu_j(xJ_j) = \#\{r(R_j \cap A_j) \in R_j/(R_j \cap A_j) : r^{-1}xJ_j \in O_j\}, \tag{3}$$

where the groups in (3) are finite adelic groups. The counted set is finite. At each class of O_j the positive integer μ_j divides some power of q; it is nonzero on every orbit satisfying (1). Every representative of every class of O_j is integral at all places not dividing q.

Let $\iota_j : \mathrm{U}(X_j) \to R_j$ be the inclusion acting trivially on the string. The set O_j is stable under $\iota_j(K_j^X)$, and $\mu_j(\iota_j(b)xJ_j) = \mu_j(xJ_j)$ for $b \in \mathrm{U}(X_j)(\mathbb{A}_{F,f})$. Consequently, for a complex vector space V, the weighted functional

$$\mathscr{Q}_\vartheta(S) = \sum_{(x_0,\dots,x_{m-1}) \in \prod_j O_j} \left(\prod_{j<m} \mu_j(x_j)^{-1} \right) S(x_0,\dots,x_{m-1}) \tag{4}$$

commutes with every linear map on V, and is unchanged when any variable is translated by $\iota_j(K_j^X)$.

All the weights in (4) belong to $\mathbb{Z}[1/q]$. If Q is empty, then $\mu_j = 1$ on O_j, $|O_j| \le B_0 = M^{fn_m^2}$, and the sum of the absolute values of the weights in (4) is at most B_0^m.

Proof. All theta models and actions used below are the fixed models of Definition 3.2.12 and Definition 3.2.18, with their laws supplied by Lemma 4.1.1. The canonical theta functionals and kernels are (WC3)–(WC4) of Definition 4.1.15. In particular, (WC3) means that the point evaluation in the displayed definition of C is

$$[\rho([T])\omega(g,h)f]([p]) = \theta(\rho([p])\rho([T])\omega(g,h)f).$$

The rational projection onto the positive-degree matrix space pushes the invariant probability measure to its invariant probability measure. Thus this C is exactly the literal theta mean (DE4) of Lemma 4.2.27. Its zero term is the canonical zero term, with the same frames and additive character; no coefficient evaluation is substituted for theta evaluation.

We first record the finite-sum consequences of the cut properties. For a class occurring in (4), positivity and $\mu_j \mid q^{e_j}$ imply $\mu_j^{-1} \in \mathbb{Z}[1/q]$. Their product has the same property. A linear map commutes with (4) because its index set is finite. Translation by $\iota_j(K_j^X)$ permutes that index set and preserves its weights, so reindexing proves invariance. Multiplicity is constant on the whole R_j-orbit: left multiplication permutes the cosets counted in (3). If Q is empty, a positive divisor of a power of one is one. Thus the sum of absolute weights equals $\prod_j |O_j| \le B_0^m$.

The bound on $|O_j|$ in this case reduces to a finite reduction map. All its representatives are integral, and J_j contains the principal congruence group at the chosen level. Distinct classes in O_j therefore inject into the reduction of the integral unitary group, hence into the full matrix ring at those places. Its cardinal is at most $M^{f(\dim Y_j)^2} \le M^{fn_m^2}$.

Duality and the exceptional lattice alternative. All lattice duals in this paragraph are hermitian duals over $\widehat{\mathcal{O}}_E$. Write P_j for the string subspace and, for a self-dual ambient lattice L, put $L^- = L \cap P_j$ and $L^+ = \mathrm{pr}_{P_j} L$. Then

$$(L^+)^\# = L^-, \qquad (L^-)^\# = L^+. \tag{5}$$

The first equality follows by pairing an element of P_j with all of L: projection does not change that pairing, and $L^{\#} = L$. Dualizing gives the second; this can also be checked over each discrete valuation ring by extending a basis of the saturated submodule $L \cap P_j$ to an integral basis.

Put $\Lambda = \Lambda_j(1)$. Formula (HP5) gives $\Lambda^- = \Lambda^+$ when s_j is even. When s_j is odd, let z_j be its middle string vector. On each nonmiddle string line both lattices have generator $u_j v_i$; on the middle line they have respectively

$$\Lambda^-_{\mathrm{mid}} = \widehat{\mathcal{O}}_E u_j z_j, \qquad \Lambda^+_{\mathrm{mid}} = \widehat{\mathcal{O}}_E N^{-1} u_j z_j. \tag{6}$$

Indeed the only mixed generator in (HP5) is $\bar{\pi}_N^{-1} y_j + u_j N^{-1} z_j$; its projection gives the second generator, whereas belonging to the string forces its y_j-coefficient to vanish and leaves the first.

The prime N splits completely in E. Here is the arithmetic check needed for this assertion. The formulas for q_1, q_2 give $dq_1 \mid q_2$: at 2 the added exponent in q_1/q_0 is s, and at 3 it is $\lfloor \log_3 s \rfloor + 5$, both bounded by the exponents provided by q_0^{s+5}; the other factors already divide q_0, and the factor d divides q_2. As $E \subseteq \mathbb{Q}(\zeta_{dq_1})$ and $N \equiv 1 \pmod{q_2^2}$, reduction of the (dq_1)th roots of unity modulo N gives degree-one residue fields and no ramification. Moreover $\pi_N \bar{\pi}_N = N$ implies that at each conjugate pair of places above N, π_N has valuation one at precisely one member and zero at the other. The conductor idele and its conjugate are units at all these places, because q is prime to N.

Suppose now $L = \Lambda_j(x)$ satisfies $\bar{\mathfrak{q}}^j \Lambda^- \subseteq L^-$. By (5),

$$\mathfrak{q}^j L^+ \subseteq \Lambda^+. \tag{7}$$

If $\mathfrak{q}^j L^- \not\subseteq \Lambda^-$, choose $y \in L^-$ witnessing the failure. Equations (6)–(7) show that s_j is odd and that at some place $w \mid N$ the image of $\mathfrak{q}^j y$ generates the one-dimensional quotient $\Lambda^+_w / \Lambda^-_w$ over k_w. Since $\mathfrak{q}$ is a unit there, the lower bound gives $\Lambda^-_w \subseteq L^-_w$. Equation (7) then gives

$$L^+_w \subseteq \Lambda^+_w = \Lambda^-_w + \mathcal{O}_{E,w} \mathfrak{q}^j y \subseteq L^-_w.$$

The reverse inclusion always holds. Thus $L^+_w = L^-_w$: the string projection preserves L_w. By (5) the same equality holds at $\bar{w}$. Replacing w by its conjugate if necessary makes w divide π_N.

Let z_w be the norm-one scaling in the definition of ramification, and let d_w act by z_w on X_j and by one on the string. At $w, \bar{w}$ the equality $L^+ = L^-$ makes both orthogonal projections integral on L. Hence

$$d_w = 1 + (z_w - 1)\,\mathrm{pr}_{X_j}, \qquad d_w^{-1} = 1 + (z_w^{-1} - 1)\,\mathrm{pr}_{X_j}$$

preserve L there. Elsewhere both are the identity. Therefore $x^{-1} d_w x \in A_j$. It is the identity at the chosen and conductor places, so in fact $x^{-1} d_w x \in J_j$. This proves the lattice sandwich alternative and the exact conjugated-level condition needed to rule out its exceptional case.

Finite Heisenberg frequency calculation. All notation in this paragraph is local. Put $\mathcal{O} = \widehat{\mathcal{O}}_E$ and let $L_0 = \mathcal{O}^n$ be the standard finite coordinate lattice, self-dual for the additive character $\Psi(\tau) = \psi_{E,f}(\jmath \alpha \bar{\alpha} \tau)$. Let $H(v)$ denote the signed finite Heisenberg operator in these scaled coordinates. Direct substitution in the cocycle gives

$$H(y)H(z) = \Psi(\bar{z}^{\,t} y)\, H(z)H(y). \tag{F1}$$

Only this commutator, the fact that $H(l)$ fixes the vacuum for $l \in L_0$, and the zero-term invariance along an isotropic subspace are used below. The signs cannot be dropped from the individual operators.

The point datum for modulus generator $b_{\mathrm{mod}} = \varpi\mathfrak{q}$ is fixed by $H(b_{\mathrm{mod}}l)$ for every $l \in L_0$. To see this for the actual residue datum, expand its function on integral position coordinates by finite Fourier inversion on the quotient by b_{mod}. Every finite additive character of this quotient is represented by a momentum vector t with $\bar{b}_{\mathrm{mod}}\, t \in L_0$, because L_0 is self-dual. Extending the function by zero outside integral position coordinates therefore expresses the point datum as a finite linear combination of $H(t)v^0$ with $\bar{b}_{\mathrm{mod}}\, t \in L_0$; coefficients and the fixed phase of each summand do not affect the assertion. Since $b_{\mathrm{mod}}l \in L_0$, it fixes v^0, and (F1) shows that it commutes with every such $H(t)$: the argument $b_{\mathrm{mod}}\bar{t}^t l$ is integral. This proves the required invariance without an assumption on the real vector or on the residue function beyond its defining modulus.

Here is the precise zero-term support argument. Let I be an isotropic subspace with finite coordinate basis matrix K_I, and let Z_I be its finite zero term. In a polarization containing I, this is restriction to zero in the position coordinates dual to I, retaining the complementary position coordinates. Consequently

$$Z_I H(u) = Z_I \qquad (u \in I). \tag{F2}$$

For matrices M, M' with $M(M')^* = 1$, suppose $H(Ml)u_{\mathrm{fin}} = u_{\mathrm{fin}}$ for every $l \in L_0$, and $Z_I H(z)u_{\mathrm{fin}} \neq 0$. Choose a change of basis T_I of I such that

$$P = (M')^* K_I T_I$$

is integral and primitive, and let B be an integral left inverse of P. Such a basis exists place by place: the intersection of the subspace $(M')^*I$ with L_0 is saturated in the free module L_0 over a discrete valuation ring, hence is a direct summand. Only finitely many places require a nonintegral change of basis, so these choices give finite adelic matrices.

For integral column v_I, the vector $u = K_I T_I v_I = MPv_I$ is a period of u_{fin}. It belongs to I, so (F1)–(F2) imply that the nonzero vector $Z_I H(z)u_{\mathrm{fin}}$ is fixed by the scalar $\Psi(\bar{v}_I^t (K_I T_I)^* z)$. Self-duality of L_0 implies $v = (K_I T_I)^* z \in \mathcal{O}^{\dim I}$. Put $l = B^* v \in L_0$. Then

$$(K_I T_I)^*(z - M'l) = v - P^* B^* v = 0, \qquad \text{hence} \qquad K_I^*(z - M'l) = 0. \tag{F3}$$

This proves $z \in M'L_0 + I^\perp$ by an explicit integral solution, including dyadic places.

Apply this to the positive-degree isotropic subspace of $\mathrm{Hom}(Y_\ell, Y_{\ell+1})$. Let g be the finite action of the pair of lattice frames and group points on Hom coordinates. The translated datum has periods ML_0 with $M = b_{\mathrm{mod}}g$; take $M' = \bar{b}_{\mathrm{mod}}^{-1}g$. A nonzero theta constant term implies that its finite zero term at the string map is nonzero. In fact choose a rational Witt splitting containing the positive-degree subspace and use its polarized theta series. Integration over that isotropic adelic quotient annihilates each nonzero rational dual-coordinate frequency; the surviving term is precisely the theta series of the restricted vector. For a real–finite pure tensor, restriction acts separately on the two factors, so zero finite restriction would make the whole constant term zero. Finite sums follow by linearity; the actual lower datum has one fixed real factor.

Write T for the rational string map and F_-, F_+ for the two finite frames including the group points. Transport (F3) to rational coordinates. The Hom scaling is α, so the difference has the form

$$\alpha^{-1}T - \bar{b}_{\mathrm{mod}}^{-1}W, \qquad W = F_+ A F_-^{-1}, \quad A \in \mathrm{Mat}(\mathcal{O}).$$

It is orthogonal to all positive-degree entries. Opposite entries are paired perfectly by the antidiagonal string Gram matrices; hence its negative-degree entries vanish. Thus W maps the

preceding lattice to the next lattice and

$$W - \frac{\bar{b}_{\mathrm{mod}}}{\alpha} T \quad \text{has no negative-degree component.} \tag{F4}$$

For Bessel translates r, r' replace W by $r'Wr^{-1}$ in rational coordinates. Both Bessel matrices have nonnegative degree, and their degree-zero actions fix the string basis and preserve the definite block. The latter assertion for negative string grades follows by adjointness from the defining fixed positive grades. Since T kills the definite block and has degree -1 on the string, the degree--1 part of the new matrix is still $(\bar{b}_{\mathrm{mod}}/\alpha)T$.

If the translated lattices are graded, every homogeneous part of a lattice-preserving map preserves them: its degree-e part on a vector v is the finite sum

$$\sum_d \mathrm{pr}'_{d+e}\, W\, \mathrm{pr}_d\, v.$$

Therefore $(\bar{b}_{\mathrm{mod}}/\alpha)T$ maps the preceding lattice into the next lattice. It kills the preceding definite block and lands in the next string. With $\beta = \alpha\bar{\varpi}^{-1}$, this is exactly

$$\frac{\bar{\mathfrak{q}}}{\beta}\, T(L_\ell^+) \subseteq L_{\ell+1}^-. \tag{F5}$$

The theta-mean identity at the beginning of the proof makes this calculation applicable to the constant term in (2).

The reference image and one conductor step. Write $\beta = \alpha/\bar{\varpi}$ for the fixed idele of (HP5). Its ideal satisfies

$$c\widehat{\mathcal{O}}_E = \beta\bar{\beta}\widehat{\mathcal{O}}_E. \tag{F6}$$

Indeed $\jmath = -1/\sqrt{-d}$ gives $c = (M\epsilon \mathrm{cyc}'_{q_1}(\zeta))^{-1}$; the self-dual coordinate lattice gives $\alpha\bar{\alpha}\widehat{\mathcal{O}}_E = \mathrm{cyc}'_{q_1}(\zeta)^{-1}\widehat{\mathcal{O}}_E$, and $\varpi\bar{\varpi}\widehat{\mathcal{O}}_E = M\widehat{\mathcal{O}}_E$. One may also check the middle equality directly at primes of the cyclotomic different: they lie above primes dividing q_1, hence q_0; these split in K by the congruences on d (including $d \equiv 7 \pmod 8$ at two), and the definition of α chooses the inverse different on one member of each conjugate pair.

The string map has the exact lattice image

$$T(\Lambda_\ell^+) = \beta\Lambda_{\ell+1}^-. \tag{F7}$$

Here is the parity check. For odd s_ℓ, $u_{\ell+1} = u_\ell$. The positive-half generators have image ideal u_ℓ, the negative-half generators have image ideal cu_ℓ, and the middle projection generator u_ℓ/N has image $-cu_\ell$. Multiplication of the next even lattice by β gives the same ideals, using (F6). For even s_ℓ, $u_\ell/u_{\ell+1} = \beta/\bar{\beta}$. The positive-half generator u_ℓ/β has image cu_ℓ/β; the negative-half generator $u_\ell\bar{\beta}$ has unchanged image. Both ideals equal $\beta u_{\ell+1}\widehat{\mathcal{O}}_E$ by (F6), including the middle line in the next odd string. These are precisely its intersection generators.

The preceding upper bound and (5) therefore give

$$\bar{\mathfrak{q}}^{\,\ell}\Lambda_\ell^+ \subseteq L_\ell^+, \qquad \bar{\mathfrak{q}}^{\,\ell+1}\Lambda_{\ell+1}^- = \frac{\bar{\mathfrak{q}}}{\beta} T(\bar{\mathfrak{q}}^{\,\ell}\Lambda_\ell^+) \subseteq L_{\ell+1}^-. \tag{F8}$$

This costs exactly one conductor power. The exceptional alternative already proved will supply the upper bound; the following argument rules out its exceptional case.

Ramified cyclic-average annihilation. Let $z = z_w$ be the actual integral norm-one scaling supported at $w, \bar{w}$ above N. Write $R_z\varphi(g) = \varphi(gz)$ on each definite group in the upper

chain. Every lift character, and the splitting character, is trivial on these finite units: the tame parts in Definition 3.2.26 are supported at the chosen places, those of the twist at Q, and both sets are disjoint from the places over N. The scalar pair (zI, zI) acts trivially on Hom coordinates, and its character multiplier is one. Thus

$$\omega(zI, zI) = 1, \qquad \omega(z^{-1}I, 1) = \omega(1, zI). \tag{R1}$$

For the actual upper theta integral $T_{F_{\mathrm{edge}}}$ with finite datum F_{edge}, change the integration variable by the central finite unit z. Haar invariance and (R1) give

$$T_{F_{\mathrm{edge}}}(R_z\varphi) = T_{\omega(1,zI)F_{\mathrm{edge}}}\varphi = R_z(T_{F_{\mathrm{edge}}}\varphi). \tag{R2}$$

The real vector is unchanged, since z has trivial infinite component. These equalities hold tensor-linearly for a finite sum of edge data; no decomposability assumption is used for an evaluated pair.

Consequently the upper chain Γ_Y associated to an evaluated tensor Y commutes with the simultaneous R_z on its source and target. If its final tensor factor satisfies

$$(1 \otimes \cdots \otimes \omega(1, zI))Y = Y, \tag{R3}$$

then its output is fixed by R_z, by applying the last equality in (R2) only at that last edge. Hence for every integer i, $\Gamma_Y(R_z^i\varphi) = \Gamma_Y\varphi$. The SAME source ramification condition is $\sum_{i=0}^{N-2} R_z^i\varphi = 0$. Applying Γ_Y gives

$$0 = (N-1)\Gamma_Y\varphi, \qquad \Gamma_Y\varphi = 0. \tag{R4}$$

No claim that a ramified vector is an eigenvector for a specified nontrivial character is required.

The norm-N sandwich already proved in the target supplies $x^{-1}\iota_j(zI)x \in J_j$ in the exceptional case. Assertion (DE8) of Lemma 4.2.27 converts this conjugated-level condition to (R3): moving the final lower point by $\iota_j(zI)$ is also right multiplication by an element of J_j, which fixes that lower point datum; its evaluated image is unchanged. On the other hand equivariance identifies its evaluated image with the last-factor action in (R3), with no scalar. Assertion (DE6) of Lemma 4.2.27 identifies the even-stage recurrence value with the coefficient of this SAME upper chain and these SAME evaluated tensors. Equation (R4) therefore makes the recurrence value zero.

Graded representatives. We use both assertions of Lemma 3.2.6. Their field hypothesis holds here: $5040 \mid q_1$, since $18480 \mid q_0$ and $9 \mid T(s)$, so the prescribed cyclotomic generator supplies a primitive 5040th root in the same E. All matrices in this paragraph are in rational coordinates unless specified otherwise, and A^* means conjugate transpose. Let J be the rational Gram matrix of Y_j, let $L = \Lambda_j(x)$, and let w_i be the grades. The form J and its inverse pair only opposite grades.

Choose a lattice basis matrix B that does not lower grades. Such a basis is obtained locally by choosing bases successively in the saturated intersections of L with the coordinate flag of grades at least d. These intersections and their successive quotients are free over each discrete valuation ring, so the choices extend to a basis of L. Only finitely many places require nonstandard choices; they give an adelic basis. Its inverse also does not lower grades, since it is block triangular. If F_x is the orthonormal frame of L, then $K_L = F_x^{-1}B$ lies in $\mathrm{GL}(\widehat{\mathcal{O}}_E)$. Put

$$G = K_L^*K_L = B^*JB, \qquad G_{0ab} = \begin{cases} G_{ab}, & w_a + w_b = 0, \\ 0, & w_a + w_b \neq 0. \end{cases}$$

Here $G_{ab} = 0$ if $w_a + w_b > 0$, and $(G^{-1})_{ab} = 0$ if $w_a + w_b < 0$, by the triangularity of B, B^{-1} and the opposite-grade property of J, J^{-1}. Extracting degree zero from $G^{-1}G = 1$ shows that

the opposite-grade part of G^{-1} is an integral inverse of G_0. Thus G_0 is integral, hermitian and invertible.

By the first assertion of Lemma 3.2.6, choose an integral invertible M_0 with $M_0^* G_0 M_0 = 1$. The columns of M_0^{-1} and K_L indexed by nonnegative grades are primitive integral matrices with the same Gram matrix: for these indices $G_0 = G$, since their weight sum is nonnegative. The second assertion supplies $U \in \mathrm{U}(\widehat{\mathcal{O}}_E)$ matching those columns. Set

$$N_0 = K_L^{-1} U M_0^{-1}.$$

Then N_0 is integral invertible, fixes every nonnegative-grade basis vector, and $N_0^* G N_0 = G_0$. Moreover $N_0 - 1$ strictly raises grades. To check the only nonimmediate case, take $w_b < 0$ and $w_a \le w_b$, and use

$$N_0 - 1 = G^{-1}(GN_0 - G).$$

A nonzero summand of its (a, b) entry requires $w_c \ge -w_a \ge -w_b > 0$. The cth column of N_0 is fixed, so the cth row of GN_0 is the cth row of G_0. Since $w_c + w_b \ge 0$, that row entry equals G_{cb}. Each summand is therefore zero.

Let D_0 be the degree-zero block of B. It is invertible and $D_0^* J D_0 = G_0$. Thus

$$V_0 = B N_0 D_0^{-1}, \qquad V_0^* J V_0 = J,$$

and $V_0 - 1$ strictly raises grades. Hence V_0 is in the finite unipotent subgroup of R_j. Since BN_0 is another basis of L, the lattice $V_0^{-1} L = D_0 \widehat{\mathcal{O}}_E^{\dim Y_j}$ is graded. This gives a graded representative in every Bessel orbit.

From string intersections to orbit representatives. Suppose two self-dual lattices L, L' have the same intersection with the string subspace P_j. Choose a basis matrix C for that common intersection. It is primitive in each ambient lattice, because intersection with a vector subspace is saturated. In orthonormal frames $F_L, F_{L'}$ for L, L', the matrices $F_L^{-1} C, F_{L'}^{-1} C$ therefore have integral left inverses. Their Gram matrices both equal $C^* J C$. The second assertion of Lemma 3.2.6 gives an integral unitary U matching them. The isometry

$$d = F_{L'} U F_L^{-1}$$

fixes C pointwise and sends L to L'. It fixes the string subspace pointwise, hence preserves its orthogonal complement X_j and belongs to the image of $\mathrm{U}(X_j)(\mathbb{A}_{F,f})$ in R_j. Thus equal string intersections are in the same Bessel–A_j double coset.

There are only finitely many string intersections satisfying (1), by the finite-quotient argument below. Choose one point for each intersection that occurs. Replace this point's components outside $Q \cup \bar{Q}$ by one. This still gives a unitary adelic point, since that set is conjugation stable. Its string intersection is unchanged: outside that set the two bounds in (1) coincide with the reference intersection, while inside it the components are unchanged. The preceding integral matching therefore puts every bounded orbit in a double coset represented by one of these finitely many points, all supported on $Q \cup \bar{Q}$. These are the representatives C_0 used in the cut construction below.

Support induction. The functions on R_j that occur in (2) have the required genericity: W_0 does by its definition, and Lemma 4.2.26 preserves that property at every step and after every fixed right translate. The integrals are absolutely convergent by Proposition 4.2.17 and the recurrence assertion of Lemma 4.2.27. A nonzero integral consequently has a point at which both factors are nonzero.

Start with the preceding lattice at such a point of the first integral. Its finite point is ry_0 for $r \in R_0$, and y_0 is integral. Hence its Bessel orbit contains the reference lattice. At a subsequent

step, nonvanishing of the preceding factor gives the preceding two bounds by induction. Choose its graded representative as above, and choose a graded representative of the next lattice by the same construction. The Bessel changes do not change the degree--1 term in (F4). Formula (F8) gives the next lower bound.

If the next upper bound fails, the exceptional alternative forces the next string length to be odd. Its index j is then even, since m is even and $s_j = m - 1 - j$. All grades of this space are even. The grading correction constructed above strictly raises grades, hence raises them by at least two; it belongs to $U_j^{(2)}$. Genericity therefore shows that the recurrence value at the graded representative is the original nonzero value multiplied by a nonzero character. At that representative the exceptional alternative gives $x^{-1}\iota_j(z_w)x \in J_j$. Apply (DE6)–(DE8) of Lemma 4.2.27 to this even index, taking its last finite point to be x and its last real point to be the real component at which the recurrence is evaluated. These last choices do not alter the earlier recurrence defining Φ_j. The upper tensors are the actual evaluations (DE3), and (DE8) gives (R3) for the last factor with scalar one. Equation (R4) contradicts nonvanishing. Thus the upper bound also holds. This proves (1) for every nonzero recurrence value, with the same constant c, the same reference lattices and exactly the exponent j.

The block level preserves the lattice. For completeness, the subgroup K_j^X in the assertion consists of integral unitary block matrices whose last row is the last standard row modulo π_N, with reduction one at the chosen places. Its inclusion lies in $R_j \cap A_j$. For an even string this follows at once from (HP5), since the definite block is unchanged and the string block is fixed. For an odd string, conjugate $\mathrm{diag}(g, 1)$ by the mixed basis in (HP5). The only possible denominators come from its last definite column, divided by $\bar{\pi}_N$, and the corresponding mixed middle row, divided by N. The row condition for g and g^{-1}, together with $g^{-1} = g^*$, gives the corresponding column condition modulo $\bar{\pi}_N$. The last diagonal entry of $g - 1$ is divisible by both π_N and $\bar{\pi}_N$, hence by N: these ideals have disjoint supports. These three congruences cancel precisely those denominators. The same argument for the inverse proves preservation, rather than just containment, of the framed lattice. This proves the claimed inclusion; the cut constructed below is therefore stable under this block level.

Bounded string intersections. Let P_j be the string subspace and $\Lambda = \Lambda_j(1)$. An intersection L satisfying (1) lies between $\bar{\mathfrak{q}}^j(\Lambda \cap P_j)$ and $\mathfrak{q}^{-j}(\Lambda \cap P_j)$. The quotient of the latter lattice by the former is finite: locally it is a finite module, and it is zero outside the finite set of places dividing the conductor and its conjugate. Each possible L is the inverse image of a submodule of this finite quotient, so there are finitely many possible intersections. Outside that set the two bounding lattices coincide.

Construction of a cut from bounded-orbit representatives. We prove the group-theoretic part explicitly. Suppose a finite set $C_0 \subset G_{j,f}$ meets, modulo R_j on the left and A_j on the right, every orbit under consideration, and each element of C_0 is integral outside the conductor and its conjugate. Put $\mathscr{D}_j = C_0 A_j$, a compact set. Its image in the discrete space $G_{j,f}/J_j$ is finite. Choose from this image one class in each R_j-orbit meeting it, obtaining C, and put

$$O_j = (R_j \cap A_j)C.$$

This is finite because it lies in the image of the compact set $A_j\mathscr{D}_j$ in $G_{j,f}/J_j$. It is stable under $R_j \cap A_j$, meets every required orbit, and all its representatives are integral at every place not dividing q.

For $x \in \mathscr{D}_j$ there is an integer n_x such that

$$(xux^{-1})^{q^{n_x}} \in A_j \qquad (u \in J_j).$$

Indeed away from the conductor and its conjugate, x and u are integral. At a conductor place,

u is congruent to one modulo the maximal ideal. If $U = 1 + B$ with every entry of B in the rth power of that ideal, $r \geq 1$, then every entry of $U^q - 1$ lies in its $(r+1)$st power: the linear term is qB, and all higher terms have valuation at least $2r \geq r + 1$. Iteration makes u^{q^n} arbitrarily close to one, uniformly in u. Choose n large enough that conjugation by the fixed x is integral at each of the finitely many conductor places. At the conjugate places the same integrality follows from the unitary relation.

For $t = xJ_j \in C$ write $\mathrm{K}_{R,j} = R_j \cap A_j$ and $Z = R_j \cap xJ_jx^{-1}$. Since C meets the orbit only once, the right cosets counted by (3) are exactly the Z-orbit of the identity coset in $R_j/\mathrm{K}_{R,j}$. Thus

$$\mu_j(t) = [Z : Z \cap A_j].$$

This index is finite because Z is compact and $Z \cap A_j$ is open. Every $z \in Z$ satisfies $z^{q^{n_x}} \in Z \cap A_j$. Applying the same assertion to all its conjugates in Z puts that power in the normal core of $Z \cap A_j$. The resulting finite quotient has every element of order dividing q^{n_x}. By Cauchy's theorem every prime divisor of its order divides q, so its order, and hence the displayed index, divides a power of q. In particular the multiplicity is positive. Translation in R_j preserves the counted cosets bijectively, proving the assertion on the entire orbit. If $q = 1$, this quotient is trivial and the index is one. This completes the cut construction from the finite representatives proved above, without a primeness assumption on q.

All the required support, stability, multiplicity and weight assertions now follow from the preceding constructions. □

Lemma 4.2.32. *Fix $0 \leq i < m/2$, put $j = 2i$ and $B_0 = M^{f n_m^2}$. There is a scalar $c_i \in \mathbb{C}$, chosen before every ordinary twist ϑ and every tuple of finite outer points, with the following property. Let ϑ have modulus Q and $q = \mathrm{N}Q$, and suppose all y_l are integral at the places not dividing q. There is a finitely supported family $(r_{i,\vartheta,y}(x,x'))$ such that*

$$\mathcal{Y}_{\vartheta,y,i} = c_i \sum_{x,x'} r_{i,\vartheta,y}(x,x')\, \mathcal{G}_i(x,x'), \tag{FE1}$$

where

$$\mathcal{G}_i(x,x') = \sum_{\eta \in \prod_{w \mid \pi_N} k_w^{n_{j+1}}} H_j(z_j(\eta))H_j(x)v_j^{\mathrm{int}} \otimes H_{j+1}(z_{j+1}(\eta))H_{j+1}(x')v_{j+1}^{\mathrm{int}}.$$

Here $z_j(\eta)$ has last column $-\eta/N$ at the selected places above N, $z_{j+1}(\eta)$ has its conjugate last row at their conjugates, and the residue representatives are the fixed $0, \dots, N-1$. In particular this is the literal gluing vector of Proposition 4.2.25, without division by its number of terms.

Every translation occurring in (FE1) satisfies $\bar{\pi}_M x, \bar{\pi}_M x'$ integral at all places not dividing q. For every prime identification above $\ell \nmid q$, every $B_0^4 r_{i,\vartheta,y}(x,x')$ is integral. If Q is empty, the coefficients can be chosen with

$$\sum_{x,x'} |\sigma r_{i,\vartheta,y}(x,x')| \leq 1 \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})). \tag{FE2}$$

Furthermore one may require $x_w = x'_w = 0$ whenever $w \nmid Mq$. The passage to this support is an equality of the individual glued vectors, not merely equality modulo a kernel or equality up to a scalar. The scalar c_i is allowed to vanish; no nonvanishing of this evaluation is asserted.

Proof. Write $\mathcal{O}_f = \widehat{\mathcal{O}}_E$. All finite coordinates in this proof are the coordinates of (WD8), so the vacuum lattice is $\mathcal{O}_f$ and the integral Heisenberg operator is the stated H. We use the actual map (DE2) of Lemma 4.2.27; in particular the polarization, stacked shift and residual output are not chosen anew.

Finite Fourier inversion. Let $\mathfrak{m}$ be a product of selected split level places and let $R_{\mathfrak{m}}$ be its residue ring. On a finite position quotient of dimension d_0, the indicator of a specified residue class is

$$\mathbf{1}_{\xi_0}(\xi) = |R_{\mathfrak{m}}|^{-d_0} \sum_{\omega \in (R_{\mathfrak{m}}^{d_0})^\vee} \omega(\xi - \xi_0). \tag{F1}$$

This is character orthogonality, including the zero-dimensional case. Under the fixed polarized datum map, each character is a Heisenberg translate of the vacuum: modulation is multiplication by that character and translation moves its position class. The scalar is its displayed value at the selected class and is a root of unity. This identification uses the fixed transport of Lemma 4.1.3 and its exact action Lemma 4.1.16. Hence (F1) is an identity of the actual coherent vectors.

Separate the chosen places dividing M from the places of Q. The latter contribute only factors whose denominators divide powers of q. At a chosen place the dual quotient is killed by $\bar{\pi}_M$: self-duality of the integral lattice identifies its character group with the inverse-conjugate-level lattice modulo the integral lattice. Thus all its translation representatives t obey $\bar{\pi}_M t \in \mathcal{O}_f^{d_0}$ away from q, and

$$\mathrm{N}(\pi_M)^{d_0} \text{ times every coefficient is integral away from } q. \tag{F2}$$

Here $\mathrm{N}(\pi_M) = M^f$. At $Q = \emptyset$ the sum of the absolute values of all conjugate coefficients in (F1) is one, since there are exactly $|R_{\mathfrak{m}}|^{d_0}$ roots of unity divided by that same integer.

Moving the two point data by the prescribed finite outer points preserves this assertion, allowing denominators at q. Indeed away from q these points are integral; the vacuum is fixed and Heisenberg covariance simply moves a translation by an integral invertible matrix. At the places above q, apply the finite vacuum formula of Lemma 4.1.13: it is a finite sum over the quotient of the integral lattice by its intersection with its transform. The order of this quotient is a power of primes dividing q, and all translation representatives are integral away from q. Its multiplier has valuation zero at every other coefficient prime. Here this last assertion also follows directly from the literal multiplier: its absolute norm factor has only primes dividing q, and its splitting-character factor is an algebraic unit away from q. To see the latter without a transcendental assumption, take a power making the fractional ideal of the idele principal. Multiply by its generator; the remaining finite idele is integral and invertible, on which the splitting character is one. The remaining archimedean expression is a product of algebraic numbers divided by square roots of their norms. Those norms are global units away from q, so the expression, and every root of it, is a unit there. Tame-character values are roots of unity. If $Q = \emptyset$, the moved points are integral everywhere; the finite vacuum formula has index one and multiplier one, and these changes preserve the coefficient sum in (FE2).

It remains to evaluate a translated joined vacuum. We do this in the same projected lattice as the fixed zero term.

The projected lattice and its fixed line. Let I be the isotropic subspace removed by (DE1), let L be the integral joined lattice, and let i_0 denote the inclusion of the residual space before applying the definite joining inverse. The residual projected lattice is

$$L_I = \{w : \ i_0 w + \xi \in L \text{ for some } \xi \in I(\mathbb{A}_{E,f})\}. \tag{F3}$$

In (F3) the vector on the right is in $I^\perp$; this is projection of $L \cap I^\perp$ along I. It is a self-dual lattice for the residual pairing. Locally $L \cap I$ is primitive, since it is an intersection with a vector subspace. A primitive isotropic basis can be completed to hyperbolic pairs inside a unimodular hermitian lattice: first choose dual vectors by unimodularity, then subtract their mutual hermitian products in triangular order; to make a dual vector isotropic subtract $c_0 h(w, w)$

times its isotropic partner. The identity $c_0 + \bar{c}_0 = 1$ makes its new norm zero, including in residue characteristic two. Orthogonal projection along these pairs is then an integral orthogonal summand, and its Gram matrix is unimodular. This proves the assertion about L_I.

A zero term of the vacuum is fixed under the residual $H(L_I)$: lift its translation to $L \cap I^{\perp}$, use that $H(L)$ fixes the original vacuum, and change the isotropic mean variable. The fixed subspace is one-dimensional. In position coordinates, invariance under integral modulations restricts support to the dual position lattice and invariance under its translations makes the function constant there. Self-duality therefore leaves precisely one lattice vacuum line. This also proves the assertion directly when a removed or residual rank is zero.

Put $L_0 = \mathcal{O}_f^{(n_j+n_{j+2})n_{j+1}}$ in the residual definite joined coordinates. The vector

$$v_I = \sum_{r \in L_I/(L_I \cap L_0)} H(r) v^0 \tag{F4}$$

spans that fixed line. Summands are independent of representatives, and the Heisenberg law reindexes the sum under $H(L_I)$. It is nonzero: its pairing with v^0 has a contribution only from the zero coset, equal to one. Consequently the zero term of the original vacuum is one scalar times (F4). That scalar depends only on the fixed splitting and frames. It may be zero. Neither the point datum nor any twist has yet been selected.

For a translated vacuum, character orthogonality in the eliminated variables gives zero unless the translation lies in $L + I^{\perp}$. If it survives, write its translation as $l + i_0 w + \xi$ with $l \in L$ and $\xi \in I$. Integral l fixes the vacuum, ξ disappears in the mean, and the remaining vector is a root of unity times $H(w)v_I$. The phases here are roots of unity because the finite additive character is evaluated on finite adeles modulo the integral lattice. Replacing w by another such representative changes it only by L_I and changes only that root of unity.

The actual neighbor lattice. We identify (F3), rather than replace it by a different self-dual lattice. Put $\delta = \varpi/\bar{\varpi}$, with ϖ from (HP5). The negative first frame has coefficient $\bar{c}\,\bar{\alpha}/\alpha$. Thus, for the string scales in (HP5),

$$(\bar{c}\,\bar{\alpha}/\alpha) u_j = \bar{c}\,\delta u_{j+2}. \tag{F5}$$

In the middle case use the plane basis in (DE1). In the last case use z_j/π_N and put the extra factor δ on this last residual target coordinate; call this diagonal factor κ. In the middle case $\kappa = 1$. It is unitary, and $\bar{\pi}_M \delta$ has the same valuations as π_M. These facts follow immediately from $\delta = \varpi/\bar{\varpi}$ and from the support of ϖ.

For clarity the membership calculation in (F3) can be made one column at a time. There is a middle-string coordinate b with $\delta b = b$. Apply the following congruences to the unscaled vector $\bar{\kappa} x$. Coordinates other than the two last definite coordinates must be integral. The last coordinates x_-, x_+ obey

$$\bar{\pi}_N x_{\pm} \in \mathcal{O}_f, \qquad b - \pi_N^{-1} x_- \in \mathcal{O}_f, \qquad \bar{c} b - \pi_N^{-1} x_+ \in \mathcal{O}_f. \tag{F6}$$

In the final case read the positive last coordinate after κ. To verify (F6), insert the two columns $\bar{\pi}_N^{-1} y + u_j N^{-1} z$ and $u_j z$ of (HP5) in the negative block and their corresponding positive columns. Their coefficients on the removed plane agree exactly when the two expressions in (F6) are integral; (F5) identifies their middle coordinate. The even middle string columns contribute no further condition because they are paired by inverse-conjugate scales. All other coordinates are the unchanged integral basis. For the final vector its scale π_N^{-1} gives the same last condition using $\pi_N \bar{\pi}_N = N$. This proves necessity and sufficiency, not only an inclusion of lattices.

At the selected places above N, (F6) leaves one residue vector $\eta \in \prod_{w|\pi_N} k_w^{n_{j+1}}$. A representative in the residual joined space is the vector $n(\eta)$ whose first last row is $c\bar{\eta}/N$ and whose

second last row is $-\bar{\eta}/N$, supported at the conjugates of the selected places. The two congruences in (F6) verify this representative and show that subtracting it leaves the integral lattice at all those places. At the other places write

$$A_M = (L_I \cap \ker e_N)/(L_I \cap L_0 \cap \ker e_N),$$

where e_N retains all components above N. Choose representatives r_ν there, zero above N. These represent every remaining coset in (F4). On the group $U = L_I \cap \ker e_N$, reduce $\overline{\bar{\kappa} x}$ modulo π_M. This is defined because $\bar{\kappa} x$ is integral away from N, and x is zero above N. If its reduction is zero, write

$$\overline{\bar{\kappa} x} = \pi_M z, \qquad x = \bar{\pi}_M \kappa \bar{z}, \qquad z \in \mathcal{O}_f^{(n_j+n_{j+2})n_{j+1}}.$$

Since $\bar{\pi}_M \kappa$ is integral, the kernel K is contained in $H = U \cap L_0$. Consequently $[U : H]$ divides $[U : K]$, which is the order of a subgroup of the residue matrix group. Thus

$$|A_M| \;\mid\; \mathrm{N}(\pi_M)^{(n_j+n_{j+2})n_{j+1}}. \tag{F7}$$

This is an index argument through $K \subset H$; no injection of U/H into that residue group is asserted. It also shows that representatives satisfy the required bound after the signed-adjoint conversion: their first-block denominator is π_M, their second-block denominator is $\bar{\pi}_M$.

The supports of $n(\eta)$ and r_ν lie in disjoint conjugation-stable sets of places, so their cross phase is one. The individual $n(\eta)$ are supported on only one side of each split pair and their mutual Heisenberg phases are also one. Undoing the signed adjoint on the first residual factor gives

$$c\,\overline{c\bar{\eta}/N} = -\eta/N.$$

The second factor remains $-\bar{\eta}/N$. Thus (F4) becomes exactly the unnormalized gluing sum in the statement, followed by the remaining translations r_ν; it has no normalization by the number of η. Write the sum over ν as its average and include $|A_M|$ in the fixed scalar. Its coefficients have conjugate absolute sum one and denominator dividing (F7).

We record explicitly how a translated input is projected, since projecting an arbitrary denominator lattice without this calculation would be invalid. Form the matrix B whose columns lift the columns of the just-constructed neighbor unitary $G = (\kappa\Gamma) \otimes 1$ by the isotropic coefficients b in (F6), and whose middle rows are the grade-zero rows of the middle frame. In the decomposition of the splitting it has the form

$$B = i_0 G + K_I Q, \qquad B^\dagger K_I = 0, \qquad B^\dagger \Sigma_{i,f} = 0. \tag{F8}$$

Here K_I consists of the removed isotropic columns. The first equality is precisely the two congruences (F6); all entries of B are integral by that calculation and the middle lattice basis (HP5). The second follows from isotropy and orthogonality. For the third, the stacked string map has nonzero degree-zero target columns only at the middle grade-one string coordinate, whereas B uses the middle grade-zero coordinates; the remaining positive entries pair only with the removed coordinates. Their product is zero entry by entry.

Remove the input translation above N first. It is integral there because its denominator divides a power of M and the twist conductor, both prime to N; conjugation-stable support decomposition gives equality on the input vacuum. Write the resulting translation as y, zero above N. The stacked shift has no residual contribution by (F8), and

$$i_0^\dagger y = G B^\dagger (\Sigma_{i,f} + y). \tag{F9}$$

For (F9), expand the first equality of (F8). The term $Q^\dagger K_I^\dagger y$ vanishes: y is zero above N, and the extra isotropic coefficients are supported there. Use $GG^\dagger = 1$. This proves the formula for the projected translation.

The same integral-column calculation proves the denominator assertion in (F9). Put $d_a = \pi_M$ on the first joined block and $d_a = \bar{\pi}_M$ on the second. A nonzero entry B_{ab} has, away from N, matching weights $\mathrm{val}(d_a) = \mathrm{val}(d_b\kappa_b)$, by (F5) and the unchanged coordinate columns; at N all these weights are units. An entry with unequal weights is zero at the conjugate place. Since B is integral, summing $\bar{B}_{ab}y_a$ therefore gives $d_b\kappa_b(B^\dagger y)_b$ integral away from q. Above N the translation is zero, and away from N the neighbor matrix Γ is the identity. Thus $i_0^\dagger y = \kappa B^\dagger y$ has the required first/second block weights. Undoing the signed adjoint changes the first weight to $\bar{\pi}_M$, as required.

For completeness, the finite zero-term translation rule used with (F9) follows from the integral columns B, not from continuity alone. In position coordinates separate the residual coordinates given by B from their orthogonal hyperbolic coordinates. Integral residual columns give the same lattice position indicator in those coordinates. Averaging the eliminated modulation characters is zero unless their restrictions to the relevant compact lattice are trivial. If they are trivial, the integral-lattice translation removes their position coordinates and contributes their single additive-character value, a root of unity. The residual position vector is exactly $GB^\dagger(\Sigma_{i,f} + y)$. Hence

$$Z_{i,f}H(\Sigma_{i,f})H(y)v^0 = \zeta\, H(i_0^\dagger y)Z_{i,f}v^0, \qquad \zeta = 0 \text{ or a root of unity.} \tag{F10}$$

The Heisenberg multiplication factor between the two input translations is another root of unity and is included in ζ. Formula (F10), followed by (F4), proves that every allowed translated input uses the SAME vacuum zero-term scalar. It also covers a zero result. This completes the actual shifted-translation check needed to apply the expansion term by term to (F1).

Denominators, size and the fixed scalar. The two point-data Fourier expansions use

$$d_1 = (\dim Y_j)(\dim Y_{j+1}) + (\dim Y_{j+1})(\dim Y_{j+2})$$

residue coordinates; the neighbor quotient uses $d_2 = (n_j + n_{j+2})n_{j+1}$ coordinates. Every rank here is at most n_m, including the smaller final lower target. Thus $d_1 + d_2 \le 4n_m^2$. Equations (F2),(F7) give the denominator $\mathrm{N}(\pi_M)^{4n_m^2} = B_0^4$, with all extra denominators supported on q. When $Q = \emptyset$, each Fourier average has conjugate absolute coefficient sum one, as does the neighbor average. The sum for their tensor product, and for its finite pushforward, is at most the product, hence at most one. This proves (FE2); collisions of translations only decrease this upper bound by the triangle inequality.

All constants absorbed so far are the fixed zero-term scalar, the fixed neighbor average cardinality, fixed frame normalizations, and fixed normalizations of the two vacua. Replacing v^0 by v^{int} in the output divides the scalar by $c_s^{n_jn_{j+1}+n_{j+1}n_{j+2}}$. It does not alter any coefficient. Their product is the single c_i in (FE1), before all twists and outer points. Vanishing evaluations or a vanishing fixed zero-term scalar cause no exception.

Exact support deletion. Let S be the set of places above the rational integer Mq. It is conjugation stable. For an allowed translation x write $x = x_S + x_{S^c}$. The two parts are orthogonal for the finite pairing, so $H(x) = H(x_S)H(x_{S^c})$ with phase one. Since π_M is a unit outside S, the second part is integral there, and $H(x_{S^c})v^{\mathrm{int}} = v^{\mathrm{int}}$. Consequently $H(x)v^{\mathrm{int}} = H(x_S)v^{\mathrm{int}}$. Apply this equality separately to the two factors before applying the gluing operators on their left. Each individual $\mathcal{G}_i(x, x')$ equals $\mathcal{G}_i(x_S, x'_S)$, even at the gluing places above N; no commutation past those gluing operators is needed. Push coefficients along this finite support map. Integrality and the absolute-sum estimate are preserved. This proves every assertion. □

Lemma 4.2.33. *Let $\mathcal{I}$ be a set with a subset $\mathcal{J} \subseteq \mathcal{I}$, let d be a positive integer, and let $a, C \in \mathbb{C}$, which may be zero. For each $\iota \in \mathcal{I}$ let $V_\iota, E_\iota, H_\iota$ be complex vector spaces, let $U_\iota \subseteq V_\iota$ be a complex linear subspace, and let*

$$R_\iota : V_\iota \longrightarrow \mathrm{Hom}_{\mathbb{C}}(E_\iota, H_\iota)$$

be complex linear. Suppose the following data are given for each ι: finite sets X_ι, Y_ι, a map $p_\iota : X_\iota \to Y_\iota$, vectors $v_{\iota,x}, w_{\iota,y} \in V_\iota$ with $v_{\iota,x} = w_{\iota,p_\iota(x)}$, a unital subring $A_\iota \subseteq \mathbb{C}$, coefficients $r_{\iota,x} \in \mathbb{C}$ with $dr_{\iota,x} \in A_\iota$, and a real number $B_\iota \geq 0$. Assume

$$\sum_{x \in X_\iota} |\sigma(r_{\iota,x})| \leq B_\iota \quad (\iota \in \mathcal{J},\ \sigma \in \mathrm{Aut}(\mathbb{C})), \qquad F_\iota = C \sum_{x \in X_\iota} r_{\iota,x} v_{\iota,x} \in U_\iota.$$

Put $L_\iota = aR_\iota(F_\iota)$. There is one $c \in \mathbb{C}^\times$, depending only on a, C, d, such that for every ι there are coefficients $s_{\iota,y} \in A_\iota$ for which

$$G_\iota = \sum_{y \in Y_\iota} s_{\iota,y} w_{\iota,y} \in U_\iota, \qquad L_\iota = cR_\iota(G_\iota), \qquad \sum_{y \in Y_\iota} |\sigma(s_{\iota,y})| \leq dB_\iota \quad (\iota \in \mathcal{J},\ \sigma \in \mathrm{Aut}(\mathbb{C})).$$

One may take $c = aC/d$ if $aC \neq 0$, and $c = 1$ otherwise. The same c therefore works for all parameters ι and all vectors of E_ι.

Proof. **First suppose $aC \neq 0$.** Set $u = d/(aC)$ and $c = u^{-1} = aC/d$. For each ι and $y \in Y_\iota$ define

$$s_{\iota,y} = \sum_{\substack{x \in X_\iota \\ p_\iota(x) = y}} dr_{\iota,x}.$$

This belongs to A_ι because that subring contains every summand and is closed under finite addition. Regrouping the finite sum and using $w_{\iota,p_\iota(x)} = v_{\iota,x}$ gives

$$G_\iota = d \sum_{x \in X_\iota} r_{\iota,x} v_{\iota,x} = uaF_\iota.$$

Since U_ι is a complex linear subspace and $F_\iota \in U_\iota$, the vector G_ι still lies in U_ι after the replacement of indices. In particular every homogeneous complex linear condition defining U_ι, including fixed-vector conditions, is preserved. Linearity of R_ι now yields

$$cR_\iota(G_\iota) = cuaR_\iota(F_\iota) = aR_\iota(F_\iota) = L_\iota.$$

For $\iota \in \mathcal{J}$, every field automorphism σ fixes the positive integer d. Applying the triangle inequality on each fiber of p_ι and then summing its disjoint fibers proves

$$\sum_{y \in Y_\iota} |\sigma(s_{\iota,y})| \leq d \sum_{x \in X_\iota} |\sigma(r_{\iota,x})| \leq dB_\iota.$$

Now suppose $aC = 0$. Take $c = 1$ and $s_{\iota,y} = 0$ for every ι, y. Then $G_\iota = 0 \in U_\iota$, its coefficients belong to A_ι, and its coefficient sum is $0 \leq dB_\iota$. The defining expansion gives

$$aF_\iota = aC \sum_x r_{\iota,x} v_{\iota,x} = 0.$$

Consequently $L_\iota = R_\iota(aF_\iota) = 0 = cR_\iota(G_\iota)$. In both cases the chosen c is nonzero and depends only on the common a, C, d, so it precedes every parameter and input vector. Empty index sets cause no exception because all sums used are finite and the empty sum is zero. □

The divisibility of the chain at every prime comes next (Proposition 4.2.34), in the four cases that the family of errors uses: split and large primes, small primes that do not split in K, the exceptional primes on the marked module, and the level prime. This is where the divisibility of theta derivatives is proved. A translated integral vacuum, projected to one place, is a product of theta functions with integral homogeneous jets (Proposition 4.2.35), and at a non-split prime these jets have a normal form in which the partner embeddings $-p^e\tau$ appear with the exponents that define the gain (Proposition 4.2.38). The normal form comes from one CM elliptic curve: the logarithm of its formal group [124, 193] (Lemma 4.2.41), a Frobenius congruence for the normalised CM theta function (Proposition 4.2.40), and integral substitutions to several variables (Lemma 4.2.46); for p-adic periods of CM abelian varieties see [159, 108]. In our reading this is the divisibility by about $p^{a/p+a/p^3+\cdots}$ that [5] describes. The jets are then counted with Schur coefficients and the hook-content formula [194] (Proposition 4.2.49), and the gains accumulate along the chain (Proposition 4.2.48, Lemma 4.2.50). At the exceptional primes and at N, separate arguments pay for the marked places and the glued pairs (Lemma 4.2.52, Proposition 4.2.53).

Proposition 4.2.34. *Fix ONCE the arithmetic realization and upper operators explicitly specified in Proposition 4.2.25, with the SAME $\alpha, \jmath, c_0, \psi_E$, splitting character, coherent reproducing carrier, rationally invariant normalized theta functional, phases and frame intertwiners, rank-one scalar c_s, period η_P, integral vacuum, residue sections, integral Weyl coordinates and branching matrix. This fixes the model before prime identifications, twists and source forms. It is not a choice of an independently rescaled abstractly isomorphic representation.*

More explicitly, in $\mathrm{Hom}(X_j, X_{j+1})$ *the vacuum is* $v_j^{\mathrm{int}} = c_s^{n_j n_{j+1}} v_j^0$ *and the finite translation is*

$$H_j(x) = \psi_{E,f}\left(\frac{c_0 \jmath \alpha \bar{\alpha}}{2} \sum_{r_{\mathrm{idx}}, c_{\mathrm{idx}}} \bar{x}_{r_{\mathrm{idx}} c_{\mathrm{idx}}} x_{r_{\mathrm{idx}} c_{\mathrm{idx}}}\right) \rho_j(0, \alpha x).$$

Use the columns $C_\tau(t_{\mathrm{ht}})$ and determinant exponents $e_j(\tau)$ displayed in Proposition 4.2.25, retaining the height-n_j source determinant minors in the rectangular Schur matrix S_C. The actual kernel matrix and class integral are

$$u(J_j(F)v) = P_{u,v}(\partial_Z)\theta_F(\pi_M \eta_P Z)\big|_0\,, \quad P_{u,v}(Z) = u(S_C(Z)v),$$

$$(T_{\vartheta,j,F}\varphi)(g_{\mathrm{out}}) = \int_{\mathrm{U}(X_j)(E^+)\backslash \mathrm{U}(X_j)(\mathbb{A}_f)} J_j(\omega_{\vartheta,j}(g, g_{\mathrm{out}})F)\varphi(g)\, dg, \qquad \mathrm{vol}(\mathrm{K}_j^X) = 1.$$

These are ordinary derivatives, with no additional factorial denominator. The pair action and its two determinant characters are exactly those of Proposition 4.2.25. For finitely supported Ψ use its displayed unnormalized finite Heisenberg gluing tensor F_Ψ, with last-column and conjugate-last-row residues $-y/N$, and put

$$B_{\vartheta,\Psi} = \mathscr{R}_\beta \mathscr{C}_{\vartheta,F_\Psi}, \qquad (\mathscr{R}_\beta A)(g_{\mathrm{out}}) = \left(\prod_\tau \tau(\bar{\pi}_N)^{-b_\tau}\right) \beta A(\mathrm{diag}(g_{\mathrm{out}}, 1)).$$

Thus this B is precisely the operator in that proposition, including its restriction scalar and its class measures.

For a prime identification $\iota \mid p$ put $O_\iota = \{z \in \mathbb{C} : |\iota z|_p \le 1\}$ and let M_ι be the full module of source forms integral at ι. For a form A on Y_m, define

$$\|A\|_\iota = \sup_{g_{\mathrm{out}}} \max_I \left|[\rho_{\lambda^Y}((\iota_\tau^f(g_{\mathrm{out}}^{-1}))_\tau)\iota(A(g_{\mathrm{out}}))]_I\right|_p, \qquad \iota_\tau(z) = \iota(F_\tau(\bar{z})).$$

The coordinates are the SAME integral target Weyl coordinates fixed above. Put

$$L_p = \frac{fa(\log b)^2}{p}\begin{cases} m, & m \mid p-1 \text{ or } m \mid p+1, \\ 1, & \text{otherwise}, \end{cases} \qquad G_p = \operatorname{gain}(p),$$

and, for small nonsplit p, put

$$C_p = \sum_{\tau\in\Phi_a} \sum_{\substack{e\in I_p \\ -p^e\tau\in\Phi_a}} \frac{a}{p^e}, \qquad I_p = \begin{cases} \{1,3,5,\ldots\}, & \chi_K(p) = -1, \\ \mathbb{Z}_{\geq 1}, & p = d. \end{cases}$$

In particular C_p has NO factor $v_p(\pi_p)$.

Then the following assertions hold, uniformly for every prime identification $\iota \mid p$ and for every ordinary twist ϑ with $p \nmid q = \mathrm{N}Q$. They hold for EVERY finitely supported Ψ on the translation tuples satisfying:

$$\bar{\pi}_M x_j \text{ integral at every } w \nmid q, \quad x_{j,w} = 0 \text{ if } w \nmid Mq \quad (x \in \operatorname{supp}\Psi),$$

all coefficients $\Psi(x)$ integral at every prime identification away from q, and the joint invariance

$$\bigotimes_{j<m} \omega_{\vartheta,j}(v_j, v_{j+1}) F_\Psi = F_\Psi \quad (v_j \in \mathrm{K}_j^X,\ v_m = 1).$$

No archimedean coefficient bound is needed for these local assertions.

1. *If $p \neq N$ and either $\chi_K(p) = 1$ or $p \geq x_1$, then $\|B_{\vartheta,\Psi}\varphi\|_\iota \leq 1$ for every $\varphi \in M_\iota$.*
2. *If $p < x_1$ and $\chi_K(p) \neq 1$, then*

   $$\|B_{\vartheta,\Psi}\varphi\|_\iota \leq p^{mC_p + L_p - G_p} \quad (\varphi \in M_\iota).$$

   Consequently if p is nonexceptional in the sense of Lemma 4.2.56, this is at most $p^{2L_p - G_p}$.
3. *At an exceptional prime take the SAME full mark, its fixed ordering, Y, j_p and $T_\iota = Y^m M_\iota + p^{j_p} M_\iota$ supplied by Proposition 4.2.57. For these choices*

   $$\|B_{\vartheta,\Psi}\varphi\|_\iota \leq p^{2L_p - G_p} \quad (\varphi \in T_\iota).$$

   The full mark and its ordering are chosen before ALL twists and Ψ.
4. *At $p = N$, use the SAME $\Sigma, \delta, \delta^\vee, K_y, U$ and T_ι as Proposition 4.2.65, specializing its fixed residue function to*

   $$\psi(u) = \psi_\alpha(-\widetilde{u}/N), \qquad \psi_\alpha(z) = \psi_{E,f}(\jmath\alpha\bar{\alpha}z).$$

   Here $\widetilde{u}/N$ is the finite adele supported at Σ, with the fixed integer representative of each residue divided by N. This is the self-dual additive character of the fixed arithmetic model; the same phase is used in gluing. Its values are roots of unity, hence integral at ι. For the resulting $T_\iota = \{\varphi \in \mathcal{S} : U^{m/2}\varphi \in M_\delta\}$,

   $$\|B_{\vartheta,\Psi}\varphi\|_\iota \leq N^{2mfb} \quad (\varphi \in T_\iota \cap \mathcal{S}^{\mathrm{ram}}).$$

   This module is not asserted to be contained in M_ι.

All modules, marked-place choices and level-prime operators in these assertions precede the twist and its chain data; the inequalities hold simultaneously for every allowed pair (ϑ, Ψ).

Proof. **The fixed arithmetic kernels.** All constructions in the proof use the single realization fixed in the statement. Its canonical Weil data and pair action are those of Definition 3.2.12 and Definition 3.2.18; the actual kernels, levels, vacuum scalar and period are the ones specified in Proposition 4.2.25. In particular no new normalization is made when the prime or twist changes. We use the prime-identification direction $\mathbb{C} \to \mathbb{C}_p$ of Definition 3.1.3.

Fix an allowed prime identification, twist and coefficient function. For $p \neq N$, Proposition 4.2.47(4) expands the unnormalized gluing into a finite sum of tensors of translated original integral vacua. Its coefficients belong to O_ι; its full tensor is jointly invariant. The step data lie in the standard module with denominator $\bar{\pi}_M$, and, if $p \nmid M$, in the module with denominator one. The finite-character factors relating the raw translations to $H_j(x)$ are roots of unity, so do not change these modules.

Every polynomial $P_{u,v}$ specifying the ordinary kernel is integral and homogeneous. This follows directly from its definition by exterior powers and the prescribed integral Weyl coordinates; its degree, denoted B_j, is the total number of boxes of the step diagram. Its full-height source determinant minors are retained. By Proposition 4.2.35, at denominator δ_{s} its jet at the original scale $\pi_M \eta_P$ has norm at most

$$\left| \iota\left(\frac{\pi_M}{\bar{\delta}_{\mathrm{s}} d} \right) \right|_p^{B_j} . \tag{1}$$

The standard module, rather than only one of its generators, is covered by that proposition. Thus the arbitrary finite integral combinations introduced by covariance remain covered.

The exact covariance formula Proposition 4.2.47(2) identifies the original endpoint gauges with an integral left and right change of coordinates and the displayed diagonal tableau factors. Its item3 gives the literal finite class formula at the original levels, each of volume one and with weight one at each class. It applies to the full invariant tensor; individual summands need not be invariant. Consequently Proposition 4.2.48 applies with its actual covariance hypothesis (K) and finite-chain hypothesis (F). All finite sums are estimated ultrametrically. There is no factor for the number of classes, tableaux, translations or Hecke cosets. The restriction at $p \neq N$ has integral matrix by Proposition 4.2.47(4).

Primes outside the small nonsplit range. In (1) take $\delta_{\mathrm{s}} = \bar{\pi}_M$. The ratio is d^{-1}, so its norm is one when $p \neq d$, including primes dividing M. With identity diagonal in the covariance formula, every step kernel is therefore nonexpanding. The finite class formula and integral restriction give a nonexpanding full composite.

Here $d < x_1$. Indeed $d \leq C_P$ by Definition 2.3.3, $10^{12} C_P \leq \log\log b$, and $(\log b)^{100} \leq x < x_1$, by Lemma 4.2.67 and Proposition 2.3.5; the strict last inequality follows from $x > 1$ and $x^{10^6} \leq x_1$. These inequalities give $d < x_1$. Since $\chi_K(d) = 0$, either alternative in item1 excludes $p = d$. This proves item1. Consistently, the gain is zero in this range: at a split prime its weight is zero, while at $p \geq x_1$ every positive partner term is zero because its numerator is at most a and every receiving nominal weight is at least a/x_1.

The unmarked small nonsplit range. Suppose $p < x_1$ is nonsplit. Both primes dividing M are congruent to 1 modulo d, so split in K; hence $p \nmid M$. These are the original level congruences of the admissible datum. Thus the expanded data above have standard denominator one. The actual nonsplit normal form is Proposition 4.2.38: its partners are $-p^e \tau$, with positive odd e at an inert prime and all $e \geq 0$ at $p = d$, at the same period η_P. Scaling the variables by the integral number $\iota(\pi_M)$ preserves the integral multiplier and coefficients, with its exact homogeneous factor. In particular this is the normal-form hypothesis for the actual Schur

kernels, not a separate formal series chosen to mimic them.

For a subset $S \subseteq \Phi$, let $D_j(S)$ and $C_p(S)$ have the exact meanings of Lemma 4.2.50; thus only the height-one columns at S are erased in the charged count. Put

$$\varepsilon_p = \frac{fa(\log b)^2}{pm}, \qquad v_{\pi,p} = \begin{cases} 1, & p \text{ inert}, \\ \frac{1}{2}, & p = d. \end{cases}$$

That lemma gives

$$D_j(S) \geq G_p/m - C_p(S) + v_{\pi,p}\#(\Phi_a \cap S)a/x_1 - \varepsilon_p. \tag{2}$$

It identifies this number with the exponent of Proposition 4.2.48, including the ramified zero-exponent subtraction and the full-height source factors.

For $S = \varnothing$ all charges and diagonal exponents are zero. The one-step bound is $p^{-D_j(\varnothing)}$. Since $C_p(\varnothing) = C_p$, summing (2) over $j < m$ gives

$$-\sum_{j<m} D_j(\varnothing) \leq mC_p - G_p + \frac{fa(\log b)^2}{p} \leq mC_p + L_p - G_p. \tag{3}$$

The adjacent gauges agree, and the class formula and restriction already checked give precisely item2. Its nonexceptional conclusion follows from Lemma 4.2.56, whose collision sum has the same unweighted positive-exponent convention: $mC_p \leq L_p$. No ramified factor is inserted into this collision sum.

Marked diagonal bounds. Now let p be exceptional and take the full mark and its ordering supplied by Proposition 4.2.57. Write this marked set as $\mathcal{L}$, put

$$P_{\mathcal{L}} = \{\tau : \mathfrak{l}_\tau \in \mathcal{L}\}, \qquad Q_{\mathcal{L}} = \{\tau : \bar{\mathfrak{l}}_\tau \in \mathcal{L}\}, \qquad h_p = \lceil \log_p m \rceil, \qquad t_p = h_p f(n_m + h_p).$$

Here $\mathfrak{l}_\tau$ is induced through $\iota \circ F_\tau$; the conjugation in the coefficient embedding is retained.

For a positive target neighbor at each $\mathfrak{l}$, let $J_{\mathfrak{l}}$ be its selected row indices, with $|J_{\mathfrak{l}}| \leq h_p$. By Lemma 4.2.51(4), the diagonal coordinate norms are $p^{E_{\tau,c}}$, where E is $+1$ on selected indices of $P_{\mathcal{L}}$, -1 on the corresponding indices of $Q_{\mathcal{L}}$, and zero elsewhere. Assign charge one to height-one boxes at $P_{\mathcal{L}}$ outside the selected indices, and to all boxes at $Q_{\mathcal{L}}$ inside them; assign zero otherwise.

These charges satisfy the exact paired-column condition of Proposition 4.2.48. An odd partner exchanges the two induced fibers and reads the same $J_{\mathfrak{l}}$. Thus, if the marked height-one endpoint has an unselected index, it is charged there; if selected, its partner is charged. The symmetric assertion applies when the marked height-one endpoint is the other endpoint. The prime is inert, so there are no even or zero partner exponents to check.

At $P_{\mathcal{L}}$ a height-one box has $E + w = 1$. In each higher column only the selected indices contribute, and their number is at most h_p, because column entries are distinct. At $Q_{\mathcal{L}}$ the negative exponent and charge cancel; elsewhere both vanish. Writing $c_{j,\tau}$ for the number of columns below n_{j+1} other than height one, the total box contribution is therefore at most

$$\sum_{\mathfrak{l} \in \mathcal{L}} A_{\mathfrak{l}} + h_p \sum_{\tau \in P_{\mathcal{L}}} c_{j,\tau}.$$

The determinant contribution is exactly zero by Lemma 4.2.52(D), with the same selected sets. Thus the diagonal tableau factor in covariance has valuation at least its charge minus this displayed bound. The one-step gauge estimate is consequently

$$p^{\sum A_{\mathfrak{l}} + h_p \sum_{P_{\mathcal{L}}} c_{j,\tau} - D_j(P_{\mathcal{L}})}. \tag{4}$$

Recovering the fixed marked operator. The actual spherical identity Lemma 4.2.51(2) is

$$\sum_{i=0}^{h_p} c_{\mathfrak{l},j,i} T_{\mathfrak{l},i}^{(j+1)} C_{j+1} = p^{t_p} C_{j+1} T_{\mathfrak{l},h_p}^{(0)}, \qquad c_{\mathfrak{l},j,i} \in O_\iota. \tag{5}$$

Its polynomial normalization, unit character value, compact fixing and tuple-independent coefficients are all part of that lemma. They hold also after the finite gluing at N. Multiplication by $p^{A_\mathfrak{l}-t_p}$ moves the prescribed scaled source operator into the datum. Each scalar has norm at most $p^{t_p-A_\mathfrak{l}}$. Expanding its finite target Hecke sum and applying (4) costs no coset cardinality.

Apply this successively in the fixed ordering. The sum of the scalar exponents cancels $\sum A_\mathfrak{l}$, leaving step exponent

$$\#\mathcal{L}\, t_p + h_p \sum_{\tau \in P_\mathcal{L}} c_{j,\tau} - D_j(P_\mathcal{L}). \tag{6}$$

The datum operators commute with the actual levels and source actions by Lemma 4.2.51(3), so the transformed full tensor remains jointly invariant. The same lemma proves the ordered partial-chain identity $C_j^t = C_j Y^j$. Hence its final value is the original chain on $Y^m\varphi$; no reordering of the factors of Y is used.

For a full mark $C_p(P_\mathcal{L}) = 0$. Indeed every a-place induces a marked prime or its conjugate. If both ends of a positive odd partner collision lay outside $P_\mathcal{L}$, the first would lie in $Q_\mathcal{L}$, whereas its odd partner induces the conjugate prime and would lie in $P_\mathcal{L}$, a contradiction. By Lemma 4.2.52(M),

$$\#\mathcal{L}\, t_p + h_p \sum_{P_\mathcal{L}} c_{j,\tau} \le L_p/m + \#(\Phi_a \cap P_\mathcal{L}) a/x_1. \tag{7}$$

Here $v_{\pi,p} = 1$. Subtracting (2) from (7), and using $\varepsilon_p \le L_p/m$, bounds (6) by $(2L_p - G_p)/m$. The finite class formula and integral restriction therefore give $p^{2L_p-G_p}$ on $Y^m M_\iota$.

For the other summand of the prescribed module, positivity of the collision exponents gives the convergent geometric bound

$$C_p \le fa \sum_{e \ge 1} p^{-e} = \frac{fa}{p-1} \le \frac{2fa}{p}.$$

Thus $mC_p \le j_p = \lceil 2mfa/p \rceil$. Equation (3) and linearity give the bound $p^{L_p-G_p}$ on $p^{j_p} M_\iota$. Since $L_p \ge 0$, the ultrametric inequality gives $p^{2L_p-G_p}$ on their sum T_ι. This is item3, with exactly the module and choices of Proposition 4.2.57.

The distinguished prime. For $p = N$, the additive character on the entire residue ring, and its restriction to the units used to define U, are exactly those in Proposition 4.2.53. Its character values are roots of unity, so the same specialization of Proposition 4.2.65 applies. All its support, coefficient and full-tensor invariance hypotheses are precisely the hypotheses in the present statement. Its item3 therefore gives

$$\|B_{\vartheta,\Psi}\varphi\|_\iota \le N^{2mfb} \qquad (\varphi \in T_\iota \cap \mathcal{S}^{\mathrm{ram}}).$$

More explicitly, its direct kernels cost N^{fb} at each of the m compatible changed bases, and its scalar converting the glued chain into the direct chain costs N^{mfb}. The original normalized restriction cancels the final diagonal-base factor. Only $U^{m/2}\varphi$ is required to be integral at its prescribed base; no integrality of φ in M_ι is assumed. This proves item4.

Finally the full mark, its ordering and module in item3 were fixed before twists by Proposition 4.2.57. The residue character, diagonal bases and U in item4 depend only on the fixed datum and prime identification. The marked transfer coefficients and datum operators may depend on the twist, but alter none of these modules or prior choices. Every argument was for an arbitrary allowed pair (ϑ, Ψ), so all four assertions have the required simultaneous quantifiers. □

Proposition 4.2.35. *Let C be a finite coordinate set, $n_C = |C|$, and $A = \sqrt{2\pi_{\mathbb{R}}/\sqrt{d}}$. For a prime p and identification $\iota : \mathbb{C} \to \mathbb{C}_p$, put $O_\iota = \{z \in \mathbb{C} : |\iota z|_p \le 1\}$ and $O = \mathcal{O}_{\mathbb{C}_p}$. Fix $0 \neq \delta \in \mathcal{O}_K$. Let F_0 belong to the following module in the actual finite Heisenberg span:*

$$\mathrm{span}_{O_\iota}\{\rho(\mathrm{fv}(x))\, c_s^{n_C} v^0 : x \in \mathbb{A}_{E,f}^C,\ \delta x_w \in \mathcal{O}_{E,w}^C \text{ for every } w \mid p\}.$$

Write $\mathscr{E}_{F_0}$ for its actual entire theta function of Lemma 4.2.2. Then there exist a finite set J, vectors $u_j \in E^C$, positive integers m_j, and $H_{F_0} \in O[[T_j : j \in J]]$, with

$$(\alpha^{-1}u_j)_w \in \mathcal{O}_{E,w}^C \quad (w \mid p), \qquad \sum_j m_j \tau(u_{j,c})\overline{\nu(u_{j,c'})} = \mathbf{1}_{\tau=\nu}\mathbf{1}_{c=c'} \quad (\tau,\nu \in \Phi,\ c,c' \in C).$$

Put $L_j(Z) = \sum_{\tau,c} \overline{\tau(u_{j,c})} Z_{\tau,c}$. The following is an identity of coefficientwise Taylor series, with ι applied to every complex coefficient:

$$\iota\Big[\mathscr{E}_{F_0}\Big(\bar\delta A Z/\Omega_{\mathrm{CM}}\Big)\Big] = \prod_j \big(\iota[\vartheta(L_j(Z))]\big)^{m_j\delta\bar\delta} H_{F_0}\big(\iota[Y(L_j(Z))] : j \in J\big).$$

Moreover, for every homogeneous polynomial $Q \in O_\iota[Z_{\tau,c}]$ of degree $B \ge 0$, and every $\zeta_{\mathrm{sc}} \in \mathbb{C}$,

$$\Big|\iota\Big[Q(\partial_Z)\mathscr{E}_{F_0}(\zeta_{\mathrm{sc}}\eta_P Z)\big|_{Z=0}\Big]\Big|_p \le \left|\iota\left(\frac{\zeta_{\mathrm{sc}}}{\bar\delta d}\right)\right|_p^B.$$

These are ordinary derivatives; no factorial is inserted. The same fixed theta functional, vacuum scalar and period factor occur throughout.

Proof. We work on the fixed standard positive space. The theta functional is the tensor functional fixed in (WC1) of Definition 4.1.15, not an independently phased functional. Put $v_{\mathrm{int}} = c_s^{n_C} v^0$. For a finite point $x = (x_c)_{c\in C}$, the tensor identity (WC1) gives

$$\theta(\rho(\mathrm{fv}(x))v_{\mathrm{int}}) = \prod_{c\in C} c_s\, \theta_+(\rho(\mathrm{fv}(x_c))v_+^0).$$

Every factor is in O_ι by Proposition 4.2.1. Thus all these actual finite theta values are integral; for $C = \varnothing$ the product is 1. No value of a coefficient function in the coherent realization has been substituted for a theta value.

1. A frame of square length prime to the specified prime. Abbreviate $q = q_1$, $U_0 = (d+1)/4$, and $D_{\mathrm{rel}} = \mathrm{cyc}_q'(\zeta)$; write cyc_r for the rth cyclotomic polynomial. By Lemma 3.2.8, every prime dividing q divides q_0. For an odd such prime, $q_0 \mid 2(d+1)$ implies that it divides U_0; at 2, $d \equiv 7 \pmod 8$ makes U_0 even. Thus every prime dividing q divides U_0. Choose z_0, $0 \le z_0 < q$, such that

$$U_0 z_0^2 + z_0 + 1 = q e_0 \tag{3}$$

for an integer $e_0 > 0$. Such a residue exists because $z \mapsto U_0 z^2 + z$ is a permutation modulo q: a difference factors as $(z - z')(1 + U_0(z + z'))$, whose second factor is a unit modulo q. Put $t_0 = \bar\omega + U_0 z_0$. Then

$$t_0 + \bar t_0 = 1 + 2U_0 z_0, \quad t_0\bar t_0 = U_0 q e_0, \quad (\omega + U_0 z_0)t_0 = U_0 q e_0. \tag{4}$$

For each rational prime $\ell \mid q$, write $q = \ell^{a_\ell+1}Q_\ell$, with $\ell \nmid Q_\ell$, and put $r_\ell = q/\ell$, $\xi_\ell = \zeta^{r_\ell}$. There is $V_\ell \in \mathcal{O}_E$, dividing Q_ℓ, such that

$$D_{\mathrm{rel}}(\xi_\ell - 1)Q_\ell = qV_\ell. \tag{5}$$

Here is an elementary verification. The cyclotomic identity $\mathrm{cyc}_{Q_\ell}(T^{\ell^{a_\ell+1}}) = \mathrm{cyc}_q(T)\mathrm{cyc}_{Q_\ell}(T^{\ell^{a_\ell}})$, differentiated at ζ, reads

$$\ell^{a_\ell+1}\zeta^{\ell^{a_\ell+1}-1}\mathrm{cyc}'_{Q_\ell}(\zeta^{\ell^{a_\ell+1}}) = D_{\mathrm{rel}}\mathrm{cyc}_{Q_\ell}(\zeta^{\ell^{a_\ell}}).$$

It shows that $\ell^{a_\ell+1}$ divides $D_{\mathrm{rel}}\mathrm{cyc}_{Q_\ell}(\zeta^{\ell^{a_\ell}})$. Evaluating the integral polynomial divisibility $\mathrm{cyc}_{Q_\ell}(T) \mid T^{Q_\ell} - 1$ at $\zeta^{\ell^{a_\ell}}$ shows that this latter cyclotomic value divides $\zeta^{q/\ell} - 1 = \xi_\ell - 1$. Thus q divides the left side of (5). On the other hand $T^q - 1 = (T^{r_\ell} - 1)\mathrm{cyc}_q(T)R(T)$ for an integral polynomial R. Differentiating and multiplying by ζ gives $q = D_{\mathrm{rel}}(\xi_\ell - 1)R(\zeta)\zeta$. Together with (5), cancellation of q gives $Q_\ell = V_\ell R(\zeta)\zeta$, proving the divisibility assertion.

Put $\gamma_\ell = t_0 Q_\ell/q$, and form the finite family

$$v_{\ell,h_{\mathrm{idx}},j_{\mathrm{idx}},i} = \zeta^{h_{\mathrm{idx}}}\left(\mathbf{1}_{j_{\mathrm{idx}}=0}(\xi_\ell)^i + \gamma_\ell((\xi_\ell)^i - (\xi_\ell)^{j_{\mathrm{idx}}})\right), \quad 0 \le h_{\mathrm{idx}} < r_\ell, \quad 0 \le j_{\mathrm{idx}}, i < \ell. \tag{6}$$

Every vector in (6) lies in $\mathfrak{a}$. If $\omega \in w$, (5) shows that $D_{\mathrm{rel}}\gamma_\ell(\xi_\ell - 1) = t_0 V_\ell$ is integral, so the required $D_{\mathrm{rel}}v$ is integral at w. If $\omega \notin w$, and $w \nmid q$, integrality is immediate. If $w \mid q$ while ω is a unit, then $\bar\omega \in w$, because every prime divisor of q divides $U_0 = \omega\bar\omega$. Hence $\omega + U_0 z_0$ is a unit at w, and the last equation in (4) shows that t_0/q is integral there. Thus γ_ℓ, and hence v, is integral at w. This verifies membership at every finite place.

The Gram sum of the ℓ-family is $C_\ell \mathbf{1}_{\tau=\sigma}$, where

$$C_\ell = q + Q_\ell(1 + 2U_0(z_0 + \ell Q_\ell e_0)). \tag{7}$$

For clarity, expand the two bracketed factors in (6). The sums of the powers of either primitive ℓ-th root are zero. If $G = \sum_{i<\ell}\tau(\xi_\ell)^i\overline{\sigma(\xi_\ell)^i}$, the sum over j_{idx}, i is

$$G + \bar\gamma_\ell G + \gamma_\ell G + \gamma_\ell\bar\gamma_\ell(\ell G + \ell G) = G(1 + \gamma_\ell + \bar\gamma_\ell + 2\ell\gamma_\ell\bar\gamma_\ell).$$

Here the first term comes from the two indicators, each mixed term uses $\sum_i \tau(\xi_\ell)^i = 0$ or its conjugate, and the last two terms are the two diagonal products in the expansion of $(\tau(\xi_\ell)^i - \tau(\xi_\ell)^j)(\overline{\sigma(\xi_\ell)^i} - \overline{\sigma(\xi_\ell)^j})$. Both off-diagonal double sums factor into zero root sums. The element γ_ℓ belongs to K, so it has the same value under all embeddings in Φ. Multiplication by the sum over $h_{\mathrm{idx}} < r_\ell$ gives

$$\sum_{h=0}^{r_\ell-1}\sum_{i=0}^{\ell-1}\left(\tau(\zeta)\overline{\sigma(\zeta)}\right)^{h+r_\ell i} = \sum_{k=0}^{q-1}\left(\tau(\zeta)\overline{\sigma(\zeta)}\right)^k = q\mathbf{1}_{\tau=\sigma}.$$

The ratio is a qth root of unity, and is 1 precisely when $\tau = \sigma$, because $E = K(\zeta)$ and both embeddings fix K. Substituting (4) in the diagonal expression gives exactly (7).

Let $C_{\mathrm{aux}} = \sum_{\ell\mid q} C_\ell$. This positive integer is prime to q. Modulo a prime $\ell_0 \mid q$, every summand with $\ell \ne \ell_0$ vanishes, since Q_ℓ is divisible by ℓ_0. The remaining summand is $Q_{\ell_0} \not\equiv 0$, since q and U_0 vanish modulo ℓ_0.

If $p \nmid q$, the vectors ζ^i, $0 \le i < q$, form a frame of length q, and repeating each q times gives a frame of length q^2. If $p \mid q$, use the union (6), of length C_{aux}, and repeat each vector C_{aux} times. In either case there are a positive integer b_{aux} prime to p and finitely many $w_i \in \mathfrak{a}$ with

$$\sum_i \tau(w_i)\overline{\nu(w_i)} = b_{\mathrm{aux}}^2\mathbf{1}_{\tau=\nu}. \tag{8}$$

All vectors are repeated explicitly, so their multiplicity in (8) is one. For the remaining argument use the arithmetic-duality conclusion of Lemma 4.2.15: for the actual $\mathfrak{a} = E \cap \alpha\widehat{\mathcal{O}}_E$,

$$\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E/K}^{-1}, \qquad \mathrm{Tr}_{E/K}(\mathfrak{a}\bar{\mathfrak{a}}) \subset \mathcal{O}_K,$$

and the lattice $\Gamma = A(\tau\mathfrak{a})_\tau$ is self-dual for

$$E_\Gamma(z,z') = \pi_{\mathbb{R}}^{-1}\operatorname{Im}\sum_\tau \bar{z}_\tau z'_\tau.$$

These statements apply coordinatewise to Γ^C. They use the very same idele and lattice as the frame just constructed. The case $q=1$, if present, uses $w=1, b_{\mathrm{aux}}=1$.

2. Integral Fourier expansion of the actual vacuum. Apply Lemma 4.2.36 with $\delta = b_{\mathrm{aux}}$, frame (8), multiplicities one, and all monomial exponents zero. Its norm hypothesis is exactly $b_{\mathrm{aux}}\bar{b}_{\mathrm{aux}} = b_{\mathrm{aux}}^2$. It supplies a vector F_* in the actual finite Heisenberg span with

$$S(Z) := \mathscr{E}_{F_*}(Z) = \prod_{i,c}\vartheta(B_{i,c}(Z)), \qquad B_{i,c}(Z) = \frac{\Omega_{\mathrm{CM}}}{b_{\mathrm{aux}}A}\sum_\tau \overline{\tau(w_i)}Z_{\tau,c}, \qquad S(0)=1.$$

It also supplies the exact original signed periods $b_{\mathrm{aux}}\Gamma^C$, not merely proportional periods.

Write $v = \mathscr{E}_{v^0}$. We check that its principal theta line is the one in the common Fourier identity. The arithmetic-duality clause of Lemma 4.2.15 says that $\jmath\alpha\bar{\alpha}\widehat{\mathcal{O}}_E$ is the absolute inverse different. Consequently

$$\psi_{E,f}(\jmath\alpha\bar{\alpha}\, r) = 1 \quad (r \in \widehat{\mathcal{O}}_E). \tag{9a}$$

To check signed invariance of the vacuum directly, write a finite variable in normalized coordinates as z, put $a_f = \jmath\alpha\bar{\alpha}$, and let $t \in \widehat{\mathcal{O}}_E^C$. Translation by $\mathrm{fv}(t)$ preserves its support. On that support the quotient of $\rho_f(t)v^0$ by v^0, using (WD5), (WD7), and (WD8), is

$$\psi_{E,f}\left(\frac{a_f}{2}[\,2\omega t^*t + (1-\omega)z^*t - \omega t^*z\,]\right).$$

Complex conjugation leaves the finite additive character unchanged, whereas $\bar{a}_f = -a_f$ and $\bar{\omega} = 1-\omega$. Conjugating the last summand inside its character therefore combines the two mixed terms. The displayed quotient is

$$\psi_{E,f}\big(a_f[\omega t^*t + \bar{\omega}z^*t]\big) = 1$$

by (9a). Off the support both sides are zero. For a rational lattice vector $\gamma \in \mathfrak{a}^C$ this applies to $t = \alpha^{-1}\gamma$. The signed translation formula of Lemma 4.2.2 therefore puts v in the theta space for the original semicharacter of Γ^C. This function is not zero: $v^0 \neq 0$ by its displayed vacuum formula, and the entire map is injective. This is precisely the vacuum and prescribed semicharacter in clauses (iii)–(iv) of Lemma 4.2.15. Its finite Fourier conclusion therefore applies to this v; no independently chosen generator of a theta line is being substituted.

Here is the application of the common finite Fourier identity in Lemma 4.2.15. Put

$$G_0 = \Gamma^C/b_{\mathrm{aux}}\Gamma^C, \qquad Y_0 = b_{\mathrm{aux}}^{-1}\Gamma^C/\Gamma^C, \qquad D_0 = b_{\mathrm{aux}}^{2fn_C}.$$

Both finite groups have order D_0. Choose representatives of each in their indicated lattices, and choose zero for the identity classes. Write R_λ for the original signed shift by λ, U_y for the raw reduced shift by y, and $\mathcal{A} = \sum_{\lambda\in G_0} R_\lambda$. The common identity, for this S and this nonzero v, gives scalars b_y with

$$\mathcal{A}U_{-y}S = b_y v, \qquad \sum_{y\in Y_0} b_y U_y v = D_0 S. \tag{9b}$$

Thus every factor and sign in this application refers to the original semicharacter. In particular its commutator pairing is $\exp(2\pi_{\mathbb{R}} i E_\Gamma(\lambda,y))$; no replacement of the theta functional or of its phase is made.

For a representative $y = A(\tau(\gamma_{y,c}))_{\tau,c}$, let $T_y = \rho(\mathrm{fv}(-\alpha^{-1}\gamma_y))$. Here the common convention is $U_y F(Z) = \exp(-\bar{y}^{\mathsf{T}} Z - \|y\|^2/2) F(Z+y)$. The raw translation identity of Lemma 4.2.2 therefore identifies the entire map of T_y with U_y. Likewise take $\widetilde{R}_\lambda = \rho_f(-\alpha^{-1}\gamma_\lambda)$, where $\lambda = A\Phi(\gamma_\lambda)$; its entire map is R_λ. The minus signs here follow from the displayed raw translation formula. Applying the value at zero to the second identity in (9b), using $S(0) = 1$, gives

$$\sum_{y \in Y_0} b_y \, \theta(T_y v^0) = D_0.$$

The first identity in (9b) and injectivity give $\sum_\lambda \widetilde{R}_\lambda T_{-y} F_* = b_y v^0$. Multiply this equality by $\theta(T_y v_{\mathrm{int}})/D_0 = c_s^{n_C} \theta(T_y v^0)/D_0$ and sum over y. The preceding scalar equality proves the exact vector identity

$$v_{\mathrm{int}} = \sum_{y \in Y_0} \frac{\theta(T_y v_{\mathrm{int}})}{D_0} \sum_{\lambda \in G_0} \widetilde{R}_\lambda T_{-y} F_*. \tag{9}$$

The coefficients outside the inner sums are in O_ι: their numerators are theta values of the same integral vacuum supplied by Proposition 4.2.1, and D_0 is a power of b_{aux}, which is prime to p. Each product of a signed shift and a raw shift is a raw shift times a finite additive-character value. Such a value is a root of unity: the character is $\exp(2\pi_{\mathbb{R}} i \,\mathrm{Tr}(a))$ for a rational approximant $a \in E$, whose trace is rational. Hence these factors are p-adic units. Representatives λ belong to $\mathfrak{a}^C$, and $b_{\mathrm{aux}}\gamma_y \in \mathfrak{a}^C$. Thus every translating point in (9), in normalized finite coordinates, becomes integral after multiplication by b_{aux}. Formula (9) is consequently a finite O_ι-linear combination of actual translates of F_*, with denominators prime to p. It makes no integrality assertion about c_s by itself.

3. Exact rational replacement and simultaneous denominator clearing. Choose a finite expression for F_0 in its stated module, and substitute (9) into every generator. The Weyl composition law, obtained by substituting (WD7), says that the product of two raw finite shifts is a raw shift by their sum times a finite additive-character value. These values are roots of unity, as in Step 2. The same is true with a signed shift, by (WD8). Thus F_0 is a finite O_ι-linear combination of $\rho(\mathrm{fv}(x))F_*$, with $\delta x_w \in \mathcal{O}_{E,w}^C$ for every $w \mid p$. Indeed the new points added by (9) become integral after multiplication by b_{aux}, and b_{aux} is a unit at such places.

We prove the exact rational replacement for any one of these finitely many x. Since F_* belongs to the finite Heisenberg span, write it as a finite complex linear combination of raw finite translates of v^0. There is an open additive subgroup $V_0 \subset \widehat{\mathcal{O}}_E^C$ for which

$$\rho(\mathrm{fv}(u))F_* = F_* \quad (u \in V_0). \tag{10a}$$

Here is why the equality, rather than convergence, holds. The signed-vacuum calculation in Step 2 gives explicitly

$$\rho(\mathrm{fv}(u))v^0 = \psi_{E,f}(a_f \omega u^* u/2)^{-1} v^0 \quad (u \in \widehat{\mathcal{O}}_E^C).$$

Choose a sufficiently small open additive subgroup on which the character in this formula is 1. Such a subgroup exists by placing $a_f \omega u^* u/2$ in the open kernel of $\psi_{E,f}$; finitely many local valuation inequalities suffice, and outside those places all coefficients are integral. For each of the finitely many translates used to express F_*, also require the Heisenberg commutator of its fixed translating point with u to be 1. Each is the kernel of a continuous finite additive character, hence contains an open additive subgroup. Their finite intersection gives (10a). Shrink V_0 once more so that the Weyl composition scalar for x, u equals 1. All these restrictions may be made inside $\widehat{\mathcal{O}}_E^C$.

The diagonal copy of E^C meets $\alpha x + \alpha V_0$. To see this directly, choose an integer $M_0 > 0$ for which $M_0 \alpha x$ is integral at every finite place, and an integral ideal $\mathfrak{c} \neq 0$ with $\widehat{\mathfrak{c}}^C \subset M_0 \alpha V_0$. The

reduction map $\mathcal{O}_E^C \to \widehat{\mathcal{O}}_E^C/\widehat{\mathfrak{c}}^C$ is surjective by the Chinese remainder theorem. Choose $a \in \mathcal{O}_E^C$ reducing to $M_0\alpha x$, and put $\gamma = a/M_0$. Then

$$u := \alpha^{-1}\gamma - x \in V_0.$$

By (10a) and the restriction on the composition scalar, this gives the exact vector equality

$$\rho(\mathrm{fv}(\alpha^{-1}\gamma))F_* = \rho(\mathrm{fv}(x))F_*. \tag{10b}$$

At every $w \mid p$, $\delta\alpha^{-1}\gamma = \delta x + \delta u$ is integral. No operation involving ι has been used in obtaining (10b).

For each such rational γ, the finite adele $\alpha^{-1}\delta\gamma$ has negative valuations at only finitely many places, none above p. Multiply, for every rational prime below one of those places, by a sufficiently high power of that prime. This gives a positive integer m, prime to p, with $m\delta\gamma \in \mathfrak{a}^C$. Take the product of the finitely many resulting integers so that one m works for all terms, and set $k_{\mathrm{aux}} = b_{\mathrm{aux}}m$. Then k_{aux} is prime to p, is divisible by b_{aux}, and

$$m\delta\gamma_c \in \mathfrak{a} \quad \text{for every term and every } c. \tag{10c}$$

We now compute the actual entire function of (10b). Put $a_\gamma = A(\tau(\gamma_c))_{\tau,c}$. The formula of Lemma 4.2.2 writes it as

$$\exp(\bar{a}_\gamma^{\mathsf{T}} Z - \|a_\gamma\|^2/2) \prod_{i,c} \vartheta(B_{i,c}(Z - a_\gamma)).$$

Let $A_0 = \mathrm{Area}(\Lambda_0) = |\Omega_{\mathrm{CM}}|^2\sqrt{d}/2$ and $H_0(v,z) = \pi_{\mathbb{R}}\bar{v}z/A_0$, as in Lemma 4.2.37. The frame identity (8) gives, by expanding the two sums in $B_{i,c}$,

$$\sum_{i,c} \overline{B_{i,c}(a_\gamma)} B_{i,c}(Z) = \frac{|\Omega_{\mathrm{CM}}|^2}{A^2}\bar{a}_\gamma^{\mathsf{T}} Z = \frac{A_0}{\pi_{\mathbb{R}}}\bar{a}_\gamma^{\mathsf{T}} Z. \tag{10d}$$

The same formula with $Z = a_\gamma$ gives the corresponding squared norm. Define

$$\xi_{i,c} = -B_{i,c}(a_\gamma) = -\frac{\Omega_{\mathrm{CM}}}{b_{\mathrm{aux}}} \operatorname{Tr}_{E/K}(\gamma_c \bar{w}_i). \tag{10}$$

The trace equality holds since Φ consists of all embeddings over K. By (10d),

$$\prod_{i,c} e^{-H_0(\xi_{i,c}, B_{i,c}(Z)) - H_0(\xi_{i,c},\xi_{i,c})/2} = e^{\bar{a}_\gamma^{\mathsf{T}} Z - \|a_\gamma\|^2/2}.$$

The entire function is therefore exactly $\prod_{i,c}(T_{\xi_{i,c}}\vartheta)(B_{i,c}(Z))$, with the original reduced translations of Lemma 4.2.37.

By (10c) and the trace-integrality clause of Lemma 4.2.15,

$$\frac{k_{\mathrm{aux}}\delta\xi_{i,c}}{\Omega_{\mathrm{CM}}} = -\operatorname{Tr}_{E/K}(m\delta\gamma_c\bar{w}_i) \in \mathcal{O}_K. \tag{10e}$$

Thus the same k_{aux} takes every translating argument in every summand into the exact torsion condition required by that section lemma.

Set

$$u_{i,c} = \frac{w_i}{b_{\mathrm{aux}}k_{\mathrm{aux}}}e_c, \qquad m_{i,c} = k_{\mathrm{aux}}^2, \qquad L_{i,c}(Z) = \sum_\tau \frac{\overline{\tau(w_i)}}{b_{\mathrm{aux}}k_{\mathrm{aux}}} Z_{\tau,c}.$$

Since $w_i \in \mathfrak{a}$ and $b_{\mathrm{aux}}k_{\mathrm{aux}}$ is prime to p, these vectors are α-integral above p. The resolution required in the statement follows explicitly from

$$\sum_i k_{\mathrm{aux}}^2 \frac{\tau(w_i)\overline{\nu(w_i)}}{b_{\mathrm{aux}}^2 k_{\mathrm{aux}}^2} = \mathbf{1}_{\tau=\nu};$$

placing each vector in its coordinate c adds the factor $\mathbf{1}_{c=c'}$.

At the scale $\bar{\delta}A/\Omega_{\mathrm{CM}}$,

$$B_{i,c}(\bar{\delta}AZ/\Omega_{\mathrm{CM}}) = \overline{k_{\mathrm{aux}}\delta}\, L_{i,c}(Z).$$

Apply the translated conclusion of Lemma 4.2.37 with its integer $k_{\mathrm{aux}}\delta$, which is nonzero, and use (10e). Each factor becomes

$$\iota\vartheta(L_{i,c}(Z))^{k_{\mathrm{aux}}^2\delta\bar{\delta}} H_{i,c}(\iota Y(L_{i,c}(Z))), \qquad H_{i,c} \in O[[T]].$$

Here and below every complex coefficient in a Taylor series is read through ι. The norm $k_{\mathrm{aux}}^2\delta\bar{\delta}$ is a positive integer. All terms have the same k_{aux}, frame, forms and exponents. The product of their integral formal series is integral, and their finite O-linear combination is integral. This gives the single series H_{F_0} and proves the product identity in the statement. Substitution is well-defined because every $Y(L_{i,c}(Z))$ has zero constant term.

4. Exact all-prime cancellation of the period factor. Define the finite multiplicative character used here by

$$\chi_{\mathrm{fin}}(x) = \upsilon(1,x)\,|x|_{\mathbb{A}_E}^{-1/2} \qquad (x \in \mathbb{A}_{E,f}^{\times}).$$

This is the character in the prescribed equation $s_0^f = \chi_{\mathrm{fin}}(\alpha)^{-1}$ of Lemma 4.2.3. For $a \in E^{\times}$, triviality of υ on principal ideles and its specified infinite part give

$$\upsilon(1,a_f) = \prod_{\tau\in\Phi} \frac{\overline{\tau(a)}}{|\tau(a)|}.$$

The product formula for the adelic norm gives $|a_f|_{\mathbb{A}_E}^{-1/2} = \prod_{\tau\in\Phi}|\tau(a)|$. Consequently

$$\chi_{\mathrm{fin}}(a_f) = \prod_{\tau\in\Phi} \overline{\tau(a)}. \tag{11a}$$

Moreover χ_{fin} is identically 1 on finite idele units: their adelic norm is 1, and (WD3), with infinite components 1, gives $\upsilon(1,x) = 1$.

For clarity we deduce its valuation formula without extending ι to a topological limit. Given a finite idele x, choose an integer $h > 0$ killing the class of its fractional ideal. There is $a \in E^{\times}$ with $x^h/a_f \in \widehat{\mathcal{O}}_E^{\times}$. Equation (11a) and triviality on these units imply

$$\chi_{\mathrm{fin}}(x)^h = \prod_{\tau\in\Phi} \overline{\tau(a)}.$$

For each τ, let w_τ be the finite place induced by $\iota\bar{\tau}$, with absolute value extending the usual p-adic one. The equality of ideals gives $|a|_{w_\tau,p} = |x_{w_\tau}|_p^h$; the intervening local unit has absolute value 1. Taking the p-adic absolute value of the preceding scalar equality and then the positive hth root proves

$$|\iota\chi_{\mathrm{fin}}(x)|_p = \prod_{\tau\in\Phi} |x_{w_\tau}|_p. \tag{11}$$

Only the algebraic equality (11a) is transported by ι.

We verify the uniformity in τ needed when $x = \alpha$. Put $D_{\mathrm{rel}} = \mathrm{cyc}'_q(\zeta)$. For $q > 1$, its factorization is

$$D_{\mathrm{rel}} = \prod_{\substack{a \in (\mathbb{Z}/q\mathbb{Z})^\times \\ a \neq 1}} (\zeta - \zeta^a).$$

An automorphism over K sends ζ to ζ^v, with v a unit modulo q. Choose positive integer representatives v, v' with $vv' \equiv 1 \pmod{q}$. For each factor put $r = \zeta^{a-1} \neq 1$. Then

$$\frac{\zeta^v - \zeta^{av}}{\zeta - \zeta^a} = \zeta^{v-1} \frac{1 - r^v}{1 - r}, \qquad \frac{1 - r^v}{1 - r} = \sum_{j=0}^{v-1} r^j.$$

Its reciprocal is integral as well, since

$$\frac{1 - r}{1 - r^v} = \sum_{j=0}^{v'-1} (r^v)^j.$$

Thus the ratio of each conjugated factor to the original is an algebraic unit, as is the ratio of the products. It follows that all K-conjugates of D_{rel} have the same absolute value at the fixed place ι. More directly, D_{rel} is a rational polynomial in ζ, and $\bar{\tau}(\zeta)$ is a primitive qth root for every $\tau \in \Phi$. The same calculation therefore shows that $|\iota\bar{\tau}(D_{\mathrm{rel}})|_p$ is independent of τ. For $q = 1$, $D_{\mathrm{rel}} = 1$ and this is immediate.

Every embedding in Φ fixes K, so $\iota\bar{\tau}(\omega) = \iota(\bar{\omega})$ is independent of τ. Therefore either ω belongs to every w_τ, or to none of them. By (WD2), in the former case each α_{w_τ} is D_{rel}^{-1} and in the latter case it is 1. The conjugate-unit calculation shows that the number

$$A_p := |\alpha_{w_\tau}|_p$$

is independent of τ. Formula (11) gives $|\iota\chi_{\mathrm{fin}}(\alpha)|_p = A_p^f$. The fixed root equation for s_0 and the all-prime unit property of the same u_{CS} in Lemma 4.2.3 now yield

$$|\iota\beta|_p = A_p^{-1}, \qquad |\iota(\beta\overline{\tau(u)})|_p = A_p^{-1}|u|_{w_\tau,p} = |(\alpha^{-1}u)_{w_\tau}|_p \leq 1 \tag{12}$$

whenever $\alpha^{-1}u$ is integral above p. This also applies coordinatewise to the u_j of Step 3, so every coefficient of βL_j is integral. If $p \nmid q$, the element D_{rel} is a unit at every place above p, since it divides q by Lemma 3.2.8; hence $A_p = 1$. In particular this holds at the nonsplit primes under consideration. At primes dividing q only the product cancellation in (12) is used, and no assertion that β itself is a unit is needed.

5. Ordinary differential integrality, including 2. We suppress ι in the following purely formal calculation. Put $g = -\log\vartheta$. The theta is even, so $g(0) = g'(0) = 0$. The logarithmic derivative of the reduced sigma translate gives

$$g''(z) = g''(0) + X(z). \tag{13}$$

This can also be checked by taking two derivatives of its defining reduced sigma expression; the constant quasi-period term disappears in the difference at zero.

Write $g(z) = g_2 z^2 + \cdots$. The CM quotient conclusion of Lemma 4.2.37, for ω and $\bar{\omega}$, has integral coefficient of Y^2. Since $Y = z + \cdots$, and g has no constant or linear term, this says that

$$-\omega\sqrt{-d}\, g_2 \in O, \qquad \bar{\omega}\sqrt{-d}\, g_2 \in O.$$

Subtracting and using $\omega+\bar{\omega}=1$ gives $\sqrt{-d}\,g_2\in O$. Thus $d\,g''(0)=2d\,g_2\in O$, without dividing by two. Also the marked integral curve equation and Lemma 4.2.37 give

$$X=X_0(Y)\in O[[Y]],\qquad Y'=3X^2+1-A_1Y\in O[[Y]]. \tag{14}$$

The derivative identity follows by differentiating the Weierstrass parametrization, equivalently from the invariant differential $dX/(2Y+A_1X)=dz$ and the curve equation.

By (12), the coefficients of each form βL_j are integral. Put $\mu_j=d\beta L_j$. Consider the O-algebra generated by all substituted series $H(Y(\mu_j):j\in J)$, by $\vartheta(\mu_j)$, and by $g'(\mu_j)$, inside the formal series ring over $\mathbb{C}_p$. It is stable under every ordinary partial derivative. Indeed differentiating a substituted integral series uses (14) and integral coefficients of μ_j. Differentiating $\vartheta(\mu_j)$ gives $-g'(\mu_j)\vartheta(\mu_j)$ times an integral coefficient. Differentiating $g'(\mu_j)$ gives a coefficient of βL_j times $d(g''(0)+X(\mu_j))$, integral by (13)–(14). The product and chain rules therefore stay in this algebra. Evaluation at zero is integral: $Y(0)=g'(0)=0$ and $\vartheta(0)=1$. All these assertions are coefficientwise; differentiating an infinite substituted series is legitimate because its parameters have zero constant term.

Substitute $d\beta Z$ in the product identity. Since $\eta_P=\beta A/\Omega_{\mathrm{CM}}$, it becomes the actual Taylor series of $\mathscr{E}_{F_0}(\bar{\delta}d\eta_P Z)$, and belongs to the preceding derivative-stable algebra. Every ordinary partial derivative at zero is therefore integral. A homogeneous Q of degree B with integral coefficients has integral evaluation there. Homogeneous rescaling from $\bar{\delta}d\eta_P$ to $\zeta_{\mathrm{sc}}\eta_P$ multiplies that evaluation by $(\zeta_{\mathrm{sc}}/(\bar{\delta}d))^B$, proving exactly the claimed bound. For $B=0$ the same argument is evaluation at zero; for $\zeta_{\mathrm{sc}}=0, B>0$ the derivative vanishes. Empty coordinate sets are constant functions and satisfy the assertions directly.

□

Lemma 4.2.36. *Put $A=\sqrt{2\pi_{\mathbb{R}}/\sqrt{d}}$, $\mathfrak{a}=E\cap\alpha\widehat{\mathcal{O}}_E$, and let C be a finite coordinate set. Write $\mathscr{E}_F$ for the actual entire finite-part theta function of Lemma 4.2.2. Fix $0\neq\delta\in\mathcal{O}_K$, $N_\delta=\delta\bar{\delta}$, a finite set I, vectors $w_i\in\mathfrak{a}$, and integers $c_i\geq 0$ with*

$$\sum_i c_i\,\tau(w_i)\overline{\sigma(w_i)}=N_\delta\mathbf{1}_{\tau=\sigma}\quad(\tau,\sigma\in\Phi).$$

For every family $e_{i,c}\in\mathbb{Z}_{\geq 0}$ with $3e_{i,c}\leq c_i$, there is a unique vector F in the finite Heisenberg span of the same standard positive vacuum on E^C satisfying

$$\mathscr{E}_F(Z)=\prod_{i,c}Y(B_{i,c}(Z))^{e_{i,c}}\,\vartheta(B_{i,c}(Z))^{c_i},\qquad B_{i,c}(Z)=\frac{\Omega_{\mathrm{CM}}}{\bar{\delta}A}\sum_{\tau\in\Phi}\overline{\tau(w_i)}Z_{\tau,c}.$$

Each factor means the entire monomial section $\vartheta^{\,c_i-3e_{i,c}}(Y\vartheta^3)^{e_{i,c}}$. For every finite adelic $x\in\mathbb{A}_{E,f}^C$, every prime p, and every prime identification $\iota:\mathbb{C}\to\mathbb{C}_p$,

$$|\iota\,\theta(\rho(\mathrm{fv}(x))F)|_p\leq 1.$$

For every $\gamma\in\mathfrak{a}^C$,

$$\rho_f(\alpha^{-1}\bar{\delta}\gamma)F=F.$$

Equivalently $\mathscr{E}_F$ has the original signed theta periods $A(\tau(\bar{\delta}\gamma_c))_{\tau,c}$, with the fixed semicharacter.

In particular take the fixed frame of Lemma 6.7.35, $C=\{1,\dots,12\}$, and $\delta=\bar{\pi}_M$. With the same $\beta=s_0/u_{\mathrm{CS}}$ and $\eta_P=\beta A/\Omega_{\mathrm{CM}}$, its Taylor series at $\pi_M\eta_P Z$ is exactly

$$\prod_{i,c}Y(L_{i,c}(Z))^{e_{i,c}}\vartheta(L_{i,c}(Z))^{c_i},\qquad L_{i,c}(Z)=\sum_{\tau\in\Phi}\beta\,\overline{\tau(w_i)}Z_{\tau,c}.$$

The signed period in this specialization is $\pi_M \mathfrak{a}^C$. The vector, its theta functional and these identities are chosen before every prime identification; they involve no independent theta normalization or additional scalar.

Proof. **1. The principal lattice and its sign.** We first establish the lattice comparisons used for the actual model. Put $\Gamma = A(\tau(\mathfrak{a}))_{\tau\in\Phi} \subset \mathbb{C}^\Phi$, and use its product over C. By Lemma 3.2.8, the relative different is $D_{\mathrm{rel}}\mathcal{O}_E$, where $D_{\mathrm{rel}} = \mathrm{cyc}'_{q_1}(\zeta)$; its prime divisors lie above split primes of K. At any such prime exactly one of $\omega = (1+\sqrt{-d})/2$ and $\bar\omega$ vanishes. Their product is $(d+1)/4$, divisible by the rational prime, while their difference $\sqrt{-d}$ is a unit there. At 2, the two roots of the integral polynomial $T^2 - T + (d+1)/4$ remain distinct. Thus the prescribed idele satisfies

$$\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E/K}^{-1}. \tag{1}$$

In particular $\mathrm{Tr}_{E/K}(\mathfrak{a}\bar{\mathfrak{a}}) \subset \mathcal{O}_K$. Since $\mathfrak{D}_{K/\mathbb{Q}} = (\sqrt{-d})$, trace duality in the tower gives $\mathfrak{D}_{E/\mathbb{Q}} = (\sqrt{-d})\mathfrak{D}_{E/K}$. For the alternating form $\mathrm{Tr}_{E/\mathbb{Q}}(\jmath\lambda\bar\mu)$, (1) therefore makes the dual of $\mathfrak{a}$ equal to itself. In normalized real coordinates this is precisely $\pi_{\mathbb{R}}^{-1}\,\mathrm{Im}\sum_{\tau,c}\bar z_{\tau,c}z'_{\tau,c}$, up to the immaterial overall sign, because $A^2 = 2\pi_{\mathbb{R}}/\sqrt{d}$. Consequently Γ^C is a full self-dual lattice.

Here and below reduced translation on entire functions is

$$U_a S(Z) = e^{-\bar a^{\mathsf T} Z - \|a\|^2/2} S(Z+a).$$

For $\lambda = A(\tau(\gamma_c)) \in \Gamma^C$, its original sign is

$$\epsilon_\Gamma(\lambda) = (-1)^{\sum_c \mathrm{Tr}_{E/K}(\gamma_c\bar\gamma_c)}. \tag{2}$$

The exponent is an integer by (1). To check this sign against the actual finite operator, its scalar is $\psi_{E,f}(\jmath\omega\sum_c \gamma_c\bar\gamma_c/2)$. Rational character triviality changes this to the inverse archimedean character. Since $\mathrm{Re}(\jmath\omega) = -1/2$, its value is exactly (2). Thus Lemma 4.2.2 identifies the action of $\rho_f(\alpha^{-1}\gamma)$ with the signed shift $R_{-\lambda} = \epsilon_\Gamma(-\lambda)U_{-\lambda}$.

These shifts fix the actual vacuum. This can be checked directly in integral finite coordinates. Put $\chi(z) = \psi_{E,f}(\jmath\alpha\bar\alpha z)$. Equation (1) and absolute trace duality say that χ kills $\widehat{\mathcal{O}}_E$. Its conjugate satisfies $\chi(\bar z) = \chi(z)^{-1}$. For an integral translating vector t, insert the finite vacuum $\chi(\omega z^* z/2)\mathbf{1}_{\widehat{\mathcal{O}}^C}(z)$ in (WD8). Its support is unchanged. On that support the ratio to the original vacuum reduces, using $\bar\omega = 1-\omega$, to $\chi(-\omega t^* z + \omega t^* t) = 1$. This proves the assertion, also at 2.

By Lemma 4.2.2, the vacuum has an entire theta function, and it is nonzero by finite-part injectivity and the vacuum norm one from Lemma 4.1.10. It is a theta function for Γ^C with precisely the sign (2). The signed shifts satisfy their usual lattice group law, either by expanding U_aU_b and (2), or by the same Heisenberg law just used. Positivity and self-duality allow Theorem A.2.16: the entire theta space for this lattice and sign is one-dimensional.

2. The monomial product has the required signed periods. Put $G = Y\vartheta^3$, an entire degree-three section of the normalized elliptic theta line, and $s_{i,c} = \vartheta^{c_i - 3e_{i,c}} G^{e_{i,c}}$. It is an entire degree-c_i section with the same theta multiplier to that power. Let $S = \prod_{i,c} s_{i,c}\circ B_{i,c}$. The full two-embedding frame identity gives

$$\sum_{i,c} c_i \frac{\pi_{\mathbb{R}}}{\mathrm{Area}(\Lambda_0)} B_{i,c}(Z)\overline{B_{i,c}(Z')} = \sum_{\tau,c} Z_{\tau,c}\overline{Z'_{\tau,c}}, \tag{3}$$

because $\mathrm{Area}(\Lambda_0) = |\Omega_{\mathrm{CM}}|^2\sqrt{d}/2$ and $|\delta|^2 = N_\delta$. Thus the sum of the pulled-back Hermitian forms is the actual positive Fock form.

For $\gamma \in \mathfrak{a}^C$, write $\lambda = A(\tau(\bar{\delta}\gamma_c))$. Then

$$B_{i,c}(\lambda) = \Omega_{\mathrm{CM}} \operatorname{Tr}_{E/K}(\gamma_c \bar{w}_i) \in \Lambda_0, \tag{4}$$

by (1). These are periods of the same elliptic curve. Its sign on $\Omega_{\mathrm{CM}} z$, $z \in \mathcal{O}_K$, is $(-1)^{\mathrm{N}_{K/\mathbb{Q}}(z)}$, as proved explicitly in Lemma 4.2.37. Their product signs agree with (2) at λ. Indeed the frame identity also gives

$$\sum_{i,c} c_i \left| \operatorname{Tr}_{E/K}(\gamma_c \bar{w}_i) \right|^2 = N_\delta \sum_c \operatorname{Tr}_{E/K}(\gamma_c \bar{\gamma}_c).$$

Both sides are integers, and the right side is $\sum_c \operatorname{Tr}_{E/K}((\bar{\delta}\gamma_c)\overline{\bar{\delta}\gamma_c})$. Equations (3)–(4) therefore prove that $R_\lambda S = S$ on the lattice $L = \bar{\delta}\Gamma^C$, with exactly the original sign, not merely some line-bundle multiplier.

3. Finite Fourier inversion realizes the product in the same model. We give the finite argument so no abstract surjectivity of a theta map is presumed. Put $G_0 = \Gamma^C/L$. Its cardinal is $D_0 = N_\delta^{f|C|}$. Self-duality identifies its full character group with $Y_0 = L^\#/\Gamma^C = \delta^{-1}\Gamma^C/\Gamma^C$, by the commutator pairing of reduced translations. The pairing is perfect: its annihilator is precisely Γ^C, and the two finite groups have the same cardinal, by unimodularity of the lattice pairing.

For $\mathcal{A} = \sum_{\lambda \in G_0} R_\lambda$, every $\mathcal{A}U_y S$, $y \in Y_0$, is a theta function for Γ^C. It is therefore $b_y S_{\mathrm{vac}}$, where S_{vac} is the actual vacuum theta function. Commuting translations and summing the characters gives the exact inversion

$$\sum_{y \in Y_0} U_{-y} \mathcal{A} U_y S = D_0 S. \tag{5}$$

Indeed the coefficient of $R_\lambda S$ is the sum of every character at λ, equal to D_0 at zero and zero elsewhere. Representatives of both finite groups can be chosen among normalized real images of rational vectors of E^C. The raw rational finite-translation formula of Lemma 4.2.2 realizes each $U_{-y}S_{\mathrm{vac}}$ by the corresponding finite Heisenberg translate of the same vacuum. Thus (5) constructs a finite-part vector F with $\mathscr{E}_F = S$. It does not choose another theta functional or rescale the prescribed one. Injectivity in that lemma proves uniqueness. It also transports $R_\lambda S = S$ back to $\rho_f(\alpha^{-1}\bar{\delta}\gamma)F = F$, including its literal (WD8) scalar.

4. All finite translated theta values are integral at every prime. First note the rank-one monomial integrality, including the prime 2. For any torsion ξ, choose a nonzero integer m with $m\xi \in \Lambda_0$. The translated assertion of Lemma 4.2.37, evaluated at zero, says that the reduced value $\vartheta[\xi]$ is integral at every prime identification. The normalized duplication identity $\vartheta[2\xi] = \vartheta[\xi]^4 - \varsigma[\xi]^4$ then proves the same for $\varsigma[\xi]$. Finally the curve equation gives the monic identity

$$G[\xi]^2 + A_1 \varsigma[\xi]^2 \vartheta[\xi] G[\xi] = \varsigma[\xi]^6 + \varsigma[\xi]^2 \vartheta[\xi]^4.$$

By Lemma 4.2.37, the coefficient A_1 is integral in this exact normalization. If $|G[\xi]|_p > 1$, its squared term dominates both the linear term and the right side, impossible. Hence $G[\xi]$, and thus every $s_{i,c}[\xi]$, is integral. This argument uses their actual reduced values.

The finite-part vector F is smooth at finite places. More explicitly it is a finite sum of finite translates of the compactly supported, locally constant vacuum; a sufficiently small open finite neighbourhood fixes each summand under raw finite Heisenberg translation. Given any x, density of E^C in the finite adeles therefore gives $\gamma \in E^C$ with $\rho(\mathrm{fv}(x))F = \rho(0, \gamma_f)F$ exactly. This equality is chosen before any potentially discontinuous ι is applied.

By the exact translation law, the theta value of this vector is $e^{-\|a_\gamma\|^2/2}S(-a_\gamma)$, with $a_\gamma = A(\tau(\gamma_c))$. Equation (3) makes this Gaussian the product of the degree-c_i Gaussians of its factors. Therefore this value is exactly the product of the reduced values of $s_{i,c}$ at

$$-\frac{\Omega_{\mathrm{CM}}}{\bar{\delta}}\,\mathrm{Tr}_{E/K}(\gamma_c\bar{w}_i).$$

Each argument is torsion, because it belongs to $\Omega_{\mathrm{CM}}K$. The rank-one assertion just proved gives integrality at every p, ι, proving the stated bound.

5. The sharpness specialization and ordinary Taylor coordinates. Take $\delta = \bar{\pi}_M$; then $N_\delta = M$, and the signed period lattice in Section2 is $\pi_M\Gamma^C$. Since $\eta_P = \beta A/\Omega_{\mathrm{CM}}$, direct substitution gives

$$B_{i,c}(\pi_M\eta_P Z) = \sum_\tau \beta\,\overline{\tau(w_i)}Z_{\tau,c}.$$

This is the displayed specialization with no conjugate ratio or scalar left over. The locally uniformly convergent entire series in Lemma 4.2.2 is the ordinary Taylor series of this same function; hence its substitution is exactly the asserted formal product. If C is empty, the product and theta function are constant one and the same conclusions hold directly in the rank-zero product model. □

Lemma 4.2.37. *Let $d \equiv 7 \pmod 8$ be prime, $K = \mathbb{Q}(\sqrt{-d})$, $\omega = (1+\sqrt{-d})/2$, and $\Lambda_0 = \Omega_{\mathrm{CM}}\mathcal{O}_K \subset \mathbb{C}$. Choose the nonzero two-torsion class t_2 for which $\omega t_2, \bar{\omega}t_2 \notin \Lambda_0$, and normalize $\wp''(t_2) = 2$. Put $A_0 = \mathrm{Area}(\Lambda_0)$ and $H_0(u,z) = \pi_{\mathbb{R}}\bar{u}z/A_0$. Let ς be the reduced sigma function with $\varsigma'(0) = 1$, and set*

$$\vartheta(z) = e^{-H_0(t_2,z)}\frac{\varsigma(z+t_2)}{\varsigma(t_2)}, \quad X(z) = \wp(z+t_2) - \wp(t_2),$$
$$Y(z) = \frac{\wp'(z+t_2) - A_1X(z)}{2}, \qquad A_1^2 = 12\wp(t_2).$$

Thus $\vartheta(0) = 1$, $Y(0) = 0$, $Y'(0) = 1$, and the marked curve is $Y^2 + A_1XY = X^3 + X$. Write χ_0 for the actual multiplier of ϑ, so $\vartheta(z+\lambda) = \chi_0(\lambda)e^{H_0(\lambda,z)+H_0(\lambda,\lambda)/2}\vartheta(z)$. Fix any prime p and $\iota : \mathbb{C} \to \mathbb{C}_p$, and put $O = \mathcal{O}_{\mathbb{C}_p}$. In this normalization, $|\iota A_1|_p \le 1$ and $|\iota(A_1^4 - 64)|_p = 1$.

Let $n \ge 0$ be an integer and let $S : \mathbb{C} \to \mathbb{C}$ be entire with the exact multiplier

$$S(z+\lambda) = \chi_0(\lambda)^n e^{nH_0(\lambda,z)+nH_0(\lambda,\lambda)/2}S(z) \quad (\lambda \in \Lambda_0).$$

For a torsion point $u \in \mathbb{Q}\Lambda_0$, define its reduced value $S[u] = e^{-n\pi_{\mathbb{R}}|u|^2/(2A_0)}S(u)$. If $|\iota S[u]|_p \le 1$ for every such u, then there is $H_S \in O[[T]]$ such that the coefficientwise Taylor identity at zero is

$$\iota S(z) = \iota\vartheta(z)^n H_S(\iota Y(z)).$$

In particular the following identities use this same theta and parameter. For every $\alpha \in \mathcal{O}_K$,

$$\frac{\iota\vartheta(\alpha z)}{\iota\vartheta(z)^{\alpha\bar{\alpha}}} \in O[[Y]].$$

For $0 \ne \delta \in \mathcal{O}_K$ and $\xi \in \mathbb{C}$ with $\delta\xi \in \Lambda_0$, put $(T_\xi\vartheta)(z) = e^{-H_0(\xi,z)-H_0(\xi,\xi)/2}\vartheta(z+\xi)$. Then

$$\frac{\iota((T_\xi\vartheta)(\bar{\delta}z))}{\iota\vartheta(z)^{\delta\bar{\delta}}} \in O[[Y]].$$

Finally, writing $g = -\log(\iota\vartheta)$ at zero, for every $a_{\mathrm{loc}} \in \mathbb{Z}_p$,

$$\exp(a_{\mathrm{loc}}^2 g(z) - g(a_{\mathrm{loc}} z)) \in O[[Y]].$$

Here $O[[Y]]$ means substitution in the displayed original Taylor parameter, and all derivatives and Taylor series have their ordinary normalization.

Proof. **1. Integral normalization at every prime.** The normalized curve has discriminant $\Delta_0 = A_1^4 - 64$ and $j = (A_1^4 - 48)^3/(A_1^4 - 64)$, by translating the Weierstrass equation to the marked point. By Theorem A.2.17, j, and hence A_1, is algebraic. Enlarge a finite local field until the curve, its CM endomorphisms and marked point are defined and it has good reduction. The CM endomorphisms extend to its good elliptic scheme by Theorem A.2.19, and their relations extend by uniqueness.

The marked point has nonzero reduction. At odd residue characteristic, reduction is injective on two-torsion. At 2, the two idempotents of $\mathcal{O}_K \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ split the divisible group into factors of height one, since both characteristic-zero Tate ranks are one. Their dimensions add to one; exactly one height-one factor is etale. The marked point is nonzero in both factors of $\mathcal{O}_K/2$, so has nonzero etale reduction. This proves the assertion also at 2.

In an integral good Weierstrass equation this point is affine. Move it to $(0, 0)$. Since it is its own inverse, $a_3 = a_6 = 0$; smoothness there forces a_4 to be a unit. After a finite integral extension choose a root of $u^2 + a_1 u = a_2$, and replace y by $y + ux$, removing the x^2 term. A unit fourth root of a_4 then gives $y^2 + A_1' xy = x^3 + x$, with A_1' integral and unit discriminant. A characteristic-zero change between this equation and the normalized equation in the statement preserves the identity and marked point, so has $x = v^2 x'$, $y = v^3 y' + u' x'$; comparison of the coefficient of x gives $v^4 = 1$. Their discriminants therefore differ by a unit. Thus

$$|\iota A_1|_p \le 1, \qquad |\iota(A_1^4 - 64)|_p = 1. \tag{1}$$

This proves integrality for the exact normalization, including 2.

2. Integral reduced torsion values and unit interpolation nodes. For a degree-one section use the reduced value in the statement. The Weierstrass addition and duplication identities, in the normalization $\wp''(t_2) = 2$, give

$$\vartheta[2u] = \vartheta[u]^4 - \varsigma[u]^4, \qquad W(2u) = D_{A_1}(W(u)),$$

where

$$W(u) = (\vartheta[u]^2, \varsigma[u]^2), \quad D_c(v, w) = ((v^2 - w^2)^2,\ vw(4v^2 + c^2 vw + 4w^2)). \tag{2}$$

They follow by substituting $X = \varsigma^2/\vartheta^2$ into the duplication formula on $Y^2 + A_1 XY = X^3 + X$; the Gaussian at $2u$ is the fourth power of that at u. Equality away from zeros extends by holomorphy. The two entries of W never vanish simultaneously, because their zero classes are the distinct classes t_2 and zero.

By (1), the reduced quartics $D_{\iota A_1}$ have no common nonzero zero. Indeed the first gives $w = \pm v$; the second then gives $A_1^2 = \mp 8$, contrary to $A_1^4 - 64 \neq 0$. This argument works in characteristic two as well. Dividing by a coordinate of maximum norm, reducing and using homogeneity proves $\|D_{\iota A_1}(v, w)\|_{\max} = \|(v, w)\|_{\max}^4$. At torsion points, reduced values at points differing by a period differ by roots of unity: the alternating lattice pairing of a torsion point and a period is rational. The doubling orbit modulo periods is finite. Thus the nonzero maximum norm of $W(u)$ has two equal distinct fourth-power iterates and must equal one. Consequently both reduced torsion values are integral and

$$\|\iota W(u)\|_{\max} = 1 \quad (u \in \mathbb{Q}\Lambda_0). \tag{3}$$

We spell out the separation argument needed at 2. If two integral pairs of norm one have a nonunit determinant, their nonzero residue pairs are proportional. Iterating the same homogeneous map $D_{\iota A_1}$ preserves proportionality. Hence they cannot have iterates respectively of the forms $(0,*)$ and $(*,0)$, with the first nonzero. For $j \geq 0$, doubling $t_2/2^j$ exactly j times gives t_2; doubling any v with $2^j v \in \Lambda_0$ gives a period. Their pairs have exactly these two coordinate patterns. Their determinant is therefore a unit.

Set $u_i = t_2/2^{i+1}$, $i \geq 0$. Apply this determinant assertion first against t_2, and then against the period $2t_2$. In conjunction with (3) it gives

$$|\iota\vartheta[u_i]|_p = 1, \qquad |\iota\varsigma[t_2/2^i]|_p = 1. \tag{4}$$

The identity $W(u)_2 = W(u)_1 X(u)$, with two distinct halving points and the same determinant argument, gives

$$X(u_i) \in O, \qquad |\,\iota X(u_i) - \iota X(u_j)\,|_p = 1 \quad (i \neq j). \tag{5}$$

The duplication identity $\varsigma[2u] = \vartheta[u]^4 X'(u)$ gives

$$|\iota X'(u_i)|_p = 1. \tag{6}$$

There is also an integral value of $Y(u_i)$. The entire cubic section $G = Y\vartheta^3$ satisfies, at torsion points,

$$G[u]^2 + A_1\varsigma[u]^2\vartheta[u]G[u] = \varsigma[u]^6 + \varsigma[u]^2\vartheta[u]^4. \tag{7}$$

This is the curve equation multiplied by ϑ^6, extended through theta zeros. Its coefficients are integral by (3). If $|\iota G[u]|_p > 1$, the quadratic term has strictly larger norm than the linear term and the right side, impossible. Thus $G[u]$ is integral; dividing by the unit in (4) gives $Y(u_i) \in O$.

3. Polynomial decomposition and integral interpolation. For $n = 0$, S is entire and elliptic, hence constant, and its torsion value at zero proves the conclusion. Let $n > 0$. The multiplier assumption makes S/ϑ^n elliptic, with no poles except at $t_2 + \Lambda_0$, of order at most n. At that point X has a double pole and Y a triple pole. Successively cancel the highest pole by X^j or YX^j, respectively of orders $2j$ and $2j+3$. These orders cover every integer at least two. A remaining elliptic function with at most one simple pole has no pole: the residue theorem on a fundamental parallelogram makes its residue zero. It is constant. Thus

$$S = \vartheta^n(R_0(X) + YR_1(X)), \tag{8}$$

where R_0, R_1 are polynomials and their allowed pole orders are at most n.

At u_i and $-u_i$, the integrality hypothesis and (4) show that $R_0(X(u_i)) + Y(\pm u_i)R_1(X(u_i))$ is integral. Since X is even and $Y(u) - Y(-u) = X'(u)$, their difference and (6) show that $R_1(X(u_i))$ is integral. As $Y(u_i)$ is integral, $R_0(X(u_i))$ is integral too. For either polynomial, apply Lagrange interpolation at one more than its degree of the nodes (5). Each denominator is a product of unit differences, and each numerator has integral coefficients. Hence $R_0, R_1 \in O[X]$.

The formal implicit equation $Y^2 + A_1XY = X^3 + X$ at $(0,0)$ has derivative -1 with respect to X, so its unique solution $X = X_0(Y)$ belongs to $O[[Y]]$. Substituting in (8) gives the asserted H_S.

4. Exact multipliers of CM pullbacks and translates. The distinguished two-torsion class is $\Omega_{\mathrm{CM}}/2$ modulo Λ_0. For $\lambda = \Omega_{\mathrm{CM}}(c + e\omega)$, the sigma sign is $(-1)^{c+e+ce}$; translation by t_2 adds $(-1)^e$. Thus

$$\chi_0(\lambda) = (-1)^{c+ce} = (-1)^{\mathrm{N}_{K/\mathbb{Q}}(c+e\omega)}, \tag{9}$$

because $(d+1)/4$ is even. It follows, including when $\mathrm{N}\alpha$ is even, that $\chi_0(\alpha\lambda) = \chi_0(\lambda)^{\alpha\bar{\alpha}}$. The Hermitian term scales by $\alpha\bar{\alpha}$. Therefore $\vartheta(\alpha z)$ has exactly the degree-$\alpha\bar{\alpha}$ multiplier required

in the statement. Its reduced torsion value is $\vartheta[\alpha u]$, integral by (3). Section 3 proves the first corollary; $\alpha = 0$ is the constant-one case.

For $S(z) = (T_\xi \vartheta)(\bar{\delta} z)$, the Hermitian term has degree $\delta\bar{\delta}$. The extra multiplier on translation by λ is

$$\exp(H_0(\bar{\delta}\lambda, \xi) - H_0(\xi, \bar{\delta}\lambda)) = \exp(2\pi_{\mathbb{R}} i \, E_0(\lambda, \delta\xi)) = 1,$$

where $E_0(u, v) = \mathrm{Im}(\bar{u}v)/A_0$ is integral on the period lattice; changing the sign of this alternating expression has the same value one. Equation (9) supplies the remaining multiplier. Since $\delta\xi$ is a period, ξ is torsion. At a torsion point u, its reduced value is a root of unity times $\vartheta[\bar{\delta}u + \xi]$: completing the Hermitian square cancels the real Gaussian, and the remaining phase is the exponential of $\pi_{\mathbb{R}} i$ times a rational alternating period pairing. It is integral by (3). This proves the translated corollary by Section 3.

In particular even scalars have not been inferred merely from the degree of a CM map. The exact duplication identity in (2) gives explicitly

$$\frac{\vartheta(2z)}{\vartheta(z)^4} = 1 - X(z)^2 \in O[[Y]],$$

with constant coefficient one. This also directly checks the even-scalar normalization.

5. All p-adic integral scalars. The integer case just proved is $\exp(a_{\mathrm{loc}}^2 g - g(a_{\mathrm{loc}} z)) \in O[[Y]]$ for $a_{\mathrm{loc}} \in \mathbb{Z}$. Choose integers tending to any prescribed $a_{\mathrm{loc}} \in \mathbb{Z}_p$. Each coefficient of the exponential is a polynomial in a_{loc}, because $g(0) = 0$; hence the series converges coefficientwise. Substitution by $Y = z + \cdots$ is a coefficientwise continuous bijection of $\mathbb{C}_p[[z]]$, with coefficientwise continuous inverse. Since O is closed in $\mathbb{C}_p$, $O[[Y]]$ is closed for this convergence. The limit gives the last assertion. □

Proposition 4.2.38. *All additional notation is local. Let C be a finite coordinate set of size n_C, and use the standard positive hermitian space E^C. For a prime identification $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, put $O_\iota = \{z \in \mathbb{C} : |\iota z|_p \le 1\}$ and*

$$\mathcal{D}_p = \mathrm{span}_{O_\iota}\{H(x) c_s^{n_C} v^0 : x \in \mathbb{A}_{E,f}^C, \ x_w \in \mathcal{O}_{E,w}^C \ (w \mid p)\}.$$

For $F_0 \in \mathcal{D}_p$, let $\mathcal{T}_{F_0}(Z)$ be the Taylor series of its actual entire theta function at $\eta_P Z$, with coefficients read through ι. Let p be nonsplit in K, put $O = \mathcal{O}_{\mathbb{C}_p}$, and set

$$(\pi, q, \rho, S) = \begin{cases} (p, p^2, (p^2-1)^{-1}, \{1, 3, 5, \ldots\}), & p \text{ inert}, \\ (\iota(\sqrt{-d}), p, [2(p-1)]^{-1}, \mathbb{Z}_{\ge 0}), & p = d. \end{cases}$$

There exist $\varpi \in \mathbb{C}_p^\times$ with $v_p(\varpi) = \rho$ and coefficients $\lambda_{e\ell} \in O$, chosen independently of F_0, such that for every $F_0 \in \mathcal{D}_p$ there exists $W_{F_0} \in O[[Z_{\tau,c} : \tau \in \Phi, c \in C]]$ satisfying

$$\mathcal{T}_{F_0}(\varpi Z) = W_{F_0}(Z) \exp\left(\frac{\varpi^{p+1}}{p} \sum_{e \in S, \ell \ge 0} \lambda_{e\ell} \Gamma_e(Z)^{p^\ell} \right), \qquad \Gamma_e(Z) = \sum_{\tau,c} Z_{\tau,c}^{p^e} Z_{u(-p^e)\tau,c}.$$

All sums are interpreted coefficientwise. At inert primes every λ_{e0} is a unit. The same assertion applies to $C = \{1, \ldots, N_{\mathrm{aux}}\} \times \{1, \ldots, r_{\mathrm{aux}}\}$ for the standard space $\mathrm{Hom}(E^{r_{\mathrm{aux}}}, E^{N_{\mathrm{aux}}})$, with its fixed orthonormal frames and $r_{\mathrm{aux}} \le N_{\mathrm{aux}}$. Primes dividing q_1 are not additionally excluded: they split in K, and hence do not occur among the primes in this assertion.

Proof. **1. The actual product expansion at the principal-period scale.** The raw finite operators and $H(x) = \rho_f(x)$ differ by the literal (WD8) finite-character scalar. It is a root of

unity, so is a unit under every prime identification. Thus the module in the statement is precisely the module in Proposition 4.2.35 with $\delta = 1$. That result supplies, for every F_0, vectors $u_j \in E^C$, positive integers m_j, the full two-index identity

$$\sum_j m_j \tau(u_{j,c})\overline{\nu(u_{j,c'})} = \mathbf{1}_{\tau=\nu}\mathbf{1}_{c=c'}, \tag{1}$$

and an actual theta expansion at scale A/Ω_{CM}, where $A = \sqrt{2\pi_{\mathbb{R}}/\sqrt{d}}$:

$$\iota[\mathscr{E}_{F_0}(AZ/\Omega_{\mathrm{CM}})] = \prod_j \vartheta_\iota(L_j(Z))^{m_j} H_{F_0}(Y_\iota(L_j(Z)) : j), \qquad L_j(Z) = \sum_{\tau,c} \iota(\overline{\tau(u_{j,c})}) Z_{\tau,c}. \tag{2}$$

Here H_{F_0} has integral coefficients. The simultaneous choice in Lemma 4.2.3 uses the same scalar c_s supplied by Proposition 4.2.1, which gives its all-prime bound at every finite theta value. The tensor choice of the standard positive theta functional is the one used in Proposition 4.2.35. Thus (2) is the entire function of the given vector F_0, with no additional scalar normalization.

Every prime dividing q_1 splits in K, by Lemma 3.2.8; hence a nonsplit p does not divide q_1. Also $d \equiv 7 \pmod 8$, so the minimal polynomial $T^2 - T + (d+1)/4$ of $\omega = (1+\sqrt{-d})/2$ has two distinct roots modulo 2. Since $\mathcal{O}_K = \mathbb{Z}[\omega]$, the prime 2 splits in K. Thus p is odd. The different generator $D_{\mathrm{rel}} = \mathrm{cyc}'_{q_1}(\zeta)$ is integral and divides q_1, again by Lemma 3.2.8; since $p \nmid q_1$, it is a unit at every place above p. Each component of the fixed α is 1 or D_{rel}^{-1}, so α is a unit at all these places. The α-integrality in Proposition 4.2.35 therefore says that every $u_{j,c}$ is integral at every place above p. Hence all coefficients of L_j lie in O.

2. The full resolution gives the exact partner coupling modulo π. Fix $e \in S$. By Lemma 4.2.45,

$$\iota(\overline{\tau(u_{j,c})})^{p^e} \equiv \iota((u(-p^e)\tau)(u_{j,c})) \pmod{\pi O}.$$

The quotient $O/\pi O$ has characteristic p, including the ramified case where $p = -\pi^2$. Frobenius is additive in this quotient, with no assumption that the quotient is reduced. Expanding $L_j^{p^e} L_j$, applying the congruence and then (1) gives

$$\sum_j m_j L_j(Z)^{p^e} L_j(Z) \equiv \sum_{\tau,c} Z_{\tau,c}^{p^e} Z_{u(-p^e)\tau,c} = \Gamma_e(Z) \pmod{\pi O[[Z]]}. \tag{3}$$

More explicitly, expansion as an ordered sum over $(\tau,c),(\nu,c')$ gives the scalar multiplying each ordered term as

$$\sum_j m_j\, \iota(\overline{\tau(u_{j,c})})^{p^e} \iota(\overline{\nu(u_{j,c'})}) \equiv \iota\left(\sum_j m_j (u(-p^e)\tau)(u_{j,c})\overline{\nu(u_{j,c'})}\right) \pmod{\pi O}.$$

By (1) this is $\mathbf{1}_{u(-p^e)\tau=\nu}\mathbf{1}_{c=c'}$. Summing these ordered terms gives (3). This argument also covers $e = 0$ at $p = d$, even when two ordered terms give the same commutative monomial: the identities are summed before equal monomials are collected.

3. Apply the actual one-curve factorization with a common scale. Choose ϖ from Lemma 4.2.41; its parameter equation gives exactly the valuation ρ in the statement. The same lemma gives $Y_\iota(\varpi L_j(Z)) \in O[[Z]]$ with zero constant term, by first applying its one-variable scaling assertion and then substituting the integral linear form L_j. Thus the auxiliary factor of (2), evaluated at ϖZ, is integral.

Use the actual one-curve factorization Lemma 4.2.39 for this ϖ. Its coefficients, denoted $\lambda^0_{e\ell}$, are independent of F_0 and of its resolution. They are integral, and λ^0_{e0} are units at inert primes.

The required error-exponential valuation is

$$v_p\left((\varpi^{p+1}/p)\pi\right) = \begin{cases} (p+1)/(p^2-1) - 1 + 1, & p \text{ inert}, \\ (p+1)/(2(p-1)) - 1 + 1/2, & p = d, \end{cases} = \frac{1}{p-1}.$$

All hypotheses of Lemma 4.2.46 now hold: its one-variable factor is the same normalized theta, its scaled multiplier is the integral auxiliary factor just checked, and its coupling congruences are (3). Consequently (2) at ϖZ has an integral prefactor and exponential

$$\exp\left(\frac{\varpi^{p+1}}{p}\sum_{e\in S,\ell\geq 0}\lambda^0_{e\ell}\Gamma_e(Z)^{p^\ell}\right). \tag{4}$$

Every sum is coefficientwise defined since the degrees $(p^e+1)p^\ell$ tend to infinity. The exponential in (4) is common to all F_0, even though their integral prefactors and finite resolutions differ.

4. Restore the exact derivative scale. Put $f = [E:K] = |\Phi| > 0$ and $\beta = s_0/u_{\mathrm{CS}}$, with exactly the simultaneous choices of Lemma 4.2.3. Define, on finite ideles,

$$\chi_{\mathrm{fin}}(x) = \upsilon(1,x)|x|^{-1/2}_{\mathbb{A}_E}.$$

The equation for the prescribed principal root in that lemma is

$$s_0^f = \chi_{\mathrm{fin}}(\alpha)^{-1}. \tag{5}$$

We check the p-adic absolute value of its right side by exact principal-idele algebra.

First χ_{fin} is identically 1 on $\widehat{\mathcal{O}}_E^\times$. Such an idele has norm 1, and its infinite components are 1, so (WD3) gives $\upsilon(1,x) = 1$. For $a \in E^\times$, triviality of υ on the principal idele and its given infinite part give

$$\upsilon(1,a_f) = \prod_{\tau\in\Phi}\frac{\overline{\tau(a)}}{|\tau(a)|}.$$

The adelic product formula gives $|a_f|^{-1/2}_{\mathbb{A}_E} = \prod_{\tau\in\Phi}|\tau(a)|$. Consequently

$$\chi_{\mathrm{fin}}(a_f) = \prod_{\tau\in\Phi}\overline{\tau(a)}. \tag{6}$$

No assertion about continuity of ι is used here.

Let $h > 0$ kill the ideal class of the fractional ideal determined by α; such h exists by finiteness of the ideal class group. Choose $a \in E^\times$ generating that ideal to its hth power. Then $\alpha^h/a_f \in \widehat{\mathcal{O}}_E^\times$. The finite-unit assertion and (6) yield the equality of complex numbers

$$\chi_{\mathrm{fin}}(\alpha)^h = \prod_{\tau\in\Phi}\overline{\tau(a)}. \tag{7}$$

For each τ, let w_τ be the place of E induced by $\iota\bar{\tau}$, with absolute value extending the usual p-adic absolute value. The relation between the ideals, or directly the local equality $\alpha^h_{w_\tau}/a \in \mathcal{O}^\times_{E,w_\tau}$, gives

$$|\iota\overline{\tau(a)}|_p = |a|_{w_\tau,p} = |\alpha_{w_\tau}|_p^h = 1.$$

The last equality uses the unit property at every place above p, proved in Step 1; it applies both to inert primes and to the ramified prime $p = d$. Apply ι to (7), take absolute values, and then the positive hth root. We obtain $|\iota\chi_{\mathrm{fin}}(\alpha)|_p = 1$. Equation (5) and $f > 0$ therefore

give $|\iota s_0|_p = 1$. Finally the same u_{CS} is a unit under every such identification by the explicit assertion of Lemma 4.2.3. Hence

$$|\iota\beta|_p = |\iota s_0|_p / |\iota u_{\mathrm{CS}}|_p = 1. \tag{8}$$

This establishes the needed unit assertion from the stated period-root choices, without imposing a further condition on c_s or changing the theta functional.

Now $\eta_P = \beta A/\Omega_{\mathrm{CM}}$. Substitute $Z \mapsto \iota(\beta)Z$ in (4) and in its integral prefactor. The latter stays integral. Since Γ_e is homogeneous of degree $p^e + 1$, set

$$\lambda_{e\ell} = \lambda^0_{e\ell}\, \iota(\beta)^{(p^e+1)p^\ell}.$$

These coefficients are integral, independent of F_0, and retain every inert leading-unit assertion. The resulting series is exactly $\mathcal{T}_{F_0}(\varpi Z)$, proving the claimed normal form with the original derivative scale and vacuum scalar. The matrix coordinate case is simply the indicated finite coordinate set in the same standard positive space; no further argument or new normalization is needed. □

Lemma 4.2.39. *Put*

$$\rho = v_p(\pi)/(q-1), \qquad A = \varpi^{p+1}/p, \qquad S = \begin{cases} \{1,3,5,\ldots\}, & p \text{ inert}, \\ \mathbb{Z}_{\geq 0}, & p = d. \end{cases}$$

Thus $\varpi^{q-1} = -\pi/b_1$ *and* $v_p(\varpi) = \rho$. *There exist* $W, \Xi \in O[[z]]$ *and* $\lambda \in O$ *such that*

$$W(0) = 1, \quad \Xi(0) = 0, \quad \theta(\varpi z) = W(z)\exp(A\Xi(z)),$$
$$\Xi(z) \equiv \sum_{e\in S, k\geq 0} \lambda_{ek}(z^{p^e} z)^{p^k} \pmod{\pi O[[z]]},$$

where

$$\lambda_{ek} = \begin{cases} \lambda, & p \text{ inert}, \\ \lambda/2, & p = d,\ e = k = 0, \\ \lambda, & p = d,\ e = 0,\ k \geq 1, \\ 2\lambda, & p = d,\ e \geq 1. \end{cases}$$

At inert p, λ *is a unit, so every* λ_{e0} *is a unit. These assertions hold for each of the scales* ϖ *satisfying the displayed parameter equation.*

Proof. Put $g = -\log\theta$, and let $g^\flat$ denote its part in degrees congruent to $p+1$ modulo $q-1$. Set $i = p+1$ in the inert case and $i = 2$ in the ramified case. By Proposition 4.2.40 there are $\ell \in O$ and $J \in O[[U]]$ with

$$qg^\flat(z) - g^\flat(\pi z) = \pi\left(\ell T(z)^i + T(z)^{i+q-1} J(T(z))\right), \qquad \exp(g^\flat - g) \in O[[T]]. \tag{1}$$

By Lemma 4.2.41,

$$T(\varpi z) = \varpi Y(z), \qquad Y \in O[[z]], \qquad Y \equiv \sum_{j\geq 0} z^{q^j} \pmod{\pi}. \tag{2}$$

Initially define $\Xi = -A^{-1}g^\flat(\varpi z)$ over $\mathbb{C}_p$. It vanishes below degree i, by the degree support of $g^\flat$ and its zero constant term. Substitute ϖz in (1), and put

$$c = -\frac{\pi\varpi^i}{qA}.$$

The resulting identity is

$$\Xi(z) - q^{-1}\Xi(\pi z) = c\ell Y(z)^i + c\varpi^{q-1}Y(z)^{i+q-1}J(\varpi Y(z)). \tag{3}$$

Here $c = -1$ in the inert case. In the ramified case $q = p$, $i = 2$ and $c = -\pi\varpi^{1-p} = b_1$, since $\varpi^{p-1} = -\pi/b_1$. Thus $c \in O^\times$ in both cases. Also $\varpi^{q-1} \in \pi O^\times$ and $J(\varpi Y) \in O[[z]]$. With $\lambda = c\ell$, (3) gives

$$(1 - \pi^n/q)[z^n]\Xi - \lambda[z^n]Y^i \in \pi O \quad (n \geq 0).$$

All hypotheses of Lemma 4.2.44 are now satisfied. It gives $\Xi \in O[[z]]$, $\Xi(0) = 0$ and exactly the stated double-series reduction, including the ramified coefficient $\lambda/2$ in degree two.

Define $W(z) = \exp(g^\flat(\varpi z) - g(\varpi z))$. By (1) this is an integral series in $T(\varpi z) = \varpi Y(z)$, so belongs to $O[[z]]$. Its constant coefficient is one. The identity

$$\theta(\varpi z) = \exp(g^\flat(\varpi z) - g(\varpi z)) \exp(-g^\flat(\varpi z))$$

is precisely the claimed factorization.

At an inert prime the coefficient of degree $p+1$ in Ξ is $-p[z^{p+1}]g$. The negative logarithm of the reduced translated theta has second derivative equal to x plus a constant, so Lemma 4.2.43 proves that this coefficient is a unit. The unit conclusion of Lemma 4.2.44 then makes λ and every required λ_{e0} units. Every step used only the displayed equation for ϖ, so no choice among its roots is excluded. □

Proposition 4.2.40. *Let $d \equiv 7 \pmod 8$ be prime, $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$, and $\Lambda \subset \mathbb{C}$ a rank-two lattice stable under $\mathcal{O}_K$. Let $A > 0$ be its Euclidean covolume and put $H(w, z) = \pi_{\mathbb{R}}\bar{w}z/A$, where $\pi_{\mathbb{R}} = 3.14159\ldots$. Let s_Λ be the odd entire reduced sigma function, uniquely normalized by $s'_\Lambda(0) = 1$ and*

$$s_\Lambda(z + \ell) = \epsilon(\ell) \exp(H(\ell, z) + H(\ell, \ell)/2)\, s_\Lambda(z) \quad (\ell \in \Lambda),$$

where $\epsilon(\ell) = 1$ if $\ell \in 2\Lambda$ and -1 otherwise. Choose $t \notin \Lambda$ with $2t \in \Lambda$, require $\omega t, \bar{\omega} t \notin \Lambda$ for $\omega = (1 + \sqrt{-d})/2$, and require

$$g_2(\Lambda) = 12\wp_\Lambda(t)^2 - 4, \qquad a_1^2 = 12\wp_\Lambda(t).$$

Define exactly

$$\theta(z) = e^{-H(t,z)}\frac{s_\Lambda(z + t)}{s_\Lambda(t)}, \quad x(z) = \wp_\Lambda(z + t) - \wp_\Lambda(t), \quad y(z) = \frac{\wp'_\Lambda(z + t) - a_1 x(z)}{2}.$$

Fix $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$ and read all Taylor series coefficientwise through ι. Let p be odd and nonsplit in K, put $O = \mathcal{O}_{\mathbb{C}_p}$, and set $(\pi, q) = (p, p^2)$ if p is inert, and $(\pi, q) = (\iota(\sqrt{-d}), p)$ if $p = d$. Let $g = -\log\theta \in z\mathbb{C}_p[[z]]$, with the formal logarithm at $\theta(0) = 1$, and let $g^\flat$ retain exactly the degrees congruent to $p + 1$ modulo $q - 1$. Then

$$qg(z) - g(\pi z) \in \pi O[[y]], \qquad \exp(g^\flat(z) - g(z)) \in O[[y]].$$

In particular these assertions use the displayed reduced-sigma normalization, without an additional quadratic Gaussian factor. With T the parameter of Lemma 4.2.41 and $i = p + 1$ in the inert case, $i = 2$ in the ramified case, the first assertion has the refined form

$$qg^\flat - g^\flat(\pi z) = \pi(\ell T^i + T^{i+q-1}J(T)), \qquad \ell \in O, \quad J \in O[[U]].$$

Proof. **1. Exact theta multipliers and the original algebraic model.** Work over $\mathbb{C}$ until explicitly applying ι. Put $E(u,v) = \mathrm{Im}(\bar{u}v)/A$. The sigma transformation in the statement gives

$$\theta(z+\lambda) = \chi(\lambda) e^{H(\lambda,z)+H(\lambda,\lambda)/2}\theta(z), \qquad \chi(\lambda) = \epsilon(\lambda) e^{2\pi_{\mathbb{R}} i E(\lambda,t)}. \tag{1}$$

The form E is integral on $\Lambda \times \Lambda$, so $2t \in \Lambda$ implies $\chi(\lambda) \in \{1,-1\}$. Since $t \notin \Lambda$, $\epsilon(2t) = -1$. Translation of the odd sigma by $2t$ gives $s_\Lambda(z+t) = -e^{2H(t,z)} s_\Lambda(z-t)$, and hence $\theta(-z) = \theta(z)$. In particular $\theta(0) = 1$ and $g_0 = g_1 = 0$.

For every nonzero $\alpha \in \mathcal{O}_K$ with $\alpha \equiv 1 \pmod{2\mathcal{O}_K}$ put $N = \alpha\bar{\alpha}$, an odd positive integer. Also $\bar{\alpha} \equiv 1 \pmod{2\mathcal{O}_K}$. Thus multiplication by α is the identity on $\Lambda/2\Lambda$ and $(\bar{\alpha}-1)t \in \Lambda$. Consequently

$$\epsilon(\alpha\lambda) = \epsilon(\lambda), \qquad E(\alpha\lambda, t) - E(\lambda, t) = E(\lambda, (\bar{\alpha}-1)t) \in \mathbb{Z},$$

and $\chi(\alpha\lambda) = \chi(\lambda)^N$. The Hermitian exponential scales by N. Equation (1) therefore shows that

$$f_\alpha(z) = \frac{\theta(\alpha z)}{\theta(z)^N} \tag{2}$$

is an elliptic meromorphic function for Λ, with $f_\alpha(0) = 1$. This calculation rules out an additional quadratic Gaussian factor.

By Lemma 4.2.7 the actual algebraic curve is

$$\mathcal{E}: \quad Y^2 + a_1 XY = X^3 + X,$$

with original group identity ∞, and the complex point parametrized by z is $(x(z), y(z))$, which corresponds to $z+t$ on $\mathbb{C}/\Lambda$. Write $Q = (0,0)$; it is the nonzero point corresponding to t. Since $\alpha t \equiv t \pmod{\Lambda}$, the map $z \mapsto \alpha z$ in (2) is exactly the ordinary CM endomorphism $[\alpha]$ of this curve in the coordinate $z+t$. The zeros of θ are simple and occur precisely at $t+\Lambda$, namely at ∞ on this curve. It follows that

$$\mathrm{div}(f_\alpha) = [\alpha]^*(\infty) - N(\infty). \tag{3}$$

2. The normalized rational quotient is integral in the actual parameter. The curve, Q and the endomorphism are defined over a number field after a finite extension, as used in Lemma 4.2.41. The normalized function (2) is algebraic over that same field after a finite extension. Here is a descent argument that also fixes its scalar. The divisor on the right of (3) is defined over that field and is principal after extension to $\mathbb{C}$. Riemann–Roch, with scalar extension for the space of sections of the corresponding divisor line bundle on the proper curve, gives a nonzero rational function over the field having that divisor: the relevant space has dimension one after scalar extension and hence before it. Its value at Q is nonzero and finite because $[\alpha]Q = Q \neq \infty$. Dividing by that value normalizes it to one at Q. Its complex ratio with (2) has zero divisor, so is constant on the proper curve, and that constant is one by evaluation at Q. Thus (2) is precisely this algebraic rational function, with no transcendental scalar remaining.

Complete the number field at the place induced by ι, and enlarge it finitely when necessary. Let R be its discrete valuation ring, ϖ_R a uniformizer, and $\mathscr{E}/R$ the projective Weierstrass model. It is a smooth elliptic scheme because the discriminant is a unit by Lemma 4.2.7. The generic CM endomorphism extends to $\mathscr{E}$ by Theorem A.2.19; its relations, including $[\bar{\alpha}][\alpha] = [N]$, extend by uniqueness. Its reduction is nonconstant: otherwise its composition with $[\bar{\alpha}]$ would be constant, whereas multiplication by the nonzero integer N on an elliptic curve in characteristic p has degree N^2 and is nonconstant. Thus $[\alpha]^*(\infty)$ has no special-fibre component.

The scheme $\mathscr{E}$ is regular, with integral smooth special fibre F. The divisor (3) on its generic fibre therefore implies

$$\operatorname{div}_{\mathscr{E}}(f_\alpha) = [\alpha]^*(\infty) - N(\infty) + cF \quad \text{for some } c \in \mathbb{Z}. \tag{4}$$

There are no other vertical prime divisors over a discrete valuation ring. Both horizontal Cartier divisors in (4) avoid the section Q: the sections $Q = (0,0)$ and ∞ are disjoint over R, and $[\alpha]Q = Q$ by extension of the generic equality. Thus $\varpi_R^{-c} f_\alpha$ has zero divisor on a neighbourhood of the closed point of Q. On a regular scheme a rational function and its reciprocal are regular where its divisor has no support; equivalently this follows from the intersection description of a normal local domain by its height-one valuation rings. It is therefore a unit near that point. Evaluation on the section Q gives the unit

$$(\varpi_R^{-c} f_\alpha)(Q) = \varpi_R^{-c},$$

since $f_\alpha(Q) = 1$. Its valuation forces $c = 0$. Thus f_α itself is regular and invertible near the closed point of Q.

The completed local parameter at Q is the original Y coordinate: Lemma 4.2.7 gives its integral implicit expansion $X = X_0(Y)$ with $X_0'(0) = 0$. Consequently the expansion of f_α belongs to $R[[Y]]$, and after substitution of the original Taylor series it belongs to $O[[y]]$. We have proved

$$\exp(\iota(N)g(z) - g(\iota(\alpha)z)) = f_\alpha(z) \in 1 + yO[[y]] \quad (\alpha \equiv 1 \pmod{2\mathcal{O}_K},\ \alpha \neq 0). \tag{5}$$

In particular no general integral theta-section expansion is needed.

3. The exact quadratic coefficient and the Frobenius congruence. Classical Weierstrass sigma and reduced sigma differ by an exponential of a quadratic polynomial. Their logarithmic second derivatives therefore differ by a constant. Hence $g'' = b + x$ for some $b \in \mathbb{C}_p$. In particular $g_0 = g_1 = 0$, already proved by evenness.

Let $\omega = (1 + \sqrt{-d})/2$, and use (5) for $\alpha_1 = 1 + 2\omega$ and $\alpha_2 = 1 + 2\bar{\omega}$. Both are nonzero and congruent to one modulo $2\mathcal{O}_K$. Since the exponent in (5) has order at least two, its exponential has quadratic coefficient

$$\iota(\alpha_j(\bar{\alpha}_j - \alpha_j))\, g_2.$$

Every series in $O[[y]]$ has integral coefficients in degrees less than q by Lemma 4.2.41, and $2 < q$. Thus these two coefficients are integral. Direct calculation gives

$$\alpha_2(\bar{\alpha}_2 - \alpha_2) - \alpha_1(\bar{\alpha}_1 - \alpha_1) = 8\sqrt{-d}.$$

Since p is odd, subtraction and division by the unit 8 show $\iota(\sqrt{-d})g_2 \in O$. Therefore $\pi g_2 \in O$: in the inert case $\iota(\sqrt{-d})$ is a unit, and at $p = d$ it equals π.

For the Frobenius multiplier take $\alpha = p$ or $\sqrt{-d}$, so $N = q$. The function f_α is even, and its only poles lie at $t + \Lambda$ with order at most $q - 1$: the denominator has order q and the numerator has a simple zero there. The even elliptic function x has its unique double pole at that same point, with leading Laurent coefficient one. Laurent expansion at t, using evenness and periodicity, contains only even powers. Successively subtracting a multiple of the appropriate power of x cancels the highest pole and lowers its order by at least two. The final remainder is entire and elliptic, hence constant. Thus

$$f_\alpha = R(x), \quad \deg R \leq (q-1)/2, \quad R(0) = 1. \tag{6}$$

For $3 \leq n < q$, the differential identity gives $n(n-1)g_n = [z^{n-2}]x$. These coefficients of x are integral by Lemma 4.2.41. For inert p, $n < p^2$ implies $v_p(n(n-1)) \leq 1$; for ramified p,

$n < p$ implies $v_p(n(n-1)) = 0$. Together with the quadratic bound and $g_0 = g_1 = 0$, this proves $\pi g_n \in O$ for all $n < q$. Put $h = qg - g(\pi z)$. Since $v_p(q) = 2v_p(\pi)$, its coefficients $h_n = (q - \pi^n)g_n$ belong to πO for $n < q$.

The coefficients below q of $\exp(h) - 1$ belong to πO. Indeed in $h^j/j!$ the relevant coefficients have valuation at least $jv_p(\pi) - v_p(j!) \geq v_p(\pi)$, because $v_p(j!) \leq (j-1)/(p-1)$ and $v_p(\pi) \geq 1/(p-1)$. Each coefficient receives finitely many contributions. By (6), $R(x) = \exp(h)$. Write $R(X) - 1 = \sum_{j=1}^{M} r_j X^j$ with $2M < q$. Induction on j, taking the coefficient of z^{2j}, proves $r_j \in \pi O$: the leading coefficient of x^j in that degree is one, larger powers contribute zero, and smaller powers have integral coefficients there and already have π-divisible scalar coefficients. Therefore

$$R(X) - 1 \in \pi O[X], \qquad R(x) - 1 \in \pi O[[y]].$$

Write $R(x) = 1 + \pi u$ with $u \in yO[[y]]$. The formal logarithm gives

$$qg - g(\pi z) = \log(1 + \pi u) \in \pi O[[y]],$$

since $\pi^{j-1}/j \in O$ for every $j \geq 1$, again by $v_p(j) \leq (j-1)/(p-1)$ and $v_p(\pi) \geq 1/(p-1)$. This proves the first assertion.

4. Density of actual CM multipliers proves averaging in the same parameter. Put $K_p = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$, a field because p is nonsplit. At inert p it is the unramified quadratic field with residue size p^2; conjugation on its (p^2-1)st roots of unity is $\zeta \mapsto \zeta^p$. This follows by reduction, where conjugation is the nontrivial residue-field automorphism, and uniqueness of Hensel lifts of roots of $X^{p^2-1} - 1$. At ramified p, its $(p-1)$st roots of unity are in $\mathbb{Q}_p$, so conjugation fixes them. Consequently at either prime type

$$\zeta\bar{\zeta} = \zeta^{p+1} \qquad (\zeta^{q-1} = 1). \tag{7}$$

The elements of $\mathcal{O}_K$ congruent to one modulo $2\mathcal{O}_K$ are dense in $\mathcal{O}_{K_p}$, because p is odd: combine the prescribed congruences modulo 2 and arbitrary powers of the unique prime above p by the Chinese remainder theorem. Choose nonzero such α_n tending to ζ^{-1}. Conjugation is continuous on K_p, so their norms tend to ζ^{-p-1} by (7). Each coefficient of the exponential in (5) is a polynomial in the multiplier and its norm, with fixed coefficients in $\mathbb{C}_p$. Thus coefficientwise limits in (5) give

$$V(z) = \exp\Big(\zeta^{-p-1} g(z) - g(\zeta^{-1} z)\Big) \in O[[y]]. \tag{8}$$

The ring $O[[y]]$ is closed for coefficientwise convergence: substitution by $y = z + \cdots$ is a coefficientwise continuous bijection with continuous inverse on zero-constant formal parameters, and O is closed in $\mathbb{C}_p$.

By the integral CM-root action in Lemma 4.2.41, $y(\zeta z) \in O[[y]]$. Substituting ζz into (8) therefore proves

$$\exp\Big(\zeta^{-p-1} g(\zeta z) - g(z)\Big) \in O[[y]]. \tag{9}$$

This substitution, rather than taking an arbitrary O-scalar power of an integral exponential, is essential. Multiplication of (9) over all $(q-1)$st roots of unity gives

$$\exp((q-1)(g^\flat - g)) \in 1 + yO[[y]],$$

by the finite root-of-unity projection formula. Since $q - 1$ is a unit, a series with constant coefficient one has a unique $(q-1)$st root with constant coefficient one in $O[[y]]$: coefficient induction divides only by $q-1$. Over $\mathbb{C}_p[[z]]$ that root is $\exp(g^\flat - g)$. Hence the second assertion follows.

5. Projection gives the refined form. The series T of Lemma 4.2.41 has degree support congruent to one modulo $q-1$, and $O[[T]] = O[[y]]$. Degree projection to any fixed residue class modulo $q-1$ therefore preserves this ring and commutes with substitution by T. Projecting the proved identity $qg - g(\pi z) \in \pi O[[T]]$ gives $qg^\flat - g^\flat(\pi z) \in \pi O[[T]]$. The projected series has support congruent to $p+1$ modulo $q-1$. It vanishes below $i = p+1$ in the inert case, and below $i = 2$ in the ramified case; the only possible lower degree in the latter case is zero, whose coefficient is zero. The inverse parameter also has degree support congruent to one, so these order and degree-support statements hold in T. Factoring its first allowed power gives

$$qg^\flat - g^\flat(\pi z) = \pi\big(\ell T^i + T^{i+q-1}J(T)\big)$$

with $\ell \in O$ and $J \in O[[U]]$, as asserted. □

Lemma 4.2.41. *Let $d \equiv 7 \pmod 8$ be prime, let $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$, and let $\Lambda \subset \mathbb{C}$ be a rank-two lattice stable under $\mathcal{O}_K$. Let $t \notin \Lambda$, $2t \in \Lambda$, and $a_1 \in \mathbb{C}$ satisfy*

$$g_2(\Lambda) = 12\wp_\Lambda(t)^2 - 4, \qquad a_1^2 = 12\wp_\Lambda(t).$$

For a fixed embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, let $y(z) \in \mathbb{C}_p[[z]]$ be the coefficientwise image of the Taylor series at zero of

$$\tfrac{1}{2}\left(\wp'_\Lambda(z+t) - a_1(\wp_\Lambda(z+t) - \wp_\Lambda(t))\right).$$

Let p be an odd prime nonsplit in K, put $O = \mathcal{O}_{\mathbb{C}_p}$ with $v_p(p) = 1$, and set

$$(\pi, q, \rho) = \begin{cases} (p, p^2, (p^2-1)^{-1}), & p \text{ inert in } K, \\ (\iota(\sqrt{-d}), p, [2(p-1)]^{-1}), & p = d. \end{cases}$$

For a zero-constant series u write $O[[u]] = \{f(u) : f \in O[[U]]\} \subset \mathbb{C}_p[[z]]$. Define

$$T(z) = \sum_{\substack{n \geq 0 \\ n \equiv 1 \pmod{q-1}}} [z^n]y(z)\, z^n.$$

Then $T(0) = 0$, $T'(0) = 1$, and $O[[T]] = O[[y]]$. There are $b_1 \in O^\times$ and $b_i \in O$ for $i \geq 2$ such that

$$T(\pi z) = g(T(z)), \qquad g(U) = \pi U + b_1 U^q + \sum_{i \geq 2} b_i U^{1+i(q-1)}.$$

For any choice of $\varpi \in \mathbb{C}_p$ satisfying

$$\varpi^{q-1} = -\pi/b_1$$

one has $v_p(\varpi) = \rho$ and

$$T(\varpi z) = \varpi Y(z), \qquad Y \in O[[z]], \quad Y(0) = 0, \quad Y'(0) = 1, \quad Y(z) \equiv \sum_{j \geq 0} z^{q^j} \pmod{\pi O[[z]]}.$$

Moreover, if $f \in O[[y]]$, all its Taylor coefficients of degrees less than q belong to O, and for every $c \in \mathbb{C}_p$ with $v_p(c) \geq \rho$ one has $f(cz) \in O[[z]]$.

Proof. **The original complex coordinate is the normalized formal logarithmic coordinate.** Put

$$x_{\mathbb{C}}(z) = \wp_\Lambda(z+t) - \wp_\Lambda(t), \qquad y_{\mathbb{C}}(z) = \tfrac{1}{2}(\wp'_\Lambda(z+t) - a_1 x_{\mathbb{C}}(z)).$$

By Lemma 4.2.7, the equation

$$E: \quad Y^2 + a_1XY = X^3 + X$$

is the given complex elliptic curve in these coordinates, a_1 is algebraic, and its discriminant is a unit at the chosen odd prime. The point $Q = (0, 0)$ corresponds to t and has order two. Its parameter Y has Taylor series $y(z) = z + \cdots$. The invariant differential is

$$\omega = \frac{dX}{2Y + a_1X}.$$

The identity $x'_{\mathbb{C}} = 2y_{\mathbb{C}} + a_1x_{\mathbb{C}}$ gives the exact pullback $\omega = dz$ under $z \mapsto (x_{\mathbb{C}}(z), y_{\mathbb{C}}(z))$. At Q the expression for ω is regular with coefficient one in the parameter Y, because $X = Y^2 + \cdots$.

Translate the group law of E so that Q is its identity. The formal completion at Q, in the parameter Y, is a one-dimensional commutative formal group. Its normalized logarithm $\ell(U)$ is the integral of its invariant differential with zero constant term. Invariance of the differential makes ℓ additive for the formal group law, and its linear coefficient is one; thus this is the normalized logarithm of the characteristic-zero theorem; see [124]. Since the differential pulls back exactly to dz, one has

$$\ell(y(z)) = z, \qquad y(z) = \ell^{-1}(z). \tag{L}$$

These identities are identities of formal series after applying ι. They identify the scale in the assertion with the original complex differential, with no unspecified unit factor.

Actual CM endomorphisms give integral action in the parameter y. The multiplication ring of Λ is exactly $\mathcal{O}_K$. Indeed its rational span is $K\omega$ for any $0 \neq \omega \in \Lambda$, so a complex scalar preserving Λ lies in K; its integral matrix on Λ makes it an algebraic integer, hence an element of the maximal order $\mathcal{O}_K$. Thus the curve has exactly this CM endomorphism ring. The curve E and all its CM endomorphisms are defined over a number field after a finite extension. Indeed the given equation is algebraic, and the algebraicity theorem for CM elliptic curves supplies a model with all endomorphisms defined over a number field; see [281], Chapter II, §4. An isomorphism between the two algebraic models can be chosen over $\overline{\mathbb{Q}}$ and then over a finite extension. Enlarge that number field to contain a_1, K and such an isomorphism. Its completion at the place induced by ι is a finite extension of $\mathbb{Q}_p$. Write R for its valuation ring, after further finite extensions when necessary. The projective Weierstrass model of E over R is an elliptic scheme, because its coefficients are integral and its discriminant is a unit by Lemma 4.2.7.

For $\alpha \in \mathcal{O}_K$ its generic-fibre endomorphism extends uniquely over R by the semiabelian extension theorem, since $\operatorname{Spec} R$ is noetherian and normal, its generic point is open and dense, and the model is an abelian scheme; see [77, Chapter I, Proposition 2.7]. With identity moved to Q, the corresponding map is

$$P \longmapsto [\alpha](P - Q) + Q.$$

Translation by the integral section Q is an isomorphism of the integral models. Hence the formal series of this map in the parameter Y has coefficients in R, and therefore in O. Its tangent multiplier is $\iota(\alpha)$, as seen on the original complex uniformization $z \mapsto \alpha z$. The logarithm identity (L) and functoriality of formal logarithms give

$$u_\alpha(U) = \ell^{-1}\big(\iota(\alpha)\ell(U)\big) \in UO[[U]], \qquad y(\iota(\alpha)z) = u_\alpha(y(z)). \tag{A}$$

Let K_p denote the completion of K inside $\mathbb{C}_p$. Since p is nonsplit, its ring of integers is the completion of $\mathcal{O}_K$, its residue field has size q, and it contains every $(q-1)$st root of unity. The last assertion follows by lifting the $q - 1$ nonzero residue elements by Hensel's lemma applied

to $X^{q-1}-1$; its derivative is a unit at each of them. For any $\zeta^{q-1}=1$, choose $\alpha_n \in \mathcal{O}_K$ with $\iota(\alpha_n) \to \zeta$. In each fixed degree, the coefficient of $\ell^{-1}(a\ell(U))$ is a polynomial in a with coefficients in $\mathbb{C}_p$. Thus the integral series u_{α_n} converge coefficientwise to

$$u_\zeta(U) = \ell^{-1}(\zeta\ell(U)).$$

The ring O is closed in $\mathbb{C}_p$, so $u_\zeta \in UO[[U]]$. By (L),

$$y(\zeta z) = u_\zeta(y(z)) \in O[[y]]. \tag{U}$$

This argument uses the actual CM maps of the curve; it does not assume an integral action from its tangent multipliers alone.

Averaging gives the prescribed generator and its exact degree support. Let μ_{q-1} be the set of $(q-1)$st roots of unity in $\mathbb{C}_p$. A finite geometric sum, coefficient by coefficient, gives

$$T(z) = \frac{1}{q-1}\sum_{\zeta\in\mu_{q-1}} \zeta^{-1}y(\zeta z). \tag{T}$$

Since $q-1$ and all ζ are units of O, (U) shows $T \in O[[y]]$. It has constant coefficient zero and linear coefficient one. Write $T = h(y)$ with $h \in UO[[U]]$ and $h'(0)=1$. Successive coefficient comparison constructs the compositional inverse of h over O; the new coefficient always has coefficient one. Hence $y = h^{-1}(T)$ and

$$O[[T]] = O[[y]]. \tag{R}$$

For each $\zeta \in \mu_{q-1}$ the degree support of T gives $T(\zeta z) = \zeta T(z)$.

The CM element giving π belongs to $\mathcal{O}_K$. Formula (A) and (R) therefore yield $T(\pi z) = g(T(z))$ for $g \in O[[U]]$. Its constant coefficient is zero and its linear coefficient is π. Moreover

$$g(\zeta T(z)) = T(\pi\zeta z) = \zeta T(\pi z) = \zeta g(T(z)).$$

Since $T = z + \cdots$, substitution by T is injective on $\mathbb{C}_p[[U]]$. Consequently $g(\zeta U) = \zeta g(U)$ for every ζ. Choosing a primitive $(q-1)$st root of unity shows that a nonzero coefficient of g can occur only in degrees congruent to one modulo $q-1$. Thus

$$g(U) = \pi U + b_1U^q + \sum_{i\geq 2} b_iU^{1+i(q-1)}, \qquad b_i \in O. \tag{G}$$

The coefficient b_1 is a unit of the actual integral model. Reduce the extended CM endomorphism and the translated group law modulo the maximal ideal of O, or first modulo a finite-extension valuation ring as above and then extend the residue field. In the inert case the endomorphism is multiplication by p, whose degree on an elliptic curve in any characteristic is $p^2 = q$. In the ramified case its square is multiplication by $-p$: the generic-fibre relation $\pi^2 = -p$ extends over R by uniqueness of extension, and therefore holds after reduction. Since multiplication by $-p$ has degree p^2, the reduced endomorphism is nonconstant and the multiplicativity of degrees gives degree $p = q$ for it. Translations preserve these degrees.

For a nonconstant map of smooth projective curves of degree q, the local multiplicity at any point is at most q: the pullback divisor of that point has total degree q, and each local multiplicity is one of its nonnegative coefficients. In the original formal parameter y, and therefore in the parameter $T = h(y)$ whose reduction is an invertible formal change of coordinate, the reduced map has order at most q. In (G) the linear coefficient π reduces to zero, and no degrees strictly between one and q occur. If b_1 also reduced to zero, the order would exceed q, a contradiction.

Hence $b_1 \in O^\times$. This use of reduction proves only the unit assertion; the stronger congruence modulo πO is obtained next by exact coefficient scaling.

Scaling and coefficient induction give the integral series modulo π. Choose $\varpi^{q-1} = -\pi/b_1$, which is possible in the algebraically closed field $\mathbb{C}_p$. It is nonzero, and

$$v_p(\varpi) = v_p(\pi)/(q-1) = \rho.$$

For $i \geq 2$ the coefficient in $g(\varpi U)/(\pi\varpi)$ of degree $1 + i(q-1)$ equals

$$b_i \, \frac{\varpi^{i(q-1)}}{\pi} = b_i(-b_1^{-1})^i \pi^{i-1}.$$

Together with the degree-one and degree-q terms this gives

$$g(\varpi U) = \pi\varpi\big(U - U^q + \pi E(U)\big), \qquad E \in U^2 O[[U]], \tag{E}$$

where explicitly the coefficient of $U^{1+i(q-1)}$ in E is $b_i(-b_1^{-1})^i\pi^{i-2}$ for $i \geq 2$, and the other coefficients are zero.

Put $Y(z) = T(\varpi z)/\varpi$, initially as a series over $\mathbb{C}_p$. Its constant and linear coefficients are zero and one. Substitution in $T(\pi z) = g(T(z))$, using (E), yields

$$Y(\pi z) = \pi\big(Y(z) - Y(z)^q + \pi E(Y(z))\big). \tag{F}$$

For degree $n \geq 2$, comparison gives

$$(\pi^{n-1} - 1)[z^n]Y = [z^n]\big(-Y^q + \pi E(Y)\big).$$

The coefficient on the right depends only on lower coefficients of Y, since $q \geq 2$, E has order at least two, and $Y(0) = 0$. The factor $\pi^{n-1} - 1$ is a unit of O because $0 < |\pi|_p < 1$. Induction proves $Y \in O[[z]]$.

Now $Y(\pi z)/\pi$ is integral and congruent to z modulo $\pi O[[z]]$, by its explicit coefficients $\pi^{n-1}[z^n]Y$ for $n \geq 1$. Equation (F) therefore gives $Y - Y^q \equiv z \pmod{\pi O[[z]]}$. The ring $O/\pi O$ has characteristic p, including in the ramified case $p = -\pi^2$. Since q is a positive power of p, its Frobenius power is additive. Iterating $\overline{Y} = z + \overline{Y}^{\,q}$ gives

$$\overline{Y} = \sum_{j=0}^{k-1} z^{q^j} + \overline{Y}^{\,q^k} \quad (k \geq 1).$$

The last term has order at least q^k because $\overline{Y}(0) = 0$. Taking each coefficient with k large proves the asserted congruence. This is an argument in $O/\pi O$, with no assumption that this quotient is a field or reduced.

The additional integrality conclusions follow in the same parameter. The defining support and linear coefficient give $T(z) = z +$ terms of degree at least q. If $f \in O[[y]] = O[[T]]$, write $f = f_0(T)$ with $f_0 \in O[[U]]$. Its coefficients in degrees less than q equal those of $f_0(z)$, and are integral. Finally suppose $v_p(c) \geq \rho$. Then $c/\varpi \in O$, so

$$T(cz) = \varpi Y((c/\varpi)z) \in O[[z]],$$

with zero constant term. Substituting this integral series into f_0 proves $f(cz) \in O[[z]]$. □

Theorem 4.2.42. *Let R be a commutative $\mathbb{Q}$-algebra and let $F(X,Y)$ be a commutative formal group law of dimension $d \geq 0$ over R. There is a unique vector of formal power series $\ell_F(X) \in R[[X_1,\ldots,X_d]]^d$ with zero constant term and linear term X such that*

$$\ell_F(F(X,Y)) = \ell_F(X) + \ell_F(Y).$$

The formal map ℓ_F has a compositional inverse. If G is a commutative formal group law of dimension e over R and $u : F \to G$ is a homomorphism of formal group laws, then

$$\ell_G(u(X)) = du(0)\,\ell_F(X),$$

where $du(0) \in M_{e,d}(R)$ is the linear term of u.

Proof. For positive dimension, the Q-theorem of [124], Chapter II, Corollary (11.1.6), supplies a strict isomorphism to the additive formal group, since R is a commutative $\mathbb{Q}$-algebra and F is finite-dimensional and commutative. Thus we obtain a zero-constant ℓ_F with identity linear term and the displayed addition identity, together with its compositional inverse ℓ_F^{-1}. When $d = 0$, there is exactly one empty tuple of formal series; it is its own inverse and satisfies that identity. These constructions also apply to G, with its dimension e.

We first prove the linearity assertion needed for uniqueness and functoriality. Let $\varphi(X) \in R[[X_1,\ldots,X_d]]^e$ have zero constant term and satisfy

$$\varphi(X+Y) = \varphi(X) + \varphi(Y).$$

Differentiate this formal identity with respect to Y_j and set $Y = 0$. The formal chain rule gives

$$\partial_j\varphi(X) = \partial_j\varphi(0) \qquad (1 \leq j \leq d).$$

Write φ_m for the homogeneous component of total degree m. For every $m \geq 2$, comparison of degree $m-1$ in this derivative identity gives $\partial_j\varphi_m = 0$ for every j. On a monomial $X_1^{a_1}\cdots X_d^{a_d}$ of total degree m, the operator $\sum_j X_j\partial_j$ is multiplication by $\sum_j a_j = m$. Applying this coefficientwise to each component gives Euler's identity

$$m\varphi_m = \sum_{j=1}^{d} X_j\partial_j\varphi_m = 0.$$

The integer m is a unit in R, so $\varphi_m = 0$. The constant component was zero, and consequently φ is exactly its linear term. This proof requires neither that R be a field nor that it be reduced. If $d = 0$ or $e = 0$, the same conclusion follows directly from the zero-constant condition or from the empty target tuple.

The inverse logarithm carries addition back to the formal group law. Indeed applying ℓ_F to $F(\ell_F^{-1}(X), \ell_F^{-1}(Y))$ gives $X+Y$, as does applying it to $\ell_F^{-1}(X+Y)$. Composing both sides with ℓ_F^{-1} therefore gives

$$F(\ell_F^{-1}(X), \ell_F^{-1}(Y)) = \ell_F^{-1}(X+Y).$$

All these substitutions are well defined because their tuples have zero constant term.

Suppose ℓ'_F is any other zero-constant tuple with identity linear term satisfying the required addition identity. The preceding inverse identity and the addition identity for ℓ'_F show that $\ell'_F \circ \ell_F^{-1}$ is additive. Its linear term is the identity: the linear term of ℓ_F^{-1} is the identity by comparison of linear terms in $\ell_F \circ \ell_F^{-1}(X) = X$. The linearity result just proved implies $\ell'_F \circ \ell_F^{-1}(X) = X$. Substituting $\ell_F(X)$ proves $\ell'_F = \ell_F$, establishing uniqueness without assuming beforehand that ℓ'_F is invertible.

Finally let $u : F \to G$ be the homomorphism in the statement. Its defining identity, the inverse identity above, and the addition identity for ℓ_G imply that $\ell_G \circ u \circ \ell_F^{-1}$ is additive: explicitly,

$$\begin{aligned}\ell_G\Big(u(\ell_F^{-1}(X+Y))\Big) &= \ell_G\Big(u(F(\ell_F^{-1}(X),\ell_F^{-1}(Y)))\Big)\\ &= \ell_G\Big(G(u(\ell_F^{-1}(X)),u(\ell_F^{-1}(Y)))\Big)\\ &= \ell_G\Big(u(\ell_F^{-1}(X))\Big)+\ell_G\Big(u(\ell_F^{-1}(Y))\Big).\end{aligned}$$

Its linear term is $du(0)$, since both logarithms and the inverse logarithm have identity linear terms. The established linearity result therefore gives

$$\ell_G(u(\ell_F^{-1}(X))) = du(0)X.$$

Substitution of $\ell_F(X)$ is legitimate and yields exactly $\ell_G(u(X)) = du(0)\ell_F(X)$. Zero source or target dimension is included by the empty-tuple conventions already checked. This proves every assertion. □

Lemma 4.2.43. *Take the lattice, marked point, coefficientwise embedding and series $x = x_p, y = y_p$ of Lemma 4.2.7. Let p be an odd prime inert in K and put $O = \mathcal{O}_{\mathbb{C}_p}$, with maximal ideal $\mathfrak{m}$. Then*

$$[z^{p-1}]x \in O^\times.$$

Consequently, for every $g \in \mathbb{C}_p[[z]]$ satisfying $g'' = b + x$ for some $b \in \mathbb{C}_p$,

$$-p[z^{p+1}]g = -\frac{[z^{p-1}]x}{p+1} \in O^\times.$$

In particular this applies to the negative logarithm of every translated sigma theta

$$\vartheta(z) = \frac{\sigma_\Lambda(t+z)}{\sigma_\Lambda(t)}\exp(-\zeta_\Lambda(t)z + cz^2), \qquad g = -\log\vartheta$$

whose Taylor coefficients are read through the fixed embedding, for any $c \in \mathbb{C}$. The assertion is independent of this quadratic normalization.

Proof. Put $a = \iota(a_1)$ and $R = O[[y]]$. By Lemma 4.2.7, $a \in O$, $x \in R$, and differentiation preserves R. The identities there give

$$(x')^2 = 4x^3 + a^2x^2 + 4x, \qquad x'' = 6x^2 + a^2x + 2.$$

The first follows directly from the curve equation and $x' = 2y + ax$, and the second by differentiating that identity using $y' = 1 + 3x^2 - ay$. Therefore, for every integer $n \geq 0$,

$$(x^n x')' = (4n+6)x^{n+2} + (n+1)a^2x^{n+1} + (4n+2)x^n. \tag{1}$$

For $n = 0$ this is the second displayed identity; for $n > 0$ use the product rule and the first one.

Write $p = 2m + 1$ and $h_n = [z^{p-1}]x^n$. Every h_n is integral: $x^n \in R$, and $p - 1 < p^2$, so Lemma 4.2.41 applies. Also $x^n x' \in R$ and $p < p^2$, whence $[z^p](x^n x') \in O$. Taking the coefficient of z^{p-1} in (1) gives

$$(4n+6)h_{n+2} + (n+1)a^2h_{n+1} + (4n+2)h_n = p[z^p](x^n x') \in \mathfrak{m}. \tag{2}$$

Suppose $h_1 \in \mathfrak{m}$. We have $h_0 = 0$ because $p - 1 > 0$. For $0 \leq n \leq m - 2$ the integer $4n + 6 = 2(2n + 3)$ is prime to p, since $0 < 2n + 3 < p$. Thus (2), integrality of a, and

induction imply $h_n \in \mathfrak{m}$ for every $0 \le n \le m$. If $m = 1$ this conclusion already follows from the supposition and needs no recurrence. But $x = z^2 +$ terms of higher degree by Lemma 4.2.7, so $h_m = [z^{2m}]x^m = 1$. This is impossible. Hence $h_1 \in O^\times$.

Since $p - 1 > 0$, the constant b contributes nothing in that degree. The equation $g'' = b + x$ gives

$$p(p+1)[z^{p+1}]g = [z^{p-1}]x.$$

As $p+1$ is a unit, the second assertion follows. Finally the classical differential identities $\sigma'/\sigma = \zeta$ and $\zeta' = -\wp$ give, directly for the displayed normalized sigma translate,

$$(-\log \vartheta)'' = \wp_\Lambda(t+z) - 2c = x(z) + \wp_\Lambda(t) - 2c.$$

These are local analytic identities near zero, where $\vartheta(0) = 1$; taking Taylor series and applying the embedding gives the required formal identity. This proves the claimed application without a Frobenius theta functional equation. □

Lemma 4.2.44. *Let p be an odd prime, put $O = \mathcal{O}_{\mathbb{C}_p}$, and normalize $v_p(p) = 1$. Choose one of the following two sets of data:*

	π	q	i	S
inert case	p	p^2	$p+1$	$\{1,3,5,\dots\}$
ramified case	$\pi^2 = -p$	p	2	$\mathbb{Z}_{\ge 0}$.

In the second row $\pi \in \mathbb{C}_p$ is a fixed square root of $-p$. Let $Y \in O[[z]]$, $\lambda \in O$, and $\xi = \sum_{n\ge 0} \xi_n z^n \in \mathbb{C}_p[[z]]$ satisfy

$$Y(z) \equiv \sum_{j\ge 0} z^{q^j} \pmod{\pi O[[z]]}, \qquad \xi_n = 0 \quad (0 \le n < i),$$

and, for every $n \ge 0$,

$$(1 - \pi^n/q)\xi_n - \lambda[z^n]Y^i \in \pi O.$$

Then $\xi \in O[[z]]$, $\xi(0) = 0$, and

$$\xi(z) \equiv \sum_{e\in S, k\ge 0} \lambda_{ek}(z^{p^e} z)^{p^k} \pmod{\pi O[[z]]},$$

where the coefficients may be taken to be

$$\lambda_{ek} = \begin{cases} \lambda, & \text{in the inert case,} \\ \lambda/2, & \text{in the ramified case, } e = k = 0, \\ \lambda, & \text{in the ramified case, } e = 0,\ k \ge 1, \\ 2\lambda, & \text{in the ramified case, } e \ge 1. \end{cases}$$

All sums are interpreted coefficientwise. In the inert case, if $|\xi_{p+1}|_p = 1$, then λ and every λ_{e0} with odd e are units of O.

Proof. **The coefficient equation forces integrality.** Let $a_n = \lambda[z^n]Y^i \in O$. In the inert case, for $n \ge p+1$ the number $\pi^n/q = p^{n-2}$ belongs to πO, because $p \ge 3$ and $n \ge 4$. Thus $1 - \pi^n/q$ is a unit of O, congruent to one modulo π. The coefficient hypothesis says that its product with ξ_n is integral and congruent to a_n. Its inverse is integral and congruent to one

modulo π, so $\xi_n \in O$ and $\xi_n \equiv a_n \pmod{\pi}$. For $n < p+1$ the coefficient ξ_n vanishes. The reduction of Y starts in degree one, so $[z^n]Y^{p+1} \equiv 0 \pmod{\pi}$ in these degrees. Consequently

$$\xi \in O[[z]], \qquad \xi \equiv \lambda Y^{p+1} \pmod{\pi O[[z]]}$$

in the inert case.

In the ramified case $\pi^2 = -p$, so for $n > 2$ one has $\pi^n/q = -\pi^{n-2} \in \pi O$. The same unit argument gives integrality and $\xi_n \equiv a_n \pmod{\pi}$ in these degrees. For $n < 2$ the coefficients vanish, and those of Y^2 also vanish modulo π. In degree two the factor is

$$1 - \pi^2/q = 2.$$

Since p is odd, two is a unit, so the coefficient hypothesis gives $\xi_2 \in O$ and $2\xi_2 \equiv \lambda[z^2]Y^2 \pmod{\pi}$. The reduction of Y is $z + z^q + \cdots$ and $q = p \geq 3$, so $[z^2]Y^2 \equiv 1 \pmod{\pi}$. Hence $\xi_2 \equiv \lambda/2 = \lambda([z^2]Y^2 - 1/2) \pmod{\pi}$. Combining all degrees yields

$$\xi \in O[[z]], \qquad \xi \equiv \lambda(Y^2 - z^2/2) \pmod{\pi O[[z]]}.$$

In either case $\xi(0) = 0$ follows from the assumed vanishing.

The reductions have the stated double-series expansions. The quotient $O/(\pi)$ has characteristic p: this is immediate for $\pi = p$, and follows from $p = -\pi^2$ in the other case. Work in its formal power-series ring. In the inert case, with $G(z) = \sum_{j\geq 0} z^{p^{2j}}$, Frobenius gives

$$G^{p+1} = \left(\sum_{a\geq 0} z^{p^{2a+1}}\right)\left(\sum_{b\geq 0} z^{p^{2b}}\right).$$

Each ordered pair of one odd exponent $2a+1$ and one even exponent $2b$ corresponds uniquely to

$$e = |(2a+1) - 2b| \text{ odd}, \qquad k = \min(2a+1, 2b).$$

Conversely, for odd e and $k \geq 0$, the ordered odd/even pair is $(k+e, k)$ if k is even and $(k, k+e)$ if k is odd. Therefore

$$G^{p+1} = \sum_{e \text{ odd}, k\geq 0} (z^{p^e} z)^{p^k}.$$

Multiplication by the reduction of λ proves the inert expansion.

In the ramified case put $G(z) = \sum_{k\geq 0} z^{p^k}$. Expanding its square into equal and unequal index pairs gives

$$G^2 - z^2/2 = \tfrac{1}{2}z^2 + \sum_{k\geq 1} z^{2p^k} + 2\sum_{e\geq 1, k\geq 0} z^{p^{k+e}+p^k}.$$

This is the desired expansion, since $z^{p^{k+e}+p^k} = (z^{p^e} z)^{p^k}$. These rearrangements are legitimate coefficientwise: for any fixed degree, the indices of all monomials contributing to that degree are bounded. Reducing Y to G in the congruences of the preceding block and lifting the resulting coefficient equalities to O proves the asserted congruence modulo $\pi O[[z]]$.

The inert leading coefficient preserves the unit output. In the inert case $q = p^2 > p+1$, so the coefficient of degree $p+1$ in the reduction of Y^{p+1} is one. The first block gives $\xi_{p+1} - \lambda \in pO$. If $|\xi_{p+1}|_p = 1$, the error has norm at most $p^{-1} < 1$, and the ultrametric inequality gives $|\lambda|_p = 1$. Thus λ is a unit of O, and so are all the coefficients $\lambda_{e0} = \lambda$ in the stated inert expansion. □

Lemma 4.2.45. *Let p be a prime with $\chi_K(p) \neq 1$, and let $e \geq 0$ be an integer such that either $p = d$, or $\chi_K(p) = -1$ and e is odd. Fix a field embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, put $O = \mathcal{O}_{\mathbb{C}_p}$ and*

$$\pi = \begin{cases} \iota(\sqrt{-d}), & p = d, \\ p, & p \neq d. \end{cases}$$

Then p is odd and $p \nmid q$. For every $y \in E$ integral at every finite place above p, and every $\tau \in \Phi$, both $\iota(\overline{\tau(y)})$ and $\iota((u(-p^e)\tau)(y))$ belong to O, and

$$\iota(\overline{\tau(y)})^{p^e} - \iota((u(-p^e)\tau)(y)) \in \pi O.$$

In particular the absolute value of this difference is at most $|\pi|_p$, where $|\pi|_p = p^{-1/2}$ if $p = d$ and $|\pi|_p = p^{-1}$ otherwise.

Proof. **The prime and local coefficient ring.** By Lemma 3.2.8, every prime dividing q splits in K; hence $p \nmid q$. As $2 \mid q$, p is odd. The same result gives $\mathcal{O}_E = \mathcal{O}_K[\zeta]$. Set $R = \mathbb{Z}_{(p)}$ and $B = \mathcal{O}_E \otimes_{\mathbb{Z}} R$. Thus $B = (\mathcal{O}_K \otimes_{\mathbb{Z}} R)[\zeta]$.

We check that the assumed y belongs to B, including its denominator. The denominator ideal

$$I = \{a \in \mathcal{O}_E : ay \in \mathcal{O}_E\}$$

is a nonzero integral ideal, and its localization at every prime above p is the unit ideal, because y is integral there. Therefore $I + p\mathcal{O}_E = \mathcal{O}_E$. Choose $a \in I$ with $a \equiv 1 \pmod{p\mathcal{O}_E}$. The determinant of multiplication by a on the free $\mathbb{Z}$-module $\mathcal{O}_E$ is its norm, and reduction modulo p gives the identity matrix. Hence its norm is an integer congruent to 1 modulo p, in particular nonzero and prime to p. The integral characteristic polynomial of a shows that $\mathrm{N}_{E/\mathbb{Q}}(a)/a \in \mathcal{O}_E$. Consequently $Ny \in \mathcal{O}_E$ for the positive integer $N = |\mathrm{N}_{E/\mathbb{Q}}(a)|$, which is prime to p. This proves $y \in B$.

An embedding of E into $\mathbb{C}_p$ sends every algebraic integer into O: a root of a monic integral polynomial cannot have absolute value greater than one, by the ultrametric inequality applied to its leading term. Integers prime to p have absolute value one. Thus both maps in the statement send B into O.

The two maps coincide modulo π on the quadratic coefficient ring. Write $s_0 = \iota(\sqrt{-d})$. If $p = d$ then $\pi = s_0$ and $p = -s_0^2$ lies in πO. Otherwise $\pi = p$. In either case $O/(\pi)$ has characteristic p. The maps

$$f(x) = \iota(\overline{\tau(x)})^{p^e} \bmod \pi, \qquad g(x) = \iota((u(-p^e)\tau)(x)) \bmod \pi$$

are ring homomorphisms $B \to O/(\pi)$; the first is a coefficient embedding followed by Frobenius. They coincide on R, since Frobenius fixes the prime field and the reductions of the allowed denominators are units.

Since $d \equiv 7 \pmod 8$, $\mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{-d})/2]$. As 2 is a unit in R, it remains to check agreement at $\sqrt{-d}$. The second embedding fixes K, so $g(\sqrt{-d}) = s_0 \bmod \pi$. If $p = d$, then $s_0 = 0$ modulo π, and $f(\sqrt{-d}) = (-s_0)^{p^e}$ is also zero, including when $e = 0$.

If $p \neq d$, the hypotheses give $\chi_K(p) = -1$ and odd e. Quadratic reciprocity for $d \equiv 3 \pmod 4$ gives $(-d/p) = (p/d) = -1$. Euler's criterion therefore yields

$$s_0^p = s_0(-d)^{(p-1)/2} \equiv -s_0 \pmod p.$$

Iterating Frobenius in characteristic p gives $s_0^{p^e} \equiv (-1)^e s_0 = -s_0$. Since p is odd, $(-s_0)^{p^e} = -s_0^{p^e} \equiv s_0$. Thus f and g agree on $\mathcal{O}_K \otimes R$ in both cases.

The cyclotomic generator completes the comparison. Complex conjugation sends ζ to ζ^{-1}. The definition of the place group and $p \nmid q$ give the exact characteristic-zero identity

$$\overline{\tau(\zeta)}^{p^e} = \tau(\zeta)^{-p^e} = (u(-p^e)\tau)(\zeta).$$

Therefore $f(\zeta) = g(\zeta)$. Since $B = (\mathcal{O}_K \otimes R)[\zeta]$, equality on its coefficient ring and generator proves $f = g$ on all of B. Applying this to y proves the congruence.

Finally $\pi^2 = -p$ when $p = d$, so $|\pi|_p = p^{-1/2}$; otherwise $\pi = p$ and $|\pi|_p = p^{-1}$. Membership in πO gives the claimed absolute-value bound. □

Lemma 4.2.46. *Let p be a prime, let $O = \mathcal{O}_{\mathbb{C}_p}$, and normalize $v_p(p) = 1$. Let $X = \Phi \times C$ for finite sets Φ, C, let J be finite, let $S \subseteq \mathbb{Z}_{\geq 0}$, and let $\sigma_e : \Phi \to \Phi$ be given for $e \in S$. For $\lambda_{ek} \in O$ put*

$$\Gamma_e(z) = \sum_{\tau\in\Phi, c\in C} z_{\tau c}^{p^e} z_{\sigma_e(\tau),c}, \qquad \Psi(z) = \sum_{e\in S, k\geq 0} \lambda_{ek}\Gamma_e(z)^{p^k}.$$

The latter sum is interpreted coefficientwise. Let $0 \neq \pi \in O$ divide p, let $\varpi \in \mathbb{C}_p^\times$, put $A = \varpi^{p+1}/p$, and suppose $v_p(A\pi) \geq 1/(p-1)$. Let $\Theta \in \mathbb{C}_p[[t]]$ and $W_0, \Xi \in O[[t]]$ satisfy

$$\Xi(0) = 0, \qquad \Theta(\varpi t) = W_0(t)\exp(A\Xi(t)), \qquad \Xi(t) - \sum_{e\in S, k\geq 0} \lambda_{ek}(t^{p^e}t)^{p^k} \in \pi O[[t]].$$

Let $L_j(z) \in O[z_x : x \in X]$ be linear forms with zero constant term, and let m_j be nonnegative integers. Suppose, for every $e \in S$,

$$\sum_{j\in J} m_j L_j(z)^{p^e} L_j(z) - \Gamma_e(z) \in \pi O[[z]].$$

For every $H \in \mathbb{C}_p[[z]]$ with $H(\varpi z) \in O[[z]]$, there exists $W \in O[[z]]$ such that

$$H(\varpi z)\prod_{j\in J} \Theta(\varpi L_j(z))^{m_j} = W(z)\exp(A\Psi(z)).$$

The same coefficients λ_{ek} occur in this equality.

Proof. **The substituted exponents agree modulo π.** The ring $O/(\pi)$ has characteristic p, since π divides p. Reduce coefficients to this ring and denote the reduction by a bar. Every series substituted below has zero constant term. The degrees $(p^e+1)p^k$ tend to infinity outside finite sets of pairs (e,k), so all the displayed infinite sums and their substitutions are coefficientwise defined. The hypothesis on Ξ gives

$$\sum_j m_j \overline{\Xi}(\overline{L_j}) = \sum_{e\in S, k\geq 0} \overline{\lambda}_{ek} \sum_j \overline{m_j}(\overline{L_j}^{p^e}\overline{L_j})^{p^k}.$$

In a ring of characteristic p, the map $x \mapsto x^{p^k}$ is additive and fixes the image of every integer. Consequently the inner sum is

$$\left(\sum_j \overline{m_j}\overline{L_j}^{p^e}\overline{L_j}\right)^{p^k} = \overline{\Gamma_e}^{p^k}.$$

Thus the preceding display equals $\overline{\Psi}$. Since reduction is coefficientwise and its kernel is πO, there is $D \in O[[z]]$ such that

$$\sum_j m_j \Xi(L_j) = \Psi + \pi D.$$

Both $\Xi(L_j)$ and Ψ have zero constant term. Since $\pi \neq 0$ and O is a domain, $D(0) = 0$.

The error exponential is integral. Consider

$$U(z) = \exp(A\pi D(z)) = \sum_{n\geq 0} \frac{(A\pi)^n}{n!} D(z)^n.$$

For a fixed monomial of degree b, only $n \leq b$ contributes, apart from the constant summand. Every coefficient of D^n belongs to O. Legendre's formula gives

$$v_p(n!) = \sum_{h\geq 1} \lfloor n/p^h \rfloor \leq n/(p-1),$$

and therefore

$$v_p((A\pi)^n/n!) \geq n/(p-1) - v_p(n!) \geq 0.$$

Every coefficient of U is a finite sum of integral elements, so $U \in O[[z]]$.

The factorization now has the required integral prefactor. Substitute $t = L_j(z)$ into the assumed one-variable identity. The zero constant term of L_j justifies substitution, and formal exponentials of zero-constant series commute with substitution. Multiplying the resulting finite family of identities and using the exponential addition identity in characteristic zero gives

$$\begin{aligned} H(\varpi z) \prod_j \Theta(\varpi L_j(z))^{m_j} &= H(\varpi z) \prod_j W_0(L_j(z))^{m_j} \exp\left(A \sum_j m_j \Xi(L_j(z)) \right) \\ &= H(\varpi z) \prod_j W_0(L_j(z))^{m_j} U(z) \exp(A\Psi(z)). \end{aligned}$$

Set $W = H(\varpi z) \prod_j W_0(L_j)^{m_j} U$. Each factor is integral, by hypothesis or the preceding block, so $W \in O[[z]]$. The exponential part retains the original coefficients λ_{ek} exactly. The proof also applies when either finite index set is empty. □

Proposition 4.2.47. *Fix an ordinary twist ϑ.*

1. *For $r_{\mathrm{in}} \in \mathrm{H}_j$, $r_{\mathrm{out}} \in \mathrm{H}_{j+1}$ and every finite datum F_j,*

$$J_j(\omega_{\vartheta,j}(r_{\mathrm{in},f}, r_{\mathrm{out},f})F_j)\rho_j(r_{\mathrm{in}}) = \rho_{j+1}(r_{\mathrm{out}})J_j(F_j).$$

   The actions use the conjugate embeddings. The source determinant exponent includes the full-height column count $C_\tau(n_j)$; these source determinant minors remain in the ordinary derivative.

2. *Fix $\iota \mid p$. For nonzero $\delta_{\mathrm{s}} \in \mathcal{O}_K$, write $A_{j,\delta_{\mathrm{s}}}$ for the $\mathcal{O}_\iota$-span of $H_j(x)v_j^{\mathrm{int}}$ with $\delta_{\mathrm{s}}x$ integral above p. Integral invertible finite pairs above p preserve this span, including the two determinant-character factors.*

   Write

$$R_j(v) = \rho_{\lambda_j}((\iota_\tau^f(v))_\tau), \qquad B_j(v) = R_j(v^{-1}).$$

   Let $u_{\mathrm{L}}, u_{\mathrm{R}} \in \mathrm{G}_{j+1}$ be integral invertible above p, and suppose $d_{\mathrm{fin}} \in \mathrm{G}_{j+1}$ has images

$$\iota_\tau^f(d_{\mathrm{fin}}) = \mathrm{diag}(u_{\tau,1}, \ldots, u_{\tau,n_{j+1}}), \qquad u_{\tau,i} \neq 0.$$

   For $F_j \in A_{j,\delta_{\mathrm{s}}}$, put

$$\mathcal{K}_{F_j}(g_{\mathrm{out}}, g_{\mathrm{in}}) = J_j(\omega_{\vartheta,j}(g_{\mathrm{in}}, g_{\mathrm{out}})\omega_{\vartheta,j}(1, u_{\mathrm{L}} d_{\mathrm{fin}} u_{\mathrm{R}})F_j)\,.$$

For every pair of endpoints there are $F'_j \in A_{j,\delta_{\mathrm{s}}}$ and integral invertible Weyl-coordinate matrices $U_{\mathrm{L}}, U_{\mathrm{R}}$, chosen before the input vector, such that

$$B_{j+1}(g_{\mathrm{out}})\,\iota(\mathcal{K}_{F_j}(g_{\mathrm{out}}, g_{\mathrm{in}}))\,B_j(g_{\mathrm{in}})^{-1} = U_{\mathrm{L}}\,\mathrm{diag}(d_T)\,\iota(J_j(F'_j))U_{\mathrm{R}},$$

where

$$d_T = \prod_{\tau}\left(\prod_{i=1}^{n_{j+1}} u_{\tau,i}\right)^{e_{j+1}(\tau)} \prod_{\tau,t_{\mathrm{ht}},c}\prod_{i=1}^{t_{\mathrm{ht}}} u_{\tau,T_{\tau,t_{\mathrm{ht}},c}(i)}.$$

Expanding the finite integral combination defining F'_j gives formula (K) of Proposition 4.2.48. No kernel estimate is assumed. A fixed change of an endpoint base is accommodated by translating the datum under the inverse base at that endpoint.

3. *Let $\mathcal{I}_{\mathrm{sum}}$ be finite, let $c_\alpha \in \mathcal{O}_\iota$, and choose finite data $F_{\alpha,j}$ for $j < m$. Suppose the full tensor*

$$F_{\mathrm{tot}} = \sum_{\alpha\in\mathcal{I}_{\mathrm{sum}}} c_\alpha \bigotimes_{j<m} F_{\alpha,j}$$

is invariant under every $\bigotimes_{j<m}\omega_{\vartheta,j}(v_j, v_{j+1})$, where $v_j \in \mathrm{K}_j$ and $v_m = 1$. Choose representatives $r_{j,c}$ of $\mathrm{H}_j\backslash\mathrm{G}_j/\mathrm{K}_j$, $j < m$. For a tuple $\mathbf{c}$ of these classes and final endpoint g_{out}, put $g_{j,\mathbf{c}} = r_{j,c_j}$ for $j < m$ and $g_{m,\mathbf{c}} = g_{\mathrm{out}}$. Then for every source form φ of level K_0,

$$(\mathscr{C}_{\vartheta,F_{\mathrm{tot}}}\varphi)(g_{\mathrm{out}}) = \sum_{\mathbf{c}}\sum_{\alpha\in\mathcal{I}_{\mathrm{sum}}} c_\alpha\left(\prod_{j=m-1}^{0} J_j(\omega_{\vartheta,j}(g_{j,\mathbf{c}}, g_{j+1,\mathbf{c}})F_{\alpha,j})\right)\varphi(g_{0,\mathbf{c}}),$$

where the product is ordered with the larger index on the left. These are precisely the classes for the fixed levels, each with weight one under $\mathrm{vol}(\mathrm{K}_j) = 1$. The formula also holds after linear changes of the data commuting with the level actions. It permits arbitrary compatible adjacent gauge bases, including diagonal bases at $p = N$. Pointwise bounds in any of these gauges may be substituted into formula (F) of Proposition 4.2.48. Individual summands need not be level invariant.

4. *Suppose the finite coefficient function Ψ has the support and coefficient conditions in Proposition 4.2.25, and its full unnormalized glued tensor is jointly invariant under the levels in item3. For its prescribed last-column and conjugate-last-row gluing tuple $z(y)$,*

$$F_\Psi = \sum_x\sum_y \Psi(x)\bigotimes_{j<m} H_j(x_j + z_j(y))v_j^{\mathrm{int}}.$$

There is no reciprocal cardinality. Collecting equal tuples adds their coefficients, preserves integrality and joint invariance, and yields the same composite on every source form simultaneously. If $p \neq N$ and $p \nmid q$, the step data in this expansion lie in $A_{j,\bar{\pi}_M}$; if also $p \nmid M$, they lie in $A_{j,1}$. At $p \neq N$ the normalized branching restriction has integral matrix: the prescribed branching matrix is integral and its scalar is a p-adic unit.

Proof. **The fixed model and ordinary coefficient covariance.** Use precisely the canonical WD/WR model specified in the statement of Proposition 4.2.25. Its displayed bicharacter is $\psi_E((\jmath/2)\bar{w}^{\mathsf{T}}Sz)$. Thus no opposite cocycle, independent theta phase or changed vacuum is being identified with it. The spaces X_j are the actual standard definite spaces, with their specified frames and levels.

For each edge put $c_j = \pi_M \eta_P \neq 0$ and $d_j = \sum_\tau \sum_{1 \le t_{\mathrm{ht}} < n_{j+1}} t_{\mathrm{ht}} C_\tau(t_{\mathrm{ht}})$. The first two assertions of Lemma 4.2.30 give

$$e_j(\tau) = e_{B,j}(\tau) + C_\tau(n_j), \qquad \Omega_{\vartheta,j,\infty}(x,y) v_{P_{u,v}} = v_{P_{u\rho_{j+1}(y)^{-1},\, \rho_j(x)v}},$$

and

$$\theta(\omega_{\vartheta,j}((x,g_f),(y,h_f))(v_{P_{u,v}} \otimes F_j)) = c_j^{-d_j}\, u\rho_{j+1}(y)^{-1} J_j(\omega_{\vartheta,j,f}(g_f,h_f)F_j)\rho_j(x)v. \tag{Clg1}$$

These statements are on the same finite datum and canonical carrier. Their proof retains every height-n_j determinant minor on the source, and uses the conjugate Weyl actions. It also identifies the ordinary scaled derivative, with its factorial cancellation, with the Fock coefficient. Thus neither a divided derivative nor a different scaling is inserted here.

The full pair action preserves this theta functional at rational points. Indeed clause3 of Lemma 4.1.1 identifies this very action and theta functional with the classical representation and a single nonzero multiple of its theta distribution. The rational splitting and theta invariance in Theorem A.5.4 therefore give the assertion. The identification includes the Schwartz domain. The vector $v_{P_{u,v}} \otimes F_j$ is in that domain: the real factor is a polynomial Gaussian and the finite factor a finite sum of vacuum translates. No rational invariance is inferred from Heisenberg covariance alone.

Apply (Clg1) with the infinite and finite components of the same rational points $r_{\mathrm{in}}, r_{\mathrm{out}}$. Its left side is then $\theta(v_{P_{u,v}} \otimes F_j) = c_j^{-d_j} u(J_j(F_j)v)$. Cancel the common nonzero $c_j^{-d_j}$. Since all target covectors u and source vectors v are allowed, it follows that

$$\rho_{j+1}(r_{\mathrm{out}}^{-1}) J_j(\omega_{\vartheta,j,f}(r_{\mathrm{in},f}, r_{\mathrm{out},f})F_j)\rho_j(r_{\mathrm{in}}) = J_j(F_j).$$

This is item1. The rational actions here are exactly the actions through the prescribed conjugate embeddings; the infinite components in (Clg1) are read in those same fixed frames. Clause2 of Lemma 4.1.1 gives the genuine representation and exact finite Heisenberg covariance used below.

Finite integral changes preserve the standard span. We record a character calculation used for every determinant factor. Let χ be an automorphic character of the norm-one group with integral infinity weight, and let t be a finite norm-one idele integral invertible above p. Its finite ideal I_t satisfies $I_t \bar{I}_t = 1$. Choose $n_{\mathrm{cl}} > 0$ such that $I_t^{n_{\mathrm{cl}}} = (z_0)$. Then $w_0 = z_0/\bar{z}_0$ has norm one and ideal $I_t^{2n_{\mathrm{cl}}}$. Both z_0 and $\bar{z}_0$ are units above p, so all embeddings of w_0 are p-adic units. The idele $t^{2n_{\mathrm{cl}}}/w_0$ is a finite integral norm-one unit. A continuous character of this compact profinite group has finite image: its image is a closed totally disconnected subgroup of the circle, hence finite. Its values are therefore roots of unity. Automorphy gives

$$\chi_f(w_0) = \chi_\infty(w_0)^{-1} = \prod_\tau \tau(w_0)^{-e_\chi(\tau)}.$$

It follows that $\chi_f(t)^{2n_{\mathrm{cl}}}$, and hence $\chi_f(t)$, is a p-adic unit at the chosen identification. This applies to the two pair characters and to v, whose restriction to the norm-one group has integral weight one. It makes no assertion about the absolute value of an arbitrary field automorphism.

Let now U be the finite unitary on the Hom space induced by a pair integral invertible above p. In the notation of Lemma 4.1.13, its lattice intersection has $I(U)$ prime to p, because its local intersection above p is the whole integral lattice. That lemma gives

$$v_f(\det U)\Delta(U)^2 = I(U), \qquad \Omega(U)v^0 = \frac{\Delta(U)}{I(U)} \sum_{y \in L/(L \cap U^{-1}L)} H(Uy)v^0.$$

The character calculation makes $v_f(\det U)$ a p-adic unit, so $\Delta(U)/I(U)$ is a p-adic unit too. Each Uy in this finite sum is integral above p. The common scalar defining v^{int} multiplies both sides and is not changed.

Exact covariance gives $\Omega(U)H(x) = H(Ux)\Omega(U)$; the trace-one scalar in H is preserved by this finite unitary. Products of finite Heisenberg operators contribute only additive-character values, which are roots of unity. If $\delta_{\mathrm{s}}x$ is integral above p, each new translation $Ux+Uy$ has the same property. Including the two pair characters, which are units by the preceding calculation, proves preservation of $A_{j,\delta_{\mathrm{s}}}$. This proof handles all nonintegral places away from p at once and does not impose an auxiliary level refinement.

Integral invertible matrices act integrally and invertibly on the Weyl modules: construct their matrices by exterior powers and tensor products; determinant powers, including negative powers, are units. A diagonal matrix acts on filling T by the product of its entries in the boxes and its determinant to the prescribed exponent. Its exact matrix is therefore $\mathrm{diag}(d_T)$ of item2.

The endpoint calculation. Choose rational unitary points $r_{\mathrm{in}}, r_{\mathrm{out}}$ for which

$$g_{\mathrm{in},0} = r_{\mathrm{in},f}\, g_{\mathrm{in}}, \qquad g_{\mathrm{out},1} = r_{\mathrm{out},f}\, g_{\mathrm{out}} u_{\mathrm{L}} d_{\mathrm{fin}}, \qquad d_{\mathrm{fin}} g_{\mathrm{out},1} d_{\mathrm{fin}}^{-1}$$

are integral invertible above p. These conditions can be imposed simultaneously. For the output, the desired neighborhood is the intersection of the integral stabilizer and its conjugate by d_{fin}, an open subgroup containing one. Approximate $(g_{\mathrm{out}} u_{\mathrm{L}} d_{\mathrm{fin}})^{-1}$ by a rational unitary point into that neighborhood; do the same independently on the input. Apply Lemma 4.2.64 separately to the actual framed spaces X_j and X_{j+1}, taking the finite set to be all places of E above p. The auxiliary finite factor in its dense products has component 1 at these places, so the rational components are dense in the corresponding local products. The input integral subgroup and the output intersection just described are nonempty open subgroups. Applying this density to the respective local inverses gives the two rational points with the stated simultaneous conditions; their infinite components are unrestricted.

Put

$$F'_j = \omega(g_{\mathrm{in},0}, g_{\mathrm{out},1} u_{\mathrm{R}}) F_j, \qquad v_{\mathrm{int}} = (d_{\mathrm{fin}} g_{\mathrm{out},1} d_{\mathrm{fin}}^{-1}) u_{\mathrm{L}}^{-1}.$$

The preceding span calculation gives $F'_j \in A_{j,\delta_{\mathrm{s}}}$. The point v_{int} is integral invertible above p, and $r_{\mathrm{out},f} g_{\mathrm{out}} = d_{\mathrm{fin}}^{-1} v_{\mathrm{int}}$. Item1 and the representation law therefore give exactly

$$B_{j+1}(g_{\mathrm{out}})\iota(\mathcal{K}_{F_j}(g_{\mathrm{out}}, g_{\mathrm{in}}))B_j(g_{\mathrm{in}})^{-1} = R_{j+1}(v_{\mathrm{int}}^{-1}) R_{j+1}(d_{\mathrm{fin}})\iota(J_j(F'_j))R_j(g_{\mathrm{in},0}).$$

This identity is obtained before applying an input vector. The first and last matrices are integral invertible; the middle diagonal matrix has precisely the factors d_T. This proves item2. Expand F'_j in its defining finite integral span to obtain formula (K) of Proposition 4.2.48. A change of endpoint base is absorbed by the inverse translate of the datum, using the same representation law; no intermediate gauge is chosen anew.

The full tensor gives the fixed-class formula. Fix g_{out} and define

$$I(g_0, \dots, g_{m-1}) = \sum_{\alpha} c_\alpha \left(\prod_{j=m-1}^{0} J_j(\omega(g_j, g_{j+1}) F_{\alpha,j}) \right) \varphi(g_0), \qquad g_m = g_{\mathrm{out}}.$$

Item1 cancels the neighboring rational Weyl actions and the source form's rational transformation. Thus the sum is well-defined on the product of rational quotients. Replacing g_j by $g_j v_j$, $v_j \in \mathrm{K}_j$, moves the full tensor by $\bigotimes_{j<m} \omega(v_j, v_{j+1})$, while fixing the source value. Joint invariance proves that I is constant on these fixed level cells. No summand is claimed to be invariant.

Apply Theorem A.4.5 to each actual X_j: it is a nonzero hermitian space over the CM extension and is definite at every infinite place. The full adelic rational quotient is compact, and its continuous projection onto the finite adelic rational quotient is compact as well. Its quotient by the open compact level is discrete and compact, hence finite. The same K_j lies in the chosen principal-level stabilizer, and Lemma 4.6.2 applies with its actual rank and that same chosen prime. Hence

$$\mathrm{H}_j \cap r_{j,c}\mathrm{K}_j r_{j,c}^{-1} = \{1\}.$$

The cell parametrization by K_j is injective, so the quotient measure of that cell is $\mathrm{vol}(\mathrm{K}_j) = 1$. This is the exact reciprocal-stabilizer weight; there is no arbitrary refinement.

Each integrand has an open finite stabilizer and is continuous on the compact rational quotients. It is bounded and integrable. Finite summation and repeated integration may therefore be interchanged. Partitioning the product quotients into their fixed cells gives exactly the ordered finite expression of item3, with weight one. It is the finite expression for the same algebraic form modules and the same levels.

A tensor product of linear datum operators commuting with these level actions preserves the joint invariance. The calculation remains valid after such operators. It is independent of the bases used to measure its values. Compatible arbitrary adjacent bases, including diagonal bases above N, are therefore allowed, with the same base on each common intermediate space. The displayed expression is formula (F) of Proposition 4.2.48; when its kernels satisfy the indicated pointwise bounds, that proposition applies. This is a finite sum, not an integration of a non-invariant pointwise estimate.

Unnormalized gluing and branching. The prescribed support of x lies above Mq, and every gluing translation $z(y)$ lies above N, with $N \nmid Mq$. These supports and their conjugates are disjoint. Every cross pairing in the finite Heisenberg multiplication law is zero. Including its trace-one semicharacter consequently gives

$$H_j(z_j(y))H_j(x_j)v_j^{\mathrm{int}} = H_j(x_j + z_j(y))v_j^{\mathrm{int}}$$

with phase one. Expand the prescribed finite tensor products of gluing sums. This proves item4 as an equality of full tensors, hence as the same operator on every source form simultaneously. No cardinality is divided out. Coincident translation tuples are collected by addition of finitely many prescribed coefficients and retain their integrality.

For $p \neq N$ the new translations vanish above p, so the original $\bar{\pi}_M x_j$-integrality is unchanged. If also $p \nmid M$, this denominator is a unit and the standard denominator is one. The prescribed coefficient condition gives integrality when $p \nmid q$. Finally the branching matrix is the fixed integral branching map, and its normalizing scalar $\prod_\tau \tau(\bar{\pi}_N)^{-b_\tau}$ is a p-adic unit at $p \neq N$, since π_N is supported above N. This proves the final assertion without a loss or an extra class weight. □

Proposition 4.2.48. *All notation in this statement is local. Fix the inert or ramified data, column counts, marked set, charges and rectangular Schur coefficients of Proposition 4.2.49, with $r \leq N$ and no columns of height greater than r. In particular retain its full-height source determinant factors. Write*

$$D = \Delta + \gamma(B - B^-) - |\Phi|N^3(1 + \log_p \max(B, 1)).$$

Let $\mathscr{T}, \mathscr{I}$ be the finite sets of semistandard output and source fillings in that proposition. Let $(F_\alpha)_{\alpha\in A}$ be any family of formal series, each satisfying the displayed normal-form hypothesis

of that proposition with the SAME $p, \Phi, r, N, \kappa, \mathcal{M}, w, \sigma_e$; the integral multiplier and integral exponential coefficients may depend on α. Set

$$J_\alpha = (J_{S_{T,I}}(F_\alpha))_{T\in\mathscr{T},I\in\mathscr{I}}.$$

Choose nonzero $u_{\tau b} \in \mathbb{C}_p$, integers e_τ, and $H \in \mathbb{R}$. Put

$$d_T = \prod_{\tau\in\Phi}\Big(\prod_{b=1}^{N} u_{\tau b}\Big)^{e_\tau} \prod_{\tau,h,j}\prod_{b=1}^{h} u_{\tau,T_{\tau hj}(b)}.$$

Suppose $v_p(d_T) \geq q(T) - H$ for every T.

Let X, Y be sets of source and output points. For $x \in X, y \in Y$ fix invertible matrices B_x on $\mathbb{C}_p^{\mathscr{I}}$ and C_y on $\mathbb{C}_p^{\mathscr{T}}$, and define the gauges

$$\|v\|_x = \|B_x v\|_\infty, \qquad \|w\|_y = \|C_y w\|_\infty.$$

Suppose a family of linear kernel maps $K(y,x)$ has the following explicit covariance property: for each (y,x) there are a finite subset $A_{y,x} \subseteq A$, numbers $a_\alpha, c_\alpha \in O$, and matrices $U_{y,x} \in M_{\mathscr{T}}(O)$, $V_{y,x} \in M_{\mathscr{I}}(O)$ such that

$$C_y K(y,x) B_x^{-1} = U_{y,x}\left(\sum_{\alpha\in A_{y,x}} a_\alpha c_\alpha^B \operatorname{diag}(d_T) J_\alpha\right) V_{y,x}. \tag{K}$$

Here $c^0 = 1$, also at $c = 0$. The choices in (K) precede the source vector. Then for every x, y, v,

$$\|K(y,x)v\|_y \leq p^{H-D}\|v\|_x. \tag{G}$$

In particular for no marks, zero charges and $u_{\tau b} = 1$, one may take $H = 0$.

The same assertion may be used at successive steps with different counts and ranks and constants D_j, H_j. More precisely, suppose the intermediate gauges at each adjacent step agree, and a composite value has a literal finite expression

$$F(y) = \sum_{\omega\in\Omega_y} a_\omega K_{m-1}(y, x_{m-1,\omega}) \cdots K_0(x_{1,\omega}, x_{0,\omega})\, f(x_{0,\omega}), \qquad a_\omega \in O, \tag{F}$$

where every kernel satisfies (K) for its own data. If $R \geq 0$ and $\|f(x)\|_x \leq R$ for every source point occurring in (F), then

$$\|F(y)\|_y \leq R\, p^{\sum_{j=0}^{m-1}(H_j - D_j)}. \tag{CG}$$

A final linear restriction whose matrix in the indicated gauges has entries of valuation at least $-R_0$ adds precisely the allowance R_0 to this exponent.

Proof. Apply (CS) of Proposition 4.2.49 to each F_α. It gives

$$v_p((J_\alpha)_{T,I}) \geq D - q(T).$$

The hypothesis on d_T therefore gives

$$v_p(d_T (J_\alpha)_{T,I}) \geq D - H.$$

Every entry of each matrix $\operatorname{diag}(d_T)J_\alpha$ consequently has absolute value at most p^{H-D}. Multiplication by $a_\alpha c_\alpha^B \in O$, finite summation and left or right multiplication by matrices over O preserve

that bound: each resulting entry is a finite sum of terms of absolute value at most p^{H-D}. This uses the ultrametric inequality and introduces no factor equal to the number of entries, generators or summands. Thus (K) implies that every entry of $C_y K(y,x) B_x^{-1}$ is bounded by p^{H-D}. Its action on $B_x v$ has sup norm at most $p^{H-D}\|B_x v\|_\infty$, proving (G), including zero-dimensional or zero-vector cases.

The factor c_α^B is exact for scaling the variables in an ordinary jet: each $S_{T,I}$ is homogeneous of degree B, so $J_{S_{T,I}}(F_\alpha(c_\alpha z)) = c_\alpha^B J_{S_{T,I}}(F_\alpha)$. No divided derivative is used. Its integrality is why such a scaling cannot worsen the estimate. The determinant factor in d_T was retained in the valuation calculation; discarding it would require an additional proof that it is a unit or that its valuation cancels. Likewise the source determinant factors belong to $S_{T,I}$ as prescribed by (SC), so no removal of full-height source columns has occurred.

In the unmarked case $q(T) = 0$, and with $u = 1$ one has $d_T = 1$, proving the stated specialization. In the ramified case the number D is, literally,

$$\frac{1}{2}\sum_{e\ge 1}\sum_{\tau}\bigl((t_\tau/p^e - t_{\sigma_e\tau} - c_{\sigma_e\tau})_+ + (L_\tau/p^e - L_{\sigma_e\tau})_+ - c_\tau/p^e\bigr)$$
$$- \frac{1}{2}D_0 + \gamma(B - B^-) - |\Phi|N^3(1 + \log_p \max(B,1)).$$

In the inert case only odd positive e occur, their prefactor is 1, and there is no D_0 term. Thus the transfer neither doubles the ramified prime weight nor loses its zero-exponent correction.

For the chain, apply (G) to each factor of a fixed summand in (F), starting with its source vector. The adjacent gauges agree, so induction gives the bound $Rp^{\sum_j (H_j - D_j)}$ for that summand. Its coefficient is integral, and the sum is finite. The ultrametric inequality proves (CG); an empty sum is zero and satisfies the bound because $R \ge 0$. Finally a restriction matrix with entry norm at most p^{R_0} multiplies a sup norm by at most p^{R_0}, giving exactly the stated extra allowance. □

Proposition 4.2.49. *Let p be an odd prime, put $O = \mathcal{O}_{\mathbb{C}_p}$, and normalize $v_p(p) = 1$ and $v_p(0) = +\infty$. Choose one of the following cases:*

	π	ρ	S
inert	p	$(p^2-1)^{-1}$	$\{1,3,5,\dots\}$
ramified	$\pi^2 = -p$	$[2(p-1)]^{-1}$	$\mathbb{Z}_{\ge 0}$.

In the second row a square root $\pi \in \mathbb{C}_p$ is fixed. Put $v_\pi = v_p(\pi)$ and $\gamma = (p-1)^{-1} - \rho$. Let Φ be a finite set, $N, r \ge 1$, and $\sigma_e : \Phi \to \Phi$ be maps for $e \in S$. Write $z_{\tau ba}$ for independent variables with $\tau \in \Phi$, $1 \le b \le N$, $1 \le a \le r$, and set

$$\Gamma_e(z) = \sum_{\tau\in\Phi}\sum_{b=1}^{N}\sum_{a=1}^{r} z_{\tau ba}^{p^e} z_{\sigma_e(\tau),b,a}.$$

Suppose $\varpi \in \mathbb{C}_p^\times$, $v_p(\varpi) = \rho$, $F \in \mathbb{C}_p[[z]]$, $W \in O[[z]]$ and $\lambda_{ej} \in O$ satisfy the coefficientwise formal identity

$$F(\varpi z) = W(z)\exp\left(A\sum_{e\in S, j\ge 0}\lambda_{ej}\Gamma_e(z)^{p^j}\right), \qquad A = \varpi^{p+1}/p.$$

Let $\kappa_{\tau h}$ be nonnegative integers for $\tau \in \Phi$ and $1 \le h < N$, extended by zero to all other heights. At each place arrange $\kappa_{\tau h}$ columns of height h in decreasing order of height. A filling T assigns to each such column an increasing list

$$T_{\tau hj}(1) < \cdots < T_{\tau hj}(h), \qquad T_{\tau hj}(k) \in \{1,\dots,N\}, \quad 1 \le j \le \kappa_{\tau h}.$$

For integral matrices $y_\tau = (y_{\tau ak})_{1\le a\le r,\,1\le k\le N} \in M_{r,N}(O)$ *put*

$$P_{T,y}(z) = \prod_{\tau\in\Phi}\prod_{h=1}^{N-1}\prod_{j=1}^{\kappa_{\tau h}} \det\left(\sum_{a=1}^{r} z_{\tau,T_{\tau hj}(b),a}\, y_{\tau ak}\right)_{1\le b,k\le h}.$$

For $Q = \sum_\alpha Q_\alpha z^\alpha$ *define*

$$J_Q(F) = Q(\partial)F(0) = \sum_\alpha Q_\alpha \alpha! [z^\alpha]F, \qquad \alpha! = \prod_{\tau,b,a} \alpha_{\tau ba}!.$$

Let $\mathcal{M}\subseteq\Phi$ *and let* $w_{\tau hb}\in\mathbb{Z}_{\ge 0}$ *be charges. Require that for every* $e\in S$, $\tau\in\Phi$, $1\le h<N$ *and* $1\le b\le N$ *for which both* $\kappa_{\tau 1}$ *and* $\kappa_{\sigma_e(\tau),h}$ *are positive,*

$$\tau\in\mathcal{M} \text{ or } (h=1 \text{ and } \sigma_e(\tau)\in\mathcal{M}) \quad\Longrightarrow\quad w_{\tau 1b}>0 \text{ or } w_{\sigma_e(\tau),h,b}>0.$$

When $N=1$, *all counts, fillings and charge conditions are empty. Define*

$$t^0_\tau = \begin{cases}\kappa_{\tau 1}, & N>1,\\ 0, & N=1,\end{cases} \qquad t_\tau = \mathbf{1}_{\tau\notin\mathcal{M}} t^0_\tau,$$

$$c_\tau = \sum_{\substack{1\le h<N\\ h\ne 1}} \kappa_{\tau h}, \qquad L_\tau = \sum_{\substack{1\le h<N\\ h\ne 1}} h\kappa_{\tau h},$$

$$B = \sum_\tau (t^0_\tau + L_\tau), \qquad B^- = \sum_\tau (t_\tau + L_\tau),$$

$$q(T) = \sum_{\tau,h,j}\sum_{b=1}^{h} w_{\tau,h,T_{\tau hj}(b)}.$$

For a real number x *write* $x_+ = \max(x,0)$. *With* $\tau' = \sigma_e(\tau)$ *put*

$$D_e = \sum_{\tau\in\Phi} \left(\min(t_\tau/p^e, t_{\tau'}+c_{\tau'}) + c_\tau/p^e + \min(L_\tau/p^e, L_{\tau'})\right).$$

The following series converges, and defines the number

$$\Delta = v_\pi \sum_{\substack{e\in S\\ e>0}}\sum_{\tau\in\Phi} \left((t_\tau/p^e - t_{\tau'} - c_{\tau'})_+ + (L_\tau/p^e - L_{\tau'})_+ - c_\tau/p^e\right) - \begin{cases}\frac12 D_0, & \textit{ramified},\\ 0, & \textit{inert}.\end{cases}$$

Then, for every T *and* y *as above,*

$$v_p(J_{P_{T,y}}(F)) + q(T) \ge \Delta + \gamma(B-B^-) - |\Phi| N^3 (1+\log_p \max(B,1)). \tag{CJ}$$

In particular $\mathcal{M}=\varnothing$ *and* $w=0$ *give the uncharged bound.*

Suppose additionally $r\le N$ *and* $\kappa_{\tau h}=0$ *for* $r<h<N$. *Put*

$$d_\tau = \begin{cases}\kappa_{\tau r}, & r<N,\\ 0, & r=N,\end{cases} \qquad \nu_{\tau i} = \sum_{i\le h<N} \kappa_{\tau h} - d_\tau \quad (1\le i\le r).$$

For semistandard fillings I_τ *of shape* ν_τ *with entries in* $\{1,\dots,r\}$, *let* $b_I(y)$ *be the product, over places and columns of these fillings, of the minors of* y_τ *on the rows in the column and its first* h *columns. Set*

$$\beta_I(y) = b_I(y)\prod_{\tau\in\Phi} \det(y_{\tau,\bullet,1:r})^{d_\tau}.$$

There is a unique expansion with integral polynomial coefficients

$$P_{T,y}(z) = \sum_{I} S_{T,I}(z)\beta_I(y), \qquad S_{T,I}(z) \in \mathbb{Z}[z]. \tag{SC}$$

For semistandard T, these are the rectangular Schur coefficients in the indicated semistandard polynomial bases, including the displayed full-height source determinant factor. For every I, the same bound holds:

$$v_p(J_{S_{T,I}}(F)) + q(T) \geq \Delta + \gamma(B - B^-) - |\Phi|N^3(1 + \log_p \max(B, 1)). \tag{CS}$$

Proof. **Finite linear congruences can be imposed with their stated determinant cost.** For a vector $a = (a_1, \dots, a_h) \in O^h$ and a nonzero modulus $\alpha \in O$, there is a matrix $u \in M_h(O)$ with nonzero determinant, $v_p(\det u) \leq v_p(\alpha)$, and every coordinate of ua divisible by α. If $h = 0$ or $a = 0$, take the identity. Otherwise choose k with $|a_k|_p$ maximal. Every a_i/a_k belongs to O. In the identity matrix replace column k by the vector whose kth entry is α and whose other entries are $-a_i/a_k$. The resulting matrix has determinant α, its kth value on a is αa_k, and its other values vanish. Apply this construction successively to a finite family of linear forms, starting with the identity matrix. At each step apply it to the values of the next form on the current rows. Old congruences persist because the new rows are integral combinations of old rows. Multiplication of determinants proves that, for any finite family of nonzero moduli α_j, all its congruences can be met by one matrix g with

$$\det g \neq 0, \qquad v_p(\det g) \leq \sum_j v_p(\alpha_j). \tag{E}$$

Three choices of congruences give exactly the deduction D_e. Write

$$H = \lfloor \log_p \max(B, 1) \rfloor$$

and retain only $e \in S$ with $e \leq H$. For every such e choose a p^eth root of π in O, and also choose a p^eth root in O of every coefficient of each original direction $y_{\tau,\bullet,1}$. Such roots exist in the algebraically closed field $\mathbb{C}_p$ and are integral by their valuations.

Fix one directed pair $\tau, \tau' = \sigma_e(\tau)$. For each place and height, a change of its first h original directions will have the form

$$\xi_{\tau hk,a} = \sum_{\ell=1}^{h} g_{\tau h,k\ell} y_{\tau a\ell}, \qquad 1 \leq k \leq h. \tag{D}$$

We prescribe congruences on the rows of $g_{\tau h}$, all linear in those rows and with coefficients depending only on the original y. A source congruence has modulus π^{1/p^e} and a partner congruence has modulus π.

First, for every source height $h \neq 1$, impose on each row the single congruence

$$\sum_a y_{\tau',a,1}^{1/p^e} \xi_{\tau hk,a} \equiv 0 \pmod{\pi^{1/p^e}}. \tag{L}$$

Its total determinant cost, counted once per source column, in units of v_π, is c_τ/p^e.

Second, choose the cheaper of the following two prescriptions: impose divisibility by π^{1/p^e} on the sole matrix entry of $g_{\tau 1}$ if $\tau \notin \mathcal{M}$; or impose

$$\sum_a y_{\tau,a,1}^{p^e} \xi_{\tau' hk,a} \equiv 0 \pmod{\pi} \tag{R}$$

on every row at the partner, for $h \neq 1$ and also for $h = 1$ if $\tau' \notin \mathcal{M}$. The costs are respectively t_τ/p^e and $c_{\tau'} + t_{\tau'}$. This contributes their minimum. A prescription of cost zero still has its stated meaning; congruences on heights having no columns do not affect the cost.

Third, choose between imposing divisibility by π^{1/p^e} on all entries of $g_{\tau h}$ for all $h \neq 1$, and imposing divisibility by π on all entries of $g_{\tau' h}$ for all $h \neq 1$. Their costs are L_τ/p^e and $L_{\tau'}$, so this contributes their minimum. Entries here mean entries of each row; there are h such congruences for a height-h matrix, as required for the determinant estimate (E).

Collect these prescriptions for every retained directed pair. Apply (E) independently to each matrix $g_{\tau h}$, imposing all its collected congruences at once. The resulting determinants are nonzero. Summing the individual costs, including repeated prescriptions when they occur, gives

$$V := \sum_{\tau,h} \kappa_{\tau h} v_p(\det g_{\tau h}) \leq v_\pi \sum_{\substack{e \in S \\ e \leq H}} D_e. \tag{C}$$

This counts two prescriptions at the same place separately when $\tau = \tau'$; (E) applies to their union, so loops cause no exception.

These congruences make

$$\sum_a \xi_{\tau hk,a}^{p^e} \xi_{\tau' h' k',a} \equiv 0 \pmod{\pi} \tag{P}$$

unless $h = 1$ and either $\tau \in \mathcal{M}$, or $h' = 1$ and $\tau' \in \mathcal{M}$. To check this assertion, first take $h \neq 1$, $h' = 1$. Raising (L) to the p^eth power and using Frobenius additivity modulo π gives $\sum_a y_{\tau',a,1} \xi_{\tau hk,a}^{p^e} \equiv 0$. The new height-one direction at τ' is a scalar multiple of $y_{\tau',\bullet,1}$, so (P) follows. Frobenius is additive here because π divides p. For $h = h' = 1$ and neither place marked, the second prescription makes either the source direction divisible by π^{1/p^e}, or its pairing with the partner direction divisible by π through (R). For $h = 1$, $h' \neq 1$, $\tau \notin \mathcal{M}$, the same second prescription applies. Finally if both heights differ from one, the third prescription makes the source direction divisible by π^{1/p^e} or the partner direction divisible by π. These cases prove (P) for every nonexempt pair. The congruences remain valid after the simultaneous changes because (L) and (R) use only an original height-one direction on the opposite side, which changes merely by a scalar.

Charges make every relevant coupling divisible by π. Changing $w_{\tau hb}$ to 1 at heights with $\kappa_{\tau h} = 0$ leaves $q(T)$ unchanged and makes the charge condition hold for every exempt pair of heights. We make that harmless replacement while constructing the substitution. Introduce variables

$$s_{\tau hkb} \quad (\tau \in \Phi,\ 1 \leq h < N,\ 1 \leq k \leq h,\ 1 \leq b \leq N)$$

and integral linear forms

$$Z_{\tau ba}(s) = \sum_{h=1}^{N-1} \sum_{k=1}^{h} p^{w_{\tau hb}} \xi_{\tau hk,a} s_{\tau hkb}.$$

Modulo π, Frobenius additivity expands $\Gamma_e(Z)$ into monomials $s_{\tau hkb}^{p^e} s_{\tau' h' k' b}$ whose coefficients are

$$p^{p^e w_{\tau hb} + w_{\tau' h' b}} \sum_a \xi_{\tau hk,a}^{p^e} \xi_{\tau' h' k',a}.$$

For nonexempt pairs (P) makes them zero modulo π. For exempt pairs the charge condition makes the displayed power of p divisible by p, hence by π. All other source and partner row indices give zero coefficients, since a coupling pairs equal row indices. Thus

$$\Gamma_e(Z) \in \pi O[[s]] \quad (e \in S,\ e \leq H). \tag{G}$$

The substituted series is integral through degree B. The valuation of $A\pi$ is $1/(p-1)$ in both cases, by substituting the displayed values of ρ and v_π. Let

$$\Psi(s) = \sum_{e\in S, j\geq 0} \lambda_{ej}\Gamma_e(Z(s))^{p^j}.$$

Its constant term is zero. The contribution with $e > H$ has degree at least $p^e + 1 > B$; this remains true when $B = 0$. For the other contributions (G) implies that every coefficient is divisible by π. Consequently the truncation of Ψ to degree at most B equals $\pi D(s)$ for a polynomial $D \in O[s]$ with $D(0) = 0$. The exponentials of $A\Psi$ and $A\pi D$ have identical coefficients through degree B.

Legendre's formula gives $v_p(n!) \leq n/(p-1)$, whence

$$v_p((A\pi)^n/n!) \geq 0.$$

The series $\exp(A\pi D)$ is therefore integral. Every coefficient involves only finitely many of its terms because $D(0) = 0$. The given normal form, after substitution $z = Z(s)$, now shows that

$$G(s) := F(\varpi Z(s))$$

is integral in every degree at most B: its other factor $W(Z)$ is integral. All substitutions and sums used here are defined coefficientwise, since the substituted linear forms have zero constant term and $(p^e + 1)p^j$ tends to infinity outside finite sets of pairs (e, j).

The determinant chain rule and factorials bound the original jet. Let

$$Q_T(s) = \prod_{\tau,h,j} \det(s_{\tau hk,T_{\tau hj}(b)})_{1\leq b,k\leq h}.$$

This is a homogeneous integral polynomial of degree B in

$$n_s = |\Phi|N\sum_{h=1}^{N-1} h \leq |\Phi|N^3$$

variables. For any multi-index α of degree B, the base-p digit formula for the factorial gives

$$v_p(\alpha!) \geq \frac{B}{p-1} - n_s(H+1).$$

Indeed $v_p(n!) = (n - s_p(n))/(p-1)$ and every $n \leq B$ has digit sum at most $(p-1)(H+1)$; the assertion also holds for $n = 0$ and $B = 0$. The coefficients of G in degree B and the coefficients of Q_T are integral, so the ultrametric inequality gives

$$v_p(J_{Q_T}(G)) \geq \frac{B}{p-1} - |\Phi|N^3(H+1). \tag{F}$$

For a linear substitution the polynomial of constant differential operators transforms by the transpose substitution. Apply this to $s \mapsto \varpi Z(s)$. In each height-h determinant, its row indexed by $T_{\tau hj}(b)$ contributes $\varpi p^{w_{\tau,h,T_{\tau hj}(b)}}$, while its directions are the rows of $g_{\tau h}$ applied to the first h columns of y_τ. Multilinearity of the determinant gives the exact identity

$$J_{Q_T}(G) = \varpi^B p^{q(T)} \left(\prod_{\tau,h} (\det g_{\tau h})^{\kappa_{\tau h}}\right) J_{P_{T,y}}(F). \tag{J}$$

This chain rule is valid for products of determinants as well, since substitution on the ring of constant differential operators is multiplicative. Its scalar is nonzero. Combining (F), (J), $v_p(\varpi) = \rho$ and (C), and taking $v_p(0) = +\infty$ if needed, yields

$$v_p(J_{P_{T,y}}(F)) + q(T) \geq \gamma B - v_\pi \sum_{\substack{e \in S \\ e \leq H}} D_e - |\Phi| N^3 (H+1). \tag{T}$$

Summing the deductions gives precisely the asserted prediction. For every $e \in S$,

$$0 \leq D_e \leq p^{-e} \sum_\tau (t_\tau + c_\tau + L_\tau).$$

Thus the series of D_e converges. The identity $x - \min(x,y) = (x-y)_+$ gives, for positive $e \in S$,

$$\sum_\tau \left((t_\tau/p^e - t_{\tau'} - c_{\tau'})_+ + (L_\tau/p^e - L_{\tau'})_+ - c_\tau/p^e \right) = B^-/p^e - D_e. \tag{A}$$

In particular the series defining Δ converges absolutely. The geometric sums are

$$\sum_{\substack{e \in S \\ e > 0}} p^{-e} = \begin{cases} p/(p^2-1), & \text{inert}, \\ 1/(p-1), & \text{ramified}, \end{cases} \qquad v_\pi \sum_{\substack{e \in S \\ e > 0}} p^{-e} = \gamma.$$

In the ramified case $v_\pi = 1/2$ and $0 \in S$, so summing (A), with the specified subtraction of $D_0/2$, gives in both cases

$$v_\pi \sum_{e \in S} D_e = \gamma B^- - \Delta. \tag{S}$$

Since all D_e are nonnegative, (T) and (S), followed by $H \leq \log_p \max(B, 1)$, prove (CJ).

The rectangular coefficient expansion retains the full-height source determinant. Assume the additional rank conditions. For one place put

$$\lambda_i = \sum_{i \leq h < N} \kappa_{\tau h} \quad (1 \leq i \leq r), \qquad d = d_\tau.$$

This is a partition, $\lambda_r = d$, and $\nu_i = \lambda_i - d$. Regard the first r columns of y_τ as an $r \times r$ matrix. For fixed z, the polynomial $P_{T,y}(z)$ is homogeneous of degree λ_i in its ith column. It is invariant under adding a multiple of an earlier column of y_τ to a later one. Indeed every minor either omits the later column or contains both columns and is unchanged by this operation. The polynomial therefore belongs to the dual Weyl polynomial space of weight λ.

Here and below this description of that space is the standard one in the classical Weyl dimension formula; see [151], Part II, Chapter 2, and [92], Lecture 15 and Theorem 24.3. A semistandard tableau of shape λ has its d full-height columns forced to be $(1, \ldots, r)$. Removing them gives a semistandard tableau of shape ν. Conversely, adjoining those columns gives a semistandard tableau: an entry in row i of the remaining tableau is at least i. Thus the polynomials associated to all tableaux of shape λ are exactly

$$b_I(y_\tau) \det(y_{\tau,\bullet,1:r})^d.$$

Order variables column by column, with smaller row index first, and take lexicographic leading terms. A flag minor with row set $a_1 < \cdots < a_h$ has leading monomial $\prod_{i=1}^h y_{\tau,a_i,i}$ with coefficient one: in the first column choose the smallest available row, and continue successively. Therefore a semistandard minor product has leading monomial recording the multiset of entries in each row

of its tableau. The weakly increasing rows recover the tableau uniquely from these multisets. Distinct such tableaux consequently have distinct leading monomials and leading coefficient one, over every field.

The number of these tableaux is the dimension of the dual Weyl polynomial space in characteristic zero. To spell out the classical count, the tableau formula for the Schur polynomial gives their number as $s_\lambda(1,\dots,1)$; see [194], Chapter I, (3.1) and (5.12). Substituting $x_i = t^{r-i}$ in its alternant expression, cancelling Vandermonde factors, and taking $t \to 1$ gives

$$s_\lambda(1,\dots,1) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

which is Weyl's dimension. Hence the independent minor products just described form a basis in characteristic zero. Multiplying the determinant factors and taking products over the disjoint place variables gives the analogous basis for all places.

Apply this basis statement to each coefficient of z in $P_{T,y}(z)$, over $\mathbb{Q}$. Every such coefficient is a polynomial in y with integer coefficients satisfying the required column homogeneities and invariances, so it is a rational linear combination of the β_I. Its combination coefficients are integers: choose the largest leading monomial among basis polynomials with nonzero coefficient, read its coefficient from the given integer polynomial, subtract that integer multiple, and continue with the finitely many remaining basis polynomials. This proves (SC) with $S_{T,I} \in \mathbb{Z}[z]$, and their uniqueness follows from independence. When $r < N$, $d = \kappa_{\tau r}$; when $r = N$, the count has no height-N columns and $d = 0$. These facts account exactly for the determinant factor in (SC).

An integral evaluation bounds each coefficient jet by a minor-product jet. Fix T and put $a_I = J_{S_{T,I}}(F) \in \mathbb{C}_p$. If all a_I vanish, (CS) is immediate. Otherwise choose I_0 with $|a_{I_0}|_p$ maximal. Then $a_I/a_{I_0} \in O$, and

$$R(y) = \sum_I (a_I/a_{I_0})\beta_I(y) \in O[y]$$

has nonzero reduction in the residue field of O. This follows from the distinct leading monomials established above, whose leading coefficients remain one after reduction. The residue field is $\overline{\mathbb{F}}_p$, hence infinite. A nonzero polynomial over an infinite field has a point where it does not vanish: induction on the number of variables reduces this to the fact that a nonzero univariate polynomial has finitely many roots. Choose such a point for the reduction of R and lift all its coordinates to O. At the resulting family y one has $|R(y)|_p = 1$. Applying the jet to (SC) gives

$$J_{P_{T,y}}(F) = \sum_I a_I \beta_I(y) = a_{I_0} R(y),$$

so $|a_I|_p \le |J_{P_{T,y}}(F)|_p$ for every I. Bound (CJ) for this integral y proves (CS), since $q(T)$ and every term on its right side are independent of y. Empty shapes and an empty set of places give the same identities with empty products equal to one. □

Lemma 4.2.50. *The constant bound C_P is that of Definition 2.3.3. All supplementary notation is local. Let $p < x_1$ be a nonsplit prime, $0 \le j < m$, $n = n_{j+1}$, and $S \subseteq \Phi$. Use the column counts*

$$C_\tau(h) = \begin{cases} a\mathbf{1}_{\Phi_a}(\tau), & h = 1, \\ \kappa_1(\tau) - n_1/2, & h = k, \\ \Delta - (t+1), & h > k,\ t+1 \mid h-k, \\ 0, & \textit{otherwise.} \end{cases}$$

Put

$$c_\tau = \sum_{\substack{1\le h<n\\ h\neq 1}} C_\tau(h), \quad l_\tau = \sum_{\substack{1\le h<n\\ h\neq 1}} hC_\tau(h), \quad t_\tau = a\mathbf{1}_{\Phi_a\setminus S}(\tau), \quad B = \sum_\tau \left(a\mathbf{1}_{\Phi_a}(\tau) + l_\tau\right).$$

Write I_p for the positive partner exponents of Definition 2.4.1, $\tau_e = -p^e\tau$, $v = v_p(\pi_p)$, and

$$\rho = \begin{cases} (p^2-1)^{-1}, & p \text{ inert},\\ [2(p-1)]^{-1}, & p = d, \end{cases} \qquad \gamma = (p-1)^{-1} - \rho.$$

Define

$$P_{\tau e} = (t_\tau/p^e - t_{\tau_e} - c_{\tau_e})_+ + (l_\tau/p^e - l_{\tau_e})_+ - c_\tau/p^e$$

and

$$R_0 = \sum_\tau \big(\min(t_\tau, t_{-\tau} + c_{-\tau}) + c_\tau + \min(l_\tau, l_{-\tau})\big).$$

Set

$$\begin{aligned} D_j(S) &= v\sum_{\tau\in\Phi}\sum_{e\in I_p} P_{\tau e} - \mathbf{1}_{p=d}R_0/2 + \gamma a\,\#(\Phi_a\cap S)\\ &\quad - fn^3(1+\log_p\max(B,1)),\\ C_p(S) &= \sum_{\tau\in\Phi_a\setminus S}\ \sum_{\substack{e\in I_p\\ \tau_e\in\Phi_a\setminus S}} a/p^e. \end{aligned}$$

All displayed series converge. The number $D_j(S)$ is precisely the charged exponent of Proposition 4.2.49 for these diagrams with height-one columns erased at S. Then

$$D_j(S) \ge \frac{\mathrm{gain}(p)}{m} - C_p(S) + v\,\#(\Phi_a\cap S)\frac{a}{x_1} - \frac{fa(\log b)^2}{pm}.$$

In particular the collision subtraction is unweighted at the ramified prime; the gain, marked surplus and positive-exponent series retain $v = 1/2$, as does the separate displayed zero-exponent correction.

Proof. We first identify the counting exponent, including its integral dimensions and truncation. Since $r = 3^{e'+1}$ is odd and $m = 2^{e+1}$ is even, $k = r + 1 - m$ and $t + 1 = 5r + 1$ are even. Thus every $n_i = k + i(t + 1)$ is an even integer and

$$\lfloor n_1/2\rfloor = n_1/2.$$

Moreover $k \ge x^5 + 1 - 2\sqrt{x} > 1$, using the scale bounds recorded below. Put $N = n_{j+1}$ and $r_{\mathrm{src}} = n_j$. Then $1 < k \le r_{\mathrm{src}} < N$, so both dimensions are positive integers and the height-one and height-k cases are disjoint. The two multiplicities with subtraction are nonnegative: $n_1 \le 7r$ gives

$$\kappa_1(\tau) - n_1/2 = \kappa(\tau) + 3r^8 - n_1/2 \ge \kappa(\tau) \ge 0, \qquad \Delta - (t+1) = 2r^8 - 5r - 1 \ge 0$$

for $r \ge 3$. They are integers. Hence replacing natural-number subtraction by ordinary subtraction, and replacing the floor by $n_1/2$, changes no column count.

In Proposition 4.2.49 set $\kappa_{\tau h} = C_\tau(h)$ for $1 \le h < N$, and zero at all other heights. This last truncation is essential: the displayed formula for $C_\tau(h)$ itself is not asserted to vanish above

N. A nonzero height in this truncated diagram is either 1 or $k + i(t+1)$ with an integer $i \geq 0$. The inequality $k + i(t+1) < k + (j+1)(t+1)$ implies $i \leq j$. Thus every such height is at most $n_j = r_{\mathrm{src}}$. In particular the additional rank and support conditions for (CS) hold with its source dimension r_{src} and target dimension N; a possible column of height exactly r_{src} is retained in the full-height determinant factor, not discarded.

Take the marked set $\mathcal{M} = S$, and the partner maps $\sigma_e(\tau) = -p^e\tau$. They are maps on Φ because the nonsplit prime is prime to q_1, by Lemma 3.2.8. The prime is odd: $d \equiv 7 \pmod 8$ implies that 2 splits in K. The inert exponent set is the positive odd integers; the ramified set is $\mathbb{Z}_{\geq 0}$, with positive part I_p and $\sigma_0(\tau) = -\tau$. The valuation of the chosen uniformizer is 1 in the inert case and 1/2 in the ramified case, and the stated ρ, γ agree with those of the counting proposition.

Its quantities t^0, t, c, L, B, B^- consequently satisfy, term by term,

$$t^0_\tau = a\mathbf{1}_{\Phi_a}(\tau), \quad t_\tau = a\mathbf{1}_{\Phi_a \setminus S}(\tau), \quad c_\tau = c_\tau, \quad L_\tau = l_\tau, \quad B - B^- = a\,\#(\Phi_a \cap S).$$

Here the repeated symbol c_τ denotes the same finite sum over the truncated heights in both formulas. Each positive-exponent summand in its Δ is therefore exactly $P_{\tau e}$. In the ramified case its D_0 is exactly R_0; in the inert case neither zero-exponent correction occurs. Finally $|\Phi| = f$ and $N = n$. Substitution into the right-hand side of (CJ), or of (CS), gives exactly $D_j(S)$, including the factor $1/2$ on R_0 and the unchanged logarithmic subtraction.

This identifies the numerical charged exponent. Application to an actual jet additionally requires the formal normal-form identity and the nonnegative charges satisfying the incidence condition in Proposition 4.2.49; those hypotheses are not consequences of this arithmetic lemma. For every such choice the displayed identification applies, independently of the filling, direction matrices, and charges. We now prove the claimed lower bound for that number.

Put $L = \log b$, $\eta = x^{60}$ and $u = a/(pm)$. We first give explicit bounds used below. By Lemma 4.2.67 and Proposition 2.3.5,

$$x \geq 10^6, \quad L \leq x^{1/100}, \quad b \geq x^{10^{12}}, \quad x_1 \geq x^{10^6}, \quad x_1 \leq b^{1/10^6}, \quad f \leq x^{12}, \quad m \leq 2\sqrt{x}, \quad k \leq 3x^5. \tag{1}$$

Also $C_P \geq 2d + 6$ by Definition 2.3.3, so $d \leq \log L/10^{12} \leq L \leq x^{1/100}$. Since $r \leq 3x^5$, $t = 5r$ and $\Delta = 2r^8$, the rank and column definitions give

$$n \leq n_m \leq 40x^6, \qquad k\kappa(\tau) \leq l_\tau \leq k\kappa(\tau) + \eta, \qquad c_\tau \leq \kappa(\tau) + \eta. \tag{2}$$

Here are direct verifications of the error terms in (2). The height-k multiplicity is $\kappa(\tau) + 3r^8 - n_1/2 \geq \kappa(\tau)$, because $n_1 \leq 7r$ and $r \geq 3$. Its excess is at most $3r^8$. Every other nonzero low height has multiplicity at most $2r^8$. Thus the column excess is at most $(3 + 2n)r^8$, and the box excess at most $(3k + 2n^2)r^8$. With $r^8 \leq 6561x^{40}$ and $n \leq 40x^6$, these are at most $600000x^{46}$ and $22000000x^{52}$ respectively, each at most x^{60} under (1). All these counts are nonnegative.

The following numerical consequences will absorb each error separately:

$$u \geq x^{1000}, \qquad \gamma c_\tau \leq u, \qquad 2\eta\,\#\{e \geq 1 : p^e < x_1\} \leq u, \qquad n^3(1 + \log_p \max(B, 1)) \leq u. \tag{3}$$

Indeed

$$u \geq \tfrac{1}{2}\sqrt{x}\, b^{1-10^{-6}} \geq \tfrac{1}{2}x^{\,10^{12}-10^6+1/2} \geq x^{1000}.$$

Since $\gamma \leq 1/(p-1) \leq 2/p$, $\kappa(\tau) \leq b$ and $\eta \leq b$, the second bound follows from $2m(b + \eta) \leq 8\sqrt{x}\, b \leq xb = a$. There are at most $1 + \log_2 x_1 \leq 3L \leq x$ exponents in the third bound, so its left side is at most $2x^{61} \leq x^{62} \leq u$. Finally (2) gives $B \leq f(a + kb + \eta) \leq x^{18}b$. Hence

$\log\max(B,1) \le L + 18\log x \le 2L$; using $\log p \ge \log 2 > 1/2$ gives $1 + \log_p \max(B,1) \le 5L \le x$. Consequently the last left side in (3) is at most $64000x^{19} \le x^{20} \le u$.

At $p = d$ one has the additional bound

$$R_0 \le 3fu. \tag{4}$$

To see this, first $c_\tau \le 2b \le a/(dm) = u$, since $dm \le 2x^{51/100}$ and $4x^{51/100} \le x$. If $\tau \notin \Phi_b$, then $\kappa(\tau) = a/x_1$ and

$$\frac{dm\, l_\tau}{a} \le \frac{6x^{551/100}}{x_1} + \frac{2x^{51/100}\eta}{a} \le 1.$$

The last inequality follows from $x_1 \ge x^{10^6}$ and $b \ge x^{10^{12}}$, each summand being at most $1/2$. The classes of τ and $-\tau$ modulo q_0 are opposite, and q_0 is divisible by 16. They cannot both be a-places or both be b-places. Therefore $\min(l_\tau, l_{-\tau}) \le u$. If $t_\tau > 0$, then $t_{-\tau} = 0$ and $\min(t_\tau, t_{-\tau} + c_{-\tau}) \le c_{-\tau} \le u$; if $t_\tau = 0$ the same assertion is immediate. Adding these three bounds proves (4).

We now compare the prime sums term by term. By Lemma 3.2.8, nonsplit p does not divide q_1, so all the partner classes here are genuine units. For $e \in I_p$, set $\tau' = \tau_e$ and let $\mathcal{C}_{\tau e}$ mean $\tau, \tau' \in \Phi_a \setminus S$. When $p^e < x_1$, the bounds (2) give

$$\begin{aligned} &\mathbf{1}_{\Phi_a}(\tau)(a/p^e - \kappa(\tau'))_+ + \mathbf{1}_{e=1}\mathbf{1}_{\Phi_a\cap S}(\tau)a/x_1 \\ &\quad \le (t_\tau/p^e - t_{\tau'} - c_{\tau'})_+ + \mathbf{1}_{\Phi_a\cap S}(\tau)a/p^e + \mathbf{1}_{\mathcal{C}_{\tau e}}a/p^e + \eta, \qquad (5)\\ &k(\kappa(\tau)/p^e - \kappa(\tau'))_+ \le (l_\tau/p^e - l_{\tau'})_+ + \eta. \qquad (6)\end{aligned}$$

For completeness, if τ is not an a-place, (5) has zero left side. If its long row is erased, its right side includes a/p^e. For $e = 1$ the additional surplus is still paid by this term, because $\kappa(\tau') \ge a/x_1$ and $p < x_1$ imply $(a/p - \kappa(\tau'))_+ + a/x_1 \le a/p$. If the row is retained and a collision occurs, the collision term pays its entire left side. In the remaining case $t_{\tau'} = 0$, and $c_{\tau'} \le \kappa(\tau') + \eta$ gives (5) by $z_+ \le (z-\eta)_+ + \eta$. For (6), use $l_\tau \ge k\kappa(\tau)$ and $l_{\tau'} \le k\kappa(\tau') + \eta$, and the same elementary positive-part inequality.

For $p^e \ge x_1$, both prime-sum terms vanish, since $\kappa(\tau') \ge a/x_1$ and $\kappa(\tau) \le a$. Here $e \ne 1$, and (5)–(6) combined hold with no error term: after the term c_τ/p^e below is added to $P_{\tau e}$, the remaining positive parts are nonnegative. It follows for all positive partner exponents that

$$\begin{aligned} &\mathbf{1}_{\Phi_a}(\tau)(a/p^e - \kappa(\tau_e))_+ + k(\kappa(\tau)/p^e - \kappa(\tau_e))_+ + \mathbf{1}_{e=1}\mathbf{1}_{\Phi_a\cap S}(\tau)a/x_1 \\ &\quad \le P_{\tau e} + (c_\tau + a\mathbf{1}_{\Phi_a\cap S}(\tau))/p^e + \mathbf{1}_{\mathcal{C}_{\tau e}}a/p^e + 2\eta\mathbf{1}_{p^e<x_1}. \end{aligned} \tag{7}$$

All series are absolutely convergent: each positive part in $P_{\tau e}$ is bounded by its numerator divided by p^e, and every other series is bounded by a geometric series or has finite support.

In both nonsplit cases,

$$\gamma = v\sum_{e\in I_p} p^{-e}, \qquad 0 < v \le 1.$$

Multiply (7) by v and sum over e. Use the two middle error bounds in (3), and bound the nonnegative collision series multiplied by v by its unweighted value. At each place this gives

$$\begin{aligned} &\mathbf{1}_{\Phi_a}(\tau)s_p(a,\tau) + ks_p(\kappa(\tau),\tau) + v\mathbf{1}_{\Phi_a\cap S}(\tau)a/x_1 \\ &\quad \le v\sum_{e\in I_p} P_{\tau e} + \gamma a\mathbf{1}_{\Phi_a\cap S}(\tau) + \sum_{e\in I_p}\mathbf{1}_{\mathcal{C}_{\tau e}}a/p^e + 2u. \end{aligned}$$

Sum over τ and use the definition of the gain:

$$\frac{\text{gain}(p)}{m} + v\,\#(\Phi_a\cap S)a/x_1 \le v\sum_{\tau,e} P_{\tau e} + \gamma a\,\#(\Phi_a\cap S) + C_p(S) + 2fu. \tag{8}$$

At an inert prime no zero-exponent term is subtracted. At $p = d$, (4) shows that its subtraction is at most $3fu/2 \le 2fu$. The last bound in (3) shows that the Schur counting subtraction is at most fu. Consequently (8) implies

$$D_j(S) \ge \mathrm{gain}(p)/m - C_p(S) + v\,\#(\Phi_a \cap S)a/x_1 - 5fu.$$

Finally $L^2 \ge 5$ and $u = a/(pm)$, so $5fu \le faL^2/(pm)$, proving the claimed estimate with every stated factor and correction retained. □

Lemma 4.2.51. *Fix the realization before all twists and source forms. Let p, a prime identification $\iota \mid p$, and an ordered marked set $\mathcal{L}$ be as in Proposition 4.2.57; write*

$$h_p = \lceil \log_p m \rceil, \qquad t_p = h_p f(n_m + h_p).$$

Let ϑ be an ordinary twist with $p \nmid q = \mathrm{N}Q$. Let $\boldsymbol{x} = (x_e)_{e<m}$ range over any finite family of translation tuples satisfying

$$\bar{\pi}_M x_e \text{ integral at every } w \nmid q, \qquad x_{e,w} = 0 \text{ for } w \nmid Mq.$$

Put $F_{\boldsymbol{x},e} = \rho_f(x_e)v_e^{\mathrm{int}}$, with the finite Heisenberg phase of (WD8), and

$$\Theta_{\boldsymbol{x},e} = T_{\vartheta,e,F_{\boldsymbol{x},e}}, \qquad C_{\boldsymbol{x},0} = 1, \qquad C_{\boldsymbol{x},e+1} = \Theta_{\boldsymbol{x},e}C_{\boldsymbol{x},e}.$$

The assertions below also hold after any fixed finite linear gluing at places above N.

For $\mathfrak{l} \in \mathcal{L}$ let $\mathrm{K}_{\mathfrak{l},e}$ be the integral maximal compact supported at $\mathfrak{l}, \bar{\mathfrak{l}}$ on X_e. On its fixed vectors define the unnormalised right Hecke sums and polynomials

$$T_{\mathfrak{l},i}^{(e)}v = \sum_{g\mathrm{K}_{\mathfrak{l},e} \in \mathrm{K}_{\mathfrak{l},e}\,\mathrm{diag}(p1_i,1)\mathrm{K}_{\mathfrak{l},e}/\mathrm{K}_{\mathfrak{l},e}} \rho_e(g)v, \quad \mathrm{H}_{\mathfrak{l},e}(U) = \sum_{i=0}^{n_e}(-U)^i q_{\mathfrak{l}}^{\binom{i}{2}} T_{\mathfrak{l},i}^{(e)}.$$

Then the following conclusions hold.

1. *There is an integer $a_{\mathfrak{l}} \ge 1$ with $q_{\mathfrak{l}} = p^{2a_{\mathfrak{l}}}$ and $2a_{\mathfrak{l}} \le f$. Put $z_{\mathfrak{l}} = p^{a_{\mathfrak{l}}}$ and*

$$\xi_{\mathfrak{l},e} = (\mu_{e+1}^0 \vartheta^{e+1})^{\mathrm{BC}}(\varpi_{\mathfrak{l}}), \qquad R_{\mathfrak{l},e}(U) = \prod_{u=0}^{t}(1 - q_{\mathfrak{l}}^u z_{\mathfrak{l}}^{n_e}\xi_{\mathfrak{l},e}U).$$

   Here the uniformizer idele is supported at $\mathfrak{l}$. One has $|\iota(\xi_{\mathfrak{l},e})|_p = 1$, so $R_{\mathfrak{l},e} \in \mathcal{O}_\iota[U]$ and $R_{\mathfrak{l},e}(0) = 1$. The datum is fixed by the target maximal compact. For every $\mathrm{K}_{\mathfrak{l},e}$-fixed form v, the form $\Theta_{\boldsymbol{x},e}v$ is $\mathrm{K}_{\mathfrak{l},e+1}$-fixed and

$$\mathrm{H}_{\mathfrak{l},e+1}(U)\Theta_{\boldsymbol{x},e}v = R_{\mathfrak{l},e}(U)\,\Theta_{\boldsymbol{x},e}\mathrm{H}_{\mathfrak{l},e}(z_{\mathfrak{l}}^{t+1}U)v.$$

   The polynomial is independent of $\boldsymbol{x}$.

2. *For every $0 \le j < m$ there are $c_{\mathfrak{l},j,i} \in \mathcal{O}_\iota$, $0 \le i \le h_p$, independent of $\boldsymbol{x}$, such that on the original source forms*

$$\sum_{i=0}^{h_p} c_{\mathfrak{l},j,i}T_{\mathfrak{l},i}^{(j+1)}C_{\boldsymbol{x},j+1} = p^{t_p}C_{\boldsymbol{x},j+1}T_{\mathfrak{l},h_p}^{(0)}. \tag{HR}$$

   The coefficients may depend on $\vartheta, \mathfrak{l}, j$; the marked set and its order precede ϑ.

3. *Write $Q^{(j)}_{\mathfrak{l},i}$ for the same right-coset sum acting on the step-j datum through $\omega_{\vartheta,j}(1,g)$. It passes through the class integral to $T^{(j+1)}_{\mathfrak{l},i}$. On the supported datum and its transforms at other marked places it commutes with every source action and every integral target action, and hence with the actual target level.*

   In the fixed composition ordering put

$$A_{\mathfrak{l}} = a\,\#\{\tau \in \Phi_a : \tau \text{ induces } \mathfrak{l}\}, \quad Y_{\mathfrak{l}} = p^{A_{\mathfrak{l}}} T^{(0)}_{\mathfrak{l},h_p}, \quad Y = \prod_{\mathfrak{l}\in\mathcal{L}} Y_{\mathfrak{l}}.$$

   There are linear datum operators t_j, independent of $\boldsymbol{x}$, built in that order from

$$S_{\mathfrak{l},j} = p^{A_{\mathfrak{l}}-t_p} \sum_{i=0}^{h_p} c_{\mathfrak{l},j,i} Q^{(j)}_{\mathfrak{l},i},$$

   such that

$$T_{\vartheta,j,t_j F_{\boldsymbol{x},j}} C_{\boldsymbol{x},j} = C_{\boldsymbol{x},j+1} Y.$$

   They retain the stated commutation. The partial chain with transformed data is exactly $C_{\boldsymbol{x},j} Y^j$ for $0 \le j \le m$, with no reordering.

4. *The target diagonal neighbor has entries p on a selected set of indices at $\mathfrak{l}$, p^{-1} on the same indices at $\bar{\mathfrak{l}}$, and 1 elsewhere. Under the arithmetic embedding $z \mapsto \iota(F_\tau(\bar{z}))$ its entries are $p^{-E_{\tau b}}$, where E is $+1$ on selected indices over the induced fibers of $\mathcal{L}$, -1 over their conjugates, and 0 elsewhere. Their norms are $p^{E_{\tau b}}$.*

Proof. **The marked vacuum and its characters.** All operators act on the canonical spaces of the statement. The genuine representation and its tensor compatibility are Proposition 4.1.14; the fixed finite-coordinate transport is Lemma 4.1.3, and its split and integral actions are Lemma 4.1.16. No second model or scalar intertwiner is chosen.

The marked-place arithmetic in Lemma 4.2.52 gives equal lift weights on the two conjugate induced fibers. These places avoid MNq: $p \nmid q$ by hypothesis, whereas $p_0, p_1, N \equiv 1 \pmod{d}$ by $d \mid q_2$ and are split in K, unlike p. The lift characters are unramified there: their finite tame parts are supported over M or Q, and the splitting character is trivial on finite units. The prime p is inert in K, whereas the primes dividing q_1 split in K by Lemma 3.2.8; also $p \neq d$. Thus p is unramified in E. Its residue degree at a marked place is even, say $2a_{\mathfrak{l}}$, since the residue field contains that of the inert quadratic subfield. The distinct places $\mathfrak{l}, \bar{\mathfrak{l}}$ have the same degree. Since $[E:\mathbb{Q}] = 2f$, one has $2a_{\mathfrak{l}} \le f$. In particular p is a uniformizer and $q_{\mathfrak{l}} = p^{2a_{\mathfrak{l}}}$.

The finite local group is identified with the general linear group by the component at $\mathfrak{l}$; write $\mathrm{K}_{\mathfrak{l}} = \mathrm{GL}(\mathcal{O}_{E_{\mathfrak{l}}})$ in its appropriate rank. At a marked pair every translation x_e vanishes, as do the gluing translations over N. The marked factor of $F_{\boldsymbol{x},e}$ is therefore its original vacuum factor, with the common integral-vacuum scalar. Normalized finite Heisenberg translations supported at this pair fix that factor; their Haar probability average is $P^{\{\mathfrak{l}\}}$. Translations away from the pair commute with it. By Lemma 4.1.3(1)–(3),

$$P^{\{\mathfrak{l}\}} F_{\boldsymbol{x},e} = F_{\boldsymbol{x},e} = \mathbf{1}_{\{\mathfrak{l}\}} \otimes_{\{\mathfrak{l}\}} F_{\boldsymbol{x},e}.$$

No phase or scalar is introduced here.

Set $n = n_e$ and $d_{\mathrm{inc}} = t+1$. All the ranks n_e are even: r is odd, m is even, $k = r+1-m$ is even and $t+1 = 5r+1$ is even. Put $\nu = \mu^0_{e+1}\vartheta^{e+1}$. The pair-character parameters are $\nu\upsilon^{n/2}$

and $\nu\upsilon^{(n+d_{\mathrm{inc}})/2}$. By (WD3), $\upsilon(\bar{u}) = \upsilon(u)^{-1}$, so the base change of its norm-one restriction is υ^2. Consequently their space characters are both $\tilde{\nu}$:

$$\widetilde{\nu\upsilon^{n/2}}\upsilon^{-n} = \nu\widetilde{\upsilon^{(n+d_{\mathrm{inc}})/2}}\upsilon^{-(n+d_{\mathrm{inc}})} = \tilde{\nu}.$$

Write $\xi_{\mathfrak{l},e} = \tilde{\nu}(\varpi_{\mathfrak{l}})$ locally. The determinant and modulus factors in Lemma 4.1.16(1) are

$$c_{\mathrm{tar}} = z_{\mathfrak{l}}^{n}\xi_{\mathfrak{l},e}, \quad c_{\mathrm{src}} = z_{\mathfrak{l}}^{n+d_{\mathrm{inc}}}\xi_{\mathfrak{l},e}, \quad q_{\mathfrak{l}}^{d_{\mathrm{inc}}}c_{\mathrm{tar}}/c_{\mathrm{src}} = z_{\mathfrak{l}}^{d_{\mathrm{inc}}}. \tag{1}$$

For integral matrices the determinants are units and the unramified characters are trivial. The integral clause of the same action law proves compact fixing.

Finite subspace counting. Use position matrices Z with $n+d_{\mathrm{inc}}$ rows and n columns over $E_{\mathfrak{l}}$. For each residue subspace U of dimension i, choose an integral matrix d_U with elementary divisors p repeated i times and 1 otherwise, and with $\ker \bar{d}_U = U$. These represent LEFT cosets of $\mathrm{K}_{\mathfrak{l}}\,\mathrm{diag}(p1_i, 1)\mathrm{K}_{\mathfrak{l}}$: left multiplication preserves the reduction kernel; conversely that kernel determines $d_U^{-1}\mathcal{O}^{d_{\mathrm{res}}}$ between $\mathcal{O}^{d_{\mathrm{res}}}$ and $p^{-1}\mathcal{O}^{d_{\mathrm{res}}}$, and hence the left coset. Transposing gives RIGHT representatives d_U^{T}.

The target action of d_U^{T} on the indicator and the inverse-source action of d_U^{-1} are, by Lemma 4.1.16(1),

$$c_{\mathrm{tar}}^{i}\mathbf{1}_{\mathrm{int}}(Z^{\mathsf{T}}d_U^{-1}), \qquad c_{\mathrm{src}}^{i}\mathbf{1}_{\mathrm{int}}(Zd_U^{-1}), \tag{2}$$

with U in residue dimension $n + d_{\mathrm{inc}}$ and n, respectively. Every other factor, including the integral-vacuum scalar, is unchanged.

For a matrix B with d_{res} columns, Bd_U^{-1} is integral exactly when B is integral and $U \subseteq \ker \bar{B}$. In an adapted basis the inverse divides just the coordinates on U by p. Also Bd_U^{-1} integral implies B integral by multiplication by d_U. For a finite residue space,

$$\sum_{U \subseteq \mathbb{F}_{q_{\mathfrak{l}}}^{d_{\mathrm{res}}}} (-V)^{\dim U} q_{\mathfrak{l}}^{\binom{\dim U}{2}} = \prod_{u=0}^{d_{\mathrm{res}}-1} (1 - q_{\mathfrak{l}}^{u}V). \tag{3}$$

Indeed splitting subspaces by containment in a fixed hyperplane gives $\begin{bmatrix} d_{\mathrm{res}} \\ i \end{bmatrix}_{q_{\mathfrak{l}}} = \begin{bmatrix} d_{\mathrm{res}}-1 \\ i \end{bmatrix}_{q_{\mathfrak{l}}} + q_{\mathfrak{l}}^{d_{\mathrm{res}}-i}\begin{bmatrix} d_{\mathrm{res}}-1 \\ i-1 \end{bmatrix}_{q_{\mathfrak{l}}}$. After weighting and summation this is $P_{d_{\mathrm{res}}}(V) = (1 - q_{\mathfrak{l}}^{d_{\mathrm{res}}-1}V)P_{d_{\mathrm{res}}-1}(V)$, with $P_0 = 1$.

If the residue rank of Z is r_{res}, (2)–(3) compute the two polynomial actions as the integral indicator times

$$\prod_{u=0}^{n+d_{\mathrm{inc}}-r_{\mathrm{res}}-1} (1 - q_{\mathfrak{l}}^{u}c_{\mathrm{tar}}U), \qquad \prod_{u=0}^{n-r_{\mathrm{res}}-1} (1 - q_{\mathfrak{l}}^{u}c_{\mathrm{src}}V).$$

Both are zero off integral matrices. Split off the first d_{inc} factors in the first product and put $V = q_{\mathfrak{l}}^{d_{\mathrm{inc}}}c_{\mathrm{tar}}U/c_{\mathrm{src}}$. This is the required datum polynomial relation.

Passing the identity to forms. For the actual class integral, right invariance of quotient Haar measure and the group law give

$$\rho_{e+1}(h)T_F\upsilon = T_{\omega_{\vartheta,e}(g,h)F}\rho_e(g)\upsilon. \tag{4}$$

Indeed the right side at h_0 is the integral of $J_e(\omega(ug, h_0h)F)\upsilon(ug)$; substitute $u' = ug$. Linearity of J_e suffices. The measures in this substitution are the original measures, so there is no level-index factor. Thus target right cosets pass directly to the datum, whereas inverse source translates d_U^{-1} pass to $T_F\rho_e(d_U)\upsilon$.

Since F is source-compact fixed, (4) gives $T_F\rho_e(k_0)w = T_F w$ for $k_0 \in \mathrm{K}_{\mathfrak{l},e}$. For spherical v, $g \mapsto T_F\rho_e(g)v$ is consequently bi-compact invariant. Its Haar integral over the double coset, with compact mass one, equals both its sum over left representatives d_U and its sum over right representatives d_U^{T}. Each coset has mass one since the local general linear group is unimodular. The latter is precisely the SAME source Hecke sum as in the statement. No equality of individual datum translates is claimed. Together with (1)–(3), this proves the polynomial relation. Equation (4) with $g = 1$ and target compact fixing proves output sphericity. All computations are independent of the translation tuple.

Arithmetic integrality and recovery. We justify the p-adic unit rather than infer it from complex unitarity. For an automorphic norm-one character of infinity weight w_τ, the norm of its base change at this unramified uniformizer is

$$p^{\sum_{\tau \text{ induces } \mathfrak{l}} w_\tau - \sum_{\tau \text{ induces } \bar{\mathfrak{l}}} w_\tau}. \tag{5}$$

To check this, choose a positive integer H with $\mathfrak{l}^H = (b_{\mathrm{pr}})$. The norm-one idele $\varpi_{\mathfrak{l}}/\bar{\varpi}_{\mathfrak{l}}$, raised to H and divided by $b_{\mathrm{pr}}/\bar{b}_{\mathrm{pr}}$, has integral finite components and infinite component $\bar{b}_{\mathrm{pr}}/b_{\mathrm{pr}}$. Its finite tame-character value is a root of unity. Rational triviality expresses the H-th power of the character value as that root times the product of these infinite components to their weights. Under ι, the norm of the factor of weight w_τ is

$$p^{w_\tau(v_p(\iota F_\tau(b_{\mathrm{pr}})) - v_p(\iota F_\tau(\bar{b}_{\mathrm{pr}})))}.$$

The valuations are H and 0 on the corresponding induced fibers, because p is unramified. Dividing by H gives (5). The calculation also proves algebraicity of the character value; roots of unity have norm one.

For the lift character $w_\tau = -\kappa_{e+1}(\tau)$, the two fiber sums agree. Multiplication by $-p$ exchanges those fibers and Lemma 4.2.52 gives equal weights there. Thus (5) is one. It follows that $R_{\mathfrak{l},e}$ has coefficients in $\mathcal{O}_\iota$ and constant term one.

Write $R_{\mathfrak{l},e}(U)^{-1} = \sum_{v\geq 0} b_{e,v}U^v$. The recurrence

$$b_{e,0} = 1, \qquad b_{e,v} = -\sum_{1\leq u\leq \min(v,\deg R_{\mathfrak{l},e})} [U^u]R_{\mathfrak{l},e}\; b_{e,v-u}$$

proves integrality of every coefficient. After multiplication by this inverse, the degree-i coefficient of the polynomial relation gives

$$\sum_{u=0}^{i}(-1)^{i+u}b_{e,i-u}q_{\mathfrak{l}}^{\binom{u}{2}}T_{\mathfrak{l},u}^{(e+1)}\Theta_{\boldsymbol{x},e} = z_{\mathfrak{l}}^{(t+1)i}q_{\mathfrak{l}}^{\binom{i}{2}}\Theta_{\boldsymbol{x},e}T_{\mathfrak{l},i}^{(e)}. \tag{6}$$

Every coefficient is tuple independent.

All indices are present. Since $m \geq 2$, $2^{m-1} \geq m$ gives $h_p \leq m-1$. By Lemma 4.2.67, $r \leq 2k$, hence $m - 1 = r - k \leq k = n_0$ and $h_p \leq n_e$. Also $m \leq r$ and $t = 5r$, so $h_p \leq t$. For $i \leq h_p$,

$$z_{\mathfrak{l}}^{(t+1)i}q_{\mathfrak{l}}^{\binom{i}{2}} = p^{a_{\mathfrak{l}}i(t+i)} \mid p^{fh_pt},$$

because $2a_{\mathfrak{l}} \leq f$ and $i(t + i) \leq 2h_pt$. Multiplying (6) by the missing nonnegative power of p recovers $p^{fh_pt}\Theta_{\boldsymbol{x},e}T_{\mathfrak{l},i}^{(e)}$ as an integral combination of $T_{\mathfrak{l},u}^{(e+1)}\Theta_{\boldsymbol{x},e}$, $u \leq i$.

Start at $C_{\boldsymbol{x},0} = 1$ with coefficients δ_{i,h_p}. If coefficients $c_{e,i}$ recover $p^{efh_pt}C_{\boldsymbol{x},e}T_{\mathfrak{l},h_p}^{(0)}$ and $d_{e,i,u}$ are the next one-step recovery coefficients, then $c_{e+1,u} = \sum_{u\leq i\leq h_p} c_{e,i}d_{e,i,u}$ recover the next

composite. Integrality and tuple independence persist. For $e \le m$, $efh_p t \le h_p f(n_m + h_p) = t_p$, since $et \le n_m$. Multiplication by the remaining power of p proves (HR).

Commutation, fixed order and the neighbor. By Proposition 4.1.14, the pair is a genuine representation of the product, so the source and target actions commute. Thus each target datum Hecke sum commutes with every source action. On a vector fixed by the marked target compact, the Hecke sum is again compact fixed: left multiplication by a compact element permutes its right cosets. It follows that $Q_{\mathfrak{l},i}^{(j)}\omega(1,k_0)F = \omega(1,k_0)Q_{\mathfrak{l},i}^{(j)}F$ for integral target k_0. The factors outside the marked pair commute. This proves the stated commutation on every transform at other marked places, and in particular for the actual target level. Equation (4) moves each datum sum onto the output.

For a list $(\mathfrak{l}, L)$ of distinct marked places define $t_{(\mathfrak{l},L),j} = S_{\mathfrak{l},j}t_{L,j}$ and $Y_{(\mathfrak{l},L)} = Y_{\mathfrak{l}}Y_L$, with empty-list operators the identity. Assume the step identity holds for L. Move $S_{\mathfrak{l},j}$ through the lift by (4), and use (HR) on $Y_L v$. This source form is spherical at the new marked place: the original source level contains its maximal compact, since it is away from MNq, and every processed Hecke operator is supported at a different marked pair. Therefore

$$T_{\vartheta,j,t_{(\mathfrak{l},L),j}F_{\boldsymbol{x},j}}C_{\boldsymbol{x},j}v = C_{\boldsymbol{x},j+1}Y_{\mathfrak{l}}Y_L v.$$

This proves item 3 for the ordered full list. Induction on j, using this identity on $Y^j v$, gives $C_{\boldsymbol{x},j}^t v = C_{\boldsymbol{x},j}Y^j v$. The original levels and the sphericities just used are preserved by the preceding commutation. No reordering of Y is involved.

The target action is $\omega(1,h_0)$, with h_0 in the positive double coset of $\mathrm{diag}(p1_i,1)$, not its inverse. In a diagonal representative its conjugate component is $\mathrm{diag}(p^{-1}1_i,1)$ by unitarity. The arithmetic coefficient embedding includes conjugation and reads the conjugate component. Thus it gives p^{-1} on selected indices over the original induced fiber and p over its conjugate. Their norms are p and p^{-1}, respectively. This is item 4, with precisely the marked determinant orientation.

Every calculation is linear in the finite datum. Fixed gluing supported over N leaves all marked factors unchanged. Applying the same identities to each summand proves the stated extension to gluing and finite linear combinations. □

Lemma 4.2.52. *Let $p < x_1$ be inert in K, with $m \nmid p-1$, $m \nmid p+1$, and with $o = \mathrm{ord}_{(\mathbb{Z}/q_0)^\times}(-p)$ odd and at most $h = \lceil \log_p m \rceil$. Fix a prime identification $\iota \mid p$. Let Λ be a finite set of places of E above p, disjoint from its conjugate, each induced through $\iota \circ F_\tau$ by an a-place. Put*

$$P = \{\tau : \mathfrak{l}_\tau \in \Lambda\}, \qquad Q = \{\tau : \bar{\mathfrak{l}_\tau} \in \Lambda\}, \qquad t_p = hf(n_m + h), \qquad L_p = fa(\log b)^2/p,$$

where $\mathfrak{l}_\tau$ is the place induced by $\iota \circ F_\tau$. For $j < m$ let $c_{j,\tau}$ be the number of columns of height below n_{j+1} different from one for the exact diagrams of Lemma 4.2.50. Then

$$\kappa(\tau) = a/x_1 \quad (\tau \in P \cup Q),$$

and, if $h = 1$, every member of $P \cup Q$ is an a-place. Moreover

$$\#\Lambda\, t_p + h\sum_{\tau\in P} c_{j,\tau} \le L_p/m + \#(\Phi_a \cap P)\, a/x_1. \tag{M}$$

For arbitrary subsets $J_{\mathfrak{l}} \subseteq \{1,\dots,n_{j+1}\}$, define

$$E_{\tau b} = \begin{cases} \mathbf{1}_{b\in J_{\mathfrak{l}_\tau}}, & \tau \in P, \\ -\mathbf{1}_{b\in J_{\bar{\mathfrak{l}_\tau}}}, & \tau \in Q, \\ 0, & \textit{otherwise}, \end{cases} \qquad e_{j+1}(\tau) = n_j/2 - \kappa_{j+1}(\tau).$$

The actual determinant exponents satisfy

$$\sum_{\tau\in\Phi}\sum_{b=1}^{n_{j+1}} E_{\tau b}e_{j+1}(\tau)=0. \tag{D}$$

All assertions hold for every such Λ; no full-mark or ordering choice is needed for these inequalities.

Proof. Since p is nonsplit, Lemma 3.2.8 gives $p\nmid q_1$. The extension E/K is the cyclotomic extension with group $(\mathbb{Z}/q_1)^\times$. At the unique prime of K above the inert prime p, it is unramified and its residue Frobenius acts on ζ by $\zeta\mapsto\zeta^{p^2}$. Its decomposition group is therefore generated by the class p^2: inertia is trivial, and the residue Galois group is generated by Frobenius. Thus two embeddings over K induce the same place precisely when their classes differ by a power of p^2.

In $E/\mathbb{Q}$, the Frobenius at p conjugates K and sends ζ to ζ^p, whereas complex conjugation sends ζ to ζ^{-1}. Their product restricts to the identity on K and acts on ζ by ζ^{-p}. Frobenius fixes each prime over p and complex conjugation interchanges conjugate primes. Consequently multiplication of an embedding class by $-p$ interchanges its induced prime and its conjugate. In particular $\tau\mapsto-p\tau$ is a bijection from P to Q, matching the fibers over $\mathfrak{l}$ and $\bar{\mathfrak{l}}$.

Each $\tau\in P\cup Q$ therefore differs from an a-place by a power of $-p$. Its class modulo q_0 lies in the odd-order subgroup generated by $-p$. It cannot equal -1, since -1 has order two modulo q_0, which is divisible by 16. Thus $\tau\notin\Phi_b$, proving $\kappa(\tau)=a/x_1$. If $h=1$, the positive integer $o\le h$ equals 1, so $-p\equiv1 \pmod{q_0}$; every such class is 1, proving the additional assertion.

The lift weight is $\kappa_{j+1}(\tau)=\kappa(\tau)+(2j+3)\Delta/2$. It is constant on $P\cup Q$. Hence $e_{j+1}(\tau)$ is constant there as well. The bijection of the preceding paragraph matches each positive fiber with its negative fiber and reads the same $J_{\mathfrak{l}}$. Their contributions to (D) cancel for each $\mathfrak{l}$ separately, proving (D) without a bound on $\#J_{\mathfrak{l}}$.

It remains to prove (M). Put $u=a/(pm)$ and $\eta=x^{60}$. The explicit bounds proved in Lemma 4.2.50, using Lemma 4.2.67 and Proposition 2.3.5, are

$$u\ge x^{1000},\quad n_m\le 40x^6,\quad f\le x^{12},\quad m\le 2\sqrt{x},\quad c_{j,\tau}\le\kappa(\tau)+\eta,\quad x_1\ge x^{10^6},\quad b\ge x^{10^{12}}.$$

Also $h\le m$: $p^m\ge 2^m\ge m$ for $m\ge2$, so the least exponent with $m\le p^h$ is at most m.

For $\tau\in P$, we claim

$$hc_{j,\tau}\le u+\mathbf{1}_{\Phi_a}(\tau)a/x_1. \tag{1}$$

If $p\le m$, then $\kappa(\tau)=a/x_1$ and

$$\frac{pm\,h(a/x_1+\eta)}{a}\le\frac{m^3}{x_1}+\frac{m^3\eta}{a}\le1.$$

Indeed $m^3\le8x^{3/2}$, and the two summands are at most $1/2$ by the displayed bounds for x_1,b and $a=xb$. This even proves $hc_{j,\tau}\le u$. If $p>m$, then $h=1$ and every $\tau\in P$ is an a-place. Therefore $hc_{j,\tau}\le a/x_1+\eta\le a/x_1+u$, again proving (1).

Since each induced fiber is nonempty and $P\subseteq\Phi$, one has $\#\Lambda\le f$ and $\#P\le f$. The transfer term satisfies

$$\#\Lambda\,t_p\le f^2m(n_m+m)\le100x^{61/2}\le x^{32}\le u.$$

Summing (1) now gives

$$\#\Lambda\,t_p+h\sum_{\tau\in P}c_{j,\tau}\le u+fu+\#(\Phi_a\cap P)a/x_1\le2fu+\#(\Phi_a\cap P)a/x_1.$$

Finally $(\log b)^2 \geq 2$ and $L_p/m = fu(\log b)^2$. This proves (M). The proof involved neither a theta realization nor a choice of full mark. □

Proposition 4.2.53. *Fix a prime identification $\iota \mid N$. Let $\Sigma, R, \jmath_\Sigma, \delta, \delta^\vee, K_y$ have exactly the meanings of Proposition 4.2.65. The finite additive character*

$$\psi_\alpha(z) = \psi_{E,f}(\jmath\alpha\bar{\alpha}z), \qquad \psi_N(u) = \psi_\alpha(-\widetilde{u}/N) \quad (u \in R)$$

is defined on all of R, using the fixed representatives at Σ. In Proposition 4.2.65 use its restriction to $R^\times$ to define U and $T_\iota = \{\varphi \in \mathcal{S} : U^{m/2}\varphi \in M_\delta\}$. For $j \leq m$, let δ_j be the unitary point with last diagonal entry N at the members of Σ dividing π_N, identity at the other members of Σ, and inverse conjugate entries at their conjugates. Thus $\delta_0 = \delta$.

Let $\Xi_j(T)$ be the indicator of integral n_{j+1}-by-n_j matrices at Σ whose last column reduces to the last basis vector. Read Ξ_j as a Schrödinger datum by the fixed transport of (WR6). Define its operator on finite data in the marked bases by

$$\mathcal{D}_j(F) = \omega_{\vartheta,j}(\delta_j, \delta_{j+1})(\Xi_j \otimes_\Sigma F).$$

For any ordinary twist ϑ, with conductor norm q, let Ψ be any finitely supported coefficient function satisfying the support, coefficient-integrality and full glued-tensor invariance conditions in Proposition 4.2.25. Put

$$F_\Psi^{\mathrm{dir}} = \sum_x \Psi(x) \bigotimes_{j<m} \mathcal{D}_j(H_j(x_j)v_j^{\mathrm{int}}).$$

Then:

1. *Each direct-datum operator commutes with the action of $\mathrm{K}_j^X \times \mathrm{K}_{j+1}^X$. Its actual kernel multiplies the transported gauge based at δ_j by at most N^{fb}, measured at the adjacent base δ_{j+1}, for every supported translation x. The full tensor F_Ψ^{dir} is jointly invariant under the same levels.*

2. *There is a scalar c, fixed before the source form, with $\|\iota(c)\| \leq N^{mfb}$, such that for every $\varphi \in \mathcal{S}^{\mathrm{ram}}$,*

$$\mathscr{C}_{\vartheta,F_\Psi}\varphi = c\,\mathscr{C}_{\vartheta,F_\Psi^{\mathrm{dir}}}(U^{m/2}\varphi).$$

   All phases, residue sections, levels and intermediate bases in this equality are the fixed ones above.

3. *For every $\varphi \in T_\iota \cap \mathcal{S}^{\mathrm{ram}}$, the normalized restricted form $B_{\vartheta,\Psi}\varphi = \mathscr{R}_\beta\mathscr{C}_{\vartheta,F_\Psi}\varphi$ has target gauge at most N^{2mfb} at every finite point, in the prescribed integral Weyl coordinates. This assertion does not require $T_\iota \subseteq M_1$.*

Proof. **The finite character and the marked levels.** The splitting of N, its residue fields, the f members of Σ, and the points $\delta, \delta^\vee$ are those established in Proposition 4.2.65. In particular $N \nmid Mdq$. The inverse-different equality in Lemma 4.1.3 shows that ψ_N is an additive character on R, independent of the representatives modulo N; its pairing on the residue matrix spaces is perfect. Its values are roots of unity, hence belong to $\mathcal{O}_\iota$. Its restriction to the units is therefore an allowed coefficient function in Proposition 4.2.65. We use exactly that U, not an operator with independently chosen characters.

Write $\epsilon_j^{\mathrm{last}}$ for the last residue basis vector in rank n_j, only within the finite calculation below. For a residue vector y in rank n_{j+1}, define the following functions, extended by zero off integral matrices:

$$\Xi_{j,y}(T) = \mathbf{1}_{\bar{T}\epsilon_j^{\mathrm{last}}=y}, \qquad \Upsilon_{j,y}(S) = \psi_N((\epsilon_{j+2}^{\mathrm{last}})^{\mathsf{T}}\bar{S}y).$$

Thus $\Xi_j = \Xi_{j,\epsilon_{j+1}^{\mathrm{last}}}$. Bars on matrices in this calculation mean reduction, not complex conjugation. The function ψ_N in Υ is its additive extension on all R; no arbitrary function defined only on $R^\times$ is evaluated at zero.

Conjugate an actual level element by δ_j. At a prime dividing π_N, its last-row congruence changes to the last-column congruence: the lower-left block is divided by N, the upper-right block multiplied by N. At the complementary prime use inverse conjugate transpose. Thus its Σ-matrix is integral invertible and fixes $\epsilon_j^{\mathrm{last}}$. Consequently

$$\Xi_j(B^{-1}TA) = \Xi_j(T)$$

for the two marked level matrices A, B. Integrality is equivalent on the two sides, and on reduction the assertion is $B^{-1}\bar{T}A\epsilon_j^{\mathrm{last}} = \epsilon_{j+1}^{\mathrm{last}}$, equivalent to $\bar{T}\epsilon_j^{\mathrm{last}} = \epsilon_{j+1}^{\mathrm{last}}$. The actual integral-pair formula of Lemma 4.1.16, transported by δ_j, δ_{j+1}, now proves that $\mathcal{D}_j$ commutes with these actual levels.

The two finite lift identities. Put $d_{\mathrm{diag},j} = \mathrm{diag}(1, \dots, 1, N^{-1})$ at every member of Σ, and let π_j denote the action in the δ_j-changed basis. Let $L_j(\Xi)$ denote the actual lift with residue function Ξ in these bases. We use the genuine canonical action and covariance of Lemma 4.1.1; the split action with its exact scalar is also the explicit Lemma 4.1.16. Change of variables in the lift, with those scalars, gives

$$\pi_{j+1}(B)L_j(\Xi(BTA))\pi_j(A) = L_j(\Xi) \tag{1}$$

for integral invertible A, B. The determinant characters on their units are one, since the lift characters are unramified above N.

For clarity we specify the nonunit scalar too. Let ν_i be the actual character of lift i, and write

$$\chi_{i,u}(G) = \widetilde{\nu}_i(\det_{\Sigma} G)|\det_{\Sigma} G|_f^{-u/2}.$$

The ranks n_j are even: r is odd, m is even, and $t+1$ is even. Hence the space-character powers in (XD2) cancel the half-rank powers in the pair parameters, leaving precisely $\widetilde{\nu}_i$ in this formula. This checks the character normalization rather than replacing it. Let $D_{\mathrm{all},j}$ have last entry N at every member of Σ, and put

$$\gamma_j = \frac{\chi_{j+1,n_{j+1}}(D_{\mathrm{all},j})}{\chi_{j+1,n_j}(D_{\mathrm{all},j+1})}.$$

The two determinants are the same idele. Their character values cancel, so $\gamma_j = N^{f(n_{j+1}-n_j)/2}$ and $\|\iota(\gamma_j)\| \le 1$. If Ξ_j^{row} is the last-row indicator, the same split-action formula gives

$$\pi_{j+1}(d_{\mathrm{diag},j+1})L_j(\Xi_j^{\mathrm{row}}) = \gamma_j L_j(\Xi_j)\pi_j(d_{\mathrm{diag},j}). \tag{2}$$

One checks its function argument block by block: conjugating by the two diagonal matrices turns the last-row condition into the last-column condition. This is the same change of marked lattices, with no new measure normalization.

We record the ramification argument needed in these formulas. If T_w is one of the central scalar operators defining ramification, $T_w^{N-1} = 1$. For a vector with $\sum_{u=0}^{N-2} T_w^u\varphi = 0$, put

$$\xi_w = -\frac{1}{N-1}\sum_{u=0}^{N-2} uT_w^u\varphi.$$

Then $(1 - T_w)\xi_w = \varphi$, by subtraction of the two finite sums. Thus at each w a ramified vector is a scalar difference. A datum unchanged by those scalars annihilates this difference under its lift, by (1). In particular $L_j(\Xi_{j,y})\varphi = 0$ when y vanishes at some member of Σ.

If y is nonzero at every member, choose an invertible residue matrix taking $\epsilon_{j+1}^{\text{last}}$ to y, and lift it integrally. Equation (1) moves that matrix from the last-column indicator to the adjacent character datum, where it cancels. Therefore every surviving term in $\sum_y L_{j+1}(\Upsilon_{j,y})L_j(\Xi_{j,y})\varphi$ equals the same mirror-after-direct lift. Writing P_{j+1} for the set of vectors nonzero at every place, and $\Upsilon_j = \Upsilon_{j,\epsilon_{j+1}^{\text{last}}}$, we obtain

$$\sum_y L_{j+1}(\Upsilon_{j,y})L_j(\Xi_{j,y})\varphi = |P_{j+1}|L_{j+1}(\Upsilon_j)L_j(\Xi_j)\varphi. \tag{3}$$

To compare mirror and direct, first sort the mirror function by its corner entry. If that entry vanishes at a place, its indicator is invariant under the scalar at that place, so its lift annihilates a ramified vector. Only unit corner entries remain. For every column y with unit last entry use the SAME K_y of Proposition 4.2.65. Sorting by the last column and using (1) yields

$$\pi_{j+1}(d_{\text{diag},j+1})L_j(\Upsilon_j)\varphi = U_{j+1}L_j(\Xi_j)\varphi.$$

Sorting instead by the last row, then applying (2), yields

$$\pi_{j+1}(d_{\text{diag},j+1})L_j(\Upsilon_j)\varphi = \gamma_j L_j(\Xi_j)U_j\varphi.$$

Here U_j is the same residue sum in rank n_j, read in the marked basis. In particular $U_0 = U$. Thus

$$U_{j+1}L_j(\Xi_j)\varphi = \gamma_j L_j(\Xi_j)U_j\varphi. \tag{4}$$

Both operators preserve ramification. Central integral scalars commute with every summand of U_j. They pass through a direct lift by (1), since simultaneous central conjugation fixes its matrix argument. Applying either operator to each scalar-difference expression proves the assertion.

The actual gluing and its scalar. Let $S_N = \Sigma \cap \{w : w \mid \pi_N\}$. The polarized formula in Lemma 4.1.3 reads a negative position translation $-y/N$ as translation of the integral box, and its conjugate momentum translation as multiplication by ψ_N paired with the position. This proves the gluing identity with exactly its stated sign. More explicitly, at $w \in S_N$, the first box requires the last column to be y/N modulo integers; multiplying that column by N gives $\Xi_{j,y}$, and the second datum gives $\Upsilon_{j,y}$. At $w \notin S_N$, the first datum is the momentum character and the second is a box translated in its last row. Sorting that row modulo N, then sorting the first last column, gives the same sum $\sum_y \Xi_{j,y}(T)\Upsilon_{j,y}(S)$, after multiplying the second last row by N. Pointwise, exactly one residue is selected by each box. No reciprocal cardinality or Fourier scalar is introduced.

Move these diagonal matrices out of the data by the exact split formula. The intermediate changes of basis cancel between the two adjacent lifts. At pair i, the remaining scalar is

$$a_i = \chi_{2i+2,n_{2i+1}}(D_{\text{all},2i+2}|_{\Sigma\setminus S_N})\,\chi_{2i+1,n_{2i+1}}(D_{\text{all},2i}|_{S_N}), \tag{5}$$

and the remaining endpoint operator is $\pi_{2i+2}(d_{\text{diag},2i+2})$. Here restriction means keep those local diagonal components and put identity elsewhere. Equations (3)–(4) therefore replace the pair by two direct lifts after U_{2i}, with scalar $a_i|P_{2i+1}|\gamma_{2i+1}\gamma_{2i}$. Induction, moving that U through all preceding direct lifts using (4), gives

$$c = \prod_{i<m/2}\left(a_i|P_{2i+1}|\prod_{\ell<2i+2}\gamma_\ell\right) \tag{6}$$

and the equality in item2. The induction is legitimate at every stage by preservation of ramification. It also proves that c is fixed before the source vector.

We check its bound without bounding a reciprocal character value separately. The scale inequalities give $b \geq x^{10^{12}}$; Lemma 4.3.5 gives $x \geq 4$ and $x_1 \geq x$. By (AD4), $\kappa(\tau)$ is either b or xb/x_1, and hence is at most b. Since $m \leq 2\sqrt{x}$, $r \leq 3x^5$, and $\Delta/2 = r^8$, for $i \leq 2m$

$$\kappa_i(\tau) \leq b + (4m+1)r^8 \leq b + x^{50} \leq 2b.$$

The middle inequality follows from $(8\sqrt{x}+1)3^8x^{40} \leq x^{50}$ for $x \geq 4$. The finite norm of the character on a diagonal point is the product of the induced components raised to these weights. To check this formula even when the diagonal idele is not principal, raise its fractional ideal to a principal power, say $I^v = (z)$. Since the unitary idele has $I\bar{I} = 1$, its $2v$-th power divided by $z/\bar{z}$ is a finite unit idele. The finite unit character values are roots of unity. The rational character relation evaluates the remaining factor by its infinity weights. Taking the N-adic norm and its $2v$-th root gives the asserted formula. A place contributing to the first factor of (5) does not contribute to the second. The modulus factors have norm at most one. Consequently each of the f embeddings contributes at most N^{2b} to (5), and

$$\|\iota(a_i)\| \leq N^{2fb}.$$

The integers $|P_{2i+1}|$ are integral at ι, and every γ_ℓ has norm at most one as already proved. There are $m/2$ factors in (6), hence $\|\iota(c)\| \leq N^{mfb}$. Linearity gives the same identity for the entire finite coefficient combination.

The direct-kernel bound. Choose a residue $u \in \{0, \ldots, N-1\}$ of c_0 at the induced prime of K, and put $\beta_0 = c_0 - u \in \mathcal{O}_K$. Then β_0 vanishes at that prime. Its conjugate residue is $1-2u$, whose square is $-d$ modulo N; it is nonzero since $N \nmid d$. Thus $\bar{\beta}_0 d$ is an ι-adic unit. Every embedding $E \to \mathbb{C}$ in the chosen CM type is over K, so β_0 vanishes at every induced prime over N. In particular $\beta_0 \neq 0$. This is an algebraic integer of K, not a rational integer.

Put $r_j = n_j n_{j+1}$ and $A_j^{\mathrm{res}} = \prod_{w \in \Sigma} \mathrm{Mat}_{n_{j+1}, n_j}(\mathbb{F}_N)$. Its order is N^{fr_j}. On the integral position box, finite character orthogonality gives

$$|A_j^{\mathrm{res}}|\, \Xi_j(T) = \sum_{\chi \in \widehat{A}_j^{\mathrm{res}}} \left(\sum_{A \in A_j^{\mathrm{res}}} \Xi_j(A)\chi(-A) \right) \chi(\bar{T}). \tag{7}$$

The inner coefficients are sums of roots of unity and are integral at ι. The perfect trace pairing identifies each χ with a pure momentum translation supported on the conjugates of Σ, with entries in $N^{-1}\mathcal{O}_{E,w}$. The exact transport formula of Lemma 4.1.3 sends the normalized vacuum under that translation to χ times the position box. Its quadratic semicharacter is one, because a pure momentum vector has $\bar{z}z = 0$. Thus (7) is an identity of the actual canonical vectors.

The supported original x_j has zero components above N, since its support is above Mq. It has no cross phase with these momentum translations. Their denominators occur only at induced primes where β_0 vanishes. Hence β_0 clears every denominator at EVERY prime above N. Equation (7), multiplied by the further integral integer N^{fr_j}, shows that N^{2fr_j} times the direct-modified datum is an $\mathcal{O}_\iota$-linear combination of the normalized standard translated vacua covered by Proposition 4.2.35, with denominator β_0.

The factors distinguishing $H_j(x)$ from $\rho(\mathrm{fv}(x))$ are values of a finite additive character, hence roots of unity and ι-adic units. The module in Proposition 4.2.35 therefore contains these same standard vectors. Apply that all-prime bound to the ordinary integral Schur polynomials, retaining the full-height determinant columns. The derivative scale is $\pi_M \eta_P$; π_M and $\bar{\beta}_0 d$ are units

at ι, so the homogeneous bound is one. The actual covariance formula of Proposition 4.2.47(2), with identity diagonal, gives the same coordinate bound at arbitrary endpoints, without an extra character denominator. Dividing by the clearing integer gives at most N^{2fr_j}. Now

$$n_m = k + m(t+1) \le 40x^6, \qquad 2n_j n_{j+1} \le 2n_m^2 \le 3200x^{12} \le x^{18} \le b.$$

Here $x \ge 4$, $k \le 3x^5$, $m \le 2\sqrt{x}$, $t \le 15x^5$, and $3200 \le 4^6$. Thus $N^{2fr_j} \le N^{fb}$. Transport by the adjacent δ_j's translates the two gauges by precisely those bases, by covariance of the lift. This proves the kernel estimate in item1.

Joint invariance, class sums and final restriction. Apply the canonical vacuum projection $P^{\{w:w|\pi_N\}}$ to every finite factor of the glued tensor. In each pair the original N-factor is the vacuum. If a residue $y \neq 0$, at some place one of the two negative translations has a nonintegral component $-y/N$; its vacuum projection vanishes by the trace orthogonality in Lemma 4.1.3. The zero residue projects to the original pair with coefficient one. Thus projection of the glued tensor is exactly the plain tensor, with no cardinality factor. The integral-pair formula of Lemma 4.1.16 shows that this projection commutes with the actual levels. The plain tensor is consequently jointly invariant. Applying the commuting operators $\mathcal{D}_j$ proved above preserves this full-tensor invariance, establishing the remaining assertion of item1.

For $\varphi \in T_\iota$, put $\varphi' = U^{m/2}\varphi$. By Proposition 4.2.65, this is a source form with gauge at most one at δ_0. Apply Proposition 4.2.47(3) to F^{dir}_{Ψ}. Its hypotheses are exactly the full-tensor joint invariance just proved, the original actual levels of volume one, and integral coefficients $\Psi(x)$, since $N \nmid q$. Its finite class formula has weight one at every class. Each ordered kernel product has compatible adjacent bases δ_j, and its bound is at most $\prod_{j<m} N^{fb} = N^{mfb}$. The ultrametric inequality gives the same bound for the finite sum; no class number or stabilizer order is paid. Equation (6) then gives N^{2mfb} for the glued chain in the final marked gauge.

Finally let g_{out} be a finite point of the target and put $\widehat{g} = \mathrm{diag}(g_{\mathrm{out}}, 1)$. The transported image of δ_m^{-1} has last entry $\tau(\bar{\pi}_N)^{-1}$. The branching relation therefore gives the exact coordinate identity

$$\rho_Y(g_{\mathrm{out}}^{-1}) \left(\prod_{\tau} \tau(\bar{\pi}_N)^{-b_\tau} \right) \beta A(\widehat{g}) = \beta\, \rho_X((\widehat{g}\delta_m)^{-1}) A(\widehat{g}),$$

read through the fixed prime identification and its conjugate embeddings. The matrix β is integral, so its coordinate maximum norm is at most one. This identity converts the final marked gauge to the target gauge with no loss and proves item3. Only φ', not φ, was required to be integral. □

The section closes with the family of divisibility errors (Proposition 4.2.54). It is an explicit family (ε_p) (Lemma 4.2.55), zero at the split and the large primes other than N, with $\sum_p \varepsilon_p \log p \le u_{\mathrm{err}}$, together with, for each prime identification, a module of integral ramified source forms of small colength, fixed before all twists, on which the composite definite lift is divisible by $p^{\mathrm{gain}(p)-\varepsilon_p}$. The gauge bounds are those of Proposition 4.2.34; what remains is to build the modules and bound their colengths. At most primes the allowance pays for the collisions of the orbit of $-p$ (Lemma 4.2.56). At the exceptional inert primes the gain is recovered on a module built from Hecke operators at marked places (Proposition 4.2.57), whose colength comes from an elementary divisor theorem for a Hecke operator along a Frobenius cycle (Proposition 4.2.59); elementary divisors go back to Smith [285, 183], and for spherical Hecke algebras see [248, 109]. At N the module is the inverse image of a diagonal module under a power of an up operator (Proposition 4.2.65). Representatives integral at the relevant places come from weak approximation [233] (Lemma 4.2.64). The last results give the ramification and the basic sizes of the datum (Lemma 4.2.66, Lemma 4.2.67).

Proposition 4.2.54. *There exists a family $(\epsilon_p)_p$ of nonnegative real numbers, indexed by rational primes and fixed before any prime identification, such that*

$$p \neq N, \quad (\chi_K(p) = 1 \text{ or } p \geq x_1) \implies \epsilon_p = 0, \qquad \sum_{p \in U} \epsilon_p \log p \leq u_{\mathrm{err}}$$

for every finite set U of rational primes, with the following uniform local property.

For each prime identification $\iota : \mathbb{C} \longrightarrow \mathbb{C}_p$, put

$$\mathcal{M}_\iota = \mathcal{S}(\iota) \cap \mathcal{S}^{\mathrm{ram}}, \qquad d_{\mathrm{src}} = \dim_{\mathbb{C}} \mathcal{S}.$$

There are an integer $j_\iota \geq 0$ and an $\mathcal{O}_\iota$-submodule $\mathcal{H}_\iota \subseteq \mathcal{M}_\iota$, fixed before all twists and admissible coefficient functions, such that

$$j_\iota \leq \epsilon_p d_{\mathrm{src}}.$$

Every finite family $(\varphi_k)_{k \in J}$ in $\mathcal{M}_\iota$ admits a matrix $A \in M_J(\mathcal{O}_\iota)$ and a scalar $c \in \mathcal{O}_\iota$ with

$$\det(A)c = p^{j_\iota}, \qquad \sum_{k \in J} A_{ik}\varphi_k \in \mathcal{H}_\iota \quad (i \in J).$$

For every ordinary twist ϑ with $p \nmid \mathrm{N}Q(\vartheta)$, every admissible Ψ for ϑ, every $\varphi \in \mathcal{H}_\iota$ and every $g \in \mathrm{U}(Y_m)(\mathbb{A}_f)$, the transported Weyl gauge satisfies

$$\max_\nu \left| \left(\rho_\iota(g_p^{-1})\, \iota(B_{\vartheta,\Psi}(\varphi)(g)) \right)_\nu \right|_p \leq p^{\,\epsilon_p - \mathrm{gain}(p)}.$$

Here ν ranges over the standard integral Young coordinates of the final module $\mathcal{V}^Y$ in Proposition 4.2.25; ρ_ι applies g_p^{-1} through its component at the place induced by $x \mapsto \iota(\tau(\bar{x}))$ on the τ factor. On m-fold twists this is exactly the gauge of (TL3) in Definition 3.2.31.

One may take the explicit family ϵ_p of Lemma 4.2.55.

Proof. Fix precisely the realization and the operator $B_{\vartheta,\Psi}$ of Proposition 4.2.25. Admissibility here means finite support, the stated away-from-NQ support and coefficient integrality conditions, and invariance under every upper level tuple with terminal element one.

Put $M = \mathcal{S}(\iota)$, $M^{\mathrm{ram}} = M \cap \mathcal{S}^{\mathrm{ram}}$, and $d_{\mathrm{src}} = \dim_{\mathbb{C}} \mathcal{S}$. Let $S_{\mathrm{ns}} = \{p < x_1 : \chi_K(p) \neq 1\}$, let L_p and C_p be exactly those of Lemma 4.2.56, and choose ϵ_p exactly as in Lemma 4.2.55. Thus $\epsilon_p = 2L_p$ on S_{ns}, $\epsilon_N = 2mfb$, and it is zero elsewhere. This choice depends only on the fixed datum, before ι, ϑ, Ψ. Its support, nonnegativity and every finite weighted sum bound follow from that lemma, which also gives $d < x_1 \leq N$; in particular $N \notin S_{\mathrm{ns}}$.

For each fixed ι, apply the four actual gauge clauses of Proposition 4.2.34: integral ramified inputs have gauge at most one outside $S_{\mathrm{ns}} \cup \{N\}$; on S_{ns} their gauge is at most $p^{mC_p + L_p - \mathrm{gain}(p)}$; at an exceptional prime the precise marked module from Proposition 4.2.57, intersected with $\mathcal{S}^{\mathrm{ram}}$, has gauge at most $p^{2L_p - \mathrm{gain}(p)}$; at N the precise up-module from Proposition 4.2.65, intersected with M^{ram}, has gauge at most N^{2mfb}. These estimates hold for the transported target Young gauge at every finite adelic evaluation, every ordinary twist prime to p, and every admissible coefficient function.

We now construct the module and exponent, before all twists and coefficient functions, in four exhaustive cases.

If $p \notin S_{\mathrm{ns}}$ and $p \neq N$, take $j_\iota = 0$ and $\mathcal{H}_\iota = M^{\mathrm{ram}}$. For every finite family take the identity carrying matrix and $c = 1$. By Lemma 4.3.5, $x \leq x_1$ and $a = xb \geq b$. The weight definition therefore gives $a/x_1 \leq \kappa(\sigma) \leq b \leq a$ for every σ. If $p \geq x_1$, $e \geq 1$, and $y = a$ or $y = \kappa(\tau)$, then $y/p^e \leq a/x_1 \leq \kappa(-p^e\sigma)$. Every relevant positive part vanishes, proving $\mathrm{gain}(p) = 0$. If

$\chi_K(p) = 1$, the defining partner index set is empty and gives the same conclusion. This also proves $\mathrm{gain}(N) = 0$. The ordinary gauge clause is consequently precisely the required bound $p^{\epsilon_p - \mathrm{gain}(p)} = 1$.

If $p \in S_{\mathrm{ns}}$ is not exceptional, again take $j_\iota = 0$, $\mathcal{H}_\iota = M^{\mathrm{ram}}$, the identity matrix and $c = 1$. The hypotheses of Lemma 4.2.56 are exactly the negation of exceptionality, including the ramified case. That lemma gives $mC_p \le L_p$. As $p > 1$, the unmarked gauge bound is at most $p^{2L_p - \mathrm{gain}(p)} = p^{\epsilon_p - \mathrm{gain}(p)}$.

If $p \in S_{\mathrm{ns}}$ is exceptional, choose once the full mark, its ordering, scaled operator Y, integer j_p, and module $T_\iota = Y^m M + p^{j_p} M$ of Proposition 4.2.57. They depend on the fixed data and ι, not on any twist or coefficient function. Let ℓ be that result's fixed integer and set

$$j_\iota = \ell, \qquad \mathcal{H}_\iota = T_\iota \cap M^{\mathrm{ram}}.$$

Then $0 \le \ell \le L_p d_{\mathrm{src}} \le \epsilon_p d_{\mathrm{src}}$. Given a finite family in M^{ram}, its carrying matrix from that result has entries in $\mathcal{O}_\iota$ and determinant dividing p^ℓ. Thus some $c \in \mathcal{O}_\iota$ satisfies $\det(A)c = p^\ell$. Each carried vector belongs to T_ι; it also belongs to M^{ram}, since this is an $\mathcal{O}_\iota$-submodule containing the original family. The exceptional gauge clause gives the asserted bound on their intersection.

Finally let $p = N$. Use the phase and up-operator of Proposition 4.2.34, with the lifts and diagonal transport of Proposition 4.2.65. Its phase values are ι-integral, so that theorem applies. Let T_ι and its fixed exponent e be supplied there, and put

$$j_\iota = e, \qquad \mathcal{H}_\iota = T_\iota \cap M^{\mathrm{ram}}.$$

The intersection is contained in M^{ram} by definition. The carrying matrix sends every finite family in $M^{\mathrm{ram}} \subseteq M$ into T_ι; its carried vectors remain in M^{ram} by linearity, giving the intersection and the same exact determinant equation. Moreover $e \le mfb\, d_{\mathrm{src}} \le \epsilon_N d_{\mathrm{src}}$. Since $N \ge x_1$, $\mathrm{gain}(N) = 0$. The level gauge clause is exactly $N^{\epsilon_N - \mathrm{gain}(N)}$. The conductor of every ordinary twist is prime to N.

These choices cover every prime and every identification and precede all twists, all admissible coefficient functions and all finite input families. A carrying matrix is selected only after its finite input family; it is independent of the later twist and coefficient function. Empty families cause no exception: the empty determinant is one and $c = p^{j_\iota}$ gives the same equation. The claimed determinant and gauge bounds follow in every case. In particular the uniform family may serve as the designated error family without changing its order of quantifiers. □

Lemma 4.2.55. *Let D be admissible data and put $L = \log b$. For each rational prime p define, locally in this statement,*

$$\epsilon_p = \begin{cases} \dfrac{2faL^2}{p} \begin{cases} m, & m \mid p-1 \text{ or } m \mid p+1, \\ 1, & \text{otherwise}, \end{cases} & p < x_1,\ \chi_K(p) \ne 1, \\ 2mfb, & p = N, \\ 0, & \text{otherwise}. \end{cases}$$

Then $d < x_1 \le N$, so the two nonzero cases are disjoint. Each ϵ_p is nonnegative,

$$p \ne N, \quad (\chi_K(p) = 1 \text{ or } x_1 \le p) \quad \Longrightarrow \quad \epsilon_p = 0,$$

and every finite set U of rational primes satisfies

$$\sum_{p \in U} \epsilon_p \log p \le faL^3 = u_{\mathrm{err}}.$$

Proof. We first establish the numerical ranges needed for the sums. By the definition of C_P in Definition 2.3.3, its summands are nonnegative and $C_P \geq 2d + 6 \geq 1$. The scale inequalities of Proposition 2.3.5 imply

$$\log L \geq 10^{12} C_P, \qquad L^{100} \leq x, \qquad x^{10^6} \leq x_1, \qquad x_1^{10^6} \leq b, \qquad (\log x_1)^6 \leq L.$$

The positivity and $L \geq 10$, $x \geq 4$ are also given by Lemma 4.3.5. In particular $L \geq 1 + \log L \geq 1 + 10^{12} > 10^6$, by $\log L \leq L - 1$. Further,

$$d \leq C_P \leq 10^{-12} \log L < L \leq x < x_1.$$

Here $L^{100} \leq x$ and $L \geq 1$ give $L \leq x$; the strict inequality $x < x_1$ follows from $x^{10^6} \leq x_1$ and $x > 1$. By Lemma 4.3.5, $\log N \geq 10L$, so $N \geq b^{10}$. Since $x_1 \leq b$, we obtain $x_1 \leq N$.

Write $z = \log x_1$. Then $z \geq \log 4 > 1$ and $z \leq L^{1/6}$. Since x_1 is a positive integer, the harmonic sum $H_{x_1} = \sum_{c=1}^{x_1} 1/c$ satisfies

$$H_{x_1} = 1 + \sum_{c=2}^{x_1} \frac{1}{c} \leq 1 + \int_1^{x_1} \frac{dt}{t} = 1 + z.$$

The integral comparison uses $1/c \leq \int_{c-1}^{c} dt/t$ for every $c \geq 2$.

Let Q be the finite set of primes $p < x_1$. Since $\log p \leq z$,

$$\sum_{p \in Q} \frac{\log p}{p} \leq z H_{x_1}.$$

For a prime $p \in Q$ with $m \mid p - 1$, write $p = mc + 1$. The integer c satisfies $c \geq 1$ because $p > 1$, and $c \leq p \leq x_1$. Also $m/p \leq 1/c \leq 2/c$. For a prime $p \in Q$ with $m \mid p + 1$, write $p = mc - 1$. Again $c \geq 1$, and $c = (p + 1)/m \leq (p + 1)/2 \leq p \leq x_1$, since $m \geq 2$ and $p \geq 2$. Moreover $mc = p + 1 \leq 2p$, hence $m/p \leq 2/c$. Within either progression distinct primes yield distinct integers c. Consequently

$$\sum_{\substack{p \in Q \\ m \mid p-1}} \frac{m \log p}{p} \leq 2 z H_{x_1}, \qquad \sum_{\substack{p \in Q \\ m \mid p+1}} \frac{m \log p}{p} \leq 2 z H_{x_1}.$$

Adding the two estimates bounds the sum on their union by $4 z H_{x_1}$. This remains valid if the progressions overlap, including when $m = 2$, because every summand is nonnegative.

Let h_p be m on that union and 1 otherwise. Then $h_p \leq 1 + m \mathbf{1}_{\{m \mid p-1 \text{ or } m \mid p+1\}}$. Removing the nonsplit restriction enlarges a nonnegative sum. We therefore have

$$\begin{aligned}
\sum_{\substack{p \in Q \\ \chi_K(p) \neq 1}} \epsilon_p \log p &\leq 2 f a L^2 \left(\sum_{p \in Q} \frac{\log p}{p} + \sum_{\substack{p \in Q \\ m \mid p-1 \text{ or } m \mid p+1}} \frac{m \log p}{p} \right) \\
&\leq 10 f a L^2 z H_{x_1} \\
&\leq 20 f a L^2 z^2 \\
&\leq 20 f a L^{7/3}.
\end{aligned}$$

The penultimate step uses $H_{x_1} \leq 1 + z \leq 2z$.

The level-prime term is controlled separately. The upper bound $\log N \leq 30 m L$ from Lemma 4.3.5, together with $m \leq 2\sqrt{x}$, gives

$$\epsilon_N \log N = 2 m f b \log N \leq 60 m^2 f b L \leq 240 x f b L = 240 f a L.$$

Since $L \geq 10^6$, we have $L^{2/3} \geq 10^4 \geq 40$ and $L^2 \geq 480$. Hence

$$20L^{7/3} \leq \tfrac{1}{2}L^3, \qquad 240L \leq \tfrac{1}{2}L^3.$$

The small nonsplit primes and N are disjoint, and all other terms vanish. The sum over the entire finite support is therefore at most faL^3. Every finite sub-sum is at most this because $\epsilon_p \geq 0$ and $\log p > 0$ for primes. Nonnegativity and the stated vanishing condition follow immediately from the defining formula. This proves all assertions. □

Lemma 4.2.56. *Let $p < x_1$ be a rational prime with $\chi_K(p) \neq 1$. Put $L = \log b$ and, locally in this statement,*

$$C_p = \sum_{\tau \in \Phi_a} \sum_{\substack{e \in I_p \\ (-p^e)\tau \in \Phi_a}} \frac{a}{p^e}, \qquad L_p = \frac{faL^2}{p} \begin{cases} m, & m \mid p-1 \text{ or } m \mid p+1, \\ 1, & \text{otherwise.} \end{cases}$$

Here $-p^e$ means $-(p^e)$, and the collision sum contains no factor $v(\pi_p)$. Let o_p be the order of $-p$ in $(\mathbb{Z}/q_0)^\times$, and let h_p be the least nonnegative integer with $m \leq p^{h_p}$. Assume that the following four conditions do not all hold:

$$\chi_K(p) = -1, \qquad m \nmid p-1 \text{ and } m \nmid p+1, \qquad o_p \text{ odd}, \qquad o_p \leq h_p.$$

Then the series defining C_p converges and

$$mC_p \leq L_p.$$

Proof. By Lemma 3.2.8, every prime dividing q_1 splits in K. Since $\chi_K(p) \neq 1$, the prime p does not divide q_1, and therefore does not divide q_0. Thus the order o_p in the statement is defined and positive. The place $u(-p^e)$ is given by the genuine unit class of $-p^e$ modulo q_1, and its reduction modulo q_0 is the unit class of that integer.

We record an explicit bound for the tails of the collision series. If every collision exponent is at least an integer $h \geq 1$, then for each $\tau \in \Phi_a$ its nonnegative series is bounded term by term by the convergent geometric series over all integers $e \geq h$. Consequently

$$0 \leq C_p \leq \#\Phi_a \sum_{e=h}^{\infty} \frac{a}{p^e} = \#\Phi_a \frac{a}{p^h(1-p^{-1})} \leq \frac{2fa}{p^h},$$

because $p \geq 2$ and $\#\Phi_a \leq f$. All partner exponents are positive, so $h = 1$ applies without further conditions. This also proves convergence of C_p.

We first dispose of the ramified prime. Suppose $p = d$, and write $q_0 = 2H$. Since $4 \mid q_0$, the integer H is even. The parameter congruence gives $d \equiv H - 1 \pmod{q_0}$ and

$$d^2 \equiv (H-1)^2 \equiv 1 \pmod{q_0};$$

indeed H^2 and $2H$ are divisible by q_0. For every positive exponent e,

$$-d^e \equiv \begin{cases} -1, & e \text{ even}, \\ 1-H, & e \text{ odd} \end{cases} \pmod{q_0}.$$

Neither residue is 1: $q_0 \geq 18480 > 2$ and $0 < H < q_0$. If $\tau \in \Phi_a$, its class modulo q_0 is 1, so $(-d^e)\tau$ is never in Φ_a. Hence $C_d = 0$, including all positive exponents in I_d. Nonnegativity of L_d proves the desired inequality in this case.

Suppose henceforth $p \neq d$. The Legendre character at a prime is 0 only at d and otherwise is 1 or -1. Therefore $\chi_K(p) = -1$ and I_p consists exactly of the positive odd integers. For $\tau \in \Phi_a$ and odd e, cancellation in the group of classes modulo q_0 gives

$$(-p^e)\tau \in \Phi_a \quad \Longleftrightarrow \quad -p^e \equiv 1 \pmod{q_0} \quad \Longleftrightarrow \quad (-p)^e \equiv 1 \pmod{q_0} \quad \Longleftrightarrow \quad o_p \mid e.$$

If $m \mid p-1$ or $m \mid p+1$, the general tail bound with $h = 1$ gives $C_p \leq 2fa/p$. Since $L \geq 10$ by Lemma 4.3.5, we have $L^2 \geq 2$, and thus

$$mC_p \leq \frac{2mfa}{p} \leq \frac{mfaL^2}{p} = L_p.$$

It remains to consider $m \nmid p-1$ and $m \nmid p+1$. If o_p is even, it divides no odd exponent; the preceding equivalence then gives $C_p = 0$. If o_p is odd, the nonexceptional hypothesis, together with $\chi_K(p) = -1$ and the two failures of divisibility by m, forces $o_p > h_p$. Every collision exponent is positive and divisible by o_p, so $e \geq o_p \geq h_p + 1$. Apply the tail bound with $h = h_p + 1$ and use $m \leq p^{h_p}$:

$$mC_p \leq \frac{2mfa}{p^{h_p+1}} \leq \frac{2fa}{p} \leq \frac{faL^2}{p} = L_p.$$

These cases exhaust all possibilities and prove the result. □

Proposition 4.2.57. *Let $p < x_1$ be prime with $\chi_K(p) = -1$, $m \nmid p-1$, $m \nmid p+1$, and with the order e_p of $-p$ modulo q_0 odd and at most $r_p = \lceil \log_p m \rceil$. Fix a prime identification $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, and put*

$$O_\iota = \{z \in \mathbb{C} : |\iota(z)|_p \leq 1\}, \qquad M_\iota = \{\varphi \in \mathcal{S} : \varphi \text{ is integral at } \iota\}, \qquad d_{\mathrm{src}} = \dim_{\mathbb{C}} \mathcal{S}.$$

There is a finite set Λ of places of E above p, every member induced by an a-place through $\iota \circ F_\tau$, disjoint from its conjugate, such that every place induced by an a-place belongs to $\Lambda \cup \overline{\Lambda}$. Choose any ordering of Λ, once and for all. Set

$$A_{\mathfrak{l}} = a\,\#\{\tau \in \Phi_a : \tau \text{ induces } \mathfrak{l}\}, \qquad Y_{\mathfrak{l}} = p^{A_{\mathfrak{l}}} T_{\mathfrak{l},r_p}, \qquad Y = \prod_{\mathfrak{l} \in \Lambda} Y_{\mathfrak{l}}, \qquad j_p = \left\lceil \frac{2mfa}{p} \right\rceil,$$

where $T_{\mathfrak{l},r_p}$ is the actual unnormalised spherical source Hecke operator for $\mathrm{diag}(pI_{r_p}, I_{k-r_p})$, with maximal compact Haar mass one. The product denotes composition in the fixed ordering. Then

$$T_\iota = Y^m M_\iota + p^{j_p} M_\iota$$

is an O_ι-submodule of M_ι. There is an integer $\ell \geq 0$, chosen before finite families, with

$$\ell \leq \frac{15fa}{p} d_{\mathrm{src}} \leq L_p d_{\mathrm{src}}, \qquad L_p = \frac{fa(\log b)^2}{p},$$

such that every finite family $\varphi_1, \ldots, \varphi_s \in M_\iota$ admits a matrix $C \in \mathrm{M}_s(O_\iota)$ with $\det C \mid p^\ell$ in O_ι and

$$\sum_{v=1}^{s} C_{uv}\varphi_v \in T_\iota \qquad (1 \leq u \leq s).$$

The set Λ, its ordering, Y, T_ι and ℓ are independent of twists and chain data.

Proof. **Choose the places and bound their total scaling.** Let A be the image of Φ_a under the map taking τ to its place induced by $\iota \circ F_\tau$. This is a finite set of places above p. By Proposition 4.2.58, every place in A is split over E^+, so it differs from its complex conjugate. Partition A by the equivalence relation whose classes are intersections with pairs $\{\mathfrak{l}, \bar{\mathfrak{l}}\}$. Choose one representative in A from each class and call the resulting set Λ. Distinct representatives cannot be conjugate, and every place in A is a representative or the conjugate of one. This proves all the asserted properties of Λ. Choose a finite ordering. These choices depend only on the fixed data, p and ι.

Every member of Λ is induced by an a-place. Its distinct fibres in Φ_a are disjoint and nonempty. Consequently

$$\#\Lambda \le \#\Phi_a \le f, \qquad \sum_{\mathfrak{l}\in\Lambda} A_{\mathfrak{l}} = a\,\#\{\tau \in \Phi_a : \tau \text{ induces a member of } \Lambda\} \le af.$$

The actual marked-place theorem Proposition 4.2.58 applies separately to every member. In particular $Y_{\mathfrak{l}} M_\iota \subset M_\iota$ and the operator is O_ι-linear. At the fixed integer j_p, choose its integer exponent $\ell_{\mathfrak{l}} \ge 0$ with

$$\ell_{\mathfrak{l}} \le \left(\frac{12A_{\mathfrak{l}}}{mp} + \frac{j_p}{m^2 f}\right) d_{\mathrm{src}}.$$

Each such exponent precedes every finite family. There are only finitely many choices to make, and none involves twists.

Compose the finite-family carrying matrices. We prove the needed composition rule directly over the commutative ring O_ι. Let U_1, U_2 be linear endomorphisms preserving M_ι with uniform carrying exponents h_1, h_2 modulo $p^{j_p} M_\iota$. Given a finite family $v = (v_i)$, choose its first matrix C_1 and elements $u_i, w_i \in M_\iota$ with

$$(C_1 v)_i = U_1 u_i + p^{j_p} w_i, \qquad \det C_1 \mid p^{h_1}.$$

Apply the second carrying property to the family (u_i), obtaining C_2 and $z_i, t_i \in M_\iota$ with

$$(C_2 u)_i = U_2 z_i + p^{j_p} t_i, \qquad \det C_2 \mid p^{h_2}.$$

By linearity,

$$(C_2 C_1 v)_i = U_1 U_2 z_i + p^{j_p}(U_1 t_i + (C_2 w)_i) \in U_1 U_2 M_\iota + p^{j_p} M_\iota.$$

The term in parentheses belongs to M_ι because U_1 preserves it. Multiplicativity of the determinant gives $\det(C_2 C_1) \mid p^{h_1+h_2}$. This proves the rule in the prescribed order without commutativity of U_1, U_2, finite generation of M_ι, or a discrete valuation.

Apply the rule along the chosen ordered list of the $Y_{\mathfrak{l}}$, and repeat that whole list m times. Its operator is exactly Y^m, and its integer exponent is

$$\ell = m \sum_{\mathfrak{l}\in\Lambda} \ell_{\mathfrak{l}}.$$

It gives a carrying matrix into $Y^m M_\iota + p^{j_p} M_\iota$ for every finite family. All factors preserve M_ι, so the image and the scalar submodule are contained in M_ι, and their sum is an O_ι-submodule. The empty product gives the identity if Λ is empty. For an empty finite family the empty matrix has determinant one and satisfies the assertion with this same exponent.

Pay for the products and the ceiling. Use the two fibre bounds to estimate

$$\ell \le m \sum_{\mathfrak{l}\in\Lambda} \left(\frac{12A_{\mathfrak{l}}}{mp} + \frac{j_p}{m^2 f}\right) d_{\mathrm{src}} \le \left(\frac{12fa}{p} + \frac{j_p}{m}\right) d_{\mathrm{src}}.$$

The definitions and Lemma 4.3.5 give $m \geq 2$, $f \geq 1$, $x \geq 4$, $a = xb$, and $\log b \geq 10$. Also $x_1 \leq b$, because the positive integer $q_1 x_1$ divides b in the admissible data. Hence $p < x_1 \leq b$ implies $p \leq mfa$, and therefore $1/m \leq fa/p$. The usual ceiling inequality gives

$$\frac{j_p}{m} \leq \frac{2fa}{p} + \frac{1}{m} \leq \frac{3fa}{p}.$$

It follows that $\ell \leq 15fa\, d_{\mathrm{src}}/p$. Since $(\log b)^2 \geq 100 \geq 15$, this is at most $L_p d_{\mathrm{src}}$. Every object and exponent used was chosen from the fixed data, p and ι, before families, twists and chain data. □

Proposition 4.2.58. *Let $p < x_1$ be a rational prime with $\chi_K(p) = -1$, $m \nmid p-1$ and $m \nmid p+1$. Put*

$$r_p = \lceil \log_p m \rceil, \qquad e_{\mathrm{mark}} = \mathrm{ord}_{(\mathbb{Z}/q_0)^\times}(-p),$$

and assume e_{mark} is odd and $e_{\mathrm{mark}} \leq r_p$. Fix a prime identification $j : \mathbb{C} \hookrightarrow \mathbb{C}_p$ and a finite place $\mathfrak{l}$ of E induced by some $\tau \in \Phi_a$ through $j \circ F_\tau$. Put

$$O_j = \{z \in \mathbb{C} : |j(z)|_p \leq 1\}, \qquad L_j = \{\varphi \in \mathcal{S} : \varphi \text{ is integral at } j\},$$
$$A_{\mathfrak{l}} = a\,\#\{\tau \in \Phi_a : \tau \text{ induces } \mathfrak{l}\}.$$

Then $\mathfrak{l}$ is split over E^+, the source level is maximal there, and $0 < r_p < k$. Let $T_{\mathfrak{l},r_p}$ be the unnormalised spherical Hecke operator for the double coset of $\mathrm{diag}(pI_{r_p}, I_{k-r_p})$ at $\mathfrak{l}$, with the maximal compact subgroup of Haar mass 1. On source forms put $Y_{\mathfrak{l}} = p^{A_{\mathfrak{l}}} T_{\mathfrak{l},r_p}$. Then $Y_{\mathfrak{l}} L_j \subseteq L_j$. For every integer $h_{\mathrm{mark}} \geq 0$ there is an integer $\ell \geq 0$ such that

$$\ell \leq \left(\frac{12 A_{\mathfrak{l}}}{mp} + \frac{h_{\mathrm{mark}}}{m^2 f} \right) \dim_{\mathbb{C}} \mathcal{S}$$

and every finite family $\varphi_1, \ldots, \varphi_s \in L_j$ admits $C \in \mathrm{M}_s(O_j)$ with $\det(C) \mid p^\ell$ in O_j and

$$\sum_{v=1}^{s} C_{uv} \varphi_v \in Y_{\mathfrak{l}} L_j + p^{h_{\mathrm{mark}}} L_j \quad (1 \leq u \leq s).$$

Proof. Write $r = r_p$, $F = E^+$ and $O = O_j$. This is a characteristic-zero valuation domain, and p is a nonunit because $|j(p)|_p = p^{-1} < 1$. All matrices below have their rows numbered by $0, \ldots, k-1$. We first determine the actual Frobenius cycle.

Every rational prime dividing q_0 splits in K. For an odd such prime, $d \equiv -1$ modulo that prime, so $-d$ is a nonzero square modulo it; for 2 use $d \equiv 7 \pmod 8$. Since the prime factors of q_1 are those of q_0, our inert prime satisfies $p \nmid q_1 d$ and $p \geq 13$. Thus $E/\mathbb{Q}$ is unramified at p, and p is a uniformiser at every place above it. Put

$$o = \mathrm{ord}_{(\mathbb{Z}/q_1)^\times}(-p).$$

We have $m \mid o$. Indeed write $m = 2^h$, so $2^s = m^{22}$ gives $s = 22h$. The two failures $m \nmid p-1, p+1$ imply $h \geq 3$ and $v_2(p-1) + v_2(p+1) \leq h$. For $t \geq 1$, repeated factorisation of $p^{2^{t+1}} - 1$ gives

$$v_2(p^{2^t} - 1) = v_2(p-1) + v_2(p+1) + t - 1.$$

The induction uses $p^{2^t} \equiv 1 \pmod 8$, so the next plus factor has valuation 1. The order of $-p$ modulo 2^s is a power 2^t with $t \geq 1$, since $-p \not\equiv 1 \pmod{2^s}$. The displayed valuation forces

$t \geq s - h + 1 \geq h$. As $2^s \mid q_1$, the order modulo q_1 is divisible by m. Reduction modulo q_0 gives $e_{\mathrm{mark}} \mid o$. Consequently $c = o/e_{\mathrm{mark}}$ is a positive even integer, $m \leq o = e_{\mathrm{mark}}c$, and $c \leq o \leq f$.

The Galois group of $E/\mathbb{Q}$ is the product of the Galois group of $K/\mathbb{Q}$ and $(\mathbb{Z}/q_1)^\times$, by Definition 2.3.1. Let Fr_p be its arithmetic Frobenius and let $\mathfrak{c}$ be complex conjugation. On K the first is conjugation, and on the cyclotomic field it is the power p. Thus the element $\mathfrak{c}\mathrm{Fr}_p$ fixes K and acts on ζ_{q_1} by the power $-p$. Since o is even, the order of Fr_p is exactly o: an exponent killing it must be even by restriction to K, and for even exponents its power equals that of $\mathfrak{c}\mathrm{Fr}_p$. The residue field at $\mathfrak{l}$ therefore has cardinality $q_{\mathfrak{l}} = p^o$. Moreover $\mathfrak{c}$ is not in its decomposition group. Otherwise $\mathfrak{c} = \mathrm{Fr}_p^t$ would require t odd on K and $o \mid t$ on multiplying by $\mathfrak{c}^t$, a contradiction. Hence $\mathfrak{l} \neq \bar{\mathfrak{l}}$ and the place of F below it is split.

Choose $\tau_0 \in \Phi_a$ inducing $\mathfrak{l}$, and define

$$\tau_i = (-p)^{e_{\mathrm{mark}} i}\tau_0 \quad (i \in \mathbb{Z}/c).$$

These places are distinct. The places inducing $\mathfrak{l}$ or $\bar{\mathfrak{l}}$ form the orbit of τ_0 under $-p$: this follows by applying the decomposition group $\langle \mathrm{Fr}_p \rangle$ to the embeddings over K, using complex conjugation for the embeddings inducing the conjugate prime. In this orbit, membership in Φ_a is exactly $(-p)^v = 1$ modulo q_0, that is, $e_{\mathrm{mark}} \mid v$. Thus the τ_i are precisely the a-places inducing $\mathfrak{l}$ or $\bar{\mathfrak{l}}$. Successive ones induce conjugate primes because e_{mark} is odd, and τ_0 induces $\mathfrak{l}$; declare these alternating parities even and odd. There are $c/2$ of each, so

$$A_{\mathfrak{l}} = ac/2.$$

Let $W = \mathcal{O}_{E,\mathfrak{l}}$, the valuation ring of E at $\mathfrak{l}$. For $x \in W$ define

$$\iota_i(x) = \begin{cases} F_{\tau_i}(x), & i \text{ even}, \\ F_{\tau_i}(\bar{x}), & i \text{ odd}. \end{cases}$$

These maps take values in O, since in both cases their induced prime is $\mathfrak{l}$. Their successive maps differ by the e_{mark}th power of Frobenius: on algebraic integers this follows from $(-p)^{e_{\mathrm{mark}}} = \mathfrak{c}\,\mathrm{Fr}_p^{e_{\mathrm{mark}}}$ and the parity change, and gives

$$p \mid \iota_{i+1}(x) - \iota_i(x)^{p^{e_{\mathrm{mark}}}}.$$

The congruence extends to W by writing its elements as fractions of algebraic integers with denominators outside $\mathfrak{l}$; their images and Frobenius images are units of O. Its divisibility is by p, rather than only by the maximal ideal of O, because p is a uniformiser of W. Choose $w \in \mathbb{C}$ with $w^2 = p$. Then $|j(w)|_p = p^{-1/2}$, so $w \in O$. These are exactly the Frobenius-cycle data of Proposition 4.2.59, with number of letters c, exponent e_{mark} and residue size $(p^{e_{\mathrm{mark}}})^c = q_{\mathfrak{l}}$.

We verify every size hypothesis of that proposition, with polynomial degree a and $\theta = (m^2 f)^{-1}$. Put $L = \log b$ and $z = \log x_1$. The scale assumptions give

$$L \geq 10, \quad x \geq L^{100}, \quad z \geq 10^6 \log x, \quad z^6 \leq L.$$

By Lemma 2.3.6, $x \geq 10^6$. Also $m \leq x$ and $k = r_{\mathrm{data}} + 1 - m \geq x^5 + 1 - x \geq x^4$, where r_{data} denotes the rank parameter called r in Definition 2.3.4. The ceiling defining r gives $r \leq \log m/\log p + 1 \leq m$. Hence $0 < r$, $2r \leq k$, and $r^2 m \leq k$. Clearly $0 < \theta \leq 1$. Since $o \geq m \geq \sqrt{x} \geq L^{50}$ and $\log p \geq 1$,

$$\log q_{\mathfrak{l}} = o \log p \geq L^{50} \geq 8L \geq 4 \log a,$$

where $\log x \le L$ was used; thus $a^4 \le q_{\mathfrak{l}}$.

Here are explicit bounds for the other size condition. The tower formula, $l_0(s) \le 81s$, $s \le 32 \log x$, $2^s = m^{22} \le 2^{22} x^{11}$ and $q_0 \le 10^{-14} \log x$ give $q_1 \le 7776 \cdot 2^{22} 10^{-14} (\log x)^2 x^{11} \le x^{13}$. Thus $f \le x^{13}$, $k \le 3x^5 \le x^6$, $c \le f$, $\theta^{-1} = m^2 f \le x^{15}$, and $\log q_{\mathfrak{l}} = o \log p \le fz \le x^{13} z$. Consequently

$$ck \log q_{\mathfrak{l}}/\theta \le x^{47} z, \qquad 5 \log(ck \log q_{\mathfrak{l}}/\theta) \le 240z.$$

Also $r \log p \le \log m + \log p \le 2z$ and $e_{\mathrm{mark}} \le r \le 2z$, so $3 e_{\mathrm{mark}} r \log p \le 12 z^2$. Since $z \ge 10^6$ and $z^6 \le L$,

$$\log((ck \log q_{\mathfrak{l}}/\theta)^5 (p^{e_{\mathrm{mark}}})^{3r}) \le 240z + 12z^2 \le 252 z^2 \le L \le \log a.$$

All size hypotheses are proved. Finally, putting $\delta = ca((p^{e_{\mathrm{mark}}})^{-2r} + r^2/(kp^{e_{\mathrm{mark}}}))$, we have

$$(p^{e_{\mathrm{mark}}})^{2r} \ge p^{2r} \ge mp, \qquad \frac{r^2}{kp^{e_{\mathrm{mark}}}} \le \frac{1}{mp}, \qquad 3\delta \le \frac{6ca}{mp} = \frac{12 A_{\mathfrak{l}}}{mp}.$$

We next construct the actual complete edge coordinates. The prime p is different from N, p_0, p_1, since $p < x_1 \le b$, whereas those primes are at least b^{10}. Therefore K_0 has the full maximal compact factor at every place above p. The local double-coset sum commutes with this compact factor and with the other level factors; right translations commute with rational left equivariance. Hence $T_{\mathfrak{l},r}$ is indeed an endomorphism of the given source space. Put

$$X = G(F) \backslash G(\mathbb{A}_{F,f}) / \mathrm{K}_0, \qquad G = \mathrm{U}(X_0).$$

This set is finite by Lemma 4.6.4, since $k \le n_m$ and K_0 contains the principal level $\pi_M \pi_N$. Its rational stabilisers are trivial by Lemma 4.6.2(c).

Choose representatives g_x integral at every prime above p. To obtain them, apply Lemma 4.2.64 to the actual framed space X_0, with the finite set consisting of all places of E above p. In its dense products $\gamma(1, k')$, the components of k' at these places are 1; hence the rational components γ_f are dense in the corresponding finite product of local unitary groups. For an initial representative g, prescribe the open conditions $\gamma_w g_w \in \mathrm{GL}_k(\mathcal{O}_{E,w})$ at these places. These conditions contain the local inverses of g, so such a rational γ exists. Replacing g by $\gamma_f g$ changes no double class and gives the required representatives.

Put $d = \mathrm{diag}(pI_r, I_{k-r})$, let $\mathcal{P}_r \subset \mathrm{GL}_k(W)$ be the matrices with upper-right r by $k-r$ block in the maximal ideal, and let $B = \mathrm{GL}_k(W)/\mathcal{P}_r$. This finite quotient is the corresponding Grassmannian over the residue field. Choose representatives $\beta_\xi \in \mathrm{GL}_k(W)$ of its cosets. At the completion, the right cosets in $\mathrm{K}_{\mathfrak{l}} d \mathrm{K}_{\mathfrak{l}}$ are exactly $\beta_\xi d \mathrm{K}_{\mathfrak{l}}$: conjugation by d^{-1} divides precisely that upper-right block by p, so

$$d^{-1} \beta^{-1} \beta' d \in \mathrm{GL}_k(\mathcal{O}_{E_{\mathfrak{l}}}) \quad \Longleftrightarrow \quad \beta^{-1}\beta' \in \mathcal{P}_r.$$

The residue map from $\mathrm{GL}_k(W)$ onto the finite general linear group is surjective, so these cosets exhaust the local ones.

For $\mathfrak{e} = (x, \xi) \in X \times B$, let $h_{\mathfrak{e}}$ be the adelic unitary element supported at $\mathfrak{l}, \bar{\mathfrak{l}}$ whose $\mathfrak{l}$-component is $(g_x)_{\mathfrak{l}}^{-1} \beta_\xi d$. Factor

$$g_x h_{\mathfrak{e}} = \gamma_{\mathfrak{e}} g_{o(\mathfrak{e})} \kappa_{\mathfrak{e}}, \quad \gamma_{\mathfrak{e}} \in G(F), \quad \kappa_{\mathfrak{e}} \in \mathrm{K}_0.$$

At $\mathfrak{l}$, the rational matrix $u_{\mathfrak{e}} = \gamma_{\mathfrak{e}}^{-1} \beta_\xi d$ and its inverse are integral, because their local value is $(g_{o(\mathfrak{e})})_{\mathfrak{l}} (\kappa_{\mathfrak{e}})_{\mathfrak{l}}$. Thus $u_{\mathfrak{e}} \in \mathrm{GL}_k(W)$ and

$$\gamma_{\mathfrak{e}} = \beta_\xi d\, u_{\mathfrak{e}}^{-1}, \qquad \gamma_{\mathfrak{e}}^{-1} = u_{\mathfrak{e}} d^{-1} \beta_\xi^{-1}.$$

The rational matrix $\gamma_{\mathfrak{e}}$ is integral with integral inverse at every prime above p other than $\mathfrak{l}, \bar{\mathfrak{l}}$, by the same adelic factorisation.

These edges satisfy the precise injectivity hypothesis of Proposition 4.2.59. Suppose two edges $\mathfrak{e} = (x, \xi)$ and $\mathfrak{e}' = (x', \xi')$ have the same origin and $u^t_{\mathfrak{e}}(u^t_{\mathfrak{e}'})^{-1} \in \mathcal{P}_r$. Taking inverse transposes in the displayed parahoric criterion shows that $d\, u_{\mathfrak{e}}^{-1} u_{\mathfrak{e}'} d^{-1}$ is integral with integral inverse at $\mathfrak{l}$. Hence so is $\gamma_{\mathfrak{e}}\gamma_{\mathfrak{e}'}^{-1} = \beta_\xi d\, u_{\mathfrak{e}}^{-1} u_{\mathfrak{e}'} d^{-1} \beta_{\xi'}^{-1}$. Put $\kappa = \kappa_{\mathfrak{e}'}^{-1}\kappa_{\mathfrak{e}}$ and $y = g_x^{-1}\gamma_{\mathfrak{e}}\gamma_{\mathfrak{e}'}^{-1} g_{x'}$. The two factorizations give

$$y\kappa = h_{\mathfrak{e}}\kappa^{-1} h_{\mathfrak{e}'}^{-1}\kappa.$$

This element is the identity away from $\mathfrak{l}, \bar{\mathfrak{l}}$ and is integral at $\mathfrak{l}$, since $g_x, g_{x'}, \kappa$ are integral there. By unitarity it is integral at the conjugate place too. Since K_0 is maximal at this split pair, $y\kappa \in \mathrm{K}_0$, hence $y \in \mathrm{K}_0$. Therefore $x = x'$. Freeness of that class makes $\gamma_{\mathfrak{e}}\gamma_{\mathfrak{e}'}^{-1} = 1$. The formulas for the rational parts then give $d^{-1}\beta_\xi^{-1}\beta_{\xi'} d = u_{\mathfrak{e}}^{-1} u_{\mathfrak{e}'} \in \mathrm{GL}_k(W)$, so $\xi = \xi'$. This proves the required edge injectivity.

It remains to identify the coefficient operator, including its transpose and its scaling. Let $\mathcal{I}$ be the fixed source monomial-coordinate set, let $\mathcal{I}_{\mathrm{off}}$ be the product of the coordinate sets at places outside $\{\tau_i\}$, and let $\mathcal{E}_a$ be the degree-a exponent set separately in the c letters. Restricting a coordinate tuple to these two subsets gives an actual bijection

$$J : \mathcal{I} \longrightarrow \mathcal{E}_a \times \mathcal{I}_{\mathrm{off}}.$$

For each edge let $t(\mathfrak{e})$ be the source-representation matrix of $\gamma_{\mathfrak{e}}$ on the off-orbit factors. It lies in $\mathrm{GL}_{\mathcal{I}_{\mathrm{off}}}(O)$: an off-orbit a-place induces neither $\mathfrak{l}$ nor its conjugate, where the rational part and inverse are integral, while at a non-a-place the factor is trivial. Polynomial monomial substitution, also applied to the inverse matrices, proves invertibility over O.

Write ρ_a for the coefficient representation in Proposition 4.2.59. In even letter i, unitarity and the rational-part formula give

$$p\,\overline{F_{\tau_i}(\gamma_{\mathfrak{e}})}^{\,t} = \iota_i(u_{\mathfrak{e}})\,\mathrm{diag}(I_r, pI_{k-r})\iota_i(\beta_\xi^{-1}).$$

In odd letter i, the same formula, inverted and transposed, gives

$$\overline{F_{\tau_i}(\gamma_{\mathfrak{e}})}^{\,t} = \iota_i(u_{\mathfrak{e}}^{-t})\,\mathrm{diag}(pI_r, I_{k-r})\iota_i(\beta_\xi^{t}).$$

Thus in each letter the diagonal is p exactly on the deep variables specified in the abstract theorem. Taking degree-a coefficient representations and tensoring, its diagonal becomes $Q_{\beta\beta} = p^{\mathrm{depth}(\beta)}$. The one-row Weyl action is the transpose of the degree-a coefficient action of the transposed matrix: this follows by comparing coefficients of the substituted monomials, or equivalently evaluating their identity at an indeterminate vector. Since there are $c/2$ even letters, their scalar factors contribute exactly $p^{ac/2} = p^{A_{\mathfrak{l}}}$. Therefore, in the same integral coordinates and for every $i, j \in \mathcal{I}$,

$$p^{A_{\mathfrak{l}}}[\rho_\lambda((\overline{F_\tau(\gamma_{\mathfrak{e}})})_\tau)]_{ij} = t(\mathfrak{e})_{J(i)_2, J(j)_2}[\rho_a(u_{\mathfrak{e}})Q\rho_a(\beta_\xi^{-1})]_{J(j)_1, J(i)_1}.$$

No change of Gram normalization or scalar is made here.

Evaluation at the g_x gives an O-linear isomorphism $f : O^{X\times\mathcal{I}} \longrightarrow L_j$. Indeed freeness extends arbitrary coordinate values at representatives uniquely to source forms. Their values there belong to O exactly when their gauges are at most 1, since the representatives are integral above p; at $\gamma g_x \kappa$ the inverse-matrix gauge cancels the rational action and the integral κ preserves the coordinate lattice. This proves the asserted equality with L_j, not only a containment. The corresponding complex evaluation isomorphism also gives $|X||\mathcal{I}| = \dim_{\mathbb{C}} \mathcal{S}$.

The spherical Hecke value at g_x is the sum over the actual right cosets $h_{\mathfrak{e}}$, with coefficient 1 because the compact subgroup has mass 1. The factorisation of each edge makes this summand $\rho_\lambda((\overline{F_\tau(\gamma_{\mathfrak{e}})})_\tau)\varphi(g_{o(\mathfrak{e})})$. The displayed coefficient identity therefore proves

$$Y_{\mathfrak{l}} f = fA,$$

where A is precisely the edge operator of Proposition 4.2.59, with $v(\xi) = \beta_\xi^{-1}$, origin o, matrices $u_{\mathfrak{e}}$, transition $t(\mathfrak{e})$ and bijection J. We checked $v(\xi)^{-1}\mathcal{P}_r = \xi$, edge injectivity, all integral transition matrices and every Frobenius size hypothesis. Apply that theorem to the ambient O-module $\mathcal{S}$ and this actual map f. It gives preservation of L_j and finite-family determinant colength at most

$$(3\delta + \theta h_{\mathrm{mark}})|X||\mathcal{I}| \le \left(\frac{12A_{\mathfrak{l}}}{mp} + \frac{h_{\mathrm{mark}}}{m^2 f}\right)\dim_{\mathbb{C}} \mathcal{S}.$$

This proves both assertions relative to the single abstract edge theorem. □

Proposition 4.2.59. *Let* R *be a commutative local ring,* O *a characteristic-zero valuation domain, and* L *a positive even integer. Let* p *be prime and a nonunit in* O*, let* $e \ge 1$ *be an integer,* $\mathsf{P} = p^e \ge 5$*, and* $\mathsf{Q} = \mathsf{P}^{\mathsf{L}}$*. Suppose the residue field of* R *has* Q *elements, unital ring homomorphisms* $\iota_s : \mathsf{R} \to O$ *indexed by* $\mathbb{Z}/\mathsf{L}$ *have alternating parities, and* $w \in O$ *satisfies* $w^2 = p$ *with* $p \mid \iota_{s+1}(x) - \iota_s(x)^{\mathsf{P}}$ *for all* $x \in \mathsf{R}$*. Let* ρ_n *on* $O^{\mathcal{E}_n}$ *be the coefficient representation on polynomials of degree* n *separately in each letter under* $z_{b,s} \mapsto \sum_d (g_s)_{db} z_{d,s}$*, where* $g_s = \iota_s(g)$ *in even letters and* $\iota_s(g^{-t})$ *in odd letters. Here* $\mathcal{E}_n$ *denotes the degree-*n *exponent vectors in* L *groups of* k *variables.*

Suppose

$$0 < r, \quad 2r \le k, \quad 0 < \theta \le 1, \quad n^4 \le \mathsf{Q}, \quad (\mathsf{L}k \log \mathsf{Q}/\theta)^5 \mathsf{P}^{3r} \le n.$$

Let $\mathcal{P}_r \subset \mathrm{GL}_k(\mathsf{R})$ *consist of matrices whose upper-right* r *by* $k-r$ *block belongs to the maximal ideal entrywise, and let* $\mathsf{B} = \mathrm{GL}_k(\mathsf{R})/\mathcal{P}_r$ *be its left coset space. Take finite sets* X, H, I *and a bijection* $J : I \to \mathcal{E}_n \times H$*. Specify an origin* $o : X \times \mathsf{B} \to X$*, matrices* $u : X \times \mathsf{B} \to \mathrm{GL}_k(\mathsf{R})$*,* $v : \mathsf{B} \to \mathrm{GL}_k(\mathsf{R})$*, and* $t : X \times \mathsf{B} \to \mathrm{GL}_H(O)$*, such that* $v(b)^{-1}\mathcal{P}_r = b$ *and*

$$o(x,b) = o(x',b'),\ u(x,b)^t (u(x',b')^t)^{-1} \in \mathcal{P}_r \quad \Longrightarrow \quad (x,b) = (x',b').$$

For an exponent β*, let* $d(\beta)$ *be its degree in rows* $b \ge r$ *of even letters and rows* $b < r$ *of odd letters, and let* $Q_{\beta\beta} = p^{d(\beta)}$*. Define the endomorphism* A *of* $O^{X \times I}$ *by*

$$(AF)(x,i) = \sum_{b \in \mathsf{B}} \sum_{j \in I} t(x,b)_{J(i)_2, J(j)_2} [\rho_n(u(x,b)) Q \rho_n(v(b))]_{J(j)_1, J(i)_1} F(o(x,b), j).$$

Let M *be any* O*-module, let* $f : O^{X \times I} \to M$ *be linear, and let* Y *be an endomorphism of* M *with* $Yf = fA$*. Put* $L = \operatorname{im} f$ *and*

$$c_{\mathrm{div}} = \mathsf{L}n\left(\mathsf{P}^{-2r} + \frac{r^2}{k\mathsf{P}}\right).$$

Then $YL \subset L$*, and for every integer* $j \ge 0$ *there is an integer* $\ell \ge 0$ *with*

$$\ell \le (3c_{\mathrm{div}} + \theta j)|X||I|$$

such that every finite family $z_1, \dots, z_m \in L$ *admits an* m *by* m *matrix* C *over* O *with* $\det C \mid p^\ell$ *and* $\sum_b C_{ab} z_b \in YL + p^j L$ *for every* a*.*

Proof. **Fix the precision and check the numerical hypotheses.** Write $L_{\mathsf{Q}} = \log \mathsf{Q}$, $c_0 = k^2 e\mathsf{L}$, $s_0 = \mathsf{L}c_0$, $c = 5s_0$, and $d_n = |\mathcal{E}_n|$. Since $\mathsf{P} \geq 5$ and $\mathsf{L} \geq 2$, $L_{\mathsf{Q}} > 3$ and $e \leq L_{\mathsf{Q}}$. Put $Z = \mathsf{L}kL_{\mathsf{Q}}/\theta$. The hypotheses give $n \geq Z^5\mathsf{P}^{3r}$, so $n \geq (\mathsf{L}k)^3$, $n > 1$ and $n < \mathsf{Q}$. Also $h = \mathsf{L}/2$ satisfies $h + 2 \leq 3\mathsf{L}/2$, and $\mathsf{L}k(h+2) + 1 + k < (\mathsf{L}k)^3 \leq n$. Hence

$$\mathsf{L}k(h+2)n + n + k < n^2 \leq \mathsf{P}^h.$$

For $t = 3kc = 15k^3 e\mathsf{L}^2$, one has $t \geq 2(k+1)$ and

$$8\mathsf{L}kt\mathsf{P}^{2r} = 120k^4 e\mathsf{L}^3\mathsf{P}^{2r} \leq \theta n.$$

Indeed the quotient of the lower bound $\mathsf{L}^5k^5L_{\mathsf{Q}}^5\theta^{-4}\mathsf{P}^{3r}$ for θn by the left side is at least

$$\mathsf{L}^2kL_{\mathsf{Q}}^4\mathsf{P}^r/120 > 1,$$

using $e \leq L_{\mathsf{Q}}$, $\mathsf{L}, k \geq 2$, $L_{\mathsf{Q}} > 3$ and $\mathsf{P}^r \geq 5$. Finally

$$s_0 = k^2 e\mathsf{L}^2 \leq \frac{\mathsf{L}n}{k\mathsf{P}} \leq c_{\mathrm{div}}.$$

For the first inequality it suffices that $n \geq \mathsf{L}k^3 e\mathsf{P}$, which follows from $n \geq \mathsf{L}^5k^5L_{\mathsf{Q}}^5\mathsf{P}^3$ and $e \leq L_{\mathsf{Q}}$. In particular $s_0 \geq 2$, and $p^{2s_0}w \mid w^{c-1}$.

The residue field κ of O has characteristic p. The maps of the cycle descend from R to its residue field and then to κ: iteration gives $\iota_s(x)^{\mathsf{Q}} - \iota_s(x) \in pO$, and for x in the maximal ideal of R, $1 - x^{\mathsf{Q}-1}$ is a unit, forcing $\iota_s(x) \in pO$. Thus the two dimension results apply to the induced alternating cycle over κ.

Choose an integral adjoint pairing and edge conditions. For $\beta \in \mathcal{E}_n$ set

$$\mathsf{h}_\beta = \prod_s \frac{p^{c_0}}{\binom{n}{\beta_s}}, \qquad \mathsf{h}'_\beta = \prod_s \binom{n}{\beta_s}.$$

All entries belong to O, because $n < \mathsf{Q} = p^{e\mathsf{L}}$ and each multinomial has p-valuation at most $ke\mathsf{L} \leq c_0$, while its factors prime to p are units. Their product is p^{s_0}. The differential monomial pairing, or its chain-rule identity, gives $\mathrm{diag}(\mathsf{h})\rho_n(g^t) = \rho_n(g)^t \,\mathrm{diag}(\mathsf{h})$.

Use Proposition 4.2.60 at precision c. Let $\mathcal{C}$ consist of the vectors whose every block-Levi translate has coefficient divisible by w^c at exponents of defect greater than $9kc$. Let $\mathcal{S}$ be the corresponding space over κ, with those coefficients zero. The averaging matrix has image in $\mathcal{C}$ and preserves the pairing with the all-edge module up to p^{s_0} modulo w^c. Every all-edge vector belongs to $\mathcal{C}$. Indeed write $w^m u = DF$ as in the all-edge condition. Every translate of F is in the Gauss filtration of depth weights at level m. Apply Lemma 4.2.61 along the root $(r-1, r)$, whose depth increments are one. At an exponent of defect $\Delta \geq 3c$, it gives coefficient divisibility by $w^{\max(0, \lceil m - d(\beta) + \Delta/3 \rceil)}$. Multiplying by $w^{d(\beta)}$ gives divisibility by w^{m+c}, even when that maximum is zero. Since D commutes with the block Levi, the latter coefficient equals w^m times the coefficient of the same Levi translate of u. Cancellation in the domain gives divisibility by w^c.

If $w^{c-1}z \in \mathcal{C}$, cancellation of w^{c-1} shows that every forbidden coefficient of every translate of z is divisible by w, so its reduction belongs to $\mathcal{S}$. The bounded-defect dimension result Proposition 4.2.62, with $t = 3kc$ and the checked inequalities, gives

$$[\mathrm{GL}_k(\mathsf{R}) : \mathcal{P}_r] \dim_\kappa \mathcal{S} \leq (1+\theta)d_n.$$

Restrictions to all edges have a saturated basis and a controlled determinant. We give the linear algebra for either set of edge origins. For a finite origin set V and a complete family of edges E with maps $o_E : E \to V$, $g_E : E \to \mathrm{GL}_k(\mathsf{R})$, completeness means that $e \mapsto (o_E(e), g_E(e)^{-1}\mathcal{P}_r)$ is onto $V \times \mathsf{B}$. Let

$$(\mathscr{R}F)(e) = D\rho_n(g_E(e))F(o_E(e)), \qquad D_{\beta\beta} = w^{d(\beta)}.$$

The map is injective because D has nonzero diagonal and every origin occurs. Its image has a basis factorization

$$\mathscr{R} = Qd, \qquad RQ = 1,$$

where Q is a rectangular matrix whose image is the saturation by powers of w of $\operatorname{im}\mathscr{R}$, and

$$\det d \mid w^{\lfloor |V| c_{\mathrm{div}} d_n \rfloor}.$$

Here and below the symbol R in a matrix factorization denotes a left inverse, not the coefficient ring.

Here are the details of the determinant bound. Smith elimination over the valuation domain gives an invertible domain matrix P_0, a rectangular split injection Q, and nonzero scalars t_j with $\mathscr{R}\mathsf{P}_0^{-1} = Q\,\mathrm{diag}(t_j)$. For every $g \in \mathrm{GL}_k(\mathsf{R})$, every origin and every coefficient, t_j divides the corresponding coefficient of $D\rho_n(g)\mathsf{P}_0^{-1}e_j$. To see why the complete coset representatives suffice, if $g_E(e)^{-1}\mathcal{P}_r = g^{-1}\mathcal{P}_r$, then $gg_E(e)^{-1} \in \mathcal{P}_r$. For $h \in \mathcal{P}_r$, $D\rho_n(h)D^{-1}$ is integral: at one letter the forbidden block entries of h map into $pO = w^2O$, compensating the possible division by w between shallow and deep rows; inverse matrices satisfy the same assertion. Symmetric powers preserve integrality. Thus divisibility at the displayed edge implies divisibility for arbitrary g.

Put $x = \max_\beta d(\beta)$. Taking $g = 1$ shows that t_j divides w^x times every entry of the primitive column $\mathsf{P}_0^{-1}e_j$, hence divides w^x. Therefore Q is exactly the w-saturation and has a left inverse; one may take $d = \mathrm{diag}(t_j)\mathsf{P}_0$. If t_j fails to divide w^u, every coefficient of depth at most u of every translate of this column is in the maximal ideal. Otherwise a unit coefficient would force $t_j \mid w^u$. The reductions of such columns therefore belong, origin by origin, to the deep-form space of depth at least $u+1$. They remain independent modulo the maximal ideal because P_0^{-1} is invertible. Consequently

$$\#\{j : t_j \nmid w^u\} \le |V| \dim_\kappa V_{u+1}.$$

For each j let a_j be the least integer with $t_j \mid w^{a_j}$; it exists and is at most x. Then

$$\det d \mid w^{\sum_j a_j}, \qquad \sum_j a_j = \sum_{u=0}^{x-1} \#\{j : t_j \nmid w^u\} \le |V| \sum_{u=1}^{x} \dim_\kappa V_u \le |V| c_{\mathrm{div}} d_n$$

by Lemma 4.2.63. The sum is integral, so it is at most $\lfloor |V| c_{\mathrm{div}} d_n \rfloor$, giving the stated bound. In particular the floor is taken only after summing over all origins. Each endpoint determinant therefore divides $w^{\lfloor |V| c_{\mathrm{div}} d_n \rfloor}$; for two endpoints their product divides $w^{2\lfloor |V| c_{\mathrm{div}} d_n \rfloor} = p^{\lfloor |V| c_{\mathrm{div}} d_n \rfloor}$.

Every value of a saturated vector belongs to the all-edge module. Indeed write $w^m\psi = \mathscr{R}F$; at an edge its value is $D\rho_n(g_E(e))F(o_E(e))/w^m$, and the complete-coset argument proves integrality after every further translate. This is precisely the all-edge condition.

Apply a near inverse and count the missing generators. Take $V = H \times X$ and edge indices $H \times X \times \mathsf{B}$. For the origin restriction use $o_E(h,x,b) = (h, o(x,b))$ and $g_E(h,x,b) = u(x,b)^t$. For the dual restriction use $o'_E(h,x,b) = (h,x)$ and $g'_E(h,x,b) = v(b)$. The first is complete because the stated edge injectivity makes $(x,b) \mapsto (o(x,b), (u(x,b)^t)^{-1}\mathcal{P}_r)$ injective between finite sets of equal size, hence bijective. The second is complete by the prescribed cosets of v. Write their factorizations as $\mathscr{R} = Qd$ and $\mathscr{R}' = Q'd'$. Both Q, Q' have left inverses,

and their columns have all-edge values. Their two determinant losses multiply to a divisor of $p^{\lfloor |V| c_{\mathrm{div}} d_n \rfloor}$.

On the edge-coordinate module let T act edgewise by $t(x,b)$ on H and let B_0 act diagonally by h_β on the form coordinate. These operators commute. Put

$$\mathscr{A} = Q'^t T \mathsf{B}_0, \qquad \mathscr{N} = T^{-1}(1_H \otimes A_c \operatorname{diag}(\mathsf{h}')) R'^t,$$

where $R'Q' = 1$. Averaging against each all-edge column of Q' gives

$$\mathscr{A}\mathscr{N} \equiv p^{2s_0} 1 \pmod{w^c}.$$

Explicitly, after cancelling T, the summand at each edge is $\langle Q' e_i, A_c \operatorname{diag}(\mathsf{h}') R'^t e_j \rangle_{\mathsf{h}}$; averaging replaces A_c by p^{s_0}, and $\mathsf{h}\mathsf{h}' = p^{s_0}$ reduces the sum to $p^{2s_0}(R'Q')_{ji}$. Both $\operatorname{im} Q$ and $\operatorname{im} \mathscr{N}$ have every edge value in $\mathcal{C}$.

The following finite-generator argument now applies with scalar $a = p^{2s_0}$ and precision $u = w^{c-1}$. If a split injection $Q : O^d \to O^K$ has left inverse R, a map $\mathscr{N} : O^d \to O^K$ has image in $\mathcal{C}^E$, and $\mathscr{A}\mathscr{N} \equiv a1 \pmod{u}$, diagonalize $(\mathscr{N} - QR\mathscr{N})R$ by Smith elimination. It yields a basis z_j of O^K and coefficients τ_j with $Rz_j = 0$ for $\tau_j \neq 0$, such that

$$\operatorname{im} Q + \operatorname{im} \mathscr{N} = \operatorname{im} Q + \sum_j O\tau_j z_j.$$

Keep those nonzero τ_j which divide u. The columns of Q and these z_j all reduce into $\mathcal{S}^E$, because multiplying them by u gives elements of $\mathcal{C}^E$. They are independent modulo the maximal ideal: use R to recover the coefficients of Q, and then the inverse basis matrix of the z_j. If s columns are kept, this gives

$$d + s \leq |H||X||\mathsf{B}| \dim_\kappa \mathcal{S} \leq (1+\theta)d, \qquad d = |H||X|d_n.$$

For every omitted coefficient valuation divisibility gives $u \mid \tau_j$. Applying $\mathscr{A}$ to the decomposition and using its near inverse gives

$$aO^d \subset \operatorname{im}(\mathscr{A}Q) + \sum_{j \text{ kept}} O\mathscr{A} z_j + uO^d.$$

Since $aw \mid u$ and w is in the maximal ideal, Nakayama applied to the quotient of aO^d by its intersection with the first two summands removes the last term. Thus at most $s \leq \theta d$ additional vectors suffice to contain $p^{2s_0}O^d$ together with $\operatorname{im}(\mathscr{A}Q)$.

Multiply the determinant losses and pass to the requested image. The elementary-divisor determinant rule gives, for every integer $j \geq 0$, colength divisor

$$\det(d'^t)\, p^{2s_0 d}\, \det(d)\, p^{js}$$

for the image of $d'^t(\mathscr{A}Q)d$ modulo p^j. To justify the rule, multiplying a free rank-d module by a scalar contributes its dth power, replacing it by the image of a square matrix contributes its determinant, and replacing the span of s generators by their scalar-z multiples contributes z^s. These assertions follow by diagonalizing the finite presentation matrices over the valuation ring; they have the finite-family carrying-matrix meaning of the statement and are preserved by linear images. Carrying matrices multiply under composition, so their determinant divisors multiply. Applying these rules successively to the two endpoint matrices, the inclusion of $p^{2s_0}O^d$ proved above and the s further generators yields the displayed divisor, without assuming that a quotient has finite cardinality.

The two endpoint determinants divide $p^{\lfloor |V| c_{\mathrm{div}} d_n \rfloor}$. Hence the total divisor is a power p^{ℓ} with

$$\ell = \lfloor |V| c_{\mathrm{div}} d_n \rfloor + 2 s_0 d + j s \le (3 c_{\mathrm{div}} + \theta j) d.$$

Here $s_0 \le c_{\mathrm{div}}$ was checked at the start.

Finally $d'^{t}(\mathscr{A} Q) d = \mathscr{R}'^{t} T \mathsf{B}_0 \mathscr{R}$. In the index order $V \times \mathcal{E}_n$ this equals the requested edge operator applied to vectors first multiplied by h. This follows term by term from the adjoint identity for ρ_n: the summand is $t(x,b) \otimes [\rho_n(u(x,b)) Q \rho_n(v(b))]^{t} \operatorname{diag}(\mathsf{h})$, with Q here the diagonal $p^{d(\beta)}$ of the statement. Thus its image maps under the regrouping and f into YL. Lift any finite family in L through f and the regrouping, use the carrying matrix just obtained, and apply f. The resulting combinations lie in $YL + p^{j} L$ with the same determinant divisor p^{ℓ}. The equality $Yf = fA$ also gives $YL \subset L$. Since $d = |X||I|$, this is the asserted bound. If X or H is empty then $L = 0$ and $\ell = 0$ suffices. □

Proposition 4.2.60. *Let R be a commutative local ring and O a characteristic-zero valuation domain. Let $\mathsf{L} = 2h > 0$, p be prime and a nonunit in O, $e \ge 1$ be an integer, $\mathsf{P} = p^{e} \ge 5$, and $\mathsf{Q} = \mathsf{P}^{\mathsf{L}}$. Assume the residue field of R has cardinality Q. Suppose unital ring homomorphisms $\iota_s : \mathsf{R} \to O$, indexed by $s \in \mathbb{Z}/\mathsf{L}$, have alternating even and odd parities, and $w \in O$ satisfies $w^2 = p$ and $p \mid \iota_{s+1}(x) - \iota_s(x)^{\mathsf{P}}$ for every $x \in \mathsf{R}$.*

Let $\mathcal{E}_n$ be the exponent vectors of degree n separately in each letter, with variables $z_{b,s}$ for $0 \le b < k$. Let ρ_n be the coefficient representation for the substitutions $z_{b,s} \mapsto \sum_d (g_s)_{db} z_{d,s}$, where $g_s = \iota_s(g)$ for even s and $g_s = \iota_s(g^{-t})$ for odd s. Choose integers a, r with $0 \le a < r < k$ and $r = a + 1$. Let $G \subset \mathrm{GL}_k(\mathsf{R})$ be the block diagonal subgroup of sizes $r, k - r$. Count as deep the rows $b \ge r$ in even letters and the rows $b < r$ in odd letters, write $d(\beta)$ for their degree in an exponent β, and set $D_{\beta\beta} = w^{d(\beta)}$. Define the submodule $\mathcal{A} \subset O^{\mathcal{E}_n}$ by

$$u \in \mathcal{A} \iff \exists m \ge 0,\ F \in O^{\mathcal{E}_n} : \quad w^{m} u = DF, \quad w^{m} \mid (D \rho_n(g) F)_{\beta} \text{ for all } g \in \mathrm{GL}_k(\mathsf{R}), \beta.$$

Let $Y_s = (r, s)$ and $X_s = (a, s)$ in even letters and reverse them in odd letters. With $[t]_{\mathsf{L}}$ the residue representative in $\{0, \dots, \mathsf{L} - 1\}$, put

$$\Delta_i(\beta) = \sum_{b,s} \frac{[(b,s) = Y_s] \mathsf{P}^{1 + [s - 1 - i]_{\mathsf{L}}} - [(b,s) = X_s] \mathsf{P}^{[s-i]_{\mathsf{L}}}}{\mathsf{Q} - 1} \beta_{b,s}.$$

Suppose

$$\mathsf{L} k (h+2) n + n + k < \mathsf{P}^{h}, \qquad c_0 = k^2 e \mathsf{L},$$

and that diagonal weights $b_\beta, b'_\beta \in O$ satisfy

$$\operatorname{diag}(b) \rho_n(g^{t}) = \rho_n(g)^{t} \operatorname{diag}(b) \quad (g \in \mathrm{GL}_k(\mathsf{R})), \qquad b_\beta b'_\beta = p^{\mathsf{L} c_0}.$$

For every integer $c \ge 0$ there is an O-linear endomorphism A_c of $O^{\mathcal{E}_n}$ such that

$$w^{c} \mid (\rho_n(g) A_c v)_{\beta} \quad \text{whenever } g \in G,\ \Delta_i(\beta) > 9kc \text{ for some } i,$$

and

$$u^{t} \operatorname{diag}(b) A_c v \equiv p^{\mathsf{L} c_0} u^{t} \operatorname{diag}(b) v \pmod{w^{c}} \quad (u \in \mathcal{A},\ v \in O^{\mathcal{E}_n}).$$

Proof. Put $c_* = 3kc$, $c_0 = k^2e\mathsf{L}$ and $D_0 = \mathsf{L}nk(h+2)\mathsf{P}^h$. The strict size assumption gives $\mathsf{L}nk(h+2) \le \mathsf{P}^h$, $h \le \mathsf{L}$, $2n \le \mathsf{Q}$, and $D_0 + k < \mathsf{Q}$. The index condition $a + 1 = r < k$ gives $k \ge 2$. The case $c = 0$ is immediate with $A_0 = 0$, so assume $c \ge 1$.

Large defect forces divisibility against the all-edge module. Use the setup of the statement. Let $d(\beta)$ be the depth counting rows at least r in even letters and below r in odd letters. Let D be diagonal with entries $w^{d(\beta)}$. Say that an integral vector u has the all-edge property if, for some $m \ge 0$ and integral F,

$$w^m u = DF, \qquad w^m \mid (D\rho_n(h)F)_\beta \quad (h \in \mathrm{GL}_k(\mathsf{R}),\ \beta \in \mathcal{E}_n).$$

Let G be the block Levi of sizes $r, k-r$. For $g \in G$ and an exponent β with $\Delta_i(\beta) \ge 3j$, one has

$$w^j \mid (\rho_n(g)u)_\beta.$$

Indeed $\rho_n(h)F$ lies in the Gauss filtration of depth weights at level m: its coefficient at γ is divisible by $w^{\max(0,m-d(\gamma))}$, obtained by cancelling the common power of the nonzero element w in the displayed all-edge divisibility. Every translate of the polynomial with coefficient vector $\rho_n(g)F$ has the same property. The depth weights rise by one along the root (a, r), so Lemma 4.2.61 yields divisibility of its coefficient at β by

$$w^{\max(0,\lceil m-d(\beta)+\Delta_i(\beta)/3\rceil)}.$$

Multiplying by $w^{d(\beta)}$ gives a multiple of w^{m+j}: if the expression inside the ceiling is nonnegative this follows from $\Delta_i(\beta)/3 \ge j$, and if it is negative then $d(\beta) > m + \Delta_i(\beta)/3 \ge m + j$. The block Levi preserves the deep and shallow rows in each letter, hence $D\rho_n(g) = \rho_n(g)D$, as follows entry by entry from its two blocks. The resulting divisibility is therefore $w^{m+j} \mid w^m(\rho_n(g)u)_\beta$. Cancellation in the domain proves the claim.

If weights b satisfy $\mathrm{diag}(b)\rho_n(g^t) = \rho_n(g)^t \mathrm{diag}(b)$, then

$$\langle u, \rho_n(g^{-1})e_\beta\rangle_b = b_\beta(\rho_n(g^{-t})u)_\beta$$

for $\langle u, v\rangle_b = u^t \mathrm{diag}(b)v$. This identity is ordinary matrix multiplication using the stated adjoint relation; it does not assume that u is invariant. Since $g^{-t} \in G$, the pairing is divisible by w^j when $\Delta_i(\beta) \ge 3j$. In particular, defect exceeding $3kc$ makes it divisible by w^c.

The actual Frobenius polynomial matrix. Let $\mathsf{L} = 2h$, and let X_j be independent generic k by k matrices for the letters j. For $s \in \mathbb{Z}/\mathsf{L}$ and $d \ge 0$, put $U_{s,d} = X_{s-d}^{[\mathsf{P}^d]}$, where brackets mean entrywise power. Write $\epsilon_{b,s} = 1$ on a deep row and 0 otherwise. Let $L_s(X)$ be the matrix whose row b is row b of

$$\begin{cases} U_{s,h+\epsilon_{b,s}-1} \cdots U_{s,0}, & s \text{ even}, \\ \mathrm{adj}(U_{s,h+\epsilon_{b,s}-1} \cdots U_{s,0})^t, & s \text{ odd}. \end{cases}$$

An empty product is the identity. Define

$$\mathscr{S}(X) = \bigotimes_{s \in \mathbb{Z}/\mathsf{L}} \mathrm{Sym}^n(L_s(X)).$$

These are row-dependent products: replacing them by one common product would change the argument.

For linear forms λ_j on the polynomial ring in the entries of X_j, their tensor contraction is defined on monomials by

$$T_\lambda\left(\prod_j X_j^{\nu_j}\right) = \prod_j \lambda_j(X_j^{\nu_j}),$$

extended linearly. The contraction omitting copy j is defined by

$$T_{\lambda,\neq j}\left(\prod_i X_i^{\nu_i}\right) = X_j^{\nu_j} \prod_{i\neq j} \lambda_i(X_i^{\nu_i}).$$

Both formulas define actual coefficient projections on a basis. They immediately give $\lambda_j T_{\lambda,\neq j} = T_\lambda$, and the same identity with λ_j replaced by an arbitrary linear form μ_j and the full family updated at j. The proof expands any polynomial as its finite monomial sum, so no passage through an infinite sum is used.

Change of frame is an identity with a conjugated polynomial matrix. For $g \in G$ let $G_s = \iota_s(g)$ and let C_g be the substitution $X_j \mapsto G_j^{-1} X_j G_j$. Define the framed factors

$$U_{s,d}^g = G_s (G_{s-d}^{-1} X_{s-d} G_{s-d})^{[\mathsf{P}^d]} G_s^{-1},$$

and let $\mathscr{S}^g$ be obtained by using these factors in the same row-dependent construction. Then

$$\mathscr{S}(C_g X) = \rho_n(g^{-1}) \mathscr{S}^g(X) \rho_n(g).$$

To prove this without commuting entrywise powers with conjugation, observe first that $U_{s,d}^g = G_s U_{s,d}(C_g X) G_s^{-1}$ by its definition. In each reversed product adjacent $G_s^{-1} G_s$ cancel. For an odd letter use $\operatorname{adj}(AMA^{-1})^t = A^{-t} \operatorname{adj}(M)^t A^t$, valid first for invertible A by the determinant formula and then directly here because $A = G_s$ is invertible. The two row selectors are preserved by the block diagonal matrices G_s, so selecting different product lengths on the two blocks respects this conjugation. Taking symmetric powers and tensor products gives the displayed identity.

Suppose now each λ_j is invariant under C_g. Applying the finite monomial contractions to the identity gives

$$\rho_n(g)\, T_\lambda(\mathscr{S}) = T_\lambda(\mathscr{S}^g) \rho_n(g).$$

The right side contains $\mathscr{S}^g$, not $\mathscr{S}$: no commutation with the original substitution is being assumed. More precisely, let $K_j(g, v)$ be the vector of polynomials left in copy j after contracting all other copies in $\mathscr{S}^g \rho_n(g) v$. Then

$$C_g K_j(1, v) = \rho_n(g^{-1}) K_j(g, v).$$

For a coefficient functional on the remaining copy, this follows from the preceding identity with the jth linear form replaced by that coefficient functional composed with C_g; equality of every coefficient proves the polynomial identity.

Consequently, if $\phi_j(X_j) = \sum_\beta u_\beta b_\beta K_j(1, v)_\beta$, then for every monomial ν in copy j,

$$[X_j^\nu] C_g \phi_j = \sum_\beta [X_j^\nu] K_j(g, v)_\beta \, \langle u, \rho_n(g^{-1}) e_\beta \rangle_b.$$

This is the precise coefficient formula needed for the pairing difference; it follows by expanding the preceding vector identity and rearranging finite sums.

Copy weights approximate the defect with an explicit error. Put $\eta_b = [b = r] - [b = a]$, and for a fixed copy i give the matrix variables weights

$$\omega(j, b, d) = \begin{cases} \eta_b / \mathsf{P}^h, & j = i, \\ \mathsf{P}^{[j-i]_{\mathsf{L}}} (\eta_b - \eta_d) / (\mathsf{Q}\mathsf{P}^h), & j \neq i. \end{cases}$$

Their weighted sum on a monomial ν is

$$\mathsf{R}_i(\nu) = \mathsf{P}^{-h}\left(\mathrm{rw}(\nu_i) + \sum_{j\neq i} \frac{\mathsf{P}^{[j-i]_{\mathsf{L}}}}{\mathsf{Q}}(\mathrm{rw}(\nu_j) - \mathrm{cw}(\nu_j))\right),$$

where $\mathrm{cw}(\nu_j) = \sum_{b,d} \nu_{j,b,d}\eta_d$. In particular $|\omega| \le 2/\mathsf{P}^h$.

Fix $\varepsilon \in \{1/2, -1/2\}$ and use Gauss weights $\varepsilon\omega$. For every factor index $0 \le d \le h$, the framed entry $(U^g_{s,d})_{bc}$ has level at least $\varepsilon\mathsf{P}^d\omega(s-d,b,c)$. Here is the complete coefficient check. Every monomial occurring in this entry has ordinary degree P^d, so has $\varepsilon\omega$-weight at least $-\mathsf{P}^d/\mathsf{P}^h \ge -1$. Its target level has absolute value at most one. Modulo $p = w^2$ the framed entry equals $X^{\mathsf{P}^d}_{s-d,b,c}$: the prime-power binomial identity and the iterated congruence $\iota_s(g) \equiv \iota_{s-d}(g)^{[\mathsf{P}^d]}$ cancel the inner and outer conjugations, for inverses as well as matrices. Thus its difference from this single monomial has every coefficient divisible by w^2. At a monomial of weight at least -1, the requested exponent for a target level at most one is at most two. This proves the stated level, since the main monomial has exactly the target weight.

Define potentials and an error unit by

$$t_s(d,b) = \frac{\varepsilon\eta_b\mathsf{P}^{d+[s-d-i]_{\mathsf{L}}}}{\mathsf{Q}\mathsf{P}^h}, \qquad u = \frac{1}{2\mathsf{P}^h}.$$

For $d \le h \le \mathsf{L}$,

$$|t_s(0,b)| \le u, \qquad \sum_b t_s(d,b) = 0, \qquad \left|t_s(d+1,b) - t_s(d,c) - \varepsilon\mathsf{P}^d\omega(s-d,b,c)\right| \le u.$$

For the last identity, if $s-d \neq i$, the residue exponent decreases by one and the expression is exactly zero. If $s-d=i$, the wrap replaces that exponent by L; after subtracting the row contribution, the remaining term is $-\varepsilon\eta_c\mathsf{P}^d/(\mathsf{Q}\mathsf{P}^h)$, of size at most u. Consequently every factor entry has level at least $t_s(d+1,b) - t_s(d,c) - u$.

An entry in a reversed product of m such factors has level at least $t_s(m,b) - t_s(0,c) - mu$: in each path of intermediate indices the potentials telescope, and addition preserves the lower bound. For $m \le h+1$ this is at least $t_s(m,b) - (h+2)u$. The (b,c) entry of the transposed adjugate is the signed minor deleting row b and column c. Each determinant term has level at least

$$\sum_{v\neq b} t_s(m,v) - \sum_{v\neq c} t_s(0,v) - (k-1)mu = -t_s(m,b) + t_s(0,c) - (k-1)mu \ge -t_s(m,b) - k(h+2)u.$$

Here the vanishing sums of the potentials are essential; the adjugate is not being replaced by an inverse over the polynomial coefficient ring.

Choose the row-dependent length $m = h + \epsilon_{b,s}$. The even-row bound is at least $t_s(m,b) - k(h+2)u$ and the odd-row bound at least $-t_s(m,b) - k(h+2)u$. The terminal potential satisfies

$$(\text{even sign } +1,\ \text{odd sign } -1)\, t_s(h+\epsilon_{b,s},b) = \varepsilon(1-1/\mathsf{Q})d_{i+h}(b,s).$$

For $b = a, r$ this is checked directly: at an even r or odd a the exponent is $1 + [s-i-h-1]_{\mathsf{L}}$, and at an even a or odd r it is $[s-i-h]_{\mathsf{L}}$ with negative sign; other rows contribute zero. Taking the nth symmetric powers and then the tensor product over L letters shows that row β of $\mathscr{S}^g$ has Gauss level at least

$$\varepsilon(1-1/\mathsf{Q})\Delta_{i+h}(\beta) - L, \qquad L = \frac{\mathsf{L}nk(h+2)}{2\mathsf{P}^h} \le \frac{1}{2}.$$

Indeed each degree-n monomial term multiplies n row entries in each letter; multinomial coefficients are integral, so cannot lower the level.

Suppose w^c does not divide the coefficient of an all-copy monomial ν in an entry of row β. Then $c \geq 1$. From both choices of ε and the definition of Gauss level,

$$|(1 - 1/\mathsf{Q})\Delta_{i+h}(\beta) - \mathsf{R}_i(\nu)| \leq 2c + 1.$$

Otherwise one of the two levels would require a power at least w^c, contradicting the assumption. Also $|\Delta_{i+h}(\beta)| \leq 2n$: per letter the total degree is n, each defect coefficient has size at most $\mathsf{Q}/(\mathsf{Q}-1)$ in the distinguished letter and at most $\mathsf{P}^d/(\mathsf{Q}-1)$ otherwise, and the geometric sum gives the bound $2n$. More explicitly, the sum of the largest absolute coefficient in each letter is at most

$$\frac{\mathsf{Q} + \sum_{d=1}^{\mathsf{L}-1} \mathsf{P}^d}{\mathsf{Q}-1} = \frac{\mathsf{Q} + (\mathsf{Q}-\mathsf{P})/(\mathsf{P}-1)}{\mathsf{Q}-1} < 2$$

for $\mathsf{P} \geq 5$; multiplying by n suffices. The size assumption gives $2n \leq \mathsf{Q}$, hence $|\Delta - (1-1/\mathsf{Q})\Delta| \leq 1$. Therefore

$$|\Delta_{i+h}(\beta) - \mathsf{R}_i(\nu)| \leq 2c + 2 \leq 3kc.$$

Finally pass from all-copy coefficients to the contracted polynomial $K_i(g, v)_\beta$. If a coefficient there is not divisible by w^c, its finite monomial expansion has a summand whose all-copy coefficient is not divisible by w^c and whose other-copy linear-form values are nonzero. Every invariant linear form annihilates a monomial with unequal row and column η-weights. To see this, p is a nonunit in R (otherwise its image in O would be a unit), so $1 + p$ is a unit in the local ring R. Conjugate by the diagonal matrix with entries $(1+p)^{\eta_b}$, which belongs to the Levi. On a monomial it acts by $(1+p)^{\mathrm{cw}\,-\,\mathrm{rw}}$. If the exponent is nonzero, this scalar differs from one in the characteristic-zero domain O; invariance and cancellation force the linear-form value to be zero. Thus in every surviving copy other than i, row and column weights agree. For the witnessing all-copy monomial one has $\mathsf{R}_i(\nu) = \mathrm{rw}(\nu_i)/\mathsf{P}^h$, and the preceding bound is exactly

$$\left|\Delta_{i+h}(\beta) - \mathrm{rw}(\nu_i)/\mathsf{P}^h\right| \leq 3kc.$$

This proves the required quantitative estimate for contracted coefficients.

Construct the invariant linear forms with integral coefficients. Fix one letter and its map $\iota : \mathsf{R} \to O$. Let $D_0 = \mathsf{L}nk(h+2)\mathsf{P}^h$, let $C = p^{c_0}$ with $c_0 = k^2 e\mathsf{L}$, let $\eta_b = [b = r] - [b = a]$, and let $T = 6kc\mathsf{P}^h$. The strict size assumption implies $D_0 + k < \mathsf{Q}$. We construct an invariant functional I on polynomials in one generic matrix, annihilating monomials of row weight greater than T, with the required ideal-compatibility up to degree D_0.

First ι induces an embedding of the residue field of R into that of O. Iterating the Frobenius cycle gives $\iota(x)^{\mathsf{Q}} - \iota(x) \in pO$. If x belongs to the maximal ideal of R, then $1 - x^{\mathsf{Q}-1}$ is a unit, so its image is a unit. Thus $p \mid \iota(x)(1 - \iota(x)^{\mathsf{Q}-1})$ implies $\iota(x) \in pO$, which lies in the maximal ideal of O. The induced residue map is unital between fields and hence injective. Write κ for the residue field of O, containing this copy of $\mathbb{F}_{\mathsf{Q}}$.

Fix $0 \leq d \leq D_0$. On the free tensor module $V_d = (O^k)^{\otimes d}$ with basis e_J, $J = (J_1, \ldots, J_d)$, form the matrix $\mathsf{B}_d(X) = (X^t_{\mathrm{block}})^{\otimes d}$, where entries outside the two diagonal blocks are zeroed. Write $S_d \subset \mathrm{End}_O(V_d)$ for the span of its coefficient matrices. These coefficient matrices are linearly independent after the zero ones are removed: their nonzero entries are ones and their supports are disjoint, since a tensor-matrix entry is exactly one monomial. Thus they form a finite free basis of S_d.

The matrices $\iota(g)^{\otimes d}$, $g \in G(\mathsf{R})$, span S_d integrally. They belong to S_d by evaluation of the transpose generic matrix. To show they span, let Q be the quotient of S_d by their span. It

is finitely generated. Any κ-linear functional on $Q/\mathfrak{m}_O Q$ gives a linear functional $\lambda : S_d \to \kappa$ annihilating the group matrices. Applying it coefficientwise to $\mathsf{B}_d(X)$ gives a homogeneous polynomial $f(X)$ of degree d in the block entries. On every invertible block matrix over $\mathbb{F}_{\mathsf{Q}}$ this polynomial vanishes: lift that matrix to R, whose block determinants are units, and apply the hypothesis on λ. Let $\delta(X)$ be the product of the two block determinants, of degree k. Then δf vanishes on every block matrix over $\mathbb{F}_{\mathsf{Q}}$, including singular ones. Its total degree is at most $d + k < \mathsf{Q}$. A polynomial over κ whose degree in every variable is below Q and which vanishes on the full $\mathbb{F}_{\mathsf{Q}}$ grid is zero: induction on the number of variables applies the one-variable root bound to every specialized coefficient polynomial. Hence $\delta f = 0$; since the polynomial ring over the field κ is a domain and $\delta \neq 0$, $f = 0$. The coefficient matrices span S_d, so $\lambda = 0$. Therefore $Q/\mathfrak{m}_O Q = 0$, and Nakayama's lemma gives $Q = 0$.

In particular S_d is an algebra: it is the span of tensor powers of group elements, and products of such tensor powers are again tensor powers. For each matrix monomial exponent ν of degree d put

$$\gamma_\nu = \frac{p^{c_0}}{\binom{d}{\nu}} \in O.$$

This quotient is integral. A multinomial is a product of at most k^2 binomial coefficients with upper indices at most $d < \mathsf{Q} = p^{e\mathsf{L}}$. The factorial valuation formula shows that the p-valuation of each such binomial is at most $e\mathsf{L}$, since each summand at a power of p is zero or one and powers at least $p^{e\mathsf{L}}$ contribute zero. All other prime factors are units in O: any integer prime to p is a unit because p lies in the maximal ideal. Thus the multinomial divides $p^{k^2 e\mathsf{L}}$ in O.

Define the operator $T_d(\phi) \in S_d$ by giving the coefficient matrix of monomial ν in B_d the scalar $\gamma_\nu \phi_\nu$. In entries,

$$T_d(\phi)_{I,J} = \gamma_\nu \phi_\nu, \qquad \nu = \sum_{\ell=1}^{d} [(J_\ell, I_\ell)],$$

if I_ℓ, J_ℓ are in the same block for every ℓ, and zero otherwise. It depends only on the homogeneous degree-d part of ϕ.

For $g \in G(\mathsf{R})$,

$$T_d(C_g \phi) = \iota(g)^{\otimes d} T_d(\phi) \iota(g^{-1})^{\otimes d}, \qquad C_g\phi(X) = \phi(\iota(g)^{-1} X \iota(g)).$$

Here is why the weights give this exact adjoint identity. Over the fraction field of O, the diagonal monomial pairing with weights γ_ν in degree d is $p^{c_0}/d!$ times the differential pairing $\langle X^\nu, X^\nu \rangle = \prod_v \nu_v!$. The chain rule shows that substitution by a linear map and by its transpose are adjoint for the latter pairing. The transpose of $X \mapsto g^{-1} X g$ on the matrix variables is $X \mapsto g^{-t} X g^t$. Applying this to X^t_{block} yields $g X^t_{\mathrm{block}} g^{-1}$, and tensoring gives the identity above. Both sides have entries in O, so the identity descends from the fraction field.

Let $U_d \subset V_d$ be the sum of $S_d e_J$ over the tensors J of weight $\sum_\ell \eta_{J_\ell} > T$. This is finitely generated, S_d-stable and torsion-free. It is free over the valuation domain: for a finite generating matrix choose a nonzero coordinate dividing every coordinate of the matrix, possible because its principal ideals are totally ordered; eliminate that coordinate from the other generators, split off the pivot vector, and induct on the number of ambient coordinates. This proves finite freeness without assuming that U_d is saturated.

Define

$$I_d(\phi) = \mathrm{tr}_{V_d} T_d(\phi) - \mathrm{tr}_{U_d}(T_d(\phi)|_{U_d}).$$

The conjugation identity and stability of U_d prove invariance under C_g. If $\phi = X^\nu$ has row weight greater than T, the only possibly nonzero columns of $T_d(\phi)$ are high-weight tensors J.

Their images lie in $S_d e_J \subset U_d$. Hence its whole image lies in U_d, and its traces on V_d and U_d agree: tensor with the fraction field, extend a basis of U_d to a basis of V_d, and the quotient block is zero. Thus I_d annihilates every bad monomial.

For arbitrary ϕ, the trace on V_d equals $C\phi_d(1)$. Indeed only monomials on the diagonal entries contribute to its trace. A diagonal exponent ν occurs in exactly $\binom{d}{\nu}$ tensor words, whose weighted contribution is $\binom{d}{\nu}\gamma_\nu\phi_\nu = C\phi_\nu$.

Finally suppose that an ideal $J_0 \subset O$ contains every bad coefficient of every conjugate $C_g\phi$. We show $T_d(\phi)U_d \subset J_0U_d$, not merely $U_d \cap J_0V_d$. For a high-weight basis tensor e_J, expand $T_d(\phi)e_J$ into monomials. A nonzero term has row weight exactly $\sum_\ell \eta_{J_\ell} > T$, so its scalar coefficient ϕ_ν lies in J_0, while its remaining vector is γ_ν times an element of $S_d e_J \subset U_d$. Therefore $T_d(\phi)e_J \in J_0U_d$. The integral group-span result says that U_d is spanned by $\iota(g)^{\otimes d}e_J$ with J high. On such a vector the conjugation identity moves $T_d(\phi)$ past the group tensor and replaces ϕ by $C_{g^{-1}}\phi$, to which the preceding argument applies. Group tensors preserve J_0U_d. This proves the desired inclusion. In any basis of the finite free module U_d, its operator matrix now has every entry in J_0, so its trace belongs to J_0. Consequently

$$I_d(\phi) \equiv C\phi_d(1) \pmod{J_0}.$$

Sum I_d over $0 \le d \le D_0$ and define it to vanish on higher homogeneous degrees. The resulting integral functional I is invariant, kills every bad monomial and satisfies $I(\phi) \equiv p^{c_0}\phi(1) \pmod{J_0}$ for $\deg\phi \le D_0$ under the stated bad-coefficient hypothesis. Apply this construction to each letter to obtain forms I_j.

Contract the copies and prove both conclusions. For $J \subset \mathbb{Z}/\mathsf{L}$, let $\lambda_i^J = I_i$ for $i \in J$ and let $\lambda_i^J = p^{c_0}\,\mathrm{ev}_1$ otherwise. Evaluation at identity is invariant under conjugation, so every λ_i^J is invariant. Put

$$A_J = T_{\lambda^J}(\mathscr{S}), \qquad A_c = A_{\mathbb{Z}/\mathsf{L}}.$$

Every entry belongs to O. The empty contraction is $A_\varnothing = p^{\mathsf{L}c_0}$ times the identity, since each product and cofactor matrix specializes to the identity when all $X_i = 1$.

For $g \in G$, a coefficient of $\rho_n(g)A_c v$ at an exponent β equals $I_j(K_j(g,v)_\beta)$ for any chosen copy j. This follows from the exact frame and contraction identity already proved. If $\Delta_{j+h}(\beta) > 3c_* = 9kc$, every monomial whose coefficient in $K_j(g,v)_\beta$ is not divisible by w^c has row weight greater than $2c_*\mathsf{P}^h$, by the copy-weight estimate. The functional I_j annihilates it. Expanding the polynomial in monomials shows that $I_j(K_j(g,v)_\beta)$ is divisible by w^c. For any letter i choose $j = i - h$; this proves the first conclusion.

For the pairing choose $j \notin J$ and contract all other copies with λ^J. Put $\phi_j = \sum_\beta u_\beta b_\beta K_j(1,v)_\beta$. Applying any linear form μ to ϕ_j gives $\langle u, T_{\lambda^J[j\mapsto\mu]}(\mathscr{S})v\rangle_b$ by the monomial contraction formulas. Every $K_j(1,v)_\beta$ has degree at most D_0 in the remaining copy. Indeed a product of at most $h+1$ factors has degree at most $(h+1)\mathsf{P}^h$, a cofactor has degree at most $(k-1)(h+1)\mathsf{P}^h$, and the nth symmetric powers over L letters give the bound $\mathsf{L}nk(h+2)\mathsf{P}^h = D_0$. Linear contractions on other copies cannot increase this degree. Thus $\deg\phi_j \le D_0$.

Consider a monomial ν of row weight greater than $2c_*\mathsf{P}^h$ in $C_g\phi_j$, $g \in G$. Its exact coefficient formula is

$$\sum_\beta [X_j^\nu]K_j(g,v)_\beta\,\langle u, \rho_n(g^{-1})e_\beta\rangle_b.$$

If the first factor of a summand is not divisible by w^c, the copy-weight estimate gives $\Delta_{j+h}(\beta) > c_* = 3kc \ge 3c$. The completed all-edge pairing argument makes the second factor divisible by w^c. Every summand is therefore divisible by w^c, as is their sum. The ideal-compatibility

property of I_j with ideal (w^c) yields

$$\langle u, A_{J\cup\{j\}}v\rangle_b \equiv \langle u, A_J v\rangle_b \pmod{w^c}.$$

Add these congruences along any finite ordering of the L copies, starting at the empty subset. The initial scalar operator is $p^{\mathsf{L}c_0}$ times the identity, so the required second conclusion follows. If $c = 0$, both divisibility conclusions are automatic; equivalently one may choose $A_0 = 0$. □

Lemma 4.2.61. *Let R be a commutative local ring and R a commutative ring with identity. Let L be a positive even integer, let p be prime, let $e \geq 0$, and put $\mathsf{P} = p^e \geq 5$ and $\mathsf{Q} = \mathsf{P}^{\mathsf{L}}$. Assume that the residue field of R has Q elements. Suppose unital ring homomorphisms $\iota_s : \mathsf{R} \to R$, indexed by $s \in \mathbb{Z}/\mathsf{L}$, have alternating even and odd parities, and that $w \in R$ satisfies*

$$w^2 = p, \qquad p \mid \iota_{s+1}(x) - \iota_s(x)^{\mathsf{P}} \quad (x \in \mathsf{R}).$$

On the variables $z_{b,s}$, $0 \leq b < k$, let $g \in \mathrm{GL}_k(\mathsf{R})$ act by

$$g \cdot z_{b,s} = \sum_d (g_s)_{db} z_{d,s}, \qquad g_s = \iota_s(g) \text{ in even letters}, \quad g_s = \iota_s(g^{-t}) \text{ in odd letters}.$$

For real weights c on the variables write $c \cdot \beta = \sum_{b,s} c_{b,s}\beta_{b,s}$, and write $f \in \mathcal{G}_c(t)$ to mean

$$w^{\max(0,\lceil t - c\cdot\beta\rceil)} \mid f_\beta \quad \text{for every exponent } \beta.$$

Fix distinct rows a, a'. Let H be the subgroup acting by $\mathrm{SL}_2(\mathsf{R})$ on these rows and as the identity elsewhere. Let $Y_s = (a', s)$ and $X_s = (a, s)$ in even letters, reversing these choices in odd letters. For $i \in \mathbb{Z}/\mathsf{L}$ set

$$d_i(b,s) = \frac{[(b,s) = Y_s]\mathsf{P}^{1+[s-1-i]_{\mathsf{L}}} - [(b,s) = X_s]\mathsf{P}^{[s-i]_{\mathsf{L}}}}{\mathsf{Q}-1}, \qquad \Delta_i(\beta) = d_i \cdot \beta,$$

where $[u]_{\mathsf{L}}$ is the representative in $\{0, \ldots, \mathsf{L}-1\}$.

If $\frac{1}{2} \leq c(Y_s) - c(X_s) \leq 1$ for every s, and f, an exponent α and an integer $\ell \geq 0$ satisfy

$$g \cdot f \in \mathcal{G}_c(\ell + c\cdot\alpha) \quad (g \in H), \qquad w^{\ell+1} \nmid f_\alpha,$$

then $\Delta_i(\alpha) \leq 0$ for every i.

Moreover, if real weights h satisfy $h(Y_s) - h(X_s) = 1$ and $g \cdot f \in \mathcal{G}_h(v)$ for every $g \in H$, then, for every i,

$$f \in \mathcal{G}_{h - d_i/3}(v).$$

Proof. Fix the letter i and the two rows in the statement. Write X_s, Y_s for its two variables, $\mathsf{P}_s = \mathsf{P}^{[s-i]_{\mathsf{L}}}$, $\delta_s = c(Y_s) - c(X_s)$, and

$$\rho = \delta_i/\mathsf{Q}, \qquad \mathsf{B}(X_s) = \mathsf{Q}\mathsf{P}_s, \quad \mathsf{B}(Y_s) = 0, \qquad A(Y_i) = \mathsf{Q}, \quad A(Y_s) = \mathsf{P}_s \ (s \neq i), \quad A(X_s) = 0.$$

Both A and B vanish off these rows. Then $\rho > 0$, $\mathsf{Q}\rho = \delta_i$, $\mathsf{P}_s\rho \leq 1/\mathsf{P} \leq 1/5$, and $\mathsf{Q}\mathsf{P}_s\rho = \mathsf{P}_s\delta_i \geq 5/2$ for $s \neq i$. The formula defining the defect gives

$$\Delta_i(\alpha) = \frac{A\cdot\alpha - (\mathsf{B}\cdot\alpha)/\mathsf{Q}}{\mathsf{Q}-1}.$$

Indeed the coefficient at Y_i is $\mathsf{Q}/(\mathsf{Q}-1)$, at Y_s for $s \neq i$ it is $\mathsf{P}_s/(\mathsf{Q}-1)$, and at X_s it is $-\mathsf{P}_s/(\mathsf{Q}-1)$.

Weighted coefficient arithmetic. For real weights λ_z on the original variables and τ on a new variable t, a polynomial $\sum_{j,\beta} u_{j,\beta} t^j z^\beta$ has level at least b if

$$w^{\max(0,\lceil b - j\tau - \lambda\cdot\beta\rceil)} \mid u_{j,\beta} \quad \text{for every } j, \beta.$$

Sums of polynomials of level at least b have level at least b, and the product of levels b, b' has level at least $b+b'$. For the product, divisibility of coefficients follows from $\max(0,\lceil u\rceil)+\max(0,\lceil v\rceil) \geq \max(0,\lceil u+v\rceil)$, followed by addition of coefficient contributions. Therefore a substitution sending each variable z to level at least $c(z)$ sends $\mathcal{G}_c(b)$ into level b. If two such substitutions differ on each variable by a polynomial of level at least $c(z)+\varepsilon$, their values on any $f \in \mathcal{G}_c(b)$ differ by level at least $b+\varepsilon$: expand the difference of the two products for each monomial, so each resulting term contains at least one changed factor. These assertions use divisibility only, and remain valid when R has zero divisors.

Construct the auxiliary polynomial and control its degree. Put $e_s = -1$ when s has the parity of i and $e_s = 1$ otherwise. Define a substitution S_t by

$$\begin{array}{r|cc} & X_s & Y_s \\ s = i & t^{\mathsf{Q}} X_i & X_i \\ s \neq i & t^{\mathsf{Q}\mathsf{P}_s} X_s - e_s t^{\mathsf{P}_s} Y_s & Y_s - e_s X_s. \end{array}$$

It fixes variables off the two rows. Let S_t^0 be the substitution obtained by retaining only $t^{\mathsf{Q}\mathsf{P}_s} X_s$ in the first column and Y_s in the second column when $s \neq i$, leaving the ith letter unchanged. Let μ merge Y_i into X_i and fix every other variable. Write C_α for the coefficient of $z^{\mu(\alpha)}$ in $S_t f$, regarded as a polynomial in t.

Use weights $\tau = -\rho$ and $\lambda = c + \mathsf{B}\rho$. The main substitution $S_t^0(z)$ has weight exactly $c(z)$ for every variable: for Y_i this is $c(X_i) + \mathsf{Q}\rho = c(Y_i)$. For $s \neq i$ the omitted term in the image of Y_s has weight $c(X_s) + \mathsf{Q}\mathsf{P}_s\rho$, exceeding $c(Y_s)$ by at least $5/2 - 1 = 3/2$. The omitted term in the image of X_s has weight $c(Y_s) - \mathsf{P}_s\rho$, exceeding $c(X_s)$ by at least $1/2 - 1/5 = 3/10$. Thus, with $\varepsilon = 1/20$, the weighted arithmetic above proves, for $b = \ell + c\cdot\alpha$,

$$S_t f \text{ has level at least } b, \qquad S_t f - S_t^0 f \text{ has level at least } b+\varepsilon.$$

Here $f \in \mathcal{G}_c(b)$ follows by taking the identity translate in the hypothesis.

The variables X_i, Y_i have equal λ-weight. Consequently $\lambda\cdot\mu(\alpha) = c\cdot\alpha + \rho\mathsf{B}\cdot\alpha$. The coefficient of t^j in C_α is therefore divisible by

$$w^{\max(0,\lceil \ell + \rho(j - \mathsf{B}\cdot\alpha)\rceil)}.$$

For $j > \mathsf{B}\cdot\alpha$ this is divisible by $w^{\ell+1}$. At $j = \mathsf{B}\cdot\alpha$, the coefficient of $S_t f - S_t^0 f$ is also divisible by $w^{\ell+1}$, because its required real order is at least $\ell+\varepsilon$.

The coefficient of $t^{\mathsf{B}\cdot\alpha} z^{\mu(\alpha)}$ in $S_t^0 f$ is exactly f_α. To check this, a monomial z^β contributes only $t^{\mathsf{B}\cdot\beta} z^{\mu(\beta)}$. If both $\mu(\beta) = \mu(\alpha)$ and $\mathsf{B}\cdot\beta = \mathsf{B}\cdot\alpha$, all exponents except those at X_i, Y_i agree, the latter equality gives $\mathsf{Q}\beta_{X_i} = \mathsf{Q}\alpha_{X_i}$, and the merged degree then gives $\beta_{Y_i} = \alpha_{Y_i}$. Thus $\beta = \alpha$.

Reduce coefficients modulo $(w^{\ell+1})$, and call the resulting polynomial $\overline{C}_\alpha \in \overline{R}[t]$, where $\overline{R} = R/(w^{\ell+1})$. It is nonzero because its coefficient at degree $\mathsf{B}\cdot\alpha$ is the nonzero image of f_α, and its degree is at most $\mathsf{B}\cdot\alpha$.

Control the multiplicity at every residue representative. For $r \in \mathsf{R}$ let $r_s = \iota_s(r)$ and define

$$E_d(t, r) = (t + r_i)^{\mathsf{P}^d} - t^{\mathsf{P}^d} - \iota_{i+d}(r).$$

Every coefficient of E_d is divisible by $p = w^2$. Indeed reduction modulo p gives the binomial identity for a power of the prime p, and iteration of the assumed Frobenius congruences gives $\iota_{i+d}(r) = \iota_i(r)^{\mathsf{P}^d}$ modulo p. This applies also to $d = 0$ and to $d = \mathsf{L} + [s - i]_{\mathsf{L}}$.

There is an element $g_r \in H_{a,a'}$ with

$$g_r X_s = \nu(X_s) Y_s, \qquad g_r Y_s = \nu(Y_s)(X_s + e_s r_s Y_s),$$

where $\nu(z) = -1$ on row a and $\nu(z) = 1$ on row a'; it fixes the other rows. Explicitly its two-by-two block in rows (a, a') is

$$\begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix} \quad \text{if } i \text{ is even}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & r \end{pmatrix} \quad \text{if } i \text{ is odd}.$$

Each block has determinant one. Applying its inverse transpose in odd letters verifies the displayed action in both parities.

Put $\Psi(z) = \nu(z) S_t(\mathrm{swap}(z))$, where the swap exchanges X_s, Y_s, and put $\nu = 1$ off these rows. Direct substitution on the variables gives

$$[S_t(g_r z)]_{t \mapsto t + r_i} - \Psi(z) = 0 \quad (z = X_s),$$

and, for $z = Y_s$, writing $d = [s - i]_{\mathsf{L}}$,

$$[S_t(g_r Y_s)]_{t \mapsto t + r_i} - \Psi(Y_s) = \nu(Y_s)(E_{\mathsf{L}+d}(t, r) X_s - e_s E_d(t, r) Y_s).$$

At $s = i$, the second error is zero since $E_0 = 0$; this formula covers that case as well. For $s \neq i$ it follows by expanding the two displayed images under S_t, using $e_s^2 = 1$. At $s = i$ it is $(t + r_i)^{\mathsf{Q}} - t^{\mathsf{Q}} - r_i$. Off the two rows both sides vanish.

Now use the finite-point weights $\tau = \rho$ and $\lambda = c - A\rho$. The substitution Ψ has level at least $c(z)$ at each variable. At $s = i$ its swapped images have weights $c(X_i)$ and $c(X_i) + \mathsf{Q}\rho = c(Y_i)$. For $s \neq i$, the two terms in $\Psi(X_s)$ have weights $c(Y_s) - \mathsf{P}_s\rho \geq c(X_s)$ and $c(X_s)$, while those in $\Psi(Y_s)$ have weights $c(X_s) + \mathsf{Q}\mathsf{P}_s\rho \geq c(Y_s)$ and $c(Y_s)$.

Every displayed error also has level at least $c(z)$. Since $\rho > 0$ and every error coefficient is divisible by w^2, a term multiplying X_s has level at least $2 + c(X_s) \geq c(Y_s) + 1$. A term multiplying Y_s has level at least $2 + c(Y_s) - A(Y_s)\rho \geq c(Y_s) + 1$, because $A(Y_i)\rho = \delta_i \leq 1$ and $A(Y_s)\rho = \mathsf{P}_s\rho \leq 1/5$ otherwise. Thus the entire substitution $z \mapsto [S_t(g_r z)]_{t \mapsto t + r_i}$ sends $\mathcal{G}_c(b)$ to finite-point level at least b.

Apply this to $g_r^{-1} \cdot f$, which lies in $\mathcal{G}_c(b)$ by hypothesis. Since $g_r(g_r^{-1} \cdot f) = f$, it follows that $[S_t f]_{t \mapsto t + r_i}$ has finite-point level at least b.

The variables X_i, Y_i have equal finite-point weight: $c(Y_i) - \mathsf{Q}\rho = c(X_i)$. Hence the coefficient of t^j in $C_\alpha(t + r_i)$ is divisible by

$$w^{\max(0, \lceil \ell + \rho(A \cdot \alpha - j) \rceil)}.$$

For every $j < A \cdot \alpha$ this is divisible by $w^{\ell+1}$. Therefore $(t - \overline{r}_i)^{A \cdot \alpha}$ divides $\overline{C}_\alpha(t)$ for every $r \in \mathsf{R}$.

Apply the root count and conclude. Here the elementary multiplicity count is valid over the possibly nonreduced ring $\overline{R}$. To recall its proof, choose representatives $r_1, \ldots, r_{\mathsf{Q}}$ of the residue classes of R. Their images under any unital map into a commutative ring have pairwise unit differences. The linear polynomial ideals at these images are therefore comaximal, and

their Eth powers remain comaximal for $E > 0$ by expanding $1 = (u+v)^{2E-1}$ when $u + v = 1$ lies in the two original ideals. Their intersection is their product: for two comaximal ideals and z in their intersection, write $z = zu + zv$; induction gives the finite assertion. If a nonzero polynomial has a root of multiplicity at least E at all these images, the monic product of the Q linear factors to the Eth power divides it, forcing its degree to be at least $\mathsf{Q}E$. For $E = 0$ the bound is immediate. Apply this argument to $\overline{C}_\alpha$, the induced map ι_i, and $E = A \cdot \alpha$. It gives

$$\mathsf{Q}(A \cdot \alpha) \le \deg \overline{C}_\alpha \le \mathsf{B} \cdot \alpha.$$

Since $\mathsf{Q} > 1$, the earlier defect identity yields $\Delta_i(\alpha) \le 0$. This proves the first conclusion.

The root inequality gives the coefficient conclusion. Fix i and write $c = h - d_i/3$. The formula for d_i gives $0 \le d_i(Y_s) - d_i(X_s) \le 3/2$: at $s = i$ the difference is $(\mathsf{Q}+1)/(\mathsf{Q}-1) \le 3/2$ because $\mathsf{Q} \ge 5$; otherwise it is $2\mathsf{P}^{[s-i]_{\mathsf{L}}}/(\mathsf{Q}-1) \le 2\mathsf{P}^{\mathsf{L}-1}/(\mathsf{Q}-1) \le 1/2$. Hence $c(Y_s) - c(X_s) \in [1/2, 1]$.

Consider pairs (α, ℓ) with $\ell \ge 0$, $\ell + c\cdot\alpha < v$, for which some translate $g \cdot f$ has coefficient at α not divisible by $w^{\ell+1}$. There are finitely many such pairs: nonzero coefficients of every translate have total degree at most the total degree of f, and for each of the finitely many possible α the inequality bounds ℓ. If there are none, applying this absence to the identity translate and $\ell = \lceil v - c \cdot \alpha \rceil - 1$ whenever that ceiling is positive proves $f \in \mathcal{G}_c(v)$.

Otherwise choose a pair and translate g_0 minimizing $t = \ell + c\cdot\alpha$. Every translate of $g_0 \cdot f$ lies in $\mathcal{G}_c(t)$. Indeed, failure at an exponent β with positive required power gives $j = \lceil t - c \cdot \beta \rceil - 1 \ge 0$, a coefficient not divisible by w^{j+1}, and $j + c \cdot \beta < t < v$, contradicting minimality. Apply the first conclusion to $g_0 \cdot f$ and the minimizing pair to obtain $\Delta_i(\alpha) \le 0$. Then $\ell + h \cdot \alpha = \ell + c \cdot \alpha + \Delta_i(\alpha)/3 < v$. But the assumed membership of $g_0 \cdot f$ in $\mathcal{G}_h(v)$ makes its coefficient at α divisible by $w^{\ell+1}$, a contradiction. Thus no pair exists, proving the consequence. This proves the second conclusion. □

Proposition 4.2.62. *Let W be a commutative local ring with residue field of cardinality $q = P^N$, where $N > 0$ is even, p is prime, and $P = p^e \ge 5$ for an integer $e \ge 1$. Let K be a field of characteristic p and let unital ring homomorphisms $\iota_s : W \to K$, $s \in \mathbb{Z}/N$, satisfy $\iota_{s+1}(x) = \iota_s(x)^P$. The even and odd letters alternate. On variables $z_{b,s}$, $0 \le b < k$, let $\mathrm{GL}_k(W)$ act by*

$$g \cdot z_{b,s} = \sum_c (g_s)_{cb} z_{c,s}, \qquad g_s = \iota_s(g) \text{ for even } s, \quad g_s = \iota_s(g^{-t}) \text{ for odd } s.$$

Write $\varepsilon_s = 1$ at even letters and -1 at odd letters, and let $[u]_N \in \{0, \dots, N-1\}$ represent $u \in \mathbb{Z}/N$. For $0 \le b < k-1$ and a real exponent vector β, set

$$\Delta_{i;b}(\beta) = \begin{cases} \beta_{b+1,i}, & i \text{ even}, \\ \beta_{b,i}, & i \text{ odd} \end{cases} + \frac{1}{q-1} \sum_{s \in \mathbb{Z}/N} \varepsilon_s P^{[s-i]_N} (\beta_{b+1,s} - \beta_{b,s}).$$

Fix integers a, r with $r = a + 1$ and $0 \le a < r < k$. Let G_r be the block diagonal subgroup with block sizes $r, k-r$, and let $\mathcal{P}_r$ be the subgroup of matrices whose upper-right r by $k-r$ block lies in the maximal ideal of W. For integers n, t and a real number η satisfying

$$n > 0, \qquad t \ge 2(k+1), \qquad \eta \le 1, \qquad 8NktP^{2r} \le \eta n,$$

let S be the space of polynomials homogeneous of degree n in each letter whose translates by every $g \in G_r$ have zero coefficient at each exponent β such that $\Delta_{i;a}(\beta) > 3t$ for some i. Then

$$[\mathrm{GL}_k(W) : \mathcal{P}_r] \dim_K S \le (1+\eta) \binom{n+k-1}{k-1}^N.$$

Proof. **Reduction to the residue field and least-height coefficients.** Iteration of the Frobenius equations shows that every $\iota_s(x)$ satisfies $X^q - X = 0$. Thus the image of ι_s is a finite subring of a field, hence a field, and its kernel is the unique maximal ideal of W. The maps therefore induce embeddings of $\mathbb{F}_q$ into K. All substitutions factor through this residue field.

Give $z_{b,s}$ height $\varepsilon_s b$, and give a monomial the sum of its variable heights. Let I be the subgroup that is lower triangular modulo the maximal ideal. For any nonzero polynomial f, the union of the supports of $g \cdot f$, $g \in G_r$, is finite, because total degree is bounded by that of f. Choose a nonzero coefficient of least height H among all these translates, at an exponent α. Write its translate as $g = yx$, with $y \in I \cap G_r$ and x a chosen representative for the right cosets $(I \cap G_r)\backslash G_r$. In an even letter, y sends each variable to a nonzero diagonal multiple of itself plus variables of larger row index; in an odd letter its inverse transpose sends it to a nonzero diagonal multiple plus variables of smaller row index. Every off-diagonal contribution strictly raises height. Since $x \cdot f$ has no term of height below H, its coefficient at α is multiplied by a nonzero scalar under y, without other contributions. Hence that coefficient in $x \cdot f$ is nonzero.

For each simple root inside one of the two blocks, all its $\mathrm{SL}_2(W)$ translates of $x \cdot f$ still have no terms below height H. Apply Lemma 4.2.61 with coefficient ring K, $w = 0$, $\ell = 0$, the height weights, and this coefficient. The weight difference from the shallow root variable to the deep root variable equals one in every letter. The root filtration hypothesis is exactly the absence of terms below H, and 0 does not divide the chosen nonzero coefficient. It follows that

$$\Delta_{i;b}(\alpha) \le 0 \quad \text{for every } i \text{ and every } b \ne a.$$

This proves coefficient detection at one of the finitely many flag representatives, with all interior simple-root defects nonpositive.

Compute the volume of the cone of nonpositive defects. Let

$$C = \{\beta : \Delta_{i;b}(\beta) \le 0 \text{ for all } i, b\}, \qquad B(D) = \{\beta \ge 0 : D_s - 1 \le \sum_b \beta_{b,s} \le D_s \text{ for all } s\}.$$

We prove, for any real D_s, that

$$F_k(q)\,\mathrm{vol}(C \cap B(D)) = \mathrm{vol}(B(D)), \qquad F_k(q) = \prod_{j=1}^{k} \frac{q^j - 1}{q - 1}.$$

All volumes are ordinary Lebesgue volumes in $\mathbb{R}^{kN}$.

For a family of permutations σ_s of the k rows, set

$$(L_\sigma \nu)_{b,s} = \varepsilon_s\big(P\nu_{b,s+1} - \nu_{\sigma_s(b),s}\big).$$

Every L_σ is invertible. Indeed, its kernel would satisfy $P\nu_{b,s+1} = \nu_{\sigma_s(b),s}$, which forces $P\|\nu\|_\infty = \|\nu\|_\infty$ and hence $\nu = 0$. More precisely,

$$|\det L_\sigma| = P^{kN} \det(1 - q^{-1}\pi_\sigma),$$

where π_σ is the product of the row permutations around the letters, acting as a permutation matrix on $\mathbb{R}^k$. To check the formula, factor out the signed shift of determinant of absolute value P^{kN}. The remaining map is $1 - P^{-1}T$ for a permutation T whose cycles have length $N\ell$ when the corresponding cycle of π_σ has length ℓ. Each such cycle contributes $1 - P^{-N\ell}$, which is exactly the contribution to $\det(1 - q^{-1}\pi_\sigma)$. In particular these determinants are positive.

Expansion and telescoping around the letters give

$$\Delta_{i;b}(L_1\nu) = (L_1\nu)_{Y_i,i} - (\nu_{b,i} - \nu_{b+1,i}), \qquad Y_i = \begin{cases} b+1, & i \text{ even}, \\ b, & i \text{ odd}. \end{cases}$$

For example, writing $u_s = \nu_{b+1,s} - \nu_{b,s}$, the sum in the defect formula becomes $\sum_s P^{[s-i]_N}(Pu_{s+1} - u_s) = (q-1)u_i$. Thus $L_1\nu \in C$ is precisely the set of inequalities

$$P\nu_{b+1,s+1} \le \nu_{b,s} \quad (s \text{ even}), \qquad \nu_{b+1,s} \le P\nu_{b,s+1} \quad (s \text{ odd}).$$

Here is the signed count that computes the volume. At a fixed ν, put

$$M_s(c,b) = \mathbf{1}_{\{\varepsilon_s(P\nu_{b,s+1} - \nu_{c,s}) > 0\}}.$$

Expanding the product of their determinants gives

$$\prod_s \det M_s = \sum_\sigma \operatorname{sgn}(\pi_\sigma) \mathbf{1}_{\{L_\sigma \nu > 0\}}.$$

Sort the k entries of ν decreasingly at each letter, using permutations τ_s. The signs from these row and column rearrangements cancel in the product, since each τ_s occurs twice. For decreasing real lists $x_0, \dots, x_{k-1}$ and $y_0, \dots, y_{k-1}$, the determinant of $(\mathbf{1}_{\{x_b > y_c\}})_{c,b}$ equals one exactly when $x_0 > y_0 \ge x_1 > y_1 \ge \cdots \ge x_{k-1} > y_{k-1}$, and otherwise zero. This follows by subtracting successive columns: their indicator supports are disjoint consecutive blocks, and the determinant is nonzero exactly when each block has one entry. Likewise, the determinant of $(\mathbf{1}_{\{y_c > x_b\}})_{c,b}$ equals one exactly when $y_0 > x_0 \ge y_1 > x_1 \ge \cdots \ge y_{k-1} > x_{k-1}$, and is zero otherwise; subtract successive rows to obtain the corresponding disjoint blocks. Applying these two assertions at the even and odd letters gives the indicator that the sorted ν satisfies the displayed interlacing inequalities and $L_1\nu > 0$.

Conversely, those inequalities together with strict positivity force strict decrease in each letter: a selected positive coordinate of $L_1\nu$ is bounded above by $\nu_{b,s} - \nu_{b+1,s}$. Thus only the decreasing arrangement can contribute. Outside the finitely many equality hyperplanes we have proved

$$\sum_\tau \mathbf{1}_{\{L_1(\nu\circ\tau) \in C,\ L_1(\nu\circ\tau) > 0\}} = \sum_\sigma \operatorname{sgn}(\pi_\sigma) \mathbf{1}_{\{L_\sigma \nu > 0\}},$$

where $\nu \circ \tau$ permutes each letter separately. The sums of coordinates in any letter of $L_\sigma\nu$ are independent of σ and of these rearrangements. Multiply by the common indicator of their membership in the intervals $[D_s - 1, D_s]$, and integrate. All sets being integrated are bounded, since they are inverse images of the bounded slab under invertible linear maps. Equality hyperplanes are null. Change of variables therefore gives

$$\frac{(k!)^N}{|\det L_1|} \operatorname{vol}(C \cap B(D)) = \operatorname{vol}(B(D)) \sum_\sigma \frac{\operatorname{sgn}(\pi_\sigma)}{|\det L_\sigma|}.$$

Each permutation π occurs $(k!)^{N-1}$ times as the product π_σ. Hence the share of the cone in every slab is the number

$$c(k,q) = \frac{\det(1 - q^{-1})}{k!} \sum_{\pi \in \mathfrak{S}_k} \frac{\operatorname{sgn}(\pi)}{\det(1 - q^{-1}\pi)}.$$

Here $\det(1 - q^{-1})$ means $(1 - q^{-1})^k$.

This calculation uses only $P > 1$, and allows any choices of signs. Evaluate the same expression for one letter, positive sign, and $P = q$. In that case $L_1 = (q-1)\,\mathrm{id}$ and the cone in the β coordinates is $q\beta_{b+1} \le \beta_b$. Put $\beta_k = 0$ and $d_b = 1 + q + \cdots + q^b$. The triangular linear map

$$\beta_b \longmapsto d_b(\beta_b - q\beta_{b+1})$$

maps this cone intersected with the nonnegative orthant bijectively to the nonnegative orthant: solve backwards from the last coordinate to see positivity of its inverse. It preserves $\sum_b \beta_b$, since $d_0 = 1$ and $d_b - q d_{b-1} = 1$, and its determinant is $\prod_b d_b = F_k(q)$. On the slab $0 \le \sum_b \beta_b \le 1$, which has positive volume, it therefore computes $c(k,q) = F_k(q)^{-1}$. This proves the required volume identity.

Amplify the coefficient space by Frobenius powers. Fix $m = q^j$. For each vector of residues $u_{b,s} \in \{0, \dots, m-1\}$, choose

$$\gamma(u)_{b,s} = u_{b,s} + m\,\mathbf{1}_{\{b=a\}} \left(k - \left\lfloor \frac{\sum_c u_{c,s}}{m} \right\rfloor \right).$$

It is nonnegative, is congruent to u modulo m, and has degree in $[mk, mk+m]$ in each letter. Pack a family $(f_u)_u \in S^{m^{kN}}$ into

$$\mathcal{F} = \sum_u f_u(z^m) z^{\gamma(u)}.$$

This is an injective linear map: exponents in different residue classes do not meet, and within one class the substitution multiplies all exponents by m and adds the fixed vector $\gamma(u)$.

Every substitution commutes with $z \mapsto z^m$, because the entries of its matrices lie in $\mathbb{F}_q$ and m is a power of q. If a coefficient at α in $g \cdot \mathcal{F}$, $g \in G_r$, is nonzero, some summand contributing to it gives

$$\alpha = m\delta + \xi,$$

where δ occurs in $g \cdot f_u$ and ξ in $g \cdot z^{\gamma(u)}$. The latter has degree at most $m(k+1)$ in each letter. For any nonnegative vector of degree at most D separately in each letter, the absolute value of its defect is at most $2D$. Indeed, at letter i the maximum absolute coefficient is $q/(q-1)$, and at letters at distance $d = 1, \dots, N-1$ it is $P^d/(q-1)$; their sum is at most two since

$$\frac{q + \sum_{d=1}^{N-1} P^d}{q-1} \le 2 \qquad (P \ge 2).$$

The defining condition on S consequently gives

$$\Delta_{i;a}(\alpha) \le m(3t + 2(k+1)) \le 4mt.$$

The degree of α in each letter lies in $[m(n+k), m(n+k+1)]$.

Shift the detected exponents into the cone. Put $\mu_b = P^{2(a-b)}$ for $b \le a$ and $\mu_b = 0$ for $b > a$. Set $\nu_{b,s} = \mu_b$ at even letters and $P\mu_b$ at odd letters, and put $\sigma = L_1 \nu$. Explicitly,

$$\sigma_{b,s} = \begin{cases} (P^2 - 1)\mu_b, & s \text{ even}, \\ 0, & s \text{ odd}. \end{cases}$$

This is a nonnegative integral vector. Since $\mu_b = P^2 \mu_{b+1} + \mathbf{1}_{\{b=a\}}$, the telescoping defect formula above gives

$$\Delta_{i;b}(\sigma) = -\mathbf{1}_{\{b=a\}} \quad (i \text{ even}), \qquad \Delta_{i;b}(\sigma) = P(\mu_{b+1} - \mu_b) \le -\mathbf{1}_{\{b=a\}} \quad (i \text{ odd}).$$

Moreover $\sum_b \sigma_{b,s}$ equals $P^{2r} - 1$ at even letters and zero at odd letters.

For a nonzero packed polynomial apply least-height detection at a flag representative x. Its detected exponent α has every interior defect nonpositive, and the preceding bound applies to its boundary defect. Therefore

$$y = 4t\sigma + \alpha/m \in C \cap B(D), \qquad D_s = n + k + 1 + 4t \sum_b \sigma_{b,s}.$$

It also belongs to $m^{-1}\mathbb{Z}^{kN}$. The degrees satisfy

$$1 \le D_s \le n + 4tP^{2r},$$

because $k + 1 \le 4t$.

Consider the linear map that lists, for every flag representative x and every lattice point y of this last set, the coefficient at $m(y - 4t\sigma)$ of $x \cdot \mathcal{F}$; set an entry to zero when its exponent is not nonnegative. The coefficient detection proves this map injective. Hence

$$m^{kN} \dim_K S \le R\,\#(C \cap B(D) \cap m^{-1}\mathbb{Z}^{kN}), \qquad R = [G_r : I \cap G_r].$$

The set $C \cap B(D)$ is bounded and convex. Its boundary has Lebesgue measure zero: a convex set with empty interior lies in a proper affine subspace, and otherwise its boundary is null. Lattice-point Riemann sums along $m = q^j \to \infty$ therefore give

$$\dim_K S \le R\ \mathrm{vol}(C \cap B(D)).$$

Cancel the flags and estimate the remaining slab. Reduction $\mathrm{GL}_k(W) \to \mathrm{GL}_k(\mathbb{F}_q)$ is surjective: lift a residue matrix and observe that its determinant is a unit. The block Levi reduces surjectively as well. Counting lower triangular matrices and invertible matrices over $\mathbb{F}_q$ gives $F_k(q)$ complete flags; counting the parabolic with its arbitrary lower-left block gives

$$[\mathrm{GL}_k(W) : \mathcal{P}_r]\, R = F_k(q).$$

The cone-volume identity thus yields

$$[\mathrm{GL}_k(W) : \mathcal{P}_r] \dim_K S \le \mathrm{vol}(B(D)).$$

Put $E = 4tP^{2r}$. For $D_s \ge 1$, the volume in one letter is $(D_s^k - (D_s - 1)^k)/k! \le D_s^{k-1}/(k-1)!$, by integrating $x^{k-1}/(k-1)!$ over $[D_s - 1, D_s]$. Consequently

$$\mathrm{vol}(B(D)) \le \left(\frac{(n+E)^{k-1}}{(k-1)!}\right)^N \le \exp\left(\frac{N(k-1)E}{n}\right)\left(\frac{n^{k-1}}{(k-1)!}\right)^N.$$

The size assumption implies $2NkE \le \eta n$, so for $x = N(k-1)E/n$ one has $0 \le 2x \le \eta \le 1$. The geometric series bound $e^x \le (1-x)^{-1} \le 1 + 2x$ for $0 \le x \le 1/2$ gives $e^x \le 1 + \eta$. Finally,

$$\frac{n^{k-1}}{(k-1)!} \le \prod_{j=1}^{k-1} \frac{n+j}{j} = \binom{n+k-1}{k-1}.$$

Combining these inequalities proves the proposition. □

Lemma 4.2.63. *Let W be a commutative local ring whose residue field has $q = P^N$ elements, where $N > 0$ is even, p is prime, and $P = p^e > 1$ for an integer $e \ge 1$. Let K be a field of characteristic p and let unital ring homomorphisms $\iota_s : W \to K$, indexed by $s \in \mathbb{Z}/N$, satisfy*

$$\iota_{s+1}(x) = \iota_s(x)^P \qquad (x \in W).$$

The even and odd letters alternate. On the polynomial variables $z_{b,s}$, $0 \le b < k$, let $g \in \mathrm{GL}_k(W)$ act by

$$g \cdot z_{b,s} = \sum_c (g_s)_{cb} z_{c,s}, \qquad g_s = \iota_s(g) \text{ for even } s, \quad g_s = \iota_s(g^{-t}) \text{ for odd } s.$$

Fix integers $n \geq 0$ and $r > 0$ with $2r \leq k$. Put

$$\epsilon_{b,s} = \begin{cases} 1, & s \text{ even and } b \geq r, \text{ or } s \text{ odd and } b < r, \\ 0, & \text{otherwise}, \end{cases} \qquad d(\beta) = \sum_{b,s} \epsilon_{b,s}\beta_{b,s}.$$

For each integer $j \geq 0$, let V_j be the subspace of polynomials homogeneous of degree n in each letter for which every translate by $\mathrm{GL}_k(W)$ has zero coefficient at every monomial z^β with $d(\beta) < j$. Then, for every integer $M \geq 0$,

$$\sum_{j=1}^{M} \dim_K V_j \leq Nn \left(P^{-2r} + \frac{r^2}{kP}\right) \binom{n+k-1}{k-1}^N.$$

Proof. **A multiplicity count for weighted monomials.** We first prove the elementary counting fact needed below. Let L be a field, let there be finitely many variables Z_v, and assign a positive integer κ_v to each variable, so that the cost of Z^α is $\sum_v \alpha_v/\kappa_v$. Suppose a finite set $\mathcal{Z}$ of distinct L-points has at least $\prod_v \kappa_v$ elements and, for each v, there is a polynomial g_v all of whose monomials have cost less than one such that $g_v(z) = z_v^{\kappa_v}$ at every $z \in \mathcal{Z}$. If a nonzero polynomial vanishes to order at least j at every point, it has a monomial of cost at least j.

Here vanishing to order j means membership in the jth power of the maximal ideal of the point. To prove the fact, put $\psi_v = Z_v^{\kappa_v} - g_v$. Repeatedly use $Z_v^{\kappa_v} = \psi_v + g_v$ to express every polynomial as a sum of

$$Z^\alpha \psi^\gamma, \qquad 0 \leq \alpha_v < \kappa_v.$$

The reduction terminates because each g_v lowers the cost of the remaining Z-factor strictly, and all costs are nonnegative multiples of the reciprocal of a common integer. It never increases $|\gamma| + \mathrm{cost}(\alpha)$. Modulo the kernels of all order-j jet evaluations, terms with $|\gamma| \geq j$ vanish. The remaining products number

$$\left(\prod_v \kappa_v\right) \binom{j + \#\{v\} - 1}{\#\{v\}},$$

for $j > 0$. The jet target has dimension $|\mathcal{Z}|\binom{j+\#\{v\}-1}{\#\{v\}}$. Its evaluation map is surjective by the Chinese remainder theorem, since the distinct point ideals, and hence their jth powers, are pairwise comaximal. The displayed products span the target, so the assumed inequality on $|\mathcal{Z}|$ forces equality of the dimensions and linear independence of their jets. A polynomial of cost strictly less than j reduces entirely into their span. If all its jets vanish, linear independence makes that polynomial zero. This is the claimed contradiction. For $j = 0$ any nonzero monomial suffices.

Construct the points from square-zero matrices. Iteration of the Frobenius equations gives $\iota_s(x)^q = \iota_s(x)$. Thus the image of ι_s is a finite subring of K, hence a field. Its kernel is the unique maximal ideal of W, so ι_s induces an embedding of the residue field $\mathbb{F}_q$ into K. We use the same notation for it.

Write $k = 2r+m$. Choose algebraically independent elements $t_{b,s}$ over K and let $L = K(t_{b,s})$. Let ϕ substitute $t_{b,s}$ for the shallow variables, meaning those with $\epsilon_{b,s} = 0$, and leave the deep variables as independent variables $Z_{b,s}$ over L. A nonzero polynomial over K stays nonzero under ϕ: regard it first as a polynomial in its deep variables, with polynomial coefficients in the shallow variables.

Let

$$\mathcal{T} = \{(a, a+r) : 0 \leq a < r\} \cup \{(a, c) : 0 \leq a < r,\ 2r \leq c < k\}.$$

This set has $r(m+1)$ elements. For $y \in \mathbb{F}_q^{\mathcal{T}}$ form a matrix $A(y)$ over W by choosing lifts of its entries at $\mathcal{T}$, with all other entries zero. Every supported row is below r and every supported column is at least r, so $A(y)^2 = 0$ and $g(y) = 1 + A(y)$ is invertible with inverse $1 - A(y)$.

Define a point $\zeta(y)$ in the variables $Z_{b,s}$, adding shallow coordinates equal to zero, by

$$\zeta_{b,s}(y) = \begin{cases} \sum_{(a,b)\in\mathcal{T}} t_{a,s}\iota_s(y_{a,b}), & s \text{ even},\ b \geq r, \\ -\sum_{(b,c)\in\mathcal{T}} t_{c,s}\iota_s(y_{b,c}), & s \text{ odd},\ b < r, \\ 0, & \epsilon_{b,s} = 0. \end{cases}$$

The matrix action gives, for each deep variable,

$$\phi(g(y)^{-1} \cdot z_{b,s}) = Z_{b,s} - \zeta_{b,s}(y).$$

It fixes every shallow variable. Consequently, if $f \in V_j$, then $\phi(f)$ vanishes to order j at every $\zeta(y)$: write $f = g(y)^{-1} \cdot (g(y) \cdot f)$ and use that each monomial of $g(y) \cdot f$ has at least j deep factors.

Recover the residue parameters with polynomials of cost less than one. Give the variable indexed by (b, s) the positive integer

$$\kappa_{b,s} = \begin{cases} P^{2r}, & \epsilon_{b,s} = 1,\ b \geq 2r, \\ P, & \epsilon_{b,s} = 1,\ b < 2r, \\ 1, & \epsilon_{b,s} = 0. \end{cases}$$

We claim that for every letter i and every $(a, c) \in \mathcal{T}$, the function $\iota_i(y_{a,c})$ on these points is represented by a polynomial in Z all of whose monomials have cost less than one.

First take $c \geq 2r$. For each $u = 0, \ldots, r-1$ choose j_u from $\{2u, 2u+1\}$ so that $i - j_u$ is even. These integers are distinct and less than $2r$. In characteristic p,

$$\zeta_{c,i-j_u}(y)^{P^{j_u}} = \sum_{a<r} t_{a,i-j_u}^{P^{j_u}} \iota_i(y_{a,c}).$$

The coefficient matrix in u, a is invertible over L. Indeed, the monomials in its determinant are distinct: a determinant term assigns exactly one exponent P^{j_u}, with its letter $i - j_u$, to the variables of each row index a; distinct assignments differ for some a, since the exponents P^{j_u} are distinct. Thus its determinant is a nonzero polynomial in the algebraically independent $t_{a,s}$. Solving this system expresses each $\iota_i(y_{a,c})$ as an L-linear combination of $Z_{c,i-j_u}^{P^{j_u}}$. Their costs are $P^{j_u}/P^{2r} < 1$.

For $c = a + r$ and even i, one has $\zeta_{a+r,i} = t_{a,i}\iota_i(y_{a,a+r})$. For odd i,

$$\zeta_{a,i} = -t_{a+r,i}\iota_i(y_{a,a+r}) - \sum_{c\geq 2r} t_{c,i}\iota_i(y_{a,c}).$$

The tail parameters have already been recovered. Division by the nonzero indicated t therefore proves the claim also for the paired parameter. The extra variable in each equation has cost $1/P < 1$, and sums and scalar multiples preserve the strict cost bound.

This recovery proves that $y \mapsto \zeta(y)$ is injective, since the embedding ι_i of $\mathbb{F}_q$ is injective. It also supplies the interpolation equations in the multiplicity count. For a deep variable $v = (b, s)$ write $\kappa_v = P^\ell$. Raising its defining linear expression to κ_v gives an L-linear combination of the

functions $\iota_{s+\ell}(y_{a,c})$, each just recovered with cost less than one. For a shallow variable take the zero polynomial. Thus each $\zeta_v^{\kappa_v}$ has the required interpolating polynomial.

Each row is deep at exactly $N/2$ letters. Hence

$$\prod_{b,s} \kappa_{b,s} = P^{(N/2)(2r+2rm)} = q^{r(m+1)} = |\mathbb{F}_q^{\mathcal{T}}|.$$

The point count and all interpolation hypotheses are now established.

A nonzero deep form has a sufficiently costly monomial. Apply the multiplicity count to $\phi(f) \neq 0$, for $0 \neq f \in V_j$. It gives a monomial in its deep variables of cost at least j. A nonzero coefficient of that monomial is a nonzero polynomial expression in the shallow t-variables; at least one original coefficient of f with those deep exponents is nonzero. Thus f has a nonzero coefficient at an exponent β for which

$$j \leq C(\beta) := \sum_{\substack{b<2r \\ s\in\mathbb{Z}/N}} \frac{\epsilon_{b,s}\beta_{b,s}}{P} + \sum_{\substack{2r\leq b<k \\ s\in\mathbb{Z}/N}} \frac{\epsilon_{b,s}\beta_{b,s}}{P^{2r}}.$$

Sum the dimension bounds. Let $\mathcal{E}_n$ be the exponents of degree n in each letter, of cardinality $d_n = \binom{n+k-1}{k-1}^N$. Restriction of coefficients from V_j to $\{\beta : C(\beta) \geq j\}$ is injective by the preceding paragraph. Therefore

$$\sum_{j=1}^{M} \dim_K V_j \leq \sum_{\beta\in\mathcal{E}_n} \#\{j : 1 \leq j \leq M,\ j \leq C(\beta)\} \leq \sum_{\beta\in\mathcal{E}_n} C(\beta).$$

By permutation symmetry among the k rows at one letter, $\sum_{\beta\in\mathcal{E}_n} \beta_{b,s} = d_n n/k$. The sum of the variable costs equals

$$\frac{N}{2}\left(\frac{2r}{P} + \frac{m}{P^{2r}}\right) \leq N\left(\frac{r^2}{P} + \frac{k}{P^{2r}}\right),$$

using $r \geq 1$ and $m \leq k$. Multiplying by $d_n n/k$ gives the asserted bound. This also covers $n = 0$ and $M = 0$. □

Lemma 4.2.64. *Let V be a space with a point with signs s on n and R a finite set of finite places of E. Then the set of products $\gamma(1,k)$, with $\gamma \in \Gamma_V$ and $k \in G_f(n)$ whose components at the places of $R \cup \bar{R}$ are 1, is dense in $\mathrm{G}(s)$.*

Proof. Write $J = J_V$. Through $g \mapsto x_V g x_V^{-1}$, which is a homeomorphism respecting components at every place, it suffices to show: the products $\gamma k'$ with $\gamma \in \mathrm{U}(V)(F)$ and $k' \in \mathrm{U}(V)(\mathbb{A}_F)$ with components 1 at the infinite places and at the places above R are dense in $\mathrm{U}(V)(\mathbb{A}_F)$. For a place v of F let $E_v = E \otimes_F F_v$ and $\mathrm{U}(V)(F_v) = \{g \in \mathrm{GL}_n(E_v) : g^* J g = J\}$. Let T be the finite set of places of F consisting of the infinite places and those below R.

Step 1: reduction to rational approximation at T. Let $g \in \mathrm{U}(V)(\mathbb{A}_F)$ and let N be a neighbourhood of g. By the definition of the restricted product topology, N contains the set of h with $h_v \in N_v$ for $v \in T$ and $h_v = g_v$ for $v \notin T$, for some neighbourhoods N_v of g_v in $\mathrm{U}(V)(F_v)$. If $\gamma \in \mathrm{U}(V)(F)$ has $\gamma_v \in N_v$ for all $v \in T$, let k' have components $\gamma^{-1} g_v$ at $v \notin T$ and 1 at $v \in T$; it lies in $\mathrm{U}(V)(\mathbb{A}_F)$, since $\gamma^{-1} g$ does and only finitely many components were changed, and $\gamma k' \in N$. So it suffices to find such a γ.

Step 2: the Cayley map. Let $\mathfrak{u} = \{X \in \mathrm{M}_n(E) : X^* J + J X = 0\}$, an F-vector space, and $\mathfrak{u}_v = \mathfrak{u} \otimes_F F_v$, the corresponding space of matrices over E_v. For a matrix X over E or E_v with $1 + X$ invertible put $c(X) = (1 - X)(1 + X)^{-1}$. If $X \in \mathfrak{u}_v$ and $1 + X$ is invertible, then

$c(X) \in \mathrm{U}(V)(F_v)$, $1 + c(X) = 2(1+X)^{-1}$ is invertible, and $c(c(X)) = X$; if $g \in \mathrm{U}(V)(F_v)$ and $1+g$ is invertible, then $c(g) \in \mathfrak{u}_v$. The map c is continuous on the open set where $1+X$ is invertible.

Step 3: a rational scalar in general position. The elements $\lambda_m = (m+\jmath)/(m-\jmath)$, $m \in \mathbb{Z}$, of E satisfy $\lambda_m \bar{\lambda}_m = 1$ (as $\bar{\jmath} = -\jmath$) and are pairwise distinct (from $\lambda_m = \lambda_{m'}$ one gets $2(m'-m)\jmath = 0$). For $v \in T$ and each factor field of E_v, $\det(\lambda + g_v) = 0$ for at most n values of λ; so there is $\lambda = \lambda_m$ with $\lambda + g_v$ invertible for every $v \in T$. Scalar matrices of modulus one are in $\mathrm{U}(V)(F)$; put $g'_v = \lambda^{-1} g_v \in \mathrm{U}(V)(F_v)$. Then $1 + g'_v = \lambda^{-1}(\lambda + g_v)$ is invertible, and $X_v = c(g'_v) \in \mathfrak{u}_v$ satisfies $c(X_v) = g'_v$.

Step 4: approximation. Choose an F-basis $B_1, \dots, B_d$ of $\mathfrak{u}$ and write $X_v = \sum_i x_{v,i} B_i$ with $x_{v,i} \in F_v$. By weak approximation in the number field F (the diagonal image of F is dense in $\prod_{v \in T} F_v$), there are $x_i \in F$ with x_i arbitrarily close to $x_{v,i}$ for all $v \in T$; put $X = \sum_i x_i B_i \in \mathfrak{u}$. As invertibility is an open condition and c is continuous at X_v, for x_i close enough $1 + X$ is invertible over E_v for every $v \in T$ (so $\det(1+X) \in E$ is nonzero and $c(X) \in \mathrm{U}(V)(F)$), and $c(X)$ lies in $\lambda^{-1} N_v$ at every $v \in T$, since $\lambda^{-1} N_v$ is a neighbourhood of $g'_v = c(X_v)$. Then $\gamma = \lambda\, c(X) \in \mathrm{U}(V)(F)$ has $\gamma_v \in N_v$ for all $v \in T$, and Step 1 concludes. □

Proposition 4.2.65. *Fix a prime identification $\iota \mid N$. Let Σ be the conjugates of the primes induced by the chosen embeddings $F_v : E \hookrightarrow \mathbb{C}$ followed by ι, for $v \in \Phi$. Write $G = \mathrm{U}(X_0)(\mathbb{A}_f)$. For $(g_w)_{w \in \Sigma} \in \prod_{w \in \Sigma} \mathrm{GL}_k(E_w)$ let $\jmath_\Sigma(g) \in G$ have component g_w at w, inverse conjugate transpose at $\bar{w}$, and identity elsewhere. Let $D_\Sigma = \jmath_\Sigma((\mathrm{diag}(1, \dots, 1, N))_{w \in \Sigma})$. Let δ have this same diagonal component at $w \in \Sigma$ dividing π_N and identity component at the other $w \in \Sigma$, with the unitary conjugate components as above. Put $\delta^\vee = \delta D_\Sigma^{-1}$.*

Let $R = \prod_{w \in \Sigma} \mathbb{F}_N$. For every $y \in R^k$ with last coordinate $y_k \in R^\times$, choose an integral invertible matrix $K_y \in \prod_{w \in \Sigma} \mathrm{GL}_k(\mathcal{O}_{E_w})$ whose reduction has both last row and last column y. Fix any function $\psi : R^\times \to \mathbb{C}$ with $\psi(u) \in \mathcal{O}_\iota$ for every u. Define on algebraic forms, with right translation $R_g \varphi(h) = \varphi(hg)$,

$$U = \sum_{\substack{y \in R^k \\ y_k \in R^\times}} \psi(y_k)\, R_{\delta^\vee \jmath_\Sigma(K_y) \delta^{-1}}.$$

For $d_0 \in \{1, \delta, \delta^\vee\}$ put

$$M_{d_0} = \{\varphi \in \mathcal{S} : \mathrm{gauge}_\iota(g d_0, \varphi(g)) \le 1 \text{ for every } g \in G\}, \qquad T_\iota = \{\varphi \in \mathcal{S} : U^{m/2} \varphi \in M_\delta\}.$$

Then U preserves $\mathcal{S}$, so T_ι is an $\mathcal{O}_\iota$-submodule of $\mathcal{S}$. There is an integer $e \ge 0$, independent of finite families, with

$$e \le mfb \dim_{\mathbb{C}} \mathcal{S},$$

such that for every finite set J and every family $(\varphi_j)_{j \in J}$ in M_1 there exist $A \in M_J(\mathcal{O}_\iota)$ and $c \in \mathcal{O}_\iota$ satisfying

$$\det(A) c = N^e, \qquad \sum_{j \in J} A_{ij} \varphi_j \in T_\iota \quad (i \in J).$$

No inclusion $T_\iota \subseteq M_1$ is asserted.

Proof. We first justify the local matrix description. By Lemma 4.2.66, $q_1 d \mid q_2$ and $E \subseteq \mathbb{Q}(\zeta_{q_1 d})$. Since $N \equiv 1 \pmod{q_2^2}$, the prime N splits completely in E. Thus its completions are $\mathbb{Q}_N$ and its residue fields are canonically $\mathbb{F}_N$. The embeddings over K induce one prime from each conjugate pair, so Σ has f elements and is disjoint from its conjugate. Since $\pi_N \bar{\pi}_N = N$ and N is unramified, exactly one prime of each pair divides π_N. Also N differs from the chosen

principal-level primes: using $\mathrm{rd}_E \geq 1$, $M \geq b^{20}$, $\log(2\pi) \leq \log b$, $m \geq 2$ and $\kappa \geq 1$, the defining interval for $\log N$ gives

$$\log N \geq 19(m-1)\log b - \tfrac{15}{2}\log b \geq \tfrac{23}{2}\log b > 11\log b.$$

Here the scale and root-discriminant bounds follow from Lemma 4.3.5. Hence $N > b^{11} \geq p_0, p_1$.

The matrices K_y exist. Over R, write $u = y_k \in R^\times$. The product of the identity with last column replaced by y and the identity with last row replaced by $u^{-1}y$ has last row and last column y and determinant u. Lift its entries at each w to $\mathcal{O}_{E_w}$; the determinant remains a unit. Any such choice is permitted.

We prove that U preserves the source level. In the basis changed by δ, the part of K_0 above N is the group of integral matrices whose last column reduces to the last basis vector. Indeed at a prime dividing π_N, conjugation by $\mathrm{diag}(1,\ldots,1,N)$ changes the original last-row congruence into this last-column congruence: lower-left entries are divided by N and upper-right entries multiplied by N. At its conjugate prime the same assertion follows from inverse conjugate transpose. At a member of Σ not dividing π_N, it follows directly from the last-row congruence at its conjugate.

Write $D = \mathrm{diag}(1,\ldots,1,N)$ at all members of Σ. For an integral matrix g with last column the last basis vector modulo N, the matrix DgD^{-1} is integral and has last row the last basis row modulo N. This follows by examining its four blocks of sizes $k-1$ and 1. Its reduction sends y to a vector with the SAME last coordinate. Moreover

$$K^{-1}_{\overline{DgD^{-1}}\,y}(DgD^{-1})K_y$$

is integral invertible and has last column the last basis vector modulo N, since the two factors compared have the same last column. It follows that left multiplication by g permutes, modulo right multiplication by the source-level subgroup, the cosets $D^{-1}K_y$, by $y \mapsto \overline{DgD^{-1}}\,y$. This permutation preserves y_k, and hence preserves the coefficients $\psi(y_k)$. Conjugating back by δ proves invariance of $U\varphi$ under the part of K_0 at N whenever φ is a source form. Away from N all translating points are the identity, so those level actions commute with U. Right translation preserves rational equivariance. Thus U preserves $\mathcal{S}$.

Put $\mathcal{O} = \mathcal{O}_\iota$. We shall compare the modules $M_1, M_{\delta^\vee}, M_\delta$. Polynomiality of the source weight, with determinant exponent zero, gives

$$M_1 \subseteq M_{\delta^\vee}.$$

In fact the inverse of $\delta^\vee$ is the diagonal with last entry N at the complementary members of Σ and identity at the others. Its Weyl matrix is integral. Applying it after the transport at g preserves coordinate integrality.

We next prove a determinant bound from $M_{\delta^\vee}$ into M_δ. The required class set is finite by a direct application of Lemma 4.6.4. Indeed X_0 is the standard definite space of rank k, and Lemma 4.3.5 gives $k \geq 2$. The formula $n_m = k + m(t+1)$ gives $1 \leq k \leq n_m \leq 2n_m$, so that lemma applies in rank k. Let $\mathrm{K}(\pi_M\pi_N)$ be the principal congruence subgroup of this level in the integral lattice stabiliser. For κ in this subgroup, all entries of $\kappa - 1$ belong to $\pi_M\pi_N\widehat{\mathcal{O}}_E$. At a chosen place this ideal is contained in the maximal ideal because π_M belongs to that place; hence κ reduces to 1 at every chosen place. Its last row is also the last basis row modulo π_N, since $\pi_M\pi_N\widehat{\mathcal{O}}_E \subseteq \pi_N\widehat{\mathcal{O}}_E$. Thus

$$\mathrm{K}(\pi_M\pi_N) \subseteq K_0.$$

All hypotheses of Lemma 4.6.4 are met, and consequently

$$\#\big(\mathrm{U}(X_0)(E^+)\backslash G/K_0\big) \leq \exp(x^{30}) < \infty.$$

This finiteness argument does not use the dimension of the source-form space.

We also verify that values on these classes are independent. The group K_0 is an open subgroup of the standard lattice stabiliser: it is defined by finitely many congruences at the places dividing $\pi_M\pi_N$. There is a chosen place: π_M is integral and $\mathrm{N}_{E/\mathbb{Q}}(\pi_M) = M^f > 1$, since $\pi_M\bar{\pi}_M = M = p_0p_1$. At any such place K_0 acts trivially on the residue lattice. Hence Lemma 4.6.2(c), applied to the definite space X_0 and its standard lattice, gives

$$\mathrm{U}(X_0)(E^+) \cap gK_0g^{-1} = \{1\} \quad (g \in G).$$

Given a vector ξ_c in the source Weyl module for each class representative r_c, define $\varphi(\gamma r_c\kappa) = \gamma \cdot \xi_c$. Two descriptions of a point have the same class, and their rational multipliers differ by the displayed trivial stabiliser. This definition is therefore well-defined and satisfies exactly the rational equivariance and right K_0-invariance required of a source form. Conversely those two laws determine a form from its values at the representatives. Evaluation is a linear isomorphism from $\mathcal{S}$ onto one copy of the Weyl module for each class.

Choose representatives r_c adapted so that both

$$r_c\delta^\vee \quad \text{and} \quad D_\Sigma^{-1}r_c\delta$$

are integral invertible above N. Apply Lemma 4.2.64 to the actual source space X_0, with the finite set of all places of E above N. The auxiliary finite factor in its dense products is 1 at every prescribed place, so the rational unitary components are dense in their product. For an initial representative c, require a rational multiplier to be sufficiently close to $(c\delta^\vee)^{-1}$ at these places. Then $r_c\delta^\vee$ is in the open integral subgroup and sufficiently close to 1 that its conjugate by D_Σ is also integral. Since $\delta = D_\Sigma\delta^\vee$, this conjugate is $D_\Sigma^{-1}r_c\delta$. Thus the same rational multiplier gives both required integral invertibility conditions; no condition is imposed at infinity.

Let $\mathcal{J}$ be the set of pairs of a class and a source monomial Weyl-coordinate index $\alpha = (\alpha_v)_{v\in\Phi}$. For such an index put

$$w_\alpha = \sum_{v\in\Phi} \alpha_{v,k}, \qquad \ell = \sum_{(c,\alpha)\in\mathcal{J}} w_\alpha.$$

Evaluation at the r_c identifies the modules as

$$M_{\delta^\vee} \cong \mathcal{O}^{\mathcal{J}}, \qquad M_\delta \cong \bigoplus_{(c,\alpha)\in\mathcal{J}} N^{w_\alpha}\mathcal{O}.$$

We check these assertions at all points, not only at the chosen representatives. The conjugates of K_0 by δ and by $\delta^\vee$ are integral above N, by the same four-block calculation used above. Rational equivariance of the source gauge and invariance under an integral invertible Weyl matrix therefore reduce integrality at every point to the representatives. At $r_c\delta^\vee$ the transport is integral invertible, yielding the ordinary coordinate module. At $r_c\delta = D_\Sigma k_c$ with k_c integral invertible, its inverse Weyl transport is an integral invertible matrix following the diagonal scaling by N^{-w_α}. This yields exactly the second displayed module.

Every finite family in the first module has a matrix over $\mathcal{O}$, determinant dividing N^ℓ, whose combinations belong to the second. Here is an explicit argument valid for this valuation ring, without a discreteness assumption. Process its coordinates one at a time. To make one coordinate divisible by N^w, do nothing if it already is so in every vector. Otherwise select a nonzero coordinate of least valuation among the finite family and move its vector to the first position. Every other coordinate is an $\mathcal{O}$-multiple of this pivot. Subtract that multiple of the pivot row, then multiply the pivot row by N^w divided by its coordinate. All row operations are over $\mathcal{O}$; the last multiplier is integral, and their determinant divides N^w because the complementary factor

is the pivot coordinate up to sign, also integral. Previous coordinate divisibilities are preserved. Multiplying these matrices over all coordinates gives determinant dividing N^{ℓ}. Empty families use the empty matrix. Denote this finite-family property by the assertion just proved; no new lattice-length interpretation is needed.

The operator sends M_{δ} into $M_{\delta^{\vee}}$. Indeed, in a summand of $U\varphi$ put $g' = g\delta^{\vee}\jmath_{\Sigma}(K_y)\delta^{-1}$. Then

$$g'\delta = g\delta^{\vee}\jmath_{\Sigma}(K_y).$$

Integral invertible Weyl transport by K_y does not change the unit coordinate bound. The coefficient $\psi(y_k)$ is integral at ι, and the gauge is ultrametric, so the finite sum has gauge at most 1 at $g\delta^{\vee}$.

It follows by iteration that finite families in M_1 can be carried into $U^{-i}M_{\delta}$ with determinant dividing $N^{\ell(i+1)}$ for every $i \geq 0$. For $i = 0$ use $M_1 \subseteq M_{\delta^{\vee}}$ and the comparison just proved. If a family already lies in $U^{-i}M_{\delta}$, its images under U^{i+1} lie in $M_{\delta^{\vee}}$ because $U(M_{\delta}) \subseteq M_{\delta^{\vee}}$. Apply the comparison to those images and pull the combinations back by linearity. Multiplying with the preceding clearing matrix multiplies determinants, adding one further exponent ℓ. This proves the induction.

It remains to bound the exponent for $i = m/2$. The monomial coordinate indices are symmetric in the k variables. Their total degree at each $v \in \Phi_a$ is a and at the other places is 0. Thus, with $d_{\mathrm{src}} = \dim_{\mathbb{C}} \mathcal{S} = \#\mathcal{J}$,

$$k\ell = d_{\mathrm{src}}\, a\, \#\Phi_a \leq d_{\mathrm{src}} f a.$$

This follows also by summing the equal total exponents of all k variables; swapping any two variables permutes the coordinate indices. We have $x \leq k$: by the rank bounds, $m \leq 2\sqrt{x} \leq x$ for $x \geq 4$, and $k = r + 1 - m \geq x^5 + 1 - x \geq x$. Also m is even and $m/2 + 1 \leq m$. Set $e = \ell(m/2 + 1)$. Since $a = xb$,

$$ke \leq d_{\mathrm{src}} f a(m/2 + 1) \leq d_{\mathrm{src}} f x b m \leq k\, m f b\, d_{\mathrm{src}}.$$

Cancel the positive integer k to obtain the claimed bound. The preceding induction supplies the clearing matrices with determinant dividing N^{e}, independently of the finite family. Source preservation of U makes its inverse-image module T_{ι} well defined, and its complex linearity makes that module an $\mathcal{O}_{\iota}$-submodule. This proves all assertions. □

Lemma 4.2.66. *For admissible data D:*

1. *$q_1 d \mid q_2$, and $E \subseteq \mathbb{Q}(\zeta_{q_1 d})$;*
2. *E is unramified at every prime $\ell \nmid q_1 d$, and the ramification index of E at every prime is at most $2f \leq h$;*
3. *p_0 and p_1 split completely in E; for each $i \in \{0, 1\}$ exactly f chosen places lie above p_i, each of norm p_i, so there are $2f$ chosen places and $\prod_{w \text{ chosen}} \mathrm{N}w = M^f$.*

Proof. **Item 1.** $q_1 d = 3q_0\, l_0(s)\, 2^s d$ with $l_0(s) = 3^{\lfloor \log_3 s \rfloor + 4}$. As $16 \cdot 3 \mid q_0$, $2^s \cdot 3^{\lfloor \log_3 s \rfloor + 5}$ divides $(16 \cdot 3)^{s+5}$, hence q_0^{s+5} (using $\lfloor \log_3 s \rfloor \leq s$). So $q_1 d \mid q_0^{s+6} d \mid (q_0 d)^{s+6} = q_2$. As $d \equiv 3 \pmod 4$ is prime, $K = \mathbb{Q}(\sqrt{-d}) \subseteq \mathbb{Q}(\zeta_d)$, so $E = K(\zeta_{q_1}) \subseteq \mathbb{Q}(\zeta_{q_1 d})$.

Item 2. $\mathbb{Q}(\zeta_{q_1 d})$ is unramified at the primes $\ell \nmid q_1 d$, hence so is its subfield E. A ramification index is at most the degree $[E : \mathbb{Q}] = 2f$, and $2f \leq 2x^{12} \leq x^{40} \leq r^8 = h$ by Lemma 4.2.67, items 3 and 4.

Item 3. By (AD5), $p_i \equiv 1 \pmod{q_2^2}$, so $p_i \equiv 1 \pmod{q_1 d}$ by item 1 and p_i splits completely in $\mathbb{Q}(\zeta_{q_1 d})$, hence in E; every place of E above p_i has norm p_i. In $\mathcal{O}_K$, $\pi_M \bar{\pi}_M = p_0 p_1$ with p_0, p_1 split in K, so $(\pi_M) = \mathfrak{p}_0 \mathfrak{p}_1$ with $\mathfrak{p}_i$ a prime of K of norm p_i. A place w of E contains π_M if and only if it lies above $\mathfrak{p}_0$ or $\mathfrak{p}_1$, and $\mathfrak{p}_i$ splits into $[E : K] = f$ places of E. Hence the product of the norms of the chosen places is $p_0^f p_1^f = M^f$. □

Lemma 4.2.67. *For admissible data D:*

1. $C_P \geq \max(64, q_0, d)$, $\log\log b \geq 6.4 \cdot 10^{13}$, *and* $x \geq e^{6.4 \cdot 10^{15}}$;
2. $(\log b)^{100} \leq x$ *and* $x^{10^{12}} \leq b$;
3. $1 \leq k \leq r \leq 3x^5$, $r \leq 2k$, $m \leq 2\sqrt{x}$, $m^2 \leq 4x$, $t \leq 15x^5$ *and* $h \leq 3^8 x^{40}$;
4. $f \leq q_1 \leq x^{12}$ *and* $q_1 d \leq x^{13}$;
5. $\log M \leq 22 \log b$.

Proof. **Item 1.** The summands $2d + 6$, q_0, $|c_{\mathrm{CS}}(d)|$ and $\sum_w |\log |\mathrm{cyc}'_{q_0}(w)||$ of C_P are non-negative and $c_{\mathrm{vac}}(P) \geq 0$, so $C_P \geq 2d + 6 + q_0$. By (2.15), $d \equiv q_0/2 - 1 \pmod{q_0}$ with d a positive prime, so $d \geq q_0/2 - 1 \geq 9239$ as $18480 \mid q_0$. Hence $C_P \geq \max(64, q_0, d)$. Then (AD1) gives $\log\log b \geq 10^{12} C_P \geq 6.4 \cdot 10^{13}$ and $\log x \geq 100 \log\log b \geq 6.4 \cdot 10^{15}$. Here $c_{\mathrm{vac}}(P) \geq 0$ is the property of the vacuum parameter assumed in Definition 2.3.3; no other property of it is used.

Item 2. This is Proposition 2.3.5: $(\log b)^{100} \leq x$, and $x^{10^{12}} = (x^{10^6})^{10^6} \leq x_1^{10^6} \leq b$.

Item 3. By (AD2), $x^5 \leq r \leq 3x^5$ and $\sqrt{x} \leq m \leq 2\sqrt{x}$, so $m^2 \leq 4x$ and $m \leq 2\sqrt{x} \leq x^5/2 \leq r/2$ by item 1. Hence $k = r + 1 - m \geq r/2 + 1 \geq 1$, $r \leq 2k$ and $k \leq r$. Then $t = 5r \leq 15x^5$ and $h = r^8 \leq 3^8 x^{40}$.

Item 4. By Definition 2.3.1, $\mathrm{Gal}(E/K) \cong (\mathbb{Z}/q_1)^\times$ is in bijection with Φ, so $f = \varphi(q_1) \leq q_1$. Now $q_1 = 3 q_0 \, l_0(s) \, 2^s$ with $2^s = m^{22} \leq 2^{22} x^{11}$, $l_0(s) \leq 81 \max(s, 1) \leq 2592 \log x$ (as $3^{\lfloor \log_3 s \rfloor} \leq s$ and $s \leq 32 \log x$), and $q_0 \leq C_P \leq \log\log b \leq \log x$ by item 1 and (AD1). So

$$q_1 \leq 3 \cdot 2592 \cdot 2^{22} \, (\log x)^2 \, x^{11} \leq x^{12},$$

since $x \geq e^{6.4 \cdot 10^{15}}$. Likewise $d \leq C_P \leq \log x \leq x$, so $q_1 d \leq x^{13}$.

Item 5. By (AD5), $p_i \leq b^{11}$, so $\log M = \log p_0 + \log p_1 \leq 22 \log b$. □

4.3 Theorem A, the spanning form, and the size of the Poincaré images

This section proves the spanning form of Theorem A (Theorem 4.3.1). Its substance is the size bound for the normalised Poincaré images, proved here through one Schur-jet kernel: for a short integral source form φ, the image $s\mathfrak{p}\,\varphi$ is bounded in terms of $h_{\mathrm{main}} - m \cdot \mathrm{rem}$ and twice the gains at the primes below x_1 that do not split in K (Proposition 4.3.2). The rest is bookkeeping. Short integral forms span the ramified source forms (Lemma 4.6.1, proved in Section 4.6); a rational factor c_B, fixed by every automorphism of $\mathbb{C}$, removes the gains again up to $u_{\mathrm{err}} + b$ (Lemma 4.3.4); and the remaining terms are absorbed by an inequality between the scales of the datum (Lemma 4.3.5).

Theorem 4.3.1. *Let D be admissible data over a parameter pair P, and let $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$ be parameters as in Definition 3.2.37 that satisfy the seesaw identity: for every ordinary twist ϑ there is a coefficient function Ψ_ϑ admissible for ϑ such that $\Theta_\vartheta(\mathcal{P}\varphi) = c_{\mathrm{ss}} \, B_{\vartheta, \Psi_\vartheta}(\varphi)$ for every $\varphi \in \mathcal{S}^{\mathrm{ram}}$, where admissibility (conditions 1–3), the normalised lower composite Θ_ϑ and the*

composite $B_{\vartheta,\Psi}$ are those of Proposition 4.2.25. Then $\mathcal{S}^{\mathrm{ram}}$ is spanned over $\mathbb{C}$ by the ramified source forms φ that are integral at (the embedding given by) every prime identification ι and satisfy

$$\|s\mathfrak{p}\,\varphi\|_\sigma^2 \;\le\; \exp\Bigl(h_{\mathrm{main}} - m\,\mathrm{rem} + f\,a\,(\log b)^4\Bigr)$$

for every $s \in \bar{G}$ and every automorphism σ of $\mathbb{C}$, the conjugate norm being that of the target lattice.

Proof. Put $L = \log b$, $u = faL^3$ and $G = \sum_{p\in S_{\mathrm{ns}}} \mathrm{gain}(p)\log p$, where S_{ns} is the set of primes $p < x_1$ with $\chi_K(p) \neq 1$. By Lemma 4.6.1, the ramified source forms φ that are integral at every prime identification and satisfy $\|\sigma(\varphi(g))\|_W^2 \le e^u$ for every field embedding $\sigma\colon \mathbb{C}\to\mathbb{C}$ and every finite adelic point g span $\mathcal{S}^{\mathrm{ram}}$. It therefore suffices to prove the asserted bound on $\|s\mathfrak{p}\,\varphi\|_\sigma^2$ for each such φ.

Fix $s \in \bar{G}$ and an automorphism σ of $\mathbb{C}$. The conjugate norm of Definition 3.1.4 is the squared norm of the conjugate coordinates of the complex-linear map Θ_1. By Lemma 4.3.4, applied to the family $(\mathrm{err}_p)_p$, which has the properties assumed there by the error conditions of Definition 3.2.37, the number c_B is a positive rational number, hence fixed by σ, and homogeneity gives

$$\|s\mathfrak{p}\,\varphi\|_\sigma^2 = c_B^2\,\|c_B^{-1}s\mathfrak{p}\,\varphi\|_\sigma^2.$$

The parameters satisfy the seesaw identity assumed in Proposition 4.3.2, by hypothesis, and the remaining hypothesis of that proposition is the bound $\|\sigma(\varphi(g))\|_W^2 \le e^u$ written in the monomial coordinates of Definition 3.2.17, so that proposition, together with the bound $\log c_B \le -G+u+b$ of Lemma 4.3.4, gives

$$\begin{aligned}\|s\mathfrak{p}\,\varphi\|_\sigma^2 &\le \exp(h_{\mathrm{main}} - m\,\mathrm{rem} + 2G + 2u + x^{99} + 2\log c_B)\\ &\le \exp(h_{\mathrm{main}} - m\,\mathrm{rem} + x^{99} + 4u + 2b)\\ &\le \exp(h_{\mathrm{main}} - m\,\mathrm{rem} + faL^4),\end{aligned}$$

the last inequality being the absorption $x^{99} + 4u + 2b \le faL^4$ of Lemma 4.3.5. The bound holds for every $s \in \bar{G}$ and every automorphism σ, so the spanning set supplied by Lemma 4.6.1 consists of forms of the kind asserted to span $\mathcal{S}^{\mathrm{ram}}$. This proves the theorem. □

The size bound (Proposition 4.3.2) is the archimedean half of Theorem A. The Poincaré map, whose classical form goes back to Poincaré's series [234], here represents a Bessel period [96], and its size is not estimated on Y_0 at all. By the seesaw identity the chain of lifts of $\mathcal{P}\varphi$ is a constant multiple of a composite of m definite lifts, which is bounded factor by factor and then carried to the target forms by the good projection. The analytic core is the bound for one Schur-jet kernel (Proposition 4.3.3): if the conjugates of the jets are Gaussian moments of bounded theta values, as Proposition 4.2.12 shows, the kernel has operator norm at most $\exp(x^{50} + R_j/2)$ at every conjugate. The section closes with the two lemmas of the assembly (Lemma 4.3.4, Lemma 4.3.5).

Proposition 4.3.2. *Let $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$ be such parameters. Assume the full seesaw identity: for every ordinary twist ϑ with modulus Q, there exists one finitely supported coefficient function Ψ_ϑ satisfying the support, coefficient integrality and empty-conductor conjugate-size bound, and joint invariance of conditions (1)–(3) of Proposition 4.2.25, such that*

$$\Theta_\vartheta(\mathcal{P}\varphi) = c_{\mathrm{ss}} B_{\vartheta,\Psi_\vartheta}(\varphi) \qquad (\varphi \in \mathcal{S}^{\mathrm{ram}}).$$

Here the arithmetic realization, the upper kernels, branching and normalized restriction defining $B_{\vartheta,\Psi}$ are precisely the fixed ones specified there; Θ_ϑ is its same normalized lower composite on

principal forms, with its index factors, measures, classical-to-arithmetic identification and target Gram form. There is no independent rescaling. The parameter c_{ss} is fixed before all twists, and Ψ_ϑ before all source forms. This is a hypothesis on the given parameter pair, not an application of that proposition's existence assertion to an arbitrary map.

Put $u = fa(\log b)^3$ and $S_{\mathrm{ns}} = \{p : p \text{ prime},\ p < x_1,\ \chi_K(p) \neq 1\}$. For $v \in \Phi$ write $d_v = a$ if $v \in \Phi_a$ and $d_v = 0$ otherwise. Let $\varphi \in \mathcal{S}^{\mathrm{ram}}$ be integral at every prime identification and satisfy, at every finite adelic point g and every field automorphism σ of $\mathbb{C}$,

$$\sum_{\substack{\alpha=(\alpha_v)_{v\in\Phi}\\ \alpha_v\in\mathbb{Z}^k_{\geq 0},\ |\alpha_v|=d_v}} \left(\prod_{v\in\Phi} \frac{d_v!}{\prod_{i=1}^k \alpha_{v,i}!}\right) |\sigma(\varphi_\alpha(g))|^2 \leq e^u,$$

where $\varphi_\alpha(g)$ are the standard monomial evaluation coordinates of the one-row Weyl module. Then for every $s \in \bar{G}$ and every field automorphism σ of $\mathbb{C}$,

$$\left\| c_B^{-1} s\mathfrak{p}\varphi \right\|_\sigma^2 \leq \exp\left(h_{\mathrm{main}} - m\,\mathrm{rem} + 2\sum_{p\in S_{\mathrm{ns}}} \mathrm{gain}(p)\log p + 2u + x^{99}\right).$$

Proof. We use the fixed, parameter-free arithmetic realization and operators specified in Proposition 4.2.25: its vacuum, derivative period, upper columns, measures, branching map, restriction scalar and classical-to-arithmetic identification are unchanged. We do not invoke its existence assertion for the parameter map $\mathcal{P}$ or identify the parameter c_{ss} with a scalar chosen for a different map. Instead we use the full seesaw hypothesis of this statement. We also use Proposition 4.2.12 and Lemma 4.6.6 on exactly that same realization.

Put $L = \log b$, $A = \#\Phi_a$, $\delta = \Delta/2$ and

$$H = h_{\mathrm{main}} - m\,\mathrm{rem} + 2\sum_{p\in S_{\mathrm{ns}}} \mathrm{gain}(p)\log p.$$

We give the row and numerical estimates before applying the kernels.

Scale and jet-size estimates. Write $w = \log x$ and $C = C_P$, where the definition and summand bounds of C_P are those of Definition 2.3.3. From Lemma 2.3.6 and (AD1),

$$w \geq 10^{14}C, \quad C \geq 18480, \quad 100\log L \leq w, \quad L \geq 10^{12}w.$$

In particular $x \geq 10^6$, $L \geq 100$, $b \geq x^{1000}$ and $L \leq x$. If s_D is the tower integer, then $l_0(s_D) \leq 81 s_D$, $s_D \leq 32w$ and $2^{s_D} = m^{22} \leq 2^{22}x^{11}$. Since $q_0 \leq C$,

$$f \leq q_1 \leq 7776\,2^{22}Cx^{11}w \leq x^{11}w^2 \leq x^{12}. \tag{1}$$

Here $7776\,2^{22} < 10^{14}$ and $w^2 \leq e^w$ for $w \geq 6$. The rank definitions give

$$m \leq x, \quad n_m \leq 4x^5 + 32x^{11/2} \leq x^6, \quad \delta = r^8 \leq 3^8 x^{40} \leq x^{41}, \quad 2m+3 \leq \delta. \tag{2}$$

The last inequality follows also from $\delta \geq x^{40}$ and $m \leq x$. Moreover $d \leq C \leq w$, so $\log d \leq w$; by Lemma 4.3.5, $1 \leq \mathrm{rd}_E \leq q_1 d$. Thus $\log \mathrm{rd}_E \leq 13w$. Set

$$G = \log(M\mathrm{rd}_E/(2\pi)), \qquad q = \log(M/\Omega).$$

Since $20L \leq \log M \leq 22L$, $|c_{\mathrm{CS}}(d)| \leq C \leq w$ and $L \geq 10^{12}w$, we have

$$19L \leq G \leq 23L, \qquad 19L \leq q = G - c_{\mathrm{CS}}(d) \leq 23L. \tag{3}$$

For example the lower bound for q follows from $q \geq 20L - 2 - C \geq 19L$, and its upper bound from $q \leq 22L + 14w \leq 23L$.

For an integer $h \geq 0$ write $s_\infty(h) = \log(h!) + hq$. This function is nonnegative and increasing. For $h \leq 2a$ its increment obeys

$$s_\infty(h+1) - s_\infty(h) = \log(h+1) + q \leq 25L, \tag{4}$$

because $2a + 1 \leq b^2$ by $b \geq x^{1000}$.

Let $c_\tau(h)$ be the actual upper column numbers in Proposition 4.2.25, and let

$$\ell_{j,\tau i} = \sum_{i<h<n_{j+1}} c_\tau(h) \quad \text{for } 0 \leq i < n_j.$$

These are the same columns and rows as in Proposition 4.3.3. Define the main row length $M_{\tau i}$ to be $a\mathbf{1}_{\Phi_a}(\tau)$ for $i = 0$, $\kappa(\tau)$ for $0 < i < k$, and 0 for $i \geq k$. Define its slack S_i to be $b + a/x_1 + \delta^2$ for $i = 0$, and δ^2 for $i > 0$. Directly from the column formula,

$$\ell_{j,\tau i} \leq M_{\tau i} + S_i \leq 2a. \tag{5}$$

Indeed the height-k columns contribute at most $\kappa(\tau) + 3\delta$, and all further nonzero columns together contribute at most $2m\delta$. For $i < k$ the excess beyond $\kappa(\tau)$ is therefore at most $(2m + 3)\delta \leq \delta^2$ by (2); for $i \geq k$ the height-k columns are absent. At $i = 0$ the remaining $\kappa(\tau)$ is bounded by $b + a/x_1$. Finally $M_{\tau i} \leq a$ and

$$S_i \leq b + a/x_1 + \delta^2 \leq 2a/x + x^{82} \leq a,$$

using $x_1 \geq x$ and $b \geq x^{1000}$. This proves (5).

Put

$$R_j = \sum_{\tau, i<n_j} s_\infty(\ell_{j,\tau i}), \qquad S_{\mathrm{ch}} = m\left(As_\infty(a) + k\sum_\tau s_\infty(\kappa(\tau))\right).$$

By monotonicity, (4)–(5) give

$$R_j \leq As_\infty(a) + (k-1)\sum_\tau s_\infty(\kappa(\tau)) + 25fL(b + a/x_1 + n_j\delta^2).$$

Here the first n_j rows contain exactly the k main rows, since $n_j \geq k$. Now $mb \leq a$, $ma/x_1 \leq a$, and $mn_m\delta^2 \leq x^{89} \leq a$ by (2). Summing, and using nonnegativity of $s_\infty(\kappa(\tau))$, yields

$$\sum_{j<m} R_j \leq S_{\mathrm{ch}} + 75faL \leq S_{\mathrm{ch}} + u/2. \tag{6}$$

The last inequality uses $L^2 \geq 150$.

Finite prime sums and the level saving. Expanding the von Mangoldt sum by prime powers gives

$$2\sum_{p\in S_{\mathrm{ns}}} \mathrm{gain}(p)\log p = m\sum_{\tau\in\Phi_a} s_{\mathrm{fin}}(a,\tau) + mk\sum_\tau s_{\mathrm{fin}}(\kappa(\tau),\tau). \tag{7}$$

To check both the truncation and constants, $0 < y \leq a$ and $\kappa(\tau) \geq a/x_1$ imply that every local term at $p \geq x_1$ is zero. Terms with $p^e > a$ also vanish, since $\kappa \geq 1$. For inert p, only odd e contribute and their von Mangoldt coefficient is $2\log p$. For $p = d$ every e contributes with coefficient $\log p$, equal to $2v(\pi_p)\log p$ because $v(\pi_d) = 1/2$. Split primes contribute zero. These

are exactly the local sums in the definition of gain, proving (7). Substituting the definitions of h_{main} and rem and cancelling the finite prime sums gives

$$H = mAs_{\mathrm{main}} + mk \sum_{\tau} s_{\infty}(\kappa(\tau)). \tag{8}$$

Write $\bar{w} = f^{-1} \sum_{\tau} \log \kappa(\tau)$. The elementary upper Stirling bound needed here is

$$\log(a!) \le a \log a - a + 1 + \tfrac{1}{2} \log a.$$

For $a > 1$ it follows by applying concavity of log on each interval $[h, h+1]$, summing the trapezoid lower bounds for its integral from 1 to a, and rearranging; for $a = 1$ it is equality. Since $c_1 = -c_{\mathrm{CS}}(d) - 1$ and $q = G - c_{\mathrm{CS}}(d)$, this gives

$$s_{\infty}(a) - s_{\mathrm{main}} \le a(\bar{w} + G) + 1 + \tfrac{1}{2} \log a.$$

The lower endpoint of (AD6) is $\log N \ge m(\bar{w} + G) - G - \frac{15}{2} L$. Therefore

$$m(s_{\infty}(a) - s_{\mathrm{main}}) - a \log N \le m(1 + \tfrac{1}{2} \log a) + a(G + \tfrac{15}{2} L) \le 32aL. \tag{9}$$

For the last step use $G \le 23L$, $\log a \le 2L$, $m \le x$ and $b \ge 2$, which imply $m(1 + \frac{1}{2} \log a) \le 2xL \le aL$. Multiplying (9) by $A \le f$ and using (8), we obtain

$$S_{\mathrm{ch}} - Aa \log N \le H + 32faL. \tag{10}$$

The other overheads fit in the remaining half unit:

$$x^{70} + fm\delta \log N + 2(mx^{30} + x^{60} + mx^{50}) + 32faL \le u/2. \tag{11}$$

Here is a direct check. By $a = xb$, $b \ge x^{1000}$ and $f \ge 1$, each of x^{70}, $2mx^{30}$, $2x^{60}$ and $2mx^{50}$ is at most fa. Also Lemma 4.3.5 gives $\log N \le 30mL$, so $m\delta \log N/a \le 30x^{42}L/b \le L$. Thus the left side of (11) is at most $fa(4 + 33L)$, which is at most $faL^3/2$ for $L \ge 10$.

The actual composite bound. Take the datum Ψ_1 supplied by the assumed seesaw identity for the trivial ordinary twist, whose modulus is empty. It is fixed before φ and every σ. Fix $\sigma \in \mathrm{Aut}(\mathbb{C})$. Its translations, coefficient bound and gluing tensor satisfy the hypotheses of part (ii) of Proposition 4.3.3. Thus there is one finite coefficient function Ψ' with $\sum_v |\sigma\Psi'(v)| \le e^{x^{60}}$, whose unglued tensor equals the original glued tensor. This equality holds before every source form and preserves joint invariance.

The actual normalized kernels satisfy (K3)–(K4) by Proposition 4.2.12. Part (i) of Proposition 4.3.3 therefore bounds each single-kernel operator on conjugate Weyl values by $\exp(x^{50} + R_j/2)$. The one-row formula in the hypothesis is exactly the column-filling squared Weyl norm of a source value. Indeed, in each place the symmetric tensor filling has $d_v! / \prod_i \alpha_{v,i}!$ equal entries for the monomial evaluation coordinate indexed by α_v; taking the tensor product over places gives the displayed multinomial formula. Its square root is at most $e^{u/2}$. Successive application of the m kernel bounds shows that every individual product, applied to a source value, has norm at most

$$\exp\left(mx^{50} + \tfrac{1}{2} \sum_{j<m} R_j + u/2 \right). \tag{12}$$

Each level K_j^X contains the principal subgroup of level $\pi_M \pi_N$: identity reduction at both levels implies both the chosen identity condition and the required last-row condition. The group

on X_j is standard definite and $n_j \le n_m$. Thus Lemma 4.6.4 bounds each class set by $e^{x^{30}}$. With $\mathrm{vol}(K_j^X) = 1$, its quotient integral is the class sum with inverse stabilizer orders, all at most one.

We justify the use of these fixed class sums even though an individual translated datum need not be fixed by the levels. In the m-fold integral, right translation of the j-th integration variable by $b_j \in K_j^X$ moves the adjacent kernel data by $\omega_j(b_j, b_{j+1})$ and leaves the source unchanged at $j = 0$. The sum tensor is jointly invariant, with $b_m = 1$, so the full integrand after summing its finite coefficients is right invariant under the product of these levels. Hence that combined integral is the sum over their class tuples. All groups in these integrals are definite; their quotients are compact, and the finite coefficient sum and locally constant data permit this regrouping. The resulting weights are rational, so applying σ commutes with the class sum. There are at most $e^{mx^{30}}$ class tuples and the product of their weights is at most one. Applying the triangle inequality to (12), and then summing the coefficient absolute values, bounds the conjugate norm of the upper composite value by

$$\exp\left(mx^{30} + x^{60} + mx^{50} + \tfrac{1}{2} \sum_{j<m} R_j + u/2 \right). \tag{13}$$

Use the SAME branching matrix and normalized restriction as in Proposition 4.2.25. The column numbers agree with those of Lemma 4.6.6; k and n_j are even, so its half-ranks agree with the floors in the prescribed weights. That lemma gives $b_\tau = a\mathbf{1}_{\Phi_a}(\tau) - m\delta$ and the conjugate restriction cost

$$\exp\bigl(x^{70} - (Aa - fm\delta)\log N\bigr).$$

Squaring (13) and applying this cost bounds every value of $B_{1,\Psi_1}\varphi$ by the exponential with exponent

$$u + \sum_{j<m} R_j - (Aa - fm\delta)\log N + x^{70} + 2(mx^{30} + x^{60} + mx^{50}).$$

Now apply (6), (10), and (11), in that order. The exponent is at most

$$H + 2u. \tag{14}$$

The assumed scalar identity and good projection. For this trivial twist, the seesaw hypothesis for the same parameter pair and datum gives

$$c_{\mathrm{ss}}^{-1}\widetilde{\Theta}_1(\mathcal{P}\varphi) = B_{1,\Psi_1}\varphi.$$

Let $z = c_{\mathrm{ss}}^{-1}\mathcal{P}\varphi$ in the principal-form space. Linearity and (14) give the pointwise bound $\|\sigma(\widetilde{\Theta}_1 z(g))\|_{\mathrm{Weyl}}^2 \le e^{H+2u}$ at every finite point g. The classical-to-arithmetic identification asserted in Proposition 4.2.25 identifies this very lower composite and its target Gram norm with those used in Lemma 4.5.2. Apply that lemma with $B = H + 2u$. It gives

$$\|\#\bar{G}\, e_0 sz\|_\sigma^2 \le e^{H+2u+x^{99}}.$$

Finally e_0 is central and $\mathfrak{p} = c_B c_{\mathrm{ss}}^{-1}\#\bar{G}\, e_0\mathcal{P}$. Consequently $\#\bar{G}\, e_0 sz = c_B^{-1} s\mathfrak{p}\varphi$, which is exactly the asserted left side. This proves the bound. The given parameter scalar is used throughout, before every twist and source form; none of the kernels, data, periods, measures or restriction maps has been independently rescaled. □

Proposition 4.3.3. *Put $\delta = \Delta/2$, and define the nonnegative column numbers*

$$c_\tau(h) = \begin{cases} a\mathbf{1}_{\Phi_a}(\tau), & h = 1, \\ \kappa_1(\tau) - \lfloor n_1/2 \rfloor, & h = k, \\ 2\delta - (t+1), & h > k, \quad t+1 \mid h-k, \\ 0, & \textit{otherwise.} \end{cases} \tag{K0}$$

For $j<m$ put $n=n_j$, $n'=n_{j+1}$,

$$\ell_{j,\tau i}=\sum_{i<h<n'} c_\tau(h)\quad(0\le i<n),\qquad D_j=fnn',\qquad A_j=C_P(q_1nn'+1)^2,$$

and

$$R_j=\sum_{\tau\in\Phi}\sum_{i<n}\left(\log(\ell_{j,\tau i}!)+\ell_{j,\tau i}\log(M/\Omega)\right). \tag{K1}$$

The Schur spaces at ranks n,n' have their column-filling Gram forms: these are the forms induced by the integral Young maps into tensor products of exterior powers with orthonormal increasing wedge bases. The rectangular Schur matrix $S_j(Z)$ is induced by the exterior powers of the $n'\times n$ matrices Z_τ. Columns of height n are retained; the one-dimensional factor $\det(\mathbb{C}^n)$ has its unit standard generator. Thus, for a covector u and vector ξ, $P_{u,\xi}(Z)=u(S_j(Z)\xi)$ is an explicitly defined polynomial.

(i) Analytic implication for a fixed normalized theta realization. *For each finite translation v and finite adelic pair (g,h), let*

$$f_{j,g,h,v}(Z)=\theta_{\omega_j(g,h)H_j(v)v_j^{\mathrm{int}}}(\pi_M\eta_PZ)$$

be the holomorphic theta germ in one fixed normalized-functional realization. Here v_j^{int}, H_j, ω_j and η_P are fixed before g,h,v; this notation is input data, not a claim that a generic theta sum has been identified with that realization. Define the linear map $J_{j,g,h,v}$ by

$$u(J_{j,g,h,v}\xi)=P_{u,\xi}(\partial_Z)f_{j,g,h,v}(Z)|_{Z=0}. \tag{K2}$$

There is no factorial divisor in this ordinary differential operator.

Assume the following exact conjugate-jet input. For every $\sigma\in\mathrm{Aut}(\mathbb{C}/K)$, every j,g,h,v, there are a permutation ϵ_σ of the CM-coordinate blocks, a number t_σ with $|t_\sigma|^2=M/\Omega$, and a measurable function $\Phi_{\sigma,j,g,h,v}$, which is the theta-value function of the corresponding conjugated vector, such that

$$|\Phi_{\sigma,j,g,h,v}(y)|\le e^{A_j} \tag{K3}$$

and, for every polynomial P in these D_j coordinates,

$$\sigma\big(P(\partial_Z)f_{j,g,h,v}(Z)|_0\big)=\pi^{-D_j}\int_{\mathbb{C}^{D_j}}P^\sigma(\epsilon_\sigma(t_\sigma\bar y))e^{-|y|^2/2}\Phi_{\sigma,j,g,h,v}(y)\,dy. \tag{K4}$$

Here P^σ means coefficientwise conjugation by σ, and dy is ordinary real Lebesgue measure. The assertions (K3)–(K4) are hypotheses on theta values and the fixed CM derivative scale, not on the underlying coherent coefficient or on a Schrödinger wave function. Then, for every $\sigma\in\mathrm{Aut}(\mathbb{C})$ and every ξ,

$$\|\sigma(J_{j,g,h,v}\xi)\|_{\mathrm{Weyl}}\le\exp(x^{50}+R_j/2)\,\|\sigma(\xi)\|_{\mathrm{Weyl}}. \tag{K5}$$

(ii) Finite gluing expansion. *For this part no hypothesis (K3) or (K4) is needed. In each Hom space use the same finite Heisenberg normalization. Explicitly, fix the common finite idele α, put $\jmath=-1/\sqrt{-d}$, $c_0=(1+\sqrt{-d})/2$, fix the finite additive character $\psi_{E,f}$, and let ρ_j have the cocycle*

$$\rho_j(0,u)\rho_j(0,v)=\psi_{E,f}\left(\frac{\jmath}{2}\sum_{a,b}\bar v_{ab}u_{ab}\right)\rho_j(0,u+v),\qquad u,v\ \text{finite matrices}.$$

For a finite matrix x put

$$H_j(x) = \psi_{E,f}\left(\frac{c_0 \jmath \alpha \bar{\alpha}}{2} \sum_{a,b} \bar{x}_{ab} x_{ab}\right) \rho_j(0, \alpha x). \tag{K6}$$

Take the fixed vectors v_j^{int} in these same representations, with no new scalar multiples. Let $\mathcal{X} = \prod_{j<m} \mathrm{Mat}_{n_{j+1},n_j}(\mathbb{A}_{E,f})$.

For each place $w \mid \pi_N$ fix the residue section once. For $y_i \in \prod_{w|\pi_N} k_w^{n_{2i+1}}$, let $z_{2i}(y_i)$ have only its last column nonzero, equal to $-y_i/N$ at those places, and let $z_{2i+1}(y_i)$ have only its last row nonzero, the conjugate of that column at the conjugate places. These matrices are zero elsewhere. Write $\mathcal{Y}$ for the product of these residue-vector spaces over $0 \le i < m/2$ and $z(y)$ for their tuple.

Let $\Psi : \mathcal{X} \to \mathbb{C}$ have finite support. Assume that every x in its support vanishes away from places above M, every entry of $\bar{\pi}_M x_j$ is integral at every finite place, and for every $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$\sum_x |\sigma\Psi(x)| \le M^{3fmn_m^2}. \tag{K7}$$

Define its glued tensor by

$$F_\Psi = \sum_x \Psi(x) \bigotimes_{i<m/2} \left(\sum_{y_i} H_{2i}(z_{2i}(y_i)) H_{2i}(x_{2i}) v_{2i}^{\mathrm{int}} \otimes H_{2i+1}(z_{2i+1}(y_i)) H_{2i+1}(x_{2i+1}) v_{2i+1}^{\mathrm{int}}\right).$$

Set, with a finite sum,

$$\Psi'(u) = \sum_{\substack{x \in \mathrm{supp}\,\Psi,\ y \in \mathcal{Y} \\ x + z(y) = u}} \Psi(x). \tag{K8}$$

Then Ψ' is fixed independently of every source form, has finite support and satisfies

$$F_\Psi = \sum_u \Psi'(u) \bigotimes_{j<m} H_j(u_j) v_j^{\mathrm{int}}, \qquad \sum_u |\sigma\Psi'(u)| \le e^{x^{60}} \quad (\sigma \in \mathrm{Aut}(\mathbb{C})). \tag{K9}$$

Every entry of $\bar{\pi}_M u_j$ is integral away from N on its support. If all coefficients of Ψ are integral at a prime identification, the same is true of Ψ'. Joint invariance of F_Ψ transfers to the right side of (K9). Applying any fixed linear composite of the single lifts, followed by the same restriction map, therefore gives the same operator on every source form. Neither the vacuum scalar, the derivative period nor the restriction scalar is altered.

Proof. We first record elementary consequences of the admissible scales, to keep both estimates explicit. Write $w = \log x$, $L_b = \log b$, $C = C_P$, and let s_D denote the tower integer. By Lemma 2.3.6,

$$w \ge 10^{14} C, \qquad C \ge 18480, \qquad x \ge 10^6.$$

Also $L_b \le x^{1/100} \le x$ and $L_b \ge 10^{12} w$, whence $b \ge x^{1000}$. Since $l_0(s_D) \le 81 s_D$, $s_D \le 32w$ and $2^{s_D} = m^{22} \le 2^{22} x^{11}$,

$$f \le q_1 \le 7776\, 2^{22} C x^{11} w \le x^{11} w^2 \le x^{12}. \tag{3}$$

The constant $7776\, 2^{22}$ is less than 10^{14} and $w^2 \le e^w$ for $w \ge 6$. Furthermore

$$m \le x, \quad r \le 3x^5, \quad n_m \le 4x^5 + 32x^{11/2} \le 36x^{11/2} \le x^6, \quad \delta = r^8 \le x^{41}. \tag{4}$$

The inequalities on x above justify the fixed numerical constants in (4).

The column numbers in (K0) are indeed nonnegative. Since $\log x_1 \le L_b/10^6$, the off weight obeys $a/x_1 \ge xb^{1-10^{-6}} \ge xb^{1/2} > n_m$. All values of κ are at least this off weight, because $x_1 \ge x$ by Lemma 4.3.5. Hence $\kappa_1 - \lfloor n_1/2 \rfloor \ge 0$. Also $2r^8 \ge 5r+1 = t+1$ for $r \ge 3$. No column has height strictly between n_j and n_{j+1}: the possible heights above k are spaced by $t+1$. The total contribution of these columns to any row is at most $\kappa(\tau) + (2m+3)\delta \le b + a/x_1 + 5m\delta$. Consequently

$$0 \le \ell_{j,\tau i} \le a + b + a/x_1 + 5m\delta \le 2a. \tag{5}$$

For the last inequality, $b + a/x_1 \le 2a/x \le a/2$ and $5m\delta \le 5x^{42} \le x^{43} \le a/2$, using $b \ge x^{1000}$.

Schur growth with the actual row lengths. For complex rectangular matrices Z_τ, let $\rho_{\tau 0} \ge \cdots \ge \rho_{\tau,n-1} \ge 0$ be their squared singular values. Then

$$\sum_{\tau,i} \rho_{\tau i} = \sum_{\tau,a,b} |Z_{\tau,ab}|^2, \qquad \|S_j(Z)\xi\|^2 \le \prod_{\tau,i} \rho_{\tau i}^{\ell_{j,\tau i}} \|\xi\|^2. \tag{6}$$

Here is a verification for the specified Gram forms. Unitary changes of bases on either side act isometrically on each exterior power and its tensor product, commute with the Young maps, and therefore act isometrically on the column-filling image. Singular-value decomposition thus reduces the assertion to a rectangular diagonal matrix. In a column of height h, a wedge basis vector uses h distinct diagonal entries, so its squared multiplier is at most $\rho_{\tau 0} \cdots \rho_{\tau,h-1}$. Tensoring over the $c_\tau(h)$ columns gives their product to those multiplicities. The rectangular inclusion of the first n basis vectors is isometric on the exterior tensors and on the Young image; a height-n column is precisely its unit determinant factor. Multiplying the column bounds and counting each box by its row gives (6), including those height-n columns. The argument also applies at zero singular values by continuity.

For nonnegative numbers ρ_ν and integers $l_\nu \ge 0$, let $L_* > 0$ and $v \in \mathbb{R}$. If $\sum_\nu \rho_\nu \le e^v(2L_*+2)R$, then

$$\prod_\nu \rho_\nu^{l_\nu/2} \le \exp\left[\frac{1}{2}\sum_\nu (l_\nu/L_* + \log(l_\nu!) + v l_\nu) + L_* R\right]. \tag{7}$$

To prove this, the l-th term in the exponential series gives $r^l \le l! \exp(l(v + 1/L_*) + re^{-v-1/L_*})$, including $l = 0$. Multiply over ν and use

$$e^{-v-1/L_*} \sum_\nu \rho_\nu \le (2L_*+2)e^{-1/L_*} R \le 2L_* R,$$

where $e^{1/L_*} \ge 1 + 1/L_*$. The sum condition forces $R \ge 0$; the same estimate holds when the sum is zero. Taking square roots proves (7).

Fix σ over K, and conjugate coefficients in (K2). For any covector u' of dual norm at most one in the conjugated target coordinates, take $u = \sigma^{-1}u'$. Since S_j has integer polynomial entries, $P^\sigma_{u,\xi}$ is the matrix-coefficient polynomial with covector u' and vector $\sigma\xi$. Apply (6)–(7) to this polynomial, with

$$v = \log(M/\Omega), \quad L_* = a+1, \quad Z = \epsilon_\sigma(t_\sigma \bar{y}), \quad R = |y|^2/(2L_*+2).$$

The permutation changes neither the total sum of squared entries nor the sum in the exponent. Its polynomial absolute value is at most

$$\|\sigma\xi\| \exp(W_j + L_*|y|^2/(2L_*+2)), \qquad W_j = \frac{1}{2}\sum_{\tau,i} \left(\ell_{j,\tau i}/L_* + \log(\ell_{j,\tau i}!) + \ell_{j,\tau i}\log(M/\Omega)\right).$$

Combine this with (K3)–(K4). The factor $e^{-|y|^2/2}$ leaves exactly $e^{-|y|^2/(2L_*+2)}$. This Gaussian is integrable for the stated normalized Lebesgue measure, with integral $(2L_*+2)^{D_j}$: this follows in one complex coordinate by polar integration, and in D_j coordinates by the product formula. Thus

$$|u'(\sigma(J_{j,g,h,v}\xi))| \le \|\sigma\xi\| \exp(A_j + W_j + D_j \log(2a+4)).$$

The estimate itself proves absolute integrability of the scalar integral used here. Taking the supremum over unit dual covectors gives the same bound for the full output norm.

We check its overhead. By (3)–(5), $nn' \le x^{12}$, $D_j \le x^{24}$ and $C \le x$, while $2a+4 \le b^2$ since $b \ge x^{1000}$. Hence

$$A_j \le 4x^{49}, \qquad D_j \log(2a+4) \le 2x^{25}, \qquad \frac{1}{2}\sum_{\tau,i} \frac{\ell_{j,\tau i}}{a+1} \le fn \le x^{18}.$$

For $x \ge 10^6$ their sum is at most x^{50}. The rest of W_j is exactly $R_j/2$, proving (K5) for σ fixing K. For an arbitrary automorphism σ, its restriction to the imaginary quadratic field K is either the identity or conjugation. In the latter case replace it by ordinary complex conjugation composed with σ. This fixes K and gives the same input and output norms because the two Gram matrices are rational real matrices. This proves (K5) for every σ.

The expansion uses only the finite Heisenberg equations. First the primes above M and N are disjoint for these admissible data. Indeed $\kappa(\tau) \ge b^{1/2}$ as proved above, $\mathrm{rd}_E \ge 1$ and $L_b \ge 10$ by Lemma 4.3.5, so

$$\log(M\mathrm{rd}_E/(2\pi)) \ge 19L_b.$$

The lower endpoint in (AD6) gives

$$\log N \ge (m/2 + 19(m-1) - 15/2)L_b \ge (25/2)L_b > 11L_b \ge \log p_i$$

for $i = 0, 1$, since $m \ge 2$. Hence $N \ne p_0, p_1$.

The supports of x_j and of $z_j(y)$, and also their conjugate supports, are disjoint: the former lie above M, the latter above the distinct prime N. Each mixed product in the quadratic semicharacter of (K6) is therefore zero. The cocycle has the same property. Thus the exact operator identity is

$$H_j(z_j(y))H_j(x_j) = H_j(x_j + z_j(y)). \tag{8}$$

In particular no phase, averaging factor or vacuum rescaling occurs. Expand the finite tensor product of the gluing sums and use (8). This yields

$$F_\Psi = \sum_{x,y} \Psi(x) \bigotimes_{j<m} H_j(x_j + z_j(y))v_j^{\mathrm{int}}.$$

Collect equal tuples. The resulting coefficient is exactly (K8), proving the tensor equality in (K9). Every coefficient is a finite sum of original coefficients, proving the stated integral assertion. Outside N, the added tuple is zero, proving the support integrality assertion.

For each gluing pair the product of the residue-field sizes above π_N is at most $|\mathrm{N}_{E/\mathbb{Q}}\pi_N| = N^f$. There are $n_{2i+1} \le n_m$ coordinates for that pair and $m/2$ pairs. Consequently $\#\mathcal{Y} \le N^{fmn_m/2}$. For any σ, the triangle inequality and (K7) now give

$$\sum_u |\sigma\Psi'(u)| \le \#\mathcal{Y} \sum_x |\sigma\Psi(x)| \le M^{3fmn_m^2} N^{fmn_m/2}.$$

By Lemma 4.3.5, $\log N \le 30mL_b$; also $\log M \le 22L_b$. Using (3)–(4) and $L_b \le x$,

$$3fmn_m^2 \log M + (fmn_m/2)\log N \le 66x^{26} + 15x^{21} \le 81x^{26} \le x^{60}.$$

This proves the coefficient bound in (K9). Joint invariance is a property of the tensor, so it is unchanged by its exact re-expression. Finally a linear map takes equal tensors to equal outputs. Composition of the single lifts is multilinear in their data and hence linear on this tensor product; applying the same restriction after it proves the all-source operator equality claimed in (ii). □

Lemma 4.3.4. *Let $(\varepsilon_p)_p$ be any family of real numbers indexed by primes such that $\varepsilon_p \geq 0$,*

$$\varepsilon_p = 0 \quad \text{if } p \neq N \text{ and } \big(\chi_K(p) = 1 \text{ or } p \geq x_1\big), \qquad \sum_{p \in U} \varepsilon_p \log p \leq u_{\mathrm{err}}$$

for every finite set U of primes. Put

$$S_{\mathrm{ns}} = \{p < x_1 : p \text{ prime},\ \chi_K(p) \neq 1\}, \quad g_p = \lfloor \mathrm{gain}(p) - \varepsilon_p \rfloor, \quad c = \prod_{p \in S_{\mathrm{ns}} \cup \{N\}} p^{-g_p}.$$

Then c is a positive rational number and, for every prime p,

$$v_p(c) = -g_p, \qquad |c|_p = p^{g_p}, \qquad |c|_p\, p^{\varepsilon_p - \mathrm{gain}(p)} \leq 1 \quad (|p|_p = p^{-1}).$$

Moreover

$$\log c \leq -\sum_{p \in S_{\mathrm{ns}}} \mathrm{gain}(p) \log p + u_{\mathrm{err}} + b.$$

Proof. Write $S = S_{\mathrm{ns}} \cup \{N\}$, a finite set. We first check the local gains outside S. The scale bounds of Lemma 4.3.5 give $x \leq x_1$ and $a = xb \geq b$. On Φ_b the weight is $b \geq a/x_1$, and off Φ_b it is a/x_1. Thus

$$a/x_1 \leq \kappa(\sigma) \leq b \leq a \quad (\sigma \in \Phi).$$

If $\chi_K(p) = 1$, the index set defining every s_p is empty, so $\mathrm{gain}(p) = 0$. If $p \geq x_1$, every $e \geq 1$ and every $y \in \{a\} \cup \{\kappa(\sigma) : \sigma \in \Phi\}$ satisfy

$$y/p^e \leq a/p \leq a/x_1 \leq \kappa(-p^e\sigma).$$

All positive parts in the definition of s_p then vanish. Hence again $\mathrm{gain}(p) = 0$. In particular $\mathrm{gain}(p) = \varepsilon_p = 0$ and $g_p = 0$ for $p \notin S$.

For each $p \in S$, the factor p^{-g_p} is a positive rational number, even when g_p is negative. Unique prime factorisation in $\mathbb{Q}^\times$ now gives $v_p(c) = -g_p$ for $p \in S$, and $v_p(c) = 0 = -g_p$ for $p \notin S$. The formula for $|c|_p$ follows. Since $g_p \leq \mathrm{gain}(p) - \varepsilon_p$, it also gives

$$|c|_p\, p^{\varepsilon_p - \mathrm{gain}(p)} = p^{g_p + \varepsilon_p - \mathrm{gain}(p)} \leq 1.$$

The other floor inequality gives $-g_p \leq -\mathrm{gain}(p) + \varepsilon_p + 1$. Therefore

$$\begin{aligned} \log c &= -\sum_{p \in S} g_p \log p \\ &\leq -\sum_{p \in S} \mathrm{gain}(p) \log p + \sum_{p \in S} \varepsilon_p \log p + \sum_{p \in S} \log p \\ &\leq -\sum_{p \in S} \mathrm{gain}(p) \log p + u_{\mathrm{err}} + b. \end{aligned}$$

The last term is bounded by Lemma 4.3.5. That lemma also gives $N \geq b^{10} \geq x_1$, because $b \geq x_1 \geq 1$. Thus $\mathrm{gain}(N) = 0$, and removing N from the gain sum proves the stated logarithmic bound. □

Lemma 4.3.5. *Put $L = \log b$, $u = faL^3$ and*

$$\mathcal{S} = \{p : \ p \text{ prime}, \ p < x_1, \ \chi_K(p) \neq 1\} \cup \{N\}.$$

Then

$$L \geq 10, \qquad x_1 \geq x \geq 4, \qquad 1 \leq f \leq q_1 \leq b, \qquad d \leq b, \qquad 2 \leq k \leq b, \qquad m \geq 2,$$

$$1 \leq \mathrm{rd}_E \leq q_1 d \leq b^2, \qquad 10L \leq \log N \leq 30mL, \qquad \frac{f}{N-1} \leq b^{-8} \leq \frac{1}{2},$$

and

$$\sum_{p \in \mathcal{S}} \log p \leq b, \qquad 4^f \left[6(q_1 d)(fk)^2\right]^{6f(k+a)} \leq e^u, \qquad x^{99} + 4u + 2b \leq faL^4.$$

Proof. By the definition of C_P in Definition 2.3.3, its summands are nonnegative and $C_P \geq 2d + 6 + q_0 \geq 1$. The scale inequalities of Proposition 2.3.5 give

$$\log L \geq 10^{12} C_P, \quad L^{100} \leq x, \quad x^{10^6} \leq x_1, \quad x_1^{10^6} \leq b.$$

In particular $L > 10$, $x \geq 4$, $x_1 \geq x$, and $b \geq x^{10^{12}}$. Since $E = K(\zeta_{q_1})$, the degree bound $[E : \mathbb{Q}] \leq 2\varphi(q_1)$ gives $1 \leq f = [E : \mathbb{Q}]/2 \leq q_1$. The divisibility $q_1 x_1 \mid b$ gives $q_1 \leq b$. Moreover

$$d \leq C_P \leq 10^{-12} \log L \leq L \leq b.$$

The rank conditions give $m \geq 2$ and $m \leq 2\sqrt{x} \leq x$. Thus

$$k = r + 1 - m \geq x^5 + 1 - x \geq 2, \qquad k \leq r \leq 3x^5 \leq x^7 \leq b.$$

The positive integral weights satisfy $1 \leq \kappa(\tau) \leq b$: on Φ_b this is immediate, and otherwise $\kappa(\tau) = xb/x_1$ is a positive integer at most b.

The quadratic Gauss-sum identity, applied to the odd prime $d \equiv 3 \pmod 4$, embeds $\mathbb{Q}(\sqrt{-d})$ in $\mathbb{Q}(\zeta_d)$. Hence

$$E \subseteq F := \mathbb{Q}(\zeta_{q_1 d}).$$

The cyclotomic discriminant formula

$$|\operatorname{disc}(\mathbb{Q}(\zeta_Q))| = \frac{Q^{\varphi(Q)}}{\prod_{\ell \mid Q} \ell^{\varphi(Q)/(\ell-1)}} \quad (Q = q_1 d)$$

gives $\mathrm{rd}_F \leq Q$. The discriminant identity

$$|\operatorname{disc}(F)| = |\operatorname{disc}(E)|^{[F:E]} \operatorname{Norm}_{E/\mathbb{Q}}(\mathfrak{d}_{F/E})$$

and the positive integral norm of the relative discriminant ideal give $\mathrm{rd}_E \leq \mathrm{rd}_F$. Also $|\operatorname{disc}(E)|$ is a nonzero integer, so $\mathrm{rd}_E \geq 1$. This proves $1 \leq \mathrm{rd}_E \leq q_1 d \leq b^2$.

We have $b^{20} \leq M \leq b^{22}$ and $\log(2\pi) \leq L$. Consequently

$$19L \leq \log \frac{M \mathrm{rd}_E}{2\pi} \leq 24L.$$

Using $0 \leq f^{-1} \sum_\tau \log \kappa(\tau) \leq L$ in the defining interval for $\log N$ gives

$$\log N \geq 19(m-1)L - \frac{15}{2}L \geq \frac{23}{2}L \geq 10L$$

and, since $\sqrt{L} \le L \le mL/2$,

$$\log N \le mL + 24(m-1)L + \sqrt{L} \le 30mL.$$

Thus $N \ge b^{10}$, and $b \ge 2$ implies $N - 1 \ge b^9$. Since $f \le b$, this yields $f/(N-1) \le b^{-8} \le 1/2$.

There are at most x_1 primes less than x_1, and each has logarithm at most x_1. Also $m \le 2\sqrt{x} \le 2x_1$ and $L \le x \le x_1$, so $\log N \le 60x_1^2$. Whether or not N occurs in the other set,

$$\sum_{p \in \mathcal{S}} \log p \le x_1^2 + \log N \le 61x_1^2 \le 64x_1^2 \le x_1^8 \le b,$$

where $x_1 \ge 2$ was used in the penultimate inequality.

Put $C = 6(q_1 d)(fk)^2$. The bounds just proved imply $C \le 6b^6 \le b^7$. Moreover $k \le b \le a$, and hence

$$\log\left(4^f C^{6f(k+a)}\right) = f\log 4 + 6f(k+a)\log C \le f\log 4 + 84faL \le fa(2+84L) \le faL^3 = u.$$

The last inequality holds for every $L \ge 10$. Finally $b \ge x^{10^{12}}$ implies $x^{99} \le xb = a$, while $2b \le 2a$. As $f \ge 1$ and $L \ge 10$,

$$x^{99} + 4u + 2b \le 3a + 4faL^3 \le fa(3 + 4L^3) \le faL^4.$$

All the inequalities are therefore valid for every admissible datum. □

4.4 Theorem A, the index form, and local determinant clearing

This section proves the index form of Theorem A (Theorem 4.4.1). Given the local determinant clearing at every prime, the index form is linear algebra over the local rings (Lemma 4.5.6). The clearing (Proposition 4.4.2) is the arithmetic input: at each prime identification $\iota \mid p$ and each translate $s \in \bar{G}$, one matrix over $\mathcal{O}_\iota$, chosen before the twist and with determinant dividing p^j for some $j \le \mathrm{err}_p \dim \mathcal{S}$, makes the images $s\mathfrak{p}\,\varphi_k$ of any finite family of ramified source forms integral at ι integral in the target lattice, for every twist allowed at p. Its proof is short once the two hypotheses are in place. By the seesaw identity the chain of lifts of $\mathcal{P}\varphi$ is c_{ss} times the composite definite lift, and by the uniform local property this lift is divisible by $p^{\mathrm{gain}(p)-\mathrm{err}_p}$ on a submodule of small colength. Division by c_B, whose valuation at p is minus the integral part of this exponent (Lemma 4.3.4), and the good projection (Lemma 4.5.1) preserve integrality.

Theorem 4.4.1. *Let D be admissible data over a parameter pair P, and let $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$ be parameters as in Definition 3.2.37 such that*

- *the family* err *has the uniform local property of Proposition 4.2.54 (with $\epsilon_p = \mathrm{err}_p$), and*
- *the seesaw identity holds: for every ordinary twist ϑ there is a coefficient function Ψ_ϑ admissible for ϑ such that $\Theta_\vartheta(\mathcal{P}\varphi) = c_{\mathrm{ss}}\, B_{\vartheta,\Psi_\vartheta}(\varphi)$ for every $\varphi \in \mathcal{S}^{\mathrm{ram}}$, where admissibility (conditions 1–3), the normalised lower composite Θ_ϑ and the composite $B_{\vartheta,\Psi}$ are those of Proposition 4.2.25.*

Let $t \subseteq \bar{G}$ be a subset, and let $y_1, \dots, y_n \in \mathcal{T}$ be target forms such that for each i there are $s \in t$ and a ramified source form φ, integral at every prime identification, with $y_i = s\mathfrak{p}\,\varphi$. Then the family $(y_1, \dots, y_n)$ has index at most

$$\#t \cdot \dim_{\mathbb{C}} \mathcal{S}^{\mathrm{ram}} \cdot 2\, u_{\mathrm{err}}$$

in the target lattice.

Proof. Let $d_{\mathrm{src}} = \dim_{\mathbb{C}} \mathcal{S}$, $d_{\mathrm{ram}} = \dim_{\mathbb{C}} \mathcal{S}^{\mathrm{ram}}$ and $u = u_{\mathrm{err}}$. By Lemma 4.6.3 and Lemma 4.3.5,

$$d_{\mathrm{ram}} \geq \Big(1 - \frac{f}{N-1}\Big) d_{\mathrm{src}} \geq \tfrac{1}{2}\, d_{\mathrm{src}}, \qquad \text{so} \qquad d_{\mathrm{src}} \leq 2 d_{\mathrm{ram}}.$$

Take $H = \mathcal{S}^{\mathrm{ram}}$ and let $H_{\mathrm{int}} \subseteq H$ be the set of ramified source forms integral at every prime identification. For each $s \in t$ put $F_s = s\mathfrak{p}$, a complex-linear map from H to $\mathcal{T}$; the set t is finite, being contained in the finite group $\bar{G}$. We apply Lemma 4.5.6 with $d = d_{\mathrm{src}}$, the family $\epsilon_p = \mathrm{err}_p$ and $u = u_{\mathrm{err}}$: the nonnegativity of this family and its finite weighted-sum bounds $\sum_{p \in U} \mathrm{err}_p \log p \leq u_{\mathrm{err}}$ are among the assumptions of Definition 3.2.37 on the parameter err (its nonnegativity and error conditions).

Let $\iota \mid p$ be a prime identification, let $s \in t$, and let $(\varphi_k)_{k \in J}$ be a finite family in H_{int}. Each φ_k is integral at ι, and the parameters satisfy the hypotheses of Proposition 4.4.2 (the uniform local property of the error family and the seesaw identity), so that proposition supplies an integer $j \leq \mathrm{err}_p\, d_{\mathrm{src}}$, a matrix $A \in M_J(\mathcal{O}_\iota)$ and $c \in \mathcal{O}_\iota$ with $\det(A)\, c = p^j$ such that every row combination $\sum_{k \in J} A_{ik}\, s\mathfrak{p}\, \varphi_k$ lies in $\mathcal{T}(\iota)$, simultaneously for every twist allowed at p, the matrix being chosen before the twist. These are exactly the hypotheses of Lemma 4.5.6. The given target forms are of the form $y_i = F_{s_i} \varphi_i$ with $s_i \in t$ and $\varphi_i \in H_{\mathrm{int}}$, so that lemma gives index at most $\#t \cdot d_{\mathrm{src}} \cdot u$ in the target lattice. Since $u \geq 0$ and $d_{\mathrm{src}} \leq 2 d_{\mathrm{ram}}$, the same witnesses give index at most

$$\#t \cdot d_{\mathrm{ram}} \cdot 2 u_{\mathrm{err}},$$

which is the assertion; the empty family and the empty set t are included. □

Proposition 4.4.2. *Let $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$ be parameters as in Definition 3.2.37 satisfying both of the following hypotheses:*

- *The family* err *has the uniform local property of Proposition 4.2.54, with $\epsilon_p = \mathrm{err}_p$.*
- *For every ordinary twist ϑ there is a coefficient function Ψ_ϑ admissible for ϑ, chosen before every source form, such that*

$$\widetilde{\Theta}_\vartheta(\mathcal{P}\varphi) = c_{\mathrm{ss}} B_{\vartheta, \Psi_\vartheta}(\varphi) \quad (\varphi \in \mathcal{S}^{\mathrm{ram}}).$$

  Here admissibility means conditions 1–3 of Proposition 4.2.25, and $B_{\vartheta,\Psi}$ and the normalised lower composite on principal-level forms $\widetilde{\Theta}_\vartheta$ are those displayed there. On m-fold twists the latter is the principal-level extension of Lemma 4.5.1, restricting to Θ_ϑ on $\mathcal{T}$.

Then the following assertions hold. For every prime identification $\iota \mid p$, every fixed $s \in \bar{G}$, every finite set J and every family $(\varphi_k)_{k \in J}$ of ramified source forms integral at ι, there exist an integer $j \geq 0$, a matrix $A \in M_J(\mathcal{O}_\iota)$ and $c \in \mathcal{O}_\iota$ such that

$$j \leq \mathrm{err}_p \dim_{\mathbb{C}} \mathcal{S}, \qquad \det(A) c = p^j,$$

and

$$\Theta_\vartheta\left(\sum_{k \in J} A_{ik}\, s\mathfrak{p}\, \varphi_k\right) \in \Lambda_\vartheta(\iota) \quad (i \in J)$$

simultaneously for every m-fold twist ϑ allowed at p. Equivalently each displayed row combination belongs to $\mathcal{T}(\iota)$. The matrix is chosen before all these twists.

Proof. Fix $\iota \mid p$, $s \in \bar{G}$, a finite set J and a family $(\varphi_k)_{k\in J}$ as in the statement. Put

$$d_{\mathrm{src}} = \dim_{\mathbb{C}} \mathcal{S}, \qquad O_\iota = \mathcal{O}_\iota, \qquad \mathcal{M}_\iota = \mathcal{S}(\iota) \cap \mathcal{S}^{\mathrm{ram}}.$$

By the assumed uniform local property of err, there are an integer $j_\iota \geq 0$ and an O_ι-submodule

$$H_\iota \subseteq \mathcal{M}_\iota, \qquad j_\iota \leq \mathrm{err}_p d_{\mathrm{src}},$$

fixed before every twist and every admissible coefficient function. The finite-family assertion in that hypothesis, applied to $(\varphi_k)_{k\in J}$, gives $A \in M_J(O_\iota)$ and $c \in O_\iota$ such that

$$\det(A)c = p^{j_\iota}, \qquad \eta_i := \sum_{k\in J} A_{ik}\varphi_k \in H_\iota \quad (i \in J).$$

These choices precede every allowed twist.

Let ϑ be any m-fold twist allowed at p. It is an ordinary twist whose conductor norm is prime to p. The assumed seesaw identity supplies an admissible Ψ_ϑ, independent of every source form, with

$$c_{\mathrm{ss}}^{-1}\widetilde{\Theta}_\vartheta(\mathcal{P}\eta) = B_{\vartheta,\Psi_\vartheta}(\eta) \quad (\eta \in \mathcal{S}^{\mathrm{ram}}).$$

The same c_{ss} occurs for every twist because it is one of the fixed parameters. Admissibility here is exactly that in the local-property hypothesis: it is conditions 1–3 for the same arithmetic composite, with its specified vacua, derivative scale, branching map, normalised restriction and unnormalised gluing.

Since $\eta_i \in H_\iota$, the gauge assertion of the assumed uniform local property applies to this Ψ_ϑ at every finite target point g. It yields

$$\max_\nu \left| \Big(\rho_\iota(g_p^{-1})\, \iota\big(B_{\vartheta,\Psi_\vartheta}(\eta_i)(g)\big)\Big)_\nu \right|_p \leq p^{\mathrm{err}_p - \mathrm{gain}(p)}.$$

The coordinates and the places used by ρ_ι are precisely those of that hypothesis: on the τ factor the place is induced by $x \mapsto \iota(\tau(\bar{x}))$. For an m-fold twist this is the target gauge of (TL3) of Definition 3.2.31.

The standing parameter assumptions of Definition 3.2.37 give nonnegativity, the required support condition and the every-finite-set error budget. Thus Lemma 4.3.4 applies to this same family err. Its rational product is exactly c_B, so

$$|c_B|_p\, p^{\mathrm{err}_p - \mathrm{gain}(p)} \leq 1.$$

In particular $c_B \in \mathbb{Q}_{>0}$, and its norm under ι is this rational p-adic norm. The transported Weyl action is linear and the maximum coordinate norm is homogeneous in scalars. Multiplying the preceding gauge inequality by $|c_B|_p$ and using the assumed seesaw identity shows that

$$(c_B/c_{\mathrm{ss}})\, \widetilde{\Theta}_\vartheta(\mathcal{P}\eta_i) = c_B B_{\vartheta,\Psi_\vartheta}(\eta_i)$$

has transported gauge at most one at every finite target point. It is therefore integral at ι.

Apply Lemma 4.5.1 with $z = \mathcal{P}\eta_i \in \mathcal{A}_M(Y_0)$ and scalar c_B/c_{ss}. It follows that

$$(c_B/c_{\mathrm{ss}})\, \Theta_\vartheta(\#\bar{G}\, e_0 s\, \mathcal{P}\eta_i)$$

is integral at ι. By Definition 3.2.16, e_0 is central in $\mathbb{C}[\bar{G}]$, so $e_0 s = s e_0$. Linearity and the definition of $\mathfrak{p}$ in Definition 3.2.37 identify this vector with

$$\Theta_\vartheta(s\mathfrak{p}\,\eta_i) = \Theta_\vartheta\left(\sum_{k\in J} A_{ik} s\mathfrak{p}\,\varphi_k\right).$$

Thus it belongs to $\Lambda_\vartheta(\iota)$.

The choices H_ι, j_ι, A, c preceded ϑ. The assumed gauge property holds for every ordinary prime-to-p twist and every admissible coefficient function, while the seesaw hypothesis supplies one such function for each allowed twist. Hence the conclusion is simultaneous in all allowed twists, with the same matrix. Taking $j = j_\iota$ gives the asserted exponent bound and determinant equation. Every row combination lies in $\mathcal{T}$, since $\mathfrak{p}$ does and centrality of e_0 makes $\mathcal{T}$ stable under $\bar{G}$. By the defining intersection of twisted integral lattices in Definition 3.1.4, the simultaneous lattice assertion is equivalent to its membership in $\mathcal{T}(\iota)$.

If J is empty, one may take $j = 0$, the empty matrix of determinant one, and $c = 1$. Nonnegativity of err_p gives the required bound, and there are no row conditions. This also proves the assertion in that case. □

4.5 The good projection

The bounds of Theorem A are proved on the chain of lifts, but they are needed for the target forms, which are cut out by the idempotent e_0 built from Green's characters [106]. This section proves that the projection carries integrality across (Lemma 4.5.1), and size at the cost of a factor $e^{x^{99}}$ (Lemma 4.5.2). Both rest on a translation identity at the chosen places (Lemma 4.5.3), which comes from the explicit Schrödinger model of the local theta correspondence at split places (Lemma 4.5.4; see [322, 172, 121]), and on a general statement about projections by finite groups (Lemma 4.5.5). The last result is the linear algebra of the index (Lemma 4.5.6), which Section 4.4 applies.

Lemma 4.5.1. *For every m-fold twist ϑ, use the lower-chain map*

$$\widetilde{\Theta}_\vartheta : \mathcal{A}_M(Y_0) \longrightarrow \mathcal{A}_\vartheta(Y_m)$$

on all forms of principal level stipulated in Proposition 6.4.6: it is the composite of the same m theta lifts with the same characters, point data, archimedean vectors and measures as the chain on $\mathcal{T}$; its restriction to $\mathcal{T}$ is Θ_ϑ.

For every m-fold twist ϑ, every prime identification ι, every $c \in \mathbb{C}$, every $z \in \mathcal{A}_M(Y_0)$ and every $s \in \bar{G}$, if $c\,\widetilde{\Theta}_\vartheta z$ is integral at ι, then

$$c\,\Theta_\vartheta(\#\bar{G}\, e_0 s z)$$

is integral at ι.

Proof. Put $n = \#\bar{G}$ and write $e_0 = \sum_t a_t t$. We first check the coefficient hypothesis, including at primes dividing n. In the regular representation of $\bar{G}$, let E be left multiplication by e_0. Since e_0 is central, $\lambda(t^{-1})$ commutes with E and preserves $\operatorname{im} E$. The trace identity is

$$n a_t = \mathrm{Tr}(\lambda(t^{-1})E) = \mathrm{Tr}(\lambda(t^{-1})|_{\operatorname{im} E}).$$

The last operator has finite order. Its eigenvalues are roots of unity, so $n a_t$ is an algebraic integer. This also covers $e_0 = 0$. Hence $n a_t \in \mathcal{O}_\iota$ for every prime identification ι.

The same classical lower-chain map in the statement is linear, and restricts to Θ_ϑ on the good forms. Apply Lemma 4.5.3 to each $ts \in \bar{G}$. For every finite adelic point g,

$$c\,\Theta_\vartheta(n e_0 s z)(g) = \sum_{t \in \bar{G}} n a_t\, c\,(\widetilde{\Theta}_\vartheta z)(g k_\vartheta(ts)).$$

The hypothesis says that $c\,\widetilde{\Theta}_{\vartheta} z$ is integral. The lattice-invariance conclusion of Lemma 4.5.3 says that each of its displayed right translates is integral at the same ι. The integral forms are an $\mathcal{O}_\iota$-module; each coefficient na_t belongs to that ring. Their sum is therefore integral. Equivalently this is the integral clause of Lemma 4.5.5, with the just-verified actual map, coefficient and lattice hypotheses.

The scalar c is the original scalar in every term, without division or conjugation. No coprimality condition on $\ell(\iota)$ or n was used. The existence and convergence of the stipulated principal-form classical composite remain in its stated setup; this proof does not supply or require an arithmetic-model normalisation. □

Lemma 4.5.2. *Use the same lower-chain composite*

$$\widetilde{\Theta}_1 : \mathcal{A}_M(Y_0) \longrightarrow \mathcal{A}_1(Y_m)$$

on principal forms stipulated in Proposition 6.4.6, with exactly the prescribed characters, point data, archimedean vectors and measures; its restriction to $\mathcal{T}$ is Θ_1.

Let $z \in \mathcal{A}_M(Y_0)$, $s \in \bar{G}$, $B \in \mathbb{R}$, and let σ be a field automorphism of $\mathbb{C}$. If every finite adelic point $g \in \mathrm{U}(Y_m)(\mathbb{A}_f)$ satisfies

$$\|\sigma((\widetilde{\Theta}_1 z)(g))\|_{\mathrm{Weyl}}^2 \le e^B,$$

where σ acts in the fixed Weyl coordinates, then

$$\|\#\bar{G}\, e_0 sz\|_\sigma^2 \le \exp(x^{99} + B).$$

Proof. Put $n = \#\bar{G}$, and write $e_0 = \sum_t a_t t$. For completeness the coefficient bound is valid for every automorphism σ of $\mathbb{C}$, without assuming that σ commutes with complex conjugation. The element $\sigma(e_0)$ is still a central idempotent of $\mathbb{C}[\bar{G}]$. In the regular unitary representation, its multiplication operator E_σ commutes with its adjoint, since both are central. A normal idempotent is an orthogonal projection: by the spectral theorem its eigenvalues are 0 or 1. Each of its matrix entries consequently has absolute value at most 1. The column at the identity consists of the coefficients $\sigma(a_t)$, so

$$|\sigma(a_t)| \le 1 \qquad (t \in \bar{G}).$$

By the actual translation identity Lemma 4.5.3, linearity and restriction to the good forms, the value of $\Theta_1(ne_0sz)$ at g is

$$\sum_{t \in \bar{G}} na_t (\widetilde{\Theta}_1 z)(gk_1(ts)).$$

Apply σ in its fixed Weyl coordinates. The Weyl Gram matrix is rational and positive definite, so its usual norm obeys the triangle inequality. The hypothesis bounds every vector on the right, after applying σ, by $e^{B/2}$. The coefficient estimate gives

$$\|\sigma(\Theta_1(ne_0sz)(g))\|_{\mathrm{Weyl}} \le n^2 e^{B/2}, \qquad \|\sigma(\Theta_1(ne_0sz)(g))\|_{\mathrm{Weyl}}^2 \le n^4 e^B. \tag{1}$$

This is precisely the pointwise clause of Lemma 4.5.5 after verification of its actual translation and coefficient hypotheses.

The class-coordinate norm in (TL4) is bounded by the sum of these squared Weyl norms over the definite classes. In its Petersson class-sum presentation the factors are inverse orders of finite stabilizers, hence at most 1; the specified Weyl Gram coordinates give the same inequality.

The untwisted level K_1 contains the principal congruence subgroup of level $\pi_M\pi_N$: an element equal to 1 modulo $\pi_M\pi_N$ is in particular equal to 1 at every chosen residue place. The space Y_m is standard definite of rank $n_m - 1$. Thus Lemma 4.6.4 applies and, using (1),

$$\|ne_0sz\|_\sigma^2 \le e^{x^{30}} n^4 e^B. \tag{2}$$

We check the order cost directly for the actual finite group. For a finite field k, $\#\mathrm{GL}_r(k) \le (\#k)^{r^2}$, simply by counting all matrices. Since the chosen prime ideals divide (π_M),

$$n \le \left(\prod_{w \text{ chosen}} \mathrm{N}w \right)^{r^2} \le |\mathrm{N}_{E/\mathbb{Q}}(\pi_M)|^{r^2} = M^{fr^2}.$$

The last equality follows by multiplying $\pi_M\bar{\pi}_M = M$ over the f complex pairs. Write $w = \log x$, $L = \log b$ and $C = C_P$. By Lemma 2.3.6, $w \ge 10^{14}C$ and $x \ge 4$. Let s_D denote the tower integer in (AD2), distinguished from the group element s in the statement. The definitions give

$$l_0(s_D) \le 81 s_D, \quad s_D \le 32w, \quad 2^{s_D} = m^{22} \le 2^{22}x^{11},$$
$$f \le q_1(s_D) \le 7776\, 2^{22} C x^{11} w \le x^{11}w^2 \le x^{12} \le x^{13}.$$

Here $7776\, 2^{22} < 10^{14}$, and $w^2 \le e^w = x$ for $w \ge 6$. Also $r \le 3x^5 \le x^6$, while $100 \log L \le \log x$ implies $L \le x^{1/100} \le x$. Therefore

$$\log n \le fr^2 \log M \le 22 fr^2 L \le 22x^{26}.$$

For $x \ge 4$ we have $88x^{26} \le x^{30}$, so

$$x^{30} + 4\log n \le x^{30} + 88x^{26} \le 2x^{30} \le x^{31} \le x^{99}.$$

Substitution in (2) proves the assertion for the given arbitrary B and σ. No normalisation of the input form or of the common theta scalar was introduced. □

Lemma 4.5.3. *For an m-fold twist ϑ, use the same convergent classical composite*

$$\widetilde{\Theta}_\vartheta : \mathcal{A}_M(Y_0) \longrightarrow \mathcal{A}_\vartheta(Y_m)$$

on principal forms stipulated in Proposition 6.4.6; no change of any prescribed datum or scalar is made. For every $s \in \bar{G}$ there is an integral finite adelic element $k_\vartheta(s) \in \mathrm{U}(Y_m)(\mathbb{A}_f)$, independent of the input form, whose reduction at each chosen place is $\mathrm{diag}(s_w, 1_{mt})$ and whose reduction at each place of Q is 1, such that

$$(\widetilde{\Theta}_\vartheta(sz))(g) = (\widetilde{\Theta}_\vartheta z)(g k_\vartheta(s)) \qquad (z \in \mathcal{A}_M(Y_0),\ g \in \mathrm{U}(Y_m)(\mathbb{A}_f)).$$

Right translation by $k_\vartheta(s)$ preserves the module of forms integral at every fixed prime identification ι. Both assertions concern this prescribed classical composite; they do not compare it with a separately normalised arithmetic theta model.

Proof. **Integral lifts in both etale components.** For $0 \le j < m$, let I_j be the rectangular inclusion matrix in the datum of the lift from Y_j to Y_{j+1}. We use the fixed finite frames throughout. By Lemma 4.6.2(a), a chosen place w and its conjugate are distinct. If v is the place of F below them, then $E \otimes_F F_v = F_v \times F_v$; call the component belonging to w the $+$ component. Choose once a lift $A_w \in \mathrm{GL}_r(\mathcal{O}_{F_v})$ of s_w. Such a lift is obtained by lifting its entries, since the determinant of any such lift is a unit.

Here is the other component of the same unitary element. In the matrix convention of Lemma 4.5.4, write a hermitian Gram matrix as $J_j = (J_{j,+}, J_{j,+}^{\mathsf{T}})$ and a unitary element as (a_+, a_-). Its unitary equation in the + component is

$$a_-^{\mathsf{T}} J_{j,+} a_+ = J_{j,+}.$$

It determines

$$a_- = J_{j,+}^{-\mathsf{T}} a_+^{-\mathsf{T}} J_{j,+}^{\mathsf{T}}, \qquad \det a_- = (\det a_+)^{-1}. \tag{1}$$

Substitution proves both component equations; the other is the transpose of the displayed equation. The fixed finite frames are orthonormal by (HP5) and (XF1), so here $J_{j,+} = 1$. In particular $a_+ \mapsto (a_+, a_+^{-\mathsf{T}})$ identifies the integral unitary group at v with $\mathrm{GL}_{r+jt}(\mathcal{O}_{F_v})$, and both the element and its inverse are integral.

Set the + component of k_j at v equal to $\mathrm{diag}(A_w, 1_{jt})$, and set its other component by (1). Do this at every chosen place and set the components at every other finite place of F equal to the identity. Thus identity outside the chosen underlying places means identity outside the pairs $\{w, \bar{w}\}$ on the E-side; the conjugate component is the one in (1), not an independently chosen identity. The twist modulus is prime to MN, so every component at a place of Q is the identity. These choices give integral finite adelic elements k_j, depending only on s. Write a^{red} for reduction of a selected component. At each chosen place,

$$k_{j+1}^{\mathrm{red}} I_j = I_j k_j^{\mathrm{red}}, \qquad \det k_{j+1}^{\mathrm{red}} = \det k_j^{\mathrm{red}} = \det s_w. \tag{2}$$

The fixed frames differ from the graded bases by matrices reducing to 1 at the chosen places. Thus the residue element is exactly s, not a conjugate of it, and k_0 realizes the given action of s on principal forms.

The local datum and its dual half. Fix an edge j and one of these split places. Put $S = Y_j$, $T = Y_{j+1}$, $a = k_j$ and $b = k_{j+1}$ locally. The matrix pairing and action are

$$B(x, y) = \mathrm{Tr}_{E_v/F_v}(\xi \,\mathrm{tr}(J_S^{-1} \bar{y}^{\mathsf{T}} J_T x)), \qquad x \longmapsto bxa^{-1}.$$

They are the pairing, additive character and selected half in Lemma 4.5.4. To check the identification of the other half explicitly, write $\xi = (\xi_+, -\xi_+)$ and set

$$L(x_+, y_-) = \xi_+ \,\mathrm{tr}(J_{S,+}^{-1} y_-^{\mathsf{T}} J_{T,+} x_+).$$

The pairing is $B(x, y) = L(x_+, y_-) - L(y_+, x_-)$, by transpose and cyclic invariance of the trace. Nondegeneracy of the Gram matrices and $\xi_+ \neq 0$ identify the $-$ half with the exact linear dual of the + half. The two transformations are

$$x_+ \longmapsto b_+ x_+ a_+^{-1}, \qquad y_- \longmapsto b_- y_- a_-^{-1}.$$

Indeed $b_-^{\mathsf{T}} J_{T,+} b_+ = J_{T,+}$ and $a_+^{-1} J_{S,+}^{-1} a_-^{-\mathsf{T}} = J_{S,+}^{-1}$ give, on substitution,

$$L(b_+ x_+ a_+^{-1},\, b_- y_- a_-^{-1}) = L(x_+, y_-).$$

Thus the latter transformation is the contragredient of the former for this same pairing. This accounts for the conjugate etale component of the unitary action; it is not a second independent operator or an extra determinant multiplier.

Formula (Sk1) of Lemma 4.5.4 says that the operator on the selected Schwartz space is

$$\varphi(x) \longmapsto |\det b_+|^{-\mathrm{rk}\, S/2} |\det a_+|^{\mathrm{rk}\, T/2} \chi_T(e \det a + \bar{e})^{-1} \chi_S(e \det b + \bar{e}) \varphi(b_+^{-1} x a_+). \tag{3}$$

The determinant of $x \mapsto b_+ x a_+^{-1}$ has absolute value $|\det b_+|^{\operatorname{rk} S} |\det a_+|^{-\operatorname{rk} T}$, so the positive factor in (3) is exactly its inverse square root. On the dual half the absolute determinant is the reciprocal, as the preceding pairing identity shows. Here all determinants are units, so both absolute determinants and the positive factor are 1. In particular the pullback preserves the local L^2 norm with the fixed Haar measure. The exact phase is already the character factor in (3).

For the edge under consideration both splitting characters have the common factor $\nu_{j+1} = \mu_{j+1}^0 \vartheta^{j+1}$; their rank corrections are powers of υ. Its tame part is trivial by (LF2). On these integral determinants the phase in (3) is consequently

$$\psi_{j+1,w}(\det b^{\mathrm{red}})\, \psi_{j+1,w}(\det a^{\mathrm{red}})^{-1} = 1, \tag{4}$$

by (2). This uses the fixed characters of the edge and their actual tame parts, without changing their splitting. Both x and $b_+^{-1} x a_+$ are integral together, since a_+, b_+ and their inverses are integral. Moreover

$$b_+^{-1} x a_+ \bmod w = I_j \iff x \bmod w = b^{\mathrm{red}} I_j (a^{\mathrm{red}})^{-1} = I_j.$$

Hence the indicator of the inclusion matrix modulo the level, extended by zero off the integral matrices, is fixed by (3). At places of Q both operators are identities. At every remaining finite place of F they are identities as well. Their infinite components are identities, so the fixed archimedean harmonic vector is unchanged. Tensoring these exact local equalities proves that the full prescribed datum φ_j is fixed by the pair (k_j, k_{j+1}). No scalar of absolute value 1 remains undetermined: (4) is the actual scalar 1.

The lift and coefficient conventions. Write $K_j(g,h) = \theta_j(\omega_j(g,h)\varphi_j)$ for the scalar kernel of the prescribed edge. The representation law gives the pointwise equality

$$K_j(g k_j, h k_{j+1}) = \theta_j(\omega_j(g,h)\omega_j(k_j,k_{j+1})\varphi_j) = K_j(g,h). \tag{5}$$

Use right translation $(R_k f)(g) = f(gk)$. The single lift in the statement, before passage to its algebraic form, is precisely the linear integral

$$(T_j f)(h) = c_j \int_{[\mathrm{U}(Y_j)]} K_j(g,h) f(g)\, dg,$$

where c_j is its fixed index factor. Absolute convergence is part of the stipulated composite. Right invariance of the quotient measure and the substitution $u = g k_j$ therefore give

$$\begin{aligned} T_j(R_{k_j} f)(h) &= c_j \int K_j(u k_j^{-1}, h) f(u)\, du \\ &= c_j \int K_j(u, h k_{j+1}) f(u)\, du = (R_{k_{j+1}} T_j f)(h). \end{aligned} \tag{6}$$

The middle equality is (5). The same c_j occurs on both sides. Thus this calculation introduces neither a complex conjugate of c_j nor a new normalization of the lift.

For completeness, the Hermitian and dual conventions in coefficient realization can be checked directly. On the Weyl module of (WM3) write $B_\kappa(v,w) = \bar{v}^{\mathsf{T}} G_\kappa w$. It is conjugate linear in its first argument and linear in its second. The Riesz map is the specified conjugate-linear map

$$\mathcal{R}(v)(w) = B_\kappa(v,w).$$

With the dual norm induced by B_κ, Cauchy–Schwarz and evaluation at $w = v/\|v\|$ show $\|\mathcal{R}(v)\| = \|v\|$, also for $v = 0$. For a unitary coefficient action $\rho(x)$, its contragredient is $\rho^\vee(x)\ell = \ell \circ \rho(x^{-1})$, and

$$\rho^\vee(x)\mathcal{R}(v)(w) = B_\kappa(v, \rho(x^{-1})w) = B_\kappa(\rho(x)v, w) = \mathcal{R}(\rho(x)v)(w).$$

Equivalently $\bar{v} \mapsto \mathcal{R}(v)$ is a linear isometry from the conjugate module to its dual. At the definite output the compact archimedean action is unitary: exterior powers preserve the tensor Hermitian form of (WM3), and determinant powers have modulus 1. These formulas fix the identification, with no free phase.

In particular coordinatewise conjugation of a coefficient function commutes with every finite right translation: $\overline{f(gk)} = (R_k\bar{f})(g)$. A fixed linear or conjugate-linear coefficient map also commutes with substitution in the finite variable. Applying such a map to (5) preserves the equality; a conjugate-linear map conjugates its scalar 1 to the same 1. For a dual action the inverse in the preceding contragredient formula is essential, and is exactly the inverse in the change of variable in (6). Thus no appeal to an unspecified duality convention is needed.

More concretely, (WM7) makes the passage to algebraic forms explicit. Its two formulas are

$$F^l_\eta(x, g) = B_\kappa(v^+_l, \rho(x^{-1})\eta(g)), \qquad F^\flat_l(g) = \#I_\kappa \int_{G_\infty(1)} F(x, g)\rho(x)v^+_l \, dx.$$

The first is linear in $\eta(g)$ under the displayed Hermitian convention. In the second, dx is the fixed probability measure and the factor $\#I_\kappa$ is fixed. Finite right translation changes only g, so direct substitution gives

$$F^l_{R_k\eta}(x, g) = F^l_\eta(x, gk), \qquad (R_kF)^\flat_l(g) = F^\flat_l(gk).$$

Thus the finite translation equality (6) survives passage to the algebraic output with exactly the same coefficient vectors and Hermitian normalization. Applying (6) for $j = 0, \dots, m-1$ and then this output identity proves the asserted equality with $k_\vartheta(s) = k_m$.

Level, norms and integral lattices. The element k_m normalizes the target level and preserves its character. At a chosen place the level is the kernel of reduction, normal in the integral group. At places of Q, $k_m = 1$, so the block upper triangular subgroup and every determinant character on its diagonal blocks are unchanged. At other places there is no change. The archimedean type is unchanged because the translation is finite. Hence R_{k_m} acts on the specified target space. On its Petersson norm, right translation merely permutes the level classes: since k_m normalizes the level, the stabilizers of g and gk_m have the same orders. If $gk_m = \gamma g_c u$ is expressed using a chosen level representative, the value changes by the rational coefficient action $\rho(\gamma_\infty)$ and the level character at u. The former is unitary for G_κ, because Y_m is definite, and the latter has absolute value 1. Stabilizer orders are unchanged under the resulting rational conjugation. Thus every term of the permuted weighted class sum has the same coefficient norm and weight, proving preservation of the Petersson norm.

Finally fix any prime identification ι. Let

$$A_\iota(h) = \prod_{\tau \in \Phi} \det f_{\iota,\tau}(h)^{e_\tau} \, S_\kappa(f_\iota(h))$$

be the coefficient representation occurring in (WM9), so that $\mathrm{gau}_\iota(g, \xi) = \|A_\iota(g^{-1})\iota(\xi)\|_{\max}$. The entries of the Schur matrices are integral polynomials by (WM1)–(WM2). For an integral invertible matrix h, every determinant is a unit even when $e_\tau < 0$. Thus both $A_\iota(h)$ and its inverse $A_\iota(h^{-1})$ have integral entries. It follows first that each is nonexpanding for the maximum nonarchimedean norm, and then, by applying the inverse, that $\|A_\iota(h)v\|_{\max} = \|v\|_{\max}$.

Both components of k_m and its inverse are integral by (1), so this applies at every place selected by $f_{\iota,\tau}$, including a conjugate place. The representation law gives

$$\begin{aligned}\mathrm{gau}_\iota(gk_m,\xi) &= \|A_\iota(k_m^{-1})A_\iota(g^{-1})\iota(\xi)\|_{\max}\\ &= \mathrm{gau}_\iota(g,\xi).\end{aligned}$$

If f is integral, use this equality with $\xi = f(gk_m)$ to obtain

$$\mathrm{gau}_\iota(g,(R_{k_m}f)(g)) = \mathrm{gau}_\iota(gk_m, f(gk_m)) \le 1.$$

The inverse translation has the same property, so the integral module is preserved. This holds for every fixed ι and requires no additional condition on its residue characteristic or on the twist conductor. □

Lemma 4.5.4. *Let k be a nonarchimedean local field of characteristic zero, $K = k \times k$ with swapping involution c, and let $e = (1,0)$, $\bar e = 1 - e$. Let J_S, J_T be invertible hermitian matrices of sizes $s,t \ge 1$ over K, let $\bar\xi = -\xi \ne 0$, and let $\psi : k \to S^1$ be nontrivial. On $\mathbb{M} = \mathrm{M}_{t\times s}(K)$ put*

$$B(x,y) = \mathrm{Tr}_{K/k}(\xi\,\mathrm{tr}(J_S^{-1}\bar y^{\mathsf T}J_T x)), \qquad (g,h)x = hxg^{-1}.$$

Let $\chi_S, \chi_T : K^\times \to S^1$ be unitary characters trivial on the diagonal $k^\times$.

Form the ordinary tensor pair with BOTH forms linear in the first variable,

$$h_V(v_1,v_2) = v_1^{\mathsf T}J_T\bar v_2, \qquad h_W(z_1,z_2) = -\xi z_1^{\mathsf T}\overline{J_S^{-1}}\bar z_2.$$

Use the full-trace tensor symplectic form and the standard Kudla circle action with characters

$$\chi_V = \chi_T^{-1}, \qquad \chi_W = \chi_S^{-1}.$$

Transport it to $\mathbb{M}$ by the k-linear map $v\otimes z \mapsto \bar v\,\bar z^{\mathsf T}$. Choose the Schrödinger model with position half $X = e\mathbb{M}$ and opposite half $Y = \bar e\mathbb{M}$. Write g_e, h_e for the first components and identify X with $\mathrm{M}_{t\times s}(k)$. The resulting action is

$$(\Omega(g,h)\varphi)(x) = |\det h_e|_k^{-s/2}|\det g_e|_k^{t/2}\chi_T(e\det g + \bar e)^{-1}\chi_S(e\det h + \bar e)\,\varphi(h_e^{-1}xg_e). \tag{Sk1}$$

The transport preserves the SAME B and ψ and sends (g,h) to $(\bar h, \overline{g^{-\mathsf T}})$ in the classical pair.

For any complex vector space N the same formula tensored with 1_N gives, for EVERY algebraic linear functional ℓ on $\mathcal{S}(X)\otimes N$ invariant under all opposite-half Heisenberg modulations,

$$\ell(\Omega(g,h)u) = |\det h_e|_k^{-s/2}|\det g_e|_k^{t/2}\chi_T(e\det g + \bar e)^{-1}\chi_S(e\det h + \bar e)\ell(u).$$

No continuity of ℓ is required. This is a formula for the specified LOCAL classical reference; no global action, nonsplit fixing or comparison with an arbitrary original datum is asserted.

Proof. **Determine the one-factor splitting in a hyperbolic acting space.** First let V_1 be any hermitian K-space of positive rank m and let the acting skew-hermitian space have hyperbolic rank $2q$. Replacing the hermitian form by twice that form identifies the full-trace, both-linear-first convention with the half-trace convention in Theorem A.5.7: conjugate the value of the skew form to read it in the conjugate-first convention. This changes neither unitary group nor splitting character. Thus the precise local formula of [191] applies, including when $K = k \times k$.

The tensor symplectic space has complementary k-Lagrangians given by its two etale components. Every element of the acting unitary group preserves both. Its positive type-II action, as in Theorem A.5.6, is therefore defined on the Schwartz space of either component. Transport the Kudla action to that model by a full Heisenberg intertwiner.

The ratio of the two operators at each group element is scalar. Indeed their Heisenberg covariance is the same, and the Schrödinger Heisenberg commutant is $\mathbb{C}$, by Theorem A.5.30. To apply that statement over k, write $\psi = \psi_p \circ \mathrm{Tr}_{k/\mathbb{Q}_p}(a\,\cdot)$ with $a \neq 0$; the trace alternating form is nondegenerate. Quotienting both centers by their character kernels, or pushing both out to their common circle-valued character image, identifies the central-character Heisenberg representations. This is not a literal identification of the unrestricted centers. The ratio is consequently a unit scalar, and the representation laws make it a character of the acting group, which is $\mathrm{GL}_{2q}(k)$ through either etale component.

Compute that scalar on a determinant representative by a partial Fourier transform. In the coordinates of the quoted formula write $x = (x_+, x_-)$ and $a = (a_+, a_-)$, with $x_\pm \in \mathrm{M}_{m\times q}(k)$ and $a_\pm \in \mathrm{GL}_q(k)$. Fourier transform in x_+ only, using the exact symplectic dual pairing and dual Haar measure; keep x_- unchanged. This moves that component of the original tensor Lagrangian to its dual, so the new position space is one ENTIRE etale component of the tensor space. Choose dual bases in this pairing and write its coordinates as (u, v). A change of variables gives

$$\begin{aligned}\mathcal{F}_+\big[|\det a_+ \det a_-|_k^{m/2}\chi(\det a)\varphi(x_+a_+, x_-a_-)\big](u,v)\\ = \chi(\det a)\,|\det a_+|_k^{-m/2}|\det a_-|_k^{m/2}(\mathcal{F}_+\varphi)(ua_+^{-\mathsf{T}}, va_-).\end{aligned} \tag{Sk2}$$

Any Gram or half-trace coefficient is in the chosen dual pairing, not an extra scalar in (Sk2). The operator on the second line after deleting $\chi(\det a)$ is exactly the positive Jacobian pullback for the SAME symplectic transformation on this position space. Its covariance also identifies the indicated linear substitution with that transformation. Changing to the opposite etale position half is a full Fourier transform, which conjugates a positive Jacobian pullback to the corresponding positive dual pullback. Hence no phase has been introduced by the choice of half.

For $z \in K^1$ define $\lambda_\chi(z) = \chi(b)$, where $b/\overline{b} = z$. The definition is independent of the lift because $\chi|_{k^\times} = 1$; explicitly $z = (d, d^{-1})$ has lift $(d, 1)$. For the quoted Levi element,

$$\det_K m(a) = \det a/\overline{\det a}, \qquad \lambda_\chi(\det_K m(a)) = \chi(\det a).$$

Thus (Sk2) computes the ratio character on the Siegel Levi. These Levi elements include a representative of EVERY determinant in $\mathrm{GL}_{2q}(k)$: choose $a_+ = \mathrm{diag}(d, 1, \ldots, 1)$ and $a_- = 1_q$. Every character of $\mathrm{GL}_{2q}(k)$ is trivial on $\mathrm{SL}_{2q}(k)$, since elementary matrices generate that group and are commutators in $\mathrm{GL}_{2q}(k)$. For example conjugating an elementary matrix by a diagonal matrix that multiplies its entry by a fixed $c \neq 1$ writes any such elementary matrix as a commutator. The ratio is therefore determined by these representatives, and equals $\lambda_\chi \circ \det_K$ on the ENTIRE acting group. No independent local twist is inferred from a global character variation.

Use defining doubling for an arbitrary acting rank. For any skew-hermitian acting space W_1, including odd rank, its double $W_1 \oplus (-W_1)$ is hyperbolic. The standard Kudla splitting on $U(W_1)$ is the restriction of that doubled splitting along $b \mapsto (b, 1)$, in the natural inclusion of the first symplectic summand's circle cover. This is the defining doubled construction, not an application of the field-only polarization theorem to an etale algebra. In the two etale Schrödinger components the symplectic direct-sum factorization is ordinary tensor factorization. The positive type-II action restricts to the positive action on the first summand tensored with

the identity; the same is true of the defining metaplectic inclusion. Also $\det_K(b,1) = \det_K b$. The doubled formula just proved consequently gives

$$\Omega_\chi(b) = \lambda_\chi(\det_K b)\Omega_{\mathrm{pos}}(b) \tag{Sk3}$$

for arbitrary positive acting rank. This derivation starts with the fixed doubled action and computes it in an etale Schrödinger model; it does not choose a new polarization-dependent action.

The other factor is treated by interchanging the two tensor factors: multiply the original skew form by δ^{-1} and the original hermitian form by δ, for any $\bar\delta = -\delta \neq 0$. Their full-trace tensor product is unchanged, the first resulting form is hermitian and the second skew-hermitian, and neither unitary group or determinant changes. This is the symmetric Kudla construction of the other factor. Applying (Sk3) gives its factor $\lambda_{\chi_W}(\det_K a)$. The two factor splittings commute, so in an ordinary both-linear-first tensor pair the total scalar relative to the positive type-II action is

$$\lambda_{\chi_W}(\det_K a)\lambda_{\chi_V}(\det_K b). \tag{Sk4}$$

All forms used here are nondegenerate. Over $k \times k$ the hyperbolic doubled space, its etale halves and their exact dual pairings exist in every residue characteristic; no odd-prime or unramified-character hypothesis entered.

Check the original matrix pairing and both determinant inverses. Let $\mathcal{A}(v \otimes z) = \bar v\, \bar z^{\mathsf T}$. It is a k-linear isomorphism, since the usual rank-one tensors span the matrix space and conjugation is a k-linear bijection. For $x_i = \mathcal{A}(v_i \otimes z_i)$ the scalar inside the original trace is the conjugate of $h_V(v_1,v_2)h_W(z_1,z_2)$:

$$\xi \operatorname{tr}(J_S^{-1}\bar x_2^{\mathsf T} J_T x_1) = \overline{h_V(v_1,v_2)h_W(z_1,z_2)}.$$

Hermitian symmetry of J_T, J_S^{-1} and $\bar\xi = -\xi$ give this identity directly. Taking $\mathrm{Tr}_{K/k}$ proves equality with B, and hence equality for the SAME additive character.

For the original (g,h) set $a = \bar h$ and $b = \overline{g^{-\mathsf T}}$. Their displayed forms are preserved: $h^* J_T h = J_T$ and $g^* J_S g = J_S$ give $a^{\mathsf T} J_T \bar a = J_T$ and $b^{\mathsf T}\overline{J_S^{-1}}\bar b = \overline{J_S^{-1}}$. Moreover

$$\mathcal{A}(av \otimes bz) = h\mathcal{A}(v \otimes z)g^{-1}, \qquad \det_K a = (\det_K h)^{-1}, \quad \det_K b = \det_K g.$$

The determinants are norm one. For any such determinant d, the element $ed + \bar e$ is a lift of d under $x \mapsto x/\bar x$. Substitution of $\chi_V = \chi_T^{-1}$ and $\chi_W = \chi_S^{-1}$ in (Sk4) gives EXACTLY

$$\chi_T(e\det g + \bar e)^{-1}\chi_S(e\det h + \bar e).$$

The inverse of the actual linear action on X is $x \mapsto h_e^{-1} x g_e$, and its positive unitary Jacobian is $|\det h_e|_k^{-s/2}|\det g_e|_k^{t/2}$. This proves (Sk1), with the original selected half and all transposes retained.

Every algebraic opposite-half functional has the same scalar. On $\mathcal{S}(X)$ the quotient by all modulation differences is evaluation at zero. To check this without a continuity assumption, let $\varphi(0) = 0$. Local constancy makes it vanish near zero. Cover its compact support by finitely many compact open sets on each of which one modulation character is a constant different from 1, refine to a disjoint compact open partition, and split φ accordingly. Each piece is a modulation difference, by dividing it by that nonzero character minus 1. The characters required on this support exist by nondegeneracy of the opposite-half pairing. Conversely every modulation difference vanishes at zero, and evaluation is onto, by a compact open indicator taking value 1 at zero. Thus the quotient is exactly $\mathbb{C}$.

Tensoring this algebraic quotient with any N identifies the quotient with N. The positive pullback in (Sk1) acts on it by its displayed positive Jacobian, because an invertible linear substitution fixes zero. The character factor is scalar on the entire space. Every invariant algebraic functional factors through this quotient, proving the last assertion with no continuity, irreducibility or multiplicity restriction on that functional. No global or nonsplit action comparison is part of this local result. □

Lemma 4.5.5. *Let G_{pr} be a finite group of order n_0, let Z_0 be a complex representation with action ρ_0, and let Ω_0 be a set with permutations r_g indexed by $g \in G_{\mathrm{pr}}$. Fix a positive definite matrix $Q_0 \in M_{d_0}(\mathbb{Q})$ and use its Hermitian norm on $\mathbb{C}^{d_0}$, with field automorphisms acting coordinatewise. Let*

$$L_0 : Z_0 \longrightarrow (\mathbb{C}^{d_0})^{\Omega_0}$$

be one complex linear map satisfying

$$L_0(\rho_0(g)z)(\omega) = (L_0 z)(r_g\omega) \quad (g \in G_{\mathrm{pr}},\ z \in Z_0,\ \omega \in \Omega_0).$$

Let $e = \sum_{t\in G_{\mathrm{pr}}} a_t t \in \mathbb{C}[G_{\mathrm{pr}}]$ satisfy $|\sigma(a_t)| \le 1$ for every field automorphism σ of $\mathbb{C}$ and every t, and suppose $n_0 a_t$ is an algebraic integer for every t. Extend ρ_0 linearly to $\mathbb{C}[G_{\mathrm{pr}}]$.

For every $c \in \mathbb{C}$, $z \in Z_0$, $s_0 \in G_{\mathrm{pr}}$ and field automorphism σ, if

$$\|\sigma(c(L_0 z)(\omega))\|_{Q_0}^2 \le e^{B_0} \quad (\omega \in \Omega_0),$$

then

$$\|\sigma(cL_0(\rho_0(n_0 e s_0)z)(\omega))\|_{Q_0}^2 \le n_0^4 e^{B_0} \quad (\omega \in \Omega_0).$$

Consequently, for any finite family $(\omega_i)_{i\in I_0}$ in Ω_0, any real weights $w_i \ge 0$ with $\sum_i w_i \le e^{H_0}$, and any real X_{pr} with $H_0 + 4\log n_0 \le X_{\mathrm{pr}}$,

$$\sum_{i\in I_0} w_i \|\sigma(cL_0(\rho_0(n_0 e s_0)z)(\omega_i))\|_{Q_0}^2 \le e^{B_0 + X_{\mathrm{pr}}}.$$

For a prime identification ι, let $M_\omega \subseteq \mathbb{C}^{d_0}$ be $\mathcal{O}_\iota$-submodules such that $M_{r_g\omega} = M_\omega$ for every g, ω. If $c(L_0 z)(\omega) \in M_\omega$ for every ω, then

$$cL_0(\rho_0(n_0 e s_0)z)(\omega) \in M_\omega \quad (\omega \in \Omega_0).$$

All symbols in this statement are local.

Proof. Linearity and the stipulated right-translation identity give, with the same scalar c on both sides,

$$cL_0(\rho_0(n_0 e s_0)z)(\omega) = \sum_{t\in G_{\mathrm{pr}}} n_0 a_t\, c(L_0 z)(r_{ts_0}\omega).$$

Apply σ coordinatewise. Since $\sigma(n_0) = n_0$, the triangle inequality in the positive definite Hermitian norm gives

$$\begin{aligned}\|\sigma(cL_0(\rho_0(n_0 e s_0)z)(\omega))\|_{Q_0} &\le \sum_{t\in G_{\mathrm{pr}}} n_0 |\sigma(a_t)| \|\sigma(c(L_0 z)(r_{ts_0}\omega))\|_{Q_0} \\ &\le n_0^2 e^{B_0/2}.\end{aligned}$$

Squaring proves the pointwise assertion. Multiplication by the nonnegative w_i and summation then give at most

$$e^{H_0} n_0^4 e^{B_0} = e^{B_0 + H_0 + 4\log n_0} \le e^{B_0 + X_{\mathrm{pr}}}.$$

For the integral assertion, $n_0 a_t$ belongs to $\mathcal{O}_\iota$, since a root of a monic integral polynomial has ι-adic absolute value at most 1. Each $c(L_0 z)(r_{ts_0}\omega)$ belongs to $M_{r_{ts_0}\omega} = M_\omega$. Each summand in the displayed expansion therefore belongs to M_ω, and closure under finite sums proves the assertion. □

Lemma 4.5.6. *Let L be a twisted lattice on a complex vector space V, let H be a complex vector space, and let $H_{\mathrm{int}} \subseteq H$ be a set. Let t be finite and let $F_s : H \to V$ be complex-linear for each $s \in t$. Let $d, u \geq 0$ and let $(\epsilon_p)_p$ be nonnegative reals indexed by primes, with*

$$\sum_{p \in U} \epsilon_p \log p \leq u$$

for every finite prime set U. Suppose that for every identification $\iota \mid p$, every $s \in t$, every finite set J, and every family $(v_k)_{k \in J}$ in H_{int} there exist an integer $j \geq 0$, a matrix $A \in M_J(\mathcal{O}_\iota)$ and $c \in \mathcal{O}_\iota$ such that

$$j \leq d\epsilon_p, \qquad \det(A)c = p^j, \qquad F_s\left(\sum_{k \in J} A_{ik} v_k\right) \in V(\iota) \quad (i \in J).$$

Membership in $V(\iota)$ is simultaneous integrality for every twist allowed at p; the matrix is one matrix for all those twists. Then every finite family $(y_i)_{i \in I}$ with $y_i = F_{s_i} v_i$ for some $s_i \in t$ and $v_i \in H_{\mathrm{int}}$ has index at most $\#t\, d\, u$ in L.

Proof. Choose representatives s_i, v_i once for each i. Put $I_s = \{i \in I : s_i = s\}$. These sets partition I. Define $\delta_p = \#t\, d\, \epsilon_p$. Then $\delta_p \geq 0$ and, for every finite prime set U,

$$\sum_{p \in U} \delta_p \log p = \#t\, d \sum_{p \in U} \epsilon_p \log p \leq \#t\, d\, u.$$

Fix an identification $\iota \mid p$. For each nonempty I_s, apply the hypothesis to the source family $(v_i)_{i \in I_s}$. Denote the resulting data by j_s, A_s, c_s. For empty I_s use the empty matrix, with determinant 1, and put $j_s = 0$. Define $A \in M_I(\mathcal{O}_\iota)$ by its block A_s on $I_s \times I_s$ and zero entries between distinct blocks. If $i \in I_s$, linearity gives

$$\sum_{k \in I} A_{ik} y_k = F_s\left(\sum_{k \in I_s} (A_s)_{ik} v_k\right) \in V(\iota).$$

This is one membership assertion for all twists allowed at p.

For each nonempty block, multiplicativity of the absolute value and $\det(A_s)c_s = p^{j_s}$ give

$$|\iota(\det A_s)|_p \, |\iota(c_s)|_p = p^{-j_s}, \qquad |\iota(c_s)|_p \leq 1,$$

hence $|\iota(\det A_s)|_p \geq p^{-j_s}$. After simultaneously permuting rows and columns to group the partition, the determinant is the product of the block determinants. The permutation introduces no determinant factor, since it is a conjugation by a permutation matrix. Therefore

$$|\iota(\det A)|_p = \prod_{s \in t} |\iota(\det A_s)|_p \geq p^{-\sum_{s \in t} j_s} \geq p^{-\#t\, d\, \epsilon_p} = p^{-\delta_p}.$$

The penultimate inequality uses $0 \leq j_s \leq d\epsilon_p$ for every nonempty block and $j_s = 0$ otherwise. Thus A carries y into $V(\iota)$ with loss δ_p. The same construction works at every identification, with no dependence on the twist. The displayed weighted-sum bound now proves the claimed index bound by (TI).

If $t = \emptyset$, the representation of every y_i forces $I = \emptyset$. The empty matrix has determinant 1 and its row conditions are vacuous, so the assertion remains valid. The construction also covers empty blocks when $t \neq \emptyset$. □

4.6 Short integral source forms and branching

This section produces a short integral spanning set of the ramified source forms (Lemma 4.6.1): forms integral at every prime identification, of squared Weyl norm at most $e^{u_{\mathrm{err}}}$ at every point and every conjugate. At each finite adelic point a self-dual hermitian lattice has k independent vectors with bounded coordinates, by Banaszczyk's transference bound [12]; since the level is neat (Lemma 4.6.2), short evaluation vectors extend to algebraic modular forms [110] supported on single double cosets, and the ramified forms are reached by the operators $1-T_w$ at the places above N. The ramified forms have small codimension (Lemma 4.6.3), which Theorems A and B use. The number of definite double cosets is at most $\exp(x^{30})$ (Lemma 4.6.4), an explicit form of the finiteness of class numbers of Borel and Harish-Chandra [37]; the count is used again for the size bound, the good projection and the local clearing.

Lemma 4.6.1. *Put $u = fa(\log b)^3$. The ramified source space $\mathcal{S}^{\mathrm{ram}}$ is spanned over $\mathbb{C}$ by the forms φ integral at every prime identification and satisfying*

$$\|\sigma(\varphi(g))\|_W^2 \le e^u \quad \text{for every field embedding } \sigma : \mathbb{C} \longrightarrow \mathbb{C} \text{ and every } g \in \mathrm{U}(X_0)(\mathbb{A}_f).$$

The norm is the source Weyl norm of Definition 3.2.17. Moreover

$$\dim_{\mathbb{C}} \mathcal{S} \le 2 \dim_{\mathbb{C}} \mathcal{S}^{\mathrm{ram}}.$$

Proof. We first construct short vectors in the lattice attached to a finite adelic point. Write $Q = q_1 d$, $C = 6Q(fk)^2$, and let $g \in G_f = \mathrm{U}(X_0)(\mathbb{A}_f)$. Write $h(x,y) = \sum_{i=1}^k \bar{x}_i y_i$ for the standard hermitian form and put

$$L_g = E^k \cap g\widehat{\mathcal{O}}_E^k.$$

Its image under the chosen complex embeddings is a full lattice in the real Euclidean space of dimension $2fk$, with inner product $\sum_{v\in\Phi} \mathrm{Re}(x_v^* y_v)$. Indeed an integer multiple of $\mathcal{O}_E^k$ is contained in L_g, and L_g is contained in a reciprocal integer multiple of $\mathcal{O}_E^k$, because g and g^{-1} are integral outside finitely many places. Local unitarity makes L_g self-dual for the $\mathcal{O}_E$-valued hermitian pairing: its localisations are $g_w\mathcal{O}_{E,w}^k$, and each is self-dual. Thus its dual under the trace pairing is $\mathfrak{D}_E^{-1}L_g$, where $\mathfrak{D}_E$ is the different, and its real Euclidean dual is $2\mathfrak{D}_E^{-1}L_g$.

We have $Q\mathcal{O}_E \subseteq \mathfrak{D}_E$. To see this, the quadratic Gauss-sum embedding gives $E \subseteq F = \mathbb{Q}(\zeta_Q)$. The cyclotomic different divides the ideal generated by Q, so $Q\mathcal{O}_F \subseteq \mathfrak{D}_{F/\mathbb{Q}}$. The different tower identity gives $\mathfrak{D}_{F/\mathbb{Q}} = \mathfrak{D}_{F/E}\mathfrak{D}_E\mathcal{O}_F \subseteq \mathfrak{D}_E\mathcal{O}_F$; contraction to $\mathcal{O}_E$ proves the claim. Consequently the Euclidean dual of $(2/Q)L_g$ is

$$((2/Q)L_g)^* = Q\mathfrak{D}_E^{-1}L_g \subseteq L_g.$$

For $0 \ne z \in L_g$, the number $h(z,z)$ is a nonzero totally positive algebraic integer of E^+. Its norm is a positive integer, so the arithmetic-geometric mean inequality gives

$$\|z\|^2 = \sum_{v\in\Phi} v(h(z,z)) \ge f \ge 1.$$

Apply Banaszczyk's covering theorem A.6.1 to $(2/Q)L_g$, whose nonzero dual vectors therefore have norm at least 1. Weakening its bound in dimension $2fk$ gives covering radius at most $(2fk)^2$. Hence every point is within

$$R = \frac{Q(2fk)^2}{2}$$

of L_g. The lattice vectors of norm at most $3R = C$ span the real ambient space: otherwise take a unit normal v to their span and approximate $2Rv$ by $z \in L_g$ within distance R. Then $\|z\| \le 3R$ but $\langle z, v\rangle \ge R > 0$, a contradiction. These vectors therefore span E^k over E. Choose k independent ones as columns of $Y \in \mathrm{GL}_k(E)$. We have proved

$$g^{-1}Y \in \mathrm{M}_k(\widehat{\mathcal{O}}_E), \qquad |\eta(Y_{ij})| \le C \quad \text{for every complex embedding } \eta : E \longrightarrow \mathbb{C}.$$

We now use the actual one-row weights to construct integral spanning vectors in the value space. At $v \in \Phi$ put $d_v = a$ on Φ_a and $d_v = 0$ otherwise. In the integral monomial coordinates indexed by $\alpha \in \mathbb{N}^k$ with $|\alpha| = d_v$, the evaluation vector is $\mathrm{ev}_v(y) = (y^\alpha)_\alpha$. The polynomial kernel $(\sum_i x_i y_i)^{d_v}$ has diagonal coefficient matrix with entries $d_v!/\prod_i \alpha_i!$. This is precisely the prescribed integral Gram matrix for the one-row Weyl module, after ordering the monomial basis. The multinomial theorem and polynomial substitution therefore give

$$\|\,\mathrm{ev}_v(y)\|_W^2 = (\sum_i |y_i|^2)^{d_v}, \qquad \rho_v(A)\,\mathrm{ev}_v(y) = \mathrm{ev}_v(Ay).$$

The tensor Gram form is the product of these forms. Denote the chosen embedding at v by F_v, and write $\overline{F}_v(z) = F_v(\bar{z})$. For tuples y_v whose entries are roots of unity, set

$$\xi_y = \bigotimes_{v\in\Phi} \mathrm{ev}_v(\overline{F}_v(Y)y_v).$$

At a prime identification $j : \mathbb{C} \to \mathbb{C}_\ell$, applying the inverse-$g$ action defining the gauge replaces $\overline{F}_v(Y)$ by the image of $g^{-1}Y$ under the finite adelic extension of $j \circ \overline{F}_v$. Its entries are integral, as are all $j(y_{v,i})$. Evaluation vectors are monomials with integral coefficients. Thus ξ_y has gauge at most 1 at g, simultaneously for all prime identifications.

Every complex embedding σ sends each root of unity to a root of unity and restricts on E to an embedding of E. Hence every coordinate of $\sigma(\overline{F}_v(Y))\sigma(y_v)$ has modulus at most kC. The displayed norm identity gives

$$\|\sigma(\xi_y)\|_W^2 \le (k^3C^2)^{\sum_v d_v} \le C^{5fa} \le C^{6f(k+a)},$$

because $C \ge k \ge 1$ and $\sum_v d_v = a\#\Phi_a \le fa$. The vectors ξ_y span W_λ: a linear functional vanishing on all of them, pulled back through the invertible action of $(\overline{F}_v(Y))_v$, is a polynomial vanishing on a product of infinite sets of roots of unity. Successive one-variable polynomial vanishing forces that polynomial to be zero. The distinct monomials are independent, so the original functional is zero.

We extend these vectors to short forms. By Lemma 4.6.2(c), the stabiliser of every double coset in $\mathrm{U}(X_0)(E^+)\backslash G_f/\mathrm{K}_0$ is trivial. A vector ξ_y at a representative g therefore extends uniquely to the form supported on that double coset and with value $\rho(\gamma)\xi_y$ at $\gamma g\kappa$. This is well-defined by triviality of the stabiliser. There are finitely many such double cosets by Lemma 4.6.4, applied with $n = k \le n_m$. Indeed an integral unitary matrix congruent to the identity modulo $\pi_M\pi_N$ reduces to the identity at every chosen place, since those places divide π_M, and its last row is congruent to $(0,\dots,0,1)$ modulo π_N. Thus the principal congruence subgroup of level $\pi_M\pi_N$ is contained in K_0, exactly as required there. Choose a finite set $\mathcal{R}$ of double-coset representatives. Evaluation gives an injective linear map

$$\mathcal{S} \longrightarrow \bigoplus_{g\in\mathcal{R}} W_\lambda, \qquad \varphi \longmapsto (\varphi(g))_{g\in\mathcal{R}},$$

because equivariance determines a form everywhere from those values. In particular $\mathcal{S}$ is finite-dimensional independently of any prior assertion of that fact. The value-space spanning just proved shows that these forms span $\mathcal{S}$.

The gauge bound holds at every point of the supporting double coset. In fact the inverse-$\gamma g\kappa$ action cancels $\rho(\gamma)$, leaving the inverse-g vector acted on by κ^{-1}; the latter preserves the integral Weyl lattice because κ is integral. At every complex embedding the rational unitary action preserves the prescribed Gram form. To check this also after applying an arbitrary σ, the entries of the rational matrices lie in the Galois CM field E, on which complex conjugation commutes with every embedding. The conjugated matrices are therefore still unitary at each place. Thus the conjugate squared norm of each constructed form at every point is at most $C^{6f(k+a)}$. By Lemma 4.3.5,

$$4^f C^{6f(k+a)} \le e^u.$$

We have therefore spanned $\mathcal{S}$ by globally integral forms with all conjugate squared norms at most $e^u/4^f$.

Finally, for each $w \mid \pi_N$ let $T_w\varphi(g) = \varphi(gz_w)$, put $n = N-1$, and set $A_w = n^{-1}\sum_{i=0}^{n-1} T_w^i$. The scalar z_w^n belongs to K_0, so $T_w^n = 1$. The commuting finite-order operators T_w are simultaneously diagonalizable over $\mathbb{C}$. On a common eigenspace A_w is 1 exactly when the T_w-eigenvalue is 1, and otherwise is 0. Consequently

$$\mathcal{S}^{\mathrm{ram}} = \mathrm{im} \prod_{w\mid\pi_N} (1 - T_w).$$

There are at most f places $w \mid \pi_N$: $N > q_1 d$ by Lemma 4.3.5, so N is unramified in E; the relation $\pi_N\bar{\pi}_N = N$ excludes both a conjugate pair and a conjugation-fixed place among the divisors of π_N, of which there can therefore be at most $[E:\mathbb{Q}]/2 = f$.

Each T_w preserves integrality because its scalar matrix is integral at all finite places. If all conjugate squared norms of φ are at most B, then

$$\|\sigma(\varphi(g) - \varphi(gz_w))\|_W^2 \le 2\|\sigma(\varphi(g))\|_W^2 + 2\|\sigma(\varphi(gz_w))\|_W^2 \le 4B.$$

Thus the product $\prod_w (1 - T_w)$ sends the spanning family just constructed to globally integral ramified forms with all conjugate squared norms at most e^u. Its image is the whole ramified space, which proves the spanning assertion. Finally Lemma 4.6.3 and Lemma 4.3.5 give

$$\dim \mathcal{S}^{\mathrm{ram}} \ge (1 - f/(N-1)) \dim \mathcal{S} \ge \tfrac{1}{2} \dim \mathcal{S},$$

as required. □

Lemma 4.6.2. *Let D be admissible data. Then:*

(a) every chosen place w lies above p_0 or p_1, is unramified over it, and satisfies $w \neq \bar{w}$; the place $\bar{w}$ is not a chosen place;

(b) if w is a chosen place, $n \ge 1$, and $\gamma \in \mathrm{GL}_n(\mathcal{O}_{E_w})$ has finite order and $\gamma \equiv 1 \pmod{w}$, then $\gamma = 1$;

(c) let V be a definite hermitian space over E with an $\mathcal{O}_E$-lattice L, let w be a chosen place, and let $K \subseteq \mathrm{U}(V)(\mathbb{A}_{E^+,f})$ be an open subgroup of the stabiliser of $\widehat{L}$ each of whose elements acts trivially on L_w/wL_w. Then $\mathrm{U}(V)(E^+) \cap gKg^{-1} = \{1\}$ for every $g \in \mathrm{U}(V)(\mathbb{A}_{E^+,f})$.

Proof. (a). By (AD5), $\pi_M\bar{\pi}_M = p_0p_1$ in $\mathcal{O}_K$ with $p_0 \neq p_1$ primes and $p_i \ge b^{10}$. The discriminant of K is $-d$ ($d \equiv 7 \bmod 8$), and $d \le C_P < b$ by Lemma 2.3.6 and (AD1), so $p_i \nmid d$ and p_i is unramified in K; as p_i is the norm of the ideal $(\pi_M) + (p_i)$, it splits, $(p_i) = \mathfrak{p}_i\bar{\mathfrak{p}}_i$ with $\mathfrak{p}_i \neq \bar{\mathfrak{p}}_i$, and comparing norms (π_M) is a product of one prime from each pair $\{\mathfrak{p}_i, \bar{\mathfrak{p}}_i\}$. Hence

$(\pi_M)+(\bar\pi_M)=\mathcal{O}_K$. A chosen place w contains π_M; if $\bar w$ were chosen, w would contain $\bar\pi_M$ too, hence 1. So $\bar w$ is not chosen and in particular $w\neq\bar w$. The prime below w divides p_0p_1. The field $E=K(\zeta_{q_1})$ is unramified outside the primes dividing dq_1, and the primes dividing $q_1=q_0T(s)$ divide q_0 or 6; all are at most $C_P<p_i$ by Lemma 2.3.6. So w is unramified over p_i.

(b). Let p be the residue characteristic of w and $\varpi=p$, a uniformiser of $\mathcal{O}_{E_w}$ by (a); $p\geq b^{10}\geq 3$. Suppose $\gamma\neq 1$. Replacing γ by a suitable power we may assume that γ has prime order ℓ; the power is still $\equiv 1 \pmod w$. Write $\gamma=1+p^aX$ with $a\geq 1$ and $X\in \mathrm{M}_n(\mathcal{O}_{E_w})$, $X\not\equiv 0 \pmod p$. Then

$$0=\gamma^\ell-1=\ell p^aX+\sum_{i=2}^{\ell}\binom{\ell}{i}p^{ia}X^i.$$

If $\ell\neq p$, all terms with $i\geq 2$ are divisible by p^{2a} and ℓ is a unit, so $X\equiv 0 \pmod{p^a}$, a contradiction. If $\ell=p$, the terms $2\leq i\leq p-1$ are divisible by p^{1+2a} and the last by p^{pa} with $pa\geq 2a+1$ as $p\geq 3$; so $p^{a+1}X\equiv 0 \pmod{p^{2a+1}}$, again $X\equiv 0 \pmod p$. Hence $\gamma=1$.

(c). Put $G=\mathrm{U}(V)$ and $\Gamma=G(E^+)$. As Γ is discrete in $G(\mathbb{A}_{E^+})$ and $G(E^+\otimes\mathbb{R})$ is compact (V definite), the projection $G(\mathbb{A}_{E^+})\to G(\mathbb{A}_{E^+,f})$ is proper and injective on Γ, so Γ is discrete in $G(\mathbb{A}_{E^+,f})$. Hence $\Gamma\cap gKg^{-1}$, discrete in a compact group, is finite. Let $\gamma\in\Gamma\cap gKg^{-1}$, $\gamma=gkg^{-1}$ with $k\in K$. Let v be the place of E^+ below w; by (a) it splits as $w\bar w$, and $G(E^+_v)\to\mathrm{GL}(V\otimes_E E_w)$, $h\mapsto h_w$, is an isomorphism. Then $k_w\in\mathrm{GL}(L_w)$ has finite order, and $k_w\equiv 1 \pmod w$ by hypothesis; by (b), in a basis of L_w, $k_w=1$. So $\gamma_w=g_wk_wg_w^{-1}=1$, and since $V\to V\otimes_E E_w$ is injective, $\gamma=1$. □

Lemma 4.6.3. *Let D be admissible data. Then*

$$\dim_{\mathbb{C}}\mathcal{S}^{\mathrm{ram}}\ \geq\ \Big(1-\frac{f}{N-1}\Big)\dim_{\mathbb{C}}\mathcal{S}\qquad\textit{and}\qquad\frac{f}{N-1}\ \leq\ b^{-8}.$$

Proof. Step 1: sizes. By (AD5), $M=p_0p_1\geq b^{20}$; $\kappa\geq 1$ by (AD4), $\mathrm{rd}_E\geq 1$, $2\pi<b$ and $m\geq 2$, so by the definition of $\log N_-$ in (AD6)

$$\log N\geq\log N_-\geq(m-1)\log\frac{M\,\mathrm{rd}_E}{2\pi}-\tfrac{15}{2}\log b\geq 19\log b-\tfrac{15}{2}\log b>11\log b.$$

So $N>b^{11}$ and $N-1\geq b^9$. Also $f=\varphi(q_1)\leq q_1\leq b$ by Lemma 2.3.7 and (AD2) ($q_1x_1\mid b$). Hence $f/(N-1)\leq b/b^9=b^{-8}$.

Step 2: the places dividing π_N. $E=K(\zeta_{q_1})$ is unramified outside the primes dividing dq_1, all at most $C_P<b<N$ (Lemma 2.3.6), so N is unramified in E and $v_w(N)=1$ for every place $w\mid N$. If $w\mid\pi_N$ then $\bar w\nmid\pi_N$: otherwise $v_w(N)=v_w(\pi_N)+v_w(\bar\pi_N)\geq 2$ (if $w=\bar w$, $v_w(N)=2v_w(\pi_N)$ is even). So the places dividing π_N contain no conjugate pair among the at most $[E:\mathbb{Q}]=2f$ places above N, and there are at most f of them. None is a chosen place (those lie above $p_0p_1\neq N$).

Step 3: one scaling. Fix $w\mid\pi_N$, write $z=z_w$, acting on $\mathcal{S}$ by $(z\varphi)(g)=\varphi(gz)$ (z as a scalar matrix, central in $\mathrm{U}(X_0)(\mathbb{A}_{E^+,f})$). A scalar z^i is integral and 1 at the chosen places and at the other places dividing π_N; its last row is $\equiv(0,\dots,0,1)$ modulo π_N if and only if $z_w^i\equiv 1 \pmod w$, that is, $(N-1)\mid i$, since z_w reduces to a generator of $k_w^\times$ of order $N-1$. So $K=K_0\langle z\rangle$ is a group containing K_0 as a normal subgroup with K/K_0 cyclic of order $N-1$, and $z^{N-1}\in K_0$ acts trivially on $\mathcal{S}$. Both K_0 and K consist of integral elements reducing to 1 at every chosen place, so by Lemma 4.6.2(c) (for X_0 with lattice $\mathcal{O}_E^k$) $\Gamma\cap gK'g^{-1}=\{1\}$ for $K'\in\{K_0,K\}$, $\Gamma=\mathrm{U}(X_0)(E^+)$ and every g. For such K' the space $\mathcal{S}(K')$ of functions φ with $\varphi(\gamma g\kappa)=\gamma\cdot\varphi(g)$

($\kappa \in K'$) has dimension $\#(\Gamma\backslash G_f/K') \cdot \dim W_\lambda$ (values at representatives are arbitrary vectors fixed by the trivial groups $\Gamma \cap g_x K' g_x^{-1}$). The fibre of $\Gamma\backslash G_f/K_0 \to \Gamma\backslash G_f/K$ over $\Gamma g K$ is $(\Gamma \cap gKg^{-1})\backslash gK/K_0 \cong K/K_0$, of size $N-1$. Since $\ker(1-z) = \mathcal{S}(K)$ and $\mathcal{S} = \mathcal{S}(K_0)$,

$$\dim \ker(1 - z_w) = \frac{\dim \mathcal{S}}{N-1}.$$

Step 4: all scalings. The operators z_w commute, and $(\sum_{i=0}^{N-2} z_w^i)(1 - z_w) = 1 - z_w^{N-1} = 0$ on $\mathcal{S}$. Hence the range of $P = \prod_{w|\pi_N}(1 - z_w)$ lies in $\bigcap_w \ker \sum_i z_w^i = \mathcal{S}^{\mathrm{ram}}$. As $\dim\ker(AB) \le \dim\ker A + \dim\ker B$, Steps 2 and 3 give $\dim\ker P \le f \dim \mathcal{S}/(N-1)$, so

$$\dim \mathcal{S}^{\mathrm{ram}} \ge \dim \mathcal{S} - \dim\ker P \ge \Big(1 - \frac{f}{N-1}\Big)\dim \mathcal{S}.$$

□

Lemma 4.6.4. *Let $1 \le n \le 2n_m$, let $G = \mathrm{U}(E^n, \sum_i \bar{z}_i z_i)$, and let $\mathrm{K} \subseteq G(\mathbb{A}_{E^+,f})$ be a subgroup containing the principal congruence subgroup of level $\pi_M\pi_N$ in the stabiliser of $\widehat{\mathcal{O}}_E^n$. Then*

$$\#\big(G(E^+)\backslash G(\mathbb{A}_{E^+,f})/\mathrm{K}\big) \le \exp(x^{30}).$$

Proof. Put $Q = q_1 d$, $C = 6Q(fn)^2$, and let $\mathrm{K}_{\max}$ be the stabiliser of $\widehat{\mathcal{O}}_E^n$. We first prove the maximal-level estimate

$$\#\big(G(E^+)\backslash G(\mathbb{A}_{E^+,f})/\mathrm{K}_{\max}\big) \le (1 + 2C^3)^{4fn^2}.$$

For each double coset choose a representative g. By Lemma 4.6.5 choose $Y \in \mathrm{GL}_n(E)$ such that $A = g^{-1}Y$ is integral at every finite place and every conjugate of every entry of Y has modulus at most C. Apply the additive assertion of that lemma entrywise to A^{-1}, obtaining $X \in \mathrm{M}_n(E)$ such that

$$A^{-1} - X \in \mathrm{M}_n(\widehat{\mathcal{O}}_E), \qquad |\eta(X_{ij})| \le 4f^2 Q.$$

In these formulas rational matrices are diagonally embedded into the finite adeles. Put $H = Y^*Y$, where $*$ is conjugate transpose. Both H and HX have entries in $\mathcal{O}_E$. Indeed Y has columns in the self-dual lattice $L_g = E^n \cap g\widehat{\mathcal{O}}_E^n$, while

$$AX = I - A(A^{-1} - X)$$

is integral, so the columns of YX also belong to L_g. Their hermitian pairings with the columns of Y are the entries of HX. All these pairings are integral at every finite place and lie in E, hence belong to $\mathcal{O}_E$.

For every complex embedding η, the entries satisfy

$$|\eta(H_{ij})| \le nC^2 \le C^3, \qquad |\eta((HX)_{ij})| \le n^2C^2(4f^2Q) \le C^3.$$

Here $C \ge n$, and $4Q(fn)^2 \le C$. We claim that the pair (H, HX) determines the double coset. Suppose another choice g', Y', A', X' gives the same pair. Then $Y'^*Y' = Y^*Y = H$, and invertibility of H gives $X' = X$. The matrix $\gamma = YY'^{-1} \in \mathrm{GL}_n(E)$ is unitary, because

$$\gamma^*\gamma = (Y'^{-1})^* H Y'^{-1} = I.$$

Moreover

$$A'A^{-1} = A'(A^{-1} - X) + A'X'$$

is integral. It equals $k = (\gamma g')^{-1} g$, which is unitary. Therefore $k^{-1} = k^*$ is integral as well, so $k \in \mathrm{K}_{\max}$ and $g = \gamma g' k$. The double cosets are equal. Choosing one such pair for each class therefore gives an injection into pairs of integral matrices whose conjugate entries are bounded by C^3.

For completeness we count these integral entries. If $0 \neq z \in \mathcal{O}_E$, its norm to $\mathbb{Q}$ is a nonzero integer; consequently $\max_\eta |\eta(z)| \geq 1$. Thus distinct elements of $\mathcal{O}_E$, in $\mathbb{C}^\Phi$ with the maximum of the complex moduli, are separated by distance at least 1. The open balls of radius $1/2$ about the elements with all conjugates bounded by a real number $B \geq 0$ are disjoint and contained in the ball of radius $B + 1/2$. The Minkowski image of $\mathcal{O}_E$ is a lattice, so there are finitely many such elements. Comparing volumes in real dimension $2f$ bounds their number by $(1 + 2B)^{2f}$. Taking $B = C^3$ and counting the $2n^2$ entries of (H, HX) proves the asserted maximal-level estimate.

We pass to the required principal level. Put $J = MN$, a positive rational integer. The relation

$$(\pi_M \pi_N)\overline{(\pi_M \pi_N)} = MN$$

shows that the principal congruence subgroup $\mathrm{K}(J)$ is contained in the principal subgroup of level $\pi_M \pi_N$, hence in K. Reduction modulo J injects $\mathrm{K}_{\max}/\mathrm{K}(J)$ into $\mathrm{GL}_n(\mathcal{O}_E/J\mathcal{O}_E)$. Therefore

$$[\mathrm{K}_{\max} : \mathrm{K}(J)] \leq \#\mathrm{M}_n(\mathcal{O}_E/J\mathcal{O}_E) = J^{2fn^2}.$$

Here $\#(\mathcal{O}_E/J\mathcal{O}_E) = J^{[E:\mathbb{Q}]} = J^{2f}$. Over each maximal-level double coset the finer $\mathrm{K}(J)$-classes are an orbit set of the rational stabiliser acting on $\mathrm{K}_{\max}/\mathrm{K}(J)$, and in particular there are at most this index many. The map from the $\mathrm{K}(J)$-class set onto the K-class set is surjective. We obtain

$$\#(G(E^+)\backslash G(\mathbb{A}_{E^+,f})/\mathrm{K}) \leq (1 + 2C^3)^{4fn^2} (MN)^{2fn^2}.$$

We finish by bounding its logarithm using only the stated scales. By Lemma 2.3.6, $\log x \geq 10^{14} C_P$, and $C_P \geq q_0, d, 1$ by Definition 2.3.3; in particular $x \geq 10^6$. The tower exponent satisfies $s \geq 1$, $s \leq 32 \log x$ and $2^s = m^{22} \leq 2^{22} x^{11}$. From the explicit tower numbers,

$$l_0(s) \leq 81s, \qquad q_1 = 3q_0 l_0(s) 2^s \leq 7776 \cdot 2^{22} q_0 (\log x) x^{11} \leq (\log x)^2 x^{11} \leq x^{13},$$

where $7776 \cdot 2^{22} < 10^{14}$ and $q_0 \leq 10^{-14} \log x$. Hence $f \leq q_1 \leq x^{13}$ and $Q = q_1 d \leq x^{14}$. The rank formula gives

$$n_m = r + 1 + 5mr \leq 3x^5 + 1 + 30x^{11/2} \leq 34x^{11/2} \leq x^6,$$

using $x \geq 34^2$. Since $n \leq 2n_m$, we have $n \leq 2x^6$, so $fn^2 \leq 4x^{25}$ and

$$C = 6Q(fn)^2 \leq 24x^{52} \leq x^{53}, \qquad \log(1 + 2C^3) \leq \log 3 + 3 \log C \leq 160 \log x.$$

Write $L = \log b$. The scale inequality $100 \log L \leq \log x$ gives $L^{100} \leq x$, so $L \leq \sqrt{x}$. Also $\log x \leq \sqrt{x}$ for $x \geq 10^6$. By (AD5) and Lemma 4.3.5,

$$\log(MN) \leq 22L + 30mL \leq (22 + 60\sqrt{x})L \leq 82x.$$

Consequently the logarithm of the preceding class bound is at most

$$4fn^2 \log(1 + 2C^3) + 2fn^2 \log(MN) \leq 2560x^{25} \log x + 656x^{26} \leq 3216x^{26} \leq x^{30},$$

since $x \geq 10^6$ implies $x^4 \geq 3216$. Exponentiating proves the result. □

Lemma 4.6.5. *Put $Q = q_1 d$. For every integer $n \geq 1$ and every finite adelic unitary matrix g for the standard hermitian form on E^n, there is $Y \in \mathrm{GL}_n(E)$ such that*

$$g^{-1}Y \in \mathrm{M}_n(\widehat{\mathcal{O}}_E), \qquad |\eta(Y_{ij})| \leq 6Q(fn)^2$$

for every complex embedding $\eta : E \hookrightarrow \mathbb{C}$ and all i, j. Moreover, for every $y \in \mathbb{A}_{E,f}$ there is $a \in E$ such that

$$y - a \in \widehat{\mathcal{O}}_E, \qquad |\eta(a)| \leq 4f^2Q \quad \textit{for every complex embedding } \eta : E \hookrightarrow \mathbb{C}.$$

Proof. The integer Q belongs to the different $\mathfrak{D}_E$. Indeed the quadratic Gauss-sum identity embeds $K = \mathbb{Q}(\sqrt{-d})$ in $\mathbb{Q}(\zeta_d)$, so $E \subseteq F = \mathbb{Q}(\zeta_Q)$. The cyclotomic different divides (Q); hence

$$Q\mathcal{O}_F \subseteq \mathfrak{D}_{F/\mathbb{Q}} = \mathfrak{D}_{F/E}\mathfrak{D}_E\mathcal{O}_F \subseteq \mathfrak{D}_E\mathcal{O}_F.$$

Contracting to $\mathcal{O}_E$ gives $Q\mathcal{O}_E \subseteq \mathfrak{D}_E$.

Write $h(z, z') = \sum_i \bar{z}_i z'_i$, and put $L_g = E^n \cap g\widehat{\mathcal{O}}_E^n$. Under the chosen complex embeddings this is a full real lattice in dimension $2fn$: an integer multiple of $\mathcal{O}_E^n$ is contained in it and it is contained in a reciprocal integer multiple of $\mathcal{O}_E^n$, by bounded denominators of g and g^{-1}. Its hermitian dual is itself, since its finite localisations are the unitary transforms of the standard self-dual lattice. Thus its trace dual is $\mathfrak{D}_E^{-1}L_g$. For the real inner product $\sum_v \operatorname{Re} h_v$, which is half the trace, its Euclidean dual is $2\mathfrak{D}_E^{-1}L_g$. Consequently

$$((2/Q)L_g)^* = Q\mathfrak{D}_E^{-1}L_g \subseteq L_g.$$

If $0 \neq z \in L_g$, the element $h(z, z)$ is a nonzero totally positive algebraic integer in E^+. Its norm is a positive integer, so

$$\|z\|^2 = \sum_v v(h(z, z)) \geq f \geq 1.$$

Therefore the nonzero dual vectors of $(2/Q)L_g$ have norm at least 1. Banaszczyk's theorem A.6.1 in dimension $2fn$ gives covering radius at most fn, which in particular is at most $(2fn)^2$. Rescaling, every point of the real ambient space is within

$$R = Q(2fn)^2/2$$

of L_g. The vectors of L_g of norm at most $3R$ span that space: if their span were proper, choose a unit normal v and approximate $2Rv$ by $z \in L_g$ within R. Then $\|z\| \leq 3R$ and $\langle z, v\rangle \geq R > 0$, a contradiction. They therefore span E^n over E. Choosing n independent ones as columns of Y gives $g^{-1}Y$ integral, and every complex coordinate is bounded by $3R = 6Q(fn)^2$. This proves the first assertion.

For the additive assertion, choose a nonzero positive integer m_0 such that $m_0 y$ is integral at every finite place. The Chinese remainder theorem in $\mathcal{O}_E$ gives $b_0 \in \mathcal{O}_E$ with $b_0 \equiv m_0 y$ modulo $m_0\mathcal{O}_{E,w}$ at every place dividing m_0. Set $a_0 = b_0/m_0$. At the places dividing m_0 the congruence gives $y - a_0$ integral; at all other places both terms are integral. Thus $y - a_0 \in \widehat{\mathcal{O}}_E$.

Apply the same dual-lattice argument with $n = 1$ and $g = 1$, now retaining Banaszczyk's bound f for the covering radius of $(2/Q)\mathcal{O}_E$. The covering radius of $\mathcal{O}_E$ in $\mathbb{C}^\Phi$ is therefore at most $Qf/2$. Choose $b_1 \in \mathcal{O}_E$ within this distance of the archimedean vector of a_0, and put $a = a_0 - b_1$. Then $y - a$ remains integral and

$$|\eta(a)| \leq Qf/2 \leq 4f^2Q$$

at every embedding, including the conjugate embeddings which have the same absolute values. This proves the second assertion. □

The composite of definite lifts ends on X_m, while the forms it is compared with live on Y_m, of rank one less. The restriction is multiplicity free, by the branching rule of Gelfand, Tsetlin and Zhelobenko [101, 334], quoted in Molev's form [211] (see [92]). The size bound needs more: a branching map that is an integer matrix in integral bases, with an explicit bound for its norm (Lemma 4.6.7, specialised in Lemma 4.6.6). The matrix is built from Gram forms, hook numbers and the straightening law for bitableaux [194, 91, 73, 71, 67] (Lemma 4.6.8), and its equivariance passes from compact unitary groups to general linear groups by Weyl's unitarian trick (Lemma 4.6.13). For the invariant theory behind these models see [135].

Lemma 4.6.6. *All supplementary symbols in this statement are local. Put $\delta = \Delta/2$, $T = t + 1$ and $n = n_m$. For $\tau \in \Phi$ and $1 \le h < n$ set*

$$c_\tau(h) = \begin{cases} a\mathbf{1}_{\tau\in\Phi_a}, & h = 1, \\ \kappa_1(\tau) - n_1/2, & h = k, \\ 2\delta - T, & h > k,\ T \mid h - k, \\ 0, & \text{otherwise}, \end{cases} \qquad v(h) = \begin{cases} \delta - T/2, & h \ge k,\ T \mid h - k, \\ 0, & \text{otherwise}. \end{cases}$$

Put $e_\tau = n_{m-1}/2 - \kappa_m(\tau)$ and, for $1 \le h < n - 1$, put

$$d_\tau(h) = v(h) + c_\tau(h + 1) - v(h + 1).$$

These are nonnegative column counts. Let W_X be the tensor over $\tau \in \Phi$ of the rank-n Weyl modules with column counts $c_\tau(h)$ and determinant exponents e_τ, and let W_Y be the tensor of the rank-$(n-1)$ Weyl modules with column counts $d_\tau(h)$ and the same determinant exponents e_τ. Give them the standard integral filling lattices and rational Weyl Gram norms.

There exists an integer coordinate matrix $\beta : W_X \to W_Y$ such that over every characteristic-zero field R,

$$\beta\, \rho_X((\mathrm{diag}(g_\tau, z_\tau))_\tau) = \left(\prod_{\tau\in\Phi} z_\tau^{a\mathbf{1}_{\tau\in\Phi_a} - m\delta} \right) \rho_Y((g_\tau)_\tau)\, \beta$$

for $g_\tau \in \mathrm{GL}_{n-1}(R)$ and $z_\tau \in R^\times$, and

$$\|\beta\xi\|_Y^2 \le e^{x^{70}} \|\xi\|_X^2 \quad (\xi \in W_X(\mathbb{C})).$$

Every complex linear map from W_X to W_Y intertwining the unitary subgroup $\prod_{\tau\in\Phi} \mathrm{U}(n-1)$ embedded by $\mathrm{diag}(g_\tau, 1)$ is a complex multiple of β.

For this same matrix put

$$A_N = \prod_{\tau\in\Phi} \tau(\bar\pi_N)^{-(a\mathbf{1}_{\tau\in\Phi_a} - m\delta)}.$$

Then, for every field automorphism σ of $\mathbb{C}$ and every $\xi \in W_X(\mathbb{C})$,

$$\|\sigma(A_N \beta\xi)\|_Y^2 \le \exp\left(x^{70} - (\#\Phi_a\, a - fm\delta) \log N \right) \|\sigma\xi\|_X^2.$$

Proof. **The columns satisfy the branching conditions.** The rank conditions make r odd, m even, k even and $T = 5r + 1$ even. In particular all n_j are even. By Lemma 4.3.5, $k \ge 2$ and $k \le r$. Since $r \ge 3$ and $\delta = r^8$, we have $\delta \ge 3r \ge T/2$. Thus $v(h) \ge 0$. At $h = k$,

$$c_\tau(k) - v(k) = \kappa(\tau) + 2\delta - k/2 \ge 0;$$

at $h > k$ with $T \mid h-k$ one has $c_\tau(h) = 2v(h)$; at other heights $v(h) = 0$. Hence $0 \le v(h) \le c_\tau(h)$ at every height. Therefore each $d_\tau(h)$ is a nonnegative integer. Also $v(h) = 0$ for $h < k$, and $v(n-1) = 0$, since $n - 1 - k = mT - 1$ is not divisible by T.

Apply Lemma 4.6.7 with the columns that keep their height equal to $v(h)$, the columns that lose one box equal to $c_\tau(h) - v(h)$, and the threshold equal to k. The target column formula is exactly $d_\tau(h)$. The target determinant exponent is $e_\tau + v(n-1) = e_\tau$. Thus every hypothesis of that branching result is satisfied.

The central character has the stated exponent. The nonzero possible column heights of c_τ are $1, k, n_1, \dots, n_{m-1}$. Write $a_\tau = a\mathbf{1}_{\tau\in\Phi_a}$ and $v_* = \delta - T/2$. The number of columns losing one box is

$$\sum_{h=1}^{n-1}(c_\tau(h) - v(h)) = a_\tau + \kappa_1(\tau) - n_1/2 + (m-2)v_*.$$

The exponent supplied by the branching theorem adds e_τ to this number. Since $\kappa_m(\tau) - \kappa_1(\tau) = 2(m-1)\delta$ and $n_{m-1} - n_1 = (m-2)T$, the result is

$$a_\tau + \frac{(m-2)T}{2} - 2(m-1)\delta + (m-2)(\delta - T/2) = a_\tau - m\delta.$$

This proves the claimed block-scalar equivariance.

For completeness, the scalar exponent is also $|\lambda_\tau^X| - |\lambda_\tau^Y|$, the difference of the sums of the highest-weight entries: each shortened column loses one box, and the determinant exponents contribute $ne_\tau - (n-1)e_\tau = e_\tau$. If a map intertwines $\mathrm{diag}(g_\tau, 1)$ for unitary g_τ, the identity

$$\mathrm{diag}(g_\tau, z_\tau) = z_\tau I_n \ \mathrm{diag}(z_\tau^{-1} g_\tau, 1)$$

for unit scalars z_τ shows that it also has the required $\mathrm{U}(1)$ character. The multiplicity assertion of Lemma 4.6.7 therefore gives the stated spanning assertion.

Only columns of height at least k enter the norm cost. For $h \ge k$ one has $v(h) \le \delta$, and $h + 1 > k$ implies $c_\tau(h+1) - v(h+1) \le \delta$. Thus $d_\tau(h) \le 2\delta = \Delta$. Consequently

$$\sum_{\tau\in\Phi}\sum_{k\le h<n-1} h\, d_\tau(h) \le fn^2\Delta.$$

Set $L_{\mathrm{br}} = fn^2\Delta \in \mathbb{Z}_{>0}$. The branching theorem gives $\|\beta\xi\|_Y^2 \le L_{\mathrm{br}}^{6L_{\mathrm{br}}}\|\xi\|_X^2$.

Here is the complete scale comparison for this particular norm cost. Put $C = C_P$, with C_P defined in Definition 2.3.3, and $L_b = \log b$. The admissible scales give $\log L_b \ge 10^{12}C$ and $\log x \ge 100 \log L_b$, so $C \le 10^{-14}\log x$ and $x \ge e^{10^{14}}$. The tower formula gives $l_0(s) \le 81s$, $q_1 = 3q_0 l_0(s) m^{22}$ and $f \le q_1$. Using $q_0 \le C$, $s \le 32\log x$ and $m \le 2\sqrt{x}$,

$$f \le 7776\, 2^{22} C\, x^{11} \log x \le x^{11}(\log x)^2 \le x^{12}.$$

The last inequality follows by setting $w = \log x \ge 6$ and using $e^w \ge w^3/6 \ge w^2$. Further, $m \le x$, $k \le r \le 3x^5$ and $t = 5r$ give

$$n = k + m(t+1) \le 19x^6 \le x^7, \qquad \Delta = 2r^8 \le 13122x^{40} \le x^{41}.$$

Hence $1 \le L_{\mathrm{br}} \le x^{67}$ and

$$6L_{\mathrm{br}} \log L_{\mathrm{br}} \le 402x^{67}\log x \le 402x^{68} \le x^{70}.$$

This proves the required operator inequality. It does not use the long height-one columns in the cost.

The normalization by the level gives the stated saving. For each complex embedding τ of the CM field, the identity $\pi_N\bar{\pi}_N = N$ gives $|\sigma(\tau(\bar{\pi}_N))|^2 = N$: every complex embedding of a CM field respects its complex conjugation. Therefore

$$|\sigma(A_N)|^2 = N^{-\sum_\tau (a\mathbf{1}_{\tau\in\Phi_a} - m\delta)} = \exp\left(-(\#\Phi_a\, a - fm\delta)\log N\right).$$

The matrix β has integer entries and so commutes with coordinatewise σ. Apply the preceding operator bound to $\sigma\xi$ and multiply by this scalar squared absolute value. The resulting inequality is exactly the final display in the statement. □

Lemma 4.6.7. *Let Φ be a finite set and, for $\tau \in \Phi$, $n_\tau \geq 1$ and $N_\tau = n_\tau + 1$; the rows $1, \dots, n_\tau$ of the rank N_τ are those of the rank n_τ, and for $g = (g_\tau)$, $g_\tau \in \mathrm{GL}_{n_\tau}(R)$, and $s = (s_\tau)$, $s_\tau \in R^\times$, write* $\mathrm{diag}(g, s)$ *for the family of block matrices* $\mathrm{diag}(g_\tau, s_\tau)$. *For a weight κ and $e\colon \Phi \to \mathbb{Z}$ put $P_{\kappa,e}(x) = \prod_\tau \det(x_\tau)^{e_\tau} S_\kappa(x)$. Let γ, κ' be weights for the ranks N, $\kappa = \gamma + \kappa'$, $e\colon \Phi \to \mathbb{Z}$, and let κ_0 be the weight for the ranks n and $e_0\colon \Phi \to \mathbb{Z}$ given by*

$$\kappa_0(\tau, h) = \kappa'(\tau, h) + \gamma(\tau, h+1) \quad (1 \leq h < n_\tau), \qquad e_{0,\tau} = e_\tau + \kappa'(\tau, n_\tau).$$

Let $q \in \mathbb{N}$ with $\kappa'(\tau, h) = 0$ for all τ and $h < q$, and put $w_\tau = \sum_{1\leq h<N_\tau} \gamma(\tau, h)$. There is a nonzero integer matrix br *with rows I_{κ_0} and columns I_κ such that:*

(a) for every field R of characteristic 0, all $g \in \prod_\tau \mathrm{GL}_{n_\tau}(R)$ and $s \in \prod_\tau R^\times$,

$$\mathrm{br}\, P_{\kappa,e}(\mathrm{diag}(g, s)) = \prod_\tau s_\tau^{w_\tau + e_\tau}\, P_{\kappa_0,e_0}(g)\, \mathrm{br};$$

(b) if $L \in \mathbb{N}$ and $\sum_\tau \sum_{q\leq h<n_\tau} h\,\kappa_0(\tau, h) \leq L$, then $\|\mathrm{br}\,\xi\|_{\kappa_0}^2 \leq (L^L)^6 \|\xi\|_\kappa^2$ for every $\xi \in \mathbb{V}_\kappa$;

(c) every complex linear map $T\colon \mathbb{C}^{I_\kappa} \to \mathbb{C}^{I_{\kappa_0}}$ with $T\,P_{\kappa,e}(\mathrm{diag}(g, 1)) = P_{\kappa_0,e_0}(g)\,T$ for all $g \in \prod_\tau \mathrm{U}(n_\tau)$ is a complex multiple of br.

Proof. **Step 1: (a) and (b).** Lemma 4.6.8, for the same Φ, n, γ, κ', e and q, gives a nonzero integer matrix br with rows I_{κ_0} and columns I_κ satisfying (a) and (b).

Step 2: from unitary to general linear equivariance. Let T be a complex linear map as in (c). Using $\det \mathrm{diag}(g_\tau, 1) = \det g_\tau$ and $e_{0,\tau} - e_\tau = \kappa'(\tau, n_\tau) \geq 0$, and dividing by $\prod_\tau \det(g_\tau)^{e_\tau} \neq 0$,

$$T\, S_\kappa(\mathrm{diag}(g, 1)) = \prod_\tau \det(g_\tau)^{\kappa'(\tau,n_\tau)}\, S_{\kappa_0}(g)\, T \tag{4.1}$$

for all $g \in \prod_\tau \mathrm{U}(n_\tau)$. The entries of $S_\kappa(z)$ are polynomials in the entries of z: (WM2) defines S_κ over every ring, compatibly with ring homomorphisms, and for the matrix of indeterminates over $\mathbb{Z}[z_{ab}]$ the coefficients of $b_i(zy)$ in the basis $(b_j(y))$ are polynomials in the z_{ab}, which specialise to $S_\kappa(z)$. So each entry of the difference of the two sides of (4.1) is a polynomial f in the entries of $g \in \prod_\tau \mathrm{M}_{n_\tau}(\mathbb{C})$ that vanishes on $\prod_\tau \mathrm{U}(n_\tau)$, so $f = 0$ by Lemma 4.6.13. Hence (4.1) holds for all $g \in \prod_\tau \mathrm{GL}_{n_\tau}(\mathbb{C})$: T is a homomorphism of representations of $G_n = \prod_\tau \mathrm{GL}_{n_\tau}(\mathbb{C})$ from $V = (\mathbb{V}_\kappa,\, g \mapsto P_{\kappa,e}(\mathrm{diag}(g, 1)))$ to $V_0 = (\mathbb{V}_{\kappa_0},\, P_{\kappa_0,e_0})$.

Step 3: the Weyl modules are irreducible. For a weight κ for ranks N, $z \mapsto S_\kappa(z)$ is a polynomial representation of $\prod_\tau \mathrm{GL}_{N_\tau}(\mathbb{C})$ ($b_i(z_1 z_2 y)$ expanded in two steps gives $S_\kappa(z_1 z_2) = S_\kappa(z_1) S_\kappa(z_2)$). For a diagonal t, $\Delta_{\tau,A}(ty) = \prod_{a\in A} t_{\tau,a}\, \Delta_{\tau,A}(y)$, so $S_\kappa(t)$ is diagonal with the weights of the b_i; these are also the weights of the span of the products of minors $\nabla_\kappa \otimes \mathbb{C}$ under $f \mapsto f(z^{\mathsf{T}}\cdot)$. That span is the irreducible polynomial representation whose highest weight at τ is λ_τ (the Schur module of the shape with row lengths $\lambda_\tau(0) \geq \lambda_\tau(1) \geq \cdots$; [91], Ch. 8, and [3]);

polynomial representations of $\prod_\tau \mathrm{GL}_{N_\tau}(\mathbb{C})$ are semisimple and determined by their characters, so $(\mathbb{V}_\kappa, S_\kappa)$ is irreducible of highest weight $(\lambda_\tau)_\tau$, and $(\mathbb{V}_\kappa, P_{\kappa,e})$ is irreducible of highest weight $(\lambda_\tau + e_\tau)_\tau$ (all entries shifted by e_τ). In particular V_0 is an irreducible representation of G_n, and $(\mathbb{V}_\kappa, P_{\kappa,e})$ is the outer tensor product over τ of the irreducible representations L_τ of $\mathrm{GL}_{N_\tau}(\mathbb{C})$ of highest weight $\mu_\tau = \lambda_\tau + e_\tau$.

Step 4: the restriction is multiplicity free. The derivative of L_τ is the irreducible $\mathfrak{gl}_{N_\tau}$-module $L(\mu_\tau)$ of Theorem A.5.23, and the derivative of $g \mapsto \mathrm{diag}(g, 1)$ is the inclusion of $\mathfrak{gl}_{n_\tau}$ as the span of the E_{ab} with $a, b \le n_\tau$. By Theorem A.5.23 (i), applicable as $N_\tau = n_\tau + 1 \ge 2$, the restriction of $L(\mu_\tau)$ to $\mathfrak{gl}_{n_\tau}$ is a direct sum of pairwise inequivalent irreducible modules $L'(\nu)$, each with multiplicity one, ν running over the integral tuples interlacing μ_τ. Since $\mathrm{GL}_{n_\tau}(\mathbb{C})$ is connected, it is generated by $\exp(\mathfrak{gl}_{n_\tau})$; hence a subspace is stable under $\mathrm{GL}_{n_\tau}(\mathbb{C})$ if and only if it is stable under $\mathfrak{gl}_{n_\tau}$, and a linear map between two representations is $\mathrm{GL}_{n_\tau}(\mathbb{C})$-equivariant if and only if it is $\mathfrak{gl}_{n_\tau}$-equivariant. So the restriction of L_τ along $g \mapsto \mathrm{diag}(g, 1)$ is a direct sum of pairwise inequivalent irreducible representations of $\mathrm{GL}_{n_\tau}(\mathbb{C})$, each with multiplicity one. Taking the outer tensor product over τ, V is the direct sum, over the families $(\nu_\tau)_\tau$, of the outer tensor products $\bigotimes_\tau L'(\nu_\tau)$; each is an irreducible representation of G_n (its endomorphism algebra is spanned by the images of G_n, by Burnside's theorem applied factor by factor), and two of them for different families are inequivalent, as their restrictions to the factor $\mathrm{GL}_{n_\tau}(\mathbb{C})$ at a τ where the families differ are multiples of the inequivalent $L'(\nu_\tau)$, $L'(\nu'_\tau)$.

Step 5: (c). By Schur's lemma, a homomorphism of G_n-representations from V to the irreducible V_0 vanishes on every summand of Step 4 not isomorphic to V_0, and on the at most one summand isomorphic to V_0 it is a scalar multiple of a fixed isomorphism; so these homomorphisms form a space of dimension at most 1. By (a) for $R = \mathbb{C}$ and $s = 1$, br (as a complex matrix) lies in this space, and $\mathrm{br} \neq 0$; so the space is $\mathbb{C}\,\mathrm{br}$. By Step 2, T lies in it, so T is a complex multiple of br. □

Lemma 4.6.8. *Let Φ be a finite set and, for $\tau \in \Phi$, $n_\tau \ge 1$ and $N_\tau = n_\tau + 1$; the rows $1, \dots, n_\tau$ of the rank N_τ are those of the rank n_τ, and for $g = (g_\tau)$, $g_\tau \in \mathrm{GL}_{n_\tau}(R)$, and $s = (s_\tau)$, $s_\tau \in R^\times$, write $\mathrm{diag}(g, s)$ for the family of block matrices $\mathrm{diag}(g_\tau, s_\tau)$. For a weight κ and $e\colon \Phi \to \mathbb{Z}$ put $P_{\kappa,e}(x) = \prod_\tau \det(x_\tau)^{e_\tau} S_\kappa(x)$. Let γ, κ' be weights for the ranks N, $\kappa = \gamma + \kappa'$, $e\colon \Phi \to \mathbb{Z}$, and let κ_0 be the weight for the ranks n and $e_0\colon \Phi \to \mathbb{Z}$ given by*

$$\kappa_0(\tau, h) = \kappa'(\tau, h) + \gamma(\tau, h+1) \quad (1 \le h < n_\tau), \qquad e_{0,\tau} = e_\tau + \kappa'(\tau, n_\tau).$$

Let $q \in \mathbb{N}$ with $\kappa'(\tau, h) = 0$ for all τ and $h < q$, and put $w_\tau = \sum_{1 \le h < N_\tau} \gamma(\tau, h)$. There is a nonzero integer matrix br *with rows I_{κ_0} and columns I_κ such that:*

(a) for every field R of characteristic 0, all $g \in \prod_\tau \mathrm{GL}_{n_\tau}(R)$ and $s \in \prod_\tau R^\times$,

$$\mathrm{br}\, P_{\kappa,e}(\mathrm{diag}(g, s)) = \prod_\tau s_\tau^{w_\tau + e_\tau}\, P_{\kappa_0, e_0}(g)\, \mathrm{br};$$

(b) if $L \in \mathbb{N}$ and $\sum_\tau \sum_{q \le h < n_\tau} h\, \kappa_0(\tau, h) \le L$, then $\|\mathrm{br}\, \xi\|^2_{\kappa_0} \le (L^L)^6 \|\xi\|^2_\kappa$ for every $\xi \in \mathbb{V}_\kappa$.

Proof. **Conventions.** All symbols below are local to this proof. Square matrices of size n_τ have rows and columns $1, \dots, n_\tau$; an $N_\tau \times n_\tau$ matrix Y_τ has rows $1, \dots, N_\tau$ (row N_τ the new row) and columns $1, \dots, n_\tau$; a polynomial in the $y_{\tau,a,c}$ is evaluated at such a family by $y_{\tau,a,c} = (Y_\tau)_{a,c+1}$ (so the column c of Lemma 4.6.11 is the column $c+1$ here); only the columns $c < n_\tau$ occur, as all heights are at most n_τ. For a matrix M and sets of rows R and columns S of equal size, $M[R; S]$ is the determinant of the submatrix, rows and columns in increasing order unless they

are displayed in another order; thus $\Delta_{\tau,A}(M) = M_\tau[A;\{1,\dots,|A|\}]$, and $M^{\mathsf T}[R;S] = M[S;R]$. For a family z of matrices, $\mathrm{Pol}_\kappa(S_\kappa(z)\xi)(y) = \mathrm{Pol}_\kappa(\xi)(z^{\mathsf T}y)$: indeed $G_\kappa S_\kappa(z) = S_\kappa(z^{\mathsf T})^{\mathsf T}G_\kappa$ by (a) of Lemma 4.6.10, so $\mathrm{Pol}_\kappa(S_\kappa(z)\xi) = \sum_j (G_\kappa\xi)_j \sum_i S_\kappa(z^{\mathsf T})_{ji} b_i = \sum_j (G_\kappa\xi)_j\, b_j(z^{\mathsf T}\cdot)$. Put $d_\tau = \kappa'(\tau, n_\tau)$, so $e_{0,\tau} = e_\tau + d_\tau$, and $|\kappa|_\tau = \sum_{1\le h<N_\tau} h\,\kappa(\tau,h)$. Products such as s^w, $\det(g)^d$ mean $\prod_\tau s_\tau^{w_\tau}$, $\prod_\tau \det(g_\tau)^{d_\tau}$.

Step 0: the branch polynomial. For a family Y of $N_\tau \times n_\tau$ matrices put

$$\mathcal{B}(Y) = \prod_\tau \prod_{1\le j\le n_\tau} Y_\tau[(N_\tau, 1, \dots, j-1); \{1,\dots,j\}]^{\gamma(\tau,j)}\, \Delta_{\tau,\{1,\dots,j\}}(Y)^{\kappa'(\tau,j)},$$

the first minors taken with the row N_τ first. Each factor is $\pm$ a minor on j rows with $j < N_\tau$, and at τ there are $\gamma(\tau,j) + \kappa'(\tau,j) = \kappa(\tau,j)$ factors of height j; so $\mathcal{B}$ is $\pm$ a product of minors of weight κ, $\mathcal{B} = \sum_i \beta_i b_i$ with $\beta \in \mathbb{Z}^{I_\kappa}$, and $\mathcal{B} \neq 0$ as a product of nonzero polynomials.

Step 1: a highest vector with a small denominator. Put $p_\tau = \min(\max(q-1,0), n_\tau - 1)$, let $\bar\kappa$ be the weight for the ranks $n-p$ with $\bar\kappa(\tau,r) = \kappa_0(\tau, r+p_\tau)$, and $c = h_{\bar\kappa}$, a positive integer. Let

$$Q(X) = \prod_\tau \prod_{1\le j<n_\tau} \Delta_{\tau,\{2,\dots,j+1\}}(X)^{\gamma(\tau,j+1)}\, \Delta_{\tau,\{1,\dots,j\}}(X)^{\kappa'(\tau,j)}.$$

We show that there is $\xi_0 \in \mathbb{Z}^{I_{\kappa_0}}$ with $\mathrm{Pol}_{\kappa_0}(\xi_0) = c\,Q$. If $1 \le h \le p_\tau$ then $q \ge 1$ and $h \le p_\tau \le q-1$, so $\kappa'(\tau,h) = 0$ and $\kappa_0(\tau,h) = \gamma(\tau,h+1)$. Let

$$\bar Q(Z) = \prod_\tau \prod_{1\le r<n_\tau - p_\tau} \Delta_{\tau,\{2,\dots,r+1\}}(Z)^{\gamma(\tau,p_\tau+r+1)}\, \Delta_{\tau,\{1,\dots,r\}}(Z)^{\kappa'(\tau,p_\tau+r)},$$

a product of minors with $\gamma(\tau,p_\tau+r+1) + \kappa'(\tau,p_\tau+r) = \bar\kappa(\tau,r)$ factors of height r, hence an element of $\nabla_{\bar\kappa}$ with coordinates $[\bar Q] \in \mathbb{Z}^{I_{\bar\kappa}}$. With $H_{\bar\kappa}$ the integral matrix of Fock pairings of the basis, Lemma 4.6.9 gives $\mathrm{Pol}_{\bar\kappa}(H_{\bar\kappa}[\bar Q]) = \sum_i (G_{\bar\kappa}H_{\bar\kappa}[\bar Q])_i b_i = c\,\bar Q$. Let $\eta = H_{\bar\kappa}[\bar Q]$ and let C be the integral matrix of (i) of Lemma 4.6.11 for κ_0 and p. Let π_τ be the permutation matrix with $(\pi_\tau X)_a = X_{\sigma(a)}$, $\sigma(a) = a+1$ for $a \le p_\tau$, $\sigma(p_\tau+1) = 1$, $\sigma(a) = a$ for $a > p_\tau+1$, and $X' = \pi X$. On the dense open set where the block A'_τ of X'_τ on the rows and columns $1,\dots,p_\tau$ (the rows $2,\dots,p_\tau+1$ of X_τ) is invertible, (ii) of Lemma 4.6.11 gives, with Z' the Schur complement of X',

$$\mathrm{Pol}_{\kappa_0}(C\eta)(X') = \prod_\tau \Big[\prod_{h\le p_\tau} \Delta_{\tau,\{1,\dots,h\}}(X')^{\kappa_0(\tau,h)} \det(A'_\tau)^{\sum_r \bar\kappa(\tau,r)} \Big]\, c\,\bar Q(Z').$$

Here $\Delta_{\tau,\{1,\dots,h\}}(X') = \Delta_{\tau,\{2,\dots,h+1\}}(X)$ and $\kappa_0(\tau,h) = \gamma(\tau,h+1)$ for $h \le p_\tau$. The row 1 of Z' sits in the row 1 of X and the row $r \ge 2$ in the row $p_\tau + r$ of X; distributing one factor $\det(A'_\tau)$ to each factor of $\bar Q$, the minor identity of (ii) of Lemma 4.6.11 gives

$$\begin{aligned}\det(A'_\tau)\Delta_{\tau,\{2,\dots,r+1\}}(Z') &= X'_\tau[\{1,\dots,p_\tau, p_\tau+2,\dots,p_\tau+r+1\}; \{1,\dots,p_\tau+r\}] \\ &= \Delta_{\tau,\{2,\dots,p_\tau+r+1\}}(X)\end{aligned}$$

and

$$\begin{aligned}\det(A'_\tau)\Delta_{\tau,\{1,\dots,r\}}(Z') &= X_\tau[(2,\dots,p_\tau+1,1,p_\tau+2,\dots,p_\tau+r); \{1,\dots,p_\tau+r\}] \\ &= (-1)^{p_\tau}\Delta_{\tau,\{1,\dots,p_\tau+r\}}(X).\end{aligned}$$

With $j = p_\tau + r$ these are the factors of Q with $j > p_\tau$, and the factors with $j \le p_\tau$ are those of the bracket. Hence $\mathrm{Pol}_{\kappa_0}(C\eta)(\pi X) = \epsilon\, c\, Q(X)$ with $\epsilon = \prod_\tau (-1)^{p_\tau \sum_{p_\tau<j<n_\tau} \kappa'(\tau,j)}$, first on a dense open set and then identically, both sides being polynomials. Put $\xi_0 = \epsilon\, S_{\kappa_0}(\pi^{\mathsf T})C\eta$, an integral vector; by the translate formula of the conventions, $\mathrm{Pol}_{\kappa_0}(\xi_0)(X) = \epsilon\,\mathrm{Pol}_{\kappa_0}(C\eta)(\pi X) = c\,Q(X)$.

Step 2: the size of c. The highest polynomial $f_{\bar\kappa}$ is a product of determinants of total degree $D=\sum_\tau\sum_{p_\tau<h<n_\tau}(h-p_\tau)\kappa_0(\tau,h)$. If $q=0$, then $p_\tau=0$ and $D=\sum_\tau\sum_{1\le h<n_\tau}h\kappa_0(\tau,h)$; if $1\le q\le n_\tau$, then $p_\tau=q-1$ and only $h\ge q$ occur, with $h-p_\tau\le h$; if $q>n_\tau$, then $p_\tau=n_\tau-1$ and no h occurs at τ. So $D\le\sum_\tau\sum_{q\le h<n_\tau}h\kappa_0(\tau,h)\le L$. Write $f_{\bar\kappa}=\sum_\alpha a_\alpha y^\alpha$; expanding the determinants, $\sum_\alpha|a_\alpha|\le\prod(\text{sizes})!\le D!$, and $\alpha!\le D!$ for $|\alpha|=D$. As $\langle y^\alpha,y^{\alpha'}\rangle_{\mathrm F}=\delta_{\alpha\alpha'}\alpha!$,

$$1\le c=\sum_\alpha\alpha!\,a_\alpha^2\le D!\Big(\sum_\alpha|a_\alpha|\Big)^2\le(D!)^3\le(L^L)^3,$$

using $D!\le D^D\le L^L$ for $1\le D\le L$, and $c=1$ for $D=0$ (with $0^0=1$).

Step 3: the kernel at a block point. For families X of $n_\tau\times n_\tau$ and Y of $N_\tau\times n_\tau$ matrices put $\mathcal K(X,Y)=\mathcal B(\mathrm{diag}(X^{\mathsf T},1)Y)$, $D(X)=\det(X)^d$ and, for $v\in\mathbb C^{I_{\kappa_0}}$, $\widehat v(X)=D(X)\,\mathrm{Pol}_{\kappa_0}(v)(X)$. The map $v\mapsto\widehat v$ is linear and injective: $D\neq0$, G_{κ_0} is invertible by (b) of Lemma 4.6.10, and the b_i are linearly independent. Let Y_τ have the old rows g_τ (an $n_\tau\times n_\tau$ matrix) and the new row $(t_\tau,0,\dots,0)$. Then $\mathrm{diag}(X^{\mathsf T},1)Y$ has the old rows $X^{\mathsf T}g$ and the same new row; expanding along the new row and using $(X^{\mathsf T}g)[R;S]=(g^{\mathsf T}X)[S;R]$, the factors of $\mathcal B$ at height j become $t_\tau\,\Delta_{\tau,\{2,\dots,j\}}(g^{\mathsf T}X)$ and $\Delta_{\tau,\{1,\dots,j\}}(g^{\mathsf T}X)$, and for $j=n_\tau$ the latter is $\det(g_\tau)\det(X_\tau)$. Collecting, $\mathcal K(X,Y)=t^w\det(g)^dD(X)\,Q(g^{\mathsf T}X)$. By Step 1 and the translate formula, $c\,Q(g^{\mathsf T}X)=\mathrm{Pol}_{\kappa_0}(S_{\kappa_0}(g)\xi_0)(X)$, so

$$c\,\mathcal K(X,Y)=\widehat v(X),\qquad v=t^w\det(g)^dS_{\kappa_0}(g)\,\xi_0.\tag{$*$}$$

This is an identity of polynomials in the entries of X, g and t with integer coefficients, hence holds with entries in any commutative ring containing $\mathbb Z$.

Step 4: integral monomial coefficients. Let now Y be a family of matrices of indeterminates, $t_\tau=Y_{\tau,N_\tau,1}$, and U_τ the upper triangular $n_\tau\times n_\tau$ matrix with first row

$$(t_\tau,-Y_{\tau,N_\tau,2},\dots,-Y_{\tau,N_\tau,n_\tau}),$$

all other diagonal entries t_τ and all other entries 0. The new row of $Y_\tau U_\tau$ is $(t_\tau^2,0,\dots,0)$. For every matrix M and set A of j rows,

$$\begin{aligned}(MYU)_\tau[A;\{1,\dots,j\}]&=(MY)_\tau[A;\{1,\dots,j\}]\,\det U_\tau[\{1,\dots,j\};\{1,\dots,j\}]\\&=t_\tau^j(MY)_\tau[A;\{1,\dots,j\}],\end{aligned}$$

since the columns $1,\dots,j$ of U_τ vanish outside the rows $1,\dots,j$; hence $\mathcal K(X,YU)=t^{|\kappa|}\mathcal K(X,Y)$. Applying $(*)$ to YU, whose old rows g' have entries in $\mathbb Z[Y]$,

$$c\,t^{|\kappa|}\mathcal K(X,Y)=\widehat{v'}(X),\qquad v'=t^{2w}\det(g')^dS_{\kappa_0}(g')\,\xi_0\in\mathbb Z[Y]^{I_{\kappa_0}}.$$

Writing $v'=\sum_\alpha v'_\alpha Y^\alpha$ with $v'_\alpha\in\mathbb Z^{I_{\kappa_0}}$, the coefficient of Y^α in the left side is $\widehat{v'_\alpha}(X)$. As $t^{|\kappa|}$ is a monomial in Y, every coefficient of a monomial Y^α in $c\,\mathcal K(X,Y)$ is $\widehat{u_\alpha}$ for some $u_\alpha\in\mathbb Z^{I_{\kappa_0}}$.

Step 5: the matrix br**.** By Step 0 and (WM2), $\mathcal K(X,Y)=\sum_i\beta_ib_i(\mathrm{diag}(X^{\mathsf T},1)Y)=\sum_j f_j(X)b_j(Y)$ with $f_j(X)=\sum_i\beta_iS_\kappa(\mathrm{diag}(X^{\mathsf T},1))_{ij}\in\mathbb Z[X]$. Apply Lemma 4.6.12 to the weight κ, the further indeterminates X and $p=c\,\mathcal K(X,Y)$: $cf_j=\sum_\alpha\varphi_j(Y^\alpha)\,\widehat{u_\alpha}=\widehat{\mathrm{br}_j}$ with $\mathrm{br}_j=\sum_\alpha\varphi_j(Y^\alpha)u_\alpha\in\mathbb Z^{I_{\kappa_0}}$. Let br be the integer matrix with columns br_j ($j\in I_\kappa$). By linearity,

$$\widehat{\mathrm{br}\,\mathrm{ev}_\kappa(Y)}(X)=\sum_j cf_j(X)b_j(Y)=c\,\mathcal K(X,Y).\tag{$**$}$$

At $X=1$, $\widehat v(1)=\mathrm{Pol}_{\kappa_0}(v)(1)=\mathrm{ev}_{\kappa_0}(1)^{\mathsf T}G_{\kappa_0}v$, so

$$\mathrm{ev}_{\kappa_0}(1)^{\mathsf T}G_{\kappa_0}\,\mathrm{br}\,\mathrm{ev}_\kappa(Y)=c\,\mathcal B(Y)\qquad\text{for all }Y.\tag{$\dagger$}$$

Since $c \neq 0$ and $\mathcal{B} \neq 0$, $\mathrm{br} \neq 0$.

Step 6: clause (a). Let g, s be complex. Each factor of $\mathcal{B}$ with a γ-exponent contains the new row once and the others do not contain it, so $\mathcal{K}(X, \mathrm{diag}(g,s)Y) = \mathcal{B}(\mathrm{diag}(X^{\mathsf{T}}g, s)Y) = s^w \mathcal{K}(g^{\mathsf{T}}X, Y)$; and $\det(g)^{\widehat{d}} S_{\kappa_0}(g) v(X) = D(g^{\mathsf{T}}X)\mathrm{Pol}_{\kappa_0}(v)(g^{\mathsf{T}}X) = \widehat{v}(g^{\mathsf{T}}X)$ by the translate formula. With $\mathrm{ev}_\kappa(\mathrm{diag}(g,s)Y) = S_\kappa(\mathrm{diag}(g,s))\mathrm{ev}_\kappa(Y)$ and $(**)$,

$$\mathrm{br}\, S_\kappa(\widehat{\mathrm{diag}(g,s)})\mathrm{ev}_\kappa(Y)(X) = c\, s^w \mathcal{K}(g^{\mathsf{T}}X, Y) = s^w \det(g)^d \widehat{S_{\kappa_0}(g)}\, \mathrm{br}\, \mathrm{ev}_\kappa(Y)(X).$$

By injectivity of $v \mapsto \widehat{v}$ and because the $\mathrm{ev}_\kappa(Y)$ span $\mathbb{C}^{I_\kappa}$ (a vector orthogonal to them gives a vanishing combination of the b_i),

$$\mathrm{br}\, S_\kappa(\mathrm{diag}(g,s)) = s^w \det(g)^d\, S_{\kappa_0}(g)\, \mathrm{br}. \tag{‡}$$

Both sides are matrices of polynomials with integer coefficients in the entries of g and s, equal on all complex points, hence equal as polynomials; so (‡) holds over every field R of characteristic 0. For $g_\tau \in \mathrm{GL}_{n_\tau}(R)$ and $s_\tau \in R^\times$ multiply by $\prod_\tau \det(\mathrm{diag}(g_\tau, s_\tau))^{e_\tau} = \prod_\tau \det(g_\tau)^{e_\tau} s_\tau^{e_\tau}$ (negative exponents only on units): this is (a), as $e_{0,\tau} = e_\tau + d_\tau$.

Step 7: clause (b). Let $\mathfrak{T} = \bigotimes_{\tau,h,j} \Lambda^h \mathbb{C}^{N_\tau}$ with the Hermitian inner product for which the basis indexed by $\mathcal{C}_\kappa$ is orthonormal, so $\|\xi\|_\kappa = \|F_\kappa \xi\|$. Call the factors (τ, h, j) with $j \leq \gamma(\tau, h)$ the γ-factors and the others the κ'-factors. For a family Y of $n_\tau \times n_\tau$ matrices let $r_a \in \mathbb{C}^{N_\tau}$ $(1 \leq a \leq N_\tau)$ be the rows of $\mathrm{diag}(Y_\tau^{\mathsf{T}}, 1)$, and let $\mathcal{A}(Y) \in \mathfrak{T}$ have the factor $r_{N_\tau} \wedge r_1 \wedge \cdots \wedge r_{h-1}$ at a γ-factor and $r_1 \wedge \cdots \wedge r_h$ at a κ'-factor of height h. For an $N_\tau \times n_\tau$ family Z with columns $z_1, z_2, \ldots$, $F_\kappa \mathrm{ev}_\kappa(Z) = (\Delta_T(Z))_T$ is the tensor with the factors $z_1 \wedge \cdots \wedge z_h$; by the Cauchy–Binet formula $\langle u_1 \wedge \cdots \wedge u_h, u'_1 \wedge \cdots \wedge u'_h \rangle_{\mathrm{bil}} = \det(u_l \cdot u'_m)$ for the bilinear pairing, and $r_a \cdot z_m$ is the entry (a,m) of $\mathrm{diag}(Y^{\mathsf{T}}, 1)Z$. Hence

$$\mathcal{A}(Y)^{\mathsf{T}} F_\kappa\, \mathrm{ev}_\kappa(Z) = \mathcal{B}(\mathrm{diag}(Y^{\mathsf{T}}, 1)Z), \qquad \mathcal{A}(Y)^{\mathsf{T}} \mathcal{A}(Y') = \det(Y)^d \det(Y')^d\, \mathrm{ev}_{\kappa_0}(Y)^{\mathsf{T}} G_{\kappa_0} \mathrm{ev}_{\kappa_0}(Y').$$

For the second identity, $r_a(Y) \cdot r_b(Y') = (Y^{\mathsf{T}}Y')_{ab}$ for $a, b \leq n_\tau$, $r_N \cdot r_N = 1$ and $r_N \cdot r_a = 0$, so a γ-factor of height h contributes $(Y^{\mathsf{T}}Y')[\{1, \ldots, h-1\}; \{1, \ldots, h-1\}]$ and a κ'-factor $(Y^{\mathsf{T}}Y')[\{1, \ldots, h\}; \{1, \ldots, h\}]$, which for $h = n_\tau$ is $\det Y_\tau \det Y'_\tau$; on the other side $\mathrm{ev}_{\kappa_0}(Y)^{\mathsf{T}} G_{\kappa_0} \mathrm{ev}_{\kappa_0}(Y') = \sum_T \Delta_T(Y)\Delta_T(Y')$ is, by Cauchy–Binet in each factor, the product over the factors of κ_0 of $(Y^{\mathsf{T}}Y')[\{1, \ldots, h\}; \{1, \ldots, h\}]$, with $\kappa_0(\tau, h) = \kappa'(\tau, h) + \gamma(\tau, h+1)$ factors of height $h < n_\tau$. The two products agree.

All entries of $\mathcal{A}$ and ev_{κ_0} are polynomials with integer coefficients, so $\overline{\mathcal{A}(Y)} = \mathcal{A}(\bar{Y})$ and similarly. For Y with all $\det Y_\tau \neq 0$ put $\mathcal{U}(Y) = \det(Y)^{-d}\mathcal{A}(Y)$; then $\langle \mathcal{U}(Y), \mathcal{U}(Y') \rangle = \langle \mathrm{ev}_{\kappa_0}(Y), \mathrm{ev}_{\kappa_0}(Y') \rangle_{\kappa_0}$. These $\mathrm{ev}_{\kappa_0}(Y)$ span $\mathbb{V}_{\kappa_0}$ (a combination of the b_i vanishing on the dense open set of invertible families vanishes), and $\langle\, , \rangle_{\kappa_0}$ is definite by (b) of Lemma 4.6.10; hence $\sum_k a_k \mathrm{ev}_{\kappa_0}(Y_k) \mapsto \sum_k a_k \mathcal{U}(Y_k)$ is a well defined linear isometry $\mathcal{I}\colon \mathbb{V}_{\kappa_0} \to \mathfrak{T}$ (the norm of the image of a combination equals the norm of the combination). Replace Y by $\mathrm{diag}(Y^{\mathsf{T}}, 1)Z$ in (†): $\mathrm{ev}_\kappa(\mathrm{diag}(Y^{\mathsf{T}}, 1)Z) = S_\kappa(\mathrm{diag}(Y^{\mathsf{T}}, 1))\mathrm{ev}_\kappa(Z)$, (‡) with $s = 1$ gives $\mathrm{br}\, S_\kappa(\mathrm{diag}(Y^{\mathsf{T}}, 1)) = \det(Y)^d S_{\kappa_0}(Y^{\mathsf{T}})\mathrm{br}$, and $\mathrm{ev}_{\kappa_0}(1)^{\mathsf{T}} G_{\kappa_0} S_{\kappa_0}(Y^{\mathsf{T}}) = (S_{\kappa_0}(Y)\mathrm{ev}_{\kappa_0}(1))^{\mathsf{T}} G_{\kappa_0} = \mathrm{ev}_{\kappa_0}(Y)^{\mathsf{T}} G_{\kappa_0}$ by (a) of Lemma 4.6.10. With the first identity above and the spanning of the $\mathrm{ev}_\kappa(Z)$,

$$\det(Y)^d\, \mathrm{ev}_{\kappa_0}(Y)^{\mathsf{T}} G_{\kappa_0}\, \mathrm{br} = c\, \mathcal{A}(Y)^{\mathsf{T}} F_\kappa.$$

Replacing Y by $\bar{Y}$ and dividing by $\det(\bar{Y})^d$: $\langle \mathrm{ev}_{\kappa_0}(Y), \mathrm{br}\, \xi \rangle_{\kappa_0} = c\, \langle \mathcal{U}(Y), F_\kappa \xi \rangle$ for all invertible Y and $\xi \in \mathbb{V}_\kappa$. Since $\mathcal{I}$ is an isometry and the $\mathrm{ev}_{\kappa_0}(Y)$ span, $\langle \mathcal{I}v, \mathcal{I}\,\mathrm{br}\, \xi \rangle = \langle \mathcal{I}v, c\, F_\kappa \xi \rangle$ for all v, so $\mathcal{I}\,\mathrm{br}\, \xi$ is the orthogonal projection of $cF_\kappa \xi$ onto the image of $\mathcal{I}$. Therefore, with Step 2,

$$\|\mathrm{br}\, \xi\|_{\kappa_0}^2 = \|\mathcal{I}\,\mathrm{br}\, \xi\|^2 \leq c^2 \|F_\kappa \xi\|^2 = c^2 \|\xi\|_\kappa^2 \leq (L^L)^6 \|\xi\|_\kappa^2.$$

□

Lemma 4.6.9. *Let κ be a weight for ranks N, and let $H_\kappa \in \mathrm{M}_{I_\kappa}(\mathbb{Z})$ be the matrix of the Fock pairings $(H_\kappa)_{ij} = \langle b_i, b_j\rangle_{\mathrm{F}}$. Then*

$$G_\kappa H_\kappa = H_\kappa G_\kappa = h_\kappa \cdot 1.$$

Proof. All polynomials are in the variables $y_{\tau,a,c}$ with complex coefficients. For a family $g = (g_\tau)$ of complex $N_\tau \times N_\tau$ matrices write $(p \circ g)(y) = p(gy)$, $(gy)_{\tau,a,c} = \sum_{a'} (g_\tau)_{aa'} y_{\tau,a',c}$; so $b_i \circ g = \sum_j S_\kappa(g)_{ij} b_j$ by (WM2). The Fock pairing is symmetric and bilinear, with $\langle y^\alpha, y^\beta\rangle_{\mathrm{F}} = \delta_{\alpha\beta}\,\alpha!$; the entries of H_κ are integers since the b_i have integer coefficients, and H_κ, $G_\kappa = F_\kappa^{\mathsf{T}} F_\kappa$ are symmetric.

Step 1: substitution is adjoint to substitution by the transpose. We show $\langle p \circ g^{\mathsf{T}}, q\rangle_{\mathrm{F}} = \langle p, q \circ g\rangle_{\mathrm{F}}$. Let $D_{\tau,a,c} = \partial/\partial y_{\tau,a,c}$ and $D'_{\tau,a',c} = \sum_a (g_\tau)_{aa'} D_{\tau,a,c}$. By the chain rule $D_{\tau,a',c}(q \circ g) = (D'_{\tau,a',c} q) \circ g$, and the D' are commuting constant-coefficient operators, so by induction on the degree $p(D)(q \circ g) = (p(D')q) \circ g$ for every polynomial p. Since $p(D') = (p \circ g^{\mathsf{T}})(D)$ and $g \cdot 0 = 0$, evaluation at $y = 0$ gives the claim.

Step 2: the polynomial of a translate. For $\xi \in \mathbb{C}^{I_\kappa}$ and z as above, $\mathrm{Pol}_\kappa(S_\kappa(z)\xi) = \mathrm{Pol}_\kappa(\xi) \circ z^{\mathsf{T}}$. Indeed $G_\kappa S_\kappa(z) = S_\kappa(z^{\mathsf{T}})^{\mathsf{T}} G_\kappa$ by (a) of Lemma 4.6.10 applied to z^{T}, so

$$\mathrm{Pol}_\kappa(S_\kappa(z)\xi) = \sum_i \big(S_\kappa(z^{\mathsf{T}})^{\mathsf{T}} G_\kappa \xi\big)_i b_i = \sum_j (G_\kappa\xi)_j \sum_i S_\kappa(z^{\mathsf{T}})_{ji} b_i = \sum_j (G_\kappa\xi)_j\, b_j \circ z^{\mathsf{T}}.$$

Step 3: $T = H_\kappa^{\mathsf{T}} G_\kappa$ commutes with all $S_\kappa(z)$. By definition $(T\xi)_k = \sum_i (H_\kappa)_{ik}(G_\kappa\xi)_i = \langle \mathrm{Pol}_\kappa(\xi), b_k\rangle_{\mathrm{F}}$. By Steps 2 and 1,

$$(T S_\kappa(z)\xi)_k = \langle \mathrm{Pol}_\kappa(\xi) \circ z^{\mathsf{T}}, b_k\rangle_{\mathrm{F}} = \langle \mathrm{Pol}_\kappa(\xi), b_k \circ z\rangle_{\mathrm{F}} = \sum_j S_\kappa(z)_{kj} (T\xi)_j.$$

Step 4: T is a scalar c. The complex span ∇ of the b_i, the span of the products of minors Δ_T, is the tensor product over τ of the spans of the products of minors at τ with $\kappa(\tau, h)$ columns of height h; after transposing the arrays y_τ this is the realisation of the Schur module of the partition with column lengths given by $\kappa(\tau, \cdot)$ by products of minors, an irreducible representation of $\mathrm{GL}_{N_\tau}(\mathbb{C})$ under substitution [91] (§8.1); a tensor product of irreducible finite-dimensional representations of the factors is irreducible for the product group (Burnside's theorem). In the basis (b_i), substitution $p \mapsto p \circ z$ acts on coefficient vectors by $S_\kappa(z)^{\mathsf{T}}$. If $W \subseteq \mathbb{C}^{I_\kappa}$ is stable under all $S_\kappa(z)$, the coefficient vectors annihilating W for the bilinear pairing $c^{\mathsf{T}} w$ form a subspace stable under all $S_\kappa(z)^{\mathsf{T}}$, hence span 0 or ∇; so $W = \mathbb{C}^{I_\kappa}$ or $W = 0$. An eigenspace of T is stable under all $S_\kappa(z)$ by Step 3, hence $T = c \cdot 1$ for some $c \in \mathbb{C}$ (Schur's lemma).

Step 5: $c = h_\kappa$. Let $y^0_{\tau,a,c} = 1$ if $c = a - 1$ and 0 otherwise, and $\xi_0 = \mathrm{ev}_\kappa(y^0)$. Since $\Delta_{\tau,A}(y^0)$ is 1 for $A = \{1, \dots, h\}$ and 0 for the other sets of h rows, $F_\kappa \xi_0 = (\Delta_T(y^0))_T$ is the basis vector of the filling T_0 with $T_0(\tau, h, j) = \{1, \dots, h\}$. Hence $G_\kappa\xi_0 = F_\kappa^{\mathsf{T}} F_\kappa \xi_0$ is the row T_0 of F_κ, $\mathrm{Pol}_\kappa(\xi_0) = \sum_i (F_\kappa)_{T_0 i} b_i = \Delta_{T_0} = f_\kappa$, and $\xi_0^{\mathsf{T}} G_\kappa \xi_0 = \|F_\kappa \xi_0\|^2 = 1$. Therefore

$$h_\kappa = \langle f_\kappa, f_\kappa\rangle_{\mathrm{F}} = \sum_k (G_\kappa\xi_0)_k \,\langle \mathrm{Pol}_\kappa(\xi_0), b_k\rangle_{\mathrm{F}} = \sum_k (G_\kappa\xi_0)_k (T\xi_0)_k = c\,\xi_0^{\mathsf{T}} G_\kappa \xi_0 = c.$$

So $H_\kappa^{\mathsf{T}} G_\kappa = h_\kappa \cdot 1$; as H_κ and G_κ are symmetric, transposing gives $G_\kappa H_\kappa = h_\kappa \cdot 1$, and $H_\kappa G_\kappa = H_\kappa^{\mathsf{T}} G_\kappa$. □

Lemma 4.6.10. *Let κ be a weight for ranks N and $e\colon \Phi \to \mathbb{Z}$. Then:*

(a) $S_\kappa(zz') = S_\kappa(z)S_\kappa(z')$, $S_\kappa(1) = 1$, *and* $S_\kappa(z)^{\mathsf T}G_\kappa = G_\kappa\, S_\kappa(z^{\mathsf T})$ *for all families* z, z' *of complex* $N_\tau \times N_\tau$ *matrices;*

(b) G_κ *is positive definite;*

(c) for $h, h' \in \prod_\tau \mathrm{U}(N_\tau)$, $\rho_{\kappa,e}(hh') = \rho_{\kappa,e}(h)\rho_{\kappa,e}(h')$, $\rho_{\kappa,e}(h)^{-1} = \prod_\tau \det(h_\tau)^{e_\tau} S_\kappa(h^{\mathsf T})$, *and* $\langle \rho_{\kappa,e}(h)\xi, \rho_{\kappa,e}(h)\xi'\rangle_\kappa = \langle \xi, \xi'\rangle_\kappa$ *for* $\xi, \xi' \in \mathbb{V}_\kappa$.

Proof. (a). Expanding

$$b_i(zz'y) = \sum_j S_\kappa(z)_{ij} b_j(z'y) = \sum_{j,k} S_\kappa(z)_{ij} S_\kappa(z')_{jk} b_k(y)$$

and using the linear independence of the b_k gives the multiplicativity, and $b_i(1y) = b_i(y)$ gives $S_\kappa(1) = 1$. Let $\Lambda(z)$ be the matrix, indexed by $\mathcal{C}_\kappa$, of $\bigotimes_{\tau,h,j} \Lambda^h z_\tau$ in the standard basis of $\bigotimes_{\tau,h,j} \Lambda^h \mathbb{C}^{N_\tau}$; its entry at (T, T') is $\prod_{\tau,h,j} \det(z_\tau)_{T(\tau,h,j),T'(\tau,h,j)}$. By the Cauchy–Binet formula a minor of zy on a set A of h rows is $\sum_B \det(z_\tau)_{A,B}\Delta_{\tau,B}(y)$ over the sets B of h rows, so $\Delta_T(zy) = \sum_{T'} \Lambda(z)_{TT'}\Delta_{T'}(y)$. Writing both sides in the basis (b_i) by (WM2) and (WM3),

$$\sum_i (F_\kappa)_{Ti} b_i(zy) = \sum_{i,j} (F_\kappa)_{Ti} S_\kappa(z)_{ij} b_j(y),$$
$$\sum_{T'} \Lambda(z)_{TT'}\Delta_{T'}(y) = \sum_{T',j} \Lambda(z)_{TT'}(F_\kappa)_{T'j} b_j(y),$$

so $F_\kappa S_\kappa(z) = \Lambda(z)F_\kappa$. As $\det(z_{A,B}) = \det((z^{\mathsf T})_{B,A})$, $\Lambda(z)^{\mathsf T} = \Lambda(z^{\mathsf T})$. Hence

$$S_\kappa(z)^{\mathsf T}G_\kappa = (F_\kappa S_\kappa(z))^{\mathsf T} F_\kappa = F_\kappa^{\mathsf T}\Lambda(z^{\mathsf T})F_\kappa = G_\kappa S_\kappa(z^{\mathsf T}).$$

(b). Each b_i is a product of minors, $b_i = \Delta_{T_i}$ for a filling T_i whose row sets, listed increasingly, make $T_i \in \mathcal{C}_\kappa$; so the row T_i of F_κ is the coordinate vector e_i, F_κ has full column rank, and $\bar\xi^{\mathsf T} G_\kappa \xi = |F_\kappa \xi|^2 > 0$ for $\xi \neq 0$.

(c). Multiplicativity follows from (a), as $\overline{hh'} = \bar h \bar h'$ and det is multiplicative. For unitary h, $\overline{h^{-1}} = h^{\mathsf T}$, so $\rho_{\kappa,e}(h^{-1}) = \prod_\tau \det(h_\tau^{\mathsf T})^{e_\tau} S_\kappa(h^{\mathsf T})$, which is $\rho_{\kappa,e}(h)^{-1}$ by multiplicativity. The entries of $S_\kappa(z)$ are polynomials with integral coefficients in the entries of z, so $\overline{S_\kappa(\bar h)} = S_\kappa(h)$, and by (a)

$$\rho_{\kappa,e}(h)^* G_\kappa \rho_{\kappa,e}(h) = \prod_\tau |\det h_\tau|^{2e_\tau}\, S_\kappa(h)^{\mathsf T} G_\kappa S_\kappa(\bar h) = G_\kappa S_\kappa(h^{\mathsf T}\bar h) = G_\kappa,$$

since $|\det h_\tau| = 1$ and $h^{\mathsf T}\bar h = \overline{h^* h} = 1$. This is the invariance of $\langle\,,\rangle_\kappa$. □

Lemma 4.6.11. *Let* $n = (n_\tau)_{\tau\in\Phi}$ *be ranks,* κ_0 *a weight for the ranks* n *and* $p = (p_\tau)$ *integers with* $0 \le p_\tau < n_\tau$. *Let* $\bar\kappa$ *be the weight for the ranks* $n - p$ *with* $\bar\kappa(\tau, r) = \kappa_0(\tau, r + p_\tau)$ $(1 \le r < n_\tau - p_\tau)$. *For* $\bar T \in \mathcal{C}_{\bar\kappa}$ *let* $J(\bar T) \in \mathcal{C}_{\kappa_0}$ *be the filling with*

$$J(\bar T)(\tau, h, j) = \{1, \dots, h\} \ \ (h \le p_\tau), \qquad J(\bar T)(\tau, h, j) = \{1, \dots, p_\tau\} \cup (p_\tau + \bar T(\tau, h - p_\tau, j)) \ \ (h > p_\tau),$$

and let $J \in \mathrm{M}_{\mathcal{C}_{\kappa_0} \times \mathcal{C}_{\bar\kappa}}(\mathbb{Z})$ *have entries* $J_{T\bar T} = 1$ *if* $T = J(\bar T)$ *and* 0 *otherwise. Then:*

(i) there is a matrix $C \in \mathrm{M}_{I_{\kappa_0} \times I_{\bar\kappa}}(\mathbb{Z})$ *with* $F_{\kappa_0} C = J\, F_{\bar\kappa}$;

(ii) let X_τ be complex $n_\tau \times n_\tau$ matrices (rows $1,\dots,n_\tau$, columns $0,\dots,n_\tau-1$) with blocks $X_\tau = \begin{pmatrix} A_\tau & B_\tau \\ C_\tau & D_\tau \end{pmatrix}$, A_τ on the rows $1,\dots,p_\tau$ and the columns $0,\dots,p_\tau-1$, every A_τ invertible, and let $Z_\tau = D_\tau - C_\tau A_\tau^{-1} B_\tau$, with rows $1,\dots,n_\tau-p_\tau$ and columns $0,\dots,n_\tau-p_\tau-1$ (row r and column c of Z_τ sit in row $p_\tau+r$ and column $p_\tau+c$ of X_τ). Then $\Delta_{\tau,\{1,\dots,p_\tau\}\cup(p_\tau+R)}(X) = \det(A_\tau)\,\Delta_{\tau,R}(Z)$ for every set R of rows of Z_τ, and for every $\eta \in \mathbb{V}_{\bar\kappa}$

$$\mathrm{Pol}_{\kappa_0}(C\eta)(X) = \prod_\tau \Big[\prod_{1\le h\le p_\tau} \Delta_{\tau,\{1,\dots,h\}}(X)^{\kappa_0(\tau,h)} \cdot \det(A_\tau)^{\sum_{p_\tau<h<n_\tau}\kappa_0(\tau,h)} \Big]\, \mathrm{Pol}_{\bar\kappa}(\eta)(Z).$$

Proof. **Step 1: the minor identity.** Fix τ and drop it from the notation; write $p = p_\tau$. Let $R = \{r_1 < \cdots < r_k\}$ be rows of Z. The minor $\Delta_{\{1,\dots,p\}\cup(p+R)}(X)$ is the determinant of the matrix of X on the rows $1,\dots,p, p+r_1,\dots,p+r_k$ and the columns $0,\dots,p+k-1$, which in blocks is $\begin{pmatrix} A & B' \\ C_R & D'_R \end{pmatrix}$, with B' the columns $0,\dots,k-1$ of B, C_R the rows R of C and D'_R the rows R and columns $0,\dots,k-1$ of D. By the determinant formula for the Schur complement, its determinant is $\det(A)\det(D'_R - C_R A^{-1} B')$, and $D'_R - C_R A^{-1} B'$ is the matrix of Z on the rows R and the columns $0,\dots,k-1$. This is the first assertion of (ii) (for $p=0$, A is empty, $\det A = 1$ and $Z = X$).

Step 2: the image of $JF_{\bar\kappa}$. For a family Z of complex $(n_\tau-p_\tau)\times(n_\tau-p_\tau)$ matrices let $X^Z_\tau = \mathrm{diag}(1_{p_\tau}, Z_\tau)$, so that $A_\tau = 1$, $B_\tau = 0$, $C_\tau = 0$ and the Schur complement is Z_τ. By (WM3), $F_{\kappa_0}\mathrm{ev}_{\kappa_0}(y) = (\Delta_T(y))_{T\in\mathcal{C}_{\kappa_0}}$ for every array y. Let $T \in \mathcal{C}_{\kappa_0}$ and consider a factor $\Delta_{\tau,T(\tau,h,j)}(X^Z)$. The columns $0,\dots,\min(h,p_\tau)-1$ of X^Z_τ are the unit vectors of the rows $1,\dots,\min(h,p_\tau)$. If $h \le p_\tau$, the factor is 1 for $T(\tau,h,j) = \{1,\dots,h\}$ and 0 otherwise (for any other set, some column restricted to its rows vanishes). If $h > p_\tau$, the factor vanishes unless $T(\tau,h,j) \supseteq \{1,\dots,p_\tau\}$, and for $T(\tau,h,j) = \{1,\dots,p_\tau\}\cup(p_\tau+R)$ it equals $\Delta_{\tau,R}(Z)$ by Step 1. As J is injective on fillings, this gives

$$F_{\kappa_0}\,\mathrm{ev}_{\kappa_0}(X^Z) = J\,F_{\bar\kappa}\,\mathrm{ev}_{\bar\kappa}(Z).$$

The vectors $\mathrm{ev}_{\bar\kappa}(Z)$ span $\mathbb{C}^{I_{\bar\kappa}}$: a vector λ orthogonal to all of them for the bilinear pairing gives a polynomial $\sum_i \lambda_i b_i$ vanishing at all complex matrices Z; since the b_i involve only the variables with $c < n_\tau - p_\tau$, it is the zero polynomial, and $\lambda = 0$ by linear independence of the b_i. Hence $JF_{\bar\kappa}\eta$ lies in $F_{\kappa_0}(\mathbb{C}^{I_{\kappa_0}})$ for every $\eta \in \mathbb{V}_{\bar\kappa}$.

Step 3: an integral left inverse of F_{κ_0} and the matrix C. By (WM1) each b_i is a product of minors; listing its sets of rows increasingly, $b_i = \Delta_{T_i}$ with $T_i \in \mathcal{C}_{\kappa_0}$, and the T_i are distinct since the b_i are. By uniqueness of coordinates in the basis (b_j), the row T_i of F_{κ_0} is the unit vector e_i. Let $P \in \mathrm{M}_{I_{\kappa_0}\times\mathcal{C}_{\kappa_0}}(\mathbb{Z})$ have $P_{iT} = 1$ if $T = T_i$ and 0 otherwise; then $PF_{\kappa_0} = 1$. Put $C = PJF_{\bar\kappa}$, an integral matrix. For $\eta \in \mathbb{V}_{\bar\kappa}$ choose ξ with $JF_{\bar\kappa}\eta = F_{\kappa_0}\xi$ (Step 2); then $F_{\kappa_0}C\eta = F_{\kappa_0}PF_{\kappa_0}\xi = F_{\kappa_0}\xi = JF_{\bar\kappa}\eta$. This is (i).

Step 4: the formula of (ii). For every weight κ and $\xi \in \mathbb{V}_\kappa$,

$$\mathrm{Pol}_\kappa(\xi) = \sum_i (F_\kappa^{\mathsf{T}} F_\kappa \xi)_i\, b_i = \sum_T (F_\kappa\xi)_T \sum_i (F_\kappa)_{Ti} b_i = \sum_{T\in\mathcal{C}_\kappa} (F_\kappa\xi)_T\, \Delta_T.$$

Applied to $\xi = C\eta$ with (i), $\mathrm{Pol}_{\kappa_0}(C\eta) = \sum_T (JF_{\bar\kappa}\eta)_T \Delta_T = \sum_{\bar T} (F_{\bar\kappa}\eta)_{\bar T}\,\Delta_{J(\bar T)}$. By the definition of $J(\bar T)$ and Step 1 applied to each factor of height $h > p_\tau$ (there are $\sum_{p_\tau<h<n_\tau}\kappa_0(\tau,h)$ of them

at τ),

$$\Delta_{J(\bar{T})}(X) = \prod_{\tau}\Big[\prod_{1\le h\le p_\tau} \Delta_{\tau,\{1,\dots,h\}}(X)^{\kappa_0(\tau,h)}\det(A_\tau)^{\sum_{p_\tau<h<n_\tau}\kappa_0(\tau,h)}\Big]\,\Delta_{\bar{T}}(Z).$$

Summing over $\bar{T}$ with the coefficients $(F_{\bar{\kappa}}\eta)_{\bar{T}}$ and using the display above for $\bar{\kappa}$ gives the formula of (ii). □

Lemma 4.6.12. *Let κ be a weight for ranks N and let $\mathbb{Z}[y]$ be the polynomial ring over $\mathbb{Z}$ in the variables $y_{\tau,a,c}$. There are $\mathbb{Z}$-linear maps $\varphi_i\colon \mathbb{Z}[y]\to\mathbb{Z}$ ($i\in I_\kappa$) with $\varphi_i(b_j)=\delta_{ij}$ for all $i,j\in I_\kappa$. Consequently, if u is a finite family of further indeterminates and $p=\sum_\alpha p_\alpha y^\alpha\in\mathbb{Z}[u][y]$ (with $p_\alpha\in\mathbb{Z}[u]$, y^α the monomials in y) satisfies $p=\sum_{j\in I_\kappa} f_j\,b_j$ with $f_j\in\mathbb{C}[u]$, then*

$$f_j=\sum_\alpha \varphi_j(y^\alpha)\,p_\alpha\in\mathbb{Z}[u]\qquad(j\in I_\kappa).$$

In particular a polynomial with integer coefficients that is a complex linear combination of the b_j has integer coordinates in the basis (b_j).

Proof. **Step 1: reduction to finitely many variables.** Every Δ_T involves only the variables $y_{\tau,a,c}$ with $1\le a\le N_\tau$ and $0\le c<N_\tau$. Let $\mathbb{Z}[y]'$ be the polynomial ring in these finitely many variables; it is spanned by part of the monomial basis of $\mathbb{Z}[y]$, hence $\mathbb{Z}[y]=\mathbb{Z}[y]'\oplus M_1$ with M_1 the span of the other monomials. It suffices to show that ∇_κ is a direct summand of $\mathbb{Z}[y]'$.

Step 2: a basis of $\mathbb{Z}[y]'$ adapted to ∇_κ. For $\tau\in\Phi$ let λ_τ be the partition whose columns are $\kappa(\tau,h)$ columns of length h for each $1\le h<N_\tau$, arranged by decreasing length, and let K_τ be the tableau of shape λ_τ whose entries in its r-th row are all $r-1$; it is standard (constant along rows, strictly increasing down columns). Apply Lemma A.5.26 with $A=\mathbb{Z}$, $\mathcal{R}=\{1,\dots,N_\tau\}$ and $\mathcal{K}=\{0,\dots,N_\tau-1\}$ for each τ: the standard bideterminants form a $\mathbb{Z}$-basis of the polynomial ring in the variables $y_{\tau,\cdot,\cdot}$. Since $\mathbb{Z}[y]'$ is the tensor product over $\mathbb{Z}$ of these rings, the products $\prod_\tau(S_\tau|T_\tau)$ of standard bideterminants form a $\mathbb{Z}$-basis $\mathfrak{B}$ of $\mathbb{Z}[y]'$. Let $\mathfrak{B}_\kappa\subset\mathfrak{B}$ consist of the products $\prod_\tau(S_\tau|K_\tau)$ with S_τ standard of shape λ_τ, and let L be its $\mathbb{Z}$-span. As $\mathfrak{B}_\kappa$ is part of a basis, $\mathbb{Z}[y]'=L\oplus M_2$ with M_2 the span of $\mathfrak{B}\smallsetminus\mathfrak{B}_\kappa$.

For a tableau S_τ of shape λ_τ with entries in $\mathcal{R}$, every column of $(S_\tau|K_\tau)$ is the determinant of $(y_{\tau,a,c})$ with a in that column of S_τ (in its order) and $0\le c<h$, h its length. Hence for a column filling T of κ with every $T(\tau,h,j)$ listed increasingly, $\Delta_T=\prod_\tau(S_{T,\tau}|K_\tau)$, where $S_{T,\tau}$ has the sets $T(\tau,h,j)$ as its columns; and every tableau with strictly increasing columns is some $S_{T,\tau}$. In particular $L\subseteq\nabla_\kappa$.

Step 3: $\nabla_\kappa=L$. Over $\mathbb{C}$, the products $(S|K_\tau)$ with S standard of shape λ_τ span the complex span of all $(S|K_\tau)$, S any filling of λ_τ with entries in $\mathcal{R}$ ([91], §8.1, where $(S|K_\tau)$ is written D_S and standard tableaux are called semistandard); taking products over τ, every Δ_T is a complex linear combination of the elements of $\mathfrak{B}_\kappa$. On the other hand $\Delta_T\in\mathbb{Z}[y]'$ is an integral combination of the elements of $\mathfrak{B}$, and $\mathfrak{B}$ is also a basis of $\mathbb{C}[y]'=\mathbb{Z}[y]'\otimes\mathbb{C}$. By uniqueness of coordinates in this basis, the integral coordinates of Δ_T vanish outside $\mathfrak{B}_\kappa$, so $\Delta_T\in L$. As ∇_κ is spanned by the Δ_T, $\nabla_\kappa\subseteq L$, and with Step 2, $\nabla_\kappa=L$. Therefore $\mathbb{Z}[y]=\nabla_\kappa\oplus M_2\oplus M_1$.

Step 4: the maps φ_i. Let $\pi\colon\mathbb{Z}[y]\to\nabla_\kappa$ be the projection along $M_2\oplus M_1$, and let $\varphi_i(p)$ be the i-th coordinate of $\pi(p)$ in the $\mathbb{Z}$-basis (b_j) of ∇_κ. Then φ_i is $\mathbb{Z}$-linear and $\varphi_i(b_j)=\delta_{ij}$, since π is the identity on ∇_κ.

Step 5: the consequence. Extend φ_j to the $\mathbb{C}[u]$-linear map $\Phi_j\colon\mathbb{C}[u][y]\to\mathbb{C}[u]$ with $\Phi_j(r\,y^\alpha)=\varphi_j(y^\alpha)\,r$ for $r\in\mathbb{C}[u]$; it maps $\mathbb{Z}[u][y]$ into $\mathbb{Z}[u]$. Since each b_i is a finite integral combination of monomials, $\Phi_j(b_i)=\varphi_j(b_i)=\delta_{ij}$. Applying Φ_j to $p=\sum_i f_ib_i$ gives $f_j=$

$\Phi_j(p) = \sum_\alpha \varphi_j(y^\alpha)p_\alpha$, a finite sum, which lies in $\mathbb{Z}[u]$. The last sentence is the case of no further indeterminates. □

Lemma 4.6.13. *Let Φ be a finite set, $n_\tau \geq 1$ for $\tau \in \Phi$, and let f be a complex polynomial in the entries of $g = (g_\tau)_{\tau\in\Phi} \in \prod_\tau \mathrm{M}_{n_\tau}(\mathbb{C})$. If $f(g) = 0$ for every $g \in \prod_\tau \mathrm{U}(n_\tau)$, then $f(g) = 0$ for every $g \in \prod_\tau \mathrm{M}_{n_\tau}(\mathbb{C})$.*

Proof. Number the elements of Φ as $\tau_1, \ldots, \tau_m$, and for $0 \leq i \leq m$ let X_i be the set of g with $g_{\tau_j} \in \mathrm{M}_{n_{\tau_j}}(\mathbb{C})$ for $j \leq i$ and $g_{\tau_j} \in \mathrm{U}(n_{\tau_j})$ for $j > i$. Then $X_0 = \prod_\tau \mathrm{U}(n_\tau)$ and $X_m = \prod_\tau \mathrm{M}_{n_\tau}(\mathbb{C})$; we show by induction on i that f vanishes on X_i. The case $i = 0$ is the hypothesis. Let $1 \leq i \leq m$, suppose f vanishes on X_{i-1}, and fix the components g_{τ_j}, $j \neq i$, of a point of X_i; write $f_1(x)$ for the value of f at the point with these components and component $x \in \mathrm{M}_n(\mathbb{C})$ at τ_i, $n = n_{\tau_i}$. Then f_1 is a polynomial in the entries of x that vanishes on $\mathrm{U}(n)$.

Let $x \in \mathrm{GL}_n(\mathbb{C})$. By the polar decomposition, $x = u\exp(H)$ with u unitary and H hermitian. The entries of $\exp(tH) = \sum_{j\geq 0} t^j H^j/j!$ are entire functions of $t \in \mathbb{C}$, so $F(t) = f_1(u\exp(tH))$ is entire. For $t \in \mathrm{i}\mathbb{R}$ the matrix tH is skew-hermitian, so $\exp(tH)$ and $u\exp(tH)$ are unitary and $F(t) = 0$. An entire function that vanishes on $\mathrm{i}\mathbb{R}$, a set with an accumulation point, vanishes identically; so $f_1(x) = F(1) = 0$. Thus f_1 vanishes on $\mathrm{GL}_n(\mathbb{C})$, which is dense in $\mathrm{M}_n(\mathbb{C})$ (for any x, $x + \epsilon \cdot 1$ is invertible for all but finitely many ϵ), and f_1 is continuous; so $f_1 = 0$, that is, f vanishes on X_i. □

5 Enough forms: spanning by a second moment (Theorem B)

This section proves Theorem B (Theorem 5.1.2): for parameters whose map satisfies two moment bounds there are target forms, $d_{\mathcal{T}} \geq 1$, and few translates of the normalised Poincaré map carry the ramified source forms onto almost all of them. It ends with the proof of the main inequality (Section 5.9), which uses Theorem B twice: for the number of translates in the bound for the index, and for the length of the family that Theorem C requires. The non-vanishing $d_{\mathcal{T}} \geq 1$ is used again in Section 6, for the cost of one lift (Proposition 6.6.13).

The idea is spanning by a second moment. After a scaling, the map followed by the good idempotent e_0 is nearly an isometry, and the second moment over $\bar{G}$ of the Gram operators of its translates is at most $(1 + 9b^{-4})$ times $D_X^2/(\mathrm{mass}(I_r) \prod_{v \in \Phi} \mathrm{d}(\varsigma(v)))$, with $D_X = \dim \mathcal{S}$, $\mathrm{mass}(I_r)$ the mass of the standard hermitian lattice and d the Weyl product (Definition 5.1.1). The trace formula bounds the target dimension by $\#\bar{G}\, \mathrm{mass}(I_r) \prod_v \mathrm{d}(\varsigma(v))$ (Proposition 5.8.54), and the constants cancel. In our reading the second moment is then within a factor $1 + O(b^{-4})$ of the least value that the Cauchy–Schwarz inequality allows, so the translates are spread evenly over the target forms. Consequently some translate sends little into any given subspace (Proposition 5.1.4), and a greedy choice of logarithmically many translates fills the target forms up to $4/b^2$ of their dimension (Lemma 5.1.3). The principle that a second moment close to its least value forces equidistribution is the one behind the large sieve and its duality [29, 213] (see [146]). In our reading the greedy step is the linear-algebra form of the greedy covering argument [154, 192].

Most of the section proves the second moment, for every combination of class Poincaré series with the source coordinates as coefficients (Proposition 5.6.48); the Bessel-period map has this form at a Bessel pair (Lemma 5.6.33). The moment is an average over the level group, bounded through the unfolding of a Poincaré series. Section 5.1 derives Theorem B from the two moments and reduces the averaged second moment, by positivity of type and an induction over fixers of frame vectors, to a main term and remaining terms. The main term is bounded in Section 5.2, once bad types are shown to be rare, by Siegel's formula used as an upper bound and a sieve over heads; the remaining terms are bounded in Section 5.3, by Schur's test over orbits, a gain in discriminants from congruences and a count of lattices. The local inputs come from an engine for frames at an abstract place and its application at the complex and the finite places (Sections 5.4 and 5.5), from the unfolding near the Bessel subgroup (Section 5.6, which also fixes the parameters of the normalised Poincaré map), and from Bessel systems built from the discrete series (Section 5.7). Section 5.8 gives the target dimension by the trace formula. For other readings of this part of the argument see [223, 75].

5.1 Theorem B, spreading and the averaged second moment

This section proves Theorem B from the two moment bounds and reduces the second of them to the bounds for the main term and for the remaining terms, which the next two sections prove. The hypotheses use the Weyl product and the discrete series parameters that open the section (Definition 5.1.1): for some inner product on $\mathcal{S}$ and some $c > 0$, the map $L = c\mathcal{P}$ is (i) a near-isometry after e_0, and (ii) satisfies the averaged second moment bound. The proof of the theorem has three steps. The near-isometry gives $d_{\mathcal{T}} \geq 1$. The translates spread (Proposition 5.1.4), by an abstract spreading lemma (Lemma 5.1.3) applied to the orthogonal projection onto the target forms, with the second moment and the bound for the target dimension from the trace formula (Proposition 5.8.54) as input. A greedy step then gives logarithmically many translates, and the passage to ramified forms costs little by the codimension bound Lemma 4.6.3.

The definitions that follow set up the objects of Section 5 in the order of their dependence:

sieve systems, the shape of each local factor of the sieve; Bessel systems at a real place; place data and gauges, the axioms under which the frames engine works at every place at once; the generalised index of lattices; filling data, the abstract object for which the two moments are proved (Definition 5.1.9); the measurements by which the remaining terms are sorted; and the terms, amplitudes and amplitude blocks into which the unfolding turns the second moment (Definition 5.1.12).

The averaged second moment (Theorem 5.1.13) holds for every map with amplitude blocks into a unitary representation of $\bar{G}$, and Section 5.6 shows that a suitably scaled map of class-series form has amplitude blocks. The key fact is that $s \mapsto \|P^{*}\rho(s)P\|^{2}$ is a function of positive type [102] (Lemma 5.1.14). The proof is a downward induction over the sets of chosen places at which frame vectors are fixed. Either the fixer of one more frame vector carries more than its share b^{-6k}, and the induction applies to it, or positivity of type forces the elements that are not open to carry little (Lemma 5.1.15), and the open elements are bounded by the main term (Theorem 5.2.15). In our reading this is a dichotomy between concentration, paid for by the index of the fixer, and spreading. The base case, frame vectors fixed at all chosen places, is covered by the bound for the remaining terms (Theorem 5.3.2) and the contribution of the scalar cosets (Proposition 5.6.42).

Definition 5.1.1. Let $N \geq 1$. A **sign vector** is an element $\varepsilon \in \{+,-\}^{N}$.

(Dp1) The groups. $J_{\varepsilon} = \mathrm{diag}(\pm 1)$ is the diagonal matrix with entries $+1$ where $\varepsilon_{a} = +$ and -1 where $\varepsilon_{a} = -$,

$$G_{\varepsilon} = \{g \in \mathrm{GL}_{N}(\mathbb{C}) : g^{*}J_{\varepsilon}g = J_{\varepsilon}\}, \qquad K_{\varepsilon} = G_{\varepsilon} \cap \mathrm{U}(N).$$

(Dp2) The Weyl product and the half-sums. For $\lambda \in \mathbb{R}^{N}$,

$$\mathrm{d}(\lambda) = \prod_{1 \leq a < b \leq N} \frac{|\lambda_{a} - \lambda_{b}|}{b-a},$$

$\rho_{c}(\varepsilon)_{a} = \frac{1}{2}(\#\{b > a : \varepsilon_{b} = \varepsilon_{a}\} - \#\{b < a : \varepsilon_{b} = \varepsilon_{a}\})$, $\rho_{n}(\varepsilon)_{a} = \frac{1}{2}(\#\{b > a : \varepsilon_{b} \neq \varepsilon_{a}\} - \#\{b < a : \varepsilon_{b} \neq \varepsilon_{a}\})$ and $\Lambda_{\varepsilon}(\lambda) = \lambda + \rho_{n}(\varepsilon) - \rho_{c}(\varepsilon)$.

(Dp3) Square-integrable with Harish-Chandra parameter. A unitary representation σ of G_{ε} on a Hilbert space $\mathcal{H} \neq 0$ is **square-integrable with Harish-Chandra parameter λ relative to ε** if λ is strictly decreasing with $\lambda_{a} - \frac{N-1}{2} \in \mathbb{Z}$, σ is strongly continuous and irreducible, all its matrix coefficients lie in $L^{2}(G_{\varepsilon})$, the K_{ε}-type with highest weight $\Lambda_{\varepsilon}(\lambda)$ (for the diagonal torus and the positive compact roots $e_{a} - e_{b}$, $a < b$, $\varepsilon_{a} = \varepsilon_{b}$) occurs in σ, and every highest weight ν of a K_{ε}-type occurring in σ satisfies $|\nu + 2\rho_{c}(\varepsilon)| \geq |\Lambda_{\varepsilon}(\lambda) + 2\rho_{c}(\varepsilon)|$ (euclidean norm on $\mathbb{R}^{N}$).

(Dp4) The matrix H. For integers $k \geq 1$, $r \geq 0$ and $n = 2r + k + 1$, H is the hermitian $n \times n$ matrix with $H_{ij} = 1$ if $i + j = n + 1$ and $\min(i,j) \leq r$, $H_{ii} = 1$ for $r < i < r + k + 1$, $H_{r+k+1,r+k+1} = -1$, and $H_{ij} = 0$ otherwise (signature $(k + r, r + 1)$);

$$\mathrm{U}(H) = \{g \in \mathrm{GL}_{n}(\mathbb{C}) : g^{*}Hg = H\}, \qquad \mathfrak{u}(H) = \{X : X^{*}H + HX = 0\}.$$

(Dp5) The same notion for $\mathrm{U}(H)$. Let $\varepsilon \in \{+,-\}^{n}$ be a sign vector and $\lambda \in \mathbb{R}^{n}$. A unitary representation σ of $\mathrm{U}(H)$ on a Hilbert space $\mathcal{H}$ is **square-integrable with Harish-Chandra parameter λ relative to ε** if, after conjugation by a unitary Cayley transform carrying H to $\mathrm{diag}(\varepsilon)$, it is square-integrable with Harish-Chandra parameter λ relative to ε in the sense of (Dp3); that is, there is a unitary matrix P with $P^{*}J_{\varepsilon}P = H$ such that $g' \mapsto \sigma(P^{-1}g'P)$, a unitary representation of G_{ε} on $\mathcal{H}$, satisfies (Dp3).

(Dp6) Smooth vectors and words. Let σ be a unitary representation of a closed subgroup G of $\mathrm{GL}_n(\mathbb{C})$ with Lie algebra $\mathfrak{g}$ (for instance $\mathrm{U}(H)$ and $\mathfrak{u}(H)$, or G_ε and its Lie algebra). A vector u is **smooth** if $g \mapsto \sigma(g)u$ is a C^∞ map $G \to \mathcal{H}$; for $X \in \mathfrak{g}$, $d\sigma(X)u = \frac{d}{dt}\big|_{t=0}\sigma(\exp tX)u$. For a **word** $w = (X_1, \dots, X_j)$ in $\mathfrak{g}$ and a smooth vector u put $d\sigma(w)u = d\sigma(X_1)\cdots d\sigma(X_j)u$.

Theorem 5.1.2. *Let D be admissible data over a parameter pair P, and let $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$ be parameters as in Definition 3.2.37 such that $\mathcal{S} \neq 0$ and, with $D_X = \dim \mathcal{S}$, there are an inner product on $\mathcal{S}$ and a real $c > 0$ for which $L = c\mathcal{P}\colon \mathcal{S} \to \mathcal{A}_M(Y_0)$ satisfies the two moment bounds*

(i) $\|e_0 L \,\mathrm{pr}_U\|_{\mathrm{HS}}^2 \geq \dim U - 5b^{-4}D_X$ *for every subspace $U \subseteq \mathcal{S}$;*

(ii) $\displaystyle\sum_{s\in\bar G} \|L^* sL\|_{\mathrm{HS}}^2 \leq \Big(1 + \frac{9}{b^4}\Big)\frac{D_X^2}{\mathrm{mass}(I_r)\prod_{v\in\Phi}\mathrm{d}(\varsigma(v))}.$

Then $d_{\mathcal{T}} \geq 1$, and there is a subset $t \subseteq \bar G$ with

$$\#t \cdot \dim_{\mathbb{C}} \mathcal{S}^{\mathrm{ram}} \;\leq\; 5\, d_{\mathcal{T}} \log(1 + 2b^2) \qquad \textit{and} \qquad \dim_{\mathbb{C}} \sum_{s\in t} s\mathfrak{p}(\mathcal{S}^{\mathrm{ram}}) \;\geq\; \Big(1 - \frac{4}{b^2}\Big) d_{\mathcal{T}}.$$

Proof. Write $D_X = \dim \mathcal{S}$, $d_{\mathrm{ram}} = \dim \mathcal{S}^{\mathrm{ram}}$, $\varepsilon = (8b^2)^{-1}$, and for $s \in \bar G$ let $\mathfrak{g}_s = \#\bar G \cdot e_0 s\mathcal{P}\colon \mathcal{S} \to \mathcal{T}$, as in Proposition 5.1.4. By Lemma 2.3.6, $\log\log b > 10^{12}$, so b is huge; we use $b^2 > 2$, $b^4 > 5$ and $1 + 2b^2 \geq e$. Let the inner product on $\mathcal{S}$, the real $c > 0$ and $L = c\mathcal{P}$ be as in the hypothesis.

The target is nonzero. By the hypothesis $\mathcal{S} \neq 0$, so $D_X \geq 1$, and the bound (i) for $U = \mathcal{S}$ gives $\|e_0 L\|_{\mathrm{HS}}^2 \geq (1 - 5b^{-4})D_X > 0$. So $e_0 L \neq 0$, and as $e_0 L$ takes values in $\mathcal{T} = e_0\mathcal{A}_M(Y_0)$, $d_{\mathcal{T}} \geq 1$.

Few translates of all source forms. By Proposition 5.1.4, whose hypothesis on $\mathcal{P}$ is the one assumed here, the family $(\mathfrak{g}_s)_{s\in\bar G}$ of maps $\mathcal{S} \to \mathcal{T}$ satisfies the hypothesis of Lemma 5.1.3(b) with this ε, since $\frac{3\varepsilon}{2} = \frac{3}{16b^2}$; and $0 < \varepsilon < \frac{1}{16}$ as $b^2 > 2$, $\mathcal{S} \neq 0$, $\mathcal{T} \neq 0$. With $\frac{1}{16\varepsilon} = \frac{b^2}{2}$, Lemma 5.1.3(b) gives $t \subseteq \bar G$ with

$$\#t \cdot D_X \leq \frac{5}{2}\Big(1 + \log\frac{b^2}{2}\Big) d_{\mathcal{T}} \leq 5\, d_{\mathcal{T}} \log(1 + 2b^2), \qquad \dim \sum_{s\in t} \mathfrak{g}_s(\mathcal{S}) \geq \Big(1 - \frac{2}{b^2}\Big) d_{\mathcal{T}},$$

where the second inequality of the first display uses $\log\frac{b^2}{2} \leq \log(1 + 2b^2)$ and $1 \leq \log(1 + 2b^2)$. As $d_{\mathrm{ram}} \leq D_X$, this gives the bound $\#t \cdot d_{\mathrm{ram}} \leq 5\, d_{\mathcal{T}} \log(1 + 2b^2)$ of the theorem.

Passing to ramified forms. By Lemma 4.6.3, $D_X - d_{\mathrm{ram}} \leq \frac{f}{N-1} D_X \leq b^{-8} D_X$. With Lemma 5.1.3(c) for $V' = \mathcal{S}^{\mathrm{ram}}$, the bound on $\#t \cdot D_X$, and $5\log(1 + 2b^2) \leq 10b^2 \leq 2b^6$,

$$\begin{aligned}\dim \sum_{s\in t} \mathfrak{g}_s(\mathcal{S}^{\mathrm{ram}}) &\geq \dim \sum_{s\in t} \mathfrak{g}_s(\mathcal{S}) - \#t\,(D_X - d_{\mathrm{ram}}) \\ &\geq \Big(1 - \frac{2}{b^2}\Big) d_{\mathcal{T}} - b^{-8} \cdot 5\, d_{\mathcal{T}} \log(1 + 2b^2) \geq \Big(1 - \frac{4}{b^2}\Big) d_{\mathcal{T}}.\end{aligned}$$

The normalised translates. For $s \in \bar G$ and $\varphi \in \mathcal{S}^{\mathrm{ram}}$, $s\mathfrak{p}\varphi = c_B c_{\mathrm{ss}}^{-1} \#\bar G \cdot s\, e_0 \mathcal{P}\varphi = c_B c_{\mathrm{ss}}^{-1} \mathfrak{g}_s\varphi$, as e_0 is central in $\mathbb{C}[\bar G]$. Since $c_B > 0$ and $c_{\mathrm{ss}} \neq 0$, $s\mathfrak{p}(\mathcal{S}^{\mathrm{ram}}) = \mathfrak{g}_s(\mathcal{S}^{\mathrm{ram}})$, and the last display is the second bound of the theorem.

□

Lemma 5.1.3. *(a) Let T be a Hilbert space, Γ a finite group acting on T by a unitary representation ρ, and pr_0 the orthogonal projection onto a subspace $T_0 \subseteq T$ with $\mathrm{pr}_0\rho(s) = \rho(s)\mathrm{pr}_0$ for all $s \in \Gamma$. Let $S \neq 0$ be a Hilbert space, $B\colon S \to T$ linear and $\varepsilon \geq 0$, and assume*

(SM1) $\|\mathrm{pr}_0 B\,\mathrm{pr}_U\|^2 \geq \dim U - \varepsilon \dim S$ *for every subspace $U \subseteq S$;*

(SM2) $\dim T_0 \cdot \sum_{s\in\Gamma} \|B^*\rho(s)B\|^2 \leq \#\Gamma\,(1+\varepsilon)(\dim S)^2$.

Then for every subspace $W \subseteq T_0$, with $\alpha = \dim W/\dim T_0$ ($\alpha = 0$ if $T_0 = 0$), there is $s \in \Gamma$ with

$$\dim\{x \in S :\ \mathrm{pr}_0\rho(s)Bx \in W\} \ \leq\ \Big(\frac{1+\alpha}{2} + \frac{3\varepsilon}{2}\Big)\dim S.$$

(b) Let $V \neq 0$ and $Y \neq 0$ be vector spaces, $0 < \varepsilon < \frac{1}{16}$, and $(B_\theta)_{\theta\in\Theta}$ a family of linear maps $B_\theta\colon V \to Y$ such that for every subspace $F \subseteq Y$ there is $\theta \in \Theta$ with $\dim B_\theta^{-1}(F) \leq (\frac{1+\dim F/\dim Y}{2} + \frac{3\varepsilon}{2})\dim V$. Then there is a finite subset $t \subseteq \Theta$ with

$$\#t \cdot \dim V \leq \frac{5}{2}\Big(1 + \log\frac{1}{16\varepsilon}\Big)\dim Y \qquad \text{and} \qquad \dim \sum_{\theta\in t} B_\theta(V) \geq (1-16\varepsilon)\dim Y.$$

(c) For linear maps $B_\theta\colon V \to Y$, a finite set t of indices and a subspace $V' \subseteq V$,

$$\dim \sum_{\theta\in t} B_\theta(V) \leq \dim \sum_{\theta\in t} B_\theta(V') + \#t\,(\dim V - \dim V').$$

Proof. **(a), the mean of $\|\mathrm{pr}_W\rho(s)B\|^2$.** Let $W \subseteq T_0$ and $w = \dim W$, so $w = \alpha\dim T_0$ in all cases. Put $Q_s = \rho(s)BB^*\rho(s)^*$ for $s \in \Gamma$, and use the Hilbert–Schmidt inner product $\langle X, Y\rangle = \mathrm{tr}(X^*Y)$ on $\mathrm{End}(T)$. As $\rho(s)^* = \rho(s^{-1})$ and the trace is cyclic, with $X = B^*\rho(s^{-1}t)B$,

$$\langle Q_s, Q_t\rangle = \mathrm{tr}\big(B^*\rho(s^{-1}t)B\,B^*\rho(t^{-1}s)B\big) = \mathrm{tr}(XX^*) = \|B^*\rho(s^{-1}t)B\|^2,$$

so $\big\|\sum_s Q_s\big\|^2 = \sum_{s,t}\|B^*\rho(s^{-1}t)B\|^2 = \#\Gamma\cdot m$ with $m = \sum_{u\in\Gamma}\|B^*\rho(u)B\|^2$. Since pr_W is a self-adjoint idempotent, $\|\mathrm{pr}_W\rho(s)B\|^2 = \mathrm{tr}(B^*\rho(s)^*\mathrm{pr}_W\rho(s)B) = \langle \mathrm{pr}_W, Q_s\rangle$, and $\|\mathrm{pr}_W\|^2 = w$. By the Cauchy–Schwarz inequality, (SM2) and $w = \alpha\dim T_0$,

$$\Big(\sum_{s\in\Gamma}\|\mathrm{pr}_W\rho(s)B\|^2\Big)^2 \leq w\,\#\Gamma\,m \leq \alpha(1+\varepsilon)\,(\#\Gamma\dim S)^2 \leq \Big(\frac{\alpha+1+\varepsilon}{2}\Big)^2(\#\Gamma\dim S)^2,$$

the last step by $4\alpha(1+\varepsilon) \leq (\alpha+1+\varepsilon)^2$. Hence some $s \in \Gamma$ has $\|\mathrm{pr}_W\rho(s)B\|^2 \leq \frac{\alpha+1+\varepsilon}{2}\dim S$.

(a), the preimage. Fix this s and let $U = \{x \in S : \mathrm{pr}_0\rho(s)Bx \in W\}$. For $u \in U$ the vector $\mathrm{pr}_0\rho(s)Bu$ lies in W, and $\rho(s)Bu - \mathrm{pr}_0\rho(s)Bu$ is orthogonal to $T_0 \supseteq W$; so $\mathrm{pr}_W\rho(s)Bu = \mathrm{pr}_0\rho(s)Bu = \rho(s)\mathrm{pr}_0 Bu$. As $\rho(s)$ is unitary, $\|\mathrm{pr}_0 B\,\mathrm{pr}_U x\| = \|\mathrm{pr}_W\rho(s)B\,\mathrm{pr}_U x\|$ for all $x \in S$, hence $\|\mathrm{pr}_0 B\,\mathrm{pr}_U\| = \|\mathrm{pr}_W\rho(s)B\,\mathrm{pr}_U\| \leq \|\mathrm{pr}_W\rho(s)B\|$, since composing on the right with an orthogonal projection does not increase the Hilbert–Schmidt norm (compute it in an orthonormal basis of S adapted to $U \oplus U^\perp$). With (SM1),

$$\dim U - \varepsilon\dim S \leq \frac{\alpha+1+\varepsilon}{2}\dim S, \qquad \text{that is} \qquad \dim U \leq \Big(\frac{1+\alpha}{2} + \frac{3\varepsilon}{2}\Big)\dim S.$$

(b), one greedy step. Let $n = \dim V \geq 1$, $d = \dim Y \geq 1$, $\lambda = 2n/(5d)$, and for a subspace $F \subseteq Y$ let $c(F) = d - \dim F$. For every θ, rank–nullity for B_θ restricted to $B_\theta^{-1}(F)$, whose image is $F \cap B_\theta(V)$, gives $\dim(F \cap B_\theta(V)) = \dim B_\theta^{-1}(F) - \dim\ker B_\theta$, and so

$$\dim\big(F + B_\theta(V)\big) = \dim F + n - \dim B_\theta^{-1}(F). \tag{5.1}$$

Suppose $c(F) > 16\varepsilon d$ and put $\alpha = \dim F/d$, so $1-\alpha = c(F)/d > 16\varepsilon$. Choose θ by the hypothesis. Then $\frac{3\varepsilon}{2} < \frac{3}{32}(1-\alpha)$ and

$$\dim B_\theta^{-1}(F) \le \Big(1 - \frac{1-\alpha}{2} + \frac{3}{32}(1-\alpha)\Big)n \le \Big(1 - \frac{2}{5}(1-\alpha)\Big)n,$$

so by (5.1) $c(F + B_\theta(V)) \le c(F) - \frac{2}{5}(1-\alpha)n = (1-\lambda)\,c(F)$.

(b), $\lambda \le 1$. Apply the step to $F = 0$, allowed as $c(0) = d > 16\varepsilon d$: then $0 \le c(B_\theta(V)) \le (1-\lambda)d$.

(b), iteration. For $l \ge 0$ there is $t \subseteq \Theta$ with $\#t \le l$ and $c(\sum_{\theta\in t} B_\theta(V)) \le \max(16\varepsilon, e^{-l\lambda})\,d$. For $l = 0$ take $t = \emptyset$. Given t for l, keep it if its codimension is at most $16\varepsilon d$; otherwise that codimension is at most $e^{-l\lambda}d$, and adding the θ of the greedy step for $F = \sum_{\theta'\in t} B_{\theta'}(V)$ gives codimension at most $(1-\lambda)e^{-l\lambda}d \le e^{-(l+1)\lambda}d$, as $0 \le 1-\lambda \le e^{-\lambda}$.

(b), the count. Let $L = \log\frac{1}{16\varepsilon} \ge 0$ and $l = \lceil L/\lambda \rceil$. Then $e^{-l\lambda} \le 16\varepsilon$, so the set t of the iteration has $\dim \sum_{\theta\in t} B_\theta(V) \ge (1-16\varepsilon)d$, and $l\lambda < L + \lambda \le L + 1$, so

$$\#t \cdot n \le l\,n = \frac{5}{2}\,l\lambda\,d \le \frac{5}{2}(1+L)\,d.$$

(c). Let C be a complement of V' in V. Then $B_\theta(V) = B_\theta(V') + B_\theta(C)$ for each θ, so $\sum_{\theta\in t} B_\theta(V) = \sum_{\theta\in t} B_\theta(V') + \sum_{\theta\in t} B_\theta(C)$ and

$$\dim \sum_{\theta\in t} B_\theta(V) \le \dim \sum_{\theta\in t} B_\theta(V') + \sum_{\theta\in t} \dim B_\theta(C) \le \dim \sum_{\theta\in t} B_\theta(V') + \#t\,(\dim V - \dim V').$$

□

Proposition 5.1.4. *Let D be admissible data, put $D_X = \dim \mathcal{S}$, and suppose that $\mathcal{S} \ne 0$ and that there are an inner product on $\mathcal{S}$ and a real $c > 0$ for which $L = c\,\mathcal{P}\colon \mathcal{S} \to \mathcal{A}_M(Y_0)$ satisfies the two moment bounds*

(i) $\|e_0 L\,\mathrm{pr}_U\|_{\mathrm{HS}}^2 \ge \dim U - 5b^{-4}D_X$ *for every subspace $U \subseteq \mathcal{S}$;*

(ii) $\displaystyle\sum_{s\in\bar G} \|L^* s L\|_{\mathrm{HS}}^2 \le \Big(1 + \frac{9}{b^4}\Big)\frac{D_X^2}{\mathrm{mass}(I_r)\prod_{v\in\Phi}\mathrm{d}(\varsigma(v))}.$

For $s \in \bar G$ let $\mathfrak{g}_s = \#\bar G \cdot e_0\, s\,\mathcal{P}\colon \mathcal{S} \to \mathcal{T}$. For every subspace $F \subseteq \mathcal{T}$, with $\alpha = \dim F/d_{\mathcal{T}}$ ($\alpha = 0$ if $d_{\mathcal{T}} = 0$), there is $s \in \bar G$ with

$$\dim\{\varphi \in \mathcal{S} :\ \mathfrak{g}_s\varphi \in F\} \ \le\ \Big(\frac{1+\alpha}{2} + \frac{3}{16b^2}\Big)D_X.$$

Proof. Write $\rho(s)$ for the action of $s \in \bar G$ on $\mathcal{A}_M(Y_0)$, $\mathrm{mass}_r = \mathrm{mass}(I_r)$, $\Pi = \prod_{v\in\Phi}\mathrm{d}(\varsigma(v))$, and $\varepsilon = (8b^2)^{-1}$. By Lemma 2.3.6, $\log\log b > 10^{12}$, so $b^2 \ge 72$, hence

$$5b^{-4} \le \varepsilon \qquad \text{and} \qquad 9b^{-4} \le \varepsilon. \tag{5.2}$$

The setting of Lemma 5.1.3. Equip $\mathcal{A}_M(Y_0)$ with the Petersson inner product and $\mathcal{S}$ with the inner product of the hypothesis, and let $c > 0$ and $L = c\mathcal{P}$ be as there; the action ρ of $\bar G$ is unitary. The element $e_0 = \sum_g e_0(g)\,g \in \mathbb{C}[\bar G]$ is a central idempotent, and its coefficients satisfy $e_0(g^{-1}) = \overline{e_0(g)}$: each e_{c_w} has the coefficients $\Xi_{c_w}(1)\overline{\Xi_{c_w}(g)}/|\mathrm{GL}_r(k_w)|$ of a virtual character Ξ_{c_w},

whose values satisfy $\Xi(g^{-1}) = \overline{\Xi(g)}$ and $\Xi(1) \in \mathbb{R}$, and this property passes to tensor products and sums. Hence

$$\rho(e_0)^* = \sum_g \overline{e_0(g)}\,\rho(g^{-1}) = \sum_g e_0(g^{-1})\rho(g^{-1}) = \rho(e_0),$$

so $\mathrm{pr}_0 = \rho(e_0)$ is a self-adjoint idempotent, that is, the orthogonal projection onto its range $\mathcal{T}$, and it commutes with every $\rho(s)$ as e_0 is central. Take $\Gamma = \bar{G}$, $T = \mathcal{A}_M(Y_0)$, $T_0 = \mathcal{T}$, $S = \mathcal{S}$ and $B = L$.

(SM1) and (SM2). By the bound (i) of the hypothesis and (5.2), $\|\mathrm{pr}_0 L\,\mathrm{pr}_U\|^2 \geq \dim U - 5b^{-4}D_X \geq \dim U - \varepsilon D_X$ for every subspace $U \subseteq \mathcal{S}$. By Proposition 5.8.54, $d_{\mathcal{T}} \leq \#\bar{G}\,\mathrm{mass}_r\,\Pi$, where $\mathrm{mass}_r > 0$, being a sum of reciprocals of orders of finite groups, and $\Pi > 0$, as the entries of each $\varsigma(v)$ are pairwise distinct (by Proposition 5.8.4(iii), ς has gap Δ for Y_0); with the bound (ii) of the hypothesis, whose left side is nonnegative, and (5.2),

$$d_{\mathcal{T}} \sum_{s\in\bar{G}} \|L^*\rho(s)L\|^2 \leq \#\bar{G}\,\mathrm{mass}_r\,\Pi \cdot (1+9b^{-4})\frac{D_X^2}{\mathrm{mass}_r\,\Pi} = \#\bar{G}\,(1+9b^{-4})D_X^2 \leq \#\bar{G}\,(1+\varepsilon)D_X^2.$$

As $\mathcal{S} \neq 0$ by the hypothesis, Lemma 5.1.3(a) applies.

Conclusion. For $F \subseteq \mathcal{T}$ with share α, Lemma 5.1.3(a) gives $s \in \bar{G}$ with

$$\dim\{\varphi \in \mathcal{S} :\ \rho(e_0)\rho(s)L\varphi \in F\} \leq \Big(\frac{1+\alpha}{2} + \frac{3\varepsilon}{2}\Big)D_X = \Big(\frac{1+\alpha}{2} + \frac{3}{16b^2}\Big)D_X.$$

Since e_0 is central, $\rho(e_0)\rho(s)L\varphi = c\,(\#\bar{G})^{-1}\mathfrak{g}_s\varphi$ with $c(\#\bar{G})^{-1} \neq 0$, and F is a subspace; so the set on the left is $\{\varphi : \mathfrak{g}_s\varphi \in F\}$. □

Definition 5.1.5. Damped powers and head weights. For $K \geq 0$ and $t > 0$ the **damped power** is $\ell_K(t) = t^K \min(1,t)^6$. For a sequence $p = (p_0, p_1, \dots)$ of nonzero complex numbers, $R > 0$, $K \geq 0$ and $i \geq 0$, put $\delta_l = p_0 p_1 \cdots p_l$ and define the **head weight**

$$\Lambda_i^{R,K}(p) = \prod_{l<i} w_l, \qquad w_l = \begin{cases} \max(1, |\delta_l|^{-1})^K, & |\delta_l| \leq R, \\ 0, & |\delta_l| > R. \end{cases}$$

Sieve system. Let C be a commutative ring with a measure, U a topological space, $u\colon C \to U$ and $\iota\colon U \to C$ maps, and $n \geq 0$. A **head** is a sequence $p\colon \mathbb{N} \to U$; $p[i \mapsto c]$ is p with its entry i replaced by c. A **sieve system** of length n on (C, U, u, ι) consists of forms $J_i(p,q) \in \mathbb{C}$ on pairs of heads ($0 \leq i \leq n+1$), a character-like function $\psi\colon C \to \mathbb{C}$, a **mass bound** $E_0 \in \mathbb{R}$, **weights** $\Lambda_i(p) \in \mathbb{R}$ and a **decay** $\beta\colon C \to \mathbb{R}$ such that:

(SV1) $\psi(0) = 1$.

(SV2) Descent. For $i \leq n$, heads p, q and $\xi \in C$, the function $c \mapsto \psi(\xi c)\,J_{i+1}(p[i \mapsto u(c)], q[i \mapsto u(c)])$ is integrable on C; its integral equals $J_i(p,q)$ if $\xi = 0$, and has absolute value at most $\beta(\xi)\Lambda_i(p)\Lambda_i(q)$ if ξ is a unit.

(SV3) Positivity of the top form. For every finite family (a_j) of heads with a common entry $a_j(n) = t$, the matrix $(J_{n+1}(a_i, a_j))_{i,j}$ is positive semidefinite.

(SV4) J_{n+1} is jointly continuous.

(SV5) Mass. $|J_0(p,q)| \leq E_0$ for all p, q.

(SV6) Unequal heads. For $i < n$, if $\xi = \iota(p_i) - \iota(q_i)$ is a unit, then

$$|J_{i+1}(p,q)| \le \beta(\xi)\Lambda_{i+1}(p)\Lambda_{i+1}(q).$$

At a real or complex place one takes $C = U = \mathbb{C}$, $u(c) = c$ for $c \neq 0$ and $u(0) = 1$, $\iota(z) = z^{-1}$, and $\psi(c) = \exp(c - \bar{c})$.

Products. Given a sieve system on C and, for each v in a finite set, a sieve system on $\mathbb{C}$ as above, all of the same length and with the same weights, the products $J_i = \prod_v J_{v,i} \cdot J_{C,i}$, $\psi = \prod_v \psi_v \cdot \psi_C$, $E_0 = \prod_v E_{0,v} \cdot E_{0,C}$, $\Lambda_i = \prod_v \Lambda_{v,i} \cdot \Lambda_{C,i}$, $\beta = \prod_v \beta_v \cdot \beta_C$ on $\mathbb{C}^v \times C$ form the **product sieve system**.

Definition 5.1.6. Setting. Fix integers $\varrho \ge 1$ and $k \ge 1$, $n = \varrho+1$, $r = 2\varrho+k+1$. Let $V_1 = \mathbb{C}^r$ with the hermitian form given by ϱ hyperbolic planes (e_a, e_a^*), the standard positive definite space $X_0 = \mathbb{C}^k$ and a vector z with $h(z,z) = -1$; $G_1 = \mathrm{U}(V_1)$, with the **frame** $(e_1, \dots, e_\varrho, z)$. Let $\mathsf{U} \subseteq G_1$ be the unipotent radical of the stabiliser of the isotropic flag $\langle e_1\rangle \subset \cdots \subset \langle e_1, \dots, e_\varrho\rangle$, $H = \mathrm{U}(X_0) \cong \mathrm{U}(k)$, and $R = \mathsf{U}H$ (the **Bessel subgroup**). For $u \in \mathsf{U}$ let $\sigma(u)$ be the sum of the entries just above the diagonal in the frame block, and $\mathrm{wave}(c) = \exp(c - \bar{c})$. The **Bessel length** is $\ell(y) = \inf_{s\in R}(\|sy\|_{\mathrm{op}} + 1)$. In the basis $e_1, \dots, e_\varrho$, the standard basis of X_0, z, $e_1^*, \dots, e_\varrho^*$ the Gram matrix J_1 of V_1 is a hermitian involution and $G_1 = \mathrm{U}(J_1)$; dy is the Cayley measure of G_1. An integral $\int_{G_1} \cdots dy$ of a vector-valued function is its Bochner integral, taken to be 0 when the integrand is not integrable. Integrals over $\mathsf{U}\backslash G_1$ are for the quotient of dy by the measure du of U fixed in the paragraph "Measure of U" below (the measure on G_1 weighted by a cutoff of total du-mass 1 on every U-orbit). For a real $b \ge 2$, y is **in the cut** if $\ell(y) \le b^{6n+2}$. Write $f = (f_0, \dots, f_\varrho) = (e_1, \dots, e_\varrho, z)$ for the frame; for $\rho \ge 0$ the **frame box** is $\mathrm{Bx}(\rho) = \{y \in G_1 : \|y^{-1}f_a\| \le \rho \ (0 \le a \le \varrho)\}$ (Euclidean norm of $\mathbb{C}^r$), and $Z(y) = (h(f_a, yf_b))_{a,b=0}^{\varrho}$ is the **frame Gram matrix** of y.

Sign frame. For a sign vector s indexed by the basis $e_1, \dots, e_\varrho$, the standard basis of X_0, z, $e_1^*, \dots, e_\varrho^*$ (values in $\{\pm 1\}$), a **sign frame** for s is a unitary complex $r \times r$ matrix p, indexed by the same basis, with $p^* J_1 p = \mathrm{diag}(s)$.

Bessel functions. Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be a unitary representation on $\mathbb{C}^I$, I finite. A **Bessel function of type** ρ is a bounded measurable $F\colon G_1 \to \mathbb{C}^I$ with $F(uy) = \mathrm{wave}(\sigma(u))F(y)$ for $u \in \mathsf{U}$ and $F(hy) = \rho(h)F(y)$ for $h \in H$.

Cut pair. Let $f \ge 1$. A pair $(W, W^\flat)$ of Bessel functions of type ρ is a **cut pair** (for b, f) if

(BS1) $W \in L^1(\mathsf{U}\backslash G_1)$, $\int_{\mathsf{U}\backslash G_1} |W|^2 = \#I$ and $\int_{\mathsf{U}\backslash G_1} |W^\flat|^2 \le \#I$;

(BS2) $W^\flat$ vanishes outside the cut, $W^\flat(y) \neq W(y)$ only if $\ell(y) > b^{6n+2}/2$, and $\int_{\mathsf{U}\backslash G_1} |W^\flat - W|^2 \le b^{-2r^3 f}\#I$;

(BS3) Decay majorant. there are a measurable left R-invariant $A\colon G_1 \to [0,\infty]$ and $a < \infty$ with $A \le a$, $\max(1, \ell(y)/b^{6n})^{3r^3 f}\,\|F(y)\| \le A(y)$ for $F \in \{W, W^\flat\}$, and $a\int_{\mathsf{U}\backslash G_1} A \le b^{400r^2}\#I$.

Schur's bound. F satisfies $\mathrm{Schur}(D)$ if $\int_{\mathsf{U}\backslash G_1} \left\|\int_{G_1} \phi(y)F(\gamma y)\,dy\right\|^2 d\gamma \le D\int |\phi|^2$ for every integrable ϕ on G_1. F has **Schur's bound** (B, D) if there are Bessel functions V_w of type ρ, indexed by words w in pairs of basis indices, with $V_\emptyset = F$, $\frac{d}{dt}V_w(\gamma(t)) = \sum_{a,b} X_{ab}V_{(a,b)w}(\gamma(t))$ whenever $\gamma'(t) = \gamma(t)X$, and V_w satisfying $\mathrm{Schur}(B^{2|w|}D)$ for $|w| \le 2n^2$.

Normalised orbital integrals. For $p = (p_0, \dots, p_\varrho)$ nonzero complex numbers let $w_p \in G_1$ send $e_{a+1} \mapsto p_a e_{a+1}^*$ and $e_{a+1}^* \mapsto e_{a+1}/\bar{p}_a$ for $0 \le a < \varrho$, and fix the other basis vectors. For

$\sigma \in G_1$,

$$N(\sigma,F,p)=\frac{\sqrt{c_G}}{\#I}\sqrt{(|p_\varrho|^2-1)_+}^{\,k-1}\prod_{a<\varrho}|p_a|^{k+1+2(\varrho-a-1)}\Big(\int_{G_1}F_i(y)\overline{F_j((w_p\sigma)^{-1}y)}\,dy\Big)_{i,j},$$

with $c_G=\operatorname{vol}\mathrm{U}(k)/\operatorname{vol}\mathrm{U}(r)\cdot\operatorname{vol}(S^{2k-1})$ (the **open-cell constant**) and $\Theta=\operatorname{vol}\mathrm{U}(k)/\operatorname{vol}\mathrm{U}(r)\cdot\pi^{(2\varrho+1)k}$, where $\operatorname{vol}(S^{2k-1})=2\pi^k/(k-1)!$.

Local bounds. Forms $(J_i)_{i\le n}$ on pairs of heads in $(\mathbb{C}^\times)^{\mathbb{N}}$ are **local bounds** for F with degree $\deg>0$ if they form a sieve system of length ϱ on $\mathbb{C}$ (with u,ι,ψ as at a complex place) with mass bound $(1+b^{-10})/\deg$, weights $\Lambda_i^{b^{60n^2},n^6}$ and decay $\beta(\xi)=\Theta\,b^{2000n^9}\ell_{k-n^7}(1/|\xi|)$, and if $J_n(p,q)=\langle N(\sigma,F,p),N(\sigma,F,q)\rangle$ for every $\sigma\in G_1$ that fixes e_a and e_a^* for $1\le a\le\varrho$ and all p,q with $h(z,\sigma z)=p_\varrho=q_\varrho$.

Open cell. For a complex $(\varrho+1)\times(\varrho+1)$ matrix Z let the **inverse corner** be $\mathrm{ic}(Z)=(Z^{-1})_{\varrho\varrho}$ when Z is invertible, and let $\mathsf{g}_1(Z)=|\det Z|^2-|D_\varrho(Z)|^2$, D_ϱ the leading principal minor of order ϱ (the **real Gram discriminant**). The **open cell** is $\Omega_{\mathrm{cell}}=\{Z:\det Z\neq0,\ |\mathrm{ic}(Z)|<1\}$; Z is **bounded in the cell** by P if $Z\in\Omega_{\mathrm{cell}}$, $\|Z\|\le P$, $\|Z^{-1}\|\le P$ (operator norms) and $(1-|\mathrm{ic}(Z)|^2)^{-1/2}\le P$. Let $\mathsf{U}^0\subseteq\mathsf{U}$ be the subgroup fixing $f_0,\dots,f_\varrho$, parametrised by $(a,b)\in\mathrm{M}_{\varrho\times k}(\mathbb{C})\times\mathrm{M}_\varrho(\mathbb{R})$ through $u(a,b)=1+XaW^*-Wa^*D^*+X\big(\mathrm{sk}(b)-\frac12aa^*\big)D^*$, where X,W,D are the matrices with columns $e_1,\dots,e_\varrho$; the basis vectors of X_0; $e_1^*,\dots,e_\varrho^*$, and $\mathrm{sk}(b)_{ll'}=\big((1+\mathrm{i})b_{ll'}-(1-\mathrm{i})b_{l'l}\big)/2$ for $l=l'$ and $\big((1+\mathrm{i})b_{ll'}-(1-\mathrm{i})b_{l'l}\big)/\sqrt2$ for $l\neq l'$ ($\mathrm{i}^2=-1$); dv is the image of Lebesgue measure under u. For $c>0$ the **fixer measure** on $\mathrm{U}(k)\times\mathsf{U}^0$ is $\mu_c=c_G\,c^{k-1}\,dh\otimes dv$, dh the Haar probability measure, and the **fixer ball** of radius R is $\{(h,v):\|v\|\le R\}$. For $y\in G_1$, $\deg,\rho,d>0$ the **pairing bound** is

$$\mathrm{PB}(y,\deg,\rho,d)=\Big(\frac{\rho^{9n}}{\mathsf{g}_1(Z(y))}\Big)^{10^5n^4}\frac{d^2}{\deg\cdot c_G\,(\mathsf{g}_1(Z(y))/2)^{k-1}}.$$

A finite family of kernels $k_i\colon\mathcal{X}\times\mathcal{Y}\to\mathbb{C}$ is an L^2 **contraction** from $(\mathcal{Y},\mu)$ to $(\mathcal{X},m)$ with bound ν (measures m on $\mathcal{X}$, μ,ν on $\mathcal{Y}$) if $\int_{\mathcal{X}}\sum_i\big|\int_{\mathcal{Y}}\phi\,k_i(x,\cdot)\,d\mu\big|^2dm(x)\le\int_{\mathcal{Y}}|\phi|^2d\nu$ for every bounded measurable ϕ. For $F\colon G_1\to\mathbb{C}^I$, maps $g,t\colon\mathcal{X}\to G_1$, reals M,C, a set $A\subseteq G_1$, a measure ν on G_1 and a measure m on $\mathcal{X}$, (A,ν,m) is a **kernel bound** for (F,g,t,M,C) if A is compact, m is a probability measure, the kernels $(x,z)\mapsto F_i(g(x)z)\overline{F_j(t(x)z)}$, $(i,j)\in I\times I$, are an L^2 contraction from $(G_1,dy|_A)$ to $(\mathcal{X},m)$ with bound ν, ν is finite with $\nu(G_1)\le C$, and for m-almost every x every $z\in\mathrm{Bx}(M)$ with $\ell(g(x)z)\le M$ lies in A. For $\gamma\in G_1$, $\rho>0$ and $\deg>0$, (A,ν,m,t) is an **open-cell bound** for $(F,\gamma,\rho,\deg)$ if t is measurable with values in the elements fixing $f_0,\dots,f_\varrho$ and (A,ν,m) is a kernel bound for $(F,\gamma t,t,\rho,\mathrm{PB}(\gamma,\deg,\rho,\#I))$.

Gram section. Rows and columns of $(\varrho+1)\times(\varrho+1)$ matrices are indexed by $0,\dots,\varrho$. Fix an index $k_0\in\{1,\dots,k\}$. Let F and F' be the complex $r\times(\varrho+1)$ matrices whose columns are the coordinate vectors, in the basis $e_1,\dots,e_\varrho$, the standard basis $x_1,\dots,x_k$ of X_0, z, $e_1^*,\dots,e_\varrho^*$, of $e_1,\dots,e_\varrho,z$, respectively of $e_1^*,\dots,e_\varrho^*,x_{k_0}$. For $Z\in\Omega_{\mathrm{cell}}$ put $\eta(Z)=(1-|\mathrm{ic}(Z)|^2)^{1/2}>0$, $\mathrm{dg}(Z)=\mathrm{diag}(1,\dots,1,\eta(Z))$, $c_b=\frac12$ for $b<\varrho$, $c_\varrho=1$, and let $\mathrm{Co}(Z)$, $\mathrm{Co}'(Z)$ be the $(\varrho+1)\times(\varrho+1)$ matrices with last rows $\mathrm{Co}(Z)_{\varrho b}=\mathrm{Co}'(Z)_{\varrho b}=-\delta_{b\varrho}$ and, for $a<\varrho$,

$$\mathrm{Co}(Z)_{ab}=-c_b\,\overline{(Z^{-1})_{\varrho a}}\,(Z^{-1})_{\varrho b},\qquad\mathrm{Co}'(Z)_{ab}=-c_b\,(Z^{-1})_{a\varrho}\,\overline{(Z^{-1})_{b\varrho}}.$$

With $F^\vee(Z)=F\,\mathrm{Co}(Z)+F'\,\mathrm{dg}(Z)$, the **Gram section** at Z is the complex $r\times r$ matrix

$$\gamma_Z=1+(F^\vee(Z)Z-F)F^*+\Big((F(Z^{-1})^*-F^\vee(Z)Z\,\mathrm{Co}'(Z))\mathrm{dg}(Z)^{-1}-F'\Big)F'^*\qquad(Z\in\Omega_{\mathrm{cell}}),$$

read as an endomorphism of V_1 in the same basis, and $\gamma_Z=1$ for $Z\notin\Omega_{\mathrm{cell}}$.

Frame-fixer measure. The **frame-fixer Haar measure** ds on G_1 is the image of $dv \otimes dh$ under $\mathsf{U}^0 \times H \to G_1$, $(v,h) \mapsto vh$, with dv as in "Open cell" and dh the Haar probability measure of H.

Tubes. Let L be real and $\mathcal{Q} \subseteq \mathsf{U}^0$. A set $\mathcal{K} \subseteq G_1$ is a **tube of size L along U^0 with base $\mathcal{Q}$** if $\mathcal{K}$ is compact; $\mathcal{Q}$ is a Borel set with $0 < dv(\mathcal{Q}) < \infty$; $\|y\|_{\mathrm{op}} \le L^{10}$ for every $y \in \mathcal{K}$; $vy \in \mathcal{K}$ for every $v \in \mathcal{Q}$ and every $y \in G_1$ with $\|y\|_{\mathrm{op}} \le L$; and $dv(\{v \in \mathsf{U}^0 : vy \in \mathcal{K}\}) \le 3\, dv(\mathcal{Q})$ for every $y \in G_1$. Here $\|\cdot\|_{\mathrm{op}}$ is the operator norm of the matrix in the basis above for the Euclidean norm of $\mathbb{C}^r$.

Bessel system. A **Bessel system** (for ϱ, k, b, f) is a finite nonempty I, an irreducible unitary $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$, continuous Bessel functions $(W, W^\flat)$ of type ρ forming a cut pair, a **degree** $\deg > 0$ and forms J, such that $8r \le b$, W has Schur's bound $(P, \deg^{-1} \#I)$ for every $P \ge b^5$, and J are local bounds for $W^\flat$ with degree $\deg$. The forms J are its **band limits**.

Measure of U. The measure of U^0 is dv (paragraph "Open cell"). For $x = (x_{ab})_{0 \le a < b \le \varrho} \in \mathbb{C}^{\{(a,b):a<b\}}$ let $T(x)$ be the upper unitriangular $(\varrho+1) \times (\varrho+1)$ matrix with the entries x_{ab} above the diagonal, $A(x)$ its upper left $\varrho \times \varrho$ block and $v(x) = (x_{a\varrho})_{a<\varrho}$ its last column above the diagonal. The **unitriangular lift** $\mathrm{Ut}(x) \in \mathsf{U}$ fixes X_0, acts on the frame by $T(x)$, $\mathrm{Ut}(x) f_b = \sum_{a \le b} T(x)_{ab} f_a$, and sends

$$e^*_{b+1} \longmapsto \sum_{c<\varrho} \big((A(x)^{-1})^*\big)_{cb} \Big(e^*_{c+1} + \overline{v_c(x)}\, z + \tfrac{1}{2}\, \overline{v_c(x)} \sum_{a<\varrho} v_a(x)\, e_{a+1} \Big) \qquad (0 \le b < \varrho).$$

The **Haar measure** du of U is the image of the product of Lebesgue measure on $\mathbb{C}^{\{(a,b):a<b\}}$ (each $\mathbb{C} = \mathbb{R}^2$) with dv under $(x, u_0) \mapsto u_0\, \mathrm{Ut}(x)$.

Definition 5.1.7. Place. A **place** is a tuple $(A, \mathrm{vol}, \tau, \delta, \nu, |\cdot|, \psi, \mathrm{kb}_A)$ consisting of: a commutative ring A with an involution $x \mapsto \bar{x}$ and a locally compact Polish topology for which addition, multiplication and the involution are continuous, with its Borel σ-algebra; a Haar measure vol of $(A,+)$, written dx in integrals (for a finite set S, A^S carries the product measure, also written vol); a **trace-one element** $\tau \in A$, $\tau + \bar{\tau} = 1$; a **skew unit** $\delta \in A^\times$, $\bar{\delta} = -\delta$; a Haar measure ν of the closed additive group $A^- = \{\beta \in A : \bar{\beta} = -\beta\}$ of **skew elements**; a measurable map $|\cdot|\colon A \to [0,\infty]$, the **modulus**, with $|1| = 1$ and $|xy| = |x|\,|y|$; a continuous homomorphism ψ from $(A,+)$ to the unit circle, the **character**; and a **kernel bound** $\mathrm{kb}_A \in [0,\infty]$; such that

(PA1) the map $A^- \times A^- \to A$, $(\alpha, \beta) \mapsto \tau\delta^{-1}\alpha + \beta$, carries $\nu \otimes \nu$ to vol;

(PA2) $|\bar{x}| = |-x| = |x|$ for every $x \in A$;

(PA3) Scaling. for every finite set S and every $S \times S$ matrix X over A whose determinant is a unit, the image of vol on A^S under $x \mapsto Xx$ is $|\det X|^{-1}\mathrm{vol}$;

(PA4) almost every element of A is a unit;

(PA5) Kernel. there is a continuous integrable $\gamma\colon A \to [0,\infty)$ with $\gamma(0) = 1$ and

$$\int_A \Big| \int_A \gamma(v)\, \psi(\bar{v}y)\, dv \Big|\, dy \le \mathrm{kb}_A;$$

(PA6) Majorant. there is a Borel function $\vartheta\colon A \to [0,\infty)$ with $\int_C \vartheta\, d\mathrm{vol} < \infty$ for every compact $C \subseteq A$ and $|u|^{-1} \le \vartheta(u)^2$ for every unit u of A.

The places used are $A = \mathbb{C}$ and $A = E_T = \prod_{w \in T} E_w$ for a finite set T of finite places, read as one ring.

Disintegrations. Let A be a place, S a finite set, J' a hermitian $S \times S$ matrix over A and μ a measure on A^S. A family $(\lambda_\beta)_{\beta \in A^-}$ of measures on A^S is a **disintegration of μ along** $\bar{x}^{\mathsf{T}} J' x$ if

(DJ1) for every continuous compactly supported $f\colon A^S \to [0,\infty)$ the function $\beta \mapsto \int f \, d\lambda_\beta$ is finite and continuous on A^-;

(DJ2) for every measurable $f\colon A \times A^S \to [0,\infty]$,

$$\int_{A^S} f(\bar{x}^{\mathsf{T}} J' x, x)\, d\mu(x) = \int_{A^-} \int_{A^S} f(\delta^{-1}\beta, x)\, d\lambda_\beta(x)\, d\nu(\beta),$$

the outer integral being the lower Lebesgue integral where its integrand is not measurable.

The split space at a place. Let A be a place, $\varrho \geq 0$ and $k \geq 1$ integers, $n = \varrho + 1$ and $r = 2\varrho + k + 1$. The **coordinate indices** are e_a, e_a^* $(0 \leq a < \varrho)$, x_i $(1 \leq i \leq k)$ and z; V is the free A-module of rank r on these indices, and the same letters denote its coordinate vectors. Let T be an invertible hermitian matrix over A indexed by $\{x_1, \ldots, x_k, z\}$, and let the **split Gram matrix** J have the entries $J_{e_a e_b^*} = J_{e_b^* e_a} = \delta_{ab}$, the entries of T on the indices $\{x_1, \ldots, x_k, z\}$, and 0 elsewhere. Put $h(v,w) = \bar{v}^{\mathsf{T}} J w$, and let

$$G = \{g \in \mathrm{GL}(V) : \bar{g}^{\mathsf{T}} J g = J\},$$

with the topology of the matrix entries and its Borel σ-algebra; g_{ij} is the entry of g in row i and column j in the coordinate basis. The **middle space** is spanned by the x_i and z. The **level** of an index is a for e_a, ϱ for x_i and z, and $2\varrho - a$ for e_a^*; its **floor** is a for e_a and e_a^*, ϱ for z and $\varrho + 1$ for x_i. The **frame** is $(f_0, \ldots, f_\varrho) = (e_0, \ldots, e_{\varrho-1}, z)$, and the **frame Gram matrix** of $\gamma \in G$ is

$$Z(\gamma) = \big(h(f_a, \gamma f_b)\big)_{0 \leq a,b \leq \varrho}. \tag{5.3}$$

For $l \geq 0$:

- V_l is the span of the coordinate vectors of floor at least l, and the **Haar measure** vol_{V_l} of V_l is the image of vol on $A^{\{\text{indices of floor} \geq l\}}$ under extension by zero;
- $d_l = k + 1 + 2\max(\varrho - l - 1, 0)$;
- the **floor group** $G^{(l)}$ is the group of the $g \in G$ fixing every coordinate vector of floor below l; so $G^{(0)} = G$, and $H = G^{(\varrho+1)}$;
- the **Bessel unipotent group** is $\mathsf{U} = \{g \in G : g_{ij} = \delta_{ij}$ whenever $\mathrm{level}(j) \leq \mathrm{level}(i)\}$;
- $N_{<l} = \{g \in G : g_{ij} = \delta_{ij}$ whenever $\mathrm{lev}_l(j) \leq \mathrm{lev}_l(i)\}$, where $\mathrm{lev}_l(i)$ is the level of i if i has floor below l and ϱ otherwise;
- $R^{(l)}$ is the group of the $g \in G$ with $g_{ij} = \delta_{ij}$ for all indices i, j with $\mathrm{level}(j) \leq \mathrm{level}(i)$ such that $l \geq \varrho + 2$, or j has floor below l and level below l.

The **simple-root sum** of $s \in G$ is $\sigma(s) = \sum_{a<\varrho} s_{f_a f_{a+1}}$. For $j \geq 0$ and a sequence $p = (p_a)_{a \geq 0}$ of units of A, the **Weyl element** $w_{j,p} \in G$ sends $e_a \mapsto p_a e_a^*$ and $e_a^* \mapsto \bar{p}_a^{-1} e_a$ for $a < \min(j, \varrho)$ and fixes the other coordinate vectors; where the length j is fixed by the statement (heads of length j), it is written w_p.

Place datum. Let ι be a finite set and $m \geq 0$ with $k + 1 = |\iota| + 2m + 2$. A **place datum of shape** (ϱ, ι, m) over (A, k, T) consists of:

- a **hyperbolic basis** of the middle space: vectors v_c, v_c^* $(0 \le c \le m)$ and o_i $(i \in \iota)$ of the middle space with $h(v_c, v_d^*) = \delta_{cd}$, $h(v_c, v_d) = h(v_c^*, v_d^*) = 0$, $h(o_i, o_{i'}) = \delta_{ii'}$ and $h(v_c, o_i) = h(v_c^*, o_i) = 0$; its **basis matrix** is the matrix of their coordinates;
- an enumeration of ι; the **free indices** $\iota' \subseteq \iota$ are its first $\max(|\iota| - 2(2m+2), 0)$ elements, and $O\colon A^{\iota'} \to V$, $O\omega = \sum_{i \in \iota'} \omega_i o_i$;
- s-finite Borel measures μ_{top} (the **top floor measure**) and μ_H (the **measure of** H) on G;
- a unit M of A (the **level unit**);
- a set $\mathcal{P}_{\mathrm{cl}} \subseteq V \times V$ of **close pairs**, a set $K \subseteq G$ (the **box**) and a set $\mathcal{P}_{\mathrm{pos}} \subseteq G$ of **positions**;

subject to (PE1)–(PE4) below, which read the following objects. For $0 \le l \le \varrho$ let $\mathcal{Q}_l = \{1, \dots, m\} \sqcup \{a : l \le a < \varrho\}$, $\epsilon_c = v_c$, $\epsilon_c^* = v_c^*$ for $1 \le c \le m$ and $\epsilon_a = e_a$, $\epsilon_a^* = e_a^*$ for $l \le a < \varrho$, and $Y_l = A^{\mathcal{Q}_l} \times A^{\iota} \times A^{\mathcal{Q}_l}$. The **floor basis** of V_l is the map

$$B_l\colon A \times A \times Y_l \to V_l, \qquad (t_1, t_2, (\zeta, w, \zeta')) \mapsto t_1 v_0 + t_2 v_0^* + \sum_{c \in \mathcal{Q}_l} (\zeta_c \epsilon_c + \zeta_c' \epsilon_c^*) + \sum_{i \in \iota} w_i o_i,$$

and $h_l'(Y, Y) = \sum_{c \in \mathcal{Q}_l} (\bar{\zeta}_c \zeta_c' + \bar{\zeta}_c' \zeta_c) + \sum_{i \in \iota} \bar{w}_i w_i$ for $Y = (\zeta, w, \zeta') \in Y_l$. For $\beta \in A^-$ the **quadric measure** $\mathrm{qv}_l(\beta)$ of floor l with parameter β is the image of $\mathrm{vol} \otimes \mathrm{vol} \otimes \nu$ on $A \times Y_l \times A^-$ under the **flat chart**

$$(t, Y, \alpha) \mapsto B_l\big(t^\dagger \tau\,(\delta^{-1}\beta - h_l'(Y, Y)) + t\alpha,\ t,\ Y\big), \tag{5.4}$$

where $t^\dagger = \bar{t}^{-1}$ if t is a unit and $t^\dagger = 0$ otherwise; the **quadric measure of floor** l is $\mathrm{qv}_l = \mathrm{qv}_l(\delta h(f_l, f_l))$ (so $\mathrm{qv}_l = \mathrm{qv}_l(0)$ for $l < \varrho$). The axioms are:

(PE1) $|\det T| = 1$, and $\bar{M} = M$;

(PE2) Top floor. $\mu_{\mathrm{top}}(G \setminus G^{(\varrho)}) = 0$, and the image of μ_{top} under $g \mapsto g^{-1}z$ is qv_ϱ;

(PE3) Measure of H**.** $\mu_H \neq 0$; $\mu_H(G \setminus H) = 0$; for every $s \in H$ the image of μ_H under $g \mapsto gs$ is μ_H; and $\mu_H(\{g \in G : ugy \in K \text{ for some } u \in \mathsf{U}\}) \le 1$ for every $y \in G$;

(PE4) $\mathcal{P}_{\mathrm{cl}}$ is open, contains $(0, 0)$ and has compact closure; K is compact; $\mathcal{P}_{\mathrm{pos}}$ is measurable.

Near points, stable products and depth. Let a place datum be given. For $j \ge 0$, $g \in G$ is **near at** j if $\xi g \in K$ for some $\xi \in R^{(j)}$. For $0 \le a < \varrho$ and $x, y \in G$, the **stable products** $\mathfrak{d}_a(x, y)$ are the numbers $h(e_a, \omega)$, $\omega \in V$, such that $h(e_b, \omega) = 0$ for $b < a$ and $(x^{-1}\omega, (xy)^{-1}\omega') \in \mathcal{P}_{\mathrm{cl}}$ for some $\omega' \in \omega + \mathrm{span}\{e_b : b < a\}$; the **depth** is $\mathrm{dp}_a(x, y) = \mathrm{vol}^*(\mathfrak{d}_a(x, y))^{-1} \in [0, \infty]$, with vol^* the outer measure, $0^{-1} = \infty$ and $\infty^{-1} = 0$. The **volume exponent** is $s = \max(|\iota'| - (3n+1+m), 0)$. The **product differences** are the numbers $M^{-1}(h(\omega_1, \omega_1') - h(\omega_2, \omega_2'))$ with $(\omega_1, \omega_2), (\omega_1', \omega_2') \in \mathcal{P}_{\mathrm{cl}}$; their set is $\mathcal{D}_{\mathrm{pr}}$. For $g, y \in G$ and a $\varrho \times \iota'$ matrix Q over A, the **common vectors** are

$$\mathcal{V}_{\mathrm{cm}}(g, y, Q) = \{\omega \in A^{\iota'} : Q\omega = 0,\ (g^{-1}O\omega, (gy)^{-1}O\omega) \in \mathcal{P}_{\mathrm{cl}}\}.$$

Sizes and heads. A **size filtration** is a family $(A_{(i)})_{i \ge 0}$ of subsets of A such that, for all $i, j \ge 0$:

(SZ1) $A_{(i)}A_{(j)} \subseteq A_{(i+j)}$;

(SZ2) every sum of at most $r + 1$ elements of $A_{(i)}$ lies in $A_{(i+1)}$;

(SZ3) $-x \in A_{(i)}$ if and only if $x \in A_{(i)}$, and $\bar{x} \in A_{(i)}$ if and only if $x \in A_{(i)}$;

(SZ4) $0, 1 \in A_{(i)}$;

(SZ5) the entries of J and of J^{-1} lie in $A_{(1)}$.

A vector or a matrix over A has **size** i if all its entries lie in $A_{(i)}$, and an element of G has size i if its matrix has. A sequence $p = (p_a)_{a\geq 0}$ of units is a **head of length** j if $(p_0 \cdots p_{l-1})^{-1} \in A_{(1)}$ for every $l \leq j$. For sequences p, q of units the **head density** is $\mathrm{hdn}_j(p,q) = \prod_{a<j} \big(|p_a q_a|^{1/2}\big)^{d_a}$. For $\mathcal{V} \subseteq A^{\iota'}$ and $\Lambda \in [0,\infty]$, a **common bound** for $\mathcal{V}$ with loss Λ is a measurable $\phi\colon A^{\iota'} \to [0,\infty]$ such that for every $\varsigma \in A^{\iota'}$

$$\mathrm{vol}^*\big(\{\bar{\varsigma}^{\mathsf{T}}\omega : \omega \in \mathcal{V}\}\big)^{-1} \leq \phi(\varsigma), \qquad \mathrm{vol}^*\big(\{c \in A : \bar{c}\,\bar{\varsigma}^{\mathsf{T}}\omega \in \mathcal{D}_{\mathrm{pr}} \text{ for all } \omega \in \mathcal{V}\}\big) \leq \phi(\varsigma),$$

and $\int_{\{\varsigma \text{ of size } j\}} \phi(\varsigma)^s \, d\varsigma \leq \Lambda^{99n(j+1)}$ for every $j \geq 0$.

Gauge. A **gauge** Γ for a place datum is a size filtration $(A_{(i)})$ with a **loss** $\Lambda \in [0,\infty]$ such that:

(GU1) every $A_{(j)}$ is compact; $2 \leq \Lambda < \infty$; $m \leq 9n$; $2(2m+2) \leq |\iota|$; $\mathrm{kb}_A \geq 1$;

(GU2) M, the basis matrix of the hyperbolic basis and $\tau\delta^{-1}$ have size 1, and every element of K has size 1;

(GU3) for every $(\omega_1, \omega_2) \in \mathcal{P}_{\mathrm{cl}}$, the vectors ω_1 and $M^{-1}(\omega_1 - \omega_2)$ have size 1;

(GU4) Rescaling. if $M^{-1} \in A_{(1)}$, then for every $i \geq 0$ there is a unit c with $c^{-1} \in A_{(i+2)}$ and $(c\omega_1, c\omega_2) \in \mathcal{P}_{\mathrm{cl}}$ for all $\omega_1, \omega_2 \in V$ of size i;

(GU5) $|c| \leq \Lambda^j$ for every unit $c \in A_{(j)}$, and $\mathrm{vol}(A_{(j)}) \leq \Lambda^{j+1}$, for every $j \geq 0$;

(GU6) Block kernel. there is an s-finite kernel $(\lambda_\beta)_{\beta \in A^-}$ from A^- to A^{2m+2} that is a disintegration of vol on A^{2m+2} along $\bar{x}^{\mathsf{T}}x$ and satisfies $\lambda_\beta(\{x \text{ of size } j\}) \leq \Lambda^{(2m+2)(j+1)}$ for all $\beta \in A^-$ and $j \geq 0$;

(GU7) Common bounds. for every $g \in K$, $y \in \mathcal{P}_{\mathrm{pos}}$ and every $\varrho \times \iota'$ matrix Q over A there is a common bound for $\mathcal{V}_{\mathrm{cm}}(g, y, Q)$ with loss Λ.

The **positions of size** Γ are the $y \in \mathcal{P}_{\mathrm{pos}}$ such that the matrix $M^{-1}(y-1)$ has size 1 for Γ.

Definition 5.1.8. Let A be either the ring $\mathbb{A}_f$ of finite adeles of a number field E, with factors $A_w = E_w$ at the finite places w of E, or a finite product $A = \prod_{w \in T} A_w$ of non-archimedean local fields; let $|\cdot|_w$ be the normalised absolute value of A_w (for a Haar measure dx of A_w and $a \in A_w$, $d(ax) = |a|_w \, dx$), and $\hat{\mathcal{O}} = \prod_w \mathcal{O}_{A_w}$ the integral elements. The **generalised index** of $\delta = (\delta_w)_w \in A$ is

$$[\delta] = \prod_w |\delta_w|_w^{-1} \quad \text{if } \delta \in A^\times, \qquad [\delta] = 0 \quad \text{otherwise,}$$

the product over the factors of A (finite for $\mathbb{A}_f$, since $|\delta_w|_w = 1$ for almost all w when δ is an idele).

Definition 5.1.9. Filling datum. Let E be a CM field with maximal totally real subfield E^+, f infinite places $v \in \Phi$, root discriminant rd_E and finite adeles $\mathbb{A}_f$, $\hat{\mathcal{O}}$ the integral finite adeles, and $\mathbb{A}_E = \mathbb{C}^\Phi \times \mathbb{A}_f$ its adeles, with E embedded diagonally by $a \mapsto ((v(a))_{v\in\Phi}, a)$ and with components x_v ($v \in \Phi$) and x_f of $x \in \mathbb{A}_E$. The **standard additive character** ψ_E of $\mathbb{A}_E$ and its finite part $\psi_f = \psi_{E,f}$ of $\mathbb{A}_f$ are those of (WD4) of Definition 3.2.12 for E and Φ (there $\hat{\mathcal{O}}$ is written $\widehat{\mathcal{O}}_E$ and $\mathbb{A}_f$ is written $\mathbb{A}_{E,f}$). A **filling datum** $\mathfrak{S}$ of **shape** (ϱ, k) over E ($k \geq 1$, $n = \varrho + 1$, $r = 2\varrho + k + 1$) consists of:

(FG1) Space. $V = E^r$ with the hermitian form h given by ϱ hyperbolic planes (e_a, e_a^*), $X_0 = E^k$ with the unit form and the standard basis $x_1, \dots, x_k$, and z with $h(z,z) = -N$; $G = \mathrm{U}(V)$; frame $(e_1, \dots, e_\varrho, z)$; U, $H = \mathrm{U}(X_0)$, $R = \mathsf{U}H$ as at each real place.

(FG2) Level data. a prime $N \geq 5$ and $\pi \in \hat{\mathcal{O}}$ with $\pi\bar{\pi} = N$, such that at each finite w either π is a unit at w or ($q_w = N$ and π is a unit at $\bar{w}$); primes $p_0, p_1 \neq N$, $M = p_0 p_1$; for $i = 0, 1$ a set S_i of f **chosen places** w, each with $q_w = p_i$, with $S_0 \cap S_1 = \emptyset$, $\bar{w} \notin S_0 \cup S_1$ for $w \in S_0 \cup S_1$, and $M \mid x$ in $\hat{\mathcal{O}}$ iff $x \equiv \bar{x} \equiv 0$ modulo every chosen w; $\varpi \in \mathcal{O}_E$ with $\bar{\varpi} = -\varpi$ generating the different of $E/\mathbb{Q}$; a real **size** $b \geq 1000$.

(FG3) Lattice and reduction. the lattice basis of $V \otimes \mathbb{A}_f$ obtained from the standard one by replacing the last basis vector x_k of X_0 by $x_k/\bar{\pi} + z/N$, and its $\hat{\mathcal{O}}$-span $L_0 \subseteq V \otimes \mathbb{A}_f$; the **lattice stabiliser** $K \subseteq G(\mathbb{A}_f)$ (integral in this basis); the **level group** $\bar{G} = \prod_{w \in S_0 \cup S_1} \mathrm{GL}_r(k_w)$; the **reduction** $\mathrm{red}\colon K \to \bar{G}$; the **principal level** $\mathrm{K}(M) = \ker \mathrm{red}$; and for $A \subseteq G(\mathbb{A}_f)$ its **reduced image** $\bar{A} = \mathrm{red}(A \cap K)$. In particular $\bar{R}_0 = \bar{R}$ and $\bar{H}$, $\bar{\mathsf{U}}$.

(FG4) Source level. an open compact $\mathrm{K}_X \subseteq \mathrm{U}(X_0)(\mathbb{A}_f)$ meeting every conjugate of $\mathrm{U}(X_0)(E^+)$ trivially; the **classes** $\mathcal{C} = \mathrm{U}(X_0)(E^+)\backslash \mathrm{U}(X_0)(\mathbb{A}_f)/\mathrm{K}_X$ (finite) with representatives h_c.

(FG5) Bessel systems. for each $v \in \Phi$ a Bessel system $\mathfrak{B}_v$ for (ϱ, k, b, f) with index set I_v, functions $(W_v, W_v^\flat)$ and degree $\deg_v$; the **Bessel coordinates** $I = \prod_v I_v$.

(FG6) Size fields. $b^{10} \leq p_i \leq b^{11}$; $b^{19} \leq M \leq b^{23}$; $10^5 \leq \varrho$; $2^{-\varrho} \leq b^{-6}$; $(\varrho+1)[E:\mathbb{Q}] \leq 2^{2\varrho}$; $1000(2\varrho+2)^9 \leq k \leq f \leq b/1000$; $(200fk\,\mathrm{rd}_E)^{1000} \leq b$; $\mathrm{rd}_E \leq b$; $\varrho + 1 \leq b$; $10^{10}n^4 \leq n^6$, $3n^6 + 12 \leq n^7 \leq k$.

Derived quantities. The **real level scale** at $v \in \Phi$ is $\mathrm{rs}_v = \mathrm{Re}(M\, v(\varpi)/2\pi i)$, so $|\mathrm{rs}_v| = M|\varpi|_v/2\pi$. The **source dimension** $D_X = \#\mathcal{C} \cdot \#I$. The **Bessel degree** $\deg = \prod_v \deg_v$. The **volume of** $\hat{\mathcal{O}}$ is the number $\mathrm{vol}(\hat{\mathcal{O}}) = 2^f |\mathrm{disc}\, E|^{-1/2}$. For $v \in \Phi$ and an integer $a \geq 0$ the **squared frame scale** is $\mathrm{fs}(v,a) = \mathrm{rs}_v^{2\max(\varrho-a,0)}/N$. The **frame scale** $\Lambda_{\mathrm{fr}} = \prod_v \prod_{a=0}^{\varrho} \mathrm{fs}(v,a)^{k+2(\varrho-a)}$ and $d = \varrho(\varrho+k)$. For $v \in \Phi$ and $l \geq 1$ the **sphere mass** and the **frame mass** are

$$\mathrm{sph}_l(v) = \frac{(2\pi)^l}{|\varpi|_v^{l-1/2}\,(l-1)!}, \qquad \mathrm{fr}_v = N^k \prod_{i=k}^{r-1} \mathrm{sph}_{i+1}(v);$$

they depend only on E, ϖ, N, k and r, and are formed with these for any such tuple (E a CM field, $\varpi \in E^\times$, $N \geq 1$, $1 \leq k \leq r$). The **unipotent covolume**

$$\nu = \Big(\frac{M^{2d+\varrho}}{N^k}\Big)^f \mathrm{vol}(\hat{\mathcal{O}})^{-d}\, |\mathrm{disc}\, E|^{\varrho/4}\, \Lambda_{\mathrm{fr}}^{-1}.$$

The **identity moment** $\mathcal{D}_{\mathrm{id}} = \#\bar{R}_0 \cdot D_X$ and the **main moment** $\mathcal{D}_{\mathrm{main}} = D_X^2/(\deg \cdot \mathrm{mass}(E,r))$. The **Siegel constant** $c_S = (\pi^k/((k-1)!N^k))^f |\mathrm{disc}\, E|^{1/4} \mathrm{vol}(\hat{\mathcal{O}})^k$ and the **scaled Siegel constant** $c_S^\sharp = \#\bar{R}_0 \cdot M^{f\varrho(2k+2\varrho+1)} \cdot \#\mathcal{C} \cdot c_S$.

Sieve constants. The **Poisson constant** of the entry $a \geq 0$ is $\mathrm{pc}(a) = \mathrm{vol}(\hat{\mathcal{O}})/\prod_v \mathrm{fs}(v,a)^2$; the **decay constant** is $\mathrm{dc} = \prod_v \Theta\left(|\mathrm{rs}_v|^{2\varrho+1}/(MN)\right)^{k-n^7}$, with Θ of Definition 5.1.6; the **normalising constant** is

$$c_{\mathrm{nm}} = \frac{1}{\#I} \prod_{v \in \Phi} \Big(\sqrt{c_G} \prod_{a=0}^{\varrho} \mathrm{fs}(v,a)^{k-1+2(\varrho-a)}\Big).$$

Let T be a finite set of finite places of E, $E_T = \prod_{w\in T} E_w$ with integers $\mathcal{O}_T$; $t \in E$ is a **T-integer** if it is integral at every finite place outside T. For $\xi \in E_T$ write $[\mathcal{O}_T : \xi\mathcal{O}_T] = [\xi]$, the generalised index of ξ in E_T (Definition 5.1.8; 0 unless $\xi \in E_T^\times$). The **character at** T is $\psi_T(x) = \psi_f(-\tilde{x}/\varpi)$ $(x \in E_T)$, where $\tilde{x} \in \mathbb{A}_f$ has components x at T and 0 elsewhere and ψ_f is the finite part of the standard additive character, (WD4) of Definition 3.2.12. The **frequency decay** is $\beta_T(\xi) = [\mathcal{O}_T : \xi\mathcal{O}_T]^{(k-n^7)/2}\,[\mathcal{O}_T : \mathcal{O}_T \cap \xi^{-1}\mathcal{O}_T]^{-3}\,b^{n^9 f}$ for $\xi \in E_T^\times$ and $\beta_T(\xi) = 0$ otherwise. The **finite head weight** of a sequence $p = (p_0, p_1, \dots)$ in $E_T^\times$ is $\Lambda_i^T(p) = \prod_{l<i}[\mathcal{O}_T : \delta_l\mathcal{O}_T]^{n^6/2}$ if every $\delta_l = p_0 \cdots p_l$ $(l < i)$ lies in $\mathcal{O}_T$, and 0 otherwise.

Admissible. $\mathfrak{S}$ is **admissible** if

(FA1) $\mathrm{K}_X = \{h \in \mathrm{U}(X_0)(\mathbb{A}_f) : h \in \mathrm{K}(M)\}$ (embedding $\mathrm{U}(X_0) \subseteq G$);

(FA2) $\mathrm{red}\colon K \to \bar{G}$ is surjective;

(FA3) $|\varpi|_v \le b$ for all v;

(FA4) $1 \le |\mathrm{rs}_v| \le b^{23}$ for all v;

(FA5) $N \le b^{60n}$;

(FA6) $b^{11kf}\mathcal{D}_{\mathrm{id}} \le \mathcal{D}_{\mathrm{main}} \le (M/b^6)^{kf}\mathcal{D}_{\mathrm{id}}$;

(FA7) $\#I \le b^{2kf}$;

(FA8) with $\mathrm{U}(X_0)(\hat{\mathcal{O}}) = \mathrm{U}(X_0)(\mathbb{A}_f) \cap \mathrm{GL}_k(\hat{\mathcal{O}})$ (matrices in the basis $x_1, \dots, x_k$; the stabiliser of $\hat{\mathcal{O}}^k$), the index of $\mathrm{U}(X_0)(\hat{\mathcal{O}}) \cap \{h \in \mathrm{U}(X_0)(\mathbb{A}_f) : h \in K\}$ in $\mathrm{U}(X_0)(\hat{\mathcal{O}})$ is finite and at least $(N^k/b)^f$;

(FA9) $c_S^\sharp \le (1 + b^{-9})\,\mathcal{D}_{\mathrm{main}}\,\deg c_G^f\,\mathrm{vol}(\hat{\mathcal{O}})^{-n}\,(\nu\Lambda_{\mathrm{fr}}/\#I)^2$.

Weil data on E means that E contains a primitive 5040-th root of unity.

Definition 5.1.10. Let $\mathfrak{S}$ be a filling datum, $A = \langle e_1, \dots, e_\varrho\rangle \subseteq V$ and $k \in K$; h is extended sesquilinearly to $V(\mathbb{A}_f)$, and $L_0 \cap A(\mathbb{A}_f) = \bigoplus_{a=1}^{\varrho} \hat{\mathcal{O}} e_a$.

(GT1) Isotropic-rows index. For $1 \le a \le \varrho$ let $r_a(k) = \big(h(kf_b, e_a)\big)_{0\le b\le \varrho} \in \mathbb{A}_f^{\varrho+1}$, the products of e_a with the frame of k. For $j \in \{0, \dots, \varrho+1\}$,

$$\mathrm{ir}_j(k) = \Big[\textstyle\sum_{a=1}^{\varrho} \hat{\mathcal{O}}\, r_a(k) :\ \sum_{a\le \min(j,\varrho)} \hat{\mathcal{O}}\, r_a(k)\Big].$$

(GT2) Torsion index. $\mathrm{ti}(k) = \big[L_0 \cap (A + kA)(\mathbb{A}_f) :\ (L_0 \cap A(\mathbb{A}_f)) + k(L_0 \cap A(\mathbb{A}_f))\big]$.

Definition 5.1.11. Let $\mathfrak{S}$ be a filling datum over E and $G_\infty = \prod_{v\in\Phi} G_1$. Write $v(x) \in \mathbb{C}$ for the image of $x \in E$ under the embedding $v \in \Phi$, and $f = (f_0, \dots, f_\varrho) = (e_1, \dots, e_\varrho, z)$ for the frame. Both V (over E) and $V_1 = \mathbb{C}^r$ carry the standard basis $e_1, \dots, e_\varrho$, the basis vectors of X_0, z, $e_1^*, \dots, e_\varrho^*$, and matrices are taken in it.

(RD1) Rescaled embedding. For $v \in \Phi$ the **basis scales** are

$$\mathrm{bs}_v(e_a) = \frac{\mathrm{rs}_v^{\varrho+1-a}}{\sqrt{N}}\ (1 \le a \le \varrho), \qquad \mathrm{bs}_v(x) = 1\ (x \text{ a basis vector of } X_0),$$
$$\mathrm{bs}_v(z) = \frac{1}{\sqrt{N}}, \qquad \mathrm{bs}_v(e_a^*) = \mathrm{bs}_v(e_a)^{-1},$$

so that $\mathrm{bs}_v(f_a) = \mathrm{rs}_v^{\varrho-a}/\sqrt{N}$ for $0 \le a \le \varrho$. The **rescaled image** of $\gamma \in G(E)$ is $\gamma_\infty = (\gamma_v)_{v\in\Phi} \in G_\infty$,

$$(\gamma_v)_{ij} = v(\gamma_{ij})\,\frac{\mathrm{bs}_v(j)}{\mathrm{bs}_v(i)} \qquad (i,j \text{ basis vectors}). \tag{5.5}$$

The group $G(E)$ acts on G_∞ by $\gamma\cdot y = \gamma_\infty y$; $\gamma^{-1}y$ means $\gamma_\infty^{-1}y$.

(RD2) Unipotent groups of a class. $\mathsf{U}(E) \subseteq G(E)$ is the unipotent radical of the stabiliser of the isotropic flag $\langle e_1\rangle \subset \cdots \subset \langle e_1,\dots,e_\varrho\rangle$ of V. For $c \in \mathcal{C}$,

$$\mathsf{U}_c(M) = \{u \in \mathsf{U}(E) : h_c^{-1}uh_c \in K(M)\}, \qquad \mathsf{U}_c^0(M) = \{u \in \mathsf{U}_c(M) : uf_a = f_a\ (0 \le a \le \varrho)\}.$$

(RD3) Residue domains. A **residue domain** is a Borel set $D_{\mathrm{res}} \subseteq \mathbb{C}^\Phi$ such that $|w_v| \le b^{26}$ for all $w \in D_{\mathrm{res}}$ and $v \in \Phi$, and for every $w \in \mathbb{C}^\Phi$ there is exactly one $x \in \mathcal{O}_E$ with $(v(Mx) + w_v)_{v\in\Phi} \in D_{\mathrm{res}}$.

(RD4) Reduced and boxed points. For $y \in G_1$ let $o_0,\dots,o_\varrho$ be the Gram–Schmidt orthogonalisation, for the standard inner product $\langle x,x'\rangle = \sum_i \bar{x}_i x'_i$ of $\mathbb{C}^r$, of the moved frame $y^{-1}f_0,\dots,y^{-1}f_\varrho$, and $T(y)$ the upper unitriangular $(\varrho+1)\times(\varrho+1)$ matrix with $y^{-1}f_b = \sum_{a\le b} T(y)_{ab}\,o_a$. For $y \in G_\infty$ and $0 \le a < b \le \varrho$ the **rational triangle** entry is $t_{ab}(y) = \big(T(y_v)_{ab}\,\mathrm{bs}_v(f_a)/\mathrm{bs}_v(f_b)\big)_{v\in\Phi} \in \mathbb{C}^\Phi$. For a residue domain D_{res}, $y \in G_\infty$ is **reduced** if $t_{ab}(y) \in D_{\mathrm{res}}$ for all $a<b$, and **boxed** if it is reduced and $\ell(y_v) \le b^{6n+2}$ for every $v \in \Phi$.

(RD5) Box partitions of unity. For $c \in \mathcal{C}$, a **box partition of unity** for c is a measurable $\omega\colon G_\infty \to [0,1]$ such that: $\omega(y) \ne 0$ implies $y_v \in \mathrm{Bx}(b^{27}\ell(y_v))$ for every v; $\omega(sy) = \omega(y)$ whenever every $s_v \in G_1$ fixes $f_0,\dots,f_\varrho$; and $\sum_{u\in\mathsf{U}_c(M)} a(u)\,\omega(u\cdot y) = 1$ for every $y \in G_\infty$ and every $a\colon \mathsf{U}_c(M) \to [0,\infty]$ with $\sum_{u\in\mathsf{U}_c^0(M)} a(uu') = 1$ for every $u' \in \mathsf{U}_c(M)$.

(RD6) Cutoff data, class cutoffs, near terms. A **cutoff datum** for $\mathfrak{S}$ is a triple

$$(D_{\mathrm{res}}, (\beta_c^0)_{c\in\mathcal{C}}, (R_{c,c'})_{c,c'\in\mathcal{C}}) :$$

a residue domain D_{res}; measurable $\beta_c^0\colon G_\infty \to [0,1]$ with

$$\sum_{u\in\mathsf{U}_c^0(M)} \beta_c^0(u\cdot y) = 1 \qquad (y \in G_\infty);$$

and for each pair (c,c') a set $R_{c,c'} \subseteq G(E)$ of representatives of the double cosets

$$\mathsf{U}_c^0(M)\backslash G(E)/\mathsf{U}_{c'}^0(M).$$

Its **class cutoffs** are $\beta_c = \beta_c^0\cdot\mathbf{1}_{\mathrm{red}}$, with $\mathbf{1}_{\mathrm{red}}$ the indicator of the reduced points. An element $\gamma \in G(E)$ is a **near term** if there is a boxed $y \in G_\infty$ with $\gamma^{-1}y$ boxed, and a **near representative** for (c,c') if moreover $\gamma \in R_{c,c'}$.

A cutoff datum is fixed with $\mathfrak{S}$ and suppressed from the notation of the objects formed with it.

Definition 5.1.12. Let $\mathfrak{S}$ be a filling datum; $G_\infty = \prod_{v\in\Phi} G_1$, with the product dy of the Cayley measures dy of the factors (as in Definition 5.1.6); an integral over G_∞ is a Bochner integral, taken to be 0 when the integrand is not integrable. The real unipotent group $\mathsf{U}_\infty = \prod_{v\in\Phi}\mathsf{U} \subseteq G_\infty$ carries the product du of the Haar measures du of U of Definition 5.1.6. The group G_∞ is the one on which $\gamma \in G(E)$ acts by left multiplication with its rescaled image γ_∞ (5.5); $\gamma^{-1}y$ means $\gamma_\infty^{-1}y$. For $W \in \{W^{\mathrm{full}}, W^{\mathrm{cut}}\}$ put $W(y)_i = \prod_v W_{v,i_v}(y_v)$ with $W_v^{\mathrm{full}} = W_v$, $W_v^{\mathrm{cut}} = W_v^\flat$.

(TA1) Terms. For $c, c' \in \mathcal{C}$ and $s \in \bar{G}$, the **terms over** (c, s, c') are the $\gamma \in G(E)$ with $h_c^{-1}\gamma h_{c'} \in K$ and $\mathrm{red}(h_c^{-1}\gamma h_{c'}) = s$.

(TA2) Amplitudes. With the class cutoffs β_c of the fixed cutoff datum, the **amplitude** of γ is $A_\gamma(c, c') = \int_{G_\infty} \beta_c(y)\beta_{c'}(\gamma^{-1}y)\, W(y) \otimes \overline{W(\gamma^{-1}y)}\, dy \in \mathbb{C}^{I\times I}$, and for $T \subseteq G(E)$ the **amplitude sum** is $A_T(c, s, c') = \sum_{\gamma\in T,\ \gamma \text{ over } (c,s,c')} A_\gamma(c, c')$ (full or cut, as W). The **amplitude majorant** is the same integral with $\|W(y) \otimes \overline{W(\gamma^{-1}y)}\|$.

(TA3) Second moments. For a family $A(c, s, c')$ and $\Omega \subseteq \bar{G}$:

$$\mathrm{SM}(A, \Omega) = \nu^{-2} \sum_{c,c'} \sum_{s\in\Omega} \|A(c, s, c')\|^2.$$

The **link moment** of T over Ω is

$$\mathrm{LM}(T, \Omega) = \sum_{c,c'} \sum_{s\in\Omega} \sum_{L} \|A^{\mathrm{cut}}_{T\cap \mathrm{link}^{-1}(L)}(c, s, c')\|^2$$

(no ν^{-2}), the inner sum over submodules L.

(TA4) Frame products, discriminant, link. $Z(\gamma) = (h(f_j, \gamma f_l))_{j,l=0}^{\varrho}$ for the frame f; $D_j(\gamma)$ its leading principal minor of order j; the **term discriminant** $\mathsf{g}(\gamma) = |D_{\varrho+1}|^2 - N^2|D_\varrho|^2$. With $X = \langle e_1, \dots, e_\varrho\rangle$, the **link** of γ is $\{([t], [\gamma^{-1}t]) : t \perp X,\ t \perp \gamma X\}$, $[\cdot]$ the coordinates in $X_0 \oplus Ez$.

(TA5) Cell rank, remaining terms. γ has **cell rank** $j \in \{0, \dots, \varrho+1\}$ if some c with $c\bar{c} = 1$ makes, for every $a \geq j$, a pair $x, y \in E^{\varrho+1}$ with $f \cdot x = \gamma f \cdot y$, $x_a = c$, $y_a = 1$, $x_b = y_b = 0$ $(b > a)$; $D_j(\gamma) \neq 0$; and $\varrho < j \iff \mathsf{g}(\gamma) \neq 0$. The cell rank of γ is a fixed choice among such j (and 0 if there is none). The **unipotent character** is $\psi(u) = \psi_f(-\sigma(u)/(M\varpi))$ for $u \in G(\mathbb{A}_f)$, where ψ_f is the finite part of the standard additive character ((WD4) of Definition 3.2.12), $\sigma(u) \in \mathbb{A}_f$ is the sum of the entries just above the diagonal in the frame block of u, M and ϖ are read in $\mathbb{A}_f$, and $u \in G(E)$ is read in $G(\mathbb{A}_f)$; thus $\psi(u) = \exp(-2\pi i \,\mathrm{Tr}_{E/\mathbb{Q}}(\sigma(u)/(M\varpi)))$ for $u \in G(E)$. γ has **agreeing characters** if $\psi(u) = \psi(u')$ whenever $u, u' \in \mathsf{U}(E)$, $u\gamma = \gamma u'$; s **agrees** if $\psi(u) = \psi(u')$ for integral $u, u' \in \mathsf{U}$ with $\bar{u}s = s\bar{u}'$. The **remaining terms** are $\Gamma^* = \{\mathsf{g} \neq 0\} \cup \{\gamma : \text{agreeing characters},\ \mathsf{g}(\gamma) = 0,\ \gamma X \neq X\}$. T is **stable** if $u^{-1}\gamma u' \in T \iff \gamma \in T$ for $u, u' \in \mathsf{U}$.

(TA6) Amplitude blocks. For a linear representation ρ of $\bar{G}$ on a complex inner-product space $\mathcal{H}$ (of any dimension; $\langle\cdot,\cdot\rangle$ conjugate-linear in the first argument), a linear $P\colon \mathbb{C}^{\mathcal{C}\times I} \to \mathcal{H}$ has **amplitude blocks** if $\langle\rho(s)Pe_y, Pe_x\rangle = \nu^{-1}A^{\mathrm{full}}_{G(E)}(c_x, s, c_y)_{i_x,i_y}$ for all s and $x = (c_x, i_x)$, $y = (c_y, i_y)$.

(TA7) Bessel translate. $r_\psi(s)$ is the matrix with entries $\nu^{-1}\overline{A^{\mathrm{full}}_{R(E)}(c_x, s, c_y)_{i_x,i_y}}$ (zero unless $s \in \bar{R}_0$).

(TA8) Reduced character. $\psi_U(s) = \psi(u)$ for an integral lift $u \in \mathsf{U}$ of $s \in \bar{\mathsf{U}}$, and 0 for $s \notin \bar{\mathsf{U}}$; its **conjugation mean** is $\psi_U^\natural(s) = \#\bar{G}^{-1}\sum_g \psi_U(gsg^{-1})$.

(TA9) Near-isometry. A linear $B\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a **near-isometry with error** ε if $\|B\,\mathrm{pr}_U\|^2_{\mathrm{HS}} \geq \dim U - \varepsilon \dim \mathcal{H}_1$ for every subspace U.

(TA10) Open elements and fixers. For a chosen w above p_0 let $Z_w(s)$ be the frame Gram matrix $(h_w(f_j, s_w f_l))$ over k_w. $s \in \bar{G}$ is **open** if for some such w every leading minor of $Z_w(s)$ of order $0, \dots, \varrho$ is nonzero. A **frame vector** at w is (ϵ, v), $\epsilon \in \{0,1\}$, $0 \neq v \in k_w^{\varrho+1}$; its **fixer** is the set of s such that g fixes $\sum_a v_a f_a$, where $g = s_w$ if $\epsilon = 0$ and g is the inverse adjoint of s_w if $\epsilon = 1$. For $S_g \subseteq S_0$ and frame vectors d, $\mathrm{Fix}(S_g, d) = \bigcap_{w \in S_g} \mathrm{Fix}(w, d_w)$.

(TA11) Meet norm. s **meets** at (w, ϵ) if some $x \neq 0$ and y satisfy $f \cdot x = g\, f \cdot y$ modulo w; the **meet norm** of s is the product of the $p_{i(w)}$ over the (w, ϵ) at which s meets.

(TA12) Gauges. $\mathfrak{g}^\sharp(j, c, c', \gamma) = |N_{E/\mathbb{Q}} \delta_*(\gamma)| / (i_\gamma i'_\gamma N^f)$, where the **subframe minor** $\delta_*(\gamma)$ is the minor of $Z(\gamma)$ on the rows and columns $\{a : a < j\} \cup \{\varrho\}$, and, for a term γ over (c, s, c'), $i_\gamma = \mathrm{ir}_j(h_c^{-1} \gamma h_{c'})$ and $i'_\gamma = \mathrm{ir}_j((h_c^{-1} \gamma h_{c'})^{-1})$, with the isotropic-rows index ir_j of (GT1) of Definition 5.1.10; the **torsion index** of such a term is $\mathrm{ti}(h_c^{-1} \gamma h_{c'})$, with ti of (GT2) of Definition 5.1.10. The **minor norm** of $\gamma \in G(E)$ at $j \in \{0, \dots, \varrho + 1\}$ is $\mathrm{mn}_j(\gamma) = |N_{E/\mathbb{Q}} D_j(\gamma)| / N^{2 \max(j - \varrho, 0) f}$. The **discriminant bound** is $b^{n^3 f}$; $\mathfrak{g}^{\max} = \max(\mathfrak{g}^\sharp, \prod_v |\mathfrak{g}(\gamma)|_v)$; the **level gauge** $\mathfrak{g}^*(j) = M^{2 \min(\varrho, j) f} \mathfrak{g}^{\max}$.

(TA13) Term size and link classes. With the **size base** $\tau_{\mathrm{sz}} = b^{kf/4}$, $n_p = \#\bar{R}_0 \# \mathcal{C} / N^{kf}$ and the **term-size constant** $\mathcal{D}_j^{\mathrm{sz}} = \mathcal{D}_{\mathrm{main}}$ for $j > \varrho$, $\mathcal{D}_j^{\mathrm{sz}} = \mathcal{D}_{\mathrm{id}} / \tau_{\mathrm{sz}}^{16}$ otherwise, the **term size** of γ of cell rank j is $\tau_{\mathrm{sz}}^2 \sqrt{\mathcal{D}_j^{\mathrm{sz}} / (n_p (M^{2 \min(\varrho, j) f} \mathfrak{g}^{\max})^k)}$. Two terms γ, γ' are in the **same link class** if $\mathfrak{g}(\gamma)\mathfrak{g}(\gamma') \neq 0$ and $\mathrm{link}\, \gamma = \mathrm{link}\, \gamma'$.

(TA14) Frame coordinates and shared vectors. Let $\gamma \in G(E)$; rows and columns of matrices with $n = \varrho + 1$ rows or columns are indexed by $0, \dots, \varrho$, as those of $Z(\gamma)$. The **frame coordinates** of γ are the matrices $Q_0(\gamma) \in \mathrm{M}_{k \times n}(E)$ and $Q_A(\gamma) \in \mathrm{M}_{\varrho \times n}(E)$ whose column b holds the coordinates of γf_b along $x_1, \dots, x_k$, respectively along $e_1, \dots, e_\varrho$, in the standard basis of V. $Z_<(\gamma) \in \mathrm{M}_{\varrho \times n}(E)$ is the matrix of the first ϱ rows of $Z(\gamma)$, $Z_<(\gamma)_{ab} = h(e_{a+1}, \gamma f_b)$. For $0 \leq j \leq \varrho + 1$ the **padded leading matrix** $Z_j^{\mathrm{pd}}(\gamma) \in \mathrm{M}_n(E)$ has the rows of $Z(\gamma)$ of index $a < j$ and the rows of the identity matrix of index $a \geq j$, and the **shared-vector matrix** $\mathrm{sh}_j(\gamma) \in \mathrm{M}_{\varrho \times n}(E)$ has the columns of $Q_A(\gamma)\, \mathrm{adj}(Z_j^{\mathrm{pd}}(\gamma))$ of index $\geq j$ and zero columns of index $< j$, where adj is the adjugate matrix.

Theorem 5.1.13. *Let $\mathfrak{S}$ be an admissible filling datum over a field E with Weil data, let ρ be a unitary representation of $\bar{G}$ on a finite-dimensional Hilbert space $\mathcal{H}$, and let $P \colon \mathbb{C}^{\mathcal{C} \times I} \to \mathcal{H}$ have amplitude blocks (TA6). Then*

$$\sum_{s \in \bar{G}} \|P^* \rho(s) P\|^2 \;\leq\; \Big(1 + \frac{9}{b^4}\Big) \mathcal{D}_{\mathrm{main}}.$$

Proof. Put $\Theta(s) = \|P^* \rho(s) P\|^2$ for $s \in \bar{G}$, and $C = (1 + 9b^{-4}) \mathcal{D}_{\mathrm{main}}$.

Step 1: Θ is a nonnegative function of positive type with $\Theta(s) = \mathrm{SM}(A^{\mathrm{full}}_{G(E)}, \{s\})$ and $\Theta(1) \leq 2D_X$. The identity is Lemma 5.1.14 (i) with the blocks of (TA6), and positivity of type is Lemma 5.1.14 (ii); $\Theta(1) = \|P^* P\|^2 \leq 2D_X$ by Proposition 5.6.42 (iii).

Step 2: the open part of a fixer. Let $S_g \subseteq S_0$, d frame vectors, and $\Omega = \{s \in \mathrm{Fix}(S_g, d) : s \text{ open}\}$. By Theorem 5.2.15, $\mathrm{LM}(\{\mathfrak{g} \neq 0\}, \Omega) \leq C_g \nu^2$ with $C_g = (1 + b^{-4}) b^{-6k \# S_g} \mathcal{D}_{\mathrm{main}}$. By Proposition 5.6.42 (i) and Step 1, $\sum_{s \in \Omega} \Theta(s) \leq (\sqrt{C_g} + b^{-15} \sqrt{\mathcal{D}_{\mathrm{id}}})^2$. By (FA6), $\mathcal{D}_{\mathrm{id}} \leq b^{-11kf} \mathcal{D}_{\mathrm{main}} \leq b^{-6k \# S_g} b^{-5kf} \mathcal{D}_{\mathrm{main}}$, so

$$\sum_{s \in \Omega} \Theta(s) \leq (1 + 2b^{-4})\, b^{-6k \# S_g}\, \mathcal{D}_{\mathrm{main}}.$$

Here Proposition 5.6.42 (i) applies because open elements are in no scalar coset, and for open s (which are not in $\bar{R}_0$, $\bar{R}_0$ lying in a scalar coset) $\Theta(s) = \mathrm{SM}(A^{\mathrm{full}}_{G(E)\setminus R(E)}, \{s\})$, both by Proposition 5.6.42 (vii).

Step 3: downward induction on $\#S_g$. Claim: $X(S_g, d) := \sum_{s\in \mathrm{Fix}(S_g,d)} \Theta(s) \le C\, b^{-6k\#S_g}$ for all $S_g \subseteq S_0$ and d. Let $\Gamma = \mathrm{Fix}(S_g, d)$, a subgroup of $\bar{G}$, and suppose the claim for all larger sets. If $S_g \neq S_0$, pick $w \in S_0 \setminus S_g$. Either some frame vector d_w at w has $\sum_{\Gamma\cap \mathrm{Fix}(w,d_w)} \Theta > b^{-6k} \sum_\Gamma \Theta$; then by induction $\sum_\Gamma \Theta < b^{6k} X(S_g \cup \{w\}, (d, d_w)) \le C b^{-6k\#S_g}$. Or every frame vector at w has fixer share $\le b^{-6k}$; then Lemma 5.1.15 (ii) gives $\sum_{s\in\Gamma \text{ not open}} \Theta \le b^{-4} \sum_\Gamma \Theta$, and with Step 2, $(1-b^{-4}) \sum_\Gamma \Theta \le (1+2b^{-4}) b^{-6k\#S_g} \mathcal{D}_{\mathrm{main}}$, so $\sum_\Gamma \Theta \le (1+4b^{-4}) b^{-6k\#S_g} \mathcal{D}_{\mathrm{main}} \le C b^{-6k\#S_g}$ (as $b \ge 1000$). Base case $S_g = S_0$. Put $T_0 = b^{-6kf} \mathcal{D}_{\mathrm{main}}$. By positivity of type, $\Theta(s) = \langle Q_1, Q_s\rangle_{\mathrm{HS}} \le \Theta(1) \le 2D_X$ for all s (Lemma 5.1.14 (ii), Cauchy–Schwarz). By Proposition 5.6.42 (vii) the elements in a scalar coset carry $\sum \Theta \le 2M^f \mathcal{D}_{\mathrm{id}} \le T_0/4$, using (FA6) and $M \le b^{23}$ (so $8M^f \mathcal{D}_{\mathrm{id}} \le 8b^{23f-11kf} \mathcal{D}_{\mathrm{main}} \le T_0$). Let Γ_{ns} be the elements of $\Gamma = \mathrm{Fix}(S_0, d)$ in no scalar coset; they are not in $\bar{R}_0$, avoid the scalar cosets and have meet norm $\ge p_0^f$ (Proposition 5.6.42 (vii)). By Theorem 5.3.2 (iv) with $q = p_0^f \ge b^{10f}$, $\mathrm{LM}(\{\mathfrak{g} \neq 0\}, \Gamma_{\mathrm{ns}}) \le \mathrm{LB}\, \nu^2$ with $\mathrm{LB} = (b^{2f}/p_0^f)^k \mathcal{D}_{\mathrm{main}} \le b^{-8kf} \mathcal{D}_{\mathrm{main}} \le T_0/8$, and by Proposition 5.6.42 (i), (vii), $\sum_{\Gamma_{\mathrm{ns}}} \Theta \le (\sqrt{\mathrm{LB}} + b^{-15}\sqrt{\mathcal{D}_{\mathrm{id}}})^2 \le 2(\mathrm{LB} + b^{-30}\mathcal{D}_{\mathrm{id}}) \le T_0/2$. Hence $X(S_0, d) \le T_0 \le C b^{-6kf}$.

Step 4. $S_g = \emptyset$ gives $\sum_{s\in\bar{G}} \Theta(s) \le C$. □

Lemma 5.1.14. *Let X be a finite set, V a Hilbert space, Γ a group acting on V by a unitary representation ρ, and $P\colon \mathbb{C}^X \to V$ linear. For $s \in \Gamma$ put $P_s = P^*\rho(s)P \in \mathrm{End}(\mathbb{C}^X)$ and $Q_s = (\rho(s)P)(\rho(s)P)^* \in \mathrm{End}(V)$.*

(i) $\|P_s\|^2 = \sum_{x,y\in X} |\langle \rho(s)Pe_y, Pe_x\rangle|^2$. In particular, if $X = C \times I$ for finite sets C, I, $\nu > 0$, and for each $c, c' \in C$ there is a matrix $A_s(c,c') \in \mathbb{C}^{I\times I}$ with $\langle \rho(s)Pe_{(c',i')}, Pe_{(c,i)}\rangle = A_s(c,c')_{i,i'}/\nu$ for all $i, i' \in I$, then

$$\|P^*\rho(s)P\|^2 = \nu^{-2} \sum_{c,c'\in C} \|A_s(c,c')\|^2.$$

(ii) For $s, t \in \Gamma$, $\langle Q_s, Q_t\rangle_{\mathrm{HS}} = \mathrm{tr}(Q_s^* Q_t) = \|P^*\rho(s^{-1}t)P\|^2$. Hence the function $\Theta(u) = \|P^*\rho(u)P\|^2$ on Γ is of positive type: for every finitely supported family $(z_s)_{s\in\Gamma}$ of complex numbers,

$$\sum_{s,t\in\Gamma} \overline{z_s}\, z_t\, \Theta(s^{-1}t) = \Big\| \sum_s z_s Q_s \Big\|^2 \ge 0,$$

and $\Theta(u^{-1}) = \Theta(u)$ for all $u \in \Gamma$.

Proof. **(i).** The matrix entries of P_s in the orthonormal basis (e_x) are

$$\langle e_x, P_s e_y\rangle = \langle Pe_x, \rho(s)Pe_y\rangle = \overline{\langle \rho(s)Pe_y, Pe_x\rangle},$$

and the squared Hilbert–Schmidt norm is the sum of the squared moduli of the matrix entries in an orthonormal basis. Under the block hypothesis the entries are $\overline{A_s(c,c')_{i,i'}}/\nu$, and summing $|A_s(c,c')_{i,i'}|^2$ over i, i' gives $\|A_s(c,c')\|^2$.

(ii). As ρ is unitary, $\rho(s)^* = \rho(s^{-1})$, and Q_s is self-adjoint. By cyclicity of the trace, with $Y = P^*\rho(s^{-1}t)P$, so that $Y^* = P^*\rho(t^{-1}s)P$,

$$\mathrm{tr}(Q_s Q_t) = \mathrm{tr}\big(\rho(s)PP^*\rho(s^{-1})\rho(t)PP^*\rho(t^{-1})\big) = \mathrm{tr}\big(P^*\rho(s^{-1}t)P\, P^*\rho(t^{-1}s)P\big) = \mathrm{tr}(YY^*) = \|Y\|^2.$$

Sesquilinearity of the Hilbert–Schmidt inner product then gives $\|\sum_s z_s Q_s\|^2 = \sum_{s,t} \overline{z_s} z_t \langle Q_s, Q_t\rangle = \sum_{s,t} \overline{z_s} z_t \Theta(s^{-1}t)$. Finally $\Theta(u^{-1}) = \|(P^*\rho(u)P)^*\|^2 = \Theta(u)$, since $\|Y^*\| = \|Y\|$. □

Lemma 5.1.15. *A function $\Theta\colon \Gamma \to \mathbb{R}$ on a group is of **positive type** if $\Theta(a^{-1}b) = \langle u(a), u(b)\rangle$ for some map u into a complex inner product space.*

(i) Let Γ be a finite group, k_0 a field with q elements, $\pi\colon \Gamma \to \mathrm{GL}_{n_0}(k_0)$ a homomorphism, $\pi'(s) = (\pi(s)^{\mathsf T})^{-1}$, $A_0, A_1 \in k_0^{n_0\times m_0}$, $\Theta \ge 0$ of positive type and $Y \ge 0$ with

$$\sum_{s:\pi(s)A_0v=A_0v} \Theta(s) \le Y \quad \text{and} \quad \sum_{s:\pi'(s)A_1v=A_1v} \Theta(s) \le Y$$

for every $0 \ne v \in k_0^{m_0}$. Then

$$q \sum_{s:\det(A_1^{\mathsf T}\pi(s)A_0)=0} \Theta(s) \le 2\sum_{\Gamma}\Theta + q^{m_0}\sqrt{q^{n_0}}\,Y.$$

(ii) Let $\mathfrak{S}$ be admissible, $\Gamma \le \bar G$, $w \in S_0$, $\beta \ge 0$ of positive type on $\bar G$ with $\sum_{s\in\Gamma\cap \mathrm{Fix}(w,d)}\beta(s) \le b^{-6k}\sum_{s\in\Gamma}\beta(s)$ for every frame vector d at w. Then for $0 \le l \le \varrho$, $\sum_{s\in\Gamma:\ \det_l Z_w(s)=0}\beta(s) \le 3b^{-10}\sum_\Gamma\beta$ ($\det_l$ the leading minor of order l), and $\sum_{s\in\Gamma\ \text{not open}}\beta(s) \le b^{-4}\sum_\Gamma\beta$.

Proof. **(i), Step 1: a character bound off the axes.** Fix a nontrivial additive character ψ of k_0 and put $\varrho_0 = \sqrt{q^{n_0}}$. Let $x, z \in k_0^{n_0}$ with $\sum_{s:\pi'(s)x=x}\Theta(s) \le Y$ and $\sum_{s:\pi(s)z=z}\Theta(s) \le Y$. We show

$$\mathrm{Re}\sum_{s\in\Gamma}\psi(x\cdot\pi(s)z)\Theta(s) \le \varrho_0 Y. \tag{5.6}$$

In $\ell^2(k_0^{n_0})$ the functions $c_x(\zeta) = \psi(-x\cdot\zeta)/\varrho_0$ ($x \in k_0^{n_0}$) are orthonormal by orthogonality of characters, and so are the indicators $e_z = \mathbf{1}_{\{z\}}$. In $\ell^2(k_0^{n_0})\otimes\mathcal{H}$ ($\mathcal{H}$ the space of u) put $X_1 = \sum_a c_{\pi'(a)x}\otimes u(a)$ and $X_0 = \sum_a e_{\pi(a)z}\otimes u(a)$. Since $(\pi'(a)x)\cdot\pi(b)z = x\cdot\pi(a)^{-1}\pi(b)z$, we get $\langle c_{\pi'(a)x}, e_{\pi(b)z}\rangle = \psi(x\cdot\pi(a^{-1}b)z)/\varrho_0$, and with $\langle u(a),u(b)\rangle = \Theta(a^{-1}b)$ and the substitution $s = a^{-1}b$,

$$\langle X_1, X_0\rangle = \frac{\#\Gamma}{\varrho_0}\sum_s \psi(x\cdot\pi(s)z)\Theta(s), \qquad \|X_1\|^2 = \#\Gamma\sum_{s:\pi'(s)x=x}\Theta(s) \le \#\Gamma Y, \qquad \|X_0\|^2 \le \#\Gamma Y,$$

the norms by orthonormality ($\langle c_{\pi'(a)x}, c_{\pi'(b)x}\rangle = \mathbf{1}[\pi'(a^{-1}b)x = x]$, and likewise for e). Then $2\,\mathrm{Re}\langle X_1, X_0\rangle \le \|X_1\|^2 + \|X_0\|^2 \le 2\#\Gamma Y$ gives (5.6).

(i), Step 2: counting the kernel. Put $Z(s) = A_1^{\mathsf T}\pi(s)A_0 \in k_0^{m_0\times m_0}$. If $\det Z(s) = 0$, the left kernel $\{y : y^{\mathsf T}Z(s) = 0\}$ contains the q multiples of a nonzero vector, so, as $\Theta \ge 0$,

$$q\sum_{\det Z(s)=0}\Theta(s) \le \sum_{y\in k_0^{m_0}}\ \sum_{s:\ y^{\mathsf T}Z(s)=0}\Theta(s) = \sum_y \mathbb{E}_{v\in k_0^{m_0}}\,\mathrm{Re}\sum_s \psi(y^{\mathsf T}Z(s)v)\Theta(s),$$

by $\mathbb{E}_v\psi(w\cdot v) = \mathbf{1}[w = 0]$. If $y = 0$ or $v = 0$ the inner sum is $\sum_\Gamma\Theta$. If $y \ne 0 \ne v$, then $y^{\mathsf T}Z(s)v = (A_1y)\cdot\pi(s)(A_0v)$ and the hypotheses give the two fixer bounds for $x = A_1y$, $z = A_0v$, so the inner sum is $\le \varrho_0 Y$ by (5.6). Bounding each term by $\mathbf{1}[y=0]\sum\Theta + \mathbf{1}[v=0]\sum\Theta + \varrho_0 Y$ (all three are ≥ 0) and summing, the right side is at most $\sum\Theta + q^{m_0}q^{-m_0}\sum\Theta + q^{m_0}\varrho_0 Y$, which is (i).

(ii), leading minors. The restriction of β to Γ is nonnegative and of positive type. Apply (i) to Γ, $k_0 = k_w$ (so $q = p_0$), $\pi(s) = s_w$, $n_0 = r$, $m_0 = l$, $A_0 = e$ the $r\times l$ matrix of the first l frame vectors and $A_1 = Je$, and $Y = b^{-6k}\sum_\Gamma\beta$. Then $A_1^{\mathsf T}\pi(s)A_0 = e^{\mathsf T}J^{\mathsf T}s_we$ is the upper-left $l\times l$ block of $Z_w(s)$ (J is symmetric), so its determinant is $\det_l Z_w(s)$. For $0 \ne v \in k_w^l$, $\pi(s)$

fixes ev iff s fixes the frame vector $(0,v)$ (padded by zeros), and $\pi'(s)=(s_w^{\mathsf T})^{-1}$ fixes Jev iff the inverse adjoint $J^{-1}(s_w^{\mathsf T})^{-1}J$ fixes ev, i.e. iff s fixes the frame vector $(1,v)$; both fixer masses are $\le Y$ by hypothesis. Hence

$$p_0\sum_{s\in\Gamma:\det_l Z_w(s)=0}\beta(s)\le 2\sum_\Gamma\beta+p_0^l\sqrt{p_0^r}\,b^{-6k}\sum_\Gamma\beta\le 3\sum_\Gamma\beta,$$

because $p_0^{2l+r}\le b^{11(2l+r)}\le b^{12k}$: indeed $l\le\varrho$, $r=2\varrho+k+1$ and $11(4\varrho+k+1)\le 12k$ as $44\varrho+11\le k$. Dividing by $p_0\ge b^{10}$ gives the first claim.

(ii), non-open elements. If s is not open, then at w some leading minor $\det_l Z_w(s)$ with $0\le l\le\varrho$ vanishes. Summing the first claim over the $\varrho+1$ orders, $\sum_{s\in\Gamma\text{ not open}}\beta(s)\le 3(\varrho+1)b^{-10}\sum_\Gamma\beta\le b^{-4}\sum_\Gamma\beta$, since $3(\varrho+1)\le 3b\le b^6$. □

5.2 Bad types, sieves and the main term

This section proves the two bounds on the second moment that concern the forms of good type and the main term. It opens with the local objects that the main term reads at a place: the measures of a place datum, lattice and filling data at finite places, partial correlations and floor forms, deep vectors, form majorants, finite Bessel vectors, fibre pairings and Kloosterman sums. The properties of the floor forms that the sieve needs are proved once, for an abstract place, in the frames engine (Theorem 5.4.1), and the band limits at the infinite places are built from it in Theorem 5.5.3.

The first statement, Proposition 5.2.10, makes the near-isometry of the Poincaré map survive the projection to good forms (Proposition 5.6.41). The idempotent e_0 is built from Green's characters of the groups GL_r of the residue fields at the chosen places [106, 194, 70], and it acts on the forms of principal level as a tensor product of two central idempotents, each of defect at most $b^{-8}/2$. The input is Green's degree formula (Theorem A.5.27); the defect is bounded by counting, place by place, the labels that are not good.

The rest of the section is the main term (Theorem 5.2.15). The lattices attached to a pair of terms and the link metric (Definition 5.2.22, Definition 5.2.27) are defined where they are first used, and the remaining terms of Section 5.3 are bounded with the same lattices and links. The architecture is that of [259]; for other readings see [223, 75].

Definition 5.2.1. Weyl elements and Heisenberg elements. For $w\in V$ write $w_l^*=h(e_l,w)$, the e_l^*-coordinate of w. For $l<\varrho$ and a unit c of A, the **Weyl element** $w_l(c)\in G$ is

$$w_l(c)e_l=c\,e_l^*,\qquad w_l(c)e_l^*=\bar c^{-1}e_l,\tag{5.7}$$

fixing the other coordinate vectors; $w_l(c)=1$ if c is not a unit. For $l<\varrho$, $v\in V_{l+1}$ and $\beta\in A^-$, the **Heisenberg element** $n_l(v,\beta)$ is the linear map

$$n_l(v,\beta)\,w=w+w_l^*\,v+\big((\beta-\tau h(v,v))\,w_l^*-h(v,w)\big)e_l;\tag{5.8}$$

it sends e_l^* to $e_l^*+v+(\beta-\tau h(v,v))e_l$ and fixes e_b,e_b^* for $b<l$ and e_l.

Floor measures. All measures below are Borel measures on G; the convolution of two s-finite measures is $\mu*\mu'=$ the image of $\mu\otimes\mu'$ under $(x,y)\mapsto xy$, so $\int f\,d(\mu*\mu')=\iint f(xy)\,d\mu(x)\,d\mu'(y)$, and δ_1 is the unit mass at 1. For $l<\varrho$ let

$$\begin{aligned}\mathrm{wl}_l&=\text{the image of }|c|^{d_l}\,dc\text{ under }c\mapsto w_l(c)^{-1},\\ \eta_l&=\text{the image of }dv\otimes\nu\text{ under }(v,\beta)\mapsto n_l(v,\beta),\end{aligned}\tag{5.9}$$

where dc is the Haar measure of A and dv the Haar measure of V_{l+1}: the **Weyl-element measure** and the **Heisenberg measure** of floor l. Put $\eta_l = \delta_1$ for $l \geq \varrho$. The **floor measure** θ_l is $\mathrm{wl}_l * \eta_l$ for $l < \varrho$, μ_{top} for $l = \varrho$, μ_H for $l = \varrho + 1$, and δ_1 for $l \geq \varrho + 2$.

Section measures. For $l < \varrho$ and $c \in A$, the **section measure** $\mathrm{Sec}_l(c)$ is the Borel measure on V

$$\mathrm{Sec}_l(c) = |c|^{d_l} \cdot (\text{the image of } \eta_l \text{ under } n \mapsto c\, n^{-1} e_l^*). \tag{5.10}$$

Tower and group measures. For $l, i \geq 0$ the **descending convolution** of the i floor measures from l on is

$$\theta_{[l,i)} = \theta_{l+i-1} * \theta_{l+i-2} * \cdots * \theta_l \qquad (\theta_{[l,0)} = \delta_1). \tag{5.11}$$

For $l \geq 0$ the **tower measure from floor** l is

$$\theta_{\geq l} = \theta_{\varrho+1} * \theta_\varrho * \cdots * \theta_l \qquad (\theta_{\geq l} = \delta_1 \text{ for } l \geq \varrho + 2).$$

For $l, i \geq 0$ the **partial group measure** of the i floors from l on is

$$\mathfrak{m}_{l,i} = (\eta_{l+i-1} * \theta_{l+i-1}) * \cdots * (\eta_l * \theta_l) \qquad (\mathfrak{m}_{l,0} = \delta_1),$$

$\mathfrak{m}_j = \mathfrak{m}_{0,j}$ is the partial group measure of the floors $0, \ldots, j-1$, and $\theta = \mathfrak{m}_{0,\varrho+2}$ is the **group measure**.

Definition 5.2.2. Finite place. A place $(A, \mathrm{vol}, \tau, \delta, \nu, |\cdot|, \psi, \mathrm{kb}_A)$ is **finite** if A is a finite product of non-archimedean local fields, with its involution, and A carries a compact open subring $\mathcal{O}_A$ (its **integers**) with: $\mathrm{vol}(\mathcal{O}_A) = 1$; $\overline{\mathcal{O}_A} = \mathcal{O}_A$, and $\mathcal{O}_A$ is the product of the valuation rings of the factors; $\mathcal{O}_A$ is **self-dual** for the character, i.e. $\psi(ra) = 1$ for all $r \in \mathcal{O}_A$ if and only if $a \in \mathcal{O}_A$; the modulus is $|x| = [x]^{-1}$ for every $x \in A$ (so $|x| = \infty$ if x is not a unit); $\mathrm{kb}_A = 1$; $\tau \in \mathcal{O}_A$; $\delta \in \mathcal{O}_A^\times$; and $\nu(A^- \cap \mathcal{O}_A) = 1$. The **ideal zeta function** is $\zeta_{\mathcal{O}_A}(s) = \sum_I [\mathcal{O}_A : I]^{-s} \in [0, \infty]$, the sum over the ideals I of $\mathcal{O}_A$ of finite index.

Congruence subgroups. For a unit c of A, $\mathrm{K}(c)$ is the set of $x \in G$ such that x and x^{-1} have entries in $\mathcal{O}_A$ and $c^{-1}(x-1)$ has entries in $\mathcal{O}_A$ (all matrices in the basis e_a, x_i, z, e_a^* of the split Gram matrix J). $\mathrm{K}(1)$ is the group of **integral isometries**; for $c \in \mathcal{O}_A$, $\mathrm{K}(c)$ is the **congruence subgroup** of level c.

Lattice datum. A place datum F over a finite place A is a **lattice datum** if:

- *integrality*: J and J^{-1} have entries in $\mathcal{O}_A$, the basis matrix of the hyperbolic basis of the middle space has entries in $\mathcal{O}_A$, and the level unit M lies in $\mathcal{O}_A$ with $\bar{M} = M$;
- *box and measures*: $K = \mathrm{K}(1)$; $\mu_{\mathrm{top}}(\mathrm{K}(1)) \geq \mathrm{vol}(\mathcal{O}_A^\times)$; and $\mu_H(\mathrm{K}(1)) = 1$;
- *close pairs*: $\mathcal{P}_{\mathrm{cl}}$ is the set of the (ω_1, ω_2) with ω_1 and $M^{-1}(\omega_1 - \omega_2)$ integral;
- *positions*: $\mathcal{P}_{\mathrm{pos}}$ is the set of the products $y^{-1} u y\, k$ with $y \in G$, $u \in \mathsf{U}$ and $k \in \mathrm{K}(M)$;
- *isotropic dual of z*: some vector u has $\bar{u}^{\mathsf{T}} J =$ the coordinate vector at z and coordinate 0 at z.

A lattice datum of shape $(\varrho, \iota, m+1)$ is **large** if moreover

$$\zeta_{\mathcal{O}_A}(m)^n \leq 1 + 2^{-n}, \qquad m < 9n, \qquad 53n \leq |\iota|,$$

$\psi(x + \bar{x}) = 1$ for all $x \in A$, and G is unimodular (some left Haar measure of G is right invariant).

Sizes of scale α. For a unit α of A with $\alpha \in \mathcal{O}_A$ and $\bar{\alpha} = \alpha$, an element $x \in A$ has **size** j **for the scale** α if $\alpha^j x \in \mathcal{O}_A$; the associated loss is $2/|\alpha|$.

Definition 5.2.3. Let $\mathfrak{S}$ be a filling datum over E and T a finite set of finite places of E stable under complex conjugation. The **lattice middle Gram matrix** J_{lat} is the Gram matrix of $X_0 \oplus Ez$ in the lattice basis of (FG3) of Definition 5.1.9, that is, in the basis made of the basis vectors x_i of X_0 with i different from the replaced index, the vector $x' = x_k/\bar{\pi} + z/N$ that replaces the last one, and z:

$$h(x_i, x_j) = \delta_{ij}, \qquad h(x_i, x') = h(x_i, z) = 0, \qquad h(x', x') = 0,$$
$$h(x', z) = h(z, x') = -1, \qquad h(z, z) = -N;$$

an integer matrix. A **filling datum at** T is a pair (P, F) of

- a structure P of finite place on E_T, for the involution $(x_w)_{w\in T} \mapsto (\overline{x_{\bar{w}}})_{w\in T}$ induced by complex conjugation ($\bar{\cdot}\colon E_{\bar{w}} \to E_w$ the continuous extension of complex conjugation of E), whose character is ψ_T, and
- a place datum F over E_T (with the structure P) of shape $(\varrho, \iota, 3\varrho+6)$, $|\iota| = k - (6\varrho+13)$, on the middle Gram matrix J_{lat} read over E_T, which is a large lattice datum and whose level unit is the image of M in E_T.

Definition 5.2.4. Let a place datum be given.

Partial correlations. For a function $\mathcal{K}\colon G \times G \to \mathbb{C}$, an integer $l \geq 0$ and $g, y \in G$, the **partial correlation of** $\mathcal{K}$ **from floor** l is

$$\mathcal{K}_l(g, y) = \int_G \mathcal{K}(xg, xy)\, d\theta_{\geq l}(x) \tag{5.12}$$

if $x \mapsto \mathcal{K}(xg, xy)$ is $\theta_{\geq l}$-integrable, and $\mathcal{K}_l(g, y) = 0$ otherwise.

Pairing kernels. Let $j \geq 0$ and $d \in G$. A function $\mathcal{K}\colon G \times G \to \mathbb{C}$ is a **pairing kernel at floor** j **for** d if

(PK1) $\mathcal{K}$ is measurable on $G \times G$;

(PK2) for all $l \geq 0$, $n, n' \in N_{<l}$ and $g, y \in G$,

$$\mathcal{K}_{l+1}(ng, n'y) = \psi\big(M^{-1}(\sigma(dn'd^{-1}) - \sigma(n))\big)\, \mathcal{K}_{l+1}(g, y);$$

(PK3) $dsd^{-1} = s$ for every $s \in G^{(j)}$;

(PK4) for every $l \geq 0$ there is $c < \infty$ with $\int_G |\mathcal{K}(xg, xy)|\, d\theta_{\geq l}(x) \leq c$ for all $g, y \in G$;

(PK5) for all $l \geq 0$ and $g, y \in G$, if $\mathcal{K}_l(g, y) \neq 0$ then g is near at l.

Cut functions. Let I be a finite set and give $\mathbb{C}^I$ the inner product $\langle u, u' \rangle = \sum_{i\in I} \bar{u}_i u'_i$. For a unit η of A, a **Bessel function of frequency** η **over the place** is a bounded measurable $W\colon G \to \mathbb{C}^I$ with $W(ug) = \psi(\eta\, \sigma(u))\, W(g)$ for all $u \in \mathsf{U}$ and $g \in G$. It is **cut** if it is a Bessel function of frequency M^{-1} over the place and, for every $g \in G$ with $W(g) \neq 0$, $ug \in K$ for some $u \in \mathsf{U}$. For $W\colon G \to \mathbb{C}^I$ and $d \in G$, the **correlation kernels** of W are

$$\mathcal{K}^d_W(g, y) = \langle W(g), W(dy) \rangle, \qquad \mathcal{K}_W = \mathcal{K}^1_W,$$

and their partial correlations (5.12) from floor l are written $\mathcal{K}^d_{W,l}$ and $\mathcal{K}_{W,l}$.

Bessel orbital integrals. Give $\mathbb{C}^{I\times I}$ the inner product of $\mathbb{C}^I$ with $I\times I$ in place of I. For $W\colon G\to\mathbb{C}^I$, a Borel measure μ on G and $\gamma\in G$, the **Bessel orbital integral** of W is the matrix $\mathrm{Orb}_\mu(\gamma)\in\mathbb{C}^{I\times I}$ with the entries

$$\mathrm{Orb}_\mu(\gamma)_{ii'}=\int_G W(y)_i\,\overline{W(\gamma^{-1}y)_{i'}}\,d\mu(y), \tag{5.13}$$

an entry being 0 where its integrand is not μ-integrable; W is fixed by the context and not written.

Form integrand and floor form. Let $W\colon G\to\mathbb{C}^I$, $j\ge 0$, let p,q be sequences of units of A, $d=w_{j,q}^{-1}w_{j,p}$, and let $\omega\colon G\to\mathbb{C}$ (the **weight**). For $x,g\in G$ the **form integrand** is

$$\mathrm{FI}_j(p,q,W,x;\omega)(g)=\int_G\int_G \omega(\xi\zeta^{-1})\,\mathcal{K}_{W,j+1}(\xi g,\xi gx)\,\overline{\mathcal{K}^d_{W,j+1}(\zeta w_{j,p}^{-1}g,\zeta w_{j,p}^{-1}gx)}\,d\theta_j(\zeta)\,d\theta_j(\xi), \tag{5.14}$$

each of the two integrals being 0 where its integrand is not integrable, and the **floor form** is

$$\mathcal{F}_j(p,q;W,\omega)=\mathrm{hdn}_j(p,q)\int_{G\times G}\mathrm{FI}_j(p,q,W,x;\omega)(g)\,d(\theta\otimes\mathfrak{m}_j)(x,g), \tag{5.15}$$

which is 0 if the integrand is not $\theta\otimes\mathfrak{m}_j$-integrable.

Definition 5.2.5. Let a place datum be given; arithmetic is in $[0,\infty]$ with $0^{-1}=\infty$, $\infty^{-1}=0$ and $t^0=1$.

Depth weights. For $j\ge 0$ and $x,y\in G$ the **depth weight** of length j is

$$\mathrm{dpw}_j(x,y)=\prod_{a<\min(j,\varrho)}\left(\mathrm{dp}_a(x,y)^{-1}\right)^s, \tag{5.16}$$

and for $x\in G$ the **inverse depth weight** at x is the function $\mathrm{dpw}^-_{j,x}\colon G\to[0,\infty]$ with $\mathrm{dpw}^-_{j,x}(g)=\mathrm{dpw}_j(g,x)^{-1}$ if g is near at j and $\mathrm{dpw}^-_{j,x}(g)=0$ otherwise.

Floor-mass bounds. For $j\ge 0$, $V\in[0,\infty]$ and $c\colon G\to[0,\infty]$, the floor j has **floor mass at most V for** c, written $\mathrm{FML}_j(V;c)$, if for every $g\in G$

$$\int_{\{\xi\in G:\ \xi g\text{ near at }j+1\}} c(\xi g)\,d\theta_j(\xi)\le V,$$

the integral being the lower Lebesgue integral where c is not measurable.

Active points and deep vectors. Let $W\colon G\to\mathbb{C}^I$, $j\ge 0$, p,q sequences of units of A and $x\in G$. A point $y\in G$ is **active** at x for (j,p,q,W) if

(AC1) $\langle W(y),W(yx)\rangle\neq 0$;

(AC2) $w_{j,p}^{-1}y$ is near at j;

(AC3) $w_{j,q}^{-1}yx$ is near at j.

For $D\in[0,\infty]$, (j,p,q,W) has a **deep vector after the head of depth at least D at** x if every point y active at x for (j,p,q,W) has some a with $j\le a<\varrho$ and $\mathrm{dp}_a(y,x)\ge D$.

Shells. Let Γ be a gauge with loss Λ, W, j, p, q as above, $S\subseteq G$ and $\Delta\in[0,\infty]$. Then S is a **shell of height** Δ for (Γ,j,p,q,W) if

(SH1) $1\le\Delta<\infty$;

(SH2) there is a finite set $\mathcal{G}$ of gauges for the same place datum with $\#\mathcal{G} \le 2\Delta^{9n}$, such that S is contained in the union of the sets of positions of size Γ', $\Gamma' \in \mathcal{G}$, and the loss of every $\Gamma' \in \mathcal{G}$ is at most $\Lambda\Delta^2$;

(SH3) for every $x \in S$ and every $\varepsilon \in [0,1)$ with $\Lambda^{1200n^2} \le \varepsilon^{400n^2}\Delta$, (j,p,q,W) has a deep vector after the head of depth at least $(\varepsilon^2)^{-1}$ at x.

Definition 5.2.6. Let a place datum be given. The floor constant $c_{\mathrm{fl}}(l)$, the constant of all floors c_{fls} and the weighted square product WSP_j are formed for a gauge of the place datum, of loss Λ; the other objects below read no gauge. Arithmetic is in $[0,\infty]$, and an integral of a function that is not measurable is its lower Lebesgue integral.

Floor moduli and constants. For $l \ge 0$ and a unit c of A the **floor modulus** is $\mathrm{fm}(l,c) = |c|^{d_l}\,|c|^{1/2}$, and $\mathrm{fm}(l) = \mathrm{fm}(l,M)$. The **floor constant** of floor l and the **constant of all floors** are

$$c_{\mathrm{fl}}(l) = \mathrm{fm}(l)\,\mathrm{kb}_A^{|\iota'|}\,\Lambda^{4900n}, \qquad c_{\mathrm{fls}} = \Lambda^{10^6n^3}.$$

Pivots, the norm of the form integrand and the form majorant. For $\mathcal{K}\colon G\times G\to\mathbb{C}$, $j\ge 0$ and $g,y\in G$ the **pivot** is

$$\mathrm{piv}_j(\mathcal{K})(g,y) = \int_G |\mathcal{K}_{j+1}(\xi g,\xi y)|\,d\theta_j(\xi).$$

Let $W\colon G\to\mathbb{C}^I$, $j\ge 0$, p,q sequences of units, $d = w_{j,q}^{-1}w_{j,p}$ and $x\in G$. For a weight $\omega\colon G\to\mathbb{C}$ the **norm of the form integrand** is

$$\mathrm{FN}_j(p,q,W,x;\omega) = \int_G |\mathrm{FI}_j(p,q,W,x;\omega)(g)|\,d\mathfrak{m}_j(g),$$

and the **form majorant** is

$$\mathrm{FM}_j(p,q,W,x) = \int_G \mathrm{piv}_j(\mathcal{K}_W)(g,gx)\;\mathrm{piv}_j(\mathcal{K}_W^d)(w_{j,p}^{-1}g, w_{j,p}^{-1}gx)\,d\mathfrak{m}_j(g).$$

Weighted squares. For $\mathcal{K}\colon G\times G\to\mathbb{C}$, $x,g\in G$ and $l\ge 0$, the **weighted square** is $\mathrm{WSq}_l(\mathcal{K},x)(g) = \mathrm{dpw}_l(g,x)\,|\mathcal{K}_{l+1}(g,gx)|^2$; the **weighted square sums** are defined by downward recursion,

$$\mathrm{WSS}_l(\mathcal{K},x)(g) = \int_G \big(\mathrm{WSq}_l(\mathcal{K},x)(\xi g) + \mathrm{WSS}_{l+1}(\mathcal{K},x)(\xi g)\big)\,d\theta_l(\xi) \quad (l<n),$$
$$\mathrm{WSS}_l(\mathcal{K},x) = 0 \quad (l\ge n);$$

and the **weighted square integral** is

$$\mathrm{WSI}_j(\mathcal{K},x) = \int_G \mathrm{WSS}_j(\mathcal{K},x)(g)\,d\theta_{[0,j)}(g) \qquad \text{(descending convolution (5.11))}.$$

For W, j, p, q, d, x as above the **weighted square product** is

$$\mathrm{WSP}_j(p,q,W,x) = \Big(\prod_{a<j}\mathrm{fm}(a,p_a^{-1})\Big)^2 c_{\mathrm{fls}}^3\,\mathrm{WSI}_0(\mathcal{K}_W,x)\,\mathrm{WSI}_j(\mathcal{K}_W^d,x).$$

Shell normalisation. For $W\colon G\to\mathbb{C}^I$ the **pairing norm** and the **shell normalisation** are

$$\mathrm{PN}(W) = \int_G\int_G |\langle W(x),W(y)\rangle|^2\,d\theta_{\ge 0}(x)\,d\theta_{\ge 0}(y), \qquad \mathrm{SN}(W) = \Big(\prod_{l<\varrho}\mathrm{fm}(l)\Big)\mathrm{kb}_A^{|\iota'|\varrho}\,\mathrm{PN}(W).$$

Definition 5.2.7. Let F be a lattice datum over a finite place A, with level unit M, and write $\mathrm{vol} = \theta$ for its group measure.

- The **extended Bessel character** $\chi_M \colon G \to \mathbb{C}$ is $\chi_M(y) = \psi(M^{-1}\sigma(u))$ if there is $u \in \mathsf{U}$ with $u^{-1}y \in \mathrm{K}(M)$, for a chosen such u, and $\chi_M(y) = 0$ if there is none.
- With the floor moduli $\mathrm{fm}(l) = \mathrm{fm}(l, M)$ of Definition 5.2.6 (so $\mathrm{fm}(l) = |M|^{k+1+2(\varrho-l-1)+1/2}$ for $l < \varrho$), the **unipotent mass** is $\mathrm{fl} = \prod_{l<\varrho} \mathrm{fm}(l)$.
- An **intermediate group** is a subgroup Z of G with $\mathrm{K}(M) \le Z \le \mathrm{K}(1)$. Its **cosets** are the right cosets $\mathrm{K}(M)c \subseteq Z$; $\mathrm{Cos}(Z)$ denotes their set, and each coset has a chosen representative c.
- The **Bessel vector** of Z is $W_Z \colon G \to \ell^2(\mathrm{Cos}(Z))$, $W_Z(y) = \big(\chi_M(yc^{-1})\big)_{\mathrm{K}(M)c \in \mathrm{Cos}(Z)}$.
- The **mass ratio** of Z is $\mathrm{mr}(Z) = \mathrm{vol}(Z)\,\mathrm{vol}(\mathrm{K}(M))\,\mathrm{fl}^{-2} \in [0, \infty]$.
- For a subgroup Z of G the **mass integral** is

$$\mathrm{mi}(Z) = \int_G \Big| \int_Z \chi_M(yxy^{-1})\, d\mathrm{vol}(y) \Big|^2 d\mathrm{vol}(x) \in [0, \infty].$$

Definition 5.2.8. Let F be a lattice datum over a finite place A, Z an intermediate group, 1 the constant sequence $(1, 1, \dots)$, and p, q sequences of units of A.

(FB1) Scaled orbital integral. For $l \in G$, $\mathrm{SO}_p(l) \in \mathbb{C}^{\mathrm{Cos}(Z)\times\mathrm{Cos}(Z)}$ is

$$\mathrm{SO}_p(l)_{c,c'} = \mathrm{hdn}_\varrho(p, 1) \int_G W_Z(y)_c\, \overline{W_Z(l\, w_{\varrho,p}^{-1} y)_{c'}}\, d\theta(y),$$

the orbital integral of W_Z at $w_{\varrho,p} l^{-1}$, scaled by the head density.

(FB2) Fibre pairing. For $e \in V$, if $e = l^{-1}z$ for some $l \in G^{(\varrho)}$, then for a chosen such l

$$\mathrm{fp}_{p,q}(e) = \int_G \big\langle \mathrm{SO}_p(kl), \mathrm{SO}_q(kl) \big\rangle\, d\mu_H(k);$$

otherwise $\mathrm{fp}_{p,q}(e) = 0$.

(FB3) Norm-fibre measures. For a finite set S, a **family of norm-fibre measures** on A^S is a family $(\mu_\beta^{\mathrm{nf}})_{\beta \in A^-}$ of measures on A^S that is a disintegration of the Haar measure vol of A^S along $\bar\omega^{\mathsf{T}}\omega$.

(FB4) Last-entry density. Let $\mathfrak{S}$ be a filling datum of shape (ϱ, k) with prime N, let F be a lattice datum on J_{lat} read over A, π a unit of A with $\pi\bar\pi = N$, and $(\mu_\beta^{\mathrm{nf}})$ a family of norm-fibre measures on A^k. For $t \in A$ and $\omega \in A^k$ let $e(\pi, t, \omega) \in V$ be the vector $\sum_{i=1}^k \omega_i x_i - N^{-1}tz$ of $X_0 \oplus Az$ (with $x_k = \bar\pi x' - \bar\pi N^{-1} z$), written in the basis $(x_i)_{i\neq k}, x', z$ and with coordinates 0 at the e_a, e_a^*; so $h(z, e) = t$ and $h(e, e) = \bar\omega^{\mathsf{T}}\omega - t\bar t/N$. The **last-entry density** is

$$\mathrm{led}(\pi, Z, p, q) = |\pi|^{-1} \int_{A^k} \mathrm{fp}_{p,q}\big(e(\pi, p_\varrho, \omega)\big)\, d\mu_\beta^{\mathrm{nf}}(\omega), \qquad \beta = \delta\Big(\frac{p_\varrho \bar p_\varrho}{N} - N\Big),$$

the integral over the sphere $\bar\omega^{\mathsf{T}}\omega = p_\varrho\bar p_\varrho/N - N$ on which $e(\pi, p_\varrho, \omega)$ has norm $-N$; it does not read q_ϱ.

Definition 5.2.9. Let F be a lattice datum over a finite place A with level unit M, and fix a representative in U of each coset in $\mathsf{U}/(\mathsf{U}\cap \mathrm{K}(M))$. For $x, s \in G$ the **Kloosterman sum** of x at s is

$$\mathrm{Kl}(x,s) = \sum_{\substack{u,u' \in \mathsf{U}/(\mathsf{U}\cap \mathrm{K}(M)) \\ s^{-1}u^{-1}xu' \in \mathrm{K}(M)}} \chi_M(u)\,\overline{\chi_M(u')},$$

each coset read through its fixed representative; the sum is taken to be 0 if it is not absolutely summable. For F a filling datum at T (Definition 5.2.3) it is written $\mathrm{Kl}_T(x,s)$.

Proposition 5.2.10. *(i) Let k_0 be a field with q elements and $1 < r < q$. Then $\#\{c \in k_0[X]$ monic of degree r, c not primitive$\} \cdot (1 - r/q)/q^r \le \exp(-1/(2r(2r\log q + 1)))$.*

(ii) Let k_w ($w \in S$, $\#S = f$) be fields with q elements, $M_0 \neq 0$, $1 < r < q$, $(1 + M_0 + 2r)/q \le \varepsilon \le 1$ and $2r(2r\log q + 1)L \le f$. Then in $\mathbb{C}[\prod_w \mathrm{GL}_r(k_w)]$, $\mathrm{Re}\,\mathrm{coeff}_1(1 - \sum_{c \text{ good for } M_0} \bigotimes_w e_{c_w}) \le f\varepsilon + e^{-L}$.

(iii) For admissible data D and $i \in \{0,1\}$ let $g_i = \sum_{c \text{ good for } q_2} \bigotimes_{w\in S_i} e_{c_w} \in \mathbb{C}[\prod_{w\in S_i} \mathrm{GL}_r(k_w)]$. Then g_i is a central idempotent, $g_0 \otimes g_1$ acts on $\mathcal{A}_M(Y_0)$ as e_0, and $\mathrm{Re}\,\mathrm{coeff}_1(1-g_i) \le b^{-8}/2$.

Proof. Throughout, k_0 is a field with q elements, $1 < r < q$ (so $q \ge 3$), and $m = (1 - r/q)/q^r \in [0, q^{-r}]$.

Step 1: primitive polynomials. Let Π be the number of primitive labels of degree r over k_0. In an algebraic closure the elements ξ of multiplicative order $q^r - 1$ are the $\varphi(q^r-1)$ generators of $\mathbb{F}_{q^r}^\times$. The minimal polynomial of such ξ over k_0 has degree d with $\xi \in \mathbb{F}_{q^d}$, so $q^r - 1 \mid q^d - 1$ and $d \ge r$, while $d \mid r$; hence it is monic irreducible of degree r with nonzero constant term and has a root of order $q^r - 1$, i.e. a primitive label. A primitive label has r roots, so $\xi \mapsto$ (minimal polynomial) is at most r-to-one and $\Pi \ge \varphi(q^r-1)/r$.

Step 2: an elementary totient bound. For $N' \ge 2$, $N' \le \varphi(N')(2\log N' + 1)$. Indeed, if $p_1 < \cdots < p_\omega$ are the primes dividing N', then $p_i \ge i+1$ and $x \mapsto x/(x-1)$ is decreasing, so $N'/\varphi(N') = \prod_i p_i/(p_i-1) \le \prod_{i=1}^{\omega} \frac{i+1}{i} = \omega + 1$, and $2^\omega \le N'$ gives $\omega \le \log N'/\log 2 \le 2\log N'$.

(i). Put $A = 2r(2r\log q + 1)$. With $N' = q^r - 1 \ge 2$ and $\log(q^r-1) \le r\log q$, Steps 1 and 2 give $q^r \le 2(q^r-1) \le 2\varphi(q^r-1)(2r\log q + 1) \le A\,\Pi$. The monic polynomials of degree r number q^r, so those that are not primitive number $q^r - \Pi$, and

$$(q^r - \Pi)\,m \le \frac{q^r - \Pi}{q^r} \le 1 - \frac{1}{A} \le e^{-1/A}.$$

Step 3: allowed labels. Let $M_0 \ge 1$. A monic c of degree r that is not a label allowed for M_0 either has a root $a \in \{0\} \cup \mu_{M_0}(k_0)$, a set of at most $1 + M_0$ elements, and then $c = (X-a)B$ with B monic of degree $r-1$; or it is not squarefree, and then $c = P^2 B$ with P monic irreducible of degree $e \ge 1$ and B monic of degree $d = r - 2e < r$, $e = (r-d)/2$. For fixed $d < r$ there are at most $q^{(r-d)/2+d} \le q^{r-1}$ such products ($(r+d)/2 < r$). So at most $(1 + M_0 + r)q^{r-1}$ monic polynomials of degree r are not allowed labels, and the number α of allowed labels satisfies, if $(1 + M_0 + 2r)/q \le \varepsilon$,

$$\alpha\, m \ge \Big(1 - \frac{1 + M_0 + r}{q}\Big)\Big(1 - \frac{r}{q}\Big) \ge 1 - \frac{1 + M_0 + 2r}{q} \ge 1 - \varepsilon.$$

Step 4: the identity coefficient of a label idempotent. Let c be a label of degree r with factor degrees $d_1, \dots, d_t$ ($\sum_j d_j = r$). By Theorem A.5.27 (c) and $|\mathrm{GL}_r(k_0)| = q^{r(r-1)/2}\prod_{i=1}^r (q^i - 1)$,

$$\mathrm{coeff}_1(e_c) = \frac{\Xi_c(1)\overline{\Xi_c(1)}}{|\mathrm{GL}_r(k_0)|} = \frac{\prod_{i=1}^r (q^i-1)}{q^{r(r-1)/2}\prod_j (q^{d_j}-1)^2} \ge \frac{q^{r(r+1)/2}\prod_{i=1}^r (1-q^{-i})}{q^{r(r-1)/2}q^{2r}} \ge \frac{1 - \sum_{i=1}^r q^{-i}}{q^r} \ge m,$$

using $(q^{d_j}-1)^2 \le q^{2d_j}$, $\prod(1-x_i) \ge 1-\sum x_i$ for $x_i \in [0,1]$, and $q^{-i} \le q^{-1}$.

(ii). For a tuple $c=(c_w)_{w\in S}$ of labels, the identity coefficient of $\bigotimes_w e_{c_w}$ in $\mathbb{C}[\prod_w \mathrm{GL}_r(k_w)]$ is $\prod_w \mathrm{coeff}_1(e_{c_w}) \ge m^f$ (Step 4). Let $\mathcal{A}_w$ be the allowed labels over k_w and $\mathcal{N}_w$ the monic non-primitive polynomials of degree r. A tuple in $\prod_w \mathcal{A}_w$ not in $\prod_w \mathcal{N}_w$ is good, so the number γ of good tuples satisfies $\gamma \ge \prod_w \#\mathcal{A}_w - \prod_w \#\mathcal{N}_w$. With $u=1/A$, Steps 3 and (i) give

$$\sum_{c \text{ good}} \mathrm{Re}\,\mathrm{coeff}_1\Big(\bigotimes_w e_{c_w}\Big) \ge \gamma\, m^f \ge \prod_w (\#\mathcal{A}_w\, m) - \prod_w (\#\mathcal{N}_w\, m) \ge (1-\varepsilon)^f - e^{-fu}.$$

By Bernoulli's inequality $(-\varepsilon \ge -1)$, $(1-\varepsilon)^f \ge 1-f\varepsilon$, and $fu \ge L$ by the hypothesis $AL \le f$. Hence $\mathrm{Re}\,\mathrm{coeff}_1(1-\sum_{c\text{ good}} \bigotimes_w e_{c_w}) \le f\varepsilon + e^{-L}$.

(iii). By Lemma 5.2.11 (c), $\#S_i = f$ and $k_w = \mathbb{F}_{p_i}$ for $w \in S_i$. By (S5) of Lemma 5.2.12, $1 < r < p_i$, $\varepsilon := (1+q_2+2r)/p_i \le 1$ and $18r(2r\log p_i + 1)\log b \le f$; so (ii) applies with $q = p_i$, $M_0 = q_2 \ge 1$ and $L = 9\log b$, and gives $\mathrm{Re}\,\mathrm{coeff}_1(1-g_i) \le f(1+q_2+2r)/p_i + b^{-9} \le b^{-8}/2$, the last step again by (S5). Each e_c is central, its coefficients being a class function; by Theorem A.5.27 (a), e_c is the central idempotent attached to the irreducible character $\epsilon_c \Xi_c$, so for a tuple c the element $\bigotimes_w e_{c_w}$ is the central idempotent of the irreducible character $\bigotimes_w \epsilon_{c_w}\Xi_{c_w}$ of $\prod_w \mathrm{GL}_r(k_w)$, and by Theorem A.5.27 (b) distinct tuples give distinct irreducible characters. The central idempotents of distinct irreducible characters are orthogonal [262] (§6.3), so g_i, a sum of such idempotents over distinct tuples, is a central idempotent. Finally $\mathbb{C}[\bar{G}] = \mathbb{C}[\prod_{w\in S_0} \mathrm{GL}_r(k_w)] \otimes \mathbb{C}[\prod_{w\in S_1} \mathrm{GL}_r(k_w)]$, and a family is good exactly when its restrictions to S_0 and to S_1 are good for q_2; expanding the product, $g_0 \otimes g_1 = \sum_{c \text{ good family}} \bigotimes_{w\in S_0\cup S_1} e_{c_w} = e_0$ in $\mathbb{C}[\bar{G}]$, so it acts on $\mathcal{A}_M(Y_0)$ as e_0. □

Lemma 5.2.11. *Let D be admissible data and $\ell \in \{N, p_0, p_1\}$.*

(a) ℓ splits completely in E: it is unramified, and there are exactly $2f$ places w of E above ℓ, each with $q_w = \ell$.

(b) The finite places w with $\pi_N \in w$ form a set T_N of exactly f places; π_N is a uniformiser at each $w \in T_N$ and a unit at every finite place not in T_N; $w \in T_N$ implies $\bar{w} \notin T_N$, and the places above N are the disjoint union of T_N and $\overline{T_N}$.

(c) For $i \in \{0,1\}$ the set S_i of chosen places above p_i has exactly f elements, $q_w = p_i$ for $w \in S_i$, $S_0 \cap S_1 = \emptyset$, the conjugate of a chosen place is not chosen, and the places above p_i are the disjoint union of S_i and $\overline{S_i}$.

(d) For $x \in \widehat{\mathcal{O}}_E$: $M \mid x$ in $\widehat{\mathcal{O}}_E$ if and only if $x \in w\widehat{\mathcal{O}}_{E,w}$ and $\bar{x} \in w\widehat{\mathcal{O}}_{E,w}$ for every chosen place w.

Proof. (a). By (AD5) and (AD6), $\ell \equiv 1 \pmod{q_2^2}$, with $q_2^2 = (q_0 d)^{2s+12}$. We claim $q_1 \mid q_2^2$. Indeed $q_1 = q_0 \cdot 3\, l_0(s) \cdot 2^s = q_0 \cdot 3^{\lfloor \log_3 s \rfloor + 5} \cdot 2^s$, and q_0^{2s+11} is divisible by $2^{4(2s+11)}$ and by 3^{2s+11} because $16\cdot 3 \mid q_0$; as $s \le 8s+44$ and $\lfloor \log_3 s \rfloor + 5 \le 2s+11$, we get $q_1 \mid q_0 \cdot q_0^{2s+11} \mid q_2^2$. Hence $\ell \equiv 1 \pmod{q_1}$, so $\ell \nmid q_1$ and the Frobenius of ℓ in $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/q_1)^\times$ is $\ell \bmod q_1 = 1$: ℓ splits completely in $\mathbb{Q}(\zeta)$ [317] (Thm. 2.13). Next, $d \mid q_2$ gives $\ell \equiv 1 \pmod{d}$, so $\ell \ne d$, and $16 \mid q_2$ gives $\ell \equiv 1 \pmod 4$. By quadratic reciprocity $\left(\frac{-d}{\ell}\right) = \left(\frac{-1}{\ell}\right)\left(\frac{d}{\ell}\right) = \left(\frac{\ell}{d}\right) = \left(\frac{1}{d}\right) = 1$; since ℓ is odd and prime to the discriminant $-d$ of K, ℓ splits in K. The field $E = K(\zeta)$ is the compositum of the Galois extensions K and $\mathbb{Q}(\zeta)$ of $\mathbb{Q}$, so $\mathrm{Gal}(E/\mathbb{Q})$ embeds into $\mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ by restriction; the decomposition group of a place above ℓ restricts to the (trivial) decomposition groups in both factors, hence is trivial. So ℓ is unramified in E with $[E:\mathbb{Q}] = 2f$ places of residue degree 1 above it.

(b). By (AD6), $\pi_N\bar{\pi}_N = N$. A finite place not above N does not contain π_N, so π_N is a unit there. For w above N, (a) gives $v_w(N) = 1$, and $v_w(\bar{\pi}_N) = v_{\bar{w}}(\pi_N)$, so

$$v_w(\pi_N) + v_{\bar{w}}(\pi_N) = 1.$$

Hence $w \neq \bar{w}$ (otherwise the left side is even), exactly one of $w, \bar{w}$ contains π_N, and there π_N has valuation 1. The $2f$ places above N therefore fall into f conjugate pairs, each contributing exactly one place to T_N; this gives $\#T_N = f$ and the remaining assertions.

(c). By (AD5), $\pi_M\bar{\pi}_M = p_0p_1$ in $\mathcal{O}_K \subseteq \mathcal{O}_E$, and the chosen places are the w with $\pi_M \in w$. As $p_0 \neq p_1$ are unramified in E by (a), $v_w(p_0p_1) = 1$ for every w above p_0 or p_1, and the argument of (b) applied to π_M shows that $w \neq \bar{w}$ and that exactly one of $w, \bar{w}$ is chosen. A place containing π_M lies above p_0 or p_1. So S_i contains exactly one place of each of the f conjugate pairs above p_i, $\#S_i = f$, the places above p_i are $S_i \sqcup \overline{S_i}$, and $q_w = p_i$ by (a). $S_0 \cap S_1 = \emptyset$ since $p_0 \neq p_1$, and the conjugate of a chosen place is not chosen (also Lemma 4.6.2 (a)).

(d). Since p_0, p_1 are distinct primes, $M \mid x$ if and only if $p_0 \mid x$ and $p_1 \mid x$ in $\widehat{\mathcal{O}}_E$. By (a), $p_i\widehat{\mathcal{O}}_{E,w} = w\widehat{\mathcal{O}}_{E,w}$ for w above p_i and p_i is a unit at the other finite places, so $p_i \mid x$ if and only if $v_w(x) \geq 1$ for all w above p_i, that is, by (c), for all $w \in S_i$ and all $w \in \overline{S_i}$. For $w \in S_i$, $v_{\bar{w}}(x) = v_w(\bar{x})$. So $M \mid x$ if and only if $v_w(x) \geq 1$ and $v_w(\bar{x}) \geq 1$ for every chosen place w. □

Lemma 5.2.12. *Let D be admissible data, $\varrho = (r-k-1)/2$ and $n = \varrho + 1$. Then, for every $v \in \Phi$ and $i \in \{0,1\}$:*

(S1) Filling fields. *$p_i < N$; $10^5 \leq \varrho$; $b^{10} \leq 2^{\varrho}$; $2f(\varrho+1) \leq 2^{2\varrho}$; $1000(2\varrho+2)^9 \leq k \leq f \leq b/1000$; $(200fk\,\mathrm{rd}_E)^{1000} \leq b$; $\mathrm{rd}_E \leq b$; $\varrho + 1 \leq b$; $b^{19} \leq M \leq b^{23}$; $5 \leq N$; $10^{10}n^4 \leq n^6$, $3n^6 + 12 \leq n^7 \leq k$.*

(S2) Admissibility. *$|\varpi|_v \leq b$; $1 \leq M|\varpi|_v/2\pi \leq b^{23}$; $\log N \leq 30(2\varrho+2)\log b$; and, with the sums over $\tau \in \Phi$,*

$$\Big|f(m-1)\Big(\varrho\log\frac{M\mathrm{rd}_E}{2\pi} + \log\frac{\sqrt{\mathrm{rd}_E}}{2\pi}\Big) - m\sum_{\tau}\log\kappa(\tau)\Big| + \log(1+b^{-9}) + m\Big(m\sum_{\tau}\log\kappa(\tau) + \frac{fr}{x}\Big) + fk\sqrt{\log b} \leq \frac{fk\log b}{4}.$$

(S3) Frame masses. *$(r-k)2^{1-\varrho} + 4f/N \leq 2b^{-9}/(b^{-9}+2)$; $8f \leq 2^{\varrho}$; $4\varrho + 12 \leq k$.*

(S4) Bessel systems. *$3a \leq b^2$; $\pi \leq b$; $(2\varrho+1)k \leq r^2$; $k \leq a$; $10^9n^3 + 3n + 6 < k$; $8nr \leq b$; $r + k + 3a \leq b^3$; $10^{30}\big((r+k)^2\log(b^3)\big)^3 \leq r^7$; $2000 \leq b$; $100 \leq r$; $r + 3a + 1 \leq b^4$; $(r+1)^4(3a + r + 1 + 2n^2) \leq b^5$; $4000n \leq f$.*

(S5) Spanning. *$1 < r < p_i$; $(1+q_2+2r)/p_i \leq 1$; $f(1+q_2+2r)/p_i + b^{-9} \leq b^{-8}/2$; $18r(2r\log p_i + 1)\log b \leq f$.*

(S6) Bessel parameters. *$2\varsigma(v) = (2\varsigma_j(v))_{0\leq j<r}$ has large gap r^7 for the signs of Y_0; $|2\varsigma_j(v)| \leq 3a$ for $0 \leq j < r$; $3\kappa_0(v) + a \leq b^2$; $10^{12} \leq b^3$; $3r^3f \leq r^7$; $10^8 \leq \varrho$.*

Proof. Write $L = \log b$. All inequalities are compared on the logarithmic scale after the following basic sizes.

Basic sizes.

(a) By Lemma 4.2.67 (1), (2), $\log L \geq 6.4 \cdot 10^{13}$, $\log x \geq 6.4 \cdot 10^{15}$, $L^{100} \leq x$ and $x^{10^{12}} \leq b$; in particular $L \geq 10^{20}$, $x \geq 10^{20}$, $x^{40000} \leq b$ and $x^{30} \leq b^{1/2}$. By Proposition 2.3.5, $x \leq x_1 \leq b$ and $(\log x_1)^6 \leq L$; by Lemma 2.3.6, $10^6 \log x \leq \log x_1$ and $10^{14} C_P \leq \log x$; by Lemma 4.2.67 (1), $q_0, d \leq C_P$.

(b) m is even, $r - k = m - 1$, so $\varrho = m/2 - 1$ and $n = m/2$ are integers, $2\varrho + 2 = m$, $2\varrho + 1 = r - k$, $\frac{1}{2}\sqrt{x} \leq n \leq \sqrt{x}$ and $\varrho \geq \frac{1}{2}\sqrt{x} - 1 \geq x^{1/3}$.

(c) $x^5 \leq r \leq 3x^5$, and $r \leq 2k$ by Lemma 4.2.67 (3), so $x^5/2 \leq k \leq r$.

(d) $x^{11} \leq f \leq x^{12}$ by Lemma 2.3.8.

(e) $x^{11} \leq \mathrm{rd}_E \leq x^{12}$ by Lemma 5.2.13 (2). By Lemma 3.2.35 (ii), $|\varpi|_v = \sqrt{d}\,|\mathrm{cyc}'_{q_1}(\zeta)|_v$ for all v, so Lemma 5.2.14 (ii) gives $x^{10} \leq |\varpi|_v \leq x^{13}$.

(f) $b^{20} \leq M \leq b^{22}$. As $x \leq x_1 \leq b$, $1 \leq xb/x_1 = a/x_1 \leq b$, so $1 \leq \kappa(\tau) \leq b$ and $0 \leq \log \kappa(\tau) \leq L$ for all τ.

(g) $mL \leq \log N \leq 25mL$, so $N \geq b^{30}$. Indeed Lemma 2.5.6 applies to $m \geq 2$, b with $L \geq 7$ and $q_1 d \leq x^{13} \leq b$ (Lemma 4.2.67 (4)), the weight κ with $1 \leq \kappa \leq b$ and M with $b^{20} \leq M \leq b^{22}$; its number λ is $\log N_-$ of (AD6), so $mL \leq \log N_- \leq 24mL$, and (AD6) gives $mL \leq \log N \leq 24mL + \sqrt{L} \leq 25mL$; finally $m \geq \sqrt{x} \geq 30$.

(h) $q_2 \leq b^{1/2}$: as $q_0 d \leq C_P^2$, $\log C_P \leq C_P$, $s + 6 \leq 38 \log x$ and $76 C_P \leq \log x$,

$$\log q_2 = (s+6)\log(q_0 d) \leq 38 \log x \cdot 2C_P \leq (\log x)^2 \leq 10^{-12}(\log x_1)^2 \leq 10^{-12} L^{1/3} \leq L/2.$$

(i) $b^{12} \leq 2^{\varrho}$: $\varrho \log 2 \geq x^{1/3}/2 \geq L^{33}/2 \geq 12L$.

(S1). $\log p_i \leq 11L < 30L \leq \log N$ by (g), so $p_i < N$. $\varrho \geq x^{1/3} \geq 10^5$ by (b). $b^{10} \leq 2^{\varrho}$ by (i). $2f(\varrho+1) \leq 2x^{12}\sqrt{x} \leq b \leq 2^{2\varrho}$ by (d), (b), (i). $1000(2\varrho+2)^9 = 1000m^9 \leq 512000\, x^{9/2} \leq x^5/2 \leq k$ by (b), (c), as $\sqrt{x} \geq 1.1 \cdot 10^6$. $k \leq 3x^5 \leq x^{11} \leq f$ and $1000f \leq 1000x^{12} \leq b$ by (c), (d), (a). $200 f k\, \mathrm{rd}_E \leq 200 \cdot x^{12} \cdot 3x^5 \cdot x^{12} \leq x^{30}$, and $x^{30000} \leq b$. $\mathrm{rd}_E \leq x^{12} \leq b$ and $\varrho + 1 = n \leq \sqrt{x} \leq b$. $b^{19} \leq M \leq b^{23}$ by (f). $N \geq b^{30} \geq 5$ by (g). As $n \geq \frac{1}{2}\sqrt{x} \geq 10^5$, $n^2 \geq 10^{10}$, so $10^{10} n^4 \leq n^6$, and $n^7 \geq 15n^6 \geq 3n^6 + 12$. $n^7 \leq x^{7/2} \leq x^5/2 \leq k$.

(S2). $|\varpi|_v \leq x^{13} \leq b$ by (e). By (e), (f), $M|\varpi|_v/2\pi \geq b^{20} x^{10}/2\pi \geq 1$ and $M|\varpi|_v/2\pi \leq b^{22} x^{13} \leq b^{23}$. $\log N \leq 25mL \leq 30(2\varrho + 2)L$ by (g) and $2\varrho + 2 = m$. For the last clause put $A = f(m-1)(\varrho \log \frac{M \mathrm{rd}_E}{2\pi} + \log \frac{\sqrt{\mathrm{rd}_E}}{2\pi})$ and $B = m \sum_\tau \log \kappa(\tau)$. By (e), (f), $2\pi \leq b^{20} \leq M \mathrm{rd}_E \leq b^{22} x^{12} \leq b^{23}$ and $4\pi^2 \leq x^{11} \leq \mathrm{rd}_E \leq b$, so $0 \leq \log \frac{M \mathrm{rd}_E}{2\pi} \leq 23L$ and $0 \leq \log \frac{\sqrt{\mathrm{rd}_E}}{2\pi} \leq L$; with $23\varrho + 1 \leq 12m$ and $m^2 \leq 4x$ this gives $0 \leq A \leq 12 f m^2 L \leq 48 f x L$. By (f), $0 \leq B \leq mfL$, so $|A - B| \leq \max(A, B) \leq 48 f x L$. Further $\log(1 + b^{-9}) \leq 1 \leq f$, and by (f), (c),

$$m\Big(m \sum_\tau \log \kappa(\tau) + \frac{fr}{x}\Big) \leq m^2 fL + \frac{2\sqrt{x}\, f \cdot 3x^5}{x} \leq 4xfL + 6 f x^{9/2}.$$

Hence the sum of all terms except $fk\sqrt{L}$ is at most $f(52xL + 1 + 6x^{9/2}) \leq 59 f x^{9/2} L \leq f x^5 L/16 \leq fkL/8$, using $\sqrt{x} \geq 944$ and $k \geq x^5/2$; and $fk\sqrt{L} \leq fkL/8$ since $L \geq 64$. The total is at most $fkL/4$.

(S3). By (b), (i), $(r-k)2^{1-\varrho} = 2(m-1)2^{-\varrho} \leq 4\sqrt{x}\, b^{-12} \leq b^{-11}$, and by (d), (g), $4f/N \leq 4x^{12} b^{-30} \leq b^{-29}$. So the left side is at most $2b^{-11} \leq \frac{2}{3} b^{-9} \leq \frac{2}{1+2b^9} = \frac{2b^{-9}}{b^{-9}+2}$. $8f \leq 8x^{12} \leq b \leq 2^{\varrho}$ by (d), (i). $4\varrho + 12 \leq 2m + 12 \leq 4\sqrt{x} + 12 \leq x^5/2 \leq k$ by (b), (c).

(S4). By (a), $b \geq x \geq 10^{20}$, so $\pi \leq b$ and $2000 \leq b$, and $3x \leq b$, so $3a = 3xb \leq b^2$. $(2\varrho{+}1)k = (r{-}k)k \leq r^2$ as $0 \leq k \leq r$. $k \leq 3x^5 \leq xb = a$. $10^9 n^3{+}3n{+}6 \leq 2{\cdot}10^9 x^{3/2} < x^5/2 \leq k$. $8nr \leq 24x^{11/2} \leq b$. $r+k+3a \leq 6x^5+3xb \leq 4xb \leq b^2 \leq b^3$. As $(r+k)^2 \leq 4r^2$ and $\log(b^3) = 3L$,

$$10^{30}\big((r+k)^2 \log(b^3)\big)^3 \leq 1728 \cdot 10^{30}\, r^6 L^3 \leq r^7,$$

since $r \geq x^5 \geq L^{500} \geq 10^{34} L^3$. $r \geq x^5 \geq 100$. $r + 3a + 1 \leq 3x^5 + 3xb + 1 \leq 4xb \leq b^4$. $(r+1)^4 \leq (4x^5)^4 = 256x^{20}$ and $3a + r + 1 + 2n^2 \leq 3xb + 3x^5 + 1 + 2x \leq 4xb$, so the product is at most $1024x^{21}b \leq b^2 \leq b^5$. $4000n \leq 4000\sqrt{x} \leq x^{11} \leq f$ by (b), (d).

(S5). $1 < x^5 \leq r \leq 3x^5 < b^{10} \leq p_i$. By (h), (c), (a), $1 + q_2 + 2r \leq 1 + b^{1/2} + 6x^5 \leq 2b^{1/2} \leq b \leq p_i$, which is the second clause; and $f(1+q_2+2r) \leq 2x^{12} b^{1/2} \leq b$, so $f(1+q_2+2r)/p_i + b^{-9} \leq b^{-9} + b^{-9} \leq b^{-8}/2$. Finally $\log p_i \leq 11L$, so

$$18r(2r \log p_i + 1)L \leq 54x^5(66x^5 L + 1)L \leq 3618\, x^{10} L^2 \leq x^{11} \leq f,$$

as $x \geq L^{100} \geq 3618 L^2$.

(S6). By (GP5) of Definition 3.2.28 and $\kappa_i(v) = \kappa(v) + (2i+1)r^8$, the vector $l = 2\varsigma(v)$ has the entries

$$l(j) = k - 2 - 2j \quad (0 \leq j \leq k-2), \qquad l(j) = -2\kappa(v) - 4(j+2-k)\, r^8 \quad (k-1 \leq j \leq r-1),$$

and the signs s of Y_0 are $s(j) = 1$ for $j \leq k-2$ and $s(j) = (-1)^{j+1-k}$ for $j \geq k-1$. As $r = 3^{e'+1}$ is odd and m is even, $k = r + 1 - m$ is even, so every $l(j)$ is even, that is $l(j) \equiv r - 1 \pmod 2$. By (c), $100 \leq x^5 \leq r$ and $k \leq r$. On $0 \leq j \leq k-2$ the entries decrease strictly and lie in $[2-k, k-2]$; on $k-1 \leq j \leq r-1$ they decrease strictly and are at most $-4r^8 < 2 - k$; so l is injective. Let $s(j) \neq s(j')$ with $j < j'$; then $j' \geq k-1$. If $j \geq k-1$, then $|l(j) - l(j')| = 4r^8(j' - j) \geq 4r^8$; if $j \leq k-2$, then $l(j) - l(j') \geq 2 - k + 4r^8 \geq 3r^8$. Since $r^2 \geq 500$ and $r \geq 4$, $2(500r^3 + r^7) \leq 2(r^5 + r^7) \leq 4r^7 \leq r^8$; hence l has gap $500r^3 + r^7$ for s, which is large gap r^7 (the index set has r elements). For the sizes: if $j \leq k-2$, $|l(j)| \leq k - 2 < k \leq 3x^5 \leq xb = a$; if $j \geq k-1$, then $1 \leq j + 2 - k \leq m \leq 2\sqrt{x}$, $\kappa(v) \leq b$ by (f) and $r^8 \leq 3^8 x^{40}$, so, as $8 \cdot 3^8 \leq \sqrt{x}$ and $x^{41} \leq b$ by (a),

$$|l(j)| \leq 2b + 8\sqrt{x} \cdot 3^8 x^{40} \leq 2b + x^{41} \leq 3b \leq 3a.$$

Next $\kappa_0(v) = \kappa(v) + r^8 \leq b + 3^8 x^{40} \leq 2b$, so $3\kappa_0(v) + a \leq 6b + xb \leq 2xb \leq b^2$, as $6 \leq x$ and $2x \leq b$ by (a). By (a), $b \geq x \geq 10^{20}$, so $10^{12} \leq b^3$. By (c), (d), $3r^3 f \leq 3x^{12} r^3 \leq x^{20} r^3 \leq r^7$. Finally $\varrho \geq x^{1/3}$ by (b), and $\log x \geq 6.4 \cdot 10^{15} \geq 24 \log 10$ by (a), so $\varrho \geq 10^8$. □

Lemma 5.2.13. *1. Let $P = (q_0, d)$ be a parameter pair and $s \geq 0$ an integer; write $q_1 = q_1(s)$ and $E = E_s$. Then $[E : \mathbb{Q}] = 2\varphi(q_1)$, every prime dividing $\operatorname{disc}(E)$ divides $q_0 d$, and*

$$T(s) \;\leq\; \mathrm{rd}_E \;\leq\; q_1 d.$$

2. If s is the exponent of admissible data over P with scale x, then

$$x^{11} \leq \mathrm{rd}_{E_s} \leq x^{12}, \qquad \log \mathrm{rd}_{E_s} + |c_{\mathrm{CS}}(d)| \leq 13 \log x.$$

Proof. Item 1: degree and the cyclotomic embedding. By Lemma 3.2.8, item 2, $[E : K] = \varphi(q_1)$, so $[E : \mathbb{Q}] = 2\varphi(q_1)$. As $d \equiv 3 \pmod 4$ (Lemma 3.2.8, item 1), the quadratic Gauss sum of d is a square root of $-d$ in $\mathbb{Q}(\zeta_d)$ [317, Chapter 4]; hence $E = \mathbb{Q}(\sqrt{-d}, \zeta)$ embeds into $\mathbb{Q}(\zeta_{q_1 d})$. So $E/\mathbb{Q}$ is abelian, and every character of $\mathrm{Gal}(E/\mathbb{Q})$, a quotient of $\mathrm{Gal}(\mathbb{Q}(\zeta_{q_1 d})/\mathbb{Q}) \cong (\mathbb{Z}/q_1 d)^\times$, corresponds to a Dirichlet character modulo $q_1 d$, whose primitive conductor divides $q_1 d$.

Item 1: discriminant. By Theorem A.2.5 over $\mathbb{Q}$, $|\operatorname{disc}(E)|$ is the product of the $[E:\mathbb{Q}]$ conductors of these characters, each a positive divisor of $q_1 d$. Hence every prime dividing $\operatorname{disc}(E)$ divides $q_1 d$, hence $q_0 d$ (every prime dividing q_1 divides q_0, Lemma 3.2.8, item 1), and $|\operatorname{disc}(E)| \le (q_1 d)^{[E:\mathbb{Q}]}$, that is $\mathrm{rd}_E \le q_1 d$. The lower bound $\mathrm{rd}_E \ge T(s)$ is Lemma 2.3.9.

Item 2. Put $L = \log x$. By Lemma 2.3.8, $x^{11} \le 2^s \le T(s)$, so $\mathrm{rd}_{E_s} \ge x^{11}$ by item 1. For the upper bound, (AD2) of Definition 2.3.4 gives $s = 22(e+1) \ge 1$, $2^s = m^{22} \le (2\sqrt{x})^{22} = 2^{22} x^{11}$ and $s \le 32L$; and $l_0(s) = 3^{\lfloor \log_3 s \rfloor + 4} \le 81 s$. Hence

$$q_1 d = 3\, q_0\, l_0(s)\, 2^s\, d \le 3 \cdot 81 \cdot 32 \cdot 2^{22}\, q_0\, d\, L\, x^{11} \le 10^{11}\, q_0\, d\, L\, x^{11}.$$

By Definition 2.3.3, $C_P \ge 2d + 6 \ge d$; by Lemma 2.3.6, $q_0 \le C_P \le 10^{-14} L$. So $10^{11} q_0 d L \le 10^{11} \cdot 10^{-28} L^3 \le L^3 \le e^L = x$, the last step because $L \ge 10^{14} \cdot 18480$. Therefore $\mathrm{rd}_{E_s} \le q_1 d \le x^{12}$. Finally $|c_{\mathrm{CS}}(d)| \le C_P \le L$, as C_P is $|c_{\mathrm{CS}}(d)|$ plus non-negative terms; so $\log \mathrm{rd}_{E_s} + |c_{\mathrm{CS}}(d)| \le 12L + L = 13L$. □

Lemma 5.2.14. *Let $P = (q_0, d)$ be a parameter pair and $s \ge 0$ an integer.*

(i) For every $v \in \Phi_s$ there is a real number $A_v > 0$ with $|\log A_v| \le C_P$ and $|\mathrm{cyc}'_{q_1}(\zeta)|_v = A_v\, T(s)$.

(ii) If s is the exponent of admissible data over P with scale x, then every $\varpi \in E_s$ with $|\varpi|_v = \sqrt{d}\, |\mathrm{cyc}'_{q_1}(\zeta)|_v$ for all $v \in \Phi_s$ satisfies $x^{10} \le |\varpi|_v \le x^{13}$ for all $v \in \Phi_s$.

Proof. The cyclotomic polynomial of order q_1. By definition $q_1 = q_0 T(s)$ with $T(s) = 3\, l_0(s)\, 2^s$, and $l_0(s)$ is a power of 3; so every prime factor of $T(s)$ is 2 or 3, and both divide q_0, since $6 \mid 18480 \mid q_0$. For an integer $n \ge 1$ and a prime $p \mid n$ one has $\mathrm{cyc}_{pn}(X) = \mathrm{cyc}_n(X^p)$: both sides are monic of degree $\varphi(pn) = p\varphi(n)$, and a complex number η is a root of the right side if and only if η^p has order n, which (as $p \mid n$) holds if and only if η has order pn; the right side has the $\varphi(pn)$ primitive pn-th roots of unity as roots, hence no multiple roots. Writing $T(s) = p_1 \cdots p_t$ as a product of primes and applying this to $n = q_0 p_1 \cdots p_{j-1}$ and $p = p_j$ for $j = 1, \dots, t$ (each p_j divides q_0, hence n),

$$\mathrm{cyc}_{q_1}(X) = \mathrm{cyc}_{q_0}\big(X^{T(s)}\big), \qquad \mathrm{cyc}'_{q_1}(X) = T(s)\, X^{T(s)-1}\, \mathrm{cyc}'_{q_0}\big(X^{T(s)}\big).$$

(i). Let $v \in \Phi_s$, let ι be an embedding inducing v and $z = \iota(\zeta)$, a primitive q_1-th root of unity in $\mathbb{C}$. As cyc_{q_1} has integer coefficients, $\iota(\mathrm{cyc}'_{q_1}(\zeta)) = \mathrm{cyc}'_{q_1}(z)$, and $|z| = 1$, so the display gives

$$|\mathrm{cyc}'_{q_1}(\zeta)|_v = T(s)\, |\mathrm{cyc}'_{q_0}(w)|, \qquad w = z^{T(s)}.$$

The order of w is $q_1/T(s) = q_0$, so w is a primitive q_0-th root of unity. Put $A_v = |\mathrm{cyc}'_{q_0}(w)|$. The polynomial cyc_{q_0} divides $X^{q_0} - 1$, which has no multiple roots in $\mathbb{C}$, so $\mathrm{cyc}'_{q_0}(w) \ne 0$ and $A_v > 0$. The number $|\log A_v|$ is one of the summands of $\sum_{w'} |\log |\mathrm{cyc}'_{q_0}(w')||$ (over the primitive q_0-th roots of unity w'), all of which are non-negative, and the other summands $c_{\mathrm{vac}}(P)$, $2d+6$, q_0, $|c_{\mathrm{CS}}(d)|$ of C_P are non-negative; hence $|\log A_v| \le C_P$.

(ii). By Lemma 2.3.8, $x^{11} \le 2^s \le T(s)$ (as $3 l_0(s) \ge 1$) and $T(s) \le q_1 \le x^{12}$. By Lemma 4.2.67 (1), $1 \le d \le C_P$, and by Lemma 2.3.6, $10^{14} C_P \le \log x$; so $e^{C_P} \le x^{1/2} \le x$ and $\sqrt{d} \le d \le C_P \le \log x \le x^{1/2}$. With (i), for every $v \in \Phi_s$,

$$|\varpi|_v = \sqrt{d}\, A_v\, T(s) \;\ge\; 1 \cdot e^{-C_P} \cdot x^{11} \;\ge\; x^{10}, \qquad |\varpi|_v \;\le\; x^{1/2} \cdot x^{1/2} \cdot x^{12} = x^{13}.$$

□

The main term is bounded by Theorem 5.2.15: for a set S_g of chosen places above p_0 and frame vectors d, the link moment of the terms with nonzero term discriminant over the open elements of the fixer $\mathrm{Fix}(S_g, d)$ is at most $(1 + b^{-4})\, b^{-6k\#S_g}\, \mathcal{D}_{\mathrm{main}}\, \nu^2$. The saving of b^{6k} for each chosen place at which a frame vector is fixed is what the downward induction in the proof of Theorem 5.1.13 consumes.

The key idea is to use Siegel's formula as an upper bound. An open element puts every term over it in the big cell, so the amplitude sum of the terms with a given link becomes a sum, over heads and one last entry, of coset amplitudes (Proposition 5.2.16 (i); for the Bruhat decomposition see [36]). Each coset amplitude unfolds into a Kloosterman sum [162] at a finite set T of places times an archimedean orbital integral. For each last entry the squared class sums are counted by Siegel's formula [280] for unitary groups in Weil's convergent range [323] (Theorem A.4.26) and bounded by a last-entry density at T (Proposition 5.2.16 (ii); for local densities and hermitian forms see [99, 275]). What remains is a sum over heads and last entries of a product of local forms. It is bounded by one-sided Poisson summation on the adeles [294, 296, 324], one head entry at a time (Proposition 5.2.97): frequency 0 gives the product of the mass bounds, and the other frequencies are small because the forms decay. Forms that descend by integration over a head entry and decay in the frequency are the sieve systems of Definition 5.1.5: the band limits of the Bessel systems at the infinite places, and at T the floor forms of a Bessel vector of an intermediate group, normalised by its mass ratio (Proposition 5.2.81). The saving enters through the mass bound of this finite system: the fixer of a frame vector at a chosen place has large index, and a subgroup of large index gains a power of b in the ratio of the mass integral to the volume (Lemma 5.2.86). In our reading the iteration over head entries, with frequency 0 giving the mass and the others an error controlled by decay, plays the part that the duality of the large sieve plays in counting problems [29, 213].

The statements below follow this order: the Siegel step, the sieve system at the finite places, and the one-sided Poisson sieve. The two closing lemmas are bookkeeping: the product of the top band limits is the Gram matrix of orbital integrals that the Siegel step requires (Lemma 5.2.105), and the constants combine to the main moment (Lemma 5.2.106).

Theorem 5.2.15. *Let $\mathfrak{S}$ be admissible with Weil data, $S_g \subseteq S_0$ and $d = (d_w)_w$ frame vectors. Then*

$$\mathrm{LM}(\{\gamma : \mathsf{g}(\gamma) \neq 0\},\ \{s \in \mathrm{Fix}(S_g, d) : s \text{ open}\}) \leq (1 + b^{-4})\, b^{-6k\#S_g}\, \mathcal{D}_{\mathrm{main}}\, \nu^2.$$

Proof. Write $\Omega = \{s \in \mathrm{Fix}(S_g, d) : s \text{ open}\}$, $\Gamma_L = \{\gamma \in G(E) : \mathsf{g}(\gamma) \neq 0,\ \mathrm{link}(\gamma) = L\}$ and $Y = (1 + b^{-4}) b^{-6k\#S_g} \mathcal{D}_{\mathrm{main}} \nu^2$. By (TA3), $\mathrm{LM}(\{\mathsf{g} \neq 0\}, \Omega) = \sum_{c,c'} \sum_{s \in \Omega} \sum_L \|A^{\mathrm{cut}}_{\Gamma_L}(c, s, c')\|^2$.

Step 1: the places T. The entry set $\mathcal{T}$ is finite: its elements are quotients of elements of $\mathcal{O}_E$ whose conjugates are all bounded (each embedding of E is some $v \in \Phi$ or its conjugate), and there are finitely many such integers. Let T be the set of finite places w such that w or $\bar{w}$ lies above p_0, p_1 or N, or some element of $\mathcal{T}$ is not a unit at w or at $\bar{w}$. Then T is finite and stable under conjugation, every element of $\mathcal{T}$ is a T-unit, and M^{-1}, N^{-1} are T-integers. The lattice basis differs from the standard basis only in the vector $e_k/\bar{\pi} + z/N$, so the entries of it and of its inverse lie in $\{0, \pm 1, \bar{\pi}^{\pm 1}, \pm 1/N, \pm\bar{\pi}/N\}$; by (FG2) π is a unit at every finite place not above N, so both are integral at every finite place outside T.

Step 2: finitely many blocks. The sum above has nonnegative terms, so it suffices to show $\sum_{j \in \mathcal{F}} \|A_j\|^2 \leq Y$ for every finite set $\mathcal{F}$ of tuples $j = (c_j, c'_j, s_j, L_j)$ with $s_j \in \Omega$ and $A_j = A^{\mathrm{cut}}_{\Gamma_{L_j}}(c_j, s_j, c'_j) \neq 0$.

Step 3: open elements put the terms in the big cell. Let $s \in \Omega$ and let γ be a term over (c, s, c'), so $\kappa = h_c^{-1} \gamma h_{c'} \in K$ and $\mathrm{red}(\kappa) = s$. The frame vectors are orthogonal to X_0, on whose

orthogonal complement $h_c, h_{c'} \in \mathrm{U}(X_0)(\mathbb{A}_f)$ act trivially; hence $h(f_a, \gamma f_b) = h(h_c f_a, \gamma h_{c'} f_b) = h(f_a, \kappa f_b)$. The frame vectors belong to the lattice basis, in which κ is integral and the Gram matrix of h is integral (the vector $e_k/\bar{\pi} + z/N$ is isotropic, pairs to -1 with z and is orthogonal to the other basis vectors). So each $D_j(\gamma)$, $0 \le j \le \varrho$, lies in $\mathcal{O}_E$, and its image in k_w at a chosen w above p_0 is the leading minor of order j of $Z_w(s)$. As s is open, at some such w all these minors are nonzero; hence $D_j(\gamma) \neq 0$ for $0 \le j \le \varrho$.

Step 4: naming the blocks. For $j \in \mathcal{F}$, Steps 1 and 3 give the hypotheses of Proposition 5.2.16 (i) for (c_j, s_j, c_j', L_j). It yields $t_j \in \mathcal{T}$, $\sigma_j \in G(E)$ fixing every e_a, e_a^*, and $\kappa_j \in K$ with $h(z, \sigma_j z) = t_j$, $N^2 < |v(t_j)|^2$ for all v, $h_{c_j}^{-1} \sigma_j h_{c_j'}$ integral off T, $\mathrm{red}(\kappa_j) = s_j$, $L_j = \mathrm{link}(w_{\mathbf{1}} \sigma_j)$ and

$$A_j = \sum_{p \in \mathcal{T}^\varrho} \mathrm{ca}(c_j, s_j, c_j', w_{\mathrm{pad}(p, t_j)} \sigma_j).$$

Since $s_j = \mathrm{red}(\kappa_j)$ and L_j is determined by σ_j, the tuples $(c_j, c_j', \mathrm{red}(\kappa_j), \sigma_j)$, $j \in \mathcal{F}$, are pairwise distinct.

Step 5: the system at T. By Proposition 5.2.81 (i) there is a filling datum (P, F) at T; by (ii), applied to it, there is an intermediate group Z of F containing the T-component of every κ_j (as $\mathrm{red}(\kappa_j) = s_j \in \mathrm{Fix}(S_g, d)$), and the forms J_i^T of (ii) are a sieve system of length ϱ on E_T with the data required in Proposition 5.2.97 (ii), mass bound $m_T = (1 + b^{-6}) b^{-6k \# S_g}$ and top form $J_n^T(p, q) = \mathrm{led}(\pi, Z, p, q)/\mathrm{mr}(Z)$.

Step 6: Siegel's step for each last entry. Let $U = \{t_j : j \in \mathcal{F}\} \subseteq \mathcal{T}$ and, for $t \in U$, $\mathcal{F}_t = \{j \in \mathcal{F} : t_j = t\}$. For $p \in \mathcal{T}^\varrho$ put $a_p = \mathrm{pad}(p, t)$, let $y(p, t)$ be the image of a_p in E_T, and

$$R_{pq}^{(t)} = \prod_{v \in \Phi} J_n^{(v)}(x_v(p, t), x_v(q, t)).$$

By Lemma 5.2.105 applied to each σ_j, $j \in \mathcal{F}_t$, $R_{pq}^{(t)} = \langle \rho_p O(w_{a_p} \sigma_j), \rho_q O(w_{a_q} \sigma_j) \rangle$. Next,

$$\rho_p \, \mathrm{hdn}_\varrho(y(p, t), \mathbf{1}) = c_{\mathrm{nm}} \big(\prod_v (|v(t)|^2 - N^2)^{k-1} \big)^{1/2}.$$

Indeed $\mathrm{fs}(v, \varrho) = 1/N$, so $x_v(p, t)_\varrho = v(t)/N$ and $(|x_v(p, t)_\varrho|^2 - 1)_+^{(k-1)/2} = N^{-(k-1)} (|v(t)|^2 - N^2)^{(k-1)/2}$; for $a < \varrho$, $|x_v(p, t)_a| = \mathrm{fs}(v, a) |v(p_a)|$; and $\mathrm{hdn}_\varrho(y(p, t), \mathbf{1}) = \prod_{a < \varrho} |p_a|_T^{(k+1+2(\varrho - a - 1))/2}$. As p_a is a T-unit, the product formula gives

$$|p_a|_T = \prod_{w \in T} |p_a|_w = \prod_{w \nmid \infty} |p_a|_w = \prod_{v \in \Phi} |v(p_a)|^{-2},$$

so the powers of $|v(p_a)|$ cancel, and with $k + 1 + 2(\varrho - a - 1) = k - 1 + 2(\varrho - a)$, $\mathrm{fs}(v, \varrho)^{k-1} = N^{-(k-1)}$ and $\prod_v \# I_v = \# I$ the product is $c_{\mathrm{nm}} (\prod_v (|v(t)|^2 - N^2)^{k-1})^{1/2}$. The remaining hypotheses of Proposition 5.2.16 (ii) for $(c_j, c_j', \kappa_j, \sigma_j)_{j \in \mathcal{F}_t}$ and the heads $(a_p)_{p \in \mathcal{T}^\varrho}$ hold by Steps 1, 4 and 5. Since $\mathrm{red}(\kappa_j) = s_j$, it gives

$$\sum_{j \in \mathcal{F}_t} c_{\mathrm{nm}}^2 \|A_j\|^2 \le (1 + b^{-6}) \, c_S^\sharp \, \mathrm{Re} \sum_{p, q \in \mathcal{T}^\varrho} R_{pq}^{(t)} \, J_n^T(y(p, t), y(q, t)).$$

Step 7: the sieve. Summing over $t \in U$ and applying Proposition 5.2.97 (ii) with $A = \mathcal{T}$ (its elements and their inverses are T-integers by Step 1), the set $U \subseteq \mathcal{T}$ and the system J^T of Step 5, and using $c_S^\sharp \ge 0$,

$$c_{\mathrm{nm}}^2 \sum_{j \in \mathcal{F}} \|A_j\|^2 \le (1 + b^{-6}) \, c_S^\sharp \prod_{a=0}^{\varrho} \mathrm{pc}(a) \Big[\prod_v \frac{1 + b^{-10}}{\deg_v} \, m_T + b^{2403 n^9 f} \mathrm{dc} \Big].$$

Step 8: the constants. As $\#S_g \le \#S_0 = f$, Lemma 5.2.106 with $s = \#S_g$ bounds the right side by $c_{\mathrm{nm}}^2 Y$. Since $c_{\mathrm{nm}} > 0$, $\sum_{j\in\mathcal{F}} \|A_j\|^2 \le Y$. □

Proposition 5.2.16. *Let $\mathfrak{S}$ be admissible with Weil data, and let T be a finite set of finite places of E, stable under complex conjugation, such that N^{-1} and M^{-1} are T-integers and the lattice basis and its inverse are integral at every finite place outside T. An element of $G(E)$ or of $G(\mathbb{A}_f)$ is **integral off** T if its matrix in the lattice basis has entries integral at every finite place outside T. For a sequence $p = (p_0, p_1, \dots)$ in $E^\times$ let $w_p \in G(E)$ send $e_{a+1} \mapsto p_a e_{a+1}^*$ and $e_{a+1}^* \mapsto \bar{p}_a^{-1} e_{a+1}$ for $0 \le a < \varrho$ and fix X_0 and z; for $p \in (E^\times)^\varrho$ and $t \in E^\times$ put $\mathrm{pad}(p,t) = (p_0, \dots, p_{\varrho-1}, t, 1, 1, \dots)$, and let $\mathbf{1}$ be the sequence with all entries 1. The **entry set** $\mathcal{T} \subseteq E^\times$ is the set of quotients δ/δ' of nonzero $\delta, \delta' \in \mathcal{O}_E$ with $|v(\delta)|^2, |v(\delta')|^2 \le b^{270n^2}$ for all $v \in \Phi$. The **coset amplitude** of $\gamma \in G(E)$ over (c, s, c') is*

$$\mathrm{ca}(c, s, c', \gamma) = \sum_{u,u' \in \mathsf{U}(E)} \mathbf{1}[u\gamma u'^{-1} \text{ is a term over } (c, s, c')]\, A^{\mathrm{cut}}_{u\gamma u'^{-1}}(c, c') \in \mathbb{C}^{I\times I},$$

*and for $\gamma \in G(E)$ the **orbital integral** is $O(\gamma) = \int_{G_\infty} W^{\mathrm{cut}}(y) \otimes \overline{W^{\mathrm{cut}}(\gamma^{-1}y)}\, dy \in \mathbb{C}^{I\times I}$.*

(i) Global to local. Assume that every element of $\mathcal{T}$ is a T-unit. Let $c, c' \in \mathcal{C}$, $s \in \bar{G}$ and a submodule L be such that $D_j(\gamma) \ne 0$ for $0 \le j \le \varrho$ and every term γ over (c, s, c'), and such that $A^{\mathrm{cut}}_{\Gamma_L}(c, s, c') \ne 0$, where $\Gamma_L = \{\gamma \in G(E) : \mathfrak{g}(\gamma) \ne 0,\ \mathrm{link}(\gamma) = L\}$. Then there are $t \in \mathcal{T}$, $\sigma \in G(E)$ and $m \in K$ such that σ fixes every e_a and e_a^*; $h(z, \sigma z) = t$; $N^2 < |v(t)|^2$ for all $v \in \Phi$; $h_c^{-1}\sigma h_{c'}$ is integral off T; $\mathrm{red}(m) = s$; $L = \mathrm{link}(w_{\mathbf{1}}\sigma)$; and

$$A^{\mathrm{cut}}_{\Gamma_L}(c, s, c') = \sum_{p \in \mathcal{T}^\varrho} \mathrm{ca}(c, s, c', w_{\mathrm{pad}(p,t)}\sigma).$$

as an upper bound. Let (P, F) be a filling datum at T and Z an intermediate group of F. Let j be a nonempty finite set, and for $i \in j$ let $c_i, c_i' \in \mathcal{C}$, $m_i \in K$ and $\gamma_i \in G(E)$ be such that the tuples $(c_i, c_i', \mathrm{red}(m_i), \gamma_i)$ are pairwise distinct, each γ_i fixes every e_a and e_a^*, $h(z, \gamma_i z) = t$ for one $t \in E^\times$ with $N^2 < |v(t)|^2$ for all $v \in \Phi$, $h_{c_i}^{-1}\gamma_i h_{c_i'}$ is integral off T, and the T-component of m_i lies in Z. Let J be a finite set and $(a_p)_{p\in J}$ sequences in $E^\times$ with $a_p(\varrho) = t$ and $a_p(l)$, $a_p(l)^{-1}$ T-integers for $l < \varrho$, and let q_p be the image of a_p in E_T. Let $R \in \mathbb{C}^{J\times J}$ and $\rho \in \mathbb{R}^J$ satisfy, for all $i \in j$ and $p, q \in J$,

$$R_{pq} = \langle \rho_p O(w_{a_p}\gamma_i),\, \rho_q O(w_{a_q}\gamma_i)\rangle, \qquad \rho_p \,\mathrm{hdn}_\varrho(q_p, \mathbf{1}) = c_{\mathrm{nm}}\Big(\prod_{v\in\Phi} (|v(t)|^2 - N^2)^{k-1}\Big)^{1/2},$$

with $\langle\cdot,\cdot\rangle$ the inner product of $\mathbb{C}^{I\times I}$, linear in the second variable, and hdn_ϱ the head density of F. Then $0 < \mathrm{mr}(Z) < \infty$ and

$$\sum_{i\in j} c_{\mathrm{nm}}^2 \Big\| \sum_{p\in J} \mathrm{ca}(c_i, \mathrm{red}(m_i), c_i', w_{a_p}\gamma_i)\Big\|^2 \le (1 + b^{-6})\, c_S^\sharp \,\mathrm{Re} \sum_{p,q\in J} R_{pq}\, \frac{\mathrm{led}(\pi, Z, q_p, q_q)}{\mathrm{mr}(Z)}.$$

Proof. **(i).** *Step 1: the entry set is finite and the sums converge.* An element of $\mathcal{O}_E$ all of whose conjugates have absolute value at most b^{135n^2} (every complex embedding of E is some $v \in \Phi$ or its conjugate) is a root of a monic integer polynomial of degree $[E : \mathbb{Q}]$ with bounded coefficients; so there are finitely many such elements, and $\mathcal{T}$ is finite. Let $\Gamma = \{\gamma \in \Gamma_L : \gamma \text{ a term over } (c, s, c')\}$. Since $\|A^{\mathrm{cut}}_\gamma(c, c')\|$ is at most the amplitude majorant of γ, Lemma 5.2.29 (ii) (with $T = \Gamma_L$ there

and the cut functions) gives $\sum_{\gamma\in\Gamma}\|A^{\mathrm{cut}}_{\gamma}(c,c')\|<\infty$; so $A^{\mathrm{cut}}_{\Gamma_L}(c,s,c')=\sum_{\gamma\in\Gamma}A^{\mathrm{cut}}_{\gamma}(c,c')$ converges absolutely and may be summed in any order and grouping.

Step 2: the element σ. As $A^{\mathrm{cut}}_{\Gamma_L}(c,s,c')\neq 0$, some $\gamma_0\in\Gamma$ has $A^{\mathrm{cut}}_{\gamma_0}(c,c')\neq 0$. By hypothesis $D_j(\gamma_0)\neq 0$ for $1\le j\le\varrho$, so by Lemma 5.2.17 (iii) $\gamma_0=u_0^{-1}w_{p^0}\sigma u_0'$ with $u_0,u_0'\in\mathsf{U}(E)$, $\sigma\in G^\circ$ and $p^0\in(E^\times)^\varrho$. By Lemma 5.2.25 (i) and Lemma 5.2.17 (v), $L=\mathrm{link}(\gamma_0)=\mathrm{link}(w_{p^0}\sigma)=\mathrm{link}(w_{\mathbf{1}}\sigma)$.

Step 3: the last entry. Put $t=h(z,\sigma z)$. By Lemma 5.2.25 (ii) and Lemma 5.2.17 (i), $\mathsf{g}(\gamma_0)=\mathsf{g}(w_{p^0}\sigma)=d\bar d\,(t\bar t-N^2)$ with $d=p^0_0\cdots p^0_{\varrho-1}$; as $\gamma_0\in\Gamma_L$, $\mathsf{g}(\gamma_0)\neq 0$, so $t\bar t\neq N^2$. Since σ fixes the e_a,e_a^*, it maps $X_0\oplus Ez$ onto itself; write $\sigma z=\omega+\alpha z$ with $\omega\in X_0$, $\alpha\in E$. Then $t=\alpha h(z,z)=-N\alpha$, and $-N=h(\sigma z,\sigma z)=h(\omega,\omega)-N\alpha\bar\alpha$, so

$$N\,h(\omega,\omega)=N^2\alpha\bar\alpha-N^2=t\bar t-N^2.$$

With $\omega=(\omega_j)_{1\le j\le k}$ and the unit form, $v(h(\omega,\omega))=\sum_j|v(\omega_j)|^2\ge 0$ for $v\in\Phi$; and $v(t\bar t-N^2)\neq 0$. Hence $|v(t)|^2-N^2=N\sum_j|v(\omega_j)|^2>0$, that is $N^2<|v(t)|^2$, for every $v\in\Phi$; in particular $t\neq 0$.

Step 4: the properties of σ, t and m. By Lemma 5.2.19 (ii) applied to γ_0, every p^0_a and t lie in $\mathcal{T}$. By hypothesis the elements of $\mathcal{T}$ are T-units, so the $(p^0_a)^{-1}$ are T-integers, and Lemma 5.2.18 (i) shows that $h_c^{-1}\sigma h_{c'}$ is integral off T. As γ_0 is a term over (c,s,c'), $m=h_c^{-1}\gamma_0 h_{c'}\in K$ and $\mathrm{red}(m)=s$. Together with Steps 2 and 3, t, σ and m have all the properties claimed.

Step 5: the terms are the big-cell elements over σ. Define $\delta\colon\mathcal{T}^\varrho\times\mathsf{U}(E)\times\mathsf{U}(E)\to G(E)$ by $\delta(p,u,u')=u\,w_{\mathrm{pad}(p,t)}\sigma\,u'^{-1}=u\,w_p\sigma\,u'^{-1}$. It is injective by Lemma 5.2.17 (ii) (applied to u^{-1}, u'^{-1}). Every value lies in Γ_L: by Lemma 5.2.25 (i), (ii) and Lemma 5.2.17 (v), (i), $\mathrm{link}(\delta(p,u,u'))=\mathrm{link}(w_p\sigma)=L$ and $\mathsf{g}(\delta(p,u,u'))=\mathsf{g}(w_p\sigma)=(\prod_a p_a\bar p_a)(t\bar t-N^2)\neq 0$. Conversely let $\gamma\in\Gamma$ with $A^{\mathrm{cut}}_{\gamma}(c,c')\neq 0$. By hypothesis and Lemma 5.2.17 (iii), $\gamma=u^{-1}w_q\tau u'$ with $u,u'\in\mathsf{U}(E)$, $\tau\in G^\circ$, $q\in(E^\times)^\varrho$; by Lemma 5.2.25 (i), $\mathrm{link}(w_q\tau)=\mathrm{link}(\gamma)=L=\mathrm{link}(w_{\mathbf{1}}\sigma)$, so $\tau=\sigma$ by Lemma 5.2.17 (v); and $h(z,\tau z)=t\neq 0$, so $q\in\mathcal{T}^\varrho$ by Lemma 5.2.19 (ii). Hence $\gamma=\delta(q,u^{-1},u'^{-1})$.

Step 6: the sum. Let $a(\gamma)=A^{\mathrm{cut}}_{\gamma}(c,c')$ if γ is a term over (c,s,c') and $a(\gamma)=0$ otherwise. By Step 5, a vanishes on Γ_L outside the image of the injection δ, whose image lies in Γ_L. With the absolute convergence of Step 1,

$$A^{\mathrm{cut}}_{\Gamma_L}(c,s,c')=\sum_{\gamma\in\Gamma_L}a(\gamma)=\sum_{(p,u,u')}a(\delta(p,u,u'))=\sum_{p\in\mathcal{T}^\varrho}\ \sum_{u,u'\in\mathsf{U}(E)}a\big(u\,w_{\mathrm{pad}(p,t)}\sigma\,u'^{-1}\big),$$

the inner sums converging absolutely. The inner sum is

$$\sum_{u,u'}\mathbf{1}[u\,w_{\mathrm{pad}(p,t)}\sigma\,u'^{-1}\text{ is a term over }(c,s,c')]\,A^{\mathrm{cut}}_{u\,w_{\mathrm{pad}(p,t)}\sigma\,u'^{-1}}(c,c')=\mathrm{ca}(c,s,c',w_{\mathrm{pad}(p,t)}\sigma),$$

and the outer sum is finite by Step 1. This proves (i).

(ii). Write t also for the image of t in E_T, $G_t=\prod_{v\in\Phi}(|v(t)|^2-N^2)^{k-1}>0$, $w=c_{\mathrm{nm}}G_t^{1/2}$ and $S_i=\sum_{p\in J}\mathrm{ca}(c_i,\mathrm{red}(m_i),c_i',w_{a_p}\gamma_i)$. Fix $i_0\in j$ and put $y_p=\rho_pO(w_{a_p}\gamma_{i_0})$, so $R_{pq}=\langle y_p,y_q\rangle$; for $u\in\mathbb{C}^J$, $\sum_{p,q}\bar u_pR_{pq}u_q=\|\sum_p u_py_p\|^2\ge 0$, so R is hermitian and positive semidefinite.

Step 7: the mass ratio. By Lemma 5.5.44 (c), (d), applied to the large lattice datum F, the set $\mathrm{Cos}(Z)$ is finite and $0<\mathrm{mr}(Z)<\infty$.

Step 8: a step function. The entries of each a_p are nonzero, so q_p is a sequence of units of E_T. For $p,q\in J$ let $\Phi_{pq}\colon V\otimes E_T\to\mathbb{C}$ be the function of Lemma 5.2.74 (v) for F, Z and q_p,q_q, and put

$$\Psi(x)=\max\Big(0,\ \mathrm{Re}\sum_{p,q\in J}R_{pq}\Phi_{pq}(x)\Big).$$

Then $\Psi \geq 0$ vanishes outside the union of the compact supports of the Φ_{pq}; $\Psi(x+v) = \Psi(x)$ for all x and all v in the intersection of the finitely many neighbourhoods of 0 given by Lemma 5.2.74 (v), so Ψ is locally constant; $\Psi(kx) = \Psi(x)$ for $k \in H \cap \mathrm{K}_T(M)$; and $\Psi = \max(0, \mathrm{Re} \sum_{p,q} R_{pq} \mathrm{fp}_{q_p,q_q})$ on Q. Let $m \in \mathrm{K}_X$. By (FA1), m lies in the principal level $\mathrm{K}(M)$, so $m_T \in \mathrm{K}_T(M)$ by Lemma 5.2.46 (i); and $m \in \mathrm{U}(X_0)(\mathbb{A}_f)$ fixes e_a, e_a^* and z, which belong to the lattice basis, so $m_T \in H$. Hence $\Psi(m_T x) = \Psi(x)$.

Step 9: positivity, and the sphere integral. First, $\mathrm{Re} \sum_{p,q} R_{pq} \mathrm{fp}_{q_p,q_q}(e) \geq 0$ for every $e \in V \otimes E_T$: this is Lemma 5.2.48 for F, Z, the positive semidefinite R, the sequences $(q_p)_{p \in J}$, the point e and the empty family ($I = \emptyset$). Next let $\beta = \delta(t\bar{t}/N - N)$ and $\mu = \mu_\beta^{\mathrm{nf}}$. Lemma 5.2.78 applies to $\mathfrak{S}$, $A = E_T$, π, the place datum F on J_{lat}, as $\pi\bar{\pi} = N$, $k \geq 2$ by (FG6) and some $c \in E_T$ has $\bar{c}c = -1$ by Lemma 5.2.80 (b); by its last assertion, μ-almost every ω has $e(\pi, t, \omega) \in Q$, where $\mathrm{fp}_{q_p,q_q} = \Phi_{pq}$. The map $\omega \mapsto e(\pi, t, \omega)$ is injective and affine, with a continuous affine left inverse (the coordinates at X_0), so each $\omega \mapsto \Phi_{pq}(e(\pi, t, \omega))$ is continuous with compact support and, by Lemma 5.2.79 (iii), μ-integrable; hence so is $\omega \mapsto \mathrm{fp}_{q_p,q_q}(e(\pi, t, \omega))$, which agrees with it μ-almost everywhere. As $q_p(\varrho) = t$ for every p, (FB4) and linearity give

$$\Sigma_R := \sum_{p,q} R_{pq} \operatorname{led}(\pi, Z, q_p, q_q) = |\pi|^{-1} \int_{E_T^k} \sum_{p,q} R_{pq} \, \mathrm{fp}_{q_p,q_q}\big(e(\pi, t, \omega)\big) \, d\mu(\omega).$$

Taking real parts, $\mathrm{Re}\, \Sigma_R = |\pi|^{-1} \int \mathrm{Re} \sum_{p,q} R_{pq} \mathrm{fp}_{q_p,q_q}(e(\pi, t, \omega)) \, d\mu(\omega)$. By the first part this integrand is ≥ 0, so μ-almost everywhere it equals $\Psi(e(\pi, t, \omega))$ (Step 8, as $e(\pi, t, \omega) \in Q$ almost everywhere). Therefore

$$\mathrm{Re}\, \Sigma_R = |\pi|^{-1} \int_{E_T^k} \Psi\big(e(\pi, t, \omega)\big) \, d\mu(\omega) \geq 0.$$

Step 10: the points, and the bound over one point. For $i \in j$ put $x_i = h_{c_i}^{-1} \gamma_i h_{c_i'}$. As γ_i fixes every e_a, e_a^*, it maps $X_0 \oplus Ez$ to itself; write $\gamma_i z = \omega_i + \alpha_i z$ with $\omega_i \in X_0$, $\alpha_i \in E$. Then $t = h(z, \gamma_i z) = -N\alpha_i$, and $h(\gamma_i z, \gamma_i z) = h(z, z) = -N$ gives $h(\omega_i, \omega_i) = N(\alpha_i \bar{\alpha}_i - 1) = \beta_0 := (t\bar{t} - N^2)/N$, so $\delta\beta_0 = \beta$. Put $g(i) = (c_i, \omega_i) \in \mathcal{C} \times X_0$ and $s = g(j)$. The h_c lie in $\mathrm{U}(X_0)(\mathbb{A}_f)$ and fix the e_a, e_a^*, z; so $x_i z = h_{c_i}^{-1} \omega_i - N^{-1} t z$, whose component at T is

$$e_{g(i)} := e\big(\pi, t, (h_{c_i}^{-1} \omega_i)_T\big) = (x_i)_T z,$$

and $(x_i)_T$ fixes every e_a, e_a^*, so $(x_i)_T \in G^{(\varrho)}$ and $e_{g(i)} \in Q$ by Lemma 5.2.75 (i). Fix $g_0 = (c, \omega) \in s$ and let $j_0 = \{i \in j : g(i) = g_0\}$. For $i \in j_0$, $c_i = c$ and $\gamma_i z = \omega - N^{-1} t z$ does not depend on i, and the triples $(c_i', \mathrm{red}(m_i), \gamma_i)$, $i \in j_0$, are pairwise distinct because the tuples of (ii) are. So Lemma 5.2.37 applies to c, the heads (a_p), ρ, w (as $\rho_p \operatorname{hdn}_\varrho(q_p, \mathbf{1}) = w$) and the family $(c_i', m_i, \gamma_i)_{i \in j_0}$, the remaining hypotheses being those of (ii):

$$\mu_H(\mathrm{K}_T(M)) \, \theta(\mathrm{K}_T(M))^2 \, \#\mathrm{Cos}(Z) \sum_{i \in j_0} w^2 \|S_i\|^2 \leq \mathrm{Re} \sum_{p,q} R_{pq} \, \mathrm{fp}_{q_p,q_q}(e_{g_0}).$$

By Lemma 5.2.66, $\mathrm{mr}(Z) \leq \mu_H(\mathrm{K}_T(M)) \, \theta(\mathrm{K}_T(M))^2 \, \#\mathrm{Cos}(Z) \, D$, where $D \geq 1$; and since $e_{g_0} \in Q$, the right side is at most $\Psi(e_{g_0})$ by Step 8. Hence

$$\frac{\mathrm{mr}(Z)}{D} \sum_{i \in j_0} w^2 \|S_i\|^2 \leq \Psi(e_{g_0}).$$

Step 11: Siegel's formula over the points. Let $(c, \omega) = g(i) \in s$. Then $h(\omega, \omega) = \beta_0$ (Step 10), and $h_c^{-1}\omega$ is integral at every finite place outside T: the column of x_i at z in the lattice

basis is integral off T, the matrix of the lattice basis in the standard basis is integral off T, so the standard coordinates of $x_i z = h_c^{-1}\omega - N^{-1}tz$ are integral off T, and those at X_0 are the coordinates of $h_c^{-1}\omega$. The function Ψ on $V \otimes E_T$ is nonnegative, locally constant, of compact support and invariant under m_T for $m \in \mathrm{K}_X$ (Step 8); E_T carries the finite place P with integers $\mathcal{O}_T$; and the norm-fibre measures on E_T^k form a disintegration of vol along $\bar{y}^{\mathsf{T}} y$ by (FB3). As $|v(t)| > N$ for all $v \in \Phi$ and N^{-1} is a T-integer, Lemma 5.2.68 applies to Ψ, t and s; with $(h_c^{-1}\omega)_T - N^{-1}tz = e_{g(i)}$, $y - N^{-1}tz = e(\pi, t, y)$ and $\delta\beta_0 = \beta$, and by Step 9,

$$\sum_{g_0 \in s} \Psi(e_{g_0}) \le (1 + b^{-6})\, \#\mathcal{C}\, c_S\, G_t\, |\pi|^{-1} \int_{E_T^k} \Psi(e(\pi, t, y))\, d\mu(y) = (1 + b^{-6})\, \#\mathcal{C}\, c_S\, G_t\, \mathrm{Re}\, \Sigma_R.$$

Step 12: conclusion. The sets j_0, $g_0 \in s$, partition j. Summing Step 10 over $g_0 \in s$ and using Step 11 and $w^2 = c_{\mathrm{nm}}^2 G_t$,

$$\frac{\mathrm{mr}(Z)}{D}\, G_t \sum_{i \in j} c_{\mathrm{nm}}^2 \|S_i\|^2 \le (1 + b^{-6})\, \#\mathcal{C}\, c_S\, G_t\, \mathrm{Re}\, \Sigma_R.$$

By Step 7, $G_t\, \mathrm{mr}(Z)/D$ is a positive real number; dividing by it and using $c_S^\sharp = D\, \#\mathcal{C}\, c_S$ (Lemma 5.2.66) gives

$$\sum_i c_{\mathrm{nm}}^2 \|S_i\|^2 \le (1 + b^{-6}) c_S^\sharp\, \mathrm{Re}\, \Sigma_R / \mathrm{mr}(Z),$$

and

$$\mathrm{Re}\, \Sigma_R / \mathrm{mr}(Z) = \mathrm{Re} \sum_{p,q} R_{pq}\, \mathrm{led}(\pi, Z, q_p, q_q)/\mathrm{mr}(Z).$$

With $0 < \mathrm{mr}(Z) < \infty$ from Step 7, this proves (ii). □

Lemma 5.2.17. *Let A be a commutative ring with an involution $a \mapsto \bar{a}$ in which 2 is a unit, and $\varrho \ge 1$. Let V be a free A-module with basis $f_0, \dots, f_{\varrho-1}, m_1, \dots, m_q, f_0^*, \dots, f_{\varrho-1}^*$ and $h\colon V \times V \to A$ a hermitian form (A-linear in the second variable, $h(y,x) = \overline{h(x,y)}$) with $h(f_a, f_b) = h(f_a^*, f_b^*) = 0$, $h(f_a, f_b^*) = \delta_{ab}$, the f_a, f_a^* orthogonal to $V^\circ = \bigoplus_i Am_i$, and $(h(m_i, m_j))_{i,j}$ invertible. Let $G = \mathrm{U}(V)$ be the group of A-linear automorphisms of V preserving h; $X = \langle f_0, \dots, f_{\varrho-1}\rangle$, $X_{<a} = \langle f_0, \dots, f_{a-1}\rangle$, $X_{>a}^* = \langle f_{a+1}^*, \dots, f_{\varrho-1}^*\rangle$;*

$$\mathsf{U}_A = \{u \in G :\ uf_a - f_a \in X_{<a},\ uf_a^* - f_a^* \in X \oplus V^\circ \oplus X_{>a}^*\ (0 \le a < \varrho),\ um - m \in X\ (m \in V^\circ)\};$$

$G^\circ \subseteq G$ the subgroup of elements fixing every f_a and f_a^*. For $0 \le j \le \varrho$ and $p = (p_0, \dots, p_{j-1}) \in (A^\times)^j$ let $w_p^{(j)} \in G$ send $f_a \mapsto p_a f_a^*$ and $f_a^* \mapsto \bar{p}_a^{-1} f_a$ for $a < j$ and fix the other basis vectors, and $w_p = w_p^{(\varrho)}$. Fix $z \in V^\circ$, put $f_\varrho = z$, $Z(\gamma) = (h(f_a, \gamma f_b))_{a,b=0}^{\varrho}$ and let $D_j(\gamma)$ be its leading principal minor of order j ($D_0 = 1$). For $t \in X^\perp = X \oplus V^\circ$ let $[t]$ be its component in V°, and $\mathrm{link}(\gamma) = \{([t], [\gamma^{-1}t]) : t \perp X,\ t \perp \gamma X\}$.

(i) For $\gamma \in G$ and $u, u' \in \mathsf{U}_A$: $D_j(u\gamma u') = D_j(\gamma)$ for $0 \le j \le \varrho + 1$. For $\sigma \in G^\circ$ and $p \in (A^\times)^\varrho$ the matrix $Z(w_p\sigma)$ is diagonal with entries $p_0, \dots, p_{\varrho-1}, t$, where $t = h(z, \sigma z)$; hence $D_j(w_p\sigma) = p_0 \cdots p_{j-1}$ for $j \le \varrho$, $D_{\varrho+1}(w_p\sigma) = dt$ with $d = p_0 \cdots p_{\varrho-1}$, and $D_{\varrho+1}\bar{D}_{\varrho+1} - N^2 D_\varrho \bar{D}_\varrho = d\bar{d}\,(t\bar{t} - N^2)$ at $w_p\sigma$ for every $N \in A$ with $\bar{N} = N$.

(ii) (Uniqueness.) If $u^{-1}w_p\sigma u' = v^{-1}w_q\tau v'$ with $u, u', v, v' \in \mathsf{U}_A$, $\sigma, \tau \in G^\circ$ and $p, q \in (A^\times)^\varrho$, then $p = q$, $\sigma = \tau$, $u = v$ and $u' = v'$.

*(iii) (Existence with integral factors.) Let $A_0 \subseteq A$ be a subring with $\overline{A_0} = A_0$, and $\mathbf{b}$ a basis of V consisting of the f_a, a basis of V° and the f_a^*, in which the Gram matrix of h has entries in A_0 and determinant in $A_0^\times$; an endomorphism of V is A_0-**integral** if its matrix in $\mathbf{b}$ has entries in A_0. If $\gamma \in G$ is A_0-integral and $D_j(\gamma) \in A_0^\times$ for $1 \le j \le \varrho$, then $\gamma = u^{-1} w_p \sigma u'$ with $u, u' \in \mathsf{U}_A$, $\sigma \in G^\circ$, $p \in (A_0^\times)^\varrho$, and u, u', σ A_0-integral. In particular, if A is a field, every $\gamma \in G$ with $D_j(\gamma) \neq 0$ for $1 \le j \le \varrho$ is $u^{-1} w_p \sigma u'$ with $u, u' \in \mathsf{U}_A$, $\sigma \in G^\circ$, $p \in (A^\times)^\varrho$.*

(iv) (Integrality of the factors.) With A_0, $\mathbf{b}$ as in (iii), let $u, u' \in \mathsf{U}_A$, $\sigma \in G^\circ$ and $p \in (A^\times)^\varrho$ with $p_a^{-1} \in A_0$ for all a. If $u^{-1} w_p \sigma u'$ is A_0-integral, then u, σ and u' are A_0-integral.

(v) (Links.) For $\sigma \in G^\circ$ and $p \in (A^\times)^\varrho$, $\operatorname{link}(w_p \sigma) = \{(m, \sigma^{-1} m) : m \in V^\circ\}$. Hence for $\sigma, \tau \in G^\circ$ and $p, q \in (A^\times)^\varrho$, $\operatorname{link}(w_p \sigma) = \operatorname{link}(w_q \tau)$ if and only if $\sigma = \tau$.

(vi) If A is a field, U_A is the unipotent radical of the stabiliser in G of the isotropic flag $\langle f_0 \rangle \subset \langle f_0, f_1 \rangle \subset \cdots \subset X$.

Proof. Throughout, vectors of V are written in the basis $f_0, \ldots, f_{\varrho-1}, m_1, \ldots, m_q, f_0^*, \ldots, f_{\varrho-1}^*$; for $y \in V$ the coefficient of f_a is $h(f_a^*, y)$ and that of f_a^* is $h(f_a, y)$ (the form is linear in the second variable and $h(f_a^*, f_b) = \overline{h(f_b, f_a^*)} = \delta_{ab}$). In particular $X^\perp = X \oplus V^\circ$ and the orthogonal complement of $X + X^*$, $X^* = \langle f_0^*, \ldots, f_{\varrho-1}^* \rangle$, is V°; so every $\sigma \in G^\circ$ maps V° onto itself.

The group U_A. The conditions defining U_A say that u maps each member of the chain of free direct summands

$$0 \subset X_{<1} \subset \cdots \subset X_{<\varrho} = X \subset X \oplus V^\circ \subset X \oplus V^\circ \oplus X^*_{>\varrho-2} \subset \cdots \subset X \oplus V^\circ \oplus X^*_{>-1} = V$$

into itself and induces the identity on each successive quotient (each quotient is spanned by the image of one basis vector, or by the images of the m_i). The elements of G with this property form a subgroup, since the property is preserved by products and, for an automorphism mapping each member onto itself with identity on the quotients, by inverses. So U_A is a subgroup of G.

(i). Let $u \in \mathsf{U}_A$. For $b < \varrho$, $u f_b - f_b \in \langle f_0, \ldots, f_{b-1} \rangle$, and $u f_\varrho - f_\varrho = uz - z \in X = \langle f_0, \ldots, f_{\varrho-1} \rangle$. So $u f_b = \sum_{a \le b} C_{ab}(u) f_a$ with an upper unitriangular $(\varrho+1) \times (\varrho+1)$ matrix $C(u)$. As $u^{-1} \in \mathsf{U}_A$, $h(f_a, u\gamma u' f_b) = h(u^{-1} f_a, \gamma u' f_b)$, and therefore

$$Z(u\gamma u') = C(u^{-1})^* \, Z(\gamma) \, C(u'),$$

C^* the conjugate transpose, which is lower unitriangular. For $j \le \varrho + 1$ the upper left $j \times j$ block of the right side is the product of the upper left $j \times j$ blocks of the three factors (the first is lower and the third upper triangular), and those of $C(u^{-1})^*$ and $C(u')$ have determinant 1. Hence $D_j(u\gamma u') = D_j(\gamma)$.

Let $\sigma \in G^\circ$, $p \in (A^\times)^\varrho$ and $\gamma = w_p \sigma$. From $w_p(\bar{p}_a f_a^*) = f_a$ we get $w_p^{-1} f_a = \bar{p}_a f_a^*$, and w_p^{-1} fixes V°. For $a, b < \varrho$: $\gamma f_b = w_p f_b = p_b f_b^*$, so $h(f_a, \gamma f_b) = p_b \delta_{ab}$ and $h(z, \gamma f_b) = 0$. Next $\sigma z \in V^\circ$ is fixed by w_p, so $h(f_a, \gamma z) = 0$ for $a < \varrho$, and $h(z, \gamma z) = h(z, \sigma z) = t$. So $Z(w_p \sigma)$ is diagonal with entries $p_0, \ldots, p_{\varrho-1}, t$, which gives the minors, and

$$D_{\varrho+1} \bar{D}_{\varrho+1} - N^2 D_\varrho \bar{D}_\varrho = dt\, \overline{dt} - N^2 d\bar{d} = d\bar{d}\,(t\bar{t} - N^2).$$

An element of U_A that fixes every f_b^* is 1. Let $y \in \mathsf{U}_A$ with $y f_b^* = f_b^*$ for all b. For $a < \varrho$, $y f_a \in X$ and $h(f_b^*, y f_a) = h(y f_b^*, y f_a) = h(f_b^*, f_a) = \delta_{ab}$; the coefficients of an element of X are its products with the f_b^*, so $y f_a = f_a$. For $m \in V^\circ$, $ym - m \in X$ and $h(f_b^*, ym) = h(f_b^*, m) = 0$ for all b, so $ym = m$. Hence y fixes a basis and $y = 1$.

(ii). Put $y = uv^{-1}$ and $y' = u'v'^{-1}$, both in U_A. The hypothesis is equivalent to $w_p\sigma y' = y\, w_q\tau$. Apply both sides to f_b, $b < \varrho$. On the left, $y'f_b = f_b + \sum_{a<b} c_a f_a$, σ fixes the f_a, and $w_p\sigma y' f_b = p_b f_b^* + \sum_{a<b} c_a p_a f_a^*$. On the right, $w_q\tau f_b = q_b f_b^*$ and $y f_b^* = f_b^* + r$ with $r \in X \oplus V^\circ \oplus X^*_{>b}$, so the right side is $q_b f_b^* + q_b r$. Comparing components: $q_b r = 0$, so $r = 0$ (q_b is a unit); $p_b = q_b$; and $c_a p_a = 0$ for $a < b$. Hence $p = q$ and $y f_b^* = f_b^*$ for every b, so $y = 1$ by the preceding paragraph, that is $u = v$. Then $w_p = w_q$ and $\sigma y' = \tau$, so $y' = \sigma^{-1}\tau$ lies in $G^\circ \cap \mathsf{U}_A$; it fixes every f_b^*, so $y' = 1$, that is $u' = v'$, and $\sigma = \tau$.

Integrality. Let S be the Gram matrix of h in $\mathbf{b}$, so $h(x,y) = x^*Sy$ in coordinates. For $\gamma \in G$, $\gamma^* S\gamma = S$, so $\gamma^{-1} = S^{-1}\gamma^* S$; as $S^{-1} = \det(S)^{-1}\mathrm{adj}(S)$ has entries in A_0 and $\overline{A_0} = A_0$, the inverse of an A_0-integral element of G is A_0-integral. For basis vectors x, y of $\mathbf{b}$ and an A_0-integral γ, $h(x, \gamma y) = (S\gamma)_{xy} \in A_0$.

Heisenberg elements. Let $0 \le j < \varrho$, $V_{>j}$ the span of the f_a, f_a^* with $a > j$ and of V°, and $P_{<j}$ the span of the f_a, f_a^* with $a < j$, so $V = P_{<j} \oplus \langle f_j, f_j^*\rangle \oplus V_{>j}$ orthogonally. For $w \in V_{>j}$ and $\beta \in A$ with $\beta + \bar\beta = 0$ put $\mu = \beta - \frac{1}{2}h(w,w)$ and let $\mathrm{hs}_j(w,\beta)$ be the A-linear map fixing $P_{<j}$ and f_j, with $y \mapsto y - h(w,y)f_j$ on $V_{>j}$ and $f_j^* \mapsto f_j^* + w + \mu f_j$. It preserves h: the vectors f_j, $y - h(w,y)f_j$ ($y \in V_{>j}$) have the products of f_j, y with each other and with $P_{<j}$, because f_j is isotropic and orthogonal to $V_{>j}$ and $P_{<j}$; $h(f_j, f_j^* + w + \mu f_j) = 1$; $h(f_j^* + w + \mu f_j, y - h(w,y)f_j) = h(w,y) - h(w,y)h(f_j^*, f_j) = 0$; and $h(f_j^* + w + \mu f_j, f_j^* + w + \mu f_j) = h(w,w) + \mu + \bar\mu = h(w,w) - h(w,w) + \beta + \bar\beta = 0$, using $\overline{h(w,w)} = h(w,w)$. It is invertible, as $\mathrm{hs}_j(w,\beta) - 1$ maps f_j^* into $V_{>j} + Af_j$, $V_{>j}$ into Af_j and f_j, $P_{<j}$ to 0, so its cube vanishes. So $\mathrm{hs}_j(w,\beta) \in G$; it fixes $P_{<j}$ pointwise, maps $V_{>j} \oplus \langle f_j, f_j^*\rangle$ into itself, and lies in U_A (it fixes the f_a with $a \le j$, sends f_a ($a > j$), $m \in V^\circ$ and f_a^* ($a > j$) to themselves plus a multiple of f_j, and f_j^* to $f_j^* + w + \mu f_j$ with $w \in X \oplus V^\circ \oplus X^*_{>j}$).

Conversely, let $y \in V$ be isotropic, orthogonal to $P_{<j}$, with $h(f_j, y) = 1$. Then $y = cf_j + f_j^* + w$ with $w \in V_{>j}$, $c \in A$, and $0 = h(y,y) = h(w,w) + c + \bar c$; with $\beta = c + \frac{1}{2}h(w,w)$ we have $\beta + \bar\beta = 0$ and $y = \mathrm{hs}_j(w,\beta)f_j^*$. If y has coordinates in A_0, then so has w, and $\mathrm{hs}_j(w,\beta)$ is A_0-integral: it fixes $P_{<j}$ and f_j, sends a basis vector y' of $\mathbf{b}$ in $V_{>j}$ to $y' - h(w,y')f_j$ with $h(w,y') = (w^*S)_{y'} \in A_0$, and sends f_j^* to y.

(iii). For $0 \le j \le \varrho$ let $G_j \subseteq G$ be the subgroup fixing f_a, f_a^* for $a < j$ (so $G_\varrho = G^\circ$). We show by induction on j: $\gamma = u^{-1} w_p^{(j)} \sigma u'$ with $u, u' \in \mathsf{U}_A$ and $\sigma \in G_j$ all A_0-integral and $p \in (A_0^\times)^j$. For $j = 0$ take $u = u' = 1$, $\sigma = \gamma$. Let $j < \varrho$ and such a decomposition be given.

The pivot $x = h(f_j, \sigma f_j)$ lies in A_0. The upper left $(j+1) \times (j+1)$ block of $Z(w_p^{(j)}\sigma)$ is $\mathrm{diag}(p_0, \dots, p_{j-1}, x)$: for $a, b < j$, $w_p^{(j)}\sigma f_b = p_b f_b^*$; for $a < j$, $h(f_a, w_p^{(j)}\sigma f_j) = h(\bar p_a f_a^*, \sigma f_j) = p_a h(\sigma^{-1}f_a^*, f_j) = p_a h(f_a^*, f_j) = 0$; $h(f_j, w_p^{(j)}\sigma f_b) = h(f_j, p_b f_b^*) = 0$ for $b < j$; and $h(f_j, w_p^{(j)}\sigma f_j) = h(f_j, \sigma f_j) = x$, as $w_p^{(j)}$ fixes f_j and preserves h. By (i), $D_{j+1}(\gamma) = D_{j+1}(w_p^{(j)}\sigma) = p_0 \cdots p_{j-1} x$, so $x^{-1} = p_0 \cdots p_{j-1} D_{j+1}(\gamma)^{-1} \in A_0$.

The vector $y_1 = x^{-1}\sigma f_j$ is isotropic, orthogonal to $P_{<j}$ (σ fixes $P_{<j}$, which is orthogonal to f_j), $h(f_j, y_1) = 1$, and has coordinates in A_0. So $y_1 = \nu_1 f_j^*$ with $\nu_1 = \mathrm{hs}_j(w_1, \beta_1)$ A_0-integral. Let $\omega = w^{(j+1)}_{(1,\dots,1,x)}$, which exchanges $f_j \mapsto x f_j^*$, $f_j^* \mapsto \bar x^{-1} f_j$ and fixes the other basis vectors; it is A_0-integral. Put $g = \sigma^{-1}\nu_1\omega \in G_j$, A_0-integral. Then $gf_j = \sigma^{-1}\nu_1(x f_j^*) = \sigma^{-1}(\sigma f_j) = f_j$. The vector $y_2 = g f_j^*$ is isotropic, orthogonal to $P_{<j}$, with $h(f_j, y_2) = h(gf_j, gf_j^*) = 1$ and coordinates in A_0, so $y_2 = \nu_2 f_j^*$ with $\nu_2 = \mathrm{hs}_j(w_2, \beta_2)$ A_0-integral. Then $\sigma' = g^{-1}\nu_2$ is A_0-integral, fixes f_j^* ($\sigma' f_j^* = g^{-1} y_2$), f_j and $P_{<j}$, so $\sigma' \in G_{j+1}$, and $\sigma = \nu_1 \omega g^{-1} = \nu_1 \omega \sigma' \nu_2^{-1}$.

The elements $w_p^{(j)}$ and ν_1 commute: both map $P_{<j}$ and $\langle f_j, f_j^*\rangle \oplus V_{>j}$ into themselves, $w_p^{(j)}$ is the identity on the second and ν_1 on the first. Also $w_p^{(j)}\omega = w_{p'}^{(j+1)}$ with $p' = (p_0, \dots, p_{j-1}, x) \in$

$(A_0^\times)^{j+1}$. Hence

$$\gamma = u^{-1} w_p^{(j)} \nu_1 \omega \sigma' \nu_2^{-1} u' = (\nu_1^{-1}u)^{-1}\, w_{p'}^{(j+1)}\, \sigma'\, (\nu_2^{-1}u'),$$

with $\nu_1^{-1}u, \nu_2^{-1}u' \in \mathsf{U}_A$ A_0-integral. This completes the induction; $j = \varrho$ is the claim. If A is a field, take $A_0 = A$: the Gram matrix in the given basis is invertible, and nonzero minors are units.

(iv). Let $\gamma = u^{-1} w_p \sigma u'$. By (i), $D_j(\gamma) = p_0 \cdots p_{j-1}$ for $1 \le j \le \varrho$. Since γ is A_0-integral and $f_0, \dots, f_{\varrho-1}$ belong to $\mathbf{b}$, the entries $h(f_a, \gamma f_b)$, $a, b < \varrho$, lie in A_0, so $D_j(\gamma) \in A_0$, and $D_j(\gamma)^{-1} = \prod_{a<j} p_a^{-1} \in A_0$. By (iii), $\gamma = v^{-1} w_q \tau v'$ with A_0-integral $v, v' \in \mathsf{U}_A$, $\tau \in G^\circ$; by (ii), $v = u$, $v' = u'$, $\tau = \sigma$.

(v). Let $\gamma = w_p \sigma$. As σ fixes X, $\gamma X = w_p X = X^*$ ($w_p f_a = p_a f_a^*$ with p_a a unit). A vector orthogonal to X and to X^* lies in V°, and then $[t] = t$ and $\gamma^{-1} t = \sigma^{-1} w_p^{-1} t = \sigma^{-1} t \in V^\circ$. So $\mathrm{link}(w_p\sigma) = \{(m, \sigma^{-1}m) : m \in V^\circ\}$. If $\mathrm{link}(w_p\sigma) = \mathrm{link}(w_q\tau)$, then for $m \in V^\circ$ the pair $(m, \sigma^{-1}m)$ is $(m', \tau^{-1}m')$ for some m', so $m' = m$ and $\sigma^{-1}m = \tau^{-1}m$; thus σ and τ agree on V° and on all f_a, f_a^*, so $\sigma = \tau$. The converse is clear.

(vi). Let A be a field, $F_a = X_{<a}$ and P the stabiliser of the flag $F_1 \subset \cdots \subset F_\varrho$. Then P also stabilises the $F_a^\perp = X \oplus V^\circ \oplus X^*_{>a-1}$, that is the chain of the first paragraph, and the kernel of the action of P on its successive quotients is U_A. This kernel is a normal unipotent subgroup (it is unitriangular in a basis adapted to the chain), and P/U_A is isomorphic to the reductive group $(\mathrm{GL}_1)^\varrho \times \mathrm{U}(X^\perp/X)$ (the Levi subgroup of elements diagonal in f_a, f_a^* and preserving V° maps isomorphically onto it). Hence U_A is the unipotent radical of P [36]. □

Lemma 5.2.18. *Let $\mathfrak{S}$ be a filling datum and T a finite set of finite places of E, stable under complex conjugation. An element of $G(E)$ or of $G(\mathbb{A}_f)$ is **integral off** T if its matrix in the lattice basis has entries integral at every finite place outside T. For $p \in (E^\times)^\varrho$ let $w_p \in G(E)$ send $e_{a+1} \mapsto p_a e_{a+1}^*$ and $e_{a+1}^* \mapsto \bar{p}_a^{-1} e_{a+1}$ ($0 \le a < \varrho$) and fix X_0 and z. Let $u, u' \in \mathsf{U}(E)$, let $\sigma \in G(E)$ fix every e_a and e_a^*, and let $c, c' \in \mathcal{C}$.*

(i) If every p_a^{-1} is a T-integer and $u^{-1} w_p \sigma u'$ is a term over (c, s, c') for some $s \in \bar{G}$, then $h_c^{-1} u h_c$, $h_c^{-1}\sigma h_{c'}$ and $h_{c'}^{-1} u' h_{c'}$ are integral off T.

(ii) If every p_a and every p_a^{-1} is a T-integer and $h_c^{-1}\sigma h_{c'}$ is integral off T, then $h_c^{-1} w_p \sigma h_{c'}$ is integral off T.

Proof. Let $A = \mathbb{A}_f$ with the involution induced by complex conjugation of E, and $A_0 = \{x \in \mathbb{A}_f : x_w \in \mathcal{O}_{E_w}$ for every finite $w \notin T\}$. It is a subring, and $\overline{A_0} = A_0$ because T is stable under conjugation; 2 is a unit of $\mathbb{A}_f$, and every element of $E^\times$ is a unit of $\mathbb{A}_f$. An element of $G(\mathbb{A}_f)$ is integral off T exactly when it is A_0-integral for the lattice basis.

The setting of Lemma 5.2.17. Take $V \otimes_E \mathbb{A}_f$ with $f_a = e_{a+1}$, $f_a^* = e_{a+1}^*$ ($0 \le a < \varrho$), $m_1, \dots, m_q$ the standard basis of X_0 followed by z, so $V^\circ = (X_0 \oplus Ez) \otimes_E \mathbb{A}_f$; its Gram matrix $\mathrm{diag}(1, \dots, 1, -N)$ is invertible (N is a unit of $\mathbb{A}_f$), and $f_\varrho = z$. The lattice basis $\mathbf{b}$ replaces the last basis vector e_k of X_0 by $v = e_k/\bar{\pi} + z/N$. Since $\pi\bar{\pi} = N$, $\bar{\pi}$ is a unit of $\mathbb{A}_f$, so $\mathbf{b}$ consists of the f_a, a basis of V° and the f_a^*. In $\mathbf{b}$ the Gram matrix of h has integer entries: the hyperbolic pairs, 1 on the other basis vectors of X_0, and on (v, z) the block

$$\begin{pmatrix} h(v,v) & h(v,z) \\ h(z,v) & h(z,z) \end{pmatrix} = \begin{pmatrix} \frac{1}{\pi\bar{\pi}} - \frac{N}{N^2} & -1 \\ -1 & -N \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & -N \end{pmatrix},$$

so its determinant is $\pm 1 \in A_0^\times$. By (vi) of Lemma 5.2.17, $\mathsf{U}(E) = \mathsf{U}_E$, and the defining conditions of U_E are linear, so the image of $\mathsf{U}(E)$ in $G(\mathbb{A}_f)$ lies in $\mathsf{U}_{\mathbb{A}_f}$; likewise $\sigma \in G^\circ$ over $\mathbb{A}_f$, and w_p is the element w_p of Lemma 5.2.17 for the image of p in $(\mathbb{A}_f^\times)^\varrho$.

Conjugation by class representatives. Let $h \in \mathrm{U}(X_0)(\mathbb{A}_f)$, acting on $X_0 \otimes \mathbb{A}_f$ and trivially on the e_a, e_a^* and z. It maps V° onto itself and fixes X, each $X_{<a}$ and each $X \oplus V^\circ \oplus X^*_{>a}$ (notation of Lemma 5.2.17) pointwise or as a set. Hence $h^{-1}\mathsf{U}_{\mathbb{A}_f} h = \mathsf{U}_{\mathbb{A}_f}$: for $u \in \mathsf{U}_{\mathbb{A}_f}$, $h^{-1}uhf_a - f_a = h^{-1}(uf_a - f_a) \in X_{<a}$, $h^{-1}uhm - m = h^{-1}(u(hm) - hm) \in X$ for $m \in V^\circ$, and $h^{-1}uhf_a^* - f_a^* = h^{-1}(uf_a^* - f_a^*) \in X \oplus V^\circ \oplus X^*_{>a}$. For $h, h' \in \mathrm{U}(X_0)(\mathbb{A}_f)$, $h^{-1}\sigma h'$ fixes every e_a, e_a^*. Finally h commutes with w_p: both preserve the decomposition $(X \oplus X^*) \oplus V^\circ$, h is the identity on the first summand and w_p on the second.

(i). Let $\gamma = u^{-1}w_p\sigma u'$ be a term over (c, s, c'). By the previous paragraph

$$h_c^{-1}\gamma h_{c'} = (h_c^{-1}uh_c)^{-1}\, w_p\, (h_c^{-1}\sigma h_{c'})\, (h_{c'}^{-1}u'h_{c'}),$$

with $h_c^{-1}uh_c, h_{c'}^{-1}u'h_{c'} \in \mathsf{U}_{\mathbb{A}_f}$ and $h_c^{-1}\sigma h_{c'} \in G^\circ$. By (TA1), $h_c^{-1}\gamma h_{c'} \in K$, so its matrix in $\mathbf{b}$ has entries in $\hat{\mathcal{O}} \subseteq A_0$. The p_a^{-1} lie in A_0. By (iv) of Lemma 5.2.17, the three factors are A_0-integral, that is, integral off T.

(ii). $h_c^{-1}w_p\sigma h_{c'} = w_p\, h_c^{-1}\sigma h_{c'}$. The matrix of w_p in $\mathbf{b}$ has entries 0, 1, p_a and $\bar{p}_a^{-1}$, since $\mathbf{b}$ agrees with the standard basis on the e_a, e_a^* and w_p fixes V°; these are in A_0, as $\bar{p}_a^{-1}$ is a T-integer when p_a^{-1} is (T is stable under conjugation). A product of A_0-integral matrices is A_0-integral. □

Lemma 5.2.19. *Let $\mathfrak{S}$ be admissible, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, and let γ be a term over (c, s, c') with $A_\gamma^{\mathrm{cut}}(c, c') \neq 0$. Let the **entry set** $\mathcal{T} \subseteq E^\times$ be the set of quotients δ/δ' of nonzero $\delta, \delta' \in \mathcal{O}_E$ with $|v(\delta)|^2, |v(\delta')|^2 \leq b^{270n^2}$ for all $v \in \Phi$.*

(i) γ is a near term, and for $0 \leq j \leq \varrho + 1$, $D_j(\gamma) \in \mathcal{O}_E$ and $|v(D_j(\gamma))|^2 \leq b^{270n^2}$ for all $v \in \Phi$.

(ii) Let $\gamma = u^{-1}w_p\sigma u'$ with $u, u' \in \mathsf{U}(E)$, $\sigma \in G(E)$ fixing every e_a and e_a^*, and $p \in (E^\times)^\varrho$, where w_p sends $e_{a+1} \mapsto p_a e_{a+1}^*$, $e_{a+1}^* \mapsto \bar{p}_a^{-1} e_{a+1}$ ($0 \leq a < \varrho$) and fixes X_0 and z. If $t = h(z, \sigma z) \neq 0$, then $p_a \in \mathcal{T}$ for $0 \leq a < \varrho$ and $t \in \mathcal{T}$.

Proof. (i), nearness. By (TA2), $A_\gamma^{\mathrm{cut}}(c, c')$ is the integral over G_∞ of $\beta_c(y)\beta_{c'}(\gamma^{-1}y)\, W^{\mathrm{cut}}(y) \otimes \overline{W^{\mathrm{cut}}(\gamma^{-1}y)}$; as it is nonzero, the integrand is nonzero at some $y \in G_\infty$. Then $\beta_c(y) \neq 0$, so y is reduced, and $\beta_{c'}(\gamma^{-1}y) \neq 0$, so $\gamma^{-1}y$ is reduced. Further some entry $W^{\mathrm{cut}}(y)_i = \prod_v W^\flat_{v,i_v}(y_v)$ is nonzero, so $W_v^\flat(y_v) \neq 0$ for every v, and by (BS2) y_v is in the cut, $\ell(y_v) \leq b^{6n+2}$; the same holds for $\gamma^{-1}y$. Hence y and $\gamma^{-1}y$ are boxed, and γ is a near term.

(i), integrality. Put $\kappa = h_c^{-1}\gamma h_{c'} \in K$. The frame vectors f_a ($e_1, \ldots, e_\varrho, z$) are orthogonal to X_0, and $h_c, h_{c'} \in \mathrm{U}(X_0)(\mathbb{A}_f)$ act trivially on $X_0^\perp$; hence, in $\mathbb{A}_f$, $h(f_a, \gamma f_b) = h(h_c f_a, \gamma h_{c'} f_b) = h(f_a, \kappa f_b)$. The f_a are members of the lattice basis, in which κ has entries in $\hat{\mathcal{O}}$ and the Gram matrix of h has integer entries (Lemma 5.2.18, proof, or directly: the replaced vector $e_k/\bar{\pi} + z/N$ is isotropic, pairs to -1 with z and is orthogonal to the other basis vectors). So $h(f_a, \gamma f_b) \in E \cap \hat{\mathcal{O}} = \mathcal{O}_E$, and every minor $D_j(\gamma)$ lies in $\mathcal{O}_E$. By Lemma 5.2.20, applied to the near term γ, $|v(D_j(\gamma))|^2 \leq b^{270n^2}$ for $0 \leq j \leq \varrho + 1$ and all v.

(ii). By Lemma 5.2.17 (i), $D_j(\gamma) = D_j(w_p\sigma) = p_0 \cdots p_{j-1}$ for $0 \leq j \leq \varrho$ and $D_{\varrho+1}(\gamma) = p_0 \cdots p_{\varrho-1}t$. These are nonzero, as the p_a and t are; by (i) each is a nonzero element of $\mathcal{O}_E$ with $|v(\cdot)|^2 \leq b^{270n^2}$ for all v. Hence $p_a = D_{a+1}(\gamma)/D_a(\gamma) \in \mathcal{T}$ for $a < \varrho$ and $t = D_{\varrho+1}(\gamma)/D_\varrho(\gamma) \in \mathcal{T}$. □

Lemma 5.2.20. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, and $\gamma \in G(E)$ a near term. Then $|D_l(\gamma)|_v^2 \leq b^{270n^2}$ for every $0 \leq l \leq \varrho + 1$ and every infinite place v.*

Proof. Fix an infinite place v and write $Z = Z(\gamma)$.

Each entry satisfies $|Z_{ab}|_v \le b^{73n}$. As γ is a near term, there is a boxed $y \in G_\infty$ (with $\gamma^{-1}y$ boxed). By Lemma 5.3.75 (iii), applied to y and v, $|Z_{ab}|_v = |v(Z(\gamma)_{ab})| \le b^{73n}$ for $0 \le a, b \le \varrho$.

The minors. $D_0(\gamma) = 1$. For $1 \le l \le \varrho + 1 = n$, $v(D_l(\gamma))$ is the leading principal minor of order l of the matrix $(v(Z_{ab}))$, a sum of $l!$ products of l entries, so $|D_l(\gamma)|_v \le l!\, b^{73nl} \le l^l b^{73nl} \le b^{l+73nl}$, because $l \le \varrho+1 \le b$ by (FG6). As $l+73nl \le 74n^2$ and $b \ge 1$, $|D_l(\gamma)|_v^2 \le b^{148n^2} \le b^{270n^2}$; for $l = 0$ the bound is $1 \le b^{270n^2}$. □

Lemma 5.2.21. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, and let D_{res} be a residue domain. Then for every class $c \in \mathcal{C}$ the indicator $\mathbf{1}_{\mathrm{red}}$ of the points of G_∞ that are reduced for D_{res} is a box partition of unity for c.*

Proof. Write $R_\Phi = \mathbb{C}^\Phi$ (a ring under componentwise operations), $\phi(x) = (v(x))_{v\in\Phi}$ for $x \in E$, and for $y \in G_\infty$ let $t(y)$ be the upper unitriangular $(\varrho+1)\times(\varrho+1)$ matrix over R_Φ with entries $t_{ab}(y)$ above the diagonal. For $u \in \mathsf{U}(E)$ let $F^\phi(u)$ be the matrix with entries $\phi(F(u)_{lb})$. Recall that an element of U (over E or over $\mathbb{C}$) maps each basis vector j to j plus a combination of basis vectors of smaller level (proof of Lemma 5.2.23, Preliminaries); as $\gamma_v = B_v^{-1} v(\gamma) B_v$ with B_v diagonal, $u \in \mathsf{U}(E)$ gives $u_v \in \mathsf{U}$.

Step 1: the triangle of a translate. For $u \in \mathsf{U}(E)$ and $y \in G_\infty$,

$$t(y) = t(u \cdot y)\, F^\phi(u).$$

Fix v and write $y = y_v$, $u = u_v$. By (5.5),

$$u f_b = \sum_l F(u)'_{lb} f_l \quad \text{with} \quad F(u)'_{lb} = v(F(u)_{lb})\, \mathrm{bs}_v(f_b)/\mathrm{bs}_v(f_l).$$

Since $u^{-1} \in \mathsf{U}$, $(uy)^{-1} f_l = y^{-1}u^{-1} f_l$ lies in $y^{-1}f_l + \langle y^{-1}f_0, \dots, y^{-1}f_{l-1}\rangle$, so the moved frames of y and uy span the same flag and have the same Gram–Schmidt vectors $o_0, \dots, o_\varrho$. Hence

$$y^{-1} f_b = y^{-1}u^{-1}u f_b = \sum_l F(u)'_{lb}\, (uy)^{-1} f_l = \sum_a \Big(\sum_l T(uy)_{al} F(u)'_{lb} \Big) o_a,$$

so $T(y) = T(uy)F(u)'$. Multiplying the entry (a,b) by $\mathrm{bs}_v(f_a)/\mathrm{bs}_v(f_b)$ and inserting $\mathrm{bs}_v(f_l)/\mathrm{bs}_v(f_l)$ in each term gives $t(y)_v = \sum_l t(u\cdot y)_{al,v}\, v(F(u)_{lb})$, the asserted identity at v (on and below the diagonal both sides are those of a unitriangular matrix).

Step 2: reduction of unitriangular matrices. Let $\Lambda = \phi(M\mathcal{O}_E) \subseteq R_\Phi$. By (RD3) and the injectivity of ϕ, every $w \in R_\Phi$ has exactly one $\lambda \in \Lambda$ with $w + \lambda \in D_{\mathrm{res}}$. Let Γ be a group of upper unitriangular matrices over R_Φ whose entries above the diagonal lie in Λ and which contains $1 + \lambda E_{pq}$ for all $p < q$ and $\lambda \in \Lambda$. We show: for every upper unitriangular n over R_Φ there is exactly one $\gamma \in \Gamma$ such that all entries of $n\gamma$ above the diagonal lie in D_{res}. First, if $\gamma \in \Gamma$ has $\gamma_{pq} = 0$ whenever $0 < q - p < d$, then

$$(n\gamma)_{pq} = n_{pq} \quad (0 < q-p < d), \qquad (n\gamma)_{pq} = n_{pq} + \gamma_{pq} \quad (q - p = d),$$

since in $(n\gamma)_{pq} = \sum_{p\le m\le q} n_{pm}\gamma_{mq}$ the terms with $p < m < q$ have $0 < q - m < d$. Existence, by induction on $d = 1, \dots, \varrho$: if $\gamma \in \Gamma$ puts the entries of $n\gamma$ at distances $< d$ from the diagonal into D_{res}, choose $\lambda_{pq} \in \Lambda$ with $(n\gamma)_{pq} + \lambda_{pq} \in D_{\mathrm{res}}$ for $q - p = d$, and let $\delta \in \Gamma$ be the product of the $1 + \lambda_{pq}E_{pq}$ over these positions; δ vanishes at distances $0 < \cdot < d$ and has entries λ_{pq} at distance d (products of two matrix units at distance d lie at distance $2d$), so by the display $n\gamma\delta$ has the required entries up to distance d. Uniqueness: if γ, γ' both work, let $\delta = \gamma^{-1}\gamma' \in \Gamma$; if

δ vanishes at distances $< d$, the display gives $(n\gamma')_{pq} = (n\gamma)_{pq} + \delta_{pq}$ at distance d, both in D_{res} with $\delta_{pq} \in \Lambda$, so $\delta_{pq} = 0$. Hence $\delta = 1$.

Step 3: exactly one coset. Let $\Gamma = F^{\phi}(\mathsf{U}_c(M))$. By Lemma 5.2.23 (i), $u \mapsto F^{\phi}(u)$ is a homomorphism into the upper unitriangular group, with entries above the diagonal in Λ and, as ϕ is injective, with kernel $\mathsf{U}_c^0(M)$; by (iii), $\Gamma \ni F^{\phi}(u) = 1 + \phi(Mt)E_{lb}$ for all $l < b$ and $t \in \mathcal{O}_E$. By Step 1, $t(u \cdot y) = t(y)F^{\phi}(u^{-1})$, so $u \cdot y$ is reduced if and only if $t(y)F^{\phi}(u^{-1})$ has its entries above the diagonal in D_{res}. Let $\gamma_0 \in \Gamma$ be the element given by Step 2 for $n = t(y)$, and $u_0 \in \mathsf{U}_c(M)$ with $F^{\phi}(u_0^{-1}) = \gamma_0$. Then $u \cdot y$ is reduced iff $F^{\phi}(u^{-1}) = F^{\phi}(u_0^{-1})$, iff $u_0 u^{-1} \in \mathsf{U}_c^0(M)$, iff $u \in \mathsf{U}_c^0(M)u_0$.

Step 4: the conditions of (RD5). The entries of $T(y)$ are continuous in y (Gram–Schmidt coefficients of the linearly independent vectors $y^{-1}f_0, \dots, y^{-1}f_{\varrho}$), so the reduced points form a Borel set and $\omega = \mathbf{1}_{\mathrm{red}}$ is measurable with values in $[0,1]$. Sum: let $a\colon \mathsf{U}_c(M) \to [0,\infty]$ with $\sum_{u \in \mathsf{U}_c^0(M)} a(uu') = 1$ for all u'. By Step 3, $\sum_{u \in \mathsf{U}_c(M)} a(u)\,\omega(u \cdot y) = \sum_{u \in \mathsf{U}_c^0(M)} a(uu_0) = 1$. Invariance: if every s_v fixes $f_0, \dots, f_{\varrho}$, then $(s_v y_v)^{-1} f_a = y_v^{-1} f_a$, so $t(sy) = t(y)$ and $\omega(sy) = \omega(y)$. Box: let y be reduced and $v \in \Phi$; write $y = y_v$. For $a < l$, $T(y)_{al} = t_{al}(y)_v\, \mathrm{bs}_v(f_l)/\mathrm{bs}_v(f_a)$ with $|\mathrm{bs}_v(f_l)/\mathrm{bs}_v(f_a)| = |\mathrm{rs}_v|^{a-l} \le 1$ by (FA4) and $|t_{al}(y)_v| \le b^{26}$ by (RD3); and $T(y)_{ll} = 1 \le b^{26}$. Next, $\|o_a\| \le \ell(y)$: for $s \in R$, s^{-1} fixes f_a modulo $\langle f_0, \dots, f_{a-1}\rangle$ (H fixes the frame, U acts unitriangularly on it), so sy has the same Gram–Schmidt vectors as y, and $\|o_a\| \le \|(sy)^{-1} f_a\| \le \|(sy)^{-1}\|_{\mathrm{op}} = \|sy\|_{\mathrm{op}}$; here $\|g^{-1}\|_{\mathrm{op}} = \|g\|_{\mathrm{op}}$ for $g \in G_1$ because $g^{-1} = J_1 g^* J_1$ with J_1 a hermitian involution, hence unitary. Taking the infimum over s gives $\|o_a\| \le \ell(y)$. So, with $\varrho + 1 \le b$ from (FG6),

$$\|y^{-1} f_l\| \le \sum_{a \le l} |T(y)_{al}|\, \|o_a\| \le (\varrho + 1)\, b^{26} \ell(y) \le b^{27} \ell(y) \qquad (0 \le l \le \varrho),$$

that is, $y_v \in \mathrm{Bx}(b^{27}\ell(y_v))$. So $\mathbf{1}_{\mathrm{red}}$ is a box partition of unity for c. □

Definition 5.2.22. Let $\mathfrak{S}$ be a filling datum over E; $\hat{\mathcal{O}}$ and $\mathbb{A}_f$ are the integral finite adeles and the finite adeles of E, $W(\mathbb{A}_f) = W \otimes_E \mathbb{A}_f$ for an E-subspace W of V, and h is extended $\mathbb{A}_f$-sesquilinearly. Let $L_0 \subseteq V(\mathbb{A}_f)$ be the $\hat{\mathcal{O}}$-span of the lattice basis ((FG3) of Definition 5.1.9), and $A = \langle e_1, \dots, e_{\varrho}\rangle$, so that $L_0 \cap A(\mathbb{A}_f) = \bigoplus_{a=1}^{\varrho} \hat{\mathcal{O}} e_a$.

(LT1) Class and term lattices. For $c \in \mathcal{C}$ the **class lattice** is $L_c = h_c L_0$, and its **definite part** is $L_c \cap X_0(\mathbb{A}_f)$. For $\gamma \in G(E)$ the **term lattice** of γ at c is

$$L_c(\gamma) = \Big\{ x \in L_c \cap X_0(\mathbb{A}_f) :\ h(\gamma f_b,\, x + Mt) = 0 \ \ (0 \le b \le \varrho) \text{ for some } t \in \textstyle\bigoplus_{a=1}^{\varrho} \hat{\mathcal{O}} e_a \Big\},$$

written L_γ when c is fixed.

(LT2) Common definite part. For $\gamma, \gamma' \in G(E)$ the **common definite part** is the E-subspace

$$X_0(\gamma, \gamma') = \{ x \in X_0 :\ (x, y) \in \mathrm{link}\,\gamma \cap \mathrm{link}\,\gamma' \text{ for some } y \in X_0 \}$$

of X_0. It is not the intersection $\mathrm{link}\,\gamma \cap \mathrm{link}\,\gamma'$, which is a subspace of $(X_0 \oplus Ez)^2$.

(LT3) Lattices of a given rank and their discriminants. An $\hat{\mathcal{O}}$-submodule $L \subseteq X_0(\mathbb{A}_f)$ is a **lattice of rank** m if $L = B\hat{\mathcal{O}}^m$ for a matrix $B = xa$ with $x \in \mathrm{M}_{k \times m}(E)$ of rank m and $a \in \mathrm{GL}_m(\mathbb{A}_f)$. Its **discriminant** is

$$\mathrm{disc}_m(L) = [\det(B^* B)]^{1/2},$$

with the generalised index $[\cdot]$ of finite adeles of Definition 5.1.8 and B^* the conjugate transpose (the form of X_0 is the unit form); $\mathrm{disc}_m(L) = 0$ if L is not a lattice of rank m.

For an E-subspace $W \subseteq X_0$ the **saturation** of W at c is $\mathrm{sat}_c(W) = L_c \cap W(\mathbb{A}_f)$.

(LT4) Pair lattices. For $c \in \mathcal{C}$ and $\gamma, \gamma' \in G(E)$, the **common part** of γ with γ' is

$$\mathrm{sat}_c(X_0(\gamma, \gamma')) \cap L_c(\gamma),$$

the **pair lattice** is

$$L_{12} = \mathrm{sat}_c(X_0(\gamma, \gamma')) \cap L_c(\gamma) \cap L_c(\gamma'),$$

and the **pair discriminant** is $\mathrm{disc}(L_{12}) = \mathrm{disc}_m(L_{12})$ with $m = \dim_E X_0(\gamma, \gamma')$.

(LT5) Link corank. The **link corank** of γ, γ' is $c^+ = \max(0,\ k + 1 - \dim_E(\mathrm{link}\,\gamma \cap \mathrm{link}\,\gamma'))$.

Lemma 5.2.23. *Let $\mathfrak{S}$ be a filling datum over a CM field E that carries Weil data, and $c \in \mathcal{C}$. For $x \in V$ and $1 \le b \le \varrho$ the e_b-coordinate of x (in the basis $e_1, \dots, e_\varrho$, the standard basis of X_0, z, $e_1^*, \dots, e_\varrho^*$) is $h(e_b^*, x)$.*

(i) Every $u \in \mathsf{U}(E)$ preserves the span of the frame; its frame matrix *$F(u) \in \mathrm{M}_{\varrho+1}(E)$, defined by $u f_b = \sum_l F(u)_{lb} f_l$, is upper unitriangular, and $F(uu') = F(u)F(u')$. An element $u \in \mathsf{U}(E)$ lies in $\mathsf{U}_c(M)$ if and only if $(u-1)L_c \subseteq ML_c$. For $u \in \mathsf{U}_c(M)$ the entries $F(u)_{lb}$, $l < b$, lie in $M\mathcal{O}_E$, and $F(u) = 1$ if and only if $u \in \mathsf{U}_c^0(M)$.*

(ii) There is $\mu \in \mathcal{O}_E$ with $\mu + \bar{\mu} = 1$; fix one. Let $1 \le a \le \varrho$, let $w \in V \cap L_c$ lie in the span of X_0, z and the e_b, e_b^* with $a < b \le \varrho$, and let $s \in \mathcal{O}_E$ with $\bar{s} = -s$. Then the map

$$E_a(w,s)\colon x \longmapsto x + M\,h(w,x)\,e_a - M\,h(e_a,x)\,w - M\big(M\mu\,h(w,w) - s\big)h(e_a,x)\,e_a$$

lies in $\mathsf{U}_c(M)$. For every standard basis vector $j \neq e_a^*$ and every $1 \le b \le \varrho$ the e_b-coordinate of $E_a(w,s)j - j$ is $M\delta_{ab}\,h(w,j)$; the e_a-coordinate of $E_a(w,s)e_a^* - e_a^*$ is $M\big(s - M\mu\,h(w,w)\big)$; and $E_a(w,s)f_b = f_b + M\,h(w,f_b)\,e_a$ for $0 \le b \le \varrho$.

(iii) For $0 \le l < b \le \varrho$ and $t \in \mathcal{O}_E$ there is $u \in \mathsf{U}_c(M)$ with $F(u) = 1 + Mt\,E_{lb}$, where E_{lb} is the matrix unit.

Proof. Preliminaries. Since $h(e_b^*, e_{b'}) = \delta_{bb'}$ and e_b^* is orthogonal to the other basis vectors, $h(e_b^*, \cdot)$ is the e_b-coordinate; likewise $h(e_b, \cdot)$ is the e_b^*-coordinate. As h_c is an isometry of $V(\mathbb{A}_f)$, Lemma 5.2.24 (i) gives $h(L_c, L_c) \subseteq \hat{\mathcal{O}}$ and $L_c = \{x : h(L_c, x) \subseteq \hat{\mathcal{O}}\}$. The element $h_c \in \mathrm{U}(X_0)(\mathbb{A}_f)$ acts trivially on $X_0^\perp$, and e_b, e_b^*, z belong to the lattice basis, so $e_b, e_b^*, z \in L_c$. Finally $E \cap M\hat{\mathcal{O}} = M\mathcal{O}_E$. Give the basis vectors *levels*: e_b has level b, the vectors of X_0 and z level $\varrho + 1$, e_b^* level $2\varrho + 2 - b$. With $F_b = \langle e_1, \dots, e_b \rangle$, the spans of the basis vectors of level at most $1, \dots, 2\varrho + 1$ are $F_1 \subset \cdots \subset F_\varrho \subset F_\varrho^\perp \subset F_{\varrho-1}^\perp \subset \cdots \subset F_1^\perp$; the stabiliser of the isotropic flag stabilises this refined flag, and its unipotent radical U consists of the isometries acting trivially on its successive quotients. So an isometry u lies in $\mathsf{U}(E)$ if and only if $(u-1)j$ is a combination of basis vectors of strictly smaller level than j, for every basis vector j.

(i). For $u \in \mathsf{U}(E)$, $(u-1)e_b$ lies in $\langle e_1, \dots, e_{b-1} \rangle$ and $(u-1)z$ in F_ϱ; with $f_l = e_{l+1}$ $(l < \varrho)$ this says $(u-1)f_b \in \langle f_0, \dots, f_{b-1} \rangle$. So the span of the frame is u-stable, $F(u)$ is the matrix of the restriction, upper unitriangular, and F is multiplicative. If $u \in \mathsf{U}_c(M)$, then $k = h_c^{-1} u h_c \in \mathrm{K}(M)$, so $(k-1)L_0 \subseteq ML_0$ by Lemma 5.2.24 (iii), and applying h_c gives $(u-1)L_c \subseteq ML_c$. Conversely let $(u-1)L_c \subseteq ML_c$. Then $uL_c \subseteq L_c$; as $u - 1$ strictly

lowers levels it is nilpotent, so $u^{-1} = \sum_{n\ge 0}(1-u)^n$ is a finite sum and $u^{-1}L_c \subseteq L_c$. Hence $k = h_c^{-1}uh_c$ satisfies $kL_0 = L_0$; its matrix in the lattice basis is integral, so $k \in K$, and $(k-1)L_0 = h_c^{-1}(u-1)L_c \subseteq ML_0$ gives $k \in \mathrm{K}(M)$, that is, $u \in \mathsf{U}_c(M)$. Let $u \in \mathsf{U}_c(M)$ and $l < b$; then $f_l = e_{l+1}$ and $F(u)_{lb} = h(e^*_{l+1}, (u-1)f_b)$. Since $f_b \in L_c$, $(u-1)f_b \in ML_c$, and $e^*_{l+1} \in L_c$, this lies in $M\hat{\mathcal{O}} \cap E = M\mathcal{O}_E$. Finally $F(u) = 1$ means that u fixes $f_0, \dots, f_\varrho$, which is the definition of $\mathsf{U}^0_c(M)$.

(ii). Weil data give a primitive 5040-th root of unity ζ; $\mu = \zeta^{840}$ is a primitive sixth root of unity, so $\mu \in \mathcal{O}_E$ and $\mu^2 - \mu + 1 = 0$. Complex conjugation of the CM field inverts roots of unity ($|v(\mu)| = 1$ for every embedding v), so $\bar\mu = \mu^{-1}$ and $\mu + \bar\mu = \mu + \mu^{-1} = 1$. Write $\gamma = E_a(w, s)$, $\gamma x = x + P(x)$ with $P(x) = Mh(w,x)e_a - Mh(e_a,x)w - c\,h(e_a,x)e_a$ and $c = M(M\mu h(w,w) - s)$. The vector e_a is isotropic, and $h(e_a, w) = 0$ because w has no e^*_a-component. Hence, using $\overline{h(y,x)} = h(x,y)$,

$$h(x, P(x')) + h(P(x), x') = -(c + \bar c)\, h(x, e_a)h(e_a, x'),$$
$$h(P(x), P(x')) = M^2 h(w,w)\, h(x, e_a)h(e_a, x'),$$

and $c + \bar c = M^2(\mu + \bar\mu)h(w,w) - M(s + \bar s) = M^2 h(w,w)$, as $h(w,w) = \overline{h(w,w)}$ and $\bar s = -s$. So $h(\gamma x, \gamma x') = h(x, x')$; an isometry of the nondegenerate form h is injective, hence $\gamma \in G(E)$. For a basis vector $j \ne e^*_a$ we have $h(e_a, j) = 0$, so $\gamma j - j = Mh(w,j)e_a$; here $h(w,j) \ne 0$ only if j pairs with a component of w, that is, j is e_b or e^*_b with $b > a$ or lies in $X_0 \oplus Ez$, all of level larger than a. Further $\gamma e^*_a - e^*_a = -Mw - c\,e_a$ (as $h(w, e^*_a)$ is the conjugate of the e_a-coordinate of w, which is 0), and w is a combination of basis vectors of level less than $2\varrho + 2 - a$. So $\gamma \in \mathsf{U}(E)$, and the coordinate statements and the formula for γf_b ($f_b \ne e^*_a$) follow; the e_a-coordinate of $-Mw - ce_a$ is $-c = M(s - M\mu h(w,w))$. For $x \in L_c$ the numbers $h(w,x)$, $h(e_a,x)$, $h(w,w)$ lie in $\hat{\mathcal{O}}$ (as $w, e_a \in L_c$), $\mu, s \in \mathcal{O}_E$, and $e_a, w \in L_c$; so $P(x) \in ML_c$, and $\gamma \in \mathsf{U}_c(M)$ by (i).

(iii). Put $a = l + 1$, so $f_l = e_a$. If $b < \varrho$, then $f_b = e_{b+1}$ with $b + 1 > a$; take $w = \bar t\, e^*_{b+1} \in L_c$. Then $h(w, f_{b'}) = t\,\delta_{bb'}$, and $u = E_a(w, 0)$ has $uf_{b'} = f_{b'} + Mt\,\delta_{bb'}f_l$ by (ii), that is, $F(u) = 1 + MtE_{lb}$. If $b = \varrho$, so $f_b = z$, we need $w \in L_c \cap (X_0 \oplus Ez)$ with $h(w,z) = t$. Let e^0 be the basis vector of X_0 that (FG3) replaces by $e' = e^0/\bar\pi + z/N$, and $L^{\mathrm{m}} = L_0 \cap (X_0 \oplus Ez)(\mathbb{A}_f)$, the $\hat{\mathcal{O}}$-span of the other basis vectors of X_0, of e' and of z. The set $Y = \{y \in X_0(\mathbb{A}_f) :\ y - (\bar t/N)z \in L^{\mathrm{m}}\}$ contains $-(\bar t/\bar\pi)e^0$ (since $-(\bar t/\bar\pi)e^0 - (\bar t/N)z = -\bar t e'$) and is a coset of the open subgroup $L^{\mathrm{m}} \cap X_0(\mathbb{A}_f)$, hence open and nonempty; so is h_cY. As E is dense in $\mathbb{A}_f$ (strong approximation), there is $y \in X_0 \cap h_cY$. Put $w = y - (\bar t/N)z \in V$. Then $h_c^{-1}w = h_c^{-1}y - (\bar t/N)z \in L^{\mathrm{m}} \subseteq L_0$, so $w \in L_c$; $h(w, e_{b'}) = 0$ for all b', and $h(w,z) = -(t/N)h(z,z) = t$. Then $u = E_a(w, 0)$ has $F(u) = 1 + MtE_{l\varrho}$ by (ii). □

Lemma 5.2.24. *Let $\mathfrak{S}$ be a filling datum.*

(i) $h(L_0, L_0) \subseteq \hat{\mathcal{O}}$, and $L_0 = \{v \in V(\mathbb{A}_f) : h(L_0, v) \subseteq \hat{\mathcal{O}}\}$.

(ii) $kL_0 = L_0$ for every $k \in K$.

(iii) For $k \in K$: $k \in K(M)$ if and only if $(k-1)L_0 \subseteq ML_0$.

Proof. Let b be the lattice basis: the e_a, e^*_a, the standard basis vectors x_i of X_0 other than the last one x_{i_0}, the vector $u = x_{i_0}/\bar\pi + z/N$, and z. Here $\bar\pi$ is a unit of $\mathbb{A}_f$, since $\pi_w\bar\pi_w = N \ne 0$ at every w. Matrices of elements of $G(\mathbb{A}_f)$ are taken in b, and J is the Gram matrix of h in b, so that $h(v, v') = c^* Jc'$ for coordinate vectors c, c'.

(i). The hyperbolic pairs give blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, orthogonal to $X_0 \oplus Ez$; the x_i $(i \neq i_0)$ are orthonormal and orthogonal to u and z; and, h being conjugate-linear in the first variable,

$$h(u,u) = \frac{1}{\pi\bar{\pi}} + \frac{h(z,z)}{N^2} = \frac{1}{N} - \frac{1}{N} = 0, \qquad h(u,z) = \frac{h(z,z)}{N} = -1, \qquad h(z,z) = -N.$$

So J is block diagonal with blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, 1 and $\begin{pmatrix} 0 & -1 \\ -1 & -N \end{pmatrix}$; its entries are integers and $\det J = \pm 1$, so $J \in \mathrm{GL}_r(\hat{\mathcal{O}})$. Hence $h(L_0, L_0) = \hat{\mathcal{O}}^{r*} J \hat{\mathcal{O}}^r \subseteq \hat{\mathcal{O}}$; and v with coordinates c' satisfies $h(L_0, v) \subseteq \hat{\mathcal{O}}$ iff $Jc' \in \hat{\mathcal{O}}^r$ iff $c' \in \hat{\mathcal{O}}^r$ iff $v \in L_0$.

(ii). $k \in K$ has an integral matrix, so $kL_0 \subseteq L_0$. As k is an isometry, $k^* J k = J$, so $k^{-1} = J^{-1} k^* J$ is integral by (i), and $k^{-1} L_0 \subseteq L_0$.

(iii). The reduction is the entrywise reduction of the matrix of k modulo the chosen places $w \in S_0 \cup S_1$. If $(k-1)L_0 \subseteq ML_0$, the entries of $k-1$ lie in $M\hat{\mathcal{O}}$, and $M = p_0 p_1$ lies in the maximal ideal at every chosen place, so $\mathrm{red}(k) = 1$. Conversely let $\mathrm{red}(k) = 1$: every entry of $k-1$ lies in the maximal ideal $\mathfrak{m}_w$ of $\hat{\mathcal{O}}_w$ for every chosen w. From $k^* = Jk^{-1}J^{-1}$,

$$k^* - 1 = J(k^{-1} - 1)J^{-1} = -J\,k^{-1}(k-1)\,J^{-1},$$

and $J^{\pm 1}$, k^{-1} are integral; so the entries of $k^* - 1$, which are the conjugates of the entries of $k-1$, also lie in $\mathfrak{m}_w$ for every chosen w. By the divisibility condition of (FG2) ($M \mid x$ iff $x \equiv \bar{x} \equiv 0$ modulo every chosen w), every entry of $k-1$ is divisible by M, that is $(k-1)L_0 \subseteq ML_0$. □

Lemma 5.2.25. *Let $\mathfrak{S}$ be a filling datum, $\gamma \in G(E)$, $u, u' \in \mathsf{U}(E)$, and $\gamma' = u^{-1}\gamma u'$.*

(i) $\mathrm{link}\,\gamma' = \mathrm{link}\,\gamma$ *and* $\Upsilon_{\gamma'} = \Upsilon_\gamma$.

(ii) $D_j(\gamma') = D_j(\gamma)$ *for* $0 \leq j \leq \varrho + 1$*; in particular* $\mathsf{g}(\gamma') = \mathsf{g}(\gamma)$.

(iii) For every $j \in \{0, \ldots, \varrho+1\}$, γ' satisfies the conditions of cell rank j if and only if γ does.

(iv) $\mathsf{g}(\gamma) = 0$ *if and only if the spans of f and of γf have a nonzero common vector.*

Proof. The action of U. U is the unipotent radical of the stabiliser of the isotropic flag $F_1 \subset \cdots \subset F_\varrho = X$, $F_a = \langle e_1, \ldots, e_a \rangle$; so $u \in \mathsf{U}(E)$ acts trivially on the quotients of $0 \subset F_1 \subset \cdots \subset F_\varrho \subset X^\perp \subset \cdots \subset V$. Hence $uX = X$, $uX^\perp = X^\perp$, $ue_a \in e_a + F_{a-1}$, and $uw - w \in X$ for $w \in X^\perp$, so that $[uw] = [w]$ for $w \perp X$. As $z \in X^\perp$, $uz \in z + X$. Thus $uf_l = \sum_{l' \leq l} U_{l'l} f_{l'}$ with an upper unitriangular matrix $U = U(u)$, i.e. $u(f \cdot x) = f \cdot (Ux)$.

(i). Let $t \perp X + \gamma X$ and put $s = u^{-1}t$. Then $s \perp u^{-1}X = X$, and $\gamma'^{-1}s = u'^{-1}\gamma^{-1}t$ lies in $u'^{-1}X^\perp = X^\perp$ since $\gamma^{-1}t \perp X$; so $s \perp X + \gamma' X$. Moreover $[s] = [t]$ and $[\gamma'^{-1}s] = [u'^{-1}(\gamma^{-1}t)] = [\gamma^{-1}t]$. Hence $([t], [\gamma^{-1}t]) \in \mathrm{link}\,\gamma'$, and $\mathrm{link}\,\gamma \subseteq \mathrm{link}\,\gamma'$. Applying this to $\gamma = u\gamma' u'^{-1}$ gives the reverse inclusion. Equal links give equal link isometries by Lemma 5.2.28 (iii).

(ii). With $U = U(u)$, $U' = U(u')$ and γ an isometry,

$$Z(\gamma')_{jl} = h(f_j, u^{-1}\gamma u' f_l) = h(uf_j, \gamma u' f_l) = \sum_{j',l'} \overline{U_{j'j}}\, Z(\gamma)_{j'l'}\, U'_{l'l},$$

that is $Z(\gamma') = U^* Z(\gamma) U'$. Since U^* is lower and U' upper triangular, the leading principal $j \times j$ block of $U^* Z(\gamma) U'$ is the product of the leading blocks of U^*, $Z(\gamma)$ and U', and the first and last have determinant 1. So $D_j(\gamma') = D_j(\gamma)$, and g is formed from D_ϱ, $D_{\varrho+1}$.

(iii). By (ii) the conditions $D_j \neq 0$ and ($\varrho < j \iff \mathsf{g} \neq 0$) are the same for γ and γ'. For the first condition, call a triple $(c, a, (x, y))$ a pair at a for γ if $f \cdot x = \gamma f \cdot y$, $x_a = c$, $y_a = 1$ and $x_b = y_b = 0$ for $b > a$. Left multiplication: if $(c, a, (x, y))$ is a pair at a for γ, then $f \cdot (Ux) = u(f \cdot x) = (u\gamma)f \cdot y$, and for $b \geq a$, $(Ux)_b = x_b + \sum_{l>b} U_{bl} x_l = x_b$; so $(c, a, (Ux, y))$ is a pair at a for $u\gamma$, for every $u \in \mathsf{U}(E)$. Inversion: from $f \cdot x = \gamma f \cdot y$ we get $f \cdot (\bar{c}y) = \gamma^{-1} f \cdot (\bar{c}x)$ with $(\bar{c}y)_a = \bar{c}$, $(\bar{c}x)_a = \bar{c}c = 1$; so $(\bar{c}, a, (\bar{c}y, \bar{c}x))$ is a pair at a for γ^{-1}, and $\bar{c}\,\bar{\bar{c}} = 1$. The first condition of cell rank j (one c with $c\bar{c} = 1$ and a pair at every $a \geq j$) is therefore preserved under $\gamma \mapsto u\gamma$ and $\gamma \mapsto \gamma^{-1}$, hence under $\gamma \mapsto u^{-1}\gamma u' = \big(u'^{-1}(u^{-1}\gamma)^{-1}\big)^{-1}$, and, applying this to $\gamma = u\gamma' u'^{-1}$, in both directions.

(iv). Write $Z = Z(\gamma)$. Since $e_1, \dots, e_\varrho$ are isotropic, mutually orthogonal and orthogonal to z, and $h(z, z) = -N$, a common vector $f \cdot x = \gamma f \cdot y$ satisfies $Zy = (h(f_j, f \cdot x))_j = -N x_\varrho\, \varepsilon_\varrho$.

Suppose $\det Z = 0$. Then $\mathsf{g}(\gamma) = 0$ by Lemma 5.2.26 (ii). Choose $y \neq 0$ with $Zy = 0$ and put $t = \gamma f \cdot y$; $t \neq 0$ as γf is linearly independent. By Lemma 5.2.28 (vi) with $c' = 0$, $\Upsilon_\gamma(y_0 + y_\varrho z^+) = 0$ for the X_0-part y_0 of t, so $y_0 = 0$ as Υ_γ is bijective. By Lemma 5.2.26 (i) with $D = 0$, t has no e^*-component and z-coordinate 0; so $t \in X \subseteq \langle f \rangle$, a nonzero common vector. Both sides of (iv) hold.

Suppose $\det Z \neq 0$, and let c be the last column of the adjugate of Z: $Zc = \det(Z)\varepsilon_\varrho$, $c_\varrho = D_\varrho(\gamma)$, $c \neq 0$. By Lemma 5.2.26 (i), $y := \gamma f \cdot c$ has no e^*-component and its X_0-part y_0 satisfies $N\, h(y_0, y_0) = |\det Z|^2 - N^2 |D_\varrho(\gamma)|^2 = \mathsf{g}(\gamma)$; as h is positive definite on X_0 at every infinite place, $\mathsf{g}(\gamma) = 0$ if and only if $y_0 = 0$. If $y_0 = 0$, then y has only e- and z-coordinates, so y is a nonzero vector of $\langle f \rangle \cap \langle \gamma f \rangle$. Conversely, let $f \cdot x = \gamma f \cdot y'$ with $y' \neq 0$. Then $Zy' = -N x_\varrho \varepsilon_\varrho$, so $y' = \tau c$ with $\tau = -N x_\varrho / \det Z \neq 0$ (Z is invertible and $y' \neq 0$). The vector $\gamma f \cdot y' = f \cdot x$ has X_0-part 0, and it equals τy; hence $y_0 = 0$ and $\mathsf{g}(\gamma) = 0$. □

Lemma 5.2.26. *Let $\mathfrak{S}$ be a filling datum and $\gamma \in G(E)$.*

(i) Let $D \in E$ and $y \in E^{\varrho+1}$ with $Z(\gamma)y = D\,\varepsilon_\varrho$, ε_ϱ the last standard basis vector, and put $t = \sum_l y_l\, \gamma f_l$. Then t has no component along $e_1^*, \dots, e_\varrho^*$, its z-coordinate is $-D/N$, and its component $x \in X_0$ satisfies

$$N\, h(x, x) = D\bar{D} - N^2\, y_\varrho \bar{y}_\varrho.$$

(ii) Either $\mathsf{g}(\gamma)$ is totally positive (as an element of E^+) and $\det Z(\gamma) \neq 0$, or $\mathsf{g}(\gamma) = 0$.

Proof. Write $Z = Z(\gamma)$, so that $Z_{jl} = h(f_j, \gamma f_l)$.

(i). The vector Zy is the vector of the products $h(f_j, t)$, $0 \leq j \leq \varrho$. Write $t = \sum_a (\alpha_a e_a + \beta_a e_a^*) + x + \tau z$ with $x \in X_0$. Since $h(e_a, e_b) = 0$, $h(e_a, e_b^*) = \delta_{ab}$ and $e_a \perp X_0 + Ez$, we get $\beta_a = h(e_a, t) = (Zy)_{a-1} = 0$ for $1 \leq a \leq \varrho$, and $-N\tau = h(z, t) = (Zy)_\varrho = D$. As γ is an isometry,

$$h(t, t) = h\Big(\sum_l y_l f_l, \sum_l y_l f_l\Big) = -N\, y_\varrho \bar{y}_\varrho,$$

because $e_1, \dots, e_\varrho$ are isotropic, mutually orthogonal and orthogonal to z. On the other hand, since all β_a vanish, $h(t, t) = h(x, x) - N\tau\bar{\tau}$. Hence $N\, h(x, x) = N^2\tau\bar{\tau} - N^2 y_\varrho \bar{y}_\varrho = D\bar{D} - N^2 y_\varrho \bar{y}_\varrho$.

(ii). Let y be the last column of the adjugate of Z. By Cramer's rule $Zy = \det(Z)\, \varepsilon_\varrho$, and y_ϱ is the cofactor of the entry (ϱ, ϱ), namely the leading principal minor $D_\varrho(\gamma)$; also $\det Z = D_{\varrho+1}(\gamma)$. By (i) with $D = \det Z$,

$$\mathsf{g}(\gamma) = |D_{\varrho+1}(\gamma)|^2 - N^2 |D_\varrho(\gamma)|^2 = N\, h(x, x) = N \sum_{i=1}^{k} x_i \bar{x}_i.$$

At every infinite place v, the embedding Φ_v maps this to $N\sum_i|\Phi_v(x_i)|^2\geq 0$. Hence $\mathtt{g}(\gamma)\in E^+$ is nonnegative at every real place of E^+; if $\mathtt{g}(\gamma)\neq 0$ its images are nonzero, so it is totally positive. Finally, if $\det Z=0$ then $\mathtt{g}(\gamma)=-N^2D_\varrho(\gamma)\overline{D_\varrho(\gamma)}$ is ≤ 0 at every real place, so with the above $\mathtt{g}(\gamma)=0$. Thus $\mathtt{g}(\gamma)\neq 0$ implies $\det Z\neq 0$. □

Definition 5.2.27. Let $\mathfrak{S}$ be a filling datum, and let $V_0=X_0\oplus Ez$ be the **middle space**, with the restriction of h; an element of V_0 is written $y_0+\lambda z$ with $y_0\in X_0$, $\lambda\in E$.

(LK1) Positive space. $V_0^+=X_0\oplus Ez^+$, a second copy of the middle space with the hermitian form

$$\langle y_0+\lambda z^+,\, y_0'+\lambda' z^+\rangle_+=h(y_0,y_0')+N\bar\lambda\lambda',$$

so that z^+ has norm N; for $a=y_0+\lambda z\in V_0$ put $a^+=y_0+\lambda z^+$.

(LK2) Pairs, source and target. A **pair** is an element $p=(p_1,p_2)$ of $V_0\times V_0$. For $p=(y_0+\lambda' z,\ y_0'+\lambda z)$ the **source** and the **target** of p are

$$p^{\mathrm{s}}=y_0+\lambda z^+\in V_0^+,\qquad p^{\mathrm{t}}=y_0'+\lambda' z^+\in V_0^+$$

(the X_0-part of one member with the z-coordinate of the other). Every link $\{([t],[\gamma^{-1}t]) : t\perp X,\ t\perp\gamma X\}$ of (TA4) is a subspace of pairs.

(LK3) Link form and link metric. On pairs,

$$\langle p,q\rangle_{\mathrm{lk}}=h(p_1,q_1)-h(p_2,q_2),\qquad \langle p,q\rangle_{\mathrm{mt}}=\tfrac12\big(\langle p_1^+,q_1^+\rangle_++\langle p_2^+,q_2^+\rangle_+\big);$$

the first is the **link form**, the second the **link metric**.

(LK4) z-forms. $\mathrm{zf}(p)=\big(h(z,p_1),h(z,p_2)\big)\in E^2$; for p as in (LK2), $\mathrm{zf}(p)=(-N\lambda',-N\lambda)$. For vectors $p^{(1)},\dots,p^{(m)}$ of pairs, $\mathrm{zf}(p^{(1)},\dots,p^{(m)})$ is the $2\times m$ matrix with the columns $\mathrm{zf}(p^{(i)})$.

(LK5) Integral and level lifts. Let $c\in\mathcal{C}$, $\gamma\in G(E)$ and σ a finite set. A family $w=(w_l)_{l\in\sigma}$ of adelic pairs ($w_l\in V_0(\mathbb{A}_f)\times V_0(\mathbb{A}_f)$) **has integral lifts** for (c,γ) if there are a finite family $(t_\rho)_\rho$ of vectors of $(X+\gamma X)^\perp\subseteq V$ and a matrix $u=(u_{\rho l})$ over $\mathbb{A}_f$ such that $\sum_\rho u_{\rho l}t_\rho\in L_c$ and $w_l=\sum_\rho u_{\rho l}\,([t_\rho],[\gamma^{-1}t_\rho])$ for every $l\in\sigma$. The family w **has level lifts** for (c,γ) if there are such (t_ρ) and u for which moreover, for every $l\in\sigma$, the coefficients of $\sum_\rho u_{\rho l}t_\rho$ on $e_1,\dots,e_\varrho$ (in the basis $e_1,\dots,e_\varrho$, the basis of X_0, z, $e_1^*,\dots,e_\varrho^*$ of V) lie in $M\hat{\mathcal{O}}$.

(LK6) Squared link discriminant. For a family $y=(y_l)_{l\in\sigma}$ of pairs over E and $\epsilon\in\mathrm{GL}_\sigma(\mathbb{A}_f)$,

$$\mathrm{lkd}(y,\epsilon)=\big|N_{E/\mathbb{Q}}\det\big(\langle y_l,y_{l'}\rangle_{\mathrm{mt}}\big)_{l,l'\in\sigma}\big|\cdot[\det\epsilon]^2,$$

the squared discriminant of the lattice with the adelic basis $y\epsilon$ for the link metric.

(LK7) Level lifts. Let M be the principal level of $\mathfrak{S}$. A family $w=(w_l)_{l\in\sigma}$ of adelic pairs **has level lifts** for (c,γ) if there are $(t_\rho)_\rho$ and u as in (LK5) such that, for every $l\in\sigma$, the vector $n_l=\sum_\rho u_{\rho l}t_\rho\in L_c$, which is orthogonal to X, has the form $n_l=\alpha_l+n_l'$ with $\alpha_l\in\bigoplus_{a=1}^{\varrho}M\hat{\mathcal{O}}e_a$ and $n_l'\in(X_0\oplus Ez)(\mathbb{A}_f)$.

Lemma 5.2.28. *Let $\mathfrak{S}$ be a filling datum and $\gamma\in G(E)$; write $W_\gamma=(X+\gamma X)^\perp$.*

(i) The map $t\mapsto([t],[\gamma^{-1}t])$ from W_γ onto $\operatorname{link}\gamma$ has kernel $X\cap\gamma X$, and $\dim_E\operatorname{link}\gamma=k+1$.

(ii) *For $p, q \in \operatorname{link}\gamma$: $\langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+ = \langle p^{\mathrm{t}}, q^{\mathrm{t}}\rangle_+$, hence $\langle p, q\rangle_{\mathrm{mt}} = \langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+$.*

(iii) *The maps $p \mapsto p^{\mathrm{s}}$ and $p \mapsto p^{\mathrm{t}}$ are bijections of $\operatorname{link}\gamma$ onto V_0^+. The* ***link isometry*** *$\Upsilon_\gamma\colon V_0^+ \to V_0^+$, $\Upsilon_\gamma(p^{\mathrm{s}}) = p^{\mathrm{t}}$ ($p \in \operatorname{link}\gamma$), is an isometry of $\langle\cdot,\cdot\rangle_+$, and $\operatorname{link}\gamma = \{p : \Upsilon_\gamma(p^{\mathrm{s}}) = p^{\mathrm{t}}\}$. In particular $\operatorname{link}\gamma = \operatorname{link}\gamma'$ if and only if $\Upsilon_\gamma = \Upsilon_{\gamma'}$.*

(iv) *For all pairs p, q: $\langle p, q\rangle_{\mathrm{lk}} = \langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+ - \langle p^{\mathrm{t}}, q^{\mathrm{t}}\rangle_+$. The link is Lagrangian for the link form: $\operatorname{link}\gamma = \{p : \langle q, p\rangle_{\mathrm{lk}} = 0$ for all $q \in \operatorname{link}\gamma\}$.*

(v) *The link metric is positive definite at every infinite place; if pairs $p^{(1)}, \dots, p^{(m)}$ are linearly independent over E, then $\det(\langle p^{(i)}, p^{(j)}\rangle_{\mathrm{mt}})_{i,j} \neq 0$.*

(vi) *If $c \in E^{\varrho+1}$ and $c' \in E$ satisfy $Z(\gamma)c = c'\varepsilon_\varrho$ (ε_ϱ the last standard basis vector) and y_0 is the X_0-part of $\sum_l c_l \gamma f_l$, then $\Upsilon_\gamma(y_0 + c_\varrho z^+) = -(c'/N)\, z^+$.*

Proof. Throughout, a vector $u \in V$ is written $u = \sum_a(\alpha_a e_a + \beta_a e_a^*) + [u]$. Since $h(e_a, e_b) = h(e_a^*, e_b^*) = 0$, $h(e_a, e_b^*) = \delta_{ab}$ and the hyperbolic planes are orthogonal to V_0, we have $h(e_a, u) = \beta_a$; so $u \perp X$ if and only if all $\beta_a(u) = 0$, and for $u, u' \perp X$

$$h(u, u') = h([u], [u']). \tag{5.17}$$

A vector t lies in W_γ if and only if $t \perp X$ and $\gamma^{-1}t \perp X$ (γ is an isometry).

(i). If $t \in W_\gamma$ and $[t] = [\gamma^{-1}t] = 0$, then t and $\gamma^{-1}t$ have neither e^*- nor middle coordinates, so $t \in X$ and $\gamma^{-1}t \in X$, that is $t \in X \cap \gamma X$. Conversely $X \cap \gamma X \subseteq W_\gamma$, as X and γX are totally isotropic, and its elements have $[t] = [\gamma^{-1}t] = 0$. Since h is nondegenerate on V ($\dim V = r = 2\varrho + k + 1$) and $\dim(X + \gamma X) = 2\varrho - \dim(X \cap \gamma X)$, rank–nullity gives $\dim \operatorname{link}\gamma = \dim W_\gamma - \dim(X \cap \gamma X) = r - 2\varrho = k + 1$.

(ii). Let $p = ([t], [\gamma^{-1}t])$, $q = ([t'], [\gamma^{-1}t'])$ with $t, t' \in W_\gamma$, and write $[t] = y_0 + \lambda' z$, $[\gamma^{-1}t] = y_0' + \lambda z$, $[t'] = w_0 + \mu' z$, $[\gamma^{-1}t'] = w_0' + \mu z$. By (5.17) for (t, t') and for $(\gamma^{-1}t, \gamma^{-1}t')$, and since γ is an isometry,

$$h(y_0, w_0) - N\bar\lambda'\mu' = h(t, t') = h(\gamma^{-1}t, \gamma^{-1}t') = h(y_0', w_0') - N\bar\lambda\mu.$$

Rearranged, $h(y_0, w_0) + N\bar\lambda\mu = h(y_0', w_0') + N\bar\lambda'\mu'$, which is $\langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+ = \langle p^{\mathrm{t}}, q^{\mathrm{t}}\rangle_+$. The two sums in the definition of $\langle p, q\rangle_{\mathrm{mt}}$ consist of the same four terms as $\langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+ + \langle p^{\mathrm{t}}, q^{\mathrm{t}}\rangle_+$, so $\langle p, q\rangle_{\mathrm{mt}} = \frac{1}{2}(\langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+ + \langle p^{\mathrm{t}}, q^{\mathrm{t}}\rangle_+) = \langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+$.

(iii). At any infinite place v, $\langle a, a\rangle_+$ maps to $\sum_i |\Phi_v(a_i)|^2 + N|\Phi_v(\lambda)|^2$ for $a = \sum_i a_i x_i + \lambda z^+$, so $\langle\cdot,\cdot\rangle_+$ is anisotropic. If $p \in \operatorname{link}\gamma$ and $p^{\mathrm{s}} = 0$, then $\langle p^{\mathrm{t}}, p^{\mathrm{t}}\rangle_+ = 0$ by (ii), so $p^{\mathrm{t}} = 0$; as p^{s} and p^{t} together contain all coordinates of p, $p = 0$. The same holds with the roles exchanged. So both maps are injective on $\operatorname{link}\gamma$, and bijective onto V_0^+ because both spaces have dimension $k + 1$ by (i). Hence Υ_γ is well defined and bijective, and it preserves $\langle\cdot,\cdot\rangle_+$ by (ii). If $p \in \operatorname{link}\gamma$, then $\Upsilon_\gamma(p^{\mathrm{s}}) = p^{\mathrm{t}}$ by definition. Conversely, if $\Upsilon_\gamma(p^{\mathrm{s}}) = p^{\mathrm{t}}$, take $q \in \operatorname{link}\gamma$ with $q^{\mathrm{s}} = p^{\mathrm{s}}$; then $q^{\mathrm{t}} = \Upsilon_\gamma(q^{\mathrm{s}}) = p^{\mathrm{t}}$, so $q = p$. The last assertion follows: equal links give equal maps by the definition, and equal maps give equal links by the graph description.

(iv). With $p = (y_0 + \lambda' z, y_0' + \lambda z)$ and $q = (w_0 + \mu' z, w_0' + \mu z)$,

$$\langle p, q\rangle_{\mathrm{lk}} = h(y_0, w_0) - N\bar\lambda'\mu' - h(y_0', w_0') + N\bar\lambda\mu = \langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+ - \langle p^{\mathrm{t}}, q^{\mathrm{t}}\rangle_+.$$

For $p, q \in \operatorname{link}\gamma$ this vanishes by (ii). Conversely, let p be a pair with $\langle q, p\rangle_{\mathrm{lk}} = 0$ for all $q \in \operatorname{link}\gamma$. Put $d = \Upsilon_\gamma(p^{\mathrm{s}}) - p^{\mathrm{t}}$ and choose, by (iii), $q \in \operatorname{link}\gamma$ with $q^{\mathrm{t}} = d$. Then, Υ_γ being an isometry with $\Upsilon_\gamma(q^{\mathrm{s}}) = q^{\mathrm{t}}$,

$$\langle d, d\rangle_+ = \langle q^{\mathrm{t}}, \Upsilon_\gamma(p^{\mathrm{s}})\rangle_+ - \langle q^{\mathrm{t}}, p^{\mathrm{t}}\rangle_+ = \langle q^{\mathrm{s}}, p^{\mathrm{s}}\rangle_+ - \langle q^{\mathrm{t}}, p^{\mathrm{t}}\rangle_+ = \langle q, p\rangle_{\mathrm{lk}} = 0.$$

So $d = 0$ by anisotropy, and $p \in \operatorname{link}\gamma$ by (iii).

(v). At an infinite place v, $\langle p,p\rangle_{\mathrm{mt}}$ maps to half the sum of the positive definite expressions $\langle p_i^+, p_i^+\rangle_+$ of (iii), which vanishes only for $p = 0$. If $p^{(1)},\dots,p^{(m)}$ are independent over E, their images under Φ_v are independent over $\mathbb{C}$ (a nonzero minor of their coordinate matrix stays nonzero), so their Gram matrix for a positive definite hermitian form has positive determinant; hence the determinant over E is nonzero.

(vi). Put $y = \sum_l c_l\gamma f_l$. By Lemma 5.2.26 (i), with $D = c'$, y has no e^*-component and z-coordinate $-c'/N$; so $y \perp X$ and $[y] = y_0 - (c'/N)z$. Moreover $\gamma^{-1}y = \sum_l c_l f_l = \sum_{a=1}^{\varrho} c_{a-1}e_a + c_\varrho z$ is orthogonal to X and $[\gamma^{-1}y] = c_\varrho z$. Hence $y \in W_\gamma$ and $p = ([y],[\gamma^{-1}y]) \in \operatorname{link}\gamma$ has $p^{\mathrm{s}} = y_0 + c_\varrho z^+$ and $p^{\mathrm{t}} = -(c'/N)z^+$. By (iii), $\Upsilon_\gamma(y_0 + c_\varrho z^+) = -(c'/N)z^+$. □

Lemma 5.2.29. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, with its fixed cutoff datum, and for $v \in \Phi$ let a_v be a constant of (BS3) for the cut pair of $\mathfrak{B}_v$. Let $c \in \mathcal{C}$, $s \in \bar{G}$, $T \subseteq G(E)$, and let $(F_v)_{v\in\Phi}$ be either $(W_v)_v$ or $(W_v^\flat)_v$, with $W(y) = \bigotimes_v F_v(y_v)$ accordingly (W^{full} or W^{cut}). Then:*

(i) for every $y \in G_\infty$,

$$\sum_{d\in\mathcal{C}} \sum_{\substack{\gamma\in T\\ \text{over } (c,s,d)}} \beta_d(\gamma^{-1}y)\prod_{v\in\Phi}\|F_v((\gamma^{-1}y)_v)\| \le 2\,b^{12n^3rf}\prod_{v\in\Phi}a_v;$$

(ii) the amplitude majorants are summable:

$$\sum_{d\in\mathcal{C}} \sum_{\substack{\gamma\in T\\ \text{over } (c,s,d)}} \int_{G_\infty} \beta_c(y)\,\beta_d(\gamma^{-1}y)\,\|W(y)\otimes\overline{W(\gamma^{-1}y)}\|\,dy < \infty.$$

Proof. For $z \in G_\infty$ put $m(z) = \prod_{v\in\Phi}\max(1,\ell(z_v)/b^{6n}) \ge 1$, and let $j = 3r^3f$. Since $r = 2n+k-1 \ge 2n$, we have $3r^2 \ge 12n^2 \ge 8n$, hence $j \ge 8nrf$.

Step 1: decay. By (BS3) there are measurable left R-invariant $A_v\colon G_1 \to [0,\infty]$ with $A_v \le a_v$ and $\max(1,\ell(x)/b^{6n})^{3r^3f}\|F_v(x)\| \le A_v(x)$ for $x \in G_1$. Multiplying over v,

$$\prod_v \|F_v(z_v)\| \le \prod_v A_v(z_v) \le \Big(\prod_v a_v\Big)m(z)^{-j} \qquad (z \in G_\infty). \tag{5.18}$$

Step 2: proof of (i). Fix y and let P_0 be the set of pairs (d,γ) with $d \in \mathcal{C}$ and $\gamma \in T$ a term over (c,s,d). By (5.18) at $z = \gamma^{-1}y$, and since the summands are nonnegative and $T \subseteq G(E)$, the left side of (i) is at most

$$\begin{aligned}\Big(\prod_v a_v\Big)\sum_{(d,\gamma)\in P_0}\beta_d(\gamma^{-1}y)\,m(\gamma^{-1}y)^{-j} &\le \Big(\prod_v a_v\Big)\sum_{d\in\mathcal{C}}\sum_{\gamma\text{ over }(c,s,d)}\beta_d(\gamma^{-1}y)\,m(\gamma^{-1}y)^{-j}\\ &\le 2\,b^{12n^3rf}\prod_v a_v,\end{aligned}$$

the last inequality by Lemma 5.3.61 with the integer $i = j \ge 8nrf$.

Step 3: proof of (ii). For vectors p,q the Hilbert–Schmidt norm of $p\otimes\bar q$ is $\|p\|\,\|q\|$, and $\|W(z)\| = \prod_v\|F_v(z_v)\|$; all norms on the finite-dimensional space $\mathbb{C}^{I\times I}$ are equivalent, so it suffices to treat the Hilbert–Schmidt norm. By Tonelli's theorem, (i) with its constant C_1, and (5.18) at $z = y$,

$$\sum_{(d,\gamma)\in P_0}\int_{G_\infty}\beta_c(y)\beta_d(\gamma^{-1}y)\|W(y)\|\,\|W(\gamma^{-1}y)\|\,dy \le C_1\int_{G_\infty}\beta_c(y)A(y)\,dy, \qquad A(y) = \prod_v A_v(y_v).$$

A is measurable, nonnegative and left invariant under $\mathsf{U}_\infty = \prod_v \mathsf{U}$, since each A_v is left R-invariant and $\mathsf{U} \subseteq R$. The groups G_∞ and U_∞ are unimodular (a product of unitary groups; a unipotent group), so $\mathsf{U}_\infty\backslash G_\infty = \prod_v \mathsf{U}\backslash G_1$ carries an invariant measure $d\dot{x}$ with $\int_{G_\infty} \phi\, dy = \int_{\mathsf{U}_\infty\backslash G_\infty} \int_{\mathsf{U}_\infty} \phi(ux)\, du\, d\dot{x}$ for measurable $\phi \geq 0$ (Weil's integration formula); by uniqueness of invariant measures on homogeneous spaces up to a factor, $d\dot{x} = \kappa \prod_v d\dot{x}_v$ for the fixed invariant measures $d\dot{x}_v$ on $\mathsf{U}\backslash G_1$ and some $\kappa > 0$. With $\phi = \beta_c A$ and Lemma 5.2.33,

$$\int_{G_\infty} \beta_c A\, dy = \int_{\mathsf{U}_\infty\backslash G_\infty} A(x) \int_{\mathsf{U}_\infty} \beta_c(ux)\, du\, d\dot{x} = \nu\, \kappa \prod_v \int_{\mathsf{U}\backslash G_1} A_v.$$

Each factor is finite: if $a_v > 0$ it is at most $b^{400r^2} \# I_v / a_v$ by (BS3), and if $a_v = 0$ then $A_v = 0$. This proves (ii). $\square$

Lemma 5.2.30. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, with its fixed cutoff datum. Let $c \in \mathcal{C}$, $s \in \bar{G}$, $y \in G_\infty$ and $R \geq 1$, and let P be the set of pairs (d, γ) with $d \in \mathcal{C}$ and γ a term over (c, s, d) such that $\ell((\gamma^{-1}y)_v) \leq b^{6n+2}R$ for every $v \in \Phi$. Then*

$$\sum_{(d,\gamma)\in P} \beta_d(\gamma^{-1}y) \leq \left(b^{6n^2} R^2\right)^{2nrf}.$$

Proof. Apply Lemma 5.2.31 with the same c, s, y, R, the set P of the statement, the weights $w(d, \gamma) = \beta_d(\gamma^{-1}y)$, $x = 1$, and V the whole space $\prod_{v\in\Phi}(\mathbb{C}^r)^{\varrho+1}$. Every pair of P is a term over (c, s, d) with $\ell((\gamma^{-1}y)_v) \leq b^{6n+2}R$ for all v, $w(d, \gamma) \leq \beta_d(\gamma^{-1}y) \cdot 1$, and the condition on differences of moved frames holds trivially. Since $\#\Phi = f$, $\dim_{\mathbb{R}} V = 2r(\varrho + 1)f = 2nrf$, and the lemma gives $\sum_{(d,\gamma)\in P} \beta_d(\gamma^{-1}y) \leq (b^{6n^2} R^2)^{2nrf}$. $\square$

Lemma 5.2.31. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, with its fixed cutoff datum. Let $c \in \mathcal{C}$, $s \in \bar{G}$, $y \in G_\infty$, $R \geq 1$, $x \in [0, \infty]$, let P be a set of pairs (d, γ) with $d \in \mathcal{C}$ and $\gamma \in G(E)$, and $w\colon P \to [0, \infty]$. Suppose that every $(d, \gamma) \in P$ with $w(d, \gamma) \neq 0$ satisfies: γ is a term over (c, s, d); $\ell((\gamma^{-1}y)_v) \leq b^{6n+2}R$ for every $v \in \Phi$; and $w(d, \gamma) \leq \beta_d(\gamma^{-1}y)\, x$. Let V be a real subspace of $\prod_{v\in\Phi}(\mathbb{C}^r)^{\varrho+1}$ such that for any two $(d, \gamma), (d', \gamma') \in P$ with nonzero weights*

$$\Big(\big((\gamma^{-1}y)_v\big)^{-1} f_b - \big((\gamma'^{-1}y)_v\big)^{-1} f_b\Big)_{v\in\Phi,\, 0\leq b\leq\varrho} \in V.$$

Then $\sum_{(d,\gamma)\in P} w(d, \gamma) \leq \left(b^{6n^2} R^2\right)^{\dim_{\mathbb{R}} V} x$.

Proof. Let $Q \subseteq P$ be the set of pairs with $w(d, \gamma) \neq 0$; only these contribute. For $(d, \gamma) \in Q$, $0 < w(d, \gamma) \leq \beta_d(\gamma^{-1}y)\, x$ forces $\beta_d(\gamma^{-1}y) \neq 0$ and $x \neq 0$. If $Q = \emptyset$ the sum is 0; assume $Q \neq \emptyset$. Put $\rho = b^{37n}R$, and for $(d, \gamma) \in Q$ write $p_\gamma = \gamma^{-1}y \in G_\infty$.

Step 1: boxes. For $(d, \gamma) \in Q$ the point p_γ is reduced ($\beta_d = \beta_d^0 \mathbf{1}_{\mathrm{red}}$), so by Lemma 5.2.21 and (RD5), $\|(p_\gamma)_v^{-1} f_a\| \leq b^{27}\ell((p_\gamma)_v) \leq b^{27} b^{6n+2} R \leq \rho$ for all v, a, as $6n + 29 \leq 37n$.

Step 2: one frame image, one class and one coset. Let $(d, \gamma), (d', \gamma') \in Q$ with $\gamma f_a = \gamma' f_a$ for all a. Then $d = d'$ and $\gamma^{-1}\gamma' \in \mathsf{U}_d^0(M)$, by Lemma 5.2.32 (ii) (both are terms over $(c, s, \cdot)$ with the same frame image). Consequently, if $Q_\varphi \subseteq Q$ is the set of pairs with a given frame image $\varphi = (\gamma f_a)_a$, fix $(d, \gamma_0) \in Q_\varphi$; every $(d', \gamma) \in Q_\varphi$ has $d' = d$ and $\gamma = \gamma_0\delta$ with $\delta \in \mathsf{U}_d^0(M)$, distinct pairs giving distinct δ, and $\gamma^{-1}y = \delta^{-1} \cdot y_0$ with $y_0 = \gamma_0^{-1}y$. For $a(u) = \beta_d^0(u \cdot y_0)$, (RD6) gives $\sum_{u_0\in\mathsf{U}_d^0(M)} a(u_0u') = 1$, so by Lemma 5.2.21 and (RD5) $\sum_{u\in\mathsf{U}_d(M)} \beta_d(u \cdot y_0) = 1$, and

$$\sum_{(d',\gamma)\in Q_\varphi} w(d', \gamma) \leq x \sum_{(d',\gamma)\in Q_\varphi} \beta_{d'}(\gamma^{-1}y) \leq x \sum_{u\in\mathsf{U}_d(M)} \beta_d(u \cdot y_0) = x.$$

Step 3: rigidity. Let $(d,\gamma),(d',\gamma')\in Q$ with $\|(p_{\gamma'})_v^{-1}f_a-(p_\gamma)_v^{-1}f_a\|<(N\rho)^{-1}$ for all $v\in\Phi$ and $0\le a\le\varrho$. We show $\gamma f=\gamma' f$. Put $g=\gamma^{-1}\gamma'$; then $h_d^{-1}gh_{d'}=k''\in K$, and for $x'\in\{f_b,e_b^*\}$ (vectors of L_0 fixed by h_d, $h_{d'}$) the numbers $h(x',gf_a)=h(x',k''f_a)$ lie in $E\cap\hat{\mathcal{O}}=\mathcal{O}_E$. An element of $\mathcal{O}_E$ with $|v(\cdot)|<1$ for all $v\in\Phi$ (hence for all embeddings, the others being conjugate) is 0, its norm being an integer of absolute value <1. Fix v, write $p=(p_\gamma)_v$, $p'=(p_{\gamma'})_v=g_v^{-1}p$. Since $g_v=B_v^{-1}v(g)B_v$ with $B_v^*H_vB_v=J_1$ ($B_v=\mathrm{diag}(\mathrm{bs}_v)$, real; H_v the Gram matrix of $V\otimes_{E,v}\mathbb{C}$),

$$v\big(h(x',gx'')\big)=\frac{h_1(x',g_vx'')}{\mathrm{bs}_v(x')\mathrm{bs}_v(x'')}\qquad(x',x''\text{ basis vectors}).$$

(a) Frame products. $h_1(f_b,g_vf_a)-h_1(f_b,f_a)=h_1(p^{-1}f_b,\ p'^{-1}f_a-p^{-1}f_a)$, of modulus $<\rho\,(N\rho)^{-1}$ by Step 1 ($|h_1(x',x'')|\le\|x'\|\|x''\|$, J_1 unitary), and $|\mathrm{bs}_v(f_b)\mathrm{bs}_v(f_a)|=|\mathrm{rs}_v|^{2\varrho-a-b}/N\ge1/N$ by (FA4). So $|v(h(f_b,gf_a)-h(f_b,f_a))|<N\rho(N\rho)^{-1}=1$ for all v, and $h(f_b,gf_a)=h(f_b,f_a)$ for all a,b. (b) Only e-coordinates move. By (a) with $f_b=e_b$, the e_b^*-coordinates $h(e_b,\cdot)$ of gf_a and f_a agree, and with $f_b=z$ so do the z-coordinates. So $gf_a=f_a+\mu_a+m_a$ with $\mu_a\in F_\varrho$, $m_a\in X_0$. Since f_a and μ_a are orthogonal to F_ϱ and to X_0 and μ_a is isotropic, $0=h(gf_a,gf_a)-h(f_a,f_a)=h(m_a,m_a)$, so $m_a=0$ (X_0 is definite). (c) The e-coordinates, by downward induction on b. Let $\Delta_b=h(e_b^*,gf_a)-h(e_b^*,f_a)\in\mathcal{O}_E$ and assume $\Delta_{b'}=0$ for $b'>b$. By (b), $g_vf_a-f_a=\sum_{b'<b}c_{b'}e_{b'}$ with $c_{b'}=v(\Delta_{b'})\mathrm{bs}_v(f_a)/\mathrm{bs}_v(e_{b'})$, and $p^{-1}(g_vf_a-f_a)=p'^{-1}f_a-p^{-1}f_a$ has norm $<(N\rho)^{-1}$. Since $\ell(p)\le b^{6n+2}R$ and $\|sp\|_{\mathrm{op}}=\|(h'^{-1}u'h')p\|_{\mathrm{op}}$ for $s=u'h'\in R$ ($h'\in H$ is unitary and normalises U), there is $u_1\in\mathsf{U}$ with $\|u_1p\|_{\mathrm{op}}<b^{6n+2}R$. As u_1 maps $e_{b'}$ to $e_{b'}$ plus a combination of $e_{b''}$, $b''<b'$, the e_b-coordinate of $u_1(g_vf_a-f_a)=(u_1p)\,p^{-1}(g_vf_a-f_a)$ is c_b, so $|c_b|<b^{6n+2}R\,(N\rho)^{-1}$. With $|\mathrm{bs}_v(e_b)/\mathrm{bs}_v(f_a)|=|\mathrm{rs}_v|^{a+1-b}\le b^{23\varrho}$ (FA4), $|v(\Delta_b)|<b^{23\varrho+6n+2}R/(Nb^{37n}R)<1$, as $23\varrho+6n+2\le37n$. Hence $\Delta_b=0$. So $gf_a=f_a$ for every a, that is, $\gamma f=\gamma' f$.

Step 4: packing in V. Let $\Sigma_0\subseteq Q$ contain exactly one pair for each frame image occurring in Q, and fix $(d_0,\gamma_0)\in\Sigma_0$. On $X=\prod_{v\in\Phi}(\mathbb{C}^r)^{\varrho+1}$ use the norm $\|\xi\|=\max_{v,a}\|\xi_{v,a}\|$, and for $(d,\gamma)\in Q$ put $\xi_\gamma=((p_\gamma)_v^{-1}f_a)_{v,a}$. By hypothesis $\eta_\gamma=\xi_\gamma-\xi_{\gamma_0}\in V$; by Step 1, $\|\eta_\gamma\|\le2\rho$; by Step 3, $\|\eta_\gamma-\eta_{\gamma'}\|=\|\xi_\gamma-\xi_{\gamma'}\|\ge(N\rho)^{-1}$ for distinct elements of Σ_0. Let $m=\dim_{\mathbb{R}}V$ and equip V with the restricted norm and a Haar measure λ; balls of radius t in V have λ-measure proportional to t^m. The open balls in V of radius $t_0=(2N\rho)^{-1}$ about the η_γ ($\gamma\in\Sigma_0$) are pairwise disjoint and lie in the ball of radius $2\rho+t_0$ about 0. Hence (for $m=0$ all η_γ are 0 and $\#\Sigma_0=1$), using $N\le b^{60n}$ (FA5), $b\ge1000$, $R\ge1$ and $6n^2\ge134n+1$ ($n\ge10^5$ by (FG6)),

$$\#\Sigma_0\le\Big(\frac{2\rho+t_0}{t_0}\Big)^m=(1+4N\rho^2)^m\le\big(5b^{134n}R^2\big)^m\le\big(b^{6n^2}R^2\big)^m.$$

By Step 2, $\sum_{(d,\gamma)\in P}w(d,\gamma)=\sum_\varphi\sum_{Q_\varphi}w\le\#\Sigma_0\cdot x$, the outer sum over the frame images of Σ_0; this is the claim. □

Lemma 5.2.32. *Let $\mathfrak{S}$ be an admissible filling datum, $c,d,d'\in\mathcal{C}$, $s,s'\in\bar G$, γ a term over (c,s,d) and γ' a term over (c,s',d') with $\gamma f_a=\gamma' f_a$ for $0\le a\le\varrho$. Then:*

(i) $s'^{-1}s\in\bar R_0$;

(ii) if $s=s'$, then $d=d'$ and $\gamma^{-1}\gamma'\in\mathsf{U}_d^0(M)$.

Proof. Put $\delta=\gamma^{-1}\gamma'\in G(E)$, $k=h_c^{-1}\gamma h_d$ and $k'=h_c^{-1}\gamma'h_{d'}$, elements of K with $\mathrm{red}(k)=s$, $\mathrm{red}(k')=s'$; then $h_d^{-1}\delta h_{d'}=k^{-1}k'\in K$.

δ lies in the Bessel subgroup. δ fixes $e_1,\dots,e_\varrho,z$. As it fixes F_b pointwise, it stabilises $F_b^\perp$, and on $F_\varrho^\perp/F_\varrho\cong X_0\oplus Ez$ it induces an isometry fixing z, hence preserving X_0; let $h\in\mathrm{U}(X_0)(E)$

be its restriction to X_0. Then $u = \delta h^{-1}$ acts trivially on all successive quotients of the refined flag $F_1 \subset \cdots \subset F_\varrho \subset F_\varrho^\perp \subset \cdots \subset F_1^\perp \subset V$ (on $V/F_\varrho^\perp$ by duality), so $u \in \mathsf{U}(E)$ and $\delta = uh$.

(i) $h_d^{-1}\delta h_{d'} = (h_d^{-1}uh_d)(h_d^{-1}hh_{d'})$ lies in the Bessel subgroup $\mathsf{U}(\mathbb{A}_f)\mathrm{U}(X_0)(\mathbb{A}_f)$, as $h_d, h_{d'} \in \mathrm{U}(X_0)(\mathbb{A}_f)$ and $\mathrm{U}(X_0)$ normalises U; it also lies in K. So $s^{-1}s' = \mathrm{red}(k^{-1}k')$ lies in the reduced image $\bar{R}_0$ of the Bessel subgroup, and so does its inverse $s'^{-1}s$.

(ii) Let $s = s'$; then $h_d^{-1}\delta h_{d'} = k^{-1}k' \in \mathrm{K}(M)$. In the lattice basis (which differs from the standard one only in the middle block) $h_d^{-1}uh_d \in \mathsf{U}(\mathbb{A}_f)$ is block unitriangular with identity middle block and $h_d^{-1}hh_{d'}$ is block diagonal with identity outside the middle block; so the middle blocks of $h_d^{-1}hh_{d'}$ and of its inverse are those of the element $h_d^{-1}\delta h_{d'}$ of $\mathrm{K}(M)$ and of its inverse. Hence $h_d^{-1}hh_{d'}$ is integral, with integral inverse, and reduces to 1 at the chosen places: it lies in $\mathrm{K}(M) \cap \mathrm{U}(X_0)(\mathbb{A}_f) = \mathrm{K}_X$ by (FA1). So $h_{d'} \in h^{-1}h_d\mathrm{K}_X$ with h rational, which says that $d = d'$ (FG4); then $h_d = h_{d'}$ and $h \in h_d\mathrm{K}_X h_d^{-1}$ is a rational point of $\mathrm{U}(X_0)$, so $h = 1$ by (FG4). Thus $\delta = u \in \mathsf{U}(E)$ with $h_d^{-1}\delta h_d \in \mathrm{K}(M)$ fixing the frame: $\delta \in \mathsf{U}_d^0(M)$. □

Lemma 5.2.33. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, with its fixed cutoff datum. Then for every class $c \in \mathcal{C}$ and every $y \in G_\infty$*

$$\int_{\mathsf{U}_\infty} \beta_c(uy)\, du = \nu.$$

Proof. Write $\Gamma = \{\gamma_\infty : \gamma \in \mathsf{U}_c(M)\}$. As each $v \in \Phi$ is injective, $\gamma \mapsto \gamma_\infty$ is injective, and $\Gamma \subseteq \mathsf{U}_\infty$ (a diagonal conjugate of an element of $\mathsf{U}(E)$ lowers levels). For $u \in \mathsf{U}_\infty$ and $1 \le a \le \varrho$ put $\mathrm{rw}_a^{\mathrm{r}}(u) = \big((u_v)_{e_aj}\big)_{j\in J_a, v\in\Phi} \in \mathbb{C}^{J_a\times\Phi}$ and $\mathrm{rw}_a^{\mathrm{i}}(u) = \big(\mathrm{Im}(u_v)_{e_ae_a^*}\big)_v \in \mathbb{R}^\Phi$, and let $\mathrm{rw}_\infty = (\mathrm{rw}_a^{\mathrm{r}}, \mathrm{rw}_a^{\mathrm{i}})_a$. By Lemma 5.2.34 at each place and Fubini, rw_∞ is continuous and injective, carries the product measure du to Lebesgue measure, and du is left invariant and invariant under inversion.

Step 1: the translates of β_c add up to 1. For every $w \in G_\infty$, by (i) of Lemma 5.2.36,

$$\sum_{\gamma\in\mathsf{U}_c(M)} \beta_c(\gamma\cdot w) = 1.$$

Step 2: lattices. Let $W_a \subseteq V$ be the span of J_a (the e_b, e_b^* with $b > a$, X_0 and z). Put

$$\Lambda_a = \Big\{\Big(M\, v(h(w,j))\,\tfrac{\mathrm{bs}_v(j)}{\mathrm{bs}_v(e_a)}\Big)_{j\in J_a,\, v\in\Phi} :\ w \in W_a \cap L_c\Big\},$$
$$I_a = \Big\{\Big(\tfrac{M\,\mathrm{Im}\, v(s)}{\mathrm{fs}(v,a-1)}\Big)_{v\in\Phi} :\ s \in \mathcal{O}_E,\ \bar{s} = -s\Big\}.$$

Note $\mathrm{bs}_v(e_a^*)/\mathrm{bs}_v(e_a) = \mathrm{bs}_v(e_a)^{-2} = \mathrm{fs}(v,a-1)^{-1}$, and that L_0, hence L_c, is the direct sum of the $\hat{\mathcal{O}}e_b$, $\hat{\mathcal{O}}e_b^*$ and $L_c \cap (X_0 \oplus Ez)(\mathbb{A}_f)$ (from the lattice basis and h_c, which fixes e_b, e_b^*). (L1) For $\gamma \in \mathsf{U}_c(M)$, $\mathrm{rw}_a^{\mathrm{r}}(\gamma_\infty) \in \Lambda_a$. Indeed, $\gamma^{-1} \in \mathsf{U}_c(M)$, so $w' = M^{-1}(\gamma^{-1} - 1)e_a^* \in V \cap L_c$ by Lemma 5.2.23 (i), and for $j \in J_a$, $\gamma_{e_aj} = h(e_a^*, \gamma j) = h(\gamma^{-1}e_a^*, j) = M\,h(w', j)$, as $h(e_a^*, j) = 0$. Let w be the component of w' in W_a; by the direct sum it lies in L_c, and $h(w,j) = h(w',j)$ for $j \in J_a$ (the dropped components are along e_b, e_b^*, $b \le a$, which pair trivially with J_a). By (5.5), $(\gamma_v)_{e_aj} = v(\gamma_{e_aj})\mathrm{bs}_v(j)/\mathrm{bs}_v(e_a)$. (L2) If moreover $\mathrm{rw}_d^{\mathrm{r}}(\gamma_\infty) = 0$, $\mathrm{rw}_d^{\mathrm{i}}(\gamma_\infty) = 0$ for $d > a$ and $\mathrm{rw}_a^{\mathrm{r}}(\gamma_\infty) = 0$, then $\mathrm{rw}_a^{\mathrm{i}}(\gamma_\infty) \in I_a$. Indeed, with w' as in (L1), $\gamma_{e_ae_a^*} = Ms$, $s = h(w', e_a^*) \in E \cap \hat{\mathcal{O}} = \mathcal{O}_E$. By Lemma 5.2.34 (v) (second claim) at v with $x = \gamma_v^{-1}$: $0 = (\gamma_v^{-1})_{e_ae_a^*} + (\gamma_v)_{e_ae_a^*}$, and $(\gamma_v^{-1})_{e_ae_a^*} = (\mathcal{J}\gamma_v^*\mathcal{J})_{e_ae_a^*} = \overline{(\gamma_v)_{e_ae_a^*}}$ ($\mathcal{J}$ the Gram matrix of V_1). So $v(s) + \overline{v(s)} = 0$ for all v, that is $v(s + \bar{s}) = 0$ and $\bar{s} = -s$; and $\mathrm{Im}(\gamma_v)_{e_ae_a^*} = M\,\mathrm{Im}\, v(s)/\mathrm{fs}(v, a-1)$. (L3) Every

point of Λ_a, resp. I_a, is attained: for $w \in W_a \cap L_c$, $\gamma = E_a(w,0) \in \mathsf{U}_c(M)$ has $\gamma_{e_a j} = Mh(w,j)$ for $j \in J_a$ and $\gamma_{e_d j} = 0$, $\gamma_{e_d e_d^*} = 0$ for $d > a$, $j \in J_d$ (Lemma 5.2.23 (ii); $e_a^* \notin J_d$); so $\mathrm{rw}_a^{\mathrm{r}}(\gamma_\infty)$ is the given point and the coordinates of the rows $d > a$ vanish. For $s \in \mathcal{O}_E$ with $\bar{s} = -s$, $\gamma = E_a(0,s)$ has $\gamma - 1 = Ms\,h(e_a, \cdot)e_a$, so all row coordinates of γ_∞ vanish except $\mathrm{rw}_a^{\mathrm{i}}(\gamma_\infty) = (M \operatorname{Im} v(s)/\mathrm{fs}(v, a-1))_v$.

Step 3: a fundamental box. The sets Λ_a, I_a are lattices (Step 5). Let P_a, Q_a be Borel fundamental domains of Λ_a in $\mathbb{C}^{J_a \times \Phi}$ and of I_a in $\mathbb{R}^\Phi$ (each point has exactly one translate by the lattice in it; for instance the half-open parallelepiped of a basis), of volumes the covolumes. Order the coordinate blocks as $\mathrm{rw}_\varrho^{\mathrm{r}}, \mathrm{rw}_\varrho^{\mathrm{i}}, \mathrm{rw}_{\varrho-1}^{\mathrm{r}}, \dots, \mathrm{rw}_1^{\mathrm{r}}, \mathrm{rw}_1^{\mathrm{i}}$, and let $\mathfrak{B} = \prod_a (P_a \times Q_a)$. We claim: for every $u \in \mathsf{U}_\infty$ there is exactly one $\gamma \in \Gamma$ with $\mathrm{rw}_\infty(u\gamma) \in \mathfrak{B}$. By Lemma 5.2.34 (v) at each place: if $\gamma \in \Gamma$ has all blocks before a block β equal to 0, then the block β of $x\gamma$ is that of x plus that of γ, for every $x \in \mathsf{U}_\infty$ (for $\beta = \mathrm{rw}_a^{\mathrm{r}}$ by the first claim, for $\beta = \mathrm{rw}_a^{\mathrm{i}}$ by the second); applied to the earlier blocks, these blocks of $x\gamma$ equal those of x. Existence: suppose $\gamma \in \Gamma$ puts the blocks before β of $u\gamma$ into their domains; take the lattice point λ with block β of $u\gamma$ plus λ in the domain, and by (L3) $\delta \in \Gamma$ with blocks before β zero and block β equal to λ; then $u\gamma\delta$ has the blocks up to β in their domains. Uniqueness: if γ, γ' both work, let $\delta = \gamma^{-1}\gamma' \in \Gamma$; if the blocks of δ before β vanish, then block β of $u\gamma' = (u\gamma)\delta$ is block β of $u\gamma$ plus that of δ, which lies in the lattice by (L1), (L2); both lie in the domain, so block β of δ vanishes. Hence $\mathrm{rw}_\infty(\delta) = 0 = \mathrm{rw}_\infty(1)$ and $\delta = 1$.

Step 4: unfolding. Let $B = \mathrm{rw}_\infty^{-1}(\mathfrak{B})$, a Borel set. By Step 3, for every u exactly one $\gamma \in \Gamma$ has $\gamma u \in B^{-1}$, so $\sum_{\gamma\in\Gamma} \mathbf{1}_{B^{-1}}(\gamma u) = 1$. With Tonelli (countable sums of nonnegative functions), left invariance, Step 1 (for $w = uy$, with $\gamma \mapsto \gamma^{-1}$ a bijection of Γ) and invariance under inversion,

$$\begin{aligned}\int \beta_c(uy)\,du &= \sum_{\gamma\in\Gamma} \int \mathbf{1}_{B^{-1}}(\gamma u)\beta_c(uy)\,du = \int_{B^{-1}} \sum_{\gamma\in\Gamma} \beta_c(\gamma^{-1}uy)\,du \\ &= du(B^{-1}) = du(B) = \mathrm{vol}(\mathfrak{B}) = \prod_{a=1}^{\varrho} \mathrm{covol}(\Lambda_a)\,\mathrm{covol}(I_a).\end{aligned}$$

Step 5: the covolumes. We use: the Minkowski image $\phi(\mathcal{O}_E)$, $\phi(x) = (v(x))_{v\in\Phi}$, has covolume $2^{-f}|\mathrm{disc}\,E|^{1/2} = \mathrm{vol}(\hat{\mathcal{O}})^{-1}$ in $\mathbb{C}^\Phi = \mathbb{R}^{2f}$; and for $A \in \mathrm{GL}_n(\mathbb{A}_f)$ the lattice $E^n \cap A\hat{\mathcal{O}}^n$ has covolume $\mathrm{vol}(\hat{\mathcal{O}})^{-n}|\det A|_{\mathbb{A}_f}^{-1}$ under ϕ, because $L \mapsto L \cap E^n$ is an index-preserving bijection from the open compact $\hat{\mathcal{O}}$-lattices of $\mathbb{A}_f^n$ onto the $\mathcal{O}_E$-lattices of E^n (strong approximation), and $[L : L'] = \mathrm{vol}(L)/\mathrm{vol}(L')$ for Haar measure on $\mathbb{A}_f^n$. Λ_a: the conjugate-linear map $w \mapsto (h(w,j))_{j\in J_a}$ is a bijection of W_a onto E^{J_a}. On the e_b, e_b^* ($b > a$) it is $w \mapsto (\overline{w_{e_b^*}}, \overline{w_{e_b}})$, and $W_a \cap L_c$ has coordinates ranging over $\mathcal{O}_E$ there. On the middle it is $m \mapsto (\overline{m_i})_{1\le i\le k} \oplus (-N\overline{m_z})$, and the middle coordinate vectors form $E^{k+1} \cap A\hat{\mathcal{O}}^{k+1}$ with $A = h_c B$, B the matrix of the middle lattice basis (the basis vectors of X_0 with x_k replaced by $x_k/\bar{\pi} + z/N$, and z), $\det B = \bar{\pi}^{-1}$. Here $|\det h_c|_{\mathbb{A}_f} = 1$ (as $\det h_c \overline{\det h_c} = 1$ and $|\bar{x}|_{\mathbb{A}_f} = |x|_{\mathbb{A}_f}$), and $|\bar{\pi}|_{\mathbb{A}_f}^2 = |\pi\bar{\pi}|_{\mathbb{A}_f} = |N|_{\mathbb{A}_f} = N^{-2f}$, so $|\det A|_{\mathbb{A}_f} = N^f$. Conjugation preserves Lebesgue measure and the factor $-N$ multiplies it by N^{2f}. So the unscaled set has covolume $\mathrm{vol}(\hat{\mathcal{O}})^{-\#J_a} N^{-f} N^{2f}$, and scaling the coordinate (j,v) by the real number $M\mathrm{bs}_v(j)/\mathrm{bs}_v(e_a)$ multiplies it by $\prod_{v,j}(M\mathrm{bs}_v(j)/\mathrm{bs}_v(e_a))^2$. Since $\mathrm{bs}_v(e_b)\mathrm{bs}_v(e_b^*) = 1$, $\mathrm{bs}_v(x_i) = 1$, $\mathrm{bs}_v(z)^2 = 1/N$ and $\mathrm{bs}_v(e_a)^2 = \mathrm{fs}(v, a-1)$,

$$\mathrm{covol}(\Lambda_a) = \Big(\mathrm{vol}(\hat{\mathcal{O}})^{-1} \prod_v \frac{M^2}{\mathrm{fs}(v, a-1)}\Big)^{\#J_a}, \qquad \#J_a = 2(\varrho - a) + k + 1.$$

I_a: let ι be a power of the primitive 5040-th root of unity of order 4; $\iota^2 = -1$, $\bar{\iota} = -\iota$, so the $s \in \mathcal{O}_E$ with $\bar{s} = -s$ are exactly ιt, $t \in \mathcal{O}_F$ ($F = E^+$), and $\operatorname{Im} v(\iota t) = \pm v(t)$. The restrictions

$v|_F$ ($v \in \Phi$) are the f real embeddings of F, so $\{(\operatorname{Im} v(s))_v\}$ has covolume $|\operatorname{disc} F|^{1/2}$. E contains a primitive 28-th root of unity, so it satisfies (Um) (Lemma 3.2.9 (1)) and $|\operatorname{disc} E| = |\operatorname{disc} F|^2$ (Lemma 5.2.35 (i)). Hence $\operatorname{covol}(I_a) = |\operatorname{disc} E|^{1/4} \prod_v M/\operatorname{fs}(v, a-1)$.

Step 6: the product. $\sum_{a=1}^{\varrho} \#J_a = \varrho(\varrho-1) + \varrho(k+1) = d$, and $\#J_a + 1 = 2(\varrho - (a-1)) + k$. So

$$\prod_a \operatorname{covol}(\Lambda_a)\operatorname{covol}(I_a) = M^{f(2d+\varrho)}\operatorname{vol}(\hat{\mathcal{O}})^{-d}|\operatorname{disc} E|^{\varrho/4} \prod_v \prod_{a'=0}^{\varrho-1} \operatorname{fs}(v,a')^{-(k+2(\varrho-a'))},$$

and

$$\prod_v \prod_{a'=0}^{\varrho-1} \operatorname{fs}(v,a')^{k+2(\varrho-a')} = \Lambda_{\mathrm{fr}} \prod_v \operatorname{fs}(v,\varrho)^{-k} = \Lambda_{\mathrm{fr}} N^{kf}.$$

The product is $(M^{2d+\varrho}/N^k)^f \operatorname{vol}(\hat{\mathcal{O}})^{-d}|\operatorname{disc} E|^{\varrho/4}\Lambda_{\mathrm{fr}}^{-1} = \nu$. □

Lemma 5.2.34. *Give the basis vectors of V_1* levels*: e_b has level b, the x_i and z level $\varrho+1$, e_b^* level $2\varrho+2-b$. For $1 \le a \le \varrho$ let J_a be the set of basis vectors of level strictly between a and $2\varrho+2-a$ (the e_b, e_b^* with $a < b \le \varrho$, the x_i and z). For $u \in \mathsf{U}$ the* row coordinates *are*

$$\mathrm{rw}(u) = \Big((u_{e_a j})_{j \in J_a},\ \operatorname{Im} u_{e_a e_a^*} \Big)_{1 \le a \le \varrho} \in \mathcal{C}_{\mathsf{U}} = \prod_{a=1}^{\varrho} (\mathbb{C}^{J_a} \times \mathbb{R}),$$

where u_{ij} is the i-coordinate of uj; $\mathcal{C}_{\mathsf{U}}$ carries Lebesgue measure ($\mathbb{C} = \mathbb{R}^2$). Then:

(i) rw *is continuous and injective.*

(ii) For every $g \in \mathsf{U}$ there is an affine bijection A_g of $\mathcal{C}_{\mathsf{U}}$ preserving Lebesgue measure with $\mathrm{rw}(ug) = A_g(\mathrm{rw}(u))$ for all $u \in \mathsf{U}$.

(iii) $du(\mathrm{rw}^{-1}(B))$ is the Lebesgue measure of B for every Borel set $B \subseteq \mathcal{C}_{\mathsf{U}}$.

(iv) du is a left and right Haar measure of U, and it is invariant under $u \mapsto u^{-1}$.

(v) Let $1 \le a \le \varrho$ and $\gamma \in \mathsf{U}$ with $\gamma_{e_d j} = 0$ for all $d > a$ and $j \in J_d$, and $\operatorname{Im} \gamma_{e_d e_d^*} = 0$ for all $d > a$. Then $(w\gamma)_{e_a j} = w_{e_a j} + \gamma_{e_a j}$ for all $w \in \mathsf{U}$ and $j \in J_a$. If moreover $\gamma_{e_a j} = 0$ for all $j \in J_a$, then also $(w\gamma)_{e_a e_a^*} = w_{e_a e_a^*} + \gamma_{e_a e_a^*}$ for all $w \in \mathsf{U}$.

Proof. Let $\mathcal{J}$ be the Gram matrix of V_1 in this basis ($\mathcal{J}e_b = e_b^*$, $\mathcal{J}e_b^* = e_b$, $\mathcal{J}x_i = x_i$, $\mathcal{J}z = -z$ on coordinate vectors), a hermitian involution; $g \in G_1$ means $g^*\mathcal{J}g = \mathcal{J}$, equivalently $g\mathcal{J}g^* = \mathcal{J}$ (as $g^{-1} = \mathcal{J}g^*\mathcal{J}$). With $F_b = \langle e_1, \dots, e_b \rangle$, the spans of the basis vectors of level at most $1, \dots, 2\varrho+1$ are $F_1 \subset \cdots \subset F_\varrho \subset F_\varrho^\perp \subset \cdots \subset F_1^\perp$, the refined flag stabilised by the stabiliser of the isotropic flag; its unipotent radical U consists of the elements acting trivially on the successive quotients. Thus $u \in G_1$ lies in U iff $u_{ij} = \delta_{ij}$ whenever $\mathrm{lev}(j) \le \mathrm{lev}(i)$, and $u \in \mathsf{U}$ lies in U^0 iff moreover $u_{e_a f} = 0$ for the frame vectors $f \ne e_a$.

Step 1: right translation. Let $u, g \in \mathsf{U}$, $1 \le a \le \varrho$ and $j \in J_a \cup \{e_a^*\}$. In $(ug)_{e_a j} = \sum_i u_{e_a i} g_{ij}$ only i with $\mathrm{lev}(i) \ge a$ contribute ($u_{e_a i} = \delta$ otherwise), and $g_{ij} = \delta_{ij}$ when $\mathrm{lev}(j) \le \mathrm{lev}(i)$. So the terms are $i = e_a$, $i = j$ and $i \in J_a$ with $\mathrm{lev}(i) < \mathrm{lev}(j)$:

$$(ug)_{e_a j} = u_{e_a j} + g_{e_a j} + \sum_{i \in J_a,\ \mathrm{lev}(i) < \mathrm{lev}(j)} u_{e_a i}\, g_{ij}. \tag{5.19}$$

For $j \in J_a$ this reads $r_a(ug) = r_a(g) + r_a(u)(1 + N_a(g))$, where $r_a(u) = (u_{e_a j})_{j \in J_a}$ and $N_a(g)$ is the $J_a \times J_a$ matrix with entries g_{ij} if $\mathrm{lev}(i) < \mathrm{lev}(j)$ and 0 otherwise; ordering J_a by level,

$N_a(g)$ is strictly upper triangular, so $1 + N_a(g)$ is a complex linear bijection of determinant 1. For $j = e_a^*$ the imaginary part of (5.19) is $\operatorname{Im} u_{e_a e_a^*} + \operatorname{Im} g_{e_a e_a^*} + \operatorname{Im} \sum_{i \in J_a} u_{e_a i} g_{i e_a^*}$. So $\mathrm{rw}(ug) = A_g(\mathrm{rw}(u))$ with an affine map A_g whose linear part is block triangular (the complex coordinates of row a are mapped by $1 + N_a(g)$, the real coordinate of row a is changed by a real-linear function of those), of determinant 1; A_g is a bijection preserving Lebesgue measure. This is (ii). The same computation for $u \in \mathsf{U}^0$ and the coordinates $\mathrm{rw}^0(u) = ((u_{e_a j})_{j \in J_a^0}, \operatorname{Im} u_{e_a e_a^*})_a$, $J_a^0 = J_a \setminus \{\text{frame vectors}\}$, gives $\mathrm{rw}^0(ug) = A_g^0(\mathrm{rw}^0(u))$ with A_g^0 affine and Lebesgue preserving: the frame columns drop out of the sum because $u_{e_a f} = 0$ for them.

Step 2: (i). Continuity is clear. If $\mathrm{rw}(u) = \mathrm{rw}(u')$, then $\mathrm{rw}(uu'^{-1}) = A_{u'^{-1}}(\mathrm{rw}(u)) = A_{u'^{-1}}(\mathrm{rw}(u')) = \mathrm{rw}(1)$; so it suffices to show that $\delta \in \mathsf{U}$ with $\mathrm{rw}(\delta) = \mathrm{rw}(1) = 0$ is 1. Row e_d of δ has the entry 1 at e_d, 0 in J_d, $g_d = \delta_{e_d e_d^*}$ at e_d^* with $\operatorname{Im} g_d = 0$, and $y_{db} = \delta_{e_d e_b^*}$ at e_b^*, $b < d$. The entry (e_d, e_d) of $\delta \mathcal{J} \delta^* = \mathcal{J}$ is $g_d + \bar{g}_d$ (the other terms contain $\delta_{e_d e_b} = 0$, $b < d$), so $g_d = 0$. For $b < d$, the entry (e_d, e_b) is $\overline{\delta_{e_b e_d^*}} + g_d \overline{\delta_{e_b e_d}} + \sum_{b' < d} y_{db'} \overline{\delta_{e_b e_{b'}}} = 0$; here $\delta_{e_b e_d^*} = 0$, $\delta_{e_b e_d} = 0$ and $\delta_{e_b e_{b'}} = 0$ for $b < b' < d$ (all in J_b), and $\delta_{e_b e_{b'}} = \delta_{bb'}$ for $b' \le b$, so $y_{db} = 0$. Thus every row e_d is the identity row. Then $\delta e_d = e_d$ and $\delta j = j$ for $j \in X_0 \oplus \mathbb{C}z$ (their columns have nonzero entries only in rows e_b, at positions of J_b, and on the diagonal), so δ is the identity on $F_\varrho^\perp$; hence $\delta e_d^* - e_d^*$ is orthogonal to $F_\varrho^\perp$, so lies in F_ϱ, and its e_b-coordinates $\delta_{e_b e_d^*}$ vanish. So $\delta = 1$.

Step 3: (iii). By the paragraph "Measure of U", du is the image of Lebesgue measure on $\mathbb{C}^{\{(p,q):p<q\}} \times \mathrm{M}_{\varrho \times k}(\mathbb{C}) \times \mathrm{M}_\varrho(\mathbb{R})$ under $\Psi(x, \alpha, \beta) = u(\alpha, \beta)\,\mathrm{Ut}(x)$, writing (α, β) for the parameters of U^0. First, $\mathrm{Ut}(x) \in \mathsf{U}$ and $u(\alpha, \beta) \in \mathsf{U}^0$:

For $u_0 \in \mathsf{U}^0$ the frame matrix of $u_0 \mathrm{Ut}(x)$ is that of $\mathrm{Ut}(x)$, namely $T(x)$, and the frame columns of J_a give exactly the entries of the frame matrix above the diagonal (row e_a is frame row $a - 1$). Split $\mathrm{rw} = (\mathrm{rw}^{\mathrm{fr}}, \mathrm{rw}^0)$ into the coordinates at the frame columns and the remaining ones (a measure-preserving regrouping). Then

$$\mathrm{rw}(\Psi(x, \alpha, \beta)) = (x,\ A^0_{\mathrm{Ut}(x)}(\mathrm{rw}^0(u(\alpha, \beta))))$$

by Step 1. Reading off row e_l of $u(\alpha, \beta)$: the entry at x_i is α_{li}, at e_m^* it is $\mathrm{sk}(\beta)_{lm} - \frac{1}{2}(\alpha\alpha^*)_{lm}$, and Im at e_l^* is β_{ll} (as $\mathrm{sk}(\beta)_{ll} = \mathrm{i}\beta_{ll}$). So $\mathrm{rw}^0(u(\alpha, \beta)) = (\alpha,\ L(\beta) + \sigma(\alpha))$, with σ continuous and $L(\beta) = ((\mathrm{sk}(\beta)_{lm})_{l<m}, (\beta_{ll})_l)$ a linear bijection preserving Lebesgue measure: for $l < m$, $(\beta_{lm}, \beta_{ml}) \mapsto \mathrm{sk}(\beta)_{lm} = ((\beta_{lm} - \beta_{ml}) + \mathrm{i}(\beta_{lm} + \beta_{ml}))/\sqrt{2}$ has real determinant 1. For fixed x the map $(\alpha, \beta) \mapsto A^0_{\mathrm{Ut}(x)}(\alpha, L(\beta) + \sigma(\alpha))$ preserves Lebesgue measure (for fixed α a translate of L, then Fubini, then the affine map of Step 1), and $\mathrm{rw} \circ \Psi$ is continuous. Hence for Borel B, by Fubini, $du(\mathrm{rw}^{-1}B) = \int \mathrm{Leb}\{(\alpha, \beta) : \mathrm{rw}(\Psi(x, \alpha, \beta)) \in B\}\, dx = \int \mathrm{Leb}(B_x)\, dx = \mathrm{Leb}(B)$, B_x the section of B at x.

Step 4: (iv). U is closed in G_1, hence a Polish group, and rw is a continuous injection, so it maps Borel sets to Borel sets (Lusin–Souslin). For a Borel set $S \subseteq \mathsf{U}$ and $g \in \mathsf{U}$, by (iii), (i), (ii): $du(Sg) = \mathrm{Leb}(\mathrm{rw}(Sg)) = \mathrm{Leb}(A_g \mathrm{rw}(S)) = \mathrm{Leb}(\mathrm{rw}(S)) = du(S)$. For compact C, $du(C) = \mathrm{Leb}(\mathrm{rw}(C)) < \infty$, and $du \ne 0$. So du is a right Haar measure. U is a connected nilpotent Lie group (its elements are unipotent), hence unimodular (its modular function is $|\det \mathrm{Ad}|$, and $\mathrm{Ad}(u)$ is unipotent); so du is also a left Haar measure, and on a unimodular group Haar measure is invariant under inversion.

Step 5: (v). By (5.19) with $u = w$, $g = \gamma$, it suffices to show $\gamma_{ij} = 0$ for all $i \in J_a$ with $\mathrm{lev}(i) < \mathrm{lev}(j)$, for $j \in J_a$ (first claim) and for $j = e_a^*$ (second claim). Write $g_d = \gamma_{e_d e_d^*}$, $y_{db} = \gamma_{e_d e_b^*}$ ($b < d$). For $d > a$, the entry (e_d, e_d) of $\gamma \mathcal{J} \gamma^* = \mathcal{J}$ gives $g_d + \bar{g}_d = 0$ as in Step 2, so $g_d = 0$ by hypothesis. Now let $i \in J_a$.

- $i = e_d$, $d > a$. The entries $\gamma_{e_d j}$ for $j \in J_d$ vanish and $g_d = 0$; the remaining columns are $j = e_b^*$ with $b < d$. For $a < b < d$ the entry (e_d, e_b) of $\gamma \mathcal{J} \gamma^*$ is $\overline{\gamma_{e_b e_d^*}} + \sum_{b' < d} y_{db'} \overline{\gamma_{e_b e_{b'}}} = 0$,

and row e_b ($b > a$) vanishes on J_b, so it reduces to $y_{db} = 0$. For $j = e_a^*$ under the second hypothesis the same entry with $b = a$ reduces to $y_{da} = 0$, as row e_a vanishes on J_a.

- $i \in X_0 \oplus \mathbb{C}z$ (level $\varrho + 1$); then $j = e_b^*$ with $b > a$, or $j = e_a^*$. The entry (i, e_b) of $\gamma\mathcal{J}\gamma^*$ is $\pm\overline{\gamma_{e_b i}} + \sum_d \gamma_{ie_d^*}\overline{\gamma_{e_b e_d}} = 0$ (row i of γ is i plus entries in the columns e_d^*); with $\gamma_{e_b i} = 0$ and $\gamma_{e_b e_d} = \delta_{bd}$ for $d \le b$, 0 for $d > b$ (row e_b vanishes on J_b; for $b = a$ under the second hypothesis) it gives $\gamma_{ie_b^*} = 0$.
- $i = e_d^*$, $d > a$; then $\mathrm{lev}(j) > \mathrm{lev}(i)$ forces $j = e_b^*$ with $b < d$, and $b > a$ or ($b = a$, second claim). The entry (e_d^*, e_b) is $\overline{\gamma_{e_b e_d}} + \sum_{b'<d} \gamma_{e_d^* e_{b'}^*}\overline{\gamma_{e_b e_{b'}}} = \delta_{db} = 0$, and the same vanishing of row e_b on J_b reduces it to $\gamma_{e_d^* e_b^*} = 0$.

So in (5.19) the sum vanishes, which gives both claims. □

Lemma 5.2.35. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field with (Um), F its maximal totally real subfield, $f = [F : \mathbb{Q}]$, d_E and d_F the discriminants of E and F, and $\mathrm{rd}_E = |d_E|^{1/[E:\mathbb{Q}]}$. Then:*

(i) E/F is unramified at every finite place, $|d_E| = |d_F|^2$ and $\mathrm{rd}_E = |d_F|^{1/f}$;

(ii) $f \ge 2$;

(iii) $\mathrm{rd}_E \ge (f^f/f!)^{2/f} \ge 2$.

Proof. **(i).** Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_F$ that ramifies in E, and $\mathfrak{P}$ the maximal ideal of $\mathcal{O}_E$ above it. Then $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}^2$, so $\bar{\mathfrak{P}} = \mathfrak{P}$ and the residue degree is 1: $\mathcal{O}_E/\mathfrak{P} = \mathcal{O}_F/\mathfrak{p}$. For $\theta \in \mathcal{O}_E$ choose $a \in \mathcal{O}_F$ with $\theta - a \in \mathfrak{P}$; conjugating, $\bar\theta - a \in \bar{\mathfrak{P}} = \mathfrak{P}$, so $\theta - \bar\theta \in \mathfrak{P}$ for every $\theta \in \mathcal{O}_E$, which contradicts (Um). Hence E/F is unramified at every finite place, the relative discriminant of E/F is $\mathcal{O}_F$, and the tower formula for discriminants gives $|d_E| = |d_F|^{[E:F]}\, N_{F/\mathbb{Q}}(\mathfrak{d}_{E/F}) = |d_F|^2$. As $[E : \mathbb{Q}] = 2f$, $\mathrm{rd}_E = |d_F|^{2/(2f)} = |d_F|^{1/f}$.

(ii). Suppose $f = 1$. Then $F = \mathbb{Q}$, $d_F = 1$ and $|d_E| = 1$ by (i). By [224] (Minkowski's bound) for E, of degree 2 with $r_2 = 1$, the trivial ideal class contains a nonzero integral ideal $\mathfrak{a}$ with

$$1 \le N\mathfrak{a} \le \frac{2!}{2^2} \cdot \frac{4}{\pi}\, |d_E|^{1/2} = \frac{2}{\pi},$$

which is false. So $f \ge 2$.

(iii). Minkowski's bound for F, of degree f with $r_2 = 0$, gives a nonzero integral ideal $\mathfrak{a}$ with $1 \le N\mathfrak{a} \le (f!/f^f)|d_F|^{1/2}$; hence $|d_F|^{1/2} \ge f^f/f!$ and, by (i), $\mathrm{rd}_E = |d_F|^{1/f} \ge (f^f/f!)^{2/f}$. Put $a_n = n^n/n!$. Then $a_1 = 1$ and $a_{n+1} = a_n(1 + 1/n)^n \ge 2a_n$, since $(1 + 1/n)^n \ge 1 + n \cdot \frac{1}{n} = 2$ (binomial expansion); so $a_f \ge 2^{f-1}$ and $(a_f)^{2/f} \ge 2^{2(f-1)/f} \ge 2$, because $2(f - 1)/f \ge 1$ for $f \ge 2$ by (ii). □

Lemma 5.2.36. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data.*

(i) For every $c \in \mathcal{C}$ and every $y \in G_\infty$, $\sum_{v \in \mathsf{U}_c(M)} \beta_c(v \cdot y) = 1$.

(ii) Let $c, c' \in \mathcal{C}$, $s \in \bar G$, $T \subseteq G(E)$ stable, X the set of terms of T over (c, s, c'), and $Z = X \times G_\infty$ with the measure $\mu =$ (counting measure)$\otimes dy$. The pairs $(a, b) \in G(E) \times G(E)$ with $aXb^{-1} = X$ form a group Γ_X under componentwise multiplication, which acts on Z by $(a, b) \cdot (\gamma, y) = (a\gamma b^{-1}, a \cdot y)$; for every $g \in \Gamma_X$ the map $z \mapsto g \cdot z$ is measurable and $\int_Z H(g \cdot z)\, d\mu(z) = \int_Z H\, d\mu$ for every measurable $H \colon Z \to [0, \infty]$. Moreover $\Delta = \mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)$ is a countable subgroup of Γ_X, and $\beta_0(\gamma, y) = \beta_c(y)\beta_{c'}(\gamma^{-1}y)$ is a measurable function on Z with $\sum_{\delta \in \Delta} \beta_0(\delta \cdot z) = 1$ for every $z \in Z$.

Proof. Since $\gamma \mapsto \gamma_\infty$ is a homomorphism (RD1), $(\gamma\gamma')\cdot y = \gamma\cdot(\gamma'\cdot y)$ for $\gamma,\gamma' \in G(E)$.

(i). Fix y and put $a(v) = \beta_c^0(v\cdot y)$ for $v \in \mathsf{U}_c(M)$. For every $v' \in \mathsf{U}_c(M)$, by (RD6) at the point $v'\cdot y$,

$$\sum_{u\in\mathsf{U}_c^0(M)} a(uv') = \sum_{u\in\mathsf{U}_c^0(M)} \beta_c^0\big(u\cdot(v'\cdot y)\big) = 1.$$

By Lemma 5.2.21, the indicator $\mathbf{1}_{\mathrm{red}}$ of the reduced points of the residue domain D_{res} is a box partition of unity for c, so (RD5) with this weight a gives $\sum_{v\in\mathsf{U}_c(M)} \beta_c^0(v\cdot y)\mathbf{1}_{\mathrm{red}}(v\cdot y) = 1$, which is the claim, as $\beta_c = \beta_c^0\mathbf{1}_{\mathrm{red}}$.

(ii), the group and the action. If $(a,b),(a',b') \in \Gamma_X$, then $(aa')X(bb')^{-1} = a(a'Xb'^{-1})b^{-1} = X$, and $a^{-1}Xb = a^{-1}(aXb^{-1})b = X$; so Γ_X is a subgroup of $G(E)\times G(E)$. For $(\gamma,y)\in Z$, $a\gamma b^{-1}\in X$, and $(a,b)\cdot\big((a',b')\cdot(\gamma,y)\big) = (aa'\gamma b'^{-1}b^{-1}, a\cdot(a'\cdot y)) = (aa',bb')\cdot(\gamma,y)$; the identity acts trivially. So this is an action of Γ_X on Z.

(ii), measurability and invariance of μ. Let $g=(a,b)\in\Gamma_X$. On each slice $\{\gamma\}\times G_\infty$ the map $z\mapsto g\cdot z$ is $y\mapsto(a\gamma b^{-1}, a_\infty y)$, continuous into the slice of $a\gamma b^{-1}$; as X is countable and carries the discrete σ-algebra, the map is measurable. For measurable $H\ge 0$, the map $y\mapsto a_\infty y$ is the product over $v\in\Phi$ of the left translations by $a_v\in G_1$, each of which preserves the Cayley measure (Lemma 5.2.41); by Tonelli's theorem for the product measure, $\int_{G_\infty} H(\gamma', a_\infty y)\,dy = \int_{G_\infty} H(\gamma',y)\,dy$ for every $\gamma'\in X$. As $\gamma\mapsto a\gamma b^{-1}$ is a bijection of X (its inverse is $\gamma\mapsto a^{-1}\gamma b$),

$$\int_Z H(g\cdot z)\,d\mu(z) = \sum_{\gamma\in X}\int_{G_\infty} H(a\gamma b^{-1}, a_\infty y)\,dy = \sum_{\gamma\in X}\int_{G_\infty} H(a\gamma b^{-1}, y)\,dy = \int_Z H\,d\mu.$$

(ii), the subgroup Δ. $\mathsf{U}_c(M)$ and $\mathsf{U}_{c'}(M)$ are subgroups of $G(E)$, which is countable since E is a number field. Let $v\in\mathsf{U}_c(M)$, $v'\in\mathsf{U}_{c'}(M)$ and $\gamma\in X$. Then

$$h_c^{-1}v\gamma v'^{-1}h_{c'} = (h_c^{-1}vh_c)(h_c^{-1}\gamma h_{c'})(h_{c'}^{-1}v'h_{c'})^{-1},$$

whose outer factors lie in $\mathrm{K}(M) = \ker\mathrm{red} \subseteq K$; so it lies in K and has reduction $\mathrm{red}(h_c^{-1}\gamma h_{c'}) = s$, that is, $v\gamma v'^{-1}$ is a term over (c,s,c'). As T is stable and $v,v'\in\mathsf{U}(E)$, $v\gamma v'^{-1}\in T$. Hence $vXv'^{-1}\subseteq X$; applied to (v^{-1},v'^{-1}) this gives $v^{-1}Xv'\subseteq X$, so $vXv'^{-1}=X$ and $(v,v')\in\Gamma_X$. Thus Δ is a countable subgroup of Γ_X.

(ii), the cutoff β_0. β_0 is measurable, as $\beta_c,\beta_{c'}$ are and $y\mapsto\gamma_\infty^{-1}y$ is continuous. For $z=(\gamma,y)\in Z$ and $(v,v')\in\Delta$, $(v\gamma v'^{-1})^{-1}(v\cdot y) = v'\cdot(\gamma^{-1}y)$. Hence, all terms being nonnegative, by (i) for c and for c',

$$\sum_{(v,v')\in\Delta}\beta_0\big((v,v')\cdot z\big) = \sum_{v\in\mathsf{U}_c(M)}\beta_c(v\cdot y)\sum_{v'\in\mathsf{U}_{c'}(M)}\beta_{c'}\big(v'\cdot\gamma^{-1}y\big) = 1.$$

□

Lemma 5.2.37. *Let $\mathfrak{S}$, T and (P,F) be as in Lemma 5.2.39, θ the group measure and μ_H the measure of H of F, Z an intermediate group of F, and $c\in\mathcal{C}$. Let J be a finite set and, for $p\in J$, $a_p=(a_p(0),a_p(1),\dots)$ a sequence in $E^\times$ with $a_p(l)$, $a_p(l)^{-1}$ T-integers for $l<\varrho$, q_p its image in E_T; let $\rho\in\mathbb{R}^J$ and $w\in\mathbb{R}$ with $\rho_p\,\mathrm{hdn}_\varrho(q_p,\mathbf{1}) = w$ for all p. Let j be a nonempty finite set and, for $i\in j$, $c_i'\in\mathcal{C}$, $\kappa_i\in K$ and $\gamma_i\in G(E)$ such that each γ_i fixes every e_a and e_a^*, the vector $\gamma_i z$ is the same for all i, $h_c^{-1}\gamma_i h_{c_i'}$ is integral off T, $(\kappa_i)_T\in Z$, and the triples $(c_i',\mathrm{red}(\kappa_i),\gamma_i)$ are pairwise distinct. Let $R\in\mathbb{C}^{J\times J}$ with $R_{pq} = \langle\rho_p O(w_{a_p}\gamma_i),\rho_q O(w_{a_q}\gamma_i)\rangle$ for all $i\in j$ and $p,q\in J$. Then $e = (h_c^{-1}\gamma_i h_{c_i'})_T\,z$ does not depend on i, and*

$$\mu_H(\mathrm{K}_T(M))\,\theta(\mathrm{K}_T(M))^2\,\#\mathrm{Cos}(Z)\sum_{i\in j} w^2\Big\|\sum_{p\in J}\mathrm{ca}\big(c,\mathrm{red}(\kappa_i),c_i',w_{a_p}\gamma_i\big)\Big\|^2 \le \mathrm{Re}\sum_{p,q\in J} R_{pq}\,\mathrm{fp}_{q_p,q_q}(e).$$

Proof. Step 1: the point e. Put $x_i = h_c^{-1}\gamma_i h_{c'_i}$ and $l_i = (x_i)_T^{-1}$. The h's act trivially on the e_a, e_a^* and z, so x_i fixes every e_a, e_a^* and $x_i z = h_c^{-1}\gamma_i z$, which does not depend on i; hence $l_i \in G_T^{(\varrho)}$ and $l_i^{-1}z = (x_i)_T z = e$ for all i. As $j \neq \emptyset$, R is a Gram matrix, hence positive semidefinite.

Step 2: one pair. Fix a pair $\pi = (c', \gamma)$ occurring as (c'_i, γ_i), let $j_\pi = \{i : (c'_i, \gamma_i) = \pi\}$ and $l_\pi = l_i$ $(i \in j_\pi)$; the reductions $\mathrm{red}(\kappa_i)$, $i \in j_\pi$, are pairwise distinct. For $i \in j_\pi$ and $b \in \mathrm{Cos}(Z)$ (with its representative, also written b), $(\kappa_i)_T^{-1} b \in Z$; let $A(i,b) \in \mathrm{Cos}(Z)$ be its coset $\mathrm{K}_T(M)(\kappa_i)_T^{-1}b$, with representative $b' = m(\kappa_i)_T^{-1}b$, $m \in \mathrm{K}_T(M)$; then $(\kappa_i)_T^{-1}bb'^{-1} = m^{-1} \in \mathrm{K}_T(M)$. By Lemma 5.2.18 (ii), $h_c^{-1}w_{a_p}\gamma h_{c'}$ is integral off T, so Lemma 5.2.38 applies to γ, a_p, $k = \kappa_i$ and the cosets b, $A(i,b)$, with $\ell^{-1} = l_\pi$:

$$\mathrm{SO}_{q_p}(l_\pi)_{b,A(i,b)}\, O(w_{a_p}\gamma) = \mathrm{hdn}_\varrho(q_p, \mathbf{1})\, \theta(\mathrm{K}_T(M))\, \mathrm{ca}(c, \mathrm{red}(\kappa_i), c', w_{a_p}\gamma).$$

Let $v = (\mathrm{SO}_{q_p}(l_\pi)_{b,A(i,b)})_{p\in J}$ and $y_p = \rho_p O(w_{a_p}\gamma)$, so that $R_{pq} = \langle y_p, y_q\rangle$. Then

$$\begin{aligned} v^*Rv = \Big\|\sum_p v_p y_p\Big\|^2 &= \Big\|\sum_p \rho_p\, \mathrm{hdn}_\varrho(q_p,\mathbf{1})\theta(\mathrm{K}_T(M))\, \mathrm{ca}(c,\mathrm{red}(\kappa_i),c',w_{a_p}\gamma)\Big\|^2 \\ &= w^2\theta(\mathrm{K}_T(M))^2\Big\|\sum_p \mathrm{ca}(c,\mathrm{red}(\kappa_i),c',w_{a_p}\gamma)\Big\|^2. \end{aligned}$$

The map $(i,b) \mapsto (b, A(i,b))$ from $j_\pi \times \mathrm{Cos}(Z)$ to $\mathrm{Cos}(Z)^2$ is injective: if $A(i,b) = A(i',b)$ with representative b', then $(\kappa_i)_T^{-1}bb'^{-1}$ and $(\kappa_{i'})_T^{-1}bb'^{-1}$ lie in $\mathrm{K}_T(M)$, hence so does $(\kappa_i)_T^{-1}(\kappa_{i'})_T$, so $\mathrm{red}(\kappa_i) = \mathrm{red}(\kappa_{i'})$ by Lemma 5.2.46 (ii), and $i = i'$. Expanding the inner products coordinate-wise, $\mathrm{HS}(l_\pi) = \sum_{x\in\mathrm{Cos}(Z)^2} v_x^* R v_x$ with $v_x = (\mathrm{SO}_{q_p}(l_\pi)_x)_{p\in J}$, a sum of nonnegative numbers as R is positive semidefinite; keeping the $x = (b, A(i,b))$ gives

$$\theta(\mathrm{K}_T(M))^2\, \#\mathrm{Cos}(Z) \sum_{i\in j_\pi} w^2\Big\|\sum_p \mathrm{ca}(c,\mathrm{red}(\kappa_i),c',w_{a_p}\gamma)\Big\|^2 \le \mathrm{HS}(l_\pi).$$

Step 3: the pairs over the point. Let $\pi = (c',\gamma) \neq \pi' = (c'',\gamma')$ be two pairs. If $l_\pi l_{\pi'}^{-1} = (x)_T^{-1}(x')_T \in \mathrm{K}_T(M)$, with $x = h_c^{-1}\gamma h_{c'}$ and $x' = h_c^{-1}\gamma' h_{c''}$, then, as $\gamma z = \gamma' z$ (so $h(z,\gamma z) = h(z,\gamma' z)$ and $\gamma z - \gamma' z = 0 \in Ez$), Lemma 5.2.65 gives $c' = c''$ and $\gamma = \gamma'$, a contradiction. So Lemma 5.2.48 applies to the family $(l_\pi)_\pi$, all over e:

$$\mu_H(\mathrm{K}_T(M)) \sum_\pi \mathrm{HS}(l_\pi) \le \mathrm{Re}\sum_{p,q} R_{pq}\, \mathrm{fp}_{q_p,q_q}(e).$$

Multiplying the inequality of Step 2 by $\mu_H(\mathrm{K}_T(M)) \ge 0$ and summing over the pairs π (the j_π partition j) proves the claim. □

Lemma 5.2.38. *Let $\mathfrak{S}$, T, (P,F), σ, $\gamma = w_p\sigma$, c, c' and k be as in Lemma 5.2.39, with $p = (p_0, p_1, \dots)$ a sequence in $E^\times$ (only $p_0, \dots, p_{\varrho-1}$ enter w_p) such that p_a^{-1} is a T-integer for $a < \varrho$. Let θ be the group measure of F, Z an intermediate group of F, and a, b the representatives of two cosets in $\mathrm{Cos}(Z)$ with $k_T^{-1}ab^{-1} \in \mathrm{K}_T(M)$. Let q be the image of p in E_T and $\ell = (h_c^{-1}\sigma h_{c'})_T$. Then*

$$\mathrm{SO}_q(\ell^{-1})_{a,b}\, O(\gamma) = \mathrm{hdn}_\varrho(q,\mathbf{1})\, \theta(\mathrm{K}_T(M))\, \mathrm{ca}(c,\mathrm{red}(k),c',\gamma).$$

Proof. By (FB1), $\mathrm{SO}_q(\ell^{-1})_{a,b} = \mathrm{hdn}_\varrho(q,\mathbf{1}) \int_{G_T} W_Z(y)\, \overline{W_Z(\ell^{-1}w_{\varrho,q}^{-1}y)_b}\, d\theta(y)$, the orbital integral of W_Z at $x = w_{\varrho,q}\ell$. The element $h_c \in \mathrm{U}(X_0)(\mathbb{A}_f)$ commutes with w_p (proof of Lemma 5.2.18),

so $h_c^{-1}\gamma h_{c'} = w_p\, h_c^{-1}\sigma h_{c'}$; the matrix of w_p in the lattice basis has the entries 0, 1, p_a, $\bar{p}_a^{-1}$ at the hyperbolic pairs, so its component at T is $w_{\varrho,q}$, and $x = (h_c^{-1}\gamma h_{c'})_T$. By Lemma 5.5.41 (1), θ is a Haar measure on G_T, and G_T is unimodular; by Lemma 5.2.47 with $s = k_T$ (the hypothesis $k_T^{-1}ab^{-1} \in \mathrm{K}_T(M)$ is the one required there),

$$\int_{G_T} W_Z(y)_a\, \overline{W_Z(x^{-1}y)_b}\, d\theta(y) = \theta(\mathrm{K}_T(M))\, \mathrm{Kl}_T(x, k_T).$$

Multiplying by $\mathrm{hdn}_\varrho(q, \mathbf{1})\, O(\gamma)$ and using Lemma 5.2.39, $\mathrm{Kl}_T(x, k_T)\, O(\gamma) = \mathrm{ca}(c, \mathrm{red}(k), c', \gamma)$, gives the claim. □

Lemma 5.2.39. *Let $\mathfrak{S}$ be admissible with Weil data, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, and (P, F) a filling datum at T. For $c, c' \in \mathcal{C}$, $s \in \bar{G}$ and $\delta \in G(E)$ put*

$$\mathrm{ca}(c, s, c', \delta) = \sum_{u,u' \in \mathsf{U}(E)} \mathbf{1}[u\delta u'^{-1} \text{ is a term over } (c, s, c')]\, A^{\mathrm{cut}}_{u\delta u'^{-1}}(c, c'),$$
$$O(\delta) = \int_{G_\infty} W^{\mathrm{cut}}(y) \otimes \overline{W^{\mathrm{cut}}(\delta^{-1}y)}\, dy,$$

an integral of a non-integrable function being read as 0. Let $\sigma \in G(E)$ fix every e_a and e_a^*, let $p \in (E^\times)^\varrho$ with every p_a^{-1} a T-integer, $\gamma = w_p\sigma$, $c, c' \in \mathcal{C}$ such that $h_c^{-1}\gamma h_{c'}$ is integral off T, and $k \in K$. Then the series defining $\mathrm{ca}(c, \mathrm{red}(k), c', \gamma)$ converges absolutely and

$$\mathrm{ca}(c, \mathrm{red}(k), c', \gamma) = \mathrm{Kl}_T((h_c^{-1}\gamma h_{c'})_T,\, k_T)\; O(\gamma).$$

Proof. Write $s = \mathrm{red}(k)$, $Q = \mathsf{U}_T/(\mathsf{U}_T \cap \mathrm{K}_T(M))$ with the fixed representatives $\hat{q}$ of Definition 5.2.9, $x = (h_c^{-1}\gamma h_{c'})_T$, and $K_\eta(y) = W^{\mathrm{cut}}(y) \otimes \overline{W^{\mathrm{cut}}(\eta^{-1}y)}$ for $\eta \in G(E)$.

Step 1: the simple-root sum and the character. For $u \in \mathsf{U}_A$ the span of $f_0, \dots, f_\varrho$ is u-stable, and u acts on it by an upper unitriangular matrix whose entries just above the diagonal are those of the frame block. Superdiagonal entries of upper unitriangular matrices add under multiplication, so $\sigma(uu') = \sigma(u) + \sigma(u')$. An $h \in \mathrm{U}(X_0)(\mathbb{A}_f)$ fixes every f_a, so $h^{-1}uhf_b = h^{-1}uf_b$ has the same coefficients on the f_a as uf_b, and $\sigma(h^{-1}uh) = \sigma(u)$. Hence ψ is a character of $\mathsf{U}(E)$ of modulus 1, and the same holds for σ on U_T.

Step 2: integrality of the unipotent factors. If $u, u' \in \mathsf{U}(E)$ and $u\gamma u'^{-1}$ is a term over (c, s, c'), then by Lemma 5.2.18 (i), applied to $u^{-1}w_p\sigma u'$ with the roles of u, u' taken by u^{-1}, u'^{-1}, the elements $h_c^{-1}u^{-1}h_c$ and $h_{c'}^{-1}u'^{-1}h_{c'}$ are integral off T.

Step 3: rational representatives of the cosets at T. Let $d \in \{c, c'\}$ and $q \in Q$. Let $\tilde{x} \in G(\mathbb{A}_f)$ have as matrix in the lattice basis 1 plus the matrix of $\hat{q} - 1$ with its entries extended by 0 off T; being an isometry and lying in $\mathsf{U}_{\mathbb{A}_f}$ are componentwise conditions, so $\tilde{x} \in \mathsf{U}_{\mathbb{A}_f}$, $\tilde{x}$ is integral off T, and $\tilde{x}_T = \hat{q}$. The set $h_d\tilde{x}\,\mathrm{K}(M)h_d^{-1} \cap \mathsf{U}_{\mathbb{A}_f}$ is open in $\mathsf{U}_{\mathbb{A}_f}$ and contains $h_d\tilde{x}h_d^{-1}$. As $\mathsf{U}(E)$ is dense in $\mathsf{U}(\mathbb{A}_f)$ (strong approximation for unipotent groups, [233], §7.1), it contains some $r \in \mathsf{U}(E)$. Then $m = \tilde{x}^{-1}h_d^{-1}rh_d \in \mathrm{K}(M)$, so by Lemma 5.2.46 (i) m is integral off T and $m_T = \hat{q}^{-1}(h_d^{-1}rh_d)_T \in \mathrm{K}_T(M)$, and $h_d^{-1}rh_d = \tilde{x}m$ is integral off T. Choose such an $r = r_d(q)$ for every q.

Let $u \in \mathsf{U}(E)$ with $h_d^{-1}uh_d$ integral off T. For $q \in Q$ and $v \in \mathsf{U}(E)$ with $u = r_d(q)v$, the element $h_d^{-1}vh_d$ is integral off T, so by Lemma 5.2.46 (i) $v \in \mathsf{U}_d(M)$ if and only if

$$(h_d^{-1}r_d(q)h_d)_T^{-1}(h_d^{-1}uh_d)_T \in \mathrm{K}_T(M),$$

that is, if and only if q is the coset of $(h_d^{-1}uh_d)_T \in \mathsf{U}_T$. Hence there is exactly one pair $(q,v) \in Q \times \mathsf{U}_d(M)$ with $u = r_d(q)v$.

Step 4: one representative in each orbit. Put $\delta(q,q') = r_c(q)^{-1}\gamma\, r_{c'}(q')$ for $(q,q') \in Q^2$ and $D = \{\delta(q,q')\}$. By Lemma 5.2.17 (ii), $(u,u') \mapsto u\gamma u'^{-1}$ is injective on $\mathsf{U}(E)^2$; so is $(q,q') \mapsto \delta(q,q')$, because the r_d are injective. The set $X = \{u\gamma u'^{-1} : u,u' \in \mathsf{U}(E)\}$ is stable. Let $\varphi(\eta,y) = 1$ for $\eta \in D$ and 0 otherwise. Let $\eta = u\gamma u'^{-1} \in X$ be a term over (c,s,c'). By Steps 2 and 3, $u^{-1} = r_c(q)v$ and $u'^{-1} = r_{c'}(q')v'$ for unique $q,q' \in Q$, $v \in \mathsf{U}_c(M)$, $v' \in \mathsf{U}_{c'}(M)$. For $(v_1,v_1') \in \mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)$ we have $v_1\eta v_1'^{-1} = (v_1u)\gamma(v_1'u')^{-1}$, and by the injectivity above this equals $\delta(q_1,q_1')$ if and only if $v_1u = r_c(q_1)^{-1}$ and $v_1'u' = r_{c'}(q_1')^{-1}$, that is $u^{-1} = r_c(q_1)v_1$ and $u'^{-1} = r_{c'}(q_1')v_1'$. By the uniqueness in Step 3 this happens exactly for $(v_1,v_1') = (v,v')$ (with $q_1 = q$, $q_1' = q'$). So $\sum_{(v_1,v_1')} \varphi(v_1\eta v_1'^{-1}, v_1 \cdot y) = 1$ for every y.

Step 5: unfolding. By Lemma 5.2.29 (ii), $\sum_\eta \|A_\eta^{\mathrm{cut}}(c,c')\| < \infty$ over the terms $\eta \in X$ over (c,s,c'), and the bijection $(u,u') \mapsto u\gamma u'^{-1}$ turns this sum into the series defining $\mathrm{ca}(c,s,c',\gamma)$; so that series converges absolutely and equals the amplitude sum $A_X^{\mathrm{cut}}(c,s,c')$. By Step 4 and Lemma 5.2.40 (with $T' = X$ and the weight φ), K_δ is integrable for every term $\delta \in D$ over (c,s,c'), and

$$\mathrm{ca}(c,s,c',\gamma) = \sum_{(q,q') \in Q^2:\ \delta(q,q') \text{ a term over } (c,s,c')} O(\delta(q,q')),$$

the series converging absolutely.

Step 6: phases. Let $u,u' \in \mathsf{U}(E)$. Substituting $y = u^{-1} \cdot z$ (left invariance of Haar measure on G_∞) and applying Lemma 5.2.43 twice,

$$\begin{aligned} K_{u^{-1}\gamma u'}(u^{-1}\cdot z) &= W^{\mathrm{cut}}(u^{-1}\cdot z) \otimes \overline{W^{\mathrm{cut}}(u'^{-1}\cdot\gamma^{-1}z)} \\ &= \psi(u^{-1})^{-1}\,\overline{\psi(u'^{-1})^{-1}}\,K_\gamma(z) = \psi(u)\overline{\psi(u')}\,K_\gamma(z), \end{aligned}$$

by Step 1. As $|\psi(u)\overline{\psi(u')}| = 1$, $K_{u^{-1}\gamma u'}$ is integrable if and only if K_γ is, and $O(u^{-1}\gamma u') = \psi(u)\overline{\psi(u')}O(\gamma)$ in all cases. In particular $O(\delta(q,q')) = \psi(r_c(q))\overline{\psi(r_{c'}(q'))}\,O(\gamma)$.

Step 7: the term condition at T. Let $\rho = (h_c^{-1}r_c(q)h_c)_T$ and $\rho' = (h_{c'}^{-1}r_{c'}(q')h_{c'})_T$; by Step 3, $\rho = \hat{q}m$ and $\rho' = \hat{q}'m'$ with $m,m' \in \mathrm{K}_T(M)$. The element $y = h_c^{-1}\delta(q,q')h_{c'} = (h_c^{-1}r_c(q)h_c)^{-1}(h_c^{-1}\gamma h_{c'})(h_{c'}^{-1}r_{c'}(q')h_{c'})$ is integral off T and $y_T = \rho^{-1}x\rho'$. Now $\delta(q,q')$ is a term over (c,s,c') if and only if $k^{-1}y \in \mathrm{K}(M)$: if $y \in K$ with $\mathrm{red}(y) = \mathrm{red}(k)$ then $k^{-1}y \in \ker \mathrm{red} = \mathrm{K}(M)$, and conversely $k^{-1}y \in \mathrm{K}(M) \subseteq K$ gives $y \in K$ and $\mathrm{red}(y) = \mathrm{red}(k)$. By Lemma 5.2.46 (i) this holds if and only if $k_T^{-1}m^{-1}\hat{q}^{-1}x\hat{q}'m' \in \mathrm{K}_T(M)$. Since $k \in K$, $k_T \in \mathrm{K}_T(1)$, which normalises $\mathrm{K}_T(M)$ (for $\kappa \in \mathrm{K}_T(1)$, $\kappa^{-1}(m-1)\kappa$ has entries in $M\mathcal{O}_T$); so $k_T^{-1}m^{-1} = (k_T^{-1}m^{-1}k_T)k_T^{-1}$ with $k_T^{-1}m^{-1}k_T \in \mathrm{K}_T(M)$, and the condition is $k_T^{-1}\hat{q}^{-1}x\hat{q}' \in \mathrm{K}_T(M)$: the condition of the pair $(\hat{q},\hat{q}')$ in $\mathrm{Kl}_T(x,k_T)$.

Step 8: the characters at T. Let $d \in \{c,c'\}$, $q \in Q$, $r = r_d(q)$ and $\rho = (h_d^{-1}rh_d)_T = \hat{q}m$ as in Step 7, so $m = \hat{q}^{-1}\rho \in \mathsf{U}_T \cap \mathrm{K}_T(M)$ and $\sigma(m) \in M\mathcal{O}_T$. Let $u_0 \in \mathsf{U}_T$ be the element chosen in the definition of $\chi_M(\hat{q})$, so $u_0^{-1}\hat{q} \in \mathrm{K}_T(M)$ and $\chi_M(\hat{q}) = \psi_T(M^{-1}\sigma(u_0))$. Then $u_0^{-1}\rho = (u_0^{-1}\hat{q})m \in \mathsf{U}_T \cap \mathrm{K}_T(M)$, so $\sigma(\rho) - \sigma(u_0) = \sigma(u_0^{-1}\rho) \in M\mathcal{O}_T$ by Step 1, and ψ_T is trivial on $\mathcal{O}_T$ by self-duality; hence $\chi_M(\hat{q}) = \psi_T(M^{-1}\sigma(\rho))$. By Step 1, $\sigma(\rho)$ is the component at T of $\sigma(r) \in E$; with $a = M^{-1}\sigma(r) \in E$, $M^{-1}\sigma(\rho) = a_T$ and $\chi_M(\hat{q}) = \psi_f(-\tilde{a}_T/\varpi)$, where $\tilde{a}_T$ has the components of a at T and 0 elsewhere. On the other hand $\psi(r) = \psi_f(-a/\varpi) = \psi_f(-\tilde{a}_T/\varpi)\,\psi_f(-(a-\tilde{a}_T)/\varpi)$. At a finite $w \notin T$ the component $a_w = M^{-1}\sigma(h_d^{-1}rh_d)_w$ is integral ($h_d^{-1}rh_d$ is integral off T and M is a unit at w), so $(a-\tilde{a}_T)/\varpi$ lies in the product of the local inverse differents, on which the standard character ψ_f is trivial. Hence $\psi(r_d(q)) = \chi_M(\hat{q})$.

Step 9: conclusion. By Steps 5 to 8,

$$\mathrm{ca}(c,s,c',\gamma) = \sum_{(q,q')\in Q^2:\ k_T^{-1}\hat{q}^{-1}x\hat{q}'\in \mathrm{K}_T(M)} \chi_M(\hat{q})\,\overline{\chi_M(\hat{q}')}\,O(\gamma),$$

the series converging absolutely. If $O(\gamma)=0$, both sides of the claim vanish. If $O(\gamma)\neq 0$, the absolute convergence shows that the series defining $\mathrm{Kl}_T(x,k_T)$ converges absolutely, and the display is $\mathrm{Kl}_T(x,k_T)\,O(\gamma)$. □

Lemma 5.2.40. *Let $\mathfrak{S}$ be admissible with Weil data, $c,c'\in\mathcal{C}$, $s\in\bar{G}$, and $T'\subseteq G(E)$ stable. Let $\varphi\colon G(E)\times G_\infty\to[0,\infty]$ be such that $\varphi(\gamma,\cdot)$ is Borel for every γ and*

$$\sum_{(v,v')\in \mathsf{U}_c(M)\times\mathsf{U}_{c'}(M)} \varphi(v\gamma v'^{-1},\, v\cdot y) = 1$$

for every $y\in G_\infty$ and every $\gamma\in T'$ that is a term over (c,s,c'). Then, for $W\in\{W^{\mathrm{full}},W^{\mathrm{cut}}\}$ and every such γ, the function $y\mapsto\varphi(\gamma,y)\,W(y)\otimes\overline{W(\gamma^{-1}y)}$ is integrable on G_∞, and

$$A_{T'}(c,s,c') = \sum_{\substack{\gamma\in T'\\ \text{over } (c,s,c')}} \int_{G_\infty} \varphi(\gamma,y)\,W(y)\otimes\overline{W(\gamma^{-1}y)}\,dy,$$

the series converging absolutely.

Proof. Fix W, write $K(\gamma,y)=W(y)\otimes\overline{W(\gamma^{-1}y)}$, and let X be the set of terms of T' over (c,s,c'). Let $Z=X\times G_\infty$, $\mu=(\text{counting measure})\otimes dy$, the group Γ_X, its countable subgroup $\Delta=\mathsf{U}_c(M)\times\mathsf{U}_{c'}(M)$ and $\beta_0(\gamma,y)=\beta_c(y)\beta_{c'}(\gamma^{-1}y)$ be as in (ii) of Lemma 5.2.36: Γ_X acts on Z by $(a,b)\cdot(\gamma,y)=(a\gamma b^{-1},a\cdot y)$, by measurable maps that preserve integrals against μ, and $\sum_{\delta\in\Delta}\beta_0(\delta\cdot z)=1$ for every $z\in Z$. By hypothesis also $\sum_{\delta\in\Delta}\varphi(\delta\cdot z)=1$ for every $z\in Z$, and φ is measurable on Z, as X is countable and each $\varphi(\gamma,\cdot)$ is Borel.

The kernel is invariant. K is measurable on Z. For $\delta=(v,v')\in\Delta$ and $z=(\gamma,y)$, $W(v\cdot y)=W(y)$ by Lemma 5.2.42, and $(v\gamma v'^{-1})^{-1}(v\cdot y)=v'\cdot(\gamma^{-1}y)$ with $W(v'\cdot x)=W(x)$, again by Lemma 5.2.42; so $K(\delta\cdot z)=K(z)$.

Integrability. By (i) of Lemma 5.2.45, applied to Γ_X, $\Lambda=\Delta$, the two cutoffs φ and β_0 and $H=\|K\|$, $\int_Z\varphi\|K\|\,d\mu=\int_Z\beta_0\|K\|\,d\mu=\sum_{\gamma\in X}\mathrm{maj}_\gamma$, the sum of the amplitude majorants of the terms of T' over (c,s,c'), which is finite by Lemma 5.2.29 (ii) (its sum over d contains the term $d=c'$). Hence $\int\varphi(\gamma,y)\|K(\gamma,y)\|\,dy<\infty$ for every $\gamma\in X$: the function $y\mapsto\varphi(\gamma,y)K(\gamma,y)$ is integrable, and φK, β_0K are integrable on Z.

The identity. By (ii) of Lemma 5.2.45, with the same data and $F=K$, $\int_Z\varphi K\,d\mu=\int_Z\beta_0K\,d\mu$. By Fubini's theorem for the integrable functions φK and β_0K on Z, the left side is $\sum_{\gamma\in X}\int\varphi(\gamma,y)K(\gamma,y)\,dy$ and the right side is $\sum_{\gamma\in X}A_\gamma(c,c')=A_{T'}(c,s,c')$, both series converging absolutely, since $\sum_\gamma\|\int\varphi(\gamma,\cdot)K(\gamma,\cdot)\|\le\int_Z\varphi\|K\|\,d\mu<\infty$. □

Lemma 5.2.41. *Let $J\in\mathrm{M}_N(\mathbb{C})$ be a hermitian involution and dy the Cayley measure of $\mathrm{U}(J)$. Then for all $a,b\in\mathrm{U}(J)$ and every Borel function $\phi\colon\mathrm{U}(J)\to[0,\infty]$,*

$$\int_{\mathrm{U}(J)}\phi(ayb)\,dy=\int_{\mathrm{U}(J)}\phi(y)\,dy.$$

Proof. Throughout, $J^* = J$ and $J^2 = 1$. Put

$$\begin{aligned} \mathfrak{l}_J &= \{Y \in \mathrm{M}_N(\mathbb{C}) : Y^*J + JY = 0\}, \\ \mathrm{cay}(Y) &= (1 + \tfrac{1}{2}Y)(1 - \tfrac{1}{2}Y)^{-1}, \\ m(Y) &= |\det(1 - \tfrac{1}{2}Y)|^{-2N}. \end{aligned}$$

Since $J^2 = 1$, $Y \in \mathfrak{l}_J$ if and only if $Y^* = -JYJ$, if and only if JY is skew-hermitian.

The chart in the coordinates Y. The map $y \mapsto S(y)$ is a real-linear bijection of $\mathrm{M}_N(\mathbb{R})$ onto the skew-hermitian matrices: the diagonal entry $S(y)_{aa} = \mathrm{i}y_{aa}$ runs over $\mathrm{i}\mathbb{R}$, for $a < b$ the entry $S(y)_{ba} = -\overline{S(y)_{ab}}$ is determined by $S(y)_{ab}$, and $(y_{ab}, y_{ba}) \mapsto S(y)_{ab}$ is a real-linear bijection $\mathbb{R}^2 \to \mathbb{C}$ because $(1+\mathrm{i})/\sqrt{2}$ and $-(1-\mathrm{i})/\sqrt{2} = \mathrm{i}(1+\mathrm{i})/\sqrt{2}$ are linearly independent over $\mathbb{R}$. Left multiplication by J maps the skew-hermitian matrices bijectively onto $\mathfrak{l}_J$. Hence $T(y) = JS(y)$ is a real-linear bijection $\mathrm{M}_N(\mathbb{R}) \to \mathfrak{l}_J$, $\dim_{\mathbb{R}} \mathfrak{l}_J = N^2$, and $D(y) = 1 - \frac{1}{2}T(y)$, $2D(y)^{-1} - 1 = \mathrm{cay}(T(y))$. Let dY be the image of Lebesgue measure under T, a Haar measure of the real vector space $\mathfrak{l}_J$. With $V = \{Y \in \mathfrak{l}_J : \det(1 - \frac{1}{2}Y) \neq 0\}$ (open), the Cayley measure is therefore

$$\int_{\mathrm{U}(J)} \phi(y)\, dy = \mathrm{vol}\,\mathrm{U}(N)^{-1} \int_V \phi(\mathrm{cay}(Y))\, m(Y)\, dY. \tag{5.20}$$

Three algebraic facts. (a) For $Y \in \mathfrak{l}_J$, $(1 + \frac{1}{2}Y)^* = 1 - \frac{1}{2}JYJ = J(1 - \frac{1}{2}Y)J$, so $\overline{\det(1 + \frac{1}{2}Y)} = \det(1 - \frac{1}{2}Y)$. Hence $V = -V$ and $m(-Y) = m(Y)$. (b) Let $\mathrm{U}^\circ = \{g \in \mathrm{U}(J) : \det(1 + g) \neq 0\}$ and $Z = \mathrm{U}(J) \setminus \mathrm{U}^\circ$. Then cay is a bijection $V \to \mathrm{U}^\circ$ with inverse $g \mapsto 2(g-1)(g+1)^{-1}$, and $\mathrm{cay}(Y)^{-1} = \mathrm{cay}(-Y)$. Indeed, for $Y \in V$, the identities $(1+\frac{1}{2}Y^*)J = J(1 - \frac{1}{2}Y)$ and $(1 - \frac{1}{2}Y^*)J = J(1 + \frac{1}{2}Y)$ give, since $1 \pm \frac{1}{2}Y$ commute,

$$\mathrm{cay}(Y)^* J\, \mathrm{cay}(Y) = (1 - \tfrac{1}{2}Y^*)^{-1} J (1 - \tfrac{1}{2}Y)(1 + \tfrac{1}{2}Y)(1 - \tfrac{1}{2}Y)^{-1} = (1 - \tfrac{1}{2}Y^*)^{-1} J (1 + \tfrac{1}{2}Y) = J,$$

and $\mathrm{cay}(Y) + 1 = 2(1 - \frac{1}{2}Y)^{-1}$ is invertible. Conversely, for $g \in \mathrm{U}^\circ$ put $Y = 2(g-1)(g+1)^{-1}$; then $1 - \frac{1}{2}Y = 2(g+1)^{-1}$ and $1 + \frac{1}{2}Y = 2g(g+1)^{-1}$, so $\det(1 - \frac{1}{2}Y) \neq 0$ and $\mathrm{cay}(Y) = g$ (the factors commute), and from $g^* = Jg^{-1}J$,

$$Y^* = 2J(g^{-1} + 1)^{-1}(g^{-1} - 1)J = 2J(1 + g)^{-1}(1 - g)J = -JYJ,$$

so $Y \in \mathfrak{l}_J$ and $Y \in V$. Finally $\mathrm{cay}(Y)^{-1} = (1 - \frac{1}{2}Y)(1 + \frac{1}{2}Y)^{-1} = \mathrm{cay}(-Y)$, and $-Y \in V$ by (a). (c) For $Y \in V$ the real-linear map $L_Y(Y') = (1 + \frac{1}{2}Y)^{-1}Y'(1 - \frac{1}{2}Y)^{-1}$ maps $\mathfrak{l}_J$ into itself and $|\det(L_Y|_{\mathfrak{l}_J})| = m(Y)$. Indeed, by (a) $(1 + \frac{1}{2}Y^*)^{-1} = J(1 - \frac{1}{2}Y)^{-1}J$ and $(1 - \frac{1}{2}Y^*)^{-1} = J(1 + \frac{1}{2}Y)^{-1}J$, so for $Y' \in \mathfrak{l}_J$

$$L_Y(Y')^* = J(1 + \tfrac{1}{2}Y)^{-1} J\, (-JY'J)\, J(1 - \tfrac{1}{2}Y)^{-1}J = -J\, L_Y(Y')\, J.$$

Next, $\mathfrak{l}_J$ is a real form of $\mathrm{M}_N(\mathbb{C})$: if Y' and $\mathrm{i}Y'$ lie in $\mathfrak{l}_J$ then $Y'^*J = JY' = -Y'^*J$, so $Y' = 0$; and $2N^2 = \dim_{\mathbb{R}} \mathrm{M}_N(\mathbb{C})$. So a real basis $E_1, \ldots, E_{N^2}$ of $\mathfrak{l}_J$ is a complex basis of $\mathrm{M}_N(\mathbb{C})$, and the complex-linear map L_Y of $\mathrm{M}_N(\mathbb{C})$ has in this basis the same (real) matrix as $L_Y|_{\mathfrak{l}_J}$. The determinant of $Y' \mapsto AY'B$ on $\mathrm{M}_N(\mathbb{C})$ is $\det(A)^N \det(B)^N$ (a Kronecker product), so by (a)

$$|\det(L_Y|_{\mathfrak{l}_J})| = |\det(1 + \tfrac{1}{2}Y)|^{-N} |\det(1 - \tfrac{1}{2}Y)|^{-N} = m(Y).$$

The translates of Z are null. Let $b \in \mathrm{U}(J)$. For $Y \in V$, $\mathrm{cay}(Y) \in bZ$ if and only if $\det(b + \mathrm{cay}(Y)) = 0$, and multiplying on the right by the invertible matrix $1 - \frac{1}{2}Y$ this holds if and only if $q(Y) = 0$, where $q(Y') = \det\big((1 + b) + \frac{1}{2}(1 - b)Y'\big)$ for $Y' \in \mathrm{M}_N(\mathbb{C})$. The function $P(z) = q(\sum_k z_k E_k)$ is a complex polynomial on $\mathbb{C}^{N^2}$, and it is not zero: at the z with

$\sum_k z_k E_k = 2 \cdot 1$ it equals $\det(2 \cdot 1) = 2^N$. A complex polynomial that vanishes on $\mathbb{R}^{N^2}$ is zero, so the real or the imaginary part of $x \mapsto P(x)$ ($x \in \mathbb{R}^{N^2}$) is a nonzero real polynomial, and the zero set of a nonzero real polynomial is a Lebesgue null set. Hence

$$\{Y \in V : \operatorname{cay}(Y) \in bZ\} \text{ is a } dY\text{-null set for every } b \in \mathrm{U}(J). \tag{5.21}$$

Left invariance. Fix $a \in \mathrm{U}(J)$ and let $V_a = \{Y \in V : a \operatorname{cay}(Y) \in \mathrm{U}^\circ\}$, an open set. By (b), $\Phi_a(Y) = 2(a \operatorname{cay}(Y) - 1)(a \operatorname{cay}(Y) + 1)^{-1}$ is a bijection $V_a \to V_{a^{-1}}$ with inverse $\Phi_{a^{-1}}$, and both are smooth (rational with nonvanishing denominators). Differentiating $\operatorname{cay}(\Phi_a(Y)) = a \operatorname{cay}(Y)$ in the direction Y' and using $\frac{d}{dt}\operatorname{cay}(Y + tY')|_{t=0} = (1 - \frac{1}{2}Y)^{-1} Y' (1 - \frac{1}{2}Y)^{-1} = \operatorname{cay}(Y) L_Y(Y')$ (from $\operatorname{cay}(Y) = 2(1 - \frac{1}{2}Y)^{-1} - 1$) together with $\operatorname{cay}(\Phi_a(Y))^{-1} a = \operatorname{cay}(Y)^{-1}$, we get $L_{\Phi_a(Y)} \circ d\Phi_a(Y) = L_Y$ on $\mathfrak{l}_J$. By (c),

$$|\det d\Phi_a(Y)|\, m(\Phi_a(Y)) = m(Y) \qquad (Y \in V_a).$$

The change of variables $Y' = \Phi_a(Y)$ gives, for Borel $\phi \geq 0$,

$$\begin{aligned} \int_{V_a} \phi(a \operatorname{cay}(Y))\, m(Y)\, dY &= \int_{V_a} \phi(\operatorname{cay}(\Phi_a(Y)))\, m(\Phi_a(Y))\, |\det d\Phi_a(Y)|\, dY \\ &= \int_{V_{a^{-1}}} \phi(\operatorname{cay}(Y'))\, m(Y')\, dY'. \end{aligned}$$

Now $V \setminus V_a = \{Y \in V : \operatorname{cay}(Y) \in a^{-1}Z\}$ and $V \setminus V_{a^{-1}} = \{Y \in V : \operatorname{cay}(Y) \in aZ\}$ are null by (5.21). With (5.20) this is $\int \phi(ay)\, dy = \int \phi(y)\, dy$.

Right invariance. By (a), (b), the substitution $Y \mapsto -Y$ in (5.20) gives $\int \phi(y^{-1})\, dy = \int \phi(y)\, dy$. For $b \in \mathrm{U}(J)$ put $\phi_1(y) = \phi(y^{-1})$; then, by inversion, left invariance under b^{-1}, and inversion again,

$$\int \phi(yb)\, dy = \int \phi(y^{-1}b)\, dy = \int \phi_1(b^{-1}y)\, dy = \int \phi_1(y)\, dy = \int \phi(y)\, dy.$$

Applying left invariance under a to $y \mapsto \phi(yb)$ and then right invariance under b gives the statement. □

Lemma 5.2.42. *Let $\mathfrak{S}$ be a filling datum, $c \in \mathcal{C}$ and $u \in \mathsf{U}_c(M)$. Then $W(u \cdot y) = W(y)$ for every $y \in G_\infty$ and $W \in \{W^{\mathrm{full}}, W^{\mathrm{cut}}\}$.*

Proof. By Lemma 5.2.43, $W(u \cdot y) = \psi(u)^{-1} W(y)$ for every $y \in G_\infty$ and $W \in \{W^{\mathrm{full}}, W^{\mathrm{cut}}\}$; it remains to show $\psi(u) = 1$. Put $g = h_c^{-1} u h_c$. As $u \in \mathsf{U}(E) \subseteq \mathsf{U}(\mathbb{A}_f)$ and $h_c \in \mathrm{U}(X_0)(\mathbb{A}_f)$, (ii) of Lemma 5.2.44 gives $g \in \mathsf{U}(\mathbb{A}_f)$ and $\psi(g) = \psi(u)$. By (RD2), $g \in K(M)$, so $g \in \mathsf{U}(\mathbb{A}_f) \cap K(M)$ and $\psi(g) = 1$ by (iii) of Lemma 5.2.44. Hence $\psi(u) = 1$ and $W(u \cdot y) = W(y)$. □

Lemma 5.2.43. *Let $\mathfrak{S}$ be a filling datum, $u \in \mathsf{U}(E)$ and $y \in G_\infty$. Then $W(u \cdot y) = \psi(u)^{-1} W(y)$ for $W \in \{W^{\mathrm{full}}, W^{\mathrm{cut}}\}$.*

Proof. The rescaled components lie in U. By (5.5), $u_v = B_v^{-1} v(u) B_v$ with B_v the real diagonal matrix of the basis scales. The element $v(u)$ preserves the flag $\langle e_1 \rangle \subset \cdots \subset X$ of $V \otimes_{E,v} \mathbb{C}$, acts trivially on its graded pieces and on $X^\perp / X$, and is unitary; conjugation by the diagonal matrix B_v, which maps each e_a, each basis vector of X_0, z and each e_a^* to a multiple of itself, preserves these three properties, so $u_v \in \mathsf{U}$.

The character at infinity. Since $\bar{\varpi} = -\varpi$, $v(\varpi)$ is purely imaginary, and it is nonzero, so $\mathrm{rs}_v = M v(\varpi)/2\pi i$ is real and nonzero. The entries of u_v just above the diagonal of the frame

block are $v(u_{f_a,f_{a+1}})\,\mathrm{bs}_v(f_{a+1})/\mathrm{bs}_v(f_a) = v(u_{f_a,f_{a+1}})/\mathrm{rs}_v$, because $\mathrm{bs}_v(f_a) = \mathrm{rs}_v^{\varrho-a}/\sqrt{N}$. With $t = \sigma(u)/(M\varpi) \in E$ this gives

$$\sigma(u_v) = \frac{v(\sigma(u))}{\mathrm{rs}_v} = 2\pi i\, v(t), \qquad \mathrm{wave}(\sigma(u_v)) = \exp\bigl(2\pi i\,(v(t) + \overline{v(t)})\bigr).$$

The $2f$ embeddings of E into $\mathbb{C}$ are the $v \in \Phi$ and their complex conjugates, so $\prod_{v\in\Phi} \mathrm{wave}(\sigma(u_v)) = \exp(2\pi i \,\mathrm{Tr}_{E/\mathbb{Q}}(t))$. On the other hand $\psi(u) = \psi_f(-t) = \exp(-2\pi i \,\mathrm{Tr}_{E/\mathbb{Q}}(t))$, $-t$ being an element of E. Hence $\prod_v \mathrm{wave}(\sigma(u_v)) = \psi(u)^{-1}$.

Conclusion. Each $F_v \in \{W_v, W_v^\flat\}$ is a Bessel function, so $F_v(u_v y_v) = \mathrm{wave}(\sigma(u_v))F_v(y_v)$. For $i \in I$, $W(u\cdot y)_i = \prod_v F_{v,i_v}(u_v y_v) = \prod_v \mathrm{wave}(\sigma(u_v))\cdot W(y)_i = \psi(u)^{-1}W(y)_i$. □

Lemma 5.2.44. *Let $\mathfrak{S}$ be a filling datum.*

(i) ψ restricted to $\mathsf{U}(\mathbb{A}_f)$ is a homomorphism to the unit circle: $\psi(gg') = \psi(g)\psi(g')$ and $|\psi(g)| = 1$ for $g, g' \in \mathsf{U}(\mathbb{A}_f)$.

(ii) For $g \in \mathsf{U}(\mathbb{A}_f)$ and $h \in \mathrm{U}(X_0)(\mathbb{A}_f)$: $h^{-1}gh \in \mathsf{U}(\mathbb{A}_f)$ and $\psi(h^{-1}gh) = \psi(g)$.

(iii) $\psi(g) = 1$ for every $g \in \mathsf{U}(\mathbb{A}_f) \cap \mathrm{K}(M)$.

(iv) If $g, g' \in \mathsf{U}(\mathbb{A}_f) \cap K$ and $\mathrm{red}(g) = \mathrm{red}(g')$, then $\psi(g) = \psi(g')$. Hence ψ_U is well defined, $\psi_U(\mathrm{red}(g)) = \psi(g)$ for every $g \in \mathsf{U}(\mathbb{A}_f) \cap K$, $\bar{\mathsf{U}}$ is a subgroup of $\bar{G}$, and ψ_U restricted to $\bar{\mathsf{U}}$ is a homomorphism to the unit circle.

(v) If $c \in \mathcal{C}$ and $u \in \mathsf{U}(E)$ with $h_c^{-1}uh_c \in K$, then $\mathrm{red}(h_c^{-1}uh_c) \in \bar{\mathsf{U}}$ and

$$\psi(u) = \psi_U(\mathrm{red}(h_c^{-1}uh_c)).$$

Proof. Write $F_b = \langle e_1, \dots, e_b\rangle$ $(0 \le b \le \varrho)$ and $P = \langle f_0, \dots, f_\varrho\rangle = F_\varrho \oplus \langle z\rangle$, over $\mathbb{A}_f$.

(i). Let $g \in \mathsf{U}(\mathbb{A}_f)$. Then g stabilises every F_b and acts trivially on F_b/F_{b-1}, so $ge_b \in e_b + F_{b-1}$; and g stabilises $F_\varrho^\perp$ and acts trivially on $F_\varrho^\perp/F_\varrho$, and $z \in F_\varrho^\perp$, so $gz \in z + F_\varrho$. Hence $gP \subseteq P$, and the matrix $T(g)$ of $g|_P$ in the basis f is upper unitriangular. Since f is part of the standard basis and $gf_{a+1} \in P$, the entry of g in the row f_a and the column f_{a+1} is $T(g)_{a,a+1}$, so $\sigma(g) = \sum_{a<\varrho} T(g)_{a,a+1}$. As P is invariant, $T(gg') = T(g)T(g')$, and for upper unitriangular matrices $(TT')_{a,a+1} = T_{a,a+1} + T'_{a,a+1}$. So $\sigma(gg') = \sigma(g) + \sigma(g')$. The map ψ_f is a homomorphism from $\mathbb{A}_f$ to the unit circle (if $x - a, x' - a' \in \hat{\mathcal{O}}$ then $x + x' - (a + a') \in \hat{\mathcal{O}}$), and $M\varpi$ is a unit of $\mathbb{A}_f$ ($M \neq 0$, $\varpi \neq 0$). Hence $\psi(gg') = \psi_f(-\sigma(g)/(M\varpi))\psi_f(-\sigma(g')/(M\varpi)) = \psi(g)\psi(g')$ and $|\psi(g)| = 1$.

(ii). The element h fixes every e_a, e_a^* and z, so it stabilises each F_b and acts trivially on each F_b/F_{b-1}; hence $h^{-1}gh$ stabilises each F_b and acts trivially on F_b/F_{b-1}, and it acts on $F_\varrho^\perp/F_\varrho$ as the conjugate of the identity by the action of h, that is trivially. So $h^{-1}gh \in \mathsf{U}(\mathbb{A}_f)$. As h fixes f, $T(h^{-1}gh) = T(g)$, so $\sigma(h^{-1}gh) = \sigma(g)$ and $\psi(h^{-1}gh) = \psi(g)$.

(iii). Let $g \in \mathsf{U}(\mathbb{A}_f) \cap \mathrm{K}(M)$. By Lemma 5.2.24 (iii), $(g-1)L_0 \subseteq ML_0$. Let $0 \le a < \varrho$. The vectors f_{a+1} and e_{a+1}^* belong to the lattice basis, hence to L_0, and the e_{a+1}-coordinate of a vector x is $h(e_{a+1}^*, x)$; as f_{a+1} has e_{a+1}-coordinate 0, $T(g)_{a,a+1} = h(e_{a+1}^*, (g-1)f_{a+1}) \in h(L_0, ML_0) \subseteq M\hat{\mathcal{O}}$ by Lemma 5.2.24 (i). So $\sigma(g) = My$ with $y \in \hat{\mathcal{O}}$ and $-\sigma(g)/(M\varpi) = -y/\varpi$. Since $\mathcal{O}_E$ is dense in $\hat{\mathcal{O}}$ and $\varpi\hat{\mathcal{O}}$ is open, $-y = t + \varpi y'$ with $t \in \mathcal{O}_E$ and $y' \in \hat{\mathcal{O}}$; so $-y/\varpi - t/\varpi = y' \in \hat{\mathcal{O}}$ and $\psi(g) = \exp(2\pi i \,\mathrm{Tr}_{E/\mathbb{Q}}(t/\varpi))$. As ϖ generates the different, $\varpi^{-1}\mathcal{O}_E$ is the inverse different $\{x \in E : \mathrm{Tr}_{E/\mathbb{Q}}(x\mathcal{O}_E) \subseteq \mathbb{Z}\}$, so $\mathrm{Tr}_{E/\mathbb{Q}}(t/\varpi) \in \mathbb{Z}$ and $\psi(g) = 1$.

(iv). $\mathsf{U}(\mathbb{A}_f) \cap K$ is a group and red is a homomorphism on K, so $\bar{\mathsf{U}}$ is a subgroup of $\bar{G}$. If $\mathrm{red}(g) = \mathrm{red}(g')$, then $g'^{-1}g \in \mathsf{U}(\mathbb{A}_f) \cap \ker \mathrm{red} = \mathsf{U}(\mathbb{A}_f) \cap \mathrm{K}(M)$, so $\psi(g'^{-1}g) = 1$ by (iii) and $\psi(g) = \psi(g')$ by (i). Thus the value $\psi_U(s) = \psi(u)$ of (TA8) does not depend on the integral lift u of $s \in \bar{\mathsf{U}}$, and $\psi_U(\mathrm{red}(g)) = \psi(g)$. For $s = \mathrm{red}(g)$, $s' = \mathrm{red}(g')$, gg' is a lift of ss', so $\psi_U(ss') = \psi(g)\psi(g') = \psi_U(s)\psi_U(s')$, of absolute value 1 by (i).

(v). Read in $G(\mathbb{A}_f)$, $u \in \mathsf{U}(\mathbb{A}_f)$. By (ii) with $h = h_c$, $g = h_c^{-1} u h_c \in \mathsf{U}(\mathbb{A}_f)$; it lies in K by hypothesis, so $\mathrm{red}(g) \in \bar{\mathsf{U}}$, and by (iv) and (ii), $\psi_U(\mathrm{red}(g)) = \psi(g) = \psi(u)$. □

Lemma 5.2.45. *Let (Z, μ) be a measure space and Γ a group acting on Z such that, for every $g \in \Gamma$, the map $z \mapsto g \cdot z$ is measurable and*

$$\int_Z H(g \cdot z)\, d\mu(z) = \int_Z H\, d\mu \qquad \text{for every measurable } H\colon Z \to [0, \infty].$$

Let $\Lambda \subseteq \Gamma$ be a countable subgroup, and let $\chi, \chi'\colon Z \to [0, \infty]$ be measurable with $\sum_{g\in\Lambda} \chi(g \cdot z) = \sum_{g\in\Lambda} \chi'(g \cdot z) = 1$ for every $z \in Z$.

(i) If $H\colon Z \to [0, \infty]$ is measurable and $H(g \cdot z) = H(z)$ for all $g \in \Lambda$ and $z \in Z$, then $\int_Z \chi H\, d\mu = \int_Z \chi' H\, d\mu$.

(ii) If $F\colon Z \to \mathbb{C}^N$ is measurable, $F(g \cdot z) = F(z)$ for all $g \in \Lambda$ and $z \in Z$, and $\int_Z \chi \|F\|\, d\mu < \infty$, then $\int_Z \chi' \|F\|\, d\mu < \infty$ and $\int_Z \chi F\, d\mu = \int_Z \chi' F\, d\mu$.

(iii) Let F be as in (ii). Suppose that there are $u \in \Gamma$ and $c_0 \in \mathbb{C}$ with $c_0 \neq 1$ and $F(u \cdot z) = c_0 F(z)$ for every $z \in Z$, and that $\Lambda \cap u^{-1}\Lambda u$ has finite index in Λ and in $u^{-1}\Lambda u$, the two indices being equal. Then $\int_Z \chi F\, d\mu = 0$.

Proof. The hypothesis on μ, applied to g^{-1}, gives $\int_Z \Phi\, d\mu = \int_Z \Phi(g^{-1} \cdot w)\, d\mu(w)$ for every measurable $\Phi\colon Z \to [0, \infty]$ and $g \in \Gamma$, and, coordinatewise, the same for every measurable $\Phi\colon Z \to \mathbb{C}^N$ with $\int \|\Phi\|\, d\mu < \infty$.

(i). Insert $1 = \sum_{g\in\Lambda} \chi'(g \cdot z)$. As Λ is countable and all functions are nonnegative and measurable, the monotone convergence theorem allows the sum and the integral to be interchanged; substituting $z = g^{-1} \cdot w$ in the term of g and using $H(g^{-1} \cdot w) = H(w)$,

$$\begin{aligned}\int_Z \chi H\, d\mu &= \sum_{g\in\Lambda} \int_Z \chi(z)\chi'(g \cdot z)H(z)\, d\mu(z) = \sum_{g\in\Lambda} \int_Z \chi(g^{-1} \cdot w)\chi'(w)H(w)\, d\mu(w) \\ &= \int_Z \chi'(w)H(w) \sum_{g\in\Lambda} \chi(g^{-1} \cdot w)\, d\mu(w).\end{aligned}$$

As $g \mapsto g^{-1}$ is a bijection of Λ, the inner sum is 1, and the right side is $\int_Z \chi' H\, d\mu$.

(ii). The function $\|F\|$ is measurable, nonnegative and Λ-invariant, so by (i) $\int \chi' \|F\|\, d\mu = \int \chi \|F\|\, d\mu < \infty$. For each coordinate F_j, the four functions $(\mathrm{Re}\, F_j)^{\pm}$, $(\mathrm{Im}\, F_j)^{\pm}$ are measurable, nonnegative, Λ-invariant and bounded by $\|F\|$; their integrals against χ and against χ' are finite, and equal by (i). Since $F_j = (\mathrm{Re}\, F_j)^+ - (\mathrm{Re}\, F_j)^- + i\big((\mathrm{Im}\, F_j)^+ - (\mathrm{Im}\, F_j)^-\big)$, linearity of the integral gives $\int \chi F_j\, d\mu = \int \chi' F_j\, d\mu$.

(iii). For a countable subgroup $\Lambda_1 \subseteq \Gamma$, call a measurable $\chi_1\colon Z \to [0, \infty]$ with $\sum_{g\in\Lambda_1} \chi_1(g \cdot z) = 1$ for all z a Λ_1-cutoff. Let $G\colon Z \to \mathbb{C}^N$ be measurable and Λ_1-invariant. If $\int \chi_1 \|G\|\, d\mu < \infty$ for one Λ_1-cutoff χ_1, then by (ii), applied to Λ_1, the same holds for every Λ_1-cutoff and $\int \chi_1 G\, d\mu$ does not depend on χ_1; we then say that $I_{\Lambda_1}(G)$ is defined and write $I_{\Lambda_1}(G) = \int \chi_1 G\, d\mu$. If $I_{\Lambda_1}(G)$ is defined, then so is $I_{\Lambda_1}(c_0 G)$, and $I_{\Lambda_1}(c_0 G) = c_0 I_{\Lambda_1}(G)$ by linearity of the integral.

Finite index. Let $\Lambda_2 \subseteq \Lambda_1$ have finite index m, and let $I_{\Lambda_1}(G)$ be defined, with a Λ_1-cutoff χ_1. Then $I_{\Lambda_2}(G)$ is defined and equals $m\, I_{\Lambda_1}(G)$. Indeed, write $\Lambda_1 = \bigsqcup_{i=1}^m g_i\Lambda_2$ and $\chi_2(z) = \sum_{i=1}^m \chi_1(g_i \cdot z)$, which is measurable; as every element of Λ_1 is $g_i h$ for exactly one pair (i,h) with $h \in \Lambda_2$,

$$\sum_{h\in\Lambda_2} \chi_2(h\cdot z) = \sum_{i=1}^m \sum_{h\in\Lambda_2} \chi_1(g_i h\cdot z) = \sum_{g\in\Lambda_1} \chi_1(g\cdot z) = 1,$$

so χ_2 is a Λ_2-cutoff, and G is Λ_2-invariant. Substituting $z = g_i^{-1}\cdot w$ in the i-th term and using $G(g_i^{-1}\cdot w) = G(w)$, $\int \chi_2\|G\|\,d\mu = \sum_i \int \chi_1(w)\|G(w)\|\,d\mu(w) = m\int \chi_1\|G\|\,d\mu < \infty$, and in the same way, coordinatewise, $\int \chi_2 G\,d\mu = m\int \chi_1 G\,d\mu$.

Conjugation. Let $v \in \Gamma$, $G_v(z) = G(v\cdot z)$, and let $I_{\Lambda_1}(G)$ be defined, with a Λ_1-cutoff χ_1. Then G_v is $v^{-1}\Lambda_1 v$-invariant, as $G_v(v^{-1}gv\cdot z) = G(g\cdot(v\cdot z)) = G(v\cdot z)$ for $g\in\Lambda_1$; the function $z\mapsto \chi_1(v\cdot z)$ is a $v^{-1}\Lambda_1 v$-cutoff, as $\sum_{g\in\Lambda_1}\chi_1(v\,v^{-1}gv\cdot z) = \sum_{g\in\Lambda_1}\chi_1(g\cdot(v\cdot z)) = 1$; and by the invariance of μ under v, $\int \chi_1(v\cdot z)\|G(v\cdot z)\|\,d\mu = \int \chi_1\|G\|\,d\mu < \infty$ and $\int \chi_1(v\cdot z)G(v\cdot z)\,d\mu = \int \chi_1 G\,d\mu$. So $I_{v^{-1}\Lambda_1 v}(G_v)$ is defined and equals $I_{\Lambda_1}(G)$.

Conclusion. Let $\Lambda' = \Lambda \cap u^{-1}\Lambda u$ and $m \geq 1$ the common index. $I_\Lambda(F)$ is defined, with the cutoff χ. By the finite-index step for $\Lambda' \subseteq \Lambda$, $I_{\Lambda'}(F)$ is defined and equals $m\, I_\Lambda(F)$; as $F_u = c_0 F$, $I_{\Lambda'}(F_u) = c_0 m\, I_\Lambda(F)$. By the conjugation step, $I_{u^{-1}\Lambda u}(F_u)$ is defined and equals $I_\Lambda(F)$, and by the finite-index step for $\Lambda' \subseteq u^{-1}\Lambda u$, $I_{\Lambda'}(F_u) = m\, I_\Lambda(F)$. Hence $(c_0 - 1)\, m\, I_\Lambda(F) = 0$, and as $c_0 \neq 1$ and $m \geq 1$, $\int_Z \chi F\,d\mu = I_\Lambda(F) = 0$. □

Lemma 5.2.46. *Let $\mathfrak{S}$ be a filling datum, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, and (P,F) a filling datum at T, with unitary group G_T, integral isometries $\mathrm{K}_T(1)$ and congruence subgroup $\mathrm{K}_T(M)$, the subgroup $\mathrm{K}(M)$ of the lattice datum F for its level unit, the image of M in E_T (also written M). For $x \in G(\mathbb{A}_f)$ let $x_T \in G_T$ be the matrix of the components at T of the entries of x in the lattice basis.*

(i) For $x \in G(\mathbb{A}_f)$: $x \in \mathrm{K}(M)$ if and only if x is integral off T and $x_T \in \mathrm{K}_T(M)$.

(ii) For $k,l \in K$: $\mathrm{red}(k) = \mathrm{red}(l)$ if and only if $k_T^{-1}l_T \in \mathrm{K}_T(M)$.

(iii) Let $B \leq G(\mathbb{A}_f)$ be a subgroup and μ a Borel measure on G_T such that μ-almost every $z \in G_T$ is x_T for some $x \in B$ integral off T, and $\mu(Ax_T) = \mu(A)$ for every Borel $A \subseteq G_T$ and $x \in B$. Then

$$\#\bar{B}\cdot\mu(\mathrm{K}_T(M)) = \mu(\mathrm{K}_T(1)).$$

Proof. Inverses. If S is the Gram matrix of h in the lattice basis, every $x \in G(\mathbb{A}_f)$ satisfies $x^{-1} = S^{-1}x^*S$ in this basis. As S and S^{-1} have integer entries, x^{-1} is integral at a finite place whenever x is, and $(x^{-1})_T = x_T^{-1}$. In particular K is the set of $x \in G(\mathbb{A}_f)$ whose matrix in the lattice basis has entries in $\hat{\mathcal{O}}$, and $x \in K$ if and only if x is integral off T and x_T has entries in $\mathcal{O}_T$, that is $x_T \in \mathrm{K}_T(1)$.

(i). By Lemma 5.2.24 (iii), $x \in \mathrm{K}(M)$ if and only if $x \in K$ and $(x-1)L_0 \subseteq ML_0$, that is, the matrix of $x-1$ in the lattice basis has entries in $M\hat{\mathcal{O}}$. At a finite place $w \notin T$, M is a unit ($M \in \mathbb{Z}$ and M^{-1} is integral at w), so the conditions at w say only that x is integral at w. At the places of T the conditions say that x_T and $x_T^{-1} = (x^{-1})_T$ have entries in $\mathcal{O}_T$ and $M^{-1}(x_T - 1)$ has entries in $\mathcal{O}_T$, that is $x_T \in \mathrm{K}_T(M)$.

(ii). red is a homomorphism on K with kernel $\mathrm{K}(M)$, so $\mathrm{red}(k) = \mathrm{red}(l)$ if and only if $k^{-1}l \in \mathrm{K}(M)$. The element $k^{-1}l \in K$ is integral off T and $(k^{-1}l)_T = k_T^{-1}l_T$; by (i) the claim follows.

(iii). The group $\bar{B} = \mathrm{red}(B \cap K) \subseteq \bar{G}$ is finite. For each $s \in \bar{B}$ choose $k_s \in B \cap K$ with $\mathrm{red}(k_s) = s$, and put $C_s = \{z \in G_T : z\,(k_s)_T \in \mathrm{K}_T(M)\} = \mathrm{K}_T(M)\,(k_s^{-1})_T$, an open set. By the invariance of μ under $(k_s^{-1})_T$, $\mu(C_s) = \mu(\mathrm{K}_T(M))$. The C_s are pairwise disjoint: if $z \in C_s \cap C_t$, then $(k_s)_T^{-1}(k_t)_T = (z(k_s)_T)^{-1}(z(k_t)_T) \in \mathrm{K}_T(M)$, so $s = t$ by (ii). Let $z = x_T$ with $x \in B$ integral off T. If $z \in C_s$, then $z \in \mathrm{K}_T(M)(k_s^{-1})_T \subseteq \mathrm{K}_T(1)$, as $k_s^{-1} \in K$. Conversely, if $z \in \mathrm{K}_T(1)$, then $x \in K$ by the first paragraph, so $x^{-1} \in B \cap K$; with $s = \mathrm{red}(x^{-1}) \in \bar{B}$, (ii) applied to x^{-1} and k_s gives $x_T(k_s)_T = ((x^{-1})_T)^{-1}(k_s)_T \in \mathrm{K}_T(M)$, that is $z \in C_s$. Hence $\bigcup_{s\in\bar{B}} C_s$ and $\mathrm{K}_T(1)$ agree outside a μ-null set, and

$$\mu(\mathrm{K}_T(1)) = \sum_{s\in\bar{B}} \mu(C_s) = \#\bar{B} \cdot \mu(\mathrm{K}_T(M)).$$

□

Lemma 5.2.47. *Let F be a lattice datum over a finite place A with level unit M, such that G is unimodular, and let θ be a Haar measure of G. Let Z be an intermediate group, $x, s \in G$, and $\mathrm{K}(M)a, \mathrm{K}(M)b \in \mathrm{Cos}(Z)$ with representatives a, b such that $s^{-1}ab^{-1} \in \mathrm{K}(M)$. Then*

$$\int_G W_Z(y)_a\, \overline{W_Z(x^{-1}y)_b}\, d\theta(y) = \theta(\mathrm{K}(M))\, \mathrm{Kl}(x, s),$$

where an integral whose integrand is not integrable is read as 0*, as is the sum defining* Kl *when it is not absolutely summable.*

Proof. Write $\mathrm{K} = \mathrm{K}(M)$ and $\chi = \chi_M$.

Step 1: invariances. K is an open compact subgroup of G, normal in K(1): for $\kappa \in \mathrm{K}(1)$ and $k \in \mathrm{K}$, $\kappa^{-1}k\kappa - 1 = \kappa^{-1}(k-1)\kappa$ has entries in $M\mathcal{O}_A$, as $\kappa^{\pm 1}$ are integral. The function χ is well defined and right K-invariant: ψ is trivial on $\mathcal{O}_A$ (self-duality with $a = 1$); the simple-root sum σ is additive on U (the entries just above the diagonal of unipotent block-triangular matrices add under multiplication), and $\sigma(v) \in M\mathcal{O}_A$ for $v \in \mathsf{U} \cap \mathrm{K}$; so two choices u, u' with $u^{-1}y, u'^{-1}y \in \mathrm{K}$ give $\psi(M^{-1}\sigma(u')) = \psi(M^{-1}\sigma(u))\psi(M^{-1}\sigma(u^{-1}u')) = \psi(M^{-1}\sigma(u))$; and $y \in u\mathrm{K}$ if and only if $yk \in u\mathrm{K}$, for $k \in \mathrm{K}$. In particular $\chi(u) = \psi(M^{-1}\sigma(u))$ for $u \in \mathsf{U}$. As $Z \subseteq \mathrm{K}(1)$, $s = ab^{-1}k_0^{-1}$ for some $k_0 \in \mathrm{K}$ lies in K(1) and normalises K. Finally, G being unimodular, θ is also right invariant.

Step 2: reduction to a sum over cosets. By definition $W_Z(y)_a = \chi(ya^{-1})$. The substitution $y \mapsto yb$ (right invariance) gives

$$\int_G \chi(ya^{-1})\overline{\chi(x^{-1}yb^{-1})}\, d\theta(y) = \int_G \chi(yba^{-1})\overline{\chi(x^{-1}y)}\, d\theta(y).$$

From $ab^{-1} \in s\mathrm{K}$ we get $ba^{-1} \in \mathrm{K}s^{-1} = s^{-1}\mathrm{K}$, so $\chi(yba^{-1}) = \chi(ys^{-1})$ by Step 1. The integrand $\Phi(y) = \chi(ys^{-1})\overline{\chi(x^{-1}y)}$ is right K-invariant: $\chi(yks^{-1}) = \chi(ys^{-1} \cdot sks^{-1}) = \chi(ys^{-1})$ and $\chi(x^{-1}yk) = \chi(x^{-1}y)$. So $\Phi = \sum_q \Phi(q)\mathbf{1}_{q\mathrm{K}}$, the sum over a set of representatives q of G/K, which is countable (K is open and G is a closed subgroup of $\mathrm{GL}_r(A)$, hence σ-compact). By left invariance $\theta(q\mathrm{K}) = \theta(\mathrm{K}) \in (0, \infty)$, so $\int |\Phi|\, d\theta = \theta(\mathrm{K}) \sum_q |\Phi(q)|$, and $\int \Phi\, d\theta = \theta(\mathrm{K}) \sum_q \Phi(q)$ when this is finite, both sides being read as 0 otherwise.

Step 3: the Kloosterman sum. Let I be the set of pairs (u, u') of the fixed representatives of $\mathsf{U}/(\mathsf{U}\cap\mathrm{K})$ with $s^{-1}u^{-1}xu' \in \mathrm{K}$, and map $(u, u') \mapsto xu'\mathrm{K} \in G/\mathrm{K}$. For $(u, u') \in I$ and $q \in xu'\mathrm{K}$ we have $x^{-1}q \in u'\mathrm{K}$ and $qs^{-1} \in xu'\mathrm{K}s^{-1} = xu's^{-1}\mathrm{K} \subseteq us\mathrm{K}s^{-1} = u\mathrm{K}$; hence $\Phi(q) = \chi(u)\overline{\chi(u')}$, the term of (u, u') in $\mathrm{Kl}(x, s)$. The map is injective: if $xu_1'\mathrm{K} = xu_2'\mathrm{K}$ then $u_1'^{-1}u_2' \in \mathsf{U}\cap\mathrm{K}$, so $u_1' = u_2'$, and then $u_1^{-1}xu_1', u_2^{-1}xu_1' \in s\mathrm{K}$ give $u_1^{-1}u_2 \in s\mathrm{K}s^{-1} = \mathrm{K}$, so $u_1 = u_2$. Every q with $\Phi(q) \neq 0$ is in the image: $\chi(qs^{-1}) \neq 0$ and $\chi(x^{-1}q) \neq 0$ give representatives u, u' with $u^{-1}qs^{-1} \in \mathrm{K}$ and

$u'^{-1}x^{-1}q \in \mathrm{K}$; then $q \in xu'\mathrm{K}$ and $s^{-1}u^{-1}xu' = (s^{-1}u^{-1}qs^{-1}s)(q^{-1}xu') \in s^{-1}\mathrm{K}s \cdot \mathrm{K} = \mathrm{K}$, so $(u,u') \in I$. The map therefore matches the nonzero terms of $\sum_q \Phi(q)$ bijectively with those of $\mathrm{Kl}(x,s)$, with equal values; so one sum converges absolutely if and only if the other does, and then they are equal. With Step 2 this gives the claim, in the convergent and in the non-convergent case. □

Lemma 5.2.48. *Let F be a large lattice datum over a finite place A with level unit M, Z an intermediate group, J a finite set, $R \in \mathbb{C}^{J\times J}$ positive semidefinite and $(\omega_a)_{a\in J}$ sequences of units of A. For $l \in G$ put $\mathrm{HS}(l) = \sum_{a,b\in J} R_{ab}\langle \mathrm{SO}_{\omega_a}(l), \mathrm{SO}_{\omega_b}(l)\rangle$. Then $\mathrm{HS}(l)$ is a nonnegative real number. Let $e \in V$, I a finite set and $l_\theta \in G^{(\varrho)}$ $(\theta \in I)$ with $l_\theta^{-1}z = e$ for all θ, such that $l_\theta l_{\theta'}^{-1} \in \mathrm{K}(M)$ only if $\theta = \theta'$. Then*

$$\mu_H(\mathrm{K}(M)) \sum_{\theta\in I} \mathrm{HS}(l_\theta) \le \mathrm{Re} \sum_{a,b\in J} R_{ab}\, \mathrm{fp}_{\omega_a,\omega_b}(e).$$

Proof. Step 1: positivity. The set $\mathrm{Cos}(Z)$ is finite ($\mathrm{K}(M)$ is open and $\mathrm{K}(1)$ compact). For $x \in \mathrm{Cos}(Z)^2$ let $v_x = (\mathrm{SO}_{\omega_a}(l)_x)_{a\in J}$. Then $\mathrm{HS}(l) = \sum_x \sum_{a,b} \overline{(v_x)_a} R_{ab} (v_x)_b = \sum_x v_x^* R v_x \ge 0$.

Step 2: invariance under $H \cap \mathrm{K}(M)$. Let $k \in H$. It fixes every e_a, e_a^* and z and maps the middle space (the orthogonal complement of the e_a, e_a^*) onto itself. For an index j let $S_{<j}$ be the span of the coordinate vectors of level below that of j; k maps each $S_{<j}$ onto itself (for $\mathrm{level}(j) \le \varrho$ it is spanned by e_b's; otherwise by the e_b, the middle space and some e_b^*). For $u \in \mathsf{U}$ and a coordinate vector j we have $uj' - j' \in S_{<j}$ for every j' in the span of the coordinate vectors of the level of j (this span is Aj, or the middle space, which $k^{\pm 1}$ preserves), so $kuk^{-1}j - j = k(u(k^{-1}j) - k^{-1}j) \in S_{<j}$: $kuk^{-1} \in \mathsf{U}$. Moreover $uf_{a+1} - f_{a+1}$ lies in the span of the e_b, which k fixes, so $kuk^{-1}f_{a+1} = kuf_{a+1}$ has the same coefficient on f_a as uf_{a+1}, and $\sigma(kuk^{-1}) = \sigma(u)$. Now let $k \in H \cap \mathrm{K}(M)$ and $y \in G$. The value $\chi_M(y) = \psi(M^{-1}\sigma(u))$ does not depend on the choice of $u \in \mathsf{U}$ with $u^{-1}y \in \mathrm{K}(M)$: two choices differ by an element of $\mathsf{U} \cap \mathrm{K}(M)$, whose simple-root sum lies in $M\mathcal{O}_A$ (the entries of $v - 1$ do, for $v \in \mathrm{K}(M)$), σ is additive on U (superdiagonal entries of unitriangular matrices add), and ψ is trivial on $\mathcal{O}_A$. If $u^{-1}y \in \mathrm{K}(M)$, then $u_1 = kuk^{-1} \in \mathsf{U}$ and $u_1^{-1}ky = ku^{-1}y \in \mathrm{K}(M)$, so $\chi_M(ky) = \psi(M^{-1}\sigma(u_1)) = \chi_M(y)$; if there is no such u for y, there is none for ky (apply the same to k^{-1}), and both values are 0. Hence $W_Z(ky) = (\chi_M(kyc^{-1}))_c = W_Z(y)$, and by (FB1), $\mathrm{SO}_p(kl) = \mathrm{SO}_p(l)$ and $\mathrm{HS}(kl) = \mathrm{HS}(l)$ for all $l \in G$.

Step 3: the fibre pairing. Let $l \in G^{(\varrho)}$ with $l^{-1}z = e$, and let l' be the element chosen in (FB2). Then $\beta = l'l^{-1} \in G^{(\varrho)}$ and $\beta z = l'e = z$, so $\beta \in H$. By (PE3) the map $k \mapsto k\beta$ preserves μ_H, hence

$$\mathrm{fp}_{p,q}(e) = \int_G \langle \mathrm{SO}_p(k\beta l), \mathrm{SO}_q(k\beta l)\rangle \, d\mu_H(k) = \int_G \langle \mathrm{SO}_p(kl), \mathrm{SO}_q(kl)\rangle \, d\mu_H(k),$$

and by Lemma 5.2.49 these integrands are integrable. Therefore

$$\sum_{a,b} R_{ab}\mathrm{fp}_{\omega_a,\omega_b}(e) = \int_G \mathrm{HS}(kl)\, d\mu_H(k),$$

a nonnegative real number by Step 1.

Step 4: the inequality. If I is empty, the left side is 0, and the right side is ≥ 0: by Step 3 if $e = l^{-1}z$ for some $l \in G^{(\varrho)}$, and because it is 0 by (FB2) otherwise. Let $\theta_0 \in I$, $l_0 = l_{\theta_0}$ and

$\beta_\theta = l_0 l_\theta^{-1}$, which lies in H as in Step 3. By (PE3), $\mu_H(\{k : k\beta_\theta \in \mathrm{K}(M)\}) = \mu_H(\mathrm{K}(M))$, which is at most $\mu_H(\mathrm{K}(1)) = 1$. So

$$\mu_H(\mathrm{K}(M)) \sum_\theta \mathrm{HS}(l_\theta) = \int_G \sum_\theta \mathbf{1}_{\mathrm{K}(M)}(k\beta_\theta)\, \mathrm{HS}(l_\theta)\, d\mu_H(k).$$

Let $k \in H$ (this holds μ_H-almost everywhere). If $k\beta_\theta \in \mathrm{K}(M)$, then $k\beta_\theta \in H \cap \mathrm{K}(M)$ and, by Step 2, $\mathrm{HS}(l_\theta) = \mathrm{HS}(k\beta_\theta l_\theta) = \mathrm{HS}(kl_0)$. At most one θ has $k\beta_\theta \in \mathrm{K}(M)$: for two such, $(k\beta_\theta)^{-1} k\beta_{\theta'} = l_\theta l_{\theta'}^{-1} \in \mathrm{K}(M)$, so $\theta = \theta'$. As $\mathrm{HS} \geq 0$, the integrand is at most $\mathrm{HS}(kl_0)$ almost everywhere, and by Step 3

$$\mu_H(\mathrm{K}(M)) \sum_\theta \mathrm{HS}(l_\theta) \leq \int_G \mathrm{HS}(kl_0)\, d\mu_H(k) = \sum_{a,b} R_{ab}\, \mathrm{fp}_{\omega_a,\omega_b}(e),$$

which is real, hence equal to its real part. □

Lemma 5.2.49. *Let F be a large lattice datum over a finite place A, Z an intermediate group, p, q sequences of units of A and $l \in G^{(\varrho)}$. Then the function $k \mapsto \langle \mathrm{SO}_p(kl), \mathrm{SO}_q(kl)\rangle$ on G is integrable for μ_H.*

Proof. This is Lemma 5.2.50, (vi), for the large lattice datum F, the intermediate group Z, the sequences p, q and $l \in G^{(\varrho)}$. □

Lemma 5.2.50. *Let F be a large lattice datum over a finite place A with level unit M, ϱ pairs and $n = \varrho + 1$, let Z be an intermediate group, $W = W_Z$, and let p, q be sequences of units of A.*

(i) There is a unit α of A with $\alpha \in \mathcal{O}_A$ and $\bar\alpha = \alpha$, and neighbourhoods of p and of q, such that every sequence in them is a head of length ϱ for the gauge of scale α.

(ii) $W(yk) = W(y)$ for $y \in G$, $k \in \mathrm{K}(M)$; $W(ky) = W(y)$ and $\mathrm{SO}_p(kl) = \mathrm{SO}_p(l)$ for $k \in H \cap \mathrm{K}(M)$ and $y, l \in G$.

(iii) There is a compact $C \subseteq G$ with $\mathrm{SO}_p(l) = 0$ for every $l \in G^{(\varrho)} \setminus C$.

(iv) There is a neighbourhood $\mathcal{N}$ of $(p, 1)$ in $(A^\times)^{\mathbb{N}} \times G$ such that $\mathrm{SO}_r(ls) = \mathrm{SO}_p(l)$ for all $(r, s) \in \mathcal{N}$ with $s \in G^{(\varrho)}$ and all $l \in G^{(\varrho)}$.

(v) For $l \in G^{(\varrho)}$,

$$\langle \mathrm{SO}_p(l), \mathrm{SO}_q(l)\rangle = \mathrm{hdn}_\varrho(p, q) \int_{G\times G} \langle W(y_1), W(y_2)\rangle \, \langle W(lw_{\varrho,q}^{-1} y_2), W(lw_{\varrho,p}^{-1} y_1)\rangle \, d(\theta\otimes\theta)(y_1, y_2),$$

the integrand being $\theta \otimes \theta$-integrable.

(vi) For $l \in G^{(\varrho)}$, the function $k \mapsto \langle \mathrm{SO}_p(kl), \mathrm{SO}_q(kl)\rangle$ is μ_H-integrable.

Proof. **(i).** For $i \leq \varrho$ the map $(a, r) \mapsto \bar a a\, (r_0 \cdots r_{i-1})^{-1}$ on $A \times (A^\times)^{\mathbb{N}}$ is continuous and vanishes at $a = 0$, so as $\mathcal{O}_A$ is open it takes values in $\mathcal{O}_A$ on a product neighbourhood of $(0, p)$ and on one of $(0, q)$; intersect over $i \leq \varrho$. The non-units form a null set (PA4) and non-empty open sets have positive Haar measure, so the neighbourhood of 0 contains a unit a. Then $\alpha = \bar a a$ is a unit fixed by the involution, $\alpha \in \mathcal{O}_A$ (the case $i = 0$), and $\alpha(r_0 \cdots r_{i-1})^{-1} \in \mathcal{O}_A$ says that $(r_0 \cdots r_{i-1})^{-1}$ has size 1 for the scale α. By Lemma 5.2.51 these sizes form a gauge Γ of F, and the sequences of the two neighbourhoods are heads of length ϱ for Γ. Below, Γ is this gauge, chosen for the two sequences at hand.

The compact set $\mathcal{S}$. $A_{(60n^2)}$ is compact by (GU1), so the matrices with entries in it form a compact set; $G = \{g : \bar{g}^{\mathsf{T}} J g = J\}$ is closed in the matrices (such g are invertible with $g^{-1} = J^{-1}\bar{g}^{\mathsf{T}} J$), carries the topology of the entries, and inversion $g \mapsto J^{-1}\bar{g}^{\mathsf{T}} J$ is continuous. So $\mathcal{S}$ is compact and G is a topological group in which $\mathcal{S}$ lies. The size clause of (DN0)(f) of Theorem 5.4.1 (for the gauge Γ, the cut function W, $\mu = \theta$ and both heads equal to the head at hand) applies: θ is a Haar measure (Lemma 5.5.41, (1)) and W is continuous (Lemma 5.5.44, (c)); it says that $W(y) \neq 0$ and $W((w_{\varrho,r}x)^{-1}y) \neq 0$, for a head r and $x \in G^{(\varrho)}$, imply $y \in \mathcal{S}$.

(ii). By Lemma 5.5.44, (a), (b): for $k \in \mathrm{K}(M)$ and a coset $\mathrm{K}(M)c \subseteq Z$, $ckc^{-1} \in \mathrm{K}(M)$, so $\chi_M(ykc^{-1}) = \chi_M(yc^{-1} \cdot ckc^{-1}) = \chi_M(yc^{-1})$; hence $W(yk) = W(y)$. Next, the level function lev_l equals the level for $l \geq \varrho$ (the indices of floor $\geq \varrho$, namely z and the x_i, have level ϱ), so $N_{<\varrho} = N_{<\varrho+1} = \mathsf{U}$. Let $k \in H \cap \mathrm{K}(M)$ and $y \in G$ with $u^{-1}y \in \mathrm{K}(M)$, $u \in \mathsf{U}$. By Lemma 5.2.58, (ii) with $l = \varrho + 1$, $u' = kuk^{-1} \in \mathsf{U}$, and by (viii) with $l = \varrho$, $\sigma(u') = \sigma(u)$; moreover $u'^{-1}ky = ku^{-1}y \in \mathrm{K}(M)$. By the independence of the choice in Lemma 5.5.44, (b), $\chi_M(ky) = \psi(M^{-1}\sigma(u')) = \chi_M(y)$; if no such u exists for y, none exists for ky (apply the argument to k^{-1}), and both values are 0. Applied to yc^{-1}, $W(ky) = W(y)$. Finally $W(kl\, w_p^{-1}y) = W(lw_p^{-1}y)$ in the integrand of (FB1), so $\mathrm{SO}_p(kl) = \mathrm{SO}_p(l)$.

(iii). Let $l \in G^{(\varrho)}$ with $\mathrm{SO}_p(l) \neq 0$. Then some y has $W(y) \neq 0$ and $W(y') \neq 0$, $y' = lw_p^{-1}y = (w_p l^{-1})^{-1}y$. As $l^{-1} \in G^{(\varrho)}$, the size clause of (DN0)(f) of Theorem 5.4.1 gives $y \in \mathcal{S}$. The sequence $\bar{p} = (\bar{p}_a)$ is a head for Γ by (SZ3), and $w_{\bar{p}} = w_p^{-1}$ (both send $e_a \mapsto \bar{p}_a e_a^*$, $e_a^* \mapsto p_a^{-1}e_a$); as w_p commutes with l (Lemma 5.2.58, (ix)), $(w_{\bar{p}}l)^{-1}y' = l^{-1}w_p\, lw_p^{-1}y = y$, so the size clause of (DN0)(f) of Theorem 5.4.1 for $\bar{p}$ and $x = l$ gives $y' \in \mathcal{S}$. Hence $l = y'y^{-1}w_p$ lies in the compact image C of $\mathcal{S} \times \mathcal{S}$ under $(y, y') \mapsto y'y^{-1}w_p$.

(iv). By Lemma 5.5.44, (a), $\mathrm{K}(M)$ is open. The map $(r, s, y) \mapsto y^{-1}w_p s\, w_r^{-1}y$ is continuous and equals 1 at $(p, 1, y)$; since $\mathcal{S}$ is compact, there is a neighbourhood of $(p, 1)$ on which it takes values in $\mathrm{K}(M)$ for every $y \in \mathcal{S}$. Shrink it so that its first components are heads for Γ (by (i)) and satisfy $p_j^{-1}r_j \in \mathcal{O}_A^\times$ for $j < \varrho$ (an open condition); then $|r_j| = |p_j|$ (both $p_j^{-1}r_j$ and its inverse are integral, of modulus ≤ 1), so $\mathrm{hdn}_\varrho(r, 1) = \mathrm{hdn}_\varrho(p, 1)$. Let (r, s) be in it, $s \in G^{(\varrho)}$, $l \in G^{(\varrho)}$, and y with $W(y) \neq 0$. If $W(lsw_r^{-1}y)$ or $W(lw_p^{-1}y)$ is non-zero, the size clause of (DN0)(f) of Theorem 5.4.1 (for r and $x = (ls)^{-1}$, or for p and $x = l^{-1}$) gives $y \in \mathcal{S}$, and then

$$lsw_r^{-1}y = lw_p^{-1}y \cdot (y^{-1}w_p s\, w_r^{-1}y) \in lw_p^{-1}y\, \mathrm{K}(M),$$

so the two values agree by (ii). The integrands of (FB1) for (r, ls) and (p, l) therefore coincide, and so do the densities.

(v). Choose Γ by (i) for p and q. For $r \in \{p, q\}$ and $c, c' \in \mathrm{Cos}(Z)$ the function $f_{r,c,c'}(y) = W(y)_c\overline{W(lw_r^{-1}y)_{c'}}$ is continuous (Lemma 5.5.44, (c)) and vanishes off $\mathcal{S}$ (the size clause of (DN0)(f) of Theorem 5.4.1 with $x = l^{-1}$); θ is a Haar measure (Lemma 5.5.41, (1)), finite on compact sets, so $f_{r,c,c'}$ is integrable and $\overline{f_{p,c,c'}(y_1)}f_{q,c,c'}(y_2)$ is $\theta \otimes \theta$-integrable. Since $\mathrm{hdn}_\varrho(p, 1)\mathrm{hdn}_\varrho(q, 1) = \mathrm{hdn}_\varrho(p, q)$ and $\mathrm{Cos}(Z)$ is finite, Fubini gives

$$\langle \mathrm{SO}_p(l), \mathrm{SO}_q(l)\rangle = \mathrm{hdn}_\varrho(p, q) \sum_{c,c'} \iint \overline{f_{p,c,c'}(y_1)}\, f_{q,c,c'}(y_2)\, d\theta(y_1)\, d\theta(y_2),$$

and summing over c and c' under the integral gives the integrand of (v).

(vi). Write $\Psi(k, y_1, y_2)$ for the integrand of (v) at kl in place of l. By Lemma 5.2.57, (ii), $\mathcal{K}_W$ is a pairing kernel, and (PK4) for the tower measure from floor $\varrho + 1$, which is $\theta_{\varrho+1} = \mu_H$, gives $c_0 < \infty$ with $\int |\langle W(ka), W(kb)\rangle|\, d\mu_H(k) \leq c_0$ for all $a, b \in G$. With $a = lw_q^{-1}y_2$, $b = lw_p^{-1}y_1$ this gives $\int |\Psi(k, y_1, y_2)|\, d\mu_H(k) \leq c_0 |\langle W(y_1), W(y_2)\rangle|$. For μ_H-almost every k, $k \in H$ (PE3), so

$kl \in G^{(\varrho)}$ and $\Psi(k, y_1, y_2) \neq 0$ forces $y_1, y_2 \in \mathcal{S}$ by the size clause of (DN0)(f) of Theorem 5.4.1. By Tonelli (μ_H s-finite, θ σ-finite, Ψ continuous),

$$\int |\Psi| \, d(\mu_H \otimes \theta \otimes \theta) \leq c_0 \int_{\mathcal{S}\times\mathcal{S}} |\langle W(y_1), W(y_2)\rangle| \, d(\theta \otimes \theta) < \infty.$$

By Fubini, $k \mapsto \int \Psi(k, \cdot)\, d(\theta \otimes \theta)$ is μ_H-integrable, and by (v), for almost every k it equals $\mathrm{hdn}_\varrho(p,q)^{-1}\langle \mathrm{SO}_p(kl), \mathrm{SO}_q(kl)\rangle$ ($\mathrm{hdn}_\varrho(p,q) \in (0,\infty)$ as the p_a, q_a are units). □

Lemma 5.2.51. *Let F be a large lattice datum over a finite place A, and let α be a unit of A with $\alpha \in \mathcal{O}_A$ and $\bar\alpha = \alpha$. Then the family $A_{(j)} = \{x \in A : \alpha^j x \in \mathcal{O}_A\}$ ($j \geq 0$) of the elements of size j for the scale α, with the loss $\Lambda = 2/|\alpha|$, is a gauge for F.*

Proof. Let the shape of F be $(\varrho, \iota, m{+}1)$, $n = \varrho{+}1$, $r = 2\varrho{+}k{+}1$, $A_{(j)} = \alpha^{-j}\mathcal{O}_A$ and $\Lambda = 2/|\alpha|$. As α is a unit in $\mathcal{O}_A$, $[\alpha] = [\mathcal{O}_A : \alpha\mathcal{O}_A] \geq 1$, so $0 < |\alpha| \leq 1$, $2 \leq \Lambda < \infty$ and $|\alpha|^{-1} \leq \Lambda$. Also $\zeta_{\mathcal{O}_A}(m) \geq 1$ (the ideal $\mathcal{O}_A$ contributes 1), so $\zeta_{\mathcal{O}_A}(m)^n \leq 2$ gives $\zeta_{\mathcal{O}_A}(m) \leq 2$.

Size filtration. (SZ1): $\alpha^i x, \alpha^j y \in \mathcal{O}_A$ give $\alpha^{i+j}xy \in \mathcal{O}_A$. (SZ2): if $\alpha^i x_l \in \mathcal{O}_A$ for $l \leq r+1$, then $\alpha^{i+1}\sum_l x_l = \alpha\sum_l \alpha^i x_l \in \mathcal{O}_A$. (SZ3): $\alpha^i(-x) = -\alpha^i x$ and $\alpha^i \bar x = \overline{\alpha^i x}$ ($\bar\alpha = \alpha$), and $\mathcal{O}_A$ is stable under -1 and the involution. (SZ4): $0 \in \mathcal{O}_A$ and $\alpha^i \in \mathcal{O}_A$. (SZ5): the entries of J and J^{-1} lie in $\mathcal{O}_A \subseteq \alpha^{-1}\mathcal{O}_A$.

(GU1)–(GU5). (GU1): $A_{(j)}$ is the image of the compact set $\mathcal{O}_A$ under $x \mapsto \alpha^{-j}x$, hence compact; $2 \leq \Lambda < \infty$; the shape parameter $m+1$ satisfies $m+1 \leq 9n$; $2(2(m+1)+2) = 4m{+}8 \leq 36n{+}4 \leq 53n \leq |\iota|$; $\mathrm{kb}_A = 1$. (GU2): M, the basis matrix, $\tau\delta^{-1}$ and the entries of the elements of K(1) lie in $\mathcal{O}_A \subseteq A_{(1)}$. (GU3): for a close pair, ω_1 and $M^{-1}(\omega_1 - \omega_2)$ are integral, hence of size 1. (GU4): suppose $\alpha M^{-1} \in \mathcal{O}_A$, let $i \geq 0$ and $c = \alpha^{i+1}$; then $\alpha^{i+2}c^{-1} = \alpha \in \mathcal{O}_A$, and for ω_1, ω_2 of size i the vector $c\omega_1 = \alpha \cdot \alpha^i\omega_1$ and $M^{-1}(c\omega_1 - c\omega_2) = (\alpha M^{-1})(\alpha^i\omega_1 - \alpha^i\omega_2)$ are integral, so $(c\omega_1, c\omega_2)$ is a close pair. (GU5): for a unit $c \in A_{(j)}$, $\alpha^j c \in \mathcal{O}_A$ gives $|\alpha|^j|c| \leq 1$, so $|c| \leq |\alpha|^{-j} \leq \Lambda^j$; and $\mathrm{vol}(A_{(j)}) = |\alpha|^{-j} \leq \Lambda^j \leq \Lambda^{j+1}$ by (PA3).

(GU6): the block kernel. Let $N = 2(m+1)+2 = 2m+4$ and $h_0(v,w) = \bar v^{\mathsf T} w$ on A^N.

An isotropic vector. There are $a, b \in A$ with $\bar a a + \bar b b = -1$; it suffices to find them in each orbit of the involution on the factors. On a pair $K_i \times K_{i'}$ ($i \neq i'$) exchanged by the involution take $b = 0$ and a with components 1 and -1: then $\bar a a = -1$. On a factor K_i mapped to itself the involution is nontrivial (it sends $\delta_i \neq 0$ to $-\delta_i$), so K_i is a quadratic extension of its fixed field F_i, a non-archimedean local field of characteristic 0, and the norm $\mathrm{N}(a) = \bar a a$ is a nondegenerate binary quadratic form over F_i. The form $\mathrm{N}(a) + \mathrm{N}(b) + z^2$ in five variables over F_i is isotropic (every quadratic form of dimension at least 5 over a non-archimedean local field is isotropic). A nontrivial zero with $z \neq 0$ gives $\mathrm{N}(a/z) + \mathrm{N}(b/z) = -1$; if $z = 0$, the nondegenerate form $\mathrm{N}(a) + \mathrm{N}(b)$ is isotropic, hence represents every element of F_i, in particular -1. Put $u = ae_1 + be_2 + e_3$ and $w = e_3 - \tau u$. Then

$$h_0(u,u) = \bar a a + \bar b b + 1 = 0, \qquad h_0(u,w) = 1, \qquad h_0(w,w) = 1 - \tau - \bar\tau = 0.$$

So the span H of u, w is a hyperbolic plane, $A^N = H \oplus H^\perp$ with the projection $v \mapsto v - h_0(w,v)u - h_0(u,v)w$ onto $H^\perp$, and $H^\perp$, a direct summand of rank $N-2$ in every factor of the product of fields A, is free. Let $f_1, \ldots, f_{N-2}$ be a basis of $H^\perp$, T' its Gram matrix (hermitian) and X the invertible $N \times N$ matrix with columns $u, w, f_1, \ldots, f_{N-2}$. Then $\bar X^{\mathsf T} X = J^{\mathrm{hyp}}_{T'}$, the matrix of $\bar c t + \bar t c + \bar v^{\mathsf T} T' v$ in the coordinates (c, t, v) of Lemma 5.2.52 (i).

The kernel. By Lemma 5.2.52 (i), the measures $\lambda_\beta^{T'}$ there are a disintegration of vol along $\bar x^{\mathsf T} J^{\mathrm{hyp}}_{T'} x$. By Lemma 5.2.53 (ii) (with $L = X^{-1}$, $J'' = 1$), the images $X_*\lambda_\beta^{T'}$ are a disintegration of $X_*\mathrm{vol} = |\det X|^{-1}\mathrm{vol}$ along $\bar y^{\mathsf T} y$; as (DJ1) and (DJ2) are linear in the measures, $\lambda_\beta =$

$|\det X|\, X_*\lambda_\beta^{T'}$ ($0 < |\det X| < \infty$, $\det X$ being a unit) is a disintegration of vol along $\bar{y}^{\mathsf{T}}y$. It is an s-finite kernel: λ_β is a constant times the image of the σ-finite measure $\mathrm{vol}\otimes\mathrm{vol}\otimes\nu$ under $q \mapsto X\,\mathrm{ch}_\beta(q)$, and $(\beta, q) \mapsto X\,\mathrm{ch}_\beta(q)$ is Borel (inversion is continuous on the open set of units), so $\beta \mapsto \lambda_\beta(B)$ is measurable for Borel B by Tonelli.

The mass bound. The vectors of size j form $\alpha^{-j}\mathcal{O}_A^N$. Apply Lemma 5.2.54 with $S = \{1, \dots, N\}$, the identity matrix (hermitian, with integral inverse), $\mu = m$ ($N = 2m+4 \geq 2m+1$), ψ trivial on traces, $g = \alpha^{-j}$ and $x_0 = 0$:

$$\lambda_\beta(\alpha^{-j}\mathcal{O}_A^N) \leq \zeta_{\mathcal{O}_A}(m)\,|\alpha|^{-j(N-1)} \leq 2\,|\alpha|^{-j(N-1)} \leq \Lambda^{1+j(N-1)} \leq \Lambda^{N(j+1)}$$

for every $\beta \in A^-$ and $j \geq 0$.

(GU7). For $g \in K = \mathrm{K}(1)$, a position y and a $\varrho \times \iota'$ matrix Q, Lemma 5.2.56 gives a common bound for $\mathcal{V}_{\mathrm{cm}}(g, y, Q)$ with loss Λ, for the sizes of scale α.

Hence $(A_{(j)})_{j\geq 0}$ with loss Λ is a gauge for F. □

Lemma 5.2.52. *Let A be a place.*

(i) Let S be a finite set and T' a hermitian $S \times S$ matrix over A. Write the points of $A^{\{1,2\}\sqcup S}$ as $x = (c, t, v)$ with $c, t \in A$, $v \in A^S$, and let $J_{T'}^{\mathrm{hyp}}$ be the hermitian matrix with $\bar{x}^{\mathsf{T}}J_{T'}^{\mathrm{hyp}}x = \bar{c}t + \bar{t}c + \bar{v}^{\mathsf{T}}T'v$. For $\beta \in A^-$ let $\lambda_\beta^{T'}$ be the image of $\mathrm{vol}\otimes\mathrm{vol}\otimes\nu$ *on $A \times A^S \times A^-$ under*

$$\mathrm{ch}_\beta(t, v, \alpha) = \big(t^\dagger\tau\,(\delta^{-1}\beta - \bar{v}^{\mathsf{T}}T'v) + t\alpha,\ t,\ v\big),$$

where $t^\dagger = \bar{t}^{-1}$ if t is a unit and $t^\dagger = 0$ otherwise. Then $(\lambda_\beta^{T'})_{\beta\in A^-}$ is a disintegration of vol *on $A^{\{1,2\}\sqcup S}$ along $\bar{x}^{\mathsf{T}}J_{T'}^{\mathrm{hyp}}x$.*

(ii) For every place datum of shape (ϱ, ι, m) over (A, k, T) and every $0 \leq l \leq \varrho$, the family $(\mathrm{qv}_l(\beta))_{\beta\in A^-}$ is a disintegration of vol_{V_l} along $h(v, v) = \bar{v}^{\mathsf{T}}Jv$.

Proof. We use that A and A^- are locally compact, second countable and metrisable, that ν is σ-finite and translation invariant, and that $t \mapsto t^\dagger$ is Borel (on the Borel set of units its graph is the closed set $\{(t, t') : \bar{t}t' = 1\}$). For a unit u, $|u|\,|u^{-1}| = |1| = 1$, so $0 < |u| < \infty$, and $|\bar{u}| = |u|$ by (PA2).

Two scaling facts. (a) For a unit u, $y_0 \in A$ and measurable $G\colon A \to [0, \infty]$, $\int_A G(uy + y_0)\,dy = |u|^{-1}\int_A G\,dy$: by (PA3) for the 1×1 matrix (u) the image of vol under $y \mapsto uy$ is $|u|^{-1}$vol, and vol is translation invariant.

(b) For a unit u with $\bar{u} = u$, the map $m_u(\alpha) = u\alpha$ is a homeomorphism of A^- and $(m_u)_*\nu = |u|^{-1/2}\nu$. Indeed, let $\Sigma(\alpha, \beta) = \tau\delta^{-1}\alpha + \beta$ on $A^- \times A^-$. Since $\overline{\delta^{-1}\alpha} = (-\delta)^{-1}(-\alpha) = \delta^{-1}\alpha$, we get $\Sigma(\alpha, \beta) + \overline{\Sigma(\alpha, \beta)} = (\tau + \bar{\tau})\delta^{-1}\alpha = \delta^{-1}\alpha$, so α, and then β, is determined by $\Sigma(\alpha, \beta)$: Σ is a continuous injection of Polish spaces, hence maps Borel sets to Borel sets (Lusin–Souslin), and a measure ρ on $A^- \times A^-$ is determined by $\Sigma_*\rho$. Now $\Sigma \circ (m_u \times m_u) = m_u \circ \Sigma$ (with m_u also denoting $y \mapsto uy$ on A), so by (PA1) and (a), $\Sigma_*\big((m_u)_*\nu \otimes (m_u)_*\nu\big) = (m_u)_*\mathrm{vol} = |u|^{-1}\mathrm{vol} = \Sigma_*(|u|^{-1}\nu \otimes \nu)$. Hence $(m_u)_*\nu \otimes (m_u)_*\nu = |u|^{-1}\nu \otimes \nu$; evaluating on $E \times E$ gives $((m_u)_*\nu(E))^2 = |u|^{-1}\nu(E)^2$ for every Borel $E \subseteq A^-$, and taking square roots in $[0, \infty]$ gives the claim.

The chart lies on the quadric. Let t be a unit, $v \in A^S$, $q = \bar{v}^{\mathsf{T}}T'v$ (so $\bar{q} = q$, as T' is hermitian) and $x = \mathrm{ch}_\beta(t, v, \alpha) = (c, t, v)$. Then $\bar{t}c = \tau(\delta^{-1}\beta - q) + t\bar{t}\alpha$ and, using $\overline{\delta^{-1}\beta} = \delta^{-1}\beta$ and $\bar{\alpha} = -\alpha$, $\bar{c}t = \overline{\bar{t}c} = \bar{\tau}(\delta^{-1}\beta - q) - t\bar{t}\alpha$. Hence

$$\bar{x}^{\mathsf{T}}J_{T'}^{\mathrm{hyp}}x = (\tau + \bar{\tau})(\delta^{-1}\beta - q) + q = \delta^{-1}\beta. \tag{5.22}$$

Moreover α is read off from x without β: with $\pi_t(x) = \bar\tau t^{-1}c - \overline{\bar\tau t^{-1}c}$ we have $\bar\tau t^{-1}c = r + \bar\tau\alpha$ with $r = \tau\bar\tau(t\bar t)^{-1}(\delta^{-1}\beta - q) = \bar r$, so

$$\pi_t(\mathrm{ch}_\beta(t,v,\alpha)) = \bar\tau\alpha - \overline{\bar\tau\alpha} = (\bar\tau + \tau)\alpha = \alpha. \tag{5.23}$$

(DJ2) in (i). Let $f\colon A \times A^{\{1,2\}\sqcup S} \to [0,\infty]$ be measurable. By Tonelli,

$$\int f(\bar x^{\mathsf T} J^{\mathrm{hyp}}_{T'} x, x)\,dx = \int_{A\times A^S} I(t,v)\,d(t,v),$$
$$I(t,v) = \int_A G_{t,v}(c)\,dc, \quad G_{t,v}(c) = f(\bar c t + \bar t c + q, (c,t,v)).$$

By (PA4), t is a unit for almost every (t,v); fix such (t,v) and put $c_0 = \tau q$. By (a) for the unit $\bar t^{-1}$ (with $|\bar t^{-1}| = |t|^{-1}$), then (PA1) with Tonelli, then (b) for the hermitian unit $u = t\bar t$ (with $|u|^{1/2} = |t|$):

$$\begin{aligned} I(t,v) &= |t|^{-1}\int_A G_{t,v}(\bar t^{-1}(y - c_0))dy \\ &= |t|^{-1}\iint G_{t,v}(\bar t^{-1}(\tau\delta^{-1}\beta + \alpha' - c_0))d\nu(\alpha')\,d\nu(\beta) \\ &= \iint G_{t,v}(\bar t^{-1}(\tau\delta^{-1}\beta + t\bar t\alpha - c_0))d\nu(\alpha)\,d\nu(\beta). \end{aligned}$$

Since $\bar t^{-1}(\tau\delta^{-1}\beta + t\bar t\alpha - \tau q) = t^\dagger\tau(\delta^{-1}\beta - q) + t\alpha$ is the first coordinate of $\mathrm{ch}_\beta(t,v,\alpha)$, (5.22) gives $G_{t,v}(\cdot) = f(\delta^{-1}\beta, \mathrm{ch}_\beta(t,v,\alpha))$ in the last integrand. Integrating over (t,v) and exchanging the order by Tonelli (the map $(\beta,t,v,\alpha) \mapsto \mathrm{ch}_\beta(t,v,\alpha)$ is Borel),

$$\begin{aligned} \int f(\bar x^{\mathsf T} J^{\mathrm{hyp}}_{T'} x, x)\,dx &= \int_{A^-}\int f(\delta^{-1}\beta, \mathrm{ch}_\beta(t,v,\alpha))\,d(t,v,\alpha)\,d\nu(\beta) \\ &= \int_{A^-}\int f(\delta^{-1}\beta, x)\,d\lambda^{T'}_\beta(x)\,d\nu(\beta). \end{aligned}$$

(DJ1) in (i). Let $f\colon A^{\{1,2\}\sqcup S} \to [0,\infty)$ be continuous with compact support C_f, and $\|f\|_\infty = \max f$. By definition of $\lambda^{T'}_\beta$ and Tonelli, $\int f\,d\lambda^{T'}_\beta = \int_{A\times A^S}\Phi_{t,v}(\beta)\,d(t,v)$ with $\Phi_{t,v}(\beta) = \int_{A^-} f(\mathrm{ch}_\beta(t,v,\alpha))\,d\nu(\alpha)$. Let $C_t \subseteq A$ and $C_v \subseteq A^S$ be the projections of C_f on the coordinates t and v, and $C_\Delta = \{\tau\delta^{-1}(x_1 - y_1) + (x_1' - y_1')\}$, where x_1, y_1, x_1', y_1' run through the first coordinates of points of C_f; all three are compact, so $\mathrm{vol}(C_\Delta) < \infty$. Fix (t,v) with t a unit.

A bound free of β. Let $F_\beta = \{\alpha \in A^- : \mathrm{ch}_\beta(t,v,\alpha) \in C_f\}$, a closed set. It is empty unless $(t,v) \in C_t \times C_v$. The first coordinate of $\mathrm{ch}_\beta(t,v,\alpha)$ is a constant plus $t\alpha$; hence for $(\alpha_1,\alpha_2), (\alpha_1',\alpha_2') \in F_\beta \times F_\beta$ the element $t(\tau\delta^{-1}(\alpha_1 - \alpha_1') + (\alpha_2 - \alpha_2'))$ lies in C_Δ. So the translate $F_\beta \times F_\beta - (\alpha_1',\alpha_2')$ lies in $\Sigma_t^{-1}(C_\Delta)$, $\Sigma_t = m_t \circ \Sigma$, and $(\Sigma_t)_*(\nu\otimes\nu) = |t|^{-1}\mathrm{vol}$ by (PA1) and (a). By translation invariance, $\nu(F_\beta)^2 \le |t|^{-1}\mathrm{vol}(C_\Delta)$. By (PA6) there is a measurable $\vartheta\colon A \to [0,\infty)$, integrable on compact sets, with $|u|^{-1} \le \vartheta(u)^2$ for every unit u. Hence, for every β,

$$\Phi_{t,v}(\beta) \le \|f\|_\infty\,\nu(F_\beta) \le b(t,v) := \|f\|_\infty\,\mathrm{vol}(C_\Delta)^{1/2}\,\vartheta(t)\,\mathbf{1}_{C_t\times C_v}(t,v),$$

and $\int b\,d(t,v) \le \|f\|_\infty \mathrm{vol}(C_\Delta)^{1/2}\mathrm{vol}(C_v)\int_{C_t}\vartheta < \infty$.

Continuity in β. By (5.23), $f(\mathrm{ch}_\beta(t,v,\alpha)) \neq 0$ forces $\alpha \in \pi_t(C_f) =: C'$, a compact set independent of β. So $\Phi_{t,v}(\beta) = \int_{C'} f(\mathrm{ch}_\beta(t,v,\alpha))\,d\nu(\alpha)$, with an integrand continuous in (β,α) and bounded by $\|f\|_\infty$ on a set of finite measure; by dominated convergence along sequences, $\Phi_{t,v}$ is continuous.

Conclusion. For almost every (t,v), $\Phi_{t,v}$ is continuous and bounded by the integrable $b(t,v)$, uniformly in β. By dominated convergence, $\beta \mapsto \int f\,d\lambda^{T'}_\beta$ is continuous, and it is at most $\int b < \infty$. This is (DJ1), and (i) is proved.

(ii). Let $S_l = \mathcal{Q}_l \sqcup \iota \sqcup \mathcal{Q}_l$ index the coordinates of Y_l, T_l' the (hermitian) Gram matrix of h_l' on Y_l, and regard B_l as the matrix X of the linear map $A^{\{1,2\}\sqcup S_l} \to V$, $(t_1, t_2, Y) \mapsto B_l(t_1, t_2, Y)$. From the relations of the hyperbolic basis, $h(e_a, e_b^*) = \delta_{ab}$, $h(e_a, e_b) = h(e_a^*, e_b^*) = 0$ and the orthogonality of the e_a, e_a^* to the middle space,

$$h(B_l(t_1,t_2,Y), B_l(t_1,t_2,Y)) = \bar{t}_1 t_2 + \bar{t}_2 t_1 + h_l'(Y,Y), \quad \text{that is,} \quad \bar{X}^{\mathsf{T}} J X = J^{\mathrm{hyp}}_{T_l'}.$$

As $(J^{\mathrm{hyp}}_{T_l'})^2 = 1$, the matrix $L = J^{\mathrm{hyp}}_{T_l'} \bar{X}^{\mathsf{T}} J$ satisfies $LX = 1$. By definition (5.4), $\mathrm{qv}_l(\beta) = X_* \lambda_\beta^{T_l'}$. It remains to see $X_*\mathrm{vol} = \mathrm{vol}_{V_l}$; then (i) and Lemma 5.2.53, (ii), with $\mu = \mathrm{vol}$, $J' = J^{\mathrm{hyp}}_{T_l'}$ and $J'' = J$, give (ii). The map X takes values in V_l and is the extension by zero of a square matrix X' from $A^{\{1,2\}\sqcup S_l}$ to $A^{\{\text{indices of floor} \geq l\}}$: both index sets have $2+2m+2(\varrho-l)+|\iota| = k+1+2(\varrho-l)$ elements. With J_l the restriction of J to the indices of floor $\geq l$, $\bar{X'}^{\mathsf{T}} J_l X' = J^{\mathrm{hyp}}_{T_l'}$; taking determinants, $\overline{\det X'} \det J_l \det X' = \pm 1$, where $\det J_l = \pm \det T$. So $\det X'$ is a unit, and by (PA2), (PE1) and multiplicativity of the modulus, $|\det X'|^2 = |\det X'|^2 |\det T| = |\pm 1| = 1$, so $|\det X'| = 1$. By (PA3), $X_*'\mathrm{vol} = \mathrm{vol}$, and extending by zero gives $X_*\mathrm{vol} = \mathrm{vol}_{V_l}$. □

Lemma 5.2.53. *Let A be a place, S and S' finite sets, J' a hermitian $S \times S$ matrix over A and μ a measure on A^S.*

(i) If $(\lambda_\beta)_{\beta \in A^-}$ and $(\lambda'_\beta)_{\beta \in A^-}$ are disintegrations of μ along $\bar{x}^{\mathsf{T}} J' x$, then $\lambda_\beta = \lambda'_\beta$ for every $\beta \in A^-$.

(ii) Let X be an $S' \times S$ matrix and L an $S \times S'$ matrix over A with $LX = 1$, and let J'' be a hermitian $S' \times S'$ matrix with $\bar{X}^{\mathsf{T}} J'' X = J'$. If $(\lambda_\beta)_{\beta \in A^-}$ is a disintegration of μ along $\bar{x}^{\mathsf{T}} J' x$, then the images $(X_* \lambda_\beta)_{\beta \in A^-}$ under $x \mapsto Xx$ form a disintegration of $X_*\mu$ along $\bar{y}^{\mathsf{T}} J'' y$.

Proof. Throughout, A is a locally compact Polish space, so A^S, $A^{S'}$ and the closed subgroup A^- of A are locally compact, second countable and metrisable. In particular A^- is σ-compact, the Haar measure ν is σ-finite and gives positive mass to every non-empty open subset of A^-, and every Borel measure on A^S that is finite on compact sets is a Radon measure.

Proof of (i). First, each λ_β is finite on compact sets: for a compact $C \subseteq A^S$, Urysohn's lemma gives a continuous compactly supported $f \colon A^S \to [0,1]$ with $f = 1$ on C, and $\lambda_\beta(C) \leq \int f \, d\lambda_\beta < \infty$ by (DJ1). The same holds for λ'_β.

Fix a continuous compactly supported $f \colon A^S \to [0, \infty)$ and put $F(\beta) = \int f \, d\lambda_\beta$, $F'(\beta) = \int f \, d\lambda'_\beta$; by (DJ1) both are finite and continuous on A^-. Let $S_0 \subseteq A^-$ be a Borel set and put $\Phi(a, x) = f(x) \, \mathbf{1}_{S_0}(\delta a)$ on $A \times A^S$; it is measurable, since $\{a \in A : \delta a \in S_0\}$ is the preimage of a Borel set under the continuous map $a \mapsto \delta a$. As $\Phi(\delta^{-1}\beta, x) = f(x) \mathbf{1}_{S_0}(\beta)$, (DJ2) for both families gives

$$\int_{S_0} F \, d\nu = \int_{A^S} \Phi(\bar{x}^{\mathsf{T}} J' x, x) \, d\mu(x) = \int_{S_0} F' \, d\nu,$$

where the outer integrals on A^- are ordinary integrals because $F \mathbf{1}_{S_0}$ and $F' \mathbf{1}_{S_0}$ are measurable. Let (C_n) be compact sets exhausting A^- and take $S_0 = \{F > F'\} \cap C_n$. On C_n the continuous functions F, F' are bounded and $\nu(C_n) < \infty$, so $\int_{S_0} (F - F') \, d\nu = 0$ with $F - F' > 0$ on S_0; hence $\nu(S_0) = 0$. Letting $n \to \infty$ and exchanging the roles of F and F', the set $\{F \neq F'\}$ is ν-null. It is open, as F and F' are continuous, so it is empty: $F = F'$ on A^-.

Thus $\int f \, d\lambda_\beta = \int f \, d\lambda'_\beta$ for every β and every continuous compactly supported $f \geq 0$. Both measures are Radon measures on A^S, so by the uniqueness part of the Riesz representation theorem $\lambda_\beta = \lambda'_\beta$.

Proof of (ii). The map $x \mapsto Xx$ is continuous and injective ($LX = 1$); its image $\{y : XLy = y\}$ is closed (if $y = Xx$ then $XLy = XLXx = y$, and conversely $y = X(Ly)$), and $y \mapsto Ly$ is a continuous inverse on it. For every $x \in A^S$,

$$\overline{Xx}^{\mathsf{T}} J'' Xx = \bar{x}^{\mathsf{T}} \bar{X}^{\mathsf{T}} J'' Xx = \bar{x}^{\mathsf{T}} J' x. \tag{5.24}$$

(DJ1) for $(X_*\lambda_\beta)$. Let $f\colon A^{S'} \to [0,\infty)$ be continuous with compact support C. Then $\int f\, d(X_*\lambda_\beta) = \int f(Xx)\, d\lambda_\beta(x)$, and $x \mapsto f(Xx)$ is continuous and vanishes outside the compact set $L(C)$: if $f(Xx) \neq 0$, then $Xx \in C$ and $x = L(Xx) \in L(C)$. By (DJ1) for (λ_β), $\beta \mapsto \int f\, d(X_*\lambda_\beta)$ is finite and continuous.

(DJ2) for $(X_*\lambda_\beta)$. Let $f\colon A \times A^{S'} \to [0,\infty]$ be measurable. The function $(a,x) \mapsto f(a, Xx)$ is measurable, and by (5.24) and (DJ2) for (λ_β),

$$\int_{A^{S'}} f(\bar{y}^{\mathsf{T}} J'' y, y)\, d(X_*\mu)(y) = \int_{A^S} f(\bar{x}^{\mathsf{T}} J' x, Xx)\, d\mu(x) = \int_{A^-} \int_{A^S} f(\delta^{-1}\beta, Xx)\, d\lambda_\beta(x)\, d\nu(\beta).$$

The inner integral equals $\int f(\delta^{-1}\beta, y)\, d(X_*\lambda_\beta)(y)$ for every β, so the outer (lower) integrals agree, which is (DJ2) for $(X_*\lambda_\beta)$ and $X_*\mu$. □

Lemma 5.2.54. *Let A be a finite place with $\psi(x + \bar{x}) = 1$ for all $x \in A$, let S be a finite set with $N = |S|$, let J be a hermitian $S \times S$ matrix such that J and J^{-1} have entries in $\mathcal{O}_A$, let $\mu \geq 0$ be an integer with $N \geq 2\mu + 1$, and let $(\lambda_\beta)_{\beta \in A^-}$ be a disintegration of* vol *on A^S along $\bar{x}^{\mathsf{T}} Jx$. Then for every unit g of A, every $x_0 \in A^S$ and every $\beta \in A^-$,*

$$\lambda_\beta(x_0 + g\mathcal{O}_A^S) \leq \zeta_{\mathcal{O}_A}(\mu)\, |g|^{N-1}.$$

Proof. Write $L = \mathcal{O}_A^S$, $Q(x) = \bar{x}^{\mathsf{T}} Jx$, $\alpha = \delta^{-1}\beta$, and for a unit u put $uL = \{ux : x \in L\}$, of volume $|u|^N$.

Step 0: ideals. The ideals of finite index of $\mathcal{O}_A$ are the products $\prod_i \varpi_i^{k_i}\mathcal{O}_{K_i}$ (ϖ_i a uniformiser of K_i, $k_i \geq 0$); they are countably many, each is $w\mathcal{O}_A$ for a unit $w \in \mathcal{O}_A$, and $[\mathcal{O}_A : w\mathcal{O}_A] = |w|^{-1}$, so $|w| \leq 1$. For $\eta \in A$ the set $I(\eta) = \{c \in \mathcal{O}_A : \eta c \in \mathcal{O}_A\}$ is such an ideal: its component in K_i is $\mathcal{O}_{K_i}$ if $\eta_i \in \mathcal{O}_{K_i}$ and $\eta_i^{-1}\mathcal{O}_{K_i}$ otherwise.

Step 1: Gauss sums. For $b \in A^-$ let

$$G(b) = \int_{gL} \Psi_b(y)\, dy, \qquad \Psi_b(y) = \psi\big(\tau b\,(Q(x_0 + y) - \alpha)\big),$$

and let $w \in \mathcal{O}_A$ be a unit with $w\mathcal{O}_A = I(g\bar{g}b)$. We show $|G(b)|^2 \leq |g|^{2N}|w|^N$. Since gL is a compact group, Fubini's theorem and the substitution $y' = y + d$ give

$$|G(b)|^2 = \int_{gL}\int_{gL} \overline{\Psi_b(y)}\Psi_b(y')\, dy\, dy' = \int_{gL} c(d)\, dd, \qquad c(d) = \int_{gL} \overline{\Psi_b(y)}\Psi_b(y + d)\, dy.$$

With $x = x_0 + y$ and $s = \bar{x}^{\mathsf{T}} Jd$ one has $Q(x + d) - Q(x) = Q(d) + s + \bar{s}$, as J is hermitian. For skew b and $t = -\bar{\tau}bs$ one has $\bar{t} = \tau b\bar{s}$ and $\tau b(s + \bar{s}) = bs + t + \bar{t}$, because $\tau + \bar{\tau} = 1$; as ψ is trivial on $t + \bar{t}$,

$$\overline{\Psi_b(y)}\Psi_b(y + d) = \psi\big(\tau bQ(d)\big)\, \psi\big(b\bar{x}_0^{\mathsf{T}} Jd\big)\, \chi_d(y), \qquad \chi_d(y) = \psi\big(b\,\bar{y}^{\mathsf{T}} Jd\big).$$

The first two factors have modulus 1 and χ_d is a continuous character of gL, so $|c(d)| = \mathrm{vol}(gL) = |g|^N$ if χ_d is trivial on gL, and $c(d) = 0$ otherwise. Let $d \in gL$ with χ_d trivial on gL. For $i \in S$

and $r \in \mathcal{O}_A$ the vector $y = g\bar{r}e_i$ lies in gL and $\bar{y}^{\mathsf{T}} Jd = \bar{g}r\,(Jd)_i$, so $\psi(r\,b\bar{g}(Jd)_i) = 1$ for all $r \in \mathcal{O}_A$, and $b\bar{g}(Jd)_i \in \mathcal{O}_A$ by self-duality. Thus $J(b\bar{g}d) \in L$, and $b\bar{g}d \in L$ as J^{-1} is integral. Writing $d = gc$ with $c \in L$, each c_i lies in $\mathcal{O}_A$ and satisfies $g\bar{g}bc_i \in \mathcal{O}_A$, so $c_i \in w\mathcal{O}_A$, and $d \in gwL$. Hence

$$|G(b)|^2 \le \int_{gL} |c(d)|\,dd \le |g|^N \operatorname{vol}(gwL) = |g|^{2N}|w|^N.$$

Step 2: summing over the ideals. For an ideal $I = w\mathcal{O}_A$ of finite index let $S_I = \{b \in A^- : g\bar{g}wb \in \mathcal{O}_A\}$ and $T_I = \{b \in A^- : I(g\bar{g}b) = I\}$. The sets T_I partition A^-, and $T_I = S_I \setminus \bigcup_{I' \supsetneq I} S_{I'}$, the union over the finitely many ideals strictly containing I (they correspond to ideals of the finite ring $\mathcal{O}_A/I$); since each $S_{I'}$ is open and closed in A^-, T_I is Borel. Let $u = g\bar{g}w$. For $a, b \in S_I$, $u\Phi(a,b) = \tau\delta^{-1}(ua) + ub \in \mathcal{O}_A$; so $S_I \times S_I \subseteq \Phi^{-1}(\{x : ux \in \mathcal{O}_A\})$ and, by (PA1), (PA3) and (PA2),

$$\nu(T_I)^2 \le \nu(S_I)^2 \le \operatorname{vol}(\{x \in A : ux \in \mathcal{O}_A\}) = |u|^{-1} = |g|^{-2}|w|^{-1}.$$

The function G is continuous on A^- (dominated convergence on the compact set gL). By Step 1, on T_I one has $|G| \le |g|^N |w|^{N/2}$, so, as $|w| \le 1$ and $(N-1)/2 \ge \mu$,

$$\int_{A^-} |G|\,d\nu = \sum_I \int_{T_I} |G|\,d\nu \le \sum_I |g|^N |w|^{N/2}\,|g|^{-1}|w|^{-1/2} \le |g|^{N-1} \sum_I |w|^\mu = \zeta_{\mathcal{O}_A}(\mu)\,|g|^{N-1},$$

with $|w|^\mu = [\mathcal{O}_A : I]^{-\mu}$.

Step 3: Fourier inversion. The class $C = x_0 + gL$ is compact and open, so $f = \mathbf{1}_C$ is continuous with compact support, and by the substitution $x = x_0 + y$ its function F in Lemma 5.2.55 is G. By Lemma 5.2.55 and Step 2,

$$\lambda_\beta(C) = \int f\,d\lambda_\beta \le \int_{A^-} |G|\,d\nu \le \zeta_{\mathcal{O}_A}(\mu)\,|g|^{N-1}.$$

□

Lemma 5.2.55. *Let A be a finite place with $\psi(x + \bar{x}) = 1$ for all $x \in A$, let S be a finite set, J a hermitian $S \times S$ matrix over A, and $(\lambda_\beta)_{\beta \in A^-}$ a disintegration of* vol *on A^S along $\bar{x}^{\mathsf{T}} Jx$. Let $f\colon A^S \to [0,\infty)$ be continuous with compact support, $\beta \in A^-$, and for $b \in A^-$*

$$F(b) = \int_{A^S} f(x)\,\psi\big(\tau b\,(\bar{x}^{\mathsf{T}} Jx - \delta^{-1}\beta)\big)\,dx.$$

If $\int_{A^-} |F|\,d\nu < \infty$, then $\int_{A^-} F\,d\nu = \int f\,d\lambda_\beta$. In every case $\int f\,d\lambda_\beta \le \int_{A^-} |F(b)|\,d\nu(b)$. Moreover, for every unit u of A with $\bar{u} = u$, the set $\{b \in A^- : ub \in \mathcal{O}_A\}$ is a compact open subgroup of A^- of ν-measure $|u|^{-1/2}$ (this holds for every finite place).

Proof. Write $Q(x) = \bar{x}^{\mathsf{T}} Jx$ and $\alpha = \delta^{-1}\beta$; both are self-adjoint (J is hermitian, $\bar{\delta} = -\delta$, $\bar{\beta} = -\beta$). The involution is a ring automorphism of A and maps $\mathcal{O}_A$ onto itself, so it permutes the factors K_i and their maximal ideals. Self-duality with $a = 1$ gives $\psi(\mathcal{O}_A) = 1$. A unit of A is an element with all components nonzero, and for a unit u, (PA3) for 1×1 matrices gives $\operatorname{vol}(\{x : ux \in E\}) = |u|^{-1}\operatorname{vol}(E)$ for Borel $E \subseteq A$.

Step 1: skew balls. Let u be a unit with $\bar{u} = u$ and $M_u = \{b \in A^- : ub \in \mathcal{O}_A\}$. Then M_u is a compact open subgroup of A^- and $\nu(M_u) = |u|^{-1/2}$. Indeed $m_u(b) = ub$ is a topological group automorphism of the closed subgroup A^- ($\overline{ub} = -ub$), so by uniqueness of Haar measure $(m_u)_*\nu = c_u\nu$ for some $c_u > 0$, and $M_u = m_u^{-1}(A^- \cap \mathcal{O}_A)$ is compact open with $\nu(M_u) = c_u\nu(A^- \cap \mathcal{O}_A) = c_u$. Let $\Phi(a,b) = \tau\delta^{-1}a + b$. Since $\Phi(ua, ub) = u\Phi(a,b)$, the image of

$\nu\otimes\nu$ under $(a,b)\mapsto u\Phi(a,b)$ is, on the one hand, the image of $c_u\nu\otimes c_u\nu$ under Φ, which is c_u^2vol by (PA1), and on the other hand the image of vol under $x\mapsto ux$, which is $|u|^{-1}$vol by (PA1) and (PA3). Hence $c_u=|u|^{-1/2}$, which is the last assertion of the statement; it uses neither J nor ψ.

Step 2: orthogonality. Let $u=\bar u$ be a unit and $y=\bar y\in A$. Then $\psi(\tau by)=1$ for all $b\in M_u$ if and only if $v=u^{-1}\delta y\in\mathcal{O}_A$. If $v\in\mathcal{O}_A$ and $b\in M_u$, then $\tau by=\tau\delta^{-1}(ub)\,v\in\mathcal{O}_A$, so $\psi(\tau by)=1$. Conversely, $\bar v=-v$. For $r\in\mathcal{O}_A$ put $b=u^{-1}\delta(r+\bar r)$; then $\bar b=-b$ and $ub=\delta(r+\bar r)\in\mathcal{O}_A$, so $b\in M_u$, and $\tau by=\tau(r+\bar r)v$. Since $\tau+\bar\tau=1$ and $\overline{\bar\tau rv}=-\tau\bar rv$,

$$rv=\tau(r+\bar r)v+\big(\bar\tau rv+\overline{\bar\tau rv}\big),$$

so $\psi(rv)=\psi(\tau by)\,\psi(\bar\tau rv+\overline{\bar\tau rv})=1$. As $r\in\mathcal{O}_A$ was arbitrary, self-duality gives $v\in\mathcal{O}_A$.

Step 3: the cut integral is a slab integral. Let $u=\bar u$ be a unit and $X_u=\{x\in A^S:u^{-1}\delta(Q(x)-\alpha)\in\mathcal{O}_A\}$, an open set. Then

$$\int_{M_u}F\,d\nu=\nu(M_u)\int_{X_u}f(x)\,dx. \tag{1}$$

The function $(b,x)\mapsto f(x)\psi(\tau b(Q(x)-\alpha))$ is continuous and bounded in modulus by $f(x)$, which is integrable on $M_u\times A^S$ as $\nu(M_u)<\infty$; so Fubini's theorem applies. For fixed x, $b\mapsto\psi(\tau b(Q(x)-\alpha))$ is a continuous character of the compact group M_u; its integral over M_u is $\nu(M_u)$ if it is trivial on M_u and 0 otherwise. By Step 2 with $y=Q(x)-\alpha$ it is trivial exactly for $x\in X_u$, which gives (1).

Step 4: the slab integral is a mean of fibre masses. Let $m(\beta')=\int f\,d\lambda_{\beta'}$, finite and continuous on A^- by (DJ1). Then

$$\int_{X_u}f(x)\,dx=\int_{M_{u^{-1}}}m(\beta+b)\,d\nu(b). \tag{2}$$

Apply (DJ2) to $g(a,x)=\mathbf{1}_W(a)f(x)$ with the open set $W=\{a\in A:u^{-1}\delta(a-\alpha)\in\mathcal{O}_A\}$: the left side becomes $\int\mathbf{1}_W(Q(x))f(x)\,dx=\int_{X_u}f$, and the right side $\int_{A^-}\mathbf{1}_W(\delta^{-1}\beta')\,m(\beta')\,d\nu(\beta')$, with a measurable integrand. Now $\delta^{-1}\beta'\in W$ if and only if $u^{-1}(\beta'-\beta)\in\mathcal{O}_A$, i.e. $\beta'-\beta\in M_{u^{-1}}$; the substitution $\beta'=\beta+b$, under which ν is invariant, gives (2).

Step 5: the limit. Choose a unit $\varpi\in\mathcal{O}_A$ whose component in each K_i is a uniformiser, and put $u_k=(\varpi\bar\varpi)^k$ for $k\ge0$, a self-adjoint unit in $\mathcal{O}_A$ whose components all lie in the maximal ideals. By (1), (2) and Step 1, with $\nu(M_{u_k})\nu(M_{u_k^{-1}})=|u_k|^{-1/2}|u_k|^{1/2}=1$,

$$\int_{M_{u_k}}F\,d\nu=\frac{1}{\nu(M_{u_k^{-1}})}\int_{M_{u_k^{-1}}}m(\beta+b)\,d\nu(b).$$

Here $M_{u_k^{-1}}=A^-\cap u_k\mathcal{O}_A$, and the sets $u_k\mathcal{O}_A$ form a neighbourhood basis of 0 in A; as m is continuous at β, the right side tends to $m(\beta)$ as $k\to\infty$. The sets M_{u_k} increase with k ($u_{k+1}=u_k\varpi\bar\varpi$ with $\varpi\bar\varpi\in\mathcal{O}_A$), and their union is A^-, since $u_kb\to0$ for every b. If $\int|F|\,d\nu<\infty$, dominated convergence gives $\int_{M_{u_k}}F\,d\nu\to\int_{A^-}F\,d\nu$, hence $\int F\,d\nu=m(\beta)=\int f\,d\lambda_\beta$. Then $\int f\,d\lambda_\beta=|\int F\,d\nu|\le\int|F|\,d\nu$; if $\int|F|\,d\nu=\infty$ the inequality is trivial. □

Lemma 5.2.56. *Let F be a large lattice datum of shape $(\varrho,\iota,m+1)$ over a finite place A, let $\alpha\in\mathcal{O}_A$ be a unit with $\bar\alpha=\alpha$, and let sizes be those of the scale α. Then for every $g\in\mathrm{K}(1)$, every position y and every $\varrho\times\iota'$ matrix Q over A there is a common bound for $\mathcal{V}_{\mathrm{cm}}(g,y,Q)$ with loss $2/|\alpha|$.*

Proof. Write $\Lambda = 2/|\alpha|$. As $\alpha \in \mathcal{O}_A$ is a unit, $0 < |\alpha| \le 1$, so $\Lambda \ge 2$ and $|\alpha|^{-1} \le \Lambda$. The shape has $m+2$ hyperbolic pairs, so $|\iota'| = |\iota| - 4m - 8$ (note $4m+8 < 53n \le |\iota|$). Put $d_0 = |\iota'| - 3\varrho$. As $m \le 9n-1$, $|\iota'| \ge 53n - 36n - 4 = 17n - 4 \ge 3\varrho + m + 5$, so $s = |\iota'| - (3n+1+(m+1))$ satisfies $d_0 = s + m + 5$ with $s \ge 0$. Also $\zeta_{\mathcal{O}_A}(m) \ge 1$ (the ideal $\mathcal{O}_A$ contributes 1), so $\zeta_{\mathcal{O}_A}(m)^n \le 2$ gives $\zeta_{\mathcal{O}_A}(m) \le 2$. Write also O for the $V \times \iota'$ matrix of $\omega \mapsto \sum_{i \in \iota'} \omega_i o_i$; its entries are entries of the basis matrix, hence integral.

Step 1: clearing columns. Let $g \in \mathrm{K}(1)$, $y = x^{-1}uxk'$ a position, Q a $\varrho \times \iota'$ matrix, and $N = xg^{-1}O$. Let C be the $3\varrho \times \iota'$ matrix whose rows are the rows of Q, the rows of N at the indices e_a^* $(a < \varrho)$, and the rows of $(u^{-1}-1)N$ at the indices e_a $(a < \varrho)$. There is $X \in \mathrm{GL}_{\iota'}(A)$ with X and X^{-1} integral whose last d_0 columns form an $\iota' \times d_0$ matrix Y with $CY = 0$. Indeed, in each factor K_i the kernel of $C_i \colon K_i^{\iota'} \to K_i^{3\varrho}$ has dimension at least d_0, and its intersection with $\mathcal{O}_{K_i}^{\iota'}$ is a saturated submodule (if $\varpi_i v$ lies in it and v is integral, so does v); by the elementary divisor theorem over the discrete valuation ring $\mathcal{O}_{K_i}$ it is a direct summand of rank at least d_0, and a basis of $\mathcal{O}_{K_i}^{\iota'}$ whose last d_0 members lie in it gives the component X_i.

Step 2: integral combinations of the columns of Y are common vectors. Let $t \in \mathcal{O}_A^{d_0}$ and $\omega = Yt$. First $Q\omega = 0$, as the rows of Q are rows of C. Next $(u^{-1}-1)NY = 0$: its rows at the e_a vanish because they are rows of CY; for an index i of level at least ϱ, the entries $(u^{-1})_{ij} - \delta_{ij}$ vanish unless j has larger level ($u^{-1} \in \mathsf{U}$), and such j are indices e_b^*, at which the rows of NY vanish. Hence $u^{-1}NY = NY$ and

$$(gy)^{-1}OY = k'^{-1}x^{-1}u^{-1}xg^{-1}OY = k'^{-1}x^{-1}u^{-1}NY = k'^{-1}x^{-1}NY = k'^{-1}W, \qquad W = g^{-1}OY.$$

The matrix W is integral, since g^{-1} ($g \in \mathrm{K}(1)$), O and Y are. So $g^{-1}O\omega = Wt$ is integral, and $M^{-1}(g^{-1}O\omega - (gy)^{-1}O\omega) = M^{-1}(1-k'^{-1})Wt = k'^{-1}M^{-1}(k'-1)Wt$ is integral, as $k' \in \mathrm{K}(M)$. Thus $(g^{-1}O\omega, (gy)^{-1}O\omega)$ is a close pair and $\omega \in \mathcal{V}_{\mathrm{cm}}(g, y, Q)$.

Step 3: product differences are integral. Let $(\omega_1, \omega_2), (\omega_1', \omega_2')$ be close pairs and $b = M^{-1}(\omega_1 - \omega_2)$, $b' = M^{-1}(\omega_1' - \omega_2')$, integral vectors. As h is conjugate linear in the first argument and $\bar{M} = M$,

$$M^{-1}(h(\omega_1, \omega_1') - h(\omega_2, \omega_2')) = h(\omega_1, b') + h(b, \omega_1') - M\,h(b, b'),$$

which is integral because J is. So $\mathcal{D}_{\mathrm{pr}} \subseteq \mathcal{O}_A$.

Step 4: the bound. For $\varsigma \in A^{\iota'}$ let $p(\varsigma) = \bar{Y}^{\mathsf{T}}\varsigma \in A^{d_0}$ and

$$\phi(\varsigma) = \mathrm{vol}(\{c \in A : c\,p(\varsigma)_l \in \mathcal{O}_A \text{ for all } l\}) \in [0, \infty].$$

Componentwise, $cp \in \mathcal{O}_A^{d_0}$ if and only if $v_i(c_i) \ge -\min_l v_i(p_{l,i})$ for every i. So $\phi(\varsigma) = \prod_i q_i^{\min_l v_i(p_{l,i})}$ (q_i the residue order, $q_i^\infty = \infty$), a Borel function of ς. If $\phi(\varsigma) < \infty$, choose for each i an index l_i attaining $\min_l v_i(p_{l,i}) < \infty$, let t be the unit with components $t_i = p_{l_i,i}$ and $r \in \mathcal{O}_A^{d_0}$ the vector with $r_l = \sum_{i : l_i = l} \epsilon_i$, ϵ_i the idempotent of K_i. Then $\sum_l r_l p_l = t$, $\{c : cp \in \mathcal{O}_A^{d_0}\} = t^{-1}\mathcal{O}_A$ and $\phi(\varsigma) = |t|^{-1}$.

First inequality. Let $\phi(\varsigma) < \infty$. For $a \in \mathcal{O}_A$ the vector $a\bar{r}$ is integral, so $\omega = Y(a\bar{r}) \in \mathcal{V}_{\mathrm{cm}}(g, y, Q)$ by Step 2, and $\bar{\varsigma}^{\mathsf{T}}\omega = a\sum_l \bar{p}_l \bar{r}_l = a\bar{t}$. So $\{\bar{\varsigma}^{\mathsf{T}}\omega : \omega \in \mathcal{V}_{\mathrm{cm}}\} \supseteq \bar{t}\mathcal{O}_A$, of volume $|\bar{t}| = |t|$, and $\mathrm{vol}^*(\{\bar{\varsigma}^{\mathsf{T}}\omega\})^{-1} \le |t|^{-1} = \phi(\varsigma)$. If $\phi(\varsigma) = \infty$ there is nothing to prove.

Second inequality. If $\bar{c}\,\bar{\varsigma}^{\mathsf{T}}\omega \in \mathcal{D}_{\mathrm{pr}}$ for all $\omega \in \mathcal{V}_{\mathrm{cm}}$, take $\omega = Y\epsilon_l$ for the unit vectors ϵ_l of A^{d_0} (Step 2): $\bar{c}\,\bar{p}_l \in \mathcal{D}_{\mathrm{pr}} \subseteq \mathcal{O}_A$ by Step 3, so $cp_l \in \mathcal{O}_A$ for all l. Hence that set of c has outer measure at most $\phi(\varsigma)$.

The integral. Fix $j \ge 0$. The substitution $\eta = \bar{X}^{\mathsf{T}}\varsigma$ preserves vol ((PA3); $|\det \bar{X}^{\mathsf{T}}| = 1$ as $\bar{X}^{\mathsf{T}}$ and its inverse are integral) and the set of vectors of size j, and $p(\varsigma) = \eta_{D_0}$, the coordinates of η at

the set D_0 of the last d_0 indices; let D_0' be the other 3ϱ indices and $\tilde\phi(\eta) = \mathrm{vol}(\{c : c\eta_{D_0} \in \mathcal{O}_A^{D_0}\})$. By (PA4) and Fubini, for almost every η a fixed coordinate of η_{D_0} is a unit, so $\tilde\phi(\eta) = |t|^{-1} < \infty$ with t as above. If moreover $\alpha^j\eta$ is integral, then $c = \alpha^j$ lies in $t^{-1}\mathcal{O}_A$, i.e. $w = t\alpha^j \in \mathcal{O}_A$, and $\eta_{D_0} \in t\mathcal{O}_A^{D_0}$. Put $I = w\mathcal{O}_A$, an ideal with $[\mathcal{O}_A : I] = |w|^{-1} \geq 1$, and

$$B_I = \{\eta : \alpha^j\eta_{D_0'} \in \mathcal{O}_A^{D_0'},\ \alpha^j\eta_{D_0} \in I^{D_0}\}, \qquad \mathrm{vol}(B_I) = |\alpha|^{-3\varrho j}\big(|\alpha|^{-j}|w|\big)^{d_0} = |\alpha|^{-3\varrho j}|t|^{s+m+5}.$$

Then $\eta \in B_I$ and, with $V_j = |\alpha|^{-j(3\varrho+m+5)}$,

$$[\mathcal{O}_A : I]^m\,\tilde\phi(\eta)^s\,\mathrm{vol}(B_I) \leq |w|^{-(m+5)}|t|^{-s}|\alpha|^{-3\varrho j}|t|^{s+m+5} = V_j.$$

For each ideal I of finite index let T_I be the Borel set of the $\eta \in B_I$ with $[\mathcal{O}_A : I]^m\tilde\phi(\eta)^s\mathrm{vol}(B_I) \leq V_j$; almost every η of size j lies in some T_I, and $\int_{T_I}\tilde\phi^s \leq \mathrm{vol}(T_I)[\mathcal{O}_A : I]^{-m}V_j/\mathrm{vol}(B_I) \leq [\mathcal{O}_A : I]^{-m}V_j$. Summing over the countably many ideals of finite index,

$$\int_{\{\varsigma \text{ of size } j\}} \phi(\varsigma)^s\,d\varsigma \leq \zeta_{\mathcal{O}_A}(m)V_j \leq 2|\alpha|^{-j(3\varrho+m+5)} \leq \Lambda^{1+j(3\varrho+m+5)} \leq \Lambda^{99n(j+1)},$$

since $3\varrho + m + 5 \leq 12n + 1 \leq 99n$ and $\Lambda \geq 2$. So ϕ is a common bound for $\mathcal{V}_{\mathrm{cm}}(g, y, Q)$ with loss Λ. □

Lemma 5.2.57. *Let a place datum be given, let $W\colon G \to \mathbb{C}^I$ be cut, let $j \geq 0$, let p, q be sequences of units of A and $d = w_{j,q}^{-1}w_{j,p}$. Then:*

(i) if $W(g) \neq 0$, then g is near at l for every $l \geq 0$;

(ii) $\mathcal{K}_W^d$ is a pairing kernel at floor j for d, and $\mathcal{K}_W$ is a pairing kernel at every floor for 1;

(iii) for all $x, g \in G$,

$$\mathrm{FI}_j(p, q, W, x; 1)(g) = \mathcal{K}_{W,j}(g, gx)\,\overline{\mathcal{K}_{W,j}^d(w_{j,p}^{-1}g,\ w_{j,p}^{-1}gx)}.$$

Proof. Let $\|W\|_\infty < \infty$ be a bound for $|W|$, and N_l the set of points near at l.

(i). If $W(g) \neq 0$, then $ug \in K$ for some $u \in \mathsf{U}$, and $\mathsf{U} \subseteq R^{(l)}$ by Lemma 5.2.58, (i).

(ii). The case of $\mathcal{K}_W$ is the case $p = q = (1, 1, \dots)$, where $d = 1$, for any floor; so it suffices to check (PK1)–(PK5) for $\mathcal{K} = \mathcal{K}_W^d$. By Lemma 5.2.58, (ix), d is diagonal.

(PK1): $(g, y) \mapsto \langle W(g), W(dy)\rangle$ is measurable, as W is measurable and the inner product is continuous.

(PK3): $dsd^{-1} = s$ for $s \in G^{(j)}$, since both $w_{j,p}$ and $w_{j,q}$ commute with $G^{(j)}$ (Lemma 5.2.58, (ix)).

(PK2): Let $l \geq 0$, $n, n' \in N_{<l}$ and $g, y \in G$. The measure $\theta_{\geq l+1}$ is carried by $G^{(l+1)}$ (Lemma 5.2.59, (ii)); let $x \in G^{(l+1)}$. Then $xnx^{-1} \in N_{<l} \subseteq \mathsf{U}$ (Lemma 5.2.58, (ii), (i)), and $d\,xn'x^{-1}d^{-1} \in \mathsf{U}$ (Lemma 5.2.58, (x)). By Lemma 5.2.58, (viii) and (x), $\sigma(xnx^{-1}) = \sigma(n)$ and

$$\sigma(d\,xn'x^{-1}d^{-1}) = \sum_{a<\varrho} d_{f_af_a}\,(xn'x^{-1})_{f_af_{a+1}}\,d^{-1}_{f_{a+1}f_{a+1}} = \sum_{a<\varrho} d_{f_af_a}\,n'_{f_af_{a+1}}\,d^{-1}_{f_{a+1}f_{a+1}} = \sigma(dn'd^{-1}).$$

Writing $xng = (xnx^{-1})\,xg$ and $dxn'y = (d\,xn'x^{-1}d^{-1})\,dxy$, the Bessel property of frequency M^{-1} gives $W(xng) = \psi(M^{-1}\sigma(n))W(xg)$ and $W(dxn'y) = \psi(M^{-1}\sigma(dn'd^{-1}))W(dxy)$, hence, the inner product being conjugate linear in the first variable and $|\psi| = 1$,

$$\mathcal{K}(xng, xn'y) = \psi\big(M^{-1}(\sigma(dn'd^{-1}) - \sigma(n))\big)\,\mathcal{K}(xg, xy) \qquad (x \in G^{(l+1)}).$$

The factor is a constant of modulus 1, so $x \mapsto \mathcal{K}(xng, xn'y)$ is $\theta_{\geq l+1}$-integrable exactly when $x \mapsto \mathcal{K}(xg, xy)$ is, and integrating gives (PK2) (both sides vanish by convention otherwise).

(PK4): By Cauchy–Schwarz $|\mathcal{K}(xg, xy)| \leq \|W\|_\infty^2$, and by (i) $\mathcal{K}(xg, xy) \neq 0$ implies $xg \in N_{\varrho+2}$. By Lemma 5.2.60, (iii), $V = \max_{a \leq \varrho} \mathrm{qv}_a(C_a)$ is finite and $\mathrm{FML}_a(V; 1)$ holds for every $a \leq \varrho$; by Lemma 5.2.60, (iv),

$$\int |\mathcal{K}(xg, xy)|\, d\theta_{\geq l}(x) \leq \|W\|_\infty^2\, \theta_{\geq l}(\{x : xg \in N_{\varrho+2}\}) \leq \|W\|_\infty^2\, V^{\max(\varrho+1-l,0)},$$

a bound independent of g and y.

(PK5): If $\mathcal{K}_l(g, y) \neq 0$, the integrand $x \mapsto \mathcal{K}(xg, xy)$ is not $\theta_{\geq l}$-almost everywhere zero; as $\theta_{\geq l}$ is carried by $G^{(l)}$, there is $x \in G^{(l)}$ with $W(xg) \neq 0$. By (i), $\xi xg \in K$ for some $\xi \in R^{(l)}$; as $x \in G^{(l)} \subseteq R^{(l)}$ (Lemma 5.2.58, (i)), $\xi x \in R^{(l)}$ and g is near at l.

(iii). By (ii), $\mathcal{K}_W$ and $\mathcal{K}_W^d$ are pairing kernels at floor j. By Lemma 5.2.64, (ii), for every ξ the function $\zeta \mapsto \mathcal{K}_{W,j+1}^d(\zeta w_{j,p}^{-1} g, \zeta w_{j,p}^{-1} gx)$ is θ_j-integrable with integral $\overline{\mathcal{K}_{W,j}^d(w_{j,p}^{-1} g, w_{j,p}^{-1} gx)}$, so with the weight 1 the inner integral in (5.14) is $\mathcal{K}_{W,j+1}(\xi g, \xi gx)\, \overline{\mathcal{K}_{W,j}^d(w_{j,p}^{-1} g, w_{j,p}^{-1} gx)}$; and $\xi \mapsto \mathcal{K}_{W,j+1}(\xi g, \xi gx)$ is θ_j-integrable with integral $\mathcal{K}_{W,j}(g, gx)$, again by Lemma 5.2.64, (ii). This gives (iii). □

Lemma 5.2.58. *Let A be a place and G the isometry group of a split space at A with ϱ pairs. For every $l \geq 0$:*

(i) $G^{(l+1)} \subseteq G^{(l)} \subseteq R^{(l)}$, $R^{(l+1)} \subseteq R^{(l)}$, $N_{<l} \subseteq N_{<l+1} \subseteq \mathsf{U} \subseteq R^{(j)}$ *for every* $j \geq 0$, *and* $G^{(l)} = \{1\}$ *if* $l \geq \varrho + 2$;

(ii) $\sigma N_{<l} \sigma^{-1} \subseteq N_{<l}$ *for every* $\sigma \in G^{(l)}$;

(iii) if $l < \varrho$, *then for* $v, v' \in V_{l+1}$ *and* $\beta, \beta' \in A^-$: $n_l(v, \beta) \in G^{(l)} \cap N_{<l+1}$,

$$n_l(v, \beta)\, n_l(v', \beta') = n_l\big(v + v',\ \beta + \beta' + \tau h(v', v) - \bar{\tau} h(v, v')\big),$$

and $u\, n_l(v, \beta)\, u^{-1} = n_l(uv, \beta)$, *with* $uv \in V_{l+1}$, *for every* $u \in G^{(l+1)}$;

(iv) if $l < \varrho$, *then for every* $t \in R^{(l+1)} \cap G^{(l)}$ *there are* $v \in V_{l+1}$ *and* $\beta \in A^-$ *with* $n_l(v, \beta)^{-1} t \in G^{(l+1)}$; *if* $l \geq \varrho$, *then* $R^{(l+1)} \cap G^{(l)} \subseteq G^{(l+1)}$;

(v) every $s \in R^{(l)}$ *is a product* $s = \gamma\nu$ *with* $\gamma \in G^{(l)}$ *and* $\nu \in N_{<l}$;

(vi) if $l \leq \varrho$ *and* $\sigma \in G^{(l)}$, *then* $\sigma \in R^{(l+1)}$ *if and only if* $\sigma f_l = f_l$;

(vii) if $l \leq \varrho$ *and* $\sigma \in G^{(l)}$, *then the rows of* σ *at the indices of floor below* l *are those of* 1, $\sigma V_l = V_l$, *and the image of the Haar measure* vol_{V_l} *under* $v \mapsto \sigma v$ *is* vol_{V_l};

(viii) if $g \in N_{<l}$ *and* $u \in G^{(l+1)}$, *then* $(ugu^{-1})_{f_a f_{a+1}} = g_{f_a f_{a+1}}$ *for every* $a < \varrho$; *in particular* $\sigma(ugu^{-1}) = \sigma(g)$;

(ix) for sequences p, q *of units of* A, $w_{l,q}^{-1} w_{l,p}$ *is the diagonal element sending* $e_a \mapsto p_a q_a^{-1} e_a$ *and* $e_a^* \mapsto \overline{q_a p_a^{-1}}\, e_a^*$ *for* $a < \min(l, \varrho)$ *and fixing the other coordinate vectors; and* $w_{l,p} t = t w_{l,p}$ *for every* $t \in G^{(l)}$;

(x) if $D \in G$ *is diagonal, then* $(DgD^{-1})_{ik} = D_{ii}\, g_{ik}\, D_{kk}^{-1}$ *for* $g \in G$ *and all indices* i, k, *and* $D\mathsf{U}D^{-1} = \mathsf{U}$, $DN_{<l}D^{-1} = N_{<l}$.

Proof. For an index i of a hyperbolic pair write i' for the opposite index ($e'_a = e^*_a$, $e^{*\prime}_a = e_a$); then $h(e_{i'}, w)$ is the i-coordinate of w. A matrix has an entry at (i, j) if row i and column j; $g \in G^{(l)}$ means that the columns of g at indices of floor below l are those of 1.

Rows of a floor group. Let $\sigma \in G^{(l)}$ and let i be an index of floor $b < l$ with $b < \varrho$. Then $e_{i'}$ has floor $b < l$, so $\sigma e_{i'} = e_{i'}$ and, for every j, $\sigma_{ij} = h(e_{i'}, \sigma e_j) = h(\sigma e_{i'}, \sigma e_j) = h(e_{i'}, e_j) = \delta_{ij}$: *the rows of σ at indices of floor below* $\min(l, \varrho)$ *are those of* 1. In particular σ fixes the span of the coordinate vectors of floor below l and, being an isometry, preserves its orthogonal complement, which is V_l when $l \le \varrho$.

(i). $G^{(l+1)} \subseteq G^{(l)}$ is clear. The conditions defining $R^{(l)}$ are among those defining $R^{(l+1)}$ (a column of floor and level below l has floor and level below $l+1$, and $l \ge \varrho + 2$ implies $l + 1 \ge \varrho + 2$), so $R^{(l+1)} \subseteq R^{(l)}$; and they are among those defining U, so $\mathsf{U} \subseteq R^{(j)}$ for all j. If $l \le \varrho + 1$, the conditions of $R^{(l)}$ concern only columns of floor below l, where an element of $G^{(l)}$ agrees with 1; if $l \ge \varrho + 2$, every index has floor below l, so $G^{(l)} = \{1\} \subseteq R^{(l)}$.

$N_{<l}$ consists of the $g \in G$ such that $g - 1$ has non-zero entries only at (i, j) with $\mathrm{lev}_l(i) < \mathrm{lev}_l(j)$; U likewise with the level. Let $\mathrm{lev}_l(i) < \mathrm{lev}_l(j)$. If i, j both have floor below l, the levels are compared. If i has floor below l and j not, then $\mathrm{level}(i) < \varrho$, so $i = e_b$ with $b < l$, and $\mathrm{level}(j) \ge l > b$. If j has floor below l and i not, then $\mathrm{level}(j) > \varrho$, so $j = e^*_b$ with $b < l$, and $\mathrm{level}(i) \le 2\varrho - l < 2\varrho - b$. (Both of floor at least l is impossible, as both get ϱ.) So $\mathrm{level}(i) < \mathrm{level}(j)$, and $N_{<l} \subseteq \mathsf{U}$. For $l \ge \varrho$, $\mathrm{lev}_l = \mathrm{lev}_{l+1} = \mathrm{level}$; for $l < \varrho$, lev_{l+1} differs from lev_l only at e_l (from ϱ to l) and e^*_l (from ϱ to $2\varrho - l$). If $\mathrm{lev}_l(i) < \mathrm{lev}_l(j)$ with $j = e_l$, then $i = e_b$ with $b < l$ and $\mathrm{lev}_{l+1}(i) = b < l$; with $i = e^*_l$, then $j = e^*_b$ with $b < l$ and $2\varrho - l < 2\varrho - b$; in the remaining cases nothing changes. So $N_{<l} \subseteq N_{<l+1}$.

(ii). Let $\sigma \in G^{(l)}$. If $\mathrm{lev}_l(i) > \mathrm{lev}_l(j)$, then $\sigma_{ij} = 0$: for j of floor below l the column is that of 1 (and $i \ne j$); for j of floor at least l, $\mathrm{lev}_l(i) > \varrho$ forces $i = e^*_b$ with $b < l$, a row of 1. So σ and $\sigma^{-1} \in G^{(l)}$ lie in the ring $\mathcal{B}$ of matrices that are block upper triangular for the preorder lev_l, and $g \in N_{<l}$ means that $g - 1$ lies in the two-sided ideal of $\mathcal{B}$ of strictly block upper triangular matrices. Hence $\sigma g \sigma^{-1} - 1 = \sigma(g-1)\sigma^{-1}$ lies in that ideal, and $\sigma g \sigma^{-1} \in G$, so $\sigma g \sigma^{-1} \in N_{<l}$.

(iii). Let $l < \varrho$ and $n = n_l(v, \beta)$. By (5.8), n fixes e_b, e^*_b ($b < l$) and e_l, sends $w \in V_{l+1}$ to $w - h(v, w)e_l$ and e^*_l to $e^*_l + v + (\beta - \tau h(v, v))e_l$; it lies in G (Definition 5.2.1), hence in $G^{(l)}$. The entries of $n - 1$ lie at (i, e^*_l) with i of floor at least $l + 1$ or $i = e_l$, and at (e_l, j) with j of floor at least $l + 1$; as $\mathrm{lev}_{l+1}(e_l) = l < \varrho < 2\varrho - l = \mathrm{lev}_{l+1}(e^*_l)$ and the indices of floor at least $l + 1$ get ϱ, $n \in N_{<l+1}$. For the product, compare both sides on the basis: on e_b, e^*_b ($b < l$) and e_l both are the identity; on $w \in V_{l+1}$ both give $w - h(v + v', w)e_l$; and

$$n_l(v, \beta)n_l(v', \beta')e^*_l = e^*_l + (v + v') + (\beta + \beta' - \tau h(v, v) - \tau h(v', v') - h(v, v'))e_l,$$

which is $n_l(v + v', \beta'')e^*_l$ for $\beta'' - \tau h(v + v', v + v')$ equal to the coefficient of e_l, that is $\beta'' = \beta + \beta' + \tau h(v', v) - \bar\tau h(v, v')$; this β'' is skew, since $\overline{\tau h(v', v)} = \bar\tau h(v, v')$. For $u \in G^{(l+1)}$: u fixes e_l and preserves V_{l+1} (first paragraph), and in (5.8) $h(e_l, u^{-1}w) = h(ue_l, w) = h(e_l, w)$, $h(v, u^{-1}w) = h(uv, w)$, $h(uv, uv) = h(v, v)$; applying u to $n\,u^{-1}w$ gives $u\,n_l(v, \beta)\,u^{-1}w = n_l(uv, \beta)w$ for every w.

(iv). Let $l < \varrho$ and $t \in R^{(l+1)} \cap G^{(l)}$. The column e_l has floor and level $l < l+1$, so $t_{ie_l} = \delta_{ie_l}$ for every i of level at least l; the other rows are those of e_b, $b < l$, rows of 1. So $te_l = e_l$. Put $w = te^*_l$. As t is an isometry fixing e_b, e^*_b ($b < l$) and e_l: w is orthogonal to the e_b, e^*_b with $b < l$, so $w \in V_l$; its e^*_l-coordinate is $h(e_l, w) = h(te_l, te^*_l) = 1$; and $h(w, w) = 0$. Write $w = \gamma e_l + e^*_l + v$ with $v \in V_{l+1}$; then $0 = h(w, w) = \gamma + \bar\gamma + h(v, v)$, so $\beta = \gamma + \tau h(v, v)$ satisfies $\beta + \bar\beta = 0$, and $w = n_l(v, \beta)e^*_l$. The element $n_l(v, \beta)^{-1}t \in G^{(l)}$ fixes e_l and e^*_l, hence lies in $G^{(l+1)}$. If $l = \varrho$, the same argument for the column z (floor and level ϱ) gives $tz = z$, so $t \in G^{(\varrho+1)}$. If $l = \varrho + 1$, then $R^{(\varrho+2)} = \mathsf{U}$, and for each x_i (level ϱ) $t_{jx_i} = \delta_{jx_i}$ for rows of level at least ϱ, while the rows of level

below ϱ are rows of 1; so t fixes every coordinate vector and $t = 1$. For $l \geq \varrho + 2$, $G^{(l)} = \{1\}$.

(v). Induction on l. For $l = 0$, take $\gamma = s$, $\nu = 1$ ($G^{(0)} = G$). Let $s \in R^{(l+1)} \subseteq R^{(l)}$ and $s = \gamma\nu$ with $\gamma \in G^{(l)}$, $\nu \in N_{<l}$. By (i), $\nu \in R^{(l+1)}$, so $\gamma = s\nu^{-1} \in R^{(l+1)} \cap G^{(l)}$, and so is γ^{-1}. If $l \geq \varrho$, $\gamma \in G^{(l+1)}$ by (iv) and $\nu \in N_{<l+1}$ by (i). If $l < \varrho$, (iv) applied to γ^{-1} gives $n = n_l(v, \beta)$ with $n^{-1}\gamma^{-1} \in G^{(l+1)}$; so $g = \gamma n \in G^{(l+1)}$ and $s = g\,(n^{-1}\nu)$, where $n^{-1} \in N_{<l+1}$ by (iii) and $\nu \in N_{<l+1}$ by (i).

(vi). Let $l \leq \varrho$ and $\sigma \in G^{(l)}$. The conditions of $R^{(l+1)}$ concern the columns of floor and level at most l: those of floor below l, where σ agrees with 1, and the single column of floor l and level at most l, which is f_l (e_l if $l < \varrho$, as e_l^* has level $2\varrho - l > l$; z if $l = \varrho$, the x_i having floor $\varrho + 1$). If $\sigma f_l = f_l$, the column f_l is that of 1 and $\sigma \in R^{(l+1)}$. Conversely, if $\sigma \in R^{(l+1)}$, the entries of the column f_l in rows of level at least l are those of 1, and the rows of level below l are those of e_b, $b < l$, rows of 1; so $\sigma f_l = f_l$.

(vii). Let $l \leq \varrho$ and $\sigma \in G^{(l)}$. The indices of floor below l have floor below $\min(l, \varrho)$, so by the first paragraph the rows and the columns of σ at these indices are those of 1, and $\sigma V_l = V_l$. Thus the matrix of σ is block diagonal, the identity on the indices of floor below l and a block σ_l on those of floor at least l, with $\det \sigma_l = \det \sigma$. From $\bar{\sigma}^{\mathsf{T}} J \sigma = J$ and $\det J$ a unit, $\overline{\det \sigma} \det \sigma = 1$; so $\det \sigma_l$ is a unit and, by (PA2) and multiplicativity of the modulus, $|\det \sigma_l|^2 = 1$, $|\det \sigma_l| = 1$. By (PA3), $v \mapsto \sigma_l v$ preserves vol on the coordinates of floor at least l; extending by zero, $\sigma_* \mathrm{vol}_{V_l} = \mathrm{vol}_{V_l}$.

(viii). Let $a < l$. The row $f_a = e_a$ of u has floor $a < \min(l + 1, \varrho)$, so it is a row of 1 (first paragraph); the column f_{a+1} of $u^{-1} \in G^{(l+1)}$ has floor $a + 1 \leq l$, so it is a column of 1; hence $(ugu^{-1})_{f_a f_{a+1}} = g_{f_a f_{a+1}}$. Let $a \geq l$. Then f_a and f_{a+1} have floor at least l, so $\mathrm{lev}_l(f_a) = \mathrm{lev}_l(f_{a+1}) = \varrho$, and both g and ugu^{-1} (which lies in $N_{<l}$ by (ii), as $u \in G^{(l)}$) have the entry 0 there.

(ix). Since $w_{l,q} e_a = q_a e_a^*$ and $w_{l,q} e_a^* = \bar{q}_a^{-1} e_a$, we have $w_{l,q}^{-1} e_a^* = q_a^{-1} e_a$ and $w_{l,q}^{-1} e_a = \bar{q}_a e_a^*$ for $a < \min(l, \varrho)$; composing with $w_{l,p}$ gives the diagonal element. Let $t \in G^{(l)}$. If $l \geq \varrho + 2$, $t = 1$. Otherwise $t \in G^{(\min(l,\varrho))}$, so by (vii) the rows and columns of t at the indices of floor below $\min(l, \varrho)$ are those of 1, while $w_{l,p}$ acts only on the span of the e_a, e_a^* with $a < \min(l, \varrho)$ and fixes the other coordinate vectors: both are block diagonal for the same two blocks, t the identity on the first and $w_{l,p}$ on the second, so they commute.

(x). The formula for the entries is matrix multiplication. The groups U and $N_{<l}$ are defined by prescribing entries $g_{ik} = \delta_{ik}$ on a set of positions, and $D_{ii}\delta_{ik}D_{kk}^{-1} = \delta_{ik}$; as $DgD^{-1} \in G$, conjugation by D and by D^{-1} preserves both groups. □

Lemma 5.2.59. *Let a place datum be given, and for $j \geq 0$ put $\eta_{<j} = \eta_0 * \eta_1 * \cdots * \eta_{j-1}$ ($\eta_{<0} = \delta_1$). Then:*

(i) $\theta_{\geq l} = \theta_{\geq l+1} * \theta_l$ for every $l \geq 0$, and $\theta_{\geq l} = \mathrm{wl}_l * \theta_{\geq l+1} * \eta_l$ for $l < \varrho$;

(ii) η_l, wl_l (for $l < \varrho$), θ_l, $\theta_{\geq l}$ and $\mathfrak{m}_{l,i}$ are carried by $G^{(l)}$, that is, the complement of $G^{(l)}$ is a null set for each of them;

(iii) for every $j \geq 0$ and $s \in G^{(j)}$, $\delta_s * \eta_{<j} = \eta_{<j} * \delta_s$;

(iv) for every $j \geq 0$, $\mathfrak{m}_j = \eta_{<j} * \theta_{[0,j)}$, with the descending convolution $\theta_{[0,j)} = \theta_{j-1} * \theta_{j-2} * \cdots * \theta_0$ of (5.11).

Proof. All measures are s-finite, so convolution is associative and Tonelli's theorem applies to $\int f\, d(\mu * \mu') = \iint f(xy)\, d\mu(x)\, d\mu'(y)$ for measurable $f \geq 0$. Each $G^{(l)}$ is closed in G (it is cut

out by conditions on matrix entries), hence measurable, and $G^{(l')} \subseteq G^{(l)}$ for $l' \geq l$. We use three general facts.

(a) *If μ and μ' are carried by a measurable subgroup S, so is $\mu * \mu'$*: the set of (x, y) with $xy \notin S$ lies in $(G \setminus S) \times G \cup G \times (G \setminus S)$, a $\mu \otimes \mu'$-null set.

(b) *If $\delta_s * \mu' = \mu' * \delta_s$ for μ-almost every s, then $\mu * \mu' = \mu' * \mu$*: for measurable $f \geq 0$, by Tonelli,

$$\int f \, d(\mu * \mu') = \int \Big(\int f \, d(\delta_s * \mu') \Big) d\mu(s) = \int \Big(\int f \, d(\mu' * \delta_s) \Big) d\mu(s) = \int f \, d(\mu' * \mu).$$

(c) *For $l < \varrho$ and $s \in G^{(l+1)}$: $s \, w_l(c) = w_l(c) \, s$ for all c, and $\delta_s * \eta_l = \eta_l * \delta_s$.* The element s fixes e_b, e_b^* for $b \leq l$, so, being an isometry, it preserves their orthogonal complement V_{l+1}; $w_l(c)$ acts only on $\langle e_l, e_l^* \rangle$ and is the identity on the other coordinate vectors. Hence the two commute. Next, for $v \in V_{l+1}$, $\beta \in A^-$ and $w \in V$, using $h(e_l, s^{-1}w) = h(se_l, w) = h(e_l, w)$, $h(v, s^{-1}w) = h(sv, w)$ and $h(sv, sv) = h(v, v)$ in (5.8),

$$s \, n_l(v, \beta) \, s^{-1} = n_l(sv, \beta).$$

The restriction of s to V_{l+1} is an invertible linear map whose determinant d equals $\det s$ (s is the identity on the coordinates of floor at most l and preserves V_{l+1}); from $\bar{s}^{\mathsf{T}} J s = J$ with $\det J$ a unit we get $\bar{d}d = 1$, so $|d|^2 = |\bar{d}| \, |d| = 1$ and $|d| = 1$. By the scaling axiom of the place, $v \mapsto sv$ preserves the Haar measure dv of V_{l+1}. Therefore, for measurable $f \geq 0$,

$$\int f \, d(\delta_s * \eta_l) = \iint f\big(n_l(sv, \beta)s\big) \, dv \, d\nu(\beta) = \iint f\big(n_l(v, \beta)s\big) \, dv \, d\nu(\beta) = \int f \, d(\eta_l * \delta_s).$$

Proof of (ii). The elements $n_l(v, \beta)$ and $w_l(c)^{-1}$ fix e_b, e_b^* for $b < l$ (Definition 5.2.1), so η_l and wl_l are carried by $G^{(l)}$; for $l \geq \varrho$, $\eta_l = \delta_1$. The top floor measure is carried by $G^{(\varrho)}$ and the measure of H by $H = G^{(\varrho+1)}$ (Definition 5.1.7), and δ_1 by every subgroup. With (a), θ_l is carried by $G^{(l)}$ for every l. Every factor θ_a of $\theta_{\geq l}$, and every factor η_a, θ_a of $\mathfrak{m}_{l,i}$, has $a \geq l$ and so is carried by $G^{(a)} \subseteq G^{(l)}$; by (a) and induction on the number of factors, $\theta_{\geq l}$ and $\mathfrak{m}_{l,i}$ are carried by $G^{(l)}$.

Proof of (i). The first identity is the definition of $\theta_{\geq l}$ when $l \leq \varrho + 1$, and both sides are δ_1 when $l \geq \varrho + 2$. Let $l < \varrho$. By (c), $\delta_s * \mathrm{wl}_l = \mathrm{wl}_l * \delta_s$ for every $s \in G^{(l+1)}$ (both are the image of $|c|^{d_l} dc$ under $c \mapsto s \, w_l(c)^{-1} = w_l(c)^{-1} s$), and $\theta_{\geq l+1}$ is carried by $G^{(l+1)}$ by (ii). By (b), $\theta_{\geq l+1} * \mathrm{wl}_l = \mathrm{wl}_l * \theta_{\geq l+1}$, and

$$\theta_{\geq l} = \theta_{\geq l+1} * \mathrm{wl}_l * \eta_l = \mathrm{wl}_l * \theta_{\geq l+1} * \eta_l.$$

Proof of (iii). Induction on j; for $j = 0$ both sides are δ_s. Let $s \in G^{(j+1)} \subseteq G^{(j)}$. By the induction hypothesis and by (c) with $l = j$ (trivial if $j \geq \varrho$),

$$\delta_s * \eta_{<j+1} = (\delta_s * \eta_{<j}) * \eta_j = \eta_{<j} * (\delta_s * \eta_j) = \eta_{<j} * \eta_j * \delta_s = \eta_{<j+1} * \delta_s.$$

Proof of (iv). Write $D_j = \theta_{j-1} * \cdots * \theta_0$ ($D_0 = \delta_1$). Induction on j; for $j = 0$ both sides are δ_1. Since θ_j and η_j are carried by $G^{(j)}$ by (ii), (iii) and (b) give $\theta_j * \eta_{<j} = \eta_{<j} * \theta_j$ and $\eta_j * \eta_{<j} = \eta_{<j} * \eta_j = \eta_{<j+1}$. Hence, by the definition of $\mathfrak{m}_{0,j+1}$ and the induction hypothesis,

$$\mathfrak{m}_{j+1} = (\eta_j * \theta_j) * \eta_{<j} * D_j = \eta_j * \eta_{<j} * \theta_j * D_j = \eta_{<j+1} * D_{j+1}.$$

□

Lemma 5.2.60. *Let a place datum of shape (ϱ, ι, m) be given. For $0 \le b \le \varrho$ let $P_b\colon V \to V$ set the coordinates of floor below b to zero, and let $C_b = \{P_b(\kappa\kappa'^{-1} f_b) : \kappa, \kappa' \in K\}$, a compact subset of V_b.*

(i) If $g \in K$, $\sigma \in G^{(b)}$ and σg is near at $b+1$, then $\sigma^{-1} f_b \in C_b$.

(ii) (Chart criterion.) Let $C \in [0,\infty]$ and let $f\colon G \to [0,\infty]$ be measurable with $f(\xi z) = f(z)$ for all $\xi \in R^{(b+1)}$ and $z \in G$. Suppose that for every $g \in K$ there is a Borel $\Psi_g\colon V \to [0,\infty]$ with $\int_{C_b} \Psi_g \, d\mathrm{qv}_b \le C$ and $f(\sigma g) \le \Psi_g(\sigma^{-1} f_b)$ for all $\sigma \in G^{(b)}$. Then $\mathrm{FML}_b(C; f)$.

(iii) $\mathrm{qv}_b(C_b) < \infty$, and $\mathrm{FML}_b(\mathrm{qv}_b(C_b); 1)$.

(iv) If $V \in [0,\infty]$ and $\mathrm{FML}_a(V; 1)$ holds for every $a \le \varrho$, then for all $l \ge 0$ and $c \in G$

$$\theta_{\ge l}(\{x \in G : xc \text{ near at } \varrho + 2\}) \le V^{\max(\varrho+1-l,\,0)}.$$

Proof. For $j \ge 0$ let $N_j \subseteq G$ be the set of points near at j. Since $\xi g \in K$ means $g \in \xi^{-1}K$, $N_j = R^{(j)}K$, the product of a closed subgroup and a compact set (PE4), so N_j is closed; and $\xi N_j = N_j$ for $\xi \in R^{(j)}$. By Lemma 5.2.58, (i), $R^{(j+1)} \subseteq R^{(j)}$ and $G^{(j)} \subseteq R^{(j)}$; hence $N_{j+1} \subseteq N_j$, and $xz \in N_i$ with $x \in G^{(j)}$, $i \ge j$, implies $z \in N_j$. The set C_b is compact as a continuous image of $K \times K$, and $C_b \subseteq V_b$.

(i). Let $\xi \in R^{(b+1)}$ with $\kappa' = \xi\sigma g \in K$. The column f_b of $\xi^{-1} \in R^{(b+1)}$ has floor and level $b < b+1$, so its entries in the rows of level at least b are those of 1, and the remaining rows are those of the e_c, $c < b$: $\xi^{-1} f_b = f_b + \sum_{c<b} t_c e_c$. As $\sigma^{-1} \in G^{(b)}$ fixes the e_c with $c < b$ and preserves V_b (Lemma 5.2.58, (vii)), $\sigma^{-1}\xi^{-1} f_b = \sigma^{-1} f_b + \sum_{c<b} t_c e_c$ with $\sigma^{-1} f_b \in V_b$, so $\sigma^{-1} f_b = P_b(\sigma^{-1}\xi^{-1} f_b) = P_b(g\kappa'^{-1} f_b) \in C_b$, using $\sigma^{-1}\xi^{-1} = g\kappa'^{-1}$.

(ii). Fix $g \in G$ and put $\Phi = f\,\mathbf{1}_{N_{b+1}}$, measurable and invariant under left translation by $R^{(b+1)}$. We must show $\int \Phi(\sigma g)\, d\theta_b(\sigma) \le C$. Recall that θ_b is carried by $G^{(b)}$ (Lemma 5.2.59, (ii)). If $g \notin N_b$, then $\Phi(\sigma g) = 0$ for every $\sigma \in G^{(b)}$ (as $\sigma g \in N_{b+1}$ would give $g \in N_b$), so the integral is 0. If $g \in N_b$, take $\xi' \in R^{(b)}$ with $g' = \xi' g \in K$. The function $\sigma \mapsto \Phi(\sigma g)$ is invariant under left translation by $R^{(b+1)}$, so Lemma 5.2.61, (i), with $S = G$ and $s = \xi'$ gives $\int \Phi(\sigma g)\, d\theta_b(\sigma) = \int \Phi(\sigma g')\, d\theta_b(\sigma)$. For $\sigma \in G^{(b)}$, (i) and the hypothesis give $\Phi(\sigma g') \le \mathbf{1}_{C_b}(\sigma^{-1} f_b)\Psi_{g'}(\sigma^{-1} f_b)$. By Lemma 5.2.63, $\sigma \mapsto \sigma^{-1} f_b$ carries θ_b to qv_b, so

$$\int \Phi(\sigma g)\, d\theta_b(\sigma) \le \int \mathbf{1}_{C_b} \Psi_{g'}\, d\mathrm{qv}_b \le C.$$

(iii). By Lemma 5.2.52, (ii), $(\mathrm{qv}_b(\beta))_\beta$ is a disintegration of vol_{V_b}; by Urysohn's lemma there is a continuous compactly supported $u\colon V \to [0,1]$ with $u = 1$ on C_b, and (DJ1) gives $\mathrm{qv}_b(C_b) \le \int u\, d\mathrm{qv}_b < \infty$, with $\mathrm{qv}_b = \mathrm{qv}_b(\delta h(f_b, f_b))$. Then (ii) with $f = 1$, $\Psi_g = 1$ and $C = \mathrm{qv}_b(C_b)$ gives $\mathrm{FML}_b(\mathrm{qv}_b(C_b); 1)$.

(iv). Put $E_c = \{x : xc \in N_{\varrho+2}\}$, a closed set. Descending induction on l. For $l \ge \varrho + 2$, $\theta_{\ge l} = \delta_1$ and $\theta_{\ge l}(E_c) \le 1$. For $l = \varrho + 1$, $\theta_{\ge \varrho+1} = \mu_H$ and $R^{(\varrho+2)} = \mathsf{U}$, so $E_c = \{x : uxc \in K \text{ for some } u \in \mathsf{U}\}$ and $\mu_H(E_c) \le 1$ by (PE3). Let $l \le \varrho$ and assume the bound at $l+1$ for every c. As $\theta_{\ge l} = \theta_{\ge l+1} * \theta_l$ (Lemma 5.2.59, (i)),

$$\theta_{\ge l}(E_c) = \int_G \theta_{\ge l+1}(\{x : x\sigma c \in N_{\varrho+2}\})\, d\theta_l(\sigma) = \int_G \theta_{\ge l+1}(E_{\sigma c})\, d\theta_l(\sigma).$$

Since $\theta_{\ge l+1}$ is carried by $G^{(l+1)}$ (Lemma 5.2.59, (ii)) and $x\sigma c \in N_{\varrho+2}$ with $x \in G^{(l+1)}$ implies $\sigma c \in N_{l+1}$, the integrand vanishes unless $\sigma c \in N_{l+1}$, and it is at most $V^{\varrho-l}$ by the induction hypothesis. By $\mathrm{FML}_l(V; 1)$, $\theta_l(\{\sigma : \sigma c \in N_{l+1}\}) \le V$. Hence $\theta_{\ge l}(E_c) \le V^{\varrho-l} \cdot V = V^{\varrho+1-l}$. □

Lemma 5.2.61. *Let a place datum of shape (ϱ, ι, m) be given, and let δ_s denote the unit mass at $s \in G$.*

(i) Let $l \geq 0$, let S be a subgroup of G containing $G^{(l)}$, and let $f\colon G \to [0,\infty]$ be measurable with $f(tx) = f(x)$ for all $t \in R^{(l+1)} \cap S$ and $x \in G$. Then $\int f(xs)\, d\theta_l(x) = \int f\, d\theta_l$ for every $s \in R^{(l)} \cap S$.

(ii) Let $l \geq 0$. If $l < \varrho$, then $\eta_l * \delta_n = \delta_n * \eta_l = \eta_l$ for every Heisenberg element $n = n_l(v,\beta)$. For every $u \in G^{(l+1)}$, $\delta_u * \eta_l = \eta_l * \delta_u$.

(iii) If $l + i = \varrho + 2$, then $\mathfrak{m}_{l,i} * \delta_s = \mathfrak{m}_{l,i}$ for every $s \in G^{(l)}$. In particular the group measure θ is right invariant: $\theta * \delta_s = \theta$ for every $s \in G$.

(iv) $\theta_l \neq 0$, $\eta_l \neq 0$ and $\mathfrak{m}_{l,i} \neq 0$ for all $l, i \geq 0$.

Proof. All measures are s-finite; convolution is associative and $\int F\, d(\mu{*}\mu') = \iint F(xy)\, d\mu(x)\, d\mu'(y)$ for measurable $F \geq 0$ (Tonelli). In particular $\int F\, d(\mu * \delta_s) = \int F(xs)\, d\mu(x)$ and $\int F\, d(\delta_s * \mu) = \int F(sx)\, d\mu(x)$.

(i). By Lemma 5.2.58, (v), $s = \gamma\nu$ with $\gamma \in G^{(l)}$, $\nu \in N_{<l}$. Let $x \in G^{(l)}$. Then $x\gamma \in G^{(l)}$, so $c = (x\gamma)\nu(x\gamma)^{-1}$ lies in $N_{<l}$ by Lemma 5.2.58, (ii), hence in $R^{(l+1)}$ by Lemma 5.2.58, (i); and $c = xs(x\gamma)^{-1} \in S$, as $x, x\gamma \in G^{(l)} \subseteq S$ and $s \in S$. So $f(xs) = f(c\,x\gamma) = f(x\gamma)$. Since θ_l is carried by $G^{(l)}$ (Lemma 5.2.59, (ii)),

$$\int f(xs)\, d\theta_l(x) = \int f(x\gamma)\, d\theta_l(x).$$

Case $l \leq \varrho$. f is invariant under left translation by $R^{(l+1)} \cap G^{(l)} \subseteq R^{(l+1)} \cap S$, so by Lemma 5.2.62 there is a Borel $f_0\colon V \to [0,\infty]$ with $f(\sigma) = f_0(\sigma^{-1} f_l)$ for $\sigma \in G^{(l)}$. For $x \in G^{(l)}$, $f(x\gamma) = f_0(\gamma^{-1}x^{-1} f_l)$. By Lemma 5.2.63, applied with $\gamma^{-1} \in G^{(l)}$ and with 1, both $x \mapsto \gamma^{-1}x^{-1} f_l$ and $x \mapsto x^{-1} f_l$ carry θ_l to qv_l; hence

$$\int f(x\gamma)\, d\theta_l(x) = \int f_0\, d\mathrm{qv}_l = \int f_0(x^{-1} f_l)\, d\theta_l(x) = \int f\, d\theta_l.$$

Case $l = \varrho + 1$. $\theta_{\varrho+1} = \mu_H$ and $\gamma \in G^{(\varrho+1)} = H$, so $x \mapsto x\gamma$ preserves μ_H by (PE3). *Case $l \geq \varrho + 2$.* $G^{(l)} = \{1\}$ (Lemma 5.2.58, (i)), so $\gamma = 1$.

(ii). If $l \geq \varrho$, $\eta_l = \delta_1$ and $u \in G^{(l+1)}$ gives $\delta_u * \delta_1 = \delta_1 * \delta_u$. Let $l < \varrho$, $n = n_l(v',\beta')$ and $F \geq 0$ measurable. By the group law of Lemma 5.2.58, (iii),

$$\int F\, d(\eta_l * \delta_n) = \int_{V_{l+1}} \int_{A^-} F\big(n_l(v + v', \beta + \beta' + \tau h(v', v) - \bar{\tau} h(v, v'))\big)\, d\nu(\beta)\, dv.$$

For fixed v the second argument is a translate of β, which ν absorbs; then $v \mapsto v + v'$ is absorbed by the Haar measure dv of V_{l+1}. So $\eta_l * \delta_n = \eta_l$, and likewise $\delta_n * \eta_l = \eta_l$ with $n\, n_l(v,\beta) = n_l(v' + v, \beta' + \beta + \tau h(v,v') - \bar{\tau} h(v',v))$. Let $u \in G^{(l+1)}$. By Lemma 5.2.58, (iii), $u\, n_l(v,\beta) = n_l(uv,\beta)\, u$. As $l + 1 \leq \varrho$, Lemma 5.2.58, (vii), shows that $v \mapsto uv$ preserves the Haar measure dv of V_{l+1}, and

$$\int F\, d(\delta_u * \eta_l) = \iint F\big(n_l(uv,\beta)u\big)\, d\nu(\beta)\, dv = \iint F\big(n_l(v,\beta)u\big)\, d\nu(\beta)\, dv = \int F\, d(\eta_l * \delta_u).$$

(iii). Induction on i, for all l with $l + i = \varrho + 2$. If $i = 0$, then $l = \varrho + 2$, $G^{(l)} = \{1\}$ and there is nothing to prove. Let $i \geq 1$, $l + i = \varrho + 2$, and put $\rho = \mathfrak{m}_{l+1,i-1} * \eta_l$, so that $\mathfrak{m}_{l,i} = \rho * \theta_l$

by the definition of $\mathfrak{m}_{l,i}$ and associativity. By the induction hypothesis, $\mathfrak{m}_{l+1,i-1} * \delta_u = \mathfrak{m}_{l+1,i-1}$ for $u \in G^{(l+1)}$.

$\rho * \delta_t = \rho$ for every $t \in R^{(l+1)} \cap G^{(l)}$. If $l < \varrho$, Lemma 5.2.58, (iv), gives a Heisenberg element $n = n_l(v, \beta)$ with $u = n^{-1}t \in G^{(l+1)}$; by (ii),

$$\rho * \delta_t = \mathfrak{m}_{l+1,i-1} * (\eta_l * \delta_n) * \delta_u = \mathfrak{m}_{l+1,i-1} * \eta_l * \delta_u = (\mathfrak{m}_{l+1,i-1} * \delta_u) * \eta_l = \rho.$$

If $l \geq \varrho$, then $t \in G^{(l+1)}$ by Lemma 5.2.58, (iv), $\eta_l = \delta_1$, and $\rho * \delta_t = \mathfrak{m}_{l+1,i-1} * \delta_t = \rho$.

Conclusion. Let $s \in G^{(l)}$, $f \geq 0$ measurable, and $a(z) = \int f(yz)\, d\rho(y)$, measurable by Tonelli. For $t \in R^{(l+1)} \cap G^{(l)}$, $a(tz) = \int f(yz)\, d(\rho * \delta_t)(y) = a(z)$. As $G^{(l)} \subseteq R^{(l)}$ (Lemma 5.2.58, (i)), (i) with $S = G^{(l)}$ gives

$$\int f\, d(\mathfrak{m}_{l,i} * \delta_s) = \int a(xs)\, d\theta_l(x) = \int a\, d\theta_l = \int f\, d\mathfrak{m}_{l,i}.$$

For $l = 0$, $i = \varrho + 2$: $\theta = \mathfrak{m}_{0,\varrho+2}$ and $G^{(0)} = G$.

(iv). For $l \leq \varrho$, by Lemma 5.2.63 (with $s = 1$), $\theta_l(G) = \mathrm{qv}_l(V)$, the total mass of the image of $\mathrm{vol} \otimes \mathrm{vol} \otimes \nu$ on $A \times Y_l \times A^-$, a product of non-zero Haar measures; so $\theta_l \neq 0$. $\theta_{\varrho+1} = \mu_H \neq 0$ by (PE3), and $\theta_l = \delta_1$ for $l \geq \varrho + 2$. The total mass of a convolution is the product of the total masses, $(\mu * \mu')(G) = \mu(G)\mu'(G)$. For $l < \varrho$, $\theta_l = \mathrm{wl}_l * \eta_l \neq 0$ forces $\eta_l(G) \neq 0$; for $l \geq \varrho$, $\eta_l = \delta_1$. Hence $\mathfrak{m}_{l,i}(G) = \prod_{a=l}^{l+i-1} \eta_a(G)\, \theta_a(G)$ is a finite product of non-zero elements of $[0, \infty]$, so it is non-zero. □

Lemma 5.2.62. *Let A be a place, G the isometry group of a split space at A with ϱ pairs, $0 \leq b \leq \varrho$, and Z a non-empty standard Borel space. Let $\Phi\colon G \to Z$ be Borel with $\Phi(t\sigma) = \Phi(\sigma)$ for all $t \in R^{(b+1)} \cap G^{(b)}$ and $\sigma \in G$. Then there is a Borel map $\Psi\colon V \to Z$ with*

$$\Phi(\sigma) = \Psi(\sigma^{-1} f_b) \qquad \text{for every } \sigma \in G^{(b)}.$$

Proof. The spaces. G is a closed subset of the Polish space of $V \times V$ matrices over A (it is cut out by the continuous equation $\bar{g}^{\mathsf{T}} J g = J$, which forces g to be invertible with inverse $J^{-1}\bar{g}^{\mathsf{T}}J$), and $G^{(b)}$ is closed in G; so $G^{(b)}$ is Polish, and $\pi(\sigma) = \sigma^{-1} f_b = J^{-1}\bar{\sigma}^{\mathsf{T}} J f_b$ is continuous on it. Let $O = \pi(G^{(b)}) \subseteq V$.

Φ is constant on the fibres of π. Let $\sigma, \tau \in G^{(b)}$ with $\sigma^{-1} f_b = \tau^{-1} f_b$. Then $t = \sigma\tau^{-1} \in G^{(b)}$ fixes f_b, so $t \in R^{(b+1)}$ by Lemma 5.2.58, (vi), and $\Phi(\sigma) = \Phi(t\tau) = \Phi(\tau)$. Hence there is a map $g\colon O \to Z$ with $\Phi = g \circ \pi$ on $G^{(b)}$.

Measurability on O. By Kuratowski's theorem, Z is Borel isomorphic to a Borel subset of $[0,1]$; we may and do assume $Z \subseteq [0,1]$ Borel. For rational $q \in [0,1]$ the sets $P_q = \{\sigma \in G^{(b)} : \Phi(\sigma) \geq q\}$ and $G^{(b)} \setminus P_q$ are Borel, so their images under π are analytic; these images are disjoint by the previous step, and their union is O. By Lusin's separation theorem there is a Borel set $E_q \subseteq V$ containing $\pi(P_q)$ and disjoint from $\pi(G^{(b)} \setminus P_q)$; thus $E_q \cap O = \pi(P_q) = \{y \in O : g(y) \geq q\}$.

Extension. Let $\Psi_0(y) = \sup\{q \in \mathbb{Q} \cap [0,1] : y \in E_q\}$ ($\sup \emptyset = 0$), a countable supremum of the Borel functions $q\mathbf{1}_{E_q}$, hence Borel. For $y \in O$, $y \in E_q$ if and only if $q \leq g(y)$, so $\Psi_0(y) = g(y)$. Fix $z_0 \in Z$ and let $\Psi = \Psi_0$ on the Borel set $\Psi_0^{-1}(Z)$ and $\Psi = z_0$ elsewhere. Then Ψ is Borel, $\Psi = g$ on O, and $\Phi(\sigma) = g(\pi(\sigma)) = \Psi(\sigma^{-1} f_b)$ for $\sigma \in G^{(b)}$. □

Lemma 5.2.63. *Let a place datum be given, let $0 \leq b \leq \varrho$ and $s \in G^{(b)}$. Then the image of the floor measure θ_b under the map $G \to V$, $\sigma \mapsto s\,\sigma^{-1} f_b$, is the quadric measure qv_b of floor b.*

Proof. The map $\sigma \mapsto s\sigma^{-1}f_b$ is continuous on G ($\sigma^{-1} = J^{-1}\bar{\sigma}^{\mathsf{T}}J$), so all images below are Borel measures. Write ρ_b for the image of θ_b under $\sigma \mapsto \sigma^{-1}f_b$; the image under $\sigma \mapsto s\sigma^{-1}f_b$ is $s_*\rho_b$. We show $\rho_b = \mathrm{qv}_b$ and then $s_*\mathrm{qv}_b = \mathrm{qv}_b$.

The top floor. For $b = \varrho$ we have $\theta_\varrho = \mu_{\mathrm{top}}$ and $f_\varrho = z$, so $\rho_\varrho = \mathrm{qv}_\varrho$ is the axiom (PE2).

A floor $a < \varrho$: the chart in the coordinate basis. Let S be the set of coordinate indices of floor at least $a+1$ (so $|S| = k+1+2(\varrho-a-1) = d_a$ and $V_{a+1} = A^S$, extended by zero), J_{a+1} the restriction of J to S, and $E\colon A^{\{1,2\}\sqcup S} \to V$, $(c_1, c_2, w) \mapsto c_1e_a + c_2e_a^* + w$. As $h(e_a, e_a^*) = 1$, $h(e_a, e_a) = h(e_a^*, e_a^*) = 0$ and e_a, e_a^* are orthogonal to V_{a+1}, we have $\bar{E}^{\mathsf{T}}JE = J^{\mathrm{hyp}}_{J_{a+1}}$ in the notation of Lemma 5.2.52, (i); this matrix is invertible (T is), so $L = (J^{\mathrm{hyp}}_{J_{a+1}})^{-1}\bar{E}^{\mathsf{T}}J$ satisfies $LE = 1$; and $E_*\mathrm{vol} = \mathrm{vol}_{V_a}$, since E is the extension by zero from the indices of floor at least a. We show $\rho_a = E_*\lambda_0^{J_{a+1}}$, with $\lambda_\beta^{J_{a+1}}$, ch_β as in Lemma 5.2.52, (i).

For $v \in V_{a+1}$ and $\beta \in A^-$ put $n = n_a(v,\beta)$. By (5.8), n fixes e_a, sends e_a^* to $e_a^* + v + (\beta - \tau h(v,v))e_a$ and $w \in V_{a+1}$ to $w - h(v,w)e_a$. Hence, for $\gamma = -\beta - \bar{\tau}h(v,v)$,

$$n(e_a^* - v + \gamma e_a) = e_a^* + (\beta - \tau h(v,v) + h(v,v) + \gamma)e_a = e_a^*,$$

using $1 - \tau = \bar{\tau}$; so $n^{-1}e_a^* = e_a^* - v + \gamma e_a$. For a unit c, $w_a(c)e_a = ce_a^*$ by (5.7), so

$$\big(w_a(c)^{-1}n\big)^{-1}e_a = n^{-1}w_a(c)e_a = c\,n^{-1}e_a^* = E(c\gamma,\ c,\ -cv).$$

Put $\alpha = (\tau - \bar{\tau})h(v,v) - \beta \in A^-$ (as $\overline{h(v,v)} = h(v,v)$). Since $h(-cv,-cv) = c\bar{c}\,h(v,v)$, the first coordinate of $\mathrm{ch}_0(c,-cv,\alpha)$ is $\bar{c}^{-1}\tau(-c\bar{c}\,h(v,v)) + c\alpha = c(-\tau h(v,v) + \alpha) = c\gamma$. Therefore

$$\big(w_a(c)^{-1}n_a(v,\beta)\big)^{-1}e_a = E\big(\mathrm{ch}_0(c,-cv,(\tau-\bar{\tau})h(v,v) - \beta)\big) \qquad (c \text{ a unit}). \tag{5.25}$$

Let $\Phi\colon V \to [0,\infty]$ be measurable. By the definitions of $\theta_a = \mathrm{wl}_a * \eta_a$, wl_a and η_a (Definition 5.2.1), Tonelli, the fact that almost every c is a unit (PA4), and (5.25),

$$\int \Phi\, d\rho_a = \int_A |c|^{d_a} \int_{V_{a+1}} \int_{A^-} \Phi\Big(E\,\mathrm{ch}_0(c,-cv,(\tau-\bar{\tau})h(v,v) - \beta)\Big)\, d\nu(\beta)\, dv\, dc.$$

The Haar measure ν of the abelian group A^- is invariant under $\beta \mapsto \alpha_0 - \beta$, so the inner integral is $\int_{A^-} \Phi(E\,\mathrm{ch}_0(c,-cv,\beta))\, d\nu(\beta)$. For a unit c, the map $v \mapsto -cv$ of A^S has determinant $(-c)^{d_a}$, a unit of modulus $|c|^{d_a}$ (PA2), so by (PA3) $\int_{V_{a+1}} F(-cv)\, dv = |c|^{-d_a}\int_{V_{a+1}} F(w)\, dw$ for measurable $F \geq 0$. Hence

$$\int \Phi\, d\rho_a = \int_A \int_{V_{a+1}} \int_{A^-} \Phi(E\,\mathrm{ch}_0(c,w,\beta))\, d\nu(\beta)\, dw\, dc = \int \Phi\, d(E_*\lambda_0^{J_{a+1}}).$$

Identification with qv_a. By Lemma 5.2.52, (i), $(\lambda_\beta^{J_{a+1}})_\beta$ is a disintegration of vol on $A^{\{1,2\}\sqcup S}$ along $\bar{x}^{\mathsf{T}}J^{\mathrm{hyp}}_{J_{a+1}}x$; by Lemma 5.2.53, (ii), applied to E and L, $(E_*\lambda_\beta^{J_{a+1}})_\beta$ is a disintegration of $E_*\mathrm{vol} = \mathrm{vol}_{V_a}$ along $h(v,v)$. By Lemma 5.2.52, (ii), so is $(\mathrm{qv}_a(\beta))_\beta$. By Lemma 5.2.53, (i), at $\beta = 0$, and since $h(f_a, f_a) = h(e_a, e_a) = 0$,

$$\rho_a = E_*\lambda_0^{J_{a+1}} = \mathrm{qv}_a(0) = \mathrm{qv}_a.$$

Invariance under $G^{(b)}$. Let $s \in G^{(b)}$. The element s fixes the coordinate vectors of floor below b and is an isometry of h, so it preserves their orthogonal complement V_b; its matrix is block diagonal, the identity on the indices of floor below b and a block s_b on those of floor at least b, with $\det s_b = \det s$. From $\bar{s}^{\mathsf{T}}Js = J$ and $\det J$ a unit, $\overline{\det s}\det s = 1$, so $\det s_b$ is a unit with $|\det s_b|^2 = 1$ by (PA2), and $|\det s_b| = 1$. By (PA3), $(s_b)_*\mathrm{vol} = \mathrm{vol}$ on the coordinates

of floor at least b, hence $s_*\mathrm{vol}_{V_b} = \mathrm{vol}_{V_b}$. By Lemma 5.2.53, (ii), with $X = s$, $L = s^{-1}$ and $J' = J'' = J$, the family $(s_*\mathrm{qv}_b(\beta))_\beta$ is a disintegration of $s_*\mathrm{vol}_{V_b} = \mathrm{vol}_{V_b}$ along $h(v,v)$; so is $(\mathrm{qv}_b(\beta))_\beta$ by Lemma 5.2.52, (ii). By Lemma 5.2.53, (i), $s_*\mathrm{qv}_b(\beta) = \mathrm{qv}_b(\beta)$ for every β, in particular $s_*\mathrm{qv}_b = \mathrm{qv}_b$. With $\rho_b = \mathrm{qv}_b$ for every $b \le \varrho$, the image of θ_b under $\sigma \mapsto s\sigma^{-1}f_b$ is $s_*\rho_b = \mathrm{qv}_b$. □

Lemma 5.2.64. *Let a place datum be given, $j \ge 0$, $d \in G$, and let $\mathcal{K}$ be a pairing kernel at floor j for d. Then for all $l \ge 0$ and $g, y \in G$:*

(i) $x \mapsto \mathcal{K}(xg, xy)$ is $\theta_{\ge l}$-integrable, so $\mathcal{K}_l(g,y) = \int \mathcal{K}(xg,xy)\,d\theta_{\ge l}(x)$, and $\mathcal{K}_l$ is measurable on $G \times G$;

(ii) $\sigma \mapsto \mathcal{K}_{l+1}(\sigma g, \sigma y)$ is θ_l-integrable and

$$\mathcal{K}_l(g,y) = \int_G \mathcal{K}_{l+1}(\sigma g, \sigma y)\,d\theta_l(\sigma); \qquad \textit{in particular} \qquad |\mathcal{K}_l(g,y)| \le \mathrm{piv}_l(\mathcal{K})(g,y);$$

(iii) if $\mathrm{piv}_l(\mathcal{K})(g,y) \ne 0$, then $\mathcal{K}(xg,xy) \ne 0$ for some $x \in G^{(l)}$;

(iv) $\mathcal{K}$ is a pairing kernel at floor l for d whenever $l \ge j$;

(v) if $l \ge j$ and $s \in R^{(l)} \cap G^{(j)}$, then $\mathcal{K}_l(sg,sy) = \mathcal{K}_l(g,y)$;

(vi) if $n \in N_{<l}$, then $|\mathcal{K}_{l+1}(g,ny)| = |\mathcal{K}_{l+1}(g,y)|$.

Proof. All measures are s-finite, so Tonelli's and Fubini's theorems apply; $\theta_{\ge l} = \delta_1$ for $l \ge \varrho + 2$, hence $\mathcal{K}_l = \mathcal{K}$ there.

(i). By (PK1) the map $(x,g,y) \mapsto \mathcal{K}(xg,xy)$ is measurable, and by (PK4) $\int |\mathcal{K}(xg,xy)|\,d\theta_{\ge l}(x)$ is bounded, in particular finite, for all g, y. So the integral defines $\mathcal{K}_l(g,y)$, and it is measurable in (g,y) by Fubini's theorem (applied to the positive and negative parts of the real and imaginary parts).

(ii). By Lemma 5.2.59, (i), $\theta_{\ge l} = \theta_{\ge l+1} * \theta_l$, so the integrability of (i) says that $(x',\sigma) \mapsto \mathcal{K}(x'\sigma g, x'\sigma y)$ is $\theta_{\ge l+1} \otimes \theta_l$-integrable. By Tonelli,

$$\int |\mathcal{K}_{l+1}(\sigma g,\sigma y)|\,d\theta_l(\sigma) \le \iint |\mathcal{K}(x'\sigma g, x'\sigma y)|\,d\theta_{\ge l+1}(x')\,d\theta_l(\sigma) < \infty,$$

and by Fubini and (i) at the points $(\sigma g, \sigma y)$,

$$\mathcal{K}_l(g,y) = \int_G \int_G \mathcal{K}(x'\sigma g, x'\sigma y)\,d\theta_{\ge l+1}(x')\,d\theta_l(\sigma) = \int_G \mathcal{K}_{l+1}(\sigma g,\sigma y)\,d\theta_l(\sigma).$$

Taking absolute values gives $|\mathcal{K}_l(g,y)| \le \int |\mathcal{K}_{l+1}(\sigma g,\sigma y)|\,d\theta_l(\sigma) = \mathrm{piv}_l(\mathcal{K})(g,y)$.

(iii). If $\mathrm{piv}_l(\mathcal{K})(g,y) \ne 0$, the set of σ with $\mathcal{K}_{l+1}(\sigma g,\sigma y) \ne 0$ has positive θ_l-measure; as θ_l is carried by $G^{(l)}$ (Lemma 5.2.59, (ii)), it contains some $\sigma \in G^{(l)}$. Then $x' \mapsto \mathcal{K}(x'\sigma g, x'\sigma y)$ is not $\theta_{\ge l+1}$-almost everywhere zero, and $\theta_{\ge l+1}$ is carried by $G^{(l+1)}$ (Lemma 5.2.59, (ii)); so $\mathcal{K}(x'\sigma g, x'\sigma y) \ne 0$ for some $x' \in G^{(l+1)}$, and $x = x'\sigma \in G^{(l)}$.

(iv). (PK1), (PK2), (PK4), (PK5) do not involve j, and (PK3) at floor $l \ge j$ follows from (PK3) at floor j because $G^{(l)} \subseteq G^{(j)}$.

(v). Descending induction on $l \ge j$. Let $l \ge \varrho + 2$. By the definitions, $R^{(l)} = \mathsf{U}$ (every entry with $\mathrm{level}(j') \le \mathrm{level}(i)$ is prescribed) and $N_{<\varrho+1} = \mathsf{U}$ (every index has floor below $\varrho + 1$ or is

some x_i of level ϱ, so $\mathrm{lev}_{\varrho+1}$ is the level). For $s \in \mathsf{U} \cap G^{(j)}$, (PK2) with $l = \varrho+1$ and $n = n' = s$, and (PK3), give

$$\mathcal{K}_{\varrho+2}(sg, sy) = \psi(M^{-1}(\sigma(dsd^{-1}) - \sigma(s)))\mathcal{K}_{\varrho+2}(g, y) = \mathcal{K}_{\varrho+2}(g, y),$$

and $\mathcal{K}_l = \mathcal{K}_{\varrho+2} = \mathcal{K}$. Now let $j \le l < \varrho + 2$ and assume (v) at $l+1$. Put $\Phi(\sigma) = \mathcal{K}_{l+1}(\sigma g, \sigma y)$, measurable by (i). By the induction hypothesis $\Phi(t\sigma) = \Phi(\sigma)$ for $t \in R^{(l+1)} \cap G^{(j)}$. Let $s \in R^{(l)} \cap G^{(j)}$ and let $\varphi\colon \mathbb{C} \to [0, \infty]$ be Borel. The function $\varphi \circ \Phi$ is invariant under left translation by $R^{(l+1)} \cap S$ for the subgroup $S = G^{(j)} \supseteq G^{(l)}$, so by Lemma 5.2.61, (i),

$$\int \varphi(\Phi(\sigma s))\, d\theta_l(\sigma) = \int \varphi(\Phi(\sigma))\, d\theta_l(\sigma).$$

Thus $\sigma \mapsto \Phi(\sigma s)$ and Φ have the same image measure on $\mathbb{C}$ under θ_l; as Φ is θ_l-integrable by (ii), so is $\sigma \mapsto \Phi(\sigma s)$, with the same integral. By (ii) at (sg, sy) and at (g, y),

$$\mathcal{K}_l(sg, sy) = \int \mathcal{K}_{l+1}(\sigma sg, \sigma sy)\, d\theta_l(\sigma) = \int \Phi(\sigma s)\, d\theta_l(\sigma) = \int \Phi\, d\theta_l = \mathcal{K}_l(g, y).$$

(vi). By (PK2) with 1 and n in place of n and n', $\mathcal{K}_{l+1}(g, ny) = \psi(M^{-1}(\sigma(dnd^{-1}) - \sigma(1)))\mathcal{K}_{l+1}(g, y)$, and $|\psi| = 1$. □

Lemma 5.2.65. *Let $\mathfrak{S}$ be admissible, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, and (P, F) a filling datum at T. Let $c, c_1', c_2' \in \mathcal{C}$ and let $\sigma_1, \sigma_2 \in G(E)$ fix every e_a and e_a^*, with $h(z, \sigma_1 z) = h(z, \sigma_2 z)$ and $\sigma_1 z - \sigma_2 z \in Ez$. Put $x_i = h_c^{-1}\sigma_i h_{c_i'}$, and suppose that x_1, x_2 are integral off T and $(x_1)_T^{-1}(x_2)_T \in \mathrm{K}_T(M)$. Then $c_1' = c_2'$ and $\sigma_1 = \sigma_2$.*

Proof. Step 1: $\sigma_1 z = \sigma_2 z$. Each σ_i fixes the e_a, e_a^*, hence maps their orthogonal complement $X_0 \oplus Ez$ onto itself; write $\sigma_i z = \omega_i + \alpha_i z$ with $\omega_i \in X_0$. The hypothesis $\sigma_1 z - \sigma_2 z \in Ez$ says $\omega_1 = \omega_2$, and $h(z, \sigma_i z) = \alpha_i h(z, z) = -N\alpha_i$ gives $\alpha_1 = \alpha_2$.

Step 2: $\delta = \sigma_1^{-1}\sigma_2$ lies in $\mathrm{U}(X_0)(E^+)$. By Step 1, δ fixes every e_a, e_a^* and z, so it preserves their orthogonal complement X_0 and is the image of an element of $\mathrm{U}(X_0)(E^+)$ (acting on X_0 and trivially on $X_0^\perp$), which we denote by δ again.

Step 3: the level. The element $y = h_{c_1'}^{-1}\delta h_{c_2'} = x_1^{-1}x_2$ is integral off T (a product of elements integral off T, Lemma 5.2.46, proof of (i), for the inverse), and $y_T = (x_1)_T^{-1}(x_2)_T \in \mathrm{K}_T(M)$. By Lemma 5.2.46 (i), $y \in \mathrm{K}(M)$. As $y \in \mathrm{U}(X_0)(\mathbb{A}_f)$, (FA1) gives $y \in \mathrm{K}_X$.

Step 4: conclusion. From $\delta h_{c_2'} = h_{c_1'} y$ with $\delta \in \mathrm{U}(X_0)(E^+)$ and $y \in \mathrm{K}_X$, the elements $h_{c_1'}$ and $h_{c_2'}$ lie in the same double coset $\mathrm{U}(X_0)(E^+)\backslash\mathrm{U}(X_0)(\mathbb{A}_f)/\mathrm{K}_X$, so $c_1' = c_2'$ and $h_{c_1'} = h_{c_2'} =: h$. Then $y = h^{-1}\delta h$ lies in $\mathrm{K}_X \cap h^{-1}\mathrm{U}(X_0)(E^+)h = \{1\}$ by (FG4), so $\delta = 1$ and $\sigma_1 = \sigma_2$. □

Lemma 5.2.66. *Let $\mathfrak{S}$ be a filling datum, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, (P, F) a filling datum at T, and Z an intermediate group of F. Put $D = \#\bar{R}_0\, M^{f\varrho(2k+2\varrho+1)}$, so that $c_S^\sharp = D\,\#\mathcal{C}\, c_S$. Then $\mathfrak{fl} = M^{-f\varrho(2k+2\varrho+1)}$, $\#\bar{H} \cdot \mu_H(\mathrm{K}_T(M)) = 1$, and*

$$\mathrm{mr}(Z) \le \mu_H(\mathrm{K}_T(M))\, \theta(\mathrm{K}_T(M))^2\, \#\mathrm{Cos}(Z) \cdot D.$$

Proof. Step 1: the unipotent mass. The level unit of F is the image of M in E_T, also written M; $|M|$ denotes its modulus. As $M \in \mathbb{Z}$ and M^{-1} is integral off T, every prime of E dividing M lies in T, so $[\mathcal{O}_T : M\mathcal{O}_T] = \#(\mathcal{O}_E/M\mathcal{O}_E) = M^{[E:\mathbb{Q}]} = M^{2f}$ and $|M| = M^{-2f}$. By the definition of the floor moduli, $\mathfrak{fl} = \prod_{l<\varrho} |M|^{k+1+2(\varrho-l-1)+1/2} = M^{-f\sum_{l<\varrho}(2k+3+4(\varrho-l-1))}$, and $\sum_{l=0}^{\varrho-1}(2k+3+4(\varrho-l-1)) = \varrho(2k+3) + 2\varrho(\varrho-1) = \varrho(2k+2\varrho+1)$.

Step 2: the measure of $\mathrm{K}_T(M)$ *in* H. Apply Lemma 5.2.46 (iii) to $B = \mathrm{U}(X_0)(\mathbb{A}_f) \leq G(\mathbb{A}_f)$ and $\mu = \mu_H$. By (PE3), μ_H-almost every $z' \in G_T$ lies in H_T; such a z' is x_T for the element $x \in B$ whose matrix in the lattice basis is 1 plus that of $z'-1$ extended by 0 off T (an isometry fixing the e_a, e_a^*, z, componentwise), which is integral off T. For $x \in B$, $x_T \in H_T$, so $\mu_H(Ax_T) = \mu_H(A)$ by (PE3). Hence $\#\bar{B} \cdot \mu_H(\mathrm{K}_T(M)) = \mu_H(\mathrm{K}_T(1)) = 1$, and $\bar{B} = \bar{H}$.

Step 3: the mass of Z. By Lemma 5.5.41 (1), θ is a Haar measure of the unimodular group G_T (F is large), hence also right invariant. Z is the union of its cosets $\mathrm{K}_T(M)c$, $c \in \mathrm{Cos}(Z)$, each of measure $\theta(\mathrm{K}_T(M))$, so $\theta(Z) \leq \#\mathrm{Cos}(Z)\,\theta(\mathrm{K}_T(M))$.

Step 4: conclusion. By Lemma 5.2.67 (b), (a), $\#\bar{R}_0 = \#\bar{H} \cdot M^{f\varrho(2k+2\varrho+1)}$. With Steps 1–3,

$$\begin{aligned}\mathrm{mr}(Z) &= \theta(Z)\,\theta(\mathrm{K}_T(M))\,\mathfrak{fl}^{-2}\\ &\leq \#\mathrm{Cos}(Z)\,\theta(\mathrm{K}_T(M))^2\,\#\bar{H}\,\mu_H(\mathrm{K}_T(M))\,M^{2f\varrho(2k+2\varrho+1)}\\ &= \mu_H(\mathrm{K}_T(M))\,\theta(\mathrm{K}_T(M))^2\,\#\mathrm{Cos}(Z)\,D.\end{aligned}$$

□

Lemma 5.2.67. *Let* $\mathfrak{S}$ *be a filling datum. Order the basis of* V *as* $e_1, \dots, e_\varrho$, *the middle block* (X_0 *and* z), $e_\varrho^*, \dots, e_1^*$, *so that the block sizes are* $1, \dots, 1, k+1, 1, \dots, 1$. *Then:*

(a) $\bar{\mathsf{U}} = \prod_w \mathsf{U}_w$, *the product over the chosen places* w *of the groups* $\mathsf{U}_w \subseteq \mathrm{GL}_r(k_w)$ *of block upper unitriangular matrices, and* $\#\bar{\mathsf{U}} = M^{f\varrho(2k+2\varrho+1)}$;

(b) $\bar{\mathsf{U}} \cap \bar{H} = \{1\}$ *and* $\bar{R}_0 = \bar{\mathsf{U}}\,\bar{H}$; *in particular* $\#\bar{R}_0 = \#\bar{\mathsf{U}} \cdot \#\bar{H}$.

Proof. The lattice at the chosen places. In the lattice basis the Gram matrix of V is that of the ϱ hyperbolic planes, the unit form on $e_1, \dots, e_{k-1}$, and on (e'_k, z), $e'_k = e_k/\bar{\pi} + z/N$, the block $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$, since $h(e'_k, e'_k) = 1/(\pi\bar{\pi}) - N/N^2 = 0$ and $h(e'_k, z) = -1$; so the lattice L spanned by the lattice basis is unimodular. The change of basis matrix only mixes the middle block, so a matrix is block upper unitriangular in the lattice basis if and only if it is so in the standard basis. Let w be a chosen place and u the place of E^+ below it. By (FG2), $q_w = p_i$ for $w \in S_i$, $\#S_0 = \#S_1 = f$, $S_0 \cap S_1 = \emptyset$, and $\bar{w}$ is not chosen; hence $w \neq \bar{w}$, and distinct chosen places lie above distinct places of E^+ (a second chosen place above u would be w or $\bar{w}$). At the split place u, $g \mapsto g_w$ is an isomorphism $G(E^+_u) \to \mathrm{GL}_r(E_w)$, written in the lattice basis, and since L is unimodular it carries the factor K_u of K onto $\mathrm{GL}_r(\mathcal{O}_{E_w})$; the component $g_{\bar{w}}$ is the inverse adjoint of g_w for the pairing $V_w \times V_{\bar{w}} \to E_w$ induced by h. A subspace $W \subseteq V$ is stable under g if and only if W_w is stable under g_w and $W_{\bar{w}}$ under $g_{\bar{w}}$, that is, W_w and its annihilator $W_w^\perp$ under g_w. Hence the stabiliser of the isotropic flag $\langle e_1\rangle \subset \cdots \subset \langle e_1, \dots, e_\varrho\rangle$ corresponds to the stabiliser in $\mathrm{GL}_r(E_w)$ of the flag of $\langle e_1, \dots, e_a\rangle$ and $\langle e_1, \dots, e_a\rangle^\perp$ ($a \leq \varrho$), the parabolic subgroup of block upper triangular matrices for the block sizes above, and $\mathsf{U}(E^+_u)$, its unipotent radical, corresponds to the group of block upper unitriangular matrices in $\mathrm{GL}_r(E_w)$.

(a). An element of $\mathsf{U}(\mathbb{A}_f) \cap K$ has, in the lattice basis, the entries of the identity on and below the block diagonal, so each component of its reduction lies in U_w. Conversely let $s = (s_w)_w$ with $s_w \in \mathsf{U}_w$. For each chosen w lift the entries of s_w above the block diagonal to $\mathcal{O}_{E_w}$, and let

$x_w \in \mathrm{GL}_r(\mathcal{O}_{E_w})$ be the block upper unitriangular matrix with these entries. Let $g \in G(\mathbb{A}_f)$ have component at the place below each chosen w the element with w-component x_w, and component 1 at all other places of E^+. Then $g \in \mathsf{U}(\mathbb{A}_f) \cap K$ and $\mathrm{red}(g) = s$. So $\bar{\mathsf{U}} = \prod_w \mathsf{U}_w$. The number of positions above the block diagonal is

$$\tfrac{1}{2}(r^2 - 2\varrho - (k+1)^2) = \tfrac{1}{2}(4\varrho^2 + 4\varrho(k+1) - 2\varrho) = \varrho(2k + 2\varrho + 1),$$

using $r = 2\varrho + k + 1$, so $\#\mathsf{U}_w = q_w^{\varrho(2k+2\varrho+1)}$ and

$$\#\bar{\mathsf{U}} = \prod_{i=0}^{1} \prod_{w \in S_i} p_i^{\varrho(2k+2\varrho+1)} = (p_0 p_1)^{f\varrho(2k+2\varrho+1)} = M^{f\varrho(2k+2\varrho+1)}.$$

(b). Elements of $H(\mathbb{A}_f)$ act trivially on the hyperbolic planes and preserve the middle block, so in the lattice basis they are block diagonal with identity blocks outside the middle; elements of U are block unitriangular, with identity middle block. A matrix of both kinds is the identity, so $\bar{\mathsf{U}} \cap \bar{H} = \{1\}$. Let $g = uh \in R(\mathbb{A}_f) \cap K$ with $u \in \mathsf{U}(\mathbb{A}_f)$, $h \in H(\mathbb{A}_f)$. As u has identity middle diagonal block and h is block diagonal, the middle diagonal block of g is that of h, and likewise the middle diagonal block of $g^{-1} = h^{-1}u^{-1}$ is that of h^{-1}. Both are integral since $g, g^{-1} \in K$, and h is the identity outside the middle block, so $h \in K$ and then $u = gh^{-1} \in K$. Hence $R(\mathbb{A}_f) \cap K = (\mathsf{U}(\mathbb{A}_f) \cap K)(H(\mathbb{A}_f) \cap K)$ and $\bar{R}_0 = \mathrm{red}(R(\mathbb{A}_f) \cap K) = \bar{\mathsf{U}}\,\bar{H}$. As $\bar{\mathsf{U}} \cap \bar{H} = \{1\}$, the map $\bar{\mathsf{U}} \times \bar{H} \to \bar{R}_0$, $(x, y) \mapsto xy$, is a bijection, and $\#\bar{R}_0 = \#\bar{\mathsf{U}} \cdot \#\bar{H}$. □

Lemma 5.2.68. *Let $\mathfrak{S}$ be admissible with Weil data, and let T be a finite set of finite places of E, stable under complex conjugation, such that N^{-1} is a T-integer. Let E_T carry the structure of a finite place with integers $\mathcal{O}_T$, and let (λ_β) be a disintegration of* vol *on $X_0 \otimes E_T = E_T^k$ along $\bar{y}^{\mathsf{T}} y$. Let $\Psi \colon V \otimes E_T \to [0, \infty)$ be locally constant with compact support and such that $\Psi(m_T u) = \Psi(u)$ for every $m \in \mathrm{K}_X$, m_T its component at T acting on $V \otimes E_T$ through $\mathrm{U}(X_0) \subseteq G$. Let $t \in E$ with $|v(t)| > N$ for all $v \in \Phi$, put $\beta = (t\bar{t} - N^2)/N \in E^+$, and let $s \subseteq \mathcal{C} \times X_0$ be a finite set such that $h(x, x) = \beta$ and $h_c^{-1} x \in X_0 \otimes \mathbb{A}_f$ is integral at every finite place outside T, for $(c, x) \in s$. Then*

$$\sum_{(c,x)\in s} \Psi\big((h_c^{-1}x)_T - N^{-1}t\,z\big) \le (1+b^{-6})\,\#\mathcal{C}\,c_S \prod_{v\in\Phi} \big(|v(t)|^2 - N^2\big)^{k-1} |\pi|^{-1} \int_{E_T^k} \Psi\big(y - N^{-1}t\,z\big)\,d\lambda_{\delta\beta}(y),$$

where $(h_c^{-1}x)_T$ is the component at T and $|\pi|$ is the modulus of the image of π in E_T.

Proof. Let $F = E^+$, Σ the set of places of F below T, $L = \mathcal{O}_E^k$, and identify the real embeddings τ of F with the $v \in \Phi$ ($\tau = v|_F$).

Step 1: the field. E contains a primitive 5040-th root of unity, hence one of order 28, so E satisfies (Um) by Lemma 3.2.9 (1), and E/F is unramified at every finite place. Hence the relative discriminant of E/F is trivial and $|\mathrm{disc}\, E| = |d_F|^2$, d_F the discriminant of F. Moreover $\tau(\beta) = (|v(t)|^2 - N^2)/N > 0$, so β is totally positive. Every place above N lies in T (N^{-1} is a T-integer), so $|N| = \prod_{w\in T} |N|_w = \prod_{w \text{ finite}} |N|_w = N^{-[E:\mathbb{Q}]} = N^{-2f}$; as $\pi\bar{\pi} = N$ and $|\bar{\pi}| = |\pi|$ ((PA2) of Definition 5.1.7), $|\pi|^{-1} = N^f$.

Step 2: the test function. Put $\Psi_\Sigma(y) = \Psi(y - N^{-1}tz)$ for $y \in X_0 \otimes E_T$; it is locally constant with compact support, as the restriction of Ψ to a closed affine subspace. Let $\phi_0 = \Psi_\Sigma \otimes \bigotimes_{w\notin\Sigma} \mathbf{1}_{L\otimes\mathcal{O}_{F_w}}$ on $X_0 \otimes \mathbb{A}_f$. Then $\phi_0(my) = \phi_0(y)$ for $m \in \mathrm{K}_X$. At T: m_T fixes z, so $\Psi_\Sigma(m_T y) = \Psi(m_T(y - N^{-1}tz)) = \Psi_\Sigma(y)$. Off T: by (FA1) $m \in K$, so m and m^{-1} are integral in the lattice basis of (FG3); at a finite $w \notin T$ we have $w \nmid N$, so by (FG2) π is a unit at w and at $\bar{w}$ (also $\bar{w} \notin T$), hence $\bar{\pi}$ and N are units at w and the lattice basis and the standard basis

span the same lattice of $V \otimes E_w$. So m_w preserves the standard lattice, and, preserving X_0, it preserves $L \otimes \mathcal{O}_{F_w}$.

Step 3: Siegel's formula and the densities. The level K_X meets every conjugate of $\mathrm{U}(X_0)(F)$ trivially by (FG4), and $k \geq 3$ by (FG6). By Lemma 5.2.69 (with $V = X_0$, $K = \mathrm{K}_X$),

$$\sum_{c \in \mathcal{C}} \sum_{h(x,x)=\beta} \phi_0(h_c^{-1}x) = \#\mathcal{C} \prod_{\tau} \frac{(2\pi)^k}{(k-1)!} \tau(\beta)^{k-1} \cdot g_\Sigma(\Psi_\Sigma, \beta) \prod_{w \notin \Sigma} g_w(\mathbf{1}_{L \otimes \mathcal{O}_{F_w}}, \beta).$$

By Lemma 5.2.71, $g_\Sigma(\Psi_\Sigma, \beta) = \prod_{w \in \Sigma} q_w^{-d_w(k-1/2)} \int \Psi_\Sigma \, d\lambda_{\delta\beta}$, and by Lemma 5.2.72 (i), (ii) (whose hypotheses $f \leq 2^{2\varrho}$, $2^{-\varrho} \leq b^{-6}$, $k \geq 3\varrho + 4$ hold by (FG6), as $(\varrho+1)[E:\mathbb{Q}] \leq 2^{2\varrho}$ and $k \geq 1000(2\varrho+2)^9$), $\prod_{w \notin \Sigma} g_w \leq (1+b^{-6})|d_F|^{-(k-1/2)} \prod_{w \in \Sigma} q_w^{d_w(k-1/2)}$. The factors at Σ cancel, and all quantities are nonnegative, so the left side is at most

$$(1+b^{-6}) \, \#\mathcal{C} \, \frac{(2\pi)^{kf}}{(k-1)!^f} \, |d_F|^{-(k-1/2)} \prod_{\tau} \tau(\beta)^{k-1} \int \Psi_\Sigma \, d\lambda_{\delta\beta}.$$

Step 4: the constants. By Step 1, $|d_F|^{-(k-1/2)} = |\mathrm{disc}\, E|^{1/4-k/2}$, so $(2\pi)^{kf}|d_F|^{-(k-1/2)} = \pi^{kf}|\mathrm{disc}\, E|^{1/4}\big(2^f|\mathrm{disc}\, E|^{-1/2}\big)^k = \pi^{kf}|\mathrm{disc}\, E|^{1/4}\mathrm{vol}(\hat{\mathcal{O}})^k$, and the constant above equals $N^{kf}c_S$. Further $\prod_\tau \tau(\beta)^{k-1} = N^{-f(k-1)} \prod_v (|v(t)|^2 - N^2)^{k-1}$, and $N^{kf}N^{-f(k-1)} = N^f = |\pi|^{-1}$ by Step 1. So the left side of Step 3 is at most the right side of the claim.

Step 5: the given points. For $(c,x) \in s$ we have $h(x,x) = \beta$ and, as $h_c^{-1}x$ is integral off T, $\phi_0(h_c^{-1}x) = \Psi_\Sigma((h_c^{-1}x)_T) = \Psi((h_c^{-1}x)_T - N^{-1}tz)$. The pairs in s are distinct terms of the double sum of Step 3, whose terms are all ≥ 0; hence the sum over s is at most that double sum, which proves the claim. □

Lemma 5.2.69. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field with maximal totally real subfield F, $k \geq 3$, $V = E^k$ with $h(x,y) = \sum_i \bar{x}_i y_i$, $G = \mathrm{U}(V)$ and $L = \mathcal{O}_E^k$. Let $K \subseteq G(\mathbb{A}_{F,f})$ be an open compact subgroup with $G(F) \cap gKg^{-1} = \{1\}$ for every $g \in G(\mathbb{A}_{F,f})$, let $\mathcal{C} = G(F)\backslash G(\mathbb{A}_{F,f})/K$ with representatives h_c, and let Σ be a finite set of finite places of F, $V_\Sigma = \prod_{w \in \Sigma} V \otimes_F F_w$. Let $\Psi \colon V_\Sigma \to [0,\infty)$ be locally constant with compact support, put $\phi_0 = \Psi \otimes \bigotimes_{w \notin \Sigma} \mathbf{1}_{L \otimes \mathcal{O}_{F_w}}$ on $V \otimes_F \mathbb{A}_{F,f}$, and assume $\phi_0(mx) = \phi_0(x)$ for $m \in K$. Let $g_\Sigma(\Psi, \cdot)$ be the continuous density, with respect to $\prod_{w \in \Sigma} db_w$, of the image of $\Psi \prod_{w \in \Sigma} dx_w$ under $x \mapsto (h(x_w, x_w))_{w \in \Sigma}$, on $\prod_{w \in \Sigma} F_w^\times$. Then $\mathcal{C}$ is finite, and for every totally positive $\beta \in F$ the sums below are finite and*

$$\sum_{c \in \mathcal{C}} \sum_{x \in V,\ h(x,x)=\beta} \phi_0(h_c^{-1}x) = \#\mathcal{C} \cdot \prod_{\tau} \frac{(2\pi)^k}{(k-1)!} \, \tau(\beta)^{k-1} \cdot g_\Sigma(\Psi, \beta) \prod_{w \notin \Sigma} g_w(\mathbf{1}_{L \otimes \mathcal{O}_{F_w}}, \beta),$$

the first product over the real embeddings τ of F and the last (convergent) product over the finite places $w \notin \Sigma$ of F.

Proof. Step 1: Ψ is a finite sum of pure tensors. Since Ψ is locally constant with compact support, there is an open compact subgroup $U = \prod_{w \in \Sigma} U_w$ of V_Σ (U_w an open compact subgroup of $V \otimes_F F_w$) with $\Psi(x+u) = \Psi(x)$ for $u \in U$: each point of the compact support has such a neighbourhood $x + U_x$, finitely many cover the support, and their intersection works. The support is then a finite union of cosets $x_j + U$, and $\Psi = \sum_j \Psi(x_j)\mathbf{1}_{x_j+U}$ with $\mathbf{1}_{x_j+U} = \bigotimes_{w \in \Sigma} \mathbf{1}_{x_{j,w}+U_w}$. For a pure tensor $\bigotimes_w \phi_w$ the image of $\prod_w \phi_w dx_w$ under the product map has the density $\prod_w g_w(\phi_w, b_w)$, by Theorem A.4.26 (2) at each $w \in \Sigma$; by linearity $g_\Sigma(\Psi, b) = \sum_j \Psi(x_j) \prod_{w \in \Sigma} g_w(\mathbf{1}_{x_{j,w}+U_w}, b_w)$ exists and is continuous.

Step 2: the archimedean test function. For each real embedding τ of F choose a smooth compactly supported $\eta_\tau\colon \mathbb{R} \to [0,1]$ equal to 1 on a neighbourhood of $\tau(\beta)$, and put $\phi_\infty(x) = \prod_\tau \eta_\tau(h_\tau(x_\tau, x_\tau))$ on $V \otimes_F F_\infty = \prod_\tau \mathbb{C}^k$; it is smooth, compactly supported (h_τ is positive definite) and invariant under $G(F_\infty) = \prod_\tau \mathrm{U}(k)$. The image of $\eta_\tau(h_\tau(x,x))\,dx_\tau$ under $x \mapsto h_\tau(x,x)$ is $\eta_\tau(b)$ times the image of dx_τ, whose density at $b > 0$ is $(2\pi)^k b^{k-1}/\Gamma(k)$ by Lemma 5.2.70 ($a = 1$). Hence $g_\infty(\phi_\infty, \beta) = \prod_\tau (2\pi)^k \tau(\beta)^{k-1}/(k-1)!$. If $h(x,x) = \beta$ and $g_\infty \in G(F_\infty)$, then $\phi_\infty(g_\infty^{-1}x) = 1$.

Step 3: Siegel's formula. The functions $\phi^{(j)} = \phi_\infty \otimes \bigotimes_{w\in\Sigma} \mathbf{1}_{x_{j,w}+U_w} \otimes \bigotimes_{w\notin\Sigma} \mathbf{1}_{L\otimes\mathcal{O}_{F_w}}$ are admissible test functions for (V, L), $a = 1 < k/2$ and $\beta \neq 0$, so Theorem A.4.26 (1)–(3) apply to each. Summing with the coefficients $\Psi(x_j)$ and writing $\phi = \phi_\infty \otimes \phi_0 = \sum_j \Psi(x_j)\phi^{(j)}$ and $\vartheta(g) = \sum_{X\in V,\ h(X,X)=\beta} \phi(g^{-1}X)$, we get for every Haar measure μ on $G(\mathbb{A}_F)$

$$\frac{1}{\mu([G])}\int_{[G]} \vartheta\, d\mu = g_\infty(\phi_\infty,\beta)\, g_\Sigma(\Psi,\beta) \prod_{w\notin\Sigma} g_w(\mathbf{1}_{L\otimes\mathcal{O}_{F_w}},\beta),$$

the product converging. By Step 2, $\vartheta(g) = \vartheta_f(g_f) := \sum_{h(x,x)=\beta} \phi_0(g_f^{-1}x)$ for $g = (g_\infty, g_f)$; the sum is finite, since the $x \in V$ with $h(x,x) = \beta$ and $g_f^{-1}x \in \operatorname{supp}\phi_0$ form a discrete subset of $V \otimes_F \mathbb{A}_F$ contained in a compact set. ϑ_f is left $G(F)$-invariant, and right K-invariant because ϕ_0 is K-invariant.

Step 4: the integral over $[G]$. Take $\mu = \mu_\infty \otimes \mu_f$ with $\mu_\infty(G(F_\infty)) = 1$ and $\mu_f(K) = 1$; $G(F)$ is discrete in $G(\mathbb{A}_F)$ and $[G]$ carries the quotient measure. For $c \in \mathcal{C}$ the map $G(F_\infty)\times K \to [G]$, $(g_\infty, m) \mapsto [(g_\infty, h_c m)]$, is injective: if $\gamma(g_\infty, h_c m) = (g'_\infty, h_c m')$ with $\gamma \in G(F)$, then $\gamma = h_c m' m^{-1} h_c^{-1} \in G(F) \cap h_c K h_c^{-1} = \{1\}$, so $g_\infty = g'_\infty$ and $m = m'$. Its image is the open set of classes of $G(F_\infty)\times G(F) h_c K$, these images are disjoint for distinct c and cover $[G]$, and each has measure $\mu_\infty(G(F_\infty))\mu_f(K) = 1$. As $\mu([G]) < \infty$ by Theorem A.4.26 (1), $\mathcal{C}$ is finite and $\mu([G]) = \#\mathcal{C}$. Moreover, by the invariances of Step 3,

$$\int_{[G]} \vartheta\, d\mu = \sum_{c\in\mathcal{C}} \int_{G(F_\infty)\times K} \vartheta_f(h_c m)\, d\mu_\infty\, d\mu_f(m) = \sum_{c\in\mathcal{C}} \vartheta_f(h_c).$$

Combining with Step 3 and $g_\infty(\phi_\infty,\beta)$ from Step 2 gives the claim, since

$$\vartheta_f(h_c) = \sum_{h(x,x)=\beta} \phi_0(h_c^{-1}x).$$

□

Lemma 5.2.70. *All symbols introduced here are local to this statement and to the results that cite it. Let (V,h) be a positive definite hermitian space over $\mathbb{C}$ of dimension k (h conjugate-linear in the first variable), $1 \le a \le k$, $\mathrm{gram}(x) = (h(x_i,x_j))_{i,j}$ on V^a, $\psi(t) = e^{2\pi i t}$, dx the Haar measure on V^a self-dual for $(x,y) \mapsto \psi(\mathrm{tr}_{\mathbb{C}/\mathbb{R}} \sum_i h(x_i,y_i))$ and db the Haar measure on the space $\mathrm{Herm}_a(\mathbb{C})$ of hermitian $a\times a$ matrices self-dual for $(b,b') \mapsto \psi(\mathrm{tr}\, bb')$. Then the image of dx under* gram*, restricted to the open set of nondegenerate matrices, has the continuous density*

$$g(W) = \frac{(2\pi)^{ka-a(a-1)/2}}{\prod_{j=0}^{a-1}\Gamma(k-j)} \det(W)^{k-a} \quad (W \textit{ positive definite}),$$
$$g(W) = 0 \quad (W \textit{ nondegenerate, not positive definite})$$

with respect to db.

Proof. **Coordinates.** Choose an h-orthonormal basis of V; then $V^a = \mathrm{M}_{k\times a}(\mathbb{C})$, $\mathrm{gram}(X) = X^*X$, and $\mathrm{tr}_{\mathbb{C}/\mathbb{R}}\sum_i h(x_i,y_i) = 2\,\mathrm{Re}\,\mathrm{tr}(X^*Y) = 2\langle X,Y\rangle$ with $\langle\,,\rangle$ the euclidean inner product of $\mathbb{R}^{2ka}$. For a real vector space $\mathbb{R}^N$ with the character $\psi(B(x,y))$ of a nondegenerate symmetric form with matrix Q, the self-dual measure is $|\det Q|^{1/2}d\lambda$ (Fourier transform $\int e^{-2\pi i\,x^{\mathsf{T}}Qy}e^{-\pi x^{\mathsf{T}}Qx}|\det Q|^{1/2}dx = e^{-\pi y^{\mathsf{T}}Qy}$ for Q positive definite, which is the case here). Hence $dx = 2^{ka}d\lambda$, $d\lambda$ Lebesgue measure on $\mathbb{R}^{2ka}$. In the coordinates $(b_{ii};\mathrm{Re}\,b_{ij},\mathrm{Im}\,b_{ij}\ (i<j))$ of $\mathrm{Herm}_a(\mathbb{C})$, $\mathrm{tr}(bb') = \sum_i b_{ii}b'_{ii} + \sum_{i<j}2(\mathrm{Re}\,b_{ij}\,\mathrm{Re}\,b'_{ij} + \mathrm{Im}\,b_{ij}\,\mathrm{Im}\,b'_{ij})$, a positive definite form with determinant $2^{a(a-1)}$; so $db = 2^{a(a-1)/2}dW$ with dW as in Theorem A.5.31.

The image of Lebesgue measure. By Theorem A.5.31 (rows independent with density $\pi^{-a}e^{-x^*x}$, i.e. X with density $\pi^{-ka}e^{-\mathrm{tr}X^*X}$ with respect to $d\lambda$), the image of $\pi^{-ka}e^{-\mathrm{tr}X^*X}d\lambda$ under $X\mapsto X^*X$ is $\tilde\Gamma_a(k)^{-1}\det(W)^{k-a}e^{-\mathrm{tr}W}\mathbf{1}_{W>0}\,dW$. Since $e^{-\mathrm{tr}X^*X}$ is a function of X^*X, multiplying both measures by $\pi^{ka}e^{\mathrm{tr}W}$ gives the image of $d\lambda$:

$$\mathrm{gram}_*(d\lambda) = \frac{\pi^{ka}}{\tilde\Gamma_a(k)}\det(W)^{k-a}\mathbf{1}_{W>0}\,dW = \frac{\pi^{ka-a(a-1)/2}}{\prod_{j=0}^{a-1}\Gamma(k-j)}\det(W)^{k-a}\mathbf{1}_{W>0}\,dW.$$

Normalisation. With $dx = 2^{ka}d\lambda$ and $dW = 2^{-a(a-1)/2}db$, the image of dx is

$$2^{ka-a(a-1)/2}\pi^{ka-a(a-1)/2}\prod_j\Gamma(k-j)^{-1}\det(W)^{k-a}\mathbf{1}_{W>0}\,db,$$

which is the asserted density; it is continuous on the nondegenerate matrices because positive definite matrices form an open and closed subset of them (the image of gram consists of positive semidefinite matrices, and a nondegenerate positive semidefinite matrix is positive definite). □

Lemma 5.2.71. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field satisfying (Um), F its maximal totally real subfield, T a finite set of finite places of E stable under complex conjugation and Σ the set of places of F below T. For $w\in\Sigma$ let q_w be the order of the residue field of F_w and d_w the exponent of w in the different of $F/\mathbb{Q}$. Let $A = E_T = \prod_{w\in\Sigma}E\otimes_F F_w$ carry the structure of a finite place with integers $\mathcal{O}_A = \prod_{w\in\Sigma}\mathcal{O}_E\otimes\mathcal{O}_{F_w}$, and identify the elements of A fixed by the involution with $F_\Sigma = \prod_{w\in\Sigma}F_w$. Let $k\geq 0$ be an integer, $(\lambda_\beta)_{\beta\in A^-}$ a disintegration of* vol *on A^k along $\bar x^{\mathsf{T}}x$, and $\Psi\colon A^k\to[0,\infty)$ continuous with compact support. Let dx_w be the Haar measure on $(E\otimes_F F_w)^k$ self-dual for $(x,y)\mapsto\psi_w(\mathrm{Tr}\,\bar x^{\mathsf{T}}y)$ and db_w that on F_w self-dual for $(b,b')\mapsto\psi_w(bb')$. Then the image of $\Psi\prod_{w\in\Sigma}dx_w$ under $x\mapsto\bar x^{\mathsf{T}}x$ has, with respect to $\prod_{w\in\Sigma}db_w$, the continuous density*

$$g_\Sigma(\Psi,\beta) = \prod_{w\in\Sigma}q_w^{-d_w(k-1/2)}\int_{A^k}\Psi\,d\lambda_{\delta\beta}\qquad(\beta\in F_\Sigma).$$

Proof. Step 1: the self-dual measures. For a Haar measure μ on a local field or a finite product of them that is self-dual for a nondegenerate pairing, and a lattice Λ with dual lattice $\Lambda^\perp$, one has $\mu(\Lambda)\mu(\Lambda^\perp) = 1$ (Fourier inversion applied to $\mathbf{1}_\Lambda$, [296], Section 2.2). Fix $w\in\Sigma$, let $\delta_w\in\mathcal{O}_{F_w}$ generate the local different and $E_w = E\otimes_F F_w$; by (Um), E_w is an unramified quadratic extension of F_w or $F_w\times F_w$, so its different over F_w is trivial. The character ψ_w is trivial exactly on $\delta_w^{-1}\mathcal{O}_{F_w}$ among fractional ideals [296], Section 2.2. Hence the dual of $\mathcal{O}_{F_w}$ for $\psi_w(bb')$ is $\delta_w^{-1}\mathcal{O}_{F_w}$, of measure $q_w^{d_w}db_w(\mathcal{O}_{F_w})$, and $db_w(\mathcal{O}_{F_w}) = q_w^{-d_w/2}$. The dual of $\mathcal{O}_{E_w}$ for $\psi_w(\mathrm{Tr}\,\bar xy)$ is $\{y:\mathrm{Tr}(\mathcal{O}_{E_w}y)\subseteq\delta_w^{-1}\mathcal{O}_{F_w}\} = \delta_w^{-1}\mathcal{O}_{E_w}$, which has $q_w^{2d_w}$ times the measure of $\mathcal{O}_{E_w}$; so the self-dual measure on E_w gives $\mathcal{O}_{E_w}$ the measure $q_w^{-d_w}$, and on E_w^k (product pairing) dx_w gives $\mathcal{O}_{E_w}^k$ the measure $q_w^{-kd_w}$. As vol is a Haar measure on $A^k = \prod_w E_w^k$ with $\mathrm{vol}(\mathcal{O}_A^k) = 1$,

$$\prod_{w\in\Sigma}dx_w = \prod_{w\in\Sigma}q_w^{-kd_w}\cdot\mathrm{vol}.$$

Step 2: the measure on the self-adjoint elements. As δ is a unit of $\mathcal{O}_A$ with $\bar{\delta} = -\delta$, the map $s\colon \beta \mapsto \delta^{-1}\beta$ is a homeomorphism of A^- onto F_Σ carrying the skew elements of $\mathcal{O}_A$ onto $\mathcal{O}_{F_\Sigma} = \prod_w \mathcal{O}_{F_w}$. So $s_*\nu$ is the Haar measure of F_Σ giving $\mathcal{O}_{F_\Sigma}$ measure 1, and by Step 1, $s_*\nu = \prod_{w\in\Sigma} q_w^{d_w/2} \cdot \prod_w db_w$.

Step 3: the density. Let $\varphi\colon F_\Sigma \to [0,\infty]$ be Borel. By (DJ2) with $f(b,x) = \varphi(b)\Psi(x)$ (and $\bar{x}^{\mathsf{T}}x \in F_\Sigma$),

$$\int_{A^k} \varphi(\bar{x}^{\mathsf{T}}x)\Psi(x)\,d\mathrm{vol}(x) = \int_{A^-} \varphi(\delta^{-1}\beta)\Big(\int \Psi\,d\lambda_\beta\Big)d\nu(\beta) = \int_{F_\Sigma} \varphi(b)\,G(b)\,d(s_*\nu)(b),$$

with $G(b) = \int \Psi\, d\lambda_{\delta b}$, which is finite and continuous in b by (DJ1) and the continuity of $b \mapsto \delta b$. Inserting Steps 1 and 2,

$$\int \varphi(\bar{x}^{\mathsf{T}}x)\Psi(x)\prod_w dx_w = \int_{F_\Sigma} \varphi(b) \prod_{w\in\Sigma} q_w^{-kd_w + d_w/2}\, G(b) \prod_w db_w,$$

so the image measure has the continuous density $\prod_w q_w^{-d_w(k-1/2)}G(b)$; a continuous density with respect to a Haar measure is unique (two of them agree almost everywhere, hence everywhere, as nonempty open sets have positive measure). □

Lemma 5.2.72. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field satisfying (Um), F its maximal totally real subfield, $f = [F:\mathbb{Q}]$, d_F the discriminant of F, and for a finite place w of F let q_w be the order of its residue field and d_w the exponent of w in the different of $F/\mathbb{Q}$. Let $k \geq 3$, $V = E^k$ with $h(x,y) = \sum_i \bar{x}_i y_i$, $L = \mathcal{O}_E^k$, Σ a finite set of finite places of F and $\beta \in F^\times$. Then*

(i) $\displaystyle\prod_{w\notin\Sigma} g_w(\mathbf{1}_{L\otimes\mathcal{O}_{F_w}}, \beta) \leq (1 + 3\cdot 2^{1-k})^f\, |d_F|^{-(k-1/2)} \prod_{w\in\Sigma} q_w^{d_w(k-1/2)}$, *all factors being ≥ 0;*

(ii) if moreover $\varrho \geq 1$ and $b \geq 1$ are real numbers with $f \leq 2^{2\varrho}$, $2^{-\varrho} \leq b^{-6}$ and $k \geq 3\varrho + 4$, then $(1 + 3\cdot 2^{1-k})^f \leq 1 + b^{-6}$.

Proof. (i) Let $w \notin \Sigma$ be a finite place of F. By (Um), $E_w = E \otimes_F F_w$ is an unramified quadratic field extension of F_w or $F_w \times F_w$, with maximal order $\mathcal{O}_{E_w} = \mathcal{O}_E \otimes \mathcal{O}_{F_w}$. Let $\delta \in \mathcal{O}_{F_w}$ generate the local different, so $v_w(\delta) = d_w$. The local component ψ_w of the standard character of $\mathbb{A}_F/F$ is trivial on $\delta^{-1}\mathcal{O}_{F_w}$ and not on $\delta^{-1}\varpi^{-1}\mathcal{O}_{F_w}$, the inverse different being the largest fractional ideal on which it is trivial [296], Section 2.2. The lattice $L_w = L \otimes \mathcal{O}_{F_w} = \mathcal{O}_{E_w}^k$ has $h(L_w, L_w) \subseteq \mathcal{O}_{E_w}$ and $L_w^\sharp = L_w$, so Lemma 5.2.73 with $a = 1$, $s = 0$ ($k \geq 2$) and the self-dual measures of Theorem A.4.26 gives

$$0 \leq g_w(\mathbf{1}_{L_w}, \beta) \leq q_w^{-d_w(k-1/2)}\big(1 - q_w^{1-k}\big)^{-1}.$$

Multiply over the finite $w \notin \Sigma$ (only finitely many have $d_w \neq 0$, and the product of the second factors converges). Since $\prod_w q_w^{d_w}$ over all finite places is the norm of the different, $|d_F|$, the product of the first factors is $|d_F|^{-(k-1/2)} \prod_{w\in\Sigma} q_w^{d_w(k-1/2)}$. For the second, above each rational prime p lie at most f places w, each with $q_w \geq p$, so

$$\prod_{w\notin\Sigma} (1 - q_w^{1-k})^{-1} \leq \prod_p (1 - p^{1-k})^{-f} = \zeta(k-1)^f,$$

and for $s = k-1 \geq 2$ one has $\zeta(s) \leq 1 + 2^{-s} + \int_2^\infty x^{-s}\,dx = 1 + 2^{-s} + \frac{2^{1-s}}{s-1} \leq 1 + 3\cdot 2^{-s} = 1 + 3\cdot 2^{1-k}$.

(ii) $u = 3f2^{1-k} \leq 2^{2\varrho+3-k} \leq 2^{-\varrho-1} \leq 1$, so $(1 + 3\cdot 2^{1-k})^f \leq \exp(u) \leq 1 + 2u \leq 1 + 2^{-\varrho} \leq 1 + b^{-6}$, using $e^u \leq 1 + 2u$ for $0 \leq u \leq 1$. □

Lemma 5.2.73. *All symbols introduced here are local to this statement and to the results that cite it. Let F_w be a nonarchimedean local field with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field of order q; let E_w be either an unramified quadratic field extension of F_w or $F_w \times F_w$, with its nontrivial F_w-involution σ, maximal order $\mathcal{O}_E$ and trace* Tr. *Let $\delta \in \mathcal{O} \setminus \{0\}$ and let ψ be a character of F_w that is trivial on $\delta^{-1}\mathcal{O}$ and not on $\delta^{-1}\varpi^{-1}\mathcal{O}$. Let V be a free E_w-module of rank k with a nondegenerate hermitian form h (conjugate-linear in the first variable), $L \subset V$ an $\mathcal{O}_E$-lattice with $h(L,L) \subseteq \mathcal{O}_E$, $L^\sharp = \{x \in V : h(L,x) \subseteq \mathcal{O}_E\}$, and $s \geq 0$ an integer such that the $\mathcal{O}_E$-module $L^\sharp/L$ is generated by s elements. Let $a \geq 1$ with $k \geq a^2 + s + 1$,* $\mathrm{gram}(x) = (h(x_i, x_j))_{i,j}$ *on V^a, dx the Haar measure on V^a self-dual for $(x,y) \mapsto \psi(\mathrm{Tr} \sum_i h(x_i, y_i))$ and db the Haar measure on the space* Herm_a *of hermitian $a \times a$ matrices over E_w self-dual for $(b,b') \mapsto \psi(\mathrm{tr}\, bb')$. Then the image of $\mathbf{1}_{L^a} dx$ under* gram *has a continuous density g with respect to db, and for every $W \in \mathrm{Herm}_a$*

$$0 \leq g(W) \leq q^{-v(\delta)(ka - a^2/2)}\, [L^\sharp : L]^{-a/2} \big(1 - q^{a^2+s-k}\big)^{-1}.$$

In particular, if $L = L^\sharp$ (take $s = 0$) and $k \geq a^2 + 1$, then $0 \leq g(W) \leq q^{-v(\delta)(ka - a^2/2)}(1 - q^{a^2-k})^{-1}$.

Proof. **Stacked bases.** Since h is nondegenerate, an $\mathcal{O}_E$-basis $f_1, \ldots, f_k$ of L has a dual basis $f_1^*, \ldots, f_k^*$ of V ($h(f_i, f_j^*) = \delta_{ij}$), and $L^\sharp = \bigoplus_j \mathcal{O}_E f_j^*$ (expand $x = \sum_j c_j f_j^*$; then $h(f_i, x) = c_i$); so $L^\sharp$ is free of rank k and contains L because $h(L,L) \subseteq \mathcal{O}_E$. We claim: there are an $\mathcal{O}_E$-basis $e_1, \ldots, e_k$ of L and non-zero-divisors $\pi_1, \ldots, \pi_k \in \mathcal{O}_E$ with $\pi_l = 1$ for $l \leq k - s$ such that $\pi_1^{-1} e_1, \ldots, \pi_k^{-1} e_k$ is an $\mathcal{O}_E$-basis of $L^\sharp$. If E_w is a field, $\mathcal{O}_E$ is a discrete valuation ring with uniformiser ϖ (E_w/F_w unramified), and the elementary divisor theorem for the free modules $L \subseteq L^\sharp$ of rank k gives a basis $g_1, \ldots, g_k$ of $L^\sharp$ and exponents $m_l \geq 0$ with $\varpi^{m_1} g_1, \ldots, \varpi^{m_k} g_k$ a basis of L; then $L^\sharp/L \simeq \bigoplus_l \mathcal{O}_E/\varpi^{m_l}$, whose minimal number of generators is $\#\{l : m_l > 0\}$ (the dimension of its reduction modulo ϖ), so at most s of the m_l are positive; order them so that $m_l = 0$ for $l \leq k - s$ and put $e_l = \varpi^{m_l} g_l$, $\pi_l = \varpi^{m_l}$. If $E_w = F_w \times F_w$, let ϵ_1, ϵ_2 be the two idempotents; every $\mathcal{O}_E$-module M is $\epsilon_1 M \oplus \epsilon_2 M$, $\epsilon_i L \subseteq \epsilon_i L^\sharp$ are free $\mathcal{O}$-modules of rank k, and $\epsilon_i(L^\sharp/L)$ is generated by s elements over $\mathcal{O}$; the field case argument over $\mathcal{O}$ for each i gives bases e_l^i and exponents m_l^i, zero for $l \leq k - s$, and $e_l = e_l^1 + e_l^2$, $\pi_l = \varpi^{m_l^1}\epsilon_1 + \varpi^{m_l^2}\epsilon_2$ have the claimed properties.

Duals and volumes. Since E_w/F_w is unramified, $\{y \in E_w : \mathrm{Tr}(y\mathcal{O}_E) \subseteq \delta^{-1}\mathcal{O}\} = \delta^{-1}\mathcal{O}_E$, and ψ is trivial on an $\mathcal{O}$-submodule J of F_w if and only if $J \subseteq \delta^{-1}\mathcal{O}$. Hence for $u \in V$, $\psi(\mathrm{Tr}\, h(y,u)) = 1$ for all $y \in L$ if and only if $h(L,u) \subseteq \delta^{-1}\mathcal{O}_E$ ($h(L,u)$ is an $\mathcal{O}_E$-module, $\mathcal{O}_E$ being stable under σ), i.e. $u \in \delta^{-1}L^\sharp$. So the dual of L^a for the pairing of dx is $(\delta^{-1}L^\sharp)^a$, and as $\mathrm{vol}(L^a)\,\mathrm{vol}((\delta^{-1}L^\sharp)^a) = 1$ for the self-dual measure and $[(\delta^{-1}L^\sharp)^a : L^a] = \big([\delta^{-1}L^\sharp : L^\sharp][L^\sharp : L]\big)^a = \big(q^{2v(\delta)k}[L^\sharp : L]\big)^a$ ($L^\sharp$ is free of rank k over $\mathcal{O}_E$ and $|\mathcal{O}_E/\delta\mathcal{O}_E| = q^{2v(\delta)}$),

$$\mathrm{vol}(L^a) = q^{-v(\delta)ka}\, [L^\sharp : L]^{-a/2}.$$

Writing $\mathrm{tr}(bb') = \sum_i b_{ii}b'_{ii} + \sum_{i<j} \mathrm{Tr}(b_{ij}b'_{ji})$, the dual of $\mathrm{Herm}_a(\mathcal{O}_E)$ is $\delta^{-1}\mathrm{Herm}_a(\mathcal{O}_E)$; as $\mathrm{Herm}_a(\mathcal{O}_E)$ is free of rank a^2 over $\mathcal{O}$, $\mathrm{vol}(\delta^{-1}\mathrm{Herm}_a(\mathcal{O}_E)) = q^{v(\delta)a^2/2}$.

Fourier transform. Let μ be the image of $\mathbf{1}_{L^a} dx$ under gram, a finite measure, and $I(b) = \hat{\mu}(b) = \int_{L^a} \psi(\mathrm{tr}(b\,\mathrm{gram}(x)))\, dx$. As $h(L,L) \subseteq \mathcal{O}_E$, $\mathrm{gram}(L^a) \subseteq \mathrm{Herm}_a(\mathcal{O}_E)$, so $\mathrm{tr}(c\,\mathrm{gram}(x)) \in \delta^{-1}\mathcal{O}$ for $c \in \delta^{-1}\mathrm{Herm}_a(\mathcal{O}_E)$, and I is constant on the cosets of $\delta^{-1}\mathrm{Herm}_a(\mathcal{O}_E)$. If $I \in L^1(db)$, Fourier inversion for finite measures on the locally compact abelian group Herm_a (identified with its dual by $(b,b') \mapsto \psi(\mathrm{tr}\, bb')$, db self-dual) gives that μ has the continuous density $g(W) = \int I(b)\psi(-\mathrm{tr}(bW))\, db$, hence $0 \leq g(W) \leq \int |I|\, db$ ($g \geq 0$ as a continuous density of a positive measure).

The square of I. Substituting $x = y + z$ ($y, z \in L^a$) and using $\mathrm{gram}(y+z) = \mathrm{gram}(y) + \mathrm{gram}(z) + H(y,z) + H(z,y)$, $H(y,z)_{ij} = h(y_i, z_j)$,

$$|I(b)|^2 = \int_{L^a} \psi(\mathrm{tr}(b\,\mathrm{gram}(z))) \int_{L^a} \psi(\ell_z(y))\,dy\,dz, \qquad \ell_z(y) = \mathrm{tr}\big(b(H(y,z) + H(z,y))\big).$$

Put $u_i = \sum_j b_{ji} z_j$. Then $\mathrm{tr}(bH(y,z)) = \sum_{i,j} b_{ji} h(y_i, z_j) = \sum_i h(y_i, u_i)$, and since $b_{ij} = \sigma(b_{ji})$ and $h(z_i, y_j) = \sigma(h(y_j, z_i))$, $\mathrm{tr}(bH(z,y)) = \sigma\big(\sum_i h(y_i, u_i)\big)$; so $\ell_z(y) = \sum_i \mathrm{Tr}\, h(y_i, u_i)$, an additive character of $y \in L^a$ whose integral is $\mathrm{vol}(L^a)$ if $u \in (\delta^{-1}L^\sharp)^a$ and 0 otherwise. Therefore $|I(b)|^2 \le \mathrm{vol}(L^a)\,\mathrm{vol}(A_b)$ with $A_b = \{z \in L^a : u \in (\delta^{-1}L^\sharp)^a\}$. Write $z_j = \sum_l c_{lj} e_l$ in the stacked basis and $c_l = (c_{l1}, \dots, c_{la})^{\mathsf T} \in \mathcal{O}_E^a$; then $u_i = \sum_l (\sum_j b_{ji} c_{lj}) e_l$, and since $\pi_l^{-1} e_l$ is a basis of $L^\sharp$, $u \in (\delta^{-1}L^\sharp)^a$ if and only if $(\delta b)^{\mathsf T} c_l \in \pi_l^{-1}\mathcal{O}_E^a$ for every l. For $l \le k - s$ ($\pi_l = 1$) this says $c_l \in C_b = \{c \in \mathcal{O}_E^a : (\delta b)^{\mathsf T} c \in \mathcal{O}_E^a\}$. For $\beta \in \mathrm{M}_a(E_w)$ put $\nu(\beta) = [\mathcal{O}_E^a + \beta\mathcal{O}_E^a : \mathcal{O}_E^a]$; the map $c \mapsto \beta c \bmod \mathcal{O}_E^a$ shows $[\mathcal{O}_E^a : \{c : \beta c \in \mathcal{O}_E^a\}] = \nu(\beta)$, and $\nu((\delta b)^{\mathsf T}) = \nu(\sigma(\delta b)) = \nu(\delta b)$ ($b^{\mathsf T} = \sigma(b)$; apply σ). The map $z \mapsto (c_l)_l$ is an isomorphism $L^a \to (\mathcal{O}_E^a)^k$ of compact groups, and A_b lies in the subgroup where $c_l \in C_b$ for $l \le k - s$, of index $\nu(\delta b)^{k-s}$. Hence $\mathrm{vol}(A_b) \le \mathrm{vol}(L^a)\,\nu(\delta b)^{-(k-s)}$ and

$$|I(b)| \le \mathrm{vol}(L^a)\,\nu(\delta b)^{-(k-s)/2}. \tag{5.26}$$

The index ν of a hermitian matrix. Let $\beta \in \mathrm{Herm}_a$ and $Q = (\mathcal{O}_E^a + \beta\mathcal{O}_E^a)/\mathcal{O}_E^a$. If E_w is a field, $\mathcal{O}_E$ is a discrete valuation ring with residue field of order q^2, $|Q| = q^{2\ell}$ with ℓ the length of Q, and $\varpi^\ell Q = 0$. If $E_w = F_w \times F_w$, then $\beta = (\beta_1, \beta_1^{\mathsf T})$ with $\beta_1 \in \mathrm{M}_a(F_w)$, $Q = Q_1 \times Q_2$ with $Q_1 = (\mathcal{O}^a + \beta_1\mathcal{O}^a)/\mathcal{O}^a$ and Q_2 the same for $\beta_1^{\mathsf T}$; by the Smith normal form $\beta_1 = UDU'$ ($U, U' \in \mathrm{GL}_a(\mathcal{O})$, D diagonal) both are isomorphic to $\bigoplus_{d_i : v(d_i) < 0} \mathcal{O}/d_i^{-1}\mathcal{O}$, so $|Q_1| = |Q_2| = q^\ell$, $|Q| = q^{2\ell}$ and $\varpi^\ell Q = 0$. In both cases $\nu(\beta) = q^{2\ell}$ with $\varpi^\ell \beta \mathcal{O}_E^a \subseteq \mathcal{O}_E^a$, i.e. $\beta \in \varpi^{-\ell}\mathrm{Herm}_a(\mathcal{O}_E)$. Hence the number of cosets $\beta + \mathrm{Herm}_a(\mathcal{O}_E)$ with $\nu(\beta) = q^{2\ell}$ (well defined: $\nu(\beta + c) = \nu(\beta)$ for integral c) is at most $[\varpi^{-\ell}\mathrm{Herm}_a(\mathcal{O}_E) : \mathrm{Herm}_a(\mathcal{O}_E)] = q^{\ell a^2}$.

Summation. By the invariance of I and the substitution $\beta = \delta b$, which maps the cosets of $\delta^{-1}\mathrm{Herm}_a(\mathcal{O}_E)$ bijectively to those of $\mathrm{Herm}_a(\mathcal{O}_E)$, and by (5.26),

$$\begin{aligned}\int |I|\,db &\le \mathrm{vol}(\delta^{-1}\mathrm{Herm}_a(\mathcal{O}_E))\mathrm{vol}(L^a) \sum_{\ell \ge 0} q^{\ell a^2} q^{-\ell(k-s)} \\ &= q^{v(\delta)a^2/2}\, q^{-v(\delta)ka}\, [L^\sharp : L]^{-a/2} \big(1 - q^{a^2+s-k}\big)^{-1} < \infty,\end{aligned}$$

using $k \ge a^2 + s + 1$. So $I \in L^1(db)$, and the bound on g follows from the third paragraph. For $s = 0$ the hypothesis says $L^\sharp = L$, and the bound is the one stated last. □

Lemma 5.2.74. *Let F be a large lattice datum over a finite place A with level unit M, Z an intermediate group, and p, q sequences of units of A.*

(i) For every $l \in G^{(\varrho)}$, $\mathrm{fp}_{p,q}(l^{-1}z) = \int_G \langle \mathrm{SO}_p(kl), \mathrm{SO}_q(kl)\rangle\, d\mu_H(k)$, the integrand being μ_H-integrable.

(ii) $\mathrm{fp}_{p,q}(ke) = \mathrm{fp}_{p,q}(e)$ for all $k \in H \cap \mathrm{K}(M)$ and $e \in V$.

(iii) There is a compact set $C \subseteq Q \cap \{l^{-1}z : l \in G^{(\varrho)}\}$ with $\mathrm{fp}_{p,q}(e) = 0$ for $e \notin C$.

(iv) Every point $e_0 = l^{-1}z$, $l \in G^{(\varrho)}$, has a neighbourhood U in V with $\mathrm{fp}_{p,q}(x) = \mathrm{fp}_{p,q}(e_0)$ for all $x \in U \cap Q$.

(v) There is $\Phi\colon V \to \mathbb{C}$ with compact support such that $\Phi(x + v) = \Phi(x)$ for all $x \in V$ and all v in a neighbourhood of 0, $\Phi(kx) = \Phi(x)$ for all $k \in H \cap \mathrm{K}(M)$ and $x \in V$, and $\Phi = \mathrm{fp}_{p,q}$ on Q; in particular Φ is continuous.

Proof. **(i).** Let $l_0 \in G^{(\varrho)}$ be the element chosen in (FB2) for $e = l^{-1}z$. By Lemma 5.2.75, (ii), $\gamma = l_0 l^{-1} \in H$. By (PE3) the image of μ_H under $k \mapsto k\gamma$ is μ_H, so with $f(g) = \langle \mathrm{SO}_p(g), \mathrm{SO}_q(g) \rangle$, $\int f(kl_0)\, d\mu_H(k) = \int f(k\gamma l)\, d\mu_H(k) = \int f(kl)\, d\mu_H(k)$. The integrand is integrable by Lemma 5.2.50, (vi).

(ii). Let $k \in H \cap \mathrm{K}(M)$ and r a sequence of units. For $m \in G$, substituting $y = ky'$ in (FB1) (θ is left invariant, being a Haar measure by Lemma 5.5.41, (1)), using $k^{-1}w_r^{-1}k = w_r^{-1}$ (Lemma 5.2.58, (ix), as $k \in H \subseteq G^{(\varrho)}$) and $W_Z(ky') = W_Z(y')$ (Lemma 5.2.50, (ii)):

$$\mathrm{SO}_r(mk^{-1})_{c,c'} = \mathrm{hdn}_\varrho(r,1) \int W_Z(ky')_c \, \overline{W_Z(mk^{-1}w_r^{-1}ky')_{c'}}\, d\theta(y') = \mathrm{SO}_r(m)_{c,c'}.$$

If $e = l^{-1}z$ with $l \in G^{(\varrho)}$, then $ke = (lk^{-1})^{-1}z$ with $lk^{-1} \in G^{(\varrho)}$, and by (i) and the identity just shown (with $m = gl$), $\mathrm{fp}_{p,q}(ke) = \int f(glk^{-1})\, d\mu_H(g) = \int f(gl)\, d\mu_H(g) = \mathrm{fp}_{p,q}(e)$. If e is not of this form, neither is ke (else $e = k^{-1}l^{-1}z = (lk)^{-1}z$), and both values are 0.

(iii). Let $C_p \subseteq G$ be the compact set of Lemma 5.2.50, (iii). The floor group $G^{(\varrho)}$ is closed, so $C_p \cap G^{(\varrho)}$ is compact, and its image C under the continuous map $g \mapsto g^{-1}z$ is compact and lies in Q by Lemma 5.2.75, (i). If $\mathrm{fp}_{p,q}(e) \neq 0$, then $e = l^{-1}z$ with $l \in G^{(\varrho)}$, and by (i) the integrand $k \mapsto f(kl)$ is non-zero on a set of positive μ_H-measure, which meets H (PE3); for such k, $kl \in G^{(\varrho)}$ and $\mathrm{SO}_p(kl) \neq 0$, so $kl \in C_p$, and $e = l^{-1}k^{-1}z = (kl)^{-1}z \in C$.

(iv). Let U and σ be as in Lemma 5.2.75, (iii), for l. By Lemma 5.2.50, (iv), applied to p and to q (with $r = p$, resp. $r = q$), there is a neighbourhood $\mathcal{V}$ of 1 in G with $\mathrm{SO}_p(ms) = \mathrm{SO}_p(m)$ and $\mathrm{SO}_q(ms) = \mathrm{SO}_q(m)$ for all $m \in G^{(\varrho)}$ and $s \in \mathcal{V} \cap G^{(\varrho)}$. As σ is continuous with $\sigma(e_0) = 1$, shrink U so that $\sigma(U) \subseteq \mathcal{V}$. For $x \in U \cap Q$, $\sigma(x) \in G^{(\varrho)}$ and $x = (l\sigma(x))^{-1}z$, so by (i), and since $kl \in G^{(\varrho)}$ for μ_H-almost every k,

$$\mathrm{fp}_{p,q}(x) = \int f(kl\sigma(x))\, d\mu_H(k) = \int f(kl)\, d\mu_H(k) = \mathrm{fp}_{p,q}(e_0).$$

(v). Apply Lemma 5.2.77 to $A^S = V$, the group $\mathcal{H} = H \cap \mathrm{K}(M)$ (its elements and their inverses have entries in $\mathcal{O}_A$, as $\mathrm{K}(M) \subseteq \mathrm{K}(1)$), the set Q, the compact set C of (iii) and $\varphi = \mathrm{fp}_{p,q}$: the support condition is (iii), the invariance is (ii), and the local constancy at the points of C, which are of the form $l^{-1}z$, is (iv). □

Lemma 5.2.75. *Let F be a lattice datum over a finite place A, and let the* ***last vector quadric*** *be*

$$Q = \{e \in V : \ e \text{ has coordinate } 0 \text{ at every } e_a \text{ and } e_a^*,\ h(e,e) = h(z,z)\}.$$

(i) Every $l \in G^{(\varrho)}$ maps Q onto Q, and $l^{-1}z \in Q$.

(ii) For $l, l' \in G^{(\varrho)}$, $l^{-1}z = l'^{-1}z$ if and only if $l'l^{-1} \in H$.

(iii) For every $l \in G^{(\varrho)}$ there are an open neighbourhood U of $l^{-1}z$ in V and a continuous map $\sigma\colon U \to G$ with $\sigma(l^{-1}z) = 1$ such that for every $x \in U \cap Q$: $\sigma(x) \in G^{(\varrho)}$ and $(l\sigma(x))^{-1}z = x$.

Proof. **(i).** By Lemma 5.2.58, (vii), the rows of $l \in G^{(\varrho)}$ at the indices e_a, e_a^* (floor below ϱ) are those of 1, so lx and x have the same coordinates there; and $h(lx, lx) = h(x,x)$. So $lQ \subseteq Q$, and as $l^{-1} \in G^{(\varrho)}$, $lQ = Q$. The coordinate vector z lies in Q, hence so does $l^{-1}z$.

(ii). If $l^{-1}z = l'^{-1}z$, then $\gamma = l'l^{-1} \in G^{(\varrho)}$ fixes $z = f_\varrho$, so $\gamma \in R^{(\varrho+1)}$ by Lemma 5.2.58, (vi), and $\gamma \in R^{(\varrho+1)} \cap G^{(\varrho)} \subseteq G^{(\varrho+1)} = H$ by Lemma 5.2.58, (iv). Conversely H fixes z (its floor is $\varrho < \varrho + 1$), so $\gamma \in H$ gives $l'^{-1}z = l^{-1}\gamma^{-1}z = l^{-1}z$.

(iii): Eichler maps. By the lattice datum there is $u \in V$ with $h(u,x) = x_z$ (the z-coordinate) for all x and $u_z = 0$. For $e \in V$ whose z-coordinate e_z is a unit let $\mathrm{ei}(e)$ be the Eichler map of Lemma 5.2.76 for this u. By (i) of that lemma, $\mathrm{ei}(e) \in G$, $\mathrm{ei}(z) = 1$, and ei is continuous on the open set of the e with e_z a unit; by its (ii), $\mathrm{ei}(e) \in G^{(\varrho)}$ and $\mathrm{ei}(e)e = z$ for $e \in Q$ with e_z a unit.

(iii): the section. Let $l \in G^{(\varrho)}$, $U = \{x \in V : (lx)_z \text{ is a unit}\}$, an open set containing $l^{-1}z$ ($(l\,l^{-1}z)_z = 1$), and $\sigma(x) = l^{-1}\,\mathrm{ei}(lx)\,l$, continuous on U, with $\sigma(l^{-1}z) = l^{-1}\mathrm{ei}(z)l = 1$. For $x \in U \cap Q$, $lx \in Q$ by (i), so $\mathrm{ei}(lx) \in G^{(\varrho)}$ and $\sigma(x) \in G^{(\varrho)}$; and $(l\sigma(x))^{-1}z = l^{-1}\mathrm{ei}(lx)^{-1}z = l^{-1}lx = x$. □

Lemma 5.2.76. *Let A be a finite place, V the split space of a split Gram matrix J over A, and $u \in V$ with $h(u,x) = x_z$ for every $x \in V$ and $u_z = 0$. For $e \in V$ such that e_z is a unit of A put $v = e - z$, $s = h(v,e) - \tau\big(h(e,e) - h(z,z)\big)$ and*

$$\mathrm{ei}(e)\,x = x + h(v,x)\,u - e_z^{-1}\,h(u,x)\,(v + su) \qquad (x \in V).$$

(i) $\mathrm{ei}(e) \in G$ *for every such e;* $\mathrm{ei}(z) = 1$*; and the entries of* $\mathrm{ei}(e)$ *are continuous functions of e on the open set of the $e \in V$ with e_z a unit.*

(ii) If moreover e has coordinate 0 at every e_a and e_a^* and $h(e,e) = h(z,z)$, then $\mathrm{ei}(e) \in G^{(\varrho)}$ *and* $\mathrm{ei}(e)\,e = z$.

(iii) If moreover J, J^{-1}, u and e have entries in $\mathcal{O}_A$ and $e_z \in \mathcal{O}_A^\times$, then $\mathrm{ei}(e) \in \mathrm{K}(1)$.

Proof. Write $\varepsilon = e_z^{-1}$, $w = v + su$, $a(x) = h(v,x)$ and $b(x) = h(u,x) = x_z$, so that $\mathrm{ei}(e)x = x + a(x)u - \varepsilon b(x)w$.

(i). The following values are read off from the hypotheses on u: $h(u,u) = u_z = 0$, $h(u,v) = v_z = e_z - 1 = \varepsilon^{-1} - 1$, $h(x,u) = \overline{b(x)}$, $h(w,x) = a(x) + \bar{s}b(x)$, and $h(w,w) = h(v,v) + s\,\overline{h(u,v)} + \bar{s}\,h(u,v)$. Moreover $h(v,v) = s + \bar{s}$: as $\tau + \bar{\tau} = 1$ and $h(e,e)$, $h(z,z)$ are fixed by the involution, $s + \bar{s} = h(v,e) + h(e,v) - h(e,e) + h(z,z)$, which equals $h(v,v)$ for $e = v + z$. Expanding $h(\mathrm{ei}(e)x, \mathrm{ei}(e)y) - h(x,y)$ as a sesquilinear expression in a and b, the coefficient of $\overline{a(x)}a(y)$ is $h(u,u) = 0$; that of $\overline{b(x)}a(y)$ is $1 - \bar{\varepsilon} - \bar{\varepsilon}(\bar{\varepsilon}^{-1} - 1) = 0$; that of $\overline{a(x)}b(y)$ is $1 - \varepsilon - \varepsilon(\varepsilon^{-1} - 1) = 0$; and that of $\overline{b(x)}b(y)$ is

$$-\varepsilon s - \bar{\varepsilon}\bar{s} + \varepsilon\bar{\varepsilon}\,h(w,w) = -\varepsilon s - \bar{\varepsilon}\bar{s} + \varepsilon\bar{\varepsilon}\big(s + \bar{s} + s(\bar{\varepsilon}^{-1} - 1) + \bar{s}(\varepsilon^{-1} - 1)\big) = 0.$$

So $\mathrm{ei}(e)$ preserves h; an isometry g of the non-degenerate form J is invertible, with $g^{-1} = J^{-1}\bar{g}^{\mathsf{T}}J$, so $\mathrm{ei}(e) \in G$. For $e = z$, $v = 0$ and $s = 0$, so $\mathrm{ei}(z) = 1$. The entries of $\mathrm{ei}(e)$ are polynomials in the coordinates of e, their conjugates and e_z^{-1}; so they are continuous on the open set of the e with e_z a unit.

(ii). Now $h(e,e) = h(z,z)$ gives $s = h(v,e)$, and $h(u,e) = e_z$; so $\mathrm{ei}(e)e = e + h(v,e)u - (v + h(v,e)u) = e - v = z$. As e and z have coordinate 0 at the e_a, e_a^*, so has v, whence $h(v,e_a) = \overline{v_{e_a^*}} = 0$ and $h(v,e_a^*) = \overline{v_{e_a}} = 0$; and $h(u,e_a) = h(u,e_a^*) = 0$, as these vectors have coordinate 0 at z. Hence $\mathrm{ei}(e)$ fixes every e_a and e_a^*, which are the coordinate vectors of floor below ϱ: $\mathrm{ei}(e) \in G^{(\varrho)}$.

(iii). As $\overline{\mathcal{O}_A} = \mathcal{O}_A$ and J is integral, h takes integral values on integral vectors; so v, s (with $\tau \in \mathcal{O}_A$) and ε are integral, and $\mathrm{ei}(e)$ has entries in $\mathcal{O}_A$. So has $\mathrm{ei}(e)^{-1} = J^{-1}\overline{\mathrm{ei}(e)}^{\mathsf{T}}J$, as J^{-1} is integral. By (i), $\mathrm{ei}(e) \in G$, hence $\mathrm{ei}(e) \in \mathrm{K}(1)$. □

Lemma 5.2.77. *Let A be a finite place, S a finite set, $\mathcal{H}$ a group of $S\times S$ matrices that have entries in $\mathcal{O}_A$ and inverses with entries in $\mathcal{O}_A$, $Q, C\subseteq A^S$ with C compact, and $\varphi\colon A^S\to\mathbb{C}$ such that: $\varphi(x)\neq 0$ implies $x\in C\cap Q$; $\varphi(kx)=\varphi(x)$ for $k\in\mathcal{H}$ whenever $\varphi(x)\neq 0$; and every $e\in C$ has a neighbourhood U_e with $\varphi(x)=\varphi(e)$ for all $x\in U_e\cap Q$. Then there is $\Phi\colon A^S\to\mathbb{C}$ with compact support such that $\Phi(x+v)=\Phi(x)$ for all x and all v in a neighbourhood of 0, $\Phi(kx)=\Phi(x)$ for all $k\in\mathcal{H}$ and all x, and $\Phi=\varphi$ on Q. In particular Φ is continuous.*

Proof. Let $L=\mathcal{O}_A^S$, a compact open additive subgroup of A^S with $kL=L$ for $k\in\mathcal{H}$.

Step 1: a uniform scale. For $e\in C$ the map $(x,b,\ell)\mapsto x+b\ell$ is continuous and sends $\{e\}\times\{0\}\times L$ into the open set U_e° (interior of U_e); as L is compact, there are an open $V_e\ni e$ and a neighbourhood N_e of 0 in A with $V_e+bL\subseteq U_e^\circ$ for $b\in N_e$. Cover C by $V_{e_1},\dots,V_{e_r}$, let $N=\bigcap_i N_{e_i}$ and choose a unit $u\in N$. If $x\in Q$, $y\in C\cap Q$ and $x-y\in uL$, then $y\in V_{e_i}$ for some i, so $x,y\in U_{e_i}\cap Q$ and

$$\varphi(x)=\varphi(e_i)=\varphi(y). \tag{5.27}$$

Step 2: the extension. Call y *near* x if $\varphi(y)\neq 0$ and $x-y\in uL$. If y,y' are both near x, then $y,y'\in C\cap Q$ and $y-y'\in uL$ (uL is a group), so $\varphi(y)=\varphi(y')$ by (5.27). Put $\Phi(x)=\varphi(y)$ for any y near x, and $\Phi(x)=0$ if there is none.

- If $\Phi(x)\neq 0$, then $x\in C+uL$, a compact set.
- uL is an open neighbourhood of 0 (multiplication by the unit u is a homeomorphism); for $v\in uL$, y is near $x+v$ if and only if it is near x, so $\Phi(x+v)=\Phi(x)$. Hence Φ is locally constant, so continuous.
- Let $k\in\mathcal{H}$. If y is near x, then $\varphi(ky)=\varphi(y)\neq 0$ and $kx-ky=k(x-y)\in kuL=uL$, so ky is near kx and $\Phi(kx)=\varphi(ky)=\Phi(x)$; if x has no near point, neither has kx (apply this to k^{-1}), and $\Phi(kx)=0=\Phi(x)$.
- Let $x\in Q$. If some y is near x, then $\Phi(x)=\varphi(y)=\varphi(x)$ by (5.27). Otherwise $\varphi(x)=0$, since x would be near itself, and $\Phi(x)=0=\varphi(x)$.

□

Lemma 5.2.78. *Let $\mathfrak{S}$ be a filling datum of shape (ϱ,k) with prime N, A a place, π a unit of A with $\pi\bar\pi=N$, and suppose that $k\geq 2$ and that some $c\in A$ satisfies $\bar c c=-1$. Let F be a place datum over A on J_{lat} read over A, let $(\mu_\beta^{\mathrm{nf}})_{\beta\in A^-}$ be the family of norm-fibre measures on A^k, and put $\beta(t)=\delta\,(t\bar t/N-N)$ for $t\in A$. Let $f\colon V\to\mathbb{C}$ be continuous with compact support. Then for every $t\in A$ the function $\omega\mapsto f(e(\pi,t,\omega))$ is $\mu_{\beta(t)}^{\mathrm{nf}}$-integrable, the function*

$$t\longmapsto|\pi|^{-1}\int_{A^k} f\,(e(\pi,t,\omega))\,d\mu_{\beta(t)}^{\mathrm{nf}}(\omega)$$

is integrable on A, and its integral is $\int_V f\,d\mathrm{qv}_\varrho$. Moreover, for every $t\in A$, $\mu_{\beta(t)}^{\mathrm{nf}}$-almost every ω has $e(\pi,t,\omega)$ on the last vector quadric: its coordinates at the e_a,e_a^* vanish and

$$h(e(\pi,t,\omega),e(\pi,t,\omega))=h(z,z).$$

Proof. **Step 1: an adapted basis of the middle space.** By the Gram matrix of Definition 5.2.3 and $\bar N=N=\pi\bar\pi$, the vectors x_i ($1\leq i\leq k$) and z satisfy $h(x_i,x_j)=\delta_{ij}$, $h(x_i,z)=0$ and $h(z,z)=-N$.

Put $u = \pi^{-1}z$, so $h(u,u) = (\pi\bar\pi)^{-1}h(z,z) = -1$. Fix $i_1 \neq i_2$ in $\{1,\dots,k\}$, $S = \{1,\dots,k\} \setminus \{i_1,i_2\}$, and let $B\colon A^{\{1,2\}\sqcup S} \to A^k$ be the map of Lemma 5.2.79, (iv). Let $S' = S \sqcup \{u\}$, $T' = \mathrm{diag}(1_S,-1)$ on S', and define $E\colon A^{\{1,2\}\sqcup S'} \to V_\varrho \subseteq V$ by

$$E(c',t',v,w) = \sum_{i=1}^{k} \big(B(c',t',v)\big)_i x_i + w\,u.$$

The vectors $b_1 = x_{i_1} + cx_{i_2}$ and $b_2 = \bar\tau x_{i_1} - \tau c x_{i_2}$, images of the first two coordinate vectors, satisfy $h(b_1,b_1) = 1 + \bar c c = 0$, $h(b_2,b_2) = \tau\bar\tau(1+\bar c c) = 0$, $h(b_1,b_2) = \bar\tau - \tau\bar c c = 1$, and are orthogonal to the x_s ($s \in S$) and to u; with the relations above, $h(Ey,Ey) = \bar y^{\mathsf T} J^{\mathrm{hyp}}_{T'} y$, with $J^{\mathrm{hyp}}_{T'}$ as in Lemma 5.2.52, (i). In the lattice coordinates of V_ϱ, E is the composite of $B \oplus \mathrm{id}$ (determinant $-c$; c is a unit as $c(-\bar c) = 1$, and $|c|^2 = |\bar c c| = |-1| = 1$ by (PA2), so $|-c| = 1$) with the map fixing the x_i, $i \neq k$, and sending $x_k \mapsto \bar\pi x' - \bar\pi N^{-1} z$, $u \mapsto \pi^{-1}z$, whose determinant is $\bar\pi\pi^{-1}$, of modulus $|\bar\pi|\,|\pi|^{-1} = 1$ by (PA2). By (PA3), E carries vol to vol_{V_ϱ}, and E has a linear left inverse (its inverse on V_ϱ composed with the coordinate projection $V \to V_\varrho$).

Step 2: the quadric measure in the adapted chart. By Lemma 5.2.52, (i), $(\lambda^{T'}_\beta)_\beta$ is a disintegration of vol along $\bar y^{\mathsf T} J^{\mathrm{hyp}}_{T'} y$; by Lemma 5.2.53, (ii), with $X = E$, $J' = J^{\mathrm{hyp}}_{T'}$ and J'' the split Gram matrix J of V (Step 1 gives $\bar E^{\mathsf T} J E = J'$), $(E_*\lambda^{T'}_\beta)_\beta$ is a disintegration of $E_*\mathrm{vol} = \mathrm{vol}_{V_\varrho}$ along $h(v,v)$. By Lemma 5.2.52, (ii), so is $(\mathrm{qv}_\varrho(\beta))_\beta$; by Lemma 5.2.53, (i), $\mathrm{qv}_\varrho(\beta) = E_*\lambda^{T'}_\beta$ for every β. With $h(z,z) = -N$ this gives $\mathrm{qv}_\varrho = E_*\lambda^{T'}_{\beta_0}$, $\beta_0 = -\delta N$: the image of $\mathrm{vol}\otimes\mathrm{vol}_{A^S}\otimes\mathrm{vol}\otimes\nu$, in the variables $(s,v,w,\alpha) \in A \times A^S \times A \times A^-$, under $(s,v,w,\alpha) \mapsto E(\mathrm{ch}_{\beta_0}(s,(v,w),\alpha))$.

Step 3: the chart along the last entry. Let ch^S_β be the chart of $\lambda^{1_S}_\beta$ (Lemma 5.2.52, (i), with $T' = 1_S$). For $t \in A$ and $w = -\bar\pi^{-1}t$ one has $\bar w^{\mathsf T}(-1)w = -t\bar t/N$, so $\delta^{-1}\beta_0 - (\bar v^{\mathsf T} v - \bar w w) = -N + t\bar t/N - \bar v^{\mathsf T} v = \delta^{-1}\beta(t) - \bar v^{\mathsf T} v$, and the first coordinates of the two charts agree:

$$\mathrm{ch}_{\beta_0}(s,(v,-\bar\pi^{-1}t),\alpha) = \big(\mathrm{ch}^S_{\beta(t)}(s,v,\alpha),\ -\bar\pi^{-1}t\big).$$

Applying E, and as $-\bar\pi^{-1}t\,u = -(\pi\bar\pi)^{-1}tz = -N^{-1}tz$,

$$E\big(\mathrm{ch}_{\beta_0}(s,(v,-\bar\pi^{-1}t),\alpha)\big) = e\big(\pi,t,B\,\mathrm{ch}^S_{\beta(t)}(s,v,\alpha)\big).$$

By (PA3) for the 1×1 matrix $-\bar\pi^{-1}$, whose modulus is $|\pi|^{-1}$, the map $t \mapsto -\bar\pi^{-1}t$ carries $|\pi|^{-1}dt$ to vol. Hence qv_ϱ is the image of $|\pi|^{-1}dt \otimes (\mathrm{vol}\otimes\mathrm{vol}_{A^S}\otimes\nu)$ under

$$\Theta(t,s,v,\alpha) = e\big(\pi,t,B\,\mathrm{ch}^S_{\beta(t)}(s,v,\alpha)\big),$$

a measurable map (sums and products of the coordinates and of $s \mapsto \bar s^{-1}$ on units, 0 elsewhere, as in the charts).

Step 4: Fubini. Let f be continuous with compact support C. By (DJ1) for the disintegration $(\mathrm{qv}_\varrho(\beta))_\beta$ and Urysohn's lemma, $\mathrm{qv}_\varrho(C) < \infty$, so f is qv_ϱ-integrable, and $f\circ\Theta$ is integrable for the product measure of Step 3, which is a product of σ-finite Haar measures (A is a locally compact Polish group, hence σ-compact). By Fubini's theorem, $t \mapsto |\pi|^{-1}\int f(\Theta(t,s,v,\alpha))\,d(s,v,\alpha)$ is integrable with integral $\int f\,d\mathrm{qv}_\varrho$, the inner integral being taken where it exists. For each t, $\omega \mapsto e(\pi,t,\omega)$ is an injective affine map with a continuous affine left inverse (read off the coordinates of X_0), so $\omega \mapsto f(e(\pi,t,\omega))$ is continuous with compact support (contained in the image of C under the left inverse); by Lemma 5.2.79, (iii), it is $\mu^{\mathrm{nf}}_{\beta(t)}$-integrable, and by Lemma 5.2.79, (iv), $\mu^{\mathrm{nf}}_{\beta(t)}$ is the image of $\mathrm{vol}\otimes\mathrm{vol}_{A^S}\otimes\nu$ under $B\circ\mathrm{ch}^S_{\beta(t)}$. So the inner integral exists for every t and equals $\int f(e(\pi,t,\omega))\,d\mu^{\mathrm{nf}}_{\beta(t)}(\omega)$, which proves the first assertion.

Step 5: the last vector quadric. By Lemma 5.2.79, (ii), $\mu^{\mathrm{nf}}_{\beta(t)}$-almost every ω satisfies $\bar\omega^{\mathsf T}\omega = \delta^{-1}\beta(t) = t\bar t/N - N$. For such ω, by Step 1 and the computation in (FB4), $h(e(\pi,t,\omega), e(\pi,t,\omega)) = \bar\omega^{\mathsf T}\omega - t\bar t/N = -N = h(z,z)$, and $e(\pi,t,\omega) \in V_\varrho$ has coordinate 0 at the e_a, e_a^*. □

Lemma 5.2.79. *Let A be a place, $k \geq 2$ an integer, and suppose that some $c \in A$ satisfies $\bar c c = -1$.*

(i) There is exactly one family $(\mu^{\mathrm{nf}}_\beta)_{\beta\in A^-}$ of norm-fibre measures on A^k.

(ii) For every $\beta \in A^-$, μ^{nf}_β is carried by the sphere $\{x \in A^k : \bar x^{\mathsf T} x = \delta^{-1}\beta\}$, and $\mu^{\mathrm{nf}}_\beta(C) < \infty$ for every compact $C \subseteq A^k$.

(iii) For every continuous compactly supported $f\colon A^k \to \mathbb{C}$, f is μ^{nf}_β-integrable for every β, and $\beta \mapsto \int f\, d\mu^{\mathrm{nf}}_\beta$ is continuous on A^-.

(iv) Let $i_1 \neq i_2$ be elements of $\{1,\dots,k\}$, $S = \{1,\dots,k\}\setminus\{i_1,i_2\}$, 1_S the identity matrix on S, and let $B\colon A^{\{1,2\}\sqcup S} \to A^k$ be the linear map sending (c',t,v) to the vector with coordinates $c' + \bar\tau t$ at i_1, $cc' - \tau ct$ at i_2 and v_s at $s \in S$. Then $\mu^{\mathrm{nf}}_\beta = B_*\lambda^{1_S}_\beta$ for every $\beta \in A^-$.

Proof. **Step 1: a hyperbolic basis.** Let e_i $(1 \leq i \leq k)$ be the coordinate vectors of A^k and put $b_1 = B(1,0,0) = e_{i_1} + ce_{i_2}$, $b_2 = B(0,1,0) = \bar\tau e_{i_1} - \tau c e_{i_2}$; B sends the coordinate vector of $s \in S$ to e_s. Since $\bar c c = -1$ and $\tau + \bar\tau = 1$,

$$h(b_1,b_1) = 1 + \bar c c = 0, \qquad h(b_2,b_2) = \tau\bar\tau + \bar\tau\tau\,\bar c c = 0, \qquad h(b_1,b_2) = \bar\tau - \tau\,\bar c c = \bar\tau + \tau = 1,$$

$h(b_2,b_1) = \overline{h(b_1,b_2)} = 1$, and b_1, b_2 are orthogonal to the e_s, $s \in S$. Hence $h(Bx,Bx) = \bar c' t + \bar t c' + \bar v^{\mathsf T} v$ for $x = (c',t,v)$, that is, $\bar B^{\mathsf T} B = J^{\mathrm{hyp}}_{1_S}$. Identify $\{1,2\}\sqcup S$ with $\{1,\dots,k\}$ by $1 \mapsto i_1$, $2 \mapsto i_2$, $s \mapsto s$; this bijection of index sets carries the product measure vol on $A^{\{1,2\}\sqcup S}$ to that on A^k, and turns B into a square matrix whose only entries off the identity are the block $\begin{pmatrix} 1 & \bar\tau \\ c & -\tau c \end{pmatrix}$ on $\{i_1,i_2\}$. Its determinant is $-\tau c - \bar\tau c = -c$. As $c(-\bar c) = 1$, c is a unit; by multiplicativity and (PA2), $|c|^2 = |\bar c|\,|c| = |\bar c c| = |-1| = |1| = 1$, so $|c| = 1$ (the modulus takes values in $[0,\infty]$) and $|\det B| = |-c| = 1$. By (PA3), $B_*\mathrm{vol} = \mathrm{vol}$. Since $J^{\mathrm{hyp}}_{1_S}$ is its own inverse, $L = J^{\mathrm{hyp}}_{1_S}\bar B^{\mathsf T}$ satisfies $LB = 1$.

Step 2: existence, uniqueness and (iv). By Lemma 5.2.52, (i), with $T' = 1_S$, the family $(\lambda^{1_S}_\beta)_\beta$ is a disintegration of vol on $A^{\{1,2\}\sqcup S}$ along $\bar x^{\mathsf T} J^{\mathrm{hyp}}_{1_S} x$. By Lemma 5.2.53, (ii), applied with $X = B$, the left inverse L, $J' = J^{\mathrm{hyp}}_{1_S}$ and $J'' = 1_k$ (Step 1 gives $\bar B^{\mathsf T} 1_k B = J'$), the family $(B_*\lambda^{1_S}_\beta)_\beta$ is a disintegration of $B_*\mathrm{vol} = \mathrm{vol}$ along $\bar y^{\mathsf T} y$, that is, a family of norm-fibre measures on A^k. Any two such families agree by Lemma 5.2.53, (i). This proves (i) and (iv).

Step 3: the spheres. Let $\beta \in A^-$, t a unit, $v \in A^S$, $\alpha \in A^-$, and $c' = \bar t^{-1}\tau(\delta^{-1}\beta - \bar v^{\mathsf T} v) + t\alpha$, so that $\mathrm{ch}_\beta(t,v,\alpha) = (c',t,v)$. Since $\bar\delta = -\delta$, $\bar\beta = -\beta$ and $\bar v^{\mathsf T} v$ is fixed by the involution, $\gamma = \delta^{-1}\beta - \bar v^{\mathsf T} v$ satisfies $\bar\gamma = \gamma$; with $\bar\alpha = -\alpha$,

$$\bar c' t + \bar t c' = \big(\bar\tau\gamma + \bar\alpha t\bar t\big) + \big(\tau\gamma + \alpha t\bar t\big) = \gamma,$$

so $\bar x^{\mathsf T} J^{\mathrm{hyp}}_{1_S} x = \gamma + \bar v^{\mathsf T} v = \delta^{-1}\beta$ for $x = \mathrm{ch}_\beta(t,v,\alpha)$, and by Step 1 Bx lies on the sphere $\bar y^{\mathsf T} y = \delta^{-1}\beta$. The set of (t,v,α) with t not a unit is null for $\mathrm{vol}\otimes\mathrm{vol}\otimes\nu$ by (PA4). By (iv), μ^{nf}_β is the image of $\mathrm{vol}\otimes\mathrm{vol}\otimes\nu$ under $B\circ\mathrm{ch}_\beta$, so the complement of the sphere has μ^{nf}_β-measure

0. For a compact $C \subseteq A^k$, Urysohn's lemma in the locally compact Hausdorff space A^k gives a continuous compactly supported $g\colon A^k \to [0,1]$ with $g = 1$ on C, and $\mu_\beta^{\mathrm{nf}}(C) \le \int g\, d\mu_\beta^{\mathrm{nf}} < \infty$ by (DJ1). This proves (ii).

Step 4: continuity. Let f be continuous with compact support C. By (ii), $|f| \le \sup|f| \cdot \mathbf{1}_C$ is μ_β^{nf}-integrable. Write $f = f_1 - f_2 + \mathrm{i}(f_3 - f_4)$ with $f_1 = \max(\mathrm{Re} f, 0)$, $f_2 = \max(-\mathrm{Re} f, 0)$, $f_3 = \max(\mathrm{Im} f, 0)$, $f_4 = \max(-\mathrm{Im} f, 0)$; each f_r is continuous, nonnegative and supported in C. Then $\int f\, d\mu_\beta^{\mathrm{nf}} = \int f_1\, d\mu_\beta^{\mathrm{nf}} - \int f_2\, d\mu_\beta^{\mathrm{nf}} + \mathrm{i}(\int f_3\, d\mu_\beta^{\mathrm{nf}} - \int f_4\, d\mu_\beta^{\mathrm{nf}})$, a combination of four finite functions of β that are continuous by (DJ1). This proves (iii). □

Lemma 5.2.80. *Let $\mathfrak{S}$ be a filling datum over E with Weil data, and let T be a finite set of finite places of E stable under complex conjugation. Give E_T the involution induced by complex conjugation, $(x_w)_{w\in T} \mapsto (\bar{x}_{\bar{w}})_{w\in T}$. Then*

(a) E_T, with this involution, its normalised Haar measure ($\mathrm{vol}(\mathcal{O}_T) = 1$) and the integers $\mathcal{O}_T$, carries a structure of finite place whose character is ψ_T;

(b) there is $c \in \mathcal{O}_T$ with $\bar{c}c = -1$.

Proof. Write $A = E_T = \prod_{w\in T} E_w$, $\mathcal{O} = \mathcal{O}_T = \prod_w \mathcal{O}_w$, $x \mapsto \bar{x}$ the involution, vol the Haar measure of A with $\mathrm{vol}(\mathcal{O}) = 1$ (the product of the normalised Haar measures of the E_w). As T is conjugation-stable, $\bar{\mathcal{O}} = \mathcal{O}$, and $\widetilde{\bar{x}} = \overline{\tilde{x}}$.

Step 1: the elements τ, δ. In the CM field E complex conjugation acts on roots of unity as inversion. Let $i = \zeta^{1260}$ (a primitive fourth root of unity) and $\omega = \zeta^{1680}$ (a primitive cube root of unity), both units of $\mathcal{O}_E$. Then $\bar{i} = i^{-1} = -i$, and $\tau = -\omega$ satisfies $\tau + \bar{\tau} = -(\omega + \omega^2) = 1$. Let τ and the skew unit $\delta = i$ also denote their images in A: $\tau \in \mathcal{O}$, $\delta \in \mathcal{O}^\times$ with $\delta^{-1} = -i \in \mathcal{O}$, and $\bar{\delta} = -\delta$.

Step 2: the skew measure (PA1). A^- is a closed subgroup of A and $\mathcal{O} \cap A^-$ is compact and open in it; let ν be the Haar measure of A^- with $\nu(\mathcal{O} \cap A^-) = 1$. Let $\Phi(\alpha, \beta) = \tau\delta^{-1}\alpha + \beta = -\tau i\alpha + \beta$ on $A^- \times A^-$. For $x \in A$ put $\alpha(x) = i(x + \bar{x})$ and $\beta(x) = x - \tau(x + \bar{x})$. Both lie in A^-: $\overline{i(x+\bar{x})} = -i(x+\bar{x})$, and $\beta + \bar{\beta} = (x+\bar{x})(1 - \tau - \bar{\tau}) = 0$. For skew α, β, $\overline{-\tau i\alpha} = -\bar{\tau} i\alpha$, so $\Phi(\alpha,\beta) + \overline{\Phi(\alpha,\beta)} = -i\alpha$, whence $\alpha(\Phi(\alpha,\beta)) = \alpha$ and $\beta(\Phi(\alpha,\beta)) = -\tau i\alpha + \beta + \tau i\alpha = \beta$; and $\Phi(\alpha(x), \beta(x)) = -\tau i \cdot i(x+\bar{x}) + x - \tau(x+\bar{x}) = x$. So Φ is an isomorphism of topological groups. As $\tau, i \in \mathcal{O}$, Φ maps $(\mathcal{O} \cap A^-)^2$ into $\mathcal{O}$, and $x \mapsto (\alpha(x), \beta(x))$ maps $\mathcal{O}$ into $(\mathcal{O} \cap A^-)^2$; so $\Phi^{-1}(\mathcal{O}) = (\mathcal{O} \cap A^-)^2$. The image of the Haar measure $\nu \otimes \nu$ under Φ is a Haar measure of A giving $\mathcal{O}$ the mass 1, hence equal to vol.

Step 3: the modulus (PA2)–(PA4). Put $|x| = [x]^{-1}$ with the generalised index $[x]$ of Definition 5.1.8: $|x| = \prod_{w\in T} |x_w|_w$ (normalised absolute values) if x is a unit, and $|x| = \infty$ otherwise. It is measurable (the units form an open set on which $|x|$ is continuous), $|1| = 1$, and multiplicative: for units this is the multiplicativity of the absolute values, and if x or y is not a unit then neither is xy, while $|x|, |y| \in (0, \infty]$, so both sides are ∞. Further $|-x| = |x|$, and $|\bar{x}| = |x|$ because conjugation maps $E_{\bar{w}}$ isometrically onto E_w and preserves units. For a unit x of A, $\mathrm{vol}(x\mathcal{O}) = \prod_w |x_w|_w \mathrm{vol}(\mathcal{O}_w) = |x|$. For a finite set S and an $S \times S$ matrix $X = (X_w)_w$ over A with unit determinant, the image of the Haar measure of E_w^S under X_w is $|\det X_w|_w^{-1}$ times it (the module of an automorphism of E_w^S is $|\det|_w$), and taking products over w gives (PA3). The non-units of A form the finite union of the null sets $\{x : x_w = 0\}$, which gives (PA4).

Step 4: self-duality. Let $a \in A$. By the product form of ψ_f,

$$\psi_T(ra) = \prod_{w\in T} \psi_{p_w}(\mathrm{Tr}_{E_w/\mathbb{Q}_{p_w}}(-r_w a_w/\varpi)),$$

the components $r_w \in \mathcal{O}_w$ being independent; so $\psi_T(ra) = 1$ for all $r \in \mathcal{O}$ if and only if, for each w, ψ_{p_w} is trivial on the $\mathbb{Z}_{p_w}$-module $\mathrm{Tr}(\mathcal{O}_w a_w/\varpi)$, i.e. $\mathrm{Tr}(\mathcal{O}_w a_w/\varpi) \subseteq \mathbb{Z}_{p_w}$, i.e. a_w/ϖ lies in the inverse of the local different of $E_w/\mathbb{Q}_{p_w}$ [296]. As ϖ generates $\mathfrak{D}_{E/\mathbb{Q}}$, the local different is $\varpi\mathcal{O}_w$, so the condition is $a_w \in \mathcal{O}_w$ for all w, i.e. $a \in \mathcal{O}$. The character ψ_T is a continuous homomorphism to the unit circle, since $x \mapsto \tilde{x}$ and multiplication by $-\varpi^{-1}$ are continuous and additive.

Step 5: the kernel (PA5) with $\mathrm{kb}_A = 1$**.** Let $\gamma = \mathbf{1}_{\mathcal{O}}$, continuous ($\mathcal{O}$ is open and closed), integrable, $\gamma(0) = 1$. For $y \in A$, $v \mapsto \psi_T(\bar{v}y)$ is a character of the compact group $\mathcal{O}$, trivial if and only if $\psi_T(ry) = 1$ for all $r \in \bar{\mathcal{O}} = \mathcal{O}$, i.e. (Step 4) if and only if $y \in \mathcal{O}$. By orthogonality, $\int_A \gamma(v)\psi_T(\bar{v}y)\,dv = \mathrm{vol}(\mathcal{O})\mathbf{1}_{\mathcal{O}}(y) = \mathbf{1}_{\mathcal{O}}(y)$, and $\int_A |\mathbf{1}_{\mathcal{O}}(y)|\,dy = 1$.

Step 6: the finite place. A is a finite product of non-archimedean local fields with a continuous involution, a locally compact Polish ring; $\mathcal{O}$ is the product of the valuation rings, compact, open, conjugation-stable, $\mathrm{vol}(\mathcal{O}) = 1$, self-dual (Step 4); the modulus is the inverse generalised index (Step 3); $\mathrm{kb}_A = 1$ (Step 5); τ is integral, δ is a unit of $\mathcal{O}$ with $\bar{\delta} = -\delta$, and $\nu(\mathcal{O} \cap A^-) = 1$ (Steps 1, 2). With (PA1)–(PA5) this is a structure of finite place on A with character ψ_T, proving (a).

Step 7: (b). It suffices to find c_w for each conjugation orbit in T. If $w \neq \bar{w}$, put $c_w = 1$, $c_{\bar{w}} = -1$; then $(\bar{c}c)_w = \overline{c_{\bar{w}}}\,c_w = -1$ and $(\bar{c}c)_{\bar{w}} = \overline{c_w}\,c_{\bar{w}} = -1$. If $w = \bar{w}$, let v be the place of E^+ below w; then $E_w = E_v^+(i)$ is a quadratic field extension with conjugation as its nontrivial automorphism, and $\bar{c}c = \mathrm{N}_{E_w/E_v^+}(c)$. So -1 is of the form $\bar{c}c$ with $c \in E_w$ if and only if the Hilbert symbol $(-1,-1)_{E_v^+}$ is 1. If the residue characteristic of v is odd, -1 is a unit and the symbol of two units is 1. If $v \mid 2$, then $\sqrt{2} = \zeta_8 + \zeta_8^{-1} \in E^+$ ($\zeta_8 = \zeta^{630}$) and $\sqrt{2} \notin \mathbb{Q}_2$, so $d = [E_v^+ : \mathbb{Q}_2]$ is even, and $(-1,-1)_{E_v^+} = (-1, \mathrm{N}_{E_v^+/\mathbb{Q}_2}(-1))_{\mathbb{Q}_2} = (-1,(-1)^d)_{\mathbb{Q}_2} = 1$ [263]. In both cases some $c_w \in E_w$ has $\bar{c}_w c_w = -1$; then $|c_w|_w^2 = |\bar{c}_w c_w|_w = 1$, so $c_w \in \mathcal{O}_w$. The element $c = (c_w)_w$ lies in $\mathcal{O}$ and $\bar{c}c = -1$. □

Proposition 5.2.81. *Let $\mathfrak{S}$ be admissible with Weil data, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, $S_g \subseteq S_0$, and $d = (d_w)_{w\in S_0}$ frame vectors.*

(i) There is a filling datum (P,F) at T.

(ii) Let (P,F) be a filling datum at T. There is an intermediate group Z of F such that $m_T \in Z$ for every $m \in K$ with $\mathrm{red}(m) \in \mathrm{Fix}(S_g, d)$, and such that the forms

$$J_j(p,q) = \frac{\mathcal{F}_j(p,q;W_Z,1)}{\mathrm{mr}(Z)} \quad (0 \le j \le \varrho), \qquad J_{\varrho+1}(p,q) = \frac{\mathrm{led}(\pi,Z,p,q)}{\mathrm{mr}(Z)},$$

on pairs of sequences p,q of units of E_T, form a sieve system of length ϱ on $(E_T, E_T^\times, u, \iota)$, with the character ψ_T, the mass bound $(1+b^{-6})/(b^{6k})^{\#S_g}$, the weights Λ_i^T and the decay β_T. Here E_T carries the Haar measure of P (for which $\mathcal{O}_T$ has measure 1), $E_T^\times$ the subspace topology, $u(c) = c$ for a unit c and $u(c) = 1$ otherwise, and $\iota(w) = M^{-1}w^{-1}$.

Proof. **(i)** is Lemma 5.2.82, $\mathfrak{S}$ being in particular a filling datum with Weil data.

(ii). *Step 1: the group Z.* By Lemma 5.2.84 there is an intermediate group Z of F with $m_T \in Z$ for every $m \in K$ with $\mathrm{red}(m) \in \mathrm{Fix}(S_g, d)$ and with

$$0 \le \frac{\mathcal{F}_0(p,q;W_Z,1)}{\mathrm{mr}(Z)} \le \frac{1+b^{-6}}{(b^{6k})^{\#S_g}} \qquad \text{for all sequences } p,q \text{ of units of } E_T.$$

By Lemma 5.5.44 (c), (d), $\mathrm{Cos}(Z)$ is finite, W_Z is cut and $0 < \mathrm{mr}(Z) < \infty$, so the forms J_j are defined.

Step 2: the data at T. P is a finite place on E_T with character ψ_T, and F is a large lattice datum over it on J_{lat} with level unit the image of M. The image of π in E_T is a unit, as $\pi\bar{\pi} = N \neq 0$ makes each of its components nonzero, and it satisfies $\pi\bar{\pi} = N$. By Lemma 5.2.80 (b) some $c \in E_T$ has $\bar{c}c = -1$, and $k \geq 2$ by (FG6); so the family of norm-fibre measures on E_T^k exists by Lemma 5.2.79 (i), and Lemma 5.2.93 and Lemma 5.2.95 apply to $\mathfrak{S}$, $A = E_T$, F, π and Z. By (FG6), $10^{10}n^4 \leq n^6 \leq n^7 \leq k$.

Step 3: the decay bound. As M^{-1} is a T-integer, every prime of E dividing M lies in T, so $[M] = [\mathcal{O}_T : M\mathcal{O}_T] = \#(\mathcal{O}_E/M\mathcal{O}_E) = M^{2f}$ and $2/|M| = 2M^{2f}$. By (FG2) and (FG6), $2 \leq b$, $M \leq b^{23}$ and $47 \leq n^3$, so

$$\Big(\frac{2}{|M|}\Big)^{n^6} \leq \big(b^{46f+1}\big)^{n^6} \leq b^{n^9 f}.$$

For a unit ξ of E_T, $[\mathcal{O}_T : \mathcal{O}_T \cap \xi^{-1}\mathcal{O}_T]$ is a positive integer, so its (-6)-th power is at most its (-3)-rd power; and $\mathrm{hw}_{n^6,j} = \Lambda_j^T$ (the two definitions agree). Hence Lemma 5.2.96, with $t = k - n^7$ and $K = n^6$ (admissible by Step 2), gives for $j \leq \varrho$, sequences p, q of units, a unit ξ and a weight ω of kind (a) or (b) there:

$$\frac{|\mathcal{F}_j(p,q;W_Z,\omega)|}{\mathrm{mr}(Z)} \leq [\xi]^{(k-n^7)/2}\,[\mathcal{O}_T : \mathcal{O}_T \cap \xi^{-1}\mathcal{O}_T]^{-3}\, b^{n^9 f}\, \Lambda_j^T(p)\,\Lambda_j^T(q) = \beta_T(\xi)\,\Lambda_j^T(p)\,\Lambda_j^T(q). \tag{5.28}$$

Step 4: (SV1). $\psi_T(0) = \psi_f(0) = 1$.

Step 5: (SV2) below the top. Let $i < \varrho$, p, q sequences of units and $\xi \in E_T$. By Lemma 5.2.91 (as $10^8 n^4 \leq k$), $\int_G \mathrm{FM}_i(p,q,W_Z,x)\,d\theta(x) < \infty$. The function $\phi(c) = \psi_T(\xi c)$ is continuous with $|\phi| = 1$. By Proposition 5.2.92 (ii), with $a = i$, $W = W_Z$, $t = \mathrm{mr}(Z)^{-1}$ and this ϕ, the function $c \mapsto \psi_T(\xi c)\,J_{i+1}(p[i \mapsto u(c)], q[i \mapsto u(c)])$ is integrable on E_T with integral $\mathcal{F}_i(p,q;W_Z,\omega)/\mathrm{mr}(Z)$, where $\omega(x) = \psi_T(\xi\, h(f_i, xf_i))$. If $\xi = 0$, then $\omega = 1$ and the integral is $J_i(p,q)$; if ξ is a unit, ω is of kind (a) of Lemma 5.2.96 for $j = i$, and (5.28) bounds the absolute value of the integral by $\beta_T(\xi)\Lambda_i^T(p)\Lambda_i^T(q)$.

Step 6: (SV2) at the top. Let p, q be sequences of units and $\xi \in E_T$. By Lemma 5.2.93, $c \mapsto \psi_T(\xi c)\,\mathrm{led}(\pi, Z, p[\varrho \mapsto u(c)], q[\varrho \mapsto u(c)])$ is integrable on E_T with integral $\mathcal{F}_\varrho(p,q;W_Z,\omega_\xi)$, $\omega_\xi(x) = \psi_T(\xi\, h(z, xz))$; dividing by $\mathrm{mr}(Z)$, the same holds for $c \mapsto \psi_T(\xi c) J_{\varrho+1}(p[\varrho \mapsto u(c)], q[\varrho \mapsto u(c)])$ with integral $\mathcal{F}_\varrho(p,q;W_Z,\omega_\xi)/\mathrm{mr}(Z)$. If $\xi = 0$, then $\omega_\xi = 1$ and the integral is $J_\varrho(p,q)$; if ξ is a unit, ω_ξ is of kind (a) for $j = \varrho$ (as $f_\varrho = z$), and (5.28) gives the bound $\beta_T(\xi)\Lambda_\varrho^T(p)\Lambda_\varrho^T(q)$.

Step 7: (SV3) and (SV4). For a finite family (a_l) of sequences of units with a common entry $a_l(\varrho) = t$, the matrix $(\mathrm{led}(\pi, Z, a_l, a_{l'}))_{l,l'}$ is positive semidefinite by Lemma 5.2.95 (i), hence so is its multiple by $\mathrm{mr}(Z)^{-1} > 0$, which is $(J_{\varrho+1}(a_l, a_{l'}))_{l,l'}$. By Lemma 5.2.95 (ii), $(p,q) \mapsto \mathrm{led}(\pi, Z, p, q)$, hence $J_{\varrho+1}$, is jointly continuous.

Step 8: (SV5) and (SV6). (SV5) is Step 1: $|J_0(p,q)| \leq (1 + b^{-6})/(b^{6k})^{\#S_g}$. For (SV6) let $i < \varrho$ and let $\xi = \iota(p_i) - \iota(q_i) = M^{-1}(p_i^{-1} - q_i^{-1})$ be a unit; the weight $\omega = 1$ is of kind (b) of Lemma 5.2.96 for $j = i + 1 \leq \varrho$, and (5.28) gives $|J_{i+1}(p,q)| \leq \beta_T(\xi)\Lambda_{i+1}^T(p)\Lambda_{i+1}^T(q)$.

Steps 4–8 are (SV1)–(SV6) of Definition 5.1.5, which proves (ii). □

Lemma 5.2.82. *Let $\mathfrak{S}$ be a filling datum over E with Weil data and T a finite set of finite places of E stable under complex conjugation. Then there is a filling datum (P, F) at T.*

Proof. Put $\iota = \{1, \ldots, k - (6\varrho + 13)\}$; this is a set of $k - (6\varrho + 13) \geq 0$ elements, as $k \geq 1000(2\varrho + 2)^9 \geq 6\varrho + 13$.

Step 1: the place and the lattice datum. By Lemma 5.2.80, E_T with the involution induced by complex conjugation carries a structure P of finite place whose character is ψ_T, and some $c \in \mathcal{O}_T$ has $\bar{c}c = -1$. As M is a nonzero rational integer, its image in E_T, also written M, is a unit of E_T, lies in $\mathcal{O}_T$ and is fixed by the involution. By Lemma 5.2.83 there is a place datum F over E_T of shape $(\varrho, \iota, 3\varrho + 6)$ on J_{lat} that is a lattice datum with level unit M. It remains to show that F is large, with $m + 1 = 3\varrho + 6$, i.e. $m = 3\varrho + 5$.

Step 2: the zeta bound. The ring $\mathcal{O}_T = \prod_{w \in T} \mathcal{O}_w$ is a finite product of discrete valuation rings, so its ideals of finite index are the products $\prod_w \mathfrak{p}_w^{a_w}$ $(a_w \geq 0)$, of index $\prod_w q_w^{a_w}$, and for real $s > 1$

$$\zeta_{\mathcal{O}_T}(s) = \prod_{w \in T} \left(1 - q_w^{-s}\right)^{-1}.$$

For a rational prime p, at most $[E : \mathbb{Q}]$ places of E lie above p, and $q_w \geq p$ for each of them; grouping the factors by p and comparing with the Euler product of the Riemann zeta function gives $\zeta_{\mathcal{O}_T}(s) \leq \zeta(s)^{[E:\mathbb{Q}]}$. For $s \geq 3$,

$$\zeta(s) - 1 \leq 2^{-s} + \int_2^\infty x^{-s}\, dx = 2^{-s}\Big(1 + \frac{2}{s-1}\Big) \leq 2^{1-s}.$$

With $s = m = 3\varrho + 5$ and $n[E : \mathbb{Q}] \leq 2^{2\varrho}$, using $1 + y \leq e^y$ and $e^y \leq 1 + 2y$ for $0 \leq y \leq 1$,

$$\zeta_{\mathcal{O}_T}(m)^n \leq \left(1 + 2^{-3\varrho-4}\right)^{2^{2\varrho}} \leq \exp\left(2^{-\varrho-4}\right) \leq 1 + 2^{-\varrho-3} \leq 1 + 2^{-n}.$$

Step 3: the ranks. $m = 3\varrho + 5 < 9\varrho + 9 = 9n$, and $|\iota| = k - (6\varrho + 13) \geq 1000(2\varrho + 2)^9 - 6\varrho - 13 \geq 53(\varrho + 1) = 53n$.

Step 4: the character is trivial on traces. For $x \in E_T$ let $\tilde{x} \in \mathbb{A}_f$ be its extension by 0. As T is stable under conjugation, $\widetilde{\bar{x}} = \overline{\tilde{x}}$. The standard character ψ_f of $\mathbb{A}_f$ is the standard character of the finite adeles of $\mathbb{Q}$ composed with the trace from E, and the trace is invariant under complex conjugation, so $\psi_f(\bar{a}) = \psi_f(a)$. Since $\bar{\varpi} = -\varpi$, $-\overline{\tilde{x}}/\varpi = \overline{\tilde{x}/\varpi}$, so

$$\psi_T(x + \bar{x}) = \psi_f(-\tilde{x}/\varpi)\, \psi_f\big(\overline{\tilde{x}/\varpi}\big) = \psi_f(-\tilde{x}/\varpi)\, \psi_f(\tilde{x}/\varpi) = 1.$$

Step 5: unimodularity. For a rational prime p let $B_p = \prod_{w \in T,\, w|p} E_w$, a finite étale $\mathbb{Q}_p$-algebra stable under the involution, so that $E_T = \prod_p B_p$ (finitely many p) and $G = \prod_p G_p$ with $G_p = \{x \in \mathrm{GL}_r(B_p) : \bar{x}^{\mathsf{T}} J x = J\}$. The modular function of a finite product of locally compact groups is the product of the modular functions, so it suffices that each G_p is unimodular. Write $X^\dagger = J^{-1}\bar{X}^{\mathsf{T}} J$ for $X \in M_r(B_p)$; then $(XY)^\dagger = Y^\dagger X^\dagger$, $X^{\dagger\dagger} = X$, and $G_p = \{x : x^\dagger x = 1\}$. Let $\mathfrak{g}_p = \{X : X^\dagger = -X\}$ and $\mathfrak{h}_p = \{X : X^\dagger = X\}$; as 2 is invertible in B_p, $M_r(B_p) = \mathfrak{g}_p \oplus \mathfrak{h}_p$. The Cayley transform $X \mapsto (1 + X)(1 - X)^{-1}$ is an analytic bijection from a neighbourhood of 0 in $\mathfrak{g}_p$ onto a neighbourhood of 1 in G_p, with inverse $x \mapsto (x - 1)(x + 1)^{-1}$ (if $X^\dagger = -X$, then $((1 + X)(1 - X)^{-1})^\dagger = (1 + X)^{-1}(1 - X)$ is the inverse, the factors being polynomials in X). Hence G_p is a $\mathbb{Q}_p$-analytic group with Lie algebra $\mathfrak{g}_p$ and adjoint action $\mathrm{Ad}(x)X = xXx^{-1}$, and its modular function is $x \mapsto |\det \mathrm{Ad}(x)|_p^{\pm 1}$, the determinant of the $\mathbb{Q}_p$-linear map $\mathrm{Ad}(x)$ of $\mathfrak{g}_p$ (Bourbaki, *Lie groups and Lie algebras*, Ch. III, §3, no. 16). For $x \in G_p$, conjugation by x preserves $\mathfrak{g}_p$ and $\mathfrak{h}_p$, since $(xXx^{-1})^\dagger = xX^\dagger x^{-1}$. Let δ_p be the component in B_p of the skew unit of P; multiplication by the central unit δ_p maps $\mathfrak{g}_p$ onto $\mathfrak{h}_p$ (as $(\delta_p X)^\dagger = \bar{\delta}_p X^\dagger = \delta_p X$ for $X \in \mathfrak{g}_p$) and commutes with conjugation by x, so the determinants of conjugation by x on $\mathfrak{g}_p$ and on $\mathfrak{h}_p$ agree. On $M_r(B_p)$, conjugation by x is left multiplication by x followed by right multiplication by x^{-1}, whose $\mathbb{Q}_p$-determinants are $\mathrm{N}_{B_p/\mathbb{Q}_p}(\det x)^r$ and $\mathrm{N}_{B_p/\mathbb{Q}_p}(\det x)^{-r}$. Therefore $(\det \mathrm{Ad}(x))^2 = 1$, $|\det \mathrm{Ad}(x)|_p = 1$, and G_p is unimodular. Hence G is unimodular: its left Haar measures are right invariant.

Conclusion. By Steps 2 to 5, F is a large lattice datum; its level unit is the image of M, and the character of P is ψ_T. So (P,F) is a filling datum at T (the involution of P is the one induced by complex conjugation, as the definition requires). □

Lemma 5.2.83. *Let $\mathfrak{S}$ be a filling datum over E, T a finite set of finite places of E stable under complex conjugation, and P a structure of finite place on E_T, for the involution induced by complex conjugation (as in Definition 5.2.3), such that some $c \in \mathcal{O}_T$ has $\bar{c}c = -1$. Let ι be a finite set with $|\iota| = k - (6\varrho + 13) \geq 0$. Then there is a place datum F over E_T (with the structure P) of shape $(\varrho, \iota, 3\varrho + 6)$ on J_{lat} read over E_T that is a lattice datum whose level unit is the image of M in E_T.*

Proof. **Step 1: the Gram matrices.** By Definition 5.2.3, J_{lat} is the identity on the x_i ($i \neq k$) plus the block $\begin{pmatrix} 0 & -1 \\ -1 & -N \end{pmatrix}$ on (x_k, z); its determinant is -1, so $|\det J_{\mathrm{lat}}| = 1$, and J_{lat}^{-1} is the identity plus $\begin{pmatrix} N & -1 \\ -1 & 0 \end{pmatrix}$. Hence J and J^{-1} are integral and hermitian, and (PE1) holds as $\bar{M} = M$.

Step 2: an integral hyperbolic basis. Let $v_0 = -x_k$ and $v_0^* = -N\tau x_k + z$. Then $h(v_0, v_0) = 0$, $h(v_0, v_0^*) = -h(x_k, z) = 1$ and $h(v_0^*, v_0^*) = N\bar{\tau} + N\tau - N = 0$. As $k - 1 = |\iota| + 2m$, split the indices $i \neq k$ into m pairs (i_c, i_c'), $1 \leq c \leq m$, and a set identified with ι. For $c \geq 1$ put $v_c = x_{i_c} + \tilde{c}x_{i_c'}$ and $v_c^* = \tau x_{i_c} - \bar{\tau}\tilde{c}x_{i_c'}$, where $\tilde{c} \in \mathcal{O}$ has $\bar{\tilde{c}}\tilde{c} = -1$; then $h(v_c, v_c) = 1 + \bar{\tilde{c}}\tilde{c} = 0$, $h(v_c^*, v_c^*) = \bar{\tau}\tau + \tau\bar{\tau}\bar{\tilde{c}}\tilde{c} = 0$ and $h(v_c, v_c^*) = \tau - \bar{\tau}\bar{\tilde{c}}\tilde{c} = \tau + \bar{\tau} = 1$. Let $o_i = x_i$ for $i \in \iota$. Vectors built on disjoint sets of indices among the x_i ($i \neq k$) are orthogonal, and these are orthogonal to x_k, z; so the relations of a hyperbolic basis hold, all coordinates lie in $\mathcal{O}$, and the Gram matrix of these $k + 1$ vectors is invertible, so they form a basis of the middle space. The z-coordinate of a vector of the middle space is its coefficient at v_0^*, the only basis vector with a z-component (equal to 1).

Step 3: an isotropic dual of z and Eichler maps. Let $u_0 = -x_k$; then $\bar{u}_0^{\mathsf{T}} J$ is the coordinate vector at z (that is, $h(u_0, y) = y_z$ for all y), and $(u_0)_z = 0$. For $e \in V$ with e_z a unit let $E_e = \mathrm{ei}(e)$ be the Eichler map of Lemma 5.2.76 for the finite place A, the split Gram matrix J and $u = u_0$. By (i) of that lemma, $E_e \in G$ and its entries are continuous functions of e on the open set of the e with e_z a unit. *(i) If e lies in the middle space and $h(e,e) = h(z,z)$, then $E_e \in G^{(\varrho)}$ and $E_e e = z$*, by (ii) of Lemma 5.2.76, as the vectors of the middle space have coordinate 0 at the e_a, e_a^*. *(ii) If moreover e is integral and $e_z \in \mathcal{O}^\times$, then $E_e \in \mathrm{K}(1)$*, by (iii) of Lemma 5.2.76, as J, J^{-1} (Step 1) and u_0 are integral.

Step 4: the top floor measure (PE2). For $l = \varrho$ the set $\mathcal{Q}_\varrho$ is $\{1, \dots, m\}$, and the floor basis B_ϱ maps into the middle space. Let ϕ be the flat chart (5.4) of floor ϱ with parameter $\beta = \delta h(z,z)$, so that qv_ϱ is the image of $\lambda = \mathrm{vol} \otimes \mathrm{vol} \otimes \nu$ on $A \times Y_\varrho \times A^-$ under ϕ. For $p = (t, Y, \alpha)$ with t a unit, $e = \phi(p)$ has $e_z = t$ (Step 2) and, with $t_1 = \bar{t}^{-1}\tau(\delta^{-1}\beta - h'_\varrho(Y,Y)) + t\alpha$,

$$h(e,e) = \bar{t}_1 t + \bar{t}t_1 + h'_\varrho(Y,Y) = (\tau + \bar{\tau})\big(\delta^{-1}\beta - h'_\varrho(Y,Y)\big) + \bar{t}(\alpha + \bar{\alpha})t + h'_\varrho(Y,Y) = \delta^{-1}\beta = h(z,z),$$

as $\delta^{-1}\beta$ and $h'_\varrho(Y,Y)$ are fixed by the involution and α is skew. Let $\Psi(p) = E_{\phi(p)}$ if t is a unit and $\Psi(p) = 1$ otherwise, a measurable map, and $\mu_{\mathrm{top}} = \Psi_* \lambda$, an s-finite measure ($\lambda$ is σ-finite). By Step 3 (i), μ_{top} is carried by $G^{(\varrho)}$ and, as the non-units of A are vol-null (PA4), $\Psi(p)^{-1} z = \phi(p)$ for λ-almost every p; so the image of μ_{top} under $g \mapsto g^{-1}z$ is $\phi_* \lambda = \mathrm{qv}_\varrho$. *The lower bound.* On $B = \mathcal{O}^\times \times \{Y \text{ integral}\} \times (\mathcal{O} \cap A^-)$, $\phi(p)$ is integral ($\bar{t}^{-1}, \tau, \delta^{-1}, \beta, \alpha$ and the basis matrix are integral) with $e_z = t \in \mathcal{O}^\times$, so $\Psi(p) \in \mathrm{K}(1)$ by Step 3 (ii). Hence $\mu_{\mathrm{top}}(\mathrm{K}(1)) \geq \lambda(B) = \mathrm{vol}(\mathcal{O}^\times) \cdot 1 \cdot \nu(\mathcal{O} \cap A^-) = \mathrm{vol}(\mathcal{O}^\times)$.

Step 5: the measure of H (PE3). An element of H fixes the e_a, e_a^* and z, so it maps the orthogonal complement of $\langle e_a, e_a^*\rangle$, the middle space, into itself: its matrix differs from 1 only in the rows of the middle indices and the columns x_i. All middle indices have level ϱ. Hence (a) for $h \in H$ and $n \in \mathsf{U}$, $h(n-1)h^{-1}$ has nonzero entries only at (i,j) with $\mathrm{level}(i) < \mathrm{level}(j)$, so H normalises U; and (b) for $n \in \mathsf{U}$, every entry of h is the entry of nh at the same place or that of 1 (for i a middle index and $j = x_l$, $(nh)_{ij} = \sum_l h_{lj} n_{il}$ over middle l, and $n_{il} = \delta_{il}$ there). If $nh \in \mathrm{K}(1)$ with $n \in \mathsf{U}$, $h \in H$, then h is integral by (b), and so is h^{-1}, applying (b) to $(nh)^{-1} = (h^{-1}n^{-1}h)h^{-1}$ with $h^{-1}n^{-1}h \in \mathsf{U}$ by (a); so $h \in \mathrm{K}(1)$. The group H is closed in G and $H \cap \mathrm{K}(1)$ is a compact open subgroup of it; let μ_H be the image in G of the right Haar measure of H giving $H \cap \mathrm{K}(1)$ the mass 1. Then $\mu_H \neq 0$, $\mu_H(G \setminus H) = 0$, μ_H is invariant under right translation by H, and $\mu_H(\mathrm{K}(1)) = 1$. Let $y \in G$ and $S = \{h \in H : nhy \in \mathrm{K}(1) \text{ for some } n \in \mathsf{U}\}$. If $h, h' \in S$, with $nhy, n'h'y \in \mathrm{K}(1)$, then by (a) $(n'h'y)(nhy)^{-1} = n'(h'h^{-1}n^{-1}(h'h^{-1})^{-1})\,h'h^{-1}$ is an element of $\mathrm{K}(1)$ of the form $n''(h'h^{-1})$ with $n'' \in \mathsf{U}$, so $h'h^{-1} \in \mathrm{K}(1)$. Hence $S \subseteq (H \cap \mathrm{K}(1))h_0$ for any $h_0 \in S$, and $\mu_H(\{g \in G : ngy \in \mathrm{K}(1) \text{ for some } n \in \mathsf{U}\}) = \mu_H(S) \leq 1$ (it is 0 if S is empty).

Step 6: the place datum and the lattice datum. Take the hyperbolic basis of Step 2 with any enumeration of ι, the measures of Steps 4 and 5, the level unit M, the box $\mathrm{K}(1)$, the close pairs $\mathcal{P}_{\mathrm{cl}} = \{(\omega_1, \omega_2) : \omega_1,\ M^{-1}(\omega_1 - \omega_2) \text{ integral}\}$ and the positions $\mathcal{P}_{\mathrm{pos}} = \{y^{-1}ny\,k : y \in G,\ n \in \mathsf{U},\ k \in \mathrm{K}(M)\}$. (PE4): $\mathcal{P}_{\mathrm{cl}} = \{(\omega_1, \omega_2) : \omega_1 \in \mathcal{O}^r,\ \omega_2 \in \omega_1 + M\mathcal{O}^r\}$ is open (as M is a unit, $M\mathcal{O}^r$ is open), contains $(0,0)$ and is contained in the compact set $\mathcal{O}^r \times \mathcal{O}^r$ ($M \in \mathcal{O}$); $\mathrm{K}(1) = G \cap \{x : x, J^{-1}\bar{x}^{\mathsf{T}}J \text{ integral}\}$ is closed in the compact set of integral matrices; $\mathcal{P}_{\mathrm{pos}}$ is a union of left translates of the open subgroup $\mathrm{K}(M)$, hence open and measurable. With Steps 1–5 this is a place datum of shape $(\underline{\varrho}, \iota, m)$ on J_{lat}, and each item of a lattice datum holds: integrality (Steps 1, 2 and $M \in \mathcal{O}$, $\bar{M} = M$), box and measures (Steps 4, 5), close pairs and positions (by construction), and the isotropic dual u_0 of z (Step 3). □

Lemma 5.2.84. *Let $\mathfrak{S}$ be an admissible filling datum over E with Weil data, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, and (P, F) a filling datum at T. Let $S_g \subseteq S_0$ and let $d = (d_w)_{w \in S_0}$ be frame vectors. Then there is an intermediate group Z of F such that $m_T \in Z$ for every $m \in K$ with $\mathrm{red}(m) \in \mathrm{Fix}(S_g, d)$, and such that for all sequences p, q of units of E_T the number $\mathcal{F}_0(p, q; W_Z, 1)$ is real and*

$$0 \leq \frac{\mathcal{F}_0(p, q; W_Z, 1)}{\mathrm{mr}(Z)} \leq \frac{1 + b^{-6}}{(b^{6k})^{\#S_g}}.$$

Proof. Let Z be the intermediate group of Proposition 5.2.85; it has the stated property (a), and with $s = \#S_g$,

$$\mathrm{mi}(Z)\,\theta(\mathrm{K})\,(b^{6k})^s \leq \mathrm{mi}(\mathrm{K})\,\theta(Z). \tag{5.29}$$

Step 1: the mass integral of K. Let c be a unit of E_T with $c \in \mathcal{O}_T$ and $c^{-1} \notin \mathcal{O}_T$. Then $c\mathcal{O}_T$ is an open subgroup of $\mathcal{O}_T$ different from $\mathcal{O}_T$ (otherwise $1 = ca$ with $a \in \mathcal{O}_T$), so it has index at least 2 and $|c| = \mathrm{vol}(c\mathcal{O}_T) \leq \frac{1}{2}$. Moreover $n + 2 \cdot 10^9 n^4 \leq n^7 \leq k$. So Lemma 5.5.42 applies to the intermediate group K with $Q = 2$ and $t = n$, and with Lemma 5.5.41 (2) and $2^{-\varrho} \leq b^{-6}$:

$$\mathrm{mi}(\mathrm{K}) \leq \theta(\mathrm{K})\,\theta(\mathrm{K}(M))(\theta(\mathrm{K}) + 2^{-n}) \leq \theta(\mathrm{K})\,\theta(\mathrm{K}(M))(1 + 2 \cdot 2^{-n}) \leq \theta(\mathrm{K})\,\theta(\mathrm{K}(M))\,(1 + b^{-6}).$$

Step 2: the form of floor 0 is the mass integral. By Lemma 5.5.44 (c) W_Z is a cut function: a bounded measurable function with $W_Z(uy) = \psi_T(M^{-1}\sigma(u))W_Z(y)$ for $u \in$

U, vanishing unless $uy \in \mathrm{K}$, the box of F, for some $u \in \mathsf{U}$. By Lemma 5.5.44 (d) for the intermediate group K, $0 < \theta(\mathrm{K}) < \infty$, and $\theta(Z) < \infty$; so Step 1 and (5.29) give $\mathrm{mi}(Z) \le \mathrm{mi}(\mathrm{K})\,\theta(Z)\,\theta(\mathrm{K})^{-1}(b^{6k})^{-s} < \infty$. By Lemma 5.2.90, $\mathrm{fl}^2 \int_G |\mathcal{K}_{W_Z,0}(1,x)|^2\,d\theta(x) = \mathrm{mi}(Z)$, and $\mathrm{fl} \in (0,\infty)$ (a product of powers of $|M| \in (0,\infty)$, M being a unit), so $\int_G |\mathcal{K}_{W_Z,0}(1,x)|^2\,d\theta(x) < \infty$ and the first case of Lemma 5.2.89 applies:

$$\mathcal{F}_0(p,q;W_Z,1)\,\mathrm{fl}^2 = \int_G \big|\mathcal{K}_{W_Z,0}(1,x)\big|^2 \mathrm{fl}^2\,d\theta(x) = \int_G \Big|\int_Z \chi_M(yxy^{-1})\,d\theta(y)\Big|^2 d\theta(x) = \mathrm{mi}(Z).$$

In particular $\mathcal{F}_0(p,q;W_Z,1)$ is real and ≥ 0, and so is its quotient by $\mathrm{mr}(Z) \in (0,\infty)$ (Lemma 5.5.44 (d)).

Step 3: the bound. By Step 2, (5.29) and Step 1,

$$\mathcal{F}_0(p,q;W_Z,1)\,\mathrm{fl}^2\,(b^{6k})^s\,\theta(\mathrm{K}) = \mathrm{mi}(Z)\,\theta(\mathrm{K})\,(b^{6k})^s \le \mathrm{mi}(\mathrm{K})\,\theta(Z) \le \theta(\mathrm{K})\,\theta(\mathrm{K}(M))\,(1+b^{-6})\,\theta(Z).$$

By Lemma 5.5.44 (d), applied to the intermediate group K, $0 < \theta(\mathrm{K}) < \infty$; dividing by $\theta(\mathrm{K})$ and by $(b^{6k})^s\,\mathrm{mr}(Z)\,\mathrm{fl}^2 = (b^{6k})^s\,\theta(Z)\,\theta(\mathrm{K}(M))$, a number in $(0,\infty)$,

$$\frac{\mathcal{F}_0(p,q;W_Z,1)}{\mathrm{mr}(Z)} \le \frac{1+b^{-6}}{(b^{6k})^s}.$$

□

Proposition 5.2.85. *Let $\mathfrak{S}$ be an admissible filling datum over E with Weil data, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, and (P,F) a filling datum at T; let θ be the group measure of F. For $m \in K$ let m_T be the matrix of the components at T of the entries of m in the lattice basis, an element of the unitary group of F. Let $S_g \subseteq S_0$ and let $d = (d_w)_{w\in S_0}$ be frame vectors. Then there is an intermediate group Z of F such that*

(a) $m_T \in Z$ for every $m \in K$ with $\mathrm{red}(m) \in \mathrm{Fix}(S_g, d)$, and

(b) $\mathrm{mi}(Z)\,\theta(\mathrm{K}(1))\,\big(b^{6k}\big)^{\#S_g} \le \mathrm{mi}(\mathrm{K}(1))\,\theta(Z)$.

Proof. Write $\mathrm{K}_U(c)$ for the congruence subgroups of a filling datum at a conjugation-stable set U, θ_U, mi_U for its group measure and mass integral, and m_U for the components of $m \in K$ at U.

Step 1: the chosen places lie in T and split it. If a chosen place w were not in T, then M^{-1} would be integral at w; but $M \in \mathfrak{p}_w$, as $q_w = p_i$ divides M, so $1 = MM^{-1} \in \mathfrak{p}_w$, which is absurd. For $s \in S_g$ put $T_s = T \cap \{s, \bar{s}\} \ni s$, and $T_0 = T \setminus \bigcup_{s\in S_g} T_s$. Since $\bar{s} \notin S_0$ for $s \in S_0$, the sets $\{s,\bar{s}\}$ ($s \in S_g$) are pairwise disjoint; so T is the disjoint union of the T_s and T_0, each stable under conjugation. Let $I = S_g$ if $T_0 = \emptyset$ and $I = S_g \sqcup \{0\}$ otherwise. By Lemma 5.2.82, for $i \in I$ there is a filling datum (P_i, F_i) at T_i; its involution, like that of P, is the one induced by complex conjugation (Definition 5.2.3).

Step 2: products. $E_T = \prod_{i\in I} E_{T_i}$, with the componentwise involution (the T_i are conjugation-stable) and $\mathcal{O}_T = \prod_i \mathcal{O}_{T_i}$. The Haar measure of P is the product of those of the P_i: both are Haar measures of E_T giving $\mathcal{O}_T$ mass 1. For $x \in E_T$ with components x_i, the extensions by zero satisfy $\tilde{x} = \sum_i \tilde{x}_i$, so $\psi_T(x) = \prod_i \psi_{T_i}(x_i)$ since ψ_f is additive. The matrix J_{lat} has integer entries, so the split Gram matrix of F_i is the i-th component of that of F, and the level unit of F_i, the image of M in E_{T_i}, is the i-th component of that of F. So Lemma 5.2.88 applies to F and the F_i.

Step 3: lifting from T_i. For each i, $m \mapsto m_{T_i}$ maps K onto $\mathrm{K}_{T_i}(1)$. Indeed, m and m^{-1} have integral entries at every finite place, so $m_{T_i} \in \mathrm{K}_{T_i}(1)$. Conversely, for $y \in \mathrm{K}_{T_i}(1)$ let $m \in \mathrm{GL}_r(\mathbb{A}_f)$ have components y_w at $w \in T_i$ and 1 elsewhere; its entries and those of its inverse are integral everywhere, and it is unitary, because the unitarity of y over E_{T_i} is the unitarity of the components at the places of T_i (the involution pairs w with $\bar{w} \in T_i$) and the identity is unitary; so $m \in K$ and $m_{T_i} = y$.

Step 4: one chosen place. Let $s \in S_g$, $d_s = (\epsilon, v)$, $u = \sum_a v_a f_a \in k_s^r$; $u \neq 0$ as $v \neq 0$ and the f_a are distinct basis vectors. Let $H_s = \{m \in K : \mathrm{red}(m) \in \mathrm{Fix}(s, d_s)\}$ and $Z_s = \{m_{T_s} : m \in H_s\}$. The map $\bar{G} \to \mathrm{GL}_r(k_s)$, $t \mapsto \phi_\epsilon(t_s)$, is a surjective homomorphism, and $\mathrm{Fix}(s, d_s)$ is the preimage of the stabiliser $\mathrm{St}(u)$ of u; so $\mathrm{Fix}(s, d_s)$ is a subgroup, $H_s = \mathrm{red}^{-1}(\mathrm{Fix}(s, d_s))$ is a subgroup of K and, by (FA2),

$$[K : H_s] = [\bar{G} : \mathrm{Fix}(s, d_s)] = [\mathrm{GL}_r(k_s) : \mathrm{St}(u)] = p_0^r - 1,$$

the last equality by the orbit–stabiliser theorem, $\mathrm{GL}_r(k_s)$ acting transitively on the $p_0^r - 1$ nonzero vectors of k_s^r ($q_s = p_0$). By Step 3, Z_s is a subgroup of $\mathrm{K}_{T_s}(1)$, the image of H_s under the surjection $\pi\colon m \mapsto m_{T_s}$ of K onto $\mathrm{K}_{T_s}(1)$. If $m_{T_s} = 1$ then $m_s = 1$ and $\mathrm{red}(m)_s = 1$, so $\ker \pi \leq H_s$, $\pi^{-1}(Z_s) = H_s$ and $[\mathrm{K}_{T_s}(1) : Z_s] = [K : H_s] = p_0^r - 1$. If $y \in \mathrm{K}_{T_s}(M)$ (the level unit of F_s is the image of M), lift it to $m \in K$ by Step 3: $M^{-1}(m_s - 1)$ is integral and $M \in \mathfrak{p}_s$, so $m_s \equiv 1$ modulo $\mathfrak{p}_s$, $\mathrm{red}(m)_s = 1$ fixes u, $m \in H_s$ and $y \in Z_s$. Hence $\mathrm{K}_{T_s}(M) \leq Z_s \leq \mathrm{K}_{T_s}(1)$: Z_s is an intermediate group of F_s.

Now apply Lemma 5.2.86 to F_s over $E_{T_s} = \prod_{w \in T_s} E_w$: $\#T_s \leq 2 \leq b$; $q_s = p_0 \geq b^{10}$ and $q_{\bar{s}} = q_s$; $3 \cdot 10^{10} n^4 \leq 3n^6 \leq n^7 \leq k$; and the index of Z_s is $p^r - 1$ with $p = p_0 \geq b^{10}$. Hence

$$\mathrm{mi}_{T_s}(Z_s)\, \theta_{T_s}(\mathrm{K}_{T_s}(1))\, b^{6k} \leq \mathrm{mi}_{T_s}(\mathrm{K}_{T_s}(1))\, \theta_{T_s}(Z_s).$$

If $0 \in I$, put $Z_0 = \mathrm{K}_{T_0}(1)$, an intermediate group of F_0, for which the same inequality with the factor 1 in place of b^{6k} is an equality.

Step 5: the product group. Let $Z = \prod_{i \in I} Z_i$, read in the unitary group of F through the components. By Lemma 5.2.88 (1), Z is an intermediate group of F, and $\prod_i \mathrm{K}_{T_i}(1) = \mathrm{K}(1)$ (integrality is componentwise). By Lemma 5.2.88 (2) with $Y_i = Z_i$, $W_i = \mathrm{K}_{T_i}(1)$, $c_s = b^{6k}$ ($s \in S_g$) and $c_0 = 1$,

$$\mathrm{mi}(Z)\, \theta(\mathrm{K}(1))\, (b^{6k})^{\#S_g} \leq \mathrm{mi}(\mathrm{K}(1))\, \theta(Z),$$

which is (b). For (a), let $m \in K$ with $\mathrm{red}(m) \in \mathrm{Fix}(S_g, d) = \bigcap_{s \in S_g} \mathrm{Fix}(s, d_s)$. Then $m \in H_s$ and $m_{T_s} \in Z_s$ for every $s \in S_g$, and $m_{T_0} \in \mathrm{K}_{T_0}(1) = Z_0$ by Step 3; so $m_T \in Z$. □

Lemma 5.2.86. *Let F be a large lattice datum over a finite place $A = \prod_{w \in T} A_w$, a product over a finite set T of non-archimedean local fields A_w with residue fields of cardinality q_w, and let* vol *be its group measure. Let $b \geq 2$ be a real number with $\#T \leq b$ and $q_w \geq b^{10}$ for all $w \in T$, and suppose $3 \cdot 10^{10} n^4 \leq k$. Let Z be an intermediate group whose index in $\mathrm{K}(1)$ is $p^r - 1$ for an integer $p \geq b^{10}$. Then*

$$\mathrm{mi}(Z)\, \mathrm{vol}(\mathrm{K}(1))\, b^{6k} \leq \mathrm{mi}(\mathrm{K}(1))\, \mathrm{vol}(Z).$$

Proof. Write $\mathrm{K} = \mathrm{K}(1)$ and $B = b^9$, so $B \geq 2^9 \geq 2$.

Step 1: the non-units are few. An element $x \in \mathcal{O}_A$ is a unit of $\mathcal{O}_A$ if and only if every component x_w is a unit of $\mathcal{O}_w$. Hence $\mathcal{O}_A \setminus \mathcal{O}_A^\times = \bigcup_{w \in T} P_w$ with $P_w = \{x \in \mathcal{O}_A : x_w \in \mathfrak{p}_w\}$. Each P_w is an additive subgroup of index q_w in $\mathcal{O}_A$, so $\mathcal{O}_A$ is the disjoint union of q_w translates

of P_w, all of measure $\mathrm{vol}(P_w)$ (the measure of A is a Haar measure), and $\mathrm{vol}(P_w) = q_w^{-1} \le b^{-10}$. Therefore

$$\mathrm{vol}(\mathcal{O}_A \setminus \mathcal{O}_A^\times) \le \#T\, b^{-10} \le b^{-9} = B^{-1} \le \tfrac{1}{2}.$$

If c is a unit of A with $c \in \mathcal{O}_A$ and $c^{-1} \notin \mathcal{O}_A$, then $c\mathcal{O}_A \subseteq \mathcal{O}_A \setminus \mathcal{O}_A^\times$ (if $ca \in \mathcal{O}_A^\times$ with $a \in \mathcal{O}_A$, then $c^{-1} = a(ca)^{-1} \in \mathcal{O}_A$), so $|c| = \mathrm{vol}(c\mathcal{O}_A) \le B^{-1}$.

Step 2: the exponent. Put $t = k - 10^{10}n^4 \ge 0$. Then $t+1+n+2\cdot 10^9 n^4 = k - 8\cdot 10^9 n^4 + 1 + n \le k$; $9t - 6k = 3k - 9\cdot 10^{10} n^4 \ge 0$ by $k \ge 3\cdot 10^{10} n^4$; and $t + 1 \le k < r$.

Step 3: Z is small. By Lemma 5.5.41 (1), vol is a Haar measure, so K, the disjoint union of $[\mathrm{K} : Z]$ left cosets of Z, has $\mathrm{vol}(\mathrm{K}) = [\mathrm{K} : Z]\,\mathrm{vol}(Z)$. As $t + 2 \le r$ and $p \ge 2$, $[\mathrm{K} : Z] = p^r - 1 \ge p^{t+2} - 1 \ge p^{t+1}$; as $p \ge b^{10} \ge B$,

$$B^{t+1}\mathrm{vol}(Z) \le \mathrm{vol}(\mathrm{K}).$$

Step 4: the upper bound for Z. By Step 1, Lemma 5.5.42 applies with $Q = B$ and the exponent $t + 1 + n$ (Step 2):

$$\mathrm{mi}(Z) \le \mathrm{vol}(Z)\,\mathrm{vol}(\mathrm{K}(M))\,\big(\mathrm{vol}(Z) + B^{-(t+1)}B^{-n}\big).$$

By Step 1 and Lemma 5.2.87, $B^{-n} \le 2^{-n} \le \mathrm{vol}(\mathrm{K})$, so $\mathrm{mi}(Z) \le \mathrm{vol}(Z)\mathrm{vol}(\mathrm{K}(M))(\mathrm{vol}(Z) + B^{-(t+1)}\mathrm{vol}(\mathrm{K}))$.

Step 5: the lower bound for K. By Lemma 5.5.44 (e) for the intermediate group K, $\mathrm{mi}(\mathrm{K}) \ge \mathrm{vol}(\mathrm{K})^2\mathrm{vol}(\mathrm{K}(M))$.

Step 6: conclusion. By Step 2 and $b \ge 1$, $b^{6k} \le b^{9t} = B^t$. With $z = \mathrm{vol}(Z)$ and $v_K = \mathrm{vol}(\mathrm{K})$, both in $(0,\infty)$ by Lemma 5.5.44 (d), Step 3 gives $zB^t \le v_K/B$, hence

$$(z + B^{-(t+1)}v_K)B^t = zB^t + v_K/B \le 2v_K/B \le v_K.$$

Therefore, by Steps 4 and 5,

$$\mathrm{mi}(Z)\, v_K\, b^{6k} \le z\,\mathrm{vol}(\mathrm{K}(M))(z + B^{-(t+1)}v_K)B^t\, v_K \le z\, v_K^2\,\mathrm{vol}(\mathrm{K}(M)) \le \mathrm{mi}(\mathrm{K})\, z.$$

□

Lemma 5.2.87. *Let F be a lattice datum over a finite place A. Then $\theta(\mathrm{K}(1)) \ge \mathrm{vol}(\mathcal{O}_A^\times)^{\varrho+1}$. In particular, if $\mathrm{vol}(\mathcal{O}_A \setminus \mathcal{O}_A^\times) \le \frac{1}{2}$, then $\theta(\mathrm{K}(1)) \ge 2^{-n}$.*

Proof. **Step 1: convolution and a group.** For s-finite measures μ, μ' on G, $(\mu * \mu')(\mathrm{K}) = (\mu \otimes \mu')\{(x,y) : xy \in \mathrm{K}\} \ge (\mu \otimes \mu')(\mathrm{K} \times \mathrm{K}) = \mu(\mathrm{K})\mu'(\mathrm{K})$, as K is a group.

Step 2: the Heisenberg measures. Let $l < \varrho$, $v \in V_{l+1}$ and $\beta \in A^-$. By (5.8) the matrix of $n_l(v,\beta)$ differs from the identity only in the column of e_l^*, which is $e_l^* + v + (\beta - \tau h(v,v))e_l$, and in the row of e_l, whose extra entries are those of $-\bar{v}^{\mathsf{T}}J$; as v has no coordinates at e_l, e_l^*, the coordinates of v and the number $\beta - \tau h(v,v)$ are entries. Hence, J and τ being integral, $n_l(v,\beta)$ is integral if and only if v is integral and β is integral; and an integral element of G has integral inverse $J^{-1}\bar{x}^{\mathsf{T}}J$. So $n_l(v,\beta) \in \mathrm{K}$ if and only if $v \in \mathcal{O}_A^{\{\text{coordinates of floor} \ge l+1\}}$ and $\beta \in \mathcal{O}_A \cap A^-$, and

$$\eta_l(\mathrm{K}) = \mathrm{vol}(\mathcal{O}_A)^{\dim V_{l+1}} \cdot \nu(\mathcal{O}_A \cap A^-) = 1.$$

For $l \ge \varrho$, $\eta_l = \delta_1$ and $\eta_l(\mathrm{K}) = 1$.

Step 3: the Weyl-element measures. Let $l < \varrho$. For $c \in \mathcal{O}_A^\times$ the matrix of $w_l(c)$ ((5.7)) has the entries c, $\bar{c}^{-1}$, 1 and 0, so $w_l(c) \in \mathrm{K}$ and $w_l(c)^{-1} \in \mathrm{K}$; and $|c| = 1$. Therefore

$$\mathrm{wl}_l(\mathrm{K}) = \int_A \mathbf{1}_{\mathrm{K}}(w_l(c)^{-1})\,|c|^{d_l}\,dc \ge \int_{\mathcal{O}_A^\times} |c|^{d_l}\,dc = \mathrm{vol}(\mathcal{O}_A^\times).$$

Step 4: the product. By definition $\theta = (\eta_{\varrho+1} * \theta_{\varrho+1}) * (\eta_\varrho * \theta_\varrho) * (\eta_{\varrho-1} * \theta_{\varrho-1}) * \cdots * (\eta_0 * \theta_0)$ with $\theta_{\varrho+1} = \mu_H$, θ_ϱ the top-floor measure and $\theta_l = \mathrm{wl}_l * \eta_l$ for $l < \varrho$. Applying Step 1 repeatedly and Steps 2, 3,

$$\theta(\mathrm{K}) \geq \eta_{\varrho+1}(\mathrm{K})\, \mu_H(\mathrm{K})\, \eta_\varrho(\mathrm{K})\, \theta_\varrho(\mathrm{K}) \prod_{l<\varrho} \eta_l(\mathrm{K})\, \mathrm{wl}_l(\mathrm{K})\, \eta_l(\mathrm{K}) \geq \mathrm{vol}(\mathcal{O}_A^\times)^{\varrho+1}.$$

If $\mathrm{vol}(\mathcal{O}_A \setminus \mathcal{O}_A^\times) \leq \frac{1}{2}$, then $\mathrm{vol}(\mathcal{O}_A^\times) \geq \mathrm{vol}(\mathcal{O}_A) - \frac{1}{2} = \frac{1}{2}$, and $\theta(\mathrm{K}) \geq 2^{-(\varrho+1)} = 2^{-n}$. □

Lemma 5.2.88. *Let I be a finite set and $A = \prod_{i\in I} A_i$ a product of finite places, with the componentwise involution, $\mathcal{O}_A = \prod_i \mathcal{O}_{A_i}$, the product Haar measure and the character $\psi(x) = \prod_i \psi_i(x_i)$. Let F be a large lattice datum over A, with split Gram matrix J, unitary group G, level unit M and group measure* vol*; for each i let F_i be a large lattice datum over A_i with split Gram matrix J_i the i-th component of J, unitary group G_i, level unit M_i the i-th component of M, and group measure* vol_i*. Identify G with $\prod_i G_i$ by taking components of matrix entries. Then:*

1. *there is $t \in (0,\infty)$ such that for all intermediate groups Y_i of F_i ($i \in I$) the group $Y = \prod_i Y_i$ is an intermediate group of F with*

$$\mathrm{vol}(Y) = t \prod_i \mathrm{vol}_i(Y_i), \qquad \mathrm{mi}(Y) = t^3 \prod_i \mathrm{mi}_i(Y_i);$$

2. *if Y_i, W_i are intermediate groups of F_i and $c_i \in [0,\infty)$ satisfy $\mathrm{mi}_i(Y_i)\,\mathrm{vol}_i(W_i)\,c_i \leq \mathrm{mi}_i(W_i)\,\mathrm{vol}_i(Y_i)$ for every i, then, with $Y = \prod_i Y_i$ and $W = \prod_i W_i$,*

$$\mathrm{mi}(Y)\,\mathrm{vol}(W) \prod_i c_i \leq \mathrm{mi}(W)\,\mathrm{vol}(Y).$$

Proof. **Step 1: the groups are products.** A matrix over A is the tuple of its components over the A_i, and $\bar{x}^\mathsf{T} J x = J$ holds if and only if it holds in every component; so $x \mapsto (x_i)_i$ is an isomorphism of topological groups $G \to \prod_i G_i$, through which we identify. The basis, the frame and the isotropic flag are the same in every component, so $\mathsf{U} = \prod_i \mathsf{U}_i$ and $\sigma(u) = (\sigma_i(u_i))_i$. Integrality is componentwise, so $\mathrm{K}(c) = \prod_i \mathrm{K}_i(c_i)$ for every unit c of A; in particular $\mathrm{K}(M) = \prod_i \mathrm{K}_i(M_i)$ and $\mathrm{K}(1) = \prod_i \mathrm{K}_i(1)$, and $Y = \prod_i Y_i$ is an intermediate group of F whenever each Y_i is one of F_i.

Step 2: the constant t. By Lemma 5.5.41 (1), vol and every vol_i are Haar measures. The product $\bigotimes_i \mathrm{vol}_i$ of Haar measures on the finitely many second countable locally compact groups G_i is a Haar measure on $\prod_i G_i = G$. By the uniqueness of Haar measure up to a positive factor, $\mathrm{vol} = t \bigotimes_i \mathrm{vol}_i$ with $t \in (0,\infty)$. For intermediate groups Y_i, $\mathrm{vol}(\prod_i Y_i) = t \prod_i \mathrm{vol}_i(Y_i)$.

Step 3: the character is a product. We claim $\chi_M(y) = \prod_i \chi_{M_i}(y_i)$ for $y \in G$. If $y \in u\mathrm{K}(M)$ with $u \in \mathsf{U}$, then $y_i \in u_i \mathrm{K}_i(M_i)$ with $u_i \in \mathsf{U}_i$ for all i, and by Lemma 5.5.44 (b) (independence of the choice) $\prod_i \chi_{M_i}(y_i) = \prod_i \psi_i(M_i^{-1}\sigma_i(u_i)) = \psi(M^{-1}\sigma(u)) = \chi_M(y)$. If $y \notin \mathsf{U}\mathrm{K}(M)$, then some $y_i \notin \mathsf{U}_i \mathrm{K}_i(M_i)$: otherwise choosing $u_i \in \mathsf{U}_i$ with $u_i^{-1} y_i \in \mathrm{K}_i(M_i)$ for every i, $u = (u_i)_i \in \mathsf{U}$ has $u^{-1}y \in \mathrm{K}(M)$. So both sides vanish.

Step 4: the mass integral. Let Y_i be intermediate groups and $Y = \prod_i Y_i$. The sets Y_i have finite measure (Lemma 5.5.44 (d)) and the integrands are continuous and bounded by 1, so Fubini's theorem and Step 3 give, for $x \in G$,

$$\int_Y \chi_M(yxy^{-1})\, d\mathrm{vol}(y) = t \prod_i \int_{Y_i} \chi_{M_i}(y_i x_i y_i^{-1})\, d\mathrm{vol}_i(y_i).$$

Put $f_i(z) = \left|\int_{Y_i} \chi_{M_i}(yzy^{-1})\, d\mathrm{vol}_i(y)\right|^2 \geq 0$, a continuous function on G_i. Then by Tonelli's theorem

$$\mathrm{mi}(Y) = \int_G t^2 \prod_i f_i(x_i)\, d\mathrm{vol}(x) = t^3 \int_{\prod_i G_i} \prod_i f_i(x_i)\, d\Big(\bigotimes_i \mathrm{vol}_i\Big)(x) = t^3 \prod_i \mathrm{mi}_i(Y_i).$$

This proves item 1.

Step 5: item 2. By item 1, and since multiplication in $[0,\infty]$ is commutative, associative and monotone,

$$\mathrm{mi}(Y)\mathrm{vol}(W)\prod_i c_i = t^4 \prod_i (\mathrm{mi}_i(Y_i)\mathrm{vol}_i(W_i)c_i) \leq t^4 \prod_i (\mathrm{mi}_i(W_i)\mathrm{vol}_i(Y_i)) = \mathrm{mi}(W)\mathrm{vol}(Y).$$

□

Lemma 5.2.89. *Let a place datum be given, $W\colon G \to \mathbb{C}^I$ cut and p, q sequences of units of A. If $\int_G |\mathcal{K}_{W,0}(1,z)|^2\, d\theta(z) < \infty$, then*

$$\mathcal{F}_0(p,q;W,1) = \int_G |\mathcal{K}_{W,0}(1,z)|^2\, d\theta(z);$$

otherwise $\mathcal{F}_0(p,q;W,1) = 0$.

Proof. For $j = 0$ the head density $\mathrm{hdn}_0(p,q)$ is an empty product, 1; the Weyl elements $w_{0,p}$, $w_{0,q}$ are 1, so $d = 1$ and $\mathcal{K}_W^d = \mathcal{K}_W$; and $\mathfrak{m}_0 = \delta_1$. By Lemma 5.2.57, (iii),

$$\mathrm{FI}_0(p,q,W,x;1)(g) = \mathcal{K}_{W,0}(g,gx)\,\overline{\mathcal{K}_{W,0}(g,gx)} = |\mathcal{K}_{W,0}(g,gx)|^2.$$

As $\theta \otimes \delta_1$ is the image of θ under $z \mapsto (z,1)$, the integrand of (5.15) is $\theta\otimes\delta_1$-integrable exactly when $z \mapsto |\mathcal{K}_{W,0}(1,z)|^2$ is θ-integrable, and then $\mathcal{F}_0(p,q;W,1) = \int |\mathcal{K}_{W,0}(1,z)|^2\, d\theta(z)$; otherwise $\mathcal{F}_0(p,q;W,1) = 0$ by the convention in (5.15). □

Lemma 5.2.90. *Let F be a large lattice datum over a finite place A, with level unit M and group measure θ, and let Z be an intermediate group. Then for every $x \in G$*

$$\Big|\int_Z \chi_M(yxy^{-1})\, d\theta(y)\Big| = |\mathcal{K}_{W_Z,0}(1,x)|\,\mathrm{fl}.$$

Proof. Put $\eta = \eta_{<\varrho+2} = \eta_0 * \cdots * \eta_{\varrho+1}$. By Lemma 5.2.59 (iv) with $j = \varrho+2$, $\theta = \mathfrak{m}_{\varrho+2} = \eta * \theta_{\geq 0}$. Each Heisenberg element $n_l(v,\beta)$ lies in U: $n_l(v,\beta) - 1$ sends e_l^* to $v + (\beta - \tau h(v,v))e_l$ and each coordinate vector w of V_{l+1} to $-h(v,w)e_l$, vectors of lower level, and kills the others. So every η_l, and hence η, is carried by U. Fix $x \in G$ and let $f(z) = \mathbf{1}_Z(z)\chi_M(zxz^{-1})$; $|f| \leq \mathbf{1}_Z$ and $\theta(Z) < \infty$ (Lemma 5.5.44 (d)).

Step 1: unfolding along U. We show that for every $y \in G$

$$\int_G f(uy)\, d\eta(u) = \mathrm{fl}\,\langle W_Z(y), W_Z(yx)\rangle. \tag{5.30}$$

For a representative c let $S_c = \{u \in G : uyc^{-1} \in \mathrm{K}(M)\}$ and $a_c = \overline{W_Z(y)_c}\, W_Z(yx)_c$. Let $u \in \mathsf{U}$. If $uy \notin Z$, then $u \notin S_c$ for every c (otherwise $uy = (uyc^{-1})c \in \mathrm{K}(M)Z = Z$), and $f(uy) = 0 = \sum_c a_c \mathbf{1}_{S_c}(u)$. If $uy \in Z$, then uy lies in exactly one coset $\mathrm{K}(M)d$, so $u \in S_c$ exactly for $c = d$. Then $yd^{-1} = u^{-1}k$ with $k = uyd^{-1} \in \mathrm{K}(M)$, so $W_Z(y)_d = \chi_M(yd^{-1}) = \psi(M^{-1}\sigma(u^{-1}))$; and by Lemma 5.5.44 (b), $W_Z(yx)_d = \chi_M(u^{-1}\cdot uyxd^{-1}) = \psi(M^{-1}\sigma(u^{-1}))\chi_M(uyx(uy)^{-1}\,k) =$

$\psi(M^{-1}\sigma(u^{-1}))\chi_M(uyx(uy)^{-1})$. As $|\psi|=1$, $a_d=f(uy)$, and again $f(uy)=\sum_c a_c\mathbf{1}_{S_c}(u)$. Since η is carried by U, $f(uy)=\sum_c a_c\mathbf{1}_{S_c}(u)$ for η-almost every u. By Lemma 5.5.43 with $\theta'=\delta_1$ and the element yc^{-1}, $\eta(S_c)=\mathrm{fl}$ if $u'yc^{-1}\in\mathrm{K}(M)$ for some $u'\in\mathsf{U}$, and $\eta(S_c)=0$ otherwise; in particular $\eta(S_c)\le\mathrm{fl}<\infty$ (Lemma 5.5.44 (d)). If $W_Z(y)_c\neq0$, then by the definition of χ_M some $u'\in\mathsf{U}$ has $u'^{-1}yc^{-1}\in\mathrm{K}(M)$, so $\eta(S_c)=\mathrm{fl}$; if $W_Z(y)_c=0$, then $a_c=0$. Integrating the finite sum term by term,

$$\int_G f(uy)\,d\eta(u)=\sum_c a_c\,\eta(S_c)=\mathrm{fl}\sum_c\overline{W_Z(y)_c}\,W_Z(yx)_c=\mathrm{fl}\,\langle W_Z(y),W_Z(yx)\rangle.$$

Step 2: integration over the tower. As $|f|\le\mathbf{1}_Z$ and $\int_G\int_G\mathbf{1}_Z(uy)\,d\eta(u)\,d\theta_{\ge0}(y)=(\eta*\theta_{\ge0})(Z)=\theta(Z)<\infty$, f is θ-integrable and Fubini's theorem for the convolution gives

$$\begin{aligned}\int_Z\chi_M(yxy^{-1})\,d\theta(y)&=\int_G f\,d(\eta*\theta_{\ge0})=\int_G\int_G f(uy)\,d\eta(u)\,d\theta_{\ge0}(y)\\&=\mathrm{fl}\int_G\langle W_Z(y),W_Z(yx)\rangle\,d\theta_{\ge0}(y)\end{aligned}$$

by (5.30). The last integrand is $\theta_{\ge0}$-integrable: by (5.30), $\mathrm{fl}\,|\langle W_Z(y),W_Z(yx)\rangle|\le\int_G\mathbf{1}_Z(uy)\,d\eta(u)$, whose $\theta_{\ge0}$-integral is $\theta(Z)<\infty$, and $\mathrm{fl}\in(0,\infty)$. Hence the last integral is $\mathcal{K}_{W_Z,0}(1,x)$, and

$$\int_Z\chi_M(yxy^{-1})\,d\theta(y)=\mathrm{fl}\,\mathcal{K}_{W_Z,0}(1,x);$$

taking absolute values gives the statement. □

Lemma 5.2.91. *Let F be a large lattice datum over a finite place A with level unit M, ϱ pairs, $n=\varrho+1$ and k middle coordinates, with $10^8n^4\le k$. Let Z be an intermediate group, $j\le\varrho$, and p,q sequences of units of A. Then*

$$\int_G\mathrm{FM}_j(p,q,W_Z,x)\,d\theta(x)<\infty.$$

Proof. If $\mathrm{FM}_j(p,q,W_Z,x)=0$ for every $x\in G$, the integral is 0. Otherwise Lemma 5.5.40 gives a unit α of A with $\alpha\in\mathcal{O}_A$ and $\bar\alpha=\alpha$ such that, for the gauge Γ of scale α, the sequences p and q are heads of length j for Γ and, if $j\neq0$, M^{-1} has size 1 for Γ. Apply Lemma 5.5.36 to Z, j, $r=0$ (as $10^8n^4\le k$), α, p, q, the function $f=\mathrm{FM}_j(p,q,W_Z,\cdot)$ and $e=1$; the condition on e is void since $e\not\le1$. With the loss $\Lambda=2/|\alpha|$ of Γ it gives

$$\mathrm{hdn}_j(p,q)\int_G\mathrm{FM}_j(p,q,W_Z,x)\,d\theta(x)\le\mathrm{mr}(Z)\,\Lambda^{10^9n^4}.$$

Here $\mathrm{mr}(Z)<\infty$ by Lemma 5.5.44 (d); $\Lambda<\infty$, as α is a unit in $\mathcal{O}_A$, so that $[\alpha]$ is a positive integer and $|\alpha|=[\alpha]^{-1}>0$; and $\mathrm{hdn}_j(p,q)=\prod_{a<j}(|p_aq_a|^{1/2})^{d_a}$ is a positive real number, since the modulus of a unit lies in $(0,\infty)$. Hence the integral is finite. □

Proposition 5.2.92. *Let a place datum be given, $W\colon G\to\mathbb{C}^I$ cut, p,q sequences of units of A, $t\in\mathbb{C}$, and $\phi\colon A\to\mathbb{C}$ measurable with $|\phi|\le1$. For $0\le b\le\varrho$ write $Z_b(x)=h(f_b,xf_b)$ for $x\in G$, and let $d=w_{b,q}^{-1}w_{b,p}$.*

(i) (General floor chart.) Let $0\le b\le\varrho$ with $\int_G\mathrm{FM}_b(p,q,W,x)\,d\theta(x)<\infty$. Let ζ be an s-finite measure on G carried by $R^{(b+1)}\cap G^{(b)}$, $\rho\colon A\to[0,\infty]$ measurable and almost everywhere

finite, and $v\colon A \to G$ *measurable, such that* θ_b *is the convolution of the image of* $\rho(c)\,dc$ *under* $c \mapsto v(c)^{-1}$ *with* ζ*. Put* $\nu = \theta \otimes ((\zeta * \theta_b) * \mathfrak{m}_b)$ *and, for* $c \in A$*,*

$$\Psi(c) = t\,\mathrm{hdn}_b(p,q)\,\rho(c)\,\phi(Z_b(v(c))) \cdot \int_{G\times G} \mathcal{K}_{W,b+1}(g,gx)\,\overline{\mathcal{K}^d_{W,b+1}(w_c^{-1}g, w_c^{-1}gx)}\,d\nu(x,g), \qquad w_c = w_{b,p}\,v(c),$$

the integral being 0 *where its integrand is not* ν*-integrable. Then* Ψ *is integrable on* A *and* $\int_A \Psi(c)\,dc = t\,\mathcal{F}_b(p,q;W,\phi\circ Z_b)$*.*

(ii) (Descent.) Let $a < \varrho$ *with* $\int_G \mathrm{FM}_a(p,q,W,x)\,d\theta(x) < \infty$*. For* $c \in A$ *let* $u(c) = c$ *if* c *is a unit and* $u(c) = 1$ *otherwise, and let* $p[a \mapsto u(c)]$ *be* p *with its entry* a *replaced by* $u(c)$*. Then* $c \mapsto \phi(c)\,t\,\mathcal{F}_{a+1}(p[a\mapsto u(c)], q[a\mapsto u(c)]; W, 1)$ *is integrable on* A*, and*

$$\int_A \phi(c)\,t\,\mathcal{F}_{a+1}(p[a\mapsto u(c)], q[a\mapsto u(c)]; W, 1)\,dc = t\,\mathcal{F}_a(p,q;W,\phi\circ Z_a).$$

Proof. **Three invariances.** Let $0 \le b \le \varrho$ and $H_b = R^{(b+1)} \cap G^{(b)}$. (a) Every $n \in H_b$ fixes f_b (Lemma 5.2.58, (vi)), so, n being an isometry, $Z_b(ny) = h(n^{-1}f_b, yf_b) = Z_b(y)$ and $Z_b(yn) = Z_b(y)$ for all y. (b) By Lemma 5.2.57, (ii), $\mathcal{K}_W$ and $\mathcal{K}^d_W$ are pairing kernels at floor b; by Lemma 5.2.64, (v) (with $l = b+1$, $j = b$) their partial correlations from floor $b+1$ are unchanged when both arguments are multiplied on the left by an element of H_b, and by Lemma 5.2.64, (i), they are measurable. (c) $w_{b,p}$ commutes with every element of $G^{(b)}$ (Lemma 5.2.58, (ix)).

(i). Write $w = w_{b,p}$, $\chi = \phi \circ Z_b$, and, for $x, g, y \in G$,

$$\Pi(w', x, g) = \mathcal{K}_{W,b+1}(g, gx)\,\overline{\mathcal{K}^d_{W,b+1}(w'^{-1}g, w'^{-1}gx)}.$$

Define on G^4 and on $G\times G\times A$ the measurable functions

$$F(x,g,\sigma,\tau) = \chi(\sigma\tau^{-1})\,\Pi(\sigma w\tau^{-1}, x, \sigma g), \qquad S(x,g',c) = \chi(v(c))\,\Pi(wv(c), x, g'),$$

and the measures $\mu_F = \theta\otimes\mathfrak{m}_b\otimes\theta_b\otimes\theta_b$ and $\mu_S = \nu\otimes\rho(c)\,dc$. Since $(\sigma w\tau^{-1})^{-1}\sigma g = \tau w^{-1} g$, (5.14) reads $\mathrm{FI}_b(p,q,W,x;\chi)(g) = \int\!\int F(x,g,\sigma,\tau)\,d\theta_b(\tau)\,d\theta_b(\sigma)$.

Invariance in τ*.* For $s \in H_b$: $F(x,g,\sigma,s\tau) = F(x,g,\sigma,\tau)$, by (a) (as $\sigma\tau^{-1}s^{-1}$) and (b) (as $(\sigma w\tau^{-1}s^{-1})^{-1}\sigma g = s\tau w^{-1} g$).

Substitution. For $\sigma \in G^{(b)}$, $n \in H_b$ and $c \in A$: $F(x,g,\sigma,v(c)^{-1}n\sigma) = S(x, n\sigma g, c)$. Indeed $\sigma\tau^{-1} = n^{-1}v(c)$ for $\tau = v(c)^{-1}n\sigma$, so the weight is $\chi(n^{-1}v(c)) = \chi(v(c))$ by (a); $\sigma w\tau^{-1} = \sigma w\sigma^{-1}n^{-1}v(c) = wn^{-1}v(c)$ by (c); and, by (b) and (c), $\mathcal{K}_{W,b+1}(\sigma g, \sigma g x) = \mathcal{K}_{W,b+1}(n\sigma g, n\sigma g x)$ and $(wn^{-1}v(c))^{-1}\sigma g = v(c)^{-1}w^{-1}n\sigma g = (wv(c))^{-1}(n\sigma g)$.

Equal laws. Let $f\colon \mathbb{C} \to [0,\infty]$ be Borel, and $x, g \in G$, $\sigma \in G^{(b)}$. Since ζ is carried by H_b, the substitution and the description of θ_b give

$$\begin{aligned}\iint f(S(x,n\sigma g,c))\,\rho(c)\,dc\,d\zeta(n) &= \iint f(F(x,g,\sigma,v(c)^{-1}n\sigma))\rho(c)\,dc\,d\zeta(n)\\ &= \int f(F(x,g,\sigma,\tau\sigma))\,d\theta_b(\tau),\end{aligned}$$

and by the invariance in τ and Lemma 5.2.61, (i) (with $S = G^{(b)}$ and $s = \sigma \in G^{(b)} \subseteq R^{(b)}$), the last integral is $\int f(F(x,g,\sigma,\tau))\,d\theta_b(\tau)$. Integrate over σ against θ_b (carried by $G^{(b)}$, Lemma 5.2.59, (ii)), over g against $\mathfrak{m}_b$ and over x against θ: as $n\sigma g$ has law $(\zeta * \theta_b) * \mathfrak{m}_b$ under

$\zeta \otimes \theta_b \otimes \mathfrak{m}_b$, Tonelli gives $\int f \circ S \, d\mu_S = \int f \circ F \, d\mu_F$. So S under μ_S and F under μ_F have the same law.

Integrability. With $|\chi| \le 1$ and Tonelli,

$$\begin{aligned}\int |F| \, d\mu_F &\le \int_G \int_G \Big(\int |\mathcal{K}_{W,b+1}(\sigma g, \sigma g x)| \, d\theta_b(\sigma) \Big) \\ &\qquad \cdot \Big(\int |\mathcal{K}^d_{W,b+1}(\tau w^{-1} g, \tau w^{-1} g x)| \, d\theta_b(\tau) \Big) d\mathfrak{m}_b(g) \, d\theta(x) \\ &= \int \mathrm{FM}_b(p,q,W,x) \, d\theta(x) < \infty.\end{aligned}$$

Hence $F \in L^1(\mu_F)$, so $S \in L^1(\mu_S)$, and $\int S \, d\mu_S = \int F \, d\mu_F$.

Both sides. By Fubini for F, the inner integrals defining $\mathrm{FI}_b(p,q,W,x;\chi)(g)$ are integrals for $\theta \otimes \mathfrak{m}_b$-almost every (x,g), the function $\mathrm{FI}_b(p,q,W,\cdot\,;\chi)(\cdot)$ is $\theta \otimes \mathfrak{m}_b$-integrable, and $\mathrm{hdn}_b(p,q) \int F \, d\mu_F = \mathcal{F}_b(p,q;W,\chi)$ by (5.15). By Fubini for S, the function $c \mapsto \rho(c) \int S(\cdot,\cdot,c) \, d\nu$ is integrable on A with integral $\int S \, d\mu_S$ (ρ being finite almost everywhere), and for almost every c the integrand $S(\cdot,\cdot,c)$ is ν-integrable, so $t \, \mathrm{hdn}_b(p,q) \rho(c) \int S(\cdot,\cdot,c) \, d\nu = \Psi(c)$. Therefore Ψ is integrable and $\int \Psi = t \, \mathrm{hdn}_b(p,q) \int S \, d\mu_S = t \, \mathcal{F}_b(p,q;W,\chi)$.

(ii). Apply (i) with $b = a$, $v(c) = w_a(c)$, $\rho(c) = |c|^{d_a}$ and $\zeta = \eta_a$. By Definition 5.2.1, $\theta_a = \mathrm{wl}_a * \eta_a$ is of the required form; ρ is finite at units, hence almost everywhere (PA4); and η_a is carried by the Heisenberg elements, which lie in $G^{(a)} \cap N_{<a+1} \subseteq H_a$ (Lemma 5.2.58, (iii), (i)). It remains to identify $\Psi(c)$ for a unit c. Put $p' = p[a \mapsto c]$, $q' = q[a \mapsto c]$. Then: $\mathrm{hdn}_{a+1}(p',q') = \mathrm{hdn}_a(p,q) \, (|c^2|^{1/2})^{d_a} = \mathrm{hdn}_a(p,q) \, \rho(c)$; $w_{a+1,p'} = w_{a,p} \, w_a(c)$ and $w_{a+1,q'} = w_{a,q} \, w_a(c)$ (the factors act on different pairs), so $w^{-1}_{a+1,q'} w_{a+1,p'} = w_a(c)^{-1} d \, w_a(c) = d$, as the diagonal element d (Lemma 5.2.58, (ix)) is the identity on e_a, e_a^*; $Z_a(w_a(c)) = h(e_a, c e_a^*) = c$; and $\mathfrak{m}_{a+1} = (\eta_a * \theta_a) * \mathfrak{m}_a$, so $\nu = \theta \otimes \mathfrak{m}_{a+1}$. By Lemma 5.2.57, (iii), at floor $a+1$,

$$\mathrm{FI}_{a+1}(p',q',W,x;1)(g) = \mathcal{K}_{W,a+1}(g, gx) \, \overline{\mathcal{K}^d_{W,a+1}(w_c^{-1} g, w_c^{-1} g x)}, \qquad w_c = w_{a,p} w_a(c),$$

so, by (5.15) and the same convention for non-integrable integrands, $\phi(c) \, t \, \mathcal{F}_{a+1}(p',q';W,1) = t \, \mathrm{hdn}_a(p,q) \, \rho(c) \, \phi(Z_a(w_a(c))) \int \mathrm{FI}_{a+1} \, d\nu = \Psi(c)$. As $u(c) = c$ for every unit c, the function in (ii) agrees with Ψ almost everywhere, and (i) gives the claim. □

Lemma 5.2.93. *Let $\mathfrak{S}$ be a filling datum of shape (ϱ, k) with prime N, A a finite place such that some $c \in A$ satisfies $\bar{c}c = -1$, F a large lattice datum over A on J_{lat} read over A, π a unit of A with $\pi\bar{\pi} = N$, Z an intermediate group, $(\mu^{\mathrm{nf}}_\beta)$ the family of norm-fibre measures on A^k, p, q sequences of units of A and $\xi \in A$. For $t \in A$ let $u(t) = t$ if t is a unit and $u(t) = 1$ otherwise. Then the function*

$$t \longmapsto \psi(\xi t) \, \mathrm{led}\big(\pi, Z, p[\varrho \mapsto u(t)], q[\varrho \mapsto u(t)]\big)$$

is integrable on A, and its integral is $\mathcal{F}_\varrho(p,q;W_Z,\omega_\xi)$ with the weight $\omega_\xi(x) = \psi(\xi \, h(z, xz))$.

Proof. Let Φ be the step function of Lemma 5.2.74, (v), for (p,q), and $g(e) = \psi(\xi h(z,e)) \Phi(e)$, a continuous compactly supported function on V.

Step 1: the last entry. As F is large, $\dim X_0 + 1 = |\iota| + 2(m+1) + 2$ and $|\iota| \ge 53n$, so $\dim X_0 \ge 2$, and Lemma 5.2.78 applies: $s(t) = |\pi|^{-1} \int g(e(\pi,t,\omega)) \, d\mu^{\mathrm{nf}}_{\beta(t)}(\omega)$ is integrable on A with $\int_A s(t) \, dt = \int_V g \, d\mathrm{qv}_\varrho$. Let t be a unit and $p' = p[\varrho \mapsto t]$, $q' = q[\varrho \mapsto t]$. The scaled orbital integrals (FB1) read a sequence only through $w_{\varrho,\cdot}$ and $\kappa_\varrho(\cdot,1)$, that is through its entries of index $< \varrho$; so $\mathrm{SO}_{p'} = \mathrm{SO}_p$, $\mathrm{SO}_{q'} = \mathrm{SO}_q$ and $\mathrm{fp}_{p',q'} = \mathrm{fp}_{p,q}$. By the last sentence of Lemma 5.2.78,

$e(\pi,t,\omega)\in Q$ for $\mu^{\mathrm{nf}}_{\beta(t)}$-almost every ω, where $\Phi=\mathrm{fp}_{p,q}$, and $h(z,e(\pi,t,\omega))=t$. By (FB4), with $p'_\varrho=t$,

$$s(t)=\psi(\xi t)\,|\pi|^{-1}\int \mathrm{fp}_{p',q'}(e(\pi,t,\omega))\,d\mu^{\mathrm{nf}}_{\beta(t)}(\omega)=\psi(\xi t)\,\mathrm{led}(\pi,Z,p',q').$$

So the function of the statement agrees with s almost everywhere: it is integrable, with integral $\int_V g\,d\mathrm{qv}_\varrho$.

Step 2: the floor form. The weight ω_ξ is measurable, $|\omega_\xi|=1$, and $\omega_\xi(gxk)=\omega_\xi(x)$ for $g,k\in H$, since H fixes z and preserves h: $h(z,gxkz)=h(g^{-1}z,xz)=h(z,xz)$. By Lemma 5.2.94, with $\mathfrak{m}_{\varrho,2}$ the image of $\mu_H\otimes\mu_{\mathrm{top}}$ under multiplication and Fubini,

$$\mathcal{F}_\varrho(p,q;W_Z,\omega_\xi)=\iint \omega_\xi((k\tau)^{-1})\langle \mathrm{SO}_p(k\tau),\mathrm{SO}_q(k\tau)\rangle\,d\mu_H(k)\,d\mu_{\mathrm{top}}(\tau).$$

For μ_{top}-almost every τ, $\tau\in G^{(\varrho)}$ (PE2), and for μ_H-almost every k, $k\in H$; then

$$\omega_\xi((k\tau)^{-1})=\psi(\xi h(z,\tau^{-1}k^{-1}z))=\psi(\xi h(z,\tau^{-1}z)),$$

and by Lemma 5.2.74, (i), the inner integral is

$$\psi(\xi h(z,\tau^{-1}z))\,\mathrm{fp}_{p,q}(\tau^{-1}z)=g(\tau^{-1}z),$$

as $\tau^{-1}z\in Q$ (Lemma 5.2.75, (i)). Since g is continuous with compact support and qv_ϱ is finite on compact sets (it is a member of a disintegration, (DJ1)), g is qv_ϱ-integrable, and by (PE2)

$$\mathcal{F}_\varrho(p,q;W_Z,\omega_\xi)=\int_G g(\tau^{-1}z)\,d\mu_{\mathrm{top}}(\tau)=\int_V g\,d\mathrm{qv}_\varrho.$$

With Step 1 the lemma follows. □

Lemma 5.2.94. *Let F be a large lattice datum over a finite place A, Z an intermediate group, p,q sequences of units of A, and $\phi\colon G\to\mathbb{C}$ measurable with $|\phi|\le 1$ and $\phi(gxk)=\phi(x)$ for all $g,k\in H$ and $x\in G$. Then $\mathfrak{m}_{\varrho,2}=\mu_H*\mu_{\mathrm{top}}$, the function $l\mapsto\phi(l^{-1})\langle\mathrm{SO}_p(l),\mathrm{SO}_q(l)\rangle$ is $\mathfrak{m}_{\varrho,2}$-integrable, and*

$$\mathcal{F}_\varrho(p,q;W_Z,\phi)=\int_G \phi(l^{-1})\,\langle\mathrm{SO}_p(l),\mathrm{SO}_q(l)\rangle\,d\mathfrak{m}_{\varrho,2}(l).$$

Proof. **Step 0: the measures.** By Definition 5.2.1, $\eta_\varrho=\eta_{\varrho+1}=\delta_1$, $\theta_\varrho=\mu_{\mathrm{top}}$, $\theta_{\varrho+1}=\mu_H$, so $\mathfrak{m}_{\varrho,2}=\mu_H*\mu_{\mathrm{top}}=\theta_{\ge\varrho}$ and $\theta_{\ge\varrho+1}=\mu_H$; and $\theta=\mathfrak{m}_{0,\varrho+2}=\mathfrak{m}_{\varrho,2}*\mathfrak{m}_\varrho$ (associativity of convolution). By Lemma 5.2.59, (ii), $\mathfrak{m}_{\varrho,2}$, μ_{top} and μ_H are carried by $G^{(\varrho)}$, $G^{(\varrho)}$ and H.

Step 1: the unfolded integrand is integrable. For $l,y_1,y_2\in G$ put

$$U(l,y_1,y_2)=\langle W(y_1),W(y_2)\rangle\,\langle W(lw_q^{-1}y_2),W(lw_p^{-1}y_1)\rangle,\qquad \Phi(l,y_1,y_2)=\phi(l^{-1})\,U(l,y_1,y_2),$$

measurable functions (W is continuous by Lemma 5.5.44, (c)). By Lemma 5.2.57, (ii), and (PK4) for $\theta_{\ge\varrho}=\mathfrak{m}_{\varrho,2}$, there is $c_0<\infty$ with $\int|\langle W(ka),W(kb)\rangle|\,d\mathfrak{m}_{\varrho,2}(k)\le c_0$ for all a,b; with $a=w_q^{-1}y_2$, $b=w_p^{-1}y_1$, $\int|U(k,y_1,y_2)|\,d\mathfrak{m}_{\varrho,2}(k)\le c_0|\langle W(y_1),W(y_2)\rangle|$. For $k\in G^{(\varrho)}$, $U(k,y_1,y_2)\ne 0$ forces $y_1,y_2\in\mathcal{S}$ by the size clause of (DN0)(f) of Theorem 5.4.1 (with $x=k^{-1}$, $\mu=\theta$ a Haar measure and W continuous). By Tonelli, $\int|\Phi|\,d(\mathfrak{m}_{\varrho,2}\otimes\theta\otimes\theta)\le c_0\int_{\mathcal{S}\times\mathcal{S}}|\langle W(y_1),W(y_2)\rangle|\,d(\theta\otimes\theta)<\infty$.

Step 2: the right side. By Fubini, $l \mapsto \int \Phi(l,\cdot)\, d(\theta\otimes\theta)$ is $\mathfrak{m}_{\varrho,2}$-integrable with integral $\int\Phi$, and for $l \in G^{(\varrho)}$, hence $\mathfrak{m}_{\varrho,2}$-almost everywhere, Lemma 5.2.50, (v), gives $\int \Phi(l,\cdot)\, d(\theta\otimes\theta) = \mathrm{hdn}_\varrho(p,q)^{-1}\phi(l^{-1})\langle \mathrm{SO}_p(l), \mathrm{SO}_q(l)\rangle$. So the function of the statement is integrable and

$$\int_G \phi(l^{-1})\langle \mathrm{SO}_p(l), \mathrm{SO}_q(l)\rangle \, d\mathfrak{m}_{\varrho,2}(l) = \mathrm{hdn}_\varrho(p,q) \int \Phi \, d(\mathfrak{m}_{\varrho,2}\otimes\theta\otimes\theta). \tag{5.31}$$

Step 3: change of variables. Let $M_6 = \mu_H\otimes\mu_{\mathrm{top}}\otimes\mu_H\otimes\mu_{\mathrm{top}}\otimes\theta\otimes\mathfrak{m}_\varrho$ in the variables (k,τ,g,σ,x,y) and $T(k,\tau,g,\sigma,x,y) = (k\tau(g\sigma)^{-1},\, g\sigma y,\, g\sigma yx)$. For every measurable $f \ge 0$ on G^3,

$$\int f\, d(\mathfrak{m}_{\varrho,2}\otimes\theta\otimes\theta) = \int f\circ T\, dM_6. \tag{5.32}$$

Indeed: by left invariance of θ, $\iint f(l,y_1,y_2)\, d\theta(y_2)\, d\theta(y_1) = \iint f(l,y_1,y_1x)\, d\theta(x)\, d\theta(y_1)$; by Step 0, y_1 may be written $g\sigma y$ with (g,σ,y) distributed by $\mu_H\otimes\mu_{\mathrm{top}}\otimes\mathfrak{m}_\varrho$, and almost surely $s = g\sigma \in G^{(\varrho)}$; for such s, Lemma 5.2.61, (iii) (with $l=\varrho$, $i=2$) gives $\int f'(l)\, d\mathfrak{m}_{\varrho,2}(l) = \int f'(ls^{-1})\, d\mathfrak{m}_{\varrho,2}(l)$, and $l = k\tau$ with (k,τ) distributed by $\mu_H\otimes\mu_{\mathrm{top}}$. Tonelli assembles these into (5.32). Now for M_6-almost every point $(k, g\in H,\, \tau,\sigma\in G^{(\varrho)})$, with $l = k\tau s^{-1}$, $s = g\sigma$ and $\zeta = k\tau w_p^{-1}y$: $\phi(l^{-1}) = \phi(g\sigma\tau^{-1}k^{-1}) = \phi(\sigma\tau^{-1})$; as the Weyl elements commute with $G^{(\varrho)}$ (Lemma 5.2.58, (ix)), $lw_p^{-1}(sy) = \zeta$ and $lw_q^{-1}(syx) = k\tau w_q^{-1}yx = d\zeta x$. Hence $\Phi\circ T = \Psi$ almost everywhere, where

$$\Psi(k,\tau,g,\sigma,x,y) = \phi(\sigma\tau^{-1})\,\langle W(g\sigma y), W(g\sigma yx)\rangle\,\langle W(d\zeta x), W(\zeta)\rangle.$$

By (5.32) for $|\Phi|$ and Step 1, Ψ is M_6-integrable; applying (5.32) to the positive and negative parts of the real and imaginary parts of Φ, $\int\Phi\, d(\mathfrak{m}_{\varrho,2}\otimes\theta\otimes\theta) = \int\Psi\, dM_6$.

Step 4: the left side. For fixed (τ,σ,x,y), (PK4) for $\mathcal{K}_W$ and $\mathcal{K}_W^d$ (Lemma 5.2.57, (ii), tower measure $\theta_{\ge\varrho+1} = \mu_H$) shows that the functions

$$g \mapsto \langle W(g\sigma y), W(g\sigma yx)\rangle \qquad \text{and} \qquad k \mapsto \langle W(k\tau w_p^{-1}y), W(dk\tau w_p^{-1}yx)\rangle$$

are μ_H-integrable, with integrals $\mathcal{K}_{W,\varrho+1}(\sigma y,\sigma yx)$ and $\overline{\mathcal{K}^d_{W,\varrho+1}(\tau w_p^{-1}y,\tau w_p^{-1}yx)}$ by (5.12). So the integral of Ψ over (k,g) is $\phi(\sigma\tau^{-1})\mathcal{K}_{W,\varrho+1}(\sigma y,\sigma yx)\overline{\mathcal{K}^d_{W,\varrho+1}(\tau w_p^{-1}y,\tau w_p^{-1}yx)}$. By Fubini for the integrable Ψ, for $(\theta\otimes\mathfrak{m}_\varrho)$-almost every (x,y) this function of (τ,σ) is $\mu_{\mathrm{top}}\otimes\mu_{\mathrm{top}}$-integrable, and its integral, which by (5.14) with $\theta_\varrho = \mu_{\mathrm{top}}$ is $\mathrm{FI}_\varrho(p,q,W,x;\phi)(y)$, is integrable in (x,y) with integral $\int\Psi\, dM_6$. By (5.15), $\mathcal{F}_\varrho(p,q;W,\phi) = \mathrm{hdn}_\varrho(p,q)\int\Psi\, dM_6$. With Step 3 and (5.31) the lemma follows. □

Lemma 5.2.95. *Let $\mathfrak{S}$ be a filling datum of shape (ϱ,k) with prime N, A a finite place such that some $c\in A$ satisfies $\bar{c}c = -1$, F a large lattice datum over A on J_{lat} read over A, π a unit with $\pi\bar{\pi} = N$, Z an intermediate group and $(\mu_\beta^{\mathrm{nf}})$ the family of norm-fibre measures on A^k.*

(i) Let J be a finite set, $(\omega_a)_{a\in J}$ sequences of units of A with a common entry $\omega_a(\varrho) = t$, and R a positive semidefinite hermitian $J\times J$ matrix. Then $\sum_{a,b\in J} R_{ab}\,\mathrm{led}(\pi,Z,\omega_a,\omega_b)$ is a real number ≥ 0. In particular the matrix $\big(\mathrm{led}(\pi,Z,\omega_a,\omega_b)\big)_{a,b\in J}$ is positive semidefinite.

(ii) The function $(p,q)\mapsto \mathrm{led}(\pi,Z,p,q)$ on pairs of sequences of units is continuous.

Proof. As F is large, $k \ge |\iota| \ge 53n \ge 2$, so Lemma 5.2.78 applies.

(i). Write $\mathrm{fp}_{ab} = \mathrm{fp}_{\omega_a,\omega_b}$. First, for every $e\in V$, $\sum_{a,b} R_{ab}\mathrm{fp}_{ab}(e)$ is real and ≥ 0. If e is not of the form $l^{-1}z$ with $l\in G^{(\varrho)}$, all terms vanish by (FB2). If $e = l^{-1}z$, then by Lemma 5.2.74, (i),

$\sum_{a,b} R_{ab}\mathrm{fp}_{ab}(e) = \int \sum_{a,b} R_{ab}\langle \mathrm{SO}_{\omega_a}(kl), \mathrm{SO}_{\omega_b}(kl)\rangle \, d\mu_H(k)$, a finite sum of integrable functions, and for each k, with the vectors $v_x = (\mathrm{SO}_{\omega_a}(kl)_x)_{a\in J}$, $x \in \mathrm{Cos}(Z)^2$,

$$\sum_{a,b} R_{ab}\langle \mathrm{SO}_{\omega_a}(kl), \mathrm{SO}_{\omega_b}(kl)\rangle = \sum_x \bar{v}_x^{\mathsf{T}} R v_x \geq 0,$$

so the integral is real and ≥ 0. Next, by (FB4) with the common last entry t, all densities are integrals against the same measure $\mu^{\mathrm{nf}}_{\beta(t)}$ at the same points $e(\pi, t, \omega)$, and each $\omega \mapsto \mathrm{fp}_{ab}(e(\pi, t, \omega))$ is integrable: it agrees almost everywhere with $\omega \mapsto \Phi_{ab}(e(\pi, t, \omega))$, where Φ_{ab} is the step function of Lemma 5.2.74, (v), because $e(\pi, t, \omega) \in Q$ almost everywhere (Lemma 5.2.78); this function is continuous with compact support ($\omega \mapsto e(\pi, t, \omega) = \sum_i \omega_i x_i - N^{-1}tz$ is affine and injective, and reading off the coefficients of the x_i is a continuous left inverse, so preimages of compact sets are compact), hence integrable by Lemma 5.2.79, (iii). So

$$\sum_{a,b} R_{ab}\,\mathrm{led}(\pi, Z, \omega_a, \omega_b) = |\pi|^{-1} \int \sum_{a,b} R_{ab}\,\mathrm{fp}_{ab}(e(\pi, t, \omega))\, d\mu^{\mathrm{nf}}_{\beta(t)}(\omega),$$

the integral of a real nonnegative integrable function, times $|\pi|^{-1} \in (0, \infty)$. For the matrix: for $c \in \mathbb{C}^J$ the matrix $R_{ab} = \bar{c}_a c_b$ is positive semidefinite, so $\sum_{a,b} \bar{c}_a c_b \,\mathrm{led}(\pi, Z, \omega_a, \omega_b) \geq 0$ for all c, which is positive semidefiniteness (and implies that the matrix is hermitian).

(ii). Fix (p_0, q_0), let Φ be the step function of Lemma 5.2.74, (v), for (p_0, q_0) and $\mathcal{V}$ a neighbourhood of 0 with $\Phi(x + v) = \Phi(x)$ for $v \in \mathcal{V}$. By Lemma 5.2.50, (iv) (with $s = 1$), there are neighbourhoods $\mathcal{N}_p \ni p_0$, $\mathcal{N}_q \ni q_0$ with $\mathrm{SO}_p = \mathrm{SO}_{p_0}$ and $\mathrm{SO}_q = \mathrm{SO}_{q_0}$ on $G^{(\varrho)}$ for $p \in \mathcal{N}_p$, $q \in \mathcal{N}_q$; by (FB2) and Lemma 5.2.74, (i), then $\mathrm{fp}_{p,q} = \mathrm{fp}_{p_0,q_0}$ on V. The vector $e(\pi, t, \omega) - e(\pi, t_0, \omega) = -N^{-1}(t - t_0)z$ depends continuously on t and vanishes at $t = t_0 = p_0(\varrho)$; shrink $\mathcal{N}_p$ so that it lies in $\mathcal{V}$ for $t = p(\varrho)$, $p \in \mathcal{N}_p$. For such (p, q), with $t = p(\varrho)$, almost every ω has $e(\pi, t, \omega) \in Q$, so

$$\mathrm{fp}_{p,q}(e(\pi, t, \omega)) = \mathrm{fp}_{p_0,q_0}(e(\pi, t, \omega)) = \Phi(e(\pi, t, \omega)) = \Phi(e(\pi, t_0, \omega)),$$

and $\mathrm{led}(\pi, Z, p, q) = |\pi|^{-1} \int \Phi(e(\pi, t_0, \omega))\, d\mu^{\mathrm{nf}}_{\beta(p(\varrho))}(\omega)$. The integrand is a fixed continuous compactly supported function of ω, so by Lemma 5.2.79, (iii), this is a continuous function of $\beta(p(\varrho))$, which depends continuously on p. So led agrees near (p_0, q_0) with a continuous function, and is continuous at (p_0, q_0). □

Lemma 5.2.96. *Let F be a large lattice datum over a finite place A with level unit M, ϱ pairs, $n = \varrho + 1$ and middle coordinates x_i, $i \in \{1, \dots, k\}$; let Z be an intermediate group, $j \leq \varrho$, and $t, K \geq 0$ integers with $t + 10^{10}n^4 \leq k$ and $10^{10}n^4 \leq K$. Let p, q be sequences of units of A, ξ a unit of A, and $\omega\colon G \to \mathbb{C}$ a weight of one of two kinds:*

(a) $\omega(x) = \psi(\xi\, h(f_j, xf_j))$; or

(b) $j = i + 1$ for some i, $\omega = 1$, and $\xi = M^{-1}(p_i^{-1} - q_i^{-1})$.

Then

$$|\mathcal{F}_j(p, q; W_Z, \omega)| \leq \mathrm{mr}(Z)\,[\xi]^{t/2}\,[\mathcal{O}_A : \mathcal{O}_A \cap \xi^{-1}\mathcal{O}_A]^{-6} \left(\frac{2}{|M|}\right)^K \mathrm{hw}_{K,j}(p)\,\mathrm{hw}_{K,j}(q).$$

Proof. Put $W = W_Z$, a cut function (Lemma 5.5.44, (c)); θ is a Haar measure (Lemma 5.5.41, (1)); $|\omega| \leq 1$.

Step 0. By (DN0)(a) of Theorem 5.4.1, $|\mathcal{F}_j(p,q;W,\omega)| \le \mathrm{hdn}_j(p,q)\int \mathrm{FN}_j(p,q,W,x;\omega)\,d\theta(x)$ and $\mathrm{FN}_j(p,q,W,\cdot;\omega) \le \mathrm{FM}_j(p,q,W,\cdot)$. If $\mathrm{FM}_j(p,q,W,\cdot)$ vanishes identically, then $\mathcal{F}_j = 0$ and there is nothing to prove. Otherwise let α, the gauge Γ of scale α and $\Lambda = 2/|\alpha|$ be as in Lemma 5.5.40, and $\beta = \alpha^{200n^2}$, a unit in $\mathcal{O}_A$; an element z has size $200n^2$ for Γ exactly if $\beta z \in \mathcal{O}_A$.

Step 1: the gain unit. We find units η, w with

$$\begin{aligned} &w \in \mathcal{O}_A, \quad \xi w \in \mathcal{O}_A, \quad \xi\eta^{-1} \in \mathcal{O}_A, \quad \min(1,e)^2 \le |w|\,|\beta|^{-1}, \\ &\mathrm{FN}_j(p,q,W,x;\omega)^2 \le e^{2r'}\,\mathrm{WSP}_j(p,q,W,x)\ (x \in \mathcal{P}_{\mathrm{pos}}), \end{aligned} \tag{5.33}$$

where $e = [\eta]^{1/2}$ and WSP_j is formed for Γ. *Case (a).* A is a finite product of local fields A_v with valuations v. Define η componentwise: $\eta_v = \xi_v$ if $v(\xi_v) \le v(\beta_v)$, and $\eta_v = \xi_v + \beta_v$ otherwise (then $v(\eta_v) = v(\beta_v) < v(\xi_v)$). So η is a unit, $\eta = \xi + c\beta$ with $c \in \mathcal{O}_A$ (components 0 or 1), $\xi\eta^{-1} \in \mathcal{O}_A$ and $\eta^{-1}\beta \in \mathcal{O}_A$. For z of size $200n^2$, $c\beta z \in \mathcal{O}_A$, so $\psi(\eta z) = \psi(\xi z)\psi(c\beta z) = \psi(\xi z)$. By (DN3)(a) of Theorem 5.4.1, applied with the unit η and the frequency ξ of ω (and p a head of length j by Lemma 5.5.40, (i)), $\mathrm{FN}_j(p,q,W,x;\omega)^2 \le (1/|\eta|)^{r'}\mathrm{WSP}_j = e^{2r'}\mathrm{WSP}_j$ at positions. Put $w = \eta^{-1}\beta$: $w \in \mathcal{O}_A$, $\xi w = (\xi\eta^{-1})\beta \in \mathcal{O}_A$, and $|w|\,|\beta|^{-1} = |\eta|^{-1} = e^2 \ge \min(1,e)^2$. *Case (b).* Put $\eta = \xi$ and $w = \beta$. By Lemma 5.5.40, (ii), M^{-1} and $p_i^{-1} - q_i^{-1}$ have size 1, so ξ has size 2 by (SZ1), hence size $200n^2$ ($\alpha \in \mathcal{O}_A$), that is $\xi w \in \mathcal{O}_A$; $\xi\eta^{-1} = 1$, $|w||\beta|^{-1} = 1 \ge \min(1,e)^2$, and (DN3)(b) gives the last inequality of (5.33).

Step 2: the shells. Apply Lemma 5.5.36 to $f = \mathrm{FN}_j(p,q,W,\cdot;\omega) \le \mathrm{FM}_j$, the gain e and $r = t + 12$ ($t + 12 + 10^8 n^4 \le k$), with the heads and the size of M^{-1} from Lemma 5.5.40, (i), (ii). If $e < 1$, then $e^{2r'} \le e^{2s}$ as $s \le r'$, so the gain hypothesis holds by (5.33). Hence

$$|\mathcal{F}_j(p,q;W,\omega)| \le \min(1,e)^{t+12}\,\mathrm{mr}(Z)\,\Lambda^{10^9 n^4}.$$

Step 3: from η to ξ. Since $\xi\eta^{-1} \in \mathcal{O}_A$, $[\xi] = [\xi\eta^{-1}][\eta] \ge [\eta]$, so $\min(1,e) \le e \le [\xi]^{1/2}$. Since $w\mathcal{O}_A \subseteq \mathcal{O}_A \cap \xi^{-1}\mathcal{O}_A$, $|w| = \mathrm{vol}(w\mathcal{O}_A) \le \mathrm{vol}(\mathcal{O}_A \cap \xi^{-1}\mathcal{O}_A) = D(\xi)^{-1}$. And $|\beta|^{-1} = |\alpha|^{-200n^2} \le \Lambda^{200n^2}$. By (5.33),

$$\min(1,e)^{t+12} = \min(1,e)^t\big(\min(1,e)^2\big)^6 \le [\xi]^{t/2}\,D(\xi)^{-6}\,\Lambda^{1200n^2}.$$

Step 4: the loss. With $E = 1200n^2 + 10^9 n^4$ one has $4E \le 4800n^4 + 4\cdot 10^9 n^4 \le 10^{10} n^4 \le K$, so by Lemma 5.5.40, (iii), $\Lambda^E \le (2/|M|)^K \mathrm{hw}_{K,j}(p)\mathrm{hw}_{K,j}(q)$. Combining Steps 2–4 gives the bound. □

Proposition 5.2.97. *Let $\mathfrak{S}$ be admissible and T a finite set of finite places.*

(i) Products. Sieve systems on $\mathbb{C}$ at the places v with common weights, times a sieve system on a ring C with an s-finite measure, form a sieve system (the product of Definition 5.1.5).

(ii) Sieve. Let $A \subseteq E^\times$ be finite with t, t^{-1} T-integers for $t \in A$, M^{-1} a T-integer, $U \subseteq A$, and J^T a sieve system of length ϱ on E_T (with the Haar measure giving $\mathcal{O}_T$ measure 1) with u the unit-or-one map, $\iota(w) = M^{-1}w^{-1}$, the character ψ_T, mass bound m_T, weights Λ^T and decay β_T. For heads $p \in A^\varrho$ and $t \in A$ let $\mathrm{pad}(p,t) = (p_0,\dots,p_{\varrho-1},t,1,1,\dots)$, $x_v(p,t)_a = \mathrm{fs}(v,a)v(\mathrm{pad}(p,t)_a)$ and $y(p,t) = \mathrm{pad}(p,t)$ in $E_T^\times$. Then

$$\begin{aligned} &\mathrm{Re}\sum_{t\in U}\sum_{p,q\in A^\varrho}\prod_v J_n^{(v)}(x_v(p,t),x_v(q,t))\,J_n^T(y(p,t),y(q,t)) \\ &\qquad \le \prod_{a<n}\mathrm{pc}(a)\Big[\prod_v \frac{1+b^{-10}}{\deg_v}\,m_T + b^{2403n^9 f}\mathrm{dc}\Big]. \end{aligned}$$

Proof. (i) Let $(J_{v,i}, \psi_v, E_{0,v}, \Lambda', \beta_v)$, $v \in \Phi$, be sieve systems of length n on $\mathbb{C}$ (with $u_\infty(c) = c$ for $c \neq 0$, $u_\infty(0) = 1$, $\iota_\infty(z) = z^{-1}$, $\psi_v(c) = \exp(c - \bar{c})$) with the common weights Λ', and $(J_{0,i}, \psi_0, E_{0,0}, \Lambda^0, \beta_0)$ one of length n on (C_0, U_0, u_0, ι_0), the measure of C_0 being s-finite. Form the product data of Definition 5.1.5 on $\mathbb{C}^\Phi \times C_0$, each form and weight being evaluated on the components of the heads. (SV1): $\psi(0) = \prod_v \psi_v(0)\psi_0(0) = 1$. (SV2): for $i \le n$, heads p, q and $\xi = ((\xi_v)_v, \xi_0)$ the integrand $\psi(\xi c)J_{i+1}(p[i \mapsto u(c)], q[i \mapsto u(c)])$ is the product of the functions $g_v(c_v) = \psi_v(\xi_v c_v)J_{v,i+1}(p_v[i \mapsto u_\infty(c_v)], q_v[i \mapsto u_\infty(c_v)])$ and of the analogous $g_0(c_0)$, each integrable by (SV2) of its factor; by Fubini's theorem (the measures are s-finite) the product is integrable and its integral is $\prod_v \int g_v \cdot \int g_0$. For $\xi = 0$ this is $\prod_v J_{v,i}(p_v, q_v)\, J_{0,i}(p_0, q_0) = J_i(p, q)$. If ξ is a unit, every component is a unit and the factor bounds multiply (all quantities involved are ≥ 0) to $\beta(\xi)\Lambda_i(p)\Lambda_i(q)$. (SV3): if the heads a_j have a common entry at n, so have their components, the factor matrices $(J_{v,n+1}(a_{j,v}, a_{l,v}))$ and $(J_{0,n+1}(a_{j,0}, a_{l,0}))$ are positive semidefinite, and so is their entrywise product by the Schur product theorem [133]. (SV4): a product of continuous functions of the components. (SV5): $|J_0| = \prod_v |J_{v,0}| \cdot |J_{0,0}| \le \prod_v E_{0,v} \cdot E_{0,0}$. (SV6): if $\iota(p_i) - \iota(q_i)$ is a unit so is each component, and the factor bounds multiply as in (SV2).

(ii) By Definition 5.1.6 the band limits $J^{(v)}$ are local bounds: sieve systems of length ϱ on $\mathbb{C}$ with mass bound $(1+b^{-10})/\deg_v$, the common weights $\Lambda^{b^{60n^2}, n^6}$ and decay $\Theta b^{2000n^9}\ell_{k-n^7}(1/|\xi|)$. By (i) (Haar measure on E_T is σ-finite), $J_i = \prod_v J_i^{(v)} \cdot J_i^T$ is a sieve system of length ϱ on (C, U, u, ι) with the character ψ of Lemma 5.2.100, mass bound $E_n = \prod_v (1 + b^{-10}) \deg_v^{-1} \cdot m_T$, the weights $\Lambda_i \ge 0$ of Lemma 5.2.104 and the decay β of Lemma 5.2.102. By Lemma 5.2.102, (BD) holds with $\varepsilon = \mathrm{dc}\, b^{2002n^9 f} \ge 0$. Put $\Gamma = b^{400n^9 f}$ and $W_i = \sum_{p \in A^i} \Lambda_i(\mathrm{hd}_i(p, 1))$; by Lemma 5.2.104, $W_i^2 \le \Gamma \prod_{a<i} \mathrm{pc}(a)$ for $i \le \varrho$.

We show $\mathrm{ds}(i) \le \prod_{a<i} \mathrm{pc}(a)\,(E_n + 2i\Gamma\varepsilon)$ for $0 \le i \le \varrho+1$. For $i = 0$, $\mathrm{ds}(0) \le |J_0(\mathrm{hd}_0, \mathrm{hd}_0)| \le E_n$ by (SV5). If it holds for some $i \le \varrho$, then Lemma 5.2.98 (with $n' = \varrho$), $\mathrm{pc}(i) \ge 0$ and $\varepsilon \ge 0$ give

$$\mathrm{ds}(i+1) \le \mathrm{pc}(i)\big(\mathrm{ds}(i) + 2\varepsilon W_i^2\big) \le \prod_{a \le i} \mathrm{pc}(a)\,(E_n + 2i\Gamma\varepsilon + 2\Gamma\varepsilon).$$

For $t \in A$ and $p \in A^\varrho$ the head $\mathrm{hd}_\varrho(p, t)$ has the components $x_v(p, t)$ (their entries are nonzero, so u changes nothing) and $y(p, t)$; as $n = \varrho + 1$,

$$\prod_v J_n^{(v)}\big(x_v(p,t), x_v(q,t)\big)\, J_n^T\big(y(p,t), y(q,t)\big) = J_{\varrho+1}\big(\mathrm{hd}_\varrho(p,t), \mathrm{hd}_\varrho(q,t)\big).$$

For each $t \in A$ the heads $\mathrm{hd}_\varrho(p, t)$, $p \in A^\varrho$, have a common entry at ϱ, so

$$\mathrm{Re} \sum_{p,q} J_{\varrho+1}(\mathrm{hd}_\varrho(p,t), \mathrm{hd}_\varrho(q,t)) \ge 0$$

by Lemma 5.2.99 (a). Since $U \subseteq A$, the left side of (ii) is at most $\mathrm{ds}(\varrho+1) \le \prod_{a<n} \mathrm{pc}(a)(E_n + 2n\Gamma\varepsilon)$. Finally $2n \le 2b \le b^2 \le b^{n^9 f}$ by (FG6) and (FG2) ($\varrho + 1 \le b$, $b \ge 1000$), so $2n\Gamma\varepsilon \le b^{(1+400+2002)n^9 f}\mathrm{dc} = b^{2403n^9 f}\mathrm{dc}$, which is (ii). □

Lemma 5.2.98. *Let $\mathfrak{S}$ be admissible and T a finite set of finite places of E. Let $U = (\mathbb{C}^\times)^\Phi \times E_T^\times$, let $u\colon C \to U$ replace every component that is not a unit by 1, and $\iota(x) = ((x_v^{-1})_v, M^{-1}x_T^{-1})$. Let $(J, \psi, E_0, \Lambda, \beta)$ be a sieve system of length n' on (C, U, u, ι) with $\Lambda \ge 0$, and let $\varepsilon \ge 0$ satisfy*

(BD) *for every $a \in \mathbb{N}$, every $c \in (\mathbb{C}^\times)^\Phi$ with $|c_v|^{-1} \le |\mathrm{rs}_v|/M$ for all v, and every finite set X of nonzero T-integers, $\sum_{y \in X} \beta\big(\mathrm{pt}((c_v/\mathrm{fs}(v,a))_v; y)\big) \le \varepsilon$.*

*Let $A \subseteq E^\times$ be finite with t and t^{-1} T-integers for $t \in A$, and let M^{-1} be a T-integer. For $i \in \mathbb{N}$, $p \in A^i$ and $t \in E^\times$ let $\mathrm{hd}_i(p,t)$ be the head with entries $u(\mathrm{pt}((\mathrm{fs}(v,a))_v; (p|t)_a))$, where $(p|t)_a = p_a$ for $a < i$, $(p|t)_i = t$ and $(p|t)_a = 1$ for $a > i$. Define the **density sums***

$$\mathrm{ds}(0) = \mathrm{Re}\, J_0(\mathrm{hd}_0, \mathrm{hd}_0), \qquad \mathrm{ds}(i+1) = \sum_{t \in A} \mathrm{Re} \sum_{p,q \in A^i} J_{i+1}\big(\mathrm{hd}_i(p,t), \mathrm{hd}_i(q,t)\big),$$

with $\mathrm{hd}_0 = \mathrm{hd}_0(\emptyset, 1)$. Then for every $i \le n'$

$$\mathrm{ds}(i+1) \le \mathrm{pc}(i)\Big(\mathrm{ds}(i) + 2\varepsilon\Big(\sum_{p \in A^i} \Lambda_i(\mathrm{hd}_i(p,1))\Big)^2\Big).$$

Proof. Fix $i \le n'$ and write $W = \sum_{p \in A^i} \Lambda_i(\mathrm{hd}_i(p,1)) \ge 0$. All $\mathrm{fs}(v,a)$ are positive, so for $t \in E^\times$ every component of $\mathrm{pt}((\mathrm{fs}(v,a))_v; t)$ is a unit and u is the identity and continuous near this point.

The function. For $x \in C$ put

$$F(x) = \sum_{p,q \in A^i} J_{i+1}\big(\mathrm{hd}_i(p,1)[i \mapsto u(x)],\ \mathrm{hd}_i(q,1)[i \mapsto u(x)]\big).$$

These heads have the common entry $u(x)$ at position i, so by Lemma 5.2.99 (a) $F(x)$ is real and ≥ 0, and by Lemma 5.2.99 (b) (with $Y = C$, first countable) F is lower semicontinuous at every point all of whose components are units, in particular at the points $\mathrm{pt}((\mathrm{fs}(v,i))_v; t)$, $t \in A$. By (SV2) for $i \le n'$, summed over the pairs (p,q), F is integrable with $\int_C F = \sum_{p,q} J_i(\mathrm{hd}_i(p,1), \mathrm{hd}_i(q,1))$, and for every unit ξ of C

$$\Big|\int_C \psi(\xi x) F(x)\, dx\Big| \le \sum_{p,q} \beta(\xi) \Lambda_i(\mathrm{hd}_i(p,1)) \Lambda_i(\mathrm{hd}_i(q,1)) = \beta(\xi) W^2.$$

Since $\mathrm{hd}_i(p,1)[i \mapsto \mathrm{pt}((\mathrm{fs}(v,i))_v; t)] = \mathrm{hd}_i(p,t)$, the definition gives

$$\mathrm{ds}(i+1) = \sum_{t \in A} F(\mathrm{pt}((\mathrm{fs}(v,i))_v; t)).$$

Poisson summation in the entry i. With $c_v = 2\pi i / v(\varpi)$ we have $|c_v|^{-1} = |v(\varpi)|/(2\pi) = |\mathrm{rs}_v|/M$, so (BD) with $a = i$ bounds the sums of β required in Lemma 5.2.100 for $r_v = \mathrm{fs}(v,i)$. Applying Lemma 5.2.100 with $V = A$, F and $M_0 = W^2$, and noting $\mathrm{vol}(\hat{\mathcal{O}}) / \prod_v \mathrm{fs}(v,i)^2 = \mathrm{pc}(i)$,

$$\mathrm{ds}(i+1) \le \mathrm{pc}(i)\Big(\varepsilon W^2 + \mathrm{Re} \sum_{p,q \in A^i} J_i\big(\mathrm{hd}_i(p,1), \mathrm{hd}_i(q,1)\big)\Big).$$

The pairs with different last entries. We show $\mathrm{Re} \sum_{p,q \in A^i} J_i(\mathrm{hd}_i(p,1), \mathrm{hd}_i(q,1)) \le \mathrm{ds}(i) + \varepsilon W^2$, which with $\mathrm{pc}(i) \ge 0$ gives the claim. For $i = 0$ there is one pair and the left side is $\mathrm{ds}(0)$. Let $i \ge 1$ and write $p = (p', s)$, $q = (q', s')$ with $p', q' \in A^{i-1}$ and $s, s' \in A$; then $\mathrm{hd}_i(p,1) = \mathrm{hd}_{i-1}(p', s)$. The terms with $s = s'$ add up to $\mathrm{ds}(i)$. Let $s \ne s'$. The entries at position $i-1$ of $\mathrm{hd}_{i-1}(p', s)$ and $\mathrm{hd}_{i-1}(q', s')$ are $\mathrm{pt}((\mathrm{fs}(v, i-1))_v; s)$ and the same with s', so

$$\xi := \iota(\cdot) - \iota(\cdot) = \mathrm{pt}\big((M/\mathrm{fs}(v, i-1))_v;\, y\big), \qquad y = M^{-1}(s^{-1} - s'^{-1}),$$

where y is a nonzero T-integer; hence ξ is a unit of C. As $i - 1 < n'$, (SV6) gives

$$|J_i(\mathrm{hd}_{i-1}(p', s), \mathrm{hd}_{i-1}(q', s'))| \le \beta(\xi) \Lambda_i(\mathrm{hd}_i(p,1)) \Lambda_i(\mathrm{hd}_i(q,1)),$$

and (BD) with $a = i - 1$, $c_v = M$ (allowed since $M^{-1} \le |\mathrm{rs}_v|/M$ by (FA4)) and $X = \{y\}$ gives $\beta(\xi) \le \varepsilon$. As $\Lambda \ge 0$ and $\varepsilon \ge 0$, the real parts of the terms with $s \ne s'$ add up to at most εW^2. □

Lemma 5.2.99. *Let $(J, \psi, E_0, \Lambda, \beta)$ be a sieve system of length n on (C, U, u, ι), let $i \le n$, and let P be a finite set.*

(a) If $(a_p)_{p\in P}$ are heads with a common entry $a_p(i) = t$, then $\sum_{p,q\in P} J_{i+1}(a_p, a_q)$ is a real number ≥ 0.

(b) Let Y be a first countable topological space, $y_0 \in Y$, and for $p \in P$ let $y \mapsto a_p(y)$ be maps from Y to heads (with the product topology on $U^{\mathbb{N}}$), continuous at y_0, such that for each y the heads $a_p(y)$ have a common entry at position i. Then $y \mapsto \operatorname{Re}\sum_{p,q\in P} J_{i+1}(a_p(y), a_q(y))$ is lower semicontinuous at y_0.

Proof. Both claims are proved by descending induction on i, for all finite families at once.

(a) For $i = n$, (SV3) says that the matrix $(J_{n+1}(a_p, a_q))_{p,q}$ is positive semidefinite; its value on the vector with all entries 1 is $\sum_{p,q} J_{n+1}(a_p, a_q)$, which is therefore real and ≥ 0. Let $i < n$ and assume (a) for $i+1$. By (SV2) for $i+1 \le n$ at $\xi = 0$, using (SV1),

$$\sum_{p,q} J_{i+1}(a_p, a_q) = \int_C \sum_{p,q} J_{i+2}\big(a_p[i+1 \mapsto u(c)], a_q[i+1 \mapsto u(c)]\big)\, dc,$$

each summand being integrable. For every c the heads $a_p[i+1 \mapsto u(c)]$ have the common entry $u(c)$ at position $i+1$, so by the induction hypothesis the integrand is real and ≥ 0; hence so is the integral.

(b) Write $S(y) = \sum_{p,q} J_{i+1}(a_p(y), a_q(y))$, real and ≥ 0 by (a). For $i = n$, S is continuous at y_0 by (SV4), since J_{n+1} is jointly continuous and the a_p are continuous at y_0. Let $i < n$ and assume (b) for $i+1$. Put $G(y, c) = \sum_{p,q} J_{i+2}(a_p(y)[i+1 \mapsto u(c)], a_q(y)[i+1 \mapsto u(c)])$. For fixed c the maps $y \mapsto a_p(y)[i+1 \mapsto u(c)]$ are continuous at y_0 and have a common entry at $i+1$, so $y \mapsto G(y, c)$, real and ≥ 0 by (a), is lower semicontinuous at y_0 by the induction hypothesis. As in (a), $S(y) = \int_C G(y, c)\, dc$ with $G(y, \cdot)$ integrable. Let $y_m \to y_0$ be a sequence. By lower semicontinuity and Fatou's lemma,

$$S(y_0) = \int_C G(y_0, c)\, dc \le \int_C \liminf_m G(y_m, c)\, dc \le \liminf_m \int_C G(y_m, c)\, dc = \liminf_m S(y_m).$$

Since Y is first countable, lower semicontinuity at y_0 follows from this inequality for all sequences $y_m \to y_0$. □

Lemma 5.2.100. *Let E be a totally complex number field with f infinite places, Φ a set of complex embeddings $v\colon E \to \mathbb{C}$ containing one for each infinite place, $\varpi \in E$ a generator of the different of $E/\mathbb{Q}$, and $\operatorname{vol}(\hat{\mathcal{O}}) = 2^f |\operatorname{disc} E|^{-1/2}$. Let ψ_f be the finite part of ψ_E, so that $\psi_f(x) = \exp(2\pi i \operatorname{Tr}_{E/\mathbb{Q}} a)$ whenever $a \in E$ and $x - a \in \hat{\mathcal{O}}$. Let T be a finite set of finite places of E, $E_T = \prod_{w\in T} E_w$ with $\mathcal{O}_T = \prod_{w\in T} \mathcal{O}_w$, call $y \in E$ a T-integer if $y \in \mathcal{O}_w$ for every finite place $w \notin T$, and put $\psi_T(x) = \psi_f(-\tilde{x}/\varpi)$ for $x \in E_T$, where $\tilde{x} \in \mathbb{A}_f$ has the components x at T and 0 elsewhere. Let $C = \mathbb{C}^\Phi \times E_T$, with the product of Lebesgue measure on each factor $\mathbb{C} = \mathbb{R}^2$ and the Haar measure of E_T giving $\mathcal{O}_T$ measure 1, and let $\psi(x) = \prod_{v\in\Phi} \exp(x_v - \bar{x}_v) \cdot \psi_T(x_T)$. For $l \in \mathbb{C}^\Phi$ and $y \in E$ put $\operatorname{pt}(l; y) = ((l_v v(y))_v,\, y) \in C$, y read in E_T. Let $r \in (0, \infty)^\Phi$, let $V \subseteq E^\times$ be a finite set of T-integers, and let $F\colon C \to [0, \infty)$ be integrable and lower semicontinuous at the points $\operatorname{pt}(r; t)$, $t \in V$. Let $M_0 \ge 0$, $\varepsilon \in \mathbb{R}$ and $\beta\colon C \to \mathbb{R}$ satisfy*

$$\Big|\int_C \psi(\xi x) F(x)\, dx\Big| \le \beta(\xi) M_0 \quad \text{for every unit } \xi \text{ of } C, \qquad \sum_{y\in X} \beta\Big(\operatorname{pt}\Big(\Big(\frac{2\pi i}{v(\varpi) r_v}\Big)_v; y\Big)\Big) \le \varepsilon$$

for every finite set X of nonzero T-integers. Then

$$\sum_{t\in V} F(\mathrm{pt}(r;t)) \le \frac{\mathrm{vol}(\hat{\mathcal{O}})}{\prod_{v\in\Phi} r_v^2}\Big(\varepsilon M_0 + \int_C F\Big).$$

Proof. Put $c = \mathrm{vol}(\hat{\mathcal{O}})\prod_v r_v^{-2} > 0$. Identify the completion E_v with $\mathbb{C}$ through v, so that $\mathbb{A}_E = \mathbb{C}^\Phi \times \mathbb{A}_f$ and $y \in E$ sits in $\mathbb{A}_E$ as $((v(y))_v, y)$. The local components of ψ_E are $\psi_v(x) = \exp(-2\pi i(x+\bar{x}))$ for $v \in \Phi$ and characters ψ_w of E_w for finite w, with $\psi_f(x) = \prod_w \psi_w(x_w)$. Since ϖ generates the different, for every finite w the set of $x \in E_w$ with $\psi_w(x\mathcal{O}_w) = 1$ is the local inverse different $\varpi^{-1}\mathcal{O}_w$ [296], Section 2.2.

The measure on the adeles. Let dz be the product of Lebesgue measure on each factor $\mathbb{C} = \mathbb{R}^2$ and the Haar measure μ_f of $\mathbb{A}_f$ with $\mu_f(\hat{\mathcal{O}}) = \mathrm{vol}(\hat{\mathcal{O}})$. Then $\mathbb{A}_E/E$ has volume 1, so dz is the measure of Lemma 5.2.101. Indeed, let $P \subseteq \mathbb{C}^\Phi = \mathbb{R}^{2f}$ be a half-open fundamental parallelotope of the lattice $\{(v(\lambda))_v : \lambda \in \mathcal{O}_E\}$, of Lebesgue measure $2^{-f}|\mathrm{disc}\, E|^{1/2}$ [224], Ch. I, Sect. 5 (there the canonical measure on a complex factor is twice Lebesgue measure). Every $z \in \mathbb{A}_E$ is $e + (p, o)$ with $e \in E$, $p \in P$, $o \in \hat{\mathcal{O}}$: choose $a \in E$ with $z_f - a \in \hat{\mathcal{O}}$ (possible as $\mathbb{A}_f = E + \hat{\mathcal{O}}$, [53], Ch. II, Sect. 15), then $\lambda \in \mathcal{O}_E$ with $(z_v - v(a) - v(\lambda))_v \in P$, and take $e = a + \lambda$. The representation is unique: a difference $e \in E$ of two of them lies in $\hat{\mathcal{O}} \cap E = \mathcal{O}_E$ and translates a point of P into P, so $e = 0$. Hence $P \times \hat{\mathcal{O}}$ is a fundamental domain for E, of measure $2^{-f}|\mathrm{disc}\, E|^{1/2}\mathrm{vol}(\hat{\mathcal{O}}) = 1$.

Transport of measure. Let $\mathbb{A}_f^{(T)} = \{x \in \mathbb{A}_f : x_w \in \mathcal{O}_w \text{ for all finite } w \notin T\}$, an open subgroup of $\mathbb{A}_f$ (it contains $\hat{\mathcal{O}}$), equal to $E_T \times \hat{\mathcal{O}}^T$ with $\hat{\mathcal{O}}^T = \prod_{w\notin T}\mathcal{O}_w$. The restriction of μ_f to it is a Haar measure of $E_T \times \hat{\mathcal{O}}^T$, hence equals $\lambda\,\mu_T \otimes \mu^T$, where μ_T is the Haar measure of E_T of the statement and μ^T the Haar probability measure of the compact group $\hat{\mathcal{O}}^T$; evaluating at $\hat{\mathcal{O}} = \mathcal{O}_T \times \hat{\mathcal{O}}^T$ gives $\lambda = \mathrm{vol}(\hat{\mathcal{O}})$. The linear map $(z_v)_v \mapsto (r_v z_v)_v$ of $\mathbb{R}^{2f}$ has determinant $\prod_v r_v^2$. Put $D = \mathbb{C}^\Phi \times \mathbb{A}_f^{(T)}$ (open in $\mathbb{A}_E$) and $m(z) = ((r_v z_v)_v, z_T) \in C$ for $z \in D$, with z_T the components of z_f at T. By Fubini's theorem and the two facts just stated,

$$\int_D h(m(z))\,dz = c\int_C h(x)\,dx \tag{5.34}$$

for every Borel function $h\colon C \to [0,\infty]$, and hence, by linearity, for every integrable $h\colon C \to \mathbb{C}$, $h \circ m$ being integrable on D.

The extension. Define $G\colon \mathbb{A}_E \to [0,\infty)$ by $G(z) = F(m(z))$ for $z \in D$ and $G(z) = 0$ otherwise. As D is open and m is continuous, G is measurable, and by (5.34) it is integrable. Let $t \in V$. As t is a T-integer, $t \in D$ and $m(t) = \mathrm{pt}(r;t)$, so $G(t) = F(\mathrm{pt}(r;t))$; on the open neighbourhood D of t we have $G = F \circ m$ with m continuous, and F is lower semicontinuous at $m(t)$, so G is lower semicontinuous at t.

Fourier coefficients. For $\xi \in E$ put $\widehat{G}(\xi) = \int_{\mathbb{A}_E}\psi_E(-\xi z)G(z)\,dz$, and write $\xi = y/\varpi$ with $y \in E$.

(a) If $\widehat{G}(y/\varpi) \neq 0$, then y is a T-integer. Let $a \in \mathbb{A}_E$ have the components 0 at the infinite places and at T, and $a_w \in \mathcal{O}_w$ at the finite $w \notin T$. Then $z + a \in D$ if and only if $z \in D$, and $m(z+a) = m(z)$, so $G(z+a) = G(z)$; by the translation invariance of dz, $\widehat{G}(\xi) = \int \psi_E(-\xi(z+a))G(z+a)\,dz = \psi_E(-\xi a)\widehat{G}(\xi)$. Hence $\psi_E(-\xi a) = 1$ for all such a. Given a finite $w \notin T$ and $u \in \mathcal{O}_w$, the choice $a_w = -u$ and $a_{w'} = 0$ for $w' \neq w$ gives $\psi_w(yu/\varpi) = 1$; so $y/\varpi \in \varpi^{-1}\mathcal{O}_w$, that is $y \in \mathcal{O}_w$.

(b) Let y be a T-integer and $\xi_y = \mathrm{pt}((2\pi i/(v(\varpi)r_v))_v; y)$. Then $\psi_E(-(y/\varpi)z) = \psi(\xi_y\, m(z))$ for $z \in D$. At $v \in \Phi$ the component of $-(y/\varpi)z$ is $-w_v$ with $w_v = v(y)z_v/v(\varpi)$, and $\psi_v(-w_v) =$

$\exp(2\pi i(w_v+\bar{w}_v))=\exp(c_v-\bar{c}_v)$ with $c_v=2\pi i w_v=\frac{2\pi i\,v(y)}{v(\varpi)r_v}\cdot r_v z_v$, the v-component of $\xi_y\,m(z)$. For the finite part, $x=yz_f$ lies in $\mathbb{A}_f^{(T)}$, both factors being integral at every finite $w\notin T$; the adele $x-\widetilde{x_T}$ has components 0 at T and in $\mathcal{O}_w$ at $w\notin T$, so $-(x-\widetilde{x_T})/\varpi$ has components in $\varpi^{-1}\mathcal{O}_w$ off T and 0 at T, where every ψ_w is trivial. Hence $\psi_f(-x/\varpi)=\psi_f(-\widetilde{x_T}/\varpi)=\psi_T(x_T)=\psi_T(y_Tz_T)$, the factor at T of $\psi(\xi_y\,m(z))$. Multiplying the components gives the identity. By (5.34) with $h(x)=\psi(\xi_y x)F(x)$ (integrable, as $|\psi|=1$),

$$\widehat{G}(y/\varpi)=\int_D \psi(\xi_y\,m(z))F(m(z))\,dz=c\int_C \psi(\xi_y x)F(x)\,dx.$$

(c) For $y=0$, (b) gives $\widehat{G}(0)=c\int_C F\ge 0$. For a nonzero T-integer y the point ξ_y is a unit of C (its components at $v\in\Phi$ are nonzero, and $y\neq 0$ in each field E_w, $w\in T$), so (b) and the hypothesis on F give $|\widehat{G}(y/\varpi)|\le c\,\beta(\xi_y)M_0$.

Partial sums. Let $Y\subseteq E$ be finite and X the set of nonzero T-integers in ϖY. By (a), (c) and $M_0\ge 0$,

$$\sum_{\xi\in Y}|\widehat{G}(\xi)|=\sum_{y\in\varpi Y}|\widehat{G}(y/\varpi)|\le c\int_C F+cM_0\sum_{y\in X}\beta(\xi_y)\le c\Big(\int_C F+\varepsilon M_0\Big),$$

the last step by the hypothesis on β for the set X.

Conclusion. G is nonnegative and integrable on $\mathbb{A}_E$, lower semicontinuous at the points of the finite set $V\subseteq E$, and its Fourier coefficients satisfy the hypothesis of Lemma 5.2.101 with $S=c(\varepsilon M_0+\int_C F)$. Hence $\sum_{t\in V}F(\mathrm{pt}(r;t))=\sum_{t\in V}G(t)\le c(\varepsilon M_0+\int_C F)$, which is the claim. □

Lemma 5.2.101. *Let E be a number field, $E\subseteq\mathbb{A}_E$ diagonally, and dx the Haar measure on $\mathbb{A}_E$ for which $\mathbb{A}_E/E$ has volume 1. Let $F\colon\mathbb{A}_E\to[0,\infty)$ be integrable, $V\subseteq E$ finite with F lower semicontinuous at every $t\in V$, and $S\in\mathbb{R}$ such that*

$$\sum_{\xi\in Y}\Big|\int_{\mathbb{A}_E}\psi_E(-\xi x)F(x)\,dx\Big|\le S\qquad\text{for every finite } Y\subseteq E.$$

Then $\sum_{t\in V}F(t)\le S$.

Proof. Write $\widehat{F}(\xi)=\int_{\mathbb{A}_E}\psi_E(-\xi x)F(x)\,dx$, $Q=\mathbb{A}_E/E$ and $[x]$ for the class of x. We use the folding formula $\int_{\mathbb{A}_E}f(x)\,dx=\int_Q\sum_{g\in E}f(g+x)\,d\mu([x])$ for integrable f (the sum is independent of the representative x of $[x]$), which holds for this normalisation of dx. Let $m<\sum_{t\in V}F(t)$; we show $m\le S$.

A bump near 0. The function $y\mapsto\sum_{t\in V}F(t+y)$ is lower semicontinuous at 0, as a finite sum of lower semicontinuous functions composed with translations. Hence $W=\{x\in\mathbb{A}_E:\ m<\sum_{t\in V}F(t+y)$ for all y in a neighbourhood of $x\}$ is open and contains 0. The quotient map is open, so $\{[x]:x\in W\}$ is an open neighbourhood of $[0]$ in the compact Hausdorff group Q, and by Urysohn's lemma there is a continuous $\eta\colon Q\to[0,1]$ with $\eta([0])=1$ and support in $\{[x]:x\in W\}$. Put $J=\int_Q\eta\,d\mu$; then $J>0$, as η is continuous, nonnegative and positive at $[0]$. Write $\eta(x)$ for $\eta([x])$.

Lower bound. By the folding formula applied to ηF (integrable, as $0\le\eta\le 1$),

$$\int_{\mathbb{A}_E}\eta(x)F(x)\,dx=\int_Q\eta([x])\sum_{g\in E}F(g+x)\,d\mu([x]).$$

If $\eta([x]) \neq 0$, choose the representative $x = u \in W$; as $F \geq 0$ and $V \subseteq E$, $\sum_{g\in E} F(g+u) \geq \sum_{t\in V} F(t+u) > m$. Hence $\int \eta F \geq Jm$.

Upper bound. Put $G(z) = \int_{\mathbb{A}_E} \eta(z+x)F(x)\,dx$. It is continuous by dominated convergence ($|\eta| \leq 1$, F integrable, η continuous) and invariant under E, so it defines a continuous function on Q, and $G(0) = \int \eta F$. By Fubini's theorem (the integrand is bounded by $F(x)$) and the substitution $z' = z + x$ on Q,

$$\widehat{G}(\xi) := \int_Q \psi_E(\xi z)G(z)\,d\mu(z) = \int_{\mathbb{A}_E} F(x)\psi_E(-\xi x)\,dx \int_Q \psi_E(\xi z')\eta(z')\,d\mu(z') = \widehat{\eta}(\xi)\,\widehat{F}(\xi),$$

with $|\widehat{\eta}(\xi)| \leq \int_Q \eta\,d\mu = J$. So $\sum_{\xi\in Y} |\widehat{G}(\xi)| \leq JS$ for every finite $Y \subseteq E$; the series $\sum_\xi |\widehat{G}(\xi)|$ converges, and Theorem A.5.2 (b) gives $G(0) = \sum_\xi \widehat{G}(\xi)$. As $G(0)$ is real, $G(0) = \operatorname{Re} G(0) \leq \sum_\xi |\widehat{G}(\xi)| \leq JS$.

Conclusion. $Jm \leq \int \eta F = G(0) \leq JS$ and $J > 0$ give $m \leq S$. □

Lemma 5.2.102. *Let $\mathfrak{S}$ be admissible and T a finite set of finite places of E. For $\xi \in C$ with all components nonzero put*

$$\beta(\xi) = \prod_{v\in\Phi} \Theta\, b^{2000n^9}\, \ell_{k-n^7}(|\xi_v|^{-1}) \cdot \beta_T(\xi_T).$$

Then for every $a \in \mathbb{N}$, every $c \in (\mathbb{C}^\times)^\Phi$ with $|c_v|^{-1} \leq |\mathrm{rs}_v|/M$ for all v, and every finite set X of nonzero T-integers,

$$\sum_{y\in X} \beta(\mathrm{pt}((c_v/\mathrm{fs}(v,a))_v; y)) \leq \mathrm{dc}\, b^{2002n^9 f}.$$

Proof. Fix a, c and X. Put $K = k - n^7$, an integer ≥ 0 by (FG6), $c'_v = c_v/\mathrm{fs}(v,a)$,

$$x_v = \frac{|\mathrm{rs}_v|^{2\varrho+1}}{MN} > 0, \qquad B = b^{23(2\varrho+1)} \geq 1,$$

so that $\mathrm{dc} = \prod_v \Theta x_v^K$.

The scales. Since $|\mathrm{rs}_v| \geq 1$ by (FA4) and the exponent $2(\varrho - a)$ (zero for $a \geq \varrho$) is at most 2ϱ, $0 < \mathrm{fs}(v,a) \leq |\mathrm{rs}_v|^{2\varrho}/N$. Hence

$$|c'_v|^{-1} = |c_v|^{-1}\mathrm{fs}(v,a) \leq \frac{|\mathrm{rs}_v|}{M} \cdot \frac{|\mathrm{rs}_v|^{2\varrho}}{N} = x_v \leq |\mathrm{rs}_v|^{2\varrho+1} \leq B,$$

using $MN \geq 1$ and $|\mathrm{rs}_v| \leq b^{23}$ (FA4).

One term. Let $y \in X$ and $\xi = \mathrm{pt}((c'_v)_v; y)$. Its components $\xi_v = c'_v v(y)$ are nonzero and $\xi_T = y$ is a unit of E_T ($y \neq 0$ in each field E_w), so

$$\beta(\xi) = \Theta^f b^{2001n^9 f} \prod_v \ell_K(|c'_v|^{-1}|v(y)|^{-1}) \cdot \sqrt{[\mathcal{O}_T : y\mathcal{O}_T]}^{\,K} [\mathcal{O}_T : \mathcal{O}_T \cap y^{-1}\mathcal{O}_T]^{-3}.$$

We bound the two finite factors. First, $|y|_w \leq 1$ for finite $w \notin T$ (y is a T-integer), so by the product formula

$$[\mathcal{O}_T : y\mathcal{O}_T] = \prod_{w\in T} |y|_w^{-1} \leq \prod_{w \text{ finite}} |y|_w^{-1} = |\mathrm{N}_{E/\mathbb{Q}}(y)| = \prod_{v\in\Phi} |v(y)|^2.$$

Second, let $D = \{s \in \mathcal{O}_T : ys \in \mathcal{O}_T\}$ and $\mathfrak{n}_y = \{a' \in \mathcal{O}_E : a'y \in \mathcal{O}_E\}$. For $a' \in \mathcal{O}_E$, the element $a'y$ is integral at every finite $w \notin T$, so its image lies in D if and only if $a'y \in \mathcal{O}_E$. Thus the

preimage of D under $\mathcal{O}_E \to \mathcal{O}_T$ is $\mathfrak{n}_y$, and $\mathcal{O}_E/\mathfrak{n}_y$ injects into $\mathcal{O}_T/D$; hence $\mathrm{N}(\mathfrak{n}_y) \le [\mathcal{O}_T : D]$. For each v put $s = |v(y)| > 0$ and $t = |c'_v|^{-1}s^{-1}$. Then $\ell_K(t)s^K = |c'_v|^{-K}\min(1,t)^6 \le x_v^K \min(1,t)^6$, and $\min(1,t) \le B\max(1,s)^{-1}$: for $s \ge 1$, $\min(1,t) \le t \le x_v s^{-1} \le Bs^{-1}$; for $s < 1$, $\min(1,t) \le 1 \le B$. Since all factors are nonnegative,

$$\beta(\xi) \le \Theta^f b^{2001n^9 f} \prod_v x_v^K B^6 \cdot \prod_v \max(1,|v(y)|)^{-6}\,\mathrm{N}(\mathfrak{n}_y)^{-3}.$$

The sum. By Lemma 5.2.103 applied to X,

$$\sum_{y\in X} \beta\big(\mathrm{pt}((c'_v)_v; y)\big) \le \mathrm{dc}\; b^{2001n^9 f}\big(B^6 \cdot 64 \cdot 38 \cdot e^4\big)^f.$$

Finally $64 \cdot 38 \cdot e^4 < 64 \cdot 38 \cdot 81 < 1000^3 \le b^3$ ($b \ge 1000$ by (FG2)) and $B^6 = b^{138(2\varrho+1)}$, so $B^6 \cdot 64 \cdot 38 \cdot e^4 \le b^{276\varrho+141} \le b^{(\varrho+1)^9} = b^{n^9}$ (as $b \ge 1$ and $\varrho \ge 1$). This gives the bound $\mathrm{dc}\; b^{2002n^9 f}$. □

Lemma 5.2.103. *Let E be a totally complex number field with f infinite places and Φ a set of complex embeddings $v\colon E \to \mathbb{C}$ containing one for each infinite place. For $\nu \in E$ let $\mathfrak{n}_\nu = \{a \in \mathcal{O}_E : a\nu \in \mathcal{O}_E\}$ be its denominator ideal. Then for every finite set $X \subseteq E$*

$$\sum_{\nu\in X} \prod_{v\in\Phi} \max(1,|v(\nu)|)^{-6} \cdot \mathrm{N}(\mathfrak{n}_\nu)^{-3} \le \big(64 \cdot 38 \cdot e^4\big)^f.$$

Proof. Write $w(z) = \max(1,|z|)^{-6}$ for $z \in \mathbb{C}$. For $m = (m_1, m_2) \in \mathbb{Z}^2$ let $Q_m = [m_1/2, (m_1+1)/2) \times [m_2/2, (m_2+1)/2) \subseteq \mathbb{C}$ (half-open squares of side $1/2$, a partition of $\mathbb{C}$) with corner $\kappa(m) = (m_1 + im_2)/2$, and let $q(z)$ be the index of the square containing z. Two points of one square differ by a complex number whose real and imaginary parts have absolute value $< 1/2$, hence by less than 1 in absolute value.

Step 1: corners. For $z \in \mathbb{C}$, $|z - \kappa(q(z))| < 1$, so $\max(1,|\kappa(q(z))|) \le \max(1,|z|) + 1 \le 2\max(1,|z|)$, and therefore $w(z) \le 64\, w(\kappa(q(z)))$.

Step 2: the corner sum. We show $\sum_{m\in\mathbb{Z}^2} w(\kappa(m)) \le 38$. For $\rho \ge 0$ let $N(\rho)$ be the number of $m \in \mathbb{Z}^2$ with $|m_1 + im_2| \le \rho$; the unit squares $m + [-\frac12, \frac12)^2$ for these m are disjoint and lie in the disc of radius $\rho + \sqrt{2}/2$, so $N(\rho) \le \pi(\rho + \sqrt{2}/2)^2$. For $s \ge 0$ we have $\max(1,s)^{-6} = \int_1^\infty 6t^{-7}\mathbf{1}[s \le t]\,dt$. Summing over m with $s = |\kappa(m)|$ and using Tonelli's theorem,

$$\sum_m w(\kappa(m)) = \int_1^\infty 6t^{-7}N(2t)\,dt \le 6\pi \int_1^\infty (4t^{-5} + 2\sqrt{2}\,t^{-6} + \tfrac12 t^{-7})dt = 6\pi\Big(1 + \frac{2\sqrt{2}}{5} + \frac{1}{12}\Big) < 32.$$

Step 3: one coset of the integers. Let $Y \subseteq E$ be finite with $\nu - \nu' \in \mathcal{O}_E$ for all $\nu, \nu' \in Y$. The map $\nu \mapsto (q(v(\nu)))_{v\in\Phi} \in (\mathbb{Z}^2)^\Phi$ is injective on Y: if $\nu \ne \nu'$ had the same image, then $|v(\nu - \nu')| < 1$ for every $v \in \Phi$, so $|\mathrm{N}_{E/\mathbb{Q}}(\nu - \nu')| = \prod_{v\in\Phi} |v(\nu - \nu')|^2 < 1$ (E is totally complex and Φ contains one embedding of each conjugate pair), which is impossible for a nonzero algebraic integer, whose norm is a nonzero rational integer. By Step 1, the injectivity, the nonnegativity of w and Step 2,

$$\sum_{\nu\in Y}\prod_v w(v(\nu)) \le \sum_{\nu\in Y}\prod_v 64\, w\big(\kappa(q(v(\nu)))\big) \le \prod_{v\in\Phi}\Big(64\sum_{m\in\mathbb{Z}^2} w(\kappa(m))\Big) \le (64\cdot 38)^f.$$

Step 4: one denominator ideal. $\mathfrak{n}_\nu$ is a nonzero ideal of $\mathcal{O}_E$ (it contains a nonzero rational integer clearing the denominator of ν). Fix a nonzero ideal $\mathfrak{b}$ and let $X_\mathfrak{b} = \{\nu \in X : \mathfrak{n}_\nu =$

$\mathfrak{b}\}$. Every $\nu \in X_\mathfrak{b}$ satisfies $\mathfrak{b}\nu \subseteq \mathcal{O}_E$, i.e. $\nu \in \mathfrak{b}^{-1}$, and $\mathfrak{b}^{-1}/\mathcal{O}_E$ has $\mathrm{N}(\mathfrak{b})$ elements (for a nonzero $a \in \mathfrak{b}$, multiplication by a identifies it with $a\mathfrak{b}^{-1}/a\mathcal{O}_E$, where $a\mathfrak{b}^{-1} \subseteq \mathcal{O}_E$ is an ideal with $(a\mathfrak{b}^{-1})\mathfrak{b} = a\mathcal{O}_E$; so its order is $[\mathcal{O}_E : a\mathcal{O}_E]/[\mathcal{O}_E : a\mathfrak{b}^{-1}] = \mathrm{N}(a\mathcal{O}_E)/\mathrm{N}(a\mathfrak{b}^{-1}) = \mathrm{N}(\mathfrak{b})$ by multiplicativity of the norm). So $X_\mathfrak{b}$ is the union of at most $\mathrm{N}(\mathfrak{b})$ sets Y as in Step 3, and $\sum_{\nu\in X_\mathfrak{b}} \prod_v w(v(\nu)) \le \mathrm{N}(\mathfrak{b})(64\cdot 38)^f$.

Step 5: the sum over denominators. Let B be the finite set of ideals $\mathfrak{n}_\nu$, $\nu \in X$, and P the finite set of prime ideals dividing some member of B. By Step 4,

$$\sum_{\nu\in X}\prod_v w(v(\nu))\,\mathrm{N}(\mathfrak{n}_\nu)^{-3} = \sum_{\mathfrak{b}\in B}\mathrm{N}(\mathfrak{b})^{-3}\sum_{\nu\in X_\mathfrak{b}}\prod_v w(v(\nu)) \le (64\cdot 38)^f\sum_{\mathfrak{b}\in B}\mathrm{N}(\mathfrak{b})^{-2}.$$

By unique factorisation and multiplicativity of the norm, the distinct ideals $\mathfrak{b} \in B$ give distinct terms $\mathrm{N}(\mathfrak{b})^{-2}$ in the expansion of the finite product of convergent geometric series with nonnegative terms $\prod_{\mathfrak{p}\in P}(1-\mathrm{N}(\mathfrak{p})^{-2})^{-1}$, so $\sum_{\mathfrak{b}\in B}\mathrm{N}(\mathfrak{b})^{-2}$ is at most this product. Above each rational prime p lie at most $[E:\mathbb{Q}] = 2f$ prime ideals, each of norm at least p, so the product is at most $\prod_p(1-p^{-2})^{-2f} = \zeta(2)^{2f} = (\pi^2/6)^{2f} \le e^{4f}$, as $\pi^2/6 < e^2$. This gives the claim. □

Lemma 5.2.104. *Let $\mathfrak{S}$ be admissible and T a finite set of finite places of E. For a head h with entries in $(\mathbb{C}^\times)^\Phi \times E_T^\times$ let h_v and h_T be the sequences of its components, and put*

$$\Lambda_i(h) = \prod_{v\in\Phi}\Lambda_i^{b^{60n^2},n^6}(h_v)\cdot\Lambda_i^T(h_T).$$

Let $A \subseteq E^\times$ be a finite set of T-integers and $i \le \varrho$. Then

$$\Big(\sum_{p\in A^i}\Lambda_i(\mathrm{hd}_i(p,1))\Big)^2 \le b^{400n^9 f}\prod_{a<i}\mathrm{pc}(a).$$

Proof. Write $R = b^{60n^2}$, $R' = b^{120n^2}$ and $K = n^6$. For $p \in A^i$ and $l < i$ put $D_l(p) = p_0p_1\cdots p_l \in E^\times$ and $c_{l,v} = \prod_{a\le l}\mathrm{fs}(v,a) > 0$. The entries of $\mathrm{hd}_i(p,1)$ at positions $a < i$ are $((\mathrm{fs}(v,a)v(p_a))_v, p_a)$ (all components are nonzero, so u changes nothing), hence the partial products of the components are $c_{l,v}v(D_l(p))$ at v and $D_l(p)$ at T, and by the definitions of the head weights

$$\Lambda_i(\mathrm{hd}_i(p,1)) = \prod_{l<i}\Big(\prod_v\omega_{l,v}(p)\Big)\,\omega_{l,T}(p),$$

with $\omega_{l,v}(p) = \max(1,|c_{l,v}v(D_l(p))|^{-1})^K$ if $|c_{l,v}v(D_l(p))| \le R$ and 0 otherwise, and $\omega_{l,T}(p) = \sqrt{[\mathcal{O}_T : D_l(p)\mathcal{O}_T]}^{\,K}$ if $D_l(p) \in \mathcal{O}_T$ and 0 otherwise. All weights are ≥ 0. Let $P \subseteq A^i$ be the set of p with $\Lambda_i(\mathrm{hd}_i(p,1)) \ne 0$.

Step 1: the scales. For $l < i \le \varrho$, $|\mathrm{rs}_v| \ge 1$ (FA4), $N \le b^{60n}$ (FA5) and $l+1 \le n$ give

$$c_{l,v}^{-1} = \frac{N^{l+1}}{\prod_{a\le l}|\mathrm{rs}_v|^{2(\varrho-a)}} \le N^{l+1} \le b^{60n^2} = R.$$

Step 2: heads of nonzero weight. Let $p \in P$ and $l < i$. As $\omega_{l,T}(p) \ne 0$, $D_l(p)$ is integral at every $w \in T$; as a product of T-integers it is integral at every finite $w \notin T$; so $D_l(p) \in \mathcal{O}_E$. As $\omega_{l,v}(p) \ne 0$, $|c_{l,v}v(D_l(p))| \le R$, so $|v(D_l(p))| \le c_{l,v}^{-1}R \le R'$ by Step 1.

Step 3: the largest weight. Let $p \in P$, $l < i$ and $D = D_l(p)$. For each v,

$$\max(1,|c_{l,v}v(D)|^{-1})\,|v(D)| = \max(|v(D)|, c_{l,v}^{-1}) \le R'$$

by Steps 1 and 2. By the product formula, as $D \neq 0$ is a T-integer,

$$[\mathcal{O}_T : D\mathcal{O}_T] = \prod_{w \in T} |D|_w^{-1} \leq |\mathrm{N}_{E/\mathbb{Q}}(D)| = \prod_v |v(D)|^2.$$

Hence

$$\prod_v \max(1, |c_{l,v} v(D)|^{-1}) \cdot \sqrt{[\mathcal{O}_T : D\mathcal{O}_T]} \leq \prod_v \max(1, |c_{l,v} v(D)|^{-1})\, |v(D)| \leq R'^f,$$

and raising to the power K and multiplying over $l < i$ gives $\Lambda_i(\mathrm{hd}_i(p,1)) \leq R'^{fKi} = b^{120n^8 fi}$.

Step 4: the number of heads of nonzero weight. The map $p \mapsto (D_l(p))_{l<i}$ is injective, since $p_0 = D_0(p)$ and $p_l = D_l(p)/D_{l-1}(p)$. Let $Z = \mathbb{C}^{\{0,\dots,i-1\} \times \Phi}$ with the norm $\|z\| = \max_{l,v} |z_{l,v}|$, a real vector space of dimension $2if$, and $F(p) = (v(D_l(p)))_{l,v} \in Z$. For $p \in P$, $\|F(p)\| \leq R'$ by Step 2. For $p \neq q$ in P there is l with $D_l(p) \neq D_l(q)$; their difference is a nonzero algebraic integer, so $\prod_v |v(D_l(p) - D_l(q))|^2 = |\mathrm{N}_{E/\mathbb{Q}}(D_l(p) - D_l(q))| \geq 1$ and some factor is ≥ 1; thus $\|F(p) - F(q)\| \geq 1$. The open balls of radius $1/2$ about the points $F(p)$, $p \in P$, are therefore disjoint and lie in the ball of radius $R' + 1/2$ about 0; comparing Lebesgue measures (the measure of a ball of radius ρ for a norm on $\mathbb{R}^{2if}$ is proportional to ρ^{2if}) gives $\#P \leq (1 + 2R')^{2if}$.

Step 5: the sum. By Steps 3 and 4, with $1 + 2R' \leq 3R' \leq b^{120n^2+1}$ ($b \geq 1000$ by (FG2)) and $i \leq \varrho < n$,

$$\sum_{p \in A^i} \Lambda_i(\mathrm{hd}_i(p,1)) = \sum_{p \in P} \Lambda_i(\mathrm{hd}_i(p,1)) \leq b^{(120n^2+1)2if + 120n^8 fi} \leq b^{fn(240n^2+2+120n^8)} \leq b^{130n^9 f},$$

since $240n^3 + 2n \leq 10n^9$.

Step 6: the Poisson constants. For every a, $\mathrm{fs}(v,a) \leq |\mathrm{rs}_v|^{2\varrho}/N \leq b^{46\varrho}$ by (FA4), and $\mathrm{vol}(\hat{\mathcal{O}}) = (2/\mathrm{rd}_E)^f \geq b^{-f}$ by $\mathrm{rd}_E \leq b$ (FG6). Hence $\mathrm{pc}(a) \geq b^{-f} b^{-92\varrho f}$ and, for $i \leq \varrho$,

$$\prod_{a<i} \mathrm{pc}(a) \geq b^{-(1+92\varrho)\varrho f} \geq b^{-100n^2 f},$$

as $(1 + 92\varrho)\varrho \leq 100(\varrho+1)^2$.

Conclusion. By Steps 5 and 6 (all quantities are nonnegative and $b \geq 1$),

$$\Big(\sum_{p \in A^i} \Lambda_i(\mathrm{hd}_i(p,1))\Big)^2 \leq b^{260n^9 f} \leq b^{260n^9 f + 100n^2 f} \prod_{a<i} \mathrm{pc}(a) \leq b^{400n^9 f} \prod_{a<i} \mathrm{pc}(a).$$

□

Lemma 5.2.105. *Let $\mathfrak{S}$ be admissible and let $\sigma \in G(E)$ fix every e_a and e_a^*, with $h(z, \sigma z) = t \in E^\times$. For $p \in (E^\times)^\varrho$ and $v \in \Phi$ let $x_v(p)$ be the sequence with entries $x_v(p)_a = \mathrm{fs}(v,a)\, v(\mathrm{pad}(p,t)_a)$, and put*

$$\rho_p = \prod_{v \in \Phi} \frac{\sqrt{c_G}}{\#I_v} \left(|x_v(p)_\varrho|^2 - 1\right)_+^{(k-1)/2} \prod_{a<\varrho} |x_v(p)_a|^{k+1+2(\varrho-a-1)}.$$

Then for all $p, q \in (E^\times)^\varrho$

$$\prod_{v \in \Phi} J_n^{(v)}(x_v(p), x_v(q)) = \big\langle \rho_p\, O(w_{\mathrm{pad}(p,t)}\sigma),\ \rho_q\, O(w_{\mathrm{pad}(q,t)}\sigma) \big\rangle,$$

the inner product of $\mathbb{C}^{I \times I}$, linear in the second variable.

Proof. Fix $p, q \in (E^\times)^\varrho$, put $P = \mathrm{pad}(p,t)$, $Q = \mathrm{pad}(q,t)$, and let σ_v be the v-component of σ_∞. All entries of $x_v(p)$ and $x_v(q)$ are nonzero, as $\mathrm{fs}(v,a) > 0$ and $P_a, Q_a \neq 0$.

Step 1: the local element. σ_v fixes every e_a and e_a^*: $\sigma_v e_a = B_v^{-1} v(\sigma) B_v e_a = \mathrm{bs}_v(e_a) B_v^{-1} v(\sigma) e_a = \mathrm{bs}_v(e_a) B_v^{-1} e_a = e_a$, and likewise for e_a^*. Moreover

$$h_1(z, \sigma_v z) = h_v(B_v z, v(\sigma) B_v z) = \mathrm{bs}_v(z)^2 \, v\big(h(z, \sigma z)\big) = v(t)/N = \mathrm{fs}(v,\varrho)\, v(P_\varrho) = x_v(p)_\varrho,$$

and equally $= x_v(q)_\varrho$, since $P_\varrho = Q_\varrho = t$ and $\mathrm{fs}(v,\varrho) = 1/N$.

Step 2: the band limits at v. By Definition 5.1.9 (FG5) and Definition 5.1.6, the band limits $J^{(v)}$ are local bounds for $W_v^\flat$, so by their top clause and Step 1

$$J_n^{(v)}\big(x_v(p), x_v(q)\big) = \big\langle N(\sigma_v, W_v^\flat, x_v(p)),\ N(\sigma_v, W_v^\flat, x_v(q))\big\rangle,$$

with $N(\sigma_v, W_v^\flat, x_v(p)) = s_v(p)\, O_v(w_{x_v(p)} \sigma_v)$, where

$$s_v(p) = \frac{\sqrt{c_G}}{\#I_v} \big(|x_v(p)_\varrho|^2 - 1\big)_+^{(k-1)/2} \prod_{a<\varrho} |x_v(p)_a|^{k+1+2(\varrho-a-1)} \geq 0$$

and

$$O_v(g) = \Big(\int_{G_1} W_{v,i}^\flat(y) \overline{W_{v,j}^\flat(g^{-1}y)}\, dy\Big)_{i,j \in I_v}.$$

Note $\rho_p = \prod_v s_v(p)$.

Step 3: the rescaled Weyl element. $(w_P \sigma)_v = (w_P)_v \sigma_v$, and $(w_P)_v = w_{x_v(p)}$. Indeed, for $0 \leq a < \varrho$, using $\mathrm{bs}_v(f_a^*) = \mathrm{bs}_v(f_a)^{-1}$, $\mathrm{bs}_v(f_a)^2 = \mathrm{fs}(v,a)$ and $v(\bar{P}_a) = \overline{v(P_a)}$ (E is CM),

$$\begin{aligned}
(w_P)_v f_a &= \mathrm{bs}_v(f_a) B_v^{-1} v(P_a) f_a^* = \mathrm{fs}(v,a) v(P_a) f_a^* = x_v(p)_a f_a^*, \\
(w_P)_v f_a^* &= \frac{\mathrm{bs}_v(f_a)^{-2}}{\overline{v(P_a)}} f_a = \frac{1}{\overline{x_v(p)_a}} f_a,
\end{aligned}$$

and $(w_P)_v$ fixes $X_0 \otimes \mathbb{C}$ and z, because w_P does and B_v is diagonal. So $(w_P\sigma)_\infty = (w_{x_v(p)}\sigma_v)_v$.

Step 4: the product. For $i = (i_v)_v$, $j = (j_v)_v \in I = \prod_v I_v$ and $g = (g_v)_v \in G_\infty$, the integrand of the (i,j)-entry of $O(g)$ is $\prod_v F_v(y_v)$ with $F_v(y) = W_{v,i_v}^\flat(y) \overline{W_{v,j_v}^\flat(g_v^{-1} y)}$. By Tonelli's theorem $\int_{G_\infty} \prod_v |F_v(y_v)|\, dy = \prod_v \int_{G_1} |F_v|$ (in $[0,\infty]$). If every F_v is integrable, Fubini's theorem gives $\int_{G_\infty} \prod_v F_v(y_v)\, dy = \prod_v \int_{G_1} F_v$. If some F_v vanishes almost everywhere, both sides are 0. In the remaining case some F_v is not integrable and none vanishes almost everywhere, so the product integrand is not integrable; with the convention that the integral of a non-integrable function is 0, under which $N(\sigma, F, p)$ and $O(\gamma)$ are defined, both sides are again 0. Hence in all cases

$$O(g)_{(i,j)} = \prod_v O_v(g_v)_{(i_v, j_v)}.$$

With $g = (w_P\sigma)_\infty$ and $g' = (w_Q\sigma)_\infty$ (Step 3), and since ρ_p, ρ_q are real,

$$\begin{aligned}
\big\langle \rho_p O(w_P\sigma), \rho_q O(w_Q\sigma)\big\rangle &= \rho_p \rho_q \sum_{i,j \in I} \overline{O(g)_{(i,j)}}\, O(g')_{(i,j)} \\
&= \prod_v \Big(s_v(p) s_v(q) \sum_{i_v, j_v \in I_v} \overline{O_v(g_v)_{(i_v,j_v)}}\, O_v(g_v')_{(i_v,j_v)} \Big),
\end{aligned}$$

the sum over $I \times I$ factoring as a product of sums over the $I_v \times I_v$. The factor at v is $\langle N(\sigma_v, W_v^\flat, x_v(p)), N(\sigma_v, W_v^\flat, x_v(q))\rangle = J_n^{(v)}(x_v(p), x_v(q))$ by Steps 2 and 3, which proves the identity. □

Lemma 5.2.106. *Let $\mathfrak{S}$ be admissible and let s be an integer with $0 \le s \le f$. Then $c_{\mathrm{nm}} > 0$ and*

$$(1+b^{-6})\,c_S^\sharp \prod_{a=0}^{\varrho} \mathrm{pc}(a)\Big[\prod_{v\in\Phi} \frac{1+b^{-10}}{\deg_v}\cdot\frac{1+b^{-6}}{b^{6ks}} + b^{2403n^9 f}\,\mathrm{dc}\Big] \le c_{\mathrm{nm}}^2\,(1+b^{-4})\,b^{-6ks}\,\mathcal{D}_{\mathrm{main}}\,\nu^2.$$

Proof. Throughout, $b \ge 1000$, $\mathrm{rd} = \mathrm{rd}_E \ge 1$, $|\mathrm{disc}\,E| = \mathrm{rd}^{2f}$ and $\mathrm{vol}(\hat{\mathcal{O}}) = (2/\mathrm{rd})^f$. Since $\mathrm{rs}_v \ne 0$ by (FA4) and $N > 0$, every $\mathrm{fs}(v,a) = \mathrm{rs}_v^{2(\varrho-a)}/N$ is positive; with $c_G > 0$ this gives $c_{\mathrm{nm}} > 0$. Put $Y = b^{-6ks}\mathcal{D}_{\mathrm{main}}\nu^2 \ge 0$ and

$$w = \mathrm{vol}(\hat{\mathcal{O}})^n\,(\#I/\Lambda_{\mathrm{fr}})^2\,c_G^{-f}.$$

The Poisson constants. As $\mathrm{pc}(a) = \mathrm{vol}(\hat{\mathcal{O}})/\prod_v \mathrm{fs}(v,a)^2$ and $\Lambda_{\mathrm{fr}} = \prod_v \prod_{a=0}^{\varrho} \mathrm{fs}(v,a)^{k+2(\varrho-a)}$,

$$\prod_{a=0}^{\varrho} \mathrm{pc}(a) = c_{\mathrm{nm}}^2\,w,$$

since in $c_{\mathrm{nm}}^2 w$ each $\mathrm{fs}(v,a)$ occurs with exponent $2(k-1+2(\varrho-a)) - 2(k+2(\varrho-a)) = -2$ and the factors c_G^f, $\#I^2$ cancel.

The mass term. Multiplying (FA9) by w gives $c_S^\sharp w \le (1+b^{-9})\mathcal{D}_{\mathrm{main}}\deg\,\nu^2$. By (FG6), $f \le b/1000$, so with $u = b^{-9}/1000 \le 1$ and $e^u \le 1+2u$ on $[0,1]$,

$$(1+b^{-10})^f \le \exp(fb^{-10}) \le \exp(b^{-9}/1000) \le 1+b^{-9}.$$

As $\prod_v \deg_v = \deg$, the first summand in the bracket times $c_S^\sharp w$ is at most $(1+b^{-9})^2(1+b^{-6})\,Y$.

The error term. We show $c_S^\sharp w\,\mathrm{dc} \le b^{n^9 f}\,\#I\,\mathcal{D}_{\mathrm{id}}\,\nu^2$. Since ϖ generates the different of $E/\mathbb{Q}$, $|N_{E/\mathbb{Q}}(\varpi)| = |\mathrm{disc}\,E|$, and as Φ contains one embedding of each conjugate pair, $\prod_v |v(\varpi)| = \mathrm{rd}^f$; hence $\prod_v |\mathrm{rs}_v| = Q^f$ with $Q = M\mathrm{rd}/(2\pi)$, and $Q \ge 1$ by (FA4). With $x_v = |\mathrm{rs}_v|^{2\varrho+1}/(MN)$ one has $\prod_v x_v = x^f$ for $x = Q^{2\varrho+1}/(MN)$, so $\mathrm{dc} = \Theta^f x^{f(k-n^7)}$ ($k \ge n^7$ by (FG6)). Put $C = \#\bar{R}_0\,\#\mathcal{C}\,(\#I/\Lambda_{\mathrm{fr}})^2$. Inserting $c_S^\sharp = \#\bar{R}_0 M^{f(2d+\varrho)}\#\mathcal{C}\,c_S$, $c_S = (\pi^k/((k-1)!N^k))^f\mathrm{rd}^{f/2}(2/\mathrm{rd})^{kf}$, $\mathcal{D}_{\mathrm{id}} = \#\bar{R}_0\#\mathcal{C}\#I$ and $\nu\Lambda_{\mathrm{fr}} = (M^{2d+\varrho}N^{-k})^f(\mathrm{rd}/2)^{df}\mathrm{rd}^{\varrho f/2}$,

$$c_S^\sharp w\,\mathrm{dc} = C\,A^f, \qquad \#I\,\mathcal{D}_{\mathrm{id}}\,\nu^2 = C\,B^f,$$

$$A = M^{2d+\varrho}\frac{\pi^k}{(k-1)!\,N^k}\,\mathrm{rd}^{1/2}\Big(\frac{2}{\mathrm{rd}}\Big)^{k+n}\frac{\Theta}{c_G}\,x^{k-n^7}, \qquad B = \Big(M^{2d+\varrho}N^{-k}(\mathrm{rd}/2)^d\,\mathrm{rd}^{\varrho/2}\Big)^2.$$

The volumes of the unitary groups cancel in Θ/c_G, and $\mathrm{vol}(S^{2k-1}) = 2\pi^k/(k-1)!$, so $\Theta/c_G = \pi^{2\varrho k}(k-1)!/2$. Expanding $x^k = M^{2\varrho k}\mathrm{rd}^{(2\varrho+1)k}(2\pi)^{-(2\varrho+1)k}N^{-k}$ and using $2d+\varrho = 2\varrho k+2\varrho^2+\varrho$,

$$\frac{A}{B} = \Big(\frac{2}{\mathrm{rd}}\Big)^{n+2\varrho^2}\frac{\mathrm{rd}^{1/2-\varrho}}{2M^{\varrho(2\varrho+1)}}\,x^{-n^7}.$$

Here $2/\mathrm{rd} \le 2 \le b$, $\mathrm{rd}^{1/2-\varrho} \le 1$, $2M^{\varrho(2\varrho+1)} \ge 1$, and $x^{-1} = MN/Q^{2\varrho+1} \le MN \le b^{23}b^{60n}$ by $Q \ge 1$, $M \le b^{23}$ (FG6) and (FA5). So $A/B \le b^{n+2\varrho^2+(23+60n)n^7} \le b^{n^9}$, because $n \ge 10^5$ gives $n+2\varrho^2+23n^7+60n^8 \le 61n^8 \le n^9$. Hence $c_S^\sharp w\,\mathrm{dc} = CA^f \le b^{n^9 f}CB^f$, the claim.

By (FA7) and (FA6), $\#I\,\mathcal{D}_{\mathrm{id}} \le b^{2kf}b^{-11kf}\mathcal{D}_{\mathrm{main}}$, so

$$c_S^\sharp w\,b^{2403n^9 f}\mathrm{dc} \le b^{2404n^9 f+2kf-11kf}\mathcal{D}_{\mathrm{main}}\nu^2 \le b^{-10}\,Y,$$

because $2404n^9f+10+6ks \le 2404n^9f+10+6kf \le 9kf$: indeed $k \ge 1000(2\varrho+2)^9 = 512000\,n^9$ by (FG6), so $3kf \ge 1536000\,n^9f \ge 2404n^9f+10$.

Conclusion. By the three displays, the left side of the claim is at most $c_{\mathrm{nm}}^2(1+b^{-6})\big[(1+b^{-9})^2(1+b^{-6})+b^{-10}\big]Y$. With $u=b^{-1}\le 10^{-3}$ and $\epsilon = u^4/100$ we have $u^6,u^9,u^{10}\le\epsilon\le 1$, hence

$$(1+u^6)\big[(1+u^9)^2(1+u^6)+u^{10}\big] \le (1+\epsilon)^4+\epsilon(1+\epsilon) = 1+\epsilon(5+7\epsilon+4\epsilon^2+\epsilon^3) \le 1+17\epsilon \le 1+u^4,$$

which is the claim. □

5.3 The remaining terms

This section bounds the terms of the second moment that the main term does not control (Theorem 5.3.2). They enter the proof of the averaged second moment twice: in the passage from the terms of nonzero discriminant to all terms (Proposition 5.6.42), and in the base case of the induction, where part (iv) bounds the link moment of the fixer of a frame vector at all chosen places by $(b^{2f}/p_0^f)^k\mathcal{D}_{\mathrm{main}}\nu^2$.

The idea is to bound the remaining terms in pairs. A cut amplitude sum over one block is split by links (Lemma 5.3.17), and each amplitude sum of a stable set of terms is at most ν times the sum of the sizes of its near representatives, by the orbit bound (Proposition 5.3.19). What remains are sums of products $\mathrm{size}(\gamma)\,\mathrm{size}(\gamma')$ over pairs near over an agreeing block, which a gain and a count bound (Lemma 5.3.3); we call this assembly the pair bound. The gain is that the discriminant of the pair lattice is small compared with the level gauges of the two terms (Proposition 5.3.64). The count is that the pairs whose common definite part has a given dimension, weighted by $\mathrm{disc}(L_{12})^{-k}$, have a small total (Proposition 5.3.66). When the links of the two terms differ, the link corank is positive, the gain is $(M/2)^{c^+f}$, and these pairs contribute a negligible multiple of the identity moment (part (i)). When the links agree and the cell rank exceeds ϱ, the gain comes from the meet norm instead, and this is the saving $(b^{2f}/q)^k$ of parts (ii) and (iv). The architecture is that of [259].

The section opens with the frame volumes, Schur weights and lattice translates that the orbit bound reads (Definition 5.3.1). After the theorem come the lemmas that assemble the pair bound: the lattice of a link and its lifts, the splitting of a link along a second one (Proposition 5.3.5), and the dimension count relating the common definite part, the link corank and the cell rank (Lemma 5.3.13). The Liouville inequality for Gram determinants (Lemma 5.3.14) is, in our reading, the form taken here by the fact that a nonzero algebraic integer has norm at least one. For hermitian lattices and their discriminants see [275].

Definition 5.3.1. Let $\mathfrak{S}$ be a filling datum. Matrices over $\mathbb{C}$ are taken in the basis $e_1,\dots,e_\varrho$, the basis of X_0, z, $e_1^*,\dots,e_\varrho^*$ of $V_1=\mathbb{C}^r$, and $\langle x,x'\rangle=\sum_l \bar{x}_l x'_l$ is the standard inner product of $\mathbb{C}^r$. For an integer $j\ge 0$ put $i_j=\min(\varrho,j)$.

(SX1) Frame volume and volume ratio. For $y\in G_1$ the **frame volume** is

$$\mathrm{Fv}(y) = \Big|\det\big(\langle (1-J_1)f_a,\, y^{-1}f_b\rangle\big)_{a,b=0}^{\varrho}\Big|,$$

the volume, up to a constant factor, of the projection of the moved frame to the (-1)-eigenspace $V_- = (1-J_1)V_1$ of J_1. For $\gamma,y\in G_1$ the **volume ratio** is $\mathrm{Vr}_\gamma(y) = \mathrm{Fv}(\gamma^{-1}y)/\mathrm{Fv}(y)$.

(SX2) Box cutoff, Schur weight and Schur integral. For $c \in \mathcal{C}$ the **box cutoff** is $\beta_c^{\mathrm{bx}} = \beta_c \cdot \mathbf{1}_{\mathrm{bx}}$, $\mathbf{1}_{\mathrm{bx}}$ the indicator of the boxed points of G_∞. For $c, c' \in \mathcal{C}$, $\gamma \in G(E)$, an integer $i \geq 0$ (the **shift**) and $y \in G_\infty$, the **Schur weight** and the **Schur integral** are

$$\mathrm{Sw}_{c',\gamma,i}(y) = \beta_{c'}^{\mathrm{bx}}(\gamma^{-1} y) \prod_{v \in \Phi} \mathrm{Vr}_{\gamma_v}(y_v)^{-(k+i)},$$

$$\mathrm{Si}(c, c', \gamma, i) = \int_{G_\infty} \beta_c^{\mathrm{bx}}(y)\, \|W^{\mathrm{cut}}(y)\|^2\, \mathrm{Sw}_{c',\gamma,i}(y)\, dy \in [0, \infty],$$

where $\|W^{\mathrm{cut}}(y)\|^2 = \prod_{v \in \Phi} \|W_v^\flat(y_v)\|^2$ (the integral of a nonnegative function).

(SX3) Eichler shifts. For $j \geq 0$ and $v \in \Phi$ let $D_{j,v}$ be the diagonal $i_j \times i_j$ matrix with entries $M/\mathrm{bs}_v(e_{l+1})$, $0 \leq l < i_j$. For $x = (x_v)_{v \in \Phi}$ with $x_v \in \mathrm{M}_{i_j \times k}(\mathbb{C})$, the **Eichler shift** with parameter x is $\mathrm{Es}_j(x) \in G_\infty$,

$$\mathrm{Es}_j(x)_v = u(a_v, 0), \qquad a_v = \begin{pmatrix} D_{j,v}^* x_v \\ 0 \end{pmatrix} \in \mathrm{M}_{\varrho \times k}(\mathbb{C}),$$

the unipotent element fixing $f_0, \ldots, f_\varrho$ that moves X_0 along $e_1, \ldots, e_{i_j}$ by the rows of $D_{j,v}^* x_v$.

(SX4) Orbit kernel bound. For $y \in G_1$ and $0 \leq i \leq \varrho$ let $Z^{(i)}(y)$ be the submatrix of $Z(y)$ on the rows and columns $\{0, \ldots, i-1\} \cup \{\varrho\}$ (the products of the subframe $e_1, \ldots, e_i, z$ with its image). For $\gamma \in G(E)$, $j \geq 0$ and $v \in \Phi$, with $i = i_j$, the **orbit kernel bound** is

$$\mathrm{Kb}_j(\gamma, v) = |\det D_{j,v}|^{-2k}\, \frac{(2b^{74n})^{(i+1)i} \left(2^{i+1}\, b^{2(6n+2)(j-i)}\right)^k}{|\det Z^{(i)}(\gamma_v)|^{k+i}}$$

if $\det Z^{(i)}(\gamma_v) \neq 0$, and $\mathrm{Kb}_j(\gamma, v) = 0$ otherwise.

(SX5) Lattice translates. Let $c \in \mathcal{C}$, $j \geq 0$, $\gamma, p_0, p \in G(E)$ and $l = (l_0, \ldots, l_{i_j - 1}) \in \mathrm{M}_{k \times i_j}(E)$ with columns $l_a \in X_0$. Then p is a **lattice translate** of p_0 by l (for c, γ, j) if every l_a lies in the definite part $L_c \cap X_0(\mathbb{A}_f)$ of the class lattice and, for $0 \leq b \leq \varrho$,

$$p f_b \equiv p_0 f_b + M \sum_{a < i_j} h(e_{a+1}, \gamma f_b)\, l_a \pmod{\langle e_1, \ldots, e_\varrho \rangle}.$$

(SX6) Column lattice at infinity. For $c \in \mathcal{C}$ and an integer $i \geq 0$, the **column lattice at infinity** is the additive group

$$\Lambda_{c,i}^\infty = \{(v(q)^*)_{v \in \Phi} :\ q \in \mathrm{M}_{k \times i}(E) \text{ with every column in } L_c \cap X_0(\mathbb{A}_f)\} \subseteq \prod_{v \in \Phi} \mathrm{M}_{i \times k}(\mathbb{C}),$$

where $v(q)$ is the entrywise image of q under v and * is the conjugate transpose; so the entry (l, a) of the component at v is $\overline{v(q_{al})}$.

Theorem 5.3.2. *Let $\mathfrak{S}$ be admissible with Weil data, and $\Omega \subseteq \bar{G}$.*

(i) $\sum_{c,c'} \sum \mathrm{size}(\gamma)\, \mathrm{size}(\gamma') \leq \mathcal{D}_{\mathrm{id}}/b^{kf}$, *over the pairs near over an agreeing block for (c, c') that are not in the same link class.*

(ii) For $q \in \mathbb{N}$, $q \geq 1$: $\sum_{c,c'} \sum \mathrm{size}(\gamma)\, \mathrm{size}(\gamma') \leq (b^{2f}/q)^k \mathcal{D}_{\mathrm{main}}$, *over the pairs near over an agreeing block in the same link class with γ over some (c, s, c') of meet norm $\geq q$.*

(iii) If $w \geq 0$ and $\mathrm{LM}(\{\mathsf{g} \neq 0\}, \Omega) \leq w\nu^2$, then $\mathrm{SM}(A^{\mathrm{cut}}_{\Gamma^*}, \Omega) \leq w + \mathcal{D}_{\mathrm{id}}/b^{kf}$.

(iv) If $q \in \mathbb{N}$, $q \geq 1$, and every $s \in \Omega$ has meet norm $\geq q$, then $\mathrm{LM}(\{\mathsf{g} \neq 0\}, \Omega) \leq (b^{2f}/q)^k \mathcal{D}_{\mathrm{main}} \nu^2$.

Proof. Throughout, $\tau_{\mathrm{sz}} \geq 1$, $\tau_{\mathrm{sz}}^4 = b^{kf}$, $\mathcal{D}_{\mathrm{id}} \geq 0$, and by (FA6) $\mathcal{D}_{\mathrm{main}} \geq b^{11kf}\mathcal{D}_{\mathrm{id}} \geq 0$; hence $\mathcal{D}^{\mathrm{sz}}_j \leq \mathcal{D}_{\mathrm{main}}$ for every cell rank j (for $j \leq \varrho$, $\mathcal{D}^{\mathrm{sz}}_j = \mathcal{D}_{\mathrm{id}}/\tau_{\mathrm{sz}}^{16} \leq \mathcal{D}_{\mathrm{id}} \leq \mathcal{D}_{\mathrm{main}}$). For a pair (γ, γ') near over the agreeing block (c, s, c') write j, j' for the cell ranks, $G = \mathsf{g}^*(j, c, c', \gamma)$, $G' = \mathsf{g}^*(j', c, c', \gamma')$, $d = \mathrm{disc}(L_{12})$, c^+ for the link corank and $l = k - \dim_E X_0(\gamma, \gamma')$. By Lemma 5.3.13 (iii),

$$l \leq c^+ + (j - \varrho)_+. \tag{5.35}$$

(i). Apply Lemma 5.3.3 with $B = \mathcal{D}_{\mathrm{id}}/\tau_{\mathrm{sz}}^{10}$ and $P(c, c')$ the set of pairs of (i). Let $(\gamma, \gamma') \in P(c, c')$.

- If $c^+ = 0$, then $\operatorname{link}\gamma = \operatorname{link}\gamma'$ by Lemma 5.3.13 (iv); as the pair is not in one link class, $\mathsf{g}(\gamma)\mathsf{g}(\gamma') = 0$, and by Lemma 5.3.13 (v) $\mathsf{g}(\gamma) = \mathsf{g}(\gamma') = 0$. By (TA5) $j, j' \leq \varrho$, so $l = 0$ by (5.35), and $\mathcal{D}^{\mathrm{sz}}_j = \mathcal{D}^{\mathrm{sz}}_{j'} = \mathcal{D}_{\mathrm{id}}/\tau_{\mathrm{sz}}^{16}$. By Proposition 5.3.64 with $x = 1$, $d^2 \leq GG'$. Take $\eta = 1$, $D = \mathcal{D}_{\mathrm{id}}/\tau_{\mathrm{sz}}^{16}$; then $\tau_{\mathrm{sz}}^{l+6} D\eta^{-k} = \mathcal{D}_{\mathrm{id}}/\tau_{\mathrm{sz}}^{10} = B$.
- If $c^+ \geq 1$, take $\eta = (M/2)^{c^+ f}$ and $D = \mathcal{D}_{\mathrm{main}}$. By Proposition 5.3.64 with $x = 1$, $(d\eta)^2 \leq GG'$, and $\mathcal{D}^{\mathrm{sz}}_j, \mathcal{D}^{\mathrm{sz}}_{j'} \leq D$. By (5.35), $l \leq c^+ + 1$. Put $y = \tau_{\mathrm{sz}}(2/M)^{kf}$. Since $M \geq b^{19} \geq 2b$ (FG6), $y \leq \tau_{\mathrm{sz}}^4(2/M)^{kf} = (2b/M)^{kf} \leq 1$, so $y^{c^+} \leq y$, and with (FA6)

$$\begin{aligned}\tau_{\mathrm{sz}}^{l+6} D\eta^{-k} &\leq \tau_{\mathrm{sz}}^{c^+ + 7}(2/M)^{c^+ kf}\mathcal{D}_{\mathrm{main}} = \tau_{\mathrm{sz}}^7 y^{c^+}\mathcal{D}_{\mathrm{main}} \\ &\leq \tau_{\mathrm{sz}}^8 (2/M)^{kf}(M/b^6)^{kf}\mathcal{D}_{\mathrm{id}} \leq \tau_{\mathrm{sz}}^8 b^{-5kf}\mathcal{D}_{\mathrm{id}} = \frac{\mathcal{D}_{\mathrm{id}}}{\tau_{\mathrm{sz}}^{12}} \leq B.\end{aligned}$$

So the hypotheses of Lemma 5.3.3 hold, and the sum of (i) is at most $\tau_{\mathrm{sz}} B = \mathcal{D}_{\mathrm{id}}/\tau_{\mathrm{sz}}^9 \leq \mathcal{D}_{\mathrm{id}}/\tau_{\mathrm{sz}}^4 = \mathcal{D}_{\mathrm{id}}/b^{kf}$.

(ii). Apply Lemma 5.3.3 with $B = \tau_{\mathrm{sz}}^7 \mathcal{D}_{\mathrm{main}}/q^k \geq 0$ and $P(c, c')$ the set of pairs of (ii). Let $(\gamma, \gamma') \in P(c, c')$, near over the agreeing block (c, s, c'), and s' with meet norm $\geq q$ such that γ is a term over (c, s', c'). As $s = \mathrm{red}(h_c^{-1}\gamma h_{c'}) = s'$, the meet norm of s is at least q. The pair is in one link class, so $\mathsf{g}(\gamma) \neq 0$ and $\operatorname{link}\gamma = \operatorname{link}\gamma'$; hence $c^+ = 0$ (Lemma 5.3.13 (iv)), $\varrho < j$ (TA5), and $l \leq 1$ by (5.35). By Proposition 5.3.64 with $x = q$, $(dq)^2 \leq GG'$. Take $\eta = q$ and $D = \mathcal{D}_{\mathrm{main}}$: then $\mathcal{D}^{\mathrm{sz}}_j, \mathcal{D}^{\mathrm{sz}}_{j'} \leq D$ and $\tau_{\mathrm{sz}}^{l+6} D\eta^{-k} \leq \tau_{\mathrm{sz}}^7 \mathcal{D}_{\mathrm{main}}/q^k = B$. Hence the sum of (ii) is at most $\tau_{\mathrm{sz}} B = \tau_{\mathrm{sz}}^8 \mathcal{D}_{\mathrm{main}}/q^k = (b^{2f}/q)^k \mathcal{D}_{\mathrm{main}}$.

Γ^* **is stable,** by Lemma 5.3.18 (ii).

(iii). Let $c, c' \in \mathcal{C}$ and $s \in \Omega$. By Lemma 5.3.17 for $T = \Gamma^*$, using $\Gamma^* \cap \{\mathsf{g} \neq 0\} = \{\mathsf{g} \neq 0\}$,

$$\|A^{\mathrm{cut}}_{\Gamma^*}(c, s, c')\|^2 \leq \sum_L \|A^{\mathrm{cut}}_{\{\mathsf{g}\neq 0\}\cap \operatorname{link}^{-1}(L)}(c, s, c')\|^2 + \nu^2 \sum_{(\gamma,\gamma') \in \mathcal{A}(c,s,c')} \mathrm{size}(c, c', \gamma)\,\mathrm{size}(c, c', \gamma'),$$

where $\mathcal{A}(c, s, c')$ is the set of pairs of near representatives of Γ^* for (c, c') over (c, s, c'), satisfying the conditions of their cell ranks, with s agreeing, that are not in one link class. Every pair in $\mathcal{A}(c, s, c')$ is near over an agreeing block for (c, c') and not in one link class, and the sets $\mathcal{A}(c, s, c')$ for different s are disjoint, because a term determines $s = \mathrm{red}(h_c^{-1}\gamma h_{c'})$. Summing over c, c' and $s \in \Omega$ and dividing by ν^2, the first terms give $\nu^{-2}\mathrm{LM}(\{\mathsf{g} \neq 0\}, \Omega) \leq w$ and the second terms give at most the sum of (i). So $\mathrm{SM}(A^{\mathrm{cut}}_{\Gamma^*}, \Omega) \leq w + \mathcal{D}_{\mathrm{id}}/b^{kf}$.

(iv). For a subspace L let $T_L = \{\mathsf{g} \neq 0\} \cap \mathrm{link}^{-1}(L)$; it lies in Γ^* and is stable by Lemma 5.2.25 (i), (ii). For $c, c' \in \mathcal{C}$ and $s \in \Omega$, Proposition 5.3.19 (ii) gives

$$\|A^{\mathrm{cut}}_{T_L}(c,s,c')\|^2 \leq \nu^2 \sum_{\gamma,\gamma' \in \mathcal{N}_L(c,s,c')} \mathrm{size}(c,c',\gamma)\,\mathrm{size}(c,c',\gamma'),$$

with $\mathcal{N}_L(c,s,c')$ the near representatives of T_L for (c,c') over (c,s,c') satisfying the conditions of their cell ranks, empty unless s agrees. A pair (γ,γ') on the right is near over an agreeing block for (c,c'), lies in one link class ($\mathsf{g} \neq 0$ for both, both links equal L), and γ is a term over (c,s,c') with meet norm of s at least q. The map $(s,L,\gamma,\gamma') \mapsto (\gamma,\gamma')$ is injective, since s is determined by γ and $L = \mathrm{link}\,\gamma$. Summing over c, c', $s \in \Omega$ and L, all terms being nonnegative, $\mathrm{LM}(\{\mathsf{g} \neq 0\}, \Omega)$ is at most ν^2 times the sum of (ii), that is, at most $(b^{2f}/q)^k \mathcal{D}_{\mathrm{main}} \nu^2$. □

Lemma 5.3.3. *Let $\mathfrak{S}$ be an admissible filling datum with Weil data, let $B \geq 0$, and for $c, c' \in \mathcal{C}$ let $P(c,c')$ be a set of pairs near over an agreeing block for (c,c'). Suppose that for every $(\gamma,\gamma') \in P(c,c')$, with cell ranks j, j' of γ, γ', $d = \mathrm{disc}(L_{12})$ and $l = k - \dim_E X_0(\gamma,\gamma')$, there are real numbers $\eta > 0$ and D with*

$$(d\eta)^2 \leq \mathsf{g}^*(j,c,c',\gamma)\,\mathsf{g}^*(j',c,c',\gamma'), \qquad \mathcal{D}^{\mathrm{sz}}_j \leq D, \qquad \mathcal{D}^{\mathrm{sz}}_{j'} \leq D, \qquad \tau_{\mathrm{sz}}^{l+6} D \eta^{-k} \leq B.$$

Then

$$\sum_{c,c' \in \mathcal{C}} \sum_{(\gamma,\gamma') \in P(c,c')} \mathrm{size}(c,c',\gamma)\,\mathrm{size}(c,c',\gamma') \leq \tau_{\mathrm{sz}} B.$$

Proof. Let $(\gamma,\gamma') \in P(c,c')$, both terms over (c,s,c') with s agreeing, of cell ranks j, j' (so each satisfies the conditions of its cell rank), and write $G = \mathsf{g}^*(j,c,c',\gamma)$, $G' = \mathsf{g}^*(j',c,c',\gamma')$, d, l as in the statement, and c^+ for the link corank of γ,γ'.

The discriminant is at least 1 and the gauges are positive. The meet norm of s is a product of primes, so it is at least 1, and $x = 1$ is admissible in Proposition 5.3.5. Applied to (γ,γ') it shows that the common part P of γ with γ' at c is a lattice of rank $\dim_E X_0(\gamma,\gamma')$; applied to (γ',γ) (the term γ' is over (c,s,c') and satisfies the conditions of cell rank j') it gives $\mathrm{ti}(h_c^{-1}\gamma' h_{c'}) \geq 1$. Hence $d \geq 1$ by Lemma 5.3.4 (ii). By Proposition 5.3.64 with $x = 1$ (so $\epsilon = 1$),

$$d^2 \leq (2/M)^{2c^+ f}\, G\, G'. \tag{5.36}$$

Both gauges are ≥ 0, since $\mathsf{g}^{\max} \geq \prod_v |\mathsf{g}(\gamma)|_v \geq 0$; as the left side of (5.36) is ≥ 1, $GG' > 0$ and so $G > 0$, $G' > 0$.

The corank and the discriminant are bounded. Both γ and γ' are near terms over (c,s,c') satisfying the conditions of their cell ranks, so $G, G' \leq b^{n^3 f}$ by Lemma 5.3.16. By (FG6) and $b \geq 2$, $M \geq b^{19} \geq 2b^{18}$, so $2/M \leq b^{-18}$, and (5.36) gives

$$1 \leq d^2 \leq b^{-36c^+ f}\, b^{2n^3 f}.$$

As $b > 1$ and $f \geq 1$, this yields $18c^+ \leq n^3$ and $d \leq b^{n^3 f}$. By Lemma 5.3.13 (iii), $k \leq \dim_E X_0(\gamma,\gamma') + c^+ + (j - \varrho)_+$ with $(j-\varrho)_+ \leq 1$ (as $j \leq \varrho + 1$), and $\dim_E X_0(\gamma,\gamma') \leq \dim_E X_0 = k$. Hence

$$0 \leq l \leq c^+ + 1 \leq n^3/18 + 1 \leq n^3.$$

One pair. Let η, D be as in the hypothesis. Then $D \geq \mathcal{D}^{\mathrm{sz}}_j \geq 0$, $n_p > 0$, and by (TA13) $\mathrm{size}(c,c',\gamma)^2 = \tau_{\mathrm{sz}}^4 \mathcal{D}^{\mathrm{sz}}_j / (n_p G^k) \leq \tau_{\mathrm{sz}}^4 D/(n_p G^k)$, and likewise for γ'. Multiplying, taking square roots and using $GG' \geq (d\eta)^2 > 0$,

$$\mathrm{size}(c,c',\gamma)\,\mathrm{size}(c,c',\gamma') \leq \frac{\tau_{\mathrm{sz}}^4 D}{n_p\,(GG')^{k/2}} \leq \frac{\tau_{\mathrm{sz}}^4 D \eta^{-k}}{n_p}\, d^{-k} = \frac{\tau_{\mathrm{sz}}^{l+6} D \eta^{-k}}{\tau_{\mathrm{sz}}^{l+2} n_p}\, d^{-k} \leq \frac{B}{\tau_{\mathrm{sz}}^{l+2} n_p}\, d^{-k}.$$

Summation. For $0 \le l \le n^3$ let $Q_l(c,c')$ be the set of pairs near over an agreeing block for (c,c') with $\dim_E X_0(\gamma,\gamma') + l = k$ and $\operatorname{disc}(L_{12}) \le b^{n^3 f}$. By the two preceding steps every pair of $P(c,c')$ lies in $Q_l(c,c')$ for its own $l \in \{0,\dots,n^3\}$ and its summand is at most $B\, d^{-k}/(\tau_{\mathrm{sz}}^{l+2} n_p)$. All summands being nonnegative, Proposition 5.3.66 gives

$$\sum_{c,c'} \sum_{P(c,c')} \text{size} \cdot \text{size} \le \sum_{l=0}^{n^3} \frac{B}{\tau_{\mathrm{sz}}^{l+2} n_p} \sum_{c,c'} \sum_{(\gamma,\gamma') \in Q_l(c,c')} \operatorname{disc}(L_{12})^{-k} \le \sum_{l=0}^{n^3} B = (n^3+1)B.$$

Finally $n \le b$ by (FG6), so $n^3 + 1 \le 2b^3 \le b^4 \le b^{kf/4} = \tau_{\mathrm{sz}}$, because $b \ge 2$ and $kf \ge 16$ by (FG6). This gives the bound $\tau_{\mathrm{sz}} B$. □

Lemma 5.3.4. *Let $\mathfrak{S}$ be a filling datum, $c,c' \in \mathcal{C}$, $s \in \bar{G}$, γ,γ' terms over (c,s,c'), $k' = h_c^{-1}\gamma' h_{c'}$, $m = \dim_E X_0(\gamma,\gamma')$, and P the common part of γ with γ' at c.*

(i) $[P : L_{12}]$ divides $\operatorname{ti}(k')$.

(ii) If P is a lattice of rank m and $\operatorname{ti}(k') \ge 1$, then L_{12} is a lattice of rank m and

$$1 \le \operatorname{disc}(L_{12}) = \operatorname{disc}_m(P)\,[P : L_{12}] \le \operatorname{disc}_m(P)\operatorname{ti}(k').$$

Proof. Write $A = \langle e_1,\dots,e_\varrho\rangle$, $L = L_c$, $A_{\hat{\mathcal{O}}} = \bigoplus_a \hat{\mathcal{O}} e_a$, and $\varphi'(x) = (h(\gamma' f_b, x))_{0\le b\le \varrho}$ for $x \in V(\mathbb{A}_f)$. Since h_c acts on X_0 only, $L \cap A(\mathbb{A}_f) = h_c(L_0 \cap A(\mathbb{A}_f)) = A_{\hat{\mathcal{O}}}$. By (LT1), $x \in L \cap X_0(\mathbb{A}_f)$ lies in $L_c(\gamma')$ iff $\varphi'(x) \in M\,R'$, where $R' = \varphi'(A_{\hat{\mathcal{O}}})$.

Step 1 (a rational identity). Let $t \in A \oplus X_0$ have X_0-part $x \in X_0(\gamma,\gamma')$ and $t \perp \gamma f_b$ for all b. Then $\gamma'\gamma^{-1}t - t \in A + \gamma' A$. Indeed, $\sigma = \gamma^{-1}t$ is orthogonal to the frame, so it has no e^*-coordinates and z-coordinate 0; t has neither. Hence $t \perp X + \gamma X$ and $p = ([t],[\sigma]) = (x,\sigma_0)$ lies in $\operatorname{link}\gamma$, σ_0 the X_0-part of σ, with source $p^{\mathrm{s}} = x$. As $x \in X_0(\gamma,\gamma')$ there is $y \in X_0$ with $(x,y) \in \operatorname{link}\gamma \cap \operatorname{link}\gamma'$, of source x; by injectivity of the source on $\operatorname{link}\gamma$ (Lemma 5.2.28 (iii)), $y = \sigma_0$. So $(x,\sigma_0) \in \operatorname{link}\gamma'$: there is $t' \perp X + \gamma' X$ with $[t'] = x$, $[\gamma'^{-1}t'] = \sigma_0$. Then $t - t'$ and $\sigma - \gamma'^{-1}t'$ are orthogonal to X with middle part 0, so they lie in A, and $\gamma'\gamma^{-1}t - t = \gamma'(\sigma - \gamma'^{-1}t') + (t' - t) \in \gamma' A + A$.

Step 2 (the image of the common part). Let W be the E-space of the $t \in A \oplus X_0(\gamma,\gamma')$ with $t \perp \gamma f_b$ for all b; as these are E-linear conditions, $W(\mathbb{A}_f)$ is the space of $y \in (A \oplus X_0(\gamma,\gamma'))(\mathbb{A}_f)$ with $h(\gamma f_b, y) = 0$ for all b, and by Step 1 the map $y \mapsto (\gamma'\gamma^{-1} - 1)y$ sends $W(\mathbb{A}_f)$ into $(A + \gamma' A)(\mathbb{A}_f)$. Let $v \in P$. Then $v \in \operatorname{sat}_c(X_0(\gamma,\gamma')) \subseteq X_0(\gamma,\gamma')(\mathbb{A}_f)$ and $v \in L_c(\gamma)$, so there is $a \in A_{\hat{\mathcal{O}}}$ with $y := v - Ma \in W(\mathbb{A}_f)$, and $y \in L$. Let $k = h_c^{-1}\gamma h_{c'}$. Then $h_c^{-1}(\gamma'\gamma^{-1})h_c = k'k^{-1} \in K$ has reduction $ss^{-1} = 1$ (K is a group by Lemma 5.2.24 (ii), and the reduction is multiplicative), so $(k'k^{-1} - 1)L_0 \subseteq ML_0$ by Lemma 5.2.24 (iii) and $(\gamma'\gamma^{-1} - 1)L \subseteq ML$. Hence $(\gamma'\gamma^{-1} - 1)y \in ML \cap (A + \gamma' A)(\mathbb{A}_f) = M(L \cap (A + \gamma' A)(\mathbb{A}_f))$. Moreover $\varphi'(\gamma'\gamma^{-1}y) = (h(f_b, \gamma^{-1}y))_b = (h(\gamma f_b, y))_b = 0$. Therefore, with $Q = \varphi'(L \cap (A + \gamma' A)(\mathbb{A}_f))$,

$$\varphi'(v) = -\varphi'((\gamma'\gamma^{-1} - 1)y) + M\varphi'(a) \in MQ + MR' = MQ,$$

since $A_{\hat{\mathcal{O}}} \subseteq L \cap (A + \gamma' A)(\mathbb{A}_f)$ gives $R' \subseteq Q$.

Step 3 (the index). By Step 2, $v \mapsto M^{-1}\varphi'(v) + R'$ is a homomorphism $P \to Q/R'$. Its kernel is $P \cap L_c(\gamma') = L_{12}$. So P/L_{12} embeds in Q/R'. The map φ' sends $L \cap (A + \gamma' A)(\mathbb{A}_f)$ onto Q and $A_{\hat{\mathcal{O}}}$ onto R', and it kills $\gamma' A_{\hat{\mathcal{O}}}$, because $h(\gamma' f_b, \gamma' e_a) = h(f_b, e_a) = 0$. Here $\gamma' A_{\hat{\mathcal{O}}} = h_c k'(L_0 \cap A(\mathbb{A}_f)) \subseteq L$. So Q/R' is a quotient of

$$T = (L \cap (A + \gamma' A)(\mathbb{A}_f))/(A_{\hat{\mathcal{O}}} + \gamma' A_{\hat{\mathcal{O}}}),$$

and h_c (which fixes A, with $h_c k' A = \gamma' A$) identifies T with the quotient defining $\mathrm{ti}(k')$. If $\mathrm{ti}(k') = 0$ there is nothing to prove; otherwise T is finite of order $\mathrm{ti}(k')$, and $|P/L_{12}|$ divides $|Q/R'|$, which divides $|T|$. This proves (i).

Step 4 (sublattices). Let $L'' \subseteq L'$ be $\hat{\mathcal{O}}$-submodules of $X_0(\mathbb{A}_f)$ with $L' = B\hat{\mathcal{O}}^m$ a lattice of rank m ($B = x\alpha$, x of rank m over E, $\alpha \in \mathrm{GL}_m(\mathbb{A}_f)$) and $d = [L' : L''] \geq 1$. Then $dL' \subseteq L''$, so L'' is determined by its components $L''_w \subseteq L'_w$, equal to L'_w for $w \nmid d$; each L''_w has finite index in the free module L'_w of rank m over the discrete valuation ring $\hat{\mathcal{O}}_w$, hence $L''_w = B\, g_w \hat{\mathcal{O}}_w^m$ with $g_w \in \mathrm{M}_m(\hat{\mathcal{O}}_w)$, $\det g_w \neq 0$, and $g_w = 1$ for $w \nmid d$. With $g = (g_w) \in \mathrm{GL}_m(\mathbb{A}_f)$, $L'' = x(\alpha g)\hat{\mathcal{O}}^m$ is a lattice of rank m, and by the elementary divisor theorem $[L' : L''] = \prod_w |\det g_w|_w^{-1} = [\det g]$. Hence, by Definition 5.1.8,

$$\mathrm{disc}_m(L'') = [\det(g^* B^* B g)]^{1/2} = [\det g]\,[\det B^* B]^{1/2} = [L' : L'']\,\mathrm{disc}_m(L').$$

(ii). By (i) and $\mathrm{ti}(k') \geq 1$, $1 \leq [P : L_{12}] \leq \mathrm{ti}(k')$. Step 4 with $L' = P$, $L'' = L_{12}$ gives that L_{12} is a lattice of rank m with $\mathrm{disc}(L_{12}) = \mathrm{disc}_m(P)[P : L_{12}] \leq \mathrm{disc}_m(P)\,\mathrm{ti}(k')$. Finally let B be a basis matrix of L_{12} as in (LT3). Its columns lie in $L \cap X_0(\mathbb{A}_f)$, and $h(L, L) = h(L_0, L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24 (i), so B^*B (the form of X_0 is the unit form) has entries in $\hat{\mathcal{O}}$. Its determinant is $\overline{\det \alpha}\det(x^*x)\det\alpha$, a unit of $\mathbb{A}_f$, since $\det(x^*x) \neq 0$ (the unit form is positive definite at an infinite place and x has rank m). So $\det(B^*B) \in \hat{\mathcal{O}} \cap \mathbb{A}_f^\times$ and $\mathrm{disc}(L_{12})^2 = [\det B^* B] = [\hat{\mathcal{O}} : \det(B^*B)\hat{\mathcal{O}}] \geq 1$. □

Proposition 5.3.5. *Let $\mathfrak{S}$ be a filling datum with Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$ agreeing, γ a term over (c, s, c') satisfying the conditions of cell rank j, $\gamma' \in G(E)$, $k = h_c^{-1}\gamma h_{c'}$, c^+ the link corank of γ, γ', $m = \dim_E X_0(\gamma, \gamma')$, P the common part of γ with γ' at c, and $x \in \mathbb{N}$ at most the meet norm of s, such that $x = 1$ unless $\varrho < j$ and $c^+ = 0$. Then $\mathrm{ti}(k) \geq 1$, P is a lattice of rank m, and there are a finite family $q = (q_\rho)_{\rho \in R}$ of pairs spanning $\mathrm{link}\,\gamma \cap \mathrm{link}\,\gamma'$, pairs $y = (y_l)_{l < c^+}$ in $\mathrm{link}\,\gamma$, $\epsilon \in \mathrm{GL}_{c^+}(\mathbb{A}_f)$ and a matrix $\sigma \in \mathrm{M}_{R \times c^+}(\mathbb{A}_f)$ such that:*

- *(i) a combination $\sum_l \tau_l y_l$ ($\tau_l \in E$) lies in $\mathrm{link}\,\gamma \cap \mathrm{link}\,\gamma'$ only if $\tau = 0$, and q and y together span $\mathrm{link}\,\gamma$;*
- *(ii) the family $\left(\sum_\rho \sigma_{\rho l} q_\rho + \sum_{l'} \epsilon_{l'l} y_{l'}\right)_{l < c^+}$ has integral lifts for (c, γ);*
- *(iii) $\left(\mathrm{ti}(k) \cdot \mathrm{disc}_m(P) \cdot x\right)^2 \cdot \mathrm{lkd}(y, \epsilon) \leq \mathfrak{g}^*(j, c, c', \gamma)^2$.*

Proof. Write $L = \mathrm{link}\,\gamma \cap \mathrm{link}\,\gamma'$ and $L^0 = \{p \in L : \mathrm{zf}(p) = 0\}$, and

$$\Gamma = \max\Big(\prod_v |\mathfrak{g}(\gamma)|_v, N^f \mathrm{mn}_j(\gamma)\Big).$$

Step 1: three orthogonal bases. Let $\beta = (\beta_i)_{i<t}$ be a basis of L^0, $\eta = (\eta_l)_{l<r}$ a basis of the orthogonal complement of L^0 in L for the link metric, and $y = (y_l)_{l<t'}$ a basis of the orthogonal complement of L in $\mathrm{link}\,\gamma$. Then $(\beta|\eta)$ is a basis of L, $(\beta|\eta|y)$ is a basis of $\mathrm{link}\,\gamma$, and the three families are mutually orthogonal for the link metric. A pair p with $\mathrm{zf}(p) = 0$ has both members in X_0 (LK4); so by (LT2), $X_0(\gamma, \gamma')$ is the set of first members of the elements of L^0, i.e. the span of the first members of the β_i. If $\mathrm{zf}(\eta)\tau = 0$ for $\tau \in E^r$, then $\eta\tau \in L^0$ is a combination of the β_i, so $\tau = 0$: the $2 \times r$ matrix $\mathrm{zf}(\eta)$ has rank r. Finally $t + r = \dim L$ and $t + r + t' = \dim \mathrm{link}\,\gamma = \dim X_0 + 1$ (Lemma 5.2.28 (i)), so $t' = \dim X_0 + 1 - \dim L = c^+$ (LT5).

Step 2: the lattice of the link. By Lemma 5.3.6 for the basis $(\beta|\eta|y)$: $\mathrm{ir}_j(k) \geq 1$, $\mathrm{ir}_j(k^{-1}) \geq 1$, $\mathrm{ti}(k) \geq 1$, and there is $g \in \mathrm{GL}_{t+r+t'}(\mathbb{A}_f)$ such that $(\beta|\eta|y)g$ has level lifts for (c,γ) and

$$\mathrm{ti}(k)^2 \,\mathrm{lkd}((\beta|\eta|y), g) \leq \Big(\frac{M^{2\min(\varrho,j)f}\,\mathfrak{g}^\sharp(j,c,c',\gamma)}{N^f}\Big)^2. \tag{5.37}$$

Step 3: two orthogonal splittings. By Lemma 5.3.11 (i) for the orthogonal families $(\beta|\eta)$ and y and the matrix g, there are a_{12}, $\epsilon \in \mathrm{GL}_{c^+}(\mathbb{A}_f)$ and σ such that $(\beta|\eta)a_{12}$ and $(\beta|\eta)\sigma + y\epsilon$ have level lifts and $\mathrm{lkd}((\beta|\eta), a_{12})\,\mathrm{lkd}(y,\epsilon) = \mathrm{lkd}((\beta|\eta|y), g)$. Again by Lemma 5.3.11 (i), for β, η and a_{12}, there are a, $a' \in \mathrm{GL}_r(\mathbb{A}_f)$ and σ_0 such that βa and $\beta\sigma_0 + \eta a'$ have level lifts and $\mathrm{lkd}(\beta, a)\,\mathrm{lkd}(\eta, a') = \mathrm{lkd}((\beta|\eta), a_{12})$. Hence

$$\mathrm{lkd}(\beta, a)\,\mathrm{lkd}(\eta, a')\,\mathrm{lkd}(y,\epsilon) = \mathrm{lkd}((\beta|\eta|y), g). \tag{5.38}$$

Step 4: the common part. The β_i are linearly independent pairs in $\mathrm{link}\,\gamma$ with $\mathrm{zf}(\beta_i) = 0$, and βa has level lifts. By Lemma 5.3.11 (ii), with W the span of the first members, which is $X_0(\gamma,\gamma')$ by Step 1: $t = \dim X_0(\gamma,\gamma') = m$, the common part $P = \mathrm{sat}_c(X_0(\gamma,\gamma')) \cap L_c(\gamma)$ (LT4) is a lattice of rank m, and $\mathrm{disc}_m(P)^2 \leq \mathrm{lkd}(\beta, a)$.

Step 5: the part seen by the z-forms. We check the hypotheses of Lemma 5.3.11 (iii) for η, a' and $w = \beta\sigma_0 + \eta a'$. The η_l lie in $\mathrm{link}\,\gamma$ and $\mathrm{zf}(\eta)$ has rank r (Step 1); w has level lifts, hence integral lifts, and $\mathrm{zf}(w) = \mathrm{zf}(\beta)\sigma_0 + \mathrm{zf}(\eta)a' = \mathrm{zf}(\eta)a'$. By Lemma 5.3.13 (iii), $m + r = \dim L \leq m + 1 + \max(j - \varrho, 0)$, so $r \leq 1 + \max(j - \varrho, 0)$. If $x \neq 1$, then $\varrho < j$ and $c^+ = 0$ by hypothesis; so $\mathfrak{g}(\gamma) \neq 0$ (conditions of cell rank) and $\mathrm{link}\,\gamma = \mathrm{link}\,\gamma'$ (Lemma 5.3.13 (iv)), and by Lemma 5.3.13 (iv) $\dim L = m + 2$, i.e. $r = 2$. Hence

$$x^2\,\mathrm{mn}_j(\gamma)^2 \leq \mathrm{lkd}(\eta, a')\,\Gamma^2. \tag{5.39}$$

Step 6: assembly. Put $q = (\beta|\eta)$, indexed by $R = \{0,\dots,t-1\} \sqcup \{0,\dots,r-1\}$; it spans L. The family y has c^+ members (Step 1), lies in $\mathrm{link}\,\gamma$, is linearly independent modulo L, and $(q|y)$ spans $\mathrm{link}\,\gamma$: this is (i). The family $q\sigma + y\epsilon$ has level lifts, hence integral lifts, for (c,γ): this is (ii). For (iii), all factors being nonnegative, Step 4, (5.39), (5.38) and (5.37) give

$$\begin{aligned}\big(\mathrm{ti}(k)\,\mathrm{disc}_m(P)\,x\big)^2 \mathrm{lkd}(y,\epsilon)\,\mathrm{mn}_j(\gamma)^2 &\leq \mathrm{ti}(k)^2\,\mathrm{lkd}(\beta,a)\,\mathrm{lkd}(y,\epsilon)\,\mathrm{lkd}(\eta,a')\,\Gamma^2\\ &\leq \Big(\frac{M^{2\min(\varrho,j)f}\,\mathfrak{g}^\sharp(j,c,c',\gamma)}{N^f}\Big)^2\Gamma^2.\end{aligned}$$

By Lemma 5.3.9 (ii), applicable since $i_\gamma i'_\gamma \geq 1$ (Step 2), $\mathfrak{g}^\sharp(j,c,c',\gamma)/N^f = \mathrm{mn}_j(\gamma)/(i_\gamma i'_\gamma)$ and $\Gamma/(i_\gamma i'_\gamma) = \mathfrak{g}^{\max}(j,c,c',\gamma)$, so the right side equals $\big(M^{2\min(\varrho,j)f}\mathfrak{g}^{\max}(j,c,c',\gamma)\big)^2\mathrm{mn}_j(\gamma)^2 = \mathfrak{g}^*(j,c,c',\gamma)^2\,\mathrm{mn}_j(\gamma)^2$. Dividing by $\mathrm{mn}_j(\gamma)^2 > 0$ (Lemma 5.3.9 (i)) gives (iii). The first two claims, $\mathrm{ti}(k) \geq 1$ and that P is a lattice of rank m, were shown in Steps 2 and 4. □

Lemma 5.3.6. *Let $\mathfrak{S}$ be a filling datum with Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, γ a term over (c,s,c') satisfying the conditions of cell rank j, $k = h_c^{-1}\gamma h_{c'}$, and $v = (v_l)_{l\in\sigma}$ a basis of $\mathrm{link}\,\gamma$ over E. Then:*

(i) $\mathrm{ir}_j(k) \geq 1$, $\mathrm{ir}_j(k^{-1}) \geq 1$ *and* $\mathrm{ti}(k) \geq 1$;

(ii) there is $e \in \mathrm{GL}_\sigma(\mathbb{A}_f)$ such that the family $ve = \big(\sum_{l'} e_{l'l}v_{l'}\big)_{l\in\sigma}$ has level lifts for (c,γ) and

$$\mathrm{ti}(k)^2\,\mathrm{lkd}(v,e) \leq \Big(\frac{M^{2\min(\varrho,j)f}\,\mathfrak{g}^\sharp(j,c,c',\gamma)}{N^f}\Big)^2.$$

Proof. Let $\omega = \mathrm{ir}_j(k)\,\mathrm{ir}_j(k^{-1})\,\mathrm{ti}(k)$. By Lemma 5.3.7, $\omega \neq 0$, and there is $e_0 \in \mathrm{GL}_\sigma(\mathbb{A}_f)$ such that ve_0 has integral lifts for (c,γ) and $\mathrm{lkd}(v,e_0)\,\omega^2 \leq \mathrm{mn}_j(\gamma)^2$.

(i). The three indices are natural numbers with nonzero product, so each is at least 1.

(ii). By Lemma 5.3.10, applied to the family ve_0, there is an integral $B \in \mathrm{GL}_\sigma(\mathbb{A}_f)$ with $[\det B] \leq M^{2\min(\varrho,j)f}$ such that ve_0B has level lifts for (c,γ). Put $e = e_0B$. By (LK6), $\mathrm{lkd}(v,e) = \mathrm{lkd}(v,e_0)\,[\det B]^2$, so

$$\mathrm{ti}(k)^2\,\mathrm{lkd}(v,e) \leq M^{4\min(\varrho,j)f}\,\frac{\mathrm{lkd}(v,e_0)\,\omega^2}{(\mathrm{ir}_j(k)\,\mathrm{ir}_j(k^{-1}))^2} \leq \Big(\frac{M^{2\min(\varrho,j)f}\,\mathrm{mn}_j(\gamma)}{\mathrm{ir}_j(k)\,\mathrm{ir}_j(k^{-1})}\Big)^2.$$

By Lemma 5.3.9 (ii) (with $h_c^{-1}\gamma h_{c'} = k \in K$ and $i_\gamma i'_\gamma \geq 1$ by (i)), $\mathrm{mn}_j(\gamma)/(\mathrm{ir}_j(k)\,\mathrm{ir}_j(k^{-1})) = \mathfrak{g}^\sharp(j,c,c',\gamma)/N^f$, which gives (ii). □

Lemma 5.3.7. *Let $\mathfrak{S}$ be a filling datum with Weil data, $c,c' \in \mathcal{C}$, $s \in \bar{G}$, γ a term over (c,s,c') satisfying the conditions of cell rank j, $k = h_c^{-1}\gamma h_{c'}$, $v = (v_l)_{l\in\sigma}$ a basis of $\mathrm{link}\,\gamma$, and $\omega = \mathrm{ir}_j(k)\,\mathrm{ir}_j(k^{-1})\,\mathrm{ti}(k)$. Then $\omega \neq 0$, and there is $e \in \mathrm{GL}_\sigma(\mathbb{A}_f)$ such that ve has integral lifts for (c,γ) and*

$$\mathrm{lkd}(v,e)\,\omega^2 \leq \mathrm{mn}_j(\gamma)^2.$$

Proof. Write $j' = \min(j,\varrho)$, $\mathcal{B} = \{a : j' \leq a < \varrho\}$, $I = X \cap \gamma X$, $W = (X+\gamma X)^\perp$, $\Phi(t) = ([t],[\gamma^{-1}t])$ for $t \in W(\mathbb{A}_f)$, and $r = 2\varrho + \dim X_0 + 1 = \dim V$. By Lemma 5.2.28 (i), Φ maps W onto $\mathrm{link}\,\gamma$ with kernel I, and $\#\sigma = \dim X_0 + 1$.

Step 0: coordinates. The element $h_c \in \mathrm{U}(X_0)(\mathbb{A}_f)$ is an isometry fixing f_a, f_a^*, z; $h_{c'}$ likewise. By Lemma 5.2.24 (i), L_0 is self-dual for h, hence so is L_c. Let $\mathrm{cc}_c\colon V(\mathbb{A}_f) \to \mathbb{A}_f^r$ be the coordinates in the $\hat{\mathcal{O}}$-basis h_c(lattice basis) of L_c, and J the Gram matrix of that basis, so $h(x,y) = \mathrm{cc}_c(x)^* J \mathrm{cc}_c(y)$, $J^* = J$, J is integral and, L_c being self-dual, $\det J \in \hat{\mathcal{O}}^\times$. The lattice basis contains the f_a ($a < \varrho$), and no other vector of it has a component along f_a; so the cc_c-coordinate of x at f_a is its f_a-coordinate in $V = X \oplus X^* \oplus X_0 \oplus Ez$, which is integral for $x \in L_c$. Moreover $f_a \in L_c$ and $\gamma f_a = h_c k h_{c'}^{-1} f_a = h_c k f_a \in L_c$, as $kL_0 = L_0$ (Lemma 5.2.24 (ii)).

Step 1: vectors of I. For $a \in \mathcal{B}$ (so $j \leq a < \varrho$) the conditions of cell rank give $x^{(a)}, y^{(a)} \in E^{\varrho+1}$ supported on $\{0,\dots,a\}$, $x_a^{(a)} \neq 0$, with $\sum_b x_b^{(a)} f_b = \gamma \sum_b y_b^{(a)} f_b$; this vector lies in $X \cap \gamma X = I$. The matrix of the $f_{a'}$-coordinates ($a' \in \mathcal{B}$) of these vectors is triangular with nonzero diagonal, so suitable combinations $w_a \in I$ ($a \in \mathcal{B}$) have $f_{a'}$-coordinate $\delta_{aa'}$ for $a' \in \mathcal{B}$; in particular $\dim I \geq \varrho - j'$.

Step 2: a rational lift. Choose $t_l \in W$ with $\Phi(t_l) = v_l$ and replace t_l by

$$t_l - \sum_{a\in\mathcal{B}} (f_a\text{-coordinate of } t_l)\, w_a;$$

as $I \subseteq W$ and $\Phi(I) = 0$, still $t_l \in W$, $\Phi(t_l) = v_l$, and now the f_a-coordinates of t_l vanish for $a \in \mathcal{B}$. The family $u = (w_a)_{a\in\mathcal{B}} \cup (t_l)_{l\in\sigma}$ is linearly independent: applying Φ to a vanishing combination kills the w_a and gives a relation among the v_l, and then the $f_{\mathcal{B}}$-coordinates give the rest.

Step 3: primitive completion. Let $Y = \mathrm{cc}_c(u)$, a matrix over $\mathbb{A}_f$ with a left inverse (a rational left inverse of u composed with cc_c^{-1}). The saturation $S_Y = \hat{\mathcal{O}}^r \cap Y\mathbb{A}_f^{\mathcal{B}\sqcup\sigma}$ is at each finite place w a direct summand of $\mathcal{O}_w^r$ (the quotient is torsion free), free of rank $\#\mathcal{B}+\#\sigma$, and equals $Y_w\mathcal{O}_w^{\mathcal{B}\sqcup\sigma}$ for almost all w; so $S_Y = Yg$ with $g \in \mathrm{GL}_{\mathcal{B}\sqcup\sigma}(\mathbb{A}_f)$, and any $\hat{\mathcal{O}}$-basis of S_Y is primitive. By the Iwasawa decomposition of GL_n over a non-archimedean local field with respect to the standard

parabolic subgroup of block upper triangular matrices (blocks $\mathcal{B}$, σ), applied at each place, with the trivial decomposition at the almost all places where g is integral, write $g = \begin{pmatrix} \alpha & \varsigma \\ 0 & e \end{pmatrix} \eta_0$ with $\alpha \in \mathrm{GL}_{\mathcal{B}}(\mathbb{A}_f)$, $e \in \mathrm{GL}_\sigma(\mathbb{A}_f)$, $\eta_0 \in \mathrm{GL}(\hat{\mathcal{O}})$; then $P := Y \begin{pmatrix} \alpha & \varsigma \\ 0 & e \end{pmatrix} = Y g \eta_0^{-1}$ is a basis of S_Y, hence primitive and integral. Its columns are $\mathrm{cc}_c(x_a)$, $x_a = \sum_{a'} \alpha_{a'a} w_{a'} \in L_c \cap I(\mathbb{A}_f)$ ($a \in \mathcal{B}$), and $\mathrm{cc}_c(n_l)$, $n_l = \sum_a \varsigma_{al} w_a + \sum_{l'} e_{l'l} t_{l'} \in L_c$. Since $\Phi(n_l) = \sum_{l'} e_{l'l} v_{l'}$, the family ve has integral lifts for (c, γ) (LK5), by the vectors u and the coefficients $\begin{pmatrix} \varsigma \\ e \end{pmatrix}$. The $f_{a'}$-coordinates ($a' \in \mathcal{B}$) of x_a are the entries $\alpha_{a'a}$; by Step 0, α is integral, and $[\det \alpha] = [\hat{\mathcal{O}}^{\mathcal{B}} : \alpha \hat{\mathcal{O}}^{\mathcal{B}}]$.

Step 4: the data of Jacobi's identity. Let $F = \mathrm{cc}_c(f_0, \dots, f_{\varrho-1}, \gamma f_0, \dots, \gamma f_{j'-1})$, integral by Step 0, with $m_1 = \varrho + j'$ columns; P has $m_2 = (\varrho - j') + \dim X_0 + 1$ columns, and $m_1 + m_2 = r$. The columns of P are coordinates of vectors of $W(\mathbb{A}_f)$, orthogonal to $X + \gamma X$, so $F^* J P = 0$. Let $Q = \mathrm{cc}_c((f_a^*)_{a \in \mathcal{B}}, z, \gamma z)$, $m_3 = \#\mathcal{B} + 2$, and $t = N$. Then $\mathcal{M}$ of Lemma 5.3.8 is the Gram matrix for h of the vectors f_a ($a < \varrho$), γf_a ($a < j'$), f_a^* ($a \in \mathcal{B}$), z, γz, with N added to the diagonal entries of the last m_3.

Step 5: $[\det \mathcal{M}] = (N^{2f} \mathrm{mn}_j(\gamma))^2$. Order the vectors as

$$(f_0, \dots, f_{\varrho-1}, z\,;\, \gamma f_0, \dots, \gamma f_{j'-1}, (f_a^*)_{a \in \mathcal{B}}, \gamma z),$$

two blocks of $\varrho + 1$ (a simultaneous permutation of rows and columns, which does not change the determinant). The first diagonal block is the Gram matrix $\mathrm{diag}(0, \dots, 0, -N)$ of the frame plus N at z, i.e. 0. The upper right block $G_{12} = (h(f_a, g_b))$ has, for $g_b = \gamma f_b$ ($b < j'$) and $g_b = \gamma z$ ($b = \varrho$), the column b of $Z(\gamma)$, and for $g_b = f_b^*$ ($b \in \mathcal{B}$) the standard column ε_b ($h(f_a, f_b^*) = \delta_{ab}$, $h(z, f_b^*) = 0$). Expanding along these standard columns, $\det G_{12}$ is the minor of $Z(\gamma)$ on the rows and columns $\{a < j'\} \cup \{\varrho\}$, which is $\underline{\delta_*(\gamma)}$ (for $j = \varrho + 1$ both sets are $\{0, \dots, \varrho\}$). The lower left block is G_{12}^*, so $\det \mathcal{M} = \pm \delta_*(\gamma) \overline{\delta_*(\gamma)}$, and $[\det \mathcal{M}] = |N_{E/\mathbb{Q}} \delta_*(\gamma)|^2 = (N^{2f} \mathrm{mn}_j(\gamma))^2$ by Lemma 5.3.9 (i); it is nonzero, so $\det \mathcal{M}$ is a unit.

Step 6: the matrix R. For $t, t' \in W(\mathbb{A}_f)$: $[t] - t$ and $[\gamma^{-1} t] - \gamma^{-1} t$ lie in $X(\mathbb{A}_f)$, which is isotropic and orthogonal to t', $\gamma^{-1} t'$ and z; so $h([t], [t']) = h(t, t') = h([\gamma^{-1} t], [\gamma^{-1} t'])$, $h(z, [t]) = h(z, t)$, $h(z, [\gamma^{-1} t]) = h(\gamma z, t)$. For $a = y_0 + \lambda z$, $b = y_0' + \lambda' z$ in V_0, (LK1) gives $\langle a^+, b^+ \rangle_+ = h(a, b) + 2N \bar{\lambda} \lambda' = h(a, b) + 2N^{-1} \overline{h(z, a)} h(z, b)$. With (LK3),

$$\langle \Phi(t), \Phi(t') \rangle_{\mathrm{mt}} = h(t, t') + N^{-1} (\overline{h(z, t)} h(z, t') + \overline{h(\gamma z, t)} h(\gamma z, t')). \tag{5.40}$$

Now

$$R = P_0^* R_u P_0 \quad \text{with} \quad P_0 = \begin{pmatrix} \alpha & \varsigma \\ 0 & e \end{pmatrix}$$

and R_u the matrix of $h(u_x, u_y) + N^{-1} \sum_q \overline{h(q, u_x)} h(q, u_y)$, q over f_a^* ($a \in \mathcal{B}$), z, γz. As $h(f_a^*, x)$ is the f_a-coordinate of x: the w_a lie in I, which is orthogonal to W and to z (as $I \subseteq X$) and to γz (as $I \subseteq \gamma X$), and have $f_{\mathcal{B}}$-coordinates δ; the t_l have $f_{\mathcal{B}}$-coordinates 0. Hence $R_u = \mathrm{diag}(N^{-1} \cdot 1_{\mathcal{B}},\ (\langle v_l, v_{l'} \rangle_{\mathrm{mt}})_{l,l'})$ by (5.40). So $\det R = \overline{\det P_0} \det P_0\, N^{-\#\mathcal{B}} \det(\langle v_l, v_{l'} \rangle_{\mathrm{mt}})$, a unit by Lemma 5.2.28 (v), and $[\det R] = [\det \alpha]^2 N^{-2f\#\mathcal{B}} \,\mathrm{lkd}(v, e)$.

Step 7: the identity. By Lemma 5.3.8, F is injective, $i(F)$ is finite, and $(N^{2f} \mathrm{mn}_j(\gamma))^2 = N^{2f(\#\mathcal{B}+2)}\, i(F)^2\, [\det \alpha]^2 N^{-2f\#\mathcal{B}} \,\mathrm{lkd}(v, e)$, i.e.

$$\mathrm{mn}_j(\gamma)^2 = i(F)^2\, [\det \alpha]^2 \,\mathrm{lkd}(v, e). \tag{5.41}$$

Step 8: $\mathrm{ir}_j(k)$ *divides* $[\det \alpha]$. Let $\varphi \colon \hat{\mathcal{O}}^{\varrho} \to \sum_a \hat{\mathcal{O}} r_a(k)$, $c \mapsto \sum_a c_a r_a(k) = (h(k f_b, \sum_a c_a f_a))_b$. It is onto, so $\mathrm{ir}_j(k) = [\hat{\mathcal{O}}^{\varrho} : \hat{\mathcal{O}}^{\{a<j'\}} + \ker \varphi] = [\hat{\mathcal{O}}^{\mathcal{B}} : \pi_{\mathcal{B}}(\ker \varphi)]$, $\pi_{\mathcal{B}}$ the projection to the

coordinates in $\mathcal{B}$. For $a \in \mathcal{B}$ let $c(x_a) \in \hat{\mathcal{O}}^\varrho$ be the f-coordinates of $x_a \in I(\mathbb{A}_f)$ (Step 0). As $kf_b = h_c^{-1}\gamma f_b$ and h_c fixes X, $h(kf_b, x_a) = h(\gamma f_b, x_a) = 0$, since $x_a \in \gamma X(\mathbb{A}_f)$ and $\gamma X \perp \gamma X + E\gamma z$. So $c(x_a) \in \ker\varphi$ and $\pi_{\mathcal{B}}(c(x_a))$ is column a of α: $\alpha\hat{\mathcal{O}}^{\mathcal{B}} \subseteq \pi_{\mathcal{B}}(\ker\varphi)$, and $\mathrm{ir}_j(k)$ divides $[\det\alpha]$.

Step 9: $\mathrm{ir}_j(k^{-1})\,\mathrm{ti}(k)$ *divides* $i(F)$. Applying h_c^{-1} (an isometry with $h_c^{-1}L_c = L_0$, $h_c^{-1}f_a = f_a$, $h_c^{-1}\gamma f_a = kf_a$), $i(F) = [S : F']$ with $F' = L_X + \sum_{a<j'}\hat{\mathcal{O}}kf_a$ and $S = L_0 \cap U$, U the $\mathbb{A}_f$-span of f_a ($a < \varrho$), kf_a ($a < j'$), of rank $\varrho + j'$ (F is injective). As $U \subseteq (X + kX)(\mathbb{A}_f)$ and $\dim(X + kX) = \dim(X + \gamma X) = 2\varrho - \dim I \le \varrho + j'$ (Step 1; $kX = h_c^{-1}\gamma X$), $U = (X + kX)(\mathbb{A}_f)$. So $F' \subseteq L_X + kL_X \subseteq S$ and $[S : F'] = \mathrm{ti}(k)\,[L_X + kL_X : F']$. As $F' \supseteq L_X$, $[L_X + kL_X : F'] = [kL_X : kL_X \cap F']$, and $kL_X \cap F' = k(L_X^{<j'} + D)$ with $L_X^{<j'} = \bigoplus_{a<j'}\hat{\mathcal{O}}f_a$ and $D = \{x \in L_X : kx \in L_X\}$; hence $[L_X + kL_X : F'] = [\hat{\mathcal{O}}^{\mathcal{B}} : \pi_{\mathcal{B}}(c(D))]$. On the other hand, as in Step 8, $\mathrm{ir}_j(k^{-1}) = [\hat{\mathcal{O}}^{\mathcal{B}} : \pi_{\mathcal{B}}(\ker\varphi')]$ for $\varphi'(c) = (h(k^{-1}f_b, \sum_a c_a f_a))_b = (h(f_b, k\sum_a c_a f_a))_b$, and $c(D) \subseteq \ker\varphi'$ because $kx \in X(\mathbb{A}_f)$ is orthogonal to the frame for $x \in D$. Hence $\mathrm{ir}_j(k^{-1})$ divides $[L_X + kL_X : F']$, and $\mathrm{ir}_j(k^{-1})\mathrm{ti}(k)$ divides $i(F)$.

Conclusion. By Steps 8 and 9, ω divides $[\det\alpha]\,i(F)$, which is a nonzero natural number; so $\omega \neq 0$ and $\omega \le [\det\alpha]\,i(F)$, and (5.41) gives $\mathrm{lkd}(v,e)\,\omega^2 \le \mathrm{mn}_j(\gamma)^2$. □

Lemma 5.3.8. *Let E be a CM field, $m_1, m_2, m_3 \ge 0$ integers and $m = m_1 + m_2$. Let $J \in \mathrm{M}_m(\hat{\mathcal{O}})$ with $J^* = J$ and $\det J \in \hat{\mathcal{O}}^\times$; $F \in \mathrm{M}_{m\times m_1}(\hat{\mathcal{O}})$; $P \in \mathrm{M}_{m\times m_2}(\hat{\mathcal{O}})$ primitive, i.e. its columns are part of an $\hat{\mathcal{O}}$-basis of $\hat{\mathcal{O}}^m$; $Q \in \mathrm{M}_{m\times m_3}(\mathbb{A}_f)$; and $t \in \mathbb{A}_f^\times$ with $\bar{t} = t$. Suppose $F^*JP = 0$, and put*

$$\mathcal{M} = \begin{pmatrix} F^*JF & F^*JQ \\ Q^*JF & Q^*JQ + t\cdot 1 \end{pmatrix}, \qquad R = P^*(J + t^{-1}JQQ^*J)P.$$

If $\det\mathcal{M}$ and $\det R$ are units of $\mathbb{A}_f$, then $y \mapsto Fy$ is injective on $\mathbb{A}_f^{m_1}$, $\{u \in \mathbb{A}_f^m : P^*Ju = 0\} = F\mathbb{A}_f^{m_1}$, the index $i(F) = [\{y \in \mathbb{A}_f^{m_1} : Fy \in \hat{\mathcal{O}}^m\} : \hat{\mathcal{O}}^{m_1}]$ is finite, and

$$[\det\mathcal{M}] = [t]^{m_3}\,i(F)^2\,[\det R].$$

Proof. All determinants are taken over the commutative ring $\mathbb{A}_f$; $\det(Z^*) = \overline{\det Z}$ and $[\bar{x}] = [x]$.

Dual bases. Complete P to $U = (C \mid P) \in \mathrm{GL}_m(\hat{\mathcal{O}})$, and put $U' = J^{-1}(U^*)^{-1} \in \mathrm{GL}_m(\hat{\mathcal{O}})$, so that $U^*JU' = 1$; write $U' = (C' \mid P')$ with C' the first m_1 columns. Every matrix Z with m rows satisfies $Z = U'(U^*JZ)$, because $U'U^*J = 1$. In particular, as $P^*JF = (F^*JP)^* = 0$,

$$U^*JF = \begin{pmatrix} G \\ 0 \end{pmatrix}, \qquad G := C^*JF \in \mathrm{M}_{m_1}(\hat{\mathcal{O}}), \qquad F = C'G. \tag{5.42}$$

An enlarged family. On $\mathbb{A}_f^m \oplus \mathbb{A}_f^{m_3}$ consider the matrix $\hat{J} = \begin{pmatrix} J & 0 \\ 0 & t\cdot 1 \end{pmatrix}$ and the square matrix of size $m + m_3$

$$T = \begin{pmatrix} F & Q & P \\ 0 & 1 & Y \end{pmatrix}, \qquad Y = -t^{-1}Q^*JP,$$

whose three column blocks we call f, q, p. Using $J^* = J$, $\bar{t} = t$ and $F^*JP = 0$: $f^*\hat{J}p = F^*JP = 0$; $q^*\hat{J}p = Q^*JP + tY = 0$; $p^*\hat{J}p = P^*JP + Y^*tY = P^*JP + t^{-1}P^*JQQ^*JP = R$; and the block of $(f \mid q)$ is $\mathcal{M}$. Hence $T^*\hat{J}T = \begin{pmatrix} \mathcal{M} & 0 \\ 0 & R \end{pmatrix}$, and

$$\overline{\det T}\,\det T\,\det J\,t^{m_3} = \det\mathcal{M}\,\det R. \tag{5.43}$$

The determinant of T. Right multiplication by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -Y \\ 0 & 0 & 1 \end{pmatrix}$ (determinant 1) turns T into $\begin{pmatrix} F & Q & P-QY \\ 0 & 1 & 0 \end{pmatrix}$, so $\det T = \pm \det(F \mid P-QY)$, with $P-QY = P+t^{-1}QQ^*JP$. By the first paragraph, $(F \mid P-QY) = U' \cdot U^*J(F \mid P-QY)$, and by (5.42) the matrix $U^*J(F \mid P-QY)$ is block upper triangular with diagonal blocks G and $P^*J(P+t^{-1}QQ^*JP) = R$. Hence $\det T = \pm \det U' \det G \det R$.

Conclusion. Inserting this into (5.43) and cancelling the unit $\det R$,

$$\det \mathcal{M} = \epsilon\, t^{m_3}\, \overline{\det G}\, \det G\, \overline{\det R}, \qquad \epsilon = \overline{\det U'}\, \det U'\, \det J \in \hat{\mathcal{O}}^\times.$$

As $\det \mathcal{M}$ is a unit, so is $\det G$; so $G \in \mathrm{GL}_{m_1}(\mathbb{A}_f)$ and $F = C'G$ is injective. Taking generalised indices, $[\det \mathcal{M}] = [t]^{m_3}[\det G]^2[\det R]$. Since $U' \in \mathrm{GL}_m(\hat{\mathcal{O}})$, a vector $C'x$ is integral exactly when x is; so $Fy = C'Gy \in \hat{\mathcal{O}}^m$ if and only if $Gy \in \hat{\mathcal{O}}^{m_1}$, i.e. $\{y : Fy \in \hat{\mathcal{O}}^m\} = G^{-1}\hat{\mathcal{O}}^{m_1}$, which contains $\hat{\mathcal{O}}^{m_1}$ because G is integral, with index $[\hat{\mathcal{O}}^{m_1} : G\hat{\mathcal{O}}^{m_1}] = [\det G]$. Thus $i(F) = [\det G]$ is finite and the identity follows. Finally, writing $u = U'x$ with $x = (x_1, x_2)$, we have $P^*Ju = P^*JU'x = x_2$ (the last m_2 rows of $U^*JU' = 1$), so $P^*Ju = 0$ if and only if $u \in C'\mathbb{A}_f^{m_1} = F\mathbb{A}_f^{m_1}$. □

Lemma 5.3.9. *Let $\mathfrak{S}$ be a filling datum and $\gamma \in G(E)$ satisfying the conditions of cell rank j.*

(i) $|N_{E/\mathbb{Q}}\delta_*(\gamma)| = N^{2f}\,\mathrm{mn}_j(\gamma)$*; in particular* $\delta_*(\gamma) \neq 0$ *and* $\mathrm{mn}_j(\gamma) > 0$.

(ii) Let $c, c' \in \mathcal{C}$ with $h_c^{-1}\gamma h_{c'} \in K$ and $i_\gamma i'_\gamma \neq 0$. Then

$$\mathrm{g}^\sharp(j,c,c',\gamma) = \frac{N^f\,\mathrm{mn}_j(\gamma)}{i_\gamma i'_\gamma}, \qquad \mathrm{g}^{\max}(j,c,c',\gamma) = \frac{\max\big(\prod_v |\mathrm{g}(\gamma)|_v,\ N^f\mathrm{mn}_j(\gamma)\big)}{i_\gamma i'_\gamma}.$$

Proof. Throughout, h is linear in its second argument, so that $Z(\gamma)y$ is the vector

$$\big(h(f_a, \gamma \sum_b y_b f_b)\big)_a$$

for $y \in E^{\varrho+1}$. The frame vectors $f_a = e_{a+1}$, $a < \varrho$, are isotropic and orthogonal to every frame vector, and $h(z,z) = -N$; hence for $x \in E^{\varrho+1}$

$$h\Big(f_a, \sum_b x_b f_b\Big) = 0 \ (a < \varrho), \qquad h\Big(z, \sum_b x_b f_b\Big) = -N x_\varrho, \qquad h\Big(\sum_b x_b f_b, \sum_b x_b f_b\Big) = -N\bar{x}_\varrho x_\varrho. \tag{5.44}$$

For $\xi \in E$ we use $|N_{E/\mathbb{Q}}\xi| = \prod_v |\xi|_v^2$ over the f complex places, and $N_{E/\mathbb{Q}}(N) = N^{2f}$.

(i), the case $\varrho < j$. Then $j = \varrho + 1$, the index set $\{a : a < j\} \cup \{\varrho\}$ is $\{0, \dots, \varrho\}$, and $\delta_*(\gamma) = \det Z(\gamma) = D_{\varrho+1}(\gamma)$. By definition $\mathrm{mn}_{\varrho+1}(\gamma) = |N_{E/\mathbb{Q}}D_{\varrho+1}(\gamma)|/N^{2f}$, which is the claim.

(i), the case $j \leq \varrho$. Let B be the square matrix of size $\varrho+1$ whose rows of index $a < j$ are those of $Z(\gamma)$ and whose rows of index $a \geq j$ are those of the identity matrix. It is block upper triangular with diagonal blocks the leading $j \times j$ block of $Z(\gamma)$ and an identity block, so $\det B = D_j(\gamma)$, which is nonzero by the conditions of cell rank j. Let $y = B^{-1}\varepsilon_\varrho$, ε_ϱ the last standard basis vector. Reading $By = \varepsilon_\varrho$ row by row:

$$(Z(\gamma)y)_a = 0 \ (a < j), \qquad y_a = 0 \ (j \leq a < \varrho), \qquad y_\varrho = 1. \tag{5.45}$$

Let B' be B with its row ϱ replaced by the row ϱ of $Z(\gamma)$. Expanding $\det B'$ successively along the rows $a = j, \dots, \varrho - 1$, each of which is the a-th standard row vector, deletes these rows and the same columns with sign $+1$; what remains is the minor of $Z(\gamma)$ on the rows and columns $\{a : a < j\} \cup \{\varrho\}$, so $\det B' = \delta_*(\gamma)$. On the other hand, expanding along row ϱ, $\det B' = \sum_b Z(\gamma)_{\varrho b} C_{\varrho b}$ with the cofactors $C_{\varrho b}$ of B, and $C_{\varrho b} = (\operatorname{adj} B)_{b\varrho} = \det B \cdot (B^{-1})_{b\varrho} = D_j(\gamma)\, y_b$. Hence

$$\delta_*(\gamma) = (Z(\gamma)y)_\varrho \cdot D_j(\gamma). \tag{5.46}$$

We show that $t = \gamma \sum_b y_b f_b$ lies in the span of the frame. By the conditions of cell rank j there are ζ with $\zeta\bar\zeta = 1$ and, for each a with $j \le a \le \varrho$, vectors $x^{(a)}, y^{(a)} \in E^{\varrho+1}$ with $\sum_b x_b^{(a)} f_b = \gamma \sum_b y_b^{(a)} f_b$, $y_a^{(a)} = 1$ and $y_b^{(a)} = 0$ for $b > a$. The vectors $y^{(\varrho)}, y^{(\varrho-1)}, \dots, y^{(j)}$ and $\varepsilon_0, \dots, \varepsilon_{j-1}$ form a basis of $E^{\varrho+1}$ (triangular with ones on the diagonal), so $y = \sum_{a=j}^{\varrho} d_a y^{(a)} + \sum_{b<j} c_b \varepsilon_b$ with $d_a, c_b \in E$. Then $t = \sum_b x_b f_b + \gamma \sum_{b<j} c_b f_b$ with $x = \sum_a d_a x^{(a)}$. For $a' < j$ the first identity of (5.44) and (5.45) give

$$0 = (Z(\gamma)y)_{a'} = h(f_{a'}, t) = \sum_{b<j} c_b\, h(f_{a'}, \gamma f_b) = \sum_{b<j} Z(\gamma)_{a'b}\, c_b,$$

so $c = 0$, the leading $j \times j$ block of $Z(\gamma)$ being invertible. Thus $t = \sum_b x_b f_b$. As γ is an isometry, the last identity of (5.44) applied to x and to y gives $-N\bar x_\varrho x_\varrho = h(t,t) = h(\sum_b y_b f_b, \sum_b y_b f_b) = -N\bar y_\varrho y_\varrho = -N$, so $\bar x_\varrho x_\varrho = 1$ and $|x_\varrho|_v = 1$ at every infinite place v. By the middle identity of (5.44), $(Z(\gamma)y)_\varrho = h(z,t) = -N x_\varrho$, and (5.46) gives $\delta_*(\gamma) = -N x_\varrho D_j(\gamma)$. Taking absolute norms, $|N_{E/\mathbb{Q}}\delta_*(\gamma)| = N^{2f}\, |N_{E/\mathbb{Q}} D_j(\gamma)| = N^{2f} \mathrm{mn}_j(\gamma)$, since $\max(j - \varrho, 0) = 0$.

In both cases $\mathrm{mn}_j(\gamma) > 0$ because $D_j(\gamma) \neq 0$, and therefore $\delta_*(\gamma) \neq 0$.

(ii). By (TA12) and (i), $\mathsf{g}^\sharp(j, c, c', \gamma) = |N_{E/\mathbb{Q}}\delta_*(\gamma)|/(i_\gamma i'_\gamma N^f) = N^f \mathrm{mn}_j(\gamma)/(i_\gamma i'_\gamma)$. For the maximal gauge, $\mathsf{g}^{\max} = \max(\mathsf{g}^\sharp, \prod_v |\mathsf{g}(\gamma)|_v)$. If $\mathsf{g}(\gamma) = 0$, both sides of the claimed identity equal $N^f \mathrm{mn}_j(\gamma)/(i_\gamma i'_\gamma)$. If $\mathsf{g}(\gamma) \neq 0$, the conditions of cell rank give $\varrho < j$, i.e. $j = \varrho + 1$; then $\min(j, \varrho) = \varrho$ and both indices of (GT1) are 1 (the two sums of row lattices coincide), so $i_\gamma = i'_\gamma = 1$ and both sides equal $\max(N^f \mathrm{mn}_j(\gamma), \prod_v |\mathsf{g}(\gamma)|_v)$.

□

Lemma 5.3.10. *Let $\mathfrak{S}$ be a filling datum, $c \in \mathcal{C}$, $\gamma \in G(E)$ satisfying the conditions of cell rank j, and $w = (w_l)_{l\in\sigma}$ a family of adelic pairs with integral lifts for (c, γ). Then there is $B \in \mathrm{GL}_\sigma(\mathbb{A}_f)$ with entries in $\hat{\mathcal{O}}$ and $[\det B] \le M^{2\min(\varrho, j) f}$ such that $wB = (\sum_{l'} B_{l'l} w_{l'})_{l\in\sigma}$ has level lifts for (c, γ).*

Proof. Let $L_X = \bigoplus_{a=1}^{\varrho} \hat{\mathcal{O}} e_a$, $I = X \cap \gamma X$, and let $\Pi\colon X(\mathbb{A}_f) \to \mathbb{A}_f^{\varrho+1}$, $\Pi(x) = (h(\gamma f_b, x))_{0\le b\le\varrho}$; Π is given by a matrix over E.

Step 1: the kernel of Π is $I(\mathbb{A}_f)$. Since Π is rational, it suffices to show $\{x \in X : \Pi(x) = 0\} = I$. If $x \in I$, then $x \perp \gamma X$ (γX is isotropic) and $x \perp \gamma z$ ($z \perp X$). Conversely let $x \in X$ with $x \perp \gamma X$ and $x \perp \gamma z$. Then $x' = \gamma^{-1} x$ is isotropic, orthogonal to X and to z. In $V = X \oplus X^* \oplus X_0 \oplus Ez$ (X^* the span of the e_a^*) we have $X^\perp = X \oplus X_0 \oplus Ez$, so $x' = x_1 + y_0 + \lambda z$ with $x_1 \in X$, $y_0 \in X_0$; $h(z, x') = -N\lambda = 0$ gives $\lambda = 0$, and then $0 = h(x', x') = h(y_0, y_0)$ gives $y_0 = 0$, the form of X_0 being the unit form. So $x' \in X$ and $x \in \gamma X$.

Step 2: $\dim_E \Pi(X) \le \min(\varrho, j)$. By Step 1, $\dim \Pi(X) = \varrho - \dim I$. If $j \ge \varrho$ there is nothing to show. If $j < \varrho$, the conditions of cell rank give, for each a with $j \le a < \varrho$, vectors $x^{(a)}, y^{(a)} \in E^{\varrho+1}$ supported on $\{0, \dots, a\}$ with $x_a^{(a)} = \zeta \neq 0$ and $\sum_b x_b^{(a)} f_b = \gamma \sum_b y_b^{(a)} f_b$. As $a < \varrho$, this vector lies in $X \cap \gamma X = I$, and the $\varrho - j$ vectors obtained are linearly independent (their coefficient vectors $x^{(a)}$ are triangular with nonzero diagonal entries). So $\dim I \ge \varrho - j$.

Step 3: the module $L_\Pi = \Pi(L_X)$. It is a finitely generated $\hat{\mathcal{O}}$-module, isomorphic to $L_X/(L_X \cap I(\mathbb{A}_f))$, which is torsion free ($I(\mathbb{A}_f)$ is an $\mathbb{A}_f$-subspace) and of rank $d_\Pi = \dim_E \Pi(X)$ at every finite place; hence $L_\Pi \cong \hat{\mathcal{O}}^{d_\Pi}$, and $\#(L_\Pi/ML_\Pi) = [\hat{\mathcal{O}} : M\hat{\mathcal{O}}]^{d_\Pi} = M^{2fd_\Pi} \le M^{2f\min(\varrho,j)}$ by Step 2, since $[\hat{\mathcal{O}} : M\hat{\mathcal{O}}] = |N_{E/\mathbb{Q}}(M)| = M^{2f}$.

Step 4: the matrix B. Let (t_ρ), u give integral lifts of w, and $n_l = \sum_\rho u_{\rho l} t_\rho \in L_c$. No vector of the lattice basis other than e_a has a component along e_a, and h_c does not change these components, so the e_a-coordinates of elements of L_c lie in $\hat{\mathcal{O}}$; as $n_l \perp X$, its e_a^*-coordinates vanish. Hence $n_l = \alpha_l + n_l'$ with $\alpha_l \in L_X$ and $n_l' \in (X_0 \oplus Ez)(\mathbb{A}_f)$; note $L_X \subseteq L_c$. Let $F\colon \hat{\mathcal{O}}^\sigma \to L_\Pi$, $F(x) = \Pi(\sum_l x_l \alpha_l)$, and $\Sigma = F^{-1}(ML_\Pi)$. Then F induces an injection $\hat{\mathcal{O}}^\sigma/\Sigma \to L_\Pi/ML_\Pi$, so $[\hat{\mathcal{O}}^\sigma : \Sigma] \le M^{2f\min(\varrho,j)}$ by Step 3. A submodule of finite index of $\hat{\mathcal{O}}^\sigma$ is free over each $\mathcal{O}_w$ of rank $\#\sigma$ and equal to $\mathcal{O}_w^\sigma$ for almost all w, so $\Sigma = B\hat{\mathcal{O}}^\sigma$ with $B \in \mathrm{GL}_\sigma(\mathbb{A}_f)$ integral, and $[\det B] = [\hat{\mathcal{O}}^\sigma : \Sigma] \le M^{2\min(\varrho,j)f}$.

Step 5: wB has level lifts. The family wB has integral lifts by (t_ρ) and uB, with vectors $n_l^B = \sum_{l'} n_{l'} B_{l'l} = \alpha_l^B + n_l'^B$, $\alpha_l^B = \sum_{l'} \alpha_{l'} B_{l'l} \in L_X$. As the column l of B lies in Σ, $\Pi(\alpha_l^B) \in ML_\Pi$: there is $\omega_l \in L_X$ with $\Pi(\alpha_l^B) = M\Pi(\omega_l)$, so $\iota_l = \alpha_l^B - M\omega_l \in I(\mathbb{A}_f)$ by Step 1. Choose a basis (b_i) of I over E and write $\iota_l = \sum_i \mu_{il} b_i$ with $\mu_{il} \in \mathbb{A}_f$. The vectors t_ρ and b_i are orthogonal to $X + \gamma X$ ($I \perp X + \gamma X$, as X and γX are isotropic and contain I), and

$$n_l'' = n_l^B - \iota_l = \sum_\rho (uB)_{\rho l} t_\rho - \sum_i \mu_{il} b_i = M\omega_l + (n_l^B - \alpha_l^B)$$

lies in L_c, with X-part $M\omega_l \in \bigoplus_a M\hat{\mathcal{O}} e_a$ and remaining part $n_l'^B \in (X_0 \oplus Ez)(\mathbb{A}_f)$. Its image under $t \mapsto ([t], [\gamma^{-1}t])$ equals that of n_l^B, which is $(wB)_l$, because this map vanishes on $I(\mathbb{A}_f)$ (Lemma 5.2.28 (i)). So $(t_\rho) \cup (b_i)$ with the coefficients uB and $-(\mu_{il})$ give level lifts of wB (LK7). □

Lemma 5.3.11. *Let $\mathfrak{S}$ be a filling datum, $c, c' \in \mathcal{C}$ and $\gamma \in G(E)$. For a family $m = (m_i)_{i\in\mathcal{I}}$ of pairs and a matrix a over $\mathbb{A}_f$ with rows $\mathcal{I}$ and columns $\mathcal{I}'$, write $ma = (\sum_i a_{il} m_i)_{l\in\mathcal{I}'}$; $(m|y)$ is the concatenation of two families.*

(i) Orthogonal splitting. *Let $m = (m_i)_{i\in\mathcal{I}_1}$ and $y = (y_l)_{l\in\mathcal{I}_2}$ be families of pairs over E with $\langle m_i, y_l\rangle_{\mathrm{mt}} = 0$ for all i, l, and $g \in \mathrm{GL}_{\mathcal{I}_1\sqcup\mathcal{I}_2}(\mathbb{A}_f)$ such that $(m|y)g$ has level lifts for (c,γ). Then there are $a_1 \in \mathrm{GL}_{\mathcal{I}_1}(\mathbb{A}_f)$, $a_2 \in \mathrm{GL}_{\mathcal{I}_2}(\mathbb{A}_f)$ and a matrix σ over $\mathbb{A}_f$ with rows $\mathcal{I}_1$ and columns $\mathcal{I}_2$ such that ma_1 and $m\sigma + ya_2$ have level lifts for (c,γ) and $\mathrm{lkd}(m, a_1)\,\mathrm{lkd}(y, a_2) = \mathrm{lkd}((m|y), g)$.*

(ii) The part in X_0. *Let $m = (m_i)_{i\in\mathcal{I}}$ be linearly independent pairs in $\operatorname{link}\gamma$ with $\mathrm{zf}(m_i) = 0$, and $a \in \mathrm{GL}_{\mathcal{I}}(\mathbb{A}_f)$ such that ma has level lifts for (c,γ). Then the first members p_i of the m_i lie in X_0; with $W = \sum_i Ep_i$, $\dim_E W = \#\mathcal{I}$, the module $\mathrm{sat}_c(W) \cap L_c(\gamma)$ is a lattice of rank $\#\mathcal{I}$, and $\mathrm{disc}_{\#\mathcal{I}}\big(\mathrm{sat}_c(W) \cap L_c(\gamma)\big)^2 \le \mathrm{lkd}(m, a)$.*

(iii) The part seen by the z-forms. *Let $s \in \bar{G}$ agree, let γ be a term over (c, s, c') satisfying the conditions of cell rank j, $x \in \mathbb{N}$ at most the meet norm of s, and $r \in \mathbb{N}$ with $r \le 1 + \max(j - \varrho, 0)$ and $r = 2$ if $x \ne 1$. Let $y = (y_l)_{l<r}$ be pairs in $\operatorname{link}\gamma$ such that the $2 \times r$ matrix $\mathrm{zf}(y)$ has rank r, $a \in \mathrm{GL}_r(\mathbb{A}_f)$, and $w = (w_l)_{l<r}$ a family of adelic pairs with integral lifts for (c,γ) and $\mathrm{zf}(w) = \mathrm{zf}(y)\,a$. Then*

$$x^2\,\mathrm{mn}_j(\gamma)^2 \le \mathrm{lkd}(y, a) \cdot \max\Big(\prod_v |\mathsf{g}(\gamma)|_v,\ N^f \mathrm{mn}_j(\gamma)\Big)^2.$$

Proof. Throughout, h is linear in the second argument and extended sesquilinearly to adelic vectors; $|N_{E/\mathbb{Q}}\xi| = \prod_v |\xi|_v^2$ for $\xi \in E$; for $\xi \in E^\times$, $[\xi] = |N_{E/\mathbb{Q}}\xi|$ (product formula), and $[\delta] = [\hat{\mathcal{O}} : \delta\hat{\mathcal{O}}] \geq 1$ for $\delta \in \hat{\mathcal{O}} \cap \mathbb{A}_f^\times$, $[\delta] = 1$ for $\delta \in \hat{\mathcal{O}}^\times$ (Definition 5.1.8).

Two remarks on lifts. (a) If w has level (or integral) lifts for (c,γ) by (t_ρ) and u, and θ is a matrix over $\hat{\mathcal{O}}$, then $w\theta$ has level (integral) lifts by (t_ρ) and $u\theta$: the new vectors $\sum_{l'} n_{l'}\theta_{l'l}$ lie in the $\hat{\mathcal{O}}$-module L_c, and their X-parts $\sum_{l'} \alpha_{l'}\theta_{l'l}$ lie in $\bigoplus_a M\hat{\mathcal{O}}e_a$. A subfamily of a family with level lifts has level lifts. (b) If w has integral lifts by (t_ρ), u, with $n_l = \sum_\rho u_{\rho l}t_\rho \in L_c$, then, as $n_l \perp X$ and $\gamma^{-1}n_l \perp X$, the members of w_l are the classes of n_l and $\gamma^{-1}n_l$ in $(X_0 \oplus Ez)(\mathbb{A}_f)$ (that is, n_l and $\gamma^{-1}n_l$ minus their X-parts), and since $z \perp X$,

$$\mathrm{zf}(w_l) = \big(h(z,n_l),\, h(z,\gamma^{-1}n_l)\big) = \big(h(z,n_l),\, h(\gamma z,n_l)\big). \tag{5.47}$$

(i). By the Iwasawa decomposition of GL_n over a non-archimedean local field with respect to a standard parabolic subgroup, at each finite place w the component g_w is a product of a block upper triangular matrix (blocks $\mathcal{I}_1$, $\mathcal{I}_2$) and an element of $\mathrm{GL}_{\mathcal{I}_1\sqcup\mathcal{I}_2}(\mathcal{O}_w)$; for almost all w, $g_w \in \mathrm{GL}_{\mathcal{I}_1\sqcup\mathcal{I}_2}(\mathcal{O}_w)$ and the triangular factor may be taken 1. Hence $g = \begin{pmatrix} a_1 & \sigma \\ 0 & a_2 \end{pmatrix}\theta$ with $a_1 \in \mathrm{GL}_{\mathcal{I}_1}(\mathbb{A}_f)$, $a_2 \in \mathrm{GL}_{\mathcal{I}_2}(\mathbb{A}_f)$ and $\theta \in \mathrm{GL}_{\mathcal{I}_1\sqcup\mathcal{I}_2}(\hat{\mathcal{O}})$. By (a), $(m|y)g\theta^{-1} = (ma_1 \,|\, m\sigma + ya_2)$ has level lifts, and so do its two subfamilies. Since $\langle m_i, y_l\rangle_{\mathrm{mt}} = 0$, the Gram matrix of $(m|y)$ for the link metric is block diagonal, with determinant the product of those of m and of y; and $[\det g] = [\det a_1][\det a_2][\det\theta] = [\det a_1][\det a_2]$. Multiplying gives $\mathrm{lkd}((m|y),g) = \mathrm{lkd}(m,a_1)\,\mathrm{lkd}(y,a_2)$ by (LK6).

(ii), the Gram matrix. Write $m_i = (p_i + \lambda_i' z,\ p_i' + \lambda_i z)$ with $p_i, p_i' \in X_0$. By (LK4), $\mathrm{zf}(m_i) = (-N\lambda_i', -N\lambda_i) = 0$, so $\lambda_i = \lambda_i' = 0$ and the source of m_i is p_i. By Lemma 5.2.28 (ii), $\langle m_i, m_{i'}\rangle_{\mathrm{mt}} = \langle p_i, p_{i'}\rangle_+ = p_i^* p_{i'}$, the form of V_0^+ on X_0 being the unit form. The m_i are linearly independent, so by Lemma 5.2.28 (v) $\det(p^*p) \neq 0$ for the matrix p with columns p_i; hence the p_i are linearly independent, $\dim W = \#\mathcal{I} =: d$, and $\mathrm{lkd}(m,a) = |N_{E/\mathbb{Q}}\det(p^*p)|\,[\det a]^2$.

(ii), the columns of pa lie in $L = \mathrm{sat}_c(W) \cap L_c(\gamma)$. Let (t_ρ), u give level lifts of ma, with $n_l = \sum_\rho u_{\rho l}t_\rho = \alpha_l + n_l'$ as in (LK7). By (b), n_l' is the first member of $(ma)_l$, i.e. $n_l' = (pa)_l := \sum_i a_{il}p_i \in W(\mathbb{A}_f)$, and by (5.47) and $\mathrm{zf}(ma) = \mathrm{zf}(m)a = 0$, $h(z,n_l) = h(\gamma z, n_l) = 0$. As $e_a \in L_0$ is fixed by $h_c \in \mathrm{U}(X_0)(\mathbb{A}_f)$, $\alpha_l \in \bigoplus_a M\hat{\mathcal{O}}e_a \subseteq L_c$, so $(pa)_l = n_l - \alpha_l \in L_c \cap W(\mathbb{A}_f) = \mathrm{sat}_c(W)$. Moreover $n_l \perp \gamma X$ (the t_ρ are orthogonal to γX) and $h(\gamma z, n_l) = 0$, so $h(\gamma f_b, (pa)_l + M\cdot M^{-1}\alpha_l) = 0$ for $0 \leq b \leq \varrho$ with $M^{-1}\alpha_l \in \bigoplus_a \hat{\mathcal{O}}e_a$; by (LT1), $(pa)_l \in L_c(\gamma)$.

(ii), the lattices. Let ℓ be a left inverse of p over E. For an $\hat{\mathcal{O}}$-module L' with $(pa)\hat{\mathcal{O}}^d \subseteq L' \subseteq \mathrm{sat}_c(W)$ put $\Lambda' = \{y \in \mathbb{A}_f^d : py \in L'\}$, so $L' = p\Lambda'$ and $a\hat{\mathcal{O}}^d \subseteq \Lambda' \subseteq \ell L_c$. At every finite place w, Λ'_w lies between the $\mathcal{O}_w$-lattices $a_w\mathcal{O}_w^d$ and $\ell L_{c,w}$ of E_w^d, so it is a finitely generated torsion-free, hence free, $\mathcal{O}_w$-module of rank d; for almost all w both bounds are $\mathcal{O}_w^d$ (then a_w, ℓ and $L_{c,w}$ are integral and standard). Thus $\Lambda' = g'\hat{\mathcal{O}}^d$ with $g' \in \mathrm{GL}_d(\mathbb{A}_f)$, and $L' = pg'\hat{\mathcal{O}}^d$ is a lattice of rank d (LT3). This applies to $L' = L$ and to $L' = P' := (pa)\hat{\mathcal{O}}^d$. With $L = pg'\hat{\mathcal{O}}^d \supseteq P'$, $a = g'\theta$ with $\theta = g'^{-1}a$ integral, and by (LT3)

$$\mathrm{disc}_d(P')^2 = \prod_w |\det(a^*p^*pa)|_w^{-1} = \mathrm{disc}_d(L)^2\,[\det\theta]^2 \geq \mathrm{disc}_d(L)^2,$$

since $[\det\theta] \geq 1$ ($\det\theta \in \hat{\mathcal{O}} \cap \mathbb{A}_f^\times$). Finally $\mathrm{disc}_d(P')^2 = \prod_w |\det(p^*p)|_w^{-1} \prod_w |\det a\,\overline{\det a}|_w^{-1} = |N_{E/\mathbb{Q}}\det(p^*p)|\,[\det a]^2 = \mathrm{lkd}(m,a)$.

(iii), a minor and Liouville's inequality. Let $\varepsilon_1, \varepsilon_2$ and u_γ be as in Lemma 5.3.12. As $\mathrm{zf}(y)$ has rank r and two rows, $r \leq 2$ and there is an injective $\varsigma\colon \{0,\dots,r-1\} \to \{1,2\}$ such that the minor δ_0 of $\mathrm{zf}(y)$ on the rows $\varsigma(0),\dots,\varsigma(r-1)$ is nonzero. Identify pairs with vectors of

$E^{2(\dim X_0+1)}$ and let Q be the matrix of the link metric; it is positive definite at every infinite place (Lemma 5.2.28 (v)). Let ε_ς be the matrix with the columns $\varepsilon_{\varsigma(i)}$, and y also the matrix with the columns y_l. By Lemma 5.3.12 (i), $\varepsilon_\varsigma^* Q y$ is the submatrix of $\mathrm{zf}(y)$ on the rows ς, so $\det(\varepsilon_\varsigma^* Q y) = \delta_0 \neq 0$, and $\det(\varepsilon_\varsigma^* Q y\, a)$ is the minor δ of $\mathrm{zf}(w) = \mathrm{zf}(y)a$ on the rows ς, a unit of $\mathbb{A}_f$. We show $x \leq [\delta]$. Let (t_ρ), u give integral lifts of w, $n_l \in L_c$ as in (b), and $n_l = h_c n_l^0$ with $n_l^0 \in L_0$; put $k = h_c^{-1}\gamma h_{c'} \in K$, with $\mathrm{red}(k) = s$. The elements h_c, $h_{c'}$ of $\mathrm{U}(X_0)(\mathbb{A}_f)$ fix e_a and z; hence by (5.47) the entries of $\mathrm{zf}(w)$ are $h(z, n_l^0)$ and $h(kz, n_l^0)$, and $h(e_a, n_l^0) = h(e_a, n_l) = 0$, $h(ke_a, n_l^0) = h(\gamma e_a, n_l) = 0$. The vector z belongs to the lattice basis, so $z \in L_0$, and $kz \in L_0$ by Lemma 5.2.24 (ii); by Lemma 5.2.24 (i) all entries of $\mathrm{zf}(w)$ lie in $\hat{\mathcal{O}}$, so $\delta \in \hat{\mathcal{O}} \cap \mathbb{A}_f^\times$ and $[\delta] \geq 1$. This settles the case $x \leq 1$. If $x \neq 1$, then $r = 2$, ς is a bijection, and $\delta = \pm \det \begin{pmatrix} h(z,n_1^0) & h(z,n_2^0) \\ h(kz,n_1^0) & h(kz,n_2^0) \end{pmatrix}$; as s agrees, Lemma 5.3.15 (iii) gives that the meet norm of s is at most $[\delta]$, so $x \leq [\delta]$. Now Lemma 5.3.14, applied to Q, the matrices ε_ς and y, the identity and a (in place of v, v') and x, gives

$$x^2 \leq \left|N_{E/\mathbb{Q}} \det(\varepsilon_\varsigma^* Q \varepsilon_\varsigma)\right| \cdot \left|N_{E/\mathbb{Q}} \det(y^* Q y)\right| [\det a]^2 = \left|N_{E/\mathbb{Q}} \det(\varepsilon_\varsigma^* Q \varepsilon_\varsigma)\right| \cdot \mathrm{lkd}(y, a). \tag{5.48}$$

(iii), the selected Gram determinant. Let $\Gamma = \max(\prod_v |\mathsf{g}(\gamma)|_v, N^f \mathrm{mn}_j(\gamma))$. By Lemma 5.3.12 (ii), $\varepsilon_\varsigma^* Q \varepsilon_\varsigma$ is the empty matrix if $r = 0$, the 1×1 matrix (N) if $r = 1$, and, if $r = 2$, the Gram matrix $N \begin{pmatrix} 1 & \bar{u}_\gamma \\ u_\gamma & 1 \end{pmatrix}$ up to a simultaneous permutation of rows and columns, with determinant $N^2(1 - \bar{u}_\gamma u_\gamma)$. For $r = 0$: $\mathrm{mn}_j(\gamma)^2 \leq (N^f \mathrm{mn}_j(\gamma))^2 \leq \Gamma^2$. For $r = 1$: $|N_{E/\mathbb{Q}}(N)|\, \mathrm{mn}_j(\gamma)^2 = (N^f \mathrm{mn}_j(\gamma))^2 \leq \Gamma^2$. For $r = 2$: $r \leq 1 + \max(j - \varrho, 0)$ forces $j = \varrho + 1$, so $\mathrm{mn}_j(\gamma) = |N_{E/\mathbb{Q}} D_{\varrho+1}(\gamma)|/N^{2f}$, and by Lemma 5.3.12 (iii)

$$\begin{aligned} |N_{E/\mathbb{Q}}(N^2(1 - \bar{u}_\gamma u_\gamma))|\, \mathrm{mn}_j(\gamma)^2 &= |N_{E/\mathbb{Q}}(D_{\varrho+1}(\gamma)\overline{D_{\varrho+1}(\gamma)}(1 - \bar{u}_\gamma u_\gamma))| \\ &= |N_{E/\mathbb{Q}}\mathsf{g}(\gamma)| = \Big(\prod_v |\mathsf{g}(\gamma)|_v\Big)^2 \leq \Gamma^2. \end{aligned}$$

In all cases $|N_{E/\mathbb{Q}} \det(\varepsilon_\varsigma^* Q \varepsilon_\varsigma)|\, \mathrm{mn}_j(\gamma)^2 \leq \Gamma^2$; multiplying (5.48) by $\mathrm{mn}_j(\gamma)^2$ gives the claim. □

Lemma 5.3.12. *Let $\mathfrak{S}$ be a filling datum and $\gamma \in G(E)$. Let $u_\gamma \in E$ be the z^+-coordinate of $\Upsilon_\gamma^{-1}(z^+) \in V_0^+ = X_0 \oplus Ez^+$, and let $\varepsilon_1, \varepsilon_2 \in \mathrm{link}\,\gamma$ be the pairs with sources $\varepsilon_1^{\mathrm{s}} = -\Upsilon_\gamma^{-1}(z^+)$ and $\varepsilon_2^{\mathrm{s}} = -z^+$.*

(i) For every $q \in \mathrm{link}\,\gamma$: $(\langle \varepsilon_1, q\rangle_{\mathrm{mt}}, \langle \varepsilon_2, q\rangle_{\mathrm{mt}}) = \mathrm{zf}(q)$.

(ii) $\big(\langle \varepsilon_a, \varepsilon_b\rangle_{\mathrm{mt}}\big)_{a,b=1,2} = N \begin{pmatrix} 1 & \bar{u}_\gamma \\ u_\gamma & 1 \end{pmatrix}$.

(iii) If γ satisfies the conditions of cell rank j, then $\mathsf{g}(\gamma) = D_j(\gamma)\overline{D_j(\gamma)}\,(1 - \bar{u}_\gamma u_\gamma)$.

Proof. The form h, hence $\langle \cdot, \cdot \rangle_+$, is linear in the second and conjugate linear in the first argument. The pairs $\varepsilon_1, \varepsilon_2$ exist and are unique because $p \mapsto p^{\mathrm{s}}$ is a bijection of $\mathrm{link}\,\gamma$ onto V_0^+ (Lemma 5.2.28 (iii)). For $p, q \in \mathrm{link}\,\gamma$ we use $\langle p, q\rangle_{\mathrm{mt}} = \langle p^{\mathrm{s}}, q^{\mathrm{s}}\rangle_+$ (Lemma 5.2.28 (ii)), $q^{\mathrm{t}} = \Upsilon_\gamma(q^{\mathrm{s}})$, and that Υ_γ is an isometry of $\langle \cdot, \cdot \rangle_+$, so $\langle \Upsilon_\gamma^{-1} a, b\rangle_+ = \langle a, \Upsilon_\gamma b\rangle_+$.

(i). Write $q = (y_0 + \lambda' z,\ y_0' + \lambda z)$ with $y_0, y_0' \in X_0$, so that $q^{\mathrm{s}} = y_0 + \lambda z^+$, $q^{\mathrm{t}} = y_0' + \lambda' z^+$ and, as $X_0 \perp z$ and $h(z, z) = -N$, $\mathrm{zf}(q) = (-N\lambda', -N\lambda)$ (LK4). Since $X_0 \perp z^+$ and $\langle z^+, z^+\rangle_+ = N$,

$$\begin{aligned} \langle \varepsilon_1, q\rangle_{\mathrm{mt}} &= -\langle \Upsilon_\gamma^{-1} z^+, q^{\mathrm{s}}\rangle_+ = -\langle z^+, \Upsilon_\gamma q^{\mathrm{s}}\rangle_+ = -\langle z^+, q^{\mathrm{t}}\rangle_+ = -N\lambda', \\ \langle \varepsilon_2, q\rangle_{\mathrm{mt}} &= -\langle z^+, q^{\mathrm{s}}\rangle_+ = -N\lambda. \end{aligned}$$

(ii). Write $\Upsilon_\gamma^{-1}z^+ = w_0 + u_\gamma z^+$ with $w_0 \in X_0$. Then $\langle\varepsilon_1,\varepsilon_1\rangle_{\mathrm{mt}} = \langle\Upsilon_\gamma^{-1}z^+,\Upsilon_\gamma^{-1}z^+\rangle_+ = \langle z^+,z^+\rangle_+ = N$, $\langle\varepsilon_2,\varepsilon_2\rangle_{\mathrm{mt}} = N$, $\langle\varepsilon_1,\varepsilon_2\rangle_{\mathrm{mt}} = \langle w_0+u_\gamma z^+, z^+\rangle_+ = N\bar{u}_\gamma$ and $\langle\varepsilon_2,\varepsilon_1\rangle_{\mathrm{mt}} = \langle z^+, w_0 + u_\gamma z^+\rangle_+ = Nu_\gamma$.

(iii), the case $\mathsf{g}(\gamma) = 0$. By the conditions of cell rank, $j \le \varrho$, so at the index $a = \varrho \ge j$ there are ζ with $\zeta\bar{\zeta} = 1$ and $x, x' \in E^{\varrho+1}$ with $\sum_b x_b f_b = \gamma\sum_b x'_b f_b$, $x_\varrho = \zeta \ne 0$, $x'_\varrho = 1$. By Lemma 5.3.13 (i), Υ_γ maps X_0 into X_0; being injective, it maps X_0 onto X_0, and being an isometry it maps the orthogonal complement Ez^+ of X_0 in V_0^+ onto itself. So $\Upsilon_\gamma^{-1}z^+ = u_\gamma z^+$ and $N = \langle\Upsilon_\gamma^{-1}z^+,\Upsilon_\gamma^{-1}z^+\rangle_+ = \bar{u}_\gamma u_\gamma N$, whence $1 - \bar{u}_\gamma u_\gamma = 0$, and both sides of (iii) vanish.

(iii), the case $\mathsf{g}(\gamma) \ne 0$. Then $\varrho < j$, i.e. $j = \varrho + 1$, and $D_{\varrho+1}(\gamma) = \det Z(\gamma) \ne 0$. Let $c \in E^{\varrho+1}$ be the column of index ϱ of the adjugate of $Z(\gamma)$; then $Z(\gamma)c = \det Z(\gamma)\,\varepsilon_\varrho$, and c_ϱ, the cofactor of the entry (ϱ,ϱ), is the leading principal minor $D_\varrho(\gamma)$. By Lemma 5.2.28 (vi) with $c' = D_{\varrho+1}(\gamma)$, $\Upsilon_\gamma(y_0 + D_\varrho(\gamma)z^+) = -(D_{\varrho+1}(\gamma)/N)\,z^+$ for some $y_0 \in X_0$. Hence $\Upsilon_\gamma^{-1}z^+ = -(N/D_{\varrho+1}(\gamma))(y_0 + D_\varrho(\gamma)z^+)$ and $u_\gamma = -ND_\varrho(\gamma)/D_{\varrho+1}(\gamma)$. Therefore

$$D_{\varrho+1}(\gamma)\overline{D_{\varrho+1}(\gamma)}\,(1-\bar{u}_\gamma u_\gamma) = D_{\varrho+1}(\gamma)\overline{D_{\varrho+1}(\gamma)} - N^2 D_\varrho(\gamma)\overline{D_\varrho(\gamma)} = \mathsf{g}(\gamma)$$

by (TA4), with $j = \varrho + 1$. □

Lemma 5.3.13. *Let* $\mathfrak{S}$ *be a filling datum and* $\gamma,\gamma' \in G(E)$. *Put* $X_0^\gamma = \{x \in X_0 : \Upsilon_\gamma(x) \in X_0\}$ *(inside* V_0^+*), and* $(j-\varrho)_+ = \max(0, j-\varrho)$.

(i) If $\mathsf{g}(\gamma) \ne 0$, *then* $\dim X_0^\gamma = k-1$. *If some* $\zeta \ne 0$ *and* $x, x' \in E^{\varrho+1}$ *satisfy* $f\cdot x = \gamma f\cdot x'$, $x_\varrho = \zeta$, $x'_\varrho = 1$, *then* $X_0^\gamma = X_0$. *Hence, if* γ *satisfies the conditions of cell rank* j, $\dim X_0^\gamma + (j-\varrho)_+ = k$.

(ii) $X_0(\gamma,\gamma') \subseteq X_0^\gamma \cap X_0^{\gamma'}$, *and* $\dim X_0^\gamma + \dim(\operatorname{link}\gamma \cap \operatorname{link}\gamma') \le \dim X_0(\gamma,\gamma') + k + 1$.

(iii) If γ *satisfies the conditions of cell rank* j, *then*

$$\dim(\operatorname{link}\gamma\cap\operatorname{link}\gamma') \le \dim X_0(\gamma,\gamma') + 1 + (j-\varrho)_+, \qquad k \le \dim X_0(\gamma,\gamma') + c^+ + (j-\varrho)_+.$$

(iv) $c^+ = 0$ *if and only if* $\operatorname{link}\gamma = \operatorname{link}\gamma'$. *If* $\mathsf{g}(\gamma) \ne 0$ *and* $\operatorname{link}\gamma = \operatorname{link}\gamma'$, *then* $X_0(\gamma,\gamma') = X_0^\gamma$ *and* $\dim(\operatorname{link}\gamma\cap\operatorname{link}\gamma') = \dim X_0(\gamma,\gamma') + 2$.

(v) If γ *and* γ' *satisfy the conditions of cell ranks* j *and* j' *and* $\operatorname{link}\gamma = \operatorname{link}\gamma'$, *then* $\mathsf{g}(\gamma) = 0$ *if and only if* $\mathsf{g}(\gamma') = 0$.

Proof. Recall from Lemma 5.2.28: $p \mapsto p^{\mathrm{s}}$ and $p \mapsto p^{\mathrm{t}}$ are bijections of $\operatorname{link}\gamma$ onto V_0^+ (of dimension $k+1$), Υ_γ is an isometry of $\langle\cdot,\cdot\rangle_+$ with $\Upsilon_\gamma(p^{\mathrm{s}}) = p^{\mathrm{t}}$ on the link, and $\Upsilon_\gamma = \Upsilon_{\gamma'}$ iff the links agree. Let K_0 be the space of pairs whose two members lie in X_0 (both z-coordinates 0). For $p = (x,y) \in K_0$, $p^{\mathrm{s}} = x$ and $p^{\mathrm{t}} = y$.

(i). X_0^γ is the kernel of the linear form $x \mapsto$ (z^+-coefficient of $\Upsilon_\gamma x$) on X_0, so its dimension is $k-1$ or k. Let $\mathsf{g}(\gamma) \ne 0$; then $\det Z(\gamma) \ne 0$ by Lemma 5.2.26 (ii). Suppose $X_0^\gamma = X_0$. Then $\Upsilon_\gamma X_0 = X_0$ (dimension), so Υ_γ maps the orthogonal complement Ez^+ of X_0 in V_0^+ to itself. Let c be the last column of the adjugate of $Z(\gamma)$, so $Z(\gamma)c = \det Z(\gamma)\,\varepsilon_\varrho$, and y_0 the X_0-part of $\gamma f\cdot c$. By Lemma 5.2.28 (vi), $\Upsilon_\gamma(y_0 + c_\varrho z^+) = -(\det Z(\gamma)/N)z^+$, so $y_0 + c_\varrho z^+ \in \Upsilon_\gamma^{-1}(Ez^+) = Ez^+$ and $y_0 = 0$. By Lemma 5.2.26 (i) (with $D = \det Z(\gamma)$, $y_\varrho = c_\varrho = D_\varrho(\gamma)$), $\mathsf{g}(\gamma) = N\,h(y_0,y_0) = 0$, a contradiction. So $\dim X_0^\gamma = k-1$.

For the second assertion let $t \perp X + \gamma X$, and write λ', λ for the z-coordinates of t and $\gamma^{-1}t$. Since $t, \gamma^{-1}t \perp X$ and the e_a are orthogonal to z, $h(f\cdot x, t) = \bar{\zeta}\,h(z,t) = -N\bar{\zeta}\lambda'$ and

$h(f\cdot x',\gamma^{-1}t)=-N\lambda$; these are equal because $h(f\cdot x,t)=h(\gamma f\cdot x',t)=h(f\cdot x',\gamma^{-1}t)$. So $\lambda=\bar\zeta\lambda'$ on the whole link. For $x_0\in X_0$ take $p\in\operatorname{link}\gamma$ with $p^{\mathrm s}=x_0$, i.e. $\lambda=0$; then $\lambda'=0$ as $\bar\zeta\neq0$, so $\Upsilon_\gamma x_0=p^{\mathrm t}\in X_0$. Thus $X_0^\gamma=X_0$.

Finally let γ satisfy the conditions of cell rank j. If $j=\varrho+1$ then $\mathsf g(\gamma)\neq0$ and $\dim X_0^\gamma=k-1$. If $j\le\varrho$, the first condition at $a=\varrho\ge j$ gives x,x' as above with $\zeta\bar\zeta=1$, so $X_0^\gamma=X_0$. In both cases $\dim X_0^\gamma+(j-\varrho)_+=k$.

(ii). If $x\in X_0(\gamma,\gamma')$, there is $y\in X_0$ with $p=(x,y)\in\operatorname{link}\gamma\cap\operatorname{link}\gamma'$; then $\Upsilon_\gamma x=\Upsilon_\gamma(p^{\mathrm s})=p^{\mathrm t}=y\in X_0$, and likewise for γ'. The map $p\mapsto p^{\mathrm s}$ is injective on $\operatorname{link}\gamma$ and sends $\operatorname{link}\gamma\cap K_0$ onto X_0^γ (for $x\in X_0^\gamma$ the pair with source x is $(x,\Upsilon_\gamma x)\in K_0$) and $\operatorname{link}\gamma\cap\operatorname{link}\gamma'\cap K_0$ onto $X_0(\gamma,\gamma')$. With $A=\operatorname{link}\gamma\cap K_0$ and $B=\operatorname{link}\gamma\cap\operatorname{link}\gamma'$, both in $\operatorname{link}\gamma$,

$$\dim X_0^\gamma+\dim B=\dim(A+B)+\dim(A\cap B)\le(k+1)+\dim X_0(\gamma,\gamma').$$

(iii). Insert $\dim X_0^\gamma=k-(j-\varrho)_+$ from (i) into (ii) for the first inequality; the second is the first with $c^+=k+1-\dim(\operatorname{link}\gamma\cap\operatorname{link}\gamma')$ (no truncation occurs, since the intersection has dimension at most $k+1$).

(iv). Both links have dimension $k+1$, so $c^+=0$ iff the intersection is the whole of each, iff they are equal. Let the links be equal and $\mathsf g(\gamma)\neq0$. Then $\Upsilon_{\gamma'}=\Upsilon_\gamma$, and for $x\in X_0^\gamma$ the pair $(x,\Upsilon_\gamma x)$ lies in $K_0\cap\operatorname{link}\gamma=K_0\cap\operatorname{link}\gamma\cap\operatorname{link}\gamma'$, so $x\in X_0(\gamma,\gamma')$; with (ii), $X_0(\gamma,\gamma')=X_0^\gamma$, of dimension $k-1$ by (i), while the intersection of the links has dimension $k+1$.

(v). Equal links give $\Upsilon_\gamma=\Upsilon_{\gamma'}$, hence $X_0^\gamma=X_0^{\gamma'}$. By (i), $\dim X_0^\gamma=k-1$ if $\mathsf g(\gamma)\neq0$ and $=k$ if $\mathsf g(\gamma)=0$ (then $j\le\varrho$ by the third condition of cell rank), and the same for γ'; so $\mathsf g(\gamma)=0$ iff $\mathsf g(\gamma')=0$. □

Lemma 5.3.14. *Let E be a CM field, τ,σ finite sets, $Q\in\mathrm M_{\tau\times\tau}(E)$ with $\Phi_w(Q)$ positive definite hermitian for every infinite place w, $a,b\in\mathrm M_{\tau\times\sigma}(E)$ with $\det(a^*Qb)\neq0$, $v,v'\in\mathrm{GL}_\sigma(\mathbb A_f)$, and $x\ge0$ a real number with $x\le[\det(v^*\,a^*Qb\,v')]$. Then*

$$x^2\le[\det v]^2\,\bigl|N_{E/\mathbb Q}\det(a^*Qa)\bigr|\,\bigl|N_{E/\mathbb Q}\det(b^*Qb)\bigr|\,[\det v']^2.$$

Proof. One place. Fix an infinite place w and put $A=\Phi_w(a^*Qa)$, $B=\Phi_w(a^*Qb)$, $C=\Phi_w(b^*Qb)$, square matrices of size $|\sigma|$. The block matrix $\begin{pmatrix}A&B\\B^*&C\end{pmatrix}=\Phi_w((a\,|\,b)^*Q(a\,|\,b))$ is positive semidefinite. We claim $|\det B|^2\le\det A\det C$. If A is positive definite, the Schur complement $C-B^*A^{-1}B$ is positive semidefinite, so $\det C\ge\det(B^*A^{-1}B)=|\det B|^2/\det A$ (determinants are monotone on positive semidefinite matrices). If A is singular, pick $u\neq0$ with $u^*Au=0$; positivity of the block matrix on vectors (su,y), $s\in\mathbb C$, gives $|u^*By|^2\le(u^*Au)(y^*Cy)=0$ for all y, so $u^*B=0$ and $\det B=0$. Hence

$$|\Phi_w\det(a^*Qb)|^2\le\Phi_w\det(a^*Qa)\cdot\Phi_w\det(b^*Qb),$$

the right side being a product of nonnegative reals.

All places. For $\alpha\in E$, $|N_{E/\mathbb Q}(\alpha)|=\prod_w|\Phi_w(\alpha)|^2$, all infinite places being complex. Raising the inequality above to the square and multiplying over w,

$$\bigl|N_{E/\mathbb Q}\det(a^*Qb)\bigr|^2\le\bigl|N_{E/\mathbb Q}\det(a^*Qa)\bigr|\,\bigl|N_{E/\mathbb Q}\det(b^*Qb)\bigr|.$$

Finite places. By Definition 5.1.8, $[\cdot]$ is multiplicative, $[\det v^*]=[\overline{\det v}]=[\det v]$, and $[\det(a^*Qb)]=|N_{E/\mathbb Q}\det(a^*Qb)|$ since $\det(a^*Qb)\in E^\times$. So

$$0\le x\le[\det v]\,\bigl|N_{E/\mathbb Q}\det(a^*Qb)\bigr|\,[\det v'],$$

and squaring and inserting the previous inequality gives the claim. □

Lemma 5.3.15. *Let $\mathfrak{S}$ be a filling datum with Weil data and $k \in K$ with $s = \mathrm{red}(k)$ agreeing. A* place above M *is $v = w$ or $v = \bar{w}$ with $w \in S_0 \cup S_1$; $\mathbb{F}_v$ is its residue field and $\bar{k}_v$ the reduction modulo v of the (integral) matrix of k.*

(i) Let v be a place above M, $0 \le a < \varrho$ and $x, y \in \mathbb{F}_v^{\varrho+1}$ with $f \cdot x = \bar{k}_v f \cdot y$, $x_b = y_b = 0$ for $b > a$. Then there are x', y' with $f \cdot x' = \bar{k}_v f \cdot y'$, $x'_{a+1} = x_a$, $y'_{a+1} = y_a$ and $x'_b = y'_b = 0$ for $b > a+1$.

(ii) s meets at (w, ϵ) if and only if the spans of f and $\bar{k}_v f$ over $\mathbb{F}_v$ have a nonzero common vector, where $v = w$ if $\epsilon = 0$ and $v = \bar{w}$ if $\epsilon = 1$.

(iii) Let $t_1, t_2 \in L_0$ with $h(e_a, t_l) = h(ke_a, t_l) = 0$ for $1 \le a \le \varrho$, $l = 1, 2$, and suppose $\delta = \det \begin{pmatrix} h(z,t_1) & h(z,t_2) \\ h(kz,t_1) & h(kz,t_2) \end{pmatrix} \in \mathbb{A}_f^\times$. Then (meet norm of s) $\le [\delta]$.

Proof. Let J be the Gram matrix of h in the lattice basis; by Lemma 5.2.24 (i) and its proof, J has integer entries, $J^{\mathsf{T}} = J$ and $J \in \mathrm{GL}(\hat{\mathcal{O}})$, and $k^{\pm 1}$ are integral. By (FG2) the places $w, \bar{w}$ ($w \in S_0 \cup S_1$) are $4f$ distinct places; the residue field of a place above p_i has at least p_i elements. By the Chinese remainder theorem, for each place v above M there is a map $\ell_v \colon \mathbb{F}_v \to \hat{\mathcal{O}}$ with $\ell_v(0) = 0$, $\ell_v(\theta) \equiv \theta \bmod v$ and $\ell_v(\theta) \equiv 0$ modulo every other place above M.

(ii). From $k^* J k = J$ we get $\bar{k} = (k^*)^{\mathsf{T}} = J^{-1}(k^{-1})^{\mathsf{T}} J$ (entrywise conjugate). Reduction modulo $\bar{w}$ of an integral α corresponds, under the conjugation isomorphism $\mathbb{F}_{\bar{w}} \cong \mathbb{F}_w$, to reduction modulo w of $\bar{\alpha}$. Hence $\bar{k}_{\bar{w}}$ corresponds to the reduction modulo w of $J^{-1}(k^{-1})^{\mathsf{T}} J$, which is the inverse adjoint g of s_w for the reduced form J. As f has integer coordinates, f and $\bar{k}_{\bar{w}} f$ have a common nonzero vector iff f and gf have one over $\mathbb{F}_w$, which is meeting at $(w, 1)$; the case $\epsilon = 0$ is the definition.

(i). Put $\mu = x_a$, $\mu' = y_a$. The vector $\mu f_{a+1} - \mu' \bar{k}_v f_{a+1}$ lies in $\langle f_b, \bar{k}_v f_b : b \le a \rangle$ if and only if every linear form m on $\mathbb{F}_v^r$ vanishing on these vectors satisfies $\mu\, m(f_{a+1}) = \mu'\, m(\bar{k}_v f_{a+1})$; and if it does, adding the resulting relation to $f \cdot x = \bar{k}_v f \cdot y$ gives x', y'. Fix such m and $r \in \hat{\mathcal{O}}$. Let $B_v = S_0 \cup S_1$ if $v = w$, and $B_v = \{\bar{w}' : w' \in S_0 \cup S_1\}$ if $v = \bar{w}$. For $v'' \in B_v$ and $\theta \in \mathbb{F}_v$, $\overline{\ell_v(\theta)} \equiv 0 \bmod v''$, because $\ell_v(\theta) \equiv 0 \bmod \bar{v}''$ and $\bar{v}'' \notin B_v \ni v$. Lifting coordinatewise by ℓ_v, let $\tilde{x}, \tilde{y} \in \hat{\mathcal{O}}^{\varrho+1}$ and the linear forms $\tilde{m}$, $\tilde{m}'$ be the lifts of x, y, m and $m \circ \bar{k}_v$. As J is unimodular there are $\tilde{v}, \tilde{v}' \in L_0$ with $h(\tilde{v}, \cdot) = \tilde{m}$, $h(\tilde{v}', \cdot) = \tilde{m}'$; and $\tilde{m}(f_b) = \tilde{m}'(f_b) = \ell_v(0) = 0$ for $b \le a$, since the f_b are basis vectors. By Weil data E contains a primitive cube root of unity ζ_3; $\tau = -\zeta_3$ is integral with $\tau + \bar{\tau} = 1$. For $(\xi, \nu) \in \{(\tilde{x}, \tilde{v}), (\tilde{y}, \tilde{v}')\}$ put $x_e = r\, f \cdot \xi$, which lies in $L_0 \cap \langle e_1, \dots, e_{a+1} \rangle(\mathbb{A}_f)$ and is isotropic, with $h(\nu, x_e) = 0$; put $\varsigma = \tau h(\nu, \nu)$, so $\varsigma + \bar{\varsigma} = h(\nu, \nu)$; and let $\mathrm{E}(q) = q + h(\nu, q) x_e - h(x_e, q)(\nu + \varsigma x_e)$. Expanding $h(\mathrm{E}q, \mathrm{E}q')$ with these three identities shows that E is an isometry of $V(\mathbb{A}_f)$. With $F_c = \langle e_1, \dots, e_c \rangle$: $h(x_e, \cdot)$ vanishes on $X^\perp \supseteq F_\varrho$ and $h(\nu, \cdot)$ on F_{a+1}, so E fixes F_{a+1} pointwise, sends $q \in F_c$ ($c > a+1$) to $q + h(\nu, q) x_e \in q + F_{c-1}$ and $q \in X^\perp$ to $q + h(\nu, q) x_e \in q + X$; an isometry stabilising $F_\bullet$ and trivial on F_c / F_{c-1} and on $X^\perp / X$ is trivial on the dual quotients too, so $\mathrm{E} \in \mathsf{U}(\mathbb{A}_f)$. Its matrix is integral ($x_e, \nu \in L_0$ and J is integral), so $\mathrm{E} \in K$. As $h(x_e, f_b) = 0$, $\mathrm{E} f_b = f_b + h(\nu, f_b) x_e$, whence $\sigma(\mathrm{E}) = r \sum_{b \ge 1} h(\nu, f_b) \xi_{b-1}$. Let u, u' be E for $(\tilde{x}, \tilde{v})$ and $(\tilde{y}, \tilde{v}')$. For $v'' \in B_v$ the coefficients $\bar{r}\,\overline{\xi_b}$ of $h(x_e, \cdot) = \sum_b \bar{r}\,\overline{\xi_b}\, h(f_b, \cdot)$ vanish modulo v'', and $x_e \equiv 0$ modulo $v'' \ne v$. Hence, modulo v, $u \equiv 1 + r(f \cdot x)\, m$ and $u' \equiv 1 + r(f \cdot y)(m \circ \bar{k}_v)$, while $u \equiv u' \equiv 1$ modulo the other places of B_v. Since $\bar{k}_v f \cdot y = f \cdot x$, $\bar{u}\bar{k}_v = \bar{k}_v + r(f \cdot x)(m \circ \bar{k}_v) = \bar{k}_v \bar{u}'$ modulo v; so $uk \equiv ku'$ modulo every place of B_v. If $B_v = S_0 \cup S_1$ this is $\mathrm{red}(u)\, s = s\, \mathrm{red}(u')$. Otherwise, by the computation in (ii), the

reduction modulo $\bar{w}'$ of an element of K determines its reduction modulo w' through the bijection $\alpha \mapsto J^{-1}(\alpha^{-1})^{\mathsf{T}}J$ (after conjugating coefficients), so again $\mathrm{red}(u)\, s = s\, \mathrm{red}(u')$. Agreement gives $\psi_f(-\sigma(u)/(M\varpi)) = \psi_f(-\sigma(u')/(M\varpi))$ for every $r \in \hat{\mathcal{O}}$, i.e. $\psi_f(-rd/\varpi) = 1$ for all $r \in \hat{\mathcal{O}}$, where $d = M^{-1}\big(\sum_b \tilde{m}(f_b)\tilde{x}_{b-1} - \sum_b \tilde{m}'(f_b)\tilde{y}_{b-1}\big)$. As ϖ generates the different, $\hat{\mathcal{O}}$ is its own dual for $(\alpha,\beta) \mapsto \psi_f(-\alpha\beta/\varpi)$ (local self-duality, Tate's thesis), so $d \in \hat{\mathcal{O}}$; by (FG2) the difference of the two sums vanishes modulo v. Reducing modulo v and using $m(f_b) = m(\bar{k}_v f_b) = 0$ for $b \le a$ and $x_b = y_b = 0$ for $b > a$, this reads $\mu\, m(f_{a+1}) = \mu'\, m(\bar{k}_v f_{a+1})$.

(iii). Let A be the set of the places v above M at which (in the sense of (ii)) s meets. Put $\delta_0 = \det\big(h(t_l,z), h(t_l,kz)\big)_l$; then $\delta = \bar{\delta}_0$, $\delta_0 \in \hat{\mathcal{O}}$ (as $t_l, z, kz \in L_0$) is a unit of $\mathbb{A}_f$, and $[\delta] = [\delta_0] = [\hat{\mathcal{O}} : \delta_0\hat{\mathcal{O}}]$. Let $v \in A$. There are x, y over $\mathbb{F}_v$, $x \neq 0$, with $f \cdot x = \bar{k}_v f \cdot y$; let a be the largest index with $(x_a, y_a) \neq (0,0)$. By (i), applied $\varrho - a$ times, there are x', y' with $f\cdot x' = \bar{k}_v f \cdot y'$ and $(x'_\varrho, y'_\varrho) = (x_a, y_a) \neq (0,0)$. Apply the reduction modulo v of $h(t_l,\cdot)$ (integral on L_0): as $h(t_l, e_a) = h(t_l, ke_a) = 0$, the identity $f\cdot x' - \bar{k}_v f\cdot y' = 0$ gives $x'_\varrho\, h(t_l,z) - y'_\varrho\, h(t_l,kz) \equiv 0 \bmod v$ for $l = 1,2$. So the reduction of the matrix of δ_0 kills the nonzero vector $(x'_\varrho, -y'_\varrho)$, and $\delta_0 \equiv 0 \bmod v$. Hence $\hat{\mathcal{O}}/\delta_0\hat{\mathcal{O}}$ maps onto $\prod_{v\in A}\mathbb{F}_v$ (Chinese remainder theorem), and

$$(\text{meet norm of } s) = \prod_{v\in A} p_{i(v)} \le \prod_{v\in A} |\mathbb{F}_v| \le [\hat{\mathcal{O}} : \delta_0\hat{\mathcal{O}}] = [\delta].$$

□

Lemma 5.3.16. *Let $\mathfrak{S}$ be an admissible filling datum with Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, and γ a near term over (c,s,c') satisfying the conditions of cell rank j. Then $\mathsf{g}^*(j,c,c',\gamma) \le b^{n^3 f}$.*

Proof. Put $k = h_c^{-1}\gamma h_{c'}$ and $B = b^{270n^2 f}$. For $\xi \in E$, $|N_{E/\mathbb{Q}}\xi| = \prod_v |\xi|_v^2$.

The minor norm and the term discriminant. By Lemma 5.2.20, $\prod_v |D_l(\gamma)|_v^2 \le B$ for every l. Hence $\mathrm{mn}_j(\gamma) = |N_{E/\mathbb{Q}} D_j(\gamma)|/N^{2\max(j-\varrho,0)f} \le |N_{E/\mathbb{Q}} D_j(\gamma)| \le B$, as $N \ge 1$. By Lemma 5.2.26 (ii), $\mathsf{g}(\gamma)$ is totally positive or 0; at each infinite place, complex conjugation of E becoming complex conjugation, $|\mathsf{g}(\gamma)|_v = |D_{\varrho+1}(\gamma)|_v^2 - N^2|D_\varrho(\gamma)|_v^2 \le |D_{\varrho+1}(\gamma)|_v^2$, so $\prod_v |\mathsf{g}(\gamma)|_v \le B$.

The maximal gauge. By Lemma 5.3.6 (i), $i_\gamma = \mathrm{ir}_j(k) \ge 1$ and $i'_\gamma = \mathrm{ir}_j(k^{-1}) \ge 1$. By Lemma 5.3.9 (ii) and (FA5), $N \le b^{60n}$,

$$\mathsf{g}^{\max}(j,c,c',\gamma) = \frac{\max\big(\prod_v |\mathsf{g}(\gamma)|_v,\ N^f \mathrm{mn}_j(\gamma)\big)}{i_\gamma i'_\gamma} \le \max(B,\ b^{60nf}B) = b^{60nf}B,$$

since $b \ge 1$.

The level gauge. By (FG6), $M \le b^{23}$, and $\min(\varrho, j) \le n$, so $M^{2\min(\varrho,j)f} \le b^{46nf}$. Hence $\mathsf{g}^*(j,c,c',\gamma) \le b^{(46n+60n+270n^2)f} \le b^{376n^2 f}$. By (FG6), $\varrho \ge 10^5$, so $n \ge 376$ and $376n^2 \le n^3$; as $b \ge 1$, $\mathsf{g}^*(j,c,c',\gamma) \le b^{n^3 f}$. □

Lemma 5.3.17. *Let $\mathfrak{S}$ be an admissible filling datum with Weil data, $T \subseteq \Gamma^*$ stable, $c, c' \in \mathcal{C}$ and $s \in \bar{G}$. Let $\mathcal{N}$ be the set of near representatives $\gamma \in T$ for (c,c') that are terms over (c,s,c') and satisfy the conditions of their cell rank if s agrees, and $\mathcal{N} = \emptyset$ otherwise. Then*

$$\|A_T^{\mathrm{cut}}(c,s,c')\|^2 \le \sum_L \|A^{\mathrm{cut}}_{T\cap\{\mathsf{g}\neq 0\}\cap \mathrm{link}^{-1}(L)}(c,s,c')\|^2 + \nu^2 \sum_{(\gamma,\gamma')} \mathrm{size}(c,c',\gamma)\,\mathrm{size}(c,c',\gamma'),$$

the first sum over all E-subspaces L, the second over the pairs $(\gamma,\gamma') \in \mathcal{N}\times\mathcal{N}$ that are not in the same link class.

Proof. **Pieces.** For $\gamma \in G(E)$ let its piece be $\pi(\gamma) = \star$ if $\mathsf{g}(\gamma) = 0$ and $\pi(\gamma) = \operatorname{link}\gamma$ otherwise, $\star$ a symbol distinct from all subspaces. For a piece p put $T_p = T \cap \pi^{-1}(p)$ and $a_p = \|A^{\mathrm{cut}}_{T_p}(c,s,c')\|$. Each T_p is stable and contained in Γ^*: for $u, u' \in \mathsf{U}(E)$, $\pi(u^{-1}\gamma u') = \pi(\gamma)$ by Lemma 5.2.25 (i), (ii), and T is stable. For a subspace L, $T_L = T \cap \{\mathsf{g} \neq 0\} \cap \operatorname{link}^{-1}(L)$.

The block is at most the sum of its pieces. For each term γ over (c,s,c'), $\|A^{\mathrm{cut}}_{\gamma}(c,c')\|$ is at most its cut amplitude majorant (the norm of a Bochner integral is at most the integral of the norm, and $A_\gamma = 0$ when the integrand is not integrable). By Lemma 5.2.29 (ii) (cut case, with $d = c'$) these majorants have a finite sum over the terms of T over (c,s,c'), so the series $A^{\mathrm{cut}}_T(c,s,c')$ converges absolutely and may be regrouped along the partition $T = \bigsqcup_p T_p$: $A^{\mathrm{cut}}_T(c,s,c') = \sum_p A^{\mathrm{cut}}_{T_p}(c,s,c')$, with only countably many nonzero summands. Hence $\|A^{\mathrm{cut}}_T(c,s,c')\| \le \sum_p a_p$ and

$$\|A^{\mathrm{cut}}_T(c,s,c')\|^2 \le \Big(\sum_p a_p\Big)^2 = \sum_L a_L^2 + \sum_{(p,p')\notin\Delta} a_p a_{p'},$$

where Δ is the set of pairs (L,L) with L a subspace. The first sum is the first sum of the statement.

The other products. By Proposition 5.3.19 (ii) applied to the stable set $T_p \subseteq \Gamma^*$, $a_p \le \nu \sum_{\gamma\in\mathcal{N}_p} \operatorname{size}(c,c',\gamma)$, where $\mathcal{N}_p = \mathcal{N} \cap \pi^{-1}(p)$ is the set of near representatives of T_p over (c,s,c') entering there. So

$$\sum_{(p,p')\notin\Delta} a_p a_{p'} \le \nu^2 \sum_{(p,p')\notin\Delta} \sum_{\gamma\in\mathcal{N}_p,\ \gamma'\in\mathcal{N}_{p'}} \operatorname{size}(c,c',\gamma)\operatorname{size}(c,c',\gamma').$$

The map $(p,p',\gamma,\gamma') \mapsto (\gamma,\gamma')$ on the index set on the right is injective, since $p = \pi(\gamma)$ and $p' = \pi(\gamma')$. Its image consists of pairs in $\mathcal{N}\times\mathcal{N}$ that are not in the same link class: if $\mathsf{g}(\gamma)\mathsf{g}(\gamma') \neq 0$ and $\operatorname{link}\gamma = \operatorname{link}\gamma' = L$, then $(p,p') = (L,L) \in \Delta$. As all summands are nonnegative, the right side is at most ν^2 times the second sum of the statement. □

Lemma 5.3.18. *Let $\mathfrak{S}$ be a filling datum, $\gamma \in G(E)$ and $u, u' \in \mathsf{U}(E)$, and put $\gamma' = u^{-1}\gamma u'$.*

(i) $D_j(\gamma') = D_j(\gamma)$ for $0 \le j \le \varrho+1$, so $\mathsf{g}(\gamma') = \mathsf{g}(\gamma)$; $\gamma' X = X$ if and only if $\gamma X = X$; and γ' has agreeing characters if and only if γ has.

(ii) The sets $R(E)$, Γ^* and $\{\gamma \in G(E) : \gamma$ has agreeing characters$\}$ are stable; complements in $G(E)$ and intersections of stable sets are stable.

(iii) Every $\gamma \in R(E)$ has $\mathsf{g}(\gamma) = 0$ and $\gamma X = X$; in particular $R(E) \cap \Gamma^* = \emptyset$.

Proof. Write $F_b = \langle e_1, \dots, e_b\rangle$, so $F_\varrho = X$. Let $g \in \mathsf{U}(E)$. It stabilises each F_b and acts trivially on F_b/F_{b-1}, so $ge_b \in e_b + F_{b-1}$; it stabilises $X^\perp \ni z$ and acts trivially on $X^\perp/X$, so $gz \in z + X$. Hence $gf_l = \sum_{a\le l} T(g)_{al} f_a$ with an upper unitriangular matrix $T(g) \in \mathrm{M}_{\varrho+1}(E)$, and $gX = X$.

(i). Since u is an isometry, $h(f_j, u^{-1}\gamma u' f_l) = h(uf_j, \gamma u' f_l)$, and expanding uf_j and $u'f_l$ in the frame,

$$Z(\gamma') = T(u)^*\, Z(\gamma)\, T(u').$$

$T(u)^*$ is lower and $T(u')$ upper unitriangular, so the leading principal $j\times j$ block of $Z(\gamma')$ is the product of the leading $j\times j$ blocks of $T(u)^*$, $Z(\gamma)$ and $T(u')$, and its determinant is $D_j(\gamma)$. Hence $D_j(\gamma') = D_j(\gamma)$ for all j, and $\mathsf{g}(\gamma') = |D_{\varrho+1}(\gamma)|^2 - N^2|D_\varrho(\gamma)|^2 = \mathsf{g}(\gamma)$. As $uX = u'X = X$, $\gamma' X = u^{-1}\gamma X$, which equals X if and only if $\gamma X = uX = X$. For agreement, it suffices to

show: if δ has agreeing characters, then so has $a\delta a'$ for all $a, a' \in \mathsf{U}(E)$ (apply it to γ with $(a, a') = (u^{-1}, u')$, and to γ' with $(a, a') = (u, u'^{-1})$). Let $w, w' \in \mathsf{U}(E)$ with $w\, a\delta a' = a\delta a'\, w'$. Then $(a^{-1}wa)\delta = \delta(a'w'a'^{-1})$ with $a^{-1}wa,\ a'w'a'^{-1} \in \mathsf{U}(E)$, so $\psi(a^{-1}wa) = \psi(a'w'a'^{-1})$. By Lemma 5.2.44 (i) ($\mathsf{U}(E) \subseteq \mathsf{U}(\mathbb{A}_f)$), ψ is multiplicative on $\mathsf{U}(E)$, so $\psi(a^{-1}wa) = \psi(w)$ and $\psi(a'w'a'^{-1}) = \psi(w')$; hence $\psi(w) = \psi(w')$.

(ii). $R(E)$ is a group containing $\mathsf{U}(E)$ ($\mathsf{U}(E)H(E)$ is a group as $H(E)$ normalises $\mathsf{U}(E)$: H fixes the e_a, e_a^* and z, so conjugation by it preserves the flag, its graded pieces and the triviality on $X^\perp/X$), so $u^{-1}\gamma u' \in R(E)$ if and only if $\gamma \in R(E)$. The set of elements with agreeing characters is stable by (i). Γ^* is defined by the conditions $\mathsf{g} \neq 0$, or agreeing characters, $\mathsf{g} = 0$ and $\gamma X \neq X$, each invariant under $\gamma \mapsto u^{-1}\gamma u'$ by (i); so Γ^* is stable. Complements and intersections of stable sets are stable, since stability says that $\gamma \mapsto u^{-1}\gamma u'$ maps the set onto itself.

(iii). Let $\gamma = gh$ with $g \in \mathsf{U}(E)$ and $h \in H(E)$. As h fixes the frame, $\gamma f_l = g f_l = \sum_{a \le l} T(g)_{al} f_a$, and $\gamma X = gX = X$. The Gram matrix of the frame is

$$(h(f_j, f_a))_{j,a} = \mathrm{diag}(0, \dots, 0, -N),$$

since the e_b are isotropic and orthogonal to each other and to z, and $h(z, z) = -N$. So $Z(\gamma) = \mathrm{diag}(0, \dots, 0, -N)\, T(g)$: its rows of index $< \varrho$ vanish. As $\varrho \ge 1$, $D_\varrho(\gamma) = 0$ and $D_{\varrho+1}(\gamma) = 0$, so $\mathsf{g}(\gamma) = 0$. Thus γ satisfies neither defining condition of Γ^*. □

The first input of the pair bound is the orbit bound Proposition 5.3.19: a large gauge of a term makes the sum of the cut amplitude majorants over its double coset small, and for a stable set of remaining terms every cut amplitude sum is at most ν times the sum of the sizes of its near representatives. The method is Schur's test [256]. The majorant of one term is at most a product of two Schur integrals (Lemma 5.3.20), which are summed over the orbit of the term under the frame-fixing unipotent groups (Lemma 5.3.22). The orbit splits into few classes of lattice translates, so that the orbit sum becomes an integral; in our reading this is the usual comparison of a lattice sum with an integral in the geometry of numbers [52]. The integral is controlled by integration in stages and by an upper bound for an integral of Hua's type, of negative powers of $\det(P + B^*B)$ over matrices B (Lemma 5.3.27; see [138]). The remaining statements of the group sort the terms: amplitude sums vanish over a block that does not agree, a term with vanishing discriminant has a cell rank, and the terms of nonzero discriminant satisfy an open-cell bound (Lemma 5.3.32).

The second input is the gain Proposition 5.3.64, and its idea is a congruence. Lifts of the links of two terms over one block pair integrally modulo M, so, in our reading, a nonzero Gram determinant of link vectors, an algebraic integer after division by the appropriate power of M, has norm at least one (Lemma 5.3.65).

Proposition 5.3.19. *Let $\mathfrak{S}$ be admissible with Weil data.*

(i) If γ is a term over (c, s, c') of cell rank $j \neq 0$, then

$$\Big(\sum_{\delta \in \mathsf{U}_c^0 \gamma \mathsf{U}_{c'}^0} \mathrm{maj}_\delta^{\mathrm{cut}}(c, c')\Big)^2 \le \Big(\nu\, \#I\, \tau_{\mathrm{sz}}^2 (b^{13}/M^2)^{\min(\varrho, j)kf}\Big)^2 \Big(\frac{N^f}{\mathsf{g}^\sharp(j, c, c', \gamma)}\Big)^k.$$

(ii) If $T \subseteq \Gamma^*$ is stable, then for all c, s, c'

$$\|A_T^{\mathrm{cut}}(c, s, c')\| \le \nu \sum_\gamma \mathrm{size}(c, c', \gamma),$$

the sum over the near representatives $\gamma \in T$ over (c, s, c') at their cell rank for which s agrees.

Proof. Write $W = W^{\mathrm{cut}}$, $\mathrm{maj}_\delta = \mathrm{maj}_\delta^{\mathrm{cut}}(c,c')$, $\mathcal{O}(\gamma) = \mathsf{U}_c^0(M)\gamma\mathsf{U}_{c'}^0(M)$, $t_\gamma = h_c^{-1}\gamma h_{c'}$ and $\operatorname{Nm}\mathsf{g}(\gamma) = \prod_v |\mathsf{g}(\gamma)|_v$.

Preliminaries.

(P1) If γ is a term over (c,s,c'), $u \in \mathsf{U}_c^0(M)$ and $u' \in \mathsf{U}_{c'}^0(M)$, then $t_{u\gamma u'} = (h_c^{-1}uh_c)\, t_\gamma\, (h_{c'}^{-1}u'h_{c'})$ with outer factors in $\mathrm{K}(M) = \ker \mathrm{red}$, so $u\gamma u'$ is a term over (c,s,c').

(P2) For $u \in \mathsf{U}_c^0(M)$ and $v \in \Phi$, $u_v = B_v^{-1}v(u)B_v$ with B_v diagonal lies in $\mathsf{U} \subseteq R$ and fixes every f_a. Hence $(u_\infty y)_v^{-1} f_b = y_v^{-1} f_b$, so $u_\infty y$ and y have the same Gram–Schmidt data and rational triangle, and $\ell(u_v y_v) = \ell(y_v)$ as ℓ is an infimum over R; thus $u_\infty y$ is boxed iff y is. If y witnesses that γ is near, $u_\infty y$ witnesses that $u\gamma u'$ is near, since $(u\gamma u')^{-1}u \cdot y = u'^{-1} \cdot (\gamma^{-1}y)$.

(P3) If $\mathrm{maj}_\delta \neq 0$, some y has $\beta_c(y)\beta_{c'}(\delta^{-1}y)\|W(y)\|\|W(\delta^{-1}y)\| \neq 0$; then y and $\delta^{-1}y$ are reduced and all their components lie in the cut by (BS2), so they are boxed and δ is near.

(P4) As γ is an isometry, $h(f_a, \gamma^{-1}f_b) = h(\gamma f_a, f_b) = \overline{h(f_b, \gamma f_a)}$, so $Z(\gamma^{-1}) = \overline{Z(\gamma)}^{\mathsf{T}}$, $D_l(\gamma^{-1}) = \overline{D_l(\gamma)}$, $\mathsf{g}(\gamma^{-1}) = \mathsf{g}(\gamma)$ and $\mathrm{mn}_j(\gamma^{-1}) = \mathrm{mn}_j(\gamma)$. If γ satisfies the conditions of cell rank j with the constant ζ, so does γ^{-1} with $\bar\zeta$: from $f \cdot x = \gamma f \cdot y$ with $x_a = \zeta$, $y_a = 1$, put $x' = \bar\zeta y$, $y' = \bar\zeta x$; then $f \cdot x' = \gamma^{-1}f \cdot y'$, $x'_a = \bar\zeta$, $y'_a = \bar\zeta\zeta = 1$, and the entries above a vanish. If γ is a term over (c,s,c'), then γ^{-1} is a term over (c', s^{-1}, c) with $t_{\gamma^{-1}} = t_\gamma^{-1}$.

(P5) A γ satisfies the conditions of at most one cell rank. Suppose it satisfies those of $j < j'$. Then $j \leq \varrho$, so $\mathsf{g}(\gamma) = 0$ and $j' \leq \varrho$. Take x, y for $a = j$: $f \cdot x = \gamma f \cdot y$, $y_j = 1$, $y_b = 0$ for $b > j$. For $l < j' \leq \varrho$, $f_l = e_{l+1}$ is orthogonal to every frame vector, so $0 = h(f_l, f \cdot x) = h(f_l, \gamma f \cdot y) = \sum_{b<j'} Z(\gamma)_{lb}\, \phi(y_b)$, with ϕ the identity or the conjugation of E according to the variable in which h is linear. The nonzero vector $(\phi(y_b))_{b<j'}$ is killed by the leading $j' \times j'$ block, so $D_{j'}(\gamma) = 0$, a contradiction. Every $\gamma \in \Gamma^*$ satisfies the conditions of some cell rank $j \geq 1$: if $\mathsf{g}(\gamma) \neq 0$, then $j = \varrho + 1$ works, the pair condition being empty and $D_{\varrho+1}(\gamma) = \det Z(\gamma) \neq 0$ by Lemma 5.2.26 (ii); if $\mathsf{g}(\gamma) = 0$, γ has agreeing characters and Lemma 5.3.31 (ii) gives some $j \leq \varrho$. The value $j = 0$ is impossible: the pairs at $a < \varrho$ give $\gamma f \cdot y^{(a)} \in \langle f_0, \dots, f_a\rangle$ for vectors $f \cdot y^{(a)}$ with $y_a^{(a)} = 1$ and $y_b^{(a)} = 0$ $(b > a)$, which span $X = \langle e_1, \dots, e_\varrho\rangle$; so $\gamma X = X$ and $\mathsf{g}(\gamma) = 0$, and $\gamma \notin \Gamma^*$. Hence the cell rank of $\gamma \in \Gamma^*$ is the unique such j, it is at least 1, and by Lemma 5.2.25 (iii) it does not change under $\gamma \mapsto u^{-1}\gamma u'$ $(u, u' \in \mathsf{U}(E))$.

Proof of (i). Let $i = i_j$. By Lemma 5.3.20 with shift i and the Cauchy–Schwarz inequality for sums in $[0,\infty]$,

$$\Big(\sum_{\delta \in \mathcal{O}(\gamma)} \mathrm{maj}_\delta\Big)^2 \leq \sum_{\delta \in \mathcal{O}(\gamma)} \mathrm{Si}(c,c',\delta,i) \cdot \sum_{\delta \in \mathcal{O}(\gamma)} \mathrm{Si}(c',c,\delta^{-1},i).$$

Inversion maps $\mathcal{O}(\gamma)$ bijectively onto $\mathsf{U}_{c'}^0(M)\gamma^{-1}\mathsf{U}_c^0(M)$. By (P4), γ^{-1} is a term over (c', s^{-1}, c) satisfying the conditions of cell rank j, with $\mathrm{mn}_j(\gamma^{-1}) = \mathrm{mn}_j(\gamma)$ and $t_{\gamma^{-1}} = t_\gamma^{-1}$. Applying Lemma 5.3.22 to γ and to γ^{-1}, the right side is at most

$$\Big(\nu\, \#I\, \tau_{\mathrm{sz}}^2 (b^{13}/M^2)^{ikf}\Big)^2 \Big(\frac{\mathrm{ir}_j(t_\gamma)\, \mathrm{ir}_j(t_\gamma^{-1})}{\mathrm{mn}_j(\gamma)}\Big)^k.$$

By Lemma 5.3.6 (i), $i_\gamma = \mathrm{ir}_j(t_\gamma) \geq 1$ and $i'_\gamma = \mathrm{ir}_j(t_\gamma^{-1}) \geq 1$, and by Lemma 5.3.9 (ii), $\mathsf{g}^\sharp(j,c,c',\gamma) = N^f \mathrm{mn}_j(\gamma)/(i_\gamma i'_\gamma)$; so the last factor is $(N^f/\mathsf{g}^\sharp(j,c,c',\gamma))^k$.

Proof of (ii). Fix c, s, c' and a stable $T \subseteq \Gamma^*$. For a term γ the norm of the Bochner integral $A^{\mathrm{cut}}_\gamma(c, c')$ is at most maj_γ, and $\sum \mathrm{maj}_\gamma < \infty$ over the terms of T over (c, s, c') by Lemma 5.2.29 (ii); so the block of every subset of T is an absolutely convergent sum, and blocks add over disjoint unions. If s does not agree, $A^{\mathrm{cut}}_T(c, s, c') = 0$ by Lemma 5.3.30, and there is nothing to prove; let s agree.

For $1 \le j \le \varrho + 1$ let T_j be the set of $\gamma \in T$ of cell rank j. By (P5) the T_j partition T, are stable, and every $\gamma \in T_j$ satisfies the conditions of cell rank j; so $\|A^{\mathrm{cut}}_T\| \le \sum_j \|A^{\mathrm{cut}}_{T_j}\|$. Fix j, let $K = \{\gamma : \mathrm{Nm}\,\mathfrak{g}(\gamma) > N^f \mathrm{mn}_j(\gamma)\}$, stable by Lemma 5.2.25 (ii), and split $T_j = (T_j \cap K) \sqcup (T_j \setminus K)$. In both parts, for a term γ, $i_\gamma, i'_\gamma \ge 1$ by Lemma 5.3.6 (i), so Lemma 5.3.9 (ii) applies.

On $T_j \cap K$: $\mathfrak{g} \ne 0$, so $j = \varrho + 1$ by (TA5). Then $\mathrm{ir}_{\varrho+1} = 1$ by (GT1) (the two lattices coincide), so $\mathfrak{g}^{\max}(\varrho + 1, c, c', \gamma) = \mathrm{Nm}\,\mathfrak{g}(\gamma)$, $i_{\varrho+1} = \varrho$ and $\mathcal{D}^{\mathrm{sz}}_{\varrho+1} = \mathcal{D}_{\mathrm{main}}$; the summand of Lemma 5.3.32 is $\mathrm{size}(c, c', \gamma)$, and that lemma gives $\|A^{\mathrm{cut}}_{T_j \cap K}\| \le \nu \sum \mathrm{size}(c, c', \gamma)$ over the near representatives $\gamma \in T_j \cap K$ over (c, s, c').

On $T' = T_j \setminus K$: by (P1) and stability, the terms of T' over (c, s, c') are the disjoint union of the orbits $\mathcal{O}(\rho)$, $\rho \in R_{c,c'}$ a term of T' over (c, s, c'). Hence $\|A^{\mathrm{cut}}_{T'}\| \le \sum_\rho \sum_{\delta \in \mathcal{O}(\rho)} \mathrm{maj}_\delta$. If the inner sum is nonzero, some $\delta \in \mathcal{O}(\rho)$ is near by (P3), so ρ is near by (P2), i.e. a near representative; ρ has cell rank $j \ge 1$, and (i) bounds the inner sum by

$$\nu \, \#I \, \tau_{\mathrm{sz}}^2 (b^{13}/M^2)^{ikf} (N^f / \mathfrak{g}^\sharp)^{k/2},$$

$i = i_j \ge 1$, $\mathfrak{g}^\sharp = \mathfrak{g}^\sharp(j, c, c', \rho)$. As $\rho \notin K$, $\mathfrak{g}^{\max} = \mathfrak{g}^\sharp$, and this bound is at most

$$\nu \, \mathrm{size}(c, c', \rho) = \nu \tau_{\mathrm{sz}}^2 \sqrt{\mathcal{D}^{\mathrm{sz}}_j / (n_p (M^{2if} \mathfrak{g}^\sharp)^k)} :$$

squaring and using $n_p N^{kf} \#I = \mathcal{D}_{\mathrm{id}}$, this is equivalent to $\#I \, (b^{13}/M)^{2ikf} \mathcal{D}_{\mathrm{id}} \le \mathcal{D}^{\mathrm{sz}}_j$, and indeed $\#I \le b^{2kf} = \tau_{\mathrm{sz}}^8$ by (FA7), $b^{13}/M \le b^{-6}$ by (FG6), so the left side is at most $\tau_{\mathrm{sz}}^{8-48i} \mathcal{D}_{\mathrm{id}} \le \tau_{\mathrm{sz}}^{-40} \mathcal{D}_{\mathrm{id}}$, while $\mathcal{D}^{\mathrm{sz}}_j = \mathcal{D}_{\mathrm{id}} / \tau_{\mathrm{sz}}^{16}$ for $j \le \varrho$ and $\mathcal{D}^{\mathrm{sz}}_j = \mathcal{D}_{\mathrm{main}} \ge b^{11kf} \mathcal{D}_{\mathrm{id}}$ for $j = \varrho + 1$ by (FA6), both at least $\tau_{\mathrm{sz}}^{-40} \mathcal{D}_{\mathrm{id}}$ as $\tau_{\mathrm{sz}} \ge 1$.

Adding the two parts and summing over j gives $\|A^{\mathrm{cut}}_T(c, s, c')\| \le \nu \sum_\gamma \mathrm{size}(c, c', \gamma)$ over the near representatives $\gamma \in T$ over (c, s, c'), each at its cell rank, with s agreeing. □

Lemma 5.3.20. *Let $\mathfrak{S}$ be a filling datum, $c, c' \in \mathcal{C}$, $\delta \in G(E)$ and $i \ge 0$ an integer. Then, in $[0, \infty]$,*

$$\mathrm{maj}^{\mathrm{cut}}_\delta(c, c')^2 \le \mathrm{Si}(c, c', \delta, i) \, \mathrm{Si}(c', c, \delta^{-1}, i).$$

Proof. Write $W = W^{\mathrm{cut}}$, so $\|W(y)\| = \prod_v \|W^\flat_v(y_v)\|$ and $\|W(y) \otimes \overline{W(x)}\| = \|W(y)\| \, \|W(x)\|$ (a rank-one tensor). Put $m(y) = \beta_c(y) \beta_{c'}(\delta^{-1} y) \|W(y)\| \, \|W(\delta^{-1} y)\|$, so that $\mathrm{maj}^{\mathrm{cut}}_\delta(c, c') = \int m$, and $\vartheta(y) = \prod_v \mathrm{Vr}_{\delta_v}(y_v)$, which is positive by Lemma 5.3.21 (i). All functions below are Borel and take values in $[0, \infty]$.

The two Schur integrals. Put

$$A(y) = \beta^{\mathrm{bx}}_c(y) \|W(y)\|^2 \beta^{\mathrm{bx}}_{c'}(\delta^{-1} y) \, \vartheta(y)^{-(k+i)}, \qquad B(y) = \beta^{\mathrm{bx}}_{c'}(\delta^{-1} y) \|W(\delta^{-1} y)\|^2 \beta^{\mathrm{bx}}_c(y) \, \vartheta(y)^{k+i}.$$

By (SX2), $\int A = \mathrm{Si}(c, c', \delta, i)$. Since $\gamma \mapsto \gamma_\infty$ is a homomorphism, $(\delta^{-1})_v = \delta_v^{-1}$ and $(\delta^{-1})^{-1} x = \delta x$, so

$$\mathrm{Si}(c', c, \delta^{-1}, i) = \int_{G_\infty} \beta^{\mathrm{bx}}_{c'}(x) \|W(x)\|^2 \beta^{\mathrm{bx}}_c(\delta x) \prod_v \mathrm{Vr}_{\delta_v^{-1}}(x_v)^{-(k+i)} \, dx.$$

The measure of G_∞ is the product of the Cayley measures of the factors, each invariant under left translations by Lemma 5.2.41; by Tonelli's theorem the substitution $x = \delta^{-1}y$ leaves the integral unchanged, and $\mathrm{Vr}_{\delta_v^{-1}}(\delta_v^{-1}y_v) = \mathrm{Vr}_{\delta_v}(y_v)^{-1}$ by Lemma 5.3.21 (ii). Hence $\mathrm{Si}(c', c, \delta^{-1}, i) = \int B$.

The pointwise bound $m^2 \le AB$. If $m(y) = 0$ there is nothing to show. Otherwise $\beta_c(y) \neq 0$, so y is reduced, and $W_v^\flat(y_v) \neq 0$ for every v, so by (BS2) every y_v lies in the cut, $\ell(y_v) \le b^{6n+2}$; hence y is boxed and $\beta_c^{\mathrm{bx}}(y) = \beta_c(y)$. In the same way $\delta^{-1}y$ is boxed and $\beta_{c'}^{\mathrm{bx}}(\delta^{-1}y) = \beta_{c'}(\delta^{-1}y)$. As $0 < \vartheta(y) < \infty$, the powers of $\vartheta(y)$ cancel and $A(y)B(y) = m(y)^2$.

Conclusion. By the Cauchy–Schwarz inequality for nonnegative functions,

$$\Big(\int m\Big)^2 \le \Big(\int \sqrt{AB}\Big)^2 \le \int A \int B,$$

which is the assertion. □

Lemma 5.3.21. *Let $\varrho \ge 1$ and κ be as in Definition 5.1.6, and $\gamma, y \in G_1$.*

(i) $\mathrm{Fv}(y) > 0$*; hence* $\mathrm{Vr}_\gamma(y) > 0$.

(ii) $\mathrm{Vr}_{\gamma^{-1}}(\gamma^{-1}y) = \mathrm{Vr}_\gamma(y)^{-1}$.

(iii) If $u \in G_1$ fixes $f_0, \dots, f_\varrho$, then $\mathrm{Fv}(uy) = \mathrm{Fv}(y)$ *and* $\mathrm{Vr}_\gamma(uy) = \mathrm{Vr}_{u^{-1}\gamma}(y)$.

Proof. In the basis $e_1, \dots, e_\varrho$, the basis of X_0, z, $e_1^*, \dots, e_\varrho^*$, the entries of J_1 are the values of h_1 on pairs of basis vectors, so $J_1 e_a = e_a^*$, $J_1 e_a^* = e_a$, $J_1 x = x$ for $x \in X_0$ and $J_1 z = -z$. Let F be the $r \times (\varrho+1)$ matrix with columns $f_0, \dots, f_\varrho$ and $P = (1 - J_1)F$, with columns $e_a - e_a^*$ $(1 \le a \le \varrho)$ and $2z$; thus $\mathrm{Fv}(y) = |\det(P^* y^{-1} F)|$.

(i) Suppose $\det(P^* y^{-1} F) = 0$, and choose $0 \neq x \in \mathbb{C}^{\varrho+1}$ with $P^* y^{-1} F x = 0$; put $w = y^{-1}Fx$. The columns of P are $\varrho+1$ nonzero, pairwise orthogonal vectors of the (-1)-eigenspace V_- of J_1, which has dimension $\varrho+1$ by the list above; so they span V_-, and w is orthogonal to V_-. As J_1 is hermitian and unitary, $V_-^\perp$ is the $(+1)$-eigenspace, so $J_1 w = w$ and $h_1(w, w) = \|w\|^2$. On the other hand $y^{-1} \in G_1$ preserves h_1, and the frame vectors satisfy $h_1(f_a, f_b) = 0$ unless $a = b = \varrho$, $h_1(z, z) = -1$; hence $h_1(w, w) = h_1(Fx, Fx) = -|x_\varrho|^2$. So $\|w\|^2 \le 0$, $w = 0$, $Fx = yw = 0$, and $x = 0$ because the columns of F are distinct basis vectors, a contradiction. Hence $\mathrm{Fv}(y) > 0$ for every y, and $\mathrm{Vr}_\gamma(y) = \mathrm{Fv}(\gamma^{-1}y)/\mathrm{Fv}(y) > 0$.

(ii) $\mathrm{Vr}_{\gamma^{-1}}(\gamma^{-1}y) = \mathrm{Fv}(\gamma\gamma^{-1}y)/\mathrm{Fv}(\gamma^{-1}y) = \mathrm{Fv}(y)/\mathrm{Fv}(\gamma^{-1}y) = \mathrm{Vr}_\gamma(y)^{-1}$, both frame volumes being positive by (i).

(iii) As u^{-1} fixes every f_b, $(uy)^{-1}F = y^{-1}u^{-1}F = y^{-1}F$, so $\mathrm{Fv}(uy) = \mathrm{Fv}(y)$. Then $\mathrm{Vr}_\gamma(uy) = \mathrm{Fv}(\gamma^{-1}uy)/\mathrm{Fv}(uy) = \mathrm{Fv}((u^{-1}\gamma)^{-1}y)/\mathrm{Fv}(y) = \mathrm{Vr}_{u^{-1}\gamma}(y)$. □

Lemma 5.3.22. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, $1 \le j \le \varrho+1$, and γ a term over (c, s, c') satisfying the conditions of cell rank j. Then*

$$\sum_{\delta \in \mathsf{U}_c^0(M)\,\gamma\,\mathsf{U}_{c'}^0(M)} \mathrm{Si}(c, c', \delta, i_j) \le \nu\, \#I\, \tau_{\mathrm{sz}}^2 \Big(\frac{b^{13}}{M^2}\Big)^{i_j k f} \Big(\frac{\mathrm{ir}_j(h_c^{-1}\gamma h_{c'})}{\sqrt{\mathrm{mn}_j(\gamma)}}\Big)^k.$$

Proof. Put $i = i_j$, $O = \mathsf{U}_c^0(M)\,\gamma\,\mathsf{U}_{c'}^0(M)$ (a countable set, as $G(E)$ is countable), $\mathrm{ir} = \mathrm{ir}_j(h_c^{-1}\gamma h_{c'})$, and

$$F(y) = \sum_{p \in O} \mathrm{Sw}_{c', p, i}(y) \qquad (y \in G_\infty),$$

a countable sum of nonnegative Borel functions, hence Borel. By monotone convergence,

$$\sum_{\delta\in O}\mathrm{Si}(c,c',\delta,i)=\int_{G_\infty}\beta_c^{\mathrm{bx}}(y)\prod_{v\in\Phi}\|W_v^\flat(y_v)\|^2\,F(y)\,dy. \tag{5.49}$$

F **is invariant under $\mathsf{U}_c^0(M)$.** Let $u\in\mathsf{U}_c^0(M)$. Then $p\mapsto up$ is a bijection of O. Each f_b has coordinates in $\{0,1\}$ and $B_vf_b=\mathrm{bs}_v(f_b)f_b$ for the diagonal matrix B_v of basis scales, so $u_vf_b=B_v^{-1}v(u)B_vf_b=f_b$ for every v and b, as $uf_b=f_b$. Hence $\mathrm{Fv}(u_vy_v)=\mathrm{Fv}(y_v)$, since $(u_vy_v)^{-1}f_b=y_v^{-1}f_b$, and $(up)_\infty^{-1}u_\infty y=p_\infty^{-1}y$; so $\mathrm{Sw}_{c',up,i}(u_\infty y)=\mathrm{Sw}_{c',p,i}(y)$, and summing over $p\in O$ gives $F(u_\infty y)=F(y)$.

The average over a fundamental domain. Let $Y=\prod_v\mathrm{M}_{i\times k}(\mathbb{C})$ with Lebesgue measure, $\Lambda=\Lambda_{c,i}^\infty\subseteq Y$ the column lattice at infinity (SX6), and let D be the Borel set of Lemma 5.3.23 for c and i, with $1\le\int_D dx<\infty$. The map $x\mapsto\mathrm{Es}_j(x)$ is continuous with values in $\prod_v\mathsf{U}^0$ (SX3). Let $u\in G_\infty$ with all $u_v\in\mathsf{U}^0$ and let y be boxed. By Lemma 5.3.24, O is the union of a family $\mathcal{B}$ of at most ir^k sets B, each containing a p_B of which every element of B is a lattice translate. For each $B\in\mathcal{B}$, Lemma 5.3.25 (with B, p_B, u, y) gives a Borel $\phi_B\ge0$ on Y with $\int_Y\phi_B\le\prod_v\mathrm{Kb}_j(\gamma,v)$ and $\sum_{p\in B}\mathrm{Sw}_{c',p,i}(u\,\mathrm{Es}_j(x)\,y)\le\tau_{\mathrm{sz}}\sum_{w\in\Lambda}\phi_B(w+x)$. Since the translates $w+D$ ($w\in\Lambda$) cover Y up to a null set and are pairwise disjoint up to null sets, $\int_D\sum_{w\in\Lambda}\phi_B(w+x)\,dx=\sum_w\int_{w+D}\phi_B=\int_Y\phi_B$. Hence, with all terms nonnegative,

$$\int_D F(u\,\mathrm{Es}_j(x)\,y)\,dx\le\sum_{B\in\mathcal{B}}\int_D\sum_{p\in B}\mathrm{Sw}_{c',p,i}(u\,\mathrm{Es}_j(x)\,y)\,dx\le\mathrm{ir}^k\tau_{\mathrm{sz}}\prod_v\mathrm{Kb}_j(\gamma,v)\le\Big(\int_D dx\Big)\,C_0,$$

where $C_0=\mathrm{ir}^k\tau_{\mathrm{sz}}\prod_v\mathrm{Kb}_j(\gamma,v)$ and the last step uses $\int_D dx\ge1$.

Integration in stages. By Lemma 5.3.28 with the measure space (Y,dx), the map Es_j in the role of t, $A=D$, $C=C_0$ and the invariant function F, the right side of (5.49) is at most $C_0\,\nu\,\#I$. By Lemma 5.3.29 ($j\ge1$),

$$\sum_{\delta\in O}\mathrm{Si}(c,c',\delta,i)\le\nu\,\#I\,\mathrm{ir}^k\,\tau_{\mathrm{sz}}\cdot\tau_{\mathrm{sz}}\Big(\frac{b^{13}}{M^2}\Big)^{ikf}\mathrm{mn}_j(\gamma)^{-k/2}=\nu\,\#I\,\tau_{\mathrm{sz}}^2\Big(\frac{b^{13}}{M^2}\Big)^{ikf}\Big(\frac{\mathrm{ir}}{\sqrt{\mathrm{mn}_j(\gamma)}}\Big)^k.$$

□

Lemma 5.3.23. *Let $\mathfrak{S}$ be a filling datum over a field E that carries Weil data, $c\in\mathcal{C}$ and $i\ge0$ an integer. Let $Y=\prod_{v\in\Phi}\mathrm{M}_{i\times k}(\mathbb{C})$ with Lebesgue measure dx in the real and imaginary parts of the entries. Then there is a Borel set $D\subseteq Y$ with $1\le\int_D dx<\infty$ whose translates $w+D$ ($w\in\Lambda_{c,i}^\infty$) are pairwise disjoint and cover Y.*

Proof. Write $x_1,\dots,x_k$ for the standard basis of $X_0=E^k$, $L=L_c\cap X_0(\mathbb{A}_f)$, and $L_E=\{x\in X_0:\ x\in L\}$ ($X_0\subseteq X_0(\mathbb{A}_f)$ diagonally). Put $\Lambda=\Lambda_{c,i}^\infty$.

The definite part L is an open compact $\hat{\mathcal{O}}$-module on which the form is integral. By (FG3), L_0 is the $\hat{\mathcal{O}}$-span of the standard basis of V with x_k replaced by $x_k/\bar\pi+z/N$; the vector z itself is a basis vector. An element $\sum_{a<k}\alpha_ax_a+\alpha_k(x_k/\bar\pi+z/N)+\alpha_zz+(\text{terms in the }e_b,e_b^*)$ with all coefficients in $\hat{\mathcal{O}}$ lies in $X_0(\mathbb{A}_f)$ if and only if the e_b,e_b^*-coefficients vanish and $\alpha_z=-\alpha_k/N$, which is integral if and only if $\alpha_k\in N\hat{\mathcal{O}}$; then $\alpha_k/\bar\pi\in\pi\hat{\mathcal{O}}$, as $N=\pi\bar\pi$. Hence

$$L_0\cap X_0(\mathbb{A}_f)=\textstyle\bigoplus_{a<k}\hat{\mathcal{O}}x_a\oplus\pi\hat{\mathcal{O}}x_k\ \subseteq\ \hat{\mathcal{O}}^k.$$

The element $h_c\in\mathrm{U}(X_0)(\mathbb{A}_f)$ acts on $V(\mathbb{A}_f)$ through the embedding $\mathrm{U}(X_0)\subseteq G$: it preserves $X_0(\mathbb{A}_f)$ and the hermitian form. So $L=h_cL_0\cap X_0(\mathbb{A}_f)=h_c(L_0\cap X_0(\mathbb{A}_f))$ is the image of an

open compact $\hat{\mathcal{O}}$-submodule of $\mathbb{A}_f^k$ under an $\mathbb{A}_f$-linear homeomorphism, hence an open compact $\hat{\mathcal{O}}$-submodule, and $h(x,y) = h(h_c^{-1}x, h_c^{-1}y) \in \hat{\mathcal{O}}$ for $x, y \in L$, since the unit form takes values in $\hat{\mathcal{O}}$ on $\hat{\mathcal{O}}^k$.

L_E is a free $\mathbb{Z}$-module of rank $2fk$. The sets $m\hat{\mathcal{O}}^k$ ($m \geq 1$ an integer) form a basis of neighbourhoods of 0 in $\mathbb{A}_f^k$, and $\mathbb{A}_f^k$ is the increasing union of the open sets $m^{-1}\hat{\mathcal{O}}^k$. As L is open and compact there are integers $m_1, m_2 \geq 1$ with $m_1\hat{\mathcal{O}}^k \subseteq L \subseteq m_2^{-1}\hat{\mathcal{O}}^k$. Intersecting with E^k and using $E \cap \hat{\mathcal{O}} = \mathcal{O}_E$ gives $m_1\mathcal{O}_E^k \subseteq L_E \subseteq m_2^{-1}\mathcal{O}_E^k$. Since $\mathcal{O}_E$ is a free $\mathbb{Z}$-module of rank $[E:\mathbb{Q}] = 2f$, the subgroup L_E of the free module $m_2^{-1}\mathcal{O}_E^k$ is free, of rank $2fk$ because it contains $m_1\mathcal{O}_E^k$. In particular a $\mathbb{Z}$-basis of L_E is a $\mathbb{Q}$-basis of E^k.

Λ is spanned by an $\mathbb{R}$-basis of Y. For $q \in \mathrm{M}_{k\times i}(E)$ put $q^\infty = (v(q)^*)_{v\in\Phi} \in Y$; the map $q \mapsto q^\infty$ is additive, and by (SX6) $\Lambda = \{q^\infty : q \in L_E^i\}$, where L_E^i denotes the matrices with all columns in L_E. The map $E \otimes_{\mathbb{Q}} \mathbb{R} \to \prod_{v\in\Phi} \mathbb{C}$, $x \otimes t \mapsto (t\,v(x))_v$, is an isomorphism of real vector spaces, because E is totally imaginary of degree $2f$ and Φ contains exactly one embedding from each pair of complex conjugate embeddings (the classical Minkowski embedding). Applied entrywise and followed by conjugate transposition, it shows that the $\mathbb{R}$-linear extension $\mathrm{M}_{k\times i}(E) \otimes_{\mathbb{Q}} \mathbb{R} \to Y$ of $q \mapsto q^\infty$ is an isomorphism. Let $g_1, \ldots, g_{2fki}$ be a $\mathbb{Z}$-basis of L_E^i (by the previous step, $L_E^i \cong L_E^i$ is free of rank $2fki$); it is a $\mathbb{Q}$-basis of $\mathrm{M}_{k\times i}(E)$, so $w_j = g_j^\infty$ ($1 \leq j \leq 2fki$) is an $\mathbb{R}$-basis of Y, and $\Lambda = \bigoplus_j \mathbb{Z}w_j$.

The euclidean products of points of Λ are integers. Let $\langle y, y'\rangle = \sum_v \sum_{l,a} \mathrm{Re}(\overline{y_{v,la}}\, y'_{v,la})$; the real and imaginary parts of the entries form an orthonormal basis for it, so dx is the Lebesgue measure of this euclidean structure. For $q, q' \in L_E^i$ the entries of the component at v of q^∞ are $\overline{v(q_{al})}$, and since every embedding of the CM field E commutes with complex conjugation,

$$\langle q^\infty, q'^\infty\rangle = \sum_{v\in\Phi} \mathrm{Re}\, v(w), \qquad w = \sum_l \sum_a q_{al}\overline{q'_{al}}.$$

Here $w = \sum_l h(q'_l, q_l)$ for the columns $q_l, q'_l \in L_E \subseteq L$ (or its conjugate, according to the side on which h is linear), so $w \in E \cap \hat{\mathcal{O}} = \mathcal{O}_E$ by the first step. The embeddings of E are the $v \in \Phi$ and their conjugates, so $\sum_v \mathrm{Re}\, v(w) = \frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(w)$. By the Weil data E contains a primitive 5040-th root of unity ζ, and $\mathrm{i} = \zeta^{1260}$ satisfies $\mathrm{i}^2 = -1$; so $\mathbb{Q}(\mathrm{i}) \subseteq E$ and $\mathrm{Tr}_{E/\mathbb{Q}}(w) = \mathrm{Tr}_{\mathbb{Q}(\mathrm{i})/\mathbb{Q}}(\mathrm{Tr}_{E/\mathbb{Q}(\mathrm{i})}(w))$ with $\mathrm{Tr}_{E/\mathbb{Q}(\mathrm{i})}(w) \in \mathcal{O}_E \cap \mathbb{Q}(\mathrm{i}) = \mathbb{Z}[\mathrm{i}]$, whose traces to $\mathbb{Q}$ are even. Hence $\langle q^\infty, q'^\infty\rangle \in \mathbb{Z}$.

The fundamental parallelepiped. The Gram matrix $(\langle w_j, w_l\rangle)_{j,l}$ is an integer matrix, positive definite as the w_j are linearly independent, so its determinant is an integer ≥ 1. Let $D = \{\sum_j t_j w_j : \ 0 \leq t_j < 1\}$, a Borel set. Every $y \in Y$ is $\sum_j s_j w_j$ for unique $s \in \mathbb{R}^{2fki}$, and $y = \sum_j \lfloor s_j \rfloor w_j + \sum_j (s_j - \lfloor s_j \rfloor) w_j$ is its unique decomposition as an element of $\Lambda = \bigoplus_j \mathbb{Z}w_j$ plus an element of D; so the translates $w + D$ ($w \in \Lambda$) are pairwise disjoint and cover Y. Finally $\int_D dx$ is the absolute value of the determinant of $(w_1, \ldots, w_{2fki})$ in an orthonormal basis, that is $\det(\langle w_j, w_l\rangle)^{1/2}$, which lies in $[1, \infty)$. (For $i = 0$, $Y = \Lambda = D = \{0\}$ and $\int_D dx = 1$.) □

Lemma 5.3.24. *Let $\mathfrak{S}$ be a filling datum with Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, and γ a term over (c, s, c') satisfying the conditions of cell rank j. Then $\mathrm{ir}_j(h_c^{-1}\gamma h_{c'})$ is a positive integer, and the double coset $\mathsf{U}_c^0(M)\,\gamma\,\mathsf{U}_{c'}^0(M)$ is the union of at most $\mathrm{ir}_j(h_c^{-1}\gamma h_{c'})^k$ sets B, each containing an element p_B such that every $p \in B$ is a lattice translate of p_B (for c, γ, j).*

Proof. Write $A = \langle e_1, \ldots, e_\varrho\rangle$, so $A^\perp = A \oplus X_0 \oplus Ez$ and $V = A^\perp \oplus \langle e_1^*, \ldots, e_\varrho^*\rangle$; for $w \in V(\mathbb{A}_f)$ the coefficient of e_a^* is $h(e_a, w)$ and the coefficient of z is $-h(z,w)/N$. Let $k_0 = h_c^{-1}\gamma h_{c'} \in K$, $i = i_j = \min(\varrho, j)$, and let X be the $\varrho \times (\varrho+1)$ matrix over E with entries $X_{ab} = h(e_a, \gamma f_b)$ ($1 \leq a \leq \varrho$, $0 \leq b \leq \varrho$), with rows X_a; $X_{\leq i}$ denotes its first i rows. As $h_c, h_{c'}$ fix the frame

vectors and preserve h, $h(f_{b''}, k_0 f_b) = h(f_{b''}, \gamma f_b)$ for all frame vectors $f_{b''}, f_b$, so X consists of the rows $0, \dots, \varrho - 1$ of $Z(\gamma)$, the rows of $f_0, \dots, f_{\varrho-1} = e_1, \dots, e_\varrho$.

Step 1: $X_{\le i}$ has rank i, and the rows X_a ($a > i$) are E-combinations of $X_1, \dots, X_i$. By (TA5), for every a with $j \le a \le \varrho$ there are $x, y \in E^{\varrho+1}$ with $f \cdot x = \gamma f \cdot y$, $y_a = 1$ and $y_b = 0$ for $b > a$, so $\gamma f_a = f \cdot x - \sum_{b<a} y_b \gamma f_b$. Since $e_{a'}$ is orthogonal to every frame vector, $X_{a'a} = -\sum_{b<a} y_b X_{a'b}$ for all a': column a of X is an E-combination, with coefficients independent of the row, of the columns $b < a$. By induction on $a = j, \dots, \varrho$, every column $a \ge j$ of X is an E-combination of the columns $b < j$. If $j \le \varrho$, then $i = j$, and the block of X on the rows $1, \dots, j$ and columns $0, \dots, j-1$ is the leading $j \times j$ block of $Z(\gamma)$, of determinant $D_j(\gamma) \ne 0$ (TA5); so $X = X_{\cdot,<j} C$ with $C = (X_{\le j,<j})^{-1} X_{\le j}$, and every row $X_a = X_{a,<j}(X_{\le j,<j})^{-1} X_{\le j}$ is an E-combination of $X_1, \dots, X_j$, and $X_{\le j}$ has rank j. If $j = \varrho + 1$, then $i = \varrho$, $X_{\le i} = X$ consists of ϱ rows of the invertible matrix $Z(\gamma)$ ($\det Z(\gamma) = D_{\varrho+1}(\gamma) \ne 0$), so it has rank ϱ, and the second claim is empty. In both cases there is $\xi' \in \mathrm{M}_{(\varrho+1)\times i}(E)$ with $X_{\le i}\, \xi' = 1_i$.

Step 2: the index. Let $R = \sum_{a=1}^{\varrho} \hat{\mathcal{O}} X_a$ and $R_i = \sum_{a \le i} \hat{\mathcal{O}} X_a$, submodules of $\mathbb{A}_f^{\varrho+1}$. The rows $r_a(k_0)$ of (GT1) are $r_a(k_0)_b = h(k_0 f_b, e_a) = \overline{X_{ab}}$, and complex conjugation is a ring automorphism of $\mathbb{A}_f$ preserving $\hat{\mathcal{O}}$; hence $\mathrm{ir}_j(k_0) = [R : R_i]$. By Step 1 there is an integer $d \ge 1$ with $dX_a \in \sum_{a' \le i} \mathcal{O}_E X_{a'}$ for all a, so $dR \subseteq R_i \subseteq R$; as R is a finitely generated $\hat{\mathcal{O}}$-module and $\hat{\mathcal{O}}/d\hat{\mathcal{O}}$ is finite, R/dR is finite. So $\mathrm{ir}_j(k_0) = [R : R_i]$ is a positive integer.

Step 3: the change of the frame along the double coset. Let $p = u\gamma u'$ with $u \in \mathsf{U}_c^0(M)$, $u' \in \mathsf{U}_{c'}^0(M)$. As u' fixes the frame, $pf_b = u\gamma f_b$. Write $\gamma f_b = w_b + \sum_a X_{ab} e_a^*$ with $w_b \in A^\perp$. The element u lies in the unipotent radical of the stabiliser of the flag $\langle e_1 \rangle \subset \cdots \subset A$, so it acts trivially on $A^\perp/A$, and $(u-1)w_b \in A$. Hence

$$pf_b - \gamma f_b \equiv \sum_{a=1}^{\varrho} X_{ab}\,(u-1)e_a^* \pmod{A}.$$

The vector $(u-1)e_a^*$ has no e^*- and no z-coefficient: $h(e_{a'}, (u-1)e_a^*) = h(u^{-1}e_{a'}, e_a^*) - h(e_{a'}, e_a^*) = 0$ and likewise for z, as u fixes $e_{a'}$ and z. Moreover $(u-1)e_a^* = h_c(t-1)e_a^*$ with $t = h_c^{-1} u h_c \in K(M)$ (RD2) and $h_c^{-1} e_a^* = e_a^*$; by (iii) of Lemma 5.2.24 and $e_a^* \in L_0$, $(t-1)e_a^* \in ML_0 \cap (A \oplus X_0)(\mathbb{A}_f)$. In the lattice basis an element of L_0 with vanishing e^*- and z-coefficients has coefficient $\alpha_k \in N\hat{\mathcal{O}}$ at $x_k/\bar{\pi} + z/N$ (x_k the last basis vector of X_0), so its A-part lies in $\bigoplus_a \hat{\mathcal{O}} e_a \subseteq L_0$ and its X_0-part in L_0; thus $L_0 \cap (A \oplus X_0)(\mathbb{A}_f) = (L_0 \cap A(\mathbb{A}_f)) \oplus (L_0 \cap X_0(\mathbb{A}_f))$, and the same holds for $ML_0 = M \cdot L_0$. As h_c preserves A and X_0, the X_0-part $\mu_a(u)$ of $(u-1)e_a^*$ lies in $M(L_c \cap X_0(\mathbb{A}_f))$. Let $Y_p \in \mathrm{M}_{k\times(\varrho+1)}(E)$ have as column b the X_0-part of $pf_b - \gamma f_b$. Then $pf_b - \gamma f_b \in A \oplus X_0$ and

$$Y_p = \mu(u)\, X, \qquad \mu(u) = (\mu_1(u), \dots, \mu_\varrho(u)), \quad \mu_a(u) \in M(L_c \cap X_0(\mathbb{A}_f)).$$

Step 4: colouring. Fix $g \in \mathrm{GL}_k(\mathbb{A}_f)$ with $L_c \cap X_0(\mathbb{A}_f) = g\,\hat{\mathcal{O}}^k$ (an open compact $\hat{\mathcal{O}}$-submodule of $\mathbb{A}_f^k$, free as $\hat{\mathcal{O}}$ is a product of principal ideal domains). By Step 3, $\mu(u) = Mg\,\omega$ with $\omega \in \mathrm{M}_{k\times\varrho}(\hat{\mathcal{O}})$, so every row of $M^{-1}g^{-1}Y_p = \omega X$ lies in R. Colour p by the classes of the k rows of $M^{-1}g^{-1}Y_p$ in R/R_i; this takes at most $[R : R_i]^k = \mathrm{ir}_j(k_0)^k$ values. (Y_p depends only on p, not on the choice of u, u'.) Let B be a colour class and $p_B \in B$. For $p \in B$ put $D = Y_p - Y_{p_B} \in \mathrm{M}_{k\times(\varrho+1)}(E)$; its rows satisfy $M^{-1}g^{-1}D = \omega' X_{\le i}$ with $\omega' \in \mathrm{M}_{k\times i}(\hat{\mathcal{O}})$. Let $l = M^{-1}D\xi' \in \mathrm{M}_{k\times i}(E)$ with ξ' of Step 1. Then $l = g\,\omega' X_{\le i}\xi' = g\,\omega'$, so every column of l lies in $L_c \cap X_0(\mathbb{A}_f)$, and $Ml\, X_{\le i} = Mg\,\omega' X_{\le i} = D$. By Step 3 for p and for p_B, $pf_b - p_B f_b \in A \oplus X_0$ with X_0-part the column b of $D = M\sum_{a<i} l_a\,(X_{a+1,b})$, where l_a is column a of l ($0 \le a < i$) and

$X_{a+1,b} = h(e_{a+1}, \gamma f_b)$; that is,

$$pf_b \equiv p_B f_b + M \sum_{a<i} h(e_{a+1}, \gamma f_b)\, l_a \pmod{A} \qquad (0 \le b \le \varrho),$$

and p is a lattice translate of p_B by l (SX5). □

Lemma 5.3.25. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, γ a term over (c, s, c') satisfying the conditions of cell rank j, and $i = i_j$. Let $B \subseteq \mathsf{U}^0_c(M)\,\gamma\,\mathsf{U}^0_{c'}(M)$ and $p_0 \in B$ be such that every $p \in B$ is a lattice translate of p_0 (for c, γ, j). Let $u \in G_\infty$ with $u_v \in \mathsf{U}^0$ for every v, and let $y \in G_\infty$ be boxed. Put $Y = \prod_{v\in\Phi} \mathrm{M}_{i\times k}(\mathbb{C})$ with Lebesgue measure dx and $\Lambda = \Lambda^\infty_{c,i} \subseteq Y$. Then there is a Borel function $\phi\colon Y \to [0,\infty]$ with $\int_Y \phi\, dx \le \prod_{v\in\Phi} \mathrm{Kb}_j(\gamma, v)$ such that for every $x \in Y$*

$$\sum_{p\in B} \mathrm{Sw}_{c',p,i}(u\,\mathrm{Es}_j(x)\,y) \le \tau_{\mathrm{sz}} \sum_{w\in\Lambda} \phi(w+x).$$

Proof. Let $O = \mathsf{U}^0_c(M)\gamma\mathsf{U}^0_{c'}(M)$; for $p \in B$ fix $l(p) \in \mathrm{M}_{k\times i}(E)$ with columns in $L_c \cap X_0(\mathbb{A}_f)$ such that p is a lattice translate of p_0 by $l(p)$. Write $\pi_0(w)$ for the X_0-coordinates of a vector w, $A = \langle e_1, \dots, e_\varrho\rangle$, and $D_v = D_{j,v}$ (SX3), a real diagonal matrix with entries $d_{v,a} = M/\mathrm{bs}_v(e_{a+1})$.

Step 0: the elements of O. Let $p = u_1\gamma u_1' \in O$. As u_1, u_1' fix the frame, $h(f_a, pf_b) = h(f_a, \gamma f_b)$, so $Z(p) = Z(\gamma)$; every pair (x', w') of (TA5) for γ is one for p ($f\cdot x' = u_1 f \cdot x' = u_1\gamma f \cdot w' = pf \cdot w'$), so p satisfies the conditions of cell rank j with the same c; and $h_c^{-1} p h_{c'} = (h_c^{-1}u_1 h_c)(h_c^{-1}\gamma h_{c'})(h_{c'}^{-1}u_1' h_{c'}) \in K(M)KK(M) = K$ has reduction s, so p is a term over (c,s,c'). By Lemma 5.3.36 (i), applied to p and to γ, $Z(p_v) = S_v\, v(Z(p))\, S_v = S_v\, v(Z(\gamma))\, S_v = Z(\gamma_v)$ with $S_v = \mathrm{diag}(\mathrm{bs}_v(f_a))_a$, and p_v has norm-one pairs from j in the sense of Lemma 5.3.26: a pair (x', w') of (TA5) with $x'_a = c_0$, $c_0\bar{c}_0 = 1$, gives $f\cdot\tilde{x} = p_v f \cdot \tilde{w}$ with $\tilde{x}_b = \mathrm{bs}_v(f_a)v(x'_b)/\mathrm{bs}_v(f_b)$, $\tilde{w}_b = \mathrm{bs}_v(f_a)v(w'_b)/\mathrm{bs}_v(f_b)$, so $\tilde{w}_a = 1$, $|\tilde{x}_a| = |v(c_0)| = 1$ and $\tilde{x}_b = \tilde{w}_b = 0$ for $b > a$. Both properties persist under left multiplication by elements of G_1 fixing the frame.

Step 1: the moved X_0-parts. For $a \in \mathrm{M}_{\varrho\times k}(\mathbb{C})$ the formula of $u(a,b)$ in Definition 5.1.6 gives $u(a,0)^{-1} = u(-a,0)$ and, for $g \in \mathsf{U}^0$ and $w \in V_1$, $\pi_0(g^{-1}w) = \pi_0(w) + a_g^*\epsilon(w)$ for a matrix a_g depending only on g, where $\epsilon(w) = (h(e_{a+1}, w))_{a<\varrho}$ are the e^*-coordinates of w; for $g = u(a,0)$, $a_g = a$. Fix v and $x \in Y$, and let $g_v = u_v \mathrm{Es}_j(x)_v \in \mathsf{U}^0$. For $p \in B$, $\epsilon(p_v f_b)$ is the column b of the rows $0, \dots, \varrho-1$ of $Z(p_v) = Z(\gamma_v)$, the same for all $p \in O$. Hence, with $\gamma_{0,v} = u_v^{-1}(p_0)_v$ and the definition (SX3) of the Eichler shift,

$$\pi_0(g_v^{-1}p_v f_b) - \pi_0(\gamma_{0,v} f_b) = \pi_0(p_v f_b) - \pi_0((p_0)_v f_b) + \sum_{a<i} Z(\gamma_v)_{ab}\,(D_v x_v)^*_a.$$

By (SX5) the X_0-part of $pf_b - p_0 f_b$ is $M\sum_{a<i} h(e_{a+1}, \gamma f_b)\, l(p)_a$; rescaling ($\mathrm{bs}_v = 1$ on X_0) and using $Z(\gamma_v)_{ab} = \mathrm{bs}_v(e_{a+1})\mathrm{bs}_v(f_b)\, v(h(e_{a+1}, \gamma f_b))$, the first difference is $\sum_{a<i} Z(\gamma_v)_{ab}\, d_{v,a}\, v(l(p)_a)$. As $l(p)^\infty_v$ has the rows $v(l(p)_a)^*$,

$$\pi_0(g_v^{-1}p_v f_b) = \pi_0(\gamma_{0,v} f_b) + \sum_{a<i} Z(\gamma_v)_{ab}\,\big(\mathcal{A}_v(p,x)\big)^*_a, \qquad \mathcal{A}_v(p,x) = D_v\big(l(p)^\infty_v + x_v\big),$$

for every frame index b. Moreover $\mathrm{Vr}_{p_v}(g_v y_v) = \mathrm{Vr}_{g_v^{-1}p_v}(y_v)$, because g_v fixes the frame, so $\mathrm{Fv}(g_v y_v) = \mathrm{Fv}(y_v)$ and $p_v^{-1} g_v y_v = (g_v^{-1}p_v)^{-1} y_v$ (SX1).

Step 2: the majorant. As y is boxed, it is reduced, so by Lemma 5.2.21 and (RD5) $y_v \in \mathrm{Bx}(b^{27}\ell(y_v)) \subseteq \mathrm{Bx}(b^{37n})$, since $\ell(y_v) \le b^{6n+2}$ and $27+6n+2 \le 37n$. By (i) of Lemma 5.3.9,

$\delta_*(\gamma) \neq 0$, so, by Lemma 5.3.36 (i), $\delta_v = \det Z^{(i)}(\gamma_v) = \prod_{a\in\{0,\dots,i-1\}\cup\{\varrho\}} \mathrm{bs}_v(f_a)^2\, v(\delta_*(\gamma)) \neq 0$. Apply Lemma 5.3.26 at each v with y_v, $\rho = b^{37n}$, $\theta = b^{6n+2}$, $\gamma_0 = \gamma_{0,v}$ (which has norm-one pairs from j and $Z(\gamma_{0,v}) = Z(\gamma_v)$ by Step 0), and $k \geq 7$ (FG6): it gives a Borel ϕ_v^0 with $\int \phi_v^0 \leq (2b^{74n})^{(i+1)i}(2^{i+1}b^{2(6n+2)(j-i)})^k/|\delta_v|^{k+i}$ and $\mathrm{Vr}_g(y_v)^{-(k+i)} \leq \phi_v^0(\mathcal{A})$ for the g described there. Put $\phi_v(X) = \phi_v^0(D_vX)$ and $\phi(X) = \prod_v \phi_v(X_v)$. The real-linear map $X \mapsto D_vX$ of $\mathrm{M}_{i\times k}(\mathbb{C})$ multiplies the row a by $d_{v,a}$ and has Jacobian $|\det D_v|^{2k}$, so $\int \phi_v = |\det D_v|^{-2k}\int \phi_v^0 \leq \mathrm{Kb}_j(\gamma, v)$ by (SX4), and $\int_Y \phi\, dx \leq \prod_v \mathrm{Kb}_j(\gamma, v)$ by Tonelli's theorem. By Steps 0 and 1 (with $g = g_v^{-1}p_v$ and $\mathcal{A} = \mathcal{A}_v(p, x)$), for $p \in B$

$$\prod_v \mathrm{Vr}_{p_v}\big((u\,\mathrm{Es}_j(x)\,y)_v\big)^{-(k+i)} \leq \prod_v \phi_v^0(D_v(l(p)_v^\infty + x_v)) = \phi(l(p)^\infty + x).$$

Step 3: packing over one lattice point. Fix x, put $y' = u\,\mathrm{Es}_j(x)\,y$, and for $w \in \Lambda$ let $B_w = \{p \in B : l(p)^\infty = w\}$; since $q \mapsto q^\infty$ is injective, l is constant on B_w, and $l(p)^\infty \in \Lambda$ for all $p \in B$ by (SX6). For $p \in B_w$, $\mathrm{Sw}_{c',p,i}(y') \leq \beta_{c'}(p^{-1}y')\,\phi(w + x)$ (Step 2 and $\beta_{c'}^{\mathrm{bx}} \leq \beta_{c'}$), and if it is nonzero then $p^{-1}y'$ is boxed, so $\ell((p^{-1}y')_v) \leq b^{6n+2}$; p is a term over (c, s, c') (Step 0). For $p, p' \in B_w$, $pf_b - p'f_b \in A$ by (SX5), so $\big((p^{-1}y')_v\big)^{-1}f_b - \big((p'^{-1}y')_v\big)^{-1}f_b = y_v'^{-1}(p_v - p_v')f_b \in y_v'^{-1}(A \otimes_{E,v} \mathbb{C})$, as $B_v^{\pm1}$ preserve the span of $e_1, \dots, e_\varrho$. Let $V \subseteq \prod_v (\mathbb{C}^r)^{\varrho+1}$ be the real subspace of the families $(y_v'^{-1}a_{v,b})_{v,b}$ with $a_{v,b} \in \langle e_1, \dots, e_\varrho\rangle_{\mathbb{C}}$; $\dim_{\mathbb{R}} V \leq 2f\varrho n$. By Lemma 5.2.31 (pairs (c', p), $p \in B_w$, $R = 1$, weights $\mathrm{Sw}_{c',p,i}(y')$, constant $\phi(w + x)$),

$$\sum_{p\in B_w} \mathrm{Sw}_{c',p,i}(y') \leq b^{12n^3\varrho f}\,\phi(w + x) \leq \tau_{\mathrm{sz}}\,\phi(w + x),$$

since $k \geq 1000(2\varrho + 2)^9 \geq 48n^3\varrho$ (FG6) gives $12n^3\varrho f \leq kf/4$. Summing over $w \in \Lambda$ (the B_w partition B, all terms are nonnegative) gives the claim. □

Lemma 5.3.26. *Let $k \geq 7$, $0 \leq j \leq \varrho+1$, $i = \min(\varrho, j)$, $\rho, \theta \geq 0$, and $y \in \mathrm{Bx}(\rho)$ with $\ell(y) \leq \theta$. Say that $g \in G_1$ has* norm-one pairs from j *if for every a with $j \leq a \leq \varrho$ there are $x, w \in \mathbb{C}^{\varrho+1}$ with $\sum_b x_b f_b = g\sum_b w_b f_b$, $|x_a| = 1$, $w_a = 1$ and $x_b = w_b = 0$ for $b > a$. For $g \in G_1$ and $0 \leq b \leq \varrho$ let $\pi_0(gf_b) \in \mathbb{C}^k$ be the coordinates of gf_b in the basis of X_0. Let $\gamma_0 \in G_1$ have norm-one pairs from j, $Z = Z(\gamma_0)$, and suppose $\delta = \det Z^{(i)}(\gamma_0) \neq 0$. Then there is a Borel function $\phi\colon \mathrm{M}_{i\times k}(\mathbb{C}) \to [0, \infty]$ with*

$$\int \phi(A)\, dA \leq \frac{(2\rho^2)^{(i+1)i}\,\big(2^{i+1}\theta^{2(j-i)}\big)^k}{|\delta|^{k+i}}$$

(dA Lebesgue measure in the real and imaginary parts of the entries) such that $\mathrm{Vr}_g(y)^{-(k+i)} \leq \phi(A)$ for every $g \in G_1$ that has norm-one pairs from j, satisfies $Z(g) = Z$, and satisfies

$$\pi_0(gf_b) = \pi_0(\gamma_0 f_b) + \sum_{a<i} Z_{ab}\,A_a^* \qquad (b \in \{0, \dots, i-1\} \cup \{\varrho\}),$$

where A_a is the row a of A and $A_a^* \in \mathbb{C}^k$ its conjugate transpose.

Proof. For $x \in \mathbb{C}^{\varrho+1}$ write $f \cdot x = \sum_b x_b f_b$. Put $\Sigma = \{0, \dots, i-1\} \cup \{\varrho\}$ and $\Sigma' = \{0, \dots, i-1\}$, ordered as integers; matrices indexed by Σ are triangular with respect to this order, and $Z^{(i)} = Z_{\Sigma,\Sigma}$, $\delta = \det Z^{(i)}$. Let $V_- = (1 - J_1)V_1$ be the (-1)-eigenspace of J_1 and $N_y = yV_-$.

Step 0: elementary facts. (a) J_1 is hermitian and $J_1^2 = 1$, so it is unitary; for $g \in G_1$, $g^*J_1g = J_1$ gives $g^{-1} = J_1g^*J_1$ and $\|g^{-1}\|_{\mathrm{op}} = \|g\|_{\mathrm{op}}$. (b) $h(f_a, f_b) = 0$ unless $a = b = \varrho$, and $h(z, z) = -1$; so $h(f \cdot x, f \cdot x) = -|x_\varrho|^2$, and for $g \in G_1$ also $h(g\,f \cdot x, g\,f \cdot x) = -|x_\varrho|^2$. Hence

the Gram matrix $(h(u_a,u_b))_{a,b}$ of any finite family in $g\langle f_0,\dots,f_\varrho\rangle$ is negative semidefinite, and $f\cdot x = g\,f\cdot w$ implies $|x_\varrho| = |w_\varrho|$. (c) J_1 exchanges e_a and e_a^*, fixes X_0 and negates z; so V_- has the basis $(1-J_1)f_b$ $(0\le b\le\varrho)$, namely $e_{b+1}-e_{b+1}^*$ and $2z$. For $x\in V_-$ and $w\in V_1$, $h(x,w) = \langle J_1x,w\rangle = -\langle x,w\rangle$; also $h(e_a,w)$ is the e_a^*-coordinate of w and $h(z,w)$ is minus its z-coordinate. (d) For $x\in V_-$, $h(yx,yx) = h(x,x) = -\|x\|^2$, so $-h$ is positive definite on N_y; for $w\in V_1$ and $\nu\in N_y$, $|h(w,\nu)| = |h(y^{-1}w,y^{-1}\nu)| = |\langle y^{-1}w,y^{-1}\nu\rangle| \le \|y^{-1}w\|\,\|y^{-1}\nu\|$ and $-h(\nu,\nu) = \|y^{-1}\nu\|^2$. (e) Let $\nu_b = y(1-J_1)f_b$, a basis of N_y, and $P(g) = (h(\nu_a,gf_b))_{a,b}$ for $g\in G_1$. By (c), $h(\nu_a,gf_b) = h((1-J_1)f_a,y^{-1}gf_b) = -\langle(1-J_1)f_a,(g^{-1}y)^{-1}f_b\rangle$, so $|\det P(g)| = \mathrm{Fv}(g^{-1}y)$; by (i) of Lemma 5.3.21 this is positive, and $\mathrm{Vr}_g(y) = |\det P(g)|/|\det P(1)|$. In particular the map $N_y\to\mathbb{C}^{\varrho+1}$, $\nu\mapsto(h(f_a,\nu))_a$, is bijective (its matrix in the basis ν_b is $P(1)^*$).

Step 1: the ratio on the subframe. Let $g\in G_1$ have norm-one pairs from j, and let $\omega_l\in N_y$ $(l\in\Sigma)$ satisfy: (i) $h(f\cdot x,\omega_l) = 0$ for all l whenever $f\cdot x = g\,f\cdot w$ with $x_\varrho = 0$; (ii) $\Phi = (h(\omega_l,f_b))_{l,b\in\Sigma}$ is invertible. We show

$$\mathrm{Vr}_g(y) = \frac{|\det(h(\omega_l,gf_b))_{l,b\in\Sigma}|}{|\det\Phi|}. \tag{5.50}$$

Let $P = P(1)$, $N = P^{-1}P(g)$, so $|\det N| = \mathrm{Vr}_g(y)$ by Step 0 (e). If $\nu = \sum_a t_a\nu_a\in N_y$, then $(h(\nu,f_b))_b = \bar t^{\mathrm T}P$, hence $(h(\nu,f_b))_b N = \bar t^{\mathrm T}P(g) = (h(\nu,gf_b))_b$; and if $f\cdot x = g\,f\cdot w$, then $Px = P(g)w$, i.e. $Nw = x$. Let $E\in\mathrm{M}_{(\varrho+1)\times\Sigma}$ be the inclusion matrix (columns the unit vectors ϵ_b, $b\in\Sigma$), $\Psi = (h(\omega_l,f_b))_{l\in\Sigma,\,0\le b\le\varrho}$, so $\Phi = \Psi E$, and $p = \Phi^{-1}\Psi$, so $pE = 1$. Put $M = 1+(NE-E)p$. For $b\in\Sigma$, $M\epsilon_b = N\epsilon_b$. For $b\notin\Sigma$ we have $i\le b<\varrho$, hence $j\le\varrho$, $i = j\le b$, and the norm-one pair at b gives x,w with $f\cdot x = g\,f\cdot w$, $|x_b| = 1$, $w_b = 1$ and $x_{b'} = w_{b'} = 0$ for $b'>b$; as $b<\varrho$, $x_\varrho = 0$, so $px = 0$ by (i) and $Mx = x = Nw$. Taking $x = w = \epsilon_b$ for $b\in\Sigma$, the matrices Ξ, Ξ' with columns x, w $(b = 0,\dots,\varrho)$ are upper triangular with $|\det\Xi| = 1 = \det\Xi'$ and $N\Xi' = M\Xi$, so $|\det N| = |\det M|$. By Sylvester's identity $\det M = \det(1+p(NE-E)) = \det(pNE)$, and $pNE = \Phi^{-1}(h(\omega_l,gf_b))_{l,b\in\Sigma}$ by the first remark applied to each ω_l. This is (5.50).

Step 2: a triangular basis for (y,Z). Let $D = Z_{\cdot,\Sigma}(Z^{(i)})^{-1}\in\mathrm{M}_{(\varrho+1)\times\Sigma}(\mathbb{C})$; its rows in Σ form 1_Σ. By Step 0 (e) there are $\omega_l^0\in N_y$ with $h(f_a,\omega_l^0) = D_{al}$; they are linearly independent, as $h(f_a,\omega_l^0) = \delta_{al}$ for $a\in\Sigma$. Their Gram matrix for $-h$ is positive definite (Step 0 (d)); by the Cholesky decomposition of its inverse there is an invertible lower triangular $T\in\mathrm{M}_\Sigma(\mathbb{C})$ with $T^*(-h(\omega_l^0,\omega_m^0))_{l,m}T = 1$. Put $\omega_m = \sum_l\omega_l^0T_{lm}$. Then $\omega_m\in N_y$, $h(\omega_l,\omega_m) = -\delta_{lm}$, $h(f_a,\omega_m) = (DT)_{am}$, and $h(f_a,\omega_m) = T_{am}$ for $a\in\Sigma$. The family ω and T depend only on y and Z. Let $g\in G_1$ have norm-one pairs from j and $Z(g) = Z$. Then ω satisfies (ii) of Step 1 with $\Phi = T^*$, and (i): if $f\cdot x = g\,f\cdot w$ with $x_\varrho = 0$, then $w_\varrho = 0$ by Step 0 (b), and $\sum_a\bar x_aZ_{ab} = h(f\cdot x,gf_b) = h(g\,f\cdot w,gf_b) = h(f\cdot w,f_b) = -\bar w_\varrho[b=\varrho] = 0$ for all b, so $h(f\cdot x,\omega_m) = (x^*DT)_m = 0$. By (5.50),

$$\mathrm{Vr}_g(y) = |\det(h(\omega_l,gf_b))_{l,b\in\Sigma}|/|\det T|. \tag{5.51}$$

Step 3: the lower bound. For $g\in G_1$ with $Z(g) = Z$ put $\omega'_m = \sum_{b\in\Sigma}gf_b\,((Z^{(i)})^{-1}T)_{bm}$ $(m\in\Sigma)$. Then $h(f_a,\omega'_m) = (Z_{\cdot,\Sigma}(Z^{(i)})^{-1}T)_{am} = h(f_a,\omega_m)$, so $d_m = \omega'_m-\omega_m$ is h-orthogonal to $f_0,\dots,f_\varrho$; by Step 0 (c) it has no e^*- and no z-coordinate, so $d_m\in\langle e_1,\dots,e_\varrho\rangle\oplus X_0$. Let $Y_g\in\mathrm{M}_{k\times\Sigma}(\mathbb{C})$ have as column m the X_0-coordinates of d_m; since the e_a are isotropic and orthogonal to X_0, on which h is the standard form, $(h(d_l,d_m))_{l,m} = Y_g^*Y_g$. With $Q = (h(\omega_l,\omega'_m))_{l,m}$, expanding $h(\omega'_l-\omega_l,\omega'_m-\omega_m)$ gives

$$(-Q)+(-Q)^*-(1+Y_g^*Y_g) = -\big(h(\omega'_l,\omega'_m)\big)_{l,m},$$

which is positive semidefinite by Step 0 (b), as $\omega'_m \in g\langle f_0,\dots,f_\varrho\rangle$. Hence the hermitian part $\frac12((-Q)+(-Q)^*)$ of $-Q$ dominates $\frac12(1+Y_g^*Y_g)$, which is positive definite. By the Ostrowski–Taussky inequality $|\det X| \ge \det\frac{X+X^*}{2}$ for a matrix X with positive definite hermitian part H_X (write $X = H_X^{1/2}(1+\mathrm{i}K)H_X^{1/2}$ with K hermitian; the eigenvalues $1+\mathrm{i}\lambda$, $\lambda\in\mathbb{R}$, have modulus ≥ 1), and by the monotonicity of the determinant on positive definite matrices ($A \ge B > 0$ gives $B^{-1/2}AB^{-1/2} \ge 1$),

$$|\det Q| \ge 2^{-(i+1)}\det(1+Y_g^*Y_g) = 2^{-(i+1)}\det(1_k + Y_gY_g^*).$$

Now $Q = (h(\omega_l, gf_b))_{l,b\in\Sigma}(Z^{(i)})^{-1}T$, so if g also has norm-one pairs from j, (5.51) gives $|\det Q| = \mathrm{Vr}_g(y)|\det T|^2/|\delta|$, and

$$|\delta|\,\det(1_k + Y_gY_g^*) \le 2^{i+1}|\det T|^2\,\mathrm{Vr}_g(y). \tag{5.52}$$

Step 4: the majorant. Let g be as in the statement, with matrix A. For $b\in\Sigma$ the hypothesis says that the X_0-coordinates of gf_b and $\gamma_0 f_b$ differ by $\sum_{a<i} Z_{ab}A_a^*$; as the rows $a<i$ lie in Σ, the X_0-coordinates of ω'_m for g and for γ_0 differ by $\sum_{a<i}A_a^*(Z^{(i)}(Z^{(i)})^{-1}T)_{am} = \sum_{a<i}A_a^*T_{am}$, while ω_m does not depend on g. So $Y_g = Y_{\gamma_0} + A^*T_{\Sigma',\Sigma}$, $A^*\in \mathrm{M}_{k\times i}(\mathbb{C})$ the matrix with columns A_a^*. As T is lower triangular, $T_{a\varrho}=0$ for $a<i$: the column ϱ of Y_g is the column y_z of Y_{γ_0}, and the columns in Σ' are $(Y_{\gamma_0})_{\cdot,\Sigma'} + A^*T_1$ with $T_1 = T_{\Sigma',\Sigma'}$. With $P_0 = 1_k + y_zy_z^*$ (positive definite), $c_0 = ((Y_{\gamma_0})_{\cdot,\Sigma'})^* \in \mathrm{M}_{i\times k}(\mathbb{C})$ and $B(A) = T_1^*A + c_0$,

$$1_k + Y_gY_g^* = P_0 + B(A)^*B(A).$$

Define the continuous, hence Borel, function

$$\phi(A) = \Big(\frac{2^{i+1}|\det T|^2}{|\delta|}\Big)^{k+i}\det\big(P_0 + B(A)^*B(A)\big)^{-(k+i)}.$$

By (5.52) and $\mathrm{Vr}_g(y)>0$, $\mathrm{Vr}_g(y)^{-(k+i)} \le \phi(A)$ for every such g.

Step 5: the integral. As T is triangular, $\det T = \det T_1\cdot T_{\varrho\varrho}$, so T_1 is invertible. By Lemma 5.3.27 with P_0, T_1^*, c_0 in the roles of P, S, c ($k\ge 7$),

$$\int\phi(A)\,dA \le \Big(\frac{2^{i+1}|\det T|^2}{|\delta|}\Big)^{k+i}\frac{|\det T_1|^{-2k}}{(\det P_0)^k} = \frac{\big(2^{i+1}|\det T|^2\big)^i\big(2^{i+1}|T_{\varrho\varrho}|^2/\det P_0\big)^k}{|\delta|^{k+i}}.$$

It remains to show (B1) $|\det T| \le \rho^{i+1}$ and (B2) $|T_{\varrho\varrho}|^2 \le \theta^{2(j-i)}\det P_0$; then

$$\int\phi \le (2\rho^2)^{(i+1)i}(2^{i+1}\theta^{2(j-i)})^k/|\delta|^{k+i}.$$

(B1) $\det T = \prod_m T_{mm}$ and $T_{mm} = h(f_m,\omega_m)$; by Step 0 (d), $\|y^{-1}\omega_m\| = 1$ and $y\in \mathrm{Bx}(\rho)$, $|T_{mm}| \le \|y^{-1}f_m\| \le \rho$.

(B2), case $j\le\varrho$, so $i=j$. Starting from the norm-one pair of γ_0 at ϱ and subtracting, for $b=\varrho-1,\dots,i$ in turn, the current b-th coordinate of w times the pair at b (which has $w_b=1$ and vanishes above b), we obtain x,w with $\gamma_0\,f\cdot w = f\cdot x$, $w_\varrho = 1$ and $w_b = 0$ for $i\le b<\varrho$; by Step 0 (b), $|x_\varrho| = 1$. Since w is supported on Σ, for $a\in\Sigma$, $(Z^{(i)}w_\Sigma)_a = h(f_a,\gamma_0\,f\cdot w) = h(f_a, f\cdot x) = -x_\varrho[a=\varrho]$, so $(Z^{(i)})^{-1}\epsilon_\varrho = -w_\Sigma/x_\varrho$. The column ϱ of T is $T_{\varrho\varrho}\epsilon_\varrho$, so for $g=\gamma_0$, $\omega'_\varrho = t\,f\cdot x$ with $t = -T_{\varrho\varrho}/x_\varrho$. Hence $h(\omega'_\varrho,\omega'_\varrho) = -|t|^2|x_\varrho|^2 = -|T_{\varrho\varrho}|^2$ and $h(\omega'_\varrho, d_\varrho) = \bar t\,h(f\cdot x, d_\varrho) = 0$, and

$$-1 = h(\omega_\varrho,\omega_\varrho) = h(\omega'_\varrho - d_\varrho, \omega'_\varrho - d_\varrho) = -|T_{\varrho\varrho}|^2 + |y_z|^2.$$

So $|T_{\varrho\varrho}|^2 = 1 + |y_z|^2 = \det P_0$, which is (B2) as $j - i = 0$.

(B2), case $j = \varrho + 1$, so $i = \varrho$, $\Sigma = \{0, \dots, \varrho\}$ and $j - i = 1$. As $\ell(y) \le \theta$, there is $s \in R$ with $\|sy\|_{\mathrm{op}} < \theta$; writing $s = u_1 h_1$ with $u_1 \in \mathsf{U}$, $h_1 \in H$, the element h_1 is unitary and normalises U, so $u = h_1^{-1} u_1 h_1 \in \mathsf{U}$ and $\|uy\|_{\mathrm{op}} = \|h_1^{-1} sy\|_{\mathrm{op}} < \theta$. The unipotent radical U of the stabiliser of the isotropic flag acts trivially on $F^\perp/F$, $F = \langle e_1, \dots, e_\varrho \rangle$, and $z \in F^\perp$; so $u^{-1}z = z + \sum_{a<\varrho} c_a f_a$, and since $T_{a\varrho} = 0$ for $a < \varrho$, $h(u^{-1}z, \omega_\varrho) = T_{\varrho\varrho}$. By Step 0 (d) and (a), $|T_{\varrho\varrho}| \le \|y^{-1}u^{-1}z\| \le \|(uy)^{-1}\|_{\mathrm{op}} = \|uy\|_{\mathrm{op}} \le \theta$. As $\det P_0 = 1 + |y_z|^2 \ge 1$, $|T_{\varrho\varrho}|^2 \le \theta^2 \det P_0$.

This proves the bound for $\int \phi$ and the lemma. (For $i = 0$, $\mathrm{M}_{0\times k}(\mathbb{C})$ is a point, T_1 is empty and the argument applies verbatim.) □

Lemma 5.3.27. *Let $k \ge 7$ and $i \ge 0$ be integers, $P \in \mathrm{M}_k(\mathbb{C})$ hermitian positive definite, $S \in \mathrm{GL}_i(\mathbb{C})$ and $c \in \mathrm{M}_{i\times k}(\mathbb{C})$. Then*

$$\int_{\mathrm{M}_{i\times k}(\mathbb{C})} \det\bigl(P + (SA + c)^*(SA + c)\bigr)^{-(k+i)}\, dA \;\le\; |\det S|^{-2k} (\det P)^{-k},$$

where dA is Lebesgue measure in the real and imaginary parts of the entries (for $i = 0$ the space is a point of mass 1).

Proof. For $Q \in \mathrm{M}_k(\mathbb{C})$ positive definite and row vectors $w \in \mathbb{C}^k$, $Q + w^*w$ is positive definite, so all determinants below are positive reals and the integrands are continuous and positive. Lebesgue measure on $\mathbb{C}^k$ is taken in the real and imaginary parts of the coordinates.

Step 1: one row. Let $r \ge k$ and Q positive definite. Then

$$\int_{\mathbb{C}^k} \det(Q + w^*w)^{-(r+1)}\, dw \le \frac{\pi^k}{k!} (\det Q)^{-r}. \tag{5.53}$$

Write $Q = C^*C$ with $C = Q^{1/2}$ invertible and substitute $w = yC$. The map $y \mapsto yC$ is $\mathbb{C}$-linear on $\mathbb{C}^k$ with complex determinant $\det C$, so its real Jacobian is $|\det C|^2 = \det Q$. Moreover $Q + C^*y^*yC = C^*(1_k + y^*y)C$ and $\det(1_k + y^*y) = 1 + yy^* = 1 + |y|^2$ (Sylvester's determinant identity), so $\det(Q + w^*w) = \det Q\,(1 + |y|^2)$. Since $r + 1 \ge k + 1$ and $1 + |y|^2 \ge 1$,

$$\int_{\mathbb{C}^k} \det(Q + w^*w)^{-(r+1)} dw = (\det Q)^{-r} \int_{\mathbb{C}^k} (1 + |y|^2)^{-(r+1)} dy \le (\det Q)^{-r} \int_{\mathbb{C}^k} (1 + |y|^2)^{-(k+1)} dy.$$

In polar coordinates on $\mathbb{C}^k = \mathbb{R}^{2k}$, with the area $2\pi^k/(k-1)!$ of the unit sphere S^{2k-1} and the substitution $s = t^2$,

$$\begin{aligned}\int_{\mathbb{C}^k} (1 + |y|^2)^{-(k+1)} dy &= \frac{2\pi^k}{(k-1)!} \int_0^\infty \frac{t^{2k-1}\, dt}{(1+t^2)^{k+1}} = \frac{\pi^k}{(k-1)!} \int_0^\infty \frac{s^{k-1}\, ds}{(1+s)^{k+1}} \\ &= \frac{\pi^k}{(k-1)!} \mathrm{B}(k, 1) = \frac{\pi^k}{k!},\end{aligned}$$

which gives (5.53).

Step 2: the rows one at a time. For an integer $m \ge 0$, Q positive definite and row vectors $b_1, \dots, b_m \in \mathbb{C}^k$ put

$$J_m(Q) = \int_{(\mathbb{C}^k)^m} \det\Bigl(Q + \sum_{a=1}^m b_a^* b_a\Bigr)^{-(k+m)} db_1 \cdots db_m.$$

We show $J_m(Q) \le (\pi^k/k!)^m (\det Q)^{-k}$ by induction on m; for $m = 0$ it is an equality. For the step from m to $m + 1$, integrate first over b_{m+1} (Tonelli's theorem, the integrand being nonnegative

and continuous): for fixed $b_1, \dots, b_m$ apply (5.53) with $Q' = Q + \sum_{a \le m} b_a^* b_a$, positive definite, and $r = k + m \ge k$; the inner integral is at most $(\pi^k/k!)\det(Q')^{-(k+m)}$. Integrating over $b_1, \dots, b_m$ gives $J_{m+1}(Q) \le (\pi^k/k!)J_m(Q) \le (\pi^k/k!)^{m+1}(\det Q)^{-k}$.

Step 3: the affine change of variables. The map $A \mapsto B = SA + c$ is an affine bijection of $\mathrm{M}_{i\times k}(\mathbb{C})$ whose linear part $A \mapsto SA$ is $\mathbb{C}$-linear and acts as S on each of the k columns, so its complex determinant is $(\det S)^k$ and its real Jacobian $|\det S|^{2k}$. If $b_1, \dots, b_i$ are the rows of B, then $B^*B = \sum_a b_a^* b_a$. Hence the left side of the claim equals $|\det S|^{-2k} J_i(P) \le |\det S|^{-2k}(\pi^k/k!)^i (\det P)^{-k}$.

Step 4: $\pi^k/k! \le 1$ for $k \ge 7$. Since $\pi < 3.15$ and $3.15^7 < 3078 < 5040 = 7!$, the claim holds for $k = 7$, and $\pi^{k+1}/(k+1)! = \frac{\pi}{k+1} \cdot \pi^k/k! < \pi^k/k!$ for $k \ge 7$. With Step 3 this proves the lemma. □

Lemma 5.3.28. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data and $c \in \mathcal{C}$. Let (X, m) be a measure space with m s-finite, $t\colon X \to G_\infty$ measurable with $t(x)_v \in \mathsf{U}^0$ for all $x \in X$ and $v \in \Phi$, $A \subseteq X$ measurable with $0 < m(A) < \infty$, $C \in [0, \infty]$, and $F\colon G_\infty \to [0, \infty]$ Borel with $F(u_\infty y) = F(y)$ for all $u \in \mathsf{U}_c^0(M)$ and $y \in G_\infty$. If*

$$\int_A F\big(u\, t(x)\, y\big)\, dm(x) \le m(A)\, C$$

for every $u \in G_\infty$ with $u_v \in \mathsf{U}^0$ for all v and every boxed $y \in G_\infty$, then

$$\int_{G_\infty} \beta_c^{\mathrm{bx}}(y) \prod_{v\in\Phi} \|W_v^\flat(y_v)\|^2\, F(y)\, dy \le C\, \nu\, \#I.$$

Proof. Write $Q(y) = \prod_{v\in\Phi} \|W_v^\flat(y_v)\|^2$, $\mathbf{1}_{\mathrm{bx}}$ for the indicator of the boxed points, and $H = \prod_{v\in\Phi} \mathsf{U}^0 \subseteq G_\infty$; we identify $u \in \mathsf{U}_c^0(M)$ with u_∞, so that $\Gamma = \{u_\infty : u \in \mathsf{U}_c^0(M)\}$ is a countable subgroup of G_∞.

Step 1: groups, measures and invariances. (a) $\Gamma \subseteq H$: for $u \in \mathsf{U}_c^0(M)$, $u_v = B_v^{-1} v(u) B_v$ with B_v the diagonal matrix of basis scales; $v(u)$ lies in the unipotent radical of the stabiliser of the flag $\langle e_1\rangle \subset \cdots \subset \langle e_1, \dots, e_\varrho\rangle$ and conjugation by the diagonal B_v preserves that flag, so $u_v \in \mathsf{U}$; and $u_v f_b = B_v^{-1} v(u) \mathrm{bs}_v(f_b) f_b = f_b$, as $u f_b = f_b$. (b) H is a closed subgroup of G_∞ and unimodular (U^0 is a connected nilpotent Lie group); fix a Haar measure dh on H, which is then invariant under right translations and under $h \mapsto h^{-1}$. The measure dy is invariant under left and right translations of G_∞, by Lemma 5.2.41 in each factor. (c) For $h \in H$ and $y \in G_\infty$: $Q(hy) = Q(y)$, since each $W_v^\flat$ is a Bessel function and $h_v \in \mathsf{U}$ gives $W_v^\flat(h_v y_v) = \mathrm{wave}(\sigma(h_v)) W_v^\flat(y_v)$ with $|\mathrm{wave}(c)| = |\exp(c - \bar{c})| = 1$; and hy is boxed if and only if y is, because $(h_v y_v)^{-1} f_b = y_v^{-1} f_b$ for all b (so the Gram–Schmidt matrices T and the rational triangle entries agree) and $\ell(h_v y_v) = \ell(y_v)$ ($s \mapsto s h_v$ is a bijection of $R \supseteq \mathsf{U}$). The same invariance of Q holds for $h \in \mathsf{U}_\infty$. (d) Boxed points are reduced, so $\beta_c^{\mathrm{bx}} = \beta_c^0 \mathbf{1}_{\mathrm{bx}}$, and $\beta_c^{\mathrm{bx}} \le \beta_c$.

Step 2: disintegration along orbits. Let $L \subseteq G_\infty$ be a closed unimodular subgroup with Haar measure dl and $\varphi\colon G_\infty \to [0, \infty)$ Borel with $\int_L \varphi(ly)\, dl = 1$ for every y. Then for every Borel $g\colon G_\infty \to [0, \infty]$

$$\int_{G_\infty} g(y)\, dy = \int_{G_\infty} \varphi(a) \int_L g(la)\, dl\, da. \tag{5.54}$$

Indeed, by Tonelli's theorem (G_∞ and L are σ-compact), the left invariance of da under $a \mapsto l^{-1}a$, and the invariance of dl under inversion, the right side is $\int_L \int \varphi(l^{-1}a) g(a)\, da\, dl = \int g(a) \int_L \varphi(l^{-1}a)\, dl\, da = \int g(a)\, da$. For $L = H$ such a φ exists: by the existence of Bruhat functions for closed subgroups (Bourbaki, Intégration, ch. VII, §2, no. 4) there is a continuous $\varphi_0 \ge 0$

with $\int_H \varphi_0(yh)\,dh = 1$ for all y, and $\varphi(y) = \varphi_0(y^{-1})$ satisfies $\int_H \varphi(hy)\,dh = \int_H \varphi_0(y^{-1}h^{-1})\,dh = 1$. For $L = \mathsf{U}_\infty$ with du (a two-sided Haar measure invariant under inversion, Definition 5.1.6) take $\varphi = \prod_v \varphi_v(y_v)$, where φ_v is a cutoff of total du-mass 1 on every U-orbit of G_1, the one by which the quotient integrals over $\mathsf{U}\backslash G_1$ are formed; then $\int_{\mathsf{U}_\infty} \varphi(uy)\,du = \prod_v \int_{\mathsf{U}} \varphi_v(u_v y_v)\,du_v = 1$.

Step 3: the bound on one orbit. Let $a \in G_\infty$ be boxed and put $p(h) = \beta_c^0(ha)$, $F_a(h) = F(ha)$ for $h \in H$. Then $\sum_{\gamma\in\Gamma} p(\gamma h) = 1$ for every h (RD6), hence also $\sum_\gamma p(\gamma^{-1}h) = 1$, and $F_a(\gamma h) = F_a(h)$ for $\gamma \in \Gamma$ by the invariance of F. For $t \in H$, by right invariance of dh, the insertion of $\sum_\gamma p(\gamma^{-1}h) = 1$, monotone convergence, the substitution $h \mapsto \gamma h$ and the Γ-invariance of F_a,

$$\int_H p\,F_a\,dh = \int_H p(ht)F_a(ht)\sum_\gamma p(\gamma^{-1}h)\,dh = \sum_\gamma \int_H p(\gamma ht)F_a(ht)\,p(h)\,dh = \int_H F_a(ht)\,p(h)\,dh.$$

Apply this with $t = t(x)$, integrate over $x \in A$ and use Tonelli's theorem (m is s-finite) and the hypothesis with $u = h$ and the boxed point a:

$$m(A)\int_H p\,F_a\,dh = \int_H p(h)\int_A F\big(h\,t(x)\,a\big)\,dm(x)\,dh \le m(A)\,C\int_H p\,dh.$$

As $0 < m(A) < \infty$, $\int_H p(h)F(ha)\,dh \le C\int_H \beta_c^0(ha)\,dh$.

Step 4: integration in stages. By (5.54) for $L = H$ and Step 1 (c), (d), the left side of the claim is

$$\begin{aligned}\int_{G_\infty} \varphi(a)\,\mathbf{1}_{\mathrm{bx}}(a)\,Q(a)\int_H \beta_c^0(ha)\,F(ha)\,dh\,da &\le C\int_{G_\infty} \varphi(a)\,\mathbf{1}_{\mathrm{bx}}(a)\,Q(a)\int_H \beta_c^0(ha)\,dh\,da\\ &= C\int_{G_\infty} \beta_c^{\mathrm{bx}}\,Q\,dy,\end{aligned}$$

using Step 3 at the boxed a (for a not boxed both integrands vanish) and (5.54) again. By Step 1 (d), $\int \beta_c^{\mathrm{bx}} Q\,dy \le \int \beta_c Q\,dy$.

Step 5: the class-cut Bessel integral. By (5.54) for $L = \mathsf{U}_\infty$, the U_∞-invariance of Q and Lemma 5.2.33,

$$\begin{aligned}\int_{G_\infty} \beta_c Q\,dy &= \int_{G_\infty} \varphi(a)Q(a)\int_{\mathsf{U}_\infty} \beta_c(ua)\,du\,da = \nu\int_{G_\infty} \varphi\,Q\,dy\\ &= \nu\prod_{v\in\Phi}\int_{G_1} \varphi_v\|W_v^\flat\|^2 dy_v = \nu\prod_v \int_{\mathsf{U}\backslash G_1} \|W_v^\flat\|^2,\end{aligned}$$

by Tonelli's theorem for the product measure. By (BS1) each factor is at most $\#I_v$, and $\#I = \prod_v \#I_v$. Hence the left side of the claim is at most $C\,\nu\,\#I$. □

Lemma 5.3.29. *Let $\mathfrak{S}$ be an admissible filling datum, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, $1 \le j \le \varrho + 1$, and γ a term over (c, s, c') satisfying the conditions of cell rank j. Then $\mathrm{mn}_j(\gamma) > 0$ and*

$$\prod_{v\in\Phi} \mathrm{Kb}_j(\gamma, v) \le \tau_{\mathrm{sz}}\Big(\frac{b^{13}}{M^2}\Big)^{i_j k f}\,\mathrm{mn}_j(\gamma)^{-k/2}.$$

Proof. Put $i = i_j$, so $1 \le i \le \varrho$ (as $j \ge 1$), and $A = \{0, \dots, i-1\} \cup \{\varrho\}$; note that $A = \{a : a < j\} \cup \{\varrho\}$ both for $j \le \varrho$ (then $i = j$) and for $j = \varrho + 1$ (then $i = \varrho$ and $A = \{0, \dots, \varrho\}$), so the minor of $Z(\gamma)$ on the rows and columns A is the subframe minor $\delta_*(\gamma)$. Write $(x)_+ = \max(x, 0)$.

$D_j(\gamma)$ **is a nonzero algebraic integer, and** $\mathrm{mn}_j(\gamma) > 0$**.** By Lemma 5.2.24 (i), $h(L_0, L_0) \subseteq \hat{\mathcal{O}}$. Let $k' = h_c^{-1}\gamma h_{c'} \in K$. The elements $h_c, h_{c'}$ of $\mathsf{U}(X_0)(\mathbb{A}_f)$ fix the frame vectors and preserve h, so $h(f_a, \gamma f_b) = h(h_c^{-1}f_a, h_c^{-1}\gamma h_{c'} f_b) = h(f_a, k' f_b)$; as $f_a, f_b \in L_0$ and $k'L_0 = L_0$ (Lemma 5.2.24 (ii)), this lies in $\hat{\mathcal{O}} \cap E = \mathcal{O}_E$. Hence $D_j(\gamma) \in \mathcal{O}_E$, and $D_j(\gamma) \neq 0$ by (TA5), so $|N_{E/\mathbb{Q}} D_j(\gamma)| \geq 1$. By (TA12) this gives $\mathrm{mn}_j(\gamma) \geq N^{-2(j-\varrho)_+ f} > 0$; positivity is also (i) of Lemma 5.3.9.

The subframe determinants. Let $v \in \Phi$ and $S_v = \mathrm{diag}(\mathrm{bs}_v(f_0), \dots, \mathrm{bs}_v(f_\varrho))$. By Lemma 5.3.36 (i), $Z(\gamma_v) = S_v\, v(Z(\gamma))\, S_v$. Hence every principal minor of $Z(\gamma_v)$ is the image under v of the same minor of $Z(\gamma)$ times the squares of the scales of its indices. As $\mathrm{bs}_v(f_a) = \mathrm{rs}_v^{\varrho-a}/\sqrt{N}$,

$$|\det Z^{(i)}(\gamma_v)| = \prod_{a \in A} \mathrm{bs}_v(f_a)^2\, |v(\delta_*(\gamma))| = \frac{W_v^2}{N^{i+1}}\, |v(\delta_*(\gamma))|, \qquad W_v = |\mathrm{rs}_v|^{\sum_{a<i}(\varrho-a)} \geq 1,$$

by (FA4). Since E is totally imaginary and Φ holds one embedding of each conjugate pair, $\prod_v |v(x)| = |N_{E/\mathbb{Q}} x|^{1/2}$ for $x \in E$, and by (i) of Lemma 5.3.9, $\prod_v |v(\delta_*(\gamma))| = N^f \mathrm{mn}_j(\gamma)^{1/2}$. In particular every $\det Z^{(i)}(\gamma_v)$ is nonzero, and

$$\prod_{v \in \Phi} |\det Z^{(i)}(\gamma_v)| = \Big(\prod_v W_v\Big)^2 N^{-if}\, \mathrm{mn}_j(\gamma)^{1/2}.$$

By (SX3), $|\det D_{j,v}| = \prod_{l<i} M\sqrt{N}/|\mathrm{rs}_v|^{\varrho-l} = (M\sqrt{N})^i / W_v$.

The product. Put $C_0 = (2b^{74n})^{(i+1)i} \big(2^{i+1} b^{2(6n+2)(j-i)}\big)^k$, the place-independent numerator of (SX4). Then

$$\begin{aligned}\prod_v \mathrm{Kb}_j(\gamma, v) &= \frac{\prod_v W_v^{2k}\, (M\sqrt{N})^{-2ikf}\, C_0^f}{(\prod_v W_v)^{2(k+i)} N^{-if(k+i)} \mathrm{mn}_j(\gamma)^{(k+i)/2}} \\ &= \Big(\prod_v W_v\Big)^{-2i} M^{-2ikf} N^{i^2 f} C_0^f\, \mathrm{mn}_j(\gamma)^{-(k+i)/2}.\end{aligned}$$

Use $W_v \geq 1$, $\mathrm{mn}_j(\gamma)^{-i/2} \leq N^{(j-\varrho)_+ fi}$ (first step) and $i + (j-\varrho)_+ = j$:

$$\prod_v \mathrm{Kb}_j(\gamma, v) \leq \Big((2b^{74n})^{i+1} N^j\Big)^{if} \Big(\frac{2^{i+1} b^{2(6n+2)(j-i)}}{M^{2i}}\Big)^{kf} \mathrm{mn}_j(\gamma)^{-k/2}.$$

The first factor is at most τ_{sz}**.** By $2 \leq b$, $N \leq b^{60n}$ (FA5), $i+1 \leq n$ and $j \leq n$: $(2b^{74n})^{i+1} N^j \leq b^{(74n+1)n + 60n^2} \leq b^{135n^2}$, so the first factor is at most $b^{135n^2 \varrho f}$. By (FG6), $k \geq 1000(2\varrho+2)^9 \geq 540n^3 \geq 540n^2\varrho$, so $135n^2\varrho f \leq kf/4$ and the first factor is at most $b^{kf/4} = \tau_{\mathrm{sz}}$.

The second factor is at most $(b^{13}/M^2)^{ikf}$**.** It suffices that $2^{i+1} b^{2(6n+2)(j-i)} \leq b^{13i}$. If $j \leq \varrho$, then $j = i \geq 1$ and $2^{i+1} \leq 4^i \leq b^{13i}$. If $j = \varrho+1$, then $i = \varrho$, $j - i = 1$ and $2(6n+2) = 12\varrho + 16$; as $2^9 \leq b$, $2^{\varrho+1} \leq b^{(\varrho+1)/9}$, and $(\varrho+1)/9 + 12\varrho + 16 \leq 13\varrho$ because $\varrho \geq 10^5$ (FG6).

Together, $\prod_v \mathrm{Kb}_j(\gamma, v) \leq \tau_{\mathrm{sz}}\, (b^{13}/M^2)^{i_j kf}\, \mathrm{mn}_j(\gamma)^{-k/2}$. □

Lemma 5.3.30. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$ and $T \subseteq G(E)$ stable. If s does not agree, then $A_T(c, s, c') = 0$, for the full and for the cut amplitude sums.*

Proof. Fix $W \in \{W^{\mathrm{full}}, W^{\mathrm{cut}}\}$ and write $K(\gamma, y) = W(y) \otimes \overline{W(\gamma^{-1} y)}$. As s does not agree, there are integral $u, u' \in \mathsf{U}(\mathbb{A}_f)$ (that is, in K) with $\bar{u} s = s \bar{u}'$ and $\psi(u) \neq \psi(u')$, where $\bar{u} = \mathrm{red}(u)$.

Step 1: the character. By Lemma 5.2.44: ψ restricted to $\mathsf{U}(\mathbb{A}_f)$ is a homomorphism to the unit circle (i); for $g \in \mathsf{U}(\mathbb{A}_f)$ and $h \in \mathrm{U}(X_0)(\mathbb{A}_f)$, $h^{-1}gh \in \mathsf{U}(\mathbb{A}_f)$ and $\psi(h^{-1}gh) = \psi(g)$ (ii); and $\psi(k) = 1$ for $k \in \mathsf{U}(\mathbb{A}_f) \cap \mathrm{K}(M)$ (iii).

Step 2: rational approximation. The set $O = \mathsf{U}(\mathbb{A}_f) \cap h_c u\, \mathrm{K}(M)\, h_c^{-1}$ is open in $\mathsf{U}(\mathbb{A}_f)$ and contains $h_c u h_c^{-1}$, which lies in $\mathsf{U}(\mathbb{A}_f)$ by Step 1 (ii) with $h = h_c^{-1}$. By strong approximation for the unipotent group U over E^+ (Theorem A.4.7, S the archimedean places) there is $x \in \mathsf{U}(E) \cap O$: $h_c^{-1} x h_c = uk$ with $k \in \mathrm{K}(M)$. As $h_c^{-1} x h_c \in \mathsf{U}(\mathbb{A}_f)$ by Step 1 (ii), $k = u^{-1} h_c^{-1} x h_c \in \mathsf{U}(\mathbb{A}_f) \cap \mathrm{K}(M)$, and by Step 1, $\psi(x) = \psi(h_c^{-1} x h_c) = \psi(u)\psi(k) = \psi(u)$; moreover $h_c^{-1} x h_c \in K$ has reduction $\bar{u}$. Likewise there is $x' \in \mathsf{U}(E)$ with $h_{c'}^{-1} x' h_{c'} \in u' \mathrm{K}(M)$, of reduction $\bar{u}'$, and $\psi(x') = \psi(u')$.

Step 3: the substitution permutes the terms. Let X be the set of terms of T over (c, s, c'). For $\gamma \in X$, $h_c^{-1} x \gamma x'^{-1} h_{c'} = (h_c^{-1} x h_c)(h_c^{-1} \gamma h_{c'})(h_{c'}^{-1} x' h_{c'})^{-1} \in K$ has reduction $\bar{u} s \bar{u}'^{-1} = s$, and $x \gamma x'^{-1} \in T$ as T is stable; so $\gamma \mapsto x \gamma x'^{-1}$ maps X to X, bijectively (its inverse is $\gamma \mapsto x^{-1} \gamma x'$, by the same argument). Thus $x X x'^{-1} = X$.

Step 4: the substitution normalises the unipotent groups. Conjugation by x is a bijection of $\mathsf{U}_c(M)$: for $v \in \mathsf{U}_c(M)$, $xvx^{-1} \in \mathsf{U}(E)$ and $h_c^{-1} x v x^{-1} h_c = k_x (h_c^{-1} v h_c) k_x^{-1} \in \mathrm{K}(M)$ with $k_x = h_c^{-1} x h_c \in K$, $\mathrm{K}(M)$ being normal in K; and likewise with x^{-1}. The same holds for x' and $\mathsf{U}_{c'}(M)$.

Step 5: the kernel on the terms. Let $Z = X \times G_\infty$, $\mu = (\text{counting measure}) \otimes dy$, the group Γ_X, its countable subgroup $\Delta = \mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)$ and $\beta_0(\gamma, y) = \beta_c(y)\beta_{c'}(\gamma^{-1} y)$ be as in (ii) of Lemma 5.2.36: Γ_X acts on Z by $(a, b) \cdot (\gamma, y) = (a \gamma b^{-1}, a \cdot y)$, by measurable maps that preserve integrals against μ, and $\sum_{\delta \in \Delta} \beta_0(\delta \cdot z) = 1$ for every $z \in Z$. By Step 3, $\delta_0 = (x, x') \in \Gamma_X$, and by Step 4, $\delta_0^{-1} \Delta \delta_0 = \Delta$. The kernel K is measurable on Z and Δ-invariant: for $(v, v') \in \Delta$, $W(v \cdot y) = W(y)$ and $W(v' \cdot w) = W(w)$ by Lemma 5.2.42, and $(v \gamma v'^{-1})^{-1}(v \cdot y) = v' \cdot \gamma^{-1} y$. For $z = (\gamma, y) \in Z$, $(x \gamma x'^{-1})^{-1}(x \cdot y) = x' \cdot \gamma^{-1} y$, and $W(x \cdot y) = \psi(x)^{-1} W(y)$, $W(x' \cdot w) = \psi(x')^{-1} W(w)$ by Lemma 5.2.43; as $|\psi(x')| = 1$ (Step 1),

$$K(\delta_0 \cdot z) = \psi(x)^{-1}\, \overline{\psi(x')^{-1}}\, K(z) = c_0\, K(z), \qquad c_0 = \psi(x)^{-1}\psi(x') = \psi(u)^{-1}\psi(u') \neq 1.$$

Step 6: conclusion. $\int_Z \beta_0 \|K\|\, d\mu = \sum_{\gamma \in X} \mathrm{maj}_\gamma$, the sum of the amplitude majorants of the terms of T over (c, s, c'), which is finite by Lemma 5.2.29 (ii). By (iii) of Lemma 5.2.45, applied to Γ_X, $\Lambda = \Delta$, the cutoff β_0, $F = K$ and the twist δ_0 with the constant c_0 (here $\Delta \cap \delta_0^{-1} \Delta \delta_0 = \Delta = \delta_0^{-1} \Delta \delta_0$, of index 1 in both), $\int_Z \beta_0 K\, d\mu = 0$. As $\beta_0 K$ is integrable on Z, Fubini's theorem gives $\int_Z \beta_0 K\, d\mu = \sum_{\gamma \in X} A_\gamma(c, c') = A_T(c, s, c')$. Hence $A_T(c, s, c') = 0$. □

Lemma 5.3.31. *Let $\mathfrak{S}$ be a filling datum and $\gamma \in G(E)$ with agreeing characters. Say that $(\mu, \mu') \in E^2$ occurs at $a \in \{0, \dots, \varrho\}$ if $f \cdot x = \gamma f \cdot y$ for some $x, y \in E^{\varrho+1}$ with $x_a = \mu$, $y_a = \mu'$ and $x_b = y_b = 0$ for $b > a$.*

(i) If (μ, μ') occurs at a, then it occurs at every l with $a \leq l \leq \varrho$, and $\mu\bar{\mu} = \mu'\bar{\mu}'$.

(ii) If $\mathsf{g}(\gamma) = 0$, let j be the least integer in $\{0, \dots, \varrho + 1\}$ such that some ζ with $\zeta\bar{\zeta} = 1$ has $(\zeta, 1)$ occurring at every $a \geq j$. Then $j \leq \varrho$, and γ satisfies the conditions of cell rank j.

Proof. Eichler elements. Let $1 \leq a \leq \varrho$, $\xi \in E^{\varrho+1}$ with $\xi_b = 0$ for $b \geq a$, $r \in E$, and $v \in V$ with $h(v, f_b) = 0$ for $b < a$. Put $x_e = r\, f \cdot \xi \in F_a = \langle e_1, \dots, e_a \rangle$, $\varsigma = \frac{1}{2} h(v, v)$ (so $\varsigma + \bar{\varsigma} = h(v, v)$) and

$$\mathrm{E}(w) = w + h(v, w)\, x_e - h(x_e, w)\, (v + \varsigma x_e) \qquad (w \in V).$$

As x_e is isotropic and $h(v, x_e) = 0$, E is the Eichler transformation of (x_e, v), an isometry of V (expand $h(\mathrm{E}w, \mathrm{E}w')$). It lies in $\mathsf{U}(E)$: for $w \in F_c$ we have $h(x_e, w) = 0$ and, if $c \leq a$, $h(v, w) = 0$,

so $\mathrm{E}w = w$ for $c \le a$ and $\mathrm{E}w \in w + F_a \subseteq w + F_{c-1}$ for $c > a$; for $w \in X^\perp$, $\mathrm{E}w \in w + X$. So E preserves the flag $F_\bullet$ and acts trivially on the quotients F_c/F_{c-1} and $X^\perp/X$, hence (being an isometry) on the dual quotients; that is, E is in the unipotent radical U of the stabiliser of $F_\bullet$. Since $h(x_e, f_b) = 0$ for all b ($x_e \in X$, X isotropic and orthogonal to z), $\mathrm{E}f_b = f_b + h(v, f_b)x_e$, whose f_{b-1}-coefficient is $r\,h(v, f_b)\xi_{b-1}$; hence

$$\sigma(\mathrm{E}) = r \sum_{b=1}^{\varrho} h(v, f_b)\, \xi_{b-1}. \tag{5.55}$$

(i), one step. Let (μ, μ') occur at $a < \varrho$ through (x, y). The vector $\mu f_{a+1} - \mu'\gamma f_{a+1}$ lies in $\langle f_b, \gamma f_b : b \le a\rangle$ if and only if every linear form m on V that kills f_b and γf_b for all $b \le a$ satisfies $\mu\, m(f_{a+1}) = \mu'\, m(\gamma f_{a+1})$; and if it does, adding the resulting relation to $f \cdot x = \gamma f \cdot y$ shows that (μ, μ') occurs at $a + 1$ (the old coefficients at $a + 1$ are 0). Let m be such a form. As h is nondegenerate, $m = h(\gamma v, \cdot)$ for some $v \in V$; then $h(\gamma v, f_b) = 0$ and $h(v, f_b) = h(\gamma v, \gamma f_b) = 0$ for $b \le a$. Let $t \in E$ and $r = -tM\varpi$. Apply the construction above with $a + 1$ in place of a: to $(\xi, v) = (x, \gamma v)$, giving $u \in \mathsf{U}(E)$, and to $(\xi, v) = (y, v)$, giving $u' \in \mathsf{U}(E)$. By (5.55), since $h(\gamma v, f_b) = h(v, f_b) = 0$ for $b \le a$ and $x_{b-1} = y_{b-1} = 0$ for $b - 1 > a$,

$$\sigma(u) = r\mu\, m(f_{a+1}), \qquad \sigma(u') = r\mu'\, h(v, f_{a+1}) = r\mu'\, m(\gamma f_{a+1}).$$

Conjugating, $\gamma u'\gamma^{-1}$ is the Eichler transformation of $(\gamma\, rf \cdot y, \gamma v) = (rf \cdot x, \gamma v)$ with $\varsigma = \frac{1}{2}h(v, v) = \frac{1}{2}h(\gamma v, \gamma v)$, that is $\gamma u'\gamma^{-1} = u$, so $u\gamma = \gamma u'$. By agreement $\psi(u) = \psi(u')$, i.e. $\psi_f(t\mu\, m(f_{a+1})) = \psi_f(t\mu'\, m(\gamma f_{a+1}))$ for every $t \in E$. Put $c = \mu\, m(f_{a+1}) - \mu'\, m(\gamma f_{a+1}) \in E$. The continuous character $t \mapsto \psi_f(tc)$ of $\mathbb{A}_f$ is trivial on the dense subgroup E, hence trivial; if $c \ne 0$, c is a unit of $\mathbb{A}_f$ and ψ_f would be trivial. So $c = 0$, as required.

(i), conclusion. By induction, (μ, μ') occurs at every $l \in \{a, \dots, \varrho\}$. At $l = \varrho$ take (x, y): since $e_1, \dots, e_\varrho$ are isotropic and orthogonal to z, $h(f \cdot x, f \cdot x) = -N\mu\bar{\mu}$, and $h(\gamma f \cdot y, \gamma f \cdot y) = h(f \cdot y, f \cdot y) = -N\mu'\bar{\mu}'$; so $\mu\bar{\mu} = \mu'\bar{\mu}'$.

Prescribed tails. Suppose $(\zeta, 1)$ occurs at every $a \ge j$. Then for every $w \in E^{\varrho+1}$ there are x, y with $f \cdot x = \gamma f \cdot y$ and $y_a = w_a$ for $a \ge j$: by descending induction on j, if (x, y') has $y'_a = w_a$ for $a > j$, add $(w - y')_j$ times a pair (X, Y) realising $(\zeta, 1)$ at j ($Y_j = 1$, $X_b = Y_b = 0$ for $b > j$). The same holds for γ^{-1}, for which $(\bar{\zeta}, 1)$ occurs at every $a \ge j$ (from $f \cdot X = \gamma f \cdot Y$ follows $f \cdot (\bar{\zeta}Y) = \gamma^{-1}f \cdot (\bar{\zeta}X)$).

(ii). The set of admissible j contains $\varrho + 1$ (the condition is empty), so j is defined. Every nonzero common vector $f \cdot x = \gamma f \cdot y$ has an index $l \ge j$ with $x_l \ne 0 \ne y_l$: let l be the largest index with $(x_l, y_l) \ne (0, 0)$; by (i), $x_l\bar{x}_l = y_l\bar{y}_l$, so both are nonzero; $\zeta = x_l/y_l$ has $\zeta\bar{\zeta} = 1$, and $(\zeta, 1)$ occurs at l (through $(x/y_l, y/y_l)$), hence at every $a \ge l$ by (i); by minimality $j \le l$.

Since $\mathsf{g}(\gamma) = 0$, Lemma 5.2.25 (iv) gives a nonzero common vector, so $j \le l \le \varrho$.

$D_j(\gamma) \ne 0$: otherwise there is $y \ne 0$ with $y_a = 0$ for $a \ge j$ and $(Z(\gamma)y)_a = 0$ for $a < j$. Let $w \in E^{\varrho+1}$. By the prescribed tails for γ^{-1} there are X, Y with $f \cdot Y = \gamma^{-1}f \cdot X$, i.e. $f \cdot X = \gamma f \cdot Y$, and $X_a = \bar{w}_a$ for $a \ge j$. Then, as the rows $a < j$ of $Z(\gamma)y$ vanish,

$$\sum_a w_a(Z(\gamma)y)_a = \sum_a \bar{X}_a(Z(\gamma)y)_a = h(f \cdot X, \gamma f \cdot y) = h(\gamma f \cdot Y, \gamma f \cdot y) = h(f \cdot Y, f \cdot y) = -N\bar{Y}_\varrho y_\varrho = 0,$$

since $y_\varrho = 0$ ($j \le \varrho$). So $Z(\gamma)y = 0$. By Lemma 5.2.28 (vi) with $c' = 0$, $\Upsilon_\gamma(y_0 + y_\varrho z^+) = 0$ for the X_0-part y_0 of $\gamma f \cdot y$, so $y_0 = 0$; by Lemma 5.2.26 (i) with $D = 0$, $\gamma f \cdot y$ has neither e^*- nor z-coordinates, so $\gamma f \cdot y = f \cdot x$ for some x, and $x \ne 0$ because γf is linearly independent and $y \ne 0$. By the previous paragraph some $l \ge j$ has $y_l \ne 0$, a contradiction.

Finally $j \le \varrho$ and $\mathsf{g}(\gamma) = 0$, so the equivalence $\varrho < j \iff \mathsf{g}(\gamma) \ne 0$ holds (both sides are false). With the choice of j and $D_j(\gamma) \ne 0$, γ satisfies the conditions of cell rank j. □

Lemma 5.3.32. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, and $T \subseteq G(E)$ stable with $\mathsf{g}(\gamma) \neq 0$ for every $\gamma \in T$. Then for all $c, c' \in \mathcal{C}$ and $s \in \bar{G}$*

$$\|A_T^{\mathrm{cut}}(c,s,c')\| \leq \nu \sum_{\gamma} \tau_{\mathrm{sz}}^2 \sqrt{\frac{\mathcal{D}_{\mathrm{main}}}{n_p (M^{2\varrho f} \prod_v |\mathsf{g}(\gamma)|_v)^k}},$$

the sum over the $\gamma \in T$ over (c,s,c') that are near representatives for (c,c').

Proof. Fix c, c' and s. For $\delta \in G(E)$ write $K_\delta^\sigma(y) = W^\sigma(y) \otimes \overline{W^\sigma(\delta^{-1} y)}$ ($\sigma \in \{\mathrm{full}, \mathrm{cut}\}$), and for $x > 0$ let $\mathrm{ob}(x) = \tau_{\mathrm{sz}}^2 \sqrt{\mathcal{D}_{\mathrm{main}}/(n_p (M^{2\varrho f} x)^k)}$, so that the summand of the statement is $\nu\, \mathrm{ob}(\prod_v |\mathsf{g}(\gamma)|_v)$. Let $\mathbf{1}_{\mathrm{red}}$ be the indicator of the reduced points; by Lemma 5.2.21 it is a box partition of unity for c.

A cutoff through chosen representatives. In each double coset $q = \mathsf{U}_c^0(M)\, \delta\, \mathsf{U}_{c'}(M)$ choose one element r_q, a near representative for (c,c') if q contains one, and let R be the set of the r_q. Put $\varphi(\delta, y) = \mathbf{1}_R(\delta) \mathbf{1}_{\mathrm{red}}(y)$; each $\varphi(\delta, \cdot)$ is Borel. For $\gamma \in T$ a term over (c,s,c') we have $\mathsf{g}(\gamma) \neq 0$, so by Lemma 5.3.33 with $\omega = \mathbf{1}_{\mathrm{red}}$, $\sum_{(u,u')} \varphi(u \gamma u'^{-1}, u \cdot y) = 1$ for every y, the sum over $\mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)$. By Lemma 5.2.40 (cut case) for the stable set T,

$$\|A_T^{\mathrm{cut}}(c,s,c')\| \leq \sum_{\substack{\gamma \in T \\ \text{over } (c,s,c')}} \|J_\gamma\|, \qquad J_\gamma = \int_{G_\infty} \varphi(\gamma, y)\, K_\gamma^{\mathrm{cut}}(y)\, dy.$$

Terms that do not contribute. If $\gamma \notin R$, then $J_\gamma = 0$. If $\gamma \in R$ is not a near representative for (c,c'), then by the choice of R its double coset $\mathsf{U}_c^0(M) \gamma \mathsf{U}_{c'}(M)$ contains no near representative for (c,c'), so by Lemma 5.3.34 $K_\gamma^{\mathrm{cut}}(y) = 0$ at every reduced y; hence the integrand of J_γ vanishes and $J_\gamma = 0$.

A near representative. Let $\gamma \in R \cap T$ be a term over (c,s,c') and a near representative for (c,c'), and $x = \prod_v |\mathsf{g}(\gamma)|_v$. By Lemma 5.3.63 (i), $x \geq 1$; put $\varepsilon = \nu\, \mathrm{ob}(x)/2 \geq 0$, so that $\prod_v \mathrm{PB}(\gamma_v, \deg_v, b^{37n}, \#I_v) \leq \varepsilon^2$ by Lemma 5.3.63 (iii). The weight $\omega = \mathbf{1}_{\mathrm{red}}$ is Borel with values in $[0,1]$. It is invariant under every $t \in G_\infty$ whose components fix $f_0, \dots, f_\varrho$, since reducedness of y depends only on the vectors $y_v^{-1} f_b$ (RD4) and $(t_v y_v)^{-1} f_b = y_v^{-1} f_b$. If $\omega(y) \neq 0$ and $\ell(y_v) \leq b^{6n+2}$, then by (RD5) $y_v \in \mathrm{Bx}(b^{27} \ell(y_v)) \subseteq \mathrm{Bx}(b^{6n+29}) \subseteq \mathrm{Bx}(b^{37n})$, as $b \geq 1$ and $6n + 29 \leq 37n$. As γ is a near term with $\mathsf{g}(\gamma) \neq 0$, Lemma 5.3.35 gives

$$\|J_\gamma\| \leq \varepsilon + \int_{G_\infty} \varphi(\gamma, y)\, \|K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y)\|\, dy.$$

The integral is one summand of the sum in Lemma 5.3.60 for the stable set T and the cutoff φ, so it is at most $\nu/(2b^{k^3 f})$, and by Lemma 5.3.63 (ii) this is at most $\nu\, \mathrm{ob}(x)/2$. Hence $\|J_\gamma\| \leq \nu\, \mathrm{ob}(x)$.

Conclusion. Summing over the terms $\gamma \in T$ over (c,s,c'), only near representatives for (c,c') contribute, each at most $\nu\, \mathrm{ob}(\prod_v |\mathsf{g}(\gamma)|_v)$, which is the statement. □

Lemma 5.3.33. *Let $\mathfrak{S}$ be a filling datum, $c, c' \in \mathcal{C}$, $R \subseteq G(E)$ a set containing exactly one element of each double coset $\mathsf{U}_c^0(M)\, \delta\, \mathsf{U}_{c'}(M)$ ($\delta \in G(E)$), ω a box partition of unity for c, and $\gamma \in G(E)$ with $\mathsf{g}(\gamma) \neq 0$. Then for every $y \in G_\infty$*

$$\sum_{(u,u') \in \mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)} \mathbf{1}_R(u \gamma u'^{-1})\, \omega(u \cdot y) = 1.$$

Proof. **Trivial stabilisers.** Let $\delta \in G(E)$ with $\mathsf{g}(\delta) \neq 0$, $p \in \mathsf{U}_c^0(M)$ and $w \in \mathsf{U}_{c'}(M)$ with $p\delta = \delta w$. We show $w = 1$, hence $p = 1$. As p fixes every f_a, $w\delta^{-1}f_a = \delta^{-1}pf_a = \delta^{-1}f_a$ for $0 \leq a \leq \varrho$. By Lemma 5.2.26 (ii), $\det Z(\delta) \neq 0$; since $h(f_a, \delta^{-1}f_b) = h(\delta f_a, f_b) = \overline{Z(\delta)_{ba}}$, the matrix $(h(f_a, \delta^{-1}f_b))_{a,b} = Z(\delta)^*$ is invertible. Hence there are $y_0, \dots, y_{\varrho-1}$ in the span of $\delta^{-1}f_0, \dots, \delta^{-1}f_\varrho$ with $h(e_{a+1}, y_b) = \delta_{ab}$ for $0 \leq a, b < \varrho$, and w fixes them. Let $Y = \langle y_0, \dots, y_{\varrho-1}\rangle$. The pairing h between X and Y is perfect; so $\dim Y = \varrho$, $Y \cap X^\perp = 0$, $X \cap Y^\perp = 0$, and $V = Y \oplus X^\perp$ as $\dim X^\perp = r - \varrho$. Now $w \in \mathsf{U}(E)$ is an isometry that preserves X and acts trivially on $X^\perp/X$. For $x \in X$ and $t \in X^\perp \cap Y^\perp$ the vectors $wx - x$ and $wt - t$ lie in X, and for $y \in Y$, $h(wx - x, y) = h(wx, wy) - h(x, y) = 0$ and likewise $h(wt - t, y) = 0$; so both lie in $X \cap Y^\perp = 0$. Since $X \cap (X^\perp \cap Y^\perp) \subseteq X \cap Y^\perp = 0$ and $\dim(X^\perp \cap Y^\perp) \geq r - 2\varrho$, $X^\perp = X \oplus (X^\perp \cap Y^\perp)$, and w is the identity on $X^\perp + Y = V$.

One representative pair per coset. For $x \in \mathsf{U}_c(M)$ put $a(x) = \sum_{u' \in \mathsf{U}_{c'}(M)} \mathbf{1}_R(x\gamma u'^{-1}) \in [0, \infty]$. Fix x and put $\delta = x\gamma$; then $\mathsf{g}(\delta) = \mathsf{g}(\gamma) \neq 0$ by Lemma 5.2.25 (ii). The double coset $\mathsf{U}_c^0(M)\delta\mathsf{U}_{c'}(M)$ contains exactly one $r \in R$, so there is a pair $(q, u') \in \mathsf{U}_c^0(M) \times \mathsf{U}_{c'}(M)$ with $q\delta u'^{-1} \in R$. If (q_1, u_1) and (q_2, u_2) are two such pairs, then $q_1\delta u_1^{-1} = q_2\delta u_2^{-1} = r$, so $p\delta = \delta w$ with $p = q_2^{-1}q_1 \in \mathsf{U}_c^0(M)$ and $w = u_2^{-1}u_1 \in \mathsf{U}_{c'}(M)$, and the first step gives $q_1 = q_2$, $u_1 = u_2$. Hence, for every $x \in \mathsf{U}_c(M)$,

$$\sum_{q \in \mathsf{U}_c^0(M)} a(qx) = \sum_{q \in \mathsf{U}_c^0(M)} \sum_{u' \in \mathsf{U}_{c'}(M)} \mathbf{1}_R(qx\gamma u'^{-1}) = 1.$$

The partition of unity. By the defining property of a box partition of unity for c (RD5), applied to the weight a, $\sum_{u \in \mathsf{U}_c(M)} a(u)\,\omega(u \cdot y) = 1$ for every y. Expanding $a(u)$ (a sum of nonnegative terms) gives the statement. □

Lemma 5.3.34. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, $c, c' \in \mathcal{C}$, $\gamma \in G(E)$, and $y \in G_\infty$ reduced with $W^{\mathrm{cut}}(y) \otimes \overline{W^{\mathrm{cut}}(\gamma^{-1}y)} \neq 0$. Then the double coset $\mathsf{U}_c^0(M)\,\gamma\,\mathsf{U}_{c'}(M)$ contains a near representative for (c, c').*

Proof. **Two invariances.** (a) $\ell(sx) = \ell(x)$ for $s \in R$, $x \in G_1$, as R is a group. For $u \in \mathsf{U}(E)$ and $v \in \Phi$, $u_v = B_v^{-1}v(u)B_v$ with B_v diagonal in a basis adapted to the flag $\langle e_1\rangle \subset \cdots \subset \langle e_1, \dots, e_\varrho\rangle$; $u_v \in G_1$ (RD1), it stabilises the flag and acts trivially on its successive quotients and on those of the refined flag, so $u_v \in \mathsf{U} \subseteq R$. Hence $\ell((u \cdot x)_v) = \ell(x_v)$ for $x \in G_\infty$. (b) If moreover $uf_b = f_b$ for all b, then $u_v f_b = B_v^{-1}v(u)\mathrm{bs}_v(f_b)f_b = f_b$, so $(u_v x_v)^{-1}f_b = x_v^{-1}f_b$; as reducedness depends only on the moved frames $x_v^{-1}f_b$ (RD4), $u \cdot x$ is reduced if and only if x is, and by (a) $u \cdot x$ is boxed if and only if x is.

Step 1: both points are in the cut. Some entry $W^{\mathrm{cut}}(y)_i W^{\mathrm{cut}}(\gamma^{-1}y)_{i'}$ of the kernel is nonzero, so $W_v^\flat(y_v) \neq 0$ and $W_v^\flat((\gamma^{-1}y)_v) \neq 0$ for every $v \in \Phi$. By (BS2) the cut functions vanish outside the cut, so $\ell(y_v) \leq b^{6n+2}$ and $\ell((\gamma^{-1}y)_v) \leq b^{6n+2}$ for every v. As y is reduced, y is boxed.

Step 2: reducing $\gamma^{-1}y$. Let $Q \subseteq \mathsf{U}_{c'}(M)$ contain exactly one element of each coset $\mathsf{U}_{c'}^0(M)u'$ and $a = \mathbf{1}_Q$; then $\sum_{u_0 \in \mathsf{U}_{c'}^0(M)} a(u_0u') = 1$ for every $u' \in \mathsf{U}_{c'}(M)$. By Lemma 5.2.21, $\mathbf{1}_{\mathrm{red}}$ is a box partition of unity for c', so (RD5) gives $\sum_{u \in \mathsf{U}_{c'}(M)} a(u)\mathbf{1}_{\mathrm{red}}(u \cdot \gamma^{-1}y) = 1$; hence there is $u \in \mathsf{U}_{c'}(M)$ with $u \cdot \gamma^{-1}y$ reduced. By (a) and Step 1 its components have Bessel length at most b^{6n+2}, so $u \cdot \gamma^{-1}y$ is boxed.

Step 3: a near element and its representative. Put $\delta_1 = \gamma u^{-1}$. Since $\gamma \mapsto \gamma_\infty$ is a homomorphism (RD1), $\delta_1^{-1}y = u \cdot \gamma^{-1}y$, which is boxed, and y is boxed: δ_1 is a near term. Let $r \in R_{c,c'}$ be the representative of $\mathsf{U}_c^0(M)\,\delta_1\,\mathsf{U}_{c'}^0(M)$, $r = a_1\delta_1 b_1$ with $a_1 \in \mathsf{U}_c^0(M)$, $b_1 \in \mathsf{U}_{c'}^0(M)$. By (b), $a_1 \cdot y$ is boxed and $r^{-1}(a_1 \cdot y) = b_1^{-1} \cdot (\delta_1^{-1}y)$ is boxed; so r is a near term, hence a near

representative for (c,c'). Finally $r = a_1\gamma(u^{-1}b_1)$ with $u^{-1}b_1 \in \mathsf{U}_{c'}(M)$, as $\mathsf{U}^0_{c'}(M) \subseteq \mathsf{U}_{c'}(M)$ and the latter is a group ($K(M)$ and $\mathsf{U}(E)$ are groups); so $r \in \mathsf{U}^0_c(M)\,\gamma\,\mathsf{U}_{c'}(M)$. □

Lemma 5.3.35. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, and $\gamma \in G(E)$ a near term with $\mathsf{g}(\gamma) \neq 0$. Let $\omega\colon G_\infty \to [0,1]$ be Borel with $\omega(ty) = \omega(y)$ whenever every $t_v \in G_1$ fixes $f_0,\dots,f_\varrho$, and such that $\omega(y) \neq 0$ and $\ell(y_v) \leq b^{6n+2}$ imply $y_v \in \mathrm{Bx}(b^{37n})$. Let $\varepsilon \geq 0$ with $\prod_{v\in\Phi} \mathrm{PB}(\gamma_v, \deg_v, b^{37n}, \#I_v) \leq \varepsilon^2$. Write $K^\sigma(y) = W^\sigma(y) \otimes \overline{W^\sigma(\gamma^{-1}y)}$ for $\sigma \in \{\mathrm{full}, \mathrm{cut}\}$. Then*

$$\Big\| \int_{G_\infty} \omega(y)\, K^{\mathrm{cut}}(y)\, dy \Big\| \leq \varepsilon + \int_{G_\infty} \omega(y)\, \|K^{\mathrm{full}}(y) - K^{\mathrm{cut}}(y)\|\, dy.$$

Proof. Put $\rho = b^{37n}$, $\gamma' = \gamma^{-1}$, so $\gamma'_v = (\gamma_v)^{-1}$, and $\mathrm{PB}_v = \mathrm{PB}(\gamma_v, \deg_v, \rho, \#I_v)$.

Step 1: hypotheses of the local bound. By Lemma 5.3.36 (iii) and (ii), $\mathsf{g}_1(Z(\gamma'_v)) = \mathsf{g}_1(Z(\gamma_v)) = s_v^2\, v(\mathsf{g}(\gamma))$ with $s_v > 0$ (the squared frame scales are positive by (FA4)), and $v(\mathsf{g}(\gamma)) > 0$ as $\mathsf{g}(\gamma) \neq 0$ is totally positive (Lemma 5.2.26 (ii)). As $\mathrm{PB}(y,\dots)$ depends on y only through $\mathsf{g}_1(Z(y))$, $\mathrm{PB}(\gamma'_v, \deg_v, \rho, \#I_v) = \mathrm{PB}_v$. Since γ is near, there is a boxed y_0 with $\gamma^{-1}y_0$ boxed. A boxed point x is reduced with $\ell(x_v) \leq b^{6n+2}$, so by Lemma 5.2.21 and (RD5) $x_v \in \mathrm{Bx}(b^{27}\ell(x_v)) \subseteq \mathrm{Bx}(\rho)$, as $6n+29 \leq 37n$. Put $y' = \gamma^{-1}y_0$: then $y'_v \in \mathrm{Bx}(\rho)$ and $(\gamma'_v)^{-1}y'_v = y_{0,v} \in \mathrm{Bx}(\rho)$; and $\rho \geq b$.

Step 2: local open-cell bounds and their product. By Lemma 5.3.37 at each v (for $\mathfrak{B}_v$, γ'_v, y'_v, ρ) there are $\mathcal{X}_v$ and an open-cell bound (A_v, ν_v, m_v, t_v) for $(W_v, \gamma'_v, \rho, \deg_v)$: A_v compact, m_v a probability measure, t_v measurable with frame-fixing values, ν_v finite with $\nu_v(G_1) \leq \mathrm{PB}_v$, the kernels

$$k^v_{(i,j)}(x,z) = W_{v,i}(\gamma'_v t_v(x) z)\, \overline{W_{v,j}(t_v(x)z)} \qquad ((i,j) \in I_v \times I_v)$$

an L^2 contraction from $(G_1, dy|_{A_v})$ to $(\mathcal{X}_v, m_v)$ with bound ν_v, and, for m_v-almost every x, every $z \in \mathrm{Bx}(\rho)$ with $\ell(\gamma'_v t_v(x) z) \leq \rho$ lies in A_v. These kernels are bounded and measurable (W_v is bounded and continuous, t_v measurable). Let $A = \prod_v A_v$, $m = \bigotimes_v m_v$, $\nu = \bigotimes_v \nu_v$, and for $x \in \mathcal{X} = \prod_v \mathcal{X}_v$

$$N(x) = \sum_{(i,j)\in I\times I} \Big| \int_A \omega(z) \prod_v k^v_{(i_v,j_v)}(x_v, z_v)\, dz \Big|^2.$$

By Lemma 5.3.59 with $\phi = \omega$ (Borel, values in $[0,1]$), $\int_{\mathcal{X}} N\, dm \leq \int \omega^2 d\nu \leq \prod_v \nu_v(G_1) \leq \prod_v \mathrm{PB}_v \leq \varepsilon^2$.

Step 3: a good shift. The set $\mathcal{X}'$ of x such that for every v each $z \in \mathrm{Bx}(\rho)$ with $\ell(\gamma'_v t_v(x_v) z) \leq \rho$ lies in A_v has full m-measure, its complement being a finite union of sets $\{x : x_v \in Q_v\}$ with $m_v(Q_v) = 0$. If $N > \varepsilon^2$ on $\mathcal{X}'$, then $\int N\, dm > \varepsilon^2$, as $m(\mathcal{X}') = 1$ and $N - \varepsilon^2 > 0$ there; so there is $x \in \mathcal{X}'$ with $N(x) \leq \varepsilon^2$. Let $s = (t_v(x_v))_v \in G_\infty$; each s_v fixes the frame, so $\omega(sz) = \omega(z)$ and $(s_v z_v)^{-1} f_a = z_v^{-1} f_a$.

Step 4: the adjoint kernel and its support. For $\sigma \in \{\mathrm{full}, \mathrm{cut}\}$ let $F^\sigma(y) = \omega(y)\, W^\sigma(\gamma' y) \otimes \overline{W^\sigma(y)}$. Since ω is real, $F^\sigma(y)$ is the conjugate transpose of $\omega(y)K^\sigma(y)$; conjugate transposition is a real linear isometry of $\mathbb{C}^{I\times I}$ for the euclidean norm and commutes with integration, so $\|\int F^{\mathrm{cut}}\| = \|\int \omega K^{\mathrm{cut}}\|$ and $\|F^{\mathrm{full}}(y) - F^{\mathrm{cut}}(y)\| = \omega(y)\|K^{\mathrm{full}}(y) - K^{\mathrm{cut}}(y)\|$. If $F^{\mathrm{cut}}(sz) \neq 0$, then $\omega(sz) \neq 0$, and $W^\flat_v(s_v z_v) \neq 0$, $W^\flat_v(\gamma'_v s_v z_v) \neq 0$ for every v; by (BS2), $\ell(s_v z_v) \leq b^{6n+2}$ and $\ell(\gamma'_v s_v z_v) \leq b^{6n+2} \leq \rho$; by the hypothesis on ω, $s_v z_v \in \mathrm{Bx}(\rho)$, hence $z_v \in \mathrm{Bx}(\rho)$; as $x \in \mathcal{X}'$, $z_v \in A_v$. So $z \mapsto F^{\mathrm{cut}}(sz)$ vanishes outside the compact set A; it is bounded and measurable, hence integrable, and so is $F^{\mathrm{full}}(s\,\cdot)$ on A.

Step 5: the bound. By the left invariance of the Cayley measures (Lemma 5.2.41) and Step 4,

$$\int_{G_\infty} F^{\mathrm{cut}}(y)\,dy = \int_A F^{\mathrm{cut}}(sz)\,dz = \int_A F^{\mathrm{full}}(sz)\,dz - \int_A (F^{\mathrm{full}} - F^{\mathrm{cut}})(sz)\,dz.$$

The entry (i,j) of $F^{\mathrm{full}}(sz)$ is $\omega(z)\prod_v k^v_{(i_v,j_v)}(x_v,z_v)$, so the first integral has norm $N(x)^{1/2} \le \varepsilon$. The second has norm at most $\int_{G_\infty} \|F^{\mathrm{full}} - F^{\mathrm{cut}}\|(sz)\,dz = \int_{G_\infty} \omega(y)\|K^{\mathrm{full}}(y) - K^{\mathrm{cut}}(y)\|\,dy$, again by left invariance. With Step 4 this is the claim. □

Lemma 5.3.36. *Let $\mathfrak{S}$ be a filling datum, $\gamma \in G(E)$ and $v \in \Phi$, and put*

$$S_v = \mathrm{diag}(\mathrm{bs}_v(f_0), \dots, \mathrm{bs}_v(f_\varrho)) \quad \textit{and} \quad s_v = \prod_{a=0}^{\varrho} \mathrm{fs}(v,a).$$

(i) $Z(\gamma_v) = S_v\, v(Z(\gamma))\, S_v$, and $\mathrm{bs}_v(f_a)^2 = \mathrm{fs}(v,a)$; hence $D_l(Z(\gamma_v)) = \prod_{a<l} \mathrm{fs}(v,a)\, v(D_l(\gamma))$ for $0 \le l \le \varrho+1$.

(ii) $\mathsf{g}_1(Z(\gamma_v)) = s_v^2\, v(\mathsf{g}(\gamma))$.

(iii) For $y \in G_1$, $Z(y^{-1}) = Z(y)^*$ and $\mathsf{g}_1(Z(y^{-1})) = \mathsf{g}_1(Z(y))$.

Proof. **(i)** Let $B_v = \mathrm{diag}(\mathrm{bs}_v)$, a real diagonal matrix in the standard basis, and H_v the Gram matrix of $V \otimes_{E,v} \mathbb{C}$ in that basis (entries 1 for the pairs (e_a, e_a^*), the identity on X_0, $-N$ at z). As $\mathrm{bs}_v(e_a)\mathrm{bs}_v(e_a^*) = 1$, $\mathrm{bs}_v = 1$ on X_0 and $\mathrm{bs}_v(z)^2 = 1/N$, the matrix $B_v^* H_v B_v$ is the Gram matrix J_1 of V_1. By (RD1), $\gamma_v = B_v^{-1} v(\gamma) B_v$. The frame vectors are standard basis vectors with $B_v f_a = \mathrm{bs}_v(f_a) f_a$, so

$$Z(\gamma_v)_{ab} = f_a^* J_1 \gamma_v f_b = (B_v f_a)^* H_v\, v(\gamma)\, (B_v f_b) = \mathrm{bs}_v(f_a)\mathrm{bs}_v(f_b)\, v(h(f_a, \gamma f_b)).$$

By (RD1), $\mathrm{bs}_v(f_a) = \mathrm{rs}_v^{\varrho-a}/\sqrt{N}$ with rs_v real, so $\mathrm{bs}_v(f_a)^2 = \mathrm{fs}(v,a)$. Taking the leading principal minor of order l of $S_v\, v(Z(\gamma))\, S_v$ gives the formula for D_l.

(ii) By (i) and $\mathrm{fs}(v,\varrho) = 1/N$, and since $v(\alpha\bar\alpha) = |v(\alpha)|^2$ for α in the CM field E,

$$\begin{aligned}\mathsf{g}_1(Z(\gamma_v)) &= s_v^2 |v(D_{\varrho+1}(\gamma))|^2 - \Big(\prod_{a<\varrho} \mathrm{fs}(v,a)\Big)^2 |v(D_\varrho(\gamma))|^2 \\ &= s_v^2 (|v(D_{\varrho+1})|^2 - N^2 |v(D_\varrho)|^2) = s_v^2\, v(\mathsf{g}(\gamma)).\end{aligned}$$

(iii) y preserves the form h_1 of V_1, so $Z(y^{-1})_{ab} = h_1(f_a, y^{-1} f_b) = h_1(y f_a, f_b) = \overline{h_1(f_b, y f_a)} = \overline{Z(y)_{ba}}$. The determinant and the leading principal minors of $Z(y)^*$ are the conjugates of those of $Z(y)$, which have the same moduli; so $\mathsf{g}_1(Z(y^{-1})) = \mathsf{g}_1(Z(y))$. □

Lemma 5.3.37. *Let $\mathfrak{B}$ be a Bessel system for (ϱ, k, b, f) with Bessel function W, index set I and degree* deg. *Let $\gamma, y \in G_1$ with $\mathsf{g}_1(Z(\gamma)) > 0$, $y \in \mathrm{Bx}(\rho)$ and $\gamma^{-1} y \in \mathrm{Bx}(\rho)$, where $\rho \ge b$. Then there are a measurable space $\mathcal{X}$ and (A, ν, m, t) that is an open-cell bound for $(W, \gamma, \rho, \deg)$; that is, $t\colon \mathcal{X} \to G_1$ is measurable with values in the elements fixing $f_0, \dots, f_\varrho$, and (A, ν, m) is a kernel bound for $(W, \gamma t, t, \rho, \mathrm{PB}(\gamma, \deg, \rho, \#I))$.*

Proof. Let $Z = Z(\gamma)$ and $n = \varrho + 1$; then $\rho \ge b \ge 8r \ge 8n \ge 8$.

Sizes. By Lemma 5.3.41 (iv), applied with y, and since $y^{-1}\gamma f_c = (\gamma^{-1}y)^{-1} f_c$, $|Z_{ac}| \le \|y^{-1} f_a\|\, \|(\gamma^{-1}y)^{-1} f_c\| \le \rho^2$ for all a, c. Hence $\|Z\| \le n\rho^2 \le \rho^3$ and $\mathsf{g}_1(Z) \le |\det Z|^2 \le \|Z\|^{2n} \le$

ρ^{6n}. Put $M = \rho^{9n}/\mathfrak{g}_1(Z)$, defined as $\mathfrak{g}_1(Z) > 0$. Then $M \geq \rho^{3n} \geq \rho \geq b$, $M^{-1} = \mathfrak{g}_1(Z)\rho^{-9n} \leq \mathfrak{g}_1(Z)$ and $|Z_{ac}| \leq \rho^2 \leq M^2$.

Reduction to the Gram section. By Lemma 5.3.47 (i), $Z \in \Omega_{\rm cell}$, and by (iii) there are $g_1, g_2 \in G_1$ fixing $f_0, \dots, f_\varrho$ with $\gamma = g_1^{-1}\gamma_Z g_2$. An element fixing the frame lies in R (Lemma 5.5.30 (i)); so $s = g_1^{-1} \in R$. Let $\mathcal{X} = \mathrm{U}(k) \times \mathsf{U}^0$ with its Borel σ-algebra and $t(h, v) = g_2^{-1}(vh)^{-1}$. Then t is continuous, hence measurable, its values fix $f_0, \dots, f_\varrho$ (so lie in R), and

$$\gamma\, t(h, v) = g_1^{-1}\gamma_Z g_2 g_2^{-1}(vh)^{-1} = s\,\gamma_Z(vh)^{-1}.$$

The kernel bound. By Lemma 5.3.38 with $M \geq b$, Z, s and t, there are A, ν and m such that (A, ν, m) is a kernel bound for $(W, \gamma t, t, M, C)$, where

$$C = M^{10^5 n^4} \frac{(\#I)^2}{\deg \cdot c_G(\mathfrak{g}_1(Z)/2)^{k-1}} = \mathrm{PB}(\gamma, \deg, \rho, \#I).$$

All clauses of a kernel bound except the last do not involve the size. For the last: for m-almost every x, every $z \in \mathrm{Bx}(M)$ with $\ell(\gamma t(x)z) \leq M$ lies in A; as $\rho \leq M$, $\mathrm{Bx}(\rho) \subseteq \mathrm{Bx}(M)$ and $\ell(\gamma t(x)z) \leq \rho$ implies $\ell(\gamma t(x)z) \leq M$. Hence (A, ν, m) is a kernel bound for $(W, \gamma t, t, \rho, \mathrm{PB}(\gamma, \deg, \rho, \#I))$, and (A, ν, m, t) is an open-cell bound for $(W, \gamma, \rho, \deg)$. □

Lemma 5.3.38. *Let $\mathfrak{B}$ be a Bessel system for (ϱ, k, b, f) with Bessel function W, index set I and degree* deg. *Let $M \geq b$, let Z be a complex $(\varrho+1) \times (\varrho+1)$ matrix with $\mathfrak{g}_1(Z) \geq M^{-1}$ and $|Z_{ac}| \leq M^2$ for all a, c, let $s \in R$ and let $t\colon \mathrm{U}(k) \times \mathsf{U}^0 \to R$ be a map; put $g(h, v) = s\gamma_Z(vh)^{-1}$. Then there are a set $A \subseteq G_1$, a measure ν on G_1 and a measure m on $\mathrm{U}(k) \times \mathsf{U}^0$ such that (A, ν, m) is a kernel bound for (W, g, t, M, C), where*

$$C = M^{10^5 n^4} \frac{(\#I)^2}{\deg \cdot c_G\,(\mathfrak{g}_1(Z)/2)^{k-1}}.$$

Proof. Scales. $M \geq b \geq 8r \geq 8n \geq 8$. Put $M_0 = M^{9n}$, $L = M_0^5$ and $P = L^n = M^{45n^2}$; then $8r \leq M \leq L \leq P$, $P \geq M^5 \geq b^5$ and $P \geq M^{9n}$.

The tube. By Lemma 5.3.39 ($L \geq 8r$) there are A and $\mathcal{Q} \subseteq \mathsf{U}^0$ such that A is a tube of size L along U^0 with base $\mathcal{Q}$: A is compact, $\|y\| \leq L^{10} \leq P^{20}$ for $y \in A$, $0 < dv(\mathcal{Q}) < \infty$, $\mathcal{Q}y \subseteq A$ when $\|y\| \leq L$, and $dv(\{v : vy \in A\}) \leq 3\,dv(\mathcal{Q})$ for every y. Taking $y = 1$, $\mathcal{Q} \subseteq A$, so $\|v\| \leq P^{20}$ for $v \in \mathcal{Q}$.

Mass of $\|W\|^2$ on the tube. Apply Lemma 5.3.42 with $w = \|W\|^2$ (Borel and left R-invariant, as wave has modulus 1 and ρ is unitary), $\phi = \mathbf{1}_A$, $K = 3\,dv(\mathcal{Q})$ and the bound P^{40}: if $s' \in \mathsf{U}$ and $y, s'y \in A$, every entry of $F^*s'F$ is an entry of $s' = (s'y)y^{-1}$, of absolute value at most $\|s'y\|\,\|y^{-1}\| = \|s'y\|\,\|y\| \leq P^{40}$ (Lemma 5.3.41 (iv)). With (BS1),

$$\int_A \|W\|^2 dy \leq 3\,dv(\mathcal{Q})\,(2P^{40})^{2n^2}\,\#I. \tag{5.56}$$

The measures. Let $c = c_G(\mathfrak{g}_1(Z)/2)^{k-1} > 0$, $\mu = \mu_{\mathfrak{g}_1(Z)/2} = c\,dh \otimes dv$, $S = \mathrm{U}(k) \times \mathcal{Q}$, so $\mu(S) = c\,dv(\mathcal{Q}) \in (0, \infty)$, $m = \mu(S)^{-1}\mu(\cdot \cap S)$, a probability measure, $K = P^{2000n^2}\deg^{-1}\#I$ and $\nu = \mu(S)^{-1}K\,\mathbf{1}_A\|W\|^2 dy$.

The contraction. We apply Lemma 5.3.43 with A, $(\mathrm{U}(k) \times \mathsf{U}^0, \mu)$, S, s, the continuous map $g_0(h, v) = \gamma_Z(vh)^{-1}$ (so $g = sg_0$), t and K. Let β be Borel, integrable and zero off A; then $\beta(y) = 0$ when $\|y\| > P^{20}$, and S lies in the fixer ball of radius P^{20}. By Lemma 5.3.56 (iii)

($M \geq 8n$, $|Z_{ac}| \leq M^2$, $\mathsf{g}_1(Z) \geq M^{-1}$, $P \geq M^{9n}$), every Y with $\|Y - Z\| < P^{-6}$ is bounded in the cell by P and has $\mathsf{g}_1(Y) \geq \mathsf{g}_1(Z)/2 > 0$. As W has Schur's bound $(P, \deg^{-1} \#I)$ and $P \geq 8r$, Lemma 5.3.44 (with $c = \mathsf{g}_1(Z)/2$, $D = \deg^{-1} \#I$, $\phi = \beta$) gives

$$\int_S \Big\| \int \beta(y) W(g_0(x)y)\, dy \Big\|^2 d\mu(x) \leq P^{2000n^2} \deg^{-1} \#I \int |\beta|^2 dy = K \int |\beta|^2 dy.$$

So the kernels $(x, y) \mapsto W_i(g(x)y)\overline{W_j(t(x)y)}$ are an L^2 contraction from $(G_1, dy|_A)$ to $(\mathrm{U}(k) \times \mathsf{U}^0, m)$ with bound ν.

The mass of ν. By (5.56), ν is finite and

$$\nu(G_1) \leq \frac{P^{2000n^2} \#I}{\deg\, c\, dv(\mathcal{Q})}\, 3\, dv(\mathcal{Q}) (2P^{40})^{2n^2} \#I = 3 \cdot 2^{2n^2} P^{2080n^2} \frac{(\#I)^2}{\deg\, c} \leq M^{10^5 n^4} \frac{(\#I)^2}{\deg\, c} = C,$$

Capture. m is carried by S. Let $x = (h, v)$ with $v \in \mathcal{Q}$, and $z \in \mathrm{Bx}(M)$ with $\ell(g(x)z) \leq M$. By Lemma 5.3.56 (i) ($M \geq n$), Z is bounded in the cell by $M_0 \geq M^{8n}$, so Z is invertible with $\|Z^{-1}\| \leq M_0$. By Lemma 5.3.57 (with M and $T = M_0$), $\|v^{-1}z\| \leq M_0^5 = L$, so $z = v\,(v^{-1}z) \in \mathcal{Q} \cdot \{\|y\| \leq L\} \subseteq A$. Finally A is compact and m is a probability measure, so (A, ν, m) is a kernel bound for (W, g, t, M, C). □

Lemma 5.3.39. *Let $L \geq 8r$. Then there are a set $\mathcal{K} \subseteq G_1$ and a set $\mathcal{Q} \subseteq \mathsf{U}^0$ such that $\mathcal{K}$ is a tube of size L along U^0 with base $\mathcal{Q}$.*

Proof. Write $\|\cdot\|$ for operator norms and $S(a, b) = \mathrm{sk}(b) - \frac{1}{2}aa^*$. For $t > 0$ let

$$B_t = \{u(a, b) :\ \|a\| \leq t,\ \|S(a, b)\| \leq t^2\} \subseteq \mathsf{U}^0.$$

Properties of B_t. (1) $B_t B_{t'} \subseteq B_{t+t'}$: by Lemma 5.3.40 (i), $u(a, b)u(a', b') = u(a + a', b'')$ with $b'' = b + b' + \beta(a, a')$, and $S(a + a', b'') = S(a, b) + S(a', b') - aa'^*$, whose norm is at most $t^2 + t'^2 + tt' \leq (t + t')^2$, while $\|a + a'\| \leq t + t'$. (2) The set $u^{-1}(B_t) \subseteq \mathcal{P}$ is closed and bounded ($\|\mathrm{sk}(b)\| \leq \frac{3}{2}t^2$ and sk is a linear bijection), hence compact; so B_t is compact and, u being injective (Lemma 5.3.40 (i)), $dv(B_t)$ is the Lebesgue measure of $u^{-1}(B_t)$, finite. (3) Under $\delta_t(a, b) = (ta, t^2 b)$ one has $S(\delta_t(a, b)) = t^2 S(a, b)$, so $u^{-1}(B_t) = \delta_t(u^{-1}(B_1))$ and $dv(B_t) = t^{d_0} dv(B_1)$ with $d_0 = 2\varrho k + 2\varrho^2$; and $dv(B_1) > 0$, as $u^{-1}(B_1)$ contains the nonempty open set $\{\|a\| < \frac{1}{2},\ \|\mathrm{sk}(b)\| < \frac{1}{2}\}$. (4) For $v = u(a, b) \in B_t$, $\|v\| \leq 1 + 2\|a\| + \|S(a, b)\| \leq (1 + t)^2$, since X, W, D are isometries. (5) If $v \in \mathsf{U}^0$, $T \geq 1$ and $\|v\| \leq T$, then $v \in B_T$: by Lemma 5.3.40 (i), $a = X^* v W$ and $S(a, b) = X^* v D$ have norms at most $\|v\| \leq T \leq T^2$.

The tube. Put $t_0 = r^2 L^2$, $\mathcal{Q} = B_{t_0}$, $Y_L = \{y \in G_1 : \|y\| \leq L\}$ (closed and bounded in the matrices, hence compact) and $\mathcal{K} = \mathcal{Q} Y_L$, the image of the compact set $\mathcal{Q} \times Y_L$ under multiplication, hence compact. $\mathcal{Q}$ is compact, so Borel, and $0 < dv(\mathcal{Q}) < \infty$ by (2), (3). For $v \in \mathcal{Q}$ and $y \in Y_L$, by (4), $\|vy\| \leq (1 + t_0)^2 L \leq L^{10}$, because $L \geq 8r \geq 8$ gives $1 + r^2 L^2 \leq 1 + L^4/64 \leq L^4$. By construction $vy \in \mathcal{K}$ for $v \in \mathcal{Q}$ and $\|y\| \leq L$.

Slices. Let $y \in G_1$ and $S_y = \{v \in \mathsf{U}^0 : vy \in \mathcal{K}\}$, a closed subset of U^0 ($\mathcal{K}$ is closed), hence dv-measurable. Suppose $S_y \neq \varnothing$ and fix $v_0 \in S_y$, $v_0 y = q_0 z_0$ with $q_0 \in \mathcal{Q}$, $z_0 \in Y_L$; put $g = q_0^{-1} v_0 \in \mathsf{U}^0$, so $gy = z_0$. Let $v \in S_y$, $vy = qz$ with $q \in \mathcal{Q}$, $z \in Y_L$. Then $q^{-1} v g^{-1} = (q^{-1}vy)(gy)^{-1} = z z_0^{-1} \in \mathsf{U}^0$ has norm at most $\|z\|\, \|z_0^{-1}\| = \|z\|\, \|z_0\| \leq L^2$ (Lemma 5.3.41 (iv)), so it lies in B_{L^2} by (5), and $v g^{-1} \in B_{t_0} B_{L^2} \subseteq B_{t_0 + L^2}$ by (1). Hence $S_y g^{-1} \subseteq B_{(1 + r^{-2})t_0}$, and by the right invariance of dv (Lemma 5.3.40 (ii)) and (3),

$$dv(S_y) = dv(S_y g^{-1}) \leq (1 + r^{-2})^{d_0}\, dv(\mathcal{Q}) \leq e^{d_0/r^2} dv(\mathcal{Q}) \leq 3\, dv(\mathcal{Q}),$$

because $d_0 = 2\varrho(k + \varrho) \leq (2\varrho + k)^2 \leq r^2$. □

Lemma 5.3.40. *(i)* sk *is an* $\mathbb{R}$*-linear bijection of* $\mathrm{M}_\varrho(\mathbb{R})$ *onto the skew-hermitian* $\varrho\times\varrho$ *matrices. For* $a,a'\in\mathrm{M}_{\varrho\times k}(\mathbb{C})$ *let* $\beta(a,a')\in\mathrm{M}_\varrho(\mathbb{R})$ *be given by* $\mathrm{sk}(\beta(a,a'))=\frac12(a'a^*-aa'^*)$. *Then* u *is injective,* $X^*u(a,b)W=a$, $X^*u(a,b)D=\mathrm{sk}(b)-\frac12aa^*$, *and*

$$u(a,b)\,u(a',b')=u(a+a',\,b+b'+\beta(a,a')),\qquad u(a,b)^{-1}=u(-a,-b).$$

(ii) For every Borel $f\colon\mathsf{U}^0\to[0,\infty]$ *and* $w\in\mathsf{U}^0$: $\int f(wv)\,dv=\int f(vw)\,dv=\int f(v^{-1})\,dv=\int f(v)\,dv$.

(iii) Let $h\in H$ *act on* X_0 *by the unitary matrix* h_0 *in the basis* $x_1,\dots,x_k$. *Then* $hu(a,b)h^{-1}=u(ah_0^*,b)$, *and* $\int f(hvh^{-1})\,dv=\int f(v)\,dv$ *for every Borel* $f\colon\mathsf{U}^0\to[0,\infty]$.

(iv) For every Borel $f\colon G_1\to[0,\infty]$,

$$\int_{G_1}f(s)\,ds=\int_H\int_{\mathsf{U}^0}f(vh)\,dv\,dh=\int_H\int_{\mathsf{U}^0}f(hv)\,dv\,dh=\int_{G_1}f(s^{-1})\,ds.$$

(v) There is a Borel $\chi\colon G_1\to[0,1]$ *with* $\int_{\mathsf{U}^0}\chi(vy)\,dv=1$ *for every* $y\in G_1$.

Proof. Throughout, X,W,D have orthonormal columns taken from one orthonormal basis, so $X^*X=1$, $W^*W=1$, $D^*D=1$ and $X^*W=X^*D=W^*D=0$. Write $S(a,b)=\mathrm{sk}(b)-\frac12aa^*$.

(i). $\mathrm{sk}(b)_{ll}=\mathrm{i}\,b_{ll}$; for $l<l'$ the map $(b_{ll'},b_{l'l})\mapsto\mathrm{sk}(b)_{ll'}=((b_{ll'}-b_{l'l})+\mathrm{i}(b_{ll'}+b_{l'l}))/\sqrt2$ is an $\mathbb{R}$-linear bijection $\mathbb{R}^2\to\mathbb{C}$, and $\mathrm{sk}(b)_{l'l}=-\overline{\mathrm{sk}(b)_{ll'}}$. A skew-hermitian matrix is the same as an arbitrary purely imaginary diagonal together with arbitrary entries above the diagonal, so sk is a bijection onto the skew-hermitian matrices; $\frac12(a'a^*-aa'^*)$ is skew-hermitian, so β is well defined and $\mathbb{R}$-bilinear. Multiplying out with the relations above, $X^*u(a,b)W=a$ and $X^*u(a,b)D=S(a,b)$; since $\mathrm{sk}(b)=S(a,b)+\frac12aa^*$, the pair (a,b) is determined by $u(a,b)$. Put $N=u(a,b)-1$ and $N'=u(a',b')-1$. Of the nine products of a term of N with a term of N' only $(XaW^*)(-Wa'^*D^*)=-Xaa'^*D^*$ survives (every other one contains W^*X, D^*X or D^*W). Hence

$$u(a,b)u(a',b')=1+X(a+a')W^*-W(a+a')^*D^*+X\big(S(a,b)+S(a',b')-aa'^*\big)D^*,$$

and $S(a,b)+S(a',b')-aa'^*=\mathrm{sk}(b+b')+\frac12(a'a^*-aa'^*)-\frac12(a+a')(a+a')^*=S(a+a',b+b'+\beta(a,a'))$. As $u(0,0)=1$ and $\beta(a,-a)=0$, $u(a,b)u(-a,-b)=1$.

(ii). Let $w=u(a_0,b_0)$. By (i), $wu(a,b)=u(\phi(a,b))$ with $\phi(a,b)=(a_0+a,\,b_0+b+\beta(a_0,a))$, an affine bijection of $\mathcal{P}$ whose linear part $(a,b)\mapsto(a,b+\beta(a_0,a))$ is block unitriangular, hence of determinant 1; so ϕ preserves Lebesgue measure and $\int f(wv)\,dv=\int f(u(\phi(a,b)))\,d(a,b)=\int f(u(a,b))\,d(a,b)=\int f\,dv$. The same argument with $(a,b)\mapsto(a+a_0,b+b_0+\beta(a,a_0))$ gives right invariance, and with $(a,b)\mapsto(-a,-b)$ inversion invariance.

(iii). In the basis, $h=1+W(h_0-1)W^*$ is unitary, $hX=X$, $hD=D$, $hW=Wh_0$, $W^*h^{-1}=h_0^*W^*$ and $D^*h^{-1}=D^*$. Hence $hu(a,b)h^{-1}=1+Xah_0^*W^*-Wh_0a^*D^*+XS(a,b)D^*=u(ah_0^*,b)$, because $(ah_0^*)^*=h_0a^*$ and $(ah_0^*)(ah_0^*)^*=aa^*$. Right multiplication by the unitary h_0^* is a Euclidean isometry of $\mathrm{M}_{\varrho\times k}(\mathbb{C})$, so $(a,b)\mapsto(ah_0^*,b)$ preserves Lebesgue measure, and $\int f(hvh^{-1})\,dv=\int f\,dv$.

(iv). The measure dv is σ-finite (u is continuous and injective, and $dv(u(C))$ is the Lebesgue measure of C for compact $C\subseteq\mathcal{P}$), so the first equality is the definition of ds with Tonelli's theorem. For fixed h, $hv=(hvh^{-1})h$ and (iii) give $\int f(hv)\,dv=\int f(vh)\,dv$. Finally $(vh)^{-1}=h^{-1}v^{-1}$, so by (ii) and the inversion invariance of the Haar measure dh of the compact group H,

$$\int f(s^{-1})\,ds=\int_H\int f(h^{-1}v^{-1})\,dv\,dh=\int_H\int f(h^{-1}v)\,dv\,dh=\int_H\int f(hv)\,dv\,dh.$$

(v). For $y \in G_1$ the $\varrho \times r$ matrix D^*y has rank ϱ, as y is invertible; let $(D^*y)^+ = (D^*y)^*\big(D^*y(D^*y)^*\big)^{-1}$, a right inverse depending continuously on y. Put

$$\begin{aligned}\alpha(y) &= -\big(W^*y(D^*y)^+\big)^*,\\ C(y) &= X^*y(D^*y)^+ + \tfrac{1}{2}\alpha(y)\alpha(y)^*,\\ \mathfrak{c}(y) &= \Big(\alpha(y),\ \mathrm{sk}^{-1}\big(\tfrac{1}{2}(C(y)-C(y)^*)\big)\Big) \in \mathcal{P},\end{aligned}$$

a continuous map $G_1 \to \mathcal{P}$. Let $v = u(a,b)$. Since $D^*v = D^*$, $W^*v = W^* - a^*D^*$ and $X^*v = X^* + aW^* + S(a,b)D^*$, we have $(D^*vy)^+ = (D^*y)^+$, $\alpha(vy) = \alpha(y) + a$, $X^*vy(D^*y)^+ = X^*y(D^*y)^+ - a\alpha(y)^* + S(a,b)$, and so

$$C(vy) = C(y) + \mathrm{sk}(b) + \tfrac{1}{2}\big(\alpha(y)a^* - a\alpha(y)^*\big) = C(y) + \mathrm{sk}\big(b + \beta(a,\alpha(y))\big).$$

The added matrix is skew-hermitian, so $\mathfrak{c}(u(a,b)y) = \big(\alpha(y)+a,\ \mathfrak{c}_2(y) + b + \beta(a,\alpha(y))\big)$, $\mathfrak{c}_2$ the second component. For fixed y this is an affine bijection of $\mathcal{P}$ with linear part $(a,b) \mapsto (a, b+\beta(a,\alpha(y)))$, which preserves Lebesgue measure. Let $E \subseteq \mathcal{P}$ be a cube of Lebesgue measure 1 and $\chi = \mathbf{1}_E \circ \mathfrak{c}$. Then χ is Borel and $\int_{U^0} \chi(vy)\,dv = \int_{\mathcal{P}} \mathbf{1}_E\big(\mathfrak{c}(u(a,b)y)\big)\,d(a,b) = 1$. □

Lemma 5.3.41. *(i) For $s \in R$ the $(\varrho+1)\times(\varrho+1)$ matrix F^*sF is upper unitriangular and $sF = F\,(F^*sF)$. For $s,t \in R$: $F^*stF = (F^*sF)(F^*tF)$ and $F^*s^{-1}F = (F^*sF)^{-1}$.*

(ii) For $s \in R$, $y \in G_1$ and $t \in G_1$ fixing $f_0,\dots,f_\varrho$: $Z(syt) = \big((F^*sF)^{-1}\big)^* Z(y)$.

(iii) For every upper unitriangular complex $(\varrho+1)\times(\varrho+1)$ matrix m and every complex $(\varrho+1)\times(\varrho+1)$ matrix Z: $\mathfrak{g}_1(m^*Z) = \mathfrak{g}_1(Z)$.

(iv) For $y \in G_1$: $y^{-1} = J_1y^*J_1$ and $\|y^{-1}\| = \|y\| \geq 1$. For $\gamma, y \in G_1$ and $0 \leq a,b \leq \varrho$: $|Z(\gamma)_{ab}| \leq \|y^{-1}f_a\|\,\|y^{-1}\gamma f_b\|$.

(v) If $s \in R$, $y \in G_1$ and $Z(y)$, $Z(sy)$ are invertible, then $F^*sF = \big(Z(y)Z(sy)^{-1}\big)^*$; in particular every entry of F^*sF has absolute value at most $\|Z(y)\|\,\|Z(sy)^{-1}\|$.

Proof. (i). By Lemma 5.5.30 (i), $sf_b - f_b \in \langle f_a : a < b\rangle$ for $s \in R$ and $0 \leq b \leq \varrho$. Hence $sF = FN$ for an upper unitriangular N, and $N = F^*FN = F^*sF$. For $s,t \in R$, $stF = s\,F(F^*tF) = F(F^*sF)(F^*tF)$; multiplying by F^* gives $F^*stF = (F^*sF)(F^*tF)$. Taking $t = s^{-1} \in R$ gives $(F^*sF)(F^*s^{-1}F) = F^*F = 1$.

(ii). As t fixes the frame, $tF = F$. From $s^*J_1s = J_1$ we get $J_1s = (s^{-1})^*J_1$. With (i) for s^{-1}, $s^{-1}F = F(F^*s^{-1}F)$, so

$$Z(syt) = F^*J_1syF = (s^{-1}F)^*J_1yF = (F^*s^{-1}F)^*F^*J_1yF = \big((F^*sF)^{-1}\big)^*Z(y).$$

(iii). The matrix m^* is lower unitriangular, so $\det(m^*Z) = \det Z$. For row and column indices $a,b < \varrho$, $(m^*Z)_{ab} = \sum_{c\leq a}(m^*)_{ac}Z_{cb}$ involves only indices $c < \varrho$; so the leading principal block of order ϱ of m^*Z is the product of the leading blocks of order ϱ of m^* (unitriangular, determinant 1) and of Z. Hence $D_\varrho(m^*Z) = D_\varrho(Z)$ and $\mathfrak{g}_1(m^*Z) = |\det Z|^2 - |D_\varrho(Z)|^2 = \mathfrak{g}_1(Z)$.

(iv). From $y^*J_1y = J_1$ and $J_1^2 = 1$, $(J_1y^*J_1)y = 1$, so $y^{-1} = J_1y^*J_1$. As J_1 is unitary, $\|y^{-1}\| = \|y^*\| = \|y\|$, and $1 = \|yy^{-1}\| \leq \|y\|^2$. Since $y^{-1} \in G_1$ preserves h, $Z(\gamma)_{ab} = h(f_a, \gamma f_b) = h(y^{-1}f_a, y^{-1}\gamma f_b)$, and $|h(v,w)| = |v^*J_1w| \leq \|v\|\,\|J_1w\| = \|v\|\,\|w\|$ by the Cauchy–Schwarz inequality.

(v). By (ii) with $t = 1$, $Z(sy) = \big((F^*sF)^{-1}\big)^*Z(y)$, so $Z(y)Z(sy)^{-1} = Z(y)Z(y)^{-1}(F^*sF)^* = (F^*sF)^*$. An entry of a matrix is bounded by its operator norm, and $\|Z(y)Z(sy)^{-1}\| \leq \|Z(y)\|\,\|Z(sy)^{-1}\|$. □

Lemma 5.3.42. *Let $w, \phi\colon G_1 \to [0,\infty]$ be Borel with $w(sy) = w(y)$ for all $s \in R$ and $y \in G_1$, and let $K \in [0,\infty]$ and $M \geq 1$. Suppose that $\int_{\mathsf{U}^0} \phi(vy)\,dv \leq K$ for every $y \in G_1$, and that every entry of $F^* sF$ has absolute value at most M whenever $s \in \mathsf{U}$ and $y \in G_1$ satisfy $\phi(y) \neq 0$ and $\phi(sy) \neq 0$. Then*

$$\int_{G_1} \phi\, w\, dy \leq K\,(2M)^{2n^2} \int_{\mathsf{U}\backslash G_1} w.$$

Proof. The inner integral. We show $\int_{\mathsf{U}} \phi(uy')\,du \leq K(2M)^{2n^2}$ for every $y' \in G_1$. If $\phi(uy') = 0$ for all $u \in \mathsf{U}$ there is nothing to prove. Otherwise fix $u_1 \in \mathsf{U}$ with $\phi(u_1 y') \neq 0$ and put $y'' = u_1 y'$. As du is right invariant (Lemma 5.2.34 (iv)), by the definition of du and Tonelli's theorem,

$$\int_{\mathsf{U}} \phi(uy')\,du = \int_{\mathsf{U}} \phi(uu_1 y')\,du = \int \int_{\mathsf{U}^0} \phi(v\,\mathrm{Ut}(x)\,y'')\,dv\,dx.$$

Let x and $v \in \mathsf{U}^0$ be such that $\phi(v\,\mathrm{Ut}(x)\,y'') \neq 0$. Then $s = v\,\mathrm{Ut}(x) \in \mathsf{U}$ and $\phi(y'') \neq 0 \neq \phi(sy'')$, so every entry of $F^* sF$ has absolute value at most M. By Lemma 5.3.41 (i), $F^* sF = (F^* vF)(F^* \mathrm{Ut}(x) F) = T(x)$, since v fixes the frame. Hence $|x_{ab}| \leq M$ for all $a < b$: the integrand vanishes unless x lies in the box $B_M = \{x : |x_{ab}| \leq M \text{ for all } a < b\}$. The inner integral over v is at most K, so

$$\int_{\mathsf{U}} \phi(uy')\,du \leq K\,\mathrm{vol}(B_M) = K\,(\pi M^2)^{n(n-1)/2} \leq K\,(4M^2)^{n^2} = K\,(2M)^{2n^2},$$

using $\pi M^2 \leq 4M^2$, $4M^2 \geq 1$ and $n(n-1)/2 \leq n^2$.

Unfolding. By Lemma 5.5.30 (vi), applied to ϕw, with its measure on R the image of $du \otimes dh$ under $(u,h) \mapsto uh$ (dh the Haar probability measure of H), and by the left R-invariance of w,

$$\int_{G_1} \phi w\, dy = \int_{\mathsf{U}\backslash G_1} \int_H \int_{\mathsf{U}} \phi(uhy)\, w(uhy)\,du\,dh\,dy = \int_{\mathsf{U}\backslash G_1} w(y) \int_H \int_{\mathsf{U}} \phi(u\,hy)\,du\,dh\,dy.$$

By the first part with $y' = hy$ the double inner integral is at most $K(2M)^{2n^2}$, as dh has mass 1. This gives the claim. □

Lemma 5.3.43. *Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be a unitary representation on $\mathbb{C}^I$, I finite, and $W\colon G_1 \to \mathbb{C}^I$ a continuous Bessel function of type ρ. Let $A \subseteq G_1$ be compact, (T,μ) a measure space, $S \subseteq T$ measurable with $0 < \mu(S) < \infty$ and $m = \mu(S)^{-1}\mu(\cdot \cap S)$; let $s \in R$, $g\colon T \to G_1$ measurable, $t\colon T \to R$ a map and $K \in [0,\infty)$. Suppose that*

$$\int_S \Big\| \int_{G_1} \beta(y)\, W(g(x)y)\,dy \Big\|^2 d\mu(x) \leq K \int_{G_1} |\beta|^2\,dy$$

for every Borel integrable $\beta\colon G_1 \to \mathbb{C}$ vanishing outside A. Then the kernels

$$(x,y) \mapsto W_i(sg(x)y)\,\overline{W_j(t(x)y)}, \qquad (i,j) \in I \times I,$$

are an L^2 contraction from $(G_1, dy|_A)$ to (T,m) with bound $\nu = \mu(S)^{-1} K\,\mathbf{1}_A \|W\|^2 dy$.

Proof. For $q \in R$ write $q = uh$ with $u \in \mathsf{U}$, $h \in H$ (Lemma 5.5.30 (i)) and $c(q) = \mathrm{wave}(\sigma(u))\rho(h)$, a unitary operator of $\mathbb{C}^I$ with $W(qy) = c(q)W(y)$ for all y. Let ϕ be bounded and Borel. As A is compact, $dy(A) < \infty$ (Lemma 5.5.17 (iii)), so $\psi = \mathbf{1}_A \phi$ is integrable, and so are $\beta_j = \psi\,\overline{W_j}$ (W is bounded); each β_j is Borel and vanishes outside A.

Removing the shifts. Fix $x \in T$ and write $g = g(x)$, $t = t(x)$. For fixed i, the vector $\bigl(\int \psi(y) W_i(sgy) \overline{W_j(ty)}\, dy\bigr)_j$ is the complex conjugate of

$$\int \overline{\psi(y) W_i(sgy)}\, W(ty)\, dy = c(t) \int \overline{\psi(y) W_i(sgy)}\, W(y)\, dy$$

(a linear map commutes with the integral); as $c(t)$ is unitary,

$$\sum_j \Bigl| \int \psi(y) W_i(sgy) \overline{W_j(ty)}\, dy \Bigr|^2 = \sum_j \Bigl| \int \beta_j(y) W_i(sgy)\, dy \Bigr|^2.$$

Summing over i and using $W(sgy) = c(s) W(gy)$ with $c(s)$ unitary,

$$\sum_{i,j} \Bigl| \int_A \phi(y) W_i(sgy) \overline{W_j(ty)}\, dy \Bigr|^2 = \sum_j \Bigl\| \int \beta_j(y) W(sgy)\, dy \Bigr\|^2 = \sum_j \Bigl\| \int \beta_j(y) W(g(x)y)\, dy \Bigr\|^2.$$

The right side is a measurable function of x: g is measurable and $g' \mapsto \int \beta_j(y) W(g'y)\, dy$ is continuous by dominated convergence (W is continuous and bounded).

Integrating. By the hypothesis applied to each β_j,

$$\begin{aligned} \int_T \sum_{i,j} \Bigl| \int_A \phi\, k_{ij}(x, \cdot)\, dy \Bigr|^2 dm(x) &= \mu(S)^{-1} \sum_j \int_S \Bigl\| \int \beta_j W(g(x)\,\cdot) \Bigr\|^2 d\mu \\ &\le \mu(S)^{-1} K \sum_j \int |\beta_j|^2 dy = \mu(S)^{-1} K \int_A |\phi|^2 \|W\|^2 dy, \end{aligned}$$

which is $\int |\phi|^2 d\nu$. □

Lemma 5.3.44. *Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be a unitary representation on $\mathbb{C}^I$, I finite, let $P \ge 8r$, $c > 0$ and $D \in [0, \infty)$, and let F be a Bessel function of type ρ with Schur's bound (P, D). Let Z be a complex $(\varrho+1) \times (\varrho+1)$ matrix such that every Y with $\|Y - Z\| < P^{-6}$ is bounded in the cell by P and satisfies $\mathsf{g}_1(Y) \ge c$. Let $\phi\colon G_1 \to \mathbb{C}$ be Borel and integrable with $\phi(y) = 0$ whenever $\|y\| > P^{20}$. Then, with $\mathcal{B}$ the fixer ball of radius P^{20},*

$$\int_{\mathcal{B}} \Bigl\| \int_{G_1} \phi(y)\, F\bigl(\gamma_Z(vh)^{-1} y\bigr)\, dy \Bigr\|^2 d\mu_c(h, v) \le P^{2000 n^2} D \int_{G_1} |\phi|^2\, dy.$$

Proof. Convolutions and their derivatives. Apply Lemma 5.3.45 to (V_w), ϕ and $M = P^{20}$: we obtain the alphabet L of quadruples of basis vectors, $\#L = r^4$, the weights ψ_i with $|\psi_i| \le |\phi|$, and the continuous functions $W_i(g) = \int \psi_i(y) V_{\pi(i)}(gy)\, dy$ with $\|W_i(sg)\| = \|W_i(g)\|$ for $s \in R$; $W_\varnothing(g) = \int \phi(y) F(gy)\, dy$. As $|\pi(i)| = |i|$ and $P \ge 1$, Schur's bound gives for $|i| \le 2n^2$

$$\int_{\mathsf{U} \setminus G_1} \|W_i\|^2 \le P^{2|i|} D \int |\psi_i|^2 \le P^{4n^2} D \int |\phi|^2 =: K_0. \tag{5.57}$$

Coordinates and the cube. Let $\mathcal{D} = \{0,1\} \times \{0, \dots, \varrho\}^2$, so $\#\mathcal{D} = 2n^2$, and for $x \in \mathbb{R}^{\mathcal{D}}$ let $E(x)$ be the matrix with entries $x_{0ab} + \mathrm{i}\, x_{1ab}$; the map $x \mapsto Z + E(x)$ carries Lebesgue measure to dY (Lebesgue measure of real and imaginary parts). Let $\varepsilon = P^{-8}$, $C = [0, \varepsilon]^{\mathcal{D}}$ and $Q = \{Y : \|Y - Z\| < P^{-6}\}$, an open set. For $x \in C$, $\|E(x)\| \le (\sum_j x_j^2)^{1/2} \le (2n^2)^{1/2} P^{-8} < P^{-6}$, as $2n^2 < P^4$ ($n \le r \le P/8$); so $Z + E(x) \in Q$. Let $\Omega = \mathrm{U}(k) \times \mathsf{U}^0$ with σ the restriction of μ_c to $\mathcal{B}$ (σ-finite), and

$$f_i(x, (h, v)) = W_i(\gamma_{Z + E(x)}(vh)^{-1}),$$

a Borel function of (x,h,v) (W_i is continuous and $Y\mapsto\gamma_Y$ is Borel, Lemma 5.3.47 (ii)). Then $f_\varnothing(0,(h,v))=\int\phi(y)F(\gamma_Z(vh)^{-1}y)\,dy$, so the left side of the claim is $\int_\Omega\|f_\varnothing(0,\omega)\|^2d\sigma(\omega)$.

The cube integrals. Let $|i|\le 2n^2$. Every $Y\in Q$ is bounded in the cell by P and has $\mathsf{g}_1(Y)\ge c$, and $\|W_i\|^2$ is Borel and left R-invariant. By the change of variables $Y=Z+E(x)$, Lemma 5.3.46 (with this $P\ge 1$, c, Q) and (5.57),

$$\int_C\int_\Omega\|f_i(x,\omega)\|^2d\sigma\,dx\le\int_Q\int_{\mathrm{U}(k)\times\mathsf{U}^0}\bigl\|W_i(\gamma_Y(vh)^{-1})\bigr\|^2d\mu_c(h,v)\,dY\le(2P^2)^{2n^2}K_0.$$

Partial derivatives. Let $\omega=(h,v)\in\mathcal{B}$ and $g_\omega=(vh)^{-1}$; as h is unitary and by Lemma 5.3.41 (iv), $\|g_\omega^{-1}\|=\|g_\omega\|=\|vh\|=\|v\|\le P^{20}$. Let i be a word, $d\in\mathcal{D}$, and x,t with $x[d\mapsto t]\in C$. The line $p(s)=Z+E(x[d\mapsto s])$ has $p'(s)=E_d$, the matrix with one entry 1 or i, so $\|E_d\|=1$; let J be the open interval of the s around t with $p(s)\in Q$ (Q is open and convex). For $s\in J$, $p(s)$ is bounded in the cell by $P\ge 4$, so by Lemma 5.3.56 (iv) $Y\mapsto\gamma_Y$ is real differentiable at $p(s)$ with derivative of norm at most P^{25}, and $\|\gamma_{p(s)}^{-1}\|=\|\gamma_{p(s)}\|\le P^{18}$ (Lemma 5.3.41 (iv)). Hence $\gamma(s)=\gamma_{p(s)}g_\omega$ is differentiable on J with $\gamma'(s)=\gamma(s)X(s)$, where

$$X(s)=g_\omega^{-1}\gamma_{p(s)}^{-1}(D\gamma_{p(s)}[E_d])g_\omega,\qquad\|X(s)\|\le P^{20}P^{18}P^{25}P^{20}=P^{83}.$$

By Lemma 5.3.45 (iii) with $M=P^{20}$ and $C=P^{83}$, $s\mapsto f_i(x[d\mapsto s],\omega)=W_i(\gamma(s))$ is differentiable at t with derivative of norm at most $P^{123}\sum_{l\in L}\|f_{li}(x[d\mapsto t],\omega)\|$.

Sobolev. By Lemma 5.3.55 with $\Lambda=P^{123}$, $\#L=r^4$, $\varepsilon=P^{-8}\le 1$ and $K=(2P^2)^{2n^2}K_0$,

$$\int_\Omega\|f_\varnothing(0,\omega)\|^2d\sigma\le\Bigl(2P^8\bigl(1+r^8P^{246}\bigr)\cdot 2P^2\Bigr)^{2n^2}K_0\le\bigl(P^{265}\bigr)^{2n^2}P^{4n^2}D\int|\phi|^2\le P^{2000n^2}D\int|\phi|^2,$$

which is the claim. □

Lemma 5.3.45. *Let $\rho\colon\mathrm{U}(k)\to\mathrm{U}(I)$ be a unitary representation on $\mathbb{C}^I$, I finite, and let (V_w) be Bessel functions of type ρ indexed by the words w in the alphabet of pairs (a,b) of basis vectors, such that $\frac{d}{dt}V_w(\gamma(t))=\sum_{a,b}X_{ab}V_{(a,b)w}(\gamma(t))$ whenever γ is a curve in G_1 with $\gamma'(t)=\gamma(t)X$. Let $M>0$ and let $\phi\colon G_1\to\mathbb{C}$ be Borel and integrable with $\phi(y)=0$ whenever $\|y\|>M$. Let L be the set of quadruples $l=(a,b,c,d)$ of basis vectors. For words i in L define $\psi_i\colon G_1\to\mathbb{C}$ and a word $\pi(i)$ in pairs by $\psi_\varnothing=\phi$, $\pi(\varnothing)=\varnothing$ and, for $l=(a,b,c,d)$,*

$$\psi_{li}(y)=\psi_i(y)\,\frac{(y^{-1})_{ad}\,y_{cb}}{M^2},\qquad\pi(li)=(a,b)\pi(i),$$

and put $W_i(g)=\int_{G_1}\psi_i(y)\,V_{\pi(i)}(gy)\,dy$ for $g\in G_1$. Then:

(i) $|\psi_i|\le|\phi|$ on G_1 for every word i;

(ii) every V_w is continuous; every W_i is continuous, and $\|W_i(sg)\|=\|W_i(g)\|$ for $s\in R$, $g\in G_1$;

(iii) if J is an open interval, $C\ge 0$ and $\gamma\colon J\to G_1$ is differentiable with $\gamma'(t)=\gamma(t)X(t)$ and $\|X(t)\|\le C$ for all $t\in J$, then for every $t_0\in J$ the function $t\mapsto W_i(\gamma(t))$ is differentiable at t_0, with derivative of norm at most $M^2C\sum_{l\in L}\|W_{li}(\gamma(t_0))\|$.

Proof. Let $B_w=\sup_{G_1}\|V_w\|<\infty$.

(i). If $\phi(y)=0$, then $\psi_i(y)=0$ for all i. If $\phi(y)\neq 0$, then $\|y\|\le M$, so $|y_{cb}|\le\|y\|\le M$ and $|(y^{-1})_{ad}|\le\|y^{-1}\|=\|y\|\le M$ (Lemma 5.3.41 (iv)); each letter multiplies by a factor of absolute value at most 1. Induction on $|i|$.

(ii). Let $\mathfrak{g}=\{Y : Y^*J_1+J_1Y=0\}$ be the Lie algebra of the closed subgroup G_1 of $\mathrm{GL}_r(\mathbb{C})$; $Y\mapsto\exp Y$ maps a neighbourhood of 0 in $\mathfrak{g}$ onto a neighbourhood of 1 in G_1. For $g\in G_1$ and $Y\in\mathfrak{g}$ the curve $t\mapsto g\exp(tY)$ lies in G_1 and has derivative $g\exp(tY)\,Y$, so by the derivative law $t\mapsto V_w(g\exp(tY))$ is differentiable with derivative of norm at most $\sum_{a,b}|Y_{ab}|B_{(a,b)w}\le r^2\|Y\|\max_{a,b}B_{(a,b)w}$. By the mean value inequality $\|V_w(g\exp Y)-V_w(g)\|\le r^2\|Y\|\max_{a,b}B_{(a,b)w}$, so V_w is continuous at g. As $|\psi_i(y)V_{\pi(i)}(gy)|\le|\phi(y)|B_{\pi(i)}$ with ϕ integrable, W_i is continuous by dominated convergence. Let $s\in R$; by Lemma 5.5.30 (i), $s=uh$ with $u\in\mathsf{U}$, $h\in H$, and $V_{\pi(i)}(sgy)=\mathrm{wave}(\sigma(u))\rho(h)V_{\pi(i)}(gy)$. A linear map commutes with the integral, so $W_i(sg)=\mathrm{wave}(\sigma(u))\rho(h)W_i(g)$, and $|\mathrm{wave}(\sigma(u))|=1$ and the unitarity of $\rho(h)$ give $\|W_i(sg)\|=\|W_i(g)\|$.

(iii). Fix i and put $w=\pi(i)$. For $y\in G_1$ the curve $t\mapsto\gamma(t)y$ in G_1 has derivative $\gamma(t)X(t)y=(\gamma(t)y)(y^{-1}X(t)y)$, so by the derivative law $t\mapsto V_w(\gamma(t)y)$ is differentiable on J with derivative $\sum_{a,b}(y^{-1}X(t)y)_{ab}V_{(a,b)w}(\gamma(t)y)$. As $(y^{-1}Xy)_{ab}=\sum_{c,d}(y^{-1})_{ad}X_{dc}y_{cb}$,

$$\frac{\partial}{\partial t}\bigl(\psi_i(y)V_w(\gamma(t)y)\bigr)=M^2\sum_{l=(a,b,c,d)\in L}X(t)_{dc}\,\psi_{li}(y)\,V_{\pi(li)}(\gamma(t)y).$$

By (i) and $|X(t)_{dc}|\le\|X(t)\|\le C$, its norm is at most $M^2C\,|\phi(y)|\sum_l B_{\pi(li)}$, an integrable function of y independent of t. By differentiation under the integral sign (the mean value inequality and dominated convergence, applied to each coordinate), $t\mapsto W_i(\gamma(t))$ is differentiable at t_0 with derivative $M^2\sum_l X(t_0)_{dc}W_{li}(\gamma(t_0))$, of norm at most $M^2C\sum_l\|W_{li}(\gamma(t_0))\|$. □

Lemma 5.3.46. *Let $P\ge1$ and $c>0$, and let Q be a Borel set of complex $(\varrho+1)\times(\varrho+1)$ matrices such that every $Z\in Q$ is bounded in the cell by P and satisfies $\mathsf{g}_1(Z)\ge c$. Let $w\colon G_1\to[0,\infty]$ be Borel with $w(sy)=w(y)$ for all $s\in R$ and $y\in G_1$. Then*

$$\int_Q\int_{\mathrm{U}(k)\times\mathsf{U}^0}w\bigl(\gamma_Z(vh)^{-1}\bigr)\,d\mu_c(h,v)\,dZ\le(2P^2)^{2n^2}\int_{\mathsf{U}\backslash G_1}w.$$

Proof. Let χ be the Borel function of Lemma 5.3.40 (v), with $\int_{\mathsf{U}^0}\chi(vy)\,dv=1$ for every y, and put $\chi_Q(y)=\chi(y)\mathbf{1}_Q(Z(y))$, a Borel function ($y\mapsto Z(y)$ is continuous). We compare both sides with $\int_{G_1}\chi_Q w\,dy$.

Unfolding. We apply Lemma 5.3.42 with $\phi=\chi_Q$, $K=1$ and $M=P^2\ge1$. First, $\int_{\mathsf{U}^0}\chi_Q(vy)\,dv\le\int_{\mathsf{U}^0}\chi(vy)\,dv=1$. Second, let $s\in\mathsf{U}$ and $y\in G_1$ with $\chi_Q(y)\neq0\neq\chi_Q(sy)$. Then $Z(y),Z(sy)\in Q$, so both are invertible with $\|Z(y)\|\le P$ and $\|Z(sy)^{-1}\|\le P$, and by Lemma 5.3.41 (v) every entry of F^*sF has absolute value at most P^2. Hence

$$\int_{G_1}\chi_Q\,w\,dy\le(2P^2)^{2n^2}\int_{\mathsf{U}\backslash G_1}w.\tag{5.58}$$

Along the fixer. Let $Z\in Q$; then $Z\in\Omega_{\mathrm{cell}}$, so $Z(\gamma_Z)=Z$ by Lemma 5.3.47 (ii). Let s and s' fix $f_0,\dots,f_\varrho$ and let $s'=vh$ with $v\in\mathsf{U}^0$, $h\in H$; then $s'\in R$ and $F^*s'F=1$, so $Z(s'\gamma_Zs)=Z(\gamma_Z)=Z$ by Lemma 5.3.41 (ii). Hence $\chi_Q(s'\gamma_Zs)=\chi(s'\gamma_Zs)$ and, by the R-invariance of w, $w(s'\gamma_Zs)=w(\gamma_Zs)$. With Lemma 5.3.40 (iv) and (v),

$$\int_{G_1}(\chi_Qw)(s'\gamma_Zs)\,ds'=w(\gamma_Zs)\int_H\int_{\mathsf{U}^0}\chi(v\,h\gamma_Zs)\,dv\,dh=w(\gamma_Zs).$$

By Tonelli's theorem and the inversion invariance of ds (Lemma 5.3.40 (iv)),

$$\int_{G_1}\int_{G_1}(\chi_Q w)(s'\gamma_Z s)\,ds\,ds' = \int_{G_1} w(\gamma_Z s)\,ds = \int_{G_1} w(\gamma_Z s^{-1})\,ds = \int_H\int_{\mathsf{U}^0} w(\gamma_Z(vh)^{-1})\,dv\,dh.$$

Mean value. By Lemma 5.3.48 for $f=\chi_Q w$, restricting the outer integral to $Q\subseteq\Omega_{\mathrm{cell}}$ and using $c_G\,\mathfrak{g}_1(Z)^{k-1}\ge c_G\,c^{k-1}$ on Q ($k\ge 1$),

$$\int_Q c_G\,c^{k-1}\int_H\int_{\mathsf{U}^0} w(\gamma_Z(vh)^{-1})\,dv\,dh\,dZ \le \int_{G_1}\chi_Q\,w\,dy.$$

The left side is the left side of the claim (definition of μ_c; the integrand is Borel in (Z,h,v) as $Z\mapsto\gamma_Z$ is Borel, Lemma 5.3.47 (ii)). Together with (5.58) this proves the claim. □

Lemma 5.3.47. *(i) For a complex $(\varrho+1)\times(\varrho+1)$ matrix Z: $Z\in\Omega_{\mathrm{cell}}$ if and only if $\mathfrak{g}_1(Z)>0$.*

(ii) For every Z, $\gamma_Z\in G_1$; for $Z\in\Omega_{\mathrm{cell}}$, $Z(\gamma_Z)=Z$; and $Z\mapsto\gamma_Z$ is Borel measurable, and continuous on the open set Ω_{cell}.

(iii) If $Z\in\Omega_{\mathrm{cell}}$ and $\gamma\in G_1$ with $Z(\gamma)=Z$, then $\gamma=g^{-1}\gamma_Z g'$ for some $g,g'\in G_1$ fixing $f_0,\dots,f_\varrho$.

Proof. Write $h(x,y)=x^*J_1y$, so that $Z(y)=F^*J_1yF$, and $n=\varrho+1$; ε_ϱ is the last unit vector of $\mathbb{C}^n$ and $E_{\varrho\varrho}=\varepsilon_\varrho\varepsilon_\varrho^{\mathsf{T}}$. The columns of F and F' are $2n$ distinct standard basis vectors, so $F^*F=F'^*F'=1$ and $F^*F'=0$, and from the Gram matrix J_1,

$$F^*J_1F=-E_{\varrho\varrho},\qquad F^*J_1F'=1-E_{\varrho\varrho},\qquad F'^*J_1F'=E_{\varrho\varrho} \tag{5.59}$$

(e_a pairs with e_a^*, z with nothing in F', $h(z,z)=-1$, $h(x_{k_0},x_{k_0})=1$, the e_a^* are isotropic and orthogonal to x_{k_0}).

(i) If $\det Z=0$, then $\mathfrak{g}_1(Z)=-|D_\varrho(Z)|^2\le 0$ and $Z\notin\Omega_{\mathrm{cell}}$. If $\det Z\ne 0$, then $\mathrm{ic}(Z)=D_\varrho(Z)/\det Z$ (cofactor formula), so $\mathfrak{g}_1(Z)=|\det Z|^2(1-|\mathrm{ic}(Z)|^2)$, positive if and only if $|\mathrm{ic}(Z)|<1$.

Dual families. For $W\in\Omega_{\mathrm{cell}}$ let $\zeta=\varepsilon_\varrho^{\mathsf{T}}W^{-1}$ be the last row of W^{-1} and $F^\vee(W)=F\,\mathrm{Co}(W)+F'\,\mathrm{dg}(W)$. We claim

$$F^*J_1F^\vee(W)=1,\qquad F^\vee(W)^*J_1F^\vee(W)=-\zeta^*\zeta. \tag{5.60}$$

By (5.59), $F^*J_1F^\vee(W)=-E_{\varrho\varrho}\mathrm{Co}(W)+(1-E_{\varrho\varrho})\mathrm{dg}(W)$; the first term has only the last row, $-\mathrm{Co}(W)_{\varrho\cdot}=\varepsilon_\varrho^{\mathsf{T}}$, and the second is $1-E_{\varrho\varrho}$; the sum is 1. Next, with $C=\mathrm{Co}(W)$, $D=\mathrm{dg}(W)$ and (5.59),

$$F^\vee(W)^*J_1F^\vee(W)=-C^*E_{\varrho\varrho}C+C^*(1-E_{\varrho\varrho})D+D(1-E_{\varrho\varrho})C+DE_{\varrho\varrho}D.$$

As the last row of C is $-\varepsilon_\varrho^{\mathsf{T}}$, $C^*E_{\varrho\varrho}C=E_{\varrho\varrho}$; $DE_{\varrho\varrho}D=\eta(W)^2E_{\varrho\varrho}$; and $(1-E_{\varrho\varrho})D=1-E_{\varrho\varrho}$. So the entry (b,c) is $[c<\varrho]\,\overline{C_{cb}}+[b<\varrho]\,C_{bc}+\delta_{b\varrho}\delta_{c\varrho}(\eta(W)^2-1)$. With $C_{ab}=-c_b\,\overline{\zeta_a}\,\zeta_b$ ($a<\varrho$), $c_b=\frac12$ ($b<\varrho$), $c_\varrho=1$ and $\eta(W)^2-1=-|\zeta_\varrho|^2$: for $b,c<\varrho$ the entry is $-\frac12\overline{\zeta_b}\zeta_c-\frac12\overline{\zeta_b}\zeta_c$; for $b<\varrho=c$ it is $-\overline{\zeta_b}\zeta_\varrho$; for $c<\varrho=b$ it is $-\overline{\zeta_c}\zeta_\varrho=-\overline{\zeta_\varrho}\zeta_c$; for $b=c=\varrho$ it is $-|\zeta_\varrho|^2$. In all cases it is $-\overline{\zeta_b}\zeta_c=(-\zeta^*\zeta)_{bc}$.

For $Z\in\Omega_{\mathrm{cell}}$ also $Z^*\in\Omega_{\mathrm{cell}}$, with $\mathrm{ic}(Z^*)=\overline{\mathrm{ic}(Z)}$, so $\eta(Z^*)=\eta(Z)$, and comparing entries $(((Z^*)^{-1})_{\varrho a}=\overline{(Z^{-1})_{a\varrho}})$, $\mathrm{Co}(Z^*)=\mathrm{Co}'(Z)$; hence $F^\vee(Z^*)=F\,\mathrm{Co}'(Z)+F'\,\mathrm{dg}(Z)$.

(ii) For $Z \notin \Omega_{\mathrm{cell}}$, $\gamma_Z = 1 \in G_1$. Let $Z \in \Omega_{\mathrm{cell}}$, $S = F^\vee(Z)Z$ and $M = \gamma_Z$, so $M = 1 + (S - F)F^* + \big((F(Z^*)^{-1} - S\,\mathrm{Co}'(Z))\mathrm{dg}(Z)^{-1} - F'\big)F'^*$. By the orthonormality relations, $MF = S$, $MF' = (F(Z^*)^{-1} - S\,\mathrm{Co}'(Z))\mathrm{dg}(Z)^{-1}$, hence

$$MF^\vee(Z^*) = S\,\mathrm{Co}'(Z) + MF'\,\mathrm{dg}(Z) = F(Z^*)^{-1},$$

and $Mx_i = x_i$ for $i \neq k_0$. Let U be the span of the columns of F and F' (that is, of e_a, e_a^*, z, x_{k_0}) and U' the span of the x_i, $i \neq k_0$; $V_1 = U \oplus U'$, orthogonally for h, and h is nondegenerate on U. The columns of $X = (F \mid F^\vee(Z^*)) = (F \mid F')\left(\begin{smallmatrix} 1 & \mathrm{Co}'(Z) \\ 0 & \mathrm{dg}(Z) \end{smallmatrix}\right)$ form a basis of U, as $\mathrm{dg}(Z)$ is invertible ($\eta(Z) > 0$). M maps them to the columns of $Y = (S \mid F(Z^*)^{-1})$, which lie in U. By (5.59) and (5.60) (for $W = Z$ and $W = Z^*$, with $\zeta' = \varepsilon_\varrho^{\mathsf{T}}(Z^*)^{-1}$, so $\zeta'^* = Z^{-1}\varepsilon_\varrho$), the two Gram matrices agree blockwise:

$$S^*J_1S = Z^*(-\zeta^*\zeta)Z = -(\zeta Z)^*(\zeta Z) = -E_{\varrho\varrho} = F^*J_1F,$$
$$S^*J_1F(Z^*)^{-1} = Z^*\big(F^\vee(Z)^*J_1F\big)(Z^*)^{-1} = 1 = F^*J_1F^\vee(Z^*),$$

$$\big(F(Z^*)^{-1}\big)^*J_1F(Z^*)^{-1} = -Z^{-1}E_{\varrho\varrho}(Z^{-1})^* = -\zeta'^*\zeta' = F^\vee(Z^*)^*J_1F^\vee(Z^*),$$

using $\zeta Z = \varepsilon_\varrho^{\mathsf{T}}$ and taking adjoints for the remaining block. Since the Gram matrix of X is invertible, so is that of Y, so the columns of Y are a basis of U; thus M maps U isometrically onto U and is the identity on $U' \perp U$, so $M^*J_1M = J_1$ and $\gamma_Z \in G_1$. Moreover $Z(\gamma_Z) = F^*J_1MF = F^*J_1F^\vee(Z)\,Z = Z$ by (5.60). Finally $\Omega_{\mathrm{cell}} = \{\mathsf{g}_1 > 0\}$ is open by (i), and on it the entries of γ_Z are polynomials in the entries of Z, Z^{-1}, their conjugates and $\eta(Z)^{\pm1} = (1 - |\mathrm{ic}(Z)|^2)^{\pm1/2}$, continuous functions; off it $\gamma_Z = 1$. So $Z \mapsto \gamma_Z$ is continuous on Ω_{cell}, constant on its closed complement, and Borel measurable.

(iii) For $s \in G_1$ with $Z(s) = Z$, the family $(F \mid sF)$ of $2n$ vectors has the Gram matrix

$$\Gamma = \begin{pmatrix} -E_{\varrho\varrho} & Z \\ Z^* & -E_{\varrho\varrho} \end{pmatrix},$$

as s is an isometry. Exchanging the two block columns and taking the Schur complement of the invertible block Z, $|\det\Gamma| = |\det Z|\,|\det(Z^* - E_{\varrho\varrho}Z^{-1}E_{\varrho\varrho})| = |\det Z|\,|\det(Z^* - \mathrm{ic}(Z)E_{\varrho\varrho})|$, and $\det(Z^* - \mathrm{ic}(Z)E_{\varrho\varrho}) = \det Z^*\,(1 - \mathrm{ic}(Z)((Z^*)^{-1})_{\varrho\varrho}) = \overline{\det Z}\,(1 - |\mathrm{ic}(Z)|^2)$; so $|\det\Gamma| = \mathsf{g}_1(Z) > 0$ and the $2n$ vectors are linearly independent. Apply this to $s = \gamma$ and to $s = \gamma_Z$ (by (ii)): the linear map sending $(F \mid \gamma F)$ to $(F \mid \gamma_Z F)$ is an isometry between the subspaces they span, and by Witt's theorem it extends to some $g \in G_1$. Then $gF = F$ and $g\gamma F = \gamma_Z F$, so $g' = \gamma_Z^{-1}g\gamma \in G_1$ fixes $f_0, \dots, f_\varrho$, and $\gamma = g^{-1}\gamma_Z g'$. □

Lemma 5.3.48. *For every Borel function $f\colon G_1 \to [0,\infty]$,*

$$\int_{\Omega_{\mathrm{cell}}} c_G\,\mathsf{g}_1(Z)^{k-1} \int_{G_1}\int_{G_1} f(s'\gamma_Z s)\,ds\,ds'\,dZ \le \int_{G_1} f(y)\,dy.$$

Proof. Let $w(Z) = \mathbf{1}_{\Omega_{\mathrm{cell}}}(Z)\,\mathrm{vol}(S^{2k-1})\,2^{-(\varrho+1)}\,\mathsf{g}_1(Z)^{k-1}$ and let λ be the image of $w(Z)\,dZ \otimes ds$ under $(Z, s') \mapsto s'\gamma_Z$; the map is Borel, as $Z \mapsto \gamma_Z$ is Borel (Lemma 5.3.47 (ii)) and multiplication is continuous. By Lemma 5.3.50, λ is s-finite and below the Gram fibre at the frame, so by Lemma 5.3.53, with $c = \mathrm{vol}\,\mathrm{U}(r)/(\mathrm{vol}\,\mathrm{U}(k)\,2^{\varrho+1})$,

$$(\lambda * ds)(B) \le c\int_{G_1} \mathbf{1}_B\,dy \qquad \text{for every Borel } B \subseteq G_1.$$

Approximating f from below by an increasing sequence of nonnegative simple Borel functions and using monotone convergence on both sides, $\int f\,d(\lambda * ds) \le c\int f\,dy$. By the definition of the convolution as an image measure and by Tonelli's theorem (all measures s-finite, the integrand Borel and nonnegative),

$$\int_{G_1} f\,d(\lambda * ds) = \int_{G_1}\int_{G_1} f(gs)\,ds\,d\lambda(g) = \int w(Z)\int_{G_1}\int_{G_1} f(s'\gamma_Z s)\,ds\,ds'\,dZ.$$

Since $w(Z)/c = \mathbf{1}_{\Omega_{\mathrm{cell}}}(Z)\operatorname{vol}\mathrm{U}(k)\operatorname{vol}(S^{2k-1})\,\mathtt{g}_1(Z)^{k-1}/\operatorname{vol}\mathrm{U}(r) = \mathbf{1}_{\Omega_{\mathrm{cell}}}(Z)\,c_G\,\mathtt{g}_1(Z)^{k-1}$, dividing by $c>0$ gives the claim. □

Definition 5.3.49. Measures are positive and may take the value ∞; in $[0,\infty]$ we put $0\cdot\infty = \infty\cdot 0 = 0$. A topological space carries its Borel σ-algebra, and a Borel measure on it is a measure on this σ-algebra. The symbols X, B, π, β, μ, ν, λ, K, C, S, m, E are local to this definition.

(BY1) Below a fibre measure. Let X and B be topological spaces, μ a Borel measure on X, ν a Borel measure on B, $\pi\colon X\to B$ a Borel measurable map and $\beta\in B$. A Borel measure λ on X is **below the fibre measure** $d\mu/d\nu$ **of** π **over** β if for every lower semicontinuous function $K\colon X\to[0,\infty]$ and every $C\in[0,\infty]$ the following holds: if β has a neighbourhood in B such that

$$\int_{\pi^{-1}(S)} K\,d\mu \le C\,\nu(S) \qquad \text{for every open set } S \text{ of } B \text{ contained in this neighbourhood,}$$

then

$$\int_X K\,d\lambda \le C.$$

(BY2) Below the Gram fibre. Let $N\ge 1$, let $J\in\mathrm{M}_N(\mathbb{C})$ be a hermitian involution and let $m\ge 0$ be an integer. Let Herm_m be the real vector space of hermitian $m\times m$ matrices. The **Gram map** of J is

$$\Gamma_J\colon \mathrm{M}_{N\times m}(\mathbb{C})\to\mathrm{Herm}_m, \qquad \Gamma_J(Y) = Y^*JY.$$

The space $\mathrm{M}_{N\times m}(\mathbb{C})$ carries the measure dY, the Lebesgue measure of the real and imaginary parts of the entries Y_{ab}, and Herm_m carries the measure dT, the Lebesgue measure of the diagonal entries T_{aa} and of the real and imaginary parts of the entries T_{ab} with $a<b$; for $m=0$ both spaces are one point and both measures are the unit point mass. The group $\mathrm{U}(J)$ carries the topology induced from $\mathrm{M}_N(\mathbb{C})$. Let $E\in\mathrm{M}_{N\times m}(\mathbb{C})$. A Borel measure λ on $\mathrm{U}(J)$ is **below the Gram fibre at** E if the image of λ under the map $\mathrm{U}(J)\to\mathrm{M}_{N\times m}(\mathbb{C})$, $g\mapsto gE$, is below the fibre measure dY/dT of Γ_J over $\Gamma_J(E)$ in the sense of (BY1).

Lemma 5.3.50. *Let $F\in\mathrm{M}_{r\times(\varrho+1)}(\mathbb{C})$ be the matrix whose columns are the coordinate vectors of $f_0,\dots,f_\varrho$, let dZ be the Lebesgue measure of the real and imaginary parts of the entries of complex $(\varrho+1)\times(\varrho+1)$ matrices, and let λ be the image of the measure*

$$\mathbf{1}_{\Omega_{\mathrm{cell}}}(Z)\,\frac{2\pi^k}{(k-1)!\,2^{\varrho+1}}\,\mathtt{g}_1(Z)^{k-1}\,dZ\otimes ds$$

under $(Z,s)\mapsto s\gamma_Z$. Then λ is an s-finite Borel measure on G_1 that is below the Gram fibre at F (for $J=J_1$).

Proof. Put $n = \varrho + 1$ and $T_0 = \Gamma_{J_1}(F) = F^* J_1 F = -E_{\varrho\varrho}$ by (5.59). For invertible Z let ζ be the last row of Z^{-1}, so $\mathrm{ic}(Z) = \zeta_\varrho$. All spaces are finite-dimensional real vector spaces or closed subsets of them, hence separable and metrizable.

Step 1: three changes of variables. (a) For $Z \in \mathrm{GL}_n(\mathbb{C})$, the map $p \mapsto pZ$ of $\mathrm{M}_{(\varrho+k)\times n}(\mathbb{C})$ multiplies each of the $\varrho + k$ rows on the right by Z, a complex-linear map of $\mathbb{C}^n$ of real determinant $|\det Z|^2$; so its real determinant is $|\det Z|^{2(\varrho+k)}$. (b) For $A \in \mathrm{GL}_n(\mathbb{C})$ let $c_A(T) = A^* T A$ on Herm_n. Since $c_{AB} = c_B \circ c_A$, $A \mapsto |\det c_A|$ is a homomorphism $\mathrm{GL}_n(\mathbb{C}) \to \mathbb{R}_{>0}$, trivial on $\mathrm{SL}_n(\mathbb{C})$, which is its own commutator subgroup. Writing $A = A' \operatorname{diag}(d, 1, \dots, 1)$ with $A' \in \mathrm{SL}_n(\mathbb{C})$ and $d = \det A$: for $\operatorname{diag}(d, 1, \dots, 1)$, $T_{00} \mapsto |d|^2 T_{00}$, $T_{0b} \mapsto \bar{d} T_{0b}$ ($b > 0$, real determinant $|d|^2$ each), the other coordinates are fixed; so $|\det c_A| = |\det A|^{2n}$ for the coordinates of dT. (c) The columns of $J_1 F$ are $e_1^*, \dots, e_\varrho^*, -z$. So the map $\Theta(Z, p) = J_1 F Z + Rp$, $\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_{(\varrho+k)\times n}(\mathbb{C}) \to \mathrm{M}_{r\times n}(\mathbb{C})$, puts the rows $0, \dots, \varrho - 1$ of Z into the rows of $e_1^*, \dots, e_\varrho^*$, minus row ϱ of Z into the row of z, and the rows of p into the rows of $e_1, \dots, e_\varrho, x_1, \dots, x_k$. These row sets partition the r rows. So Θ permutes the real coordinates up to sign and preserves Lebesgue measure.

Step 2: tuples with given products with the frame. Let $Z \in \Omega_{\mathrm{cell}}$, $K \colon \mathrm{M}_{r\times n}(\mathbb{C}) \to [0, \infty]$ Borel, and $T' = (Z^{-1})^* T Z^{-1}$ for $T \in \mathrm{Herm}_n$. We claim

$$\int K(J_1 F Z + Rp)\, dp = \frac{\pi^k |\det Z|^{2(k-1)}}{2^\varrho (k-1)!} \int_{\mathrm{Herm}_n} \mathbf{1}_{\{T'_{\varrho\varrho} > -1\}} (1 + T'_{\varrho\varrho})^{k-1} \int_{G_1} K\big(sY^\vee(T')Z\big)\, ds\, dT. \tag{5.61}$$

By (a), substituting $p = p'Z$, the left side is $|\det Z|^{2(\varrho+k)} \int K\big((J_1 F + Rp')Z\big)\, dp'$. By Lemma 5.3.51 (ii), applied to $Y \mapsto K(YZ)$, this is $|\det Z|^{2(\varrho+k)} \int G(T')\, dT'$ with $G(T') = \frac{\pi^k}{2^\varrho (k-1)!} \mathbf{1}_{\{T'_{\varrho\varrho} > -1\}} (1 + T'_{\varrho\varrho})^{k-1} \int K(sY^\vee(T')Z)\, ds$. By (b) for $A = Z^{-1}$, $\int G(T')\, dT' = |\det Z|^{-2n} \int G\big((Z^{-1})^* T Z^{-1}\big)\, dT$, and $2(\varrho + k) - 2n = 2(k-1)$.

Step 3: a parametrisation of the fibres. Apply Lemma 5.5.13 (i) to $\pi = \Gamma_{J_1}$ on the space $\mathrm{M}_{r\times n}(\mathbb{C})$ (in the role of the space X there) with $\mu = dY$, $B = V = \mathrm{Herm}_n$ with $\nu = dT$, $\beta = T_0$, and $P = \mathrm{M}_n(\mathbb{C}) \times G_1$ with $\mu_P = \mathbf{1}_{\Omega_{\mathrm{cell}}}(Z)\, dZ \otimes ds$, an s-finite Borel measure. For $(T, (Z, s)) \in B \times P$ let $T' = (Z^{-1})^* T Z^{-1}$ if $\det Z \neq 0$ and $T' = 0$ otherwise, and

$$\psi(T, (Z, s)) = sY^\vee(T')Z, \qquad w(T, (Z, s)) = \mathbf{1}_{\Omega_{\mathrm{cell}}}(Z)\, \frac{\pi^k |\det Z|^{2(k-1)}}{2^\varrho (k-1)!}\, \mathbf{1}_{\{T'_{\varrho\varrho} > -1\}} (1 + T'_{\varrho\varrho})^{k-1};$$

both are Borel (Ω_{cell} is open by Lemma 5.3.47 (i), inversion is continuous on $\mathrm{GL}_n(\mathbb{C})$, and $Y^\vee$ is continuous). The hypotheses of (i): dT is positive on nonempty open sets and finite on bounded ones. If $w(T, (Z, s)) \neq 0$, then $Z \in \Omega_{\mathrm{cell}}$ and $T'_{\varrho\varrho} > -1$, so by Lemma 5.3.51 (i) and since s preserves J_1, $\Gamma_{J_1}(\psi(T, (Z, s))) = Z^* \Gamma_{J_1}(Y^\vee(T'))Z = Z^* T' Z = T$. For $Z \in \Omega_{\mathrm{cell}}$ (so for μ_P-almost every point), $T \mapsto \psi(T, (Z, s))$ is continuous, and at $T = T_0$ one has $T' = -(Z^{-1})^* E_{\varrho\varrho} Z^{-1} = -\zeta^* \zeta$, so $T'_{\varrho\varrho} = -|\mathrm{ic}(Z)|^2 > -1$; the indicator is then 1 near T_0 and $T \mapsto w(T, (Z, s))$ is continuous at T_0. Finally, for Borel $K \geq 0$, by Tonelli's theorem, (5.61) and Step 1 (c),

$$\int_V \int_P w\, K(\psi)\, d\mu_P\, dT = \int_{\Omega_{\mathrm{cell}}} \int K(J_1 F Z + Rp)\, dp\, dZ \leq \int K(\Theta(Z, p))\, d(Z, p) = \int K\, dY.$$

Hence the image of $w(T_0, \cdot)\, \mu_P$ under $\psi(T_0, \cdot)$ is below the fibre measure dY/dT of Γ_{J_1} over $T_0 = \Gamma_{J_1}(F)$.

Step 4: the point and the density at T_0. Let $Z \in \Omega_{\mathrm{cell}}$. Then $\eta(Z)^2 = 1 - |\zeta_\varrho|^2 = 1 + T'_{\varrho\varrho}$ at $T = T_0$, and $\mathsf{g}_1(Z) = |\det Z|^2 (1 - |\mathrm{ic}(Z)|^2)$ (proof of Lemma 5.3.47 (i)); so

$$w(T_0, (Z, s)) = \frac{\pi^k}{2^\varrho (k-1)!} \big(|\det Z|^2 \eta(Z)^2\big)^{k-1} = \frac{2\pi^k}{(k-1)!\, 2^{\varrho+1}}\, \mathsf{g}_1(Z)^{k-1},$$

the density in the statement. Next, $Y^\vee(-\zeta^*\zeta) = F^\vee(Z)$: for $b < \varrho$, column b of $F\,\mathrm{Co}(Z) + F'\,\mathrm{dg}(Z)$ is $e^*_{b+1} + \sum_{a<\varrho}\mathrm{Co}(Z)_{ab}e_{a+1}$ with $\mathrm{Co}(Z)_{ab} = -\frac12\overline{\zeta_a}\zeta_b = \frac12(-\zeta^*\zeta)_{ab}$, and column ϱ is $-z + \sum_{a<\varrho}(-\overline{\zeta_a}\zeta_\varrho)e_{a+1} + \eta(Z)x_{k_0}$, with $\eta(Z) = \max(0, 1-|\zeta_\varrho|^2)^{1/2}$; these are the columns of $Y^\vee(-\zeta^*\zeta)$. Since the columns of F and F' are distinct standard basis vectors, $F^*F = 1$ and $F'^*F = 0$, and the formula for γ_Z gives $\gamma_Z F = F + (F^\vee(Z)Z - F) = F^\vee(Z)Z$. Hence $\psi(T_0,(Z,s)) = sF^\vee(Z)Z = s\gamma_Z F$ for $Z \in \Omega_{\mathrm{cell}}$.

Conclusion. The map $(Z,s)\mapsto s\gamma_Z$ is Borel by Lemma 5.3.47 (ii) and the continuity of multiplication, and λ is its image of $w(T_0,\cdot)\,\mu_P$ by Step 4; a measure with a Borel density with respect to an s-finite measure is s-finite, and so is its image, so λ is an s-finite Borel measure on G_1. The image of λ under $g\mapsto gF$ is the image of $w(T_0,\cdot)\,\mu_P$ under $(Z,s)\mapsto s\gamma_Z F$, which agrees μ_P-almost everywhere with $\psi(T_0,\cdot)$ (Step 4), so it is the measure of Step 3, below the fibre measure dY/dT of Γ_{J_1} over $\Gamma_{J_1}(F)$. By (BY2), λ is below the Gram fibre at F. □

Lemma 5.3.51. *Index the rows and columns of matrices in* $\mathrm{Herm}_{\varrho+1}$ *by* $0,\dots,\varrho$. *For* $T\in\mathrm{Herm}_{\varrho+1}$ *let* $Y^\vee(T)\in\mathrm{M}_{r\times(\varrho+1)}(\mathbb{C})$ *be the matrix whose columns are the coordinate vectors of*

$$e^*_{b+1} + \tfrac12\sum_{a<\varrho}T_{ab}\,e_{a+1}\quad(0\le b<\varrho),\qquad -z+\sum_{a<\varrho}T_{a\varrho}\,e_{a+1} + \max(0,1+T_{\varrho\varrho})^{1/2}\,x_{k_0}.$$

(i) $F^*J_1Y^\vee(T) = 1$, *and* $\Gamma_{J_1}(Y^\vee(T)) = T$ *if* $T_{\varrho\varrho}\ge -1$.

(ii) Give $\mathrm{M}_{(\varrho+k)\times(\varrho+1)}(\mathbb{C})$ *the Lebesgue measure* dp *of the real and imaginary parts of its entries. For every Borel function* $K\colon \mathrm{M}_{r\times(\varrho+1)}(\mathbb{C})\to[0,\infty]$,

$$\int K(J_1F+Rp)\,dp = \frac{\pi^k}{2^\varrho(k-1)!}\int_{\mathrm{Herm}_{\varrho+1}}\mathbf{1}_{\{T_{\varrho\varrho}>-1\}}\,(1+T_{\varrho\varrho})^{k-1}\int_{G_1}K\big(sY^\vee(T)\big)\,ds\,dT.$$

Proof. Write $T_1\in\mathrm{Herm}_\varrho$ for the upper left block of T, $c=(T_{a\varrho})_{a<\varrho}\in\mathbb{C}^\varrho$ and $\eta=\max(0,1+T_{\varrho\varrho})^{1/2}$. Then

$$Y^\vee(T) = \big(D + \tfrac12 XT_1 \;\big|\; -z + Xc + \eta\,x_{k_0}\big), \tag{5.62}$$

the first block having ϱ columns.

(i) For $a<\varrho$, $h(e_{a+1},y)$ is the coefficient of e^*_{a+1} in y, and $h(z,y)$ is minus the coefficient of z. So $h(f_a, Y^\vee(T)_b) = \delta_{ab}$ for all a,b, that is $F^*J_1Y^\vee(T) = 1$. The Gram matrix of the first ϱ columns is T_1 by Lemma 5.3.52 (i) (with $v=1$). The e_a are isotropic and orthogonal to z and X_0, $z\perp X_0$, and $h(e^*_{b+1},e_{a+1}) = \delta_{ab}$, $h(e^*_{b+1},z) = h(e^*_{b+1},x_{k_0}) = 0$; hence for $b<\varrho$ the product of column b with the last column is $c_b = T_{b\varrho}$, and the last column has h-square $h(z,z)+\eta^2h(x_{k_0},x_{k_0}) = -1+\eta^2$, which is $T_{\varrho\varrho}$ when $T_{\varrho\varrho}\ge -1$. As $\Gamma_{J_1}(Y^\vee(T))$ and T are hermitian, they agree.

The action of the frame fixer on a last column. Let $v = u(a,b)\in\mathsf{U}^0$ and $g\in H$, the latter acting by a unitary $k\times k$ matrix, also written g, on the coordinates of X_0 and trivially on e_l, z, e_l^*. From the formula for $u(a,b)$ and $W^*W=1$, $D^*W=0$, $vW = W+Xa$; also $vX=X$ and $vz=z$ ((5.66) and v fixes the frame). Hence, for $c\in\mathbb{C}^\varrho$ and $x\in\mathbb{C}^k$,

$$vg\,(-z+Xc+Wx) = -z + X(c+a\,gx) + W\,gx,\qquad vg\,(D+\tfrac12XT_1) = v(D+\tfrac12XT_1). \tag{5.63}$$

(ii) Write $p=(p'\mid q)$ with $p'\in\mathrm{M}_{(\varrho+k)\times\varrho}(\mathbb{C})$ and last column $q=\binom{c}{x}$, $c\in\mathbb{C}^\varrho$, $x\in\mathbb{C}^k$. The columns of J_1F are $e_1^*,\dots,e_\varrho^*,-z$, so $J_1F+Rp = (D+Rp'\mid -z+Xc+Wx)$. By Tonelli's theorem and Lemma 5.3.52 (ii) applied to the Borel function $L(Y') = \int K(Y'\mid -z+Xc+Wx)\,dc\,dx$,

$$\int K(J_1F+Rp)\,dp = \int L(D+Rp')\,dp' = 2^{-\varrho}\int_{\mathrm{Herm}_\varrho}\int_{\mathsf{U}^0}L\big(v(D+\tfrac12XT_1)\big)\,dv\,dT_1. \tag{5.64}$$

Fix T_1 and $v = u(a,b)$, and put $Y' = v(D + \frac{1}{2}XT_1)$. Translating c by ax for fixed x, $L(Y') = \int_{\mathbb{C}^k}\int_{\mathbb{C}^\varrho} K\big(Y' \mid -z + X(c+ax) + Wx\big)\,dc\,dx$. Polar coordinates in $\mathbb{C}^k$: Lebesgue measure is $\rho^{2k-1}d\rho\,d\sigma$, with σ the surface measure of the unit sphere, of total mass $2\pi^k/(k-1)!$; the normalised σ is the unique $\mathrm{U}(k)$-invariant Borel probability measure on the sphere, on which $\mathrm{U}(k)$ acts transitively, so it is the image of the Haar probability measure dg under $g \mapsto g\varepsilon_{k_0}$, ε_{k_0} the k_0-th unit vector; with $t = \rho^2$, so that $\rho^{2k-1}d\rho = \frac{1}{2}t^{k-1}dt$,

$$\int_{\mathbb{C}^k}\phi(x)\,dx = \frac{\pi^k}{(k-1)!}\int_0^\infty t^{k-1}\int_H \phi(\sqrt{t}\,g\varepsilon_{k_0})\,dg\,dt \qquad (\phi \geq 0 \text{ Borel}).$$

Apply it to the inner c-integral as a function of x, then substitute $t = 1+\tau$, and use (5.63) with $x = \sqrt{1+\tau}\,\varepsilon_{k_0}$ (so $Wx = \sqrt{1+\tau}\,x_{k_0}$) and Tonelli:

$$L(Y') = \frac{\pi^k}{(k-1)!}\int_{\mathbb{C}^\varrho}\int_{\mathbb{R}} \mathbf{1}_{\{\tau>-1\}}(1+\tau)^{k-1} \int_H K\Big(vg\big(D + \tfrac{1}{2}XT_1 \mid -z + Xc + \max(0,1+\tau)^{1/2}x_{k_0}\big)\Big)\,dg\,d\tau\,dc.$$

By (5.62) the argument of K is $vg\,Y^\vee(T)$, where $T \in \mathrm{Herm}_{\varrho+1}$ has upper left block T_1, entries $T_{a\varrho} = c_a$ above the corner and corner $T_{\varrho\varrho} = \tau$. The map $(T_1, c, \tau) \mapsto T$ is a bijection $\mathrm{Herm}_\varrho \times \mathbb{C}^\varrho \times \mathbb{R} \to \mathrm{Herm}_{\varrho+1}$ carrying $dT_1\,dc\,d\tau$ to dT, since the coordinates of dT (diagonal entries, real and imaginary parts of the entries above the diagonal) are those of T_1, c and τ. Inserting this into (5.64), using Tonelli's theorem once more and $\int_{\mathsf{U}^0}\int_H K(vgY)\,dg\,dv = \int_{G_1} K(sY)\,ds$ (definition of ds as the image of $dv \otimes dg$ under $(v,g) \mapsto vg$), we obtain (ii). □

Lemma 5.3.52. *Let $F \in \mathrm{M}_{r\times(\varrho+1)}(\mathbb{C})$ be the matrix whose columns are the coordinate vectors of $f_0, \dots, f_\varrho$, let $R = (X \mid W) \in \mathrm{M}_{r\times(\varrho+k)}(\mathbb{C})$ be the matrix with the columns of X followed by those of W, and give $\mathrm{M}_{(\varrho+k)\times\varrho}(\mathbb{C})$ the Lebesgue measure dp of the real and imaginary parts of its entries.*

(i) For $T \in \mathrm{Herm}_\varrho$ and $v \in \mathsf{U}^0$ the matrix $Y = v(D + \frac{1}{2}XT)$ satisfies $F^*J_1Y = F^*J_1D$ and $\Gamma_{J_1}(Y) = T$.

(ii) For every Borel function $K\colon \mathrm{M}_{r\times\varrho}(\mathbb{C}) \to [0,\infty]$,

$$\int_{\mathrm{M}_{(\varrho+k)\times\varrho}(\mathbb{C})} K(D+Rp)\,dp = 2^{-\varrho}\int_{\mathrm{Herm}_\varrho}\int_{\mathsf{U}^0} K\big(v(D+\tfrac{1}{2}XT)\big)\,dv\,dT.$$

Proof. The columns of X, W, D are distinct standard basis vectors, so $X^*X = W^*W = D^*D = 1$ and $X^*W = X^*D = W^*D = 0$. From the Gram matrix J_1 (e_a pairs with e_a^*, J_1 is the identity on X_0, $h(z,z) = -1$) we have $J_1X = D$, $J_1D = X$, $J_1W = W$, hence

$$X^*J_1X = D^*J_1D = X^*J_1W = 0, \qquad X^*J_1D = D^*J_1X = 1, \qquad F^*J_1X = 0, \tag{5.65}$$

the last because $h(e_a, e_b) = h(z, e_b) = 0$.

The action of U^0. Let $(a,b) \in \mathrm{M}_{\varrho\times k}(\mathbb{C}) \times \mathrm{M}_\varrho(\mathbb{R})$ and $u = u(a,b) = 1 + XaW^* - Wa^*D^* + X\big(\mathrm{sk}(b) - \frac{1}{2}aa^*\big)D^*$. Since $W^*X = D^*X = 0$, $uX = X$; since $W^*D = 0$ and $D^*D = 1$, $uD = D - Wa^* + X\big(\mathrm{sk}(b) - \frac{1}{2}aa^*\big)$. Hence for every $\alpha \in \mathrm{M}_\varrho(\mathbb{C})$

$$u(a,b)\,(D + X\alpha) = D + X\big(\mathrm{sk}(b) + \alpha - \tfrac{1}{2}aa^*\big) - Wa^*. \tag{5.66}$$

(i) An element $v \in \mathsf{U}^0$ fixes $f_0, \dots, f_\varrho$ and preserves h, so $h(f_a, vy) = h(v^{-1}f_a, y) = h(f_a, y)$ for all y; that is, $F^*J_1vY_0 = F^*J_1Y_0$ and $\Gamma_{J_1}(vY_0) = \Gamma_{J_1}(Y_0)$ for every $Y_0 \in \mathrm{M}_{r\times\varrho}(\mathbb{C})$. For $Y_0 = D + \frac{1}{2}XT$, (5.65) gives $F^*J_1Y_0 = F^*J_1D$ and, as $T^* = T$,

$$\Gamma_{J_1}(Y_0) = D^*J_1D + \tfrac{1}{2}D^*J_1XT + \tfrac{1}{2}T^*X^*J_1D + \tfrac{1}{4}T^*X^*J_1XT = \tfrac{1}{2}T + \tfrac{1}{2}T = T.$$

A linear change of variables. Let $\Phi\colon \mathrm{M}_\varrho(\mathbb{R}) \times \mathrm{Herm}_\varrho \to \mathrm{M}_\varrho(\mathbb{C})$, $\Phi(b,T) = \mathrm{sk}(b) + \frac{1}{2}T$. We claim that Φ is a real-linear bijection with $|\det\Phi| = 2^{-\varrho}$ for the coordinates of db (the entries of b), dT and the Lebesgue measure $d\alpha$ of $\mathrm{M}_\varrho(\mathbb{C})$. The map decomposes into blocks. For each l, $(b_{ll}, T_{ll}) \mapsto \Phi(b,T)_{ll} = \mathrm{i}b_{ll} + \frac{1}{2}T_{ll}$, with real part $\frac{1}{2}T_{ll}$ and imaginary part b_{ll}: determinant of absolute value $\frac{1}{2}$. For each pair $l < l'$, put $\xi = \mathrm{sk}(b)_{ll'} = \big((1+\mathrm{i})b_{ll'} - (1-\mathrm{i})b_{l'l}\big)/\sqrt{2}$; then $\mathrm{sk}(b)_{l'l} = -\bar{\xi}$, and since $(1+\mathrm{i})\overline{(1-\mathrm{i})} = 2\mathrm{i}$ is purely imaginary, $|\xi|^2 = b_{ll'}^2 + b_{l'l}^2$, so $(b_{ll'}, b_{l'l}) \mapsto \xi$ is a linear isometry $\mathbb{R}^2 \to \mathbb{C}$. With $\tau = T_{ll'}$ the block sends (ξ,τ) to $(\Phi_{ll'}, \overline{\Phi_{l'l}}) = (\xi + \frac{1}{2}\tau,\ -\xi + \frac{1}{2}\tau)$, a complex-linear map of $\mathbb{C}^2$ of determinant 1, hence of real determinant 1, followed by complex conjugation in the second coordinate, of real determinant -1. So each pair block has determinant of absolute value 1, and $|\det\Phi| = 2^{-\varrho}$. Consequently, for Borel $g\colon \mathrm{M}_\varrho(\mathbb{C}) \to [0,\infty]$,

$$\int_{\mathrm{M}_\varrho(\mathbb{C})} g(\alpha)\, d\alpha = 2^{-\varrho} \int_{\mathrm{Herm}_\varrho} \int_{\mathrm{M}_\varrho(\mathbb{R})} g\big(\mathrm{sk}(b) + \tfrac{1}{2}T\big)\, db\, dT. \tag{5.67}$$

(ii) Write $p = \binom{\alpha}{c}$ with $\alpha \in \mathrm{M}_\varrho(\mathbb{C})$ its first ϱ rows and $c \in \mathrm{M}_{k\times\varrho}(\mathbb{C})$ the others; then $Rp = X\alpha + Wc$ and $dp = d\alpha\, dc$. The map $a \mapsto -a^*$, $\mathrm{M}_{\varrho\times k}(\mathbb{C}) \to \mathrm{M}_{k\times\varrho}(\mathbb{C})$, permutes the real coordinates up to sign and so preserves Lebesgue measure. By Tonelli's theorem, the invariance of $d\alpha$ under the translation $\alpha \mapsto \alpha - \frac{1}{2}aa^*$ and (5.67),

$$\begin{aligned}\int K(D+Rp)\, dp &= \int_{\mathrm{M}_{\varrho\times k}(\mathbb{C})} \int_{\mathrm{M}_\varrho(\mathbb{C})} K\big(D + X\big(\alpha - \tfrac{1}{2}aa^*\big) - Wa^*\big)\, d\alpha\, da \\ &= 2^{-\varrho} \int_{\mathrm{Herm}_\varrho} \int_{\mathrm{M}_{\varrho\times k}(\mathbb{C})\times\mathrm{M}_\varrho(\mathbb{R})} K\big(D + X\big(\mathrm{sk}(b) + \tfrac{1}{2}T - \tfrac{1}{2}aa^*\big) - Wa^*\big)\, d(a,b)\, dT.\end{aligned}$$

By (5.66) with $\alpha = \frac{1}{2}T$ the integrand is $K\big(u(a,b)(D + \frac{1}{2}XT)\big)$, and as dv is the image of the Lebesgue measure $d(a,b)$ under u, the inner integral is $\int_{\mathsf{U}^0} K\big(v(D + \frac{1}{2}XT)\big)\, dv$. All integrands are compositions of K with continuous maps, hence Borel. □

Lemma 5.3.53. *Let $F \in \mathrm{M}_{r\times(\varrho+1)}(\mathbb{C})$ be the matrix whose columns are the coordinate vectors of $f_0, \dots, f_\varrho$, and let m be an s-finite Borel measure on G_1 that is below the Gram fibre at F (for $J = J_1$). Then for every Borel set $B \subseteq G_1$*

$$(m * ds)(B) \le \frac{\mathrm{vol}\,\mathrm{U}(r)}{\mathrm{vol}\,\mathrm{U}(k)\, 2^{\varrho+1}} \int_{G_1} \mathbf{1}_B(y)\, dy.$$

Proof. Let E^* be the matrix of the coordinate vectors of $e_1^*, \dots, e_\varrho^*$ and $\nu = 2^{-\varrho}\, dv$, a σ-finite Borel measure on G_1 carried by U^0.

Step 1: $m * \nu$ is below the Gram fibre at $(F\ E^*)$. Every $v \in \mathsf{U}^0$ fixes $f_0, \dots, f_\varrho$, so $gF = F$ for ν-almost every g; and the image of ν under $g \mapsto gE^*$ is below the fibre measure $dY/(dM \otimes dT)$ of $Y \mapsto (F^*J_1Y, \Gamma_{J_1}(Y))$ over the value at E^*, by Lemma 5.3.54. As m is below the Gram fibre at F, Lemma 5.5.12 (i) with $E_1 = F$, $E_2 = E^*$, $\lambda = m$, $\lambda' = \nu$ shows that $m * \nu$ is below the Gram fibre at the matrix $(F\ E^*)$ with the columns $e_1, \dots, e_\varrho, z, e_1^*, \dots, e_\varrho^*$.

Step 2: the definite complement. Reordering the columns to $e_1, e_1^*, \dots, e_\varrho, e_\varrho^*, z$ is right multiplication by a permutation matrix, so by Lemma 5.5.12 (ii) $m * \nu$ is below the Gram fibre

at the matrix E^{out} of Lemma 5.5.16. The measure $m * \nu$ is s-finite (the image of the s-finite measure $m \otimes \nu$), so Lemma 5.5.16 gives, with $\mu_H = dh$,

$$((m * \nu) * dh)(B) \le \frac{\operatorname{vol} \mathrm{U}(r)}{\operatorname{vol} \mathrm{U}(k)\, 2^{2\varrho+1}} \int_{G_1} \mathbf{1}_B \, dy \qquad (B \subseteq G_1 \text{ Borel}).$$

Step 3: conclusion. Convolution of s-finite measures is associative (Tonelli for the image of $m \otimes \nu \otimes dh$ under $(g, g', g'') \mapsto gg'g''$) and bilinear, so $(m*\nu)*dh = m*(\nu*dh) = 2^{-\varrho}\, m*(dv*dh) = 2^{-\varrho}\, m * ds$. Multiplying the display by 2^{ϱ} gives the claim, as $2^{\varrho}/2^{2\varrho+1} = 1/2^{\varrho+1}$. □

Lemma 5.3.54. *Let $F \in \mathrm{M}_{r\times(\varrho+1)}(\mathbb{C})$ and $E^* \in \mathrm{M}_{r\times\varrho}(\mathbb{C})$ be the matrices whose columns are the coordinate vectors of $f_0, \dots, f_\varrho$, respectively of $e_1^*, \dots, e_\varrho^*$, give $\mathrm{M}_{(\varrho+1)\times\varrho}(\mathbb{C})$ the Lebesgue measure dM of the real and imaginary parts of its entries, and let*

$$\pi \colon \mathrm{M}_{r\times\varrho}(\mathbb{C}) \to \mathrm{M}_{(\varrho+1)\times\varrho}(\mathbb{C}) \times \mathrm{Herm}_\varrho, \qquad \pi(Y) = \big(F^* J_1 Y,\ \Gamma_{J_1}(Y)\big).$$

Then the image of the measure $2^{-\varrho}\, dv$ under $\mathsf{U}^0 \to \mathrm{M}_{r\times\varrho}(\mathbb{C})$, $v \mapsto vE^*$, is below the fibre measure $dY/(dM \otimes dT)$ of π over $\pi(E^*)$.

Proof. Write $D = E^*$, $R = (X \mid W)$ and $M_0 = F^* J_1 D \in \mathrm{M}_{(\varrho+1)\times\varrho}(\mathbb{C})$; then $\pi(D) = (M_0, 0)$, since $D^* J_1 D = 0$ by (5.65). All spaces below are finite-dimensional real vector spaces, hence separable and metrizable, and all measures are Lebesgue measures or images of them, hence s-finite.

Products with the frame. The columns of F and of R are distinct standard basis vectors, so $F^*F = 1$ and $R^*R = 1$, and $F^*J_1R = 0$ because $h(f_a, e_b) = h(f_a, x_i) = 0$ for every a, b, i (the e_b are isotropic and orthogonal to z, and $X_0 \perp e_b, z$). The matrix J_1F has the columns $e_1^*, \dots, e_\varrho^*, -z$, and $(M_0)_{ab} = h(f_a, e_{b+1}^*)$ is δ_{ab} for $a < \varrho$ and 0 for $a = \varrho$; hence $J_1 F M_0 = D$. By Lemma 5.5.20 (with $J = J_1$, $N = r$, $E = F$, $n_1 = \varrho + 1$, $n_2 = \varrho + k$, so that $n_1 + n_2 = r$, $n = \varrho$, $M = M_0$), the measure μ_0, the image of the Lebesgue measure dp of $\mathrm{M}_{(\varrho+k)\times\varrho}(\mathbb{C})$ under $p \mapsto D + Rp$, is below the fibre measure dY/dM of $\pi_1 \colon Y \mapsto F^* J_1 Y$ over M_0.

Gram matrices. Apply Lemma 5.5.13 (i) with $X = \mathrm{M}_{r\times\varrho}(\mathbb{C})$, $\mu = \mu_0$, $B = V = \mathrm{Herm}_\varrho$, $\nu = dT$, $\beta = 0$, $P = \mathrm{M}_{\varrho\times k}(\mathbb{C}) \times \mathrm{M}_\varrho(\mathbb{R})$ with Lebesgue measure, $\psi(T, (a, b)) = u(a, b)(D + \frac{1}{2}XT)$ and $w \equiv 2^{-\varrho}$. Its hypotheses hold: dT is positive on nonempty open sets and finite on bounded ones; ψ and w are continuous; $\Gamma_{J_1}(\psi(T, p)) = T$ for all (T, p) by Lemma 5.3.52 (i); and for every Borel $K \ge 0$, by Lemma 5.3.52 (ii) and since dv is the image of the Lebesgue measure of P under u,

$$\int_V \int_P w\, K(\psi(T,p))\, dp\, dT = 2^{-\varrho} \int_{\mathrm{Herm}_\varrho} \int_{\mathsf{U}^0} K\big(v(D + \tfrac{1}{2}XT)\big)\, dv\, dT = \int K(D + Rp)\, dp = \int K\, d\mu_0.$$

Hence the image of $2^{-\varrho}$ times Lebesgue measure on P under $(a, b) \mapsto \psi(0, (a, b)) = u(a, b)D$, which is the image λ' of $2^{-\varrho} dv$ under $v \mapsto vD$, is below the fibre measure $d\mu_0/dT$ of Γ_{J_1} over 0.

Both together. The map Γ_{J_1} is continuous and dM, dT are s-finite. By Lemma 5.5.13 (iv), applied to $\lambda = \mu_0$ (below dY/dM of π_1 over M_0) and to λ' (below $d\mu_0/dT$ of Γ_{J_1} over 0), the measure λ' is below the fibre measure $dY/(dM \otimes dT)$ of $Y \mapsto (\pi_1(Y), \Gamma_{J_1}(Y)) = \pi(Y)$ over $(M_0, 0) = \pi(E^*)$. □

Lemma 5.3.55. *Let $\mathcal{D}$, L and I be finite sets, (Ω, σ) a σ-finite measure space, $0 < \varepsilon \le 1$, $\Lambda \ge 0$ and $K \in [0, \infty]$. For every word i (finite sequence) in the alphabet L let $f_i \colon \mathbb{R}^{\mathcal{D}} \times \Omega \to \mathbb{C}^I$ be measurable; li denotes the word with first letter l followed by i, $|i|$ the length of i, $\varnothing$ the empty word, $x[d \mapsto t]$ the point x with its coordinate d replaced by t, and $C = [0, \varepsilon]^{\mathcal{D}}$. Assume:*

(a) for σ-almost every ω, for every word i, every $d \in \mathcal{D}$, every $x \in \mathbb{R}^{\mathcal{D}}$ and every $t \in \mathbb{R}$ with $x[d \mapsto t] \in C$, the function $s \mapsto f_i(x[d \mapsto s], \omega)$ is differentiable at t, with derivative of norm at most $\Lambda \sum_{l \in L} \|f_{li}(x[d \mapsto t], \omega)\|$;

(b) $\int_C \int_\Omega \|f_i(x,\omega)\|^2 \, d\sigma(\omega)\, dx \le K$ for every word i with $|i| \le \#\mathcal{D}$.

Then

$$\int_\Omega \|f_\varnothing(0,\omega)\|^2 \, d\sigma(\omega) \le \Big(\frac{2(1+(\#L\,\Lambda)^2)}{\varepsilon}\Big)^{\#\mathcal{D}} K.$$

Proof. Norms are Euclidean; dx and dt are Lebesgue measures.

One variable. Let $g\colon \mathbb{R} \to \mathbb{C}^I$ be differentiable at every point of $[0,\varepsilon]$. We show

$$\|g(0)\|^2 \le \frac{2}{\varepsilon} \int_0^\varepsilon \big(\|g(t)\|^2 + \|g'(t)\|^2\big)\, dt. \tag{5.68}$$

The derivative g' is Borel on $[0,\varepsilon]$, as a pointwise limit of difference quotients of the continuous function g. Put $\varphi(t) = \|g(t)\|^2 (t-\varepsilon)/\varepsilon$; it is differentiable at every point of $[0,\varepsilon]$ with $\varphi'(t) = \big(2\operatorname{Re}\langle g(t), g'(t)\rangle (t-\varepsilon) + \|g(t)\|^2\big)/\varepsilon$. As $|t-\varepsilon| \le \varepsilon \le 1$ on $[0,\varepsilon]$, the Cauchy–Schwarz inequality gives

$$|\varphi'(t)| \le \frac{2\|g(t)\|\,\|g'(t)\| + \|g(t)\|^2}{\varepsilon} \le \frac{(\|g(t)\| + \|g'(t)\|)^2}{\varepsilon} \le \frac{2}{\varepsilon}\big(\|g(t)\|^2 + \|g'(t)\|^2\big).$$

If $\int_0^\varepsilon |\varphi'| = \infty$, (5.68) holds trivially. Otherwise φ' is integrable on $[0,\varepsilon]$ and φ is differentiable at every point, so $\varphi(\varepsilon) - \varphi(0) = \int_0^\varepsilon \varphi'$ (Rudin, *Real and Complex Analysis*, Theorem 7.21); as $\varphi(\varepsilon) = 0$ and $\varphi(0) = -\|g(0)\|^2$, $\|g(0)\|^2 \le \int_0^\varepsilon |\varphi'|$, and (5.68) follows.

One coordinate, integrated. Let ω lie in the set of full measure of (a), i a word, $d \in \mathcal{D}$ and $x \in C$ with $x_d = 0$. Then $g(s) = f_i(x[d \mapsto s], \omega)$ satisfies the hypothesis of (5.68), $g(0) = f_i(x,\omega)$, and $\|g'(t)\|^2 \le \#L\,\Lambda^2 \sum_l \|f_{li}(x[d \mapsto t], \omega)\|^2$ by (a) and the Cauchy–Schwarz inequality. Hence

$$\|f_i(x,\omega)\|^2 \le \frac{2}{\varepsilon} \int_0^\varepsilon \Big(\|f_i(x[d \mapsto t],\omega)\|^2 + \#L\,\Lambda^2 \sum_{l \in L} \|f_{li}(x[d \mapsto t],\omega)\|^2\Big) dt. \tag{5.69}$$

Induction over the coordinates. For $S \subseteq \mathcal{D}$ and $x \in [0,\varepsilon]^S$ let $\bar{x} \in \mathbb{R}^{\mathcal{D}}$ have the coordinates of x in S and 0 outside S, and put $N_i(S) = \int_{[0,\varepsilon]^S} \int_\Omega \|f_i(\bar{x},\omega)\|^2 \, d\sigma\, dx$ (measurable integrands; so $N_i(\varnothing) = \int_\Omega \|f_i(0,\omega)\|^2 d\sigma$ and $N_i(\mathcal{D})$ is the integral in (b)). Let $c = 2(1+(\#L\,\Lambda)^2)/\varepsilon$. We prove by induction on m: *if $\#S = m$, $M \in [0,\infty]$ and $N_i(S) \le M$ for all words i with $|i| \le m$, then $N_\varnothing(\varnothing) \le c^m M$.* For $m = 0$ this is the hypothesis for $i = \varnothing$. Let $\#S = m+1$, pick $d \in S$ and put $S_0 = S \setminus \{d\}$. For $x \in [0,\varepsilon]^{S_0}$ the point $\bar{x}$ (formed for S_0) lies in C and has $\bar{x}_d = 0$. Integrating (5.69) over ω (the exceptional set is null) and over $x \in [0,\varepsilon]^{S_0}$, and using Tonelli's theorem to combine the integrals over x and t into one over $[0,\varepsilon]^S$, we get for every word i with $|i| \le m$

$$N_i(S_0) \le \frac{2}{\varepsilon}\Big(N_i(S) + \#L\,\Lambda^2 \sum_{l \in L} N_{li}(S)\Big) \le \frac{2}{\varepsilon}\big(M + (\#L\,\Lambda)^2 M\big) = cM,$$

since $|i| \le m+1$ and $|li| \le m+1$. The induction hypothesis for S_0 and cM gives $N_\varnothing(\varnothing) \le c^m \cdot cM$. Applying the statement with $S = \mathcal{D}$ and $M = K$, which is allowed by (b), proves the lemma. □

Lemma 5.3.56. *Let Z be a complex $(\varrho+1) \times (\varrho+1)$ matrix.*

(i) If $M \ge n$, $|Z_{ab}| \le M^2$ for all a, b and $\mathfrak{g}_1(Z) \ge M^{-1}$, then Z is bounded in the cell by every $T \ge M^{8n}$.

(ii) If $T \ge 8$, $T \ge n$ and Z is bounded in the cell by T, then every Y with $\|Y - Z\| < T^{-6}$ is bounded in the cell by $2T$ and satisfies $\mathfrak{g}_1(Y) \ge \mathfrak{g}_1(Z)/2$.

(iii) If $M \ge 8n$, $|Z_{ab}| \le M^2$ for all a, b, $\mathfrak{g}_1(Z) \ge M^{-1}$ and $P \ge M^{9n}$, then every Y with $\|Y - Z\| < P^{-6}$ is bounded in the cell by P and satisfies $\mathfrak{g}_1(Y) \ge \mathfrak{g}_1(Z)/2$.

(iv) The map $Y \mapsto \gamma_Y$ is real differentiable on the open set Ω_{cell}. If $T \ge 4$ and Z is bounded in the cell by T, then $\|\gamma_Z\| \le T^{18}$ and the real derivative of $Y \mapsto \gamma_Y$ at Z has norm at most T^{25}.

Proof. **Step 0: $\mathfrak{g}_1(Y) = |\det Y|^2 \eta(Y)^2$ for invertible Y.** By Cramer's rule $\mathrm{ic}(Y) = (Y^{-1})_{\varrho\varrho} = D_\varrho(Y)/\det Y$, so $|D_\varrho(Y)|^2 = |\det Y|^2 |\mathrm{ic}(Y)|^2$. We use $\|A\| \le (\sum_{a,b} |A_{ab}|^2)^{1/2} \le l \max_{a,b} |A_{ab}|$ for an $l \times l$ matrix A, $|\det A| \le \|A\|^l$, $|A_{ab}| \le \|A\|$, and $\|F\|, \|F'\| \le 1$ (their columns are distinct vectors of an orthonormal basis).

Proof of (i). $M \ge n \ge 1$. Every square submatrix A of Z of size $l \le n$ has $\|A\| \le lM^2 \le M^3$, so $|\det A| \le M^{3l}$; in particular $\|Z\| \le M^3$. As $\mathfrak{g}_1(Z) \ge M^{-1} > 0$, Z is invertible and lies in Ω_{cell}, and by Step 0 $|\det Z|^2 \eta(Z)^2 \ge M^{-1} \ge M^{-2}$, so $|\det Z|\,\eta(Z) \ge M^{-1}$. Since $\eta(Z) \le 1$, $|\det Z|^{-1} \le M$; and $\eta(Z)^{-1} \le M |\det Z| \le M^{3n+1}$. The entries of the adjugate are minors of order ϱ, of modulus $\le M^{3\varrho}$, so $\|Z^{-1}\| = |\det Z|^{-1} \|\mathrm{adj}\, Z\| \le M \cdot nM^{3\varrho} \le M^{3\varrho+2}$. All four bounds are $\le M^{8n} \le T$.

Proof of (ii). Let $\|Y - Z\| < T^{-6}$. Then $\|Z^{-1}(Y - Z)\| \le T^{-5} \le \frac{1}{2}$, so $Y = Z(1 + Z^{-1}(Y - Z))$ is invertible; from $Y^{-1} - Z^{-1} = Y^{-1}(Z - Y)Z^{-1}$ we get $\|Y^{-1}\| \le T + \frac{1}{2}\|Y^{-1}\|$, so $\|Y^{-1}\| \le 2T$ and $\|Y^{-1} - Z^{-1}\| \le 2T \cdot T^{-6} \cdot T = 2T^{-4}$. Hence $|\mathrm{ic}(Y)| \le |\mathrm{ic}(Z)| + 2T^{-4}$ and, as $|\mathrm{ic}(Z)| < 1$, $|\mathrm{ic}(Y)|^2 \le |\mathrm{ic}(Z)|^2 + 8T^{-4}$. With $\eta(Z)^2 \ge T^{-2}$,

$$\eta(Y)^2 \ge \eta(Z)^2 - 8T^{-4} \ge (1 - 8T^{-2})\,\eta(Z)^2 \ge \tfrac{7}{8}\,\eta(Z)^2 \qquad (T \ge 8).$$

So $\eta(Y) > 0$, $Y \in \Omega_{\mathrm{cell}}$, $\eta(Y)^{-1} \le (8/7)^{1/2} T \le 2T$, and $\|Y\| \le T + T^{-6} \le 2T$: Y is bounded in the cell by $2T$. For the discriminant write $Z = (1 - (Y - Z)Y^{-1})Y$; as $\|(Y - Z)Y^{-1}\| \le 2T^{-5}$ and $n \le T$, $|\det Z| \le (1 + 2T^{-5})^n |\det Y| \le e^{2T^{-4}} |\det Y| \le \frac{9}{8} |\det Y|$. By Step 0,

$$\mathfrak{g}_1(Y) = |\det Y|^2 \eta(Y)^2 \ge \Big(\frac{8}{9}\Big)^2 |\det Z|^2 \cdot \frac{7}{8}\,\eta(Z)^2 = \frac{56}{81}\,\mathfrak{g}_1(Z) \ge \frac{1}{2}\,\mathfrak{g}_1(Z).$$

Proof of (iii). $M \ge 8n \ge 8$, so $M^{9n} \ge M \cdot M^{8n} \ge 2M^{8n}$ and $T = P/2 \ge M^{8n}$. By (i), Z is bounded in the cell by T; moreover $T \ge M \ge 8n$, so $T \ge 8$ and $T \ge n$. As $P^{-6} \le T^{-6}$, (ii) gives that Y is bounded in the cell by $2T = P$ and $\mathfrak{g}_1(Y) \ge \mathfrak{g}_1(Z)/2$.

Proof of (iv). On Ω_{cell} the maps $Y \mapsto Y^{-1}$, $Y \mapsto \mathrm{ic}(Y)$ and $Y \mapsto \eta(Y) = (1 - |\mathrm{ic}(Y)|^2)^{1/2}$ (with $1 - |\mathrm{ic}(Y)|^2 > 0$) are real differentiable, hence so are Co, Co′ (polynomials in the entries of the last row and column of Y^{-1} and their conjugates), dg, dg^{-1} and the formula for γ_Y of Definition 5.1.6, which is built from these by sums and products. Let Z be bounded in the cell by $T \ge 4$; write ζ, ζ' for the last row and the last column of Z^{-1} ($\|\zeta\|, \|\zeta'\| \le T$), and let H be a direction with $\|H\| \le 1$, ∂ the derivative at Z in the direction H. Then $\partial(Y^{-1}) = -Z^{-1} H Z^{-1}$ has norm $\le T^2$, so $\|\partial\zeta\|, \|\partial\zeta'\|, |\partial\,\mathrm{ic}| \le T^2$. Off its last row, $\mathrm{Co}(Z)$ is $-\bar{\zeta}^{\top}\zeta$ restricted to the rows $a < \varrho$, with column b multiplied by $c_b \in \{\frac{1}{2}, 1\}$; hence $\|\mathrm{Co}(Z)\| \le 1 + T^2$ and $\|\partial\mathrm{Co}\| \le 2T^3$, and likewise for Co′. Next $\partial\eta = -\mathrm{Re}(\overline{\mathrm{ic}}\,\partial\,\mathrm{ic})/\eta$, so $|\partial\eta| \le T^3$ and $|\partial(\eta^{-1})| = |\partial\eta|/\eta^2 \le T^5$; thus $\|\mathrm{dg}\| \le 1$, $\|\mathrm{dg}^{-1}\| \le T$, $\|\partial\,\mathrm{dg}\| \le T^3$, $\|\partial\,\mathrm{dg}^{-1}\| \le T^5$. For

$F^\vee = F\mathrm{Co} + F'\mathrm{dg}$: $\|F^\vee\| \le 2 + T^2 \le 2T^2$ and $\|\partial F^\vee\| \le 3T^3$. Put $Q = F(Z^{-1})^* - F^\vee Z\,\mathrm{Co}'$, so that $\gamma_Z = 1 + (F^\vee Z - F)F^* + (Q\,\mathrm{dg}^{-1} - F')F'^*$. Then, using $1 + T^2 \le 2T^2$ and $T \ge 4$,

$$\|Q\| \le T + 2T^2 \cdot T \cdot 2T^2 \le 5T^5, \qquad \|\partial Q\| \le T^2 + (3T^4 + 2T^2) \cdot 2T^2 + 2T^3 \cdot 2T^3 \le 13T^6.$$

Hence $\|\gamma_Z\| \le 1 + (2T^3 + 1) + (5T^6 + 1) \le 6T^6 \le T^{18}$, and

$$\|\partial\gamma\| \le \|\partial F^\vee\|\,\|Z\| + \|F^\vee\| + \|\partial Q\|\,\|\mathrm{dg}^{-1}\| + \|Q\|\,\|\partial\,\mathrm{dg}^{-1}\| \le 3T^4 + 2T^2 + 13T^7 + 5T^{10} \le 7T^{10} \le T^{25}.$$

□

Lemma 5.3.57. *Let $M \ge 8n$ and $T \ge M^{9n}$ be reals and Z an invertible complex $(\varrho+1)\times(\varrho+1)$ matrix with $\mathsf{g}_1(Z) \ge M^{-1}$ and $\|Z^{-1}\| \le T$. Let $s \in R$, $v \in \mathsf{U}^0$, $h \in H$ and $z \in \mathrm{Bx}(M)$ with $\ell(s\gamma_Z(vh)^{-1}z) \le M$. Then $\|v^{-1}z\| \le T^5$.*

Proof. Reduction. Put $z' = (vh)^{-1}z$. As vh fixes the frame, $z'^{-1}f_a = z^{-1}vhf_a = z^{-1}f_a$, so $z' \in \mathrm{Bx}(M)$. As R is a group, $\ell(qy) = \ell(y)$ for $q \in R$, so $\ell(\gamma_Z z') = \ell(s\gamma_Z z') \le M$. Finally $v^{-1}z = hz'$ and h is a unitary matrix, so $\|v^{-1}z\| = \|z'\|$. We show $\|z'\| \le T^5$.

A short point. As the infimum is $\le M < M + 1$, there is $q \in R$ with $\|q\gamma_Z z'\| \le M$. Write $q = uh'$ with $u \in \mathsf{U}$, $h' \in H$ (Lemma 5.5.30 (i)) and $w = h'^{-1}uh' \in \mathsf{U}$ (Lemma 5.5.29 (iii)); then $q = h'w$ and $y := w\gamma_Z z'$ satisfies $\|y\| = \|h'y\| = \|q\gamma_Z z'\| \le M$.

The Gram matrix at $w\gamma_Z$. By Lemma 5.3.47 (i), (ii), $Z \in \Omega_{\mathrm{cell}}$ and $Z(\gamma_Z) = Z$. Let $m = (F^*wF)^{-1}$, upper unitriangular by Lemma 5.3.41 (i), and $Z' = Z(w\gamma_Z)$. By Lemma 5.3.41 (ii), (iii), $Z' = m^*Z$ and $\mathsf{g}_1(Z') = \mathsf{g}_1(Z) \ge M^{-1}$. Since $y^{-1}w\gamma_Z = z'^{-1}$, Lemma 5.3.41 (iv) gives $|Z'_{ab}| \le \|y^{-1}f_a\|\,\|z'^{-1}f_b\| \le \|y\|M \le M^2$. By Lemma 5.3.56 (i) ($M \ge n$), Z' is bounded in the cell by every number $\ge M^{8n}$, in particular by T; so $\|Z'^{-1}\| \le T$.

The dual columns of z'^{-1}. As the columns of F and F' are distinct vectors of one orthonormal basis, $F^*F = F'^*F' = 1$ and $F^*F' = F'^*F = 0$; the formula for γ_Z gives $\gamma_Z F = F^\vee(Z)Z$ and $\gamma_Z F'\mathrm{dg}(Z) = F(Z^{-1})^* - \gamma_Z F\,\mathrm{Co}'(Z)$, so

$$F'\mathrm{dg}(Z) = \gamma_Z^{-1}F(Z^{-1})^* - F\,\mathrm{Co}'(Z), \qquad z'^{-1}F'\mathrm{dg}(Z) = y^{-1}wF(Z^{-1})^* - z'^{-1}F\,\mathrm{Co}'(Z),$$

using $(\gamma_Z z')^{-1} = y^{-1}w$. Now $wF = Fm^{-1}$ (Lemma 5.3.41 (i)) and $(Z^{-1})^* = (Z'^{-1}m^*)^* = m(Z'^{-1})^*$, so $wF(Z^{-1})^* = F(Z'^{-1})^*$. Column $a < \varrho$ of F' is e^*_{a+1} and $\mathrm{dg}(Z)_{aa} = 1$; hence

$$z'^{-1}e^*_{a+1} = \sum_{b=0}^{\varrho} \overline{(Z'^{-1})_{ab}}\,y^{-1}f_b + \sum_{b<\varrho} \tfrac{1}{2}(Z^{-1})_{b\varrho}\overline{(Z^{-1})_{a\varrho}}\,z'^{-1}f_b.$$

Entries are bounded by operator norms, $\|y^{-1}f_b\| \le \|y^{-1}\| = \|y\| \le M$ (Lemma 5.3.41 (iv)) and $\|z'^{-1}f_b\| \le M$, so with $M \le T$ and $\varrho + 1 = n \le M/8 \le T/8$,

$$\|z'^{-1}e^*_{a+1}\| \le (\varrho+1)TM + \tfrac{1}{2}\varrho T^2 M \le \tfrac{1}{8}T^3 + \tfrac{1}{16}T^4 \le T^4.$$

Conclusion. Also $\|z'^{-1}f_a\| \le M \le T^4$. By Lemma 5.3.58 with the bound $T^4 \ge 1$, $\|z'\| + 1 \le 4(\varrho+1)T^4 \le 8(\varrho+1)T^4 \le MT^4 \le T^5$. □

Lemma 5.3.58. *Let $M \ge 1$ and $y \in G_1$ with $\|y^{-1}f_a\| \le M$ for $0 \le a \le \varrho$ and $\|y^{-1}e^*_a\| \le M$ for $1 \le a \le \varrho$. Then $\|y\| + 1 \le 4(\varrho+1)M$.*

Proof. Let $A = y^{-1} \in G_1$. By Lemma 5.3.41 (iv), $\|A\| = \|y\| \geq 1$ and $A^{-1} = J_1 A^* J_1$, so $AJ_1A^* = J_1$ and

$$AA^* = AJ_1A^* + A(1-J_1)A^* = J_1 + A(1-J_1)A^*, \qquad \|A\|^2 = \|AA^*\| \leq 1 + \|A(1-J_1)\|\,\|A\|.$$

The matrix $1 - J_1$ sends $x_i \mapsto 0$, $z \mapsto 2z$, $e_a \mapsto e_a - e_a^*$ and $e_a^* \mapsto e_a^* - e_a$. For any matrix B, $\|B\| \leq \sum_q \|Bq\|$, the sum over the basis vectors q (as $B = \sum_q (Bq)q^*$). With $e_a = f_{a-1}$ and $z = f_\varrho$,

$$\|A(1-J_1)\| \leq 2\|Az\| + 2\sum_{a=1}^{\varrho}(\|Ae_a\| + \|Ae_a^*\|) \leq (4\varrho+2)M.$$

As $\|A\| \geq 1$, $\|A\|^2 \leq (1 + (4\varrho+2)M)\|A\|$, so $\|y\| = \|A\| \leq 1 + (4\varrho+2)M$ and $\|y\| + 1 \leq 2 + (4\varrho+2)M \leq (4\varrho+4)M = 4(\varrho+1)M$, using $M \geq 1$. □

Lemma 5.3.59. *Let Φ be a finite set. For $v \in \Phi$ let $(\mathcal{X}_v, m_v)$ be a probability space, $\mathcal{Y}_v$ a measurable space with finite measures μ_v, ν_v, I_v a finite set, and $k_i^v \colon \mathcal{X}_v \times \mathcal{Y}_v \to \mathbb{C}$ $(i \in I_v)$ bounded measurable functions forming an L^2 contraction from $(\mathcal{Y}_v, \mu_v)$ to $(\mathcal{X}_v, m_v)$ with bound ν_v. Then the kernels*

$$k_i(x,z) = \prod_{v\in\Phi} k_{i_v}^v(x_v, z_v) \qquad \Big(i \in \prod_v I_v\Big)$$

on $\prod_v \mathcal{X}_v \times \prod_v \mathcal{Y}_v$ form an L^2 contraction from $(\prod_v \mathcal{Y}_v, \bigotimes_v \mu_v)$ to $(\prod_v \mathcal{X}_v, \bigotimes_v m_v)$ with bound $\bigotimes_v \nu_v$.

Proof. By induction on $\#\Phi$; for $\Phi = \emptyset$ both sides are $|\phi|^2$ at the one point. For the step, write the factors as two: $(\mathcal{X}_1, m_1, \mathcal{Y}_1, \mu_1, \nu_1, I_1, k^1)$ for one place, and $(\mathcal{X}_2, m_2, \mathcal{Y}_2, \mu_2, \nu_2, I_2, k^2)$ for the product over the remaining places, which by the induction hypothesis is again a probability space, finite measures and bounded measurable kernels forming an L^2 contraction. All measures are finite, so Tonelli's and Fubini's theorems apply. Let ϕ be bounded measurable on $\mathcal{Y}_1 \times \mathcal{Y}_2$, $|\phi| \leq c$, and let $|k_i^2| \leq c_2$ for all i. For $x_2 \in \mathcal{X}_2$, $i_2 \in I_2$ and $z_1 \in \mathcal{Y}_1$ put

$$\psi_{x_2,i_2}(z_1) = \int_{\mathcal{Y}_2} \phi(z_1,z_2)\, k_{i_2}^2(x_2,z_2)\, d\mu_2(z_2);$$

it is jointly measurable in (x_2, z_1) and bounded by $c\, c_2\, \mu_2(\mathcal{Y}_2)$. By Fubini's theorem, for $i = (i_1, i_2)$,

$$\int_{\mathcal{Y}_1\times\mathcal{Y}_2} \phi\, k_i(x,\cdot)\, d(\mu_1\otimes\mu_2) = \int_{\mathcal{Y}_1} \psi_{x_2,i_2}(z_1)\, k_{i_1}^1(x_1,z_1)\, d\mu_1(z_1).$$

Hence, by Tonelli's theorem, the contraction property at the first factor applied to the bounded measurable ψ_{x_2,i_2}, Tonelli's theorem again, and the contraction property at the second factor applied to $\phi(z_1,\cdot)$,

$$\begin{aligned}
\int \sum_i \Big|\int \phi\, k_i(x,\cdot)\, d\mu\Big|^2 dm(x) &= \int_{\mathcal{X}_2} \sum_{i_2} \int_{\mathcal{X}_1} \sum_{i_1} \Big|\int_{\mathcal{Y}_1} \psi_{x_2,i_2} k_{i_1}^1(x_1,\cdot)\, d\mu_1\Big|^2 dm_1\, dm_2 \\
&\leq \int_{\mathcal{X}_2} \sum_{i_2} \int_{\mathcal{Y}_1} |\psi_{x_2,i_2}(z_1)|^2\, d\nu_1(z_1)\, dm_2(x_2) \\
&= \int_{\mathcal{Y}_1} \int_{\mathcal{X}_2} \sum_{i_2} \Big|\int_{\mathcal{Y}_2} \phi(z_1,\cdot)\, k_{i_2}^2(x_2,\cdot)\, d\mu_2\Big|^2 dm_2\, d\nu_1(z_1) \\
&\leq \int_{\mathcal{Y}_1}\int_{\mathcal{Y}_2} |\phi|^2\, d\nu_2\, d\nu_1.
\end{aligned}$$

The product kernel is bounded and measurable, which completes the induction. □

Lemma 5.3.60. *Let $\mathfrak{S}$ be admissible with Weil data, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, and $T' \subseteq G(E)$ stable. Let $\varphi\colon G(E) \times G_\infty \to [0,\infty]$ be such that $\varphi(\gamma,\cdot)$ is Borel for every γ and*

$$\sum_{(v,v') \in \mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)} \varphi\big(v\gamma v'^{-1},\, v \cdot y\big) = 1$$

for every $y \in G_\infty$ and every $\gamma \in T'$ that is a term over (c,s,c'). Then

$$\sum_{\substack{\gamma \in T' \\ over\ (c,s,c')}} \int_{G_\infty} \varphi(\gamma,y) \left\| W^{\mathrm{full}}(y) \otimes \overline{W^{\mathrm{full}}(\gamma^{-1}y)} - W^{\mathrm{cut}}(y) \otimes \overline{W^{\mathrm{cut}}(\gamma^{-1}y)} \right\| dy \le \frac{\nu}{2\, b^{k^3 f}}.$$

Proof. Write $K^\sigma(\gamma,y) = W^\sigma(y) \otimes \overline{W^\sigma(\gamma^{-1}y)}$ ($\sigma \in \{\mathrm{full}, \mathrm{cut}\}$) and $D(\gamma,y) = \|K^{\mathrm{full}}(\gamma,y) - K^{\mathrm{cut}}(\gamma,y)\|$, a Borel function of y. Let X be the set of terms of T' over (c,s,c'), and let $Z = X \times G_\infty$, $\mu =$ (counting measure) $\otimes\, dy$, the group Γ_X, its countable subgroup $\Delta = \mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)$ and $\beta_0(\gamma,y) = \beta_c(y)\beta_{c'}(\gamma^{-1}y)$ be as in (ii) of Lemma 5.2.36: Γ_X acts on Z by $(a,b)\cdot(\gamma,y) = (a\gamma b^{-1}, a \cdot y)$, by measurable maps that preserve integrals against μ, and $\sum_{\delta \in \Delta} \beta_0(\delta \cdot z) = 1$ for every $z \in Z$.

Step 1: exchange of the cutoff. D is Δ-invariant: for $\delta = (v,v') \in \Delta$ and $z = (\gamma,y)$, $W^\sigma(v \cdot y) = W^\sigma(y)$ by Lemma 5.2.42, and $(v\gamma v'^{-1})^{-1}(v \cdot y) = v' \cdot \gamma^{-1} y$ with $W^\sigma(v' \cdot x) = W^\sigma(x)$, so $K^\sigma(\delta \cdot z) = K^\sigma(z)$ and $D(\delta \cdot z) = D(z)$. By hypothesis $\sum_{\delta \in \Delta} \varphi(\delta \cdot z) = 1$ for every $z \in Z$, and φ is measurable on Z, as X is countable. By (i) of Lemma 5.2.45, applied to Γ_X, $\Lambda = \Delta$, the two cutoffs φ and β_0 and $H = D$,

$$\int_Z \varphi D \, d\mu = \int_Z \beta_0 D \, d\mu.$$

The left side is the left side of the claim. Adding the nonnegative terms of all classes d and all terms of $G(E)$ over (c,s,d),

$$\sum_{\gamma \in X} \int \varphi(\gamma,y) D(\gamma,y)\, dy \le \int_{G_\infty} \beta_c(y) \sum_{d \in \mathcal{C}} \sum_{\gamma \text{ over } (c,s,d)} \beta_d(\gamma^{-1}y)\, D(\gamma,y)\, dy. \tag{5.70}$$

Step 2: the bound. By Tonelli's theorem the right side of (5.70) is

$$\sum_{d \in \mathcal{C}} \sum_{\substack{\gamma \in G(E) \\ \text{over } (c,s,d)}} \int_{G_\infty} \beta_c(y)\, \beta_d(\gamma^{-1}y)\, \|K^{\mathrm{full}}(\gamma,y) - K^{\mathrm{cut}}(\gamma,y)\|\, dy,$$

which is the left side of (i) of Lemma 5.3.62 for the set $G(E)$ of terms; by that clause it is at most $\nu/(2b^{k^3 f})$. □

Lemma 5.3.61. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, with its fixed cutoff datum. For $z \in G_\infty$ put $m(z) = \prod_{v \in \Phi} \max(1, \ell(z_v)/b^{6n})$. Let $c \in \mathcal{C}$, $s \in \bar{G}$, $y \in G_\infty$ and let $i \ge 8nrf$ be an integer. Then*

$$\sum_{d \in \mathcal{C}} \sum_{\gamma \ over\ (c,s,d)} \beta_d(\gamma^{-1}y)\, m(\gamma^{-1}y)^{-i} \le 2\, b^{12n^3 rf}.$$

Proof. Let P_0 be the set of pairs (d,γ) with $d \in \mathcal{C}$ and γ a term over (c,s,d), and for $(d,\gamma) \in P_0$ put $w(d,\gamma) = \beta_d(\gamma^{-1}y) \in [0,1]$ and $m(d,\gamma) = m(\gamma^{-1}y) \geq 1$. Let $m_0 = 2nrf$ and $K = b^{6n^2}$.

Layers. Let $R \geq 1$. If $m(d,\gamma) \leq R$, every factor of $m(\gamma^{-1}y)$, being at least 1, is at most R, so $\ell((\gamma^{-1}y)_v) \leq b^{6n}R \leq b^{6n+2}R$ for every $v \in \Phi$. By Lemma 5.2.30,

$$\sum_{(d,\gamma)\in P_0,\ m(d,\gamma)\leq R} w(d,\gamma) \leq (KR^2)^{m_0}.$$

Summation. As $0 < m^{-i} \leq 1$, Tonelli's theorem for the counting measure on P_0 gives, with $R = t^{-1/i} \geq 1$ for $0 < t \leq 1$,

$$\begin{aligned}\sum_{P_0} w\, m^{-i} &= \int_0^1 \sum_{P_0} w\, \mathbf{1}[m^{-i} > t]\, dt \leq \int_0^1 \sum_{P_0} w\, \mathbf{1}[m \leq t^{-1/i}]\, dt \\ &\leq \int_0^1 K^{m_0} t^{-2m_0/i}\, dt = \frac{K^{m_0}}{1 - 2m_0/i} \leq 2K^{m_0},\end{aligned}$$

since $i \geq 8nrf = 4m_0$ gives $2m_0/i \leq \frac{1}{2}$. Finally $K^{m_0} = b^{12n^3rf}$. □

Lemma 5.3.62. *Let $\mathfrak{S}$ be admissible with Weil data, $T \subseteq G(E)$, $c \in \mathcal{C}$ and $s \in \bar{G}$. For $\gamma \in G(E)$ and $y \in G_\infty$ put $K^\sigma(\gamma,y) = W^\sigma(y) \otimes \overline{W^\sigma(\gamma^{-1}y)}$ ($\sigma \in \{\mathrm{full}, \mathrm{cut}\}$). Then:*

(i)

$$\sum_{d\in\mathcal{C}} \sum_{\substack{\gamma\in T \\ over\ (c,s,d)}} \int_{G_\infty} \beta_c(y)\, \beta_d(\gamma^{-1}y)\, \|K^{\mathrm{full}}(\gamma,y) - K^{\mathrm{cut}}(\gamma,y)\|\, dy \leq \frac{\nu}{2\, b^{k^3 f}};$$

(ii)

$$\sum_{c'\in\mathcal{C}} \|A_T^{\mathrm{full}}(c,s,c') - A_T^{\mathrm{cut}}(c,s,c')\| \leq \frac{\nu}{2\, b^{k^3 f}}.$$

Proof. Write $D(\gamma,y) = \|K^{\mathrm{full}}(\gamma,y) - K^{\mathrm{cut}}(\gamma,y)\|$, a Borel function of y (the functions of a Bessel system are continuous), and for $z \in G_\infty$ let $m(z) = \prod_v \max(1, \ell(z_v)/b^{6n}) \geq 1$. All sums over terms are countable, as $G(E)$ is. Since the summands of (i) are nonnegative, it suffices to prove (i) with T replaced by $G(E)$.

Step 1: a pointwise bound for the difference of the kernels. For $v \in \Phi$ let (A_v, a_v) be a decay majorant (BS3) of the cut pair $(W_v, W_v^\flat)$ of $\mathfrak{B}_v$: $A_v \colon G_1 \to [0,\infty]$ is measurable and left R-invariant, $A_v \leq a_v < \infty$, $\max(1, \ell(x)/b^{6n})^{3r^3 f}\|F(x)\| \leq A_v(x)$ for $F \in \{W_v, W_v^\flat\}$, and $a_v \int_{\mathsf{U}\setminus G_1} A_v \leq b^{400r^2} \# I_v$. Put $i = r^3 f$, $A(y) = \prod_v A_v(y_v)$ and $a = \prod_v a_v$. Taking the product over v of the bounds of (BS3),

$$\|W^\sigma(x)\| \leq m(x)^{-3i} A(x) \leq m(x)^{-3i} a \qquad (x \in G_\infty,\ \sigma \in \{\mathrm{full}, \mathrm{cut}\}).$$

Let $\gamma \in G(E)$, $y \in G_\infty$ and $z = \gamma^{-1}y$. If $D(\gamma,y) \neq 0$, then $W^{\mathrm{full}}(y) \neq W^{\mathrm{cut}}(y)$ or $W^{\mathrm{full}}(z) \neq W^{\mathrm{cut}}(z)$, so $W_v \neq W_v^\flat$ at y_v or at z_v for some v; by (BS2) that point has Bessel length $> b^{6n+2}/2$, so its factor in m exceeds $b^2/2 \geq b$, and as all factors are at least 1, $m(y)m(z) \geq b$. Since $\|K^\sigma(\gamma,y)\| = \|W^\sigma(y)\|\, \|W^\sigma(z)\| \leq m(y)^{-3i} A(y)\, m(z)^{-3i} a$, and $m(y)^{-3i} m(z)^{-3i} \leq (m(y)m(z))^{-2i} m(z)^{-i}$, we get in all cases

$$D(\gamma,y) \leq \|K^{\mathrm{full}}(\gamma,y)\| + \|K^{\mathrm{cut}}(\gamma,y)\| \leq 2\, b^{-2i}\, a\, A(y)\, m(\gamma^{-1}y)^{-i}.$$

Step 2: the sum over the terms. By (FG6), $k \geq 1000(2\varrho+2)^9 \geq 8n$, and $r \geq k$, so $i = r^3 f \geq 8nrf$. Hence Lemma 5.3.61 applies with this i and gives, for every $y \in G_\infty$,

$$\sum_{d\in\mathcal{C}} \sum_{\gamma \text{ over } (c,s,d)} \beta_d(\gamma^{-1}y)\, D(\gamma,y) \leq 2b^{-2i} a A(y) \sum_d \sum_\gamma \beta_d(\gamma^{-1}y)\, m(\gamma^{-1}y)^{-i} \leq 4\, b^{12n^3rf-2i}\, a\, A(y).$$

Step 3: unfolding along U_∞. For each v fix a Borel $\chi_v \colon G_1 \to [0,\infty]$ with $\int_{\mathsf{U}} \chi_v(ux)\, du = 1$ for every $x \in G_1$, so that $\int_{\mathsf{U}\backslash G_1} F = \int_{G_1} \chi_v F\, dx$ for every left U-invariant Borel $F \geq 0$ (Definition 5.1.6), and put $\chi(y) = \prod_v \chi_v(y_v)$; by Tonelli's theorem $\int_{\mathsf{U}_\infty} \chi(uy)\, du = 1$ for every y. A is left U_∞-invariant, as $\mathsf{U} \subseteq R$. Inserting this identity, substituting $y \mapsto u^{-1}y$ in the inner integral (the Cayley measures are left invariant, Lemma 5.2.41), interchanging the integrals (Tonelli), using that du is invariant under $u \mapsto u^{-1}$ (Lemma 5.2.34 (iv)) and then Lemma 5.2.33,

$$\begin{aligned}\int_{G_\infty} \beta_c A\, dy &= \int_{\mathsf{U}_\infty} \int_{G_\infty} \beta_c(u^{-1}y)\, \chi(y) A(y)\, dy\, du = \int_{G_\infty} \chi(y) A(y) \int_{\mathsf{U}_\infty} \beta_c(uy)\, du\, dy \\ &= \nu \int_{G_\infty} \chi A\, dy = \nu \prod_v \int_{\mathsf{U}\backslash G_1} A_v.\end{aligned}$$

Step 4: proof of (i). By Tonelli's theorem, Step 2 and Step 3, and then (BS3) and (FA7) ($\#I \leq b^{2kf}$),

$$\begin{aligned}\sum_d \sum_{\gamma \text{ over } (c,s,d)} \int \beta_c(y)\beta_d(\gamma^{-1}y) D(\gamma,y)\, dy &\leq 4\nu\, b^{12n^3rf-2i} \prod_v \Big(a_v \int_{\mathsf{U}\backslash G_1} A_v\Big) \\ &\leq 4\nu\, b^{12n^3rf+400r^2f-2r^3f}\, \#I \\ &\leq 4\nu\, b^{12n^3rf+400r^2f+2kf-2r^3f}.\end{aligned}$$

By (FG6), $k \geq 1000(2\varrho+2)^9 \geq 12n^3+500$ and $r \geq k+1 \geq 402$; so $12n^3r \leq r^2$, $2k+1 \leq r^2$, $402r^2 \leq r^3$ and $k^3 \leq r^3$, whence, with $f \geq 1$, $1 + (12n^3r + 400r^2 + 2k + k^3)f \leq (402r^2 + r^3)f \leq 2r^3 f$. As $8 \leq b$ (FG2), $8\, b^{12n^3rf+400r^2f+2kf+k^3f} \leq b^{2r^3f}$, and the right side above is at most $\nu/(2b^{k^3f})$. This proves (i).

Step 5: proof of (ii). Let $c' \in \mathcal{C}$ and $\sigma \in \{\mathrm{full}, \mathrm{cut}\}$. By Lemma 5.2.29 (ii), applied with $(F_v)_v = (W_v)_v$ and with $(F_v)_v = (W_v^\flat)_v$, the amplitude majorants $\int \beta_c(y)\beta_{c'}(\gamma^{-1}y)\|K^\sigma(\gamma,y)\|\, dy$ of the terms $\gamma \in T$ over (c,s,c') have a finite sum. So for each such γ the measurable function $y \mapsto \beta_c(y)\beta_{c'}(\gamma^{-1}y)K^\sigma(\gamma,y)$ is integrable, its Bochner integral is the amplitude $A^\sigma_\gamma(c,c')$ with $\|A^\sigma_\gamma(c,c')\|$ at most its majorant, and the series $A^\sigma_T(c,s,c') = \sum_\gamma A^\sigma_\gamma(c,c')$ converges absolutely. Subtracting the two absolutely convergent series term by term, and using linearity of the Bochner integral and $\|\int g\| \leq \int \|g\|$,

$$\begin{aligned}\|A^{\mathrm{full}}_T(c,s,c') - A^{\mathrm{cut}}_T(c,s,c')\| &= \Big\|\sum_\gamma \int \beta_c(y)\beta_{c'}(\gamma^{-1}y)(K^{\mathrm{full}} - K^{\mathrm{cut}})(\gamma,y)\, dy\Big\| \\ &\leq \sum_{\substack{\gamma\in T \\ \text{over } (c,s,c')}} \int \beta_c(y)\beta_{c'}(\gamma^{-1}y) D(\gamma,y)\, dy.\end{aligned}$$

Summing over $c' \in \mathcal{C}$ and applying (i) gives (ii). □

Lemma 5.3.63. *Let $\mathfrak{S}$ be an admissible filling datum and $\gamma \in G(E)$. Put $x = \prod_{v\in\Phi} |\mathsf{g}(\gamma)|_v$ and, for $x > 0$, $\mathrm{ob}(x) = \tau_{\mathrm{sz}}^2 \sqrt{\mathcal{D}_{\mathrm{main}}/(n_p(M^{2\varrho f}x)^k)}$.*

(i) If γ is a term over some (c,s,c') and $\mathsf{g}(\gamma) \neq 0$, then $x \geq 1$.

(ii) If E carries Weil data, γ is a near term and $x \geq 1$, then $b^{-k^3 f} \leq \mathrm{ob}(x)$.

(iii) If $x \geq 1$, then $\prod_{v\in\Phi} \mathrm{PB}(\gamma_v, \deg_v, b^{37n}, \#I_v) \leq \big(\nu\,\mathrm{ob}(x)/2\big)^2$.

Proof. Write $e_1 = 10^5 n^4$ and $g_v = v(\mathsf{g}(\gamma))$.

(i) Let γ be a term over (c, s, c'), $k_0 = h_c^{-1}\gamma h_{c'} \in K$. The frame vectors are lattice basis vectors fixed by h_c and $h_{c'}$, so $h(f_a, \gamma f_b) = h(h_c^{-1}f_a, k_0 h_{c'}^{-1} f_b) = h(f_a, k_0 f_b) \in h(L_0, L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24 (i), (ii). So the entries of $Z(\gamma)$ lie in $E \cap \hat{\mathcal{O}} = \mathcal{O}_E$, hence $D_{\varrho+1}(\gamma), D_\varrho(\gamma) \in \mathcal{O}_E$ and $\mathsf{g}(\gamma) \in \mathcal{O}_E$. If $\mathsf{g}(\gamma) \neq 0$, its norm is a nonzero rational integer, and since the embeddings of E are the $v \in \Phi$ and their conjugates, $x^2 = \prod_v |\mathsf{g}(\gamma)|_v^2 = |\mathrm{N}_{E/\mathbb{Q}}\mathsf{g}(\gamma)| \geq 1$.

(ii) As $x \geq 1$, $\mathsf{g}(\gamma) \neq 0$, so it is totally positive by Lemma 5.2.26 (ii): $0 < g_v = |v(D_{\varrho+1})|^2 - N^2|v(D_\varrho)|^2 \leq |D_{\varrho+1}(\gamma)|_v^2 \leq b^{270n^2}$ by Lemma 5.2.20. So $x \leq b^{270n^2 f}$. Next $n_p = \#\bar{R}_0 \#\mathcal{C}/N^{kf} \leq \#\bar{R}_0\#\mathcal{C}\#I = \mathcal{D}_{\mathrm{id}} \leq \mathcal{D}_{\mathrm{main}}$ by (FA6), and $\tau_{\mathrm{sz}} \geq 1$; with $M \leq b^{23}$ (FG6),

$$\mathrm{ob}(x) \geq (M^{2\varrho f} x)^{-k/2} \geq b^{-(46\varrho + 270n^2)fk/2} \geq b^{-k^3 f},$$

since $(46\varrho + 270n^2)k \leq 316n^2 k \leq 2k^3$ by $k \geq 1000(2\varrho+2)^9$ (FG6).

(iii) *The real Gram discriminant.* By Lemma 5.3.36 (ii),

$$\mathsf{g}_1(Z(\gamma_v)) = s_v^2\, g_v, \qquad s_v = \prod_{a=0}^{\varrho} \mathrm{fs}(v, a).$$

As $x \geq 1$, $\mathsf{g}(\gamma)$ is totally positive (Lemma 5.2.26 (ii)), so $g_v > 0$ and $\prod_v g_v = x$.

Splitting off x. With $\rho = b^{37n}$ and $d_v = \#I_v$, the definition of PB gives $\mathrm{PB}(\gamma_v, \deg_v, \rho, d_v) = C_v\, g_v^{-(e_1+k-1)}$ with

$$C_v = \Big(\frac{\rho^{9n}}{s_v^2}\Big)^{e_1} \frac{2^{k-1} d_v^2}{\deg_v c_G\, s_v^{2(k-1)}}.$$

Since $x \geq 1$ and $e_1 + k - 1 \geq k$, $\prod_v \mathrm{PB}(\gamma_v, \deg_v, \rho, d_v) \leq x^{-k} \prod_v C_v$.

Frame scales. By (FA4) and (FA5), $b^{-60n} \leq N^{-1} \leq \mathrm{fs}(v,a) \leq b^{46n}$, so $b^{-60n^2} \leq s_v \leq b^{46n^2}$ and $\mathrm{fs}(v,a)^{2(\varrho-a)} \leq b^{92n^2}$; hence $\Lambda_v = \prod_a \mathrm{fs}(v,a)^{k+2(\varrho-a)} \leq s_v^k b^{92n^3}$, and $\Lambda_{\mathrm{fr}} = \prod_v \Lambda_v$. Using $\rho^{9n}/s_v^2 \leq b^{453n^2}$, $s_v^{-2(k-1)} = s_v^2 s_v^{-2k} \leq b^{92n^2} b^{184n^3} \Lambda_v^{-2}$ and $2^{k-1} \leq 2^k$,

$$C_v \leq 2^k b^e \frac{d_v^2}{\deg_v c_G \Lambda_v^2}, \qquad e = 453n^2 e_1 + 92n^2 + 184n^3 \leq 10^5 n^9 \leq k/4,$$

the last two by $n \geq 10^5$ and $k \geq 1000(2n)^9$ (FG6). Multiplying over the f places, with $b^{ef} \leq b^{kf/4} = \tau_{\mathrm{sz}}$,

$$\prod_v C_v \leq 2^{kf} \tau_{\mathrm{sz}} \frac{\#I^2}{\deg\, c_G^f\, \Lambda_{\mathrm{fr}}^2} \leq 2^{kf} \tau_{\mathrm{sz}}\,(1 + b^{-9}) \frac{\mathcal{D}_{\mathrm{main}}\, \nu^2}{c_S^\sharp\, \mathrm{vol}(\hat{\mathcal{O}})^n},$$

the second inequality being (FA9) rearranged.

The Siegel constant. By the definitions, $c_S^\sharp = \#\bar{R}_0 M^{f\varrho(2k+2\varrho+1)} \#\mathcal{C}\, c_S$ and $n_p = \#\bar{R}_0\#\mathcal{C}/N^{kf}$, so, with $M \geq 1$, $\pi^k \geq 1$, $|\mathrm{disc}\, E| \geq 1$ and $\mathrm{vol}(\hat{\mathcal{O}}) = (2/\mathrm{rd}_E)^f \geq \mathrm{rd}_E^{-f}$,

$$\frac{c_S^\sharp\, \mathrm{vol}(\hat{\mathcal{O}})^n}{n_p M^{2\varrho fk}} \geq N^{kf} c_S\, \mathrm{vol}(\hat{\mathcal{O}})^n = \Big(\frac{\pi^k}{(k-1)!}\Big)^f |\mathrm{disc}\, E|^{1/4} \mathrm{vol}(\hat{\mathcal{O}})^{k+n} \geq \frac{1}{k^{kf}\, \mathrm{rd}_E^{f(k+n)}}.$$

Let $X = 200fk\,\mathrm{rd}_E$; then $2, k, \mathrm{rd}_E \leq X$ (as $\mathrm{rd}_E \geq 1$), $n \leq k$ and $X^{1000} \leq b$ (FG6), so $2^{kf} k^{kf} \mathrm{rd}_E^{f(k+n)} \leq X^{4kf} \leq b^{kf/4} = \tau_{\mathrm{sz}}$. Hence $2^{kf} n_p M^{2\varrho fk} \leq \tau_{\mathrm{sz}}\, c_S^\sharp \mathrm{vol}(\hat{\mathcal{O}})^n$.

Conclusion. Combining, $\prod_v \mathrm{PB}(\gamma_v, \deg_v, \rho, d_v) \le (1+b^{-9})\tau_{\mathrm{sz}}^2\nu^2\mathcal{D}_{\mathrm{main}}/(n_p M^{2\varrho fk}x^k)$, while $(\nu\,\mathrm{ob}(x)/2)^2 = \nu^2\tau_{\mathrm{sz}}^4\mathcal{D}_{\mathrm{main}}/(4n_pM^{2\varrho fk}x^k)$. As $\tau_{\mathrm{sz}} \ge b \ge 1000$ (FG2), $4(1+b^{-9}) \le \tau_{\mathrm{sz}}^2$, which proves (iii). □

Proposition 5.3.64. *Let $\mathfrak{S}$ be a filling datum with Weil data, $s \in \bar{G}$ agreeing, $x \in \mathbb{N}$ with $x \le$ the meet norm of s, and γ, γ' terms over (c, s, c') of cell ranks j, j'. Put $\epsilon = x$ if $\varrho < j$ and $c^+ = 0$, and $\epsilon = 1$ otherwise. Then*

$$\big(\mathrm{disc}(L_{12})\,\epsilon\big)^2 \le (2/M)^{2c^+f}\,\mathfrak{g}^*(j,c,c',\gamma)\,\mathfrak{g}^*(j',c,c',\gamma').$$

Proof. Write $k = h_c^{-1}\gamma h_{c'}$, $k' = h_c^{-1}\gamma' h_{c'}$, $m = \dim_E X_0(\gamma,\gamma')$, and let P, P' be the common parts of γ with γ' and of γ' with γ at c. The space $X_0(\gamma,\gamma')$, the link corank c^+ and the pair lattice L_{12} are symmetric in γ, γ'. Put $\epsilon = x$ if $\varrho < j$ and $c^+ = 0$, and $\epsilon = 1$ otherwise; then ϵ is at most the meet norm of s (which is a product of primes, hence ≥ 1).

Step 1 (splittings). By Proposition 5.3.5 for $(\gamma, \gamma', j, \epsilon)$: $\mathrm{ti}(k) \ge 1$, P is a lattice of rank m, and there are q, y, ϵ_1, σ as there with

$$\big(\mathrm{ti}(k)\,\mathrm{disc}_m(P)\,\epsilon\big)^2\,\mathrm{lkd}(y,\epsilon_1) \le \mathfrak{g}^*(j,c,c',\gamma)^2. \tag{5.71}$$

The same applies to $(\gamma', \gamma, j', \epsilon)$: if $\epsilon \ne 1$ then $\varrho < j$ and $c^+ = 0$, so $\mathfrak{g}(\gamma) \ne 0$ by the conditions of cell rank j and $\mathrm{link}\,\gamma = \mathrm{link}\,\gamma'$ by Lemma 5.3.13 (iv); then $\mathfrak{g}(\gamma') \ne 0$ by Lemma 5.3.13 (v), hence $\varrho < j'$. This gives $\mathrm{ti}(k') \ge 1$, P' a lattice of rank m, and $q', y', \epsilon_1', \sigma'$ with

$$\big(\mathrm{ti}(k')\,\mathrm{disc}_m(P')\,\epsilon\big)^2\,\mathrm{lkd}(y',\epsilon_1') \le \mathfrak{g}^*(j',c,c',\gamma')^2.$$

Step 2 (congruence). Let $w_l = \sum_\rho \sigma_{\rho l} q_\rho + \sum_a (\epsilon_1)_{al} y_a$ and w_l' likewise; they have integral lifts for (c,γ) and (c,γ'), so $M^{-1}\langle w_l, w_{l'}'\rangle_{\mathrm{lk}} \in \hat{\mathcal{O}}$ by Lemma 5.3.65 (i). The q_ρ, q_ρ' lie in both links, the y_a in $\mathrm{link}\,\gamma$ and the y_b' in $\mathrm{link}\,\gamma'$; as each link is isotropic for the link form (Lemma 5.2.28 (iv)), $\langle q_\rho, q_{\rho'}'\rangle_{\mathrm{lk}} = \langle q_\rho, y_b'\rangle_{\mathrm{lk}} = \langle y_a, q_{\rho'}'\rangle_{\mathrm{lk}} = 0$. Hence the matrix $(\langle w_l, w_{l'}'\rangle_{\mathrm{lk}})$ is $\epsilon_1^*\Lambda\epsilon_1'$ with $\Lambda = (\langle y_a, y_b'\rangle_{\mathrm{lk}})_{a,b<c^+}$, and $\epsilon_1^*(M^{-1}\Lambda)\epsilon_1'$ is integral. Moreover $\det\Lambda \ne 0$: if $\Lambda\tau = 0$, then $u = \sum_b \tau_b y_b' \in \mathrm{link}\,\gamma'$ is orthogonal for the link form to every y_a and to every $q_\rho \in \mathrm{link}\,\gamma'$, hence to $\mathrm{link}\,\gamma$, which q and y span; by Lemma 5.2.28 (iv) $u \in \mathrm{link}\,\gamma$, so $u \in \mathrm{link}\,\gamma \cap \mathrm{link}\,\gamma'$ and $\tau = 0$ by (i) of Proposition 5.3.5 for γ'. By Lemma 5.3.65 (ii) with $i = c^+$,

$$1 \le (2/M)^{4c^+f}\,\mathrm{lkd}(y,\epsilon_1)\,\mathrm{lkd}(y',\epsilon_1').$$

Step 3 (combination). With $d = \mathrm{ti}(k)\mathrm{disc}_m(P)\epsilon \ge 0$ and $d' = \mathrm{ti}(k')\mathrm{disc}_m(P')\epsilon \ge 0$, Steps 1 and 2 give

$$(dd')^2 \le (dd')^2(2/M)^{4c^+f}\mathrm{lkd}(y,\epsilon_1)\mathrm{lkd}(y',\epsilon_1') \le (2/M)^{4c^+f}\,\mathfrak{g}^*(j,c,c',\gamma)^2\,\mathfrak{g}^*(j',c,c',\gamma')^2,$$

so, both sides being squares of nonnegative numbers, $dd' \le (2/M)^{2c^+f}\mathfrak{g}^*(j,c,c',\gamma)\,\mathfrak{g}^*(j',c,c',\gamma')$.

Step 4 (index). By Lemma 5.3.4 (ii) for (γ,γ') and for (γ',γ) (the pair lattice is the same), $0 \le \mathrm{disc}(L_{12}) \le \mathrm{disc}_m(P)\,\mathrm{ti}(k')$ and $\mathrm{disc}(L_{12}) \le \mathrm{disc}_m(P')\,\mathrm{ti}(k)$. Multiplying, $(\mathrm{disc}(L_{12})\,\epsilon)^2 \le dd'$, and Step 3 gives the claim. □

Lemma 5.3.65. *Let $\mathfrak{S}$ be a filling datum, $c, c' \in \mathcal{C}$, $s \in \bar{G}$, and γ, γ' terms over (c, s, c').*

(i) If the family $w = (w_l)_{l\in\sigma}$ has integral lifts for (c,γ) and $w' = (w_{l'}')_{l'\in\sigma'}$ has integral lifts for (c,γ'), then $M^{-1}\langle w_l, w_{l'}'\rangle_{\mathrm{lk}} \in \hat{\mathcal{O}}$ for all l, l'.

(ii) Let $i \geq 0$, let $y = (y_l)_{l<i}$ be pairs in $\operatorname{link}\gamma$ and $y' = (y'_l)_{l<i}$ pairs in $\operatorname{link}\gamma'$, and $\epsilon, \epsilon' \in \mathrm{GL}_i(\mathbb{A}_f)$. Suppose that the matrix $\Lambda = (\langle y_l, y'_{l'}\rangle_{\mathrm{lk}})_{l,l'}$ has $\det\Lambda \neq 0$ and that $\epsilon^*(M^{-1}\Lambda)\epsilon'$ is integral. Then

$$1 \leq \Big(\frac{2}{M}\Big)^{4if} \operatorname{lkd}(y,\epsilon)\operatorname{lkd}(y',\epsilon').$$

Proof. (i). By (LK5) there are vectors $t_\rho \perp X + \gamma X$, $t'_{\rho'} \perp X + \gamma' X$ and adelic coefficients u, u' with $n_l = \sum_\rho u_{\rho l} t_\rho \in L_c$, $n'_{l'} = \sum_{\rho'} u'_{\rho' l'} t'_{\rho'} \in L_c$, $w_l = \sum_\rho u_{\rho l}([t_\rho],[\gamma^{-1}t_\rho])$ and similarly for w'. For vectors q, q' of V orthogonal to $X = \langle e_1, \dots, e_\varrho\rangle$ the e_a^*-coordinates vanish, and the e_a-coordinates pair only with them, so $h(q,q') = h([q],[q'])$. Applied to (t,t') and to $(\gamma^{-1}t, \gamma'^{-1}t')$ for $t \perp X + \gamma X$, $t' \perp X + \gamma' X$, and with $\delta = \gamma\gamma'^{-1}$,

$$\big\langle([t],[\gamma^{-1}t]),([t'],[\gamma'^{-1}t'])\big\rangle_{\mathrm{lk}} = h(t,t') - h(\gamma^{-1}t, \gamma'^{-1}t') = h(t,t') - h(t,\delta t').$$

By sesquilinearity, $\langle w_l, w'_{l'}\rangle_{\mathrm{lk}} = h(n_l, n'_{l'}) - h(n_l, \delta n'_{l'}) = -h\big(n_l, (\delta-1)n'_{l'}\big)$. Let $k = h_c^{-1}\gamma h_{c'}$, $k' = h_c^{-1}\gamma' h_{c'}$; they lie in K with $\operatorname{red}(k) = \operatorname{red}(k') = s$. By Lemma 5.2.24 (ii) K is a group, and red (entrywise reduction) is multiplicative, so $kk'^{-1} = h_c^{-1}\delta h_c \in K$ with reduction 1, i.e. $kk'^{-1} \in K(M)$. By Lemma 5.2.24 (iii), $(kk'^{-1}-1)L_0 \subseteq ML_0$, so $(\delta-1)L_c = h_c(kk'^{-1}-1)L_0 \subseteq ML_c$. Since h_c is an isometry, $h(L_c,L_c) = h(L_0,L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24 (i). Hence $\langle w_l, w'_{l'}\rangle_{\mathrm{lk}} \in h(L_c, ML_c) \subseteq M\hat{\mathcal{O}}$.

(ii), the pairing in the positive space. Put $a_l = y_l^{\mathrm{s}}$, $b_l = y_l'^{\mathrm{s}} \in V_0^+$, and $\tilde{b}_l = \Upsilon_\gamma^{-1}\Upsilon_{\gamma'} b_l$, with the link isometries of Lemma 5.2.28 (iii); so $y_l^{\mathrm{t}} = \Upsilon_\gamma a_l$ and $y_l'^{\mathrm{t}} = \Upsilon_{\gamma'} b_l = \Upsilon_\gamma \tilde{b}_l$. By Lemma 5.2.28 (iv) and since Υ_γ is an isometry,

$$\Lambda_{ll'} = \langle a_l, b_{l'}\rangle_+ - \langle \Upsilon_\gamma a_l, \Upsilon_\gamma \tilde{b}_{l'}\rangle_+ = \langle a_l,\, b_{l'} - \tilde{b}_{l'}\rangle_+.$$

For a family $x = (x_l)$ in V_0^+ write $G(x) = (\langle x_l, x_{l'}\rangle_+)$. By Lemma 5.2.28 (ii), $G(a)$ and $G(b)$ are the Gram matrices of y and y' for the link metric, so $\operatorname{lkd}(y,\epsilon) = |N\det G(a)|[\det\epsilon]^2$ and likewise for y' ($N = N_{E/\mathbb{Q}}$).

Liouville. In the basis of V_0^+ formed by the standard basis of X_0 and z^+, $\langle\cdot,\cdot\rangle_+$ has the matrix $\operatorname{diag}(1,\dots,1,N)$, positive definite at every infinite place. Apply Lemma 5.3.14 to the coordinate matrices of $(M^{-1}a_l)_l$ and $(b_l - \tilde{b}_l)_l$, with $v = \epsilon$, $v' = \epsilon'$ and $x = 1$: the pairing matrix is $M^{-1}\Lambda$ (M is real), with $\det(M^{-1}\Lambda) \neq 0$; and $\delta = \det(\epsilon^* M^{-1}\Lambda\,\epsilon')$ is a unit of $\mathbb{A}_f$ lying in $\hat{\mathcal{O}}$ by hypothesis, so $[\delta] = [\hat{\mathcal{O}} : \delta\hat{\mathcal{O}}] \geq 1$. Hence

$$1 \leq [\det\epsilon]^2\,\big|N\det(M^{-2}G(a))\big|\,\big|N\det G(b-\tilde{b})\big|\,[\det\epsilon']^2,$$
$$\big|N\det(M^{-2}G(a))\big| = M^{-4if}|N\det G(a)|.$$

The difference. $\Upsilon_\gamma^{-1}\Upsilon_{\gamma'}$ is an isometry, so $G(\tilde{b}) = G(b)$ and $G(b-\tilde{b}) + G(b+\tilde{b}) = 2G(b) + 2G(\tilde{b}) = 4G(b)$. At an infinite place w, $\Phi_w G(b+\tilde{b})$ is positive semidefinite, so $0 \leq \Phi_w G(b-\tilde{b}) \leq 4\,\Phi_w G(b)$ in the order of hermitian matrices, and the determinant is monotone on positive semidefinite matrices: $0 \leq \det\Phi_w G(b-\tilde{b}) \leq 4^i \det\Phi_w G(b)$. As $|N\alpha| = \prod_w |\Phi_w\alpha|^2$, $|N\det G(b-\tilde{b})| \leq 4^{2if}|N\det G(b)|$. Inserting,

$$1 \leq M^{-4if}\,4^{2if}\,|N\det G(a)|[\det\epsilon]^2\,|N\det G(b)|[\det\epsilon']^2 = \Big(\frac{2}{M}\Big)^{4if}\operatorname{lkd}(y,\epsilon)\operatorname{lkd}(y',\epsilon').$$

□

The third input of the pair bound is the count Proposition 5.3.66: weighted by $\operatorname{disc}(L_{12})^{-k}$, the pairs near over an agreeing block whose common definite part has dimension $k-l$ and whose pair discriminant is at most $b^{n^3 f}$ have total at most $\tau^{l+2} n_p$. The count has two steps. The first passes from pairs of terms to lattices: few near terms of one label contain a given lattice of bounded discriminant in their term lattice (Lemma 5.3.67), by the geometry of the frame-fixing moves and a count of algebraic integers in a box. The second sums inverse powers of the discriminant over the lattices of given corank in the definite class lattices (Lemma 5.3.78). After an averaging of the classes, the sum over subspaces is a height zeta sum, bounded by Siegel's lemma relative to a unimodular adelic lattice (Lemma 5.3.82) through an adelic form of Minkowski's second theorem [52] (Lemma 5.3.83), and the sum over sublattices of finite index is a sublattice zeta bound (Lemma 5.3.84). For the adelic form of Siegel's lemma and heights of subspaces see [32, 31]; zeta functions counting sublattices are a classical tool of integral representation theory [286], and the counting of subspaces of bounded height is in the tradition of Schmidt [252].

Proposition 5.3.66. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data. For $c, c' \in \mathcal{C}$ call a pair (γ, γ')* near over an agreeing block *for (c, c') if γ and γ' are near representatives for (c, c'), each satisfying the conditions of some cell rank, and both are terms over one (c, s, c') with s agreeing. Then for every integer $0 \le l \le n^3$,*

$$\sum_{c,c' \in \mathcal{C}} \sum_{(\gamma,\gamma')} \operatorname{disc}(L_{12})^{-k} \le \tau_{\mathrm{sz}}^{l+2}\, n_p,$$

the inner sum over the pairs near over an agreeing block for (c, c') with $\dim_E X_0(\gamma, \gamma') + l = k$ and $\operatorname{disc}(L_{12}) \le b^{n^3 f}$.

Proof. Fix $l \le n^3$. If $l > k$ no pair satisfies $\dim_E X_0(\gamma, \gamma') + l = k$ and the sum is 0; let $l \le k$ and $r = k - l$. Put $B = b^{n^3 f}$ and $Q = b^{2000 n^6 (l+1) f}$. Fix a representative of every left coset of $\bar{R}_0$ in $\bar{G}$, and labels as in Lemma 5.3.67.

Each pair gives a labelled lattice. Let (γ, γ') be near over the agreeing block (c, s, c'), of cell ranks j, j', with $\dim_E X_0(\gamma, \gamma') = r$ and $\operatorname{disc}(L_{12}) \le B$, and let t be the label of s. By Proposition 5.3.5 for (γ, γ') with $x = 1$, the common part P of γ with γ' at c is a lattice of rank r; by Proposition 5.3.5 for (γ', γ) with $x = 1$, $\operatorname{ti}(h_c^{-1} \gamma' h_{c'}) \ge 1$. By Lemma 5.3.4 (ii), L_{12} is a lattice of rank r and $\operatorname{disc}_r(L_{12}) = \operatorname{disc}(L_{12}) \ge 1$. By (LT4), $L_{12} \subseteq \operatorname{sat}_c(X_0(\gamma, \gamma')) \subseteq L_c \cap X_0(\mathbb{A}_f)$ and $L_{12} \subseteq L_c(\gamma) \cap L_c(\gamma')$. So L_{12} lies in the set $\mathcal{L}_c$ of lattices $L \subseteq L_c \cap X_0(\mathbb{A}_f)$ of rank r with $\operatorname{disc}_r(L) \le B$, and both (c', γ) and (c', γ') lie in the set $S(t, L_{12})$ of pairs counted in Lemma 5.3.67 for c, t and $L = L_{12}$.

Counting. Fix c. The map $(c', \gamma, \gamma') \mapsto (t, L_{12}, (c', \gamma), (c', \gamma'))$ is injective, and for fixed $(t, L) \in \bar{R}_0 \times \mathcal{L}_c$ the last two entries of an image lie in $S(t, L)$, which has at most Q elements by Lemma 5.3.67. Hence at most Q^2 triples have the image (t, L) in the first two entries, all with summand $\operatorname{disc}_r(L)^{-k}$, and

$$\sum_{c'} \sum_{(\gamma,\gamma')} \operatorname{disc}(L_{12})^{-k} \le \#\bar{R}_0\, Q^2 \sum_{L \in \mathcal{L}_c} \operatorname{disc}_r(L)^{-k}.$$

As $l + 1 \le 2n^3$, $Q^2 \le b^{8000 n^9 f}$, and by (FG6) $k \ge 1000(2\varrho + 2)^9 = 2^9 \cdot 1000\, n^9 \ge 32000\, n^9$, so $Q^2 \le b^{kf/4} = \tau_{\mathrm{sz}}$ ($b \ge 1$). Summing over c and applying Lemma 5.3.78,

$$\sum_{c,c'} \sum_{(\gamma,\gamma')} \operatorname{disc}(L_{12})^{-k} \le \#\bar{R}_0\, \tau_{\mathrm{sz}}\, \frac{\#\mathcal{C}}{N^{kf}}\, \tau_{\mathrm{sz}}^{l+1} = \tau_{\mathrm{sz}}^{l+2}\, n_p.$$

□

Lemma 5.3.67. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data. Fix a representative of every left coset of $\bar{R}_0$ in $\bar{G}$, and call* label *of $s \in \bar{G}$ the element $t \in \bar{R}_0$ with $s = r_0 t$, r_0 the representative of $s\bar{R}_0$. Let $c \in \mathcal{C}$, $t \in \bar{R}_0$, integers $r, l \geq 0$ with $r + l = k$, and $L \subseteq L_c \cap X_0(\mathbb{A}_f)$ a lattice of rank r with $\mathrm{disc}_r(L) \leq b^{n^3 f}$. Then at most $b^{2000n^6(l+1)f}$ pairs $(c', \gamma) \in \mathcal{C} \times G(E)$ have the following properties: γ is a near representative for (c, c') satisfying the conditions of some cell rank; γ is a term over (c, s, c') for some s of label t; and $L \subseteq L_c(\gamma)$.*

Proof. Let P be the set of pairs of the statement. For $(c', \gamma) \in P$ the cell rank $j(\gamma)$ of γ satisfies the conditions of cell rank $j(\gamma)$ (TA5), and γ is a near term. For $0 \leq j \leq \varrho + 1$ and a matrix Z let $P_{j,Z}$ be the set of $(c', \gamma) \in P$ with $j(\gamma) = j$ and $Z(\gamma) = Z$.

Each class is small. By Lemma 5.3.68 (all pairs of P have the label t), if $(c_1, \gamma_1), (c_2, \gamma_2) \in P$ and $u_1, u_2 \in \mathsf{U}_c^0(M)$ satisfy $u_1 \gamma_1 f_a = u_2 \gamma_2 f_a$ for all a, the pairs are equal. So each $P_{j,Z}$ satisfies the hypotheses of Lemma 5.3.70 for c, j, r, l and L, and

$$\#P_{j,Z} \leq b^{ef}, \qquad e = 366n^4 + 2n^3(\varrho + ln + n\varrho).$$

Few classes. There are $\varrho + 2$ values of j, and $\varrho + 2 \leq b + 1 \leq b^2 \leq b^{2f}$ as $\varrho + 1 \leq b$ (FG6) and $b \geq 2$. By Lemma 5.3.69 at most $b^{270n^3 f}$ matrices Z occur, since every γ is a near term over some (c, s, c'). Hence

$$\#P \leq b^{2f}\, b^{270n^3 f}\, b^{ef}.$$

As $\varrho < n$, $\varrho + ln + n\varrho \leq n^2(l+2)$, so $2n^3(\varrho + ln + n\varrho) \leq 4n^5(l+1)$, and $2 + 270n^3 + 366n^4 \leq 638n^4 \leq 638n^5(l+1)$. So the exponent is at most $642n^5(l+1)f \leq 2000n^6(l+1)f$, and $\#P \leq b^{2000n^6(l+1)f}$ as $b \geq 1$. □

Lemma 5.3.68. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, with a fixed representative of every left coset of $\bar{R}_0$ in $\bar{G}$, and let $c \in \mathcal{C}$, $t \in \bar{R}_0$. For $i = 1, 2$ let $(c_i, \gamma_i) \in \mathcal{C} \times G(E)$ be such that γ_i is a near representative for (c, c_i) satisfying the conditions of some cell rank, and a term over (c, s_i, c_i) for some s_i of label t. If $u_1, u_2 \in \mathsf{U}_c^0(M)$ satisfy $u_1 \gamma_1 f_a = u_2 \gamma_2 f_a$ for $0 \leq a \leq \varrho$, then $c_1 = c_2$ and $\gamma_1 = \gamma_2$.*

Proof. **Step 1: the moved elements are terms.** For $u \in \mathsf{U}_c^0(M)$, $h_c^{-1} u h_c \in \mathrm{K}(M) \subseteq K$ has reduction 1, so $h_c^{-1}(u\gamma_i)h_{c_i} = (h_c^{-1} u h_c)(h_c^{-1}\gamma_i h_{c_i}) \in K$ has reduction s_i: $u_i\gamma_i$ is a term over (c, s_i, c_i), $i = 1, 2$.

Step 2: equal reductions. As $u_1\gamma_1$ and $u_2\gamma_2$ have the same frame image, Lemma 5.2.32 (i) gives $s_2^{-1} s_1 \in \bar{R}_0$, so $s_1\bar{R}_0 = s_2\bar{R}_0$ and s_1, s_2 have the same coset representative r_0. Both have the label t, so $s_1 = r_0 t = s_2$.

Step 3: one class and one double coset. By Lemma 5.2.32 (ii) for $u_1\gamma_1$ and $u_2\gamma_2$, terms over (c, s_1, c_1) and (c, s_1, c_2) with the same frame image, $c_1 = c_2 =: c'$ and $\delta = (u_1\gamma_1)^{-1} u_2 \gamma_2 \in \mathsf{U}_{c'}^0(M)$. Hence $\gamma_2 = u_2^{-1} u_1 \gamma_1 \delta \in \mathsf{U}_c^0(M)\, \gamma_1\, \mathsf{U}_{c'}^0(M)$. Both γ_1 and γ_2 are near representatives for (c, c'), so both lie in $R_{c,c'}$, which contains exactly one element of each double coset $\mathsf{U}_c^0(M) \backslash G(E) / \mathsf{U}_{c'}^0(M)$. Therefore $\gamma_1 = \gamma_2$. □

Lemma 5.3.69. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, and $c \in \mathcal{C}$. The set of matrices $Z(\gamma)$, γ a near term that is a term over (c, s, c') for some $s \in \bar{G}$ and $c' \in \mathcal{C}$, has at most $b^{270n^3 f}$ elements.*

Proof. Let γ be a near term that is a term over (c, s, c'), and $0 \leq a, b' \leq \varrho$.

Integrality. With $k_0 = h_c^{-1}\gamma h_{c'} \in K$, and since the frame vectors are lattice basis vectors fixed by $h_c, h_{c'}$, $h(f_a, \gamma f_{b'}) = h(f_a, k_0 f_{b'}) \in h(L_0, L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24 (i), (ii); so $Z(\gamma)_{ab'} \in E \cap \hat{\mathcal{O}} = \mathcal{O}_E$.

Size. As γ is a near term, there is a boxed $y \in G_\infty$ (with $\gamma^{-1}y$ boxed), and Lemma 5.3.75 (iii), applied to y, gives $|v(Z(\gamma)_{ab'})| \le B := b^{73n}$ for every $v \in \Phi$.

Counting integers. By Lemma 5.3.77, at most $(1+2B)^{2f}$ elements $\alpha \in \mathcal{O}_E$ satisfy $|v(\alpha)| \le B$ for all $v \in \Phi$.

Conclusion. Each of the n^2 entries of $Z(\gamma)$ is one of at most $(1+2B)^{2f}$ integers, and $1 + 2b^{73n} \le 3b^{73n} \le b^{74n}$ ($b \ge 1000$, (FG2)). So at most $(b^{74n})^{2fn^2} = b^{148n^3 f} \le b^{270n^3 f}$ matrices occur. □

Lemma 5.3.70. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, $c \in \mathcal{C}$, $0 \le j \le \varrho + 1$, integers $r, l \ge 0$ with $r + l = k$, and $L \subseteq L_c \cap X_0(\mathbb{A}_f)$ a lattice of rank r with $\mathrm{disc}_r(L) \le b^{n^3 f}$. Let $B \subseteq \mathcal{C} \times G(E)$ be a set of pairs (c', γ) such that: every γ is a near term, a term over (c, s, c') for some s, and satisfies the conditions of cell rank j; all the matrices $Z(\gamma)$ are equal; $L \subseteq L_c(\gamma)$ for all; and whenever $(c_1, \gamma_1), (c_2, \gamma_2) \in B$ and $u_1, u_2 \in \mathsf{U}_c^0(M)$ satisfy $u_1\gamma_1 f_a = u_2\gamma_2 f_a$ for $0 \le a \le \varrho$, the two pairs are equal. Then*

$$\#B \le b^{(366n^4 + 2n^3(\varrho + ln + n\varrho))f}.$$

Proof. For $p = (c', \gamma) \in B$ write $\gamma_p = \gamma$, and let s_p be such that γ_p is a term over (c, s_p, c'). Let Z be the common matrix $Z(\gamma_p)$, $Z_<$ its first ϱ rows, and $D = D_j(Z)$, nonzero by (TA5). By Lemma 5.3.74 (iii), Z has entries in $\mathcal{O}_E$. Put $d = MD$ and $d' = MD\bar{D}$, nonzero elements of $\mathcal{O}_E$.

The three moduli. By (FG6), $M \le b^{23}$, and by Lemma 5.2.20 (each γ_p is a near term) $|v(D)|^2 \le b^{270n^2}$ for all $v \in \Phi$. Hence

$$[d] = \prod_v M^2 |v(D)|^2 \le b^{(46+270n^2)f} \quad \text{and} \quad [d'] = \prod_v M^2 |v(D)|^4 \le b^{(46+540n^2)f};$$

as $n \ge 10^5$, $46 + 540n^2 \le 2n^3$, so

$$[d] \le b^{2n^3 f}, \qquad [d'] \le b^{2n^3 f}, \qquad \mathrm{disc}_r(L)^2 \le b^{2n^3 f}.$$

First round: lifts of L. Write $L = ya\,\hat{\mathcal{O}}^r$ with $y \in \mathrm{M}_{k\times r}(E)$ of rank r and $a \in \mathrm{GL}_r(\mathbb{A}_f)$ (LT3). For each $p \in B$, as $L \subseteq L_c(\gamma_p)$, Lemma 5.3.71 (i) gives $w_p \in \mathrm{M}_{\varrho\times r}(E)$ with $w_p a$ integral and $Q_0(\gamma_p)^* y = -MZ_<^* w_p$. Colour p by the classes of the ϱ rows of $w_p a$ in $\hat{\mathcal{O}}^r / K_L$; by Lemma 5.3.71 (ii) there are at most $\mathrm{disc}_r(L)^{2\varrho}$ colours. Fix one colour class B_1 and $p_1 \in B_1$. For $p \in B_1$ the rows of $(w_{p_1} - w_p)a$ lie in K_L, so Lemma 5.3.71 (iii) gives $u_p \in \mathsf{U}_c^0(M)$ with

$$Q_0(u_p\gamma_p)^* y = Q_0(\gamma_p)^* y - MZ_<^*(w_{p_1} - w_p) = -MZ_<^* w_{p_1}.$$

Put $\gamma_p^{(1)} = u_p\gamma_p$. By Lemma 5.3.72 (i), $Z(\gamma_p^{(1)}) = Z$, and $\gamma_p^{(1)}$ satisfies the conditions of cell rank j and is a term over (c, s_p, c').

Second round: the definite parts. Let $W = \{t \in X_0 : y^* t = 0\}$, of dimension $k - r = l$, and $P = L_c \cap W(\mathbb{A}_f)$. For $p \in B_1$ let $\Delta_p = Q_0(\gamma_p^{(1)}) - Q_0(\gamma_{p_1}^{(1)})$. Its columns lie in P: first, $y^*\Delta_p = 0$ by the first round, so they lie in W. Second, for each b the vector $\gamma_p^{(1)} f_b - \gamma_{p_1}^{(1)} f_b$ lies in L_c, since a term γ over (c, s, c') satisfies $\gamma f_b = h_c(h_c^{-1}\gamma h_{c'}) f_b \in h_c L_0 = L_c$; its coordinates along z and along $e_1^*, \dots, e_\varrho^*$ are $-h(z, \cdot)/N$ and $h(e_a, \cdot)$, determined by Z, hence 0. As $L_c = \bigoplus_a \hat{\mathcal{O}} e_a \oplus h_c L_0^{\mathrm{m}} \oplus \bigoplus_a \hat{\mathcal{O}} e_a^*$, with L_0^{m} the span of the middle basis vectors in $(X_0 \oplus Ez)(\mathbb{A}_f)$ and h_c preserving $(X_0 \oplus Ez)(\mathbb{A}_f)$, the component of this vector in $X_0(\mathbb{A}_f)$, which is the column b of Δ_p, lies in $L_c \cap X_0(\mathbb{A}_f)$.

At each finite place w the $\mathcal{O}_w$-module $P_w = L_{c,w} \cap W_w$ is finitely generated and torsion-free, hence free, and it spans W_w, so it has rank l and $[P_w : dP_w] = |d|_w^{-l}$; as $P = \prod_w P_w$, $[P : dP] = [d]^l$. Colour $p \in B_1$ by the classes of the n columns of Δ_p in P/dP: at most $[d]^{ln}$ colours. Fix a colour class $B_2 \subseteq B_1$ and $p_2 \in B_2$. For $p \in B_2$, $Q_0(\gamma_p^{(1)}) - Q_0(\gamma_{p_2}^{(1)}) = \Delta_p - \Delta_{p_2} = MD\,T_p$ with a rational matrix T_p whose columns lie in $P \subseteq L_c \cap X_0(\mathbb{A}_f)$. By Lemma 5.3.73, applied to $\gamma_{p_2}^{(1)}$ and $\gamma_p^{(1)}$ (cell rank j, the same integral Z, $D_j = D$), there is $u'_p \in \mathsf{U}_c^0(M)$ with $Q_0(u'_p\gamma_p^{(1)}) = Q_0(\gamma_{p_2}^{(1)})$. Put $\gamma_p^{(2)} = u'_p\gamma_p^{(1)}$; by Lemma 5.3.72 (i), $Z(\gamma_p^{(2)}) = Z$ and $\gamma_p^{(2)}$ is a term over (c, s_p, c'), and all $\gamma_p^{(2)}$, $p \in B_2$, have the same matrix Q_0.

Third round: shared vectors and isotropic parts. By Lemma 5.3.74 (ii), $\mathrm{sh}_j(\gamma_p^{(2)}) = \mathrm{sh}_j(\gamma_p)$, and by Lemma 5.3.74 (iii), (iv) its ϱn entries are algebraic integers with $|v(\cdot)| \le b^{182n^2}$ for all $v \in \Phi$. By Lemma 5.3.77 each entry takes at most $(1 + 2b^{182n^2})^{2f} \le b^{366n^2 f}$ values ($b \ge 1000$), so $\mathrm{sh}_j(\gamma_p)$ takes at most $b^{366n^2 f\varrho n} \le b^{366n^4 f}$ values. Colour B_2 by $\mathrm{sh}_j(\gamma_p)$ and fix a colour class B_3. By Lemma 5.3.74 (iii), $Q_A(\gamma_p^{(2)}) \in \mathrm{M}_{\varrho\times n}(\mathcal{O}_E)$; colour B_3 by the class of $Q_A(\gamma_p^{(2)})$ modulo $d'\mathrm{M}_{\varrho\times n}(\mathcal{O}_E)$: at most $\#(\mathcal{O}_E/d'\mathcal{O}_E)^{\varrho n} = [d']^{n\varrho}$ colours. Let B_4 be a colour class and $p, q \in B_4$. Then $\gamma_p^{(2)}$ and $\gamma_q^{(2)}$ have the same integral Z with $D_j(Z) = D \ne 0$, the same shared-vector matrix, the same Q_0, and $Q_A(\gamma_q^{(2)}) - Q_A(\gamma_p^{(2)}) = MD\bar{D}\,\Psi$ with Ψ integral. By Lemma 5.3.76 there is $u'' \in \mathsf{U}_c^0(M)$ with $u''\gamma_p^{(2)} f_b = \gamma_q^{(2)} f_b$ for all b, that is,

$$(u''u'_pu_p)\,\gamma_p f_b = (u'_qu_q)\,\gamma_q f_b \qquad (0 \le b \le \varrho),$$

with both factors in the group $\mathsf{U}_c^0(M)$. By the separation hypothesis on B, $p = q$. So $\#B_4 \le 1$.

Count. Multiplying the numbers of colours of the four colourings,

$$\#B \le \mathrm{disc}_r(L)^{2\varrho}\,[d]^{ln}\,b^{366n^4 f}\,[d']^{n\varrho} \le b^{2n^3 f\varrho}\,b^{2n^3 fln}\,b^{366n^4 f}\,b^{2n^3 fn\varrho} = b^{(366n^4+2n^3(\varrho+ln+n\varrho))f}.$$

□

Lemma 5.3.71. *Let $\mathfrak{S}$ be a filling datum over a field E that carries Weil data, and $c \in \mathcal{C}$. Let $L \subseteq L_c \cap X_0(\mathbb{A}_f)$ be a lattice of rank r, $L = ya\,\hat{\mathcal{O}}^r$ with $y \in \mathrm{M}_{k\times r}(E)$ of rank r and $a \in \mathrm{GL}_r(\mathbb{A}_f)$. Conjugate transposes $(\cdot)^*$ are formed with the conjugation of E and of $\mathbb{A}_f$.*

(i) If $\gamma \in G(E)$ and $L \subseteq L_c(\gamma)$, there is $w \in \mathrm{M}_{\varrho\times r}(E)$ with $wa \in \mathrm{M}_{\varrho\times r}(\hat{\mathcal{O}})$ and $Q_0(\gamma)^*y + M\,Z_<(\gamma)^*w = 0$.

(ii) The Gram matrix $G_L = (ya)^*(ya)$ has entries in $\hat{\mathcal{O}}$, and the $\hat{\mathcal{O}}$-module $K_L \subseteq \hat{\mathcal{O}}^r$ of row vectors spanned by the rows of G_L has index $[\hat{\mathcal{O}}^r : K_L] = \mathrm{disc}_r(L)^2$.

(iii) If $w, w' \in \mathrm{M}_{\varrho\times r}(E)$ and every row of $(w' - w)a$ lies in K_L, there is $u \in \mathsf{U}_c^0(M)$ with

$$Q_0(u\gamma)^*y = Q_0(\gamma)^*y - M\,Z_<(\gamma)^*(w' - w) \qquad \text{for every } \gamma \in G(E).$$

Proof. Write $B = ya$, with columns $B_1, \dots, B_r \in L$.

(i) By (LT1), for each i there is $t_i = \sum_{a<\varrho} W_{ai}e_{a+1}$ with $W \in \mathrm{M}_{\varrho\times r}(\hat{\mathcal{O}})$ such that $h(\gamma f_b, B_i + Mt_i) = 0$ for $0 \le b \le \varrho$. The components of γf_b outside X_0 are orthogonal to X_0, and X_0 carries the unit form, so $h(\gamma f_b, B_i) = (Q_0(\gamma)^*B)_{bi}$; and $h(\gamma f_b, e_{a+1}) = \overline{h(e_{a+1}, \gamma f_b)} = \overline{Z_<(\gamma)_{ab}}$, so $h(\gamma f_b, Mt_i) = M(Z_<(\gamma)^*W)_{bi}$. Hence $S = Wa^{-1}$ solves the linear system

$$C\,s = d, \qquad C = M\,Z_<(\gamma)^* \in \mathrm{M}_{n\times\varrho}(E), \quad d = -Q_0(\gamma)^*y \in \mathrm{M}_{n\times r}(E),$$

over $\mathbb{A}_f$. Its component at one finite place solves it over a field containing E, so the system is consistent over E: there is $s_0 \in \mathrm{M}_{\varrho\times r}(E)$ with $Cs_0 = d$. Let $\kappa_1, \dots, \kappa_q$ be a basis of the E-space of $\kappa \in \mathrm{M}_{\varrho\times r}(E)$ with $C\kappa = 0$; as $\mathbb{A}_f$ is an E-vector space, the solutions over $\mathbb{A}_f$ are the $s_0 + \sum_m \lambda_m\kappa_m$ with $\lambda_m \in \mathbb{A}_f$, so $S = s_0 + \sum_m \lambda_m\kappa_m$ for some λ_m. The set $\mathcal{U} = \{s : sa \in \mathrm{M}_{\varrho\times r}(\hat{\mathcal{O}})\}$ is open in $\mathrm{M}_{\varrho\times r}(\mathbb{A}_f)$ and contains S. The field E is dense in $\mathbb{A}_f$: for $x \in \mathbb{A}_f$ and an integer $m \geq 1$, write $x/m = \alpha + o$ with $\alpha \in E$, $o \in \hat{\mathcal{O}}$; then $x - m\alpha \in m\hat{\mathcal{O}}$, and the $m\hat{\mathcal{O}}$ form a basis of neighbourhoods of 0. Choose $\lambda'_m \in E$ so close to λ_m that $w = s_0 + \sum_m \lambda'_m\kappa_m \in \mathcal{U}$. Then w is rational, wa is integral and $M\,Z_<(\gamma)^*w = -Q_0(\gamma)^*y$.

(ii) $(G_L)_{ii'} = h(B_i, B_{i'}) \in h(L_c, L_c) = h(L_0, L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24. Further $\det G_L = \overline{\det a}\,\det a\,\det(y^*y)$, and $\det(y^*y) \in E^\times$, as y has rank r and the unit form is definite; so $G_L \in \mathrm{GL}_r(\mathbb{A}_f)$, and G_L lies in $\mathrm{GL}_r(\mathcal{O}_w)$ at almost all finite places w. At each w, by the elementary divisor theorem over the principal ideal domain $\mathcal{O}_w$, $[\mathcal{O}_w^r : \mathcal{O}_w^r G_{L,w}] = |\det G_{L,w}|_w^{-1}$. As $\hat{\mathcal{O}} = \prod_w \mathcal{O}_w$, $K_L = \hat{\mathcal{O}}^r G_L = \prod_w \mathcal{O}_w^r G_{L,w}$ and

$$[\hat{\mathcal{O}}^r : K_L] = \prod_w |\det G_{L,w}|_w^{-1} = [\det(B^*B)] = \mathrm{disc}_r(L)^2$$

by (LT3).

(iii) Write $(w' - w)a = tG_L = t\,a^*y^*y\,a$ with $t \in \mathrm{M}_{\varrho\times r}(\hat{\mathcal{O}})$; then $w' - w = t\,a^*y^*y$. The matrix $y^*y \in \mathrm{GL}_r(E)$ is hermitian. Put $\eta = y\,(y^*y)^{-1}(w' - w)^* \in \mathrm{M}_{k\times\varrho}(E)$. Then $\eta^*y = (w' - w)(y^*y)^{-1}y^*y = w' - w$, and over $\mathbb{A}_f$, using $(w' - w)^* = y^*y\,a\,t^*$,

$$\eta = y\,(y^*y)^{-1}y^*y\,a\,t^* = ya\,t^*,$$

whose columns are $\hat{\mathcal{O}}$-combinations of the columns of ya (t^* is integral); so they lie in $L \subseteq L_c \cap X_0(\mathbb{A}_f)$. By Lemma 5.3.72 (ii) there is $u \in \mathsf{U}_c^0(M)$ with $Q_0(u\gamma) = Q_0(\gamma) - M\eta Z_<(\gamma)$ for every γ; then $Q_0(u\gamma)^*y = Q_0(\gamma)^*y - M\,Z_<(\gamma)^*\eta^*y = Q_0(\gamma)^*y - M\,Z_<(\gamma)^*(w' - w)$. □

Lemma 5.3.72. *Let $\mathfrak{S}$ be a filling datum over a field E that carries Weil data, and $c \in \mathcal{C}$.*

(i) If $u \in G(E)$ fixes $f_0, \dots, f_\varrho$, then $Z(u\gamma) = Z(\gamma)$ for every $\gamma \in G(E)$, and $u\gamma$ satisfies the conditions of cell rank j whenever γ does. If $u \in \mathsf{U}_c^0(M)$ and γ is a term over (c, s, c'), then $u\gamma$ is a term over (c, s, c').

(ii) Let $\eta \in \mathrm{M}_{k\times\varrho}(E)$ have its columns in $L_c \cap X_0(\mathbb{A}_f)$. There is $u \in \mathsf{U}_c^0(M)$ with $Q_0(u\gamma) = Q_0(\gamma) - M\,\eta\,Z_<(\gamma)$ for every $\gamma \in G(E)$.

(iii) Let $\sigma \in \mathrm{M}_\varrho(\mathcal{O}_E)$ with $\sigma^* = -\sigma$ (σ^* the conjugate transpose). There is $u \in \mathsf{U}_c^0(M)$ with

$$u\gamma f_b = \gamma f_b + M\sum_{a=0}^{\varrho-1}\big(\sigma Z_<(\gamma)\big)_{ab}\,e_{a+1} \qquad (\gamma \in G(E),\ 0 \leq b \leq \varrho);$$

in particular $Q_A(u\gamma) = Q_A(\gamma) + M\sigma Z_<(\gamma)$, and the other coordinates of the frame vectors $u\gamma f_b$ are those of γf_b.

Proof. **(i)** As u^{-1} fixes the frame, $Z(u\gamma)_{ab} = h(f_a, u\gamma f_b) = h(u^{-1}f_a, \gamma f_b) = Z(\gamma)_{ab}$. The conditions of cell rank j (TA5) are: the pairs (x, y) with $f \cdot x = \gamma f \cdot y$ and prescribed entries, $D_j(\gamma) \neq 0$, and $\varrho < j \iff \mathsf{g}(\gamma) \neq 0$. The last two depend on $Z(\gamma)$ only, and a pair for γ is a pair for $u\gamma$, as $u\gamma f \cdot y = u(f \cdot x) = f \cdot x$. If $u \in \mathsf{U}_c^0(M)$ and γ is a term over (c, s, c'), then $h_c^{-1}u\gamma h_{c'} = (h_c^{-1}uh_c)(h_c^{-1}\gamma h_{c'})$ with $h_c^{-1}uh_c \in \mathrm{K}(M) = \ker \mathrm{red} \subseteq K$, so it lies in K and has reduction s.

(ii) Fix μ as in Lemma 5.2.23 (ii), and for $1 \le a \le \varrho$ let $w_a \in X_0$ be the column of η of index $a-1$. Then $w_a \in V \cap L_c$ lies in X_0, so Lemma 5.2.23 (ii) with $s=0$ gives $E_a = E_a(w_a, 0) \in \mathsf{U}_c(M)$,

$$E_a x = x + M\,h(w_a, x)\,e_a - M\,h(e_a, x)\,w_a - M^2\mu\,h(w_a, w_a)\,h(e_a, x)\,e_a \qquad (x \in V),$$

and $E_a f_b = f_b + M\,h(w_a, f_b)\,e_a = f_b$, as X_0 is orthogonal to $e_1, \dots, e_\varrho, z$. So $E_a \in \mathsf{U}_c^0(M)$, and $u = E_1 E_2 \cdots E_\varrho \in \mathsf{U}_c^0(M)$, a group. Since the $e_{a'}$ are isotropic, mutually orthogonal and orthogonal to X_0, $h(e_{a'}, E_a x) = h(e_{a'}, x)$ for all a', and the coordinate vector of $E_a x$ along $x_1, \dots, x_k$ is that of x minus $M\,h(e_a, x)\,w_a$. By induction over the factors, the coordinate vector of ux along $x_1, \dots, x_k$ is that of x minus $M\sum_a h(e_a, x) w_a$. For $x = \gamma f_b$, $h(e_a, \gamma f_b) = Z_<(\gamma)_{a-1,b}$, so $Q_0(u\gamma) = Q_0(\gamma) - M\eta Z_<(\gamma)$.

(iii) Define $u\colon V \to V$ by $ux = x + M\sum_{a,a'=0}^{\varrho-1} \sigma_{aa'}\,h(e_{a'+1}, x)\,e_{a+1}$. As h is conjugate-linear in the first variable, the e_a are isotropic and mutually orthogonal, and $\overline{h(e_{a'+1}, x)} = h(x, e_{a'+1})$,

$$h(ux, ux') - h(x, x') = M\sum_{a,a'} (\overline{\sigma_{a'a}} + \sigma_{aa'}) h(x, e_{a+1})\,h(e_{a'+1}, x') = 0,$$

because $\sigma^* = -\sigma$. So u preserves h; it is invertible, since $(u-1)V \subseteq A$ and $(u-1)A = 0$ give $(u-1)^2 = 0$; hence $u \in G(E)$. It is the identity on $A^\perp$ (where all $h(e_{a'+1}, \cdot)$ vanish) and $(u-1)V \subseteq A \subseteq A^\perp$, so it preserves every member of the flag above and acts trivially on its successive quotients: $u \in \mathsf{U}(E)$. For $x \in L_c$ write $x = h_c x'$ with $x' \in L_0$; as h_c fixes $e_{a'+1}$ and is unitary, $h(e_{a'+1}, x) = h(e_{a'+1}, x') \in h(L_0, L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24, and $e_{a+1} = h_c e_{a+1} \in L_c$; with σ integral, $(u-1)L_c \subseteq ML_c$, so $u \in \mathsf{U}_c(M)$ by Lemma 5.2.23 (i). Since $h(e_{a'+1}, f_b) = 0$ for all a', b, u fixes the frame: $u \in \mathsf{U}_c^0(M)$. Finally $h(e_{a'+1}, \gamma f_b) = Z_<(\gamma)_{a'b}$, so $u\gamma f_b = \gamma f_b + M\sum_a (\sigma Z_<(\gamma))_{ab} e_{a+1}$, which changes only the coordinates along $e_1, \dots, e_\varrho$. □

Lemma 5.3.73. *Let $\mathfrak{S}$ be a filling datum over a field E that carries Weil data, $c \in \mathcal{C}$ and $0 \le j \le \varrho + 1$. Let $\gamma, \gamma' \in G(E)$ satisfy the conditions of cell rank j, with $Z(\gamma) = Z(\gamma')$ having entries in $\mathcal{O}_E$, and suppose that*

$$Q_0(\gamma') - Q_0(\gamma) = M\,D_j(\gamma)\,\Psi$$

for a matrix $\Psi \in \mathrm{M}_{k\times n}(E)$ whose columns lie in $L_c \cap X_0(\mathbb{A}_f)$. Then there is $u \in \mathsf{U}_c^0(M)$ with $Q_0(u\gamma') = Q_0(\gamma)$.

Proof. Write $Z = Z(\gamma)$, $D = D_j(\gamma)$, which is nonzero by (TA5), $Z_< = Z_<(\gamma)$ and $Z^{\mathrm{pd}} = Z_j^{\mathrm{pd}}(\gamma)$ ((TA14) of Definition 5.1.12); then $\det Z^{\mathrm{pd}} = D$ and Z^{pd}, $\mathrm{adj}(Z^{\mathrm{pd}})$ have entries in $\mathcal{O}_E$. Put $\Delta = Q_0(\gamma') - Q_0(\gamma)$ and

$$\eta' = \Delta\,(Z^{\mathrm{pd}})^{-1} = M\,\Psi\,\mathrm{adj}(Z^{\mathrm{pd}}).$$

The columns of $\Psi\,\mathrm{adj}(Z^{\mathrm{pd}})$ are $\mathcal{O}_E$-combinations of columns of Ψ, hence lie in the $\hat{\mathcal{O}}$-module $L_c \cap X_0(\mathbb{A}_f)$; so do those of η'.

The columns of η' of index $m \ge j$ vanish. The column m of $Q_0(\gamma)(Z^{\mathrm{pd}})^{-1}$ is D^{-1} times the column m of $Q_0(\gamma)\mathrm{adj}(Z^{\mathrm{pd}})$, which is 0 for $j \le m \le \varrho$ by Lemma 5.3.74 (i); the same holds for γ', which has the same Z^{pd}.

The column of η' of index ϱ vanishes. For $j \le \varrho$ this is the previous step; let $j = \varrho + 1$, so $Z^{\mathrm{pd}} = Z$. Let $t = Z^{-1}\varepsilon_\varrho$ and $q = \sum_l t_l \gamma f_l$. Then $h(f_i, q) = (Zt)_i = \delta_{i\varrho}$, so the coordinates of q along $e_1^*, \dots, e_\varrho^*$ (which are $h(e_a, q)$) vanish and its coordinate along z is $-h(z, q)/N = -1/N$: $q = q_A + q_0 - z/N$ with $q_A \in \langle e_1, \dots, e_\varrho\rangle$ and $q_0 \in X_0$ the column ϱ of $Q_0(\gamma)Z^{-1}$. As γ is an

isometry and in the frame only $h(z,z) = -N$ is nonzero, $h(q,q) = h(f\cdot t, f\cdot t) = -N\, t_\varrho \bar{t}_\varrho$; on the other hand $h(q,q) = h(q_0,q_0) - 1/N$. So for every $v \in \Phi$,

$$\sum_i |v(q_{0,i})|^2 = \frac{1}{N} - N|v(t_\varrho)|^2 \le \frac{1}{N}.$$

The same holds for the column q_0' of $Q_0(\gamma')Z^{-1}$, and the column $w = q_0' - q_0$ of η' satisfies $\sum_i |v(w_i)|^2 \le 2(\frac{1}{N} + \frac{1}{N}) = \frac{4}{N} < 1$ for all v, by (FG2). Now $w \in E^k$ lies in L_c, and $h(L_c, L_c) = h(L_0, L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24 ($L_c = h_c L_0$ with h_c unitary); so $h(w,w) \in E \cap \hat{\mathcal{O}} = \mathcal{O}_E$, with $|v(h(w,w))| < 1$ for all $v \in \Phi$. Then $|\mathrm{N}_{E/\mathbb{Q}}(h(w,w))| = \prod_v |v(h(w,w))|^2 < 1$, so $h(w,w) = 0$ and, the form being definite, $w = 0$.

The move. Let $\eta \in \mathrm{M}_{k\times\varrho}(E)$ be the matrix of the first ϱ columns of $\Psi \operatorname{adj}(Z^{\mathrm{pd}})$; its columns lie in $L_c \cap X_0(\mathbb{A}_f)$, and those of index $m \ge j$ vanish by the first step ($M \ne 0$). Since the columns of η' of index $\ge j$ and of index ϱ vanish, and the row of Z^{pd} of index $m < \min(j,\varrho)$ is the row m of $Z_<$,

$$\Delta = \eta' Z^{\mathrm{pd}} = \sum_{m<\min(j,\varrho)} \eta' \varepsilon_m \varepsilon_m^{\mathsf{T}} Z^{\mathrm{pd}} = M\,\eta\, Z_<.$$

By Lemma 5.3.72 (ii) there is $u \in \mathsf{U}_c^0(M)$ with $Q_0(u\gamma') = Q_0(\gamma') - M\eta Z_< = Q_0(\gamma') - \Delta = Q_0(\gamma)$, the first ϱ rows of $Z(\gamma')$ being those of Z. □

Lemma 5.3.74. *Let $\mathfrak{S}$ be a filling datum over E, $0 \le j \le \varrho+1$, and let $\gamma \in G(E)$ satisfy the conditions of cell rank j.*

(i) For $j \le m \le \varrho$ the vector $\sum_l \operatorname{adj}(Z_j^{\mathrm{pd}}(\gamma))_{lm}\, \gamma f_l$ lies in the E-span of $f_0,\dots,f_\varrho$; in particular the column of index m of $Q_0(\gamma)\operatorname{adj}(Z_j^{\mathrm{pd}}(\gamma))$ is 0.

(ii) If $u \in G(E)$ fixes $f_0,\dots,f_\varrho$, then $Z_j^{\mathrm{pd}}(u\gamma) = Z_j^{\mathrm{pd}}(\gamma)$ and $\mathrm{sh}_j(u\gamma) = \mathrm{sh}_j(\gamma)$.

(iii) If γ is a term over (c,s,c') for some $c,c' \in \mathcal{C}$ and s, then $Z(\gamma)$, $Q_A(\gamma)$, $Z_j^{\mathrm{pd}}(\gamma)$ and $\mathrm{sh}_j(\gamma)$ have entries in $\mathcal{O}_E$.

(iv) If $\mathfrak{S}$ is admissible, E carries Weil data and γ is a near term, then $|v(\mathrm{sh}_j(\gamma)_{am})| \le b^{182n^2}$ for all $v \in \Phi$, a and m.

Proof. Expanding along the rows of index $\ge j$, $\det Z^{\mathrm{pd}} = D_j(\gamma)$, which is nonzero by (TA5).

(i) We may assume $j \le \varrho$. For $j \le a \le \varrho$ let $(x^{(a)}, y^{(a)})$ be the pair of (TA5): $f\cdot x^{(a)} = \gamma f \cdot y^{(a)}$ with $y_a^{(a)} = 1$ and $y_b^{(a)} = 0$ for $b > a$. Then $(Z(\gamma)y^{(a)})_i = h(f_i, \gamma f\cdot y^{(a)}) = h(f_i, f\cdot x^{(a)})$, which vanishes for $i < \varrho$, because $f_i = e_{i+1}$ is isotropic and orthogonal to $e_1,\dots,e_\varrho,z$. The rows of Z^{pd} of index $i < j$ are rows of $Z(\gamma)$ with $i < \varrho$, so $Z^{\mathrm{pd}}y^{(a)} = \sum_{j\le i\le a} y_i^{(a)}\varepsilon_i$, with coefficient 1 at ε_a. Hence the vectors $Z^{\mathrm{pd}}y^{(a)}$, $j \le a \le \varrho$, span $\langle \varepsilon_m : m \ge j\rangle$, and for $m \ge j$ the vector $\operatorname{adj}(Z^{\mathrm{pd}})\varepsilon_m = D_j(\gamma)\,(Z^{\mathrm{pd}})^{-1}\varepsilon_m$ lies in the span of the $y^{(a)}$. So $\sum_l \operatorname{adj}(Z^{\mathrm{pd}})_{lm}\gamma f_l = \gamma f\cdot(\operatorname{adj}(Z^{\mathrm{pd}})\varepsilon_m)$ lies in the span of the $\gamma f\cdot y^{(a)} = f\cdot x^{(a)}$, hence in the span of the frame, whose vectors have no coordinates along $x_1,\dots,x_k$.

(ii) As u^{-1} fixes the frame, $Z(u\gamma)_{ab} = h(u^{-1}f_a, \gamma f_b) = Z(\gamma)_{ab}$, so $Z_j^{\mathrm{pd}}(u\gamma) = Z^{\mathrm{pd}}$. For $m \ge j$, by (i) the vector $t_m = \sum_l \operatorname{adj}(Z^{\mathrm{pd}})_{lm}\gamma f_l$ lies in the span of the frame, which u fixes; so $\sum_l \operatorname{adj}(Z^{\mathrm{pd}})_{lm} u\gamma f_l = ut_m = t_m$. The column of index m of sh_j consists of the coordinates of this vector along $e_1,\dots,e_\varrho$, for $u\gamma$ and for γ alike.

(iii) Let $k_0 = h_c^{-1}\gamma h_{c'} \in K$, so $k_0 L_0 = L_0$. As $h_{c'}$ fixes f_b, $\gamma f_b = h_c k_0 f_b$; as h_c is unitary and fixes f_a and e_a^*,

$$Z(\gamma)_{ab} = h(f_a, k_0 f_b), \qquad Q_A(\gamma)_{ab} = h(e_{a+1}^*, \gamma f_b) = h(e_{a+1}^*, k_0 f_b),$$

both in $h(L_0, L_0) \subseteq \hat{\mathcal{O}}$ by Lemma 5.2.24, and in E; so they lie in $E \cap \hat{\mathcal{O}} = \mathcal{O}_E$. The entries of Z^{pd} are entries of $Z(\gamma)$, 0 or 1, so Z^{pd} and $\mathrm{adj}(Z^{\mathrm{pd}})$ are integral, and so is $\mathrm{sh}_j(\gamma)$, whose nonzero columns are columns of $Q_A(\gamma)\mathrm{adj}(Z^{\mathrm{pd}})$.

(iv) Fix v, a and m; for $m < j$ there is nothing to prove, so let $m \geq j$. Put $\xi = \mathrm{adj}(Z^{\mathrm{pd}})\varepsilon_m \in E^n$. By (i), $\gamma f \cdot \xi = f \cdot w$ for some $w \in E^n$, and $\mathrm{sh}_j(\gamma)_{am} = w_a$ is its coordinate along $f_a = e_{a+1}$. Fix a boxed $y \in G_\infty$ such that $y' = \gamma^{-1} y$ is boxed; then $y_v^{-1}\gamma_v = y_v'^{-1}$. Let $B_v = \mathrm{diag}(\mathrm{bs}_v)$; by (RD1), $B_v^{-1} v(\gamma) = \gamma_v B_v^{-1}$ and $B_v^{-1} f_l = \mathrm{bs}_v(f_l)^{-1} f_l$. Applying $y_v^{-1} B_v^{-1} v$ to $f \cdot w = \gamma f \cdot \xi$ gives

$$\sum_l \frac{v(w_l)}{\mathrm{bs}_v(f_l)}\, y_v^{-1} f_l = \sum_l \frac{v(\xi_l)}{\mathrm{bs}_v(f_l)}\, y_v'^{-1} f_l.$$

By Lemma 5.3.75 (ii) at y and (i) at y',

$$\frac{|v(w_a)|}{|\mathrm{bs}_v(f_a)|} \leq b^{13n^2} \sum_l \frac{|v(\xi_l)|}{|\mathrm{bs}_v(f_l)|}\, b^{6n+29}.$$

By (RD1) and (FA4), $|\mathrm{bs}_v(f_a)|/|\mathrm{bs}_v(f_l)| = |\mathrm{rs}_v|^{l-a} \leq b^{23\varrho}$. The entries of $v(Z^{\mathrm{pd}})$ have modulus at most b^{73n} (Lemma 5.3.75 (iii) for the rows of index $< j$; the others are 0 or 1), so the entries $v(\xi_l)$ of $\mathrm{adj}(v(Z^{\mathrm{pd}}))$, minors of order ϱ up to sign, have modulus at most $\varrho!\, b^{73n\varrho} \leq b^{\varrho + 73n\varrho}$ ($\varrho + 1 \leq b$ by (FG6)). With $n \leq b$ and $\varrho < n$,

$$|v(w_a)| \leq b^{13n^2} \cdot n \cdot b^{\varrho + 73n\varrho} \cdot b^{23\varrho} \cdot b^{6n+29} \leq b^{87n^2 + 29n + 30} \leq b^{182n^2}.$$

□

Lemma 5.3.75. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, $y \in G_\infty$ a boxed point and $v \in \Phi$.*

(i) $\|y_v^{-1} f_a\| \leq b^{6n+29}$ *for* $0 \leq a \leq \varrho$.

(ii) For every $t \in \mathbb{C}^n$ *and* $0 \leq a \leq \varrho$,

$$|t_a| \leq b^{13n^2} \Big\| \sum_{l=0}^{\varrho} t_l\, y_v^{-1} f_l \Big\|.$$

(iii) If $\gamma \in G(E)$ *is a near term, then* $|v(Z(\gamma)_{ab})| \leq b^{73n}$ *for* $0 \leq a, b \leq \varrho$.

Proof. **(i)** The set $\mathcal{C}$ of classes is a quotient of $\mathrm{U}(X_0)(\mathbb{A}_f)$, hence non-empty; fix $c \in \mathcal{C}$. By Lemma 5.2.21 the indicator $\mathbf{1}_{\mathrm{red}}$ of the reduced points (for the residue domain of the fixed cutoff datum) is a box partition of unity for c. A boxed point is reduced, so $\mathbf{1}_{\mathrm{red}}(y) = 1 \neq 0$, and the first condition of (RD5) gives $y_v \in \mathrm{Bx}(b^{27}\ell(y_v))$. As y is boxed, $\ell(y_v) \leq b^{6n+2}$, so $\|y_v^{-1} f_a\| \leq b^{6n+29}$.

(ii) By the definition of $\ell(y_v)$ as an infimum over R and $\ell(y_v) \leq b^{6n+2}$, there is $s \in R$ with $\|s y_v\|_{\mathrm{op}} + 1 < b^{6n+2} + 1$; write $s = uh$ with $u \in \mathsf{U}$, $h \in H = \mathrm{U}(X_0)$. The element h acts trivially on $X_0^{\perp}$, which contains $f_0, \ldots, f_\varrho$, so $s f_b = u f_b$. The element u acts trivially on the successive quotients of the flag $\langle e_1 \rangle \subset \cdots \subset \langle e_1, \ldots, e_\varrho \rangle \subset \langle e_1, \ldots, e_\varrho \rangle^{\perp}$, and $z \in \langle e_1, \ldots, e_\varrho \rangle^{\perp}$; hence

$uf_b - f_b \in \langle f_0, \dots, f_{b-1}\rangle$ for $0 \le b \le \varrho$, that is, $uf_b = \sum_{a\le b} T_{ab} f_a$ with an upper unitriangular matrix $T \in \mathrm{M}_n(\mathbb{C})$. The frame vectors are standard basis vectors of $\mathbb{C}^r$, hence orthonormal for $\langle\cdot,\cdot\rangle$, so by (i)

$$|T_{ab}| = |\langle f_a, (sy_v)(y_v^{-1} f_b)\rangle| \le \|sy_v\|_{\mathrm{op}}\, \|y_v^{-1} f_b\| \le b^{6n+2}\, b^{6n+29} = b^{12n+31}.$$

Let $q = \sum_l t_l\, y_v^{-1} f_l$. Then $sy_v q = \sum_l t_l\, sf_l = \sum_a (Tt)_a f_a$, so $|(Tt)_a| \le \|sy_v q\| \le b^{6n+2}\|q\|$ for every a. As $\det T = 1$, $t = \mathrm{adj}(T)\,(Tt)$; the entries of $\mathrm{adj}(T)$ are minors of order ϱ of T up to sign, so they have modulus at most $\varrho!\,(b^{12n+31})^\varrho \le b^{(12n+32)\varrho}$, because $\varrho! \le \varrho^\varrho \le b^\varrho$ by $\varrho + 1 \le b$ (FG6). Hence, using $n \le b$,

$$|t_a| \le n\, b^{(12n+32)\varrho}\, b^{6n+2}\|q\| \le b^{(12n+32)(n-1)+6n+3}\|q\| = b^{12n^2+26n-29}\|q\| \le b^{13n^2}\|q\|,$$

since $26n \le n^2$ ($n \ge 10^5$ by (FG6)) and $b \ge 1$.

(iii) Fix a boxed $y' \in G_\infty$ such that $x = \gamma^{-1} y'$ is boxed; then $x_v = \gamma_v^{-1} y'_v$. By Lemma 5.3.36 (i), $Z(\gamma_v) = S\, v(Z(\gamma))\, S$ with $S = \mathrm{diag}(\mathrm{bs}_v(f_0), \dots, \mathrm{bs}_v(f_\varrho))$, so

$$v(Z(\gamma)_{ab}) = Z(\gamma_v)_{ab} / (\mathrm{bs}_v(f_a)\mathrm{bs}_v(f_b)),$$

and $|\mathrm{bs}_v(f_a)\mathrm{bs}_v(f_b)| = |\mathrm{rs}_v|^{2\varrho-a-b}/N \ge 1/N$ by (RD1) and (FA4). As $y'_v \in G_1$ preserves h_1 and $y_v'^{-1}\gamma_v = x_v^{-1}$,

$$Z(\gamma_v)_{ab} = h_1(f_a, \gamma_v f_b) = h_1(y_v'^{-1} f_a,\ x_v^{-1} f_b).$$

The matrix J_1 is a hermitian involution, hence unitary, so $|h_1(w, w')| = |\langle w, J_1 w'\rangle| \le \|w\|\,\|w'\|$. With (i) for the boxed points y' and x and with (FA5),

$$|v(Z(\gamma)_{ab})| \le N\, b^{6n+29}\, b^{6n+29} \le b^{60n+12n+58} \le b^{73n}.$$

□

Lemma 5.3.76. *Let $\mathfrak{S}$ be a filling datum over a field E that carries Weil data, $c \in \mathcal{C}$ and $0 \le j \le \varrho + 1$. Let $\gamma, \gamma' \in G(E)$ satisfy: $Z(\gamma) = Z(\gamma')$ with entries in $\mathcal{O}_E$; $D_j(\gamma) \ne 0$; $\mathrm{sh}_j(\gamma) = \mathrm{sh}_j(\gamma')$; $Q_0(\gamma) = Q_0(\gamma')$; and*

$$Q_A(\gamma') - Q_A(\gamma) = M\, D_j(\gamma)\, \overline{D_j(\gamma)}\, \Psi \qquad \text{for some } \Psi \in \mathrm{M}_{\varrho\times n}(\mathcal{O}_E).$$

Then there is $u \in \mathsf{U}_c^0(M)$ with $u\gamma f_b = \gamma' f_b$ for $0 \le b \le \varrho$.

Proof. Write $Z = Z(\gamma)$, $D = D_j(\gamma)$, $Z^{\mathrm{pd}} = Z_j^{\mathrm{pd}}(\gamma) = Z_j^{\mathrm{pd}}(\gamma')$, $Z_< = Z_<(\gamma) \in \mathrm{M}_{\varrho\times n}(\mathcal{O}_E)$ (the first ϱ rows of Z), $d = MD\bar{D}$ and $\xi = Q_A(\gamma') - Q_A(\gamma) = d\Psi$. Then $d \in \mathcal{O}_E$, $\bar{d} = d$ and $d \ne 0$; $\det Z^{\mathrm{pd}} = D$, and Z^{pd}, $\mathrm{adj}(Z^{\mathrm{pd}})$ have entries in $\mathcal{O}_E$.

The frames differ only along $e_1, \dots, e_\varrho$. For $x \in V$, the coordinate of x along z is $-h(z,x)/N$ and its coordinate along e_{a+1}^* is $h(e_{a+1}, x)$. For $x = \gamma f_b$ these are $-Z_{\varrho b}/N$ and Z_{ab}, the same for γ'; the coordinates along $x_1, \dots, x_k$ agree as $Q_0(\gamma) = Q_0(\gamma')$. Hence

$$\gamma' f_b = \gamma f_b + \sum_{a=0}^{\varrho-1} \xi_{ab}\, e_{a+1} \qquad (0 \le b \le \varrho).$$

$\Psi^* Z_<$ is skew. Both frames γf and $\gamma' f$ have the Gram matrix of f. Expanding $h(\gamma' f_b, \gamma' f_{b'})$ with the display, using that the e_{a+1} are isotropic and mutually orthogonal and that $h(e_{a+1}, \gamma f_{b'}) = (Z_<)_{ab'}$, the difference $h(\gamma' f_b, \gamma' f_{b'}) - h(\gamma f_b, \gamma f_{b'})$ is $(\xi^* Z_< + Z_<^* \xi)_{bb'}$. So $\xi^* Z_< = -Z_<^* \xi$, and dividing by the real number $d \ne 0$,

$$\Psi^* Z_< = -Z_<^* \Psi. \tag{1}$$

The columns of $\Psi\,\mathrm{adj}(Z^{\mathrm{pd}})$ of index $\geq j$ and of index ϱ vanish. For $m \geq j$, the column m of $\xi\,\mathrm{adj}(Z^{\mathrm{pd}}) = Q_A(\gamma')\mathrm{adj}(Z^{\mathrm{pd}}) - Q_A(\gamma)\mathrm{adj}(Z^{\mathrm{pd}})$ is $\mathrm{sh}_j(\gamma')\varepsilon_m - \mathrm{sh}_j(\gamma)\varepsilon_m = 0$; divide by d. If $j = \varrho+1$, then $Z^{\mathrm{pd}} = Z$; let α be the matrix of the first ϱ columns of $\mathrm{adj}(Z)$. From $Z\,\mathrm{adj}(Z) = D\cdot 1$, $Z_<\alpha = D\cdot 1_\varrho$ and $Z_<\mathrm{adj}(Z)\varepsilon_\varrho = 0$. By (1), $Z_<^*\Psi\,\mathrm{adj}(Z)\varepsilon_\varrho = -\Psi^* Z_<\mathrm{adj}(Z)\varepsilon_\varrho = 0$, and $\alpha^* Z_<^* = \bar{D}\cdot 1_\varrho$, so $\bar{D}\,\Psi\mathrm{adj}(Z)\varepsilon_\varrho = \alpha^* Z_<^*\Psi\mathrm{adj}(Z)\varepsilon_\varrho = 0$; as $D \neq 0$ the column ϱ vanishes.

A solution of $\Psi\alpha Z_< = D\Psi$. Let now, for every j, $\alpha \in \mathrm{M}_{n\times\varrho}(\mathcal{O}_E)$ be the matrix of the first ϱ columns of $\mathrm{adj}(Z^{\mathrm{pd}})$. As $\mathrm{adj}(Z^{\mathrm{pd}})Z^{\mathrm{pd}} = D\cdot 1$,

$$D\Psi = \Psi\,\mathrm{adj}(Z^{\mathrm{pd}})\,Z^{\mathrm{pd}} = \sum_{m=0}^{\varrho}\left(\Psi\mathrm{adj}(Z^{\mathrm{pd}})\varepsilon_m\right)\varepsilon_m^{\mathsf{T}} Z^{\mathrm{pd}} = \sum_{m<\min(j,\varrho)}\left(\Psi\mathrm{adj}(Z^{\mathrm{pd}})\varepsilon_m\right)\varepsilon_m^{\mathsf{T}} Z_< = \Psi\alpha Z_<, \tag{2}$$

because the terms with $m \geq j$ or $m = \varrho$ vanish, the row $m < j$ of Z^{pd} is the row m of Z, which for $m < \varrho$ is the row m of $Z_<$, and the terms with $j \leq m < \varrho$ added in the last step vanish.

The skew-hermitian matrix. Put

$$\sigma = \bar{D}\,\Psi\alpha - D\,(\Psi\alpha)^* - \alpha^* Z_<^*\Psi\alpha \;\in \mathrm{M}_\varrho(\mathcal{O}_E).$$

By (1), $\alpha^*\Psi^* Z_<\alpha = -\alpha^* Z_<^*\Psi\alpha$, so $\sigma^* = D\,(\Psi\alpha)^* - \bar{D}\,\Psi\alpha + \alpha^* Z_<^*\Psi\alpha = -\sigma$. By (2) and (1),

$$\sigma Z_< = \bar{D}\,\Psi\alpha Z_< - D\,\alpha^*\Psi^* Z_< - \alpha^* Z_<^*(\Psi\alpha Z_<) = D\bar{D}\,\Psi + D\,\alpha^* Z_<^*\Psi - D\,\alpha^* Z_<^*\Psi = D\bar{D}\,\Psi.$$

By Lemma 5.3.72 (iii) there is $u \in \mathsf{U}_c^0(M)$ with $u\gamma f_b = \gamma f_b + M\sum_a(\sigma Z_<)_{ab}e_{a+1} = \gamma f_b + \sum_a \xi_{ab}e_{a+1} = \gamma' f_b$. □

Lemma 5.3.77. *Let E be a totally complex number field of degree $2f$, $\mathcal{O}_E$ its ring of integers, and Φ a set of f embeddings $E \to \mathbb{C}$ containing exactly one of each pair of complex conjugate embeddings. For every real $B \geq 0$,*

$$\#\{\alpha \in \mathcal{O}_E :\ |v(\alpha)| \leq B \text{ for all } v \in \Phi\} \leq (1+2B)^{2f}.$$

Proof. Embed E in $\mathbb{C}^\Phi \cong \mathbb{R}^{2f}$ by $\alpha \mapsto (v(\alpha))_{v\in\Phi}$ and give $\mathbb{C}^\Phi$ the norm $\|w\| = \max_v |w_v|$. For distinct $\alpha, \beta \in \mathcal{O}_E$ we have $\|\alpha - \beta\| \geq 1$: otherwise the nonzero algebraic integer $\alpha - \beta$ would have $|\mathrm{N}_{E/\mathbb{Q}}(\alpha-\beta)| = \prod_{v\in\Phi}|v(\alpha-\beta)|^2 < 1$, while the norm of a nonzero algebraic integer is a nonzero rational integer. Hence the open balls of radius $\frac{1}{2}$ (for $\|\cdot\|$) about the points α of the set are pairwise disjoint, and they lie in the ball of radius $B + \frac{1}{2}$ about 0. The balls of $\|\cdot\|$ are products of f discs, so the ball of radius t has Lebesgue measure $(\pi t^2)^f$; comparing measures, the number of points is at most $((B+\frac{1}{2})/\frac{1}{2})^{2f} = (1+2B)^{2f}$. □

Lemma 5.3.78. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data, and $r, l \geq 0$ integers with $r + l = k$. Then*

$$\sum_{c\in\mathcal{C}}\sum_{L}\mathrm{disc}_r(L)^{-k} \leq \frac{\#\mathcal{C}}{N^{kf}}\,\tau_{\mathrm{sz}}^{l+1},$$

the inner sum over the lattices $L \subseteq L_c \cap X_0(\mathbb{A}_f)$ of rank r with $\mathrm{disc}_r(L) \leq b^{n^3 f}$.

Proof. Put $H = b^{n^3 f}$, $\mu = 2\varrho + 4$, $X = 200fk\,\mathrm{rd}_E$, $K_\circ = \mathrm{U}(X_0)(\hat{\mathcal{O}})$ and $K' = \mathrm{U}(X_0)(\mathbb{A}_f) \cap K$. For $c \in \mathcal{C}$ let

$$F(c) = \sum_L \mathrm{disc}_r(L)^{-k} \in [0,\infty],$$

the sum over the lattices $L \subseteq L_c \cap X_0(\mathbb{A}_f)$ of rank r with $\mathrm{disc}_r(L) \le H$; each term is finite, as $\det(B^*B)$ is a unit of $\mathbb{A}_f$ in (LT3). We show $\sum_c F(c) \le \#\mathcal{C}\, N^{-kf} \tau_{\mathrm{sz}}^{l+1}$.

Step 1: the cosets of K'. By Lemma 5.3.80 (ii), K' is a subgroup of $K_\circ$. By (FA1), $\mathrm{K}_X \subseteq \mathrm{K}(M) \subseteq K$, so $K' \supseteq \mathrm{K}_X$ is open in the compact group $K_\circ$, and $\mathcal{A} = K_\circ/K'$ is finite; by (FA8), $\#\mathcal{A} \ge (N^k/b)^f > 0$. Choose $y_\varphi \in \varphi$ for each $\varphi \in \mathcal{A}$. For $\varphi \ne \varphi'$ the k-th rows of y_φ^{-1} and $y_{\varphi'}^{-1}$ do not differ by an element of $\pi\hat{\mathcal{O}}^{1\times k}$: otherwise $y_\varphi^{-1} y_{\varphi'} \in K \cap \mathrm{U}(X_0)(\mathbb{A}_f) = K'$ by Lemma 5.3.80 (ii), that is, $\varphi = \varphi'$.

Step 2: the translates of one u. Fix $u \in \mathrm{U}(X_0)(\mathbb{A}_f)$ and let $\mathcal{L}$ be the set of lattices $L \subseteq u\hat{\mathcal{O}}^k$ of rank r with $\mathrm{disc}_r(L) \le H$. By Lemma 5.3.80 (iii) (with $y = y_\varphi$; $\Lambda(u, y_\varphi) \subseteq u\hat{\mathcal{O}}^k$), $F([uy_\varphi]) = \sum_{L\in\mathcal{L},\, L\subseteq\Lambda(u,y_\varphi)} \mathrm{disc}_r(L)^{-k}$. Summing over φ and exchanging the sums of nonnegative terms,

$$\sum_{\varphi\in\mathcal{A}} F([uy_\varphi]) = \sum_{L\in\mathcal{L}} n(L)\,\mathrm{disc}_r(L)^{-k}, \qquad n(L) = \#\{\varphi :\ L \subseteq \Lambda(u, y_\varphi)\}.$$

By Step 1 and Lemma 5.3.81 (iii), $n(L) \le i(L) N^{fl}$; by Lemma 5.3.81 (i), $\mathrm{disc}_r(L) = i(L)\, d(W_L)$ with $i(L) \ge 1$, $d(W_L) \ge 1$, and $i(L) d(W_L) \le H$.

Step 3: an elementary inequality. For reals $d, i \ge 1$ with $di \le H$,

$$i\,(di)^{-k} \le H^{\mu+2}\, d^{-(k+1)}\, i^{-(r+\mu)}\ :$$

the quotient of the right side by the left side is $H^{\mu+2} d^{-1} i^{\,l-\mu-1}$, and $d \le H$, $i^{\,l-\mu-1} \ge i^{-(\mu+1)} \ge H^{-(\mu+1)}$ (as $1 \le i \le H$ and $l \ge 0$), so the quotient is at least 1. With Step 2,

$$\sum_{\varphi} F([uy_\varphi]) \le N^{fl} H^{\mu+2} \sum_{L\in\mathcal{L}} d(W_L)^{-(k+1)}\, i(L)^{-(r+\mu)}.$$

Step 4: the sum over L, sorted by W_L. Each W_L is a subspace of E^k of dimension r, i.e. of codimension l (Lemma 5.3.81 (i)). Fix such a W that occurs, $W = W_{L_1}$. The $L \in \mathcal{L}$ with $W_L = W$ are lattices of rank r contained in $W(\mathbb{A}_f) \cap u\hat{\mathcal{O}}^k = \mathrm{sat}(L_1) = \mathrm{sat}(L)$, with $i(L) = [\mathrm{sat}(L_1) : L]$. By Lemma 5.3.81 (ii) and Lemma 5.3.84 (degree $m = 2f$, $s = r + \mu$, $t = \mu + 1 = 2\varrho + 5 \ge 3$),

$$\sum_{L\in\mathcal{L},\ W_L = W} i(L)^{-(r+\mu)} \le \sum_{M''} [\hat{\mathcal{O}}^r : M'']^{-(r+\mu)} \le \exp\bigl(2f \cdot 2^{-2\varrho-3}\bigr) \le e^{1/8} < 2,$$

the middle sum over the submodules of $\hat{\mathcal{O}}^r$ of finite index; here $2f \le (\varrho+1)[E:\mathbb{Q}] \le 2^{2\varrho}$ by (FG6), so $2f \cdot 2^{-2\varrho-3} \le \frac{1}{8}$. Hence, by Lemma 5.3.82 (ii) with $s = l$ and $\Lambda = u\hat{\mathcal{O}}^k$ ($u^*u = 1$),

$$\sum_{\varphi} F([uy_\varphi]) \le 2N^{fl} H^{\mu+2} \sum_{W} d(W)^{-(k+1)} \le 2N^{fl} H^{2\varrho+6} X^{f(k+1)l} =: \#\mathcal{A} \cdot \beta,$$

the sum over the subspaces W of codimension l.

Step 5: the classes. As u was arbitrary and $y_\varphi \in K_\circ$, Lemma 5.3.79 gives, with $\#\mathcal{A} \ge (N^k/b)^f$,

$$\sum_{c\in\mathcal{C}} F(c) \le \#\mathcal{C}\,\beta \le \#\mathcal{C}\, N^{-kf} \cdot \bigl(2b^f H^{2\varrho+6}\bigr) \cdot \bigl(N^f X^{f(k+1)}\bigr)^l.$$

Step 6: the constants. By (FG6), $k \ge 1000(2\varrho+2)^9 \ge 5\cdot 10^5 n^9$. First, $2 \le b^f$ and $n^3(2\varrho+6) = n^3(2n+4) \le 6n^4$, so $2b^f H^{2\varrho+6} \le b^{f(2+6n^4)} \le b^{kf/4} = \tau_{\mathrm{sz}}$. Second, $N \le b^{60n}$ by (FA5), and $1 \le X \le b^{1/1000}$ by (FG6), so $N^f X^{f(k+1)} \le b^{f(60n+(k+1)/1000)} \le b^{kf/4} = \tau_{\mathrm{sz}}$, since $60n \le k/8$ and $(k+1)/1000 \le k/8$. Hence $\sum_c F(c) \le \#\mathcal{C}\, N^{-kf} \tau_{\mathrm{sz}}^{l+1}$. □

Lemma 5.3.79. *Let $\mathfrak{S}$ be a filling datum, and for $g \in \mathrm{U}(X_0)(\mathbb{A}_f)$ let $[g] \in \mathcal{C}$ be its double coset. Let $\mathrm{U}(X_0)(\hat{\mathcal{O}}) = \mathrm{U}(X_0)(\mathbb{A}_f) \cap \mathrm{GL}_k(\hat{\mathcal{O}})$, matrices taken in the basis $x_1, \dots, x_k$. Let A be a finite nonempty set, $(a_\varphi)_{\varphi \in A}$ a family in $\mathrm{U}(X_0)(\hat{\mathcal{O}})$, $F \colon \mathcal{C} \to [0, \infty]$ and $\beta \in [0, \infty)$. If*

$$\sum_{\varphi \in A} F([u\, a_\varphi]) \le \#A \cdot \beta \qquad \text{for every } u \in \mathrm{U}(X_0)(\mathbb{A}_f),$$

then $\sum_{c \in \mathcal{C}} F(c) \le \#\mathcal{C} \cdot \beta$.

Proof. Write $\Gamma = \mathrm{U}(X_0)(E^+)$, $G_f = \mathrm{U}(X_0)(\mathbb{A}_f)$ and $K_\circ = \mathrm{U}(X_0)(\hat{\mathcal{O}})$. As $\mathrm{GL}_k(\hat{\mathcal{O}})$ is compact and open in $\mathrm{GL}_k(\mathbb{A}_f)$ and G_f is closed in it, $K_\circ$ is a compact open subgroup of G_f.

A small normal subgroup. For an integer $m \ge 1$ let $K_\circ(m) = \{g \in K_\circ : g \equiv 1 \bmod m\hat{\mathcal{O}}\}$, the kernel of reduction $K_\circ \to \mathrm{GL}_k(\hat{\mathcal{O}}/m\hat{\mathcal{O}})$; it is an open normal subgroup of $K_\circ$, and these groups form a basis of neighbourhoods of 1 in $K_\circ$. Since K_X is open, there is m with $K_1 := K_\circ(m) \subseteq \mathrm{K}_X \cap K_\circ$. As K_X is compact and K_1 is open in it, the index $e = [\mathrm{K}_X : K_1]$ is finite.

The finite set Y. Let $Y = \Gamma \backslash G_f / K_1$ and $p \colon Y \to \mathcal{C}$, $\Gamma g K_1 \mapsto [g]$, well defined as $K_1 \subseteq \mathrm{K}_X$. Fix $c \in \mathcal{C}$ with representative h_c. Every g with $[g] = c$ is $\gamma h_c k$ with $\gamma \in \Gamma$, $k \in \mathrm{K}_X$, so $kK_1 \mapsto \Gamma h_c k K_1$ is a surjection from the set K_X/K_1 of left cosets onto $p^{-1}(c)$. It is injective: if $\Gamma h_c k K_1 = \Gamma h_c k' K_1$, then $h_c k' = \gamma h_c k k_1$ with $\gamma \in \Gamma$, $k_1 \in K_1$, so $h_c^{-1} \gamma h_c = k' k_1^{-1} k^{-1} \in \mathrm{K}_X \cap h_c^{-1} \Gamma h_c = \{1\}$ by (FG4), whence $\gamma = 1$ and $k' \in kK_1$. Hence $\#p^{-1}(c) = e$ for every c, Y is finite, $\#Y = e\,\#\mathcal{C}$, and for every $G \colon \mathcal{C} \to [0, \infty]$

$$\sum_{y \in Y} G(p(y)) = e \sum_{c \in \mathcal{C}} G(c). \tag{5.72}$$

Right translation. For $a \in K_\circ$ the map $\Gamma g K_1 \mapsto \Gamma g a K_1$ is well defined, since $gK_1 a = gaK_1$ (K_1 is normal in $K_\circ$), and it is a bijection of Y, with inverse given by a^{-1}. Hence, for each $\varphi \in A$, $\sum_y F(p(y\, a_\varphi)) = \sum_y F(p(y))$, where $y\, a_\varphi$ denotes the translate.

Conclusion. For each $y \in Y$ choose $u_y \in G_f$ with $y = \Gamma u_y K_1$; then $p(y\, a_\varphi) = [u_y a_\varphi]$. Summing the translation identity over φ and applying the hypothesis with $u = u_y$,

$$\#A \sum_{y \in Y} F(p(y)) = \sum_{y \in Y} \sum_{\varphi \in A} F([u_y a_\varphi]) \le \#Y \cdot \#A \cdot \beta = e\,\#\mathcal{C}\,\#A\,\beta.$$

By (5.72) with $G = F$ the left side is $\#A \cdot e \sum_c F(c)$; all quantities lie in $[0, \infty]$ and the right side is finite, so dividing by $\#A \cdot e > 0$ gives $\sum_c F(c) \le \#\mathcal{C}\,\beta$. □

Lemma 5.3.80. *Let $\mathfrak{S}$ be a filling datum; $\mathrm{U}(X_0) \subseteq G$ acts trivially on $X_0^\perp$, vectors of $X_0(\mathbb{A}_f) = \mathbb{A}_f^k$ are columns in the basis $x_1, \dots, x_k$, and for $u \in \mathrm{U}(X_0)(\mathbb{A}_f)$, $y \in \mathrm{U}(X_0)(\hat{\mathcal{O}})$ put*

$$\Lambda(u, y) = \{v \in u\hat{\mathcal{O}}^k : \ (y^{-1}u^{-1}v)_k \in \pi\hat{\mathcal{O}}\}.$$

(i) $L_0 \cap X_0(\mathbb{A}_f) = \{v \in \hat{\mathcal{O}}^k : \ v_k \in \pi\hat{\mathcal{O}}\}$.

(ii) $\mathrm{U}(X_0)(\mathbb{A}_f) \cap K \subseteq \mathrm{U}(X_0)(\hat{\mathcal{O}})$. *If $y, y' \in \mathrm{U}(X_0)(\hat{\mathcal{O}})$ and the k-th rows of y^{-1} and y'^{-1} differ by an element of $\pi\hat{\mathcal{O}}^{1\times k}$, then $y^{-1}y' \in K$.*

(iii) Suppose (FA1). Let $u \in \mathrm{U}(X_0)(\mathbb{A}_f)$, $y \in \mathrm{U}(X_0)(\hat{\mathcal{O}})$ and $c = [uy]$. Then there is $\eta \in \mathrm{U}(X_0)(E^+)$ such that for every $r \ge 0$ the map $L \mapsto \eta^{-1}L$ is a bijection from the lattices of rank r contained in $L_c \cap X_0(\mathbb{A}_f)$ onto the lattices of rank r contained in $\Lambda(u, y)$, and $\mathrm{disc}_r(\eta^{-1}L) = \mathrm{disc}_r(L)$.

Proof. Write $L_0' = L_0 \cap X_0(\mathbb{A}_f)$ and $K_\circ = \mathrm{U}(X_0)(\hat{\mathcal{O}})$. As $\pi \in \hat{\mathcal{O}}$ and $\pi\bar{\pi} = N$ is a unit of $\mathbb{A}_f$, π and $\bar{\pi}$ are units of $\mathbb{A}_f$. By (FG2), at every finite place w one of π, $\bar{\pi}$ is a unit of $\mathcal{O}_w$ (if π is not, then π is a unit at $\bar{w}$, i.e. $\bar{\pi}$ is a unit at w); hence

$$\pi\hat{\mathcal{O}} \cap \bar{\pi}\hat{\mathcal{O}} = N\hat{\mathcal{O}}. \tag{5.73}$$

(i) Let $v \in L_0'$. In the lattice basis, $v = \sum_a(\alpha_a e_a + \alpha_a^* e_a^*) + \sum_{i<k} a_i x_i + a'x' + bz$ with all coefficients in $\hat{\mathcal{O}}$ and $x' = x_k/\bar{\pi} + z/N$. As $v \in X_0(\mathbb{A}_f)$, the α_a, α_a^* vanish and the coefficient $a'/N + b$ of z is 0; so $a' = -Nb$, $v_i = a_i \in \hat{\mathcal{O}}$ for $i < k$ and $v_k = a'/\bar{\pi} = -\pi b \in \pi\hat{\mathcal{O}}$. Conversely, if $v \in \hat{\mathcal{O}}^k$ and $v_k = \pi c$ with $c \in \hat{\mathcal{O}}$, then $v = \sum_{i<k} v_i x_i + Nc\,x' - c\,z \in L_0$.

(ii), first claim. Let $g \in \mathrm{U}(X_0)(\mathbb{A}_f) \cap K$. Then g preserves L_0 and $X_0(\mathbb{A}_f)$, hence L_0', and fixes z. For $i < k$, $x_i \in L_0'$ by (i), so $gx_i \in L_0' \subseteq \hat{\mathcal{O}}^k$. Further $gx' - x' = (gx_k - x_k)/\bar{\pi}$ lies in $L_0 \cap X_0(\mathbb{A}_f) = L_0' \subseteq \hat{\mathcal{O}}^k$, so $gx_k \in x_k + \bar{\pi}\hat{\mathcal{O}}^k \subseteq \hat{\mathcal{O}}^k$. Thus $g\hat{\mathcal{O}}^k \subseteq \hat{\mathcal{O}}^k$; the same holds for $g^{-1} \in K$, so $g \in \mathrm{GL}_k(\hat{\mathcal{O}})$.

(ii), second claim. Put $\kappa = y^{-1}y' \in K_\circ$. Then $e_k^T\kappa - e_k^T = (e_k^T y^{-1} - e_k^T y'^{-1})y' \in \pi\hat{\mathcal{O}}^{1\times k}$, as y' is integral. We show: every $\kappa \in K_\circ$ with $e_k^T(\kappa - 1) \in \pi\hat{\mathcal{O}}^{1\times k}$ satisfies $\kappa L_0 \subseteq L_0$. Write $e_k^T\kappa = e_k^T + \pi\rho$ with $\rho \in \hat{\mathcal{O}}^{1\times k}$. Since $\kappa^*\kappa = 1$, $\kappa^{-1} = \kappa^*$, so $e_k^T = (e_k^T + \pi\rho)\kappa^*$ and $e_k^T\kappa^* = e_k^T - \pi\rho\kappa^* \in e_k^T + \pi\hat{\mathcal{O}}^{1\times k}$. The entries of $e_k^T\kappa^*$ are $\overline{\kappa_{ik}}$, so $\kappa_{ik} - \delta_{ik} \in \bar{\pi}\hat{\mathcal{O}}$ for all i; in particular $\kappa_{kk} - 1 \in \pi\hat{\mathcal{O}} \cap \bar{\pi}\hat{\mathcal{O}} = N\hat{\mathcal{O}}$ by (5.73). Now κ fixes the e_a, e_a^* and z. For $i < k$, κx_i has integral coordinates with k-th coordinate $\kappa_{ki} \in \pi\hat{\mathcal{O}}$, so $\kappa x_i \in L_0$ by (i). Finally $\kappa x' - x' = (\kappa x_k - x_k)/\bar{\pi}$ has integral coordinates $(\kappa_{ik} - \delta_{ik})/\bar{\pi}$ and k-th coordinate $(\kappa_{kk} - 1)/\bar{\pi} \in (N/\bar{\pi})\hat{\mathcal{O}} = \pi\hat{\mathcal{O}}$, so it lies in L_0 by (i), and $\kappa x' \in L_0$. Hence $\kappa L_0 \subseteq L_0$. As $e_k^T(\kappa^{-1} - 1) = -e_k^T(\kappa - 1)\kappa^{-1} \in \pi\hat{\mathcal{O}}^{1\times k}$, also $\kappa^{-1}L_0 \subseteq L_0$, so $\kappa L_0 = L_0$ and $\kappa \in K$.

(iii) As $[uy] = c = [h_c]$, $h_c = \eta uyg$ with $\eta \in \mathrm{U}(X_0)(E^+)$ and $g \in \mathrm{K}_X$. By (FA1), $g \in \mathrm{K}(M) \subseteq K$, so g preserves L_0 and $X_0(\mathbb{A}_f)$, hence $gL_0' = L_0'$; and h_c preserves $X_0(\mathbb{A}_f)$. Therefore

$$L_c \cap X_0(\mathbb{A}_f) = h_c L_0 \cap h_c X_0(\mathbb{A}_f) = h_c L_0' = \eta\, uy\, L_0'.$$

By (i), $v \in uyL_0'$ if and only if $y^{-1}u^{-1}v \in \hat{\mathcal{O}}^k$ and $(y^{-1}u^{-1}v)_k \in \pi\hat{\mathcal{O}}$; as $y \in K_\circ$, the first condition is equivalent to $u^{-1}v \in \hat{\mathcal{O}}^k$, so $uyL_0' = \Lambda(u, y)$. Thus η^{-1} maps $L_c \cap X_0(\mathbb{A}_f)$ bijectively onto $\Lambda(u, y)$, and $L \mapsto \eta^{-1}L$ is a bijection between their $\hat{\mathcal{O}}$-submodules. If $L = B\hat{\mathcal{O}}^r$ with $B = xa$, $x \in \mathrm{M}_{k\times r}(E)$ of rank r and $a \in \mathrm{GL}_r(\mathbb{A}_f)$, then $\eta^{-1}L = (\eta^{-1}x)a\hat{\mathcal{O}}^r$ with $\eta^{-1}x \in \mathrm{M}_{k\times r}(E)$ of rank r, and conversely with η in place of η^{-1}; so lattices of rank r correspond. Since $\eta^*\eta = 1$, $(\eta^{-1})^*\eta^{-1} = \eta\eta^* = 1$ and $(\eta^{-1}B)^*(\eta^{-1}B) = B^*B$, so $\mathrm{disc}_r(\eta^{-1}L) = \mathrm{disc}_r(L)$. □

Lemma 5.3.81. *Let $\mathfrak{S}$ be a filling datum, $u \in \mathrm{U}(X_0)(\mathbb{A}_f)$, $0 \le r \le k$, $l = k - r$, and $L \subseteq u\hat{\mathcal{O}}^k$ a lattice of rank r. Let $W = E^k \cap \mathbb{A}_f L$, where $\mathbb{A}_f L$ is the $\mathbb{A}_f$-span of L, and* $\mathrm{sat}(L) = W(\mathbb{A}_f) \cap u\hat{\mathcal{O}}^k$.

(i) W is an E-subspace of dimension r; $\mathrm{sat}(L)$ *is a lattice of rank r containing L, the index $i(L) = [\mathrm{sat}(L) : L]$ is a positive integer, $d(W) := \mathrm{disc}_r(\mathrm{sat}(L)) \ge 1$, and* $\mathrm{disc}_r(L) = i(L)\,d(W)$.

(ii) If $\mathrm{sat}(L) = B_s\hat{\mathcal{O}}^r$ *with $B_s \in \mathrm{M}_{k\times r}(\mathbb{A}_f)$ as in (LT3), then $M \mapsto B_s^{-1}M$ is a bijection from the lattices of rank r contained in* $\mathrm{sat}(L)$ *onto the $\hat{\mathcal{O}}$-submodules of $\hat{\mathcal{O}}^r$ of finite index, with* $[\mathrm{sat}(L) : M] = [\hat{\mathcal{O}}^r : B_s^{-1}M]$*; every such M has $E^k \cap \mathbb{A}_f M = W$.*

(iii) Let $y_1, \dots, y_m \in \mathrm{U}(X_0)(\hat{\mathcal{O}})$ be such that the k-th rows of y_j^{-1} and $y_{j'}^{-1}$ do not differ by an element of $\pi\hat{\mathcal{O}}^{1\times k}$ for $j \ne j'$, and $L \subseteq \Lambda(u, y_j)$ for all j. Then $m \le i(L)\,N^{fl}$.

Proof. Write $\Lambda = u\hat{\mathcal{O}}^k$; as $u^*u = 1$, $u^{-1} = u^*$ and $(u^{-1})^*u^{-1} = 1$.

(i) Let $L = xa\hat{\mathcal{O}}^r$ as in (LT3), $x \in \mathrm{M}_{k\times r}(E)$ of rank r, with left inverse $x^+ \in \mathrm{M}_{r\times k}(E)$. Then $\mathbb{A}_f L = x\mathbb{A}_f^r$, and $xz \in E^k$ with $z \in \mathbb{A}_f^r$ forces $z = x^+xz \in E^r$; so $W = xE^r$ has dimension r and $W(\mathbb{A}_f) = x\mathbb{A}_f^r$. By the first assertion of Lemma 5.3.82 (for the subspace W and Λ), $\mathrm{sat}(L) = B_s\hat{\mathcal{O}}^r$ with $B_s = x_s a_s$, $x_s \in \mathrm{M}_{k\times r}(E)$ a basis of W and $a_s \in \mathrm{GL}_r(\mathbb{A}_f)$: a lattice of rank r, with $d(W)$ the quantity of that lemma. Clearly $L \subseteq \mathrm{sat}(L)$. As $x = x_s g$ for some $g \in \mathrm{GL}_r(E)$, $L = B_s A\hat{\mathcal{O}}^r$ with $A = a_s^{-1}ga \in \mathrm{GL}_r(\mathbb{A}_f)$, and $B_s\colon \mathbb{A}_f^r \to W(\mathbb{A}_f)$ is an isomorphism; so $L \subseteq \mathrm{sat}(L)$ gives $A\hat{\mathcal{O}}^r \subseteq \hat{\mathcal{O}}^r$, i.e. $A \in \mathrm{M}_r(\hat{\mathcal{O}})$. By elementary divisors at each place, $[\mathrm{sat}(L) : L] = [\hat{\mathcal{O}}^r : A\hat{\mathcal{O}}^r] = \prod_w |\det A_w|_w^{-1} = [\det A]$, a finite product of powers of residue cardinalities, hence a positive integer. Since $B^*B = A^*(B_s^*B_s)A$ for $B = B_sA$ and $[\overline{\det A}] = [\det A]$, $\mathrm{disc}_r(L) = [\det A]\,\mathrm{disc}_r(\mathrm{sat}(L)) = i(L)\,d(W)$. Finally $m_s = u^{-1}B_s$ has columns in $\hat{\mathcal{O}}^k$ and $B_s^*B_s = m_s^*m_s$; so $\delta = \det(B_s^*B_s)$ is integral, and a unit of $\mathbb{A}_f$ by (LT3), whence $d(W)^2 = [\delta] = \prod_w |\delta_w|_w^{-1} \geq 1$.

(ii) If $M \subseteq \mathrm{sat}(L)$ is a lattice of rank r, $M = x_M a_M \hat{\mathcal{O}}^r$, then $x_M E^r = E^k \cap \mathbb{A}_f M \subseteq E^k \cap W(\mathbb{A}_f) = W$, and both have dimension r, so $E^k \cap \mathbb{A}_f M = W$; as in (i), $M = B_s A_M \hat{\mathcal{O}}^r$ with $A_M \in \mathrm{M}_r(\hat{\mathcal{O}}) \cap \mathrm{GL}_r(\mathbb{A}_f)$, so $B_s^{-1}M = A_M\hat{\mathcal{O}}^r$ has finite index $[\det A_M] = [\mathrm{sat}(L) : M]$. Conversely, a submodule $M'' \subseteq \hat{\mathcal{O}}^r$ of finite index n contains $n\hat{\mathcal{O}}^r$, so it is open and compact, hence at each place a free $\mathcal{O}_w$-module of rank r equal to $\mathcal{O}_w^r$ for $w \nmid n$: $M'' = A\hat{\mathcal{O}}^r$ with $A \in \mathrm{GL}_r(\mathbb{A}_f)$, and $B_sM'' = x_s(a_sA)\hat{\mathcal{O}}^r$ is a lattice of rank r in $\mathrm{sat}(L)$. The two maps are inverse to each other, as B_s is injective.

(iii) Put $m_s = u^{-1}B_s \in \mathrm{M}_{k\times r}(\hat{\mathcal{O}})$. At each finite place w, $m_{s,w}\mathcal{O}_w^r = V_w \cap \mathcal{O}_w^k$ with $V_w = u_w^{-1}W(E_w)$ an E_w-subspace, so $\mathcal{O}_w^k/m_{s,w}\mathcal{O}_w^r$ embeds into E_w^k/V_w, is torsion-free, hence free, and the columns of $m_{s,w}$ extend to a basis of $\mathcal{O}_w^k$. This gives $U \in \mathrm{GL}_k(\hat{\mathcal{O}})$ whose first r columns are those of m_s. With A as in (i), $u^{-1}L = m_sA\hat{\mathcal{O}}^r$. Let $\rho_j = e_k^T y_j^{-1} \in \hat{\mathcal{O}}^{1\times k}$. Since $L \subseteq \Lambda$, the condition $L \subseteq \Lambda(u, y_j)$ says $\rho_j u^{-1}v \in \pi\hat{\mathcal{O}}$ for $v \in L$, i.e.

$$\rho_j m_s A \in \pi\hat{\mathcal{O}}^{1\times r}.$$

Write $\rho_j U = (s_j, t_j)$ with $s_j = \rho_j m_s \in \hat{\mathcal{O}}^{1\times r}$ and $t_j \in \hat{\mathcal{O}}^{1\times l}$, and put $W_j = \pi^{-1}s_jA \in \hat{\mathcal{O}}^{1\times r}$. Colour j by the class of W_j in $\hat{\mathcal{O}}^{1\times r}/\hat{\mathcal{O}}^{1\times r}A$ and the class of t_j in $\hat{\mathcal{O}}^{1\times l}/\pi\hat{\mathcal{O}}^{1\times l}$. If j, j' have the same colour, then $W_j - W_{j'} = zA$ with z integral, so $s_j - s_{j'} = \pi z$ (A is invertible), and $t_j - t_{j'} \in \pi\hat{\mathcal{O}}^{1\times l}$; hence $(\rho_j - \rho_{j'})U \in \pi\hat{\mathcal{O}}^{1\times k}$ and $\rho_j - \rho_{j'} \in \pi\hat{\mathcal{O}}^{1\times k}U^{-1} = \pi\hat{\mathcal{O}}^{1\times k}$, so $j = j'$. Therefore m is at most the number of colours,

$$m \leq [\hat{\mathcal{O}}^{1\times r} : \hat{\mathcal{O}}^{1\times r}A]\,[\hat{\mathcal{O}} : \pi\hat{\mathcal{O}}]^l = [\det A]\,[\pi]^l.$$

Finally $[\pi] = N^f$: $[\pi][\bar{\pi}] = [N] = |N_{E/\mathbb{Q}}(N)| = N^{2f}$ and $[\bar{\pi}] = [\pi]$. With $[\det A] = i(L)$ this is (iii). □

Lemma 5.3.82. *Let E be a CM field of degree $2f$ with root discriminant rd_E, Φ a set of f embeddings $E \to \mathbb{C}$ containing one of each pair of complex conjugate embeddings, $k \geq 1$, and $u \in \mathrm{GL}_k(\mathbb{A}_f)$ with $u^*u = 1$; put $\Lambda = u\hat{\mathcal{O}}^k$. For $x \in E^k$ let $\|x\| = (\sum_{v\in\Phi}\sum_i |v(x_i)|^2)^{1/2}$, and $h(x,t) = \sum_i \bar{x}_i t_i$. For an E-subspace $T \subseteq E^k$ of dimension m the $\hat{\mathcal{O}}$-module $T(\mathbb{A}_f) \cap \Lambda$ equals $B\hat{\mathcal{O}}^m$ for some $B = xa$ with $x \in \mathrm{M}_{k\times m}(E)$ of rank m and $a \in \mathrm{GL}_m(\mathbb{A}_f)$, and $d(T) = [\det(B^*B)]^{1/2}$ does not depend on B. Let $0 \leq s \leq k$.*

(i) For every subspace T of codimension s there are nonzero $X_1, \ldots, X_s \in E^k \cap \Lambda$ with $T = \{t \in E^k : h(X_l, t) = 0\ (1 \leq l \leq s)\}$ and

$$\Big(\prod_{l=1}^{s} \|X_l\|\Big)^{2f} \leq (fs\,\mathrm{rd}_E)^{fs}\,d(T).$$

(ii) $\sum_T d(T)^{-(k+1)} \le (200fk\,\mathrm{rd}_E)^{f(k+1)s}$, *the sum over all subspaces* $T \subseteq E^k$ *of codimension* s.

Proof. Write $\Lambda_E = E^k \cap \Lambda$. As u is unitary, $h(x,t)$, extended to $\mathbb{A}_f^k$, takes values in $\hat{\mathcal{O}}$ on Λ, so $h(x,x) \in \mathcal{O}_E$ for $x \in \Lambda_E$.

Bases and $d(T)$. Let T have dimension m and a basis matrix $x \in \mathrm{M}_{k\times m}(E)$. Then $T(\mathbb{A}_f) = x\mathbb{A}_f^m$ and $T(\mathbb{A}_f) \cap \Lambda = x\mathcal{M}$ with $\mathcal{M} = \{z \in \mathbb{A}_f^m : xz \in \Lambda\}$; Λ is open and compact and x is injective with a left inverse over E, so $\mathcal{M}$ is an open compact $\hat{\mathcal{O}}$-submodule of $\mathbb{A}_f^m$, at each place a finitely generated torsion-free $\mathcal{O}_w$-module of rank m, hence $\mathcal{M} = a\hat{\mathcal{O}}^m$ with $a \in \mathrm{GL}_m(\mathbb{A}_f)$. Two such B differ by an element of $\mathrm{GL}_m(\hat{\mathcal{O}})$ on the right, so $d(T)$ is well defined; and $d(T)^2 = [\det a]^2\,|\mathrm{N}_{E/\mathbb{Q}} \det(x^*x)|$, with $\det(x^*x) \in E^\times$ totally positive.

Duality: $d(T) = d(T^\perp)$, $T^\perp = \{t : h(x',t) = 0\ \forall x' \in T\}$. Let $B = xa$ and $B' = ya'$ be bases of $T(\mathbb{A}_f) \cap \Lambda$ and $T^\perp(\mathbb{A}_f) \cap \Lambda$ as above, $m' = k - m$, and put $F = u^{-1}B$, $P = u^{-1}B'$. As $uu^* = 1$, $F^*P = B^*B' = 0$, $F^*F = B^*B$ and $P^*P = B'^*B'$, both with unit determinant. The columns of P are an $\hat{\mathcal{O}}$-basis of $u^{-1}T^\perp(\mathbb{A}_f) \cap \hat{\mathcal{O}}^k$, a saturated submodule of $\hat{\mathcal{O}}^k$, hence locally a direct summand: P is primitive. Apply Lemma 5.3.8 with $J = 1$, $m_1 = m$, $m_2 = m'$, $m_3 = 0$ and $t = 1$: then $\mathcal{M} = F^*F$, $R = P^*P$, and $[\det(B^*B)] = i(F)^2[\det(B'^*B')]$. Here $Fz \in \hat{\mathcal{O}}^k$ means $xaz \in \Lambda$, that is $az \in a\hat{\mathcal{O}}^m$, so $i(F) = 1$, and $d(T) = d(T^\perp)$.

(i) Let $W = T^\perp$, of dimension s, and $B' = ya'$ a basis of $W(\mathbb{A}_f) \cap \Lambda$ ($y \in \mathrm{M}_{k\times s}(E)$ of rank s). For each $v \in \Phi$ the complex matrix $v(y)$ has rank s, so (Gram–Schmidt) there is $r_v \in \mathrm{GL}_s(\mathbb{C})$ with $(v(y)r_v)^*(v(y)r_v) = 1$; then $|\det r_v|^{-2} = \det(v(y)^*v(y)) = v(\det(y^*y))$. By Lemma 5.3.83 (ii) for (r, a') there is a basis $B_1, \dots, B_s$ of E^s with $a'^{-1}B_l$ integral and

$$\prod_v |\det r_v|^2\,[\det a']^{-1} \prod_l \|B_l\|_r^{2f} \le (fs\,\mathrm{rd}_E)^{fs}.$$

Put $X_l = yB_l \in E^k$. Then $u^{-1}X_l = (u^{-1}ya')(a'^{-1}B_l)$ is integral, so $X_l \in \Lambda_E$, and $X_l \ne 0$ as y is injective. The X_l span $yE^s = W$, so $T = W^\perp = \{t : h(X_l, t) = 0\ \forall l\}$. Since $v(y)r_v$ has orthonormal columns, $\|v(y)v(B_l)\| = \|r_v^{-1}v(B_l)\|$, so $\|X_l\| = \|B_l\|_r$. Finally $\prod_v v(\det(y^*y)) = |\mathrm{N}_{E/\mathbb{Q}} \det(y^*y)|^{1/2}$ ($\det(y^*y)$ lies in the totally real subfield and is totally positive), so by the first step and duality

$$\prod_v |\det r_v|^2 [\det a']^{-1} = \left([\det a']^2 |\mathrm{N}_{E/\mathbb{Q}} \det(y^*y)|\right)^{-1/2} = d(W)^{-1} = d(T)^{-1},$$

which gives (i).

Short vectors are separated. For $0 \ne x \in \Lambda_E$, $\alpha = h(x,x) \in \mathcal{O}_E$ is nonzero and totally positive, so $|\mathrm{N}_{E/\mathbb{Q}}\alpha| = \prod_v v(\alpha)^2 \ge 1$ and, by the inequality of arithmetic and geometric means, $\|x\|^2 = \sum_v v(\alpha) \ge f \ge 1$. Hence distinct points of Λ_E have distance at least 1 in $V_\infty = \prod_v \mathbb{C}^k \cong \mathbb{R}^{2fk}$, the open balls of radius $\frac{1}{2}$ about them are disjoint, and $\#\{x \in \Lambda_E : \|x\| \le R\} \le (1+2R)^{2fk} \le (3R)^{2fk}$ for $R \ge 1$. With $N = 2f(k+1)$ and $D = 2fk$, by partial summation (layer cake) over the nonzero x, all of norm ≥ 1,

$$\begin{aligned}\sum_{0\ne x\in\Lambda_E} \|x\|^{-N} &= \int_1^\infty NR^{-N-1}\#\{0 \ne x : \|x\| \le R\}\,dR \\ &\le N3^D \int_1^\infty R^{D-N-1}dR = \frac{N3^D}{N-D} = (k+1)\,3^{2fk} \le 5^{2f(k+1)},\end{aligned}$$

the last step as $k + 1 \le 1 + \frac{4}{3}fk \le (5/3)^{2fk}$.

(ii) For each T of codimension s fix a tuple $X(T)=(X_1,\dots,X_s)$ of nonzero vectors of Λ_E as in (i). As T is the orthogonal of its tuple, $T\mapsto X(T)$ is injective. Raising (i) to the power $k+1$,

$$d(T)^{-(k+1)}\le (fs\,\mathrm{rd}_E)^{f(k+1)s}\prod_{l=1}^{s}\|X_l\|^{-N}.$$

Summing over T and then over all s-tuples of nonzero vectors of Λ_E (all terms are nonnegative),

$$\sum_T d(T)^{-(k+1)}\le (fs\,\mathrm{rd}_E)^{f(k+1)s}\Big(\sum_{0\neq x\in\Lambda_E}\|x\|^{-N}\Big)^s\le (25fs\,\mathrm{rd}_E)^{f(k+1)s}\le (200fk\,\mathrm{rd}_E)^{f(k+1)s},$$

as $s\le k$. (For $s=0$ the tuple is empty, $T=E^k$ and $d(E^k)=[\det u^*u]^{1/2}=1$.) □

Lemma 5.3.83. *Let E be a CM field of degree $2f$ with discriminant d_E and root discriminant $\mathrm{rd}_E=|d_E|^{1/(2f)}$, and Φ a set of f embeddings $E\to\mathbb{C}$ containing one of each pair of complex conjugate embeddings. Let $s\ge 1$, $a\in\mathrm{GL}_s(\mathbb{A}_f)$ and $r=(r_v)_{v\in\Phi}$ with $r_v\in\mathrm{GL}_s(\mathbb{C})$. Put*

$$\Lambda=\{B\in E^s:\ a^{-1}B\in\hat{\mathcal{O}}^s\},\qquad \|B\|_r=\Big(\sum_{v\in\Phi}\|r_v^{-1}v(B)\|^2\Big)^{1/2},$$

where $v(B)\in\mathbb{C}^s$ is the entrywise image and $\|\cdot\|$ the standard hermitian norm of $\mathbb{C}^s$. Give $V_\infty=\prod_{v\in\Phi}\mathbb{C}^s$ the real inner product $\langle x,y\rangle=\mathrm{Re}\sum_v\sum_i\bar{x}_{v,i}y_{v,i}$ and its Lebesgue measure, and put $j(B)=(r_v^{-1}v(B))_{v\in\Phi}$ for $B\in E^s$. Then:

(i) $j(\Lambda)$ is a lattice in V_∞ of covolume $2^{-fs}|d_E|^{s/2}\,[\det a]\prod_{v\in\Phi}|\det r_v|^{-2}$;

(ii) there is a basis $B_1,\dots,B_s$ of E^s over E with $B_i\in\Lambda$ for all i and

$$\prod_{v\in\Phi}|\det r_v|^2\cdot[\det a]^{-1}\cdot\prod_{i=1}^{s}\|B_i\|_r^{2f}\le(fs\,\mathrm{rd}_E)^{fs}.$$

Proof. Let $V_\infty=\prod_{v\in\Phi}\mathbb{C}^s$ with the real inner product $\langle x,y\rangle=\mathrm{Re}\sum_v\sum_i\bar{x}_{v,i}y_{v,i}$, a euclidean space of dimension $2fs$ whose Lebesgue measure is the standard one, and $j\colon E^s\to V_\infty$, $j(B)=(r_v^{-1}v(B))_v$, so that $\|B\|_r=\|j(B)\|$. Put $L=a\hat{\mathcal{O}}^s$, so $\Lambda=E^s\cap L$.

(i): $j(\Lambda)$ is a lattice of covolume $2^{-fs}|d_E|^{s/2}[\det a]\prod_v|\det r_v|^{-2}$. The $\hat{\mathcal{O}}$-module L is open and compact in $\mathbb{A}_f^s$, so there are integers $m_1,m_2\ge1$ with $m_1\hat{\mathcal{O}}^s\subseteq L\subseteq m_2^{-1}\hat{\mathcal{O}}^s$, and intersecting with E^s ($E\cap\hat{\mathcal{O}}=\mathcal{O}_E$), $m_1\mathcal{O}_E^s\subseteq\Lambda\subseteq m_2^{-1}\mathcal{O}_E^s$. By Minkowski's theorem on $\mathcal{O}_E$ (applied coordinatewise), $j_1(\mathcal{O}_E^s)$ is a lattice of covolume $(2^{-f}|d_E|^{1/2})^s$ for $j_1=j$ with all $r_v=1$; hence $j_1(\Lambda)$ is a lattice, of covolume $(2^{-f}|d_E|^{1/2})^s\,[\mathcal{O}_E^s:\Lambda\cap\mathcal{O}_E^s]/[\Lambda:\Lambda\cap\mathcal{O}_E^s]$. The field E is dense in $\mathbb{A}_f$ ($\mathbb{A}_f=E+m\hat{\mathcal{O}}$ for every integer $m\ge1$, by $\mathbb{A}_f=E+\hat{\mathcal{O}}$ applied to x/m), and $L\cap\hat{\mathcal{O}}^s$ is open, so the injective maps $\Lambda/(\Lambda\cap\mathcal{O}_E^s)\to L/(L\cap\hat{\mathcal{O}}^s)$ and $\mathcal{O}_E^s/(\Lambda\cap\mathcal{O}_E^s)\to\hat{\mathcal{O}}^s/(L\cap\hat{\mathcal{O}}^s)$ are surjective: for $x\in L$ choose $B\in E^s$ with $B-x\in L\cap\hat{\mathcal{O}}^s$; then $B\in\Lambda$. So the quotient of the two indices is $[\hat{\mathcal{O}}^s:L\cap\hat{\mathcal{O}}^s]/[L:L\cap\hat{\mathcal{O}}^s]=\prod_w[\mathcal{O}_w^s:L_w\cap\mathcal{O}_w^s]/[L_w:L_w\cap\mathcal{O}_w^s]=\prod_w|\det a_w|_w^{-1}=[\det a]$, where at each finite place w the last equality follows from the elementary divisor theorem for a_w over the principal ideal domain $\mathcal{O}_w$ (write $a_w=k_1\mathrm{diag}(\varpi_w^{e_i})k_2$ with $k_1,k_2\in\mathrm{GL}_s(\mathcal{O}_w)$). Finally $j=(r_v^{-1})_v\circ j_1$, and the complex linear map r_v^{-1} of $\mathbb{C}^s$ has real determinant $|\det r_v|^{-2}$.

(ii), successive minima. By Theorem A.6.3 in $V_\infty\cong\mathbb{R}^{2fs}$, with $\mathrm{vol}(B_{2fs})=\pi^{fs}/(fs)!$, there are $\mathbb{R}$-linearly independent $y_1,\dots,y_{2fs}\in j(\Lambda)$ with $\|y_m\|=\lambda_m$, $\lambda_1\le\cdots\le\lambda_{2fs}$ the successive minima, and

$$\prod_{m=1}^{2fs}\lambda_m\le\frac{2^{2fs}(fs)!}{\pi^{fs}}\,2^{-fs}|d_E|^{s/2}[\det a]\prod_v|\det r_v|^{-2}.$$

An E-basis among the minima. Write $y_m = j(Y_m)$ with $Y_m \in \Lambda$ (j is injective). Choose indices $m_1 < m_2 < \dots$ greedily: $m_1 = 1$, and m_i is the least m such that Y_m is not in the E-span of $Y_{m_1}, \dots, Y_{m_{i-1}}$. An E-subspace of E^s of dimension d is mapped by j into a real subspace of dimension $2fd$; as the y_m are $\mathbb{R}$-independent, $y_1, \dots, y_{m_i-1}$, which lie in the image of an E-subspace of dimension $i-1$, number at most $2f(i-1)$, so $m_i \le 2f(i-1)+1$. The y_m span $V_\infty = j(E^s) \otimes \mathbb{R}$, so the Y_m span E^s and the choice produces s indices; put $B_i = Y_{m_i}$, a basis of E^s in Λ. As the minima increase, $\|B_i\|_r = \lambda_{m_i} \le \lambda_m$ for $2f(i-1) < m \le 2fi$, so

$$\prod_{i=1}^{s} \|B_i\|_r^{2f} \le \prod_{i=1}^{s} \prod_{m=2f(i-1)+1}^{2fi} \lambda_m = \prod_{m=1}^{2fs} \lambda_m.$$

The constant. With $|d_E|^{s/2} = \mathrm{rd}_E^{fs}$, $(fs)! \le (fs)^{fs}$ and $2/\pi < 1$,

$$\prod_v |\det r_v|^2 [\det a]^{-1} \prod_i \|B_i\|_r^{2f} \le \Big(\frac{2}{\pi}\Big)^{fs} (fs)! \, \mathrm{rd}_E^{fs} \le (fs \, \mathrm{rd}_E)^{fs}.$$

□

Lemma 5.3.84. *Let E be a number field of degree m, $\hat{\mathcal{O}}$ its ring of integral finite adeles, $r \ge 0$ an integer and s a real number with $t = s - r + 1 \ge 3$. Then*

$$\sum_M [\hat{\mathcal{O}}^r : M]^{-s} \le \exp\big(m \, 2^{2-t}\big),$$

the sum over all $\hat{\mathcal{O}}$-submodules $M \subseteq \hat{\mathcal{O}}^r$ of finite index.

Proof. For $r = 0$ the only module is $M = 0$, of index 1, and the bound holds. Let $r \ge 1$ and put $\sigma_i = s - r + i = t - 1 + i$ for $1 \le i \le r$; so $\sigma_i \ge t \ge 3$. Write $\mathcal{O} = \mathcal{O}_E$, embedded diagonally in $\hat{\mathcal{O}} = \prod_w \mathcal{O}_w$.

Adelic and global submodules. Let $n \ge 1$ be an integer. By the Chinese remainder theorem, $\mathcal{O}/n\mathcal{O} \cong \prod_{w|n} \mathcal{O}/\mathfrak{p}_w^{v_w(n)}$, and $\mathcal{O}/\mathfrak{p}_w^{v_w(n)} \to \mathcal{O}_w/n\mathcal{O}_w$ is an isomorphism for each finite place w with prime ideal $\mathfrak{p}_w$; as $\mathcal{O}_w/n\mathcal{O}_w = 0$ for $w \nmid n$, the diagonal map $\mathcal{O}/n\mathcal{O} \to \hat{\mathcal{O}}/n\hat{\mathcal{O}}$ is a ring isomorphism. Hence $\mathcal{O}^r/n\mathcal{O}^r \to \hat{\mathcal{O}}^r/n\hat{\mathcal{O}}^r$ is an isomorphism of $\mathcal{O}$-modules, and, since $\hat{\mathcal{O}}$ acts on $\hat{\mathcal{O}}^r/n\hat{\mathcal{O}}^r$ through $\hat{\mathcal{O}}/n\hat{\mathcal{O}}$, which is the image of $\mathcal{O}$, the $\mathcal{O}$-submodules and the $\hat{\mathcal{O}}$-submodules of $\hat{\mathcal{O}}^r/n\hat{\mathcal{O}}^r$ are the same. A submodule $M \subseteq \hat{\mathcal{O}}^r$ of index n contains $n\hat{\mathcal{O}}^r$ (the quotient is a group of order n, so it is killed by n), hence is the preimage of a submodule of index n of $\hat{\mathcal{O}}^r/n\hat{\mathcal{O}}^r$, and conversely; the same holds for submodules $L \subseteq \mathcal{O}^r$ of index n and $\mathcal{O}^r/n\mathcal{O}^r$. Composing these bijections with the isomorphism above gives, for each n, a bijection between the $\hat{\mathcal{O}}$-submodules of $\hat{\mathcal{O}}^r$ of index n and the $\mathcal{O}$-submodules of $\mathcal{O}^r$ of index n. All terms being nonnegative, the sums may be grouped by the index, and

$$\sum_M [\hat{\mathcal{O}}^r : M]^{-s} = \sum_L [\mathcal{O}^r : L]^{-s},$$

the second sum over the $\mathcal{O}$-submodules $L \subseteq \mathcal{O}^r$ of finite index.

Solomon's formula. Since $s = t + r - 1 > r$, Solomon's formula [286] gives

$$\sum_L [\mathcal{O}^r : L]^{-s} = \prod_{j=0}^{r-1} \zeta_E(s-j) = \prod_{i=1}^{r} \zeta_E(\sigma_i),$$

with $j = r - i$, each factor given by its absolutely convergent Euler product, as $\sigma_i > 1$.

The bound. Let q_w be the residue cardinality of the finite place w, so that $\zeta_E(\sigma) = \prod_w (1-q_w^{-\sigma})^{-1}$ for $\sigma > 1$. Above each rational prime p lie at most m places w, each with $q_w \geq p$, so $\zeta_E(\sigma) \leq \prod_p (1-p^{-\sigma})^{-m} = \zeta(\sigma)^m$ for $\sigma > 1$. For $\sigma \geq 3$, $\zeta(\sigma) - 1 \leq 2^{-\sigma} + \int_2^\infty x^{-\sigma} dx = 2^{-\sigma}(1 + \frac{2}{\sigma-1}) \leq 2^{1-\sigma}$. Hence, with $\log(1+x) \leq x$,

$$\log \prod_{i=1}^{r} \zeta_E(\sigma_i) \leq m \sum_{i=1}^{r} 2^{1-\sigma_i} = m\, 2^{2-t} \sum_{i=1}^{r} 2^{-i} \leq m\, 2^{2-t}.$$

□

5.4 The frames engine

The sieve of the main term needs, at every place, forms on pairs of heads that descend by integration over a head entry, decay in the frequency and have a controlled mass. This section proves these properties once, for an abstract place, in one theorem (Theorem 5.4.1). It is applied at the finite places of the main term and, in the next section, at the complex place; treating both kinds of place at once is the point of the abstraction. Every estimate is proved from the axioms of a place datum and a gauge (Definition 5.1.7). The objects are a frame with its flag of isotropic lines, the unipotent radical of the stabiliser of the flag, a tower of floor groups and the Bessel subgroup [96, 36, 42], and the floor forms obtained by pairing a cut function with its Weyl translates for two heads (Definition 5.2.4).

The core of the section is the gains. A character of the last frame product, or two different heads, makes the integrand oscillate along a floor or along a Heisenberg group, and Plancherel's inequality turns the oscillation into a decay factor (Proposition 5.4.34, Proposition 5.4.35); a deep vector after the head gives a further factor (Proposition 5.4.5). The shell sum adds the gains over dyadic shells of positions: a remote shell pays for its height with the deep-vector gain, because far positions have deep vectors (Proposition 5.4.38, Proposition 5.4.21). The descent is Fubini's theorem on the tower (Proposition 5.2.92, printed with the sieve system at the finite places, where it is first used). Plancherel's theorem for local fields is in Tate's thesis [294, 324], and harmonic analysis on Heisenberg groups goes back to the theorem of Stone and von Neumann [195].

Theorem 5.4.1. *Let Γ be a gauge of loss Λ for a place datum over a place A, with size filtration $(A_{(i)})_{i\geq 0}$; sizes refer to Γ unless another gauge is named. Put $r' = |\iota'|$, let $Z(\gamma)$ be the frame Gram matrix (5.3) of $\gamma \in G$, and let $W\colon G \to \mathbb{C}^I$ be cut. In (DN1)–(DN3) the group measure θ is a Haar measure of G.*

ro, orbital integrals. Let p, q be sequences of units of A.

(a) For $j \geq 0$ and $\omega\colon G \to \mathbb{C}$, $|\mathcal{F}_j(p,q;W,\omega)| \leq \mathrm{hdn}_j(p,q) \int_G \mathrm{FN}_j(p,q,W,x;\omega)\, d\theta(x)$; if $|\omega| \leq 1$ on G, then $\mathrm{FN}_j(p,q,W,x;\omega) \leq \mathrm{FM}_j(p,q,W,x)$ for every $x \in G$.

(b) For $0 \leq b \leq \varrho$, $\mathrm{FML}_b(\Lambda^{1287n} \operatorname{vol}(A_{(11)})^{r'}; 1)$.

(c) If

$$\int_G |\mathcal{K}_{W,0}(1,z)|^2\, d\theta(z) < \infty,$$

then $\mathcal{F}_0(p,q;W,1)$ equals this integral; otherwise $\mathcal{F}_0(p,q;W,1) = 0$.

(d) If $x \in \mathcal{P}_{\mathrm{pos}}$, $\sigma \in [0,\infty]$ and $(0,1,1,W)$ has a deep vector after the head of depth at least σ^{-1} at x, then $\mathrm{FM}_0(1,1,W,x)^2 \leq \sigma^{2s}\, \mathrm{WSP}_0(1,1,W,x)$.

(e) *If $l \geq 0$, $\gamma, y \in G$ and both y and $\gamma^{-1}y$ are near at l, then for $0 \leq i \leq \min(l,n)$ the leading principal minor of order i of $Z(\gamma)$ lies in $A_{(9n)}$.*

(f) *For a Borel measure μ on G and $\gamma \in G$ let $\mathrm{Orb}_\mu(\gamma) \in \mathbb{C}^{I\times I}$ be the Bessel orbital integral (5.13) of W. If $\mathrm{Orb}_\mu(\gamma) \neq 0$, then every leading principal minor of $Z(\gamma)$ lies in $A_{(9n)}$. If μ is left invariant, then $\mathrm{Orb}_\mu(u^{-1}\gamma u') = \overline{\psi(M^{-1}\sigma(u))}\,\psi(M^{-1}\sigma(u'))\,\mathrm{Orb}_\mu(\gamma)$ for all $u, u' \in \mathsf{U}$ and $\gamma \in G$. If μ is a Haar measure, W is continuous, p, q are heads of length ϱ and $x \in G^{(\varrho)}$, then every $y \in G$ with $W(y) \neq 0$ and $W((w_{\varrho,p}x)^{-1}y) \neq 0$ has size $60n^2$, and*

$$\langle \mathrm{Orb}_\mu(w_{\varrho,p}x), \mathrm{Orb}_\mu(w_{\varrho,q}x)\rangle = \int_{G\times G} \langle W(y), W(y')\rangle \overline{\langle W((w_{\varrho,p}x)^{-1}y), W((w_{\varrho,q}x)^{-1}y')\rangle}\, d(\mu\otimes\mu)(y,y').$$

(DN1) Descent. *For $a < \varrho$, $t \in \mathbb{C}$, $\eta \in A$ and heads p, q with $\int \mathrm{FM}_a(p,q,W,x)\,d\theta(x) < \infty$, the function $c \mapsto \psi(\eta c)\, t\,\mathcal{F}_{a+1}(p[a \mapsto u(c)], q[a \mapsto u(c)]; W, 1)$, where $u(c) = c$ if c is a unit and $u(c) = 1$ otherwise, is integrable on A and its integral is $t\,\mathcal{F}_a(p,q;W, x \mapsto \psi(\eta\, h(f_a, x f_a)))$.*

(DN2) Shells. *Let $j \leq \varrho$, p, q heads of length j, $V \geq 1$ with $\mathrm{FML}_j(V;1)$, $(S_i)_{i\geq 0}$ shells of heights 2^i for (j,p,q,W) covering the support of $\phi \geq 0$, $\phi \leq \mathrm{FM}_j(p,q,W,\cdot)$, $e \in [0,\infty]$ with $\phi(x)^2 \leq (Ve^s)^2 \mathrm{WSP}_j(p,q,W,x)$ for $x \in \mathcal{P}_{\mathrm{pos}}$ when $e < 1$, and an integer $t \geq 0$ with $9\cdot 10^7 n^4 + t \leq s$. Then*

$$\mathrm{hdn}_j(p,q) \int_G \phi\, d\theta \leq V \min(1,e)^t \Lambda^{4\cdot 10^8 n^4}\, \mathrm{SN}(W).$$

(DN3) Gains. *Let $j \leq \varrho$, p a head of length j, q a sequence of units and $x \in \mathcal{P}_{\mathrm{pos}}$.*

(a) *If ξ is a unit, $\eta \in A$ and $\psi(\eta z) = \psi(\xi z)$ for every $z \in A_{(200n^2)}$, then*

$$\mathrm{FN}_j(p,q,W,x; g \mapsto \psi(\eta\, h(f_j, g f_j)))^2 \leq (\mathrm{kb}_A/|\xi|)^{r'}\, \mathrm{WSP}_j(p,q,W,x).$$

(b) *If $j = i+1$ and $\xi = M^{-1}(p_i^{-1} - q_i^{-1})$ is a unit, then*

$$\mathrm{FN}_j(p,q,W,x;1)^2 \leq (\mathrm{kb}_A/|\xi|)^{r'}\, \mathrm{WSP}_j(p,q,W,x).$$

(c) *If $V \in [0,\infty]$ and $\mathrm{FML}_j(V;1)$, then $\mathrm{FM}_j(p,q,W,x)^2 \leq V^2\, \mathrm{WSP}_j(p,q,W,x)$; if moreover $V \geq 1$, $\sigma \in [0,\infty]$ and (j,p,q,W) has a deep vector after the head of depth at least σ^{-1} at x, then $\mathrm{FM}_j(p,q,W,x)^2 \leq \sigma^s V^2\, \mathrm{WSP}_j(p,q,W,x)$.*

Moreover, for $x \in \mathcal{P}_{\mathrm{pos}}$ and $a < \varrho$, $\mathrm{FML}_a(\Lambda^{10^4 n}; g \mapsto \mathrm{dp}_a(g,x)^s)$; and for $j \leq \varrho$, $d \in G$ and a pairing kernel $\mathcal{K}$ at floor j for d,

$$\int_{\{\text{positions of size } \Gamma\}} \mathrm{WSI}_j(\mathcal{K}, x)\, d\theta(x) \leq n\Big(\prod_{l<\varrho} c_{\mathfrak{fl}}(l)\Big) \int_G \int_G |\mathcal{K}(g,y)|^2\, d\theta_{\geq 0}(y)\, d\theta_{\geq 0}(g).$$

(DN4) Far means deep. *Let S be a second gauge for the same place datum, with size filtration $(A^S_{(i)})$, let $j \leq \varrho$ and let p, q be heads of length j for S; if $j \neq 0$, assume $M^{-1} \in A^S_{(1)}$ and $A^S_{(10n)} \subseteq A_{(1)}$. Let $x, y \in G$ and $u, v \in \mathsf{U}$ with $ux \in K$, $vxy \in K$ and $M^{-1}(vxy - ux)$ of size 2, and let $w^{-1}_{j,p}x$ and $w^{-1}_{j,q}xy$ be near at j. If for every a with $j \leq a < \varrho$ the set $\mathfrak{d}_a(x,y)$ contains a unit c with $c^{-1} \in A_{(1)}$, then $M^{-1}(y-1)$ has size $99n$.*

Proof. Each item is one of the following results, applied to the gauge Γ (and, in (DN4), the second gauge S), the cut function W and, in (DN1)–(DN3), the Haar measure θ.

- (DN0)(a): Lemma 5.4.2 (i) and (ii).
- (DN0)(b): Lemma 5.4.3 (ii).
- (DN0)(c): Lemma 5.2.89.
- (DN0)(d): Proposition 5.4.5 (iii).
- (DN0)(e): Lemma 5.4.13 (ii).
- (DN0)(f): Lemma 5.4.20: the minors by (ii); the equivariance by (i), as a cut function is a Bessel function of frequency M^{-1}; the size $60n^2$ by (iii); the product formula by (iv).
- (DN1): Proposition 5.2.92 (ii) with $\phi(c) = \psi(\eta c)$, which is measurable with $|\phi| = 1$; then $\phi \circ Z_a = (x \mapsto \psi(\eta\, h(f_a, x f_a)))$. (The heads are not needed.)
- (DN2): Proposition 5.4.21.
- (DN3)(a): Proposition 5.4.34; (DN3)(b): Proposition 5.4.35; (DN3)(c): Proposition 5.4.5 (i) and (ii).
- (DN3), the floor-mass bound of the depth: Lemma 5.4.19; the one-frame bound: Proposition 5.4.25.
- (DN4): Proposition 5.4.38.

In each case the hypotheses of the item are those of the cited result, or contain them. □

Lemma 5.4.2. *Let a place datum be given, $W\colon G \to \mathbb{C}^I$, $j \geq 0$, p, q sequences of units of A and $\omega\colon G \to \mathbb{C}$.*

(i) $|\mathcal{F}_j(p, q; W, \omega)| \leq \mathrm{hdn}_j(p, q) \int_G \mathrm{FN}_j(p, q, W, x; \omega)\, d\theta(x)$.

(ii) If W is cut and $|\omega| \leq 1$ on G, then $\mathrm{FN}_j(p, q, W, x; \omega) \leq \mathrm{FM}_j(p, q, W, x)$ for every $x \in G$.

Proof. **(i).** By (5.15), $\mathcal{F}_j(p, q; W, \omega)$ is either 0 or $\mathrm{hdn}_j(p, q) \int \mathrm{FI}_j(p, q, W, x; \omega)(g)\, d(\theta \otimes \mathfrak{m}_j)(x, g)$; in both cases its modulus is at most $\mathrm{hdn}_j(p, q) \int |\mathrm{FI}_j(p, q, W, x; \omega)(g)|\, d(\theta \otimes \mathfrak{m}_j)(x, g)$, which by Tonelli is $\mathrm{hdn}_j(p, q) \int \mathrm{FN}_j(p, q, W, x; \omega)\, d\theta(x)$.

(ii). Let $d = w_{j,q}^{-1} w_{j,p}$. By Lemma 5.2.57, (ii), $\mathcal{K}_W$ and $\mathcal{K}_W^d$ are pairing kernels at floor j, so by Lemma 5.2.64, (i), their partial correlations from floor $j+1$ are measurable. Fix x, g and put $u(\xi) = |\mathcal{K}_{W,j+1}(\xi g, \xi g x)|$ and $u'(\zeta) = |\mathcal{K}^d_{W,j+1}(\zeta w_{j,p}^{-1} g, \zeta w_{j,p}^{-1} g x)|$, measurable functions with $\int u\, d\theta_j = \mathrm{piv}_j(\mathcal{K}_W)(g, gx)$ and $\int u'\, d\theta_j = \mathrm{piv}_j(\mathcal{K}_W^d)(w_{j,p}^{-1} g, w_{j,p}^{-1} g x)$. Each of the two integrals in (5.14) is 0 or an integral, so its modulus is at most the integral of the modulus of its integrand; with $|\omega| \leq 1$ and Tonelli,

$$|\mathrm{FI}_j(p, q, W, x; \omega)(g)| \leq \int_G \int_G u(\xi)\, u'(\zeta)\, d\theta_j(\zeta)\, d\theta_j(\xi) = \mathrm{piv}_j(\mathcal{K}_W)(g, gx)\, \mathrm{piv}_j(\mathcal{K}_W^d)(w_{j,p}^{-1} g, w_{j,p}^{-1} g x).$$

Integrating over g against $\mathfrak{m}_j$ gives $\mathrm{FN}_j(p, q, W, x; \omega) \leq \mathrm{FM}_j(p, q, W, x)$. □

Lemma 5.4.3. *Let Γ be a gauge of loss Λ, with size filtration $(A_{(i)})_{i \geq 0}$, for a place datum of shape (ϱ, ι, m), and put $r' = |\iota'|$.*

(i) Let $i \geq 0$, let $g \in G$ have size i, let $0 \leq b \leq \varrho$, $\sigma \in G^{(b)}$, and let σg be near at $b+1$. Then $\sigma^{-1} f_b$ has size $i+6$.

(ii) For every $0 \leq b \leq \varrho$, $\mathrm{FML}_b(\Lambda^{1287n} \operatorname{vol}(A_{(11)})^{r'}; 1)$.

Proof. **Inverses and products.** By (SZ2) with one summand, $A_{(i)} \subseteq A_{(i+1)}$. If $g \in G$ has size i, then $g^{-1} = J^{-1}\bar{g}^{\mathsf{T}} J$ (from $\bar{g}^{\mathsf{T}} J g = J$) has size $i+4$: each entry of $J^{-1}\bar{g}^{\mathsf{T}}$ is a sum of at most r products of an entry of J^{-1} (size 1, (SZ5)) and the conjugate of an entry of g (size i, (SZ3)), hence lies in $A_{(i+2)}$ by (SZ1), (SZ2), and multiplying by J gives in the same way entries in $A_{(i+4)}$. Likewise the product of a matrix of size i and one of size i' has size $i+i'+1$. Elements of K have size 1 (GU2), so their inverses have size 5.

(i). As σg is near at $b+1$, there is $\xi \in R^{(b+1)}$ with $\kappa = \xi \sigma g \in K$. The column of f_b in ξ: the index f_b has floor $b < b+1$ and level $b < b+1$, so by the definition of $R^{(b+1)}$, $\xi_{i f_b} = \delta_{i f_b}$ for every index i of level at least b; the indices of level below b are the e_a with $a < b$. Hence $\xi f_b \in f_b + E_{<b}$, $E_{<b} = \operatorname{span}\{e_a : a < b\}$. In the same way $\xi e_a \in e_a + \operatorname{span}\{e_{a'} : a' < a\}$ for $a < b$, so ξ maps $E_{<b}$ bijectively onto itself, and therefore $\xi^{-1} f_b \in f_b + E_{<b}$. As $\sigma \in G^{(b)}$ fixes every e_a with $a < b$,

$$\sigma^{-1}\xi^{-1} f_b \in \sigma^{-1} f_b + E_{<b}.$$

Now $\sigma^{-1}\xi^{-1} = g\kappa^{-1}$ has size $i+5+1 = i+6$, so all coordinates of $\sigma^{-1}\xi^{-1} f_b$, a column of $g\kappa^{-1}$, lie in $A_{(i+6)}$; and the coordinates of floor at least b of $\sigma^{-1} f_b$ are those of $\sigma^{-1}\xi^{-1} f_b$, since $E_{<b}$ has none. The coordinates of floor below b of $\sigma^{-1} f_b$ vanish: the vectors v with $h(v, e_a) = h(v, e_a^*) = 0$ for all $a < b$ are those whose coordinates e_a^*, e_a ($a < b$) vanish, i.e. V_b; the vector f_b lies in V_b, and σ^{-1} preserves h and fixes the e_a, e_a^* with $a < b$, so $\sigma^{-1} f_b \in V_b$. As $0 \in A_{(i+6)}$ (SZ4), $\sigma^{-1} f_b$ has size $i+6$.

(ii). Let C_b and the projection P_b be as in Lemma 5.2.60. For $\kappa, \kappa' \in K$ the element $\kappa\kappa'^{-1}$ has size $1+5+1 = 7$, so the coordinates of $\kappa\kappa'^{-1} f_b$ lie in $A_{(7)}$, and P_b replaces some of them by 0; hence C_b lies in the set of vectors of V of size 7. By Lemma 5.2.60, (iii), $\mathrm{FML}_b(\mathrm{qv}_b(C_b); 1)$ holds, and by Lemma 5.4.4, (iv), with $\rho = \mathrm{qv}_b$, $j = 7$ and $\varphi = 1$,

$$\mathrm{qv}_b(C_b) \leq \mathrm{qv}_b(\{y \in V \text{ of size } 7\}) \leq \Lambda^{99n \cdot 13} \operatorname{vol}(A_{(11)}^{\iota'}) = \Lambda^{1287n} \operatorname{vol}(A_{(11)})^{r'}.$$

A floor-mass bound with a constant V holds with every larger constant, which gives (ii). □

Lemma 5.4.4. *Let Γ be a gauge of loss Λ, with size filtration $(A_{(i)})_{i \geq 0}$, for a place datum of shape (ϱ, ι, m), and let $0 \leq l \leq \varrho$. Write the points of $A \times A \times Y_l$ as $x = (c, t, Y)$ with $Y = (\zeta, w, \zeta')$, and for $\beta \in A^-$ let $\lambda_l(\beta)$ be the image of $\mathrm{vol} \otimes \mathrm{vol} \otimes \nu$ on $A \times Y_l \times A^-$ under $(t, Y, \alpha) \mapsto \big(t^{\dagger}\tau(\delta^{-1}\beta - h_l'(Y,Y)) + t\alpha,\, t,\, Y\big)$, so that $\mathrm{qv}_l(\beta)$ is the image of $\lambda_l(\beta)$ under B_l.*

(i) Let $B \subseteq \iota$ with $|B| = 2m+2$. Write $x = (x^\circ, z)$ with $z = (w_i)_{i \in B} \in A^B$ and $x^\circ \in X^\circ$ the other coordinates, put $q^\circ(x^\circ) = \bar{c}t + \bar{t}c + h_l'(Y,Y) - \bar{z}^{\mathsf{T}} z$ (a function of x°), and let $(\lambda_\gamma^B)_{\gamma \in A^-}$ be the kernel of (GU6) carried to A^B by a bijection $\{1, \dots, 2m+2\} \to B$. Then for every $\beta \in A^-$ and every measurable $f \colon A \times A \times Y_l \to [0, \infty]$,

$$\int f \, d\lambda_l(\beta) = \int_{X^\circ} \int_{A^B} f(x^\circ, z) \, d\lambda^B_{\beta - \delta q^\circ(x^\circ)}(z) \, dx^\circ,$$

and $\lambda^B_{\beta - \delta q^\circ(x^\circ)}(A^B_{(j)}) \leq \Lambda^{(2m+2)(j+1)}$ for all $x^\circ \in X^\circ$ and $j \geq 0$; moreover $\mathrm{vol}(A^B_{(j)}) \leq \Lambda^{(2m+2)(j+1)}$.

(ii) Let ρ_0 be vol *or $\lambda_l(\beta)$ for some $\beta \in A^-$. For every $j \geq 0$ and every measurable $G\colon A^{\iota'} \to [0,\infty]$,*

$$\int_{\{x:\ \text{all coordinates in } A_{(j)}\}} G((w_i)_{i\in\iota'})\, d\rho_0(x) \leq \Lambda^{99n(j+2)} \int_{A^{\iota'}_{(j)}} G\, d\mathrm{vol}.$$

(iii) For $x = (c,t,Y)$ and $x' = (c',t',Y')$ in $A \times A \times Y_l$, $h(B_l(x'), B_l(x)) = \bar{c}'t + \bar{t}'c + h'_l(Y',Y)$ and $h(o_i, B_l(x)) = w_i$ for $i \in \iota$; the image of vol *under B_l is vol_{V_l}; and if $B_l(x)$ has size j, then all coordinates of x lie in $A_{(j+4)}$.*

(iv) Let ρ be vol_{V_l} or $\mathrm{qv}_l(\beta)$ for some $\beta \in A^-$. For every $j \geq 0$ and every measurable $\varphi\colon A^{\iota'} \to [0,\infty]$,

$$\int_{\{y\in V \text{ of size } j\}} \varphi((h(o_i,y))_{i\in\iota'})\, d\rho(y) \leq \Lambda^{99n(j+6)} \int_{A^{\iota'}_{(j+4)}} \varphi\, d\mathrm{vol}.$$

Proof. Throughout, A and A^- are locally compact, second countable and metrisable; vol on a finite power of A is the product of copies of a σ-finite Haar measure; each $A_{(j)}$ is compact by (GU1), hence Borel. Recall from (SZ2) with one summand that $A_{(i)} \subseteq A_{(i+1)}$. For $r \in A$ put $\mathrm{sk}(r) = \tau\delta r - \overline{\tau\delta r} \in A^-$, a continuous map; if $\bar{r} = r$ then $\overline{\tau\delta r} = \bar{\tau}\bar{\delta}r = -\bar{\tau}\delta r$, so $\mathrm{sk}(r) = (\tau + \bar{\tau})\delta r = \delta r$. For $\gamma \in A^-$ the element $\delta^{-1}\gamma$ is hermitian ($\overline{\delta^{-1}\gamma} = (-\delta)^{-1}(-\gamma)$), so $\mathrm{sk}(\delta^{-1}\gamma) = \gamma$. The Haar measure ν of the abelian group A^- is invariant under $\gamma \mapsto \beta - \gamma$ (translation invariance, and invariance under the involutive automorphism $\gamma \mapsto -\gamma$, whose image of ν is $c\nu$ with $c^2 = 1$).

(i). Let $S^\circ = \mathcal{Q}_l \sqcup (\iota \setminus B) \sqcup \mathcal{Q}_l$, so that $X^\circ = A \times A \times A^{S^\circ}$, and let T° be the Gram matrix of the restriction h° of h'_l to the coordinates in S°. As the ι-part of h'_l is $\sum_{i\in\iota} \bar{w}_i w_i$, we have $h'_l(Y,Y) = h^\circ(Y^\circ,Y^\circ) + \bar{z}^{\mathsf{T}}z$, where Y° is Y without z; hence $q^\circ(x^\circ) = \bar{c}t + \bar{t}c + h^\circ(Y^\circ,Y^\circ)$ depends only on x° and is hermitian, so $\delta q^\circ(x^\circ) \in A^-$. For $\gamma \in A^-$ let λ°_γ be the image of $\mathrm{vol} \otimes \mathrm{vol} \otimes \nu$ on $A \times A^{S^\circ} \times A^-$ under $\mathrm{ch}^\circ_\gamma(t, Y^\circ, \alpha) = (t^\dagger\tau(\delta^{-1}\gamma - h^\circ(Y^\circ,Y^\circ)) + t\alpha, t, Y^\circ)$. By Lemma 5.2.52, (i), with $S = S^\circ$ and $T' = T^\circ$, the family (λ°_γ) is a disintegration of vol on X° along q°. The transported kernel (λ^B_γ) is an s-finite kernel and a disintegration of vol on A^B along $\bar{z}^{\mathsf{T}}z$ with $\lambda^B_\gamma(A^B_{(j)}) \leq \Lambda^{(2m+2)(j+1)}$ for all γ and j, because a permutation of coordinates preserves vol, the form $\bar{z}^{\mathsf{T}}z$ and the box $A^B_{(j)}$ (GU6). This gives the mass bound in (i); and $\mathrm{vol}(A^B_{(j)}) = \mathrm{vol}(A_{(j)})^{2m+2} \leq \Lambda^{(j+1)(2m+2)}$ by (GU5).

The chart splits. For $t \in A$, $\alpha \in A^-$ and $Y = (Y^\circ, z)$ the first coordinate of the chart of $\lambda_l(\beta)$ is $t^\dagger\tau(\delta^{-1}\beta - h^\circ(Y^\circ,Y^\circ) - \bar{z}^{\mathsf{T}}z) + t\alpha$, and $\delta^{-1}\beta - \bar{z}^{\mathsf{T}}z = \delta^{-1}(\beta - \delta\bar{z}^{\mathsf{T}}z)$ with $\beta - \delta\bar{z}^{\mathsf{T}}z \in A^-$. So the chart of $\lambda_l(\beta)$ at $(t,(Y^\circ,z),\alpha)$ is $(\mathrm{ch}^\circ_{\beta-\delta\bar{z}^{\mathsf{T}}z}(t,Y^\circ,\alpha), z)$, and by Tonelli (the measure vol on A^{S_l}, $S_l = \mathcal{Q}_l \sqcup \iota \sqcup \mathcal{Q}_l$, is the product of those on A^{S° and A^B)

$$\int f\, d\lambda_l(\beta) = \int_{A^B} \int_{X^\circ} f(x^\circ, z)\, d\lambda^\circ_{\beta-\delta\bar{z}^{\mathsf{T}}z}(x^\circ)\, dz. \tag{5.74}$$

Exchange of the two fibrations. Put

$$F(r,z) = \int f(x^\circ,z)\, d\lambda^\circ_{\beta-\mathrm{sk}(r)}(x^\circ) = \int f(\mathrm{ch}^\circ_{\beta-\mathrm{sk}(r)}(t,Y^\circ,\alpha), z)\, d(t,Y^\circ,\alpha)$$

for $(r,z) \in A \times A^B$, and $F'(r,x^\circ) = \int f(x^\circ,z)\, d\lambda^B_{\beta-\mathrm{sk}(r)}(z)$ for $(r,x^\circ) \in A \times X^\circ$. Both are measurable: $(\gamma,t,Y^\circ,\alpha) \mapsto \mathrm{ch}^\circ_\gamma(t,Y^\circ,\alpha)$ is Borel ($t \mapsto t^\dagger$ is Borel), so F is measurable by Tonelli, and F' is the integral of a measurable function against an s-finite kernel composed with the measurable map $(r,x^\circ) \mapsto \beta - \mathrm{sk}(r)$. Since $\bar{z}^{\mathsf{T}}z$ and $q^\circ(x^\circ)$ are hermitian, $F(\bar{z}^{\mathsf{T}}z, z)$ is the

inner integral of (5.74) and $F'(q^\circ(x^\circ),x^\circ)$ is the inner integral of the right side of (i). By (DJ2) for (λ^B_γ) and for (λ°_γ), using $\mathrm{sk}(\delta^{-1}\gamma)=\gamma$,

$$\int_{A^B} F(\bar z^{\mathsf T} z, z)\,dz = \int_{A^-}\int_{A^B}\int_{X^\circ} f\,d\lambda^\circ_{\beta-\gamma}\,d\lambda^B_\gamma\,d\nu(\gamma),$$
$$\int_{X^\circ} F'(q^\circ(x^\circ),x^\circ)\,dx^\circ = \int_{A^-}\int_{X^\circ}\int_{A^B} f\,d\lambda^B_{\beta-\gamma}\,d\lambda^\circ_\gamma\,d\nu(\gamma);$$

the outer integrands are measurable (Tonelli for the kernel λ^B and for the charts of λ°), so no lower integral occurs. In the second identity substitute $\gamma\mapsto\beta-\gamma$, which preserves ν; its inner double integral becomes $\int_{X^\circ}\int_{A^B} f\,d\lambda^B_\gamma\,d\lambda^\circ_{\beta-\gamma}$, which equals $\int_{A^B}\int_{X^\circ} f\,d\lambda^\circ_{\beta-\gamma}\,d\lambda^B_\gamma$ by Tonelli, as λ^B_γ is s-finite and $\lambda^\circ_{\beta-\gamma}$ is the image of a σ-finite measure. So the two right sides agree, and with (5.74) this is the identity of (i).

(ii). By (GU1), $|\iota|-|\iota'|=2(2m+2)$, so there is $B\subseteq\iota\setminus\iota'$ with $|B|=2m+2$; fix it. For $j\ge 0$ let $\mathbf{1}_j$ be the indicator of $A_{(j)}$ in each coordinate; the indicator of the box of $A\times A\times Y_l$ is $\mathbf{1}_j(x^\circ)\mathbf{1}_j(z)$, and the coordinates w_i, $i\in\iota'$, are coordinates of x°. By (i) (for $\rho_0=\mathrm{vol}$ by Tonelli and $\mathrm{vol}(A^B_{(j)})\le\Lambda^{(2m+2)(j+1)}$),

$$\begin{aligned}\int_{\mathrm{box}} G\big((w_i)_{i\in\iota'}\big)\,d\rho_0 &\le \Lambda^{(2m+2)(j+1)}\int_{X^\circ}\mathbf{1}_j(x^\circ)\,G\big((w_i)_{i\in\iota'}\big)\,dx^\circ\\ &= \Lambda^{(2m+2)(j+1)}\,\mathrm{vol}(A_{(j)})^d\int_{A^{\iota'}_{(j)}} G\,d\mathrm{vol},\end{aligned}$$

by Tonelli on $X^\circ=A^{\iota'}\times A^d$, where d is the number of coordinates of X° outside ι'. Here $d+2m+2=2+2|\mathcal{Q}_l|+|\iota|-|\iota'|=2+2(m+\varrho-l)+2(2m+2)\le 6m+6+2\varrho\le 62n\le 99n$, using $m\le 9n$ (GU1) and $n=\varrho+1\ge 1$. By (GU5) and $\Lambda\ge 2$, $\mathrm{vol}(A_{(j)})^d\Lambda^{(2m+2)(j+1)}\le\Lambda^{(d+2m+2)(j+1)}\le\Lambda^{99n(j+2)}$.

(iii). Let $S'_l=\{1,2\}\sqcup\mathcal{Q}_l\sqcup\iota\sqcup\mathcal{Q}_l$ index the coordinates $x=(c,t,Y)$, and regard B_l as the matrix X of the linear map $A^{S'_l}\to V$, $x\mapsto B_l(x)=cv_0+tv_0^*+\sum_{c'\in\mathcal{Q}_l}(\zeta_{c'}\epsilon_{c'}+\zeta'_{c'}\epsilon^*_{c'})+\sum_{i\in\iota}w_io_i$. Let J° be the hermitian matrix with $\bar x^{\mathsf T}J^\circ x'=\bar c t'+\bar t c'+h'_l(Y,Y')$; it is a permutation matrix of order two, $(J^\circ)^2=1$. From the relations of the hyperbolic basis, $h(e_a,e_b^*)=\delta_{ab}$, $h(e_a,e_b)=h(e_a^*,e_b^*)=0$ and the orthogonality of the e_a,e_a^* to the middle space, $\bar X^{\mathsf T}JX=J^\circ$, and $h(o_i,B_l(x))=w_i$ for $i\in\iota$.

The image of vol. The map X takes values in V_l and is the extension by zero of a square matrix X' from $A^{S'_l}$ to the coordinates of floor $\ge l$ (both index sets have $k+1+2(\varrho-l)$ elements). With J_l the restriction of J to these coordinates, $\bar X'^{\mathsf T}J_lX'=J^\circ$; taking determinants, $\overline{\det X'}\det J_l\det X'=\det J^\circ=\pm1$ and $\det J_l=\pm\det T$, so $\det X'$ is a unit and, by (PA2), (PE1) and multiplicativity, $|\det X'|^2=|\pm1|=1$ ($|-1|^2=|1|=1$), so $|\det X'|=1$. By (PA3), $X'_*\mathrm{vol}=\mathrm{vol}$, and extending by zero, $X_*\mathrm{vol}=\mathrm{vol}_{V_l}$.

Coordinates cost size 4. Since $(J^\circ)^2=1$, $x=J^\circ\bar X^{\mathsf T}J\,(Xx)$. The entries of X are 0, 1 or entries of the basis matrix, hence of size 1 by (GU2), (SZ4); those of J have size 1 (SZ5). By (SZ3) and (SZ1), each entry of $\bar X^{\mathsf T}J$ is a sum of at most r products of two elements of $A_{(1)}$, hence lies in $A_{(3)}$ by (SZ2); if Xx has size j, each entry of $\bar X^{\mathsf T}J(Xx)$ is a sum of at most r elements of $A_{(j+3)}$, hence lies in $A_{(j+4)}$; and J° permutes the entries. So if $B_l(x)$ has size j, then all coordinates of x lie in $A_{(j+4)}$.

(iv). The set of $y\in V$ of size j is Borel, and by (iii) and the definition of $\mathrm{qv}_l(\beta)$, $\rho=X_*\rho_0$

with $\rho_0 = \mathrm{vol}$ or $\rho_0 = \lambda_l(\beta)$. By (iii) and (ii) at $j+4$,

$$\begin{aligned}\int_{\{y \text{ of size } j\}} \varphi((h(o_i,y))_{i\in\iota'})\,d\rho(y) &= \int \mathbf{1}_{\{B_l(x) \text{ of size } j\}}\,\varphi((w_i)_{i\in\iota'})\,d\rho_0(x)\\ &\le \int_{\{x\text{: coordinates in } A_{(j+4)}\}} \varphi((w_i)_{i\in\iota'})\,d\rho_0(x)\\ &\le \Lambda^{99n(j+6)} \int_{A^{\iota'}_{(j+4)}} \varphi\,d\mathrm{vol}.\end{aligned}$$

□

Proposition 5.4.5. *Let Γ be a gauge of loss Λ for a place datum, let $W\colon G \to \mathbb{C}^I$ be cut and $x \in \mathcal{P}_{\mathrm{pos}}$; WSP_j is formed for Γ, and arithmetic is in $[0,\infty]$.*

(i) Suppose that the group measure θ is a Haar measure of G. Let $j \le \varrho$, p a head of length j, q a sequence of units, and $V \in [0,\infty]$ with $\mathrm{FML}_j(V;1)$. Then $\mathrm{FM}_j(p,q,W,x)^2 \le V^2\,\mathrm{WSP}_j(p,q,W,x)$.

(ii) If moreover $V \ge 1$, $\sigma \in [0,\infty]$ and (j,p,q,W) has a deep vector after the head of depth at least σ^{-1} at x, then $\mathrm{FM}_j(p,q,W,x)^2 \le \sigma^s V^2\,\mathrm{WSP}_j(p,q,W,x)$. The same conclusion holds under the hypotheses of (i) with $V \ge 1$ and $\sigma \in [0,\infty]$ such that, if $\sigma < 1$, (j,p,q,W) has a deep vector after the head of depth at least σ^{-1} at x.

(iii) If $\sigma \in [0,\infty]$ and $(0,1,1,W)$ has a deep vector after the head of depth at least σ^{-1} at x, then $\mathrm{FM}_0(1,1,W,x)^2 \le \sigma^{2s}\,\mathrm{WSP}_0(1,1,W,x)$.

Proof. Put $w_p = w_{j,p}$, $d = w_{j,q}^{-1}w_p$, $\mathcal{K}_1 = \mathcal{K}_W$, $\mathcal{K}_2 = \mathcal{K}_W^d$; by Lemma 5.2.57 (ii) these are pairing kernels at floor j (for 1 and for d). By definition $\mathrm{FM}_j(p,q,W,x) = \int f\,d\mathfrak{m}_j$ with

$$f(g) = \mathrm{piv}_j(\mathcal{K}_1)(g,gx)\,\mathrm{piv}_j(\mathcal{K}_2)(z,zx), \qquad z = w_p^{-1}g.$$

A bound for one pivot. Let $\mathcal{K}'$ be a pairing kernel at floor $j \le \varrho$ and $g' \in G$. By Lemma 5.4.16 (i) with $\mathrm{FML}_j(V;1)$, and (iv),

$$\mathrm{piv}_j(\mathcal{K}')(g',g'x)^2 \le V \int |\mathcal{K}'_{j+1}(\sigma'g',\sigma'g'x)|^2\,d\theta_j(\sigma') \le V\,\frac{\mathrm{WSS}_j(\mathcal{K}',x)(g')}{\mathrm{dpw}_j(g',x)}. \tag{5.75}$$

(i). By (5.75) for both pivots, and $c_{\mathrm{fls}} \ge 1$, $f(g)^2 \le V^2 c_{\mathrm{fls}} \frac{\mathrm{WSS}_j(\mathcal{K}_1,x)(g)}{\mathrm{dpw}_j(g,x)} \frac{\mathrm{WSS}_j(\mathcal{K}_2,x)(z)}{\mathrm{dpw}_j(z,x)}$ for every g, and Lemma 5.4.6 (ii) with $C = V^2$ gives the claim.

(ii). Let g with $f(g) \ne 0$. Then $\mathrm{piv}_j(\mathcal{K}_2)(z,zx) \ne 0$, so by Lemma 5.2.64 (iii) there is $\tau \in G^{(j)}$ with $\langle W(\tau z), W(d\tau zx)\rangle \ne 0$. As τ commutes with w_p and $w_{j,q}$ (Lemma 5.2.58 (ix)), $\tau z = w_p^{-1}\tau g$ and $d\tau zx = w_{j,q}^{-1}\tau gx$; so $W(w_p^{-1}\tau g) \ne 0$ and $W(w_{j,q}^{-1}\tau gx) \ne 0$, and both points are near at j by Lemma 5.2.57 (i). Let $\rho \in G^{(j)}$ with $\mathcal{K}_1(\rho g, \rho gx) \ne 0$. Then $w_p^{-1}\rho g = (\rho\tau^{-1})w_p^{-1}\tau g$ and $w_{j,q}^{-1}\rho gx = (\rho\tau^{-1})w_{j,q}^{-1}\tau gx$ with $\rho\tau^{-1} \in G^{(j)} \subseteq R^{(j)}$, so both are near at j; together with $\langle W(\rho g), W(\rho gx)\rangle \ne 0$ this says that ρg is active at x for (j,p,q,W), so some a with $j \le a < \varrho$ has $\mathrm{dp}_a(\rho g,x) \ge \sigma^{-1}$. By Lemma 5.4.17 (with $l = j$ and $\varepsilon_1 = \sigma$), $\mathrm{piv}_j(\mathcal{K}_1)(g,gx)^2 \le \sigma^s c_{\mathrm{fls}} \mathrm{WSS}_j(\mathcal{K}_1,x)(g)/\mathrm{dpw}_j(g,x)$. With (5.75) for the second pivot and $V \le V^2$,

$$f(g)^2 \le \sigma^s V^2\,c_{\mathrm{fls}}\,\frac{\mathrm{WSS}_j(\mathcal{K}_1,x)(g)}{\mathrm{dpw}_j(g,x)}\,\frac{\mathrm{WSS}_j(\mathcal{K}_2,x)(z)}{\mathrm{dpw}_j(z,x)},$$

and Lemma 5.4.6 (ii) with $C = \sigma^s V^2$ gives the first claim. For the second claim: if $\sigma < 1$ this is the first claim; if $\sigma \geq 1$, then $\sigma^s \geq 1$ and (i) suffices.

(iii). Here $j = 0$, $p = q = 1$, so $w_{0,1} = 1$, $d = 1$, $\mathfrak{m}_0 = \delta_1$, and $\mathrm{FM}_0(1,1,W,x) = \mathrm{piv}_0(\mathcal{K}_W)(1,x)^2$. The group $R^{(0)}$ is G (no index has floor below 0), so a point is near at 0 if and only if $K \neq \emptyset$. Let $\rho \in G = G^{(0)}$ with $\mathcal{K}_W(\rho, \rho x) = \langle W(\rho), W(\rho x)\rangle \neq 0$; then $W(\rho) \neq 0$, so ρ is near at 0 (Lemma 5.2.57 (i)), $K \neq \emptyset$, and ρ is active at x for $(0,1,1,W)$; the deep vector gives $a < \varrho$ with $\mathrm{dp}_a(\rho, x) \geq \sigma^{-1}$. By Lemma 5.4.17 with $l = 0$, $g = 1$ and $\mathrm{dpw}_0 = 1$, $\mathrm{piv}_0(\mathcal{K}_W)(1,x)^2 \leq \sigma^s c_{\mathrm{fls}} \mathrm{WSS}_0(\mathcal{K}_W, x)(1)$. Since the convolution in WSI_0 is δ_1, $\mathrm{WSP}_0(1,1,W,x) = c_{\mathrm{fls}}^3 \mathrm{WSS}_0(\mathcal{K}_W, x)(1)^2$, and therefore

$$\mathrm{FM}_0(1,1,W,x)^2 = \mathrm{piv}_0(\mathcal{K}_W)(1,x)^4 \leq \sigma^{2s} c_{\mathrm{fls}}^2 \mathrm{WSS}_0(\mathcal{K}_W, x)(1)^2 \leq \sigma^{2s} \mathrm{WSP}_0(1,1,W,x).$$

□

Lemma 5.4.6. *Let Γ be a gauge of loss Λ for a place datum, and suppose that the group measure θ is a Haar measure of G. Let $j \leq \varrho$, p a head of length j, $w_p = w_{j,p}$, $x \in \mathcal{P}_{\mathrm{pos}}$, $C \in [0,\infty]$ and $\Theta = \theta_{j-1} * \cdots * \theta_0$ ($\Theta = \delta_1$ if $j = 0$); arithmetic is in $[0,\infty]$, the integral of a function that is not measurable is its lower Lebesgue integral, and c_{fls} and WSP_j are formed for Γ.*

(i) Let $B, B' \colon G \to [0,\infty]$ be measurable, with $B(\xi g) = B(g)$ and $B'(\xi g) = B'(g)$ for $\xi \in R^{(j)}$, $g \in G$, and $B(g) = B'(g) = 0$ unless g is near at j. Let $f \colon G \to [0,\infty]$ satisfy

$$f(g)^2 \leq C \, \frac{B(g)}{\mathrm{dpw}_j(g,x)} \, \frac{B'(w_p^{-1} g)}{\mathrm{dpw}_j(w_p^{-1} g, x)} \qquad \text{whenever } f(g) \neq 0.$$

Then

$$\Big(\int_G f \, d\mathfrak{m}_j\Big)^2 \leq C \Big(\Big(\prod_{a<j} \mathrm{fm}(a, p_a^{-1})\Big) c_{\mathrm{fls}}\Big)^2 \int_G B \, d\Theta \int_G B' \, d\Theta.$$

(ii) Let $W \colon G \to \mathbb{C}^I$ be cut, q a sequence of units, $d = w_{j,q}^{-1} w_p$, and let $f \colon G \to [0,\infty]$ satisfy

$$f(g)^2 \leq C \, c_{\mathrm{fls}} \, \frac{\mathrm{WSS}_j(\mathcal{K}_W, x)(g)}{\mathrm{dpw}_j(g,x)} \, \frac{\mathrm{WSS}_j(\mathcal{K}_W^d, x)(w_p^{-1} g)}{\mathrm{dpw}_j(w_p^{-1} g, x)} \qquad \text{whenever } f(g) \neq 0.$$

Then $\big(\int_G f \, d\mathfrak{m}_j\big)^2 \leq C \, \mathrm{WSP}_j(p,q,W,x)$.

Proof. Put $D = \big(\prod_{a<j} \mathrm{fm}(a, p_a^{-1})\big) c_{\mathrm{fls}}$.

(i). For $\Phi \in \{B, B'\}$ and every g, $\Phi(g)/\mathrm{dpw}_j(g,x) = \Phi(g)\,\mathrm{dpw}_{j,x}^-(g)$: both sides are 0 if g is not near at j, and $\mathrm{dpw}_{j,x}^-(g) = \mathrm{dpw}_j(g,x)^{-1}$ otherwise. Hence for every g (trivially where $f(g) = 0$)

$$f(g)^2 \leq \big[B'(w_p^{-1} g)\,\mathrm{dpw}_{j,x}^-(g)\big] \cdot C\big[B(g)\,\mathrm{dpw}_{j,x}^-(w_p^{-1} g)\big],$$

so f is bounded by the square root of the measurable function on the right, and by the Cauchy–Schwarz inequality for that function $(\int f \, d\mathfrak{m}_j)^2 \leq I_1 \cdot C I_2$ with $I_1 = \int B'(w_p^{-1} g) \mathrm{dpw}_{j,x}^-(g) \, d\mathfrak{m}_j(g)$ and $I_2 = \int B(g) \mathrm{dpw}_{j,x}^-(w_p^{-1} g) \, d\mathfrak{m}_j(g)$; the inverse depth weights are measurable by Lemma 5.4.8 (iii). By Lemma 5.2.59 (iv), $\mathfrak{m}_j = \eta_{<j} * \Theta$.

The first integral. Put $\Phi(g) = B'(g)\,\mathrm{dpw}_{j,x}^-(w_p g)$, so the integrand of I_1 is $\Phi(w_p^{-1} g)$. For $t \in G^{(j)} \subseteq R^{(j)}$, $B'(tg) = B'(g)$, and $w_p t = t w_p$ (Lemma 5.2.58 (ix)), so $\mathrm{dpw}_{j,x}^-(w_p t g) =$

$\mathrm{dpw}^-_{j,x}(tw_pg) = \mathrm{dpw}^-_{j,x}(w_pg)$ by Lemma 5.4.8 (iii); so Φ is invariant under left multiplication by $G^{(j)}$. By Lemma 5.4.15 (ii) and then Lemma 5.4.7 (iv) (with $b = j$, $\rho = \Theta$ and the function B'),

$$I_1 = \int \Phi \, d\mathfrak{m}_j = \int B'(g) \, \mathrm{dpw}^-_{j,x}(w_pg) \, d(\eta_{<j} * \Theta)(g) \le D \int B' \, d\Theta.$$

The second integral. $w_p^{-1} = w_{j,\bar{p}}$ with $\bar{p} = (\bar{p}_a)_a$: indeed $w_p^{-1}e_a = \bar{p}_a e_a^*$ and $w_p^{-1}e_a^* = p_a^{-1}e_a$ for $a < j$. The sequence $\bar{p}$ is a head of length j (conjugation preserves $A_{(1)}$, (SZ3)), and $\mathrm{fm}(a, \bar{p}_a^{-1}) = \mathrm{fm}(a, p_a^{-1})$ by (PA2). So Lemma 5.4.7 (iv) for the head $\bar{p}$ and the function B gives $I_2 \le D \int B \, d\Theta$. Hence $(\int f \, d\mathfrak{m}_j)^2 \le D \int B' \, d\Theta \cdot CD \int B \, d\Theta$.

(ii). By Lemma 5.2.57 (ii), $\mathcal{K}_W$ and $\mathcal{K}_W^d$ are pairing kernels at floor j (for 1 and for d). Put $B = \mathrm{WSS}_j(\mathcal{K}_W, x)$ and $B' = \mathrm{WSS}_j(\mathcal{K}_W^d, x)$; by Lemma 5.4.16 (ii), (iii) they are measurable, invariant under left multiplication by $R^{(j)}$ and zero off the points near at j. By (i) with Cc_{fls} in place of C, the definition $\mathrm{WSI}_j(\mathcal{K}, x) = \int \mathrm{WSS}_j(\mathcal{K}, x) \, d\Theta$ and Lemma 5.4.16 (vi) ($\mathrm{WSI}_j \le \mathrm{WSI}_0$, as $j \le n$),

$$\begin{aligned}\Big(\int f \, d\mathfrak{m}_j\Big)^2 &\le Cc_{\mathrm{fls}} D^2 \, \mathrm{WSI}_j(\mathcal{K}_W, x) \, \mathrm{WSI}_j(\mathcal{K}_W^d, x) \\ &\le C\Big(\prod_{a<j} \mathrm{fm}(a, p_a^{-1})\Big)^2 c_{\mathrm{fls}}^3 \, \mathrm{WSI}_0(\mathcal{K}_W, x) \, \mathrm{WSI}_j(\mathcal{K}_W^d, x),\end{aligned}$$

which is $C \, \mathrm{WSP}_j(p, q, W, x)$. □

Lemma 5.4.7. *Let Γ be a gauge of loss Λ for a place datum, and let $x \in \mathcal{P}_{\mathrm{pos}}$.*

(i) Let $a < \varrho$, p a head of length $a+1$, $y \in K$, $\nu \in N_{<a}$, and write $z(n) = w_{a+1,p} n \nu y$. Then

$$\int_G \mathbf{1}\{z(n) \text{ near at } a+1\} \, \mathrm{dp}_a(z(n), x)^s \, d\eta_a(n) \le \mathrm{fm}(a, p_a^{-1}) \, \Lambda^{10^6 n^2}.$$

(ii) Under the hypotheses of (i),

$$\int_G \mathrm{dpw}^-_{a+1,x}(z(n)) \, d\eta_a(n) \le \mathrm{dpw}^-_{a,x}(w_{a,p}\nu y) \, \mathrm{fm}(a, p_a^{-1}) \, \Lambda^{10^6 n^2}.$$

(iii) Let $j \le \varrho$, p a head of length j, and $g \in G$ near at j. Then

$$\int_G \mathrm{dpw}^-_{j,x}(w_{j,p}\nu g) \, d\eta_{<j}(\nu) \le \Big(\prod_{a<j} \mathrm{fm}(a, p_a^{-1})\Big) c_{\mathrm{fls}}.$$

(iv) Let $j \le b$ with $j \le \varrho$, p a head of length j, ρ an s-finite measure on G, and $\Phi \colon G \to [0, \infty]$ Borel with $\Phi(\xi g) = \Phi(g)$ for $\xi \in R^{(b)}$ and $\Phi(g) = 0$ unless g is near at b. Then

$$\int_G \Phi(g) \, \mathrm{dpw}^-_{j,x}(w_{j,p}g) \, d(\eta_{<j} * \rho)(g) \le \Big(\prod_{a<j} \mathrm{fm}(a, p_a^{-1})\Big) c_{\mathrm{fls}} \int_G \Phi \, d\rho.$$

Proof. The set of points near at l is closed (it is $R^{(l)}K$), and depth and inverse depth weights are measurable (Lemma 5.4.8), so all integrands are Borel.

(i). Let Q be the $\varrho \times \iota'$ matrix with $(Q\omega)_b = (\nu O\omega)_{e_b}$ for $b < \varrho$. By (GU7) (with $y \in K$, $x \in \mathcal{P}_{\mathrm{pos}}$) there is a common bound ϕ for $\mathcal{V}_{\mathrm{cm}}(y, x, Q)$ with loss Λ. Let n be a Heisenberg element of floor a and $\sigma = w_{a+1,p} n \nu$. For $\omega \in \mathcal{V}_{\mathrm{cm}}(y, x, Q)$ and $b < a$: $h(e_b, \sigma O\omega)$ is the e_b^*-coordinate of $\sigma O\omega$, which is p_b times the e_b-coordinate of $n\nu O\omega$ (as $w_{a+1,p}e_b = p_b e_b^*$), which equals

$(\nu O\omega)_{e_b} = (Q\omega)_b = 0$, since the rows of $n \in G^{(a)}$ at the e_b, $b < a$, are those of 1 (Lemma 5.2.58, (vii)). By Lemma 5.4.9, $\mathrm{dp}_a(z(n), x) \le \phi(\varsigma)$ with $\varsigma = (h(o_i, \sigma^{-1}e_a))_{i\in\iota'}$; by Lemma 5.4.10, (ii), $\sigma^{-1}e_a - \bar{p}_a n^{-1} e_a^* \in \mathrm{span}\{e_b : b < a\}$, and $h(o_i, e_b) = 0$, so $\varsigma = (h(o_i, \bar{p}_a n^{-1} e_a^*))_i$. Let $\Psi(w) = \mathbf{1}_{20n}(w)\, \phi((h(o_i, w))_i)^s$, with $\mathbf{1}_{20n}$ the indicator of the vectors of size $20n$. If $z(n)$ is near at $a+1$, then $\bar{p}_a n^{-1} e_a^*$ has size $20n$ (Lemma 5.4.10, (iii)); hence the integrand of (i) at n is at most $\Psi(\bar{p}_a n^{-1} e_a^*)$. As η_a is carried by the Heisenberg elements, with $c = \bar{p}_a$ and the section measure $\mathrm{Sec}_a(c)$ of (5.10) in Definition 5.2.1,

$$\int \mathbf{1}\{\dots\}\, \mathrm{dp}_a(z(n), x)^s\, d\eta_a(n) \le \int \Psi(cn^{-1}e_a^*)\, d\eta_a(n) = |c|^{-d_a} \int \Psi\, d\mathrm{Sec}_a(c).$$

By Lemma 5.4.10, (iv), with $j = 20n$, $\varphi = \phi^s$ and $B = \Lambda^{99n(20n+6)} \ge \int_{A^{\iota'}_{(20n+5)}} \phi^s$ (property of a common bound), $\int \Psi\, d\mathrm{Sec}_a(c) \le |c|^{-1/2} \Lambda^{20n+3} \Lambda^{99n(20n+7)} \Lambda^{99n(20n+6)}$. The exponent is $3960n^2 + 1307n + 3 \le 10^6 n^2$, and $|c| = |p_a|$ (PA2), so $|c|^{-d_a - 1/2} = \mathrm{fm}(a, p_a^{-1})$.

(ii). Let n be a Heisenberg element of floor a and $z = z(n)$. If z is not near at $a+1$, the integrand is 0. Otherwise z is near at a ($R^{(a+1)} \subseteq R^{(a)}$, Lemma 5.2.58, (i)), and by Lemma 5.4.8, (iii), $\mathrm{dpw}^-_{a+1,x}(z) = \mathrm{dpw}_a(z,x)^{-1}\mathrm{dp}_a(z,x)^s = \mathrm{dpw}^-_{a,x}(z)\, \mathrm{dp}_a(z,x)^s$. Now $w_{a+1,p} = w_{a,p}\, w_a(p_a)$, with $w_a(p_a), n \in G^{(a)}$, and $w_{a,p}$ commutes with $G^{(a)}$ (Lemma 5.2.58, (ix)); so $z = (w_a(p_a)n)\, w_{a,p}\nu y$, and as $\mathrm{dpw}^-_{a,x}$ is invariant under left translation by $R^{(a)} \supseteq G^{(a)}$ (Lemma 5.4.8, (iii)), $\mathrm{dpw}^-_{a,x}(z) = \mathrm{dpw}^-_{a,x}(w_{a,p}\nu y)$. Hence the integrand is at most $\mathrm{dpw}^-_{a,x}(w_{a,p}\nu y)\, \mathbf{1}\{z \text{ near at } a{+}1\}\, \mathrm{dp}_a(z,x)^s$, and (i) gives (ii).

(iii). Let $\xi \in R^{(j)}$ with $y_0 = \xi g \in K$; by Lemma 5.2.58, (v), $\xi = \gamma\nu_0$ with $\gamma \in G^{(j)}$, $\nu_0 \in N_{<j}$. Then $\nu g = \gamma^{-1}(\gamma\nu\nu_0^{-1}\gamma^{-1})y_0$, and since $w_{j,p}$ commutes with $\gamma^{-1} \in G^{(j)} \subseteq R^{(j)}$ and $\mathrm{dpw}^-_{j,x}$ is left $R^{(j)}$-invariant, $\mathrm{dpw}^-_{j,x}(w_{j,p}\nu g) = \mathrm{dpw}^-_{j,x}(w_{j,p}(\gamma\nu\nu_0^{-1}\gamma^{-1})y_0)$. By Lemma 5.4.14, (iii), $\eta_{<j}$ is invariant under $\nu \mapsto \nu\nu_0^{-1}$, and by Lemma 5.2.59, (iii), under $\nu \mapsto \gamma\nu\gamma^{-1}$; hence

$$I_j := \int \mathrm{dpw}^-_{j,x}(w_{j,p}\nu g)\, d\eta_{<j}(\nu) = \int \mathrm{dpw}^-_{j,x}(w_{j,p}\nu y_0)\, d\eta_{<j}(\nu).$$

We show by induction on $a \le j$ that $\int \mathrm{dpw}^-_{a,x}(w_{a,p}\nu y_0)\, d\eta_{<a}(\nu) \le \prod_{l<a} \mathrm{fm}(l, p_l^{-1}) \cdot \Lambda^{10^6 n^2 a}$ (p is a head of length a). For $a = 0$ the integral is $\mathrm{dpw}^-_{0,x}(y_0) \le 1$. For $a+1 \le j$ (so $a < \varrho$), by Lemma 5.4.14, (ii), $\eta_{<a+1} = \eta_a * \eta_{<a}$ with $\eta_{<a}$ carried by $N_{<a}$, so by Tonelli and (ii),

$$\begin{aligned}\int \mathrm{dpw}^-_{a+1,x}(w_{a+1,p}\nu' y_0)\, d\eta_{<a+1}(\nu') &= \iint \mathrm{dpw}^-_{a+1,x}(w_{a+1,p} n\nu y_0)\, d\eta_a(n)\, d\eta_{<a}(\nu) \\ &\le \mathrm{fm}(a, p_a^{-1}) \Lambda^{10^6 n^2} \int \mathrm{dpw}^-_{a,x}(w_{a,p}\nu y_0)\, d\eta_{<a}(\nu).\end{aligned}$$

At $a = j$: $I_j \le \prod_{a<j} \mathrm{fm}(a, p_a^{-1})\, \Lambda^{10^6 n^2 j} \le \prod_{a<j} \mathrm{fm}(a, p_a^{-1})\, c_{\mathrm{fls}}$, as $j < n$ and $\Lambda \ge 1$.

(iv). By Tonelli, the left side is $\iint \Phi(\nu y)\, \mathrm{dpw}^-_{j,x}(w_{j,p}\nu y)\, d\eta_{<j}(\nu)\, d\rho(y)$. For $\eta_{<j}$-almost every ν, $\nu \in N_{<j} \subseteq R^{(b)}$ (Lemma 5.4.14, (ii); Lemma 5.2.58, (i)), so $\Phi(\nu y) = \Phi(y)$. If y is near at b, it is near at j ($R^{(b)} \subseteq R^{(j)}$), and (iii) bounds the inner integral of $\mathrm{dpw}^-_{j,x}(w_{j,p}\nu y)$; otherwise $\Phi(y) = 0$. This gives (iv). □

Lemma 5.4.8. *Let a place datum be given and $0 \le a < \varrho$.*

(i) The set $\{(x, y, c) \in G \times G \times A : c \in \mathfrak{d}_a(x, y)\}$ is open. For all $x, y \in G$, $\mathfrak{d}_a(x, y)$ is open, contains 0 and has compact closure; hence $0 < \mathrm{vol}(\mathfrak{d}_a(x, y)) < \infty$, $0 < \mathrm{dp}_a(x, y) < \infty$, and $(x, y) \mapsto \mathrm{dp}_a(x, y)$ is measurable.

(ii) $\mathrm{dp}_a(\xi x, y) = \mathrm{dp}_a(x, y)$ for $\xi \in R^{(a+1)}$, and $\mathrm{dp}_a(x, x^{-1}\nu y) = \mathrm{dp}_a(x, x^{-1}y)$ for $\nu \in N_{<a}$, for all $x, y \in G$.

(iii) For $j \geq 0$: $\mathrm{dpw}_{a+1} = \mathrm{dpw}_a \cdot (\mathrm{dp}_a^{-1})^s$; $0 < \mathrm{dpw}_j(x,y) < \infty$; $\mathrm{dpw}_j(\xi x, y) = \mathrm{dpw}_j(x,y)$ for $\xi \in R^{(j)}$; dpw_j is measurable on $G \times G$; and for every x, $\mathrm{dpw}^-_{j,x}$ is measurable with $\mathrm{dpw}^-_{j,x}(\xi g) = \mathrm{dpw}^-_{j,x}(g)$ for $\xi \in R^{(j)}$.

Proof. Recall that $c \in \mathfrak{d}_a(x,y)$ means $c = h(e_a, \omega)$, the e_a^*-coordinate of some $\omega \in V$ with $h(e_b, \omega) = 0$ (the e_b^*-coordinate) for $b < a$, for which there is $\omega' \in \omega + \langle e_b : b < a\rangle$ with $(x^{-1}\omega, (xy)^{-1}\omega') \in \mathcal{P}_{\mathrm{cl}}$; call (ω, ω') a witness. Note that ω and ω' have the same e_b^*-coordinates for all b.

(i). *Openness.* Let (ω, ω') witness $c \in \mathfrak{d}_a(x,y)$. For $c_1 \in A$ let ω_{c_1}, ω'_{c_1} be ω, ω' with the e_a^*-coordinate replaced by c_1. Their e_b^*-coordinates for $b < a$ are unchanged, $\omega'_{c_1} - \omega_{c_1} = \omega' - \omega$, and $h(e_a, \omega_{c_1}) = c_1$. The map $(x_1, y_1, c_1) \mapsto (x_1^{-1}\omega_{c_1}, (x_1 y_1)^{-1}\omega'_{c_1})$ is continuous and sends (x,y,c) into the open set $\mathcal{P}_{\mathrm{cl}}$ (PE4); so a neighbourhood of (x,y,c) lies in the set of (i), each of its points being witnessed by $(\omega_{c_1}, \omega'_{c_1})$. Its sections $\mathfrak{d}_a(x,y)$ are open. $0 \in \mathfrak{d}_a(x,y)$, witnessed by $(0,0) \in \mathcal{P}_{\mathrm{cl}}$ (PE4). *Boundedness.* With $w_1 = x^{-1}\omega$, $c = (xw_1)_{e_a^*}$; so $\mathfrak{d}_a(x,y)$ lies in the image of the compact set $\overline{\mathcal{P}_{\mathrm{cl}}}$ (PE4) under the continuous map $(w_1, w_2) \mapsto (xw_1)_{e_a^*}$. A Haar measure is finite on compact sets and positive on non-empty open sets, so $0 < \mathrm{vol}(\mathfrak{d}_a(x,y)) < \infty$ (the outer measure of an open set is its measure), and $0 < \mathrm{dp}_a(x,y) < \infty$. *Measurability.* $\mathrm{vol}(\mathfrak{d}_a(x,y)) = \int_A \mathbf{1}_O(x,y,c)\,dc$ with O the open set above; it is measurable in (x,y) by Tonelli, and so is its inverse.

(ii). Let $\xi \in R^{(a+1)}$. For $b \leq a$ the column e_b of ξ and of ξ^{-1} has floor and level $b < a+1$, so its entries in rows of level at least b are those of 1: $\xi^{\pm 1} e_b \in e_b + \langle e_c : c < b\rangle$. Hence ξ preserves $\langle e_b : b < a\rangle$, and for ω with $h(e_b, \omega) = 0$ $(b < a)$: $h(e_b, \xi\omega) = h(\xi^{-1}e_b, \omega) = 0$ for $b < a$, and $h(e_a, \xi\omega) = h(\xi^{-1}e_a, \omega) = h(e_a, \omega)$. So if (ω, ω') witnesses $c \in \mathfrak{d}_a(x,y)$, then $(\xi\omega, \xi\omega')$ witnesses $c \in \mathfrak{d}_a(\xi x, y)$, as $((\xi x)^{-1}\xi\omega, (\xi xy)^{-1}\xi\omega') = (x^{-1}\omega, (xy)^{-1}\omega')$. Thus $\mathfrak{d}_a(x,y) \subseteq \mathfrak{d}_a(\xi x, y)$, and with ξ^{-1} equality holds; so $\mathrm{dp}_a(\xi x, y) = \mathrm{dp}_a(x,y)$.

Let $\nu \in N_{<a}$; the two points of $\mathfrak{d}_a(x, x^{-1}y)$ are x and y, those of $\mathfrak{d}_a(x, x^{-1}\nu y)$ are x and νy. Let ω' have e_b^*-coordinate 0 for $b < a$. The entries of $\nu - 1$ lie at (i,k) with $\mathrm{lev}_a(i) < \mathrm{lev}_a(k)$; if i has floor at least a, or $i = e_b^*$ with $b < a$, every such k is some e_c^* with $c < a$, where ω' vanishes. So $\nu\omega' - \omega' \in \langle e_b : b < a\rangle$. Hence if (ω, ω') witnesses c for (x,y), then $(\omega, \nu\omega')$ witnesses c for $(x, \nu y)$, as $(\nu y)^{-1}\nu\omega' = y^{-1}\omega'$; with ν^{-1} the converse holds, so the two sets of stable products are equal.

(iii). The recursion and $0 < \mathrm{dpw}_j < \infty$ follow from (5.16) and (i). If $\xi \in R^{(j)}$ and $a < \min(j, \varrho)$, then $\xi \in R^{(a+1)}$ (Lemma 5.2.58, (i)), so each factor of $\mathrm{dpw}_j(\xi x, y)$ equals that of $\mathrm{dpw}_j(x,y)$ by (ii). Measurability follows from (i). The set N_j of points near at j is $R^{(j)}K$ ($\xi g \in K$ means $g \in \xi^{-1}K$), the product of a closed subgroup and a compact set (PE4), hence closed, and $\xi N_j = N_j$ for $\xi \in R^{(j)}$. So $\mathrm{dpw}^-_{j,x} = \mathbf{1}_{N_j} \cdot \mathrm{dpw}_j(\cdot, x)^{-1}$ is measurable and left $R^{(j)}$-invariant. □

Lemma 5.4.9. *Let a place datum be given, $0 \leq a < \varrho$, $g, x, \sigma \in G$, Q a $\varrho \times \iota'$ matrix over A, and let ϕ be a common bound for $\mathcal{V}_{\mathrm{cm}}(g, x, Q)$ with some loss, for some size filtration. If $h(e_b, \sigma O\omega) = 0$ for all $\omega \in \mathcal{V}_{\mathrm{cm}}(g, x, Q)$ and $b < a$, then*

$$\mathrm{dp}_a(\sigma g, x) \leq \phi((h(o_i, \sigma^{-1}e_a))_{i \in \iota'}).$$

Proof. Put $\varsigma = (h(o_i, \sigma^{-1}e_a))_{i \in \iota'}$ and let $\omega \in \mathcal{V}_{\mathrm{cm}}(g, x, Q)$, $w = \sigma O\omega$. By hypothesis $h(e_b, w) = 0$ for $b < a$, and $((\sigma g)^{-1}w, (\sigma g x)^{-1}w) = (g^{-1}O\omega, (gx)^{-1}O\omega) \in \mathcal{P}_{\mathrm{cl}}$; taking $\omega' = w$ in the definition

of stable products, $h(e_a, w) \in \mathfrak{d}_a(\sigma g, x)$. As σ preserves h and h is hermitian,

$$h(e_a, \sigma O\omega) = h(\sigma^{-1}e_a, O\omega) = \sum_{i\in\iota'} h(\sigma^{-1}e_a, o_i)\,\omega_i = \sum_{i\in\iota'} \overline{h(o_i, \sigma^{-1}e_a)}\,\omega_i = \bar{\varsigma}^{\mathsf{T}}\omega.$$

Hence $\{\bar{\varsigma}^{\mathsf{T}}\omega : \omega \in \mathcal{V}_{\mathrm{cm}}(g, x, Q)\} \subseteq \mathfrak{d}_a(\sigma g, x)$, so by monotonicity of the outer measure and the first property of a common bound,

$$\mathrm{dp}_a(\sigma g, x) = \mathrm{vol}^*(\mathfrak{d}_a(\sigma g, x))^{-1} \le \mathrm{vol}^*\big(\{\bar{\varsigma}^{\mathsf{T}}\omega : \omega \in \mathcal{V}_{\mathrm{cm}}(g, x, Q)\}\big)^{-1} \le \phi(\varsigma).$$

□

Lemma 5.4.10. *Let a place datum be given, let $a < \varrho$ and $E_{<a} = \mathrm{span}\{e_b : b < a\}$. For $c \in A$ let $\mathrm{Sec}_a(c)$ be the section measure,* (5.10) *of Definition 5.2.1 ($|c|^{d_a}$ times the image of η_a under $n \mapsto c\,n^{-1}e_a^*$).*

(i) For every Borel $f\colon V \to [0,\infty]$, $\int_V f\,d\mathrm{qv}_a = \int_A \int_V f\,d\mathrm{Sec}_a(c)\,dc$.

(ii) For $v \in V_{a+1}$, $\beta \in A^-$ and $c \in A$, $c\,n_a(v,\beta)^{-1}e_a^* = c(e_a^* - v - (\beta + \bar{\tau}h(v,v))e_a)$. For $n \in G^{(a)}$ the coordinates of $n^{-1}e_a^*$ of floor below a vanish, and for $\nu \in N_{<a}$ and a sequence p of units, $(w_{a+1,p}n\nu)^{-1}e_a - \bar{p}_a\,n^{-1}e_a^* \in E_{<a}$.

Let now Γ be a gauge of loss Λ for the place datum, with size filtration $(A_{(i)})$.

(iii) Let p be a head of length $a+1$, $y \in K$, n a Heisenberg element of floor a and $\nu \in N_{<a}$. If $w_{a+1,p}n\nu y$ is near at $a+1$, then $\bar{p}_a\,n^{-1}e_a^*$ has size $20n$.

(iv) Let c be a unit, $j \ge 0$, $B \in [0,\infty]$ and $\varphi\colon A^{\iota'} \to [0,\infty]$ Borel with $\int_{A^{\iota'}_{(j+5)}} \varphi\,d\mathrm{vol} \le B$. Then

$$\int_{\{y\in V \text{ of size } j\}} \varphi((h(o_i,y))_{i\in\iota'})\,d\mathrm{Sec}_a(c)(y) \le |c|^{-1/2}\Lambda^{j+3}\Lambda^{99n(j+7)}B.$$

Proof. **(i).** By Lemma 5.2.63 with $b = a$ and $s = 1$, qv_a is the image of θ_a under $\sigma \mapsto \sigma^{-1}e_a$ ($f_a = e_a$). As $\theta_a = \mathrm{wl}_a * \eta_a$ and wl_a is the image of $|c|^{d_a}dc$ under $c \mapsto w_a(c)^{-1}$, Tonelli gives

$$\int f\,d\mathrm{qv}_a = \iint f((xn)^{-1}e_a)\,d\mathrm{wl}_a(x)\,d\eta_a(n) = \int_A |c|^{d_a} \int f(n^{-1}w_a(c)e_a)\,d\eta_a(n)\,dc.$$

For a unit c, $w_a(c)e_a = ce_a^*$ (5.7), so the inner integral is $\int f\,d\mathrm{Sec}_a(c)$ divided by $|c|^{d_a}$; almost every c is a unit (PA4).

(ii). The first formula is c times the second formula of Lemma 5.4.11, (i), plus ce_a^*. For $n \in G^{(a)}$, $n^{-1} \in G^{(a)}$, whose rows at the indices of floor below a are those of 1 (Lemma 5.2.58, (vii)); as e_a^* has floor a, these coordinates of $n^{-1}e_a^*$ vanish. The inverse of $w = w_{a+1,p}$ sends e_a to $\bar{p}_a e_a^*$, so $(wn\nu)^{-1}e_a = \bar{p}_a\,\nu^{-1}n^{-1}e_a^*$; by the last claim of Lemma 5.4.11, (i), applied to $n^{-1} \in G^{(a)}$ and $\nu^{-1} \in N_{<a}$, $\nu^{-1}n^{-1}e_a^* - n^{-1}e_a^* \in E_{<a}$.

(iii). The conjugate sequence $\bar{p}$ is a head of length $a+1$ (SZ3), and $w_{a+1,\bar{p}}^{-1} = w_{a+1,p}$ (both send $e_b \mapsto p_b e_b^*$ and $e_b^* \mapsto \bar{p}_b^{-1}e_b$ for $b \le a$). Put $g = n\nu y$ and $s = (n\nu)^{-1}$. By Lemma 5.2.58, (iii) and (i), $n \in N_{<a+1}$ and $\nu \in N_{<a}$ lie in $R^{(a+1)}$, so $s \in R^{(a+1)}$, $sg = y \in K$, and $w_{a+1,\bar{p}}^{-1}g = w_{a+1,p}n\nu y$ is near at $a+1$. By Lemma 5.4.12, (i), with $j = a+1$ and the head $\bar{p}$, both $\bar{p}_a$ and the entries of $se_a^* = \nu^{-1}n^{-1}e_a^*$ lie in $A_{(10n)}$, so $(w_{a+1,p}n\nu)^{-1}e_a = \bar{p}_a\,\nu^{-1}n^{-1}e_a^*$ has size $20n$ (SZ1). By (ii), $\bar{p}_a n^{-1}e_a^*$ has the same coordinates as this vector at all indices of level at least a (the

indices of $E_{<a}$ are the e_b, $b<a$, of level $b<a$), and its coordinates at the remaining indices, the e_b with $b<a$, vanish. Hence it has size $20n$ (SZ4).

(iv). Put $u=(-c)^{-1}$, so $u^{-1}=-c$ and $|u|=|c|^{-1}$ by (PA2). Let

$$\Phi(n)=\mathbf{1}_j(cn^{-1}e_a^*)\,\varphi((h(o_i,cn^{-1}e_a^*))_{i\in\iota'}),$$

with $\mathbf{1}_j$ the indicator of the vectors of size j, and $\Psi(v')=\varphi((h(o_i,v'))_{i\in\iota'})$; both are Borel, and the left side of (iv) is $|c|^{d_a}\int\Phi\,d\eta_a$. Let $n=n_a(v,\beta)$ with $\Phi(n)\neq 0$. As $h(o_i,e_a)=h(o_i,e_a^*)=0$, (ii) gives $h(o_i,cn^{-1}e_a^*)=h(o_i,-cv)$, so $\Phi(n)\leq\Psi(u^{-1}v)$. Moreover $cn^{-1}e_a^*$ has size j, and its e_a^*-coordinate is c by (ii) ($v\in V_{a+1}$ has no e_a^*-coordinate); so $w=(n^{-1}-1)e_a^*$ satisfies $u^{-1}w=ce_a^*-cn^{-1}e_a^*$, of size $j+1$ (SZ2, SZ3). By Lemma 5.4.4, (iv), with $l=a+1$, $\rho=\mathrm{vol}_{V_{a+1}}$ and $j+1$ in place of j, $\int_{\{v'\text{ of size }j+1\}}\Psi\,d\mathrm{vol}_{V_{a+1}}\leq\Lambda^{99n(j+7)}B$. Hence Lemma 5.4.11, (iii), with $C=1$, $j+1$ and $I=\Lambda^{99n(j+7)}B$ gives $\int\Phi\,d\eta_a\leq\mathrm{fm}(a,u)\Lambda^{j+3}\Lambda^{99n(j+7)}B$, with $\mathrm{fm}(a,u)=|u|^{d_a}|u|^{1/2}=|c|^{-d_a}|c|^{-1/2}$. Multiplying by $|c|^{d_a}$ ($0<|c|<\infty$ for a unit) gives (iv). □

Lemma 5.4.11. *Let a place datum be given, let $a<\varrho$, and write $E_{<a}=\mathrm{span}\{e_b:b<a\}$.*

(i) For $v\in V_{a+1}$ and $\beta\in A^-$: $(n_a(v,\beta)-1)e_a^*=v+(\beta-\tau h(v,v))e_a$ and $(n_a(v,\beta)^{-1}-1)e_a^*=-v-(\beta+\bar\tau h(v,v))e_a$. For $n\in G^{(a)}$ and $\nu\in N_{<a}$, $(\nu n-1)e_a^*-(n-1)e_a^*\in E_{<a}$.

Let now Γ be a gauge of loss Λ for the place datum, with size filtration $(A_{(i)})$.

(ii) Let u be a unit, $i\geq 0$, $v\in V_{a+1}$, $\beta\in A^-$, and let $w\in V$ with $w-(n_a(v,\beta)-1)e_a^*\in E_{<a}$ and $u^{-1}w$ of size i. Then $u^{-1}v$ has size i and $u^{-1}(\beta-\tau h(v,v))\in A_{(i)}$. The same holds with $n_a(v,\beta)^{-1}$ in place of $n_a(v,\beta)$ and $-u^{-1}(\beta+\bar\tau h(v,v))$ in place of $u^{-1}(\beta-\tau h(v,v))$.

(iii) Let u be a unit, $j\geq 0$, $C,I\in[0,\infty]$, let $\Psi\colon V\to[0,\infty]$ be Borel with $\int_{\{v'\text{ of size }j\}}\Psi\,d\mathrm{vol}_{V_{a+1}}\leq I$, and let $\Phi\colon G\to[0,\infty]$ be Borel. Suppose that for all $v\in V_{a+1}$ and $\beta\in A^-$ with $\Phi(n_a(v,\beta))\neq 0$ there is $w\in V$ with $w-(n_a(v,\beta)^{-1}-1)e_a^*\in E_{<a}$ and $u^{-1}w$ of size j, and $\Phi(n_a(v,\beta))\leq C\,\Psi(u^{-1}v)$. Then

$$\int_G\Phi\,d\eta_a\leq C\,\mathrm{fm}(a,u)\,\Lambda^{j+2}\,I.$$

Proof. **(i).** Write $n=n_a(v,\beta)$. In (5.8) take $w=e_a^*$: its e_a^*-coordinate is $h(e_a,e_a^*)=1$ and $h(v,e_a^*)=0$ as $v\in V_{a+1}$, so $ne_a^*=e_a^*+v+(\beta-\tau h(v,v))e_a$. By Lemma 5.2.58, (iii), $n_a(v,\beta)\,n_a(-v,\beta')=n_a(0,\beta+\beta'-\tau h(v,v)+\bar\tau h(v,v))$, and $n_a(0,0)=1$; so $n^{-1}=n_a(-v,\beta')$ with $\beta'=-\beta+(\tau-\bar\tau)h(v,v)\in A^-$, and the first formula gives $(n^{-1}-1)e_a^*=-v+(\beta'-\tau h(v,v))e_a=-v-(\beta+\bar\tau h(v,v))e_a$. For the last claim, $(\nu n-1)e_a^*-(n-1)e_a^*=(\nu-1)w_0$ with $w_0=ne_a^*$. For $b<a$, $h(e_b,w_0)=h(n^{-1}e_b,e_a^*)=h(e_b,e_a^*)=0$, so the e_b^*-coordinates of w_0 vanish ($h(e_b,w)$ is the e_b^*-coordinate of w). By the definition of $N_{<a}$, the columns of ν at the e_c ($c<a$) and at the indices of floor at least a are the coordinate columns plus elements of $E_{<a}$ (the indices of modified level below that of the column are among the e_b, $b<a$). Hence $(\nu-1)w_0\in E_{<a}$.

(ii). By (i), w and $v+(\beta-\tau h(v,v))e_a$ have the same coordinates outside the e_b, $b<a$. The coordinates of v at floors at most a vanish, so the coordinates of $u^{-1}v$ are 0 or coordinates of $u^{-1}w$, hence in $A_{(i)}$ (SZ4), and $u^{-1}(\beta-\tau h(v,v))$ is the e_a-coordinate of $u^{-1}w$. The second statement follows in the same way from the second formula of (i), using (SZ3) for the signs.

A bound for the skew measure. Let u be a unit and $S\subseteq A^-$ a Borel set with $u^{-1}(\beta-\beta')\in A_{(j+1)}$ for all $\beta,\beta'\in S$. Then $\nu(S)\leq|u|^{1/2}\Lambda^{j+2}$. Indeed, let $\Sigma(\alpha,\beta)=\tau\delta^{-1}\alpha+\beta$, which carries $\nu\otimes\nu$ to vol (PA1). For $p,p'\in S\times S$, $u^{-1}(\Sigma(p)-\Sigma(p'))=\tau\delta^{-1}\cdot u^{-1}(\alpha-\alpha')+u^{-1}(\beta-\beta')\in A_{(j+3)}$,

as $\tau\delta^{-1} \in A_{(1)}$ (GU2), by (SZ1), (SZ2). Fix $p_0 \in S\times S$ (if $S=\emptyset$ there is nothing to prove). Then $\Sigma(S\times S - p_0) \subseteq uA_{(j+3)}$, so by translation invariance and (PA1), $\nu(S)^2 \le \mathrm{vol}(uA_{(j+3)})$. By (PA3) for the 1×1 matrix (u^{-1}), $\mathrm{vol}(uC) = |u|\,\mathrm{vol}(C)$; with (GU5), $\nu(S)^2 \le |u|\Lambda^{j+4} \le (|u|^{1/2}\Lambda^{j+2})^2$.

(iii). By the definition of η_a and Tonelli, $\int \Phi\, d\eta_a = \int_{V_{a+1}} \int_{A^-} \Phi(n_a(v,\beta))\, d\nu(\beta)\, dv$. Fix v and let $S_v = \{\beta : \Phi(n_a(v,\beta)) \neq 0\}$, a Borel set. For $\beta \in S_v$, (ii) (second form, with $i=j$) gives that $u^{-1}v$ has size j and $u^{-1}(\beta + \bar\tau h(v,v)) \in A_{(j)}$; hence for $\beta,\beta' \in S_v$ the difference $u^{-1}(\beta-\beta')$ lies in $A_{(j+1)}$ (SZ2, SZ3), and by the bound for the skew measure,

$$\int_{A^-} \Phi(n_a(v,\beta))\, d\nu(\beta) \le \nu(S_v)\, C\, \Psi(u^{-1}v)\, \mathbf{1}_j(u^{-1}v) \le |u|^{1/2}\Lambda^{j+2}\, C\, \Psi(u^{-1}v)\, \mathbf{1}_j(u^{-1}v),$$

where $\mathbf{1}_j$ is the indicator of the vectors of size j (the left side is 0 if $S_v=\emptyset$). The space V_{a+1} has d_a coordinates, so by (PA3) for the diagonal matrix $u\cdot 1$ the image of $\mathrm{vol}_{V_{a+1}}$ under $v' \mapsto uv'$ is $|u|^{-d_a}\mathrm{vol}_{V_{a+1}}$, and

$$\int_{V_{a+1}} \Psi(u^{-1}v)\mathbf{1}_j(u^{-1}v)\, dv = |u|^{d_a} \int_{\{v' \text{ of size } j\}} \Psi\, d\mathrm{vol}_{V_{a+1}} \le |u|^{d_a} I.$$

Hence $\int \Phi\, d\eta_a \le C\,|u|^{d_a}|u|^{1/2}\Lambda^{j+2} I = C\,\mathrm{fm}(a,u)\,\Lambda^{j+2} I$. □

Lemma 5.4.12. *Let Γ be a gauge, with size filtration $(A_{(i)})_{i\ge 0}$, for a place datum; let $j \le \varrho$, let p be a head of length j, write $w_p = w_{j,p}$, and let $g \in G$.*

(i) Let $s \in R^{(j)}$ with $sg \in K$, and let $w_p^{-1}g$ be near at j. Then $p_0\cdots p_{l-1} \in A_{(9n)}$ for $0 \le l \le j$, and for every $a < j$: $p_0\cdots p_a\, s_{ke_a^*} \in A_{(8\varrho)}$ for every index k, the column se_a^* has size $8\varrho+1$, and p_a, p_a^{-1} and the entries of se_a^* lie in $A_{(10n)}$.

(ii) If moreover $s \in N_{<j}$, then g has size $60n^2$ and $w_p^{-1}g$ has size $99n^2$.

(iii) If g and $w_p^{-1}g$ are near at j, then there is $\sigma \in G^{(j)}$ such that σg has size $60n^2$ and $\sigma w_p^{-1}g$ has size $99n^2$.

Proof. We use the size rules of the proof of Lemma 5.4.13, which follow from (SZ1)–(SZ5): $A_{(i)} \subseteq A_{(i+1)}$; a product of matrices of sizes i and i' has size $i+i'+1$; the inverse of an element of G of size i has size $i+4$, so elements of K (size 1, (GU2)) have inverses of size 5; a determinant of order $i \le \varrho$ with entries in $A_{(7)}$ lies in $A_{(8i)}$. Recall also that for $t \in R^{(j)}$ and $d < j$ the column te_d is $e_d + \sum_{c\le d}(\cdot)\, e_c$ ((5.76) of Lemma 5.4.13, as $e_d = f_d$), and that $t^{-1} = J^{-1}\bar t^{\mathsf T} J$, which gives $t_{e_b^* e_c^*} = \overline{(t^{-1})_{e_c e_b}}$ for all $b,c < \varrho$, since the row e_b^* of J^{-1} and the column e_c^* of J are coordinate vectors at e_b and e_c.

(i). Let $\eta \in R^{(j)}$ with $\eta w_p^{-1} g \in K$. As $sg \in K$, g is near at j; so $y = g$ and $\gamma = w_p$ satisfy the hypothesis of Lemma 5.4.13, (ii), with $l = j$, and with (iii) there, $p_0\cdots p_{l-1} = D_l(w_p) \in A_{(9n)}$ for $l \le j$.

Cramer. Fix $a < j$ and an index k. The element $y = sw_p\eta^{-1} = (sg)(\eta w_p^{-1} g)^{-1}$ has size $1+5+1 = 7$. Let Z be the $(a+1)\times(a+1)$ matrix $Z_{bd} = y_{R_b e_d}$ $(b,d \le a)$ with rows $R_b = e_b^*$ for $b < a$ and $R_a = k$; its entries lie in $A_{(7)}$, so $\det Z \in A_{(8(a+1))} \subseteq A_{(8\varrho)}$. Since $\eta^{-1} \in R^{(j)}$ and $d \le a < j$, $\eta^{-1}e_d = \sum_{c\le d} C_{cd} e_c$ with C upper unitriangular; and $w_p e_c = p_c e_c^*$ for $c < j$. Hence $ye_d = \sum_{c\le d} C_{cd}\, p_c\, se_c^*$, that is $Z = LC$ with $L_{bc} = p_c\, s_{R_b e_c^*}$, and $\det Z = \det L$. For $b \le c < j$, $s_{e_b^* e_c^*} = \overline{(s^{-1})_{e_c e_b}} = \delta_{bc}$, because the column e_b of $s^{-1} \in R^{(j)}$ is e_b plus a combination of the $e_{c'}$ with $c' < b$. So the rows $b < a$ of L are p_b times the coordinate row at b on and to the right of the diagonal, L is lower triangular, and

$$\det Z = \det L = p_0\cdots p_{a-1}\cdot p_a\, s_{ke_a^*} \in A_{(8\varrho)}.$$

As p is a head of length j and $a+1 \le j$, $(p_0 \cdots p_a)^{-1} \in A_{(1)}$, so $s_{ke_a^*} \in A_{(8\varrho+1)} \subseteq A_{(10n)}$ by (SZ1). Finally $p_a = (p_0 \cdots p_{a-1})^{-1}(p_0 \cdots p_a)$ and $p_a^{-1} = (p_0 \cdots p_a)^{-1}(p_0 \cdots p_{a-1})$ lie in $A_{(9n+1)} \subseteq A_{(10n)}$.

(ii). Put $d = 10n$; by (i), every column se_b^*, $b < j$, has size d. We bound the other columns of s, using $s \in N_{<j}$ and that s preserves h. With lev_j as in the definition of $N_{<j}$, the e_c ($c < j$) have $\mathrm{lev}_j = c < \varrho$, the e_c^* ($c < j$) have $\mathrm{lev}_j = 2\varrho - c > \varrho$, and the indices of floor at least j have $\mathrm{lev}_j = \varrho$. Hence: $se_a = e_a + \sum_{c<a} u_{ca} e_c$ for $a < j$; $sv = v + \sum_{c<j} t_c(v) e_c$ for every coordinate vector v of floor at least j; and $s_{e_c^* e_b^*} = \delta_{cb}$ for $c \le b < j$. For $b, c < j$ put $N_{bc} = h(se_b^*, e_c) = \overline{s_{e_c^* e_b^*}}$ (the column e_c of J is the coordinate vector at e_c^*). Then $N_{bb} = 1$, $N_{bc} = 0$ for $c < b$, and $N_{bc} \in A_{(d)}$ (SZ3).

Columns of floor at least j. Let v be such a coordinate vector and $h_b = h(se_b^*, v)$, a sum of at most r products of the conjugate of an entry of se_b^* and an entry of J, so $h_b \in A_{(d+2)}$. Since $h(e_b^*, v) = v_{e_b} = 0$, preservation of h gives $0 = h(se_b^*, sv) = h_b + \sum_{c<j} N_{bc} t_c(v)$, that is $t_b(v) = -h_b - \sum_{b<c<j} N_{bc} t_c(v)$. By descending induction on b, each step being a sum of at most $j \le r+1$ terms, $t_b(v) \in A_{(d+2+(d+1)(j-1-b))} \subseteq A_{((d+1)j+1)}$.

Columns e_a. Put $u_{aa} = 1$ and $u_{ca} = 0$ for $c > a$. Since $h(e_b^*, e_a) = \delta_{ab}$, preservation of h gives $\delta_{ab} = h(se_b^*, se_a) = \sum_{c \le a} N_{bc} u_{ca}$; for $b < a$ this reads $u_{ba} = -\sum_{b<c\le a} N_{bc} u_{ca}$, and descending induction on b gives $u_{ba} \in A_{((d+1)(a-b))} \subseteq A_{((d+1)j+1)}$.

All other entries of s are 0 or 1. So s has size $D = (d+1)j + 1$ (for $j = 0$, $s = 1$), s^{-1} has size $D+4$, and $g = s^{-1}(sg)$ has size $D + 6 = (10n+1)j + 7 \le (10n+1)(n-1) + 7 \le 60n^2$. The inverse w_p^{-1} sends $e_a^* \mapsto p_a^{-1} e_a$ and $e_a \mapsto \bar{p}_a e_a^*$ for $a < j$ and fixes the other coordinate vectors, so by (i) and (SZ3), (SZ4) it has size $10n$; hence $w_p^{-1} g$ has size $10n + 60n^2 + 1 \le 99n^2$.

(iii). Let $\xi \in R^{(j)}$ with $\xi g \in K$ and $\zeta \in R^{(j)}$ with $\zeta w_p^{-1} g \in K$. By Lemma 5.2.58, (v), $\xi = \sigma\nu$ with $\sigma \in G^{(j)}$ and $\nu \in N_{<j}$. Put $g' = \sigma g$ and $s = \sigma\nu\sigma^{-1}$; by Lemma 5.2.58, (ii) and (i), $s \in N_{<j} \subseteq R^{(j)}$, and $sg' = \xi g \in K$. By Lemma 5.2.58, (ix), $w_p^{-1} g' = \sigma w_p^{-1} g$, and $(\zeta\sigma^{-1})(\sigma w_p^{-1} g) \in K$ with $\zeta\sigma^{-1} \in R^{(j)}$ (as $G^{(j)} \subseteq R^{(j)}$), so $w_p^{-1} g'$ is near at j. Now (ii) for g' and s gives the claim. □

Lemma 5.4.13. *Let a place datum be given. For $\gamma \in G$ let $Z(\gamma) = (h(f_a, \gamma f_b))_{0\le a,b\le\varrho}$, and for $0 \le i \le n$ let $D_i(\gamma)$ be the determinant of its upper left $i \times i$ block ($D_0(\gamma) = 1$).*

(i) For $l \ge 0$, $\xi, \xi' \in R^{(l)}$, $\gamma \in G$ and $0 \le i \le \min(l, n)$: $D_i(\xi^{-1}\gamma\xi') = D_i(\gamma)$.

(ii) Let Γ be a gauge for the place datum, with size filtration $(A_{(i)})$. If $l \ge 0$, $\gamma, y \in G$ and both y and $\gamma^{-1} y$ are near at l, then $D_i(\gamma) \in A_{(9n)}$ for $0 \le i \le \min(l, n)$.

(iii) For $j \le \varrho$, a sequence p of units of A and $0 \le i \le j$: $D_i(w_{j,p}) = p_0 p_1 \cdots p_{i-1}$.

Proof. **Frame columns of $R^{(l)}$.** Let $s \in R^{(l)}$ and $b < \min(l, n)$. The index f_b has level b; if $l \le \varrho + 1$ it has floor $b < l$ and level $b < l$, and if $l \ge \varrho + 2$ no condition on the column is needed. In both cases the definition of $R^{(l)}$ gives $s_{kf_b} = \delta_{kf_b}$ for every index k of level at least b. The indices of level below b are the $e_c = f_c$ with $c < b$ (the x_i and z have level $\varrho \ge b$, the e_c^* level $2\varrho - c \ge \varrho + 1$). Hence

$$sf_b = f_b + \sum_{c<b} t_{cb}(s)\, f_c \qquad (b < \min(l, n)) \tag{5.76}$$

for some $t_{cb}(s) \in A$.

(i). Fix $i \le \min(l, n)$, let $Z_i(\gamma)$ be the upper left $i \times i$ block of $Z(\gamma)$, and let $U_i(s)$ be the $i \times i$ matrix with entries $t_{cb}(s)$ for $c < b$, 1 on the diagonal and 0 below it; $\det U_i(s) = 1$. As h is linear in its second argument, (5.76) for ξ' gives $h(f_a, \gamma\xi' f_b) = \sum_{c\le b} h(f_a, \gamma f_c)\, t_{cb}(\xi')$ for $a, b < i$,

that is $Z_i(\gamma\xi') = Z_i(\gamma)U_i(\xi')$. As ξ preserves h, $h(f_a, \xi^{-1}w) = h(\xi f_a, w)$, and h is conjugate linear in its first argument, so (5.76) for ξ gives $Z_i(\xi^{-1}\gamma') = \overline{U_i(\xi)}^{\mathsf{T}} Z_i(\gamma')$. Taking determinants with $\gamma' = \gamma\xi'$ gives $D_i(\xi^{-1}\gamma\xi') = \overline{\det U_i(\xi)}\, D_i(\gamma) \det U_i(\xi') = D_i(\gamma)$.

Sizes of inverses, products and determinants. By (SZ2) with one summand, $A_{(i)} \subseteq A_{(i+1)}$. A product of matrices of sizes i and i' (of order at most r) has size $i + i' + 1$, by (SZ1), (SZ2). If $g \in G$ has size i, then $g^{-1} = J^{-1}\bar{g}^{\mathsf{T}}J$ has size $i + 4$, by (SZ3), (SZ5) and the rule for products. A square matrix of order $i \leq \varrho$ with entries in $A_{(7)}$ has determinant in $A_{(8i)}$: for $i = 0$ it is $1 \in A_{(0)}$ (SZ4), and by Laplace expansion along the first row the determinant of order $i \geq 1$ is a sum of $i \leq r + 1$ terms $\pm$(entry)(minor of order $i - 1$), each in $A_{(7+8(i-1))}$ by induction, (SZ1) and (SZ3), hence lies in $A_{(8i)}$ by (SZ2).

(ii). Let $\xi, \xi' \in R^{(l)}$ with $\xi y \in K$ and $\xi'\gamma^{-1}y \in K$, and put $x = (\xi y)(\xi'\gamma^{-1}y)^{-1} = \xi\gamma\xi'^{-1}$. Elements of K have size 1 (GU2), so x has size $1 + 5 + 1 = 7$, and $\gamma = \xi^{-1}x\xi'$, so $D_i(\gamma) = D_i(x)$ by (i). Since f_a is a coordinate vector, $Z(x)_{ab} = (Jx)_{f_a f_b}$. For $a < \varrho$ the row e_a of J is the coordinate row of e_a^*, so $Z(x)_{ab} = x_{e_a^* f_b} \in A_{(7)}$; the row $f_\varrho = z$ of Jx has entries in $A_{(9)}$ by the rule for products. If $i \leq \varrho$, $D_i(x)$ is a determinant of order i with entries in $A_{(7)}$, so $D_i(x) \in A_{(8i)} \subseteq A_{(9n)}$. If $i = n$ and $\varrho = 0$, $D_1(x) = Z(x)_{00} \in A_{(9)} = A_{(9n)}$. If $i = n$ and $\varrho \geq 1$, expanding along the last row, $D_n(x)$ is a sum of $n \leq r + 1$ terms $\pm$(entry of the row of z)(determinant of order ϱ with entries in $A_{(7)}$), each in $A_{(9+8\varrho)}$, so $D_n(x) \in A_{(8\varrho+10)} \subseteq A_{(9n)}$, as $8\varrho + 10 \leq 9\varrho + 9$.

(iii). For $b < i \leq j \leq \varrho$, $f_b = e_b$ and $w_{j,p}e_b = p_b e_b^*$, so $h(f_a, w_{j,p}f_b) = p_b\, h(e_a, e_b^*) = p_b\delta_{ab}$ for $a, b < i$. The block is the diagonal matrix with entries $p_0, \ldots, p_{i-1}$. □

Lemma 5.4.14. *Let a place datum be given and $j \geq 0$. Call $n_j(v, \beta)$ ($v \in V_{j+1}$, $\beta \in A^-$) the Heisenberg elements of floor j if $j < \varrho$, and 1 the only one if $j \geq \varrho$.*

(i) Every $s \in N_{<j+1}$ is a product $s = n\nu$ of a Heisenberg element n of floor j and some $\nu \in N_{<j}$.

(ii) $\eta_{<j+1} = \eta_j * \eta_{<j}$, and $\eta_{<j}$ is carried by $N_{<j}$.

(iii) $\eta_{<j} * \delta_s = \delta_s * \eta_{<j} = \eta_{<j}$ for every $s \in N_{<j}$.

Proof. **Two facts on $N_{<l}$.** The group $N_{<0}$ is $\{1\}$: for $l = 0$ every index has modified level ϱ, so every entry is that of 1. If $l \geq \varrho$, the modified level lev_{l+1} agrees with lev_l (the indices that change from level ϱ to their level have level ϱ already), so $N_{<l+1} = N_{<l}$.

(i). If $j \geq \varrho$, take $n = 1$, $\nu = s$. Let $j < \varrho$. By Lemma 5.2.58, (i), $s \in N_{<j+1} \subseteq R^{(j)}$, so by (v) there, $s = \gamma\nu$ with $\gamma \in G^{(j)}$ and $\nu \in N_{<j}$; then $\gamma = s\nu^{-1} \in N_{<j+1}$, as $N_{<j} \subseteq N_{<j+1}$ and $N_{<j+1}$ is a group. Since $N_{<j+1} \subseteq R^{(j+1)}$, $\gamma \in R^{(j+1)} \cap G^{(j)}$, and by Lemma 5.2.58, (iv), there is a Heisenberg element n of floor j with $\gamma' = n^{-1}\gamma \in G^{(j+1)}$; also $\gamma' \in N_{<j+1}$ (Heisenberg elements of floor j lie in $N_{<j+1}$, Lemma 5.2.58, (iii)). We show $\gamma' = 1$. It fixes the e_b, e_b^* with $b \leq j$. A coordinate vector w of floor at least $j + 1$ has $\mathrm{lev}_{j+1}(w) = \varrho$, and the indices of smaller modified level are the e_c, $c \leq j$; so $\gamma' w = w + \sum_{c \leq j} t_c e_c$; but the rows of $\gamma' \in G^{(j+1)}$ at the indices of floor below $j + 1$ are those of 1 (Lemma 5.2.58, (vii)), so all $t_c = 0$ and $\gamma' w = w$. Hence $\gamma' = 1$ and $s = n\nu$.

Commutation. If a measure μ is carried by a set of elements s with $\delta_s * \mu' = \mu' * \delta_s$, then $\mu * \mu' = \mu' * \mu$: for measurable $f \geq 0$, by Tonelli, $\int f\, d(\mu * \mu') = \iint f(sy)\, d\mu'(y)\, d\mu(s) = \iint f(ys)\, d\mu'(y)\, d\mu(s) = \int f\, d(\mu' * \mu)$.

(ii). By definition $\eta_{<j+1} = \eta_{<j} * \eta_j$. The measure η_j is carried by $G^{(j)}$ (Lemma 5.2.59, (ii)), and $\delta_s * \eta_{<j} = \eta_{<j} * \delta_s$ for $s \in G^{(j)}$ (Lemma 5.2.59, (iii)); by the commutation fact,

$\eta_{<j} * \eta_j = \eta_j * \eta_{<j}$. For the carrier, induct on j: $\eta_{<0} = \delta_1$ and $1 \in N_{<0}$; if $\eta_{<j}$ is carried by $N_{<j} \subseteq N_{<j+1}$, then as η_j is carried by the Heisenberg elements of floor j, which lie in $N_{<j+1}$ (for $j \geq \varrho$, $\eta_j = \delta_1$), the image $\eta_{<j} * \eta_j$ of $\eta_{<j} \otimes \eta_j$ under multiplication is carried by the group $N_{<j+1}$, which is closed (it is defined by equations on entries), hence Borel.

(iii). Induction on j. For $j = 0$, $N_{<0} = \{1\}$. Assume (iii) for j and let $s \in N_{<j+1}$; by (i), $s = n\nu$ with n a Heisenberg element of floor j and $\nu \in N_{<j}$, so $\delta_s = \delta_n * \delta_\nu$. By Lemma 5.2.61, (ii) (trivially if $j \geq \varrho$), $\eta_j * \delta_n = \delta_n * \eta_j = \eta_j$. Using (ii) and associativity of convolution,

$$\eta_{<j+1} * \delta_s = \eta_{<j} * (\eta_j * \delta_n) * \delta_\nu = (\eta_{<j} * \eta_j) * \delta_\nu = \eta_j * (\eta_{<j} * \delta_\nu) = \eta_j * \eta_{<j} = \eta_{<j+1},$$

$$\delta_s * \eta_{<j+1} = \delta_n * (\delta_\nu * \eta_{<j}) * \eta_j = \delta_n * \eta_{<j} * \eta_j = \delta_n * \eta_j * \eta_{<j} = (\delta_n * \eta_j) * \eta_{<j} = \eta_j * \eta_{<j} = \eta_{<j+1}.$$

□

Lemma 5.4.15. *Let a place datum of shape (ϱ, ι, m) be given, and let $p = (p_a)_{a\geq 0}$, $q = (q_a)_{a\geq 0}$ be sequences of units of A.*

(i) For $0 \leq j \leq \varrho$ and every measurable $\Phi\colon G \to [0,\infty]$,

$$\int \Phi(w_{j,q}^{-1} w_{j,p}\, x)\, d\theta_{\geq 0}(x) = \prod_{a<j} |p_a q_a^{-1}|^{k+2(\varrho-a)} \int \Phi\, d\theta_{\geq 0}.$$

(ii) Suppose that θ is a left Haar measure of G. For $0 \leq j \leq \varrho + 2$ and every measurable $\Phi\colon G \to [0,\infty]$ with $\Phi(tg) = \Phi(g)$ for all $t \in G^{(j)}$ and $g \in G$,

$$\int \Phi(w_{j,p}^{-1} g)\, d\mathfrak{m}_j(g) = \int \Phi\, d\mathfrak{m}_j.$$

Proof. All measures are s-finite; convolution is associative and $\int F\, d(\mu{*}\mu') = \iint F(xy)\, d\mu(x)\, d\mu'(y)$ for measurable $F \geq 0$.

(i). Put $R_j = w_{j,q}^{-1} w_{j,p}$. Since $w_{j,q} e_a = q_a e_a^*$ and $w_{j,q} e_a^* = \bar{q}_a^{-1} e_a$, we have $w_{j,q}^{-1} e_a^* = \overline{q_a p_a^{-1}}\, e_a^*$ and $w_{j,q}^{-1} e_a = \bar{q}_a e_a^*$ for $a < j$; hence R_j is diagonal, $R_j e_a = p_a q_a^{-1} e_a$ and $R_j e_a^* = \overline{q_a p_a^{-1}}\, e_a^*$ for $a < j$, and R_j fixes the other coordinate vectors. For $l < \varrho$ and a unit c, (5.7) gives $w_l(c)^{-1} e_l = \bar{c} e_l^*$ and $w_l(c)^{-1} e_l^* = c^{-1} e_l$.

We argue by induction on j; for $j = 0$, $R_0 = 1$. Let $j < \varrho$ and put $r = p_j q_j^{-1}$. Then $D = R_j^{-1} R_{j+1}$ scales e_j by r and e_j^* by $\bar{r}^{-1}$ and fixes the other coordinate vectors, and for every unit c

$$D\, w_j(rc)^{-1} = w_j(c)^{-1}, \qquad \text{so} \qquad R_{j+1}\, w_j(rc)^{-1} = R_j\, w_j(c)^{-1}, \tag{5.77}$$

as both sides of the first identity send $e_j \mapsto \overline{rc}\,\bar{r}^{-1} e_j^* = \bar{c} e_j^*$, $e_j^* \mapsto r(rc)^{-1} e_j = c^{-1} e_j$ and fix the other coordinate vectors. By Lemma 5.2.59, (i), applied repeatedly, $\theta_{\geq 0} = \theta_{\geq j} * \theta_{j-1} * \cdots * \theta_0$ and $\theta_{\geq j} = \mathrm{wl}_j * \theta_{\geq j+1} * \eta_j$; so $\theta_{\geq 0} = \mathrm{wl}_j * \mu$ with $\mu = \theta_{\geq j+1} * \eta_j * \theta_{j-1} * \cdots * \theta_0$, and by the definition of wl_j,

$$\int \Psi\, d\theta_{\geq 0} = \int_A |c|^{d_j} \int_G \Psi(w_j(c)^{-1} y)\, d\mu(y)\, dc \qquad (\Psi \geq 0 \text{ measurable}). \tag{5.78}$$

Apply (5.78) to $\Psi = \Phi(R_{j+1}\,\cdot)$ and put $F(c) = \int \Phi(R_{j+1} w_j(c)^{-1} y)\, d\mu(y)$. By (PA3) for the 1×1 matrix (r), $\int_A H(c)\, dc = |r| \int_A H(rc)\, dc$ for measurable $H \geq 0$; with $|rc|^{d_j} = |r|^{d_j} |c|^{d_j}$, the fact

that almost every c is a unit (PA4) and (5.77),

$$\begin{aligned}\int \Phi(R_{j+1}x)\,d\theta_{\geq 0} &= |r|^{d_j+1}\int_A |c|^{d_j} F(rc)\,dc \\ &= |r|^{d_j+1}\int_A |c|^{d_j}\int_G \Phi\big(R_j w_j(c)^{-1}y\big)\,d\mu(y)\,dc \\ &= |r|^{d_j+1}\int \Phi(R_j x)\,d\theta_{\geq 0},\end{aligned}$$

the last step by (5.78) for $\Psi = \Phi(R_j\,\cdot)$. As $d_j + 1 = k + 2(\varrho - j)$ for $j < \varrho$, the induction hypothesis gives (i) for $j+1$.

(ii). Let $w = w_{j,p}$, $s = w^{-1}$, $i = \varrho + 2 - j$ and $\rho = \mathfrak{m}_{j,i}$; by the definition of the partial group measures and associativity, $\theta = \mathfrak{m}_{0,\varrho+2} = \rho * \mathfrak{m}_j$.

Properties of ρ. It is carried by $G^{(j)}$ (Lemma 5.2.59, (ii)), satisfies $\rho * \delta_t = \rho$ for $t \in G^{(j)}$ (Lemma 5.2.61, (iii)), and $\rho \neq 0$ (Lemma 5.2.61, (iv)). Moreover $wt = tw$ for every $t \in G^{(j)}$: if $j \geq \varrho + 2$ then $t = 1$; otherwise $t \in G^{(\min(j,\varrho))}$, so by Lemma 5.2.58, (vii), the rows and columns of t at the indices of floor below $\min(j,\varrho)$ are those of 1, while w acts only on the span of the e_a, e_a^* with $a < \min(j,\varrho)$ and fixes the other coordinate vectors; both matrices are block diagonal for the same two blocks, t being the identity on the first and w on the second. Hence $\int F(sy)\,d\rho(y) = \int F(ys)\,d\rho(y)$ for measurable $F \geq 0$, that is, $\delta_s * \rho = \rho * \delta_s$.

Conclusion. The floor group $G^{(j)}$ is a closed subgroup of G, being cut out by the closed conditions of fixing the coordinate vectors of floor below j. Apply Lemma 5.4.33 to the s-finite measures ρ and $\mu = \mathfrak{m}_j$ (so that $\theta = \rho * \mathfrak{m}_j$), the closed subgroup $S = G^{(j)}$, by which ρ is carried and under whose right translations it is invariant, and the element $s = w^{-1}$, with $\delta_s * \rho = \rho * \delta_s$: for the left $G^{(j)}$-invariant Φ it gives $\int \Phi(w^{-1}g)\,d\mathfrak{m}_j(g) = \int \Phi\,d\mathfrak{m}_j$. □

Lemma 5.4.16. *Let a place datum be given, $j \geq 0$, $d \in G$, let $\mathcal{K}$ be a pairing kernel at floor j for d, and let $x \in G$; write* $\mathrm{WSq}_l = \mathrm{WSq}_l(\mathcal{K}, x)$ *and* $\mathrm{WSS}_l = \mathrm{WSS}_l(\mathcal{K}, x)$.

(i) Let $l \geq 0$, $g, y \in G$, $V \in [0,\infty]$, and let $u, v\colon G \to [0,\infty]$ be measurable with $\mathrm{FML}_l(V; u)$ *and $|\mathcal{K}_{l+1}(\sigma g, \sigma y)|^2 \leq u(\sigma)\,v(\sigma)$ for every $\sigma \in G^{(l)}$. Then* $\mathrm{piv}_l(\mathcal{K})(g,y)^2 \leq V\int_G v\,d\theta_l$. *In particular, if* $\mathrm{FML}_l(V;1)$, *then* $\mathrm{piv}_l(\mathcal{K})(g,y)^2 \leq V\int_G |\mathcal{K}_{l+1}(\sigma g, \sigma y)|^2\,d\theta_l(\sigma)$.

(ii) For every $l \geq 0$, $(x', g) \mapsto \mathrm{WSS}_l(\mathcal{K}, x')(g)$ is measurable on $G \times G$.

(iii) Let $l \geq j$. Then $\mathrm{WSq}_l(sg) = \mathrm{WSq}_l(g)$ for $s \in R^{(l+1)}$, and $\mathrm{WSq}_l(g) = 0$ if g is not near at $l+1$; and $\mathrm{WSS}_l(sg) = \mathrm{WSS}_l(g)$ for $s \in R^{(l)}$, and $\mathrm{WSS}_l(g) = 0$ if g is not near at l.

(iv) For $l \leq \varrho$ and $g \in G$, $\displaystyle\int_G |\mathcal{K}_{l+1}(\sigma g, \sigma g x)|^2\,d\theta_l(\sigma) \leq \mathrm{WSS}_l(g)/\mathrm{dpw}_l(g,x)$.

(v) For $l, i \geq 0$ with $l + i \leq \varrho$ and $g \in G$, $\displaystyle\int_G \mathrm{WSq}_{l+i}(\sigma g)\,d(\theta_{l+i} * \theta_{l+i-1} * \cdots * \theta_l)(\sigma) \leq \mathrm{WSS}_l(g)$.

(vi) For $l \leq \varrho$, $\mathrm{WSI}_l(\mathcal{K}, x) = \displaystyle\int_G \mathrm{WSq}_l\,d(\theta_l * \cdots * \theta_0) + \mathrm{WSI}_{l+1}(\mathcal{K}, x)$; hence $\mathrm{WSI}_l(\mathcal{K}, x) \leq \mathrm{WSI}_0(\mathcal{K}, x)$ for $l \leq n$.

Proof. The floor measures are s-finite (images and convolutions of s-finite measures: Haar measures of second countable locally compact groups, μ_{top}, μ_H), so Tonelli's theorem applies to all iterated integrals below. $R^{(l)}$ is a closed subgroup of G, and the set of points near at l is $R^{(l)}K$, closed (Lemma 5.4.8, proof of (iii)). By Lemma 5.2.64 (i), every $\mathcal{K}_l$ is measurable on $G \times G$,

and by Lemma 5.2.64 (iv), $\mathcal{K}$ is a pairing kernel at floor l for every $l \geq j$. By the definition of WSq, $\mathrm{WSq}_l(\sigma g) = \mathrm{dpw}_l(\sigma g, x)|\mathcal{K}_{l+1}(\sigma g, \sigma g x)|^2$.

(i). By Lemma 5.2.59 (ii), θ_l is carried by $G^{(l)}$, so $\mathrm{piv}_l(\mathcal{K})(g,y) = \int f\, d\theta_l$ with $f(\sigma) = \mathbf{1}_{G^{(l)}}(\sigma)|\mathcal{K}_{l+1}(\sigma g, \sigma y)|$, a measurable function. By (PK5) at $l+1$, f vanishes off the closed set $N = \{\sigma : \sigma g \text{ near at } l+1\}$, and $f(\sigma)^2 \leq u(\sigma g)v(\sigma)$ for all σ. By the Cauchy–Schwarz inequality and $\mathrm{FML}_l(V;u)$,

$$\mathrm{piv}_l(\mathcal{K})(g,y)^2 = \Big(\int_N f\, d\theta_l\Big)^2 \leq \int_N u(\sigma g)\, d\theta_l(\sigma) \int_N v\, d\theta_l \leq V \int_G v\, d\theta_l.$$

The second assertion is the case $u = 1$, $v(\sigma) = |\mathcal{K}_{l+1}(\sigma g, \sigma y)|^2$.

(ii). $(x', g) \mapsto \mathrm{WSq}_l(\mathcal{K}, x')(g)$ is measurable, as dpw_l is measurable (Lemma 5.4.8 (iii)) and $\mathcal{K}_{l+1}$ is. $\mathrm{WSS}_l = 0$ for $l \geq n$, and for $l < n$ the recursion $\mathrm{WSS}_l(\mathcal{K}, x')(g) = \int(\mathrm{WSq}_l(\mathcal{K}, x') + \mathrm{WSS}_{l+1}(\mathcal{K}, x'))(\xi g)\, d\theta_l(\xi)$ preserves joint measurability by Tonelli's theorem; descending induction on l.

(iii), the weighted squares. Let $l \geq j$ and $s \in R^{(l+1)} \subseteq R^{(l)}$ (Lemma 5.2.58 (i)). By Lemma 5.2.58 (v), $s = \gamma\nu$ with $\gamma \in G^{(l)}$, $\nu \in N_{<l}$; as $N_{<l} \subseteq R^{(l+1)}$ (Lemma 5.2.58 (i)), $\gamma = s\nu^{-1} \in R^{(l+1)} \cap G^{(l)}$. By Lemma 5.4.8 (iii), $\mathrm{dpw}_l(sg, x) = \mathrm{dpw}_l(g, x)$. By Lemma 5.2.64 (v) at floor $l+1$ (as $\gamma \in R^{(l+1)} \cap G^{(j)}$, since $G^{(l)} \subseteq G^{(j)}$), $\mathcal{K}_{l+1}(\gamma\nu g, \gamma\nu g x) = \mathcal{K}_{l+1}(\nu g, \nu g x)$, and by (PK2) with $n = n' = \nu \in N_{<l}$ this has the modulus of $\mathcal{K}_{l+1}(g, gx)$, as ψ takes values on the unit circle. So $\mathrm{WSq}_l(sg) = \mathrm{WSq}_l(g)$. If $\mathcal{K}_{l+1}(g, gx) \neq 0$, then g is near at $l+1$ by (PK5).

(iii), the sums. Descending induction on $l \geq j$; for $l \geq n$, $\mathrm{WSS}_l = 0$. Let $j \leq l \leq \varrho$, and let $F = \mathrm{WSq}_l + \mathrm{WSS}_{l+1}$, which by the first part and the induction hypothesis is measurable, invariant under left multiplication by $R^{(l+1)}$ and zero at the points that are not near at $l+1$. For $g \in G$, $f(\xi) = F(\xi g)$ satisfies $f(t\xi) = f(\xi)$ for $t \in R^{(l+1)}$, so Lemma 5.2.61 (i) with $S = G$ gives, for $s \in R^{(l)}$,

$$\mathrm{WSS}_l(sg) = \int F(\xi s g)\, d\theta_l(\xi) = \int f(\xi s)\, d\theta_l(\xi) = \int f\, d\theta_l = \mathrm{WSS}_l(g).$$

If g is not near at l and $\xi \in G^{(l)}$, then ξg is not near at $l+1$: otherwise $t\xi g \in K$ with $t \in R^{(l+1)}$, and $t\xi \in R^{(l)}$ as $G^{(l)} \subseteq R^{(l)}$; so $F(\xi g) = 0$ for $\xi \in G^{(l)}$, and $\mathrm{WSS}_l(g) = 0$ because θ_l is carried by $G^{(l)}$.

(iv). By Lemma 5.4.8 (iii), $0 < \mathrm{dpw}_l(g,x) < \infty$ and $\mathrm{dpw}_l(\sigma g, x) = \mathrm{dpw}_l(g,x)$ for $\sigma \in G^{(l)} \subseteq R^{(l)}$. As θ_l is carried by $G^{(l)}$ and $l < n$,

$$\begin{aligned}\mathrm{dpw}_l(g,x) \int |\mathcal{K}_{l+1}(\sigma g, \sigma g x)|^2\, d\theta_l(\sigma) &= \int \mathrm{WSq}_l(\sigma g)\, d\theta_l(\sigma) \\ &\leq \int (\mathrm{WSq}_l + \mathrm{WSS}_{l+1})(\sigma g)\, d\theta_l(\sigma) = \mathrm{WSS}_l(g).\end{aligned}$$

(v). Induction on i. For $i = 0$ the integral is $\int \mathrm{WSq}_l(\sigma g)\, d\theta_l(\sigma) \leq \mathrm{WSS}_l(g)$, as in (iv). For $i+1$ (with $l+i+1 \leq \varrho$) write $\theta_{l+i+1} * \cdots * \theta_l = (\theta_{l+i+1} * \cdots * \theta_{l+1}) * \theta_l$; by Tonelli's theorem and the induction hypothesis at $l+1$,

$$\begin{aligned}\int \mathrm{WSq}_{l+i+1}(\sigma g)\, d(\theta_{l+i+1} * \cdots * \theta_l)(\sigma) &= \iint \mathrm{WSq}_{(l+1)+i}(\sigma'\tau g)\, d(\theta_{l+i+1} * \cdots * \theta_{l+1})(\sigma')\, d\theta_l(\tau) \\ &\leq \int \mathrm{WSS}_{l+1}(\tau g)\, d\theta_l(\tau) \leq \mathrm{WSS}_l(g).\end{aligned}$$

(vi). For $l \leq \varrho$, by the recursion of WSS_l ($l < n$) and the definition of convolution,

$$\mathrm{WSI}_l(\mathcal{K}, x) = \iint (\mathrm{WSq}_l + \mathrm{WSS}_{l+1})(\xi g)\, d\theta_l(\xi)\, d(\theta_{l-1} * \cdots * \theta_0)(g) = \int (\mathrm{WSq}_l + \mathrm{WSS}_{l+1})\, d(\theta_l * \cdots * \theta_0),$$

which is the asserted sum, as $\int \mathrm{WSS}_{l+1}\, d(\theta_l * \cdots * \theta_0) = \mathrm{WSI}_{l+1}(\mathcal{K}, x)$. In particular $\mathrm{WSI}_{l+1} \le \mathrm{WSI}_l$ for $l \le \varrho$, and $\mathrm{WSI}_l \le \mathrm{WSI}_0$ for $l \le n$ by induction. $\square$

Lemma 5.4.17. *Let Γ be a gauge of loss Λ for a place datum, $x \in \mathcal{P}_{\mathrm{pos}}$, $j \ge 0$, $d \in G$, and let $\mathcal{K}$ be a pairing kernel at floor j for d. Let $l \ge 0$, $g \in G$ and $\varepsilon_1 \in [0, \infty]$, and suppose that for every $\sigma \in G^{(l)}$ with $\mathcal{K}(\sigma g, \sigma g x) \ne 0$ there is a with $l \le a < \varrho$ and $\mathrm{dp}_a(\sigma g, x) \ge \varepsilon_1^{-1}$. Then, with arithmetic in $[0, \infty]$,*

$$\mathrm{piv}_l(\mathcal{K})(g, gx)^2 \le \varepsilon_1^s\, c_{\mathrm{fls}}\, \frac{\mathrm{WSS}_l(\mathcal{K}, x)(g)}{\mathrm{dpw}_l(g, x)}.$$

Proof. Write $\Theta_{[l,i)} = \theta_{l+i-1} * \cdots * \theta_l$ ($\Theta_{[l,0)} = \delta_1$), $\mathrm{WSS}_l = \mathrm{WSS}_l(\mathcal{K}, x)$, $\mathrm{WSq}_a = \mathrm{WSq}_a(\mathcal{K}, x)$, and for $a < \varrho$, $D \in [0, \infty]$ and $y \in G$ let

$$\mathrm{DM}_{a,D}(y) = \mathbf{1}\{\mathrm{dp}_a(y, x) \ge D\}\, |\mathcal{K}_{a+1}(y, yx)|,$$

a measurable function (Lemma 5.4.8 (i), Lemma 5.2.64 (i)). The measures are s-finite and Tonelli's theorem is used freely.

Step 1: the cut. Let $D \in [0, \infty]$. We show, by descending induction on l, for all g: if for every $\sigma \in G^{(l)}$ with $\mathcal{K}(\sigma g, \sigma g x) \ne 0$ some a with $l \le a < \varrho$ has $\mathrm{dp}_a(\sigma g, x) \ge D$, then

$$\mathrm{piv}_l(\mathcal{K})(g, gx) \le \sum_{l \le a < \varrho} \int_G \mathrm{DM}_{a,D}(\sigma g)\, d\Theta_{[l,a-l+1)}(\sigma). \tag{5.79}$$

If $l \ge \varrho$, no a exists, so $\mathcal{K}(\sigma g, \sigma g x) = 0$ for all $\sigma \in G^{(l)}$, and $\mathrm{piv}_l(\mathcal{K})(g, gx) = 0$ by Lemma 5.2.64 (iii). Let $l < \varrho$. As θ_l is carried by $G^{(l)}$ (Lemma 5.2.59 (ii)), it suffices to bound $|\mathcal{K}_{l+1}(\sigma g, \sigma g x)|$ for $\sigma \in G^{(l)}$ by the sum over $l \le a < \varrho$ of $\mathrm{DM}_{l,D}(\sigma g)$ for $a = l$ and of $\int \mathrm{DM}_{a,D}(\tau \sigma g)\, d\Theta_{[l+1,a-l)}(\tau)$ for $a > l$: integrating over σ against θ_l then gives (5.79), since $\Theta_{[l,a-l+1)} = \Theta_{[l+1,a-l)} * \theta_l$. If $\mathrm{dp}_l(\sigma g, x) \ge D$, then $|\mathcal{K}_{l+1}(\sigma g, \sigma g x)| = \mathrm{DM}_{l,D}(\sigma g)$. Otherwise, let $\tau \in G^{(l+1)}$ with $\mathcal{K}(\tau \sigma g, \tau \sigma g x) \ne 0$. Since $\tau \sigma \in G^{(l)}$, some $a \ge l$ has $\mathrm{dp}_a(\tau \sigma g, x) \ge D$; and $\mathrm{dp}_l(\tau \sigma g, x) = \mathrm{dp}_l(\sigma g, x) < D$ by Lemma 5.4.8 (ii), as $\tau \in G^{(l+1)} \subseteq R^{(l+1)}$ (Lemma 5.2.58 (i)); so $a \ge l + 1$. The induction hypothesis at $l + 1$ and the point σg, with Lemma 5.2.64 (ii), gives

$$|\mathcal{K}_{l+1}(\sigma g, \sigma g x)| \le \mathrm{piv}_{l+1}(\mathcal{K})(\sigma g, \sigma g x) \le \sum_{l+1 \le a < \varrho} \int \mathrm{DM}_{a,D}(\tau \sigma g)\, d\Theta_{[l+1,a-l)}(\tau).$$

Step 2: one piece. Let $l \le a < \varrho$, write $a = l + i$, and let $D = \varepsilon_1^{-1}$. We show

$$\Big(\int \mathrm{DM}_{a,D}(\sigma g)\, d\Theta_{[l,i+1)}(\sigma) \Big)^2 \le \varepsilon_1^s (\Lambda^{10^4 n})^{\varrho}\, \frac{\mathrm{WSS}_l(g)}{\mathrm{dpw}_l(g, x)}. \tag{5.80}$$

First, for every y: $\mathrm{DM}_{a,D}(y)^2 \le \varepsilon_1^s\, \mathrm{dpw}^-_{a+1,x}(y)\, \mathrm{WSq}_a(y)$. Indeed, if $\mathrm{DM}_{a,D}(y) \ne 0$, then $\mathcal{K}_{a+1}(y, yx) \ne 0$, so y is near at $a + 1$ by (PK5) and $\mathrm{dpw}^-_{a+1,x}(y) = \mathrm{dpw}_{a+1}(y, x)^{-1}$; and $\mathrm{dp}_a(y, x) \ge \varepsilon_1^{-1}$ with $0 < \mathrm{dp}_a(y, x) < \infty$ (Lemma 5.4.8 (i)) gives $1 \le \varepsilon_1 \mathrm{dp}_a(y, x)$. Since $\mathrm{dpw}_{a+1} = \mathrm{dpw}_a \cdot \mathrm{dp}_a^{-s}$ (Lemma 5.4.8 (iii)),

$$\begin{aligned} \mathrm{DM}_{a,D}(y)^2 &\le (\varepsilon_1 \mathrm{dp}_a(y, x))^s |\mathcal{K}_{a+1}(y, yx)|^2 = \varepsilon_1^s\, \mathrm{dpw}_{a+1}(y, x)^{-1}\, \mathrm{dpw}_a(y, x)\, |\mathcal{K}_{a+1}(y, yx)|^2 \\ &= \varepsilon_1^s\, \mathrm{dpw}^-_{a+1,x}(y)\, \mathrm{WSq}_a(y). \end{aligned}$$

By the Cauchy–Schwarz inequality the left side of (5.80) is at most $\varepsilon_1^s \int \mathrm{dpw}^-_{a+1,x}(\sigma g)\, d\Theta_{[l,i+1)}(\sigma) \cdot \int \mathrm{WSq}_a(\sigma g)\, d\Theta_{[l,i+1)}(\sigma)$. By Lemma 5.4.18 (ii) (as $l + i + 1 = a + 1 \le \varrho$), the first integral is at most $(\Lambda^{10^4 n})^{i+1} \mathrm{dpw}^-_{l,x}(g) \le (\Lambda^{10^4 n})^{\varrho} \mathrm{dpw}_l(g, x)^{-1}$, using $i + 1 \le \varrho$, $\Lambda \ge 2$, and $\mathrm{dpw}^-_{l,x}(g) \le \mathrm{dpw}_l(g, x)^{-1}$. By Lemma 5.4.16 (v) (as $l + i \le \varrho$), the second is at most $\mathrm{WSS}_l(g)$.

Step 3: conclusion. Let B be the right side of (5.80). By the hypothesis, Step 1 with $D = \varepsilon_1^{-1}$ applies, and with Step 2, $\mathrm{piv}_l(\mathcal{K})(g,gx) \le \varrho B^{1/2}$. As $\varrho \le 2^\varrho \le \Lambda^\varrho$ and $2\varrho + 10^4 n\varrho \le 10^6 n^3$,

$$\mathrm{piv}_l(\mathcal{K})(g,gx)^2 \le \Lambda^{2\varrho} B = \varepsilon_1^s \Lambda^{2\varrho+10^4 n\varrho} \frac{\mathrm{WSS}_l(g)}{\mathrm{dpw}_l(g,x)} \le \varepsilon_1^s c_{\mathrm{fls}} \frac{\mathrm{WSS}_l(g)}{\mathrm{dpw}_l(g,x)}.$$

□

Lemma 5.4.18. *Let Γ be a gauge of loss Λ for a place datum and $x \in \mathcal{P}_{\mathrm{pos}}$.*

(i) For $a < \varrho$ and $g \in G$, $\int_G \mathrm{dpw}^-_{a+1,x}(\sigma g)\, d\theta_a(\sigma) \le \Lambda^{10^4 n}\, \mathrm{dpw}^-_{a,x}(g)$.

(ii) For $l, i \ge 0$ with $l + i \le \varrho$ and $g \in G$, with the descending convolution $\theta_{[l,i)}$ of (5.11) in Definition 5.2.1,

$$\int_G \mathrm{dpw}^-_{l+i,x}(\sigma g)\, d\theta_{[l,i)}(\sigma) \le \left(\Lambda^{10^4 n}\right)^i \mathrm{dpw}^-_{l,x}(g).$$

Proof. **(i).** The measure θ_a is carried by $G^{(a)}$ (Lemma 5.2.59, (ii)). Let $\sigma \in G^{(a)}$. If σg is not near at $a+1$, then $\mathrm{dpw}^-_{a+1,x}(\sigma g) = 0$. If $\xi\sigma g \in K$ with $\xi \in R^{(a+1)}$, then $\xi\sigma \in R^{(a)}$ (Lemma 5.2.58, (i): $R^{(a+1)} \subseteq R^{(a)}$ and $G^{(a)} \subseteq R^{(a)}$), so g is near at a; and by Lemma 5.4.8, (iii), $\mathrm{dpw}_{a+1} = \mathrm{dpw}_a \cdot (\mathrm{dp}_a^{-1})^s$ and $\mathrm{dpw}_a(\sigma g, x) = \mathrm{dpw}_a(g,x)$ (left invariance under $R^{(a)}$), all values in $(0,\infty)$. Hence

$$\mathrm{dpw}^-_{a+1,x}(\sigma g) = \mathbf{1}\{\sigma g \text{ near at } a+1\}\, \mathrm{dp}_a(\sigma g, x)^s\, \mathrm{dpw}^-_{a,x}(g) \qquad (\sigma \in G^{(a)}),$$

and by Lemma 5.4.19 (the floor-mass bound $\mathrm{FML}_a(\Lambda^{10^4 n}; g' \mapsto \mathrm{dp}_a(g',x)^s)$ at the point g), $\int \mathrm{dpw}^-_{a+1,x}(\sigma g)\, d\theta_a(\sigma) \le \Lambda^{10^4 n}\, \mathrm{dpw}^-_{a,x}(g)$.

(ii). Induction on i, for all l and g. For $i = 0$ both sides are $\mathrm{dpw}^-_{l,x}(g)$. Let $l + i + 1 \le \varrho$ and put $C = \Lambda^{10^4 n}$. As $\theta_{[l,i+1)} = \theta_{[l+1,i)} * \theta_l$, Tonelli, the induction hypothesis at $(l+1, i)$ and the point τg, and (i) with $a = l < \varrho$ give

$$\begin{aligned}\int \mathrm{dpw}^-_{l+i+1,x}(\sigma g)\, d\theta_{[l,i+1)}(\sigma) &= \iint \mathrm{dpw}^-_{(l+1)+i,x}(\sigma\tau g)\, d\theta_{[l+1,i)}(\sigma)\, d\theta_l(\tau) \\ &\le C^i \int \mathrm{dpw}^-_{l+1,x}(\tau g)\, d\theta_l(\tau) \le C^{i+1}\, \mathrm{dpw}^-_{l,x}(g).\end{aligned}$$

□

Lemma 5.4.19. *Let Γ be a gauge of loss Λ for a place datum, let $x \in \mathcal{P}_{\mathrm{pos}}$ and $0 \le a < \varrho$. Then*

$$\mathrm{FML}_a(\Lambda^{10^4 n};\ g \mapsto \mathrm{dp}_a(g,x)^s).$$

Proof. Put $f(g) = \mathrm{dp}_a(g,x)^s$. By Lemma 5.4.8, (i) and (ii), f is measurable and $f(\xi g) = f(g)$ for $\xi \in R^{(a+1)}$. We apply the chart criterion Lemma 5.2.60, (ii), with $b = a$ and its compact set $C_a = \{P_a(\kappa\kappa'^{-1} f_a) : \kappa, \kappa' \in K\}$.

Fix $g \in K$. By (GU7) with $y = x$ and $Q = 0$ there is a common bound ϕ for $\mathcal{V}_{\mathrm{cm}}(g,x,0)$ with loss Λ; it is measurable, and $\int_{\{\varsigma \text{ of size } 11\}} \phi^s \le \Lambda^{99n\cdot 12}$. Put $\Psi_g(v) = \phi\big((h(o_i, v))_{i\in\iota'}\big)^s$ for $v \in V$, a Borel function. For $\sigma \in G^{(a)}$ and $\omega \in A^{\iota'}$, the vector $O\omega$ lies in the middle space, so $h(e_b, \sigma O\omega) = h(\sigma^{-1} e_b, O\omega) = h(e_b, O\omega) = 0$ for $b < a$; by Lemma 5.4.9, $f(\sigma g) \le \Psi_g(\sigma^{-1} f_a)$.

It remains to bound $\int_{C_a} \Psi_g \, d\mathrm{qv}_a$. Elements of K have size 1 (GU2), their inverses size 5 (the inverse of an element of G of size i, $J^{-1}\bar{g}^{\mathsf{T}}J$, has size $i+4$ by (SZ1)–(SZ3), (SZ5)), and $\kappa\kappa'^{-1}$ has size 7; the projection P_a only replaces coordinates by 0 (SZ4). So C_a lies in the set of vectors of size 7, and by Lemma 5.4.4, (iv), with $\rho = \mathrm{qv}_a$ and $j = 7$,

$$\int_{C_a} \Psi_g \, d\mathrm{qv}_a \le \Lambda^{99n\cdot 13} \int_{A'_{(11)}} \phi^s \, d\mathrm{vol} \le \Lambda^{99n\cdot 13}\Lambda^{99n\cdot 12} = \Lambda^{2475n} \le \Lambda^{10^4 n},$$

as $\Lambda \ge 1$. The criterion gives the claim. □

Lemma 5.4.20. *Let a place datum be given, let $W\colon G \to \mathbb{C}^I$, and for a measure μ on G and $\gamma \in G$ let $\mathrm{Orb}_\mu(\gamma) \in \mathbb{C}^{I\times I}$ be the Bessel orbital integral of W,* (5.13) *of Definition 5.2.4, with the inner product of $\mathbb{C}^{I\times I}$ fixed there.*

(i) If W is a Bessel function of frequency η over the place and μ is left invariant, then $\mathrm{Orb}_\mu(u^{-1}\gamma u') = \overline{\psi(\eta\,\sigma(u))}\,\psi(\eta\,\sigma(u'))\,\mathrm{Orb}_\mu(\gamma)$ for all $u, u' \in \mathsf{U}$ and $\gamma \in G$.

Let now Γ be a gauge for the place datum, with size filtration $(A_{(i)})$, and let W be cut.

(ii) If $\mathrm{Orb}_\mu(\gamma) \ne 0$, then every leading principal minor of the frame Gram matrix $Z(\gamma)$, (5.3) *of Definition 5.1.7, lies in $A_{(9n)}$.*

(iii) If p is a head of length ϱ, $x \in G^{(\varrho)}$ and $y \in G$ satisfies $W(y) \ne 0$ and $W((w_{\varrho,p}x)^{-1}y) \ne 0$, then y has size $60n^2$.

(iv) If μ is a Haar measure of G, W is continuous, p, q are heads of length ϱ and $x \in G^{(\varrho)}$, then

$$\begin{aligned}&\langle \mathrm{Orb}_\mu(w_{\varrho,p}x), \mathrm{Orb}_\mu(w_{\varrho,q}x)\rangle\\ &\quad= \int_{G\times G} \langle W(y), W(y')\rangle\,\overline{\langle W((w_{\varrho,p}x)^{-1}y), W((w_{\varrho,q}x)^{-1}y')\rangle}\, d(\mu\otimes\mu)(y,y').\end{aligned}$$

Proof. **(i).** Put $\chi(v) = \psi(\eta\,\sigma(v))$ for $v \in \mathsf{U}$, a number of modulus 1. For $v \in \mathsf{U}$ and $y \in G$, $W(y) = W(v\cdot v^{-1}y) = \chi(v)W(v^{-1}y)$, so $W(v^{-1}y) = \overline{\chi(v)}\,W(y)$. As μ is left invariant, the map $y \mapsto u^{-1}y$ preserves μ, so it preserves integrability and Bochner integrals; with $(u^{-1}\gamma u')^{-1}u^{-1}y = u'^{-1}(\gamma^{-1}y)$,

$$\mathrm{Orb}_\mu(u^{-1}\gamma u')_{ii'} = \int W(u^{-1}y)_i\,\overline{W(u'^{-1}\gamma^{-1}y)_{i'}}\,d\mu(y) = \overline{\chi(u)}\,\chi(u')\int W(y)_i\,\overline{W(\gamma^{-1}y)_{i'}}\,d\mu(y),$$

where the integrand on the right is integrable exactly when the one on the left is (they differ by a constant of modulus 1), so the identity also holds when both entries are 0 by convention.

Near points of a cut function. If W is cut and $W(y) \ne 0$, then $uy \in K$ for some $u \in \mathsf{U}$, and $\mathsf{U} \subseteq R^{(l)}$ for every l (Lemma 5.2.58, (i)); so y is near at every l.

(ii). Some entry of $\mathrm{Orb}_\mu(\gamma)$ is a nonzero integral, so its integrand does not vanish identically: there is y with $W(y) \ne 0$ and $W(\gamma^{-1}y) \ne 0$. Then y and $\gamma^{-1}y$ are near at n, and Lemma 5.4.13, (ii), with $l = n$ gives $D_i(\gamma) \in A_{(9n)}$ for $i \le n$, where $D_i(\gamma)$ is the leading principal minor of order i of $(h(f_a, \gamma f_b))_{a,b}$.

(iii). Let $u \in \mathsf{U}$ with $uy \in K$. Since $W((w_{\varrho,p}x)^{-1}y) \ne 0$, the point $x^{-1}w_{\varrho,p}^{-1}y$ is near at ϱ: $\zeta x^{-1}w_{\varrho,p}^{-1}y \in K$ for some $\zeta \in R^{(\varrho)}$; as $x^{-1} \in G^{(\varrho)} \subseteq R^{(\varrho)}$ (Lemma 5.2.58, (i)), $w_{\varrho,p}^{-1}y$ is near at ϱ. For $l = \varrho$ the modified level lev_ϱ agrees with the level (the indices of floor at least ϱ are z and

the x_i, of level ϱ), so $N_{<\varrho} = \mathsf{U}$ and $u \in R^{(\varrho)} \cap N_{<\varrho}$. By Lemma 5.4.12, (ii), with $j = \varrho$, $s = u$ and $g = y$, y has size $60n^2$.

(iv). For a head r of length ϱ and $(i, i') \in I \times I$ put $F_{r,ii'}(y) = W(y)_i \overline{W((w_{\varrho,r}x)^{-1}y)_{i'}}$, so that $\mathrm{Orb}_\mu(w_{\varrho,r}x)_{ii'} = \int F_{r,ii'}\, d\mu$. The function $F_{r,ii'}$ is continuous, and by (iii) it vanishes off the set C of elements of G of size $60n^2$. As $A_{(60n^2)}$ is compact (GU1) and G is closed in the space of matrices, C is compact; a Haar measure is finite on compact sets, so $F_{r,ii'}$ is μ-integrable. The group G is second countable and μ is σ-finite, so by Fubini

$$\overline{\int F_{p,ii'}\, d\mu} \int F_{q,ii'}\, d\mu = \int_{G\times G} \overline{F_{p,ii'}(y)}\, F_{q,ii'}(y')\, d(\mu\otimes\mu)(y,y').$$

Summing over $(i, i') \in I \times I$, the left side becomes $\langle \mathrm{Orb}_\mu(w_{\varrho,p}x), \mathrm{Orb}_\mu(w_{\varrho,q}x)\rangle$, and the integrand on the right becomes

$$\begin{aligned}\sum_{i,i'} &\overline{W(y)_i}\, W(y')_i\; W((w_{\varrho,p}x)^{-1}y)_{i'}\, \overline{W((w_{\varrho,q}x)^{-1}y')_{i'}} \\ &= \langle W(y), W(y')\rangle\, \overline{\langle W((w_{\varrho,p}x)^{-1}y), W((w_{\varrho,q}x)^{-1}y')\rangle},\end{aligned}$$

which is the claim. □

Proposition 5.4.21. *Let Γ be a gauge of loss Λ for a place datum, and suppose that the group measure θ is a Haar measure of G. Let $W\colon G \to \mathbb{C}^I$ be cut, $j \le \varrho$, p, q heads of length j for Γ, $V \ge 1$ with $\mathrm{FML}_j(V;1)$, and $(S_i)_{i\ge 0}$ shells of heights 2^i for (Γ, j, p, q, W) covering the support of $\phi\colon G \to [0,\infty]$, with $\phi \le \mathrm{FM}_j(p,q,W,\cdot)$. Let $e \in [0,\infty]$ with $\phi(x)^2 \le (Ve^s)^2\, \mathrm{WSP}_j(p,q,W,x)$ for $x \in \mathcal{P}_{\mathrm{pos}}$ when $e < 1$ (WSP_j formed for Γ), and let $t \ge 0$ be an integer with $9\cdot 10^7 n^4 + t \le s$. Then, $\int \phi\, d\theta$ being the lower integral if ϕ is not measurable,*

$$\mathrm{hdn}_j(p,q) \int_G \phi\, d\theta \le V \min(1,e)^t\, \Lambda^{4\cdot 10^8 n^4}\, \mathrm{SN}(W).$$

Proof. Arithmetic is in $[0,\infty]$. Put $e' = \min(1,e) \in [0,1]$ and $B = Ve'^t \Lambda^{3\cdot 10^8 n^4} \mathrm{SN}(W)$. Every S_i lies in $\mathcal{P}_{\mathrm{pos}}$, by (SH2).

One shell. Let $i \ge 0$ and $x \in S_i$. If $e < 1$, then $e' = e$ and $\phi(x)^2 \le (Ve'^s)^2 \mathrm{WSP}_j(p,q,W,x)$ by hypothesis. If $e \ge 1$, then $e' = 1$ and $\phi(x)^2 \le \mathrm{FM}_j(p,q,W,x)^2 \le V^2 \mathrm{WSP}_j(p,q,W,x)$ by Proposition 5.4.5 (i). So Lemma 5.4.22 applies to S_i (height $\Delta = 2^i$), e', t and ϕ, and gives a measurable $M_i\colon G \to [0,\infty]$ with $\phi \le M_i$ on S_i and $\mathrm{hdn}_j(p,q) \int M_i\, d\theta \le 2^{-i}B$.

The sum. $M = \sum_{i\ge 0} M_i$ is measurable, and $\phi \le M$ on G: off $\bigcup_i S_i$, $\phi = 0$; on S_i, $\phi \le M_i \le M$. By monotone convergence,

$$\mathrm{hdn}_j(p,q) \int_G \phi\, d\theta \le \mathrm{hdn}_j(p,q) \int_G M\, d\theta = \sum_{i\ge 0} \mathrm{hdn}_j(p,q) \int_G M_i\, d\theta \le \sum_{i\ge 0} 2^{-i} B = 2B \le \Lambda^{10^8 n^4} B,$$

as $\Lambda \ge 2$ (GU1). This is $V \min(1,e)^t \Lambda^{4\cdot 10^8 n^4} \mathrm{SN}(W)$. □

Lemma 5.4.22. *Let Γ be a gauge of loss Λ for a place datum, and suppose that the group measure θ is a Haar measure of G. Let $W\colon G \to \mathbb{C}^I$ be cut, $j \le \varrho$, p, q heads of length j for Γ, S a shell of height Δ for (Γ, j, p, q, W), $V \ge 1$ with $\mathrm{FML}_j(V;1)$, $e \in [0,1]$, and $t \ge 0$ an integer with $9\cdot 10^7 n^4 + t \le s$. Let $f\colon G \to [0,\infty]$ with $f(x) \le \mathrm{FM}_j(p,q,W,x)$ and $f(x)^2 \le (Ve^s)^2\, \mathrm{WSP}_j(p,q,W,x)$ for every $x \in S$ (WSP_j formed for Γ). Then there is a measurable $M\colon G \to [0,\infty]$ with $f \le M$ on S and*

$$\mathrm{hdn}_j(p,q) \int_G M\, d\theta \le \Delta^{-1}\, V\, e^t\, \Lambda^{3\cdot 10^8 n^4}\, \mathrm{SN}(W).$$

Proof. Arithmetic is in $[0,\infty]$; put $N = 9\cdot 10^7 n^4$ and $c = \Lambda^{2\cdot 10^6 n^3}$. By (SH1), $1 \le \Delta < \infty$. For $g \in [0,1]$, $g^s \le g^N g^t$, as $s \ge N + t$. Note $2\cdot 10^6 n^3 + 29\cdot 10^7 n^4 \le 3\cdot 10^8 n^4$, so it suffices to find M with $f \le M$ on S and $\mathrm{hdn}_j(p,q)\int M\,d\theta \le \Delta^{-1} V e^t\, c\,\Lambda^{29\cdot 10^7 n^4}\mathrm{SN}(W)$.

Case $\Delta \le \Lambda^{2400n^2}$. By Lemma 5.4.23 with the gain Ve^s there is M with $f \le M$ on S and $\mathrm{hdn}_j(p,q)\int M \le Ve^s c\,\Delta^{10^5 n^2}\mathrm{SN}(W)$. Here $e^s \le e^t$ and $\Delta^{10^5 n^2} = \Delta^{-1}\Delta^{1+10^5 n^2} \le \Delta^{-1}\Lambda^{2400n^2(1+10^5 n^2)} \le \Delta^{-1}\Lambda^{29\cdot 10^7 n^4}$.

Case $\Delta > \Lambda^{2400n^2}$. Let $\tau \in (0,1)$ with $\tau^{800n^2} = \Delta^{-1}$ (as $\Delta > 1$). Since $800n^2\cdot 112500n^2 = N$, $1 + 10^5 n^2 \le 112500 n^2$ and $\Delta \ge 1$,

$$\tau^N \Delta^{1+10^5 n^2} \le \left(\tau^{800n^2}\Delta\right)^{112500n^2} = 1. \tag{5.81}$$

If $e \le \tau$: Lemma 5.4.23 with the gain Ve^s gives M with $f \le M$ on S and $\mathrm{hdn}_j(p,q)\int M \le Ve^t e^N c\,\Delta^{10^5 n^2}\mathrm{SN}(W) \le \Delta^{-1} Ve^t c\,\tau^N \Delta^{1+10^5 n^2}\mathrm{SN}(W) \le \Delta^{-1} Ve^t c\,\mathrm{SN}(W)$, by (5.81).

If $\tau < e$: let $x \in S$. Since $\Lambda^{2400n^2} < \Delta = \tau^{800n^2}\Delta^2$, we have $\Lambda^{1200n^2} \le \tau^{400n^2}\Delta$, and $\tau < 1$; so by (SH3) with $\varepsilon = \tau$, (j,p,q,W) has a deep vector after the head of depth at least $(\tau^2)^{-1}$ at x. By (SH2), x is a position of size some gauge, so $x \in \mathcal{P}_{\mathrm{pos}}$. By Proposition 5.4.5 (ii) with $\sigma = \tau^2$, $\mathrm{FM}_j(p,q,W,x)^2 \le \tau^{2s} V^2 \mathrm{WSP}_j(p,q,W,x) = (V\tau^s)^2 \mathrm{WSP}_j(p,q,W,x)$. Lemma 5.4.23 for the function $\mathrm{FM}_j(p,q,W,\cdot)$ with the gain $V\tau^s$ gives M with $\mathrm{FM}_j(p,q,W,\cdot) \le M$ on S, hence $f \le M$ on S, and, as $\tau^s \le \tau^N\tau^t \le \tau^N e^t$, $\mathrm{hdn}_j(p,q)\int M \le V\tau^N e^t c\,\Delta^{10^5 n^2}\mathrm{SN}(W) \le \Delta^{-1} Ve^t c\,\mathrm{SN}(W)$, by (5.81). □

Lemma 5.4.23. *Let Γ be a gauge of loss Λ for a place datum, and suppose that the group measure θ is a Haar measure of G. Let $W\colon G \to \mathbb{C}^I$ be cut, $j \le \varrho$, p a sequence of units, q a head of length j for Γ, and S a shell of height Δ for (Γ, j, p, q, W). Let $e \in [0,\infty]$ and $f\colon G \to [0,\infty]$ with $f(x)^2 \le e^2\,\mathrm{WSP}_j(p,q,W,x)$ for every $x \in S$ (WSP_j formed for Γ). Then there is a measurable $M\colon G \to [0,\infty]$ with $f \le M$ on S and*

$$\mathrm{hdn}_j(p,q)\int_G M\,d\theta \le e\,\Lambda^{2\cdot 10^6 n^3}\,\Delta^{10^5 n^2}\,\mathrm{SN}(W);$$

in particular the lower integral of $f\mathbf{1}_S$ satisfies the same bound.

Proof. Arithmetic is in $[0,\infty]$. By (SH1), $1 \le \Delta < \infty$; by (SH2) there is a finite set $\mathcal{G}$ of gauges for the place datum with $\#\mathcal{G} \le 2\Delta^{9n}$, such that every $x \in S$ is a position of size Γ' for some $\Gamma' \in \mathcal{G}$, and the loss Λ' of every $\Gamma' \in \mathcal{G}$ satisfies $\Lambda' \le \Lambda\Delta^2$. Put $u = (e^2\,\mathrm{WSP}_j(p,q,W,\cdot))^{1/2}$, a measurable function (its factors WSI_0, WSI_j are measurable by Lemma 5.4.16 (ii) and Tonelli's theorem), and let $P_{\Gamma'}$ be the (measurable) set of positions of size Γ'. Put

$$M = \sum_{\Gamma' \in \mathcal{G}} u\,\mathbf{1}_{P_{\Gamma'}},$$

a measurable function with $f \le u \le M$ on S. For $\Gamma' \in \mathcal{G}$, Proposition 5.4.24 for the gauges Γ, Γ', the set $P_{\Gamma'}$ and the function u (for which $u^2 = e^2\mathrm{WSP}_j$) gives

$$\mathrm{hdn}_j(p,q)\int_{P_{\Gamma'}} u\,d\theta \le e\,\Lambda^{1500000n^3+1}\Lambda'^{25000n^2}\mathrm{SN}(W) \le e\,\Lambda^{1500000n^3+25000n^2+1}\Delta^{50000n^2}\mathrm{SN}(W).$$

Summing over $\mathcal{G}$ and using $2 \le \Lambda$ (GU1) and $\Delta \ge 1$,

$$\mathrm{hdn}_j(p,q)\int_G M\,d\theta \le 2\Delta^{9n}\,e\,\Lambda^{1500000n^3+25000n^2+1}\Delta^{50000n^2}\mathrm{SN}(W) \le e\,\Lambda^{2\cdot 10^6 n^3}\Delta^{10^5 n^2}\mathrm{SN}(W),$$

since $1500000n^3 + 25000n^2 + 2 \le 2\cdot 10^6 n^3$ and $9n + 50000n^2 \le 10^5 n^2$. The lower integral of $f\mathbf{1}_S$ is at most $\int M\,d\theta$, as $f\mathbf{1}_S \le M$ and M is measurable. □

Proposition 5.4.24. *Let Γ and Γ' be gauges of losses Λ and Λ' for a place datum, and suppose that the group measure θ is a Haar measure of G. Let $W\colon G \to \mathbb{C}^I$ be cut, $j \le \varrho$, p a sequence of units and q a head of length j for Γ; let S be a set of positions of size Γ', $e \in [0,\infty]$, and $f\colon G \to [0,\infty]$ with $f(x)^2 \le e^2\,\mathrm{WSP}_j(p,q,W,x)$ for every $x \in S$, where WSP_j is formed for Γ. Then, with $\int_S f\,d\theta$ the lower integral of $f\mathbf{1}_S$,*

$$\mathrm{hdn}_j(p,q)\int_S f\,d\theta \le e\,\Lambda^{1500000n^3+1}\,\Lambda'^{25000n^2}\,\mathrm{SN}(W).$$

Proof. Arithmetic is in $[0,\infty]$ with $0\cdot\infty = 0$. Put $d = w_{j,q}^{-1}w_{j,p}$, $D = \big(\prod_{a<j}\mathrm{fm}(a,p_a^{-1})\big)^2$, $J = \prod_{a<j}|p_aq_a^{-1}|^{k+2(\varrho-a)}$, $V = \prod_{l<\varrho}c'_{\mathrm{fl}}(l)$ with $c'_{\mathrm{fl}}(l) = \mathrm{fm}(l)\mathrm{kb}_A^{|\iota'|}\Lambda'^{4900n}$ the floor constant of Γ', $N_1(x) = \mathrm{WSI}_0(\mathcal{K}_W,x)$, $N_2(x) = \mathrm{WSI}_j(\mathcal{K}_W^d,x)$, and let P' be the set of positions of size Γ'; so $\mathrm{WSP}_j(p,q,W,x) = D\,c_{\mathrm{fls}}^3N_1(x)N_2(x)$, with $c_{\mathrm{fls}} = \Lambda^{10^6n^3}$.

The head density. For $a < j \le \varrho$, $d_a = k + 2(\varrho-a) - 1$, and all moduli of units lie in $(0,\infty)$. So the a-th factor of $\mathrm{hdn}_j(p,q)^2DJ$ is

$$|p_aq_a|^{d_a}\,|p_a|^{-2d_a-1}\,|p_aq_a^{-1}|^{d_a+1} = |q_a|^{-1},$$

and $\mathrm{hdn}_j(p,q)^2DJ = |(q_0\cdots q_{j-1})^{-1}| \le \Lambda$ by (GU5), as q is a head for Γ, i.e. $(q_0\cdots q_{j-1})^{-1} \in A_{(1)}$.

Cauchy–Schwarz. By Lemma 5.2.57 (ii), $\mathcal{K}_W$ is a pairing kernel at floor 0 for 1 and $\mathcal{K}_W^d$ one at floor j for d; N_1, N_2 are measurable (Lemma 5.4.16 (ii) and Tonelli's theorem). Put $u = (e^2Dc_{\mathrm{fls}}^3N_1N_2)^{1/2}$, a measurable function with $f\mathbf{1}_S \le u\mathbf{1}_{P'}$. Hence, by the Cauchy–Schwarz inequality,

$$\Big(\int_S f\,d\theta\Big)^2 \le \Big(\int_{P'} u\,d\theta\Big)^2 \le e^2Dc_{\mathrm{fls}}^3\int_{P'}N_1\,d\theta\int_{P'}N_2\,d\theta.$$

One frame each. By Proposition 5.4.25 for the gauge Γ', with

$$\int\int|\mathcal{K}_W(g,y)|^2\,d\theta_{\ge0}(y)\,d\theta_{\ge0}(g) = \mathrm{PN}(W),$$

$\int_{P'}N_1\,d\theta \le nV\,\mathrm{PN}(W)$. For the second frame, Lemma 5.4.15 (i) applied for fixed g to $\Phi(y) = |\langle W(g),W(y)\rangle|^2$ (measurable) gives

$$\int|\langle W(g),W(dy)\rangle|^2\,d\theta_{\ge0}(y) = J\int|\langle W(g),W(y)\rangle|^2\,d\theta_{\ge0}(y);$$

integrating over g, the L^2 norm of $\mathcal{K}_W^d$ is $J\,\mathrm{PN}(W)$, and Proposition 5.4.25 gives $\int_{P'}N_2\,d\theta \le nVJ\,\mathrm{PN}(W)$.

Collecting constants. Combining,

$$\mathrm{hdn}_j(p,q)^2\Big(\int_S f\,d\theta\Big)^2 \le \big(\mathrm{hdn}_j(p,q)^2DJ\big)\,e^2c_{\mathrm{fls}}^3\,(nV)^2\,\mathrm{PN}(W)^2 \le \Lambda\,e^2\,\Lambda^{3\cdot10^6n^3}(nV)^2\,\mathrm{PN}(W)^2.$$

As $\Lambda' \ge 2$, $n \le 2^n \le \Lambda'^n$, and

$$nV \le \Lambda'^n\Big(\prod_{l<\varrho}\mathrm{fm}(l)\Big)\mathrm{kb}_A^{|\iota'|\varrho}\Lambda'^{4900n\varrho} \le \Big(\prod_{l<\varrho}\mathrm{fm}(l)\Big)\mathrm{kb}_A^{|\iota'|\varrho}\Lambda'^{25000n^2},$$

since $n + 4900n(n-1) \le 25000n^2$. Taking square roots, and using $\Lambda^{1/2} \le \Lambda$ and

$$\big(\prod_{l<\varrho}\mathrm{fm}(l)\big)\mathrm{kb}_A^{|\iota'|\varrho}\mathrm{PN}(W) = \mathrm{SN}(W),$$

$$\mathrm{hdn}_j(p,q)\int_S f\,d\theta \le e\,\Lambda^{1500000n^3+1}\Lambda'^{25000n^2}\,\mathrm{SN}(W).$$

□

Proposition 5.4.25. *Let Γ be a gauge for a place datum, and suppose that the group measure θ is a Haar measure of G. Let $j \le \varrho$, $d \in G$, and let $\mathcal{K}$ be a pairing kernel at floor j for d. Then*

$$\int_{\{\text{positions of size }\Gamma\}} \mathrm{WSI}_j(\mathcal{K},x)\,d\theta(x) \le n\Big(\prod_{l<\varrho} c_{\mathrm{fl}}(l)\Big)\int_G\int_G |\mathcal{K}(g,y)|^2\,d\theta_{\ge 0}(y)\,d\theta_{\ge 0}(g).$$

Proof. Write P_Γ for the set of positions of size Γ (measurable, Lemma 5.4.27), $\mathbf{1}_\Gamma$ for its indicator, $\Theta_l = \theta_l * \theta_{l-1} * \cdots * \theta_0$, and $B = \big(\prod_{l<\varrho} c_{\mathrm{fl}}(l)\big)\int\int |\mathcal{K}(g,y)|^2\,d\theta_{\ge 0}(y)\,d\theta_{\ge 0}(g)$. Let clm_l, $\mathrm{csq}_l = \mathrm{csq}_l(\mathcal{K})$ and $c_{\mathrm{up}}(l)$ be as in Lemma 5.4.26, and $C_l = \prod_{a<l} c_{\mathrm{up}}(a)$; so $C_{\varrho+1} = \prod_{a<\varrho} c_{\mathrm{fl}}(a)$, and $C_l = \prod_{a<l} c_{\mathrm{fl}}(a)$ for $l \le \varrho$. All measures are s-finite, and Tonelli's theorem is used without comment; the functions integrated are measurable by Lemma 5.4.26 (i), Lemma 5.4.16 (ii) and Lemma 5.2.64 (i).

Step 1: factors of θ. We show by descending induction on l, for $j \le l \le \varrho+1$: there is an s-finite measure ρ_l with $\theta = \eta_{<l} * \rho_l$ and

$$C_l\int_G\int_G \mathrm{csq}_l(g,y)\,d\rho_l(y)\,d\Theta_l(g) \le B. \tag{5.82}$$

For $l = \varrho+1$: by Lemma 5.2.59 (iv), $\theta = \mathfrak{m}_{\varrho+2} = \eta_{<\varrho+2} * \Theta_{\varrho+1}$, and $\eta_{<\varrho+2} = \eta_{<\varrho+1}$ as $\eta_{\varrho+1} = \delta_1$; and $\Theta_{\varrho+1} = \theta_{\ge 0}$. Take $\rho_{\varrho+1} = \theta_{\ge 0}$. Since $\theta_{\ge\varrho+2} = \delta_1$, $\mathcal{K}_{\varrho+2} = \mathcal{K}$, so $\mathrm{csq}_{\varrho+1} \le |\mathcal{K}|^2$ and (5.82) holds.

Let $j \le l \le \varrho$ and let $\rho = \rho_{l+1}$ satisfy the claim at $l+1$. Put $\rho_l = \eta_l * \rho$; since $\eta_{<l+1} = \eta_{<l} * \eta_l$, $\theta = \eta_{<l} * \rho_l$. For $\sigma \in G^{(l+1)}$ and $g' \in G$, Lemma 5.4.33 applies to $\theta = \eta_{<l+1} * \rho$, $S = N_{<l+1}$ (closed; $\eta_{<l+1}$ is carried by S and $\eta_{<l+1} * \delta_t = \eta_{<l+1}$ for $t \in S$ by Lemma 5.4.14 (ii), (iii)), $s = \sigma$ ($\delta_\sigma * \eta_{<l+1} = \eta_{<l+1} * \delta_\sigma$ by Lemma 5.2.59 (iii)) and $\Phi = \mathrm{csq}_{l+1}(g',\cdot)$, which is invariant under left multiplication by $N_{<l+1}$ (Lemma 5.4.26 (i)); so

$$\int_G \mathrm{csq}_{l+1}(g',\sigma y)\,d\rho(y) = \int_G \mathrm{csq}_{l+1}(g',y)\,d\rho(y). \tag{5.83}$$

By Lemma 5.4.26 (ii) for each y, then (5.83) for θ_{l+1}-almost every σ (Lemma 5.2.59 (ii)), $\Theta_{l+1} = \theta_{l+1} * \Theta_l$, and the claim at $l+1$,

$$\begin{aligned}
C_l\iint \mathrm{csq}_l(g,y)\,d\rho_l(y)\,d\Theta_l(g) &= C_l\iiint \mathrm{csq}_l(g,ny)\,d\eta_l(n)\,d\rho(y)\,d\Theta_l(g)\\
&\le C_{l+1}\iiint \mathrm{csq}_{l+1}(\sigma g,\sigma y)\,d\rho(y)\,d\theta_{l+1}(\sigma)\,d\Theta_l(g)\\
&= C_{l+1}\iiint \mathrm{csq}_{l+1}(\sigma g,y)\,d\rho(y)\,d\theta_{l+1}(\sigma)\,d\Theta_l(g)\\
&= C_{l+1}\iint \mathrm{csq}_{l+1}(g,y)\,d\rho(y)\,d\Theta_{l+1}(g) \le B.
\end{aligned}$$

Step 2: one floor of weighted squares. Let $j \le l \le \varrho$. We show

$$\int_{P_\Gamma}\int_G \mathrm{WSq}_l(\mathcal{K},x)(g)\,d\Theta_l(g)\,d\theta(x) \le B. \tag{5.84}$$

Fix $g \in G$ and put $F(x) = \mathbf{1}_\Gamma(x)\,\mathrm{WSq}_l(\mathcal{K},x)(g) = \mathbf{1}_\Gamma(x)\,\mathrm{dpw}_l(g,x)\,|\mathcal{K}_{l+1}(g,gx)|^2$. As θ is left invariant and $\theta = \eta_{<l} * \rho_l$ (Step 1),

$$\int F\,d\theta = \int F(g^{-1}x)\,d\theta(x) = \iint F(g^{-1}\nu y)\,d\eta_{<l}(\nu)\,d\rho_l(y).$$

For $\nu \in N_{<l}$, $|\mathcal{K}_{l+1}(g,\nu y)| = |\mathcal{K}_{l+1}(g,y)|$ (Lemma 5.2.64 (vi)), and $\eta_{<l}$ is carried by $N_{<l}$, so the inner integral is $|\mathcal{K}_{l+1}(g,y)|^2 D(g,y)$ with $D(g,y) = \int \mathbf{1}_\Gamma(g^{-1}\nu y)\,\mathrm{dpw}_l(g,g^{-1}\nu y)\,d\eta_{<l}(\nu)$. We claim $|\mathcal{K}_{l+1}(g,y)|^2 D(g,y) \le C_l\,\mathrm{csq}_l(g,y)$. If $\mathcal{K}_{l+1}(g,y) = 0$ this is clear. If $\mathrm{clm}_l(g,y) = 0$, then $\mathbf{1}_\Gamma(g^{-1}\nu y) = 0$ for $\eta_{<l}$-almost every ν and $D(g,y) = 0$. Otherwise $\mathrm{csq}_l(g,y) = |\mathcal{K}_{l+1}(g,y)|^2$, and g is near at $l+1$ by (PK5) (Lemma 5.2.64 (iv): $\mathcal{K}$ is a pairing kernel at floor $l+1$), hence near at l as $R^{(l+1)} \subseteq R^{(l)}$ (Lemma 5.2.58 (i)); by Lemma 5.4.27 (ii), $D(g,y) \le \prod_{a<l} c_{\mathrm{fl}}(a) = C_l$. Integrating over y and then over g against Θ_l, and exchanging the integrals in x and g, the left side of (5.84) is at most $C_l \int\int \mathrm{csq}_l(g,y)\,d\rho_l(y)\,d\Theta_l(g) \le B$ by (5.82).

Step 3: summing the floors. By Lemma 5.4.16 (vi), $\mathrm{WSI}_l(\mathcal{K},x) = \int \mathrm{WSq}_l(\mathcal{K},x)\,d\Theta_l + \mathrm{WSI}_{l+1}(\mathcal{K},x)$ for $l \le \varrho$, and $\mathrm{WSI}_{\varrho+1}(\mathcal{K},x) = 0$ because $\mathrm{WSS}_n = 0$. Hence $\mathrm{WSI}_j(\mathcal{K},x) = \sum_{l=j}^{\varrho} \int \mathrm{WSq}_l(\mathcal{K},x)\,d\Theta_l$, and by (5.84) its integral over P_Γ is at most $(\varrho+1-j)B \le nB$. □

Lemma 5.4.26. *Let Γ be a gauge for a place datum and $\mathbf{1}_\Gamma$ the indicator function of the set of positions of size Γ. For $l \ge 0$, $g,y \in G$ and $\mathcal{K}\colon G \times G \to \mathbb{C}$ put*

$$\mathrm{clm}_l(g,y) = \int_G \mathbf{1}_\Gamma(g^{-1}\nu y)\,d\eta_{<l}(\nu), \qquad \mathrm{csq}_l(\mathcal{K})(g,y) = \begin{cases} |\mathcal{K}_{l+1}(g,y)|^2 & \text{if } \mathrm{clm}_l(g,y) > 0, \\ 0 & \text{if } \mathrm{clm}_l(g,y) = 0. \end{cases}$$

Let $j \ge 0$, $d \in G$ and let $\mathcal{K}$ be a pairing kernel at floor j for d.

(i) For every $l \ge 0$, clm_l and $\mathrm{csq}_l(\mathcal{K})$ are measurable on $G \times G$; $\mathrm{clm}_l(g,\nu y) = \mathrm{clm}_l(g,y)$ and $\mathrm{csq}_l(\mathcal{K})(g,\nu y) = \mathrm{csq}_l(\mathcal{K})(g,y)$ for $\nu \in N_{<l}$; and $\mathrm{clm}_l(\sigma g,\sigma y) = \mathrm{clm}_l(g,y)$ for $\sigma \in G^{(l)}$.

(ii) Let $j \le l \le \varrho$, and put $c_{\mathrm{up}}(l) = c_{\mathrm{fl}}(l)$ for $l < \varrho$ and $c_{\mathrm{up}}(\varrho) = 1$. Then for all $g,y \in G$

$$\int_G \mathrm{csq}_l(\mathcal{K})(g,ny)\,d\eta_l(n) \le c_{\mathrm{up}}(l) \int_G \mathrm{csq}_{l+1}(\mathcal{K})(\sigma g,\sigma y)\,d\theta_{l+1}(\sigma).$$

Proof. All measures are s-finite, so Tonelli's theorem applies; the set of positions of size Γ is measurable (Lemma 5.4.27, proof of (ii)), and $\mathcal{K}_l$ is measurable (Lemma 5.2.64 (i)).

(i). $(\nu,g,y) \mapsto \mathbf{1}_\Gamma(g^{-1}\nu y)$ is measurable, so clm_l is measurable by Tonelli's theorem, and $\mathrm{csq}_l(\mathcal{K}) = \mathbf{1}_{\{\mathrm{clm}_l>0\}}|\mathcal{K}_{l+1}|^2$ is measurable. For $\nu \in N_{<l}$, $\eta_{<l} * \delta_\nu = \eta_{<l}$ (Lemma 5.4.14 (iii)), so $\mathrm{clm}_l(g,\nu y) = \int \mathbf{1}_\Gamma(g^{-1}\nu' \nu y)\,d\eta_{<l}(\nu') = \mathrm{clm}_l(g,y)$; with $|\mathcal{K}_{l+1}(g,\nu y)| = |\mathcal{K}_{l+1}(g,y)|$ (Lemma 5.2.64 (vi)) this gives the invariance of $\mathrm{csq}_l(\mathcal{K})$. For $\sigma \in G^{(l)}$, $\delta_\sigma * \eta_{<l} = \eta_{<l} * \delta_\sigma$ (Lemma 5.2.59 (iii)), hence $\delta_{\sigma^{-1}} * \eta_{<l} * \delta_\sigma = \eta_{<l}$ and $\mathrm{clm}_l(\sigma g,\sigma y) = \int \mathbf{1}_\Gamma(g^{-1}\sigma^{-1}\nu\sigma y)\,d\eta_{<l}(\nu) = \mathrm{clm}_l(g,y)$.

(ii), the case $\mathrm{clm}_{l+1}(g,y) = 0$. As $\eta_{<l+1} = \eta_{<l} * \eta_l$, $0 = \mathrm{clm}_{l+1}(g,y) = \int \mathrm{clm}_l(g,ny)\,d\eta_l(n)$, so $\mathrm{clm}_l(g,ny) = 0$, hence $\mathrm{csq}_l(\mathcal{K})(g,ny) = 0$, for η_l-almost every n; the left side is 0.

(ii), the case $\mathrm{clm}_{l+1}(g,y) > 0$. By (i) and since θ_{l+1} is carried by $G^{(l+1)}$ (Lemma 5.2.59 (ii)), $\mathrm{clm}_{l+1}(\sigma g,\sigma y) > 0$ and so $\mathrm{csq}_{l+1}(\mathcal{K})(\sigma g,\sigma y) = |\mathcal{K}_{l+2}(\sigma g,\sigma y)|^2$ for θ_{l+1}-almost every σ. Since $\mathrm{csq}_l(\mathcal{K}) \le |\mathcal{K}_{l+1}|^2$, it suffices to show

$$\int_G \mathrm{csq}_l(\mathcal{K})(g,ny)\,d\eta_l(n) \le c_{\mathrm{up}}(l) \int_G |\mathcal{K}_{l+2}(\sigma g,\sigma y)|^2\,d\theta_{l+1}(\sigma). \tag{5.85}$$

The top floor $l = \varrho$. Here $\eta_\varrho = \delta_1$, and the left side of (5.85) is at most $|\mathcal{K}_{\varrho+1}(g,y)|^2 \le \mathrm{piv}_{\varrho+1}(\mathcal{K})(g,y)^2$ by Lemma 5.2.64 (ii). By the definition of $R^{(l)}$, $R^{(\varrho+2)} = \mathsf{U}$, so for every $g' \in G$ the set of ξ with $\xi g'$ near at $\varrho+2$ is $\{\xi : u\xi g' \in K \text{ for some } u \in \mathsf{U}\}$, of μ_H-measure at most 1 by (PE3); as $\theta_{\varrho+1} = \mu_H$, this is $\mathrm{FML}_{\varrho+1}(1;1)$. Lemma 5.4.16 (i) gives $\mathrm{piv}_{\varrho+1}(\mathcal{K})(g,y)^2 \le \int |\mathcal{K}_{\varrho+2}(\sigma g, \sigma y)|^2 \, d\theta_{\varrho+1}(\sigma)$, which is (5.85) with $c_{\mathrm{up}}(\varrho) = 1$.

A floor $l < \varrho$. We apply Proposition 5.4.28 with $a = l$, $\varepsilon_0 = 0$, $\xi = M^{-1}$, $u = M$, the points $x = g$ and y, $i = 1$, the size index 14 in place of its j, $C = 1$, the Borel function $\Phi(n) = \mathrm{csq}_l(\mathcal{K})(g, ny)$, and

$$S = \{n \text{ a Heisenberg element of floor } l : \ g \text{ is near at } l+1, \\ g^{-1}\nu_0 ny \text{ is a position of size } \Gamma \text{ for some } \nu_0 \in N_{<l}\}.$$

By Lemma 5.2.64 (iv), $\mathcal{K}$ is a pairing kernel at floor $l+1 \ge j$ for d. For a Heisenberg element s of floor l, $s \in G^{(l)} \subseteq G^{(j)}$, so $dsd^{-1} = s$ by (PK3), and $\sigma(s^0) = \sigma(1) = 0$; so $M^{-1}(\sigma(dsd^{-1}) - \sigma(s^0)) = \xi\,\sigma(s)$.

Hypothesis (a): if $\Phi(n) \ne 0$ for a Heisenberg element n, then $\mathrm{clm}_l(g, ny) > 0$, so, $\eta_{<l}$ being carried by $N_{<l}$ (Lemma 5.4.14 (ii)), $g^{-1}\nu_0 ny$ is a position of size Γ for some $\nu_0 \in N_{<l}$; and $\mathcal{K}_{l+1}(g, ny) \ne 0$, so g is near at $l+1$ by (PK5). Thus $n \in S$, and $\Phi(n) \le |\mathcal{K}_{l+1}(n^0 g, ny)|^2$.

Hypothesis (b): let $n_0 \in S$, with $\nu_0 \in N_{<l}$ such that $g^{-1}\nu_0 n_0 y$ is a position of size Γ, and let $\xi_0 g \in K$ with $\xi_0 \in R^{(l+1)} \subseteq R^{(l)}$. By Lemma 5.2.58 (v), $\xi_0 = t\nu'$ with $t \in G^{(l)}$, $\nu' \in N_{<l}$; as $N_{<l} \subseteq \mathsf{U} \subseteq R^{(l+1)}$ and $N_{<l} \subseteq N_{<l+1}$ (Lemma 5.2.58 (i)), $t \in R^{(l+1)} \cap G^{(l)}$, $\nu' \in N_{<l+1}$, and $t\nu' g = \xi_0 g \in K$ has size 1 (GU2). Let $n \in S$, with $\mu \in N_{<l}$ such that $g^{-1}\mu ny$ is a position of size Γ. Put $g' = t\nu' g \in K$, $Y_0 = t\nu'\nu_0 n_0 y$, $\delta = n n_0^{-1} \in G^{(l)}$, $n^* = tn(tn_0)^{-1} = t\delta t^{-1} \in G^{(l)}$ and

$$\kappa = t\,\nu'\mu\,\delta(\nu'\nu_0)^{-1}\delta^{-1}\,t^{-1}.$$

By Lemma 5.2.58 (ii) (conjugation by δ and by t, elements of $G^{(l)}$), $\kappa \in N_{<l}$. Now $g'^{-1}Y_0 = g^{-1}\nu_0 n_0 y$ is a position of size Γ, and $\kappa n^* Y_0 = t\nu'\mu\delta(\nu'\nu_0)^{-1}\delta^{-1}t^{-1} \cdot t\delta\nu'\nu_0 n_0 y = t\nu'\mu\, n n_0^{-1} n_0 y = t\nu'\mu n y$, so $g'^{-1}\kappa n^* Y_0 = g^{-1}\mu n y$ is a position of size Γ. By Lemma 5.4.27 (i) (with the box point g', the point Y_0, $n^* \in G^{(l)}$ and $\kappa \in N_{<l}$), $w = (n^{*-1}\kappa^{-1} - 1)e_l^*$ satisfies $w - (n^{*-1} - 1)e_l^* \in E_{<l}$ and $M^{-1}w$ has size 14; and $n^{*-1} = tn_0(tn)^{-1}$. This is (b) with $t_0 = t$ and $q = \nu'$.

Proposition 5.4.28 gives, as $|\xi u| = |1| = 1$,

$$\int_G \Phi \, d\eta_l \le \mathrm{kb}_A^{|\iota'|}\,\mathrm{fm}(l)\,\Lambda^{100n(1+14+31)} \int_G |\mathcal{K}_{l+2}(\sigma g, \sigma y)|^2 \, d\theta_{l+1}(\sigma),$$

and $\mathrm{kb}_A^{|\iota'|}\mathrm{fm}(l)\Lambda^{4600n} \le c_{\mathrm{fl}}(l)$ as $\Lambda \ge 2$. This is (5.85). □

Lemma 5.4.27. *Let Γ be a gauge of loss Λ for a place datum, with size filtration $(A_{(i)})_{i\ge 0}$; let $\mathbf{1}_\Gamma$ be the indicator function of the set of positions of size Γ, and write $E_{<a} = \mathrm{span}\{e_b : b < a\}$.*

(i) Let $a < \varrho$, $g \in K$, $y \in G$, $n \in G^{(a)}$ and $\nu_0 \in N_{<a}$, and let $g^{-1}y$ and $g^{-1}\nu_0 ny$ be positions of size Γ. Then $w = (n^{-1}\nu_0^{-1} - 1)e_a^*$ satisfies $w - (n^{-1} - 1)e_a^* \in E_{<a}$, and $M^{-1}w$ has size 14.

(ii) Let $j \le \varrho$ and let $g \in G$ be near at j. Then for every $y \in G$

$$\int_G \mathbf{1}_\Gamma(g^{-1}\nu y)\,\mathrm{dpw}_j(g, g^{-1}\nu y)\,d\eta_{<j}(\nu) \le \prod_{l<j} \mathrm{fm}(l)\,\Lambda^{4600n} \le \prod_{l<j} c_{\mathrm{fl}}(l).$$

Proof. Throughout, products and sums of matrices are formed in the coordinate basis, and we use: if X has size i and Y size i', then XY has size $i+i'+1$ (each entry is a sum of r elements of $A_{(i+i')}$, by (SZ1), (SZ2)); a sum $X+Y$ of two matrices of size i has size $i+1$ and $-X$, $\bar{X}$ have size i (SZ2), (SZ3); $A_{(i)} \subseteq A_{(i+1)}$ (a sum of one element, (SZ2)); for $x \in G$, $x^{-1} = J^{-1}\bar{x}^{\mathsf{T}}J$, so if x has size i then x^{-1} has size $i+4$ by (SZ5).

(i). Put $x = g^{-1}y$ and $x' = g^{-1}\nu_0 ny$. Since $M^{-1}(x'-1)$ has size 1 and $M \in A_{(1)}$ (GU2), the entries of $x' = M \cdot M^{-1}(x'-1)+1$ lie in $A_{(3)}$ by (SZ1), (SZ4), (SZ2); as $g \in K$ has size 1 (GU2), gx' has size 5 and $(gx')^{-1}$ size 9. The matrix $D = M^{-1}(x-1) - M^{-1}(x'-1)$ has size 2, so gD has size 4 and $gD(gx')^{-1}$ has size 14. Since M is a scalar,

$$\begin{aligned} gD(gx')^{-1} &= M^{-1}\, g(x-x')(gx')^{-1} = M^{-1}(gx(gx')^{-1} - 1) \\ &= M^{-1}(y(\nu_0 ny)^{-1} - 1) = M^{-1}(n^{-1}\nu_0^{-1} - 1). \end{aligned}$$

The vector $M^{-1}w$ is the column of this matrix at e_a^*, so it has size 14. Put $\mu = n^{-1}\nu_0^{-1}n$. As $n^{-1} \in G^{(a)}$ and $\nu_0^{-1} \in N_{<a}$, Lemma 5.2.58 (ii) gives $\mu \in N_{<a}$, and $n^{-1}\nu_0^{-1} = \mu n^{-1}$. By Lemma 5.4.11 (i), applied to $n^{-1} \in G^{(a)}$ and $\mu \in N_{<a}$, $w - (n^{-1}-1)e_a^* = (\mu n^{-1}-1)e_a^* - (n^{-1}-1)e_a^* \in E_{<a}$.

Inverse depth along the Heisenberg group of floor a. Let $a < \varrho$, $g \in K$ and $y \in G$ with $g^{-1}y \in \mathcal{P}_{\mathrm{pos}}$. By (GU7) with $Q = 0$ there is a common bound ϕ with loss Λ for $\mathcal{V} = \mathcal{V}_{\mathrm{cm}}(g, g^{-1}y, 0) = \{\omega \in A^{\iota'} : (g^{-1}O\omega, y^{-1}O\omega) \in \mathcal{P}_{\mathrm{cl}}\}$. We show: for $v \in V_{a+1}$, $\beta \in A^-$, $n = n_a(v,\beta)$ and $\varsigma(v) = (h(o_i, M^{-1}v))_{i\in\iota'}$,

$$\mathrm{dp}_a(g, g^{-1}ny)^{-1} \le \phi(\varsigma(v)). \tag{5.86}$$

By Lemma 5.4.8 (i), $\mathrm{dp}_a(g,g^{-1}ny)^{-1} = \mathrm{vol}^*(\mathfrak{d}_a(g,g^{-1}ny))$, and the second clause of a common bound bounds vol^* of $\{c \in A : \bar{c}\,\overline{\varsigma(v)}^{\mathsf{T}}\omega \in \mathcal{D}_{\mathrm{pr}} \text{ for all } \omega \in \mathcal{V}\}$ by $\phi(\varsigma(v))$; so it suffices that $\mathfrak{d}_a(g,g^{-1}ny)$ lies in this set. Let $c \in \mathfrak{d}_a(g,g^{-1}ny)$: $c = h(e_a,\zeta)$ with $h(e_b,\zeta) = 0$ for $b < a$, and $(g^{-1}\zeta, (ny)^{-1}\zeta') \in \mathcal{P}_{\mathrm{cl}}$ for some $\zeta' \in \zeta + E_{<a}$. Let $\omega \in \mathcal{V}$ and $o = O\omega$. As o lies in the middle space, $h(e_b,o) = 0$ for all $b < \varrho$; so (5.8) gives $no = o - h(v,o)e_a$, and since n is an isometry,

$$h(n^{-1}\zeta', o) = h(\zeta', no) = h(\zeta', o) - h(v,o)\,h(\zeta', e_a).$$

Here $h(\zeta',o) = h(\zeta,o)$, because $\zeta' - \zeta \in E_{<a}$ and $h(e_b,o) = 0$; and $h(\zeta',e_a) = \overline{h(e_a,\zeta')} = \overline{h(e_a,\zeta)} = \bar{c}$, because $h(e_a,e_b) = 0$. Hence $h(\zeta,o) - h(n^{-1}\zeta',o) = \bar{c}\,h(v,o)$. The pairs $(g^{-1}\zeta, (ny)^{-1}\zeta')$ and $(g^{-1}o, y^{-1}o)$ are close, and $h(g^{-1}\zeta, g^{-1}o) = h(\zeta,o)$, $h((ny)^{-1}\zeta', y^{-1}o) = h(n^{-1}\zeta', o)$; so $M^{-1}\bar{c}\,h(v,o) \in \mathcal{D}_{\mathrm{pr}}$. Finally, since $\bar{M} = M$ (PE1),

$$\overline{\varsigma(v)}^{\mathsf{T}}\omega = \sum_{i\in\iota'} M^{-1}h(v,o_i)\omega_i = M^{-1}h(v,o),$$

so $\bar{c}\,\overline{\varsigma(v)}^{\mathsf{T}}\omega = M^{-1}\bar{c}\,h(v,o) \in \mathcal{D}_{\mathrm{pr}}$. This proves (5.86).

Put $\Psi(v') = \phi((h(o_i,v'))_{i\in\iota'})^s$ for $v' \in V$; it is Borel, as ϕ is measurable and $v' \mapsto (h(o_i,v'))_i$ is continuous. By Lemma 5.4.4 (iv) with $l = a+1 \le \varrho$, $\rho = \mathrm{vol}_{V_{a+1}}$ and $j = 14$, and the integral clause of a common bound at size 18,

$$\int_{\{v' \text{ of size } 14\}} \Psi \, d\mathrm{vol}_{V_{a+1}} \le \Lambda^{99n\cdot 20} \int_{A^{\iota'}_{(18)}} \phi^s \, d\mathrm{vol} \le \Lambda^{99n\cdot 20}\Lambda^{99n\cdot 19} = \Lambda^{3861n}. \tag{5.87}$$

(ii) for g in the box. Fix $g \in K$. For $a \le \varrho$ put $F_a(x) = \mathbf{1}_\Gamma(x)\,\mathrm{dpw}_a(g,x)$, $I_a(z) = \int_G F_a(g^{-1}\nu z)\,d\eta_{<a}(\nu)$ and $C_a = \prod_{l<a}\mathrm{fm}(l)\Lambda^{4600n}$; $C_a < \infty$, as M is a unit (so $0 < |M| < \infty$)

and $\Lambda < \infty$ (GU1). The set of positions of size Γ is measurable: $\mathcal{P}_{\mathrm{pos}}$ is measurable (PE4), and $\{x : M^{-1}(x-1)$ has size $1\}$ is closed because $A_{(1)}$ is compact (GU1). With Lemma 5.4.8 (iii), F_a is Borel. The measures η_l are images of products of Haar measures of second countable locally compact groups, hence s-finite, and so are their convolutions; by Tonelli's theorem I_a is Borel and the iterated integrals below may be exchanged. We show $I_a \le C_a$ on G by induction on a. For $a = 0$, $\eta_{<0} = \delta_1$ and $\mathrm{dpw}_0 = 1$, so $I_0(z) = \mathbf{1}_\Gamma(g^{-1}z) \le 1$.

Let $a < \varrho$, assume $I_a \le C_a$, and let $z \in G$ with $I_{a+1}(z) \neq 0$. As $\eta_{<a+1}$ is carried by $N_{<a+1}$ (Lemma 5.4.14 (ii)), there is $\nu_1 \in N_{<a+1}$ with $F_{a+1}(g^{-1}\nu_1 z) \neq 0$; put $y = \nu_1 z$, so that $g^{-1}y$ is a position of size Γ. By Lemma 5.4.14 (iii), $\eta_{<a+1} * \delta_{\nu_1} = \eta_{<a+1}$, hence $I_{a+1}(z) = \int F_{a+1}(g^{-1}\nu\nu_1 z)\, d\eta_{<a+1}(\nu) = I_{a+1}(y)$. By associativity of convolution $\eta_{<a+1} = \eta_{<a} * \eta_a$, so

$$I_{a+1}(y) = \int_G J(n)\, d\eta_a(n), \qquad J(n) = \int_G F_{a+1}(g^{-1}\nu n y)\, d\eta_{<a}(\nu).$$

For $\nu \in N_{<a}$, Lemma 5.4.8 (iii) and (ii) give $F_{a+1}(g^{-1}\nu n y) = F_a(g^{-1}\nu n y)\,\mathrm{dp}_a(g, g^{-1}ny)^{-s}$. Since $\eta_{<a}$ is carried by $N_{<a}$, $J(n) = \mathrm{dp}_a(g, g^{-1}ny)^{-s} I_a(ny) \le C_a\,\mathrm{dp}_a(g, g^{-1}ny)^{-s}$, and $J(n) \neq 0$ implies $g^{-1}\nu_0 n y$ is a position of size Γ for some $\nu_0 \in N_{<a}$. We apply Lemma 5.4.11 (iii) with $u = M$, $j = 14$, Ψ, $I = \Lambda^{3861n}$ (5.87), $\Phi = J$ (Borel) and $C = C_a$. Its hypothesis: let $J(n_a(v,\beta)) \neq 0$ and $n = n_a(v,\beta) \in G^{(a)}$ (Lemma 5.2.58 (iii)); with ν_0 as above, (i) gives w with $w - (n^{-1} - 1)e_a^* \in E_{<a}$ and $M^{-1}w$ of size 14; and by (5.86), $J(n) \le C_a\,\phi(\varsigma(v))^s = C_a\Psi(M^{-1}v)$. Hence, as $\Lambda \ge 2$ and $16 + 3861n \le 4600n$,

$$I_{a+1}(z) = I_{a+1}(y) \le C_a\,\mathrm{fm}(a)\,\Lambda^{16}\Lambda^{3861n} \le C_a\,\mathrm{fm}(a)\Lambda^{4600n} = C_{a+1}.$$

(ii) in general. Let g be near at j: $\xi g \in K$ for some $\xi \in R^{(j)}$; by Lemma 5.2.58 (v), $\xi = \gamma\nu'$ with $\gamma \in G^{(j)}$ and $\nu' \in N_{<j}$. Put $g' = \xi g \in K$. By Lemma 5.4.8 (iii), $\mathrm{dpw}_j(g,x) = \mathrm{dpw}_j(g',x)$, and $g^{-1} = g'^{-1}\gamma\nu'$; let $F'(x) = \mathbf{1}_\Gamma(x)\mathrm{dpw}_j(g',x)$. Using $\delta_{\nu'} * \eta_{<j} = \eta_{<j}$ (Lemma 5.4.14 (iii)) and then $\delta_\gamma * \eta_{<j} = \eta_{<j} * \delta_\gamma$ (Lemma 5.2.59 (iii)),

$$\int F'(g'^{-1}\gamma\nu'\nu y)\, d\eta_{<j}(\nu) = \int F'(g'^{-1}\gamma\nu y)\, d\eta_{<j}(\nu) = \int F'(g'^{-1}\nu\gamma y)\, d\eta_{<j}(\nu),$$

which is at most $\prod_{l<j}\mathrm{fm}(l)\Lambda^{4600n}$ by the case of the box at g' and the point γy. The last inequality of (ii) holds termwise, as $\mathrm{kb}_A \ge 1$ (GU1) and $\Lambda \ge 2$, so $\mathrm{fm}(l)\Lambda^{4600n} \le \mathrm{fm}(l)\mathrm{kb}_A^{|\iota'|}\Lambda^{4900n} = c_{\mathfrak{fl}}(l)$. □

Proposition 5.4.28. *Let Γ be a gauge of loss Λ for a place datum, put $r' = |\iota'|$ and $E_{<a} = \mathrm{span}\{e_b : b < a\}$, let $a < \varrho$, $\varepsilon_0 \in \{0,1\}$, $d \in G$, let $\mathcal{K}$ be a pairing kernel at floor $a+1$ for d, and let ξ, u be units of A with $M^{-1}(\sigma(dsd^{-1}) - \sigma(s^{\varepsilon_0})) = \xi\,\sigma(s)$ for every Heisenberg element $s = n_a(v,\beta)$. Let $x, y \in G$, $i, j \ge 0$, $C \in [0,\infty]$, $S \subseteq G$, and let $\Phi\colon G \to [0,\infty]$ be Borel. Assume:*

(a) for every Heisenberg element $n = n_a(v,\beta)$ with $\Phi(n) \neq 0$: $n \in S$ and

$$\Phi(n) \le C\,|\mathcal{K}_{a+1}(n^{\varepsilon_0}x, ny)|^2;$$

(b) for every Heisenberg element $n_0 \in S$ there are $t_0 \in R^{(a+1)} \cap G^{(a)}$ and $q \in N_{<a+1}$ such that $t_0 q x$ has size i and, for every Heisenberg element $n \in S$, some $w \in V$ with $w - (t_0 n_0 (t_0 n)^{-1} - 1)e_a^* \in E_{<a}$ has $u^{-1}w$ of size j.

Then

$$\int_G \Phi\, d\eta_a \le C\Big(\frac{\mathrm{kb}_A}{|\xi u|}\Big)^{r'} \mathrm{fm}(a,u)\,\Lambda^{100n(i+j+31)} \int_G |\mathcal{K}_{a+2}(\varsigma x, \varsigma y)|^2\, d\theta_{a+1}(\varsigma).$$

Proof. If $\int \Phi\, d\eta_a = 0$ there is nothing to prove. Otherwise, as η_a is the image of a measure under $(v,\beta) \mapsto n_a(v,\beta)$, there is a Heisenberg element $n_0 = n_a(z,\gamma)$ with $\Phi(n_0) \neq 0$; by (a), $n_0 \in S$. Let t_0, q be as in (b) for n_0.

Splitting t_0. By Lemma 5.2.58, (iv), there is a Heisenberg element r' with $r'^{-1}t_0 \in G^{(a+1)}$; put $t = r'^{-1}t_0$ and $r = t^{-1}r't$. By Lemma 5.2.58, (iii), r is a Heisenberg element (conjugation by $t^{-1} \in G^{(a+1)}$ maps $n_a(v',\beta')$ to $n_a(t^{-1}v',\beta')$), and $t_0 = r't = tr$ with $t \in G^{(a+1)}$.

Substitution. For a Heisenberg element n put $N(n) = r^{-1}(t^{-1}nt)(rn_0)$, again a Heisenberg element by Lemma 5.2.58, (iii). By Lemma 5.2.61, (ii), η_a is invariant under left and right translation by Heisenberg elements and under $n \mapsto t^{-1}nt$; hence $\int \Phi\, d\eta_a = \int \Phi(N(n))\, d\eta_a(n)$. For $n = n_a(v,\beta)$, the product formula of Lemma 5.2.58, (iii), shows that $N(n)$ has vector part $t^{-1}v + z$, so by Lemma 5.4.29, (i), and since t preserves h,

$$\begin{aligned}\mathcal{K}_{a+1}(N(n)^{\varepsilon_0}x, N(n)y) &= \int_G \psi(-\xi\, h(t^{-1}v + z, \varsigma^{-1}f_{a+1}))\kappa(\varsigma)\, d\theta_{a+1}(\varsigma)\\ &= \int_G \psi(\xi'\, h(u^{-1}v + v_0, t\varsigma^{-1}f_{a+1}))\kappa(\varsigma)\, d\theta_{a+1}(\varsigma),\end{aligned}$$

where $\kappa(\varsigma) = \mathcal{K}_{a+2}(\varsigma x, \varsigma y)$, $\xi' = -\xi\bar{u}$ (a unit, with $|\xi'| = |\xi u|$ by (PA2)) and $v_0 = u^{-1}tz$; here $h(v + tz, \cdot) = \bar{u}\, h(u^{-1}v + u^{-1}tz, \cdot)$. Call the right side $T(u^{-1}v)$, and put $\Psi(v') = C|T(v')|^2$, a Borel function of $v' \in V$.

Plancherel one floor up. Put $q' = t_0 q = trq$. Then $t^{-1}q' = rq \in N_{<a+1}$ (Heisenberg elements of floor a lie in $N_{<a+1}$ by Lemma 5.2.58, (iii)), $q'x = t_0 qx$ has size i, $t \in G^{(a+1)}$, and $\mathcal{K}$ is a pairing kernel at floor $a+1$. By Lemma 5.4.30, (ii), with $b = a+1$, $\rho = \mathrm{vol}_{V_{a+1}}$, the frequency ξ', the shift v_0 and j in place of i,

$$\int_{\{v' \text{ of size } j\}} \Psi\, d\mathrm{vol}_{V_{a+1}} \leq C\, \Lambda^{99n(j+6)}\Lambda^{99n(i+12)}\Big(\frac{\mathrm{kb}_A}{|\xi u|}\Big)^{r'} \int_G |\kappa|^2\, d\theta_{a+1} =: I.$$

Integration over the Heisenberg group. Let $n = n_a(v,\beta)$ with $\Phi(N(n)) \neq 0$. By (a), $N(n) \in S$ and $\Phi(N(n)) \leq C|\mathcal{K}_{a+1}(N(n)^{\varepsilon_0}x, N(n)y)|^2 = \Psi(u^{-1}v)$. Moreover $t_0 N(n) = tr\, r^{-1}t^{-1}nt\, rn_0 = n\, t_0 n_0$, so $t_0 n_0 (t_0 N(n))^{-1} = n^{-1}$. By (b) applied to $N(n) \in S$, some $w \in V$ with $w - (n^{-1} - 1)e_a^* \in E_{<a}$ has $u^{-1}w$ of size j. So Lemma 5.4.11, (iii), applies to $\Phi \circ N$ with the constant 1, the function Ψ and the bound I:

$$\int \Phi\, d\eta_a = \int \Phi(N(n))\, d\eta_a(n) \leq \mathrm{fm}(a,u)\, \Lambda^{j+2}\, I.$$

Finally $j + 2 + 99n(j+6) + 99n(i+12) \leq 100n(i+j+31)$, as $n \geq 1$, and $\Lambda \geq 1$. □

Lemma 5.4.29. *Let a place datum be given, $a < \varrho$, $\varepsilon_0 \in \{0,1\}$, $d \in G$, $\xi \in A$, and let $\mathcal{K}$ be a pairing kernel (at some floor) for d.*

(i) Suppose that $M^{-1}(\sigma(dsd^{-1}) - \sigma(s^{\varepsilon_0})) = \xi\, \sigma(s)$ for every Heisenberg element $s = n_a(v,\beta)$. Then for all $v \in V_{a+1}$, $\beta \in A^-$ and $x, y \in G$,

$$\mathcal{K}_{a+1}(n_a(v,\beta)^{\varepsilon_0}x,\ n_a(v,\beta)\, y) = \int_G \psi(-\xi\, h(v, \varsigma^{-1}f_{a+1}))\, \mathcal{K}_{a+2}(\varsigma x, \varsigma y)\, d\theta_{a+1}(\varsigma).$$

(ii) If p, q are sequences of units, $j \leq a+1$ and $d = w_{j,q}^{-1}w_{j,p}$, the hypothesis of (i) holds with $\xi = M^{-1}(p_a q_a^{-1} - \varepsilon_0)$ if $a < j$, and with $\xi = M^{-1}(1 - \varepsilon_0)$ if $a \geq j$.

Proof. **The simple-root sum on the Heisenberg group.** Let $s = n_a(v,\beta)$. By (5.8), s fixes the coordinate vectors of floor at most a other than e_a^*, sends $w \in V_{a+1}$ to $w - h(v,w)e_a$, and $se_a^* - e_a^* \in \langle e_a\rangle + V_{a+1}$. Among the entries $s_{f_b f_{b+1}}$ ($b < \varrho$), only $b = a$ can be non-zero: for $b < a$ the column f_{b+1} has floor at most a and is not e_a^*, and for $b > a$ the column $f_{b+1} \in V_{a+1}$ has $sf_{b+1} - f_{b+1} \in \langle e_a\rangle$ with $e_a \neq f_b$. Hence

$$\sigma(n_a(v,\beta)) = s_{e_a f_{a+1}} = -h(v, f_{a+1}). \tag{5.88}$$

(i). Fix v, β, x, y and $n = n_a(v,\beta)$. By Lemma 5.2.64, (ii), at floor $a+1$,

$$\mathcal{K}_{a+1}(n^{\varepsilon_0}x, ny) = \int_G \mathcal{K}_{a+2}(\varsigma n^{\varepsilon_0}x, \varsigma ny)\, d\theta_{a+1}(\varsigma),$$

and θ_{a+1} is carried by $G^{(a+1)}$ (Lemma 5.2.59, (ii)). Let $\varsigma \in G^{(a+1)}$ and $s = \varsigma n \varsigma^{-1}$. By Lemma 5.2.58, (iii), $s = n_a(\varsigma v, \beta)$ is a Heisenberg element, so $s, s^{\varepsilon_0} \in N_{<a+1}$, and $\varsigma n^{\varepsilon_0} = s^{\varepsilon_0}\varsigma$, $\varsigma n = s\varsigma$. By (PK2) with $l = a+1$ and the hypothesis,

$$\mathcal{K}_{a+2}(s^{\varepsilon_0}\varsigma x, s\varsigma y) = \psi\big(M^{-1}(\sigma(dsd^{-1}) - \sigma(s^{\varepsilon_0}))\big)\mathcal{K}_{a+2}(\varsigma x, \varsigma y) = \psi(\xi\,\sigma(s))\,\mathcal{K}_{a+2}(\varsigma x, \varsigma y),$$

and by (5.88), $\sigma(s) = -h(\varsigma v, f_{a+1}) = -h(v, \varsigma^{-1}f_{a+1})$, ς being an isometry. Substituting gives (i).

(ii). By Lemma 5.2.58, (ix), d is diagonal with $d_{e_a e_a} = p_a q_a^{-1}$ if $a < j$ and $d_{e_a e_a} = 1$ otherwise, and $d_{f_{a+1} f_{a+1}} = 1$, as f_{a+1} is e_{a+1} with $a+1 \geq j$ or is z. By Lemma 5.2.58, (x), $(dsd^{-1})_{f_b f_{b+1}} = d_{f_b f_b} s_{f_b f_{b+1}} d^{-1}_{f_{b+1} f_{b+1}}$, and only $b = a$ contributes, so $\sigma(dsd^{-1}) = r\,\sigma(s)$ with $r = p_a q_a^{-1}$ if $a < j$ and $r = 1$ otherwise. Moreover $\sigma(s^{\varepsilon_0}) = \varepsilon_0\,\sigma(s)$, as $\sigma(1) = 0$. Hence $M^{-1}(\sigma(dsd^{-1}) - \sigma(s^{\varepsilon_0})) = M^{-1}(r - \varepsilon_0)\,\sigma(s)$. □

Lemma 5.4.30. *Let Γ be a gauge of loss Λ for a place datum, put $r' = |\iota'|$, let $0 \leq b \leq \varrho$, let $\mathcal{K}$ be a pairing kernel at floor b for some $d \in G$, let $t \in G^{(b)}$ and $q \in G$ with $t^{-1}q \in N_{<b}$, let $j \geq 0$ and let $x, y \in G$ with qx of size j.*

(i) For every $\sigma \in G^{(b)}$ with $\mathcal{K}_{b+1}(\sigma x, \sigma y) \neq 0$, the vector $t\sigma^{-1}f_b$ has size $j+6$.

(ii) For every unit ξ, every $v_0 \in V$, every $i \geq 0$ and $\rho = \mathrm{vol}_{V_b}$ or $\rho = \mathrm{qv}_b$,

$$\int_{\{v \in V \text{ of size } i\}} \Big| \int_G \psi(\xi\, h(v + v_0, t\sigma^{-1}f_b))\, \mathcal{K}_{b+1}(\sigma x, \sigma y)\, d\theta_b(\sigma)\Big|^2 d\rho(v)$$
$$\leq \Lambda^{99n(i+6)}\Lambda^{99n(j+12)}\Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'} \int_G |\mathcal{K}_{b+1}(\sigma x, \sigma y)|^2\, d\theta_b(\sigma).$$

Proof. **(i).** By Lemma 5.2.64, (iv), $\mathcal{K}$ is a pairing kernel at floor $b+1$, so by (PK5) the point σx is near at $b+1$. Put $c = \sigma(t^{-1}q)\sigma^{-1}$; by Lemma 5.2.58, (ii) and (i), $c \in N_{<b} \subseteq R^{(b+1)}$, and $(\sigma t^{-1})(qx) = c\,\sigma x$. As $R^{(b+1)}$ is a group, $c\sigma x$ is near at $b+1$. Now $\sigma t^{-1} \in G^{(b)}$ and qx has size j, so Lemma 5.4.3, (i), gives that $(\sigma t^{-1})^{-1}f_b = t\sigma^{-1}f_b$ has size $j+6$.

(ii). Put $\kappa(\sigma) = \mathcal{K}_{b+1}(\sigma x, \sigma y)$. By Lemma 5.2.64, (i) and (ii), κ is measurable and θ_b-integrable, and by (v) there (with $l = b+1 \geq b$), $\kappa(s\sigma) = \kappa(\sigma)$ for $s \in R^{(b+1)} \cap G^{(b)}$. By Lemma 5.2.62 (with $Z = \mathbb{C}$) there is a Borel $f_0 \colon V \to \mathbb{C}$ with $\kappa(\sigma) = f_0(\sigma^{-1}f_b)$ for $\sigma \in G^{(b)}$. Let $f(w) = f_0(t^{-1}w)$ if w has size $j+6$ and $f(w) = 0$ otherwise, and $\phi(w) = \psi(\xi\, h(v_0, w))\, f(w)$; both are Borel, as the set of vectors of size $j+6$ is compact (GU1). For $\sigma \in G^{(b)}$: if $t\sigma^{-1}f_b$ has

size $j+6$, then $f(t\sigma^{-1}f_b) = f_0(\sigma^{-1}f_b) = \kappa(\sigma)$; otherwise $\kappa(\sigma) = 0 = f(t\sigma^{-1}f_b)$ by (i). As θ_b is carried by $G^{(b)}$ (Lemma 5.2.59, (ii)),

$$\kappa(\sigma) = f(t\sigma^{-1}f_b) \qquad \text{for } \theta_b\text{-almost every } \sigma.$$

By Lemma 5.2.63 with $s = t$, the map $\sigma \mapsto t\sigma^{-1}f_b$ carries θ_b to qv_b. Hence f and ϕ are qv_b-integrable, ϕ vanishes outside the vectors of size $j+6$, $\int |\phi|^2\,d\mathrm{qv}_b = \int |f|^2\,d\mathrm{qv}_b = \int |\kappa|^2\,d\theta_b$, and, as $h(v+v_0,w) = h(v,w) + h(v_0,w)$ and ψ is a character,

$$\int_G \psi(\xi\, h(v+v_0, t\sigma^{-1}f_b))\kappa(\sigma)\,d\theta_b(\sigma) = \int_V \psi(\xi\, h(v,w))\,\phi(w)\,d\mathrm{qv}_b(w) \qquad (v \in V).$$

Now Proposition 5.4.31 with $j+6$ in place of j gives the claim. □

Proposition 5.4.31. *Let Γ be a gauge of loss Λ for a place datum of shape (ϱ, ι, m), put $r' = |\iota'|$, let $0 \le b \le \varrho$, let ξ be a unit of A and let ρ be vol_{V_b} or qv_b. For $i, j \ge 0$ and every measurable $\phi\colon V \to \mathbb{C}$ that is qv_b-integrable and vanishes outside the vectors of size j,*

$$\int_{\{v\in V \text{ of size } i\}} \Big|\int_V \psi(\xi\, h(v,y))\,\phi(y)\,d\mathrm{qv}_b(y)\Big|^2 d\rho(v) \le \Lambda^{99n(i+6)}\,\Lambda^{99n(j+6)}\Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'} \int_V |\phi|^2\,d\mathrm{qv}_b.$$

Proof. Arithmetic is in $[0,\infty]$. Let $x = (c,t,Y)$, $Y = (\zeta, w, \zeta')$, be the coordinates of $A \times A \times Y_b$, and $\lambda_b(\beta)$, B_b as in Lemma 5.4.4; put $\lambda = \lambda_b(\delta h(f_b,f_b))$, so that $\mathrm{qv}_b = (B_b)_*\lambda$, and let $\rho_0 = \mathrm{vol}$ if $\rho = \mathrm{vol}_{V_b}$ and $\rho_0 = \lambda$ if $\rho = \mathrm{qv}_b$, so that $\rho = (B_b)_*\rho_0$ (Lemma 5.4.4, (iii)). Put $\tilde\phi = \phi \circ B_b$, which is λ-integrable, and let $\square_l$ be the set of x with all coordinates in $A_{(l)}$, a compact set.

Two blocks. By (GU1), $|\iota \setminus \iota'| = 2(2m+2)$; choose disjoint $B, B' \subseteq \iota \setminus \iota'$ with $|B| = |B'| = 2m+2$. For the variable $y = B_b(x)$ write $x = (u, w_{\iota'}, z)$, with $z = (w_k)_{k\in B}$ the block, $w_{\iota'} = (w_k)_{k\in\iota'}$ the inner coordinates and $u \in A^U$ the other (outer) coordinates. For the variable $v = B_b(x')$ write likewise $x' = (u', w'_{\iota'}, z')$ with the block B' and outer coordinates $u' \in A^{U'}$. By Lemma 5.4.4, (i), there are s-finite kernels $(\zeta_{u,w})$ on A^B and $(\zeta'_{u',w'})$ on $A^{B'}$ with $\lambda = \mathrm{vol}\otimes\zeta$, $\rho_0 = \mathrm{vol}\otimes\zeta'$ in the sense of (i) there, and $\zeta_{u,w}(A^B_{(l)}) \le \Lambda^{(2m+2)(l+1)}$, $\zeta'_{u',w'}(A^{B'}_{(l)}) \le \Lambda^{(2m+2)(l+1)}$ for all l (for $\rho_0 = \mathrm{vol}$, $\zeta' = \mathrm{vol}$).

The phase. By Lemma 5.4.4, (iii), $h(B_b(x'), B_b(x)) = \bar c' t + \bar t' c + h'_b(Y', Y)$; in h'_b the coordinates indexed by ι are orthonormal and orthogonal to the others. As B' is disjoint from $\iota' \cup B$, the coordinates w_k, $k \in B'$, of x are outer coordinates, components of u; write $u_{B'}$ for them. Hence

$$h(B_b(x'), B_b(x)) = \overline{w'_{\iota'}}^{\mathsf T} w_{\iota'} + \overline{z'}^{\mathsf T} u_{B'} + a(u', x),$$

where $a(u',x)$ is the sum of the terms of $\bar c' t + \bar t' c + h'_b(Y', Y)$ that contain a coordinate of u'.

Cauchy–Schwarz in the outer coordinates of y. Fix x'. If $\tilde\phi(x) \ne 0$ then $B_b(x)$ has size j, so $x \in \square_{j+4}$ (Lemma 5.4.4, (iii)) and $u \in A^U_{(j+4)}$. The integrand $\psi(\xi h(B_b(x'), B_b(x)))\tilde\phi(x)$ is λ-integrable, so by Fubini for $\lambda = \mathrm{vol}\otimes\zeta$,

$$\int_V \psi(\xi h(v,y))\phi(y)\,d\mathrm{qv}_b(y) = \int_{A^U} \psi(\xi\,\overline{z'}^{\mathsf T} u_{B'})\,F_{u'}(w'_{\iota'}, u)\,du,$$

$$F_{u'}(w', u) = \int_{A^{\iota'}} \psi(\xi\,\overline{w'}^{\mathsf T} w)\,G_{u'}(u,w)\,dw,$$

with $G_{u'}(u,w) = \int_{A^B} \psi(\xi\, a(u', (u,w,z)))\,\tilde\phi(u,w,z)\,d\zeta_{u,w}(z)$, a measurable function that vanishes unless $u \in A^U_{(j+4)}$. As $|\psi| = 1$, Cauchy–Schwarz on $A^U_{(j+4)}$ gives, with $H(u',w') = \int_{A^U} |F_{u'}(w',u)|^2\,du$,

$$\Big|\int_V \psi(\xi h(B_b(x'), y))\phi(y)\,d\mathrm{qv}_b(y)\Big|^2 \le \mathrm{vol}(A_{(j+4)})^{|U|}\,H(u', w'_{\iota'}), \tag{5.89}$$

a bound that does not depend on z'.

Plancherel in the inner coordinates, Cauchy–Schwarz in the block of y. Fix u'. Since $\tilde\phi$ is λ-integrable, $(u,w)\mapsto\int|\tilde\phi(u,w,z)|\,d\zeta_{u,w}(z)$ is integrable, so for almost every u the function $w\mapsto G_{u'}(u,w)$ is integrable on $A^{\iota'}$. By Lemma 5.4.32 (its hypotheses (a)–(c) are (PA3) for 1×1 matrices, (PA4) and (PA5)) with $\eta=\xi$, for almost every u,

$$\int_{A^{\iota'}}|F_{u'}(w',u)|^2\,dw'\le\Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'}\int_{A^{\iota'}}|G_{u'}(u,w)|^2\,dw.$$

By Cauchy–Schwarz in z, as $\tilde\phi(u,w,z)\neq0$ forces $z\in A^B_{(j+4)}$,

$$|G_{u'}(u,w)|^2\le\zeta_{u,w}(A^B_{(j+4)})\int|\tilde\phi(u,w,z)|^2\,d\zeta_{u,w}(z)\le\Lambda^{(2m+2)(j+5)}\int|\tilde\phi|^2\,d\zeta_{u,w}.$$

Integrating over u and using Tonelli,

$$\int_{A^{\iota'}}H(u',w')\,dw'\le\Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'}\Lambda^{(2m+2)(j+5)}\int|\tilde\phi|^2\,d\lambda=\Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'}\Lambda^{(2m+2)(j+5)}\int_V|\phi|^2\,d\mathrm{qv}_b.$$

Integration over v. If $v=B_b(x')$ has size i, then $x'\in\square_{i+4}$ (Lemma 5.4.4, (iii)). By (5.89) and $\rho_0=\mathrm{vol}\otimes\zeta'$,

$$\int_{\{v\text{ of size }i\}}\Big|\int\psi(\xi h(v,y))\phi(y)\,d\mathrm{qv}_b(y)\Big|^2d\rho(v)\le\mathrm{vol}(A_{(j+4)})^{|U|}\cdot\int_{A^{U'}_{(i+4)}}\int_{A^{\iota'}}H(u',w')\,\zeta'_{u',w'}(A^{B'}_{(i+4)})\,dw'\,du',$$

which, by the two bounds above, is at most $\mathrm{vol}(A_{(i+4)})^{|U'|}\Lambda^{(2m+2)(i+5)}\cdot\mathrm{vol}(A_{(j+4)})^{|U|}\Lambda^{(2m+2)(j+5)}\cdot(\mathrm{kb}_A/|\xi|)^{r'}\int|\phi|^2d\mathrm{qv}_b$. Counting coordinates, $|U|+2m+2=|U'|+2m+2=2+2(m+\varrho-b)+|\iota|-|\iota'|=6m+6+2(\varrho-b)\le62n\le99n$ by (GU1) ($m\le9n$, $|\iota|-|\iota'|=2(2m+2)$), $\mathrm{vol}(A_{(l)})\le\Lambda^{l+1}$ (GU5) and $\Lambda\ge2$, so $\mathrm{vol}(A_{(l+4)})^{|U|}\Lambda^{(2m+2)(l+5)}\le\Lambda^{99n(l+5)}\le\Lambda^{99n(l+6)}$ for $l=i,j$, which gives the claim. □

Lemma 5.4.32. *Let A be a commutative ring with an involution $x\mapsto\bar x$, carrying a locally compact Polish topology for which addition, multiplication and the involution are continuous, and let dx be a Haar measure of $(A,+)$. For a finite set ι give A^ι the product measure and write $v^*w=\sum_{i\in\iota}\bar v_iw_i$ for $v,w\in A^\iota$. Let ψ be a continuous homomorphism from $(A,+)$ to the unit circle, and $|\cdot|\colon A\to[0,\infty]$ a measurable map with $|1|=1$, $|xy|=|x|\,|y|$ and $|\bar x|=|x|$. Assume:*

(a) for every unit u of A the image of dx under $x\mapsto ux$ is $|u|^{-1}dx$;

(b) almost every element of A is a unit;

(c) there is a continuous integrable $G\colon A\to[0,\infty)$ with $G(0)=1$ and

$$\int_A\Big|\int_AG(v)\,\psi(\bar vz)\,dv\Big|\,dz\le\mathrm{kb}_A,\qquad\mathrm{kb}_A\in[0,\infty].$$

Then for every unit η of A and every integrable $f\colon A^\iota\to\mathbb{C}$,

$$\int_{A^\iota}\Big|\int_{A^\iota}\psi(\eta\,v^*w)\,f(w)\,dw\Big|^2dv\ \le\ \Big(\frac{\mathrm{kb}_A}{|\eta|}\Big)^{|\iota|}\int_{A^\iota}|f(w)|^2\,dw,$$

in $[0,\infty]$ with $0\cdot\infty=0$.

Proof. For a unit u we have $|u|\,|u^{-1}| = |1| = 1$, so $0 < |u| < \infty$. For $\xi \in A$ and $v \in A^\iota$ put $\mathcal{F}_\xi f(v) = \int_{A^\iota} \psi(\xi\, v^* w) f(w)\, dw$; the integrand is bounded by $|f|$, so $\mathcal{F}_\xi f$ is defined and, by dominated convergence, continuous in v. Let G be as in (c) and put $\widehat{G}_\xi(a) = \int_A G(v)\psi(\bar{v}\xi a)\, dv$ for $a \in A$.

Step 1: the damped inequality. For every $\xi \in A$,

$$\int_{A^\iota} |\mathcal{F}_\xi f(v)|^2 \prod_{i\in\iota} G(v_i)\, dv \le \Big(\int_A |\widehat{G}_\xi(a)|\, da\Big)^{|\iota|} \int_{A^\iota} |f|^2. \tag{5.90}$$

Put $g(v,w,w') = \psi(\xi\, v^*(w'-w)) \prod_i G(v_i)\, \overline{f(w)} f(w')$ on $(A^\iota)^3$. Since $|\psi| = 1$, $|g|$ is the product of $\prod_i G(v_i) \in L^1(A^\iota)$ and $|f(w)||f(w')| \in L^1(A^\iota \times A^\iota)$, so g is integrable. Writing $|\mathcal{F}_\xi f(v)|^2 = \overline{\mathcal{F}_\xi f(v)}\, \mathcal{F}_\xi f(v)$ and using $\overline{\psi(x)} = \psi(-x)$ and additivity of ψ,

$$\Phi(v) := |\mathcal{F}_\xi f(v)|^2 \prod_i G(v_i) = \iint g(v,w,w')\, dw\, dw' \ \ge 0.$$

By Fubini, Φ is integrable and $\int \Phi\, dv = \iint \big(\int g(v,w,w')\, dv\big) dw\, dw'$; as $\int \Phi \ge 0$ this gives $\int \Phi \le \iint \big|\int g(v,w,w')\, dv\big|\, dw\, dw'$. Since $\psi(\xi v^* z) = \prod_i \psi(\bar{v}_i \xi z_i)$, Fubini on the product A^ι gives

$$\int_{A^\iota} \psi(\xi v^* z) \prod_i G(v_i)\, dv = \prod_{i\in\iota} \widehat{G}_\xi(z_i), \qquad z \in A^\iota,$$

so $\big|\int g(v,w,w')\, dv\big| = \prod_i |\widehat{G}_\xi(w_i' - w_i)|\, |f(w)|\, |f(w')|$. The map $(w,z) \mapsto (w, w+z)$ preserves the Haar measure of $A^\iota \times A^\iota$, and by Cauchy–Schwarz and translation invariance $\int |f(w)||f(w+z)|\, dw \le \int |f|^2$ for every z. Hence, by Tonelli,

$$\int \Phi \le \int_{A^\iota} \prod_i |\widehat{G}_\xi(z_i)| \int_{A^\iota} |f(w)||f(w+z)|\, dw\, dz \le \Big(\int_A |\widehat{G}_\xi|\Big)^{|\iota|} \int |f|^2.$$

Step 2: the damped integral at a unit. For every unit u of A,

$$\int_{A^\iota} |\mathcal{F}_\eta f(v)|^2 \prod_i G(\bar{u} v_i)\, dv \le \Big(\frac{\mathrm{kb}_A}{|\eta|}\Big)^{|\iota|} \int |f|^2. \tag{5.91}$$

By (a) applied coordinatewise, the map $v' \mapsto \bar{u}^{-1} v'$ carries the measure of A^ι to $|\bar{u}^{-1}|^{-|\iota|} dv = |u|^{|\iota|} dv$. Since $(\bar{u}^{-1}v')^* w = u^{-1} v'^* w$, we get $\mathcal{F}_\eta f(\bar{u}^{-1}v') = \mathcal{F}_{\eta u^{-1}} f(v')$, and therefore the left side of (5.91) equals

$$|u|^{-|\iota|} \int_{A^\iota} |\mathcal{F}_{\eta u^{-1}} f(v')|^2 \prod_i G(v_i')\, dv' \le |u|^{-|\iota|} \Big(\int_A |\widehat{G}_{\eta u^{-1}}(a)|\, da\Big)^{|\iota|} \int |f|^2$$

by (5.90). Now $\widehat{G}_{\eta u^{-1}}(a) = \widehat{G}_1(\eta u^{-1} a)$, and by (a) for the unit ηu^{-1} and (c),

$$\int_A |\widehat{G}_1(\eta u^{-1} a)|\, da = |\eta u^{-1}|^{-1} \int_A |\widehat{G}_1(z)|\, dz \le |u|\, |\eta|^{-1} \mathrm{kb}_A.$$

Inserting this, the powers of $|u|$ cancel and (5.91) follows.

Step 3: removing the damping. By (b) the non-units form a null set; a Haar measure gives positive mass to every nonempty open set, so every neighbourhood of 0 contains a unit. As A is metrizable, there are units $u_k \to 0$. For each v, continuity of G, of the involution and of multiplication give $\prod_i G(\bar{u}_k v_i) \to G(0)^{|\iota|} = 1$, hence $|\mathcal{F}_\eta f(v)|^2 = \liminf_k |\mathcal{F}_\eta f(v)|^2 \prod_i G(\bar{u}_k v_i)$. The functions on the right are nonnegative and measurable, so Fatou's lemma and (5.91) give

$$\int_{A^\iota} |\mathcal{F}_\eta f|^2 \le \liminf_k \int_{A^\iota} |\mathcal{F}_\eta f(v)|^2 \prod_i G(\bar{u}_k v_i)\, dv \le \Big(\frac{\mathrm{kb}_A}{|\eta|}\Big)^{|\iota|} \int |f|^2.$$

□

Lemma 5.4.33. *Let a place datum be given and suppose that the group measure θ is a left Haar measure of G. Let ρ, μ be s-finite Borel measures on G with $\theta = \rho * \mu$, let S be a closed subgroup of G with $\rho(G \setminus S) = 0$ and $\rho * \delta_t = \rho$ for every $t \in S$, and let $s \in G$ with $\delta_s * \rho = \rho * \delta_s$. Then for every measurable $\Phi\colon G \to [0,\infty]$ with $\Phi(tg) = \Phi(g)$ for all $t \in S$ and $g \in G$,*

$$\int_G \Phi(sg)\, d\mu(g) = \int_G \Phi\, d\mu.$$

Proof. G is a closed subset of a finite product of copies of the locally compact Polish space A, hence a second countable locally compact group, and its Haar measure θ is σ-finite and not zero. Choose a measurable $f\colon G \to (0,\infty)$ with $\int f\, d\theta < \infty$, and put $\mathrm{av}(g) = \int f(yg)\, d\rho(y) \in [0,\infty]$, a measurable function by Tonelli's theorem.

Properties of av. For $t \in S$, $\mathrm{av}(tg) = \int f(ytg)\, d\rho(y) = \int f(yg)\, d(\rho * \delta_t)(y) = \mathrm{av}(g)$. Since $\theta = \rho * \mu \neq 0$, $\rho \neq 0$, and as $f > 0$, $\mathrm{av}(g) > 0$ for every g.

A formula for μ. Let μ' be any s-finite measure with $\theta = \rho * \mu'$. Then $\int \mathrm{av}\, d\mu' = \int f\, d\theta < \infty$, so $\mathrm{av} < \infty$ μ'-almost everywhere, and by Tonelli's theorem, using $\rho(G \setminus S) = 0$ and the S-invariance of Φ and av,

$$\int_G \frac{\Phi f}{\mathrm{av}}\, d\theta = \int_G \int_G \frac{\Phi(yg)\, f(yg)}{\mathrm{av}(yg)}\, d\rho(y)\, d\mu'(g) = \int_G \frac{\Phi(g)}{\mathrm{av}(g)}\, \mathrm{av}(g)\, d\mu'(g) = \int_G \Phi\, d\mu'.$$

The left side does not depend on μ'.

Conclusion. The measure $\delta_s * \mu$ is s-finite and, by associativity of convolution, $\rho * (\delta_s * \mu) = (\rho * \delta_s) * \mu = (\delta_s * \rho) * \mu = \delta_s * \theta = \theta$, the last equality by left invariance of θ. Applying the formula to $\mu' = \delta_s * \mu$ and to $\mu' = \mu$ gives $\int \Phi(sg)\, d\mu(g) = \int \Phi\, d(\delta_s * \mu) = \int \Phi\, d\mu$. □

Proposition 5.4.34. *Let Γ be a gauge of loss Λ for a place datum, with size filtration $(A_{(i)})$, and suppose that the group measure θ is a Haar measure of G; put $r' = |\iota'|$. Let $W\colon G \to \mathbb{C}^I$ be cut, $j \le \varrho$, p a head of length j, q a sequence of units and $x \in \mathcal{P}_{\mathrm{pos}}$. If ξ is a unit, $\eta \in A$ and $\psi(\eta z) = \psi(\xi z)$ for every $z \in A_{(200n^2)}$, then*

$$\mathrm{FN}_j(p,q,W,x;\, g \mapsto \psi(\eta\, h(f_j, gf_j)))^2 \le \Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'} \mathrm{WSP}_j(p,q,W,x),$$

with WSP_j *formed for* Γ.

Proof. Write $w_p = w_{j,p}$, $d = w_{j,q}^{-1} w_p$, $\omega(g) = \psi(\eta\, h(f_j, gf_j))$, $\mathcal{K}_1 = \mathcal{K}_W$, $\mathcal{K}_2 = \mathcal{K}_W^d$ (pairing kernels at floor j, Lemma 5.2.57 (ii), hence at floor $j+1$, Lemma 5.2.64 (iv)). By Lemma 5.4.6 (ii) with $C = (\mathrm{kb}_A/|\xi|)^{r'}$ and $f(g) = |\mathrm{FI}_j(p,q,W,x;\omega)(g)|$ (so that $\mathrm{FN}_j = \int f\, d\mathfrak{m}_j$), and by Lemma 5.4.16 (iv) at g and at $z = w_p^{-1} g$, it suffices to show for every g with $f(g) \neq 0$:

$$\begin{aligned} f(g)^2 &\le \Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'} c_{\mathrm{fls}} \int_G |F_1|^2\, d\theta_j \int_G |F_2|^2\, d\theta_j, \\ F_1(\sigma) &= \mathcal{K}_{1,j+1}(\sigma g, \sigma g x),\ F_2(\tau) = \mathcal{K}_{2,j+1}(\tau z, \tau z x). \end{aligned} \tag{5.92}$$

Fix such a g. Since σ, τ are isometries, $\omega(\sigma\tau^{-1}) = \psi(\eta\, h(\sigma^{-1} f_j, \tau^{-1} f_j))$, so by (5.14)

$$\mathrm{FI}_j(p,q,W,x;\omega)(g) = \int_G \int_G \psi(\eta\, h(\sigma^{-1} f_j, \tau^{-1} f_j))\, F_1(\sigma)\, \overline{F_2(\tau)}\, d\theta_j(\tau)\, d\theta_j(\sigma),$$

each integral being 0 where its integrand is not integrable.

A common translate of bounded size. As the double integral is not 0 and θ_j is carried by $G^{(j)}$ (Lemma 5.2.59 (ii)), there are $\sigma_0, \tau_0 \in G^{(j)}$ with $F_1(\sigma_0) \neq 0 \neq F_2(\tau_0)$. By (PK5) at $j+1$, $\sigma_0 g$ and $\tau_0 z$ are near at $j+1$, hence g and $z = w_p^{-1} g$ are near at j (as $\sigma_0, \tau_0 \in G^{(j)} \subseteq R^{(j)}$ and $R^{(j+1)} \subseteq R^{(j)}$, Lemma 5.2.58 (i)). By Lemma 5.4.12 (iii) there is $t \in G^{(j)}$ with tg of size $60n^2$ and tz of size $99n^2$.

Supports and the phase. By Lemma 5.4.30 (i) (floor $b = j$, $q = t$), for $\sigma, \tau \in G^{(j)}$: if $F_1(\sigma) \neq 0$ then $t\sigma^{-1} f_j$ has size $60n^2+6$, and if $F_2(\tau) \neq 0$ then $t\tau^{-1} f_j$ has size $99n^2+6$. For such σ, τ, $h(\sigma^{-1} f_j, \tau^{-1} f_j) = h(t\sigma^{-1} f_j, t\tau^{-1} f_j)$ has size $159n^2+15 \leq 200n^2$ (a pairing of vectors of sizes i, i' has size $i+i'+3$), so the hypothesis on η gives $\psi(\eta\, h(\sigma^{-1} f_j, \tau^{-1} f_j)) = \psi(\xi\, h(t\sigma^{-1} f_j, t\tau^{-1} f_j))$. Put, for $v \in V$,

$$G_2(v) = \int_G \psi(-\xi\, h(v, t\tau^{-1} f_j)) F_2(\tau)\, d\theta_j(\tau) \qquad \text{(0 if the integrand is not integrable).}$$

For θ_j-almost every σ with $F_1(\sigma) \neq 0$, the integrands of $\int \psi(\eta h(\sigma^{-1} f_j, \tau^{-1} f_j)) F_1(\sigma) \overline{F_2(\tau)}\, d\theta_j(\tau)$ and of $F_1(\sigma) \overline{G_2(t\sigma^{-1} f_j)}$ (as $\overline{\psi(a)} = \psi(-a)$) agree for θ_j-almost every τ and have the same modulus $|F_1(\sigma) F_2(\tau)|$; so the inner integral equals $F_1(\sigma) \overline{G_2(t\sigma^{-1} f_j)}$, and both vanish where $F_1(\sigma) = 0$. With $S = \{\sigma : t\sigma^{-1} f_j \text{ has size } 60n^2 + 6\}$, which contains almost every σ with $F_1(\sigma) \neq 0$, and the Cauchy–Schwarz inequality,

$$f(g)^2 \leq \Big(\int_S |F_1(\sigma)|\, |G_2(t\sigma^{-1} f_j)|\, d\theta_j(\sigma) \Big)^2 \leq \int_G |F_1|^2\, d\theta_j \cdot \int_S |G_2(t\sigma^{-1} f_j)|^2\, d\theta_j(\sigma).$$

Plancherel. By Lemma 5.2.63, the image of θ_j under $\sigma \mapsto t\sigma^{-1} f_j$ is qv_j, so the last integral is $\int_{\{v \text{ of size } 60n^2+6\}} |G_2|^2\, d\mathrm{qv}_j$. By Lemma 5.4.30 (ii) for the kernel $\mathcal{K}_2$ at floor j, t, $q = t$, the points z, zx (with tz of size $99n^2$), the unit $-\xi$ ($|-\xi| = |\xi|$ by (PA2)), $v_0 = 0$, $i = 60n^2 + 6$ and $\rho = \mathrm{qv}_j$, it is at most

$$\Lambda^{99n(60n^2+12)} \Lambda^{99n(99n^2+12)} \Big(\frac{\mathrm{kb}_A}{|\xi|} \Big)^{r'} \int_G |F_2|^2\, d\theta_j \leq c_{\mathrm{fls}} \Big(\frac{\mathrm{kb}_A}{|\xi|} \Big)^{r'} \int_G |F_2|^2\, d\theta_j,$$

since $99n(159n^2 + 24) \leq 10^6 n^3$. This proves (5.92). □

Proposition 5.4.35. *Let Γ be a gauge of loss Λ for a place datum, and suppose that the group measure θ is a Haar measure of G; put $r' = |\iota'|$. Let $W \colon G \to \mathbb{C}^I$ be cut, $j = i+1 \leq \varrho$, p a head of length j, q a sequence of units and $x \in \mathcal{P}_{\mathrm{pos}}$. If $\xi = M^{-1}(p_i^{-1} - q_i^{-1})$ is a unit, then*

$$\mathrm{FN}_j(p, q, W, x; 1)^2 \leq \Big(\frac{\mathrm{kb}_A}{|\xi|} \Big)^{r'} \mathrm{WSP}_j(p, q, W, x),$$

with WSP_j formed for Γ.

Proof. Put $w = w_{j,p}$, $d = w_{j,q}^{-1} w$, $\mathcal{K}_1 = \mathcal{K}_W$, $\mathcal{K}_2 = \mathcal{K}_W^d$ (pairing kernels at floor i and at floor j, Lemma 5.2.57 (ii)), $\Theta = \theta_i * \cdots * \theta_0$, $C = (\mathrm{kb}_A/|\xi|)^{r'}$, $\varphi_1(g) = |\mathcal{K}_{1,i+1}(g, gx)|$ and $\varphi_2(g) = |\mathcal{K}_{2,i+1}(g, gx)|$, measurable by Lemma 5.2.64 (i). By Lemma 5.2.57 (iii), $|\mathrm{FI}_j(p, q, W, x; 1)(g)| = \varphi_1(g) \varphi_2(w^{-1} g)$, so $\mathrm{FN}_j(p, q, W, x; 1) = \int \varphi_1(g) \varphi_2(w^{-1} g)\, d\mathfrak{m}_{i+1}(g)$. We apply Lemma 5.4.36 with $B = \mathrm{WSS}_{i+1}(\mathcal{K}_2, x)$.

Invariance and support. $\mathrm{dpw}_i(g, x) \varphi_1(g)^2 = \mathrm{WSq}_i(\mathcal{K}_1, x)(g)$ and B are invariant under left multiplication by $R^{(i+1)}$ and vanish off the points near at $i+1$, by Lemma 5.4.16 (iii) (at the floors i and $i+1$). For $\sigma \in G^{(i+1)} \subseteq R^{(i+1)} \cap G^{(j)}$, $\varphi_2(\sigma g) = \varphi_2(g)$ by Lemma 5.2.64 (v), and $\varphi_2(g) \neq 0$ forces g near at $i+1$ by (PK5). All three are measurable (Lemma 5.4.16 (ii)).

The section bound. For $g \in G$, Lemma 5.4.37 with $y = gx$, $C = 1$ and $\Phi(n) = \mathbf{1}\{wng \text{ near at } i+1\}\varphi_2(ng)^2$ (note $\varphi_2(ng) = |\mathcal{K}_{2,i+1}(ng, n(gx))|$), followed by Lemma 5.4.16 (iv) at $i+1 \le \varrho$, gives

$$\int_{\{n:\, wng \text{ near at } i+1\}} \varphi_2(ng)^2 \, d\eta_i(n) \le C \,\mathrm{fm}(i, p_i^{-1})\, c_{\mathrm{fls}} \int |\mathcal{K}_{2,i+2}(\sigma g, \sigma g x)|^2 \, d\theta_{i+1}(\sigma)$$
$$\le C \,\mathrm{fm}(i, p_i^{-1})\, c_{\mathrm{fls}} \frac{B(g)}{\mathrm{dpw}_{i+1}(g,x)}.$$

Conclusion. Lemma 5.4.36 gives

$$\mathrm{FN}_j(p,q,W,x;1)^2 \le C\Big(\prod_{a<j} \mathrm{fm}(a, p_a^{-1})\Big)^2 c_{\mathrm{fls}}^3 \int \mathrm{WSq}_i(\mathcal{K}_1, x)\, d\Theta \int \mathrm{WSS}_{i+1}(\mathcal{K}_2, x)\, d\Theta.$$

By Lemma 5.4.16 (vi),

$$\int \mathrm{WSq}_i(\mathcal{K}_1, x)\, d\Theta \le \mathrm{WSI}_i(\mathcal{K}_1, x) \le \mathrm{WSI}_0(\mathcal{K}_1, x),$$

and $\int \mathrm{WSS}_{i+1}(\mathcal{K}_2, x)\, d\Theta = \mathrm{WSI}_j(\mathcal{K}_2, x)$ by definition; so the right side is at most $C\,\mathrm{WSP}_j(p,q,W,x)$. □

Lemma 5.4.36. *Let Γ be a gauge of loss Λ for a place datum, and suppose that the group measure θ is a Haar measure of G. Let $x \in \mathcal{P}_{\mathrm{pos}}$, $i+1 \le \varrho$, p a head of length $i+1$, $w = w_{i+1,p}$, $\Theta = \theta_i * \cdots * \theta_0$, $C \in [0,\infty]$, and let $\varphi_1, \varphi_2, B \colon G \to [0,\infty]$ be measurable with:*

- *$g \mapsto \mathrm{dpw}_i(g,x)\varphi_1(g)^2$ and B are invariant under left multiplication by $R^{(i+1)}$, φ_2 is invariant under left multiplication by $G^{(i+1)}$, and all three vanish at the points that are not near at $i+1$;*
- *for every $g \in G$,*

$$\int_{\{n:\ wng \text{ near at } i+1\}} \varphi_2(ng)^2 \, d\eta_i(n) \le C \,\mathrm{fm}(i, p_i^{-1})\, c_{\mathrm{fls}} \, \frac{B(g)}{\mathrm{dpw}_{i+1}(g,x)}.$$

Then, with arithmetic in $[0,\infty]$,

$$\Big(\int_G \varphi_1(g)\, \varphi_2(w^{-1}g)\, d\mathfrak{m}_{i+1}(g)\Big)^2 \le C\Big(\prod_{a\le i} \mathrm{fm}(a, p_a^{-1})\Big)^2 c_{\mathrm{fls}}^3 \int_G \mathrm{dpw}_i(g,x)\varphi_1(g)^2\, d\Theta(g) \int_G B\, d\Theta.$$

Proof. Write $\mathrm{dpw}_l^- = \mathrm{dpw}_{l,x}^-$, $w' = w_{i,p}$ and N_l for the closed set of points near at l. All functions below are measurable (Lemma 5.4.8 (iii)), and the measures are s-finite.

Two facts on Weyl elements. (1) $w = w' w_i(p_i)$, where the floor Weyl element $w_i(p_i)$ fixes the coordinate vectors of floor below i; so $w'w^{-1} = w_i(p_i)^{-1} \in G^{(i)} \subseteq R^{(i)}$, and for $g \in N_{i+1} \subseteq N_i$, by Lemma 5.4.8 (iii), $\mathrm{dpw}_i^-(w'w^{-1}g) = \mathrm{dpw}_i^-(g) = \mathrm{dpw}_i(g,x)^{-1}$. (2) $w^{-1} = w_{i+1,\bar{p}}$ with $\bar{p} = (\bar{p}_a)_a$, a head of length $i+1$ with $\mathrm{fm}(a, \bar{p}_a^{-1}) = \mathrm{fm}(a, p_a^{-1})$ (SZ3), (PA2). Elements of $G^{(i+1)}$ (resp. $G^{(i)}$) commute with w (resp. w') by Lemma 5.2.58 (ix).

Crossing the weights. Put $r(g) = \mathrm{dpw}_{i+1}(g,x)\,\mathrm{dpw}_i^-(w'g)$ and $u = \mathbf{1}_{\{g:\, wg \in N_{i+1}\}}\, r\, \varphi_2^2$. We claim for every g

$$\big(\varphi_1(g)\varphi_2(w^{-1}g)\big)^2 \le \big[\mathrm{dpw}_i(g,x)\varphi_1(g)^2\, \mathrm{dpw}_{i+1}^-(w^{-1}g)\big] \cdot u(w^{-1}g).$$

If $g \notin N_{i+1}$, then $\mathrm{dpw}_i(g,x)\varphi_1(g)^2 = 0$ and $0 < \mathrm{dpw}_i < \infty$, so $\varphi_1(g) = 0$. If $w^{-1}g \notin N_{i+1}$, then $\varphi_2(w^{-1}g) = 0$. Otherwise $\mathrm{dpw}^-_{i+1}(w^{-1}g) = \mathrm{dpw}_{i+1}(w^{-1}g,x)^{-1}$, and by (1)

$$u(w^{-1}g) = \mathrm{dpw}_{i+1}(w^{-1}g,x)\,\mathrm{dpw}_i(g,x)^{-1}\varphi_2(w^{-1}g)^2;$$

the right side is then $\varphi_1(g)^2\varphi_2(w^{-1}g)^2$. By the Cauchy–Schwarz inequality the square of the integral is at most $I_1 I_2$, with

$$I_1 = \int \mathrm{dpw}_i(g,x)\varphi_1(g)^2 \mathrm{dpw}^-_{i+1}(w^{-1}g)\,d\mathfrak{m}_{i+1}(g) \quad \text{and} \quad I_2 = \int u(w^{-1}g)\,d\mathfrak{m}_{i+1}(g).$$

By Lemma 5.2.59 (iv) and Lemma 5.4.14 (ii), $\mathfrak{m}_{i+1} = \eta_{<i+1} * \Theta = \eta_i * \eta_{<i} * \Theta$.

The first integral. By (2) and Lemma 5.4.7 (iv) (length and b equal to $i+1$, head $\bar{p}$, $\rho = \Theta$, the function $\mathrm{dpw}_i(\cdot,x)\varphi_1^2$), $I_1 \le (\prod_{a\le i} \mathrm{fm}(a,p_a^{-1}))c_{\mathrm{fls}} \int \mathrm{dpw}_i(g,x)\varphi_1(g)^2\,d\Theta(g)$.

The second integral. u is invariant under left multiplication by $G^{(i+1)}$: for $\sigma \in G^{(i+1)} \subseteq R^{(i+1)}$, $w\sigma g = \sigma w g$ and N_{i+1} is $R^{(i+1)}$-invariant; $\mathrm{dpw}_{i+1}(\sigma g,x) = \mathrm{dpw}_{i+1}(g,x)$; $\mathrm{dpw}^-_i(w'\sigma g) = \mathrm{dpw}^-_i(\sigma w' g) = \mathrm{dpw}^-_i(w'g)$; and $\varphi_2(\sigma g) = \varphi_2(g)$. By Lemma 5.4.15 (ii) and the factorisation of $\mathfrak{m}_{i+1}$,

$$I_2 = \int u\,d\mathfrak{m}_{i+1} = \int_G \int_G u(ng)\,d\eta_i(n)\,d(\eta_{<i} * \Theta)(g).$$

η_i is carried by the Heisenberg elements n of floor i, which lie in $G^{(i)} \cap N_{<i+1}$ (Lemma 5.2.58 (iii)); for them $r(ng) = r(g)$ (as $n \in R^{(i+1)}$, and $w'n = nw'$ with $n \in R^{(i)}$). Hence, by the hypothesis on φ_2,

$$\int u(ng)\,d\eta_i(n) = r(g)\int_{\{n:\, wng \in N_{i+1}\}} \varphi_2(ng)^2\,d\eta_i(n) \le C\,\mathrm{fm}(i,p_i^{-1})\,c_{\mathrm{fls}}\,B(g)\,\mathrm{dpw}^-_i(w'g),$$

using $0 < \mathrm{dpw}_{i+1}(g,x) < \infty$. By Lemma 5.4.7 (iv) (length i, $b = i+1$, the head p of length i, $\rho = \Theta$, the function B),

$$I_2 \le C\,\mathrm{fm}(i,p_i^{-1})\,c_{\mathrm{fls}}\Big(\prod_{a<i} \mathrm{fm}(a,p_a^{-1})\Big)c_{\mathrm{fls}} \int B\,d\Theta.$$

Multiplying the bounds for I_1 and I_2 gives the claim. □

Lemma 5.4.37. *Let Γ be a gauge of loss Λ for a place datum, put $r' = |\iota'|$, let $W\colon G \to \mathbb{C}^I$ be cut, $i+1 \le \varrho$, p a head of length $i+1$, q a sequence of units, $w_p = w_{i+1,p}$, $d = w^{-1}_{i+1,q}w_p$, and let $\xi = M^{-1}(p_i^{-1} - q_i^{-1})$ be a unit. Let $g, y \in G$, $C \in [0,\infty]$ and $\Phi\colon G \to [0,\infty]$ Borel such that for every Heisenberg element n of floor i with $\Phi(n) \ne 0$, $w_p ng$ is near at $i+1$ and $\Phi(n) \le C\,|\mathcal{K}^d_{W,i+1}(ng,ny)|^2$. Then*

$$\int_G \Phi\,d\eta_i \le C\Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'} \mathrm{fm}(i,p_i^{-1})\,c_{\mathrm{fls}} \int_G |\mathcal{K}^d_{W,i+2}(\sigma g,\sigma y)|^2\,d\theta_{i+1}(\sigma).$$

Proof. Write $\mathcal{K} = \mathcal{K}^d_W$, a pairing kernel at floor $i+1$ for d (Lemma 5.2.57 (ii)), and $E_{<i} = \mathrm{span}\{e_b : b < i\}$. We apply Proposition 5.4.28 with $a = i$, $\varepsilon_0 = 1$, the units $\xi' = -\xi p_i$ and $u = p_i^{-1}$, the points g, y, the size indices $99n^2$ (for i there) and $999n^2$ (for j there), the constant C, the function Φ, and

$$S = \{n \text{ a Heisenberg element of floor } i :\ w_p ng \text{ and } ng \text{ are near at } i+1\}.$$

The frequency. By Lemma 5.4.29 (ii) with $j = i+1$, $a = i < j$ and $\varepsilon_0 = 1$, the hypothesis of Proposition 5.4.28 holds with $M^{-1}(p_i q_i^{-1} - 1)$, and $M^{-1}(p_i q_i^{-1} - 1) = -M^{-1}(p_i^{-1} - q_i^{-1})p_i = \xi'$.

Hypothesis (a). If $\Phi(n) \neq 0$, then $\mathcal{K}_{i+1}(ng, ny) \neq 0$, so ng is near at $i+1$ by (PK5); with the hypothesis, $n \in S$, and $\Phi(n) \leq C|\mathcal{K}_{i+1}(n^1 g, ny)|^2$.

Sizes on S. Let $n \in S$. As $w_p ng$ and $w_p^{-1}(w_p ng) = ng$ are near at $i+1$, Lemma 5.4.12 (iii) (length $i+1$, head p, point $w_p ng$) gives $t_n \in G^{(i+1)}$ with $t_n ng$ of size $99n^2$, and Lemma 5.4.12 (i) gives $p_i \in A_{(10n)}$.

Hypothesis (b). Let $n_0 \in S$ and $t_0 = t_{n_0}$; then $t_0 \in G^{(i+1)} \subseteq R^{(i+1)} \cap G^{(i)}$, $n_0 \in N_{<i+1}$ (Lemma 5.2.58 (iii)), and $t_0 n_0 g$ has size $99n^2$. Let $n \in S$, $t = t_n$, and $v = (t_0 n_0 (t_0 n)^{-1} - 1)e_i^*$; we take $w = v$, so $w - v = 0 \in E_{<i}$. As t^{-1} and t_0^{-1} lie in $G^{(i+1)}$, they fix e_i^* (floor i), so

$$v = (t_0 n_0 n^{-1} t_0^{-1} - 1)e_i^* = (t_0 n_0 n^{-1} - 1)e_i^* = \big((t_0 n_0 g)(tng)^{-1} - 1\big)e_i^*.$$

The matrix $(t_0 n_0 g)(tng)^{-1} - 1$ has size $99n^2 + (99n^2 + 4) + 1 + 1$ (inverses of elements of G add 4, products add 1, a difference adds 1), so v has size $198n^2 + 6$ and $u^{-1}v = p_i v$ has size $10n + 198n^2 + 6 \leq 999n^2$.

Conclusion. Proposition 5.4.28 gives, with $|\xi' u| = |-\xi| = |\xi|$ (PA2),

$$\int_G \Phi \, d\eta_i \leq C\Big(\frac{\mathrm{kb}_A}{|\xi|}\Big)^{r'} \mathrm{fm}(i, p_i^{-1}) \Lambda^{100n(99n^2+999n^2+31)} \int_G |\mathcal{K}_{i+2}(\sigma g, \sigma y)|^2 \, d\theta_{i+1}(\sigma),$$

and $100n(1098n^2 + 31) \leq 10^6 n^3$. □

Proposition 5.4.38. *Let Γ and S be gauges for a place datum, with size filtrations $(A_{(i)})$ and $(A^S_{(i)})$; sizes refer to Γ unless S is named. Let $j \leq \varrho$ and let p, q be heads of length j for S; if $j \neq 0$, assume $M^{-1} \in A^S_{(1)}$ and $A^S_{(10n)} \subseteq A_{(1)}$. Let $x, y \in G$ and $u, v \in \mathsf{U}$ with $ux \in K$, $vxy \in K$ and $M^{-1}(vxy - ux)$ of size 2, and let $w_{j,p}^{-1}x$ and $w_{j,q}^{-1}xy$ be near at j. If for every a with $j \leq a < \varrho$ the set $\mathfrak{d}_a(x, y)$ contains a unit c with $c^{-1} \in A_{(1)}$, then $M^{-1}(y-1)$ has size $99n$.*

Proof. Write $E_{<l} = \mathrm{span}\{e_b : b < l\}$. We use the size rules of Lemma 5.4.39 (a matrix of size i maps vectors of size i' to vectors of size $i + i' + 1$; inverses of elements of G of size i have size $i + 4$), for either size filtration. Elements of K have size 1 for Γ and for S (GU2), so their inverses have size 5.

Step 1: every $\mathfrak{d}_a(x, y)$, $a < \varrho$, contains a unit c with $c^{-1} \in A_{(1)}$. For $a \geq j$ this is the hypothesis. Let $a < j$; then $j \neq 0$. Let (z, r, s) be (x, p, u) or (xy, q, v). By Lemma 5.2.58 (i), $\mathsf{U} \subseteq R^{(j)}$, so $s \in R^{(j)}$ with $sz \in K$, and $w_{j,r}^{-1}z$ is near at j; by Lemma 5.4.12 (i) for the gauge S and the head r, the column se_a^* has S-size $8\varrho + 1$. Hence $z^{-1}e_a^* = (sz)^{-1}(se_a^*)$ has S-size $8\varrho + 7$. As $M^{-1} \in A^S_{(1)}$, (GU4) for S gives a unit c with $c^{-1} \in A^S_{(8\varrho+9)}$ and $(c\omega_1, c\omega_2) \in \mathcal{P}_{\mathrm{cl}}$ for all ω_1, ω_2 of S-size $8\varrho + 7$. For $\omega = ce_a^*$ we get $(x^{-1}\omega, (xy)^{-1}\omega) \in \mathcal{P}_{\mathrm{cl}}$, $h(e_b, \omega) = 0$ for $b < a$ and $h(e_a, \omega) = c$; so $c \in \mathfrak{d}_a(x, y)$ (with $\omega' = \omega$). Finally $c^{-1} \in A^S_{(8\varrho+9)} \subseteq A^S_{(10n)} \subseteq A_{(1)}$.

Step 2: every e_a, $a < \varrho$, is 5-balanced for (x, xy) modulo M (in the sense of Lemma 5.4.39; $M \in A_{(1)}$ by (GU2)). *Lifts.* Put $f = u^{-1}e_a$ and $f' = v^{-1}e_a$. Since $u^{-1} \in \mathsf{U}$, its column at e_a agrees with that of 1 in every row of level at least a, and the indices of level below a are the e_b, $b < a$; so $f \in e_a + E_{<a}$, and likewise f'. Moreover $x^{-1}f = (ux)^{-1}e_a$ and $(xy)^{-1}f' = (vxy)^{-1}e_a$ are columns of inverses of elements of K, of size 5. *Dual.* By Step 1 take a unit $c \in \mathfrak{d}_a(x, y)$ with $c^{-1} \in A_{(1)}$: $c = h(e_a, \omega)$ with $h(e_b, \omega) = 0$ for $b < a$ and $(x^{-1}\omega, (xy)^{-1}\omega') \in \mathcal{P}_{\mathrm{cl}}$ for some $\omega' \in \omega + E_{<a}$. By (GU3), $x^{-1}\omega$ and $M^{-1}(x^{-1}\omega - (xy)^{-1}\omega')$

have size 1. Put $\zeta = c^{-1}\omega$ and $\zeta' = c^{-1}\omega' \in \zeta + E_{<a}$: then $h(e_b, \zeta) = 0$ for $b < a$, $h(e_a, \zeta) = 1$, and $x^{-1}\zeta$, $M^{-1}(x^{-1}\zeta - (xy)^{-1}\zeta')$ have size $2 \le 5$ (SZ1).

Step 3: the lengths at x and xy differ by at most 9 modulo M from ϱ on. Let $\omega \in V$ with $h(e_b, \omega) = 0$ for $b < \varrho$, and $x^{-1}\omega$ of size i. Put $\omega' = v^{-1}u\omega$. As $v^{-1}u \in \mathsf{U}$, the row of $v^{-1}u$ at an index k of level at least ϱ agrees with that of 1 except at columns of level above that of k, which are among the e_b^*; and the e_b^*-coordinate of ω is $h(e_b, \omega) = 0$. So ω' and ω agree at all indices of level at least ϱ, that is, $\omega' - \omega \in E_{<\varrho}$. Moreover

$$M^{-1}(x^{-1}\omega - (xy)^{-1}\omega') = (vxy)^{-1}\, M^{-1}(vxy - ux)\, x^{-1}\omega,$$

since $(vxy)^{-1}(vxy - ux)x^{-1}\omega = x^{-1}\omega - (vxy)^{-1}u\omega$ and $(vxy)^{-1}u = (xy)^{-1}v^{-1}u$. The matrix $(vxy)^{-1}M^{-1}(vxy - ux)$ has size $5 + 2 + 1 = 8$, so the vector has size $i + 9$.

Step 4: conclusion. By Lemma 5.4.39 (ii) with $g = x$, $g' = xy$, the index ϱ, $c = 5$ and $t = 9$ (Steps 2 and 3), the lengths at x and xy differ by at most $9 + 32\varrho$ modulo M from 0 on; by Lemma 5.4.39 (iii), $M^{-1}(y - 1)$ has size $13 + 32\varrho \le 99n$. □

Lemma 5.4.39. *Let a place datum be given, let $(A_{(i)})_{i\ge 0}$ be a size filtration, assume that the level unit M lies in $A_{(1)}$, and let $g, g' \in G$; write $E_{<l} = \operatorname{span}\{e_b : b < l\}$. For $0 \le l \le \varrho$ and $t \ge 0$ say that the* lengths at g and g' differ by at most t modulo M from l on *if for every $i \ge 0$ and every $\omega \in V$ with $h(e_b, \omega) = 0$ for all $b < l$ and $g^{-1}\omega$ of size i there is $\omega' \in \omega + E_{<l}$ such that $M^{-1}(g^{-1}\omega - g'^{-1}\omega')$ has size $i + t$. For $a < \varrho$ and $c \ge 0$ say that e_a is* c-balanced for (g, g') modulo M *if*

- *there are $f, f' \in e_a + E_{<a}$ such that $g^{-1}f$ and $g'^{-1}f'$ have size c, and*
- *there is $\zeta \in V$ with $h(e_b, \zeta) = 0$ for $b < a$, $h(e_a, \zeta) = 1$ and $g^{-1}\zeta$ of size c, and some $\zeta' \in \zeta + E_{<a}$ such that $M^{-1}(g^{-1}\zeta - g'^{-1}\zeta')$ has size c.*

(i) Let $a < \varrho$, $c, t \ge 0$, and let e_a be c-balanced for (g, g') modulo M. If the lengths at g and g' differ by at most t modulo M from $a + 1$ on, they differ by at most $t + 4c + 12$ modulo M from a on.

(ii) Let $j \le \varrho$, $c, t \ge 0$, and let e_a be c-balanced for (g, g') modulo M for every $a < j$. If the lengths at g and g' differ by at most t modulo M from j on, they differ by at most $t + (4c + 12)j$ modulo M from 0 on.

(iii) Let $x \in G$ and $t \ge 0$. If the lengths at g and gx differ by at most t modulo M from 0 on, then $M^{-1}(x - 1)$ has size $t + 4$.

Proof. We use the size rules: a product of entries of sizes i and i' has size $i + i'$ (SZ1); a sum of at most $r + 1$ elements of size i has size $i + 1$, so $A_{(i)} \subseteq A_{(i+1)}$, and negatives and conjugates keep the size (SZ2), (SZ3); hence a matrix of size i maps vectors of size i' to vectors of size $i + i' + 1$, and, as J has size 1 (SZ5), $h(v, w) = \bar{v}^{\mathsf{T}} J w$ has size $i + i' + 3$ when v has size i and w size i'. Since $J_{e_a e_b} = 0$ and $h(e_b, w)$ is the e_b^*-coordinate of w, we have: $h(\epsilon, w) = 0$ for $\epsilon \in E_{<l}$ whenever $h(e_b, w) = 0$ for all $b < l$, and $h(e_a, \epsilon) = 0$ for $\epsilon \in E_{<\varrho}$.

(i). Let f, f', ζ, ζ' be as in the definition of balance, let $i \ge 0$, and let $\omega \in V$ with $h(e_b, \omega) = 0$ for $b < a$ and $g^{-1}\omega$ of size i. Put $\alpha = h(e_a, \omega)$. As $f - e_a \in E_{<a}$, $\alpha = h(f, \omega) = h(g^{-1}f, g^{-1}\omega)$, so $\alpha \in A_{(c+i+3)}$. Put $\omega_0 = \omega - \alpha\zeta$. Then $h(e_b, \omega_0) = 0$ for $b < a$, and $h(e_a, \omega_0) = \alpha - \alpha h(e_a, \zeta) = 0$; and $g^{-1}\omega_0 = g^{-1}\omega - \alpha g^{-1}\zeta$ has size $2c + i + 4$. By hypothesis there is $\omega_0' \in \omega_0 + E_{<a+1}$ such that $M^{-1}(g^{-1}\omega_0 - g'^{-1}\omega_0')$ has size $m = 2c + i + 4 + t$. Write $\omega_0' - \omega_0 = \mu e_a + \epsilon$ with $\mu \in A$ and $\epsilon \in E_{<a}$.

The coefficient μ is small. We have $h(\zeta', e_a) = \overline{h(e_a, \zeta')} = \overline{h(e_a, \zeta)} = 1$, as $\zeta' - \zeta \in E_{<a}$; $h(\zeta', \epsilon) = 0$, as $h(e_b, \zeta') = h(e_b, \zeta) = 0$ for $b < a$; and $h(\zeta' - \zeta, \omega_0) = 0$, as $h(e_b, \omega_0) = 0$ for $b \le a$. Hence

$$\mu = h(\zeta', \omega_0') - h(\zeta, \omega_0) = h(v', w') - h(v, w), \qquad v = g^{-1}\zeta,\ v' = g'^{-1}\zeta',\ w = g^{-1}\omega_0,\ w' = g'^{-1}\omega_0',$$

since g, g' are isometries. Here v and $M^{-1}(v - v')$ have size c, w has size $2c + i + 4 \le m$ and $M^{-1}(w - w')$ has size m. As $\bar{M} = M$ (PE1), $h(M^{-1}u, u') = M^{-1}h(u, u')$, and

$$M^{-1}\big(h(v', w') - h(v, w)\big) = -h(v', M^{-1}(w - w')) - h(M^{-1}(v - v'), w).$$

Since $v' = v - M \cdot M^{-1}(v - v')$ has size $c + 2$ ($M \in A_{(1)}$), the two terms have sizes $c + m + 5$ and $c + m + 3$, so $M^{-1}\mu \in A_{(c+m+6)}$.

The lift. Put $\omega' = \alpha\zeta' + \omega_0' - \mu f'$. Then $\omega' - \omega = \alpha(\zeta' - \zeta) + \epsilon - \mu(f' - e_a) \in E_{<a}$, and

$$M^{-1}(g^{-1}\omega - g'^{-1}\omega') = M^{-1}(g^{-1}\omega_0 - g'^{-1}\omega_0') + \alpha\, M^{-1}(g^{-1}\zeta - g'^{-1}\zeta') + (M^{-1}\mu)\, g'^{-1} f'.$$

The three terms have sizes m, $2c + i + 3$ and $2c + m + 6$; the sum of the last two has size $2c + m + 7$, and the whole sum size $2c + m + 8 = i + t + 4c + 12$. So the lengths differ by at most $t + 4c + 12$ from a on.

(ii). By induction on j: for $j = 0$ there is nothing to show; for $j \ge 1$, (i) at $a = j - 1$ turns the bound t from j on into $t + 4c + 12$ from $j - 1$ on, and the induction hypothesis for $j - 1$ (balance holds for all $a < j - 1$) gives $t + 4c + 12 + (4c + 12)(j - 1) = t + (4c + 12)j$ from 0 on.

(iii). From 0 on there is no orthogonality condition and $E_{<0} = 0$, so the lift of ω is ω itself. For an index k take $\omega = ge_k$: $g^{-1}\omega = e_k$ has size 0 (SZ4), hence $M^{-1}(e_k - (gx)^{-1}ge_k) = -M^{-1}(x^{-1} - 1)e_k$ has size t. So $N = M^{-1}(x^{-1} - 1)$ has size t. As $x^{-1} \in G$, $x = (x^{-1})^{-1} = J^{-1}\overline{x^{-1}}^{\mathsf{T}} J$, and since $\bar{M} = M$,

$$M^{-1}(x - 1) = J^{-1}\bar{N}^{\mathsf{T}} J.$$

$\bar{N}^{\mathsf{T}} J$ has size $t + 2$ and $J^{-1}\bar{N}^{\mathsf{T}} J$ size $t + 4$, by (SZ5). □

5.5 Local bounds at the complex and finite places

At each infinite place the sieve of the main term needs band limits: a sieve system with mass bound $(1 + b^{-10})/\deg$ whose top form is the Gram matrix of the normalised orbital integrals of the cut Bessel function. The theorem of this section, Theorem 5.5.3, produces them for the group $G_1 = \mathrm{U}(V_1)$ of signature $(\varrho + k, \varrho + 1)$ at a real place, under an L^2 bound for the matrix coefficients and polynomial growth of balls. It is applied once, in the construction of the Bessel systems of the data (Proposition 5.7.3), with deg a product of formal degrees. The frames engine sees the real place through the field $\mathbb{C}$ with complex conjugation, and this is the complex place of the statements.

The idea is that the complex place is one more place for the frames engine. Its data form a place datum with a gauge (Lemma 5.5.5), so that the descent and the decay come from Section 5.4 (Lemma 5.5.4, Lemma 5.5.33). Two statements are particular to the archimedean place and carry the weight of the section. The mass bound on the bottom floor is where the L^2 hypothesis is used (Lemma 5.5.34); in our reading it is the orthogonality relation of the discrete series, with the formal degree in the role of deg, made effective. On the top floor the Gram form of the normalised orbital integrals does not depend on the frame-fixing element (Lemma 5.5.32). For the discrete series, their characters and formal degrees see [119, 250, 163, 315]; for Whittaker and Bessel functions on real groups see [316, 153].

The section closes with the corresponding statements at a finite place, for a large lattice datum, which feed the sieve system at T of the main term: the shell sum, the integrality of head products, the masses of the integral isometries and of congruence cosets, and the mass integral of the finite Bessel vector, whose main part comes from the congruence subgroup (Lemma 5.5.42).

Definition 5.5.1. Let $\varrho \geq 1$ and $k \geq 2$ be integers, $n = \varrho + 1$, $r = 2\varrho + k + 1$, and fix an index $k_0 \in \{1, \dots, k\}$. In this definition $|x|$ is the usual absolute value of $x \in \mathbb{C}$ and $\|\cdot\|$ the Euclidean norm of $\mathbb{C}^r$ (operator norm on matrices).

(RP1) The complex place. The **complex place** is $\mathbb{C}$ with complex conjugation and the tuple $(\mathrm{vol}, \tau, \delta, \nu, |\cdot|_{\mathbb{C}}, \psi, \mathrm{kb}_{\mathbb{C}})$: vol the Lebesgue measure of $\mathbb{C} = \mathbb{R}^2$ (coordinates Re, Im); $\tau = 1/2$; $\delta = \mathrm{i}/2$; ν the image of Lebesgue measure of $\mathbb{R}$ under $t \mapsto t\mathrm{i}$ on $\mathbb{C}^- = \mathrm{i}\mathbb{R}$; the modulus $|x|_{\mathbb{C}} = |x|^2$; the character $\psi =$ wave; and $\mathrm{kb}_{\mathbb{C}} = \pi^2$.

(RP2) Boosts and the top-floor chart. Let T be the Gram matrix of $X_0 \oplus \mathbb{C}z$ with $T = 1$ on $x_1, \dots, x_k$ and $h(z, z) = -1$. For $a \in \mathbb{C}$ with $|a| \geq 1$ and $s_a = \sqrt{|a|^2 - 1}$, the **boost** $\mathrm{bt}(a) \in G$ is

$$\mathrm{bt}(a)\, x_{k_0} = \bar{a}\, x_{k_0} - s_a\, z, \qquad \mathrm{bt}(a)\, z = s_a\, x_{k_0} - a\, z,$$

fixing every other coordinate vector. For $c \in \mathbb{C}^\times$ the **top-floor chart** is

$$\sigma_c = \mathrm{bt}(c/\min(1, |c|)).$$

(RP3) Real-place sizes. Real numbers ρ, b and an integer $s \geq 0$ are **real-place sizes** if $\rho = b^{6n+2}$, $8r \leq \rho$, $b \geq 2$, $k = s + 3n + 6$, $10^9 n^3 < s$, $n^7 \leq k$ and $10^8 \leq \varrho$.

(RP4) The place datum at the complex place. For $\rho > 0$, the **data at $\mathbb{C}$ with box radius** ρ are the following data of a place datum over $(\mathbb{C}, k, T)$ of shape $(\varrho, \{1, \dots, k\} \setminus \{k_0\}, 0)$: hyperbolic basis $v_0 = \frac{1+\mathrm{i}}{2}(x_{k_0} + z)$, $v_0^* = \frac{1+\mathrm{i}}{2}(x_{k_0} - z)$, $o_i = x_i$ $(i \neq k_0)$; μ_H the image of the Haar probability measure of $\mathrm{U}(k)$ under its embedding in G (acting on X_0, fixing e_a, e_a^*, z); μ_{top} the image of $\mu_{\mathrm{cell}} \otimes \mu_H$ under $(g, g') \mapsto gg'$, where μ_{cell} is the image under $c \mapsto \sigma_c^{-1}$ of the measure

$$\frac{2\pi^k}{(k-1)!}\big((|c|^2 - 1)_+\big)^{k-1}\, dc \qquad \text{on } \mathbb{C}^\times;$$

level unit $M = 1$; close pairs $\mathcal{P}_{\mathrm{cl}} = \{(\omega_1, \omega_2) : \|\omega_1\|^2 + \|\omega_2\|^2 < 1\}$; box $K = \{x \in G : \|x\| \leq \rho\}$; positions $\mathcal{P}_{\mathrm{pos}} = G$.

(RP5) Gauges of scale L. For $L \geq \rho$, the **size filtration of scale** L is $A_{(j)} = \{x \in \mathbb{C} : |x| \leq L^j\}$ $(j \geq 0)$, with loss $\Lambda = L^2$.

Definition 5.5.2. Let I be a finite set, $F\colon G_1 \to \mathbb{C}^I$ continuous, and $p = (p_a)_{a \geq 0}$ a sequence in $\mathbb{C}^\times$. The **normalised cell integral** of F at p is

$$\mathrm{Ci}_F(p) = N(\sigma_{p_\varrho}, F, p) \in \mathbb{C}^{I \times I},$$

the normalised orbital integral of F at the element $w_p \sigma_{p_\varrho}$; it depends only on $p_0, \dots, p_\varrho$.

Theorem 5.5.3. *Let $b \geq 2$, $f \geq 1$, $8r \leq b^{6n+2}$, $k = s + 3n + 6$ with an integer $10^9 n^3 < s$, $n^7 \leq k$ and $10^8 \leq \varrho$. Let $(W, W^\flat)$ be a cut pair for b, f of continuous Bessel functions whose type is an irreducible unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$. Let $\deg > 0$ with*

$$\int_{G_1} \Big| \int_{\mathsf{U} \backslash G_1} \langle W(y), W(yx) \rangle dy \Big|^2 dx \leq \frac{(\#I)^2}{\deg}, \qquad \deg \cdot \Theta \leq b^{4r^2},$$

$\deg\cdot\mathrm{vol}\{g\in G_1:\|g\|_{\mathrm{op}}\le T\}\le T^{9r^2}b^{99r^2}$ *for all* $T\ge 1$, *and* $4000n\le f$. *Then* $W^\flat$ *has local bounds with degree* deg. *More precisely, let* $\mathcal{F}_i$ *be the floor forms and* λ *the Haar ratio of the data at* $\mathbb{C}$ *with box radius* b^{6n+2}, *and* $c_F=\mathrm{vol}\,\mathrm{U}(k)/\mathrm{vol}\,\mathrm{U}(r)/(\lambda\#I)^2$; *there is a form* $J_{\varrho+1}$ *such that the forms* $J_i=c_F\mathcal{F}_i(\cdot,\cdot;W^\flat,1)$ *(*$i\le\varrho$*) and* $J_{\varrho+1}$ *are local bounds for* $W^\flat$ *with degree* deg*; in particular* $J_{\varrho+1}(p,q)=\langle N(\sigma,W^\flat,p),N(\sigma,W^\flat,q)\rangle$ *for every* $\sigma\in G_1$ *that fixes* e_a *and* e_a^* *for* $1\le a\le\varrho$ *and all* p,q *with* $h(z,\sigma z)=p_\varrho=q_\varrho$, *and* $|c_F\mathcal{F}_0(p,q;W^\flat,1)|\le(1+b^{-10})/\deg$ *for all* p,q.

Proof. Put $\rho=b^{6n+2}$. The hypotheses say that ρ,b,s are real-place sizes; consider the data at $\mathbb{C}$ with box radius ρ, a place datum by Lemma 5.5.5, (i). Write $F=W^\flat$; it is cut for this place datum (Lemma 5.5.9). The type is irreducible, so $I\ne\emptyset$. By Lemma 5.5.10, $0<\lambda<\infty$, so $0<c_F<\infty$. Define the forms on pairs of sequences of nonzero complex numbers

$$J_i(p,q)=c_F\,\mathcal{F}_i(p,q;F,1)\quad(i\le\varrho),\qquad J_{\varrho+1}(p,q)=\langle\mathrm{Ci}_F(p),\mathrm{Ci}_F(q)\rangle,$$

and take the mass bound $E_0=(1+b^{-10})/\deg$, the weights $\Lambda_i=\Lambda_i^{R,K}$ with $R=b^{60n^2}$, $K=n^6$, and the decay $\beta(\xi)=\Theta\,b^{2000n^9}\ell_{k-n^7}(1/|\xi|)$, which is $\mathrm{Dec}(\xi)$ of Lemma 5.5.33. We check (SV1)–(SV6) for length ϱ and the top clause. Since $\#I\ge1$, the bounds of Lemma 5.5.33, (a) and (b), give the same bounds without the factor $\#I$.

(SV1). $\psi(0)=\exp(0)=1$.

(SV2) Descent. Let $i\le\varrho$, p,q sequences of nonzero numbers and $\xi\in\mathbb{C}$; put $\phi(c)=\mathrm{wave}(\xi c)$, a continuous function of modulus 1. We show that $c\mapsto\phi(c)J_{i+1}(p[i\mapsto u(c)],q[i\mapsto u(c)])$ is integrable with

$$\int_{\mathbb{C}}\phi(c)\,J_{i+1}(p[i\mapsto u(c)],q[i\mapsto u(c)])\,dc=c_F\,\mathcal{F}_i(p,q;F,\,x\mapsto\mathrm{wave}(\xi\,h(f_i,xf_i))).\tag{5.93}$$

If $i=\varrho$, this is Lemma 5.5.31 with this ϕ. Let $i<\varrho$. Choose $L\ge\rho$ with $|p_0\cdots p_{l-1}|^{-1}\le L$ and $|q_0\cdots q_{l-1}|^{-1}\le L$ for $l\le i$; then p,q are heads of length i for the gauge Γ_L, and Lemma 5.5.4, (ii), with $j=i$, $t=1$ and the majorant $\mathrm{FM}_i(p,q,F,\cdot)$ itself as the bounded function, gives $\#I\,c_F\,\mathrm{hdn}_i(p,q)\int\mathrm{FM}_i(p,q,F,x)\,d\theta(x)<\infty$; as $\#I\,c_F\,\mathrm{hdn}_i(p,q)>0$, the integral of the majorant is finite. So Proposition 5.2.92, (ii), applies with $a=i$, $t=c_F$ and this ϕ, and gives the integrability and (5.93), its weight being $\phi(h(f_i,xf_i))$. If $\xi=0$, the weight in (5.93) is 1 and the integral is $J_i(p,q)$. If $\xi\ne0$, Lemma 5.5.33, (a), with $\nu=\xi$, bounds the absolute value of (5.93) by $\beta(\xi)\Lambda_i(p)\Lambda_i(q)$.

(SV3) Positivity of the top form. For finitely many sequences $a_1,\dots,a_N$ and $x\in\mathbb{C}^N$, $\sum_{i,j}\bar{x}_ix_jJ_{\varrho+1}(a_i,a_j)=\|\sum_jx_j\mathrm{Ci}_F(a_j)\|^2\ge0$; so the matrix $(J_{\varrho+1}(a_i,a_j))_{i,j}$ is positive semidefinite (with or without a common last entry).

(SV4). $J_{\varrho+1}$ is jointly continuous by Lemma 5.5.32, (i), as F is continuous.

(SV5) Mass. $|J_0(p,q)|=|c_F\,\mathcal{F}_0(p,q;F,1)|\le(1+b^{-10})/\deg=E_0$ for all p,q, by Lemma 5.5.34, whose hypotheses are those of the theorem.

(SV6) Unequal heads. Let $i<\varrho$ with $\xi=\iota(p_i)-\iota(q_i)=p_i^{-1}-q_i^{-1}\ne0$. By Lemma 5.5.33, (b), with $\nu=\xi$, $|J_{i+1}(p,q)|\le\beta(\xi)\Lambda_{i+1}(p)\Lambda_{i+1}(q)$.

The top clause. Let $\sigma\in G_1$ fix e_a and e_a^* for $1\le a\le\varrho$ (the vectors of all ϱ hyperbolic planes), and let $h(z,\sigma z)=p_\varrho=q_\varrho$. By Lemma 5.5.32, (ii), $J_{\varrho+1}(p,q)=\langle\mathrm{Ci}_F(p),\mathrm{Ci}_F(q)\rangle=\langle N(\sigma,F,p),N(\sigma,F,q)\rangle$.

Hence the forms $J_0,\dots,J_{\varrho+1}$ with ψ, E_0, Λ_i and β form a sieve system of length ϱ on $\mathbb{C}$ with the data of a complex place, mass bound $(1+b^{-10})/\deg$, weights $\Lambda_i^{b^{60n^2},n^6}$ and decay $\Theta\,b^{2000n^9}\ell_{k-n^7}(1/|\xi|)$, and they satisfy the top clause: they are local bounds for $W^\flat$ with degree deg, and the mass clause of the statement is (SV5). □

Lemma 5.5.4. *Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, let λ be its Haar ratio, and put*

$$V = (\pi\rho^{24})^k \rho^{9999n}, \qquad c_F = \frac{\operatorname{vol} \mathrm{U}(k)}{\operatorname{vol} \mathrm{U}(r)\,(\lambda \#I)^2}.$$

(i) $V \geq 1$, and $\mathrm{FML}_j(V;1)$ holds for every $j \leq \varrho$.

(ii) Let $(W, W^\flat)$ be a cut pair for b, f whose type is an irreducible unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$, I nonempty. Let $L \geq \rho$, $j \leq \varrho$, p, q heads of length j for Γ_L, $t \geq 1$ real, and $\phi\colon G \to [0,\infty]$ with $\phi \leq \mathrm{FM}_j(p, q, W^\flat, \cdot)$ and, if $t > 1$, $\phi(x)^2 \leq (V/t^s)^2\,\mathrm{WSP}_j(p, q, W^\flat, x)$ for all $x \in G$. Then

$$\#I \cdot c_F\, \mathrm{hdn}_j(p,q) \int_G \phi\, d\theta \leq \Theta\, L^{10^9 n^4} \rho^{24k}\, \frac{t^{10^8 n^4}}{t^s}.$$

Proof. **Numbers.** $\rho \geq 8r \geq 8$. From $k = s + 3n + 6 \geq n^7$ and $n > \varrho \geq 10^8$: $s \geq n^7 - 3n - 6 \geq 10^{10} n^4$; so $s = 10^8 n^4 + r_0$ with an integer $r_0 \geq 0$, and $9 \cdot 10^7 n^4 + r_0 \leq s$.

(i). $V \geq 1$ as $\pi, \rho \geq 1$. By Lemma 5.5.5, (ii), Γ_ρ is a gauge of loss ρ^2, and by (GU5), $\operatorname{vol}(A_{(11)}) \leq (\rho^2)^{12}$. By (DN0)(b) of Theorem 5.4.1 for Γ_ρ, $\mathrm{FML}_j\big((\rho^2)^{1287n}\operatorname{vol}(A_{(11)})^{k-5}; 1\big)$ for $j \leq \varrho$, and $(\rho^2)^{1287n}(\rho^{24})^{k-5} \leq \rho^{9999n}(\pi\rho^{24})^k = V$; floor-mass bounds pass to larger constants.

(ii). By Lemma 5.5.9 ($\rho = b^{6n+2}$), $W^\flat$ is cut for the data at $\mathbb{C}$, and by Lemma 5.5.10 the group measure $\theta = \lambda\, dy$ ($\lambda > 0$) is a Haar measure. By Lemma 5.5.24 the dyadic shells D_i ($i \geq 0$) are shells of heights 2^i for $(\Gamma_L, j, p, q, W^\flat)$ and cover G, hence the support of ϕ. Put $e = t^{-1} \in (0,1]$; if $e < 1$ then $t > 1$ and $\phi(x)^2 \leq (Ve^s)^2 \mathrm{WSP}_j(p, q, W^\flat, x)$ for every $x \in \mathcal{P}_{\mathrm{pos}}$. With (i) and the integer r_0, (DN2) of Theorem 5.4.1 for Γ_L (loss L^2) gives

$$\mathrm{hdn}_j(p,q) \int_G \phi\, d\theta \leq V\, t^{-r_0}\, (L^2)^{4\cdot 10^8 n^4}\, \mathrm{SN}(W^\flat), \qquad \mathrm{SN}(W^\flat) = \pi^{2(k-5)\varrho}\, \mathrm{PN}(W^\flat) \leq \pi^{2(k-5)\varrho} \lambda^2 \#I$$

by Lemma 5.5.27. Multiplying by $\#I \cdot c_F$, the factors $\lambda^2(\#I)^2$ cancel, and with $t^{-r_0} = t^{10^8 n^4}/t^s$,

$$\#I\, c_F\, \mathrm{hdn}_j(p,q) \int_G \phi\, d\theta \leq \frac{\operatorname{vol} \mathrm{U}(k)}{\operatorname{vol} \mathrm{U}(r)}\, \pi^{k+2(k-5)\varrho} \rho^{24k} \rho^{9999n} L^{8\cdot 10^8 n^4}\, \frac{t^{10^8 n^4}}{t^s} \leq \Theta\, L^{10^9 n^4} \rho^{24k}\, \frac{t^{10^8 n^4}}{t^s},$$

since $\pi^{k+2(k-5)\varrho} \leq \pi^{(2\varrho+1)k}$, $\rho^{9999n} \leq L^{9999n}$ and $9999n + 8 \cdot 10^8 n^4 \leq 10^9 n^4$. □

Lemma 5.5.5. *(i) For every $\rho > 0$ and every enumeration of $\{1, \dots, k\} \setminus \{k_0\}$, the data at $\mathbb{C}$ with box radius ρ satisfy (PE1)–(PE4), so they form a place datum of shape $(\varrho, \{1, \dots, k\} \setminus \{k_0\}, 0)$ over $(\mathbb{C}, k, T)$.*

(ii) Let ρ, b, s be real-place sizes and $L \geq \rho$. Then the size filtration of scale L, with loss L^2, is a gauge for the data at $\mathbb{C}$ with box radius ρ. In (GU4) one may take $c = L^{-(i+2)}$, and in (GU6) a kernel with $\lambda_\beta(\{x \text{ of size } j\}) \leq 4\pi^2 L^{2j}$ for all β and j.

Proof. **(i) The hyperbolic basis and the shape.** $|\iota| + 2m + 2 = (k-1) + 0 + 2 = k + 1$. The vectors v_0, v_0^*, o_i lie in the middle space, and, h being conjugate-linear in the first variable, $h(x_{k_0} \pm z, x_{k_0} \pm z) = 1 - 1 = 0$ and $h(x_{k_0} + z, x_{k_0} - z) = 1 + 1 = 2$, so $h(v_0, v_0) = h(v_0^*, v_0^*) = 0$ and $h(v_0, v_0^*) = \left|\frac{1+\mathrm{i}}{2}\right|^2 \cdot 2 = 1$; for $i, i' \neq k_0$, $h(o_i, o_{i'}) = \delta_{ii'}$ and $h(v_0, o_i) = h(v_0^*, o_i) = 0$. The measures μ_H (finite) and μ_{top} (an image of the product of a σ-finite measure with density on $\mathbb{C}^\times$ and a finite measure) are s-finite Borel measures, and $M = 1$ is a unit.

(PE1). $|\det T| = |-1|^2 = 1$ and $\bar{M} = M = 1$.

(PE2) is Lemma 5.5.6.

(PE3). μ_H is a probability measure, so $\mu_H \neq 0$ and $\mu_H(E) \leq 1$ for every set E; in particular the last clause holds. The embedding ϵ of $\mathrm{U}(k)$ acts on $X_0 = \mathrm{span}\{x_i\}$ and fixes e_a, e_a^*, z, so it takes values in $H = G^{(\varrho+1)}$, the group of the $g \in G$ fixing the coordinate vectors of floor below $\varrho+1$, i.e. the e_a, e_a^* and z; hence $\mu_H(G \setminus H) = 0$. Conversely let $s \in H$. For $w \in X_0$ and $y \in \{e_a, e_a^*, z\}$, $h(sw, y) = h(sw, sy) = h(w, y) = 0$; as J is nondegenerate, the h-orthogonal complement of $\mathrm{span}\{e_a, e_a^*, z\}$ is X_0, so s preserves X_0, and its restriction preserves the standard form $h|_{X_0}$: $s = \epsilon(u)$ with $u \in \mathrm{U}(k)$. Right invariance of the Haar measure of $\mathrm{U}(k)$ gives that the image of μ_H under $g \mapsto gs$ is μ_H.

(PE4). $\mathcal{P}_{\mathrm{cl}}$ is the preimage of $[0,1)$ under the continuous map $(\omega_1, \omega_2) \mapsto \|\omega_1\|^2 + \|\omega_2\|^2$, hence open; it contains $(0,0)$; its closure lies in the compact set $\{\|\omega_1\|^2 + \|\omega_2\|^2 \leq 1\}$. G is closed in the matrices (defined by $\bar{g}^{\mathsf{T}} J g = J$, which forces invertibility), so K is closed and bounded, hence compact. $\mathcal{P}_{\mathrm{pos}} = G$ is measurable.

(ii) Numbers. From $\rho \geq 8r$: $L \geq 8r \geq 8$; so $r+1 \leq L$, $2r < L^4$, $\pi \leq L^2$ and $4\pi^2 \leq L^4$.

(SZ1)–(SZ5). If $|x| \leq L^i$ and $|y| \leq L^j$ then $|xy| \leq L^{i+j}$. A sum of at most $r+1$ elements of modulus at most L^i has modulus at most $(r+1)L^i \leq L^{i+1}$. The discs are invariant under $x \mapsto -x$ and $x \mapsto \bar{x}$, and contain 0 and 1 ($L \geq 1$). The entries of $J = J^{-1}$ have modulus at most $1 \leq L$.

(GU1). Each $A_{(j)}$ is a closed disc, hence compact; $2 \leq L^2 < \infty$; $m = 0 \leq 9n$; $2(2m+2) = 4 \leq k - 1 = |\iota|$; $\mathrm{kb}_{\mathbb{C}} = \pi^2 \geq 1$.

(GU2). $|M| = 1 \leq L$. The coordinates of v_0, v_0^* are 0 and $\pm\frac{1+\mathrm{i}}{2}$, of modulus at most $2^{-1/2}$, and those of the o_i are $0, 1$. $\tau\delta^{-1} = \frac{1}{2} \cdot \frac{2}{\mathrm{i}} = -\mathrm{i}$ has modulus 1. For $x \in K$ every entry satisfies $|x_{ij}| \leq \|x\| \leq \rho \leq L$.

(GU3). If $\|\omega_1\|^2 + \|\omega_2\|^2 < 1$, every entry of ω_1 and of ω_2 has modulus less than 1, so the entries of ω_1 and of $M^{-1}(\omega_1 - \omega_2) = \omega_1 - \omega_2$ have modulus at most $2 \leq L$.

(GU4). $M^{-1} = 1 \in A_{(1)}$. Let $i \geq 0$ and $c = L^{-(i+2)}$, a unit with $c^{-1} = L^{i+2} \in A_{(i+2)}$. If $\omega_1, \omega_2 \in V$ have size i, each has r entries of modulus at most L^i, so $\|c\omega_1\|^2 + \|c\omega_2\|^2 \leq 2rL^{2i}/L^{2i+4} = 2r/L^4 < 1$, and $(c\omega_1, c\omega_2) \in \mathcal{P}_{\mathrm{cl}}$.

(GU5). For a unit c with $|c| \leq L^j$, its modulus is $|c|^2 \leq (L^2)^j$. $\mathrm{vol}(A_{(j)}) = \pi L^{2j} \leq (L^2)^{j+1}$.

(GU6). Here $2m+2 = 2$. Let e be the first coordinate vector of $\mathbb{C}^2$, du the Haar probability measure of $\mathrm{U}(2)$, and for $\beta = t\mathrm{i} \in \mathrm{i}\mathbb{R}$ let λ_β be $2\pi^2(2t)_+$ times the image of du under $u \mapsto \sqrt{(2t)_+}\, ue$. For a Borel set B, $\beta \mapsto \lambda_\beta(B) = 2\pi^2(2t)_+ \int \mathbf{1}_B(\sqrt{(2t)_+}\, ue)\, du$ is measurable by Tonelli, so (λ_β) is a kernel; $\lambda_\beta(\mathbb{C}^2) = 4\pi^2 t_+$ is bounded on each set $\{N-1 \leq |t| < N\}$, so λ is the countable sum of the finite kernels $\mathbf{1}_{\{N-1\leq|t|<N\}}\lambda$ and is s-finite. (DJ1): for g continuous with compact support, $\int g\, d\lambda_{t\mathrm{i}} = 2\pi^2(2t)_+ \int g(\sqrt{(2t)_+}\, ue)\, du$ is finite and continuous in t, the integrand being continuous on $\mathbb{R} \times \mathrm{U}(2)$ and $\mathrm{U}(2)$ compact. (DJ2): by polar coordinates on $\mathbb{C}^2$ (Lebesgue measure is $\mathrm{U}(2)$-invariant, $\mathrm{U}(2)$ is transitive on spheres, and $\|x\|^2$ has density $\pi^2 r$ on $r > 0$), then $r = 2t$,

$$\begin{aligned}\int_{\mathbb{C}^2} f(\bar{x}^{\mathsf{T}}x, x)\, dx &= \int_0^\infty \pi^2 r \int f(r, \sqrt{r}\, ue)\, du\, dr = \int_{\mathbb{R}} 2\pi^2(2t)_+ \int f(2t, \sqrt{(2t)_+}\, ue)\, du\, dt \\ &= \int_{\mathrm{i}\mathbb{R}} \int f(\delta^{-1}\beta, x)\, d\lambda_\beta(x)\, d\nu(\beta),\end{aligned}$$

as $\delta^{-1}(t\mathrm{i}) = 2t$. Mass: if $\lambda_{t\mathrm{i}}$ charges $\{x : |x_1|, |x_2| \leq L^j\}$, then $t > 0$ and the sphere $\|x\|^2 = 2t$ meets this box, so $2t \leq 2L^{2j}$ and $\lambda_{t\mathrm{i}}(\{x \text{ of size } j\}) \leq \lambda_{t\mathrm{i}}(\mathbb{C}^2) = 2\pi^2 \cdot 2t \leq 4\pi^2 L^{2j} \leq L^4 L^{4j} = (L^2)^{(2m+2)(j+1)}$.

(GU7) is Lemma 5.5.8, applied to $g \in K \subseteq G$ and $y \in \mathcal{P}_{\mathrm{pos}} = G$. □

Lemma 5.5.6. *Let $\rho > 0$ and consider the data at $\mathbb{C}$ with box radius ρ. Put $\mathrm{sph}_k(r) = 2\pi^k r^{k-1}/(k-1)!$ for $r \geq 0$, and for $q \in \mathbb{R}$ let hy_q be the image of the measure $\mathrm{sph}_k((q+|c|^2)_+)\,dc\,du$ on $\mathbb{C}\times\mathrm{U}(k)$ under*

$$(c,u) \mapsto \sqrt{(q+|c|^2)_+}\,u\,x_{k_0} + c\,z \in V.$$

Then $\mathrm{qv}_\varrho(\beta) = \mathrm{hy}_{\delta^{-1}\beta}$ for every $\beta \in \mathrm{i}\mathbb{R}$ (note $\delta^{-1}\beta = 2\,\mathrm{Im}\,\beta$); in particular $\mathrm{qv}_\varrho = \mathrm{hy}_{-1}$. Moreover $\mu_{\mathrm{top}}(G \setminus G^{(\varrho)}) = 0$ and the image of μ_{top} under $g \mapsto g^{-1}z$ is qv_ϱ; that is, μ_{top} satisfies (PE2).

Proof. **Step 0: polar coordinates on $\mathbb{C}^k$.** For measurable $g\colon \mathbb{C}^k \to [0,\infty]$ and e the coordinate vector of index k_0,

$$\int_{\mathbb{C}^k} g(\omega)\,d\omega = \int_0^\infty \frac{\pi^k r^{k-1}}{(k-1)!}\int_{\mathrm{U}(k)} g(\sqrt{r}\,ue)\,du\,dr. \tag{5.94}$$

Indeed, Lebesgue measure is $\mathrm{U}(k)$-invariant, so by Tonelli $\int g = \int_{\mathbb{C}^k} G(\omega)\,d\omega$ with $G(\omega) = \int_{\mathrm{U}(k)} g(u\omega)\,du$. If $\|\omega\|^2 = r$, then $\omega = u_0\sqrt{r}\,e$ for some $u_0 \in \mathrm{U}(k)$, and right invariance of du gives $G(\omega) = \int g(\sqrt{r}\,ue)\,du$; so $G(\omega) = \Phi(\|\omega\|^2)$ for a measurable Φ. The law of $\|\omega\|^2$ under Lebesgue measure of $\mathbb{C}^k = \mathbb{R}^{2k}$ has density $\frac{d}{dr}\frac{\pi^k r^k}{k!} = \frac{\pi^k r^{k-1}}{(k-1)!}$ on $r > 0$ (volume of the ball of radius $\sqrt{r}$ in $\mathbb{R}^{2k}$), which gives (5.94).

Step 1: the quadric measures disintegrate vol_{V_ϱ}. At floor ϱ and $m = 0$ the index set $\mathcal{Q}_\varrho$ is empty, so $Y_\varrho = \mathbb{C}^\iota$, $h'_\varrho(w,w) = \sum_{i\in\iota}|w_i|^2$ and $B_\varrho(t_1,t_2,w) = t_1v_0 + t_2v_0^* + \sum_{i\in\iota} w_ix_i$. Let X be the matrix of B_ϱ, from $\mathbb{C}^{\{1,2\}\sqcup\iota}$ to V, and let J^{hyp} be the hermitian matrix with $\bar{x}^{\mathsf{T}}J^{\mathrm{hyp}}x = \bar{t}_1t_2 + \bar{t}_2t_1 + \sum_i |w_i|^2$ for $x = (t_1,t_2,w)$. From $h(v_0,v_0^*) = 1$, $h(v_0,v_0) = h(v_0^*,v_0^*) = 0$, $h(x_i,x_{i'}) = \delta_{ii'}$ and $h(v_0,x_i) = h(v_0^*,x_i) = 0$ $(i \neq k_0)$ we get $\bar{X}^{\mathsf{T}}JX = J^{\mathrm{hyp}}$; as $(J^{\mathrm{hyp}})^2 = 1$, $L = J^{\mathrm{hyp}}\bar{X}^{\mathsf{T}}J$ satisfies $LX = 1$. By definition (5.4), $\mathrm{qv}_\varrho(\beta)$ is the image under X of the measure λ^1_β of Lemma 5.2.52, (i), for $S = \iota$ and $T' = 1$; the complex place is a place (Lemma 5.5.7), so by Lemma 5.2.52, (i), $(\lambda^1_\beta)_\beta$ is a disintegration of vol on $\mathbb{C}^{\{1,2\}\sqcup\iota}$ along $\bar{x}^{\mathsf{T}}J^{\mathrm{hyp}}x$, and by Lemma 5.2.53, (ii), $(\mathrm{qv}_\varrho(\beta))_\beta$ is a disintegration of $X_*\mathrm{vol}$ along $h(v,v)$. Finally X is the extension by zero of the square matrix X' onto the coordinates $x_1,\dots,x_k,z$ of floor at least ϱ; X' is the identity on the x_i, $i \neq k_0$, and on $(t_1,t_2) \mapsto (x_{k_0},z)$-coordinates it is $\frac{1+\mathrm{i}}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, of determinant $-\frac{(1+\mathrm{i})^2}{2} = -\mathrm{i}$. So $|\det X'|_{\mathbb{C}} = 1$, (PA3) gives $X'_*\mathrm{vol} = \mathrm{vol}$, and $X_*\mathrm{vol} = \mathrm{vol}_{V_\varrho}$.

Step 2: the hyperboloid measures disintegrate vol_{V_ϱ}. Write

$$p_q(c,u) = \sqrt{(q+|c|^2)_+}\,ux_{k_0} + cz,$$

and for $\beta \in \mathrm{i}\mathbb{R}$ put $\lambda'_\beta = \mathrm{hy}_{\delta^{-1}\beta}$; for $\beta = t\mathrm{i}$, $\delta^{-1}\beta = 2t$.

(DJ2). Let $f\colon \mathbb{C}\times V \to [0,\infty]$ be measurable. A point of V_ϱ is $v = \omega + cz$ with $\omega \in X_0 = \mathbb{C}^k$, and $h(v,v) = \|\omega\|^2 - |c|^2$. By Tonelli and (5.94),

$$\int f(h(v,v),v)\,d\mathrm{vol}_{V_\varrho}(v) = \int_{\mathbb{C}}\int_0^\infty \frac{\pi^k r^{k-1}}{(k-1)!}\int_{\mathrm{U}(k)} f\big(r-|c|^2, \sqrt{r}\,ux_{k_0} + cz\big)\,du\,dr\,dc.$$

Substitute $r = 2t+|c|^2$ $(dr = 2\,dt)$: the range $r > 0$ becomes $2t+|c|^2 > 0$, the factor $2\pi^k r^{k-1}/(k-1)!$ is $\mathrm{sph}_k(r)$, and $\mathrm{sph}_k(0) = 0$ as $k \geq 2$, so the range may be extended to all $t \in \mathbb{R}$ with the density $\mathrm{sph}_k((2t+|c|^2)_+)$ and the point $p_{2t}(c,u)$. Exchanging the integrals over c and t (Tonelli),

$$\begin{aligned}\int f(h(v,v),v)\,d\mathrm{vol}_{V_\varrho} &= \int_{\mathbb{R}}\int_{\mathbb{C}\times\mathrm{U}(k)} f\big(2t, p_{2t}(c,u)\big)\,\mathrm{sph}_k\big((2t+|c|^2)_+\big)\,dc\,du\,dt\\ &= \int_{\mathrm{i}\mathbb{R}}\int f(\delta^{-1}\beta,v)\,d\lambda'_\beta(v)\,d\nu(\beta).\end{aligned}$$

(DJ1). Let $g\colon V \to [0,\infty)$ be continuous with compact support C_g, and let $C_z \subseteq \mathbb{C}$ be the (compact) set of z-coordinates of points of C_g. The z-coordinate of $p_q(c,u)$ is c, so

$$\int g\,d\mathrm{hy}_q = \int_{C_z\times \mathrm{U}(k)} \mathrm{sph}_k((q+|c|^2)_+)\, g(p_q(c,u))\,dc\,du.$$

The integrand is continuous in (q,c,u), so it is bounded on $[q_0-1,q_0+1]\times C_z\times \mathrm{U}(k)$, and $C_z\times \mathrm{U}(k)$ has finite measure; by dominated convergence $q\mapsto \int g\,d\mathrm{hy}_q$ is finite and continuous, and so is $\beta\mapsto \int g\,d\lambda'_\beta$.

Step 3: the identification. By Steps 1 and 2 and Lemma 5.2.53, (i), $\mathrm{qv}_\varrho(\beta) = \mathrm{hy}_{\delta^{-1}\beta}$ for every β. As $h(f_\varrho, f_\varrho) = h(z,z) = -1$, $\mathrm{qv}_\varrho = \mathrm{qv}_\varrho(-\delta) = \mathrm{hy}_{-1}$.

Step 4: (PE2). Write $\epsilon\colon \mathrm{U}(k)\to G$ for the embedding (acting on X_0, fixing the e_a, e_a^* and z). The boost $\mathrm{bt}(a)$ fixes every e_a, e_a^*, so $\sigma_c^{-1}\in G^{(\varrho)}$ for $c\neq 0$, and $\epsilon(u)\in G^{(\varrho)}$; as μ_{top} is the image of $\mu_{\mathrm{cell}}\otimes\mu_H$ under $(g,g')\mapsto gg'$, with μ_{cell} carried by $\{\sigma_c^{-1}\}$ and μ_H by $\epsilon(\mathrm{U}(k))$, it is carried by $G^{(\varrho)}$. For measurable $f\colon V\to[0,\infty]$, by the definitions of μ_{top}, μ_{cell}, μ_H,

$$\int f(g^{-1}z)\,d\mu_{\mathrm{top}}(g) = \int_{\mathbb{C}^\times} \mathrm{sph}_k((|c|^2-1)_+)\int_{\mathrm{U}(k)} f\big(\epsilon(u)^{-1}\sigma_c z\big)\,du\,dc,$$

since $(\sigma_c^{-1}\epsilon(u))^{-1} = \epsilon(u)^{-1}\sigma_c$ and the density of μ_{cell} is $\frac{2\pi^k}{(k-1)!}((|c|^2-1)_+)^{k-1} = \mathrm{sph}_k((|c|^2-1)_+)$. The density vanishes for $|c|\le 1$ ($k\ge 2$). For $|c|>1$, $\sigma_c = \mathrm{bt}(c)$ and $\mathrm{bt}(c)z = s_c x_{k_0} - cz$ with $s_c = \sqrt{|c|^2-1}$, so $\epsilon(u)^{-1}\sigma_c z = s_c\,u^{-1}x_{k_0} - cz$. The substitutions $u\mapsto u^{-1}$ (Haar measure of the compact group $\mathrm{U}(k)$ is invariant under inversion) and $c\mapsto -c$ (Lebesgue measure and the density are invariant) turn the integral into

$$\int_{\mathbb{C}} \mathrm{sph}_k((-1+|c|^2)_+)\int_{\mathrm{U}(k)} f\big(p_{-1}(c,u)\big)\,du\,dc = \int f\,d\mathrm{hy}_{-1},$$

where the set $\{|c|\le 1\}$, on which both densities vanish, has been added back. So the image of μ_{top} under $g\mapsto g^{-1}z$ is $\mathrm{hy}_{-1} = \mathrm{qv}_\varrho$ by Step 3. □

Lemma 5.5.7. *The complex place is a place. Moreover (PA5) holds with $\gamma(v) = e^{-|v|^2}$, for which*

$$\int_{\mathbb{C}} \gamma(v)\,\psi(\bar v y)\,dv = \pi e^{-|y|^2} \qquad (y\in\mathbb{C}),$$

and (PA6) holds with $\vartheta(u) = |u|^{-1}$ for $u\neq 0$ and $\vartheta(0)=0$.

Proof. **The structure.** $\mathbb{C}$ with complex conjugation is a commutative ring with an involution; its usual topology is that of $\mathbb{R}^2$, locally compact and Polish, and addition, multiplication and conjugation are continuous. Lebesgue measure is a Haar measure of $(\mathbb{C},+)$. $\tau+\bar\tau = \frac12+\frac12 = 1$; $\delta = \mathrm{i}/2$ is a unit with $\bar\delta = -\delta$. The skew elements form the closed subgroup $\mathrm{i}\mathbb{R}$, and $t\mapsto t\mathrm{i}$ is an isomorphism of topological groups $\mathbb{R}\to\mathrm{i}\mathbb{R}$, so ν is a Haar measure of $\mathrm{i}\mathbb{R}$. The modulus $x\mapsto|x|^2$ is continuous, hence measurable, with $|1|^2=1$ and $|xy|^2 = |x|^2|y|^2$. For $x = a+b\mathrm{i}$, $\psi(x) = \exp(x-\bar x) = e^{2b\mathrm{i}}$ is continuous, has modulus 1, and $\psi(x+x') = \psi(x)\psi(x')$. Finally $\mathrm{kb}_{\mathbb{C}} = \pi^2\in[0,\infty]$.

(PA1). For $\alpha = s\mathrm{i}$ and $\beta = t\mathrm{i}$ ($s,t\in\mathbb{R}$), $\tau\delta^{-1}\alpha+\beta = \frac12\cdot\frac{2}{\mathrm{i}}\cdot s\mathrm{i} + t\mathrm{i} = s+t\mathrm{i}$. Hence the map $(\alpha,\beta)\mapsto\tau\delta^{-1}\alpha+\beta$, composed with the parametrisation $(s,t)\mapsto(s\mathrm{i},t\mathrm{i})$ that carries Lebesgue measure of $\mathbb{R}^2$ to $\nu\otimes\nu$, is the identification $(s,t)\mapsto s+t\mathrm{i}$ of $\mathbb{R}^2$ with $\mathbb{C}$; it carries Lebesgue measure to vol, and so $\nu\otimes\nu$ is carried to vol.

(PA2). $|\bar x|^2 = |-x|^2 = |x|^2$.

(PA3). Let X be a complex $S \times S$ matrix with $\det X \neq 0$. Viewed as a real-linear map of $\mathbb{C}^S = \mathbb{R}^{2S}$, X has real determinant $|\det X|^2$ (classical: over $\mathbb{C}$ the real-linear map $x \mapsto Xx$ is conjugate to $\mathrm{diag}(X, \bar{X})$). By the change of variables formula for a linear automorphism of $\mathbb{R}^{2S}$, the image of Lebesgue measure under $x \mapsto Xx$ is $|\det X|^{-2}\mathrm{vol} = |\det X|_{\mathbb{C}}^{-1}\mathrm{vol}$.

(PA4). The non-units form the null set $\{0\}$.

(PA5). $\gamma(v) = e^{-|v|^2}$ is continuous, nonnegative, integrable (with integral π), and $\gamma(0) = 1$. Write $v = a + b\mathrm{i}$ and $y = c + d\mathrm{i}$. Then $\bar{v}y = (ac + bd) + (ad - bc)\mathrm{i}$, so $\psi(\bar{v}y) = e^{2\mathrm{i}(ad-bc)}$, and by Fubini and the Fourier transform of the Gaussian, $\int_{\mathbb{R}} e^{-a^2} e^{2\mathrm{i}au}\, da = \sqrt{\pi}\, e^{-u^2}$ $(u \in \mathbb{R})$,

$$\int_{\mathbb{C}} \gamma(v)\psi(\bar{v}y)\, dv = \int_{\mathbb{R}} e^{-a^2} e^{2\mathrm{i}ad}\, da \int_{\mathbb{R}} e^{-b^2} e^{-2\mathrm{i}bc}\, db = \sqrt{\pi} e^{-d^2} \cdot \sqrt{\pi} e^{-c^2} = \pi e^{-|y|^2}.$$

Hence $\int_{\mathbb{C}} \left| \int_{\mathbb{C}} \gamma(v)\psi(\bar{v}y)\, dv \right| dy = \pi \int_{\mathbb{C}} e^{-|y|^2}\, dy = \pi^2 = \mathrm{kb}_{\mathbb{C}}$.

(PA6). ϑ is Borel with values in $[0, \infty)$. For a unit u, $|u|_{\mathbb{C}}^{-1} = |u|^{-2} = \vartheta(u)^2$. A compact $C \subseteq \mathbb{C}$ lies in a disc $\{|u| \leq R\}$, and in polar coordinates $\int_{|u| \leq R} |u|^{-1}\, du = \int_0^{2\pi} \int_0^R r^{-1}\, r\, dr\, d\theta = 2\pi R < \infty$, so $\int_C \vartheta\, d\mathrm{vol} < \infty$. □

Lemma 5.5.8. *Let ρ, b, s be real-place sizes and consider the data at $\mathbb{C}$ with box radius ρ. Let $L \geq \rho$ and let sizes be those of the size filtration of scale L. Then for all $x, y \in G$ and every $\varrho \times \iota'$ matrix Q over $\mathbb{C}$ there is a common bound for $\mathcal{V}_{\mathrm{cm}}(x, y, Q)$ with loss L^2.*

Proof. **Step 0: numbers.** As $m = 0$, $|\iota| = k - 1$, $|\iota'| = k - 5$ and the volume exponent is $|\iota'| - (3n + 1) = k - 3n - 6 = s$; so $|\iota'| = s + 1 + 3n$. Also $r \geq 1$, so $L \geq \rho \geq 8r \geq 8$, and $|\iota'| \leq k \leq r \leq L$.

Step 1: a space of common vectors. Fix x, y, Q. Let N be the matrix of the n linear forms $u \mapsto u_{e_a} - u_{e_a^*}$ $(a < \varrho)$ and $u \mapsto u_z$ on V; if $Nu = 0$ then $Ju = u$. Let C be the matrix with the rows of Q, of $Nx^{-1}O$ and of $N(xy)^{-1}O$; it has $\varrho + 2n \leq 3n$ rows, so its kernel in $\mathbb{C}^{\iota'}$ has dimension at least $|\iota'| - 3n = s + 1$. Let Y be an $\iota' \times (s + 1)$ matrix whose columns are orthonormal and lie in $\ker C$, and complete them to a unitary matrix $U = (Y\ Y')$. For $t \in \mathbb{C}^{s+1}$ and $w = Yt$: $Qw = 0$, and for $g \in \{x^{-1}, (xy)^{-1}\}$, $JgOw = gOw$, so

$$\|gOw\|^2 = (gOw)^* J(gOw) = h(gOw, gOw) = h(Ow, Ow) = \|w\|^2 = \|t\|^2,$$

as g is an isometry of h and h is the standard form on the span of the x_i. Hence $Yt \in \mathcal{V}_{\mathrm{cm}}(x, y, Q)$ whenever $\|t\|^2 < 1/2$.

Step 2: product differences. For close pairs (ω_1, ω_2), (ω_1', ω_2'), as J is unitary, $|h(u, v)| \leq \|u\|\|v\| \leq (\|u\|^2 + \|v\|^2)/2$, so $|h(\omega_1, \omega_1') - h(\omega_2, \omega_2')| < 1$: $\mathcal{D}_{\mathrm{pr}}$ lies in the unit disc.

Step 3: the two inequalities. For $\varsigma \in \mathbb{C}^{\iota'}$ put $q(\varsigma) = \|Y^*\varsigma\|^2$ and $\phi(\varsigma) = 36/q(\varsigma)$ $(= \infty$ if $q(\varsigma) = 0)$, a Borel function; let $q = q(\varsigma) > 0$ (otherwise there is nothing to prove). For $c \in \mathbb{C}$ with $|c| \leq \sqrt{q}/2$ let $t = (c/q)Y^*\varsigma$; then $\|t\|^2 = |c|^2/q \leq 1/4$, so $w = Yt$ is a common vector, and $\bar{\varsigma}^{\mathsf{T}} w = (c/q)(Y^*\varsigma)^*(Y^*\varsigma) = c$. So $\{\bar{\varsigma}^{\mathsf{T}} w : w \in \mathcal{V}_{\mathrm{cm}}\}$ contains the disc of radius $\sqrt{q}/2$, of area $\pi q/4 \geq q/36$, which gives the first inequality. If $\bar{c}\,\bar{\varsigma}^{\mathsf{T}} w \in \mathcal{D}_{\mathrm{pr}}$ for all common vectors w, take w with $\bar{\varsigma}^{\mathsf{T}} w = \sqrt{q}/2$: Step 2 gives $|c|\sqrt{q}/2 \leq 1$, so these c lie in the disc of radius $2/\sqrt{q}$, of area $4\pi/q \leq 36/q$.

Step 4: the integral. Let $j \geq 0$ and let B_j be the set of ς with $|\varsigma_i| \leq L^j$ for all i. The unitary substitution $(p, p') = (Y^*\varsigma, Y'^*\varsigma) \in \mathbb{C}^{s+1} \times \mathbb{C}^{3n}$ preserves Lebesgue measure; on B_j each coordinate has modulus at most $|\iota'|L^j \leq L^{j+1}$, so $\|p\|^2 \leq (s+1)L^{2j+2} \leq R := L^{2j+3}$. Hence, by polar coordinates on $\mathbb{C}^{s+1}$,

$$\int_{B_j} \phi^s \leq \int_0^R \Big(\frac{36}{u}\Big)^s \frac{\pi^{s+1} u^s}{s!}\, du \cdot (\pi L^{2j+2})^{3n} = \pi \frac{(36\pi)^s}{s!}\, R\, (\pi L^{2j+2})^{3n} \leq 4e^{144} R\, (4L^{2j+2})^{3n},$$

using $\pi \le 4$ and $(144)^s/s! \le e^{144}$. With $4e^{144} \le 4 \cdot 3^{144} \le L^{145}$ and $4 \le L$, this is at most $L^{145+(2j+3)(3n+1)} \le L^{198n(j+1)} = (L^2)^{99n(j+1)}$, since $145 + (2j+3)(3n+1) \le 198n(j+1)$ for $n \ge 1$. So ϕ is a common bound with loss L^2. □

Lemma 5.5.9. *Let $b \ge 2$ and $f \ge 1$, and let $(W, W^\flat)$ be a cut pair for b, f (of any type). Then $W^\flat$ is cut for the data at $\mathbb{C}$ with box radius b^{6n+2}: it is a Bessel function of frequency $M^{-1} = 1$ over the place, and for every g with $W^\flat(g) \ne 0$ there is $u \in \mathsf{U}$ with $\|ug\|_{\mathrm{op}} < b^{6n+2}$.*

Proof. $W^\flat$ is a Bessel function of the type of the pair: bounded, measurable, $W^\flat(uy) = \mathrm{wave}(\sigma(u))W^\flat(y) = \psi(M^{-1}\sigma(u))W^\flat(y)$ for $u \in \mathsf{U}$. Let $W^\flat(g) \ne 0$; by (BS2), $\ell(g) \le b^{6n+2}$. Every $s \in R$ is $s = uh$ with $u \in \mathsf{U}$, $h \in H$; h is unitary for the Euclidean inner product of $\mathbb{C}^r$ (it acts unitarily on X_0 and fixes the other basis vectors), so $\|uhg\|_{\mathrm{op}} = \|h(h^{-1}uh)g\|_{\mathrm{op}} = \|u'g\|_{\mathrm{op}}$ with $u' = h^{-1}uh \in \mathsf{U}$ (H normalises U, as it preserves the span of the basis vectors of each level). Hence $\inf_{u \in \mathsf{U}} \|ug\|_{\mathrm{op}} = \ell(g) - 1 \le b^{6n+2} - 1 < b^{6n+2}$, and some $u \in \mathsf{U}$ has $\|ug\|_{\mathrm{op}} < b^{6n+2}$, so $ug \in K$. □

Lemma 5.5.10. *Let $\rho > 0$ and let θ, $\theta_{\ge 0}$ be the group measure and the tower measure from floor 0 of the data at $\mathbb{C}$ with box radius ρ. There is a unique real number λ (the* Haar ratio*) with $\theta(E) = \lambda \int_{G_1} \mathbf{1}_E \, dy$ for every Borel set $E \subseteq G_1$, and*

$$0 < \lambda \le \frac{\mathrm{vol}\,\mathrm{U}(r)}{\mathrm{vol}\,\mathrm{U}(k)}.$$

Moreover, for every Borel $f \colon G_1 \to [0, \infty]$ with $f(uy) = f(y)$ for all $u \in \mathsf{U}$ and $y \in G_1$,

$$\int_{G_1} f \, d\theta_{\ge 0} = \lambda \int_{\mathsf{U} \backslash G_1} f.$$

Proof. **Step 1: the Cayley measure is a nonzero Haar measure, invariant under inversion.** By Lemma 5.2.41, dy is invariant under left and right translations. The Cayley transform $g \mapsto (g-1)(g+1)^{-1}$ inverts κ ($\kappa(y) = (1 + \frac{1}{2}J_1S(y))(1 - \frac{1}{2}J_1S(y))^{-1}$, and $y \mapsto S(y)$ is a linear bijection onto the skew-hermitian matrices), so κ is a homeomorphism of Ω onto the open neighbourhood $\{g : \det(g+1) \ne 0\}$ of 1 in G_1. For a compact ball $B \subseteq \Omega$ around 0, $\kappa(B)$ is a neighbourhood of 1, and by injectivity of κ, $\int \mathbf{1}_{\kappa(B)} \, dy = \mathrm{vol}\,\mathrm{U}(r)^{-1} \int_B |\det D(y)|^{-2r} \, dy \in (0, \infty)$, the density being continuous and positive on B. By Lemma 5.5.17 (iii), applied to the hermitian involution J_1 with $\mathrm{U}(J_1) = G_1$, dy is finite on compact sets; it is a Radon measure on the locally compact, second countable group G_1, hence a left and right Haar measure, nonzero. Its image under $g \mapsto g^{-1}$ is again a left Haar measure, so equals $c\,dy$ with $c > 0$ by uniqueness of Haar measure; applying inversion twice gives $c^2 = 1$, so dy is invariant under inversion.

Step 2: $\theta = \lambda\, dy$ with $0 < \lambda < \infty$. By Lemma 5.2.61, (iii), θ is right invariant, and by (iv) there, $\theta \ne 0$. For compact C, C^{-1} is compact, and by Lemma 5.5.11 and Step 1, $\theta(C) = \theta^{-1}(C^{-1}) \le \frac{\mathrm{vol}\,\mathrm{U}(r)}{\mathrm{vol}\,\mathrm{U}(k)} \int \mathbf{1}_{C^{-1}} dy < \infty$. So θ is a right-invariant Radon measure on G_1, and by uniqueness of right Haar measure (Weil) there is $\lambda \in [0, \infty)$ with $\theta = \lambda\, dy$; λ is unique as $dy \ne 0$, and $\lambda > 0$ as $\theta \ne 0$.

Step 3: the bound. By Step 1, $\theta^{-1} = \lambda\, dy$. With $E = \kappa(B)$ of Step 1, Lemma 5.5.11 gives $\lambda \int \mathbf{1}_E \, dy \le \frac{\mathrm{vol}\,\mathrm{U}(r)}{\mathrm{vol}\,\mathrm{U}(k)} \int \mathbf{1}_E \, dy$ with $0 < \int \mathbf{1}_E \, dy < \infty$, so $\lambda \le \mathrm{vol}\,\mathrm{U}(r)/\mathrm{vol}\,\mathrm{U}(k)$.

Step 4: unfolding. By Lemma 5.2.59, (iv), with $j = \varrho + 2$, $\theta = \mathfrak{m}_{\varrho+2} = \eta_{<\varrho+2} * \theta_{[0,\varrho+2)}$; here $\eta_{<\varrho+2} = \eta_{<\varrho}$ since $\eta_\varrho = \eta_{\varrho+1} = \delta_1$, and $\theta_{[0,\varrho+2)} = \theta_{\varrho+1} * \cdots * \theta_0 = \theta_{\ge 0}$. By Lemma 5.5.23,

$\eta_{<\varrho}$ is du carried to G_1. Let f be Borel, $f \geq 0$, left U-invariant, and χ the cutoff. By Step 2 and Tonelli (both measures s-finite),

$$\lambda \int_{\mathsf{U}\backslash G_1} f = \lambda \int f\chi \, dy = \int f\chi \, d\theta = \int_{G_1} \int_{\mathsf{U}} f(ug)\chi(ug) \, du \, d\theta_{\geq 0}(g)$$
$$= \int_{G_1} f(g) \int_{\mathsf{U}} \chi(ug) \, du \, d\theta_{\geq 0}(g) = \int f \, d\theta_{\geq 0}.$$

□

Lemma 5.5.11. *Let $\rho > 0$ and let θ be the group measure of the data at $\mathbb{C}$ with box radius ρ. Then the image θ^{-1} of θ under $g \mapsto g^{-1}$ satisfies*

$$\theta^{-1}(B) \leq \frac{\operatorname{vol} \mathrm{U}(r)}{\operatorname{vol} \mathrm{U}(k)} \int_{G_1} \mathbf{1}_B(y) \, dy \qquad \text{for every Borel set } B \subseteq G_1.$$

Proof. All measures below are Borel measures on G. Each of η_l, θ_l, μ_{top}, μ_H is s-finite (images of σ-finite measures, and μ_{top} by Definition 5.1.7); images, finite multiples and convolutions of s-finite measures are s-finite, so all measures that occur are, and Tonelli's theorem applies to all iterated integrals. For a measure μ write μ^{-1} for its image under $g \mapsto g^{-1}$. For a single column $y \in \mathbb{C}^r$, $\Gamma_J(y) = y^*Jy = h(y,y)$, so $\mathrm{Herm}_1 = \mathbb{R}$ with dT the Lebesgue measure, and for a matrix E with columns $\varepsilon_1, \ldots, \varepsilon_n$ the entries of E^*Jy are $h(\varepsilon_j, y)$. For $0 \leq l \leq \varrho$ let $E_l \in \mathrm{M}_{r\times 2l}(\mathbb{C})$ have the columns $e_0, e_0^*, \ldots, e_{l-1}, e_{l-1}^*$, and for $a < \varrho$ let $E'_a = (E_a \; e_a)$; thus $E_{a+1} = (E'_a \; e_a^*)$ and $(E_\varrho \; z) = E^{\mathrm{out}}$ is the matrix of Lemma 5.5.16. The data at $\mathbb{C}$ form a place datum by Lemma 5.5.5, (i).

Step 1: rules for inversion. For s-finite μ, μ' and Borel $f \geq 0$, by Tonelli

$$\int f \, d(\mu * \mu')^{-1} = \iint f(y^{-1}x^{-1}) \, d\mu(x) \, d\mu'(y) = \int f \, d(\mu'^{-1} * \mu^{-1}),$$

so $(\mu * \mu')^{-1} = \mu'^{-1} * \mu^{-1}$; convolution is associative and $(c\mu) * \mu' = c(\mu * \mu') = \mu * (c\mu')$ for $c \in [0, \infty)$. Moreover $\eta_l^{-1} = \eta_l$ for every l: for $l \geq \varrho$, $\eta_l = \delta_1$; for $l < \varrho$, Lemma 5.2.58, (iii), with $\tau = \bar{\tau} = \frac{1}{2}$ gives $n_l(v,\beta)\, n_l(-v,-\beta) = n_l(0, -\frac{1}{2}h(v,v) + \frac{1}{2}h(v,v)) = n_l(0,0) = 1$ (by (5.8)), so $n_l(v,\beta)^{-1} = n_l(-v,-\beta)$, and $(v,\beta) \mapsto (-v,-\beta)$ preserves $dv \otimes \nu$. Finally $\mu_H^{-1} = \mu_H$, because μ_H is the image of the Haar probability measure of the compact group $\mathrm{U}(k)$, which is invariant under inversion, under a homomorphism.

Step 2: the inverse of the group measure. By definition $\mathfrak{m}_{0,0} = \delta_1$ and $\mathfrak{m}_{0,j+1} = (\eta_j * \theta_j) * \mathfrak{m}_{0,j}$, so by Step 1

$$\mathfrak{m}_{0,j+1}^{-1} = \mathfrak{m}_{0,j}^{-1} * \theta_j^{-1} * \eta_j \qquad (j \geq 0).$$

As $\eta_\varrho = \eta_{\varrho+1} = \delta_1$, $\theta_\varrho = \mu_{\mathrm{top}}$ and $\theta_{\varrho+1} = \mu_H$, this gives $\theta^{-1} = \mathfrak{m}_{0,\varrho+2}^{-1} = \mathfrak{m}_{0,\varrho}^{-1} * \mu_{\mathrm{top}}^{-1} * \mu_H$.

Step 3: adding the frame vector of a floor. Let $0 \leq b \leq \varrho$ and let λ be s-finite and below the Gram fibre at E_b. We show that $\lambda * \frac{1}{2}\theta_b^{-1}$ is below the Gram fibre at $(E_b \; f_b)$, by Lemma 5.5.12, (i), with $E_1 = E_b$, $E_2 = f_b$, $\lambda' = \frac{1}{2}\theta_b^{-1}$. By Lemma 5.2.59, (ii), θ_b is carried by the group $G^{(b)}$, hence so is θ_b^{-1}; elements of $G^{(b)}$ fix the coordinate vectors of floor below b, which are the columns of E_b, so $g'E_b = E_b$ for λ'-almost every g'. The image of θ_b^{-1} under $g' \mapsto g'f_b$ is the image of θ_b under $\sigma \mapsto \sigma^{-1}f_b$, which is qv_b by Lemma 5.2.63 with $s = 1$. So the image of λ' under $g' \mapsto g'f_b$ is $\frac{1}{2}\mathrm{qv}_b$, which by Lemma 5.5.14 is below the fibre measure of $y \mapsto (E_b^*Jy, h(y,y))$ over its value at f_b; this is the hypothesis of Lemma 5.5.12, (i), for $n_2 = 1$.

Step 4: adding the dual vector of a floor. Let $0 \le a < \varrho$ and let λ be s-finite and below the Gram fibre at E'_a. Then $\lambda * \frac{1}{2}\eta_a$ is below the Gram fibre at E_{a+1}, by Lemma 5.5.12, (i), with $E_1 = E'_a$, $E_2 = e_a^*$, $\lambda' = \frac{1}{2}\eta_a$: each $n_a(v,\beta)$ fixes e_b, e_b^* $(b < a)$ and e_a (Definition 5.2.1), so the closed set $\{g' : g'E'_a = E'_a\}$ contains the image of n_a and has η_a-null complement; and the image of λ' under $g' \mapsto g'e_a^*$ is below the fibre measure of $y \mapsto (E_a'^* J y, h(y,y))$ over its value at e_a^* by Lemma 5.5.15.

Step 5: induction over the floors. We show that $4^{-l}\mathfrak{m}_{0,l}^{-1}$ is below the Gram fibre at E_l for $0 \le l \le \varrho$. For $l = 0$, $E_0 \in \mathrm{M}_{r\times 0}(\mathbb{C})$ and Herm_0 are points carrying unit masses, and the image of δ_1 under $g \mapsto gE_0$ is the unit mass; if $K \ge 0$ and C satisfy the hypothesis of (BY1), then the open set $S = \mathrm{Herm}_0$ lies in every neighbourhood of $\Gamma_J(E_0)$ and gives $\int K\, d\delta_{E_0} = K(E_0) \le C$. For the step from $a < \varrho$ to $a+1$, Steps 1 and 2 give

$$4^{-(a+1)}\mathfrak{m}_{0,a+1}^{-1} = \left((4^{-a}\mathfrak{m}_{0,a}^{-1}) * \tfrac{1}{2}\theta_a^{-1}\right) * \tfrac{1}{2}\eta_a;$$

by the induction hypothesis and Step 3 with $b = a$ ($f_a = e_a$, so $(E_a\ f_a) = E'_a$) the inner convolution is below the Gram fibre at E'_a, and by Step 4 the whole is below the Gram fibre at E_{a+1}.

Step 6: the top floor and the definite complement. By Step 5 for $l = \varrho$ and Step 3 with $b = \varrho$ ($f_\varrho = z$, $\theta_\varrho = \mu_{\mathrm{top}}$), $\lambda_0 = (4^{-\varrho}\mathfrak{m}_{0,\varrho}^{-1}) * \frac{1}{2}\mu_{\mathrm{top}}^{-1}$ is below the Gram fibre at $(E_\varrho\ z) = E^{\mathrm{out}}$. By Steps 1 and 2, $\theta^{-1} = 2^{2\varrho+1}\,\lambda_0 * \mu_H$, and Lemma 5.5.16 gives, for every Borel set $B \subseteq G_1$,

$$\theta^{-1}(B) = 2^{2\varrho+1}(\lambda_0 * \mu_H)(B) \le 2^{2\varrho+1}\,\frac{\mathrm{vol}\,\mathrm{U}(r)}{\mathrm{vol}\,\mathrm{U}(k)\,2^{2\varrho+1}}\int_{G_1} \mathbf{1}_B\, dy = \frac{\mathrm{vol}\,\mathrm{U}(r)}{\mathrm{vol}\,\mathrm{U}(k)}\int_{G_1}\mathbf{1}_B\, dy.$$

□

Lemma 5.5.12. *Let $J \in \mathrm{M}_N(\mathbb{C})$ be a hermitian involution.*

(i) Let $n_1, n_2 \ge 0$, $E_1 \in \mathrm{M}_{N\times n_1}(\mathbb{C})$, $E_2 \in \mathrm{M}_{N\times n_2}(\mathbb{C})$, and let $E = (E_1\ E_2) \in \mathrm{M}_{N\times(n_1+n_2)}(\mathbb{C})$ have the columns of E_1 followed by those of E_2. Give $\mathrm{M}_{n_1\times n_2}(\mathbb{C})$ the Lebesgue measure dM of the real and imaginary parts of its entries, $\mathrm{M}_{N\times n_2}(\mathbb{C})$ the measure dY and Herm_{n_2} the measure dT, and let

$$\pi\colon \mathrm{M}_{N\times n_2}(\mathbb{C}) \to \mathrm{M}_{n_1\times n_2}(\mathbb{C}) \times \mathrm{Herm}_{n_2}, \qquad \pi(Y) = \big(E_1^* J Y,\ \Gamma_J(Y)\big).$$

Let λ, λ' be s-finite Borel measures on $\mathrm{U}(J)$ such that λ is below the Gram fibre at E_1, $g'E_1 = E_1$ for λ'-almost every g', and the image of λ' under $g' \mapsto g'E_2$ is below the fibre measure $dY/(dM \otimes dT)$ of π over $\pi(E_2)$. Then the convolution $\lambda * \lambda'$ is below the Gram fibre at E.

(ii) Let $m \ge 0$, $E \in \mathrm{M}_{N\times m}(\mathbb{C})$ and let P be an $m \times m$ permutation matrix. If a Borel measure λ on $\mathrm{U}(J)$ is below the Gram fibre at E, then it is below the Gram fibre at EP.

Proof. Throughout, Lebesgue measures on matrix spaces are those of the real and imaginary parts of the entries, dT_j is the measure dT of Herm_j, and for $g \in \mathrm{U}(J)$ we have $|\det g| = 1$ (from $g^*Jg = J$ and $|\det J| = 1$), so left multiplication by g, acting on each column, preserves Lebesgue measure on $\mathrm{M}_{N\times j}(\mathbb{C})$ (its real determinant is $|\det g|^{2j}$), and $\Gamma_J(gY) = \Gamma_J(Y)$.

(i), Step 1: the first block. Let $n = n_1 + n_2$ and write $Y \in \mathrm{M}_{N\times n}(\mathbb{C})$ as $Y = (Y_1\ Y_2)$ with Y_1 its first n_1 columns. Put $\pi_1(Y) = \Gamma_J(Y_1)$, $\pi_2(Y) = (Y_1^* J Y_2, \Gamma_J(Y_2))$ and $\mathrm{mv}(g,y) = (gE_1\ gy)$

for $g \in \mathrm{U}(J)$, $y \in \mathrm{M}_{N\times n_2}(\mathbb{C})$. Let ℓ_1 be the image of λ under $g \mapsto gE_1$, and κ the image of $\lambda \otimes dy$ under mv, dy the Lebesgue measure of $\mathrm{M}_{N\times n_2}(\mathbb{C})$. Substituting gy for y for fixed g (Tonelli),

$$\int f\,d\kappa = \iint f((gE_1\ gy))\,dy\,d\lambda(g) = \iint f((gE_1\ y))\,dy\,d\lambda(g) = \int f\,d\varphi_*(dy\otimes\ell_1)$$

for Borel $f \geq 0$, where $\varphi(y, Y_1) = (Y_1\ y)$. By hypothesis ℓ_1 is below the fibre measure dY_1/dT_{n_1} of Γ_J over $\Gamma_J(E_1)$. By Lemma 5.5.13, (v), with the parameter space $\mathrm{M}_{N\times n_2}(\mathbb{C})$ and dy, the continuous maps $Y_1 \mapsto \varphi(y, Y_1)$ and $\pi_0 = \pi_1$ (indeed $\pi_1(\varphi(y, Y_1)) = \Gamma_J(Y_1)$), the measure $\kappa = \varphi_*(dy \otimes \ell_1)$ is below the fibre measure of π_1 over $\Gamma_J(E_1)$ for $\varphi_*(dy \otimes dY_1)$ and dT_{n_1}; and $\varphi_*(dy \otimes dY_1)$ is the Lebesgue measure dY of $\mathrm{M}_{N\times n}(\mathbb{C})$, φ being a permutation of coordinates.

Step 2: the law of gE. Let ℓ_2 be the image of λ' under $g' \mapsto g'E_2$. As $g'E_1 = E_1$ for λ'-almost every g', $\mathrm{mv}(g, g'E_2) = (gg'E_1\ gg'E_2) = gg'E$ for $\lambda \otimes \lambda'$-almost every (g, g'), so the image of $\lambda \otimes \ell_2$ under mv is the image of $\lambda * \lambda'$ under $h \mapsto hE$; call it ℓ.

Step 3: the second block. By hypothesis ℓ_2 is below the fibre measure $dy/(dM \otimes dT_{n_2})$ of π over $\pi(E_2)$. For $g \in \mathrm{U}(J)$, $\pi_2(\mathrm{mv}(g, y)) = (E_1^* g^* Jgy, \Gamma_J(gy)) = \pi(y)$, and $y \mapsto \mathrm{mv}(g, y)$ is continuous. By Lemma 5.5.13, (v), with the parameter space $\mathrm{U}(J)$ and λ, $\ell = \mathrm{mv}_*(\lambda \otimes \ell_2)$ is below the fibre measure of π_2 over $\pi(E_2)$ for $\mathrm{mv}_*(\lambda \otimes dy) = \kappa$ and $dM \otimes dT_{n_2}$.

Step 4: composition and the blocks of the Gram matrix. As π_2 is continuous and the base measures are s-finite, Steps 1 and 3 and Lemma 5.5.13, (iv), show that ℓ is below the fibre measure $dY/(dT_{n_1} \otimes dM \otimes dT_{n_2})$ of $Y \mapsto (\pi_1(Y), \pi_2(Y))$ over $(\Gamma_J(E_1), \pi(E_2))$. Let $\mathrm{bl}\colon \mathrm{Herm}_n \to \mathrm{Herm}_{n_1} \times \mathrm{M}_{n_1\times n_2}(\mathbb{C}) \times \mathrm{Herm}_{n_2}$ send T to its blocks (T_{11}, T_{12}, T_{22}). It is a linear bijection ($T_{21} = T_{12}^*$), and it permutes coordinates: the diagonal entries of T are those of T_{11} and T_{22}, and the entries T_{ab} with $a < b$ are the entries above the diagonal of T_{11} or T_{22}, or, when $a \leq n_1 < b$, the entries of T_{12}. So bl is a homeomorphism carrying dT_n to $dT_{n_1} \otimes dM \otimes dT_{n_2}$. Since $\Gamma_J(Y) = \left(\begin{smallmatrix} Y_1^*JY_1 & Y_1^*JY_2 \\ Y_2^*JY_1 & Y_2^*JY_2 \end{smallmatrix}\right)$, $(\pi_1, \pi_2) = \mathrm{bl} \circ \Gamma_J$, and $(\Gamma_J(E_1), \pi(E_2)) = \mathrm{bl}(\Gamma_J(E))$. By Lemma 5.5.13, (iii), with the homeomorphism bl^{-1}, ℓ is below the fibre measure dY/dT_n of Γ_J over $\Gamma_J(E)$, that is, $\lambda * \lambda'$ is below the Gram fibre at E.

(ii). Let ℓ be the image of λ under $g \mapsto gE$, $r_P(Y) = YP$ on $\mathrm{M}_{N\times m}(\mathbb{C})$ and $c_P(T) = P^{\mathsf{T}}TP$ on Herm_m. The map r_P permutes columns, so it is continuous and preserves Lebesgue measure. If P sends the j-th standard basis vector to the $p(j)$-th, then $c_P(T)_{ab} = T_{p(a)p(b)}$: c_P permutes the diagonal entries, and sends each entry above the diagonal to an entry above the diagonal or to the conjugate of one; so c_P is a homeomorphism carrying dT to dT. As P is real, $\Gamma_J(YP) = c_P(\Gamma_J(Y))$. By Lemma 5.5.13, (iii), ℓ is below the fibre measure dY/dT of $c_P \circ \Gamma_J = \Gamma_J \circ r_P$ over $c_P(\Gamma_J(E)) = \Gamma_J(EP)$, and by Lemma 5.5.13, (vi), the image of ℓ under r_P, which is the image of λ under $g \mapsto gEP$, is below the fibre measure of Γ_J over $\Gamma_J(EP)$ for $(r_P)_* dY = dY$ and dT. □

Lemma 5.5.13. *In this lemma every topological space is separable and metrizable and carries its Borel σ-algebra; products carry the product topology, whose Borel σ-algebra is the product σ-algebra. Let $\pi\colon X \to B$ be Borel, μ a Borel measure on X, ν a Borel measure on B and $\beta \in B$; "below $d\mu/d\nu$" means below the fibre measure $d\mu/d\nu$ of π over β when π, β are clear.*

(i) (Parametrisations.) Suppose $\nu(O) > 0$ for every nonempty open $O \subseteq B$ and every point of B has a neighbourhood of finite ν-measure. Let V be a neighbourhood of β, P a space with an s-finite Borel measure μ_P, and $\psi\colon B \times P \to X$, $w\colon B \times P \to [0, \infty]$ Borel maps such that: for every $t \in V$, $\pi(\psi(t, p)) = t$ for μ_P-almost every p with $w(t, p) \neq 0$; for μ_P-almost every p the maps $t \mapsto \psi(t, p)$ and $t \mapsto w(t, p)$ are continuous at β; and

$$\int_V \int_P w(t,p)\,K(\psi(t,p))\,d\mu_P(p)\,d\nu(t) \leq \int_X K\,d\mu \qquad \text{for every Borel } K\colon X \to [0,\infty].$$

Then the image of the measure $w(\beta,p)\,d\mu_P(p)$ under $p\mapsto\psi(\beta,p)$ is below the fibre measure $d\mu/d\nu$ of π over β.

(ii) (Measure-preserving parametrisations.) Suppose ν is as in (i), μ_P is an s-finite Borel measure on a space P, and $\varphi\colon B\times P\to X$ is a Borel map carrying $\nu\otimes\mu_P$ to μ, with $\pi(\varphi(t,p))=t$ for all (t,p) and $t\mapsto\varphi(t,p)$ continuous at β for every p. Then the image of μ_P under $p\mapsto\varphi(\beta,p)$ is below the fibre measure $d\mu/d\nu$ of π over β.

(iii) (Change of base.) Let $e\colon B\to B'$ be a homeomorphism carrying ν to a measure ν'. If λ is below the fibre measure $d\mu/d\nu$ of π over β, then λ is below the fibre measure $d\mu/d\nu'$ of $e\circ\pi$ over $e(\beta)$.

(iv) (Two steps.) Let ν, ν' be s-finite, ν' a Borel measure on B', $\pi_2\colon X\to B'$ continuous and $\beta'\in B'$. If λ is below the fibre measure $d\mu/d\nu$ of π over β and λ' is below the fibre measure $d\lambda/d\nu'$ of π_2 over β', then λ' is below the fibre measure $d\mu/(\nu\otimes\nu')$ of $x\mapsto(\pi(x),\pi_2(x))$ over (β,β').

(v) (Families.) Let μ and λ be s-finite, μ_P an s-finite Borel measure on a space P, $\varphi\colon P\times X\to X'$ a Borel map with $x\mapsto\varphi(p,x)$ continuous for every p, and $\pi_0\colon X'\to B$ Borel with $\pi_0(\varphi(p,x))=\pi(x)$ for all (p,x). If λ is below the fibre measure $d\mu/d\nu$ of π over β, then the image of $\mu_P\otimes\lambda$ under φ is below the fibre measure of π_0 over β for the image of $\mu_P\otimes\mu$ under φ and ν.

(vi) (Images.) Let $\varphi\colon X\to X'$ be continuous and $\pi_0\colon X'\to B$ Borel. If λ is below the fibre measure $d\mu/d\nu$ of $\pi_0\circ\varphi$ over β, then the image of λ under φ is below the fibre measure of π_0 over β for the image of μ under φ and ν.

Proof. All functions below take values in $[0,\infty]$, with $0\cdot\infty=0$; Tonelli's theorem is used for s-finite measures, and jointly Borel integrands, on products of the spaces above. A function A is lower semicontinuous at x_0 if for every $c<A(x_0)$ the set $\{A>c\}$ is a neighbourhood of x_0; in a metrizable space this is equivalent to $A(x_0)\le\liminf A(x_j)$ for every sequence $x_j\to x_0$.

Two auxiliary facts. (A) Let μ_P be a measure on P and $F\colon X\times P\to[0,\infty]$ with $F(x,\cdot)$ Borel for every x and $x\mapsto F(x,p)$ lower semicontinuous at x_0 for μ_P-almost every p. Then $x\mapsto\int F(x,p)\,d\mu_P(p)$ is lower semicontinuous at x_0: for $x_j\to x_0$, Fatou's lemma gives $\int F(x_0,p)\,d\mu_P\le\int\liminf_j F(x_j,p)\,d\mu_P\le\liminf_j\int F(x_j,p)\,d\mu_P$. (B) Let ν be as in (i), let $A\colon B\to[0,\infty]$ be lower semicontinuous at β and Borel, $C\in[0,\infty]$, and U a neighbourhood of β with $\int_S A\,d\nu\le C\,\nu(S)$ for every open $S\subseteq U$. Then $A(\beta)\le C$: for $c<A(\beta)$ there is an open $S\ni\beta$ inside U, inside $\{A>c\}$ and of finite measure; then $0<\nu(S)<\infty$ and $c\,\nu(S)\le\int_S A\,d\nu\le C\,\nu(S)$, so $c\le C$.

(i). Let $K\colon X\to[0,\infty]$ be lower semicontinuous, $C\in[0,\infty]$ and U a neighbourhood of β with $\int_{\pi^{-1}(S)}K\,d\mu\le C\nu(S)$ for open $S\subseteq U$. Put $A(t)=\int_P w(t,p)K(\psi(t,p))\,d\mu_P(p)$, a Borel function of t by Tonelli; the integral of K against the image measure of (i) is $A(\beta)$. For almost every p, $t\mapsto K(\psi(t,p))$ is lower semicontinuous at β and $t\mapsto w(t,p)$ is continuous there, so their product is lower semicontinuous at β (for $t_j\to\beta$, $w(\beta,p)K(\psi(\beta,p))\le\lim_j w(t_j,p)\cdot\liminf_j K(\psi(t_j,p))\le\liminf_j w(t_j,p)K(\psi(t_j,p))$, where the second inequality holds as the middle product is 0 if a factor is 0, and otherwise, for finite $0<a'<\lim_j w(t_j,p)$ and $0<b'<\liminf_j K(\psi(t_j,p))$, the products $w(t_j,p)K(\psi(t_j,p))$ eventually exceed $a'b'$); by (A), A is lower semicontinuous at β. Let $S\subseteq U\cap V$ be open. For $t\in S$ and almost every p, either $w(t,p)=0$ or $\psi(t,p)\in\pi^{-1}(S)$, so $w(t,p)K(\psi(t,p))=w(t,p)(\mathbf{1}_{\pi^{-1}(S)}K)(\psi(t,p))$; hence,

applying the hypothesis to the Borel function $\mathbf{1}_{\pi^{-1}(S)}K$,

$$\int_S A\,d\nu \le \int_V \int_P w(t,p)\,(\mathbf{1}_{\pi^{-1}(S)}K)(\psi(t,p))\,d\mu_P\,d\nu \le \int_{\pi^{-1}(S)} K\,d\mu \le C\,\nu(S).$$

By (B) with the neighbourhood $U\cap V$, $A(\beta)\le C$.

(ii). The measure ν is σ-finite: B is Lindelöf and covered by open sets of finite measure. Apply (i) with $V=B$, $\psi=\varphi$ and $w\equiv 1$: the fibre and continuity conditions hold for all t, p, and for Borel K, by Tonelli and as φ carries $\nu\otimes\mu_P$ to μ, $\int_B\int_P K(\varphi(t,p))\,d\mu_P\,d\nu = \int K\circ\varphi\,d(\nu\otimes\mu_P) = \int K\,d\mu$.

(iii). Let K, C and a neighbourhood U' of $e(\beta)$ be such that $\int_{(e\circ\pi)^{-1}(S')} K\,d\mu \le C\nu'(S')$ for open $S'\subseteq U'$. Then $e^{-1}(U')$ is a neighbourhood of β, and for open $S\subseteq e^{-1}(U')$ the set $S'=e(S)$ is open in U', $(e\circ\pi)^{-1}(S') = \pi^{-1}(S)$ and $\nu'(S') = \nu(e^{-1}(S')) = \nu(S)$. So the hypothesis of (BY1) for λ, π, β holds, and $\int K\,d\lambda\le C$.

(iv). Let K be lower semicontinuous, $C\in[0,\infty]$, and U a neighbourhood of (β,β') with $\int_{(\pi,\pi_2)^{-1}(S)} K\,d\mu \le C\,(\nu\otimes\nu')(S)$ for open $S\subseteq U$. Choose open $W\ni\beta$, $W'\ni\beta'$ with $W\times W'\subseteq U$, and let $S'\subseteq W'$ be open. The function $K'=\mathbf{1}_{\pi_2^{-1}(S')}K$ is lower semicontinuous ($\pi_2^{-1}(S')$ is open), and for open $S\subseteq W$

$$\int_{\pi^{-1}(S)} K'\,d\mu = \int_{(\pi,\pi_2)^{-1}(S\times S')} K\,d\mu \le C\,\nu(S)\,\nu'(S') = \big(C\nu'(S')\big)\,\nu(S),$$

as $S\times S'\subseteq U$ is open. Since λ is below $d\mu/d\nu$, $\int_{\pi_2^{-1}(S')} K\,d\lambda = \int K'\,d\lambda \le C\nu'(S')$ for every open $S'\subseteq W'$, and since λ' is below $d\lambda/d\nu'$, $\int K\,d\lambda'\le C$.

(v). Let $K\colon X'\to[0,\infty]$ be lower semicontinuous, C and U as in (BY1) for π_0 and the image $\varphi_*(\mu_P\otimes\mu)$. Put $K'(x)=\int_P K(\varphi(p,x))\,d\mu_P(p)$. For each p, $x\mapsto K(\varphi(p,x))$ is lower semicontinuous, so K' is lower semicontinuous by (A), and $\int K\,d\varphi_*(\mu_P\otimes\lambda) = \int K'\,d\lambda$ by Tonelli. For open $S\subseteq U$, as $\mathbf{1}_{\pi^{-1}(S)}(x) = \mathbf{1}_{\pi_0^{-1}(S)}(\varphi(p,x))$,

$$\int_{\pi^{-1}(S)} K'\,d\mu = \int \mathbf{1}_{\pi_0^{-1}(S)}(\varphi(p,x))\,K(\varphi(p,x))\,d(\mu_P\otimes\mu)(p,x) = \int_{\pi_0^{-1}(S)} K\,d\varphi_*(\mu_P\otimes\mu) \le C\,\nu(S),$$

so $\int K'\,d\lambda\le C$.

(vi). For lower semicontinuous K on X', $K\circ\varphi$ is lower semicontinuous, $\int_{(\pi_0\circ\varphi)^{-1}(S)} K\circ\varphi\,d\mu = \int_{\pi_0^{-1}(S)} K\,d\varphi_*\mu$, and $\int K\circ\varphi\,d\lambda = \int K\,d\varphi_*\lambda$; so the hypothesis of (BY1) for $\varphi_*\lambda$ is the hypothesis for λ, and the conclusion follows. □

Lemma 5.5.14. *Let $\rho>0$, consider the data at $\mathbb{C}$ with box radius ρ, and let $0\le b\le \varrho$. Let $E_b\in \mathrm{M}_{r\times 2b}(\mathbb{C})$ be the matrix with the columns $e_0, e_0^*, \dots, e_{b-1}, e_{b-1}^*$, and*

$$\pi_b\colon V\to\mathbb{C}^{2b}\times\mathbb{R},\qquad \pi_b(y) = \big(E_b^*Jy,\ h(y,y)\big),$$

so that the entries of E_b^*Jy are the products $h(e_a,y)$, $h(e_a^*,y)$, $a<b$. Give V the Lebesgue measure dy of the real and imaginary parts of the coordinates, and $\mathbb{C}^{2b}\times\mathbb{R}$ the Lebesgue measure $dc\otimes dt$. Then $\frac12\mathrm{qv}_b$ is below the fibre measure $dy/(dc\otimes dt)$ of π_b over $\pi_b(f_b)=(0,h(f_b,f_b))$.

Proof. For $a<\varrho$, $Je_a=e_a^*$ and $Je_a^*=e_a$, so $h(e_a,y)$ and $h(e_a^*,y)$ are the coordinates of y at e_a^* and at e_a. In particular $E_b^*Jf_b=0$, as $f_b\in\{e_b,z\}$ has no coordinate of floor below b, and $\pi_b(f_b)=(0,h(f_b,f_b))$. Put $t_0=h(f_b,f_b)$.

Step 1: the products with the lower basis vectors. Let I_b be the set of indices of floor below b and I_b' its complement, so $V=\mathbb{C}^{I_b}\times\mathbb{C}^{I_b'}$ and dy is the product of the Lebesgue

measures. For $c \in \mathbb{C}^{2b}$ and $v \in \mathbb{C}^{I'_b}$ let $\phi(c,v) \in V$ have the coordinates c_{2a+1} at e_a^* and c_{2a+2} at e_a ($a < b$) and v at the indices in I'_b. Then ϕ is a permutation of coordinates, so it carries $dc \otimes dv$ to dy; $E_b^* J \phi(c,v) = c$; and ϕ is continuous. By Lemma 5.5.13, (ii), at the base point 0 of $\mathbb{C}^{2b}$, the image of dv under $v \mapsto \phi(0,v)$, which is vol_{V_b} (as I'_b is the set of indices of floor at least b), is below the fibre measure dy/dc of $y \mapsto E_b^* J y$ over 0.

Step 2: the norm. We show that $\frac{1}{2}\mathrm{qv}_b$ is below the fibre measure $d\mathrm{vol}_{V_b}/dt$ of $y \mapsto h(y,y)$ over t_0, by Lemma 5.5.13, (i), with the space V and its measure vol_{V_b}, the base space $\mathbb{R}$ with the measure dt, the base point t_0 and its neighbourhood $\mathbb{R}$, $P = \mathbb{C} \times Y_b \times \mathrm{i}\mathbb{R}$ with $\mu_P = \mathrm{vol} \otimes \mathrm{vol} \otimes \nu$, $w \equiv \frac{1}{2}$, and

$$\psi(t,(s,Y,\alpha)) = B_b(s^\dagger \tfrac{1}{2}(t - h'_b(Y,Y)) + s\alpha,\ s,\ Y),$$

the flat chart (5.4) with parameter δt (as $\tau\delta^{-1} \cdot \delta t = t/2$); ψ is Borel and affine in t, and the image of $\frac{1}{2}\mu_P$ under $\psi(t_0,\cdot)$ is $\frac{1}{2}\mathrm{qv}_b(\delta t_0) = \frac{1}{2}\mathrm{qv}_b$. Fibres: for $s \neq 0$, so for μ_P-almost every p, $h(\psi(t,p),\psi(t,p)) = t$.

Bound: for Borel $K \geq 0$, by Tonelli, then the substitution $\beta = \delta t = \mathrm{i}t/2$, which carries $\frac{1}{2}dt$ to ν, and the disintegration (DJ2) of vol_{V_b} along $h(y,y)$ by $(\mathrm{qv}_b(\beta))_\beta$ (Lemma 5.2.52, (ii), for the place $\mathbb{C}$ of Lemma 5.5.7 and the place datum of Lemma 5.5.5, (i)) applied to $f(c,y) = K(y)$,

$$\int_{\mathbb{R}} \int_P \tfrac{1}{2} K(\psi(t,p))\, d\mu_P\, dt = \int_{\mathbb{R}} \int K\, d\mathrm{qv}_b(\delta t)\, \tfrac{1}{2} dt = \int_{\mathrm{i}\mathbb{R}} \int K\, d\mathrm{qv}_b(\beta)\, d\nu(\beta) = \int K\, d\mathrm{vol}_{V_b}$$

(the function $\beta \mapsto \int K\, d\mathrm{qv}_b(\beta)$ is Borel by Tonelli, so the outer integral in (DJ2) is the ordinary one). Lebesgue measure on $\mathbb{R}$ is positive on nonempty open sets and locally finite, so Lemma 5.5.13, (i), applies.

Step 3: composition. As $y \mapsto h(y,y)$ is continuous, Steps 1 and 2 and Lemma 5.5.13, (iv), show that $\frac{1}{2}\mathrm{qv}_b$ is below the fibre measure $dy/(dc \otimes dt)$ of π_b over $(0,t_0) = \pi_b(f_b)$. □

Lemma 5.5.15. *Let $\rho > 0$, consider the data at $\mathbb{C}$ with box radius ρ, and let $0 \leq a < \varrho$. Let $E'_a \in \mathrm{M}_{r \times (2a+1)}(\mathbb{C})$ be the matrix with the columns $e_0, e_0^*, \dots, e_{a-1}, e_{a-1}^*, e_a$, and*

$$\pi'_a \colon V \to \mathbb{C}^{2a+1} \times \mathbb{R}, \qquad \pi'_a(y) = (E_a'^* J y,\ h(y,y)).$$

Give V the Lebesgue measure dy of the real and imaginary parts of the coordinates, and $\mathbb{C}^{2a+1} \times \mathbb{R}$ the Lebesgue measure $dc \otimes dt$. Then one half of the image of η_a under $G \to V$, $u \mapsto ue_a^*$, is below the fibre measure $dy/(dc \otimes dt)$ of π'_a over $\pi'_a(e_a^*) = (\epsilon, 0)$, where $\epsilon \in \mathbb{C}^{2a+1}$ has last entry $h(e_a, e_a^*) = 1$ and all other entries 0.

Proof. For $b < \varrho$, $h(e_b, y)$ and $h(e_b^*, y)$ are the coordinates of y at e_b^* and e_b ($Je_b = e_b^*$, $Je_b^* = e_b$). So the entries of $E_a'^* J y$ are the coordinates of y at $e_0^*, e_0, \dots, e_{a-1}^*, e_{a-1}, e_a^*$, and $E_a'^* J e_a^* = \epsilon$; also $h(e_a^*, e_a^*) = 0$, so $\pi'_a(e_a^*) = (\epsilon, 0)$.

Step 1: the products. The indices not listed above are e_a and the indices of floor at least $a+1$. For $c \in \mathbb{C}^{2a+1}$, $\mu' \in \mathbb{C}$ and $v \in V_{a+1}$ let $\phi(c,(\mu',v)) \in V$ have the coordinates c_{2b+1} at e_b^*, c_{2b+2} at e_b ($b < a$), c_{2a+1} at e_a^*, μ' at e_a, and the coordinates of v at the indices of floor at least $a+1$. It is a permutation of coordinates, hence carries $dc \otimes d\mu' \otimes dv$ to dy, it is continuous, and $E_a'^* J \phi(c,(\mu',v)) = c$. By Lemma 5.5.13, (ii), at the base point ϵ, the image λ_a of $d\mu' \otimes dv$ under $(\mu',v) \mapsto \phi(\epsilon,(\mu',v)) = e_a^* + v + \mu' e_a$ is below the fibre measure dy/dc of $y \mapsto E_a'^* J y$ over ϵ.

Step 2: the norm. For $\mu' \in \mathbb{C}$ and $v \in V_{a+1}$, as e_a, e_a^* are isotropic with $h(e_a, e_a^*) = 1$ and orthogonal to V_{a+1},

$$h(e_a^* + v + \mu' e_a,\ e_a^* + v + \mu' e_a) = 2\,\mathrm{Re}\,\mu' + h(v,v).$$

Apply Lemma 5.5.13, (i), with the space V and the measure λ_a, the map $y \mapsto h(y,y)$, the base space $\mathbb{R}$ with dt, the base point 0 and its neighbourhood $\mathbb{R}$, the parameter space $V_{a+1} \times \mathrm{i}\mathbb{R}$ with $dv \otimes \nu$, $w \equiv \frac{1}{2}$ and

$$\psi(t,(v,\beta)) = e_a^* + v + (\tfrac{t}{2} + \beta - \tfrac{1}{2}h(v,v))e_a.$$

Since $\operatorname{Re}\beta = 0$ and $h(v,v)$ is real, $h(\psi(t,(v,\beta)),\psi(t,(v,\beta))) = t$ for all t, v, β; ψ is continuous; and $\psi(0,(v,\beta)) = n_a(v,\beta)e_a^*$, so the image of $\frac{1}{2}dv \otimes \nu$ under $\psi(0,\cdot)$ is one half of the image of η_a under $u \mapsto ue_a^*$. For Borel $K \geq 0$, by Tonelli, then because $(t,\beta) \mapsto \frac{t}{2} + \beta$ carries $\frac{1}{2}dt \otimes \nu$ to Lebesgue measure $d\mu'$ of $\mathbb{C}$ (real part $t/2$, imaginary part the coordinate of β), and by the translation invariance of $d\mu'$,

$$\int_{\mathbb{R}} \iint \tfrac{1}{2}K(\psi(t,(v,\beta)))\,d\nu(\beta)\,dv\,dt = \int_{V_{a+1}} \int_{\mathbb{C}} K(e_a^* + v + (\mu' - \tfrac{1}{2}h(v,v))e_a)\,d\mu'\,dv = \int K\,d\lambda_a.$$

So one half of the image of η_a under $u \mapsto ue_a^*$ is below the fibre measure $d\lambda_a/dt$ of $y \mapsto h(y,y)$ over 0.

Step 3: composition. As $y \mapsto h(y,y)$ is continuous, Steps 1 and 2 and Lemma 5.5.13, (iv), give the claim over $(\epsilon, 0) = \pi'_a(e_a^*)$. □

Lemma 5.5.16. *Let $E^{\mathrm{out}} \in \mathrm{M}_{r\times(2\varrho+1)}(\mathbb{C})$ be the matrix with the columns $e_0, e_0^*, \dots, e_{\varrho-1}, e_{\varrho-1}^*, z$. If λ is an s-finite Borel measure on G_1 below the Gram fibre at E^{out} (for $J = J_1$), then*

$$(\lambda * \mu_H)(B) \leq \frac{\operatorname{vol}\mathrm{U}(r)}{\operatorname{vol}\mathrm{U}(k)\,2^{2\varrho+1}} \int_{G_1} \mathbf{1}_B(y)\,dy \qquad \text{for every Borel set } B \subseteq G_1.$$

Proof. Write $J = J_1$, let $X \in \mathrm{M}_{r\times k}(\mathbb{C})$ have the columns $x_1, \dots, x_k$, let $c = \operatorname{vol}\mathrm{U}(k)/2^k$, and let $\iota\colon \mathrm{U}(k) \to G_1$ be the embedding of (RP4), $\iota(h)x_j = \sum_i h_{ij}x_i$, $\iota(h)$ fixing e_a, e_a^*, z; so $\iota(h)X = Xh$ and $\iota(h)E^{\mathrm{out}} = E^{\mathrm{out}}$. Matrix spaces carry the Lebesgue measures of the real and imaginary parts of the entries, Herm_k the measure dT_k. As $T = 1$ on the x_i and the x_i are orthogonal to the other coordinate vectors, $JX = X$; and $X^*X = 1$, $E^{\mathrm{out}*}E^{\mathrm{out}} = 1$, $E^{\mathrm{out}*}X = 0$, the columns being distinct standard basis vectors.

Step 1: bases of X_0. The Cayley measure dh of $\mathrm{U}(1_k) = \mathrm{U}(k)$ is invariant under left translations (Lemma 5.2.41), has total mass 1 (Lemma 5.5.21) and is finite on compact sets and outer regular (Lemma 5.5.17, (iii)); by uniqueness of Haar measure it is the Haar probability measure of $\mathrm{U}(k)$, so μ_H is the image of dh under ι. By Lemma 5.5.17, (ii), with $N = k$ and $J = 1_k$, the measure $c\,dh$, viewed on $\mathrm{M}_k(\mathbb{C})$, is below the fibre measure dx/dT_k of $x \mapsto x^*x$ over 1_k. Let $\varphi(x) = Xx$; then $\Gamma_J(\varphi(x)) = x^*X^*JXx = x^*x$. By Lemma 5.5.13, (vi), the image of $c\,dh$ under $h \mapsto Xh = \iota(h)X$, which is the image of $c\mu_H$ under $g \mapsto gX$, is below the fibre measure of Γ_J over $1_k = \Gamma_J(X)$ for $\varphi_* dx$ and dT_k.

Step 2: the products with E^{out}. By Lemma 5.5.20 with $E = E^{\mathrm{out}}$, $R = X$, $n_1 = 2\varrho + 1$, $n_2 = n = k$ and $M = 0$, $\varphi_* dx$ is below the fibre measure dy/dM of $y \mapsto E^{\mathrm{out}*}Jy$ on $\mathrm{M}_{r\times k}(\mathbb{C})$ over $0 = E^{\mathrm{out}*}JX$. As Γ_J is continuous, Lemma 5.5.13, (iv), and Step 1 show that the image of $c\mu_H$ under $g \mapsto gX$ is below the fibre measure $dy/(dM \otimes dT_k)$ of $y \mapsto (E^{\mathrm{out}*}Jy, \Gamma_J(y))$ over $(E^{\mathrm{out}*}JX, \Gamma_J(X))$.

Step 3: the full basis. Every element of the image of ι fixes E^{out}, so $gE^{\mathrm{out}} = E^{\mathrm{out}}$ for μ_H-almost every g. By Step 2 and Lemma 5.5.12, (i), with $E_1 = E^{\mathrm{out}}$, $E_2 = X$ and the s-finite measures λ, $c\mu_H$, the convolution $\lambda * c\mu_H$ is below the Gram fibre at $P_0 = (E^{\mathrm{out}}\ X)$. The columns of P_0 are the r coordinate vectors, each once, so P_0 is a permutation matrix, and by

Lemma 5.5.12, (ii), with $P = P_0^{-1}$, $\lambda * c\mu_H$ is below the Gram fibre at 1_r. By Lemma 5.5.17, (i), for $J = J_1$, for every Borel $B \subseteq G_1$

$$c\,(\lambda * \mu_H)(B) = (\lambda * c\mu_H)(B) \leq \frac{\operatorname{vol} \mathrm{U}(r)}{2^r} \int \mathbf{1}_B \, dy,$$

and as $r - k = 2\varrho + 1$, $(\lambda * \mu_H)(B) \leq \frac{\operatorname{vol} \mathrm{U}(r)}{\operatorname{vol} \mathrm{U}(k)\, 2^{2\varrho+1}} \int \mathbf{1}_B \, dy$. □

Lemma 5.5.17. *Let $J \in \mathrm{M}_N(\mathbb{C})$ be a hermitian involution, dg the Cayley measure of $\mathrm{U}(J)$ and $c_N = \operatorname{vol} \mathrm{U}(N)/2^N$.*

(i) If a Borel measure λ on $\mathrm{U}(J)$ is below the Gram fibre at the identity matrix 1_N, then $\lambda(B) \leq c_N \int \mathbf{1}_B \, dg$ for every Borel set $B \subseteq \mathrm{U}(J)$.

(ii) The measure $c_N \, dg$ is below the Gram fibre at 1_N.

(iii) dg is finite on compact subsets of $\mathrm{U}(J)$, and $\int \mathbf{1}_B \, dg = \inf\{\int \mathbf{1}_O \, dg : O \supseteq B \text{ open}\}$ for every Borel set $B \subseteq \mathrm{U}(J)$.

Proof. **(iii).** By Lemma 5.5.19, (i), γ is a continuous injection of the open set $\mathcal{D}_J \ni 0$ onto the open subset $\mathrm{U}^\circ(J) = \{g : \det(1+g) \neq 0\}$ of $\mathrm{U}(J)$, with continuous inverse $g \mapsto S^{-1}(2J(g-1)(g+1)^{-1})$: if $g = \gamma(y) = 2D(y)^{-1} - 1$, then $1 - \frac{1}{2}JS(y) = D(y) = 2(g+1)^{-1}$, so $S(y) = 2J(1 - 2(g+1)^{-1}) = 2J(g-1)(g+1)^{-1}$ as $J^2 = 1$. Let $\bar{\Delta} \subseteq \mathcal{D}_J$ be a closed ball around 0. Then $\gamma(\bar{\Delta})$ is compact, contains the open neighbourhood $\gamma(\bar{\Delta}^\circ)$ of $\gamma(0) = 1$, and by injectivity $dg(\gamma(\bar{\Delta})) = \operatorname{vol} \mathrm{U}(N)^{-1} \int_{\bar{\Delta}} |\det D(y)|^{-2N} dy < \infty$, the density being continuous on the compact $\bar{\Delta}$. A compact $C \subseteq \mathrm{U}(J)$ is covered by finitely many open sets $g\gamma(\bar{\Delta}^\circ)$, each of measure at most $dg(\gamma(\bar{\Delta}))$ by Lemma 5.2.41; so $dg(C) < \infty$. As $\mathrm{U}(J)$ is closed in $\mathrm{M}_N(\mathbb{C})$, it is locally compact, separable and metrizable, and a Borel measure on such a space that is finite on compact sets is outer regular (Rudin, Real and Complex Analysis, Theorem 2.18). In particular dg is σ-finite.

(ii). Let $\Omega \ni J$ and φ be as in Lemma 5.5.18, (i). Apply Lemma 5.5.13, (i), with the space $\mathrm{M}_N(\mathbb{C})$ and dY, the base space Herm_N and dT (Lebesgue measure: positive on nonempty open sets, locally finite), the map Γ_J, the base point J and its neighbourhood Ω, the parameter space $\mathrm{U}(J)$ with dg, $w \equiv c_N$, and $\psi(T, g) = g\varphi(T)$ for $T \in \Omega$, $\psi(T, g) = g$ otherwise (Borel, as Ω is open and φ continuous). For $T \in \Omega$, $\Gamma_J(g\varphi(T)) = \varphi(T)^* g^* J g \varphi(T) = \Gamma_J(\varphi(T)) = T$; $T \mapsto g\varphi(T)$ is continuous at J; and for Borel $K \geq 0$, by Lemma 5.5.18, (ii),

$$\int_\Omega \int_{\mathrm{U}(J)} c_N \, K(g\varphi(T)) \, dg \, dT = \int_{\Gamma_J^{-1}(\Omega)} K \, dY \leq \int K \, dY.$$

So the image of $c_N \, dg$ under $g \mapsto g\varphi(J) = g$ is below the fibre measure dY/dT of Γ_J over J, which is (ii).

(i). Let $O \subseteq \mathrm{U}(J)$ be open, $O = O' \cap \mathrm{U}(J)$ with O' open in $\mathrm{M}_N(\mathbb{C})$, and Ω, φ as in (ii). The map $Y \mapsto Y\varphi(\Gamma_J(Y))^{-1}$ is continuous on the open set $\Gamma_J^{-1}(\Omega)$ (φ is continuous with invertible values), so

$$W = \{Y \in \Gamma_J^{-1}(\Omega) : Y\varphi(\Gamma_J(Y))^{-1} \in O'\}$$

is open and $K = \mathbf{1}_W$ is lower semicontinuous. For $T \in \Omega$ and $g \in \mathrm{U}(J)$, $\Gamma_J(g\varphi(T)) = T$ and $g\varphi(T)\varphi(T)^{-1} = g$, so $g\varphi(T) \in W$ if and only if $g \in O$; hence $\int K(g\varphi(T)) \, d\mu'(g) = \mu'(O)$ for every measure μ' on $\mathrm{U}(J)$. For an open $S \subseteq \Omega$, Lemma 5.5.18, (ii), applied to S and $\varphi|_S$, gives

$$\int_{\Gamma_J^{-1}(S)} K \, dY = c_N \int_S \int_{\mathrm{U}(J)} K(g\varphi(T)) \, dg \, dT = \big(c_N \, dg(O)\big) \, dT(S).$$

As λ is below the Gram fibre at 1_N and Ω is a neighbourhood of $J = \Gamma_J(1_N)$,

$$\lambda(O) = \int K(g\varphi(J))\, d\lambda(g) = \int K\, d\lambda \le c_N\, dg(O).$$

By (iii), for Borel B,

$$\lambda(B) \le \inf_{O \supseteq B} \lambda(O) \le c_N \inf_{O \supseteq B} dg(O) = c_N\, dg(B).$$

□

Lemma 5.5.18. *Let $J \in \mathrm{M}_N(\mathbb{C})$ be a hermitian involution, $\Gamma_J(Y) = Y^*JY$ for $Y \in \mathrm{M}_N(\mathbb{C})$ its Gram map, and dY on $\mathrm{M}_N(\mathbb{C})$ and dT on Herm_N the measures of (BY2), in the case $m = N$.*

(i) There are an open neighbourhood Ω of J in Herm_N and a continuously differentiable map $\varphi\colon \Omega \to \mathrm{GL}_N(\mathbb{C})$ with $\varphi(J) = 1$ and $\Gamma_J(\varphi(T)) = T$ for $T \in \Omega$.

(ii) Let $\Omega \subseteq \mathrm{Herm}_N$ be open and $\varphi\colon \Omega \to \mathrm{GL}_N(\mathbb{C})$ differentiable with $\Gamma_J(\varphi(T)) = T$ for $T \in \Omega$. Then for every Borel $f\colon \mathrm{M}_N(\mathbb{C}) \to [0,\infty]$

$$\int_{\Gamma_J^{-1}(\Omega)} f(Y)\, dY = \frac{\mathrm{vol}\,\mathrm{U}(N)}{2^N} \int_\Omega \int_{\mathrm{U}(J)} f(g\varphi(T))\, dg\, dT.$$

Proof. Throughout, $|\det h| = 1$ for $h \in \mathrm{U}(J)$ (from $h^*Jh = J$ and $|\det J| = 1$, as $J^2 = 1$), so $Y \mapsto hY$ preserves dY (its real determinant is $|\det h|^{2N}$) and $\Gamma_J(hY) = \Gamma_J(Y)$. The Cayley measure is σ-finite and nonzero: it is the image under the injective chart γ of the measure $\mathrm{vol}\,\mathrm{U}(N)^{-1}|\det D(y)|^{-2N}dy$ on the open set $\mathcal{D}_J \ni 0$, whose density is continuous and positive.

(i). The map $Q(X) = \Gamma_J(1 + \frac{1}{2}JX)$ of Herm_N into itself is polynomial, $Q(0) = J$, and $dQ_0(X) = (\frac{1}{2}JX)^*J + J(\frac{1}{2}JX) = \frac{1}{2}XJ^2 + \frac{1}{2}J^2X = X$. By the inverse function theorem there is an open neighbourhood U_0 of 0 such that $Q(U_0)$ is an open neighbourhood of J and $Q\colon U_0 \to Q(U_0)$ is a diffeomorphism of class C^1. Put $\varphi(T) = 1 + \frac{1}{2}JQ^{-1}(T)$; it is C^1, $\varphi(J) = 1$ and $\Gamma_J(\varphi(T)) = Q(Q^{-1}(T)) = T$. As φ is continuous and $\varphi(J)$ invertible, $\Omega = \{T \in Q(U_0) : \det\varphi(T) \ne 0\}$ is an open neighbourhood of J.

(ii), Step 1: a parametrisation up to a null set. Let $\Phi(T,y) = \gamma(y)\varphi(T)$ on the open set $\Omega \times \mathcal{D}_J \subseteq \mathrm{Herm}_N \times \mathrm{M}_N(\mathbb{R})$; Φ is continuous (φ is differentiable, hence continuous). As $\gamma(y) \in \mathrm{U}(J)$, $\Gamma_J(\Phi(T,y)) = T$, so Φ maps into $\Gamma_J^{-1}(\Omega)$, and Φ is injective: equal values have equal T, then equal $\gamma(y)$, then equal y (Lemma 5.5.19, (i)). Let $Y \in \Gamma_J^{-1}(\Omega)$, $T = \Gamma_J(Y)$ and $g_Y = Y\varphi(T)^{-1}$; then $g_Y^*Jg_Y = \varphi(T)^{-*}T\varphi(T)^{-1} = J$, so $g_Y \in \mathrm{U}(J)$ and $Y = g_Y\varphi(T)$. Since $Y + \varphi(T) = (g_Y + 1)\varphi(T)$, Lemma 5.5.19, (i), shows that Y lies in the image of Φ if and only if $Y \notin B$, where

$$B = \{Y \in \Gamma_J^{-1}(\Omega) : \det(Y + \varphi(\Gamma_J(Y))) = 0\},$$

a Borel set. We show $dY(B) = 0$. Fix $Y \in \mathrm{M}_N(\mathbb{C})$. If $\Gamma_J(Y) \notin \Omega$, then $hY \notin B$ for all $h \in \mathrm{U}(J)$. Otherwise $hY \in B$ if and only if $\det((hg_Y + 1)\varphi(T)) = 0$, that is $hg_Y \notin \mathrm{U}^\circ(J)$, that is $h \in (\mathrm{U}(J) \setminus \mathrm{U}^\circ(J))g_Y^{-1}$, a null set by Lemma 5.5.19, (i), and the right invariance of dg (Lemma 5.2.41). So $dg\{h : hY \in B\} = 0$ for every Y, and by Tonelli on $\mathrm{U}(J) \times \mathrm{M}_N(\mathbb{C})$ (the set $\{(h,Y) : hY \in B\}$ is Borel)

$$0 = \iint \mathbf{1}_B(hY)\, dg(h)\, dY = \int_{\mathrm{U}(J)} dY(h^{-1}B)\, dg(h) = dY(B) \cdot dg(\mathrm{U}(J)),$$

and $dg(\mathrm{U}(J)) > 0$. Hence $\Gamma_J^{-1}(\Omega)$ and $\Phi(\Omega \times \mathcal{D}_J)$ differ by the null set B.

Step 2: the Jacobian. Let $(T,y)\in\Omega\times\mathcal{D}_J$, $\gamma=\gamma(y)$, $\varphi=\varphi(T)$, $P=(\gamma\varphi)^*J$. Then Φ is differentiable at (T,y) with $d\Phi(T',y')=\gamma\,d\varphi_T(T')+d\gamma_y(y')\,\varphi$, and, by $\gamma^*J\gamma=J$,

$$P\,d\Phi(T',y')=\varphi^*J\,d\varphi_T(T')+\varphi^*W_y(y')\varphi.$$

Let Skew_N carry the measure dS, the image of Lebesgue measure under $y\mapsto S(y)$; it is the Lebesgue measure of the coordinates $\operatorname{Im}S_{aa}$ and $\operatorname{Re}S_{ab}$, $\operatorname{Im}S_{ab}$ $(a<b)$, since the pair $(y_{ab},y_{ba})\mapsto S(y)_{ab}$ has the real matrix $\frac{1}{\sqrt2}\left(\begin{smallmatrix}1&-1\\1&1\end{smallmatrix}\right)$ of determinant 1. Let $\kappa_0(Z)=(Z+Z^*,\frac12(Z-Z^*))$, a real-linear bijection $\mathrm{M}_N(\mathbb{C})\to\mathrm{Herm}_N\times\mathrm{Skew}_N$ with inverse $(H,S)\mapsto\frac12H+S$. For the measures dY and $dT\otimes dS$ its determinant has absolute value 2^N: on a diagonal entry it is $(\operatorname{Re}Z_{aa},\operatorname{Im}Z_{aa})\mapsto(2\operatorname{Re}Z_{aa},\operatorname{Im}Z_{aa})$; on a pair $a<b$ it is $(Z_{ab},Z_{ba})\mapsto(Z_{ab}+\overline{Z_{ba}},\frac12(Z_{ab}-\overline{Z_{ba}}))$, the conjugation of Z_{ba} (determinant -1) followed by a complex-linear map of $\mathbb{C}^2$ of complex determinant -1, hence of real determinant 1.

The first component of $\kappa_0(P\,d\Phi(T',y'))$ is $\varphi^*J\,d\varphi_T(T')+d\varphi_T(T')^*J\varphi$, the term $\varphi^*(W_y+W_y^*)\varphi$ vanishing as $W_y(y')$ is skew-hermitian (Lemma 5.5.19, (ii)); it is the derivative of $\Gamma_J\circ\varphi$ at T in the direction T', which is T' because $\Gamma_J\circ\varphi$ is the identity of the open set Ω. So $\kappa_0\circ P\,d\Phi$ is block triangular,

$$(T',y')\longmapsto\big(T',\ s(T')+\varphi^*W_y(y')\varphi\big)$$

for a linear s, and its determinant, for $dT\otimes dy$ and $dT\otimes dS$, has absolute value $|\det(X\mapsto\varphi^*X\varphi)|\cdot|\det(S^{-1}\circ W_y)|$. The map $X\mapsto\varphi^*X\varphi$ is the restriction to the real form Skew_N of $\mathrm{M}_N(\mathbb{C})=\mathrm{Skew}_N\oplus\mathrm{i}\,\mathrm{Skew}_N$ of a complex-linear map of complex determinant $\det(\varphi^*)^N\det(\varphi)^N=|\det\varphi|^{2N}$, so its determinant on Skew_N is $|\det\varphi|^{2N}$; and $|\det(S^{-1}\circ W_y)|=|\det D(y)|^{-2N}$ by Lemma 5.5.19, (ii). Left multiplication by P on $\mathrm{M}_N(\mathbb{C})$ is complex-linear of complex determinant $\det(P)^N$, so its real determinant has absolute value $|\det P|^{2N}=|\det\varphi|^{2N}$ ($|\det\gamma|=|\det J|=1$). Writing $d\Phi=P^{-1}\circ\kappa_0^{-1}\circ(\kappa_0\circ P\,d\Phi)$, the Jacobian of Φ for $dT\otimes dy$ and dY is

$$|\det d\Phi_{(T,y)}|=|\det\varphi|^{-2N}\cdot2^{-N}\cdot|\det\varphi|^{2N}|\det D(y)|^{-2N}=2^{-N}|\det D(y)|^{-2N}.$$

Step 3: change of variables. Φ is continuous and injective on the open set $\Omega\times\mathcal{D}_J$ and differentiable at each of its points, so, by the change of variables formula for such maps (Rudin, Real and Complex Analysis, Theorem 7.26), $\Phi(\Omega\times\mathcal{D}_J)$ is Borel and, with Step 1 and Step 2,

$$\int_{\Gamma_J^{-1}(\Omega)}f\,dY=\int_{\Phi(\Omega\times\mathcal{D}_J)}f\,dY=2^{-N}\int_\Omega\int_{\mathcal{D}_J}f(\gamma(y)\varphi(T))\,|\det D(y)|^{-2N}\,dy\,dT$$

(Tonelli). By the definition of the Cayley measure the inner integral is $\operatorname{vol}\mathrm{U}(N)\int_{\mathrm{U}(J)}f(g\varphi(T))\,dg$, which gives the formula. □

Lemma 5.5.19. *Let $J\in\mathrm{M}_N(\mathbb{C})$ be a hermitian involution, $\mathcal{D}_J=\{y\in\mathrm{M}_N(\mathbb{R}):\det D(y)\neq0\}$, $\gamma(y)=2D(y)^{-1}-1$ for $y\in\mathcal{D}_J$, and Skew_N the real vector space of skew-hermitian $N\times N$ matrices; $y\mapsto S(y)$ is a real-linear bijection $\mathrm{M}_N(\mathbb{R})\to\mathrm{Skew}_N$.*

(i) $\mathcal{D}_J$ is open, γ is a smooth injective map $\mathcal{D}_J\to\mathrm{U}(J)$ with image $\mathrm{U}^\circ(J)=\{g\in\mathrm{U}(J):\det(1+g)\neq0\}$, and $\mathrm{U}(J)\setminus\mathrm{U}^\circ(J)$ is a null set of the Cayley measure.

(ii) For $y\in\mathcal{D}_J$ and $y'\in\mathrm{M}_N(\mathbb{R})$ the matrix $W_y(y')=\gamma(y)^*J\,d\gamma_y(y')$ is skew-hermitian, and the real-linear map $y'\mapsto S^{-1}(W_y(y'))$ of $\mathrm{M}_N(\mathbb{R})$ has determinant of absolute value $|\det D(y)|^{-2N}$.

Proof. **The chart in the coordinates Y.** Put $\mathfrak{l}_J = \{Y \in \mathrm{M}_N(\mathbb{C}) : Y^*J + JY = 0\}$. As $J^* = J$ and $J^2 = 1$, $Y \in \mathfrak{l}_J$ if and only if JY is skew-hermitian, so $X \mapsto JX$ maps Skew_N bijectively onto $\mathfrak{l}_J$, and $\Psi(y) = JS(y)$ is a real-linear bijection $\mathrm{M}_N(\mathbb{R}) \to \mathfrak{l}_J$ with inverse $S^{-1}(J\,\cdot)$. (S is a bijection onto Skew_N: the diagonal entry $\mathrm{i}y_{aa}$ runs over $\mathrm{i}\mathbb{R}$, and for $a < b$ the entry $S(y)_{ab}$, which determines $S(y)_{ba} = -\overline{S(y)_{ab}}$, is the image of (y_{ab}, y_{ba}) under the real-linear bijection with matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ in the coordinates Re, Im.) For $Y \in \mathfrak{l}_J$ write $A_\pm = 1 \pm \frac{1}{2}Y$; they commute, and $D(y) = A_-$ for $Y = \Psi(y)$.

(a) For $Y \in \mathfrak{l}_J$, $A_+^* = 1 - \frac{1}{2}JYJ = JA_-J$, so $\overline{\det A_+} = \det A_-$ and $|\det A_+| = |\det A_-|$.

(i). $\mathcal{D}_J$ is open as $y \mapsto \det D(y)$ is continuous, and γ is smooth there (a rational map with nonvanishing denominator). Let $y \in \mathcal{D}_J$, $Y = \Psi(y)$; then $\gamma(y) = 2A_-^{-1} - 1 = A_+A_-^{-1}$. From $A_+^*J = JA_-$ and $A_-^*J = JA_+$ (by (a)) we get $\gamma(y)^*J\gamma(y) = (A_-^*)^{-1}JA_-A_+A_-^{-1} = (A_-^*)^{-1}JA_+ = J$, so $\gamma(y) \in \mathrm{U}(J)$, and $\gamma(y)+1 = 2A_-^{-1}$ is invertible, so $\gamma(y) \in \mathrm{U}^\circ(J)$. Conversely, for $g \in \mathrm{U}^\circ(J)$ put $Y = 2(g-1)(g+1)^{-1}$; then $1 - \frac{1}{2}Y = 2(g+1)^{-1}$ is invertible, $1 + \frac{1}{2}Y = 2g(g+1)^{-1}$, so $A_+A_-^{-1} = g$, and from $g^* = Jg^{-1}J$, $Y^* = 2J(g^{-1}+1)^{-1}(g^{-1}-1)J = 2J(1+g)^{-1}(1-g)J = -JYJ$, so $Y \in \mathfrak{l}_J$ and $g = \gamma(\Psi^{-1}(Y))$. As $\gamma(y)$ determines $A_-^{-1} = \frac{1}{2}(\gamma(y)+1)$, hence Y, hence y, γ is injective. The Cayley measure is the image under γ of a measure on $\mathcal{D}_J$ (Definition 3.2.1), and $\gamma(\mathcal{D}_J) = \mathrm{U}^\circ(J)$, so the complement of $\mathrm{U}^\circ(J)$ is null.

(ii). Differentiating $\gamma = 2A_-^{-1} - 1$ with $A_- = 1 - \frac{1}{2}\Psi(y)$ gives $d\gamma_y(y') = A_-^{-1}Y'A_-^{-1}$ with $Y' = \Psi(y') \in \mathfrak{l}_J$. As $\gamma \in \mathrm{U}(J)$, $\gamma^*J = J\gamma^{-1}$ and $\gamma^{-1} = A_-A_+^{-1}$, so, $A_\pm^{-1}$ commuting with A_-,

$$W_y(y') = J\gamma^{-1}d\gamma_y(y') = J\,L_Y(Y'), \qquad L_Y(Y') = A_+^{-1}Y'A_-^{-1}.$$

Differentiating $\gamma^*J\gamma = J$ gives $d\gamma^*J\gamma + \gamma^*Jd\gamma = 0$, so $W_y(y')^* = -W_y(y')$. The map L_Y preserves $\mathfrak{l}_J$: by (a), $(A_+^*)^{-1} = JA_-^{-1}J$ and $(A_-^*)^{-1} = JA_+^{-1}J$, so for $Y' \in \mathfrak{l}_J$, $L_Y(Y')^* = JA_-^{-1}J(-JY'J)JA_+^{-1}J = -J\,L_Y(Y')\,J$. Hence $S^{-1}(W_y(\cdot)) = S^{-1}(J\,L_Y(\Psi(\cdot))) = \Psi^{-1} \circ L_Y|_{\mathfrak{l}_J} \circ \Psi$, whose determinant is $\det(L_Y|_{\mathfrak{l}_J})$. Now $\mathfrak{l}_J$ is a real form of $\mathrm{M}_N(\mathbb{C})$: if Y' and $\mathrm{i}Y'$ lie in $\mathfrak{l}_J$ then JY' and $\mathrm{i}JY'$ are skew-hermitian, so $Y' = 0$, and $\dim_{\mathbb{R}} \mathfrak{l}_J = N^2$. So a real basis of $\mathfrak{l}_J$ is a complex basis of $\mathrm{M}_N(\mathbb{C})$, in which the complex-linear map $Y' \mapsto A_+^{-1}Y'A_-^{-1}$ of $\mathrm{M}_N(\mathbb{C})$ has the same matrix as $L_Y|_{\mathfrak{l}_J}$. Its complex determinant is $\det(A_+)^{-N}\det(A_-)^{-N}$ (a Kronecker product), so by (a), $|\det(L_Y|_{\mathfrak{l}_J})| = |\det A_-|^{-2N} = |\det D(y)|^{-2N}$. □

Lemma 5.5.20. *Let $J \in \mathrm{M}_N(\mathbb{C})$ be a hermitian involution, $n_1, n_2, n \geq 0$ integers with $n_1+n_2 = N$, and $E \in \mathrm{M}_{N\times n_1}(\mathbb{C})$, $R \in \mathrm{M}_{N\times n_2}(\mathbb{C})$ with $E^*E = 1$, $R^*R = 1$ and $E^*JR = 0$. Give each space $\mathrm{M}_{a\times b}(\mathbb{C})$ the Lebesgue measure of the real and imaginary parts of its entries. Then for every $M \in \mathrm{M}_{n_1\times n}(\mathbb{C})$ the image of Lebesgue measure on $\mathrm{M}_{n_2\times n}(\mathbb{C})$ under $x \mapsto JEM + Rx$ is below the fibre measure of $\mathrm{M}_{N\times n}(\mathbb{C}) \to \mathrm{M}_{n_1\times n}(\mathbb{C})$, $y \mapsto E^*Jy$, over M.*

Proof. Let $U = (JE\ R) \in \mathrm{M}_N(\mathbb{C})$. As $J^* = J$ and $J^2 = 1$,

$$U^*U = \begin{pmatrix} E^*E & E^*JR \\ R^*JE & R^*R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

using $R^*JE = (E^*JR)^* = 0$; so U is unitary. The map $\Phi(M', x) = JEM' + Rx = U\left(\begin{smallmatrix} M' \\ x \end{smallmatrix}\right)$ from $\mathrm{M}_{n_1\times n}(\mathbb{C}) \times \mathrm{M}_{n_2\times n}(\mathbb{C})$ to $\mathrm{M}_{N\times n}(\mathbb{C})$ is the identification of a pair with the stacked matrix, which preserves Lebesgue measure, followed by left multiplication by U, which acts on each column by the unitary map U of $\mathbb{C}^N = \mathbb{R}^{2N}$, a Euclidean isometry; so Φ carries the product of the Lebesgue measures to Lebesgue measure. Moreover $E^*J\Phi(M', x) = E^*J^2EM' + E^*JRx = M'$, and $M' \mapsto \Phi(M', x)$ is continuous. By Lemma 5.5.13, (ii), with $B = \mathrm{M}_{n_1\times n}(\mathbb{C})$ (Lebesgue measure is positive on nonempty open sets and locally finite), $\beta = M$, $P = \mathrm{M}_{n_2\times n}(\mathbb{C})$ and $\varphi = \Phi$, the image of Lebesgue measure under $x \mapsto \Phi(M, x)$ is below the fibre measure of $y \mapsto E^*Jy$ over M. □

Lemma 5.5.21. *Let $N \geq 1$. The Cayley measure of $\mathrm{U}(1_N) = \{g \in \mathrm{GL}_N(\mathbb{C}) : g^*g = 1\}$ has total mass 1.*

Proof. Write μ for the Cayley measure of $\mathrm{U}(N) = \mathrm{U}(1_N)$ and $|y|$ for the euclidean norm of $y \in \mathrm{M}_N(\mathbb{R}) = \mathbb{R}^{N\times N}$. Give $\mathrm{M}_N(\mathbb{C})$ the inner product $\langle Y, Z\rangle = \mathrm{Re}\,\mathrm{tr}(YZ^*)$ and let ν be the Riemannian volume of the induced bi-invariant metric on $\mathrm{U}(N)$: the Haar measure of Lemma 5.5.22, of total mass

$$\nu(\mathrm{U}(N)) = \frac{(2\pi)^{N(N+1)/2}}{\prod_{j=1}^{N-1} j!}.$$

The chart. For $y \in \mathrm{M}_N(\mathbb{R})$ the matrix $S(y)$ is skew-hermitian, so its eigenvalues lie in $\mathrm{i}\mathbb{R}$ and $D(y) = 1 - \frac{1}{2}S(y)$ is invertible. Hence μ is the image of $\mathrm{vol}\,\mathrm{U}(N)^{-1}|\det D(y)|^{-2N}dy$ on all of $\mathrm{M}_N(\mathbb{R})$ under $\psi(y) = 2D(y)^{-1} - 1$. The map ψ is smooth and injective: $y \mapsto S(y)$ is injective and linear, and $\psi(y)$ determines $D(y)^{-1}$, hence $S(y)$. From $\psi(y) = 2(1 - \frac{1}{2}S(y))^{-1} - 1$ we get $d\psi_0 = S$. The squared norm of $S(y)$ is

$$\mathrm{tr}\big(S(y)S(y)^*\big) = \sum_a y_{aa}^2 + 2\sum_{a<b} |S(y)_{ab}|^2 = \sum_a y_{aa}^2 + 2\sum_{a\neq b} y_{ab}^2,$$

since $S(y)_{ba} = -\overline{S(y)_{ab}}$ and $|(1+\mathrm{i})y_{ab} - (1-\mathrm{i})y_{ba}|^2 = 2y_{ab}^2 + 2y_{ba}^2$ (the cross term is $-2y_{ab}y_{ba}\,\mathrm{Re}((1+\mathrm{i})\overline{(1-\mathrm{i})}) = -2y_{ab}y_{ba}\,\mathrm{Re}(2\mathrm{i}) = 0$). So S is injective, $\dim_{\mathbb{R}} \mathrm{U}(N) = N^2$, and the Gram matrix of S in the standard basis of $\mathbb{R}^{N\times N}$ is diagonal with N entries 1 and $N(N-1)$ entries 2. By the inverse function theorem there is $r > 0$ such that ψ maps the ball $B_r = \{|y| < r\}$ diffeomorphically onto an open neighbourhood W of 1 in $\mathrm{U}(N)$. In this chart the Riemannian volume has a continuous density: $\nu(\psi(A)) = \int_A \rho(y)\,dy$ for Borel $A \subseteq B_r$, where $\rho(y)$ is the square root of the Gram determinant of $d\psi_y$, and

$$\rho(0) = \big(\det \mathrm{Gram}(S)\big)^{1/2} = 2^{N(N-1)/2}.$$

Since ψ is injective on $\mathrm{M}_N(\mathbb{R})$, also $\mu(\psi(A)) = \mathrm{vol}\,\mathrm{U}(N)^{-1}\int_A |\det D(y)|^{-2N}dy$ for Borel $A \subseteq B_r$.

μ is a finite Haar measure. Since $\psi^{-1}(W) = B_r$,

$$\mu(W) \leq \mathrm{vol}\,\mathrm{U}(N)^{-1}\int_{\overline{B_r}} |\det D(y)|^{-2N}dy < \infty,$$

the integrand being continuous. By compactness $\mathrm{U}(N) = \bigcup_{i=1}^k a_i W$ for some $a_1, \ldots, a_k \in \mathrm{U}(N)$, and $\mu(a_iW) = \mu(W)$ by Lemma 5.2.41 applied to the indicator of a_iW (whose composite with $y \mapsto a_i y$ is the indicator of W). Hence μ is a finite, nonzero, left-invariant Borel measure on the compact metrisable group $\mathrm{U}(N)$; it is regular, being finite on a compact metric space, and by the uniqueness of Haar measure $\mu = c\,\nu$ for some $c > 0$.

The constant. For Borel $A \subseteq B_r$ the two descriptions above give

$$\int_A \big(\mathrm{vol}\,\mathrm{U}(N)^{-1}|\det D(y)|^{-2N} - c\rho(y)\big)dy = 0.$$

The integrand is continuous on B_r, so it vanishes identically there; at $y = 0$, where $D(0) = 1$,

$$\mathrm{vol}\,\mathrm{U}(N)^{-1} = c\,2^{N(N-1)/2}.$$

The mass. Since $0! = 1$, $\mathrm{vol}\,\mathrm{U}(N) = \prod_{i=0}^{N-1} 2\pi^{i+1}/i! = 2^N\pi^{N(N+1)/2}/\prod_{j=1}^{N-1} j!$, and therefore

$$\mu(\mathrm{U}(N)) = c\,\nu(\mathrm{U}(N)) = \frac{(2\pi)^{N(N+1)/2}}{2^{N(N-1)/2}\,2^N\,\pi^{N(N+1)/2}} = 2^{N(N+1)/2 - N(N-1)/2 - N} = 1.$$

□

Lemma 5.5.22. *All symbols introduced here are local to this statement and to the results that cite it. Equip* $\mathrm{M}_n(\mathbb{C})$ *with the euclidean inner product* $\langle Y, Z\rangle = \operatorname{Re}\operatorname{tr}(YZ^*)$*, and give the compact groups* $\mathrm{U}(n)$ *and* $\mathrm{SU}(n)$ *the Haar measures whose density at* 1*, with respect to the Lebesgue measure of the restriction of this inner product to their Lie algebras* $\mathfrak{u}(n)$*,* $\mathfrak{su}(n)$*, equals* 1*. Then for every integer* $n \geq 1$

$$\operatorname{vol}\mathrm{U}(n) = \frac{(2\pi)^{n(n+1)/2}}{\prod_{j=1}^{n-1} j!}, \qquad \operatorname{vol}\mathrm{SU}(n) = \frac{\sqrt{n}\,(2\pi)^{n(n+1)/2-1}}{\prod_{j=1}^{n-1} j!} \quad (n \geq 2).$$

In particular $\operatorname{vol}\mathrm{SU}(n) \cdot \prod_{j=1}^{n-1} \dfrac{j!}{(2\pi)^{j+1}} = \sqrt{n}$ *for* $n \geq 2$.

Proof. **Haar measure as Riemannian volume.** The inner product is invariant under left and right multiplication by unitary matrices ($\operatorname{tr}(gYZ^*g^*) = \operatorname{tr}(YZ^*)$ for g unitary, and similarly on the right). Hence the euclidean metric of $\mathrm{M}_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$ restricted to the submanifolds $\mathrm{U}(n)$, $\mathrm{SU}(n)$ is a bi-invariant Riemannian metric, its Riemannian volume is a Haar measure, and its density at 1 with respect to the Lebesgue measure of the inner product on the Lie algebra is 1. So the measures of the statement are the Riemannian volumes of these submanifolds. We use the coarea formula for a smooth submersion $\pi\colon M \to N$ of compact Riemannian manifolds: $\int_M J\pi \, d\mathrm{vol}_M = \int_N \operatorname{vol}(\pi^{-1}(y))\, d\mathrm{vol}_N(y)$, where $J\pi(x)$ is the absolute value of the determinant of $d\pi_x$ restricted to $(\ker d\pi_x)^\perp$, computed in orthonormal bases.

The unitary group. For $n = 1$, $\mathrm{U}(1) = \{e^{i\theta}\}$ is the unit circle of $\mathbb{C}$, of length 2π, which is the formula. Let $n \geq 2$ and $\pi\colon \mathrm{U}(n) \to S^{2n-1} \subseteq \mathbb{C}^n$, $\pi(g) = ge_1$, where $\mathbb{C}^n$ carries the euclidean metric $\operatorname{Re}(y^*x)$. It is a surjective submersion, equivariant for left multiplication by $h \in \mathrm{U}(n)$, which is an isometry of $\mathrm{U}(n)$ and of S^{2n-1}; hence $J\pi$ is constant, equal to $J\pi(1)$. The fibre $\pi^{-1}(ge_1) = g\,\mathrm{U}(n-1)$, with $\mathrm{U}(n-1)$ embedded as the stabiliser of e_1 (lower right block), is isometric to $\mathrm{U}(n-1)$ by left translation. At 1: $\ker d\pi_1 = \{X \in \mathfrak{u}(n) : Xe_1 = 0\}$ consists of the skew-hermitian matrices with vanishing first column, hence vanishing first row; its orthogonal complement in $\mathfrak{u}(n)$ consists of the matrices

$$X(a,b) = \begin{pmatrix} ia & -b^* \\ b & 0 \end{pmatrix}, \qquad a \in \mathbb{R},\ b \in \mathbb{C}^{n-1}, \qquad |X(a,b)|^2 = a^2 + 2|b|^2,$$

and $d\pi_1 X(a,b) = X(a,b)e_1 = (ia, b)$, of squared length $a^2 + |b|^2$. In the orthonormal basis $X(1,0)$, $X(0, e_k/\sqrt{2})$, $X(0, ie_k/\sqrt{2})$ $(1 \leq k \leq n-1)$ the images are pairwise orthogonal of lengths 1 and $2(n-1)$ times $1/\sqrt{2}$, so $J\pi = 2^{-(n-1)}$. The coarea formula and $\operatorname{vol}(S^{2n-1}) = 2\pi^n/(n-1)!$ give

$$\operatorname{vol}\mathrm{U}(n) = 2^{n-1}\,\frac{2\pi^n}{(n-1)!}\operatorname{vol}\mathrm{U}(n-1) = \frac{(2\pi)^n}{(n-1)!}\operatorname{vol}\mathrm{U}(n-1),$$

and by induction $\operatorname{vol}\mathrm{U}(n) = \prod_{k=1}^{n}(2\pi)^k/(k-1)! = (2\pi)^{n(n+1)/2}/\prod_{j=1}^{n-1} j!$.

The special unitary group. Let $n \geq 2$ and $\det\colon \mathrm{U}(n) \to \mathrm{U}(1)$, a surjective submersion with fibres $\det^{-1}(\det g) = g\,\mathrm{SU}(n)$, isometric to $\mathrm{SU}(n)$. Since $\det(hg) = \det h \det g$ and multiplication by $\det h$ is an isometry of the circle, $J\det$ is constant. At 1, $d\det_1(X) = \operatorname{tr} X$, $\ker d\det_1 = \mathfrak{su}(n)$, whose orthogonal complement in $\mathfrak{u}(n)$ is $i\mathbb{R}\,I_n$ (as $\operatorname{Re}\operatorname{tr}(Y(itI_n)^*) = \operatorname{Re}(-it\operatorname{tr} Y) = 0$ for $\operatorname{tr} Y = 0$); the unit vector $iI_n/\sqrt{n}$ is mapped to $i\sqrt{n}$, so $J\det = \sqrt{n}$. The coarea formula gives $\sqrt{n}\operatorname{vol}\mathrm{U}(n) = 2\pi \operatorname{vol}\mathrm{SU}(n)$, which is the second formula.

The product. Since $\sum_{j=1}^{n-1}(j+1) = n(n+1)/2 - 1$,

$$\operatorname{vol}\mathrm{SU}(n) \prod_{j=1}^{n-1} \frac{j!}{(2\pi)^{j+1}} = \sqrt{n}\,(2\pi)^{n(n+1)/2-1-\sum_{j=1}^{n-1}(j+1)} = \sqrt{n}.$$

□

Lemma 5.5.23. *Let $\rho > 0$ and consider the data at $\mathbb{C}$ with box radius ρ. Then $\eta_{<\varrho} = \eta_0 * \eta_1 * \cdots * \eta_{\varrho-1}$ is the image of the Haar measure du of U under the inclusion $\mathsf{U} \subseteq G$.*

Proof. Write g_{ij} for the entries of $g \in G$ in the coordinate basis and, for $0 \le l \le \varrho$ and $g \in G$,

$$\mathcal{G}_l(g) = \exp\Big(-\pi \sum_{b<l} \Big(\sum_{i:\,\mathrm{floor}(i)>b} |g_{i\,e_b^*}|^2 + (\operatorname{Im} g_{e_b\,e_b^*})^2 \Big)\Big),$$

a continuous function with values in $(0,1]$; $\mathcal{G}_0 = 1$.

Step 1: $N_{<\varrho} = \mathsf{U}$. For $l = \varrho$ the indices of floor below ϱ are the e_a, e_a^*, and the others (x_i, z) have level ϱ; so lev_ϱ is the level, and the definitions of $N_{<\varrho}$ and U coincide.

Step 2: $\int \mathcal{G}_\varrho \, d\eta_{<\varrho} = 1$. We show $\int \mathcal{G}_a \, d\eta_{<a} = 1$ for $0 \le a \le \varrho$ by induction; for $a = 0$ both sides are 1. Let $a < \varrho$, $\nu' \in N_{<a}$ and $n = n_a(v,\beta)$ with $v \in V_{a+1}$, $\beta = t\mathrm{i}$. For $b < a$, n fixes e_b^*, so column e_b^* of $\nu' n$ is that of ν'. Column e_a^* of $\nu' n$ is $\nu' w$ with $w = e_a^* + v + (\beta - \tau h(v,v)) e_a$. Let i have floor $\ge a$; then $\mathrm{lev}_a(i) = \varrho$, so $\nu'_{ij} = \delta_{ij}$ unless $\mathrm{lev}_a(j) > \varrho$, i.e. unless $j = e_c^*$ with $c < a$; as w has no e_c^*-coordinate for $c < a$, $(\nu' w)_i = w_i$. Hence the entries of column e_a^* of $\nu' n$ in the rows of floor $> a$ are the coordinates of v, and its (e_a, e_a^*) entry is $\beta - \frac{1}{2} h(v,v)$, of imaginary part t ($h(v,v)$ is real). So

$$\mathcal{G}_{a+1}(\nu' n_a(v, t\mathrm{i})) = e^{-\pi(\|v\|^2 + t^2)} \, \mathcal{G}_a(\nu') \qquad (\nu' \in N_{<a}).$$

By Lemma 5.4.14, (ii), $\eta_{<a}$ is carried by $N_{<a}$, and $\eta_{<a+1} = \eta_{<a} * \eta_a$ by definition of the convolution. By Tonelli,

$$\int \mathcal{G}_{a+1} \, d\eta_{<a+1} = \iint e^{-\pi(\|v\|^2 + t^2)} \, dv \, dt \cdot \int \mathcal{G}_a \, d\eta_{<a} = 1 \cdot 1,$$

since $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$ in each real coordinate.

Step 3: $\int \mathcal{G}_\varrho(y^{-1}) \, du(y) = 1$. For $y \in G$, $\bar{y}^{\mathsf{T}} J y = J$ and $J^2 = 1$ give $y^{-1} = J \bar{y}^{\mathsf{T}} J$. As J exchanges e_c and e_c^*, fixes the x_j and sends z to $-z$, the entry of y^{-1} in row i and column e_b^* is $\pm\overline{y_{e_b\,i'}}$, where i' is the image of i under the exchange $e_c \leftrightarrow e_c^*$ (and $i' = i$ for $i = x_j, z$); for $i = e_b$ the sign is $+$. The indices of floor $> b$ are exchanged among themselves, and they are the set J_{b+1} of Lemma 5.2.34 in its numbering. Hence $\mathcal{G}_\varrho(y^{-1}) = \exp(-\pi \|\mathrm{rw}(y)\|^2)$ for $y \in \mathsf{U}$, and by Lemma 5.2.34, (iii), the image of du under rw is Lebesgue measure of $\mathcal{C}_\mathsf{U}$, so $\int \mathcal{G}_\varrho(y^{-1}) \, du(y) = \int_{\mathcal{C}_\mathsf{U}} e^{-\pi\|c\|^2} \, dc = 1$.

Step 4: uniqueness. Write $\eta = \eta_{<\varrho}$ and $\mathcal{G} = \mathcal{G}_\varrho$. By Step 1 and Lemma 5.4.14, (ii) and (iii), η is carried by U and $\int f(xs) \, d\eta(x) = \int f(sx) \, d\eta(x) = \int f \, d\eta$ for measurable $f \ge 0$ and $s \in \mathsf{U}$; by Lemma 5.2.34, (iv), du is left invariant. Both are s-finite, so Tonelli applies. For measurable $f \colon G \to [0,\infty]$, using Step 3, right invariance of η, Tonelli, then for η-almost every $x \in \mathsf{U}$ the substitution $y \mapsto x^{-1} y$ in du, Tonelli again, left invariance of η and Step 2:

$$\begin{aligned} \int f \, d\eta &= \int_\mathsf{U} \mathcal{G}(y^{-1}) \int f(xy) \, d\eta(x) \, du(y) = \int \int_\mathsf{U} \mathcal{G}(y^{-1}x) f(y) \, du(y) \, d\eta(x) \\ &= \int_\mathsf{U} f(y) \int \mathcal{G}(y^{-1}x) \, d\eta(x) \, du(y) = \int_\mathsf{U} f \, du. \end{aligned}$$

So η is the image of du under the inclusion. □

Lemma 5.5.24. *For $i \ge 0$ let the* dyadic shell *D_i be the set of $x \in G$ such that 2^i is the least of the numbers $1, 2, 4, \ldots$ with $|(x-1)_{ac}| \le 2^i$ for all indices a, c. Then the D_i are Borel sets and*

G is their disjoint union. Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, let $W\colon G \to \mathbb{C}^I$ be cut, $L \geq \rho$, $j \leq \varrho$, and let p, q be heads of length j for the size filtration Γ_L of scale L. Then for every $i \geq 0$, D_i is a shell of height 2^i for (Γ_L, j, p, q, W).

Proof. **The dyadic shells.** Let $m(x) = \max_{a,c} |(x-1)_{ac}|$, a continuous function. Then $D_0 = \{m \leq 1\}$ and $D_i = \{m \leq 2^i\} \setminus \{m \leq 2^{i-1}\}$ for $i \geq 1$, Borel sets; as $m(x) < \infty$ there is a least i with $m(x) \leq 2^i$, so every x lies in exactly one D_i.

Fix i and put $\Delta = 2^i$. As $\rho \geq 8r \geq 8$, $L \geq 8$.

(SH1). $1 \leq \Delta < \infty$.

(SH2). Let $L' = L\Delta \geq \rho$; by Lemma 5.5.5, (ii), the size filtration Γ' of scale L' is a gauge for the same place datum, with loss $L'^2 = L^2\Delta^2$, the loss of Γ_L times Δ^2. Take $\mathcal{G} = \{\Gamma'\}$; $\#\mathcal{G} = 1 \leq 2\Delta^{9n}$. For $x \in D_i$ every entry of $M^{-1}(x-1) = x - 1$ has modulus at most $\Delta \leq L'$, so x is a position of size Γ'.

(SH3). Let $x \in D_i$ and $\varepsilon \in [0, 1)$ with $(L^2)^{1200n^2} \leq \varepsilon^{400n^2}\Delta$. The left side is positive, so $\varepsilon > 0$. Put $T = L^{11n+1}/\varepsilon$. Then $T \geq L^{11n+1} \geq L^{11n}$, and $T^{-1} = \varepsilon/L^{11n+1} \leq \varepsilon/2$. Moreover $2T^{99n} \leq \Delta$: as $\varepsilon \leq 1$, $400n^2 \geq 99n$ and $2 \leq L$,

$$\varepsilon^{400n^2} \cdot 2T^{99n} \leq \varepsilon^{99n} L\Big(\frac{L^{11n+1}}{\varepsilon}\Big)^{99n} = L^{(11n+1)99n+1} \leq L^{2400n^2} \leq \varepsilon^{400n^2}\Delta,$$

using $(11n+1)99n + 1 = 1089n^2 + 99n + 1 \leq 2400n^2$ for $n \geq 1$; divide by $\varepsilon^{400n^2} > 0$. Now some entry of $x - 1$ has modulus above T^{99n}: if $i = 0$, then $2T^{99n} \leq 1$ contradicts $T \geq 1$, so this case does not occur; if $i \geq 1$ and all entries had modulus at most $T^{99n} \leq \Delta/2 = 2^{i-1}$, then $x \notin D_i$. By Lemma 5.5.25, (ii), with $\tau = \varepsilon$, (j, p, q, W) has a deep vector after the head of depth at least $\varepsilon^{-2} = (\varepsilon^2)^{-1}$ at x. □

Lemma 5.5.25. *Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, let $W\colon G \to \mathbb{C}^I$ be cut, $L \geq \rho$, $j \leq \varrho$, and let p, q be heads of length j for the size filtration of scale L. Let $T \geq L^{11n}$ and let $x \in G$ have an entry with $|(x-1)_{ac}| > T^{99n}$. Then:*

(i) every point y active at x for (j, p, q, W) has some a with $j \leq a < \varrho$ and $d_a(y, x) \geq T^2$, hence $\mathrm{dp}_a(y, x) \geq T^2/\pi$;

(ii) for every $\tau > 0$ with $T^{-1} \leq \tau/2$, (j, p, q, W) has a deep vector after the head of depth at least τ^{-2} at x.

Proof. **(i).** As $\rho \geq 8r \geq 8$, $L \geq 8$ and $T \geq L^{11n} \geq L$. By Lemma 5.5.5, (ii), the size filtrations of scales L and T (losses L^2, T^2) are gauges S and Γ for the data at $\mathbb{C}$, which form a place datum by Lemma 5.5.5, (i). Let y be active at x and suppose, for a contradiction, that $d_a(y, x) < T^2$ for every a with $j \leq a < \varrho$. We check the hypotheses of (DN4) of Theorem 5.4.1 for the gauge Γ, the second gauge S, the points y and x (in the roles of x and y there), and the heads p, q.

- p, q are heads of length j for S. If $j \neq 0$: $M^{-1} = 1$ has modulus $1 \leq L$, so $M^{-1} \in A^S_{(1)}$; and a number of modulus at most L^{10n} has modulus at most T, so $A^S_{(10n)} \subseteq A^\Gamma_{(1)}$.
- By (AC1), $W(y) \neq 0$ and $W(yx) \neq 0$; as W is cut there are $u, v \in \mathsf{U}$ with $uy \in K$ and $vyx \in K$. Every entry of an element of K has modulus at most $\rho \leq T$, so every entry of $M^{-1}(vyx - uy)$ has modulus at most $2\rho \leq T^2$: it has size 2 for Γ.
- By (AC2) and (AC3), $w_{j,p}^{-1}y$ and $w_{j,q}^{-1}yx$ are near at j.

- For $j \le a < \varrho$ let $c = T^{-1}$, a unit with $c^{-1} = T$ of size 1 for Γ. By Lemma 5.5.26, $\mathfrak{d}_a(y,x) = \{c' : |c'|^2 d_a(y,x) < 1\}$, and $|c|^2 d_a(y,x) = d_a(y,x)/T^2 < 1$, so $c \in \mathfrak{d}_a(y,x)$.

By (DN4), $M^{-1}(x-1) = x - 1$ has size $99n$ for Γ: every entry has modulus at most T^{99n}, against the hypothesis on x. So some a with $j \le a < \varrho$ has $d_a(y,x) \ge T^2$, and $\mathrm{dp}_a(y,x) = d_a(y,x)/\pi \ge T^2/\pi$ by Lemma 5.5.26.

(ii). If $T^{-1} \le \tau/2$ then $T^2 \ge 4\tau^{-2} \ge \pi\tau^{-2}$, so the a of (i) has $\mathrm{dp}_a(y,x) \ge T^2/\pi \ge \tau^{-2}$ for every active y; this is a deep vector after the head of depth at least τ^{-2} at x. □

Lemma 5.5.26. *Consider the data at $\mathbb{C}$ with box radius $\rho > 0$. For $a < \varrho$ and $x, y \in G$ let $d_a(x,y) \in [0,\infty)$ be the infimum of $\|x^{-1}\zeta\|^2 + \|(xy)^{-1}\zeta'\|^2$ over the pairs (ζ,ζ') of vectors with $h(e_b,\zeta) = 0$ for $b < a$, $h(e_a,\zeta) = 1$ and $\zeta' - \zeta \in \mathrm{span}\{e_b : b < a\}$. Then*

$$\mathfrak{d}_a(x,y) = \{c \in \mathbb{C} : |c|^2 d_a(x,y) < 1\}, \qquad \mathrm{dp}_a(x,y) = d_a(x,y)/\pi.$$

Proof. Write $d = d_a(x,y)$; it is finite, as $\zeta = \zeta' = e_a^*$ is admissible. Call (ω,ω') admissible for c if $h(e_b,\omega) = 0$ for $b < a$, $h(e_a,\omega) = c$ and $\omega' - \omega \in \mathrm{span}\{e_b : b < a\}$; all these conditions are linear, so (ω,ω') is admissible for $c \ne 0$ exactly when $(c^{-1}\omega, c^{-1}\omega')$ is admissible for 1.

The set. For $c = 0$ the pair $(0,0)$ is admissible and close, so $0 \in \mathfrak{d}_a(x,y)$, and $|0|^2 d < 1$. Let $c \ne 0$. Then $c \in \mathfrak{d}_a(x,y)$ if and only if some pair (ζ,ζ') admissible for 1 has $|c|^2(\|x^{-1}\zeta\|^2 + \|(xy)^{-1}\zeta'\|^2) < 1$, i.e. $\|x^{-1}\zeta\|^2 + \|(xy)^{-1}\zeta'\|^2 < |c|^{-2}$; such a pair exists if and only if the infimum d is below $|c|^{-2}$, that is $|c|^2 d < 1$.

The depth. If $d = 0$, then $\mathfrak{d}_a(x,y) = \mathbb{C}$ has infinite area and $\mathrm{dp}_a(x,y) = 0 = d/\pi$. If $d > 0$, $\mathfrak{d}_a(x,y)$ is the open disc of radius $d^{-1/2}$, of area π/d, so $\mathrm{dp}_a(x,y) = d/\pi$. □

Lemma 5.5.27. *Let $\rho > 0$, consider the data at $\mathbb{C}$ with box radius ρ, and let λ be its Haar ratio. Let $(W, W^\flat)$ be a cut pair whose type is an irreducible unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$, I nonempty. Then*

$$\mathrm{PN}(W^\flat) = \int_{G_1}\int_{G_1} |\langle W^\flat(x), W^\flat(y)\rangle|^2 \, d\theta_{\ge 0}(x)\, d\theta_{\ge 0}(y) \le \lambda^2 \, \#I.$$

Proof. Fix $x \in G_1$. The function $y \mapsto |\langle W^\flat(x), W^\flat(y)\rangle|^2$ is Borel, nonnegative and left U-invariant, since $|\mathrm{wave}| = 1$. By the unfolding identity of Lemma 5.5.10 and by Lemma 5.5.28 with $v = W^\flat(x)$,

$$\int_{G_1} |\langle W^\flat(x), W^\flat(y)\rangle|^2 \, d\theta_{\ge 0}(y) = \lambda \int_{\mathsf{U}\backslash G_1} |\langle W^\flat(x), W^\flat\rangle|^2 \le \lambda \, \|W^\flat(x)\|^2.$$

The function $\|W^\flat\|^2$ is also Borel and left U-invariant, so integrating over x (Tonelli) and unfolding once more,

$$\mathrm{PN}(W^\flat) \le \lambda \int_{G_1} \|W^\flat(x)\|^2 \, d\theta_{\ge 0}(x) = \lambda^2 \int_{\mathsf{U}\backslash G_1} \|W^\flat\|^2 \le \lambda^2 \, \#I,$$

by (BS1). □

Lemma 5.5.28. *Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be an irreducible unitary representation on $\mathbb{C}^I$, I finite and nonempty, and let F be a Bessel function of type ρ with $\int_{\mathsf{U}\backslash G_1} \|F\| < \infty$. Then for all $i, i' \in I$ and $v \in \mathbb{C}^I$*

$$\int_{\mathsf{U}\backslash G_1} F_i\,\overline{F_{i'}} = \frac{\delta_{ii'}}{\#I}\int_{\mathsf{U}\backslash G_1}\|F\|^2, \qquad \int_{\mathsf{U}\backslash G_1}|\langle v, F\rangle|^2 = \frac{\|v\|^2}{\#I}\int_{\mathsf{U}\backslash G_1}\|F\|^2.$$

In particular, if $(W, W^\flat)$ is a cut pair of type ρ, then $\int_{\mathsf{U}\backslash G_1}|\langle v, W^\flat\rangle|^2 \le \|v\|^2$ for every $v \in \mathbb{C}^I$.

Proof. **Step 1: the matrix of integrals is scalar.** For $i, i' \in I$ the function $\phi_{ii'} = F_i\overline{F_{i'}}$ is Borel and left U-invariant ($|\text{wave}| = 1$), and $|\phi_{ii'}| \le \|F\|_\infty\|F\|$, so it is integrable over $\mathsf{U}\backslash G_1$. Let $P = (\int_{\mathsf{U}\backslash G_1}\phi_{ii'})_{i,i'}$. For $h \in H$, $F(hy)F(hy)^* = \rho(h)F(y)F(y)^*\rho(h)^*$, so Lemma 5.5.29, (iii), gives $\rho(h)P\rho(h)^* = P$; as $\rho(h)$ is unitary, P commutes with $\rho(\mathrm{U}(k))$, and by Schur's lemma for the irreducible representation ρ, $P = c\cdot 1$. Taking traces, $c\,\#I = \int_{\mathsf{U}\backslash G_1}\|F\|^2$. This is the first identity.

Step 2: the second identity. $|\langle v, F\rangle|^2 = \sum_{i,i'} v_i\bar{v}_{i'}\overline{F_i}F_{i'}$; integrating term by term, $\int_{\mathsf{U}\backslash G_1}|\langle v, F\rangle|^2 = \sum_{i,i'} v_i\bar{v}_{i'}\overline{P_{ii'}} = c\|v\|^2$.

Step 3: cut pairs. Let $(W, W^\flat)$ be a cut pair of type ρ and take A, a of (BS3). If $a = 0$ then $A = 0$ and $W^\flat = 0$. Otherwise $\|W^\flat\| \le A$ and $\int_{\mathsf{U}\backslash G_1} A \le a^{-1}b^{400r^2}\#I < \infty$, so $\int_{\mathsf{U}\backslash G_1}\|W^\flat\| < \infty$; and $\int_{\mathsf{U}\backslash G_1}\|W^\flat\|^2 \le \#I$ by (BS1). The second identity for $F = W^\flat$ gives $\int_{\mathsf{U}\backslash G_1}|\langle v, W^\flat\rangle|^2 \le \|v\|^2$. □

Lemma 5.5.29. *Call a Borel function $\chi\colon G_1 \to [0,\infty]$ with $\int_{\mathsf{U}}\chi(uy)\,du = 1$ for every $y \in G_1$ a* cutoff. *Let $\phi\colon G_1 \to [0,\infty]$ be Borel with $\phi(uy) = \phi(y)$ for all $u \in \mathsf{U}$, $y \in G_1$. Then:*

(i) $\int_{G_1}\phi\chi\,dy = \int_{G_1}\phi\chi'\,dy$ for any two cutoffs χ, χ'; this common value is $\int_{\mathsf{U}\backslash G_1}\phi$;

(ii) $\int_{\mathsf{U}\backslash G_1}\phi(yz)\,dy = \int_{\mathsf{U}\backslash G_1}\phi$ for every $z \in G_1$;

(iii) $h\mathsf{U}h^{-1} = \mathsf{U}$ and $\int_{\mathsf{U}\backslash G_1}\phi(hy)\,dy = \int_{\mathsf{U}\backslash G_1}\phi$ for every $h \in H$.

The same identities hold for complex Borel ϕ, left U-invariant, with $\int_{\mathsf{U}\backslash G_1}|\phi| < \infty$.

Proof. **(i).** By Tonelli, the left invariance of dy (Lemma 5.2.41) applied to $y \mapsto uy$ for fixed u, and the U-invariance of ϕ,

$$\begin{aligned}\int\phi\chi\,dy &= \int_{\mathsf{U}}\int\phi(y)\chi(y)\chi'(uy)\,dy\,du = \int_{\mathsf{U}}\int\phi(y)\chi(u^{-1}y)\chi'(y)\,dy\,du\\ &= \int\phi(y)\chi'(y)\int_{\mathsf{U}}\chi(u^{-1}y)\,du\,dy = \int\phi\chi'\,dy,\end{aligned}$$

as du is invariant under $u \mapsto u^{-1}$ (Lemma 5.2.34, (iv)), so $\int_{\mathsf{U}}\chi(u^{-1}y)\,du = 1$.

(ii). $y \mapsto \phi(yz)$ is again U-invariant, and $\chi_z(y) = \chi(yz^{-1})$ is a cutoff. By right invariance of dy (Lemma 5.2.41) and (i), $\int\phi(yz)\chi(y)\,dy = \int\phi(y)\chi(yz^{-1})\,dy = \int\phi\chi\,dy$.

(iii). h fixes the e_a, e_a^*, z and preserves X_0, so it preserves the span of the basis vectors of each level; as $u \in \mathsf{U}$ means that $u - 1$ maps each basis vector into the span of basis vectors of lower level, so does $h(u-1)h^{-1} = huh^{-1} - 1$, and $h\mathsf{U}h^{-1} = \mathsf{U}$. Conjugation by h preserves du (Lemma 5.5.30, (vi)), so $\chi_h(y) = \chi(h^{-1}y)$ is a cutoff: $\int_{\mathsf{U}}\chi(h^{-1}uy)\,du = \int_{\mathsf{U}}\chi(u'h^{-1}y)\,du' = 1$. The function $y \mapsto \phi(hy)$ is U-invariant, and by left invariance of dy and (i), $\int\phi(hy)\chi(y)\,dy = \int\phi\chi_h\,dy = \int\phi\chi\,dy$.

For complex ϕ with $\int|\phi|\chi\,dy < \infty$, apply the above to the positive and negative parts of $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$, which are Borel, nonnegative and U-invariant. □

Lemma 5.5.30. *Fix the setting of Definition 5.1.6. For a complex $r \times r$ matrix g and basis vectors i, j of V_1 write g_{ij} for the i-coordinate of gj. Give the basis vectors* levels*: e_b has level b, the x_i and z level $\varrho+1$, e_b^* level $2\varrho+2-b$. Let dh be the Haar probability measure of H, and ds the image of $du \otimes dh$ under $(u,h) \mapsto uh$.*

(i) R is the set of $g \in G_1$ with $gf_b - f_b \in \langle f_a : a < b\rangle$ for $0 \le b \le \varrho$; it is a closed subgroup of G_1. Every $s \in R$ is uniquely $s = uh$ with $u \in \mathsf{U}$, $h \in H$; writing $h_s = h$, the matrix of h_s in $\mathrm{U}(k)$ is $(s_{x_i x_j})_{1\le i,j\le k}$, the map $s \mapsto h_s$ is a continuous homomorphism $R \to H$, and $s \mapsto (sh_s^{-1}, h_s)$ is a homeomorphism $R \to \mathsf{U} \times H$.

(ii) For $s \in R$ put $\sigma(s) = \sigma(sh_s^{-1})$. Then $\sigma(s) = \sum_{a<\varrho} s_{f_a f_{a+1}}$, $\sigma(st) = \sigma(s) + \sigma(t)$ and $\sigma(s^{-1}) = -\sigma(s)$ for $s, t \in R$, and $\sigma = 0$ on H.

(iii) If Z is a complex $r \times r$ matrix with $Z_{ij} = 0$ whenever the level of i exceeds the level of j, then $\mathrm{tr}\big(Z(s-h_s)\big) = 0$ for every $s \in R$. In particular $\mathrm{tr}(s-1) = \mathrm{tr}(h_s - 1)$, the right side computed in $\mathrm{U}(k)$.

(iv) Let C be the chain matrix*: $Ce_b = f_b$ $(1 \le b \le \varrho)$, $Cz = -e_\varrho^*$, $Ce_b^* = e_{b-1}^*$ $(2 \le b \le \varrho)$, $Ce_1^* = 0$ and $Cx_i = 0$ $(1 \le i \le k)$. Then $\mathrm{tr}\big(C(s-1)\big) = \sigma(s) - \overline{\sigma(s)}$ for every $s \in R$.*

(v) With the topology of G_1 on R, left and right translations of R are continuous, and ds is a right-invariant Borel measure on R that is finite on compact sets.

(vi) Conjugation by every $h \in H$ preserves du, and for every Borel $f\colon G_1 \to [0,\infty]$, with dy the Cayley measure of G_1,

$$\int_{\mathsf{U}\backslash G_1} \int_R f(sy)\, ds\, dy = \int_{G_1} f(g)\, dg.$$

Proof. Here $h(v,w) = v^* J_1 w$, linear in w, and J_1 exchanges e_b and e_b^*, fixes the x_i and sends z to $-z$. Hence the coordinates of $w \in V_1$ are $w_{e_b^*} = h(e_b, w)$, $w_{e_b} = h(e_b^*, w)$, $w_{x_i} = h(x_i, w)$ and $w_z = -h(z,w)$. Write $F_c = \langle e_1, \dots, e_c\rangle$ $(0 \le c \le \varrho)$, $F = F_\varrho$, and $\perp$ for the orthogonal for h. Then $F_c^\perp$ is spanned by the e_b, the x_i, z and the e_b^* with $b > c$, and the span of the basis vectors of level at most m is F_m for $m \le \varrho$ and $F_{2\varrho+1-m}^\perp$ for $\varrho+1 \le m \le 2\varrho+1$. So the refined flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_\varrho \subset F_\varrho^\perp \subset F_{\varrho-1}^\perp \subset \cdots \subset F_0^\perp = V_1$$

is the flag of the level spans. The stabiliser P of $F_1 \subset \cdots \subset F_\varrho$ in G_1 stabilises the refined flag, and its unipotent radical U consists of the elements of G_1 that act trivially on each of its successive quotients (the standard description of the unipotent radical of a parabolic subgroup of a unitary group, [36]). Equivalently: $(u-1)j$ lies in the span of the basis vectors of level smaller than that of j, for every basis vector j; that is, $u_{ij} = \delta_{ij}$ whenever the level of i is at least the level of j. The group H fixes F pointwise and preserves X_0, so $H \subseteq P$; and P normalises U.

(i). Let Q be the set of $g \in G_1$ with $gf_b - f_b \in \langle f_a : a < b\rangle$ for all b. On the span $W = \langle f_0, \dots, f_\varrho\rangle$ such a g acts by an upper unitriangular matrix, so Q is a subgroup; the conditions say that finitely many coordinates of gf_b are 0 or 1, so Q is closed. Every $u \in \mathsf{U}$ lies in Q: $ue_b - e_b \in F_{b-1}$ because u is trivial on F_b/F_{b-1}, and $uz - z \in F$ because u is trivial on $F^\perp/F$ and $z \in F^\perp$. Every $h \in H$ fixes the frame. Hence $R = \mathsf{U}H \subseteq Q$.

Conversely let $g \in Q$. Then $gF_c = F_c$ for every c, hence $gF^\perp = F^\perp$. For $1 \le j \le k$ write $gx_j = w_j + \sum_i m_{ij} x_i + \lambda_j z$ with $w_j \in F$. Since $g^{-1} \in Q$, $g^{-1}z \in z + F$, so $-\lambda_j = h(z, gx_j) = h(g^{-1}z, x_j) = h(z, x_j) = 0$, as $F \perp X_0$. As F is isotropic and orthogonal to X_0,

$\delta_{jj'} = h(gx_j, gx_{j'}) = \sum_i \overline{m_{ij}} m_{ij'}$, so (m_{ij}) is unitary; let $h \in H$ be the element with this matrix. Put $u = gh^{-1}$; then $u \in G_1$, $ue_b = ge_b \in e_b + F_{b-1}$, $uz = gz \in z + F$, and $ux_j = g(h^{-1}x_j) \in x_j + F$ by the decomposition of $gx_{j'}$. So u preserves the flag $F_1 \subset \cdots \subset F_\varrho$, acts trivially on F_c/F_{c-1} and on $F^\perp/F = \langle x_i, z\rangle + F$. It also acts trivially on $F_{c-1}^\perp/F_c^\perp$ $(1 \le c \le \varrho)$: for $w \in F_{c-1}^\perp$, $uw - w \in F_{c-1}^\perp$ and $h(e_c, uw - w) = h(u^{-1}e_c - e_c, w) = 0$ because $u^{-1}e_c - e_c \in F_{c-1}$; so $uw - w \in F_c^\perp$. Hence $u \in \mathsf{U}$ and $g = uh \in R$. Thus $R = Q$, a closed subgroup.

Uniqueness: if $h \in H \cap \mathsf{U}$, then h preserves X_0 and acts trivially on $F^\perp/F$, so $hx_i \in (x_i + F) \cap X_0 = \{x_i\}$ and $h = 1$; hence $uh = u'h'$ forces $u'^{-1}u = h'h^{-1} = 1$. For $s = uh$ and $1 \le j \le k$, $sx_j = u(hx_j) \in hx_j + F$, so $s_{x_i x_j}$ is the (i,j) entry of the matrix of h; thus $s \mapsto h_s$ is the restriction of a linear map, hence continuous. For $s = uh$, $t = u'h'$ in R, $st = (u \cdot hu'h^{-1})(hh')$ with $hu'h^{-1} \in \mathsf{U}$, so $h_{st} = h_s h_t$. The map $s \mapsto (sh_s^{-1}, h_s)$ is continuous with continuous inverse $(u,h) \mapsto uh$.

(ii). By (i), each $s \in R$ acts on W by an upper unitriangular matrix $[s] = (s_{f_a f_b})_{a,b \le \varrho}$, and $[st] = [s][t]$. For unitriangular matrices $([s][t])_{a,a+1} = [s]_{a,a+1} + [t]_{a,a+1}$, so $\tau(s) = \sum_{a<\varrho} s_{f_a f_{a+1}}$ is additive on R, $\tau(1) = 0$ and $\tau(s^{-1}) = -\tau(s)$. Elements of H fix W, so $\tau = 0$ on H, and $\tau(s) = \tau(sh_s^{-1}) + \tau(h_s) = \tau(sh_s^{-1}) = \sigma(sh_s^{-1})$, the last equality being the definition of σ on U. So $\sigma = \tau$ on R, which gives (ii).

(iii). Let $s = uh$ with $h = h_s$; then $s - h = (u-1)h$ and $\operatorname{tr}(Z(s-h)) = \operatorname{tr}(hZ(u-1))$. The element h maps the level-$(\varrho+1)$ span to itself and fixes the other basis vectors, so $Y = hZ$ again satisfies $Y_{ij} = 0$ when the level of i exceeds that of j. By the description of U, $(u-1)_{ji} \ne 0$ only if the level of j is smaller than that of i. Hence every term of $\operatorname{tr}(Y(u-1)) = \sum_{i,j} Y_{ij}(u-1)_{ji}$ vanishes. For $Z = 1$ this gives $\operatorname{tr}(s-1) = \operatorname{tr}(h_s - 1)$; as $h_s - 1$ vanishes outside X_0, its trace is that of its matrix in $\mathrm{U}(k)$.

(iv). Let $g \in G_1$. Computing $\operatorname{tr}(Cg) = \operatorname{tr}(gC)$ in the basis,

$$\operatorname{tr}(gC) = \sum_{b=1}^{\varrho} g_{e_b f_b} - g_{z,e_\varrho^*} + \sum_{b=2}^{\varrho} g_{e_b^*, e_{b-1}^*}.$$

By the coordinate formulas and $h(gv, w) = h(v, g^{-1}w)$: $g_{e_b^*, e_{b-1}^*} = h(e_b, ge_{b-1}^*) = \overline{h(e_{b-1}^*, g^{-1}e_b)} = \overline{(g^{-1})_{e_{b-1} f_{b-1}}}$, and $-g_{z,e_\varrho^*} = h(z, ge_\varrho^*) = \overline{h(e_\varrho^*, g^{-1}z)} = \overline{(g^{-1})_{e_\varrho f_\varrho}}$. Since $f_a = e_{a+1}$ for $a < \varrho$, the first sum is $\sum_{a<\varrho} g_{f_a f_{a+1}}$ and the last two terms together are $\overline{\sum_{a<\varrho}(g^{-1})_{f_a f_{a+1}}}$. For $g = s \in R$ and $g = 1$, (ii) gives $\operatorname{tr}(Cs) = \sigma(s) + \overline{\sigma(s^{-1})} = \sigma(s) - \overline{\sigma(s)}$ and $\operatorname{tr}(C) = 0$, whence (iv).

(v). Translations are restrictions of matrix multiplication, hence continuous. By (i), $m\colon (u,h) \mapsto uh$ is a homeomorphism $\mathsf{U} \times H \to R$, so $ds = m_*(du \otimes dh)$ is a Borel measure on R, and for compact $C \subseteq R$, $ds(C) = (du \otimes dh)(m^{-1}C) < \infty$, because $m^{-1}C$ is compact, dh is finite and du is finite on compact sets (by Lemma 5.2.34 (i), (iii), a compact set of U has compact image under the row-coordinate map, of finite Lebesgue measure). Let $t = u'h' \in R$ and $\phi \ge 0$ Borel on R. For $s = uh$, $st = u\, c_h\, hh'$ with $c_h = hu'h^{-1} \in \mathsf{U}$. By Tonelli's theorem, the right invariance of du (Lemma 5.2.34 (iv)) applied for fixed h to c_h, and then the right invariance of dh applied for fixed u to h',

$$\int_R \phi(st)\, ds = \int_H \int_{\mathsf{U}} \phi(u\, c_h\, hh')\, du\, dh = \int_H \int_{\mathsf{U}} \phi(u\, hh')\, du\, dh = \int_{\mathsf{U}} \int_H \phi(uh)\, dh\, du = \int_R \phi\, ds.$$

(vi). Let $h \in H$, and write $\mathrm{rw}\colon \mathsf{U} \to \mathcal{C}_{\mathsf{U}}$ for the row-coordinate map of Lemma 5.2.34. Since h and h^{-1} fix the e_b, e_b^* and z and preserve X_0, the e_a-coordinate of hw is that of w for every w; hence $(huh^{-1})_{e_a j} = u_{e_a, h^{-1}j}$, which is $u_{e_a j}$ for $j \in \{e_b, e_b^*, z\}$ and $\sum_{i'} u_{e_a x_{i'}} (h^{-1})_{x_{i'} x_i}$ for $j = x_i$. So $\mathrm{rw}(huh^{-1}) = \Theta_h(\mathrm{rw}(u))$, where Θ_h is the linear map of $\mathcal{C}_{\mathsf{U}}$ that multiplies each block $(u_{e_a x_i})_{1 \le i \le k}$ by the unitary matrix h^{-1} and fixes the other coordinates; it preserves Lebesgue

measure. Let $A \subseteq \mathsf{U}$ be Borel. As rw is continuous and injective on the closed subgroup U of G_1 (Lemma 5.2.34 (i)), $B = \mathrm{rw}(A)$ is Borel (Lusin–Souslin) and $A = \mathrm{rw}^{-1}(B)$. By Lemma 5.2.34 (iii), $du(\{u : huh^{-1} \in A\}) = du(\mathrm{rw}^{-1}(\Theta_h^{-1}B)) = \mathrm{Leb}(\Theta_h^{-1}B) = \mathrm{Leb}(B) = du(A)$; so conjugation by h preserves du.

For the unfolding, recall that $\int_{\mathsf{U}\backslash G_1} \Phi = \int_{G_1} \Phi(y)\chi(y)\,dy$ for left U-invariant Borel $\Phi \geq 0$, where $\chi \geq 0$ is a Borel cutoff with $\int_{\mathsf{U}} \chi(uy)\,du = 1$ for every y (Definition 5.1.6). Put $\Phi(y) = \int_R f(sy)\,ds$; by (v) it is left R-invariant, in particular left U-invariant. By Tonelli's theorem and the left invariance of dy (Lemma 5.2.41), substituting $g = uhy$,

$$\int_{\mathsf{U}\backslash G_1} \Phi = \int_H \int_{\mathsf{U}} \int_{G_1} \chi(y) f(uhy)\,dy\,du\,dh = \int_{G_1} f(g) \int_H \int_{\mathsf{U}} \chi(h^{-1}u^{-1}g)\,du\,dh\,dg.$$

Here $h^{-1}u^{-1} = (h^{-1}u^{-1}h)h^{-1}$, and $u \mapsto h^{-1}u^{-1}h$ preserves du (inversion by Lemma 5.2.34 (iv), conjugation by the first part); so the inner integral over U is $\int_{\mathsf{U}} \chi(u'h^{-1}g)\,du' = 1$, and $\int_H dh = 1$ gives the claim. □

Lemma 5.5.31. *Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, let λ be its Haar ratio and $c_F = \mathrm{vol}\,\mathrm{U}(k)/(\mathrm{vol}\,\mathrm{U}(r)(\lambda\#I)^2)$, and let $(W, W^\flat)$ be a cut pair for b, f whose type is an irreducible unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$, I nonempty. Let p, q be sequences of nonzero complex numbers and $\phi\colon \mathbb{C} \to \mathbb{C}$ measurable with $|\phi| \leq 1$. For $c \in \mathbb{C}$ let $u(c) = c$ if $c \neq 0$ and $u(c) = 1$ if $c = 0$, let p_c, q_c be p, q with the entry of index ϱ replaced by $u(c)$, and*

$$\Phi(c) = \phi(c)\,\langle \mathrm{Ci}_{W^\flat}(p_c), \mathrm{Ci}_{W^\flat}(q_c)\rangle.$$

Then Φ is Lebesgue integrable on $\mathbb{C}$ and

$$\int_{\mathbb{C}} \Phi(c)\,dc = c_F\,\mathcal{F}_\varrho(p, q; W^\flat,\ x \mapsto \phi(h(f_\varrho, xf_\varrho))).$$

Proof. Write $F = W^\flat$, $\mathrm{dn}(c) = \frac{2\pi^k}{(k-1)!}((|c|^2 - 1)_+)^{k-1}$ for $c \in \mathbb{C}$, and $Z_\varrho(x) = h(f_\varrho, xf_\varrho)$. Since $k = s + 3n + 6 \geq 2$, $\mathrm{dn}(c) = 0$ for $|c| \leq 1$.

Step 1: the setting of the descent. By Lemma 5.5.5, (i), the data at $\mathbb{C}$ with box radius $\rho = b^{6n+2}$ form a place datum, and by Lemma 5.5.9 F is cut for it. Choose $L \geq \rho$ with $|p_0 \cdots p_{l-1}|^{-1} \leq L$ and $|q_0 \cdots q_{l-1}|^{-1} \leq L$ for all $l \leq \varrho$; then p, q are heads of length ϱ for the gauge Γ_L (Lemma 5.5.5, (ii)), whose size filtration is $A_{(j)} = \{|x| \leq L^j\}$. By Lemma 5.5.4, (ii), with $j = \varrho$, $t = 1$ and the majorant $\mathrm{FM}_\varrho(p, q, F, \cdot)$ itself as the function bounded there (Θ as in Definition 5.1.6),

$$\#I\,c_F\,\mathrm{hdn}_\varrho(p, q) \int_G \mathrm{FM}_\varrho(p, q, F, x)\,d\theta(x) \leq \Theta\,L^{10^9 n^4}\rho^{24k} < \infty;$$

as $0 < \lambda < \infty$ (Lemma 5.5.10) and the p_a, q_a are nonzero, $\#I\,c_F\,\mathrm{hdn}_\varrho(p, q) > 0$, so

$$\int \mathrm{FM}_\varrho(p, q, F, x)\,d\theta(x) < \infty.$$

Put $\zeta = \mu_H$, $v(c) = \sigma_c$ for $c \neq 0$ and $v(0) = 1$; v is measurable (continuous on $\mathbb{C}^\times$ by the formula (RP2)). By (RP4), $\theta_\varrho = \mu_{\mathrm{top}}$ is the image of $\mu_{\mathrm{cell}} \otimes \mu_H$ under $(g, g') \mapsto gg'$, and μ_{cell} is the image of $\mathrm{dn}(c)\,dc$ under $c \mapsto v(c)^{-1}$ (the point $c = 0$ is Lebesgue-null); so θ_ϱ is the convolution of that image with ζ. The density dn is measurable and finite. The finite measure ζ

is carried by $H = G^{(\varrho+1)}$, and $G^{(\varrho+1)} \subseteq G^{(\varrho)} \cap R^{(\varrho+1)}$ (Lemma 5.2.58, (i)). So Proposition 5.2.92, (i), applies with $b = \varrho$, the weight ϕ, $t = c_F$, the density dn and the maps v: the function

$$\Psi(c) = c_F \,\mathrm{hdn}_\varrho(p,q)\,\mathrm{dn}(c)\,\phi(Z_\varrho(v(c)))\int_{G\times G} \mathcal{K}_{F,\varrho+1}(g,gx)\,\overline{\mathcal{K}^d_{F,\varrho+1}(w_c^{-1}g, w_c^{-1}gx)}\,d\nu(x,g),$$

with $w_c = w_{\varrho,p}v(c)$, $d = w_{\varrho,q}^{-1}w_{\varrho,p}$ and $\nu = \theta\otimes((\mu_H * \theta_\varrho) * \mathfrak{m}_\varrho)$, is integrable on $\mathbb{C}$ with $\int\Psi = c_F\,\mathcal{F}_\varrho(p,q;F,\phi\circ Z_\varrho)$. It remains to show $\Phi(c) = \Psi(c)$ for every $c \neq 0$.

Step 2: the measures and the kernels above the top floor. By Definition 5.2.1, $\eta_l = \delta_1$ for $l \geq \varrho$, so $\theta = \mathfrak{m}_{0,\varrho+2} = \mu_H * \mu_{\mathrm{top}} * \mathfrak{m}_\varrho$ (convolution of s-finite measures is associative, by Tonelli), that is $(\mu_H * \theta_\varrho) * \mathfrak{m}_\varrho = \theta$, and $\nu = \theta\otimes\theta = \lambda^2\,dy\otimes dy$ by Lemma 5.5.10. The tower measure from floor $\varrho+1$ is $\theta_{\geq\varrho+1} = \mu_H$, the image of the Haar probability measure of $\mathrm{U}(k)$. Let $\mathrm{em}(h) \in H$ be the image of $h \in \mathrm{U}(k)$. For $g, y \in G$: $F(\mathrm{em}(h)g)$ is the image of $F(g)$ under the unitary operator by which h acts in the type of F, so $\langle F(\mathrm{em}(h)g), F(\mathrm{em}(h)y)\rangle = \langle F(g), F(y)\rangle$; and d commutes with $\mathrm{em}(h) \in G^{(\varrho)}$ (Lemma 5.2.58, (ix)), so $\langle F(\mathrm{em}(h)g), F(d\mathrm{em}(h)y)\rangle = \langle F(g), F(dy)\rangle$. The integrands being bounded and μ_H a probability measure,

$$\mathcal{K}_{F,\varrho+1}(g,y) = \langle F(g), F(y)\rangle, \qquad \mathcal{K}^d_{F,\varrho+1}(g,y) = \langle F(g), F(dy)\rangle.$$

Step 3: the inner integral is a Gram product of orbital integrals. Let $c \neq 0$ and put $\gamma_p = w_{\varrho,p}\sigma_c$, $\gamma_q = w_{\varrho,q}\sigma_c$. As $\sigma_c \in G^{(\varrho)}$ (the boosts fix every e_a, e_a^*) and $w_{\varrho,p}$ commutes with $G^{(\varrho)}$ (Lemma 5.2.58, (ix)), $dw_c^{-1} = w_{\varrho,q}^{-1}w_{\varrho,p}\sigma_c^{-1}w_{\varrho,p}^{-1} = \gamma_q^{-1}$, while $w_c^{-1} = \gamma_p^{-1}$. By Step 2 the integrand of $\Psi(c)$ is

$$\Pi(x,g) = \langle F(g), F(gx)\rangle\,\overline{\langle F(\gamma_p^{-1}g), F(\gamma_q^{-1}gx)\rangle}.$$

The map $(x,g) \mapsto (y,y') = (g,gx)$ is a homeomorphism of $G\times G$ carrying $dy\otimes dy$ to itself (for fixed g, $x \mapsto gx$ preserves dy, Lemma 5.2.41; then Tonelli), and turns Π into $\Pi'(y,y') = \langle F(y), F(y')\rangle\,\overline{\langle F(\gamma_p^{-1}y), F(\gamma_q^{-1}y')\rangle}$. By Lemma 5.4.20, (iii), for Γ_L and $x = \sigma_c \in G^{(\varrho)}$: if $F(y) \neq 0$ and $F(\gamma_p^{-1}y) \neq 0$ then y lies in the set C_0 of elements of size $60n^2$, and likewise for y' with q. The set C_0 is compact ($A_{(60n^2)}$ is compact by (GU1), and G is closed), and $dy(C_0) < \infty$: a compact neighbourhood of 1 contained in the image of a compact neighbourhood of 0 under the Cayley chart (a homeomorphism onto an open neighbourhood of 1 with continuous density) has finite measure, and finitely many of its left translates cover C_0 (Lemma 5.2.41). Since F is bounded and measurable, Π' is integrable for $dy\otimes dy$, vanishing outside $C_0\times C_0$, so Π is ν-integrable, and by Fubini, expanding $\langle u,u'\rangle = \sum_i \bar{u}_i u'_i$,

$$\int\Pi\,d\nu = \lambda^2\sum_{i,j\in I}\overline{\int_G F_i(y)\overline{F_j(\gamma_p^{-1}y)}\,dy}\int_G F_i(y')\overline{F_j(\gamma_q^{-1}y')}\,dy' = \lambda^2\langle \mathrm{Orb}_{dy}(\gamma_p), \mathrm{Orb}_{dy}(\gamma_q)\rangle,$$

each single integral being absolutely convergent for the same reason.

Step 4: comparison of the constants. If $0 < |c| \leq 1$, then $\mathrm{dn}(c) = 0$, so $\Psi(c) = 0$; and the factor $\sqrt{(|c|^2-1)_+}^{k-1}$ of both normalised orbital integrals vanishes ($k \geq 2$), so $\Phi(c) = 0$. Let $|c| > 1$. Then $u(c) = c$, $\sigma_c = \mathrm{bt}(c)$ and $\mathrm{bt}(c)z = s_c x_{k_0} - cz$, so $Z_\varrho(v(c)) = h(z, s_c x_{k_0} - cz) = -c\,h(z,z) = c$ and the weight in $\Psi(c)$ is $\phi(c)$. The Weyl element w_{p_c} depends only on the entries of index below ϱ, so $w_{p_c} = w_{\varrho,p}$ and, by Definition 5.5.2 and the definition of N,

$$\mathrm{Ci}_F(p_c) = \frac{\sqrt{c_G}}{\#I}\,(|c|^2-1)^{(k-1)/2}\prod_{a<\varrho}|p_a|^{d_a}\,\mathrm{Orb}_{dy}(\gamma_p),$$

and likewise for q_c with γ_q. The scalar factors are real, so

$$\Phi(c) = \phi(c)\,\frac{c_G}{(\#I)^2}\,(|c|^2-1)^{k-1}\,\mathrm{hdn}_\varrho(p,q)\,\langle \mathrm{Orb}_{dy}(\gamma_p), \mathrm{Orb}_{dy}(\gamma_q)\rangle,$$

using $\prod_{a<\varrho}(|p_a||q_a|)^{d_a} = \mathrm{hdn}_\varrho(p,q)$. By Step 3,

$$\Psi(c) = c_F\,\lambda^2\,\mathrm{dn}(c)\,\mathrm{hdn}_\varrho(p,q)\,\phi(c)\,\langle \mathrm{Orb}_{dy}(\gamma_p), \mathrm{Orb}_{dy}(\gamma_q)\rangle,$$

and

$$c_F\,\lambda^2\,\mathrm{dn}(c) = \frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)\,(\#I)^2}\cdot\frac{2\pi^k}{(k-1)!}(|c|^2-1)^{k-1} = \frac{c_G}{(\#I)^2}(|c|^2-1)^{k-1},$$

since $c_G = \mathrm{vol}\,\mathrm{U}(k)/\mathrm{vol}\,\mathrm{U}(r)\cdot 2\pi^k/(k-1)!$. So $\Phi(c) = \Psi(c)$.

Conclusion. $\Phi = \Psi$ outside the null set $\{0\}$, so Φ is integrable and $\int\Phi = \int\Psi = c_F\,\mathcal{F}_\varrho(p,q;F,\phi\circ Z_\varrho)$. □

Lemma 5.5.32. *Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, and let $(W, W^\flat)$ be a cut pair for b, f whose type is a unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$, I nonempty. Give the set $\mathcal{S}$ of sequences $(p_a)_{a\geq 0}$ in $\mathbb{C}^\times$ the product topology.*

(i) If $W^\flat$ is continuous, then $p \mapsto \mathrm{Ci}_{W^\flat}(p)$ is continuous on $\mathcal{S}$; hence $(p,q) \mapsto \langle \mathrm{Ci}_{W^\flat}(p), \mathrm{Ci}_{W^\flat}(q)\rangle$ is continuous on $\mathcal{S}\times\mathcal{S}$.

(ii) Let $\sigma \in G_1$ fix every e_a and e_a^* $(a<\varrho)$, and let $p, q \in \mathcal{S}$ with $p_\varrho = q_\varrho = h(z, \sigma z)$. Then

$$\langle N(\sigma, W^\flat, p), N(\sigma, W^\flat, q)\rangle = \langle \mathrm{Ci}_{W^\flat}(p), \mathrm{Ci}_{W^\flat}(q)\rangle.$$

Proof. Write $F = W^\flat$ and $\mathrm{em}(u) \in G_1$ for the image of $u \in \mathrm{U}(k)$ (acting on X_0 through the orthonormal coordinates $x_1, \dots, x_k$, fixing every e_a, e_a^* and z); $F(\mathrm{em}(u)y) = \Pi(u)F(y)$, where $\Pi(u)$ is the unitary matrix by which u acts in the type of F. For $p \in \mathcal{S}$ put

$$s(p) = \frac{\sqrt{c_G}}{\#I}\left((|p_\varrho|^2-1)_+\right)^{(k-1)/2}\prod_{a<\varrho}|p_a|^{d_a},$$

so that $N(\sigma, F, p) = s(p)\,\mathrm{Orb}_{dy}(w_p\sigma)$ for every $\sigma \in G_1$, and $\mathrm{Ci}_F(p) = s(p)\,\mathrm{Orb}_{dy}(w_p\sigma_{p_\varrho})$ (Definition 5.1.6, Definition 5.5.2).

Step 0: compact support. By Lemma 5.5.5, (i), and Lemma 5.5.9, F is cut for the place datum at $\mathbb{C}$ with box radius $\rho = b^{6n+2}$. Let $L \geq \rho$, let $p \in \mathcal{S}$ be a head of length ϱ for Γ_L and $x \in G^{(\varrho)}$. By Lemma 5.4.20, (iii), the function $y \mapsto F_i(y)\overline{F_j((w_px)^{-1}y)}$ vanishes outside the set C_L of the elements of G_1 of size $60n^2$ for Γ_L. The set C_L is compact ((GU1) and G_1 closed), and $dy(C_L) < \infty$: a compact neighbourhood of 1 contained in the image of a compact neighbourhood of 0 under the Cayley chart (a homeomorphism onto an open neighbourhood of 1, with continuous density) has finite measure, and finitely many of its left translates cover C_L (Lemma 5.2.41). As F is bounded and measurable, every entry of $\mathrm{Orb}_{dy}(w_px)$ is an absolutely convergent integral over C_L. Every $p \in \mathcal{S}$ is a head of length ϱ for Γ_L as soon as $L \geq \rho$ and $L \geq |p_0\cdots p_{l-1}|^{-1}$ for $l \leq \varrho$.

(i). Fix $p \in \mathcal{S}$ and choose $L \geq \rho$ with $|p_0\cdots p_{l-1}|^{-1} < L$ for all $l \leq \varrho$. The set $\mathcal{S}_L$ of the $p' \in \mathcal{S}$ with $|p'_0\cdots p'_{l-1}|^{-1} < L$ for all $l \leq \varrho$ is open (it is defined by finitely many continuous coordinate conditions), contains p, and consists of heads of length ϱ for Γ_L. On $\mathcal{S}$, s is continuous (it depends continuously on $p_0, \dots, p_\varrho$). The map $c \mapsto \sigma_c = \mathrm{bt}(c/\min(1,|c|))$ is continuous on $\mathbb{C}^\times$: $c \mapsto c/\min(1,|c|)$ is continuous with values in $\{|a| \geq 1\}$, where the matrix entries $\bar a$, $\pm a$, $\sqrt{|a|^2-1}$ of $\mathrm{bt}(a)$ are continuous. The entries p'_a, $\bar p'^{-1}_a$ of $w_{p'}$ are continuous in p', and inversion and multiplication are continuous on G_1. Hence, F being continuous, $(p', y) \mapsto F_i(y)\overline{F_j((w_{p'}\sigma_{p'_\varrho})^{-1}y)}$ is continuous on $\mathcal{S}_L \times G_1$; by Step 0 (with $x = \sigma_{p'_\varrho} \in G^{(\varrho)}$, the

boosts fixing every e_a, e_a^*) it vanishes outside $\mathcal{S}_L \times C_L$, and it is bounded by $(\sup\|F\|)^2$. The space $\mathcal{S}$ is metrizable (a countable product of metric spaces), so continuity at p is sequential continuity, and dominated convergence with the integrable majorant $(\sup\|F\|)^2\mathbf{1}_{C_L}$ shows that $p' \mapsto \mathrm{Orb}_{dy}(w_{p'}\sigma_{p'_\varrho})$ is continuous at p. So $\mathrm{Ci}_F = s \cdot \mathrm{Orb}_{dy}(w.\sigma_{._\varrho})$ is continuous at p, and the Gram form is continuous as the composite with the continuous inner product.

(ii). Put $c = h(z, \sigma z)$; the normalising factors of both sides are $s(p)s(q)$, real. If $|c| \le 1$, then $s(p) = s(q) = 0$ (as $k \ge 2$), and both sides vanish. Let $|c| > 1$ and $s_c = \sqrt{|c|^2 - 1}$.

Cartan decomposition: $\sigma = \mathrm{em}(u_1)\,\mathrm{bt}(c)\,\mathrm{em}(u_2)$ for some $u_1, u_2 \in \mathrm{U}(k)$. As σ is an isometry fixing the vectors of the hyperbolic planes, it preserves their h-orthogonal complement $X_0 \oplus \mathbb{C}z$. Write $\sigma z = \omega + c'z$ with $\omega \in X_0$. Then $c = h(z, \sigma z) = c'h(z, z) = -c'$, and $-1 = h(\sigma z, \sigma z) = \|\omega\|^2 - |c|^2$, so $\|\omega\| = s_c > 0$. As $\mathrm{U}(k)$ acts transitively on the sphere of radius s_c in $X_0 = \mathbb{C}^k$, there is u_1 with $u_1(s_c x_{k_0}) = \omega$; then $\mathrm{em}(u_1)\mathrm{bt}(c)z = \mathrm{em}(u_1)(s_c x_{k_0} - cz) = \omega - cz = \sigma z$. So $\tau = (\mathrm{em}(u_1)\mathrm{bt}(c))^{-1}\sigma \in G_1$ fixes z and every e_a, e_a^*; it preserves their orthogonal complement X_0, on which h is the standard positive form, so $\tau = \mathrm{em}(u_2)$ with $u_2 = \tau|_{X_0} \in \mathrm{U}(k)$. Since $\sigma_c = \mathrm{bt}(c)$ for $|c| \ge 1$, $\sigma = \mathrm{em}(u_1)\sigma_c\mathrm{em}(u_2)$.

Transport of the orbital integrals. The elements $\mathrm{em}(u_1)$, σ and σ_c lie in $G^{(\varrho)}$, which commutes with w_p and w_q (Lemma 5.2.58, (ix)); so $w_p\sigma = \mathrm{em}(u_1)\,\gamma_p\,\mathrm{em}(u_2)$ with $\gamma_p = w_p\sigma_c$, and likewise for q. Take L so large that p, q are heads of length ϱ for Γ_L; by Step 0 all entries of $\mathrm{Orb}_{dy}(w_p\sigma)$ and $\mathrm{Orb}_{dy}(\gamma_p)$ (and those for q) are absolutely convergent. Substituting $y = \mathrm{em}(u_1)y''$, which preserves dy (Lemma 5.2.41), and writing $U = \Pi(u_1)$, $V = \Pi(u_2)^{-1}$,

$$\mathrm{Orb}_{dy}(w_p\sigma)_{ij} = \int_{G_1} F_i(\mathrm{em}(u_1)y'')\,\overline{F_j(\mathrm{em}(u_2)^{-1}\gamma_p^{-1}y'')}\,dy'' = \sum_{a,b\in I} U_{ia}\,\mathrm{Orb}_{dy}(\gamma_p)_{ab}\,\overline{V_{jb}},$$

that is $\mathrm{Orb}_{dy}(w_p\sigma) = U\,\mathrm{Orb}_{dy}(\gamma_p)\,V^*$, and the same with q and the same unitary U, V. The inner product of $\mathbb{C}^{I\times I}$ is $\langle A, B\rangle = \mathrm{tr}(A^*B)$, so $\langle UAV^*, UBV^*\rangle = \mathrm{tr}(VA^*BV^*) = \langle A, B\rangle$. Therefore

$$\langle N(\sigma, F, p), N(\sigma, F, q)\rangle = s(p)s(q)\langle \mathrm{Orb}_{dy}(\gamma_p), \mathrm{Orb}_{dy}(\gamma_q)\rangle = \langle \mathrm{Ci}_F(p), \mathrm{Ci}_F(q)\rangle,$$

the last equality because $p_\varrho = q_\varrho = c$, so $\gamma_p = w_p\sigma_{p_\varrho}$ and $\gamma_q = w_q\sigma_{q_\varrho}$. □

Lemma 5.5.33. *Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, and let $(W, W^\flat)$ be a cut pair for b, f whose type is an irreducible unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$, I nonempty. Put $R = b^{60n^2}$, $K = n^6$ and $\mathrm{Dec}(\nu) = \Theta\, b^{2000n^9}\ell_{k-n^7}(|\nu|^{-1})$ for $\nu \in \mathbb{C}^\times$. Let p, q be sequences of nonzero complex numbers.*

(a) For $i \le \varrho$ and $\nu \in \mathbb{C}^\times$,

$$\#I \cdot |c_F\,\mathcal{F}_i(p, q; W^\flat, x \mapsto \psi(\nu\, h(f_i, xf_i)))| \le \mathrm{Dec}(\nu)\,\Lambda_i^{R,K}(p)\,\Lambda_i^{R,K}(q).$$

(b) For $i + 1 \le \varrho$ with $\nu = p_i^{-1} - q_i^{-1} \ne 0$,

$$\#I \cdot |c_F\,\mathcal{F}_{i+1}(p, q; W^\flat, 1)| \le \mathrm{Dec}(\nu)\,\Lambda_{i+1}^{R,K}(p)\,\Lambda_{i+1}^{R,K}(q).$$

Proof. Treat (a) and (b) together: let $j = i$ and $\omega(x) = \psi(\nu h(f_i, xf_i))$ in case (a), $j = i + 1$ and $\omega = 1$ in case (b); in both cases $|\omega| \le 1$ and $j \le \varrho$. Write $H = \Lambda_j^{R,K}(p)\Lambda_j^{R,K}(q)$. By Lemma 5.5.9, $W^\flat$ is cut, and by Lemma 5.5.10, θ is a Haar measure. By (DN0)(a) of Theorem 5.4.1 and $c_F > 0$,

$$|c_F\,\mathcal{F}_j(p, q; W^\flat, \omega)| \le c_F\,\mathrm{hdn}_j(p, q)\int_G \mathrm{FN}_j(p, q, W^\flat, x; \omega)\,d\theta(x), \qquad \mathrm{FN}_j(\cdot;\omega) \le \mathrm{FM}_j. \quad (5.95)$$

Step 1: large head products give the form 0. As $n \ge 3$, $\rho^{9n} = b^{(6n+2)9n} \le b^{60n^2} = R$. Suppose some $l \le j$ has $|\delta_{l-1}(p)| > R$ or $|\delta_{l-1}(q)| > R$ (with $l \ge 1$). Let y be active at some x for $(j,p,q,W^\flat)$. By (AC1), $W^\flat(y) \ne 0$ and $W^\flat(yx) \ne 0$, so $uy \in K$ and $vyx \in K$ for some $u, v \in \mathsf{U}$; as $\mathsf{U} \subseteq R^{(j)}$, y and yx are near at j. By (AC2), (AC3), $w_{j,p}^{-1}y$ and $w_{j,q}^{-1}yx$ are near at j. The frame Gram matrix of $w_{j,p}$ is diagonal in its first j rows with entries $p_0, \dots, p_{j-1}$ ($h(e_a, w_{j,p}e_a) = p_a$), so its leading principal minor of order l is $\delta_{l-1}(p)$; by (DN0)(e) of Theorem 5.4.1 for the gauge Γ_ρ, $|\delta_{l-1}(p)| \le \rho^{9n} \le R$, and likewise for q. This contradicts the assumption, so no point is active at any x: $(j,p,q,W^\flat)$ has a deep vector after the head of depth at least $\infty = \sigma^{-1}$, $\sigma = 0$, at every x. With $V \ge 1$ and $\mathrm{FML}_j(V;1)$ from Lemma 5.5.4, (i), and p a head of length j for Γ_L with $L = \max(\rho, \max_{l \le j} |\delta_{l-1}(p)|^{-1})$, (DN3)(c) gives $\mathrm{FM}_j(p,q,W^\flat,x)^2 \le 0^s V^2 \mathrm{WSP}_j = 0$ ($s > 0$). By (5.95) the form is 0, and the claim holds.

Step 2: a gauge for both heads. Assume now $|\delta_l(p)|, |\delta_l(q)| \le R$ for $l < j$. Let

$$L = \max\bigl(\rho, \max_{l \le j} |\delta_{l-1}(p)|^{-1}, \max_{l \le j} |\delta_{l-1}(q)|^{-1}\bigr)$$

(empty products = 1). Then $L \ge \rho$ and p, q are heads of length j for Γ_L. Each term of the maximum is at most

$$\rho \prod_{l<j} \max(1, |\delta_l(p)|^{-1}) \prod_{l<j} \max(1, |\delta_l(q)|^{-1}),$$

as every factor is ≥ 1 and $\rho \ge 1$; raising to the power K and using the definition of the head weights,

$$L^K \le \rho^K H.$$

Step 3: the local gain. Let $x \in G$. In case (a), (DN3)(a) of Theorem 5.4.1 with $\xi = \eta = \nu$ gives $\mathrm{FN}_j(p,q,W^\flat,x;\omega)^2 \le (\pi^2/|\nu|^2)^{k-5} \mathrm{WSP}_j(p,q,W^\flat,x)$ for Γ_L; in case (b), (DN3)(b) with $\xi = M^{-1}(p_i^{-1} - q_i^{-1}) = \nu$ gives the same. Put $t = \max(1, \rho^{24}|\nu|)$ and V as in Lemma 5.5.4. If $t > 1$, then $t = \rho^{24}|\nu|$, and since $\pi\rho^{24} \ge 1$, $\rho^{24}|\nu| \ge 1$ and $s \le k - 5 \le k$,

$$\Bigl(\frac{\pi^2}{|\nu|^2}\Bigr)^{k-5} = \Bigl(\frac{(\pi\rho^{24})^{k-5}}{(\rho^{24}|\nu|)^{k-5}}\Bigr)^2 \le \Bigl(\frac{(\pi\rho^{24})^k \rho^{9999n}}{(\rho^{24}|\nu|)^s}\Bigr)^2 = \Bigl(\frac{V}{t^s}\Bigr)^2.$$

So $\phi = \mathrm{FN}_j(p,q,W^\flat,\cdot;\omega)$ satisfies the hypotheses of Lemma 5.5.4, (ii), and with (5.95),

$$\#I\, |c_F\, \mathcal{F}_j(p,q;W^\flat,\omega)| \le \Theta\, L^{10^9 n^4} \rho^{24k}\, \frac{t^{10^8 n^4}}{t^s}.$$

Step 4: the damped power. Write $k = n^7 + K'$ ($K' = k - n^7 \ge 0$). As $n > 10^8$: $10^9 n^4 \le n^6 = K$, and $s = k - 3n - 6 \ge 10^8 n^4 + K' + 6$. As $L, t \ge 1$, Step 2 gives $L^{10^9 n^4} \le L^K \le \rho^K H$, and $t^{10^8 n^4}/t^s \le t^{-K'-6}$. Moreover $\rho^{24}/t \le |\nu|^{-1}$ and $t^{-1} \le \min(1, |\nu|^{-1})$ (as $t \ge 1$ and $t \ge \rho^{24}|\nu| \ge |\nu|$). Hence

$$\begin{aligned} L^{10^9 n^4} \rho^{24k} \frac{t^{10^8 n^4}}{t^s} &\le \rho^{n^6} \rho^{24n^7} \Bigl(\frac{\rho^{24}}{t}\Bigr)^{K'} t^{-6} H \\ &\le \rho^{n^6 + 24n^7}\, |\nu|^{-K'} \min(1, |\nu|^{-1})^6\, H = \rho^{n^6+24n^7}\, \ell_{K'}(|\nu|^{-1})\, H, \end{aligned}$$

and $\rho^{n^6+24n^7} = b^{(6n+2)(n^6+24n^7)} \le b^{2000n^9}$. With Step 3 this is the claim. □

Lemma 5.5.34. *Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, and let $(W, W^\flat)$ be a cut pair for b, f whose type is an irreducible unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$, I nonempty. Let* $\deg > 0$ *with*

$$\int_{G_1} \Bigl| \int_{\mathsf{U} \setminus G_1} \langle W(y), W(yx) \rangle\, dy \Bigr|^2 dx \le \frac{(\#I)^2}{\deg}, \qquad \deg \cdot \Theta \le b^{4r^2},$$

$\deg\cdot\mathrm{vol}\{g\in G_1:\|g\|_{\mathrm{op}}\le T\}\le T^{9r^2}b^{99r^2}$ *for all* $T\ge 1$, *and* $4000n\le f$. *Then for all sequences* p,q *of nonzero complex numbers*

$$|c_F\,\mathcal{F}_0(p,q;W^\flat,1)|\le\frac{1+b^{-10}}{\deg}.$$

Proof. **Step 0: numbers.** $n\ge 3$, so $\rho=b^{6n+2}\le b^{7n}$; $r=s+5n+5$; $s>10^9n^3$, and $s\ge 10^{10}n^4$ (from $k=s+3n+6\ge n^7$, $n>10^8$). Put $\varepsilon=b^{-r^3f}$.

Step 1: the matrix coefficients. For $A,B\in\{W,W^\flat,W^\flat-W\}$ (Bessel functions of the type of the pair) and $x\in G_1$, the function $y\mapsto\langle A(y),B(yx)\rangle$ is left U-invariant ($|\text{wave}|=1$), and by Cauchy–Schwarz and Lemma 5.5.29, (ii), applied to the U-invariant $\|B\|^2$,

$$\int_{\mathsf{U}\backslash G_1}|\langle A(y),B(yx)\rangle|\,dy\le\Big(\int_{\mathsf{U}\backslash G_1}\|A\|^2\Big)^{1/2}\Big(\int_{\mathsf{U}\backslash G_1}\|B\|^2\Big)^{1/2}.$$

Put $Q(x)=\int_{\mathsf{U}\backslash G_1}\langle W(y),W(yx)\rangle\,dy$ and $Q^\flat(x)=\int_{\mathsf{U}\backslash G_1}\langle W^\flat(y),W^\flat(yx)\rangle\,dy$, both finite and Borel in x (Fubini). Since $\langle W^\flat(y),W^\flat(yx)\rangle-\langle W(y),W(yx)\rangle=\langle(W^\flat-W)(y),W^\flat(yx)\rangle+\langle W(y),(W^\flat-W)(yx)\rangle$, (BS1) and (BS2) give $|Q^\flat(x)-Q(x)|\le 2\varepsilon\#I$ for every x.

Step 2: $\mathrm{FN}_0(p,q,W^\flat,x;1)=\lambda^2|Q^\flat(x)|^2$**.** Fix x and put $g_x(y)=\langle W^\flat(y),W^\flat(yx)\rangle=\mathcal{K}_{W^\flat}(y,yx)$. By the unfolding identity of Lemma 5.5.10, applied to $|g_x|$ and to the positive and negative parts of $\mathrm{Re}\,g_x$, $\mathrm{Im}\,g_x$ (all Borel, ≥ 0, U-invariant), g_x is $\theta_{\ge 0}$-integrable with $\int g_x\,d\theta_{\ge 0}=\lambda Q^\flat(x)$. As $\theta_{\ge 0}=\theta_{\ge 1}*\theta_0$ (Lemma 5.2.59, (i)), Fubini gives: for θ_0-almost every ξ, $y'\mapsto g_x(y'\xi)$ is $\theta_{\ge 1}$-integrable, so $A(\xi):=\mathcal{K}_{W^\flat,1}(\xi,\xi x)=\int g_x(y'\xi)\,d\theta_{\ge 1}(y')$, and $\int A\,d\theta_0=\lambda Q^\flat(x)$. At floor 0 the Weyl elements are 1, so $d=1$ and $\mathcal{K}^d_{W^\flat}=\mathcal{K}_{W^\flat}$, and by definition

$$\mathrm{FI}_0(p,q,W^\flat,x;1)(1)=\iint A(\xi)\,\overline{A(\zeta)}\,d\theta_0(\zeta)\,d\theta_0(\xi)=\Big|\int A\,d\theta_0\Big|^2=\lambda^2|Q^\flat(x)|^2;$$

as $\mathfrak{m}_0=\delta_1$, $\mathrm{FN}_0(p,q,W^\flat,x;1)=|\mathrm{FI}_0(p,q,W^\flat,x;1)(1)|=\lambda^2|Q^\flat(x)|^2$.

Step 3: positions near 1. Let $T_0=b^{6r}\rho^{11n}$ and $C=\{x:|(x-1)_{ac}|\le T_0^{99n}\ \forall a,c\}$, a closed set. For $x\in C$, $\|x-1\|_{\mathrm{op}}\le rT_0^{99n}$, so $\|x\|_{\mathrm{op}}\le(r+1)T_0^{99n}\le\rho^2T_0^{99n}\le b^{14n+594nr+7623n^3}\le b^{700nr}=:T$, as $14n+7623n^3\le 106nr$ ($r>10^9n^3$). By hypothesis $\mathrm{vol}(C)\le T^{9r^2}b^{99r^2}/\deg$. By Step 1 and Minkowski's inequality in $L^2(C,dx)$,

$$\Big(\int_C|Q^\flat|^2dx\Big)^{1/2}\le\frac{\#I}{\sqrt{\deg}}+2\varepsilon\#I\Big(\frac{T^{9r^2}b^{99r^2}}{\deg}\Big)^{1/2}\le(1+b^{-12})\frac{\#I}{\sqrt{\deg}},$$

since $(2\varepsilon)^2T^{9r^2}b^{99r^2}\le b^{2+6300nr^3+99r^2-2r^3f}\le b^{-24}$, using $2r^3f\ge 8000nr^3\ge 6300nr^3+99r^2+26$. By Step 2, $\theta=\lambda\,dx$ and Lemma 5.5.10 ($\lambda\,\mathrm{vol}\,\mathrm{U}(k)/\mathrm{vol}\,\mathrm{U}(r)\le 1$),

$$c_F\int_C\mathrm{FN}_0(p,q,W^\flat,x;1)\,d\theta=c_F\lambda^3\int_C|Q^\flat|^2dx\le\frac{\mathrm{vol}\,\mathrm{U}(k)}{\mathrm{vol}\,\mathrm{U}(r)}\,\lambda\,\frac{(1+b^{-12})^2}{\deg}\le\frac{(1+b^{-12})^2}{\deg}.$$

Step 4: remote positions. With $L=\rho$ and $t=b^{6r}\ge 1$, $C=B(tL^{11n})$ in the notation of Lemma 5.5.35; every sequence is a head of length 0. By Lemma 5.5.9, $W^\flat$ is cut, so (DN0)(a) of Theorem 5.4.1 gives $\mathrm{FN}_0(\cdot;1)\le\mathrm{FM}_0$, and Lemma 5.5.35 with $j=0$ gives

$$c_F\int_{G\backslash C}\mathrm{FN}_0\,d\theta\le\Theta\,\rho^{10^9n^4}\rho^{24k}\frac{t^{10^8n^4}}{t^s}\le\frac{b^E}{\deg\,b^{6rs}},\qquad E=4r^2+7\cdot10^9n^5+168nk+6\cdot10^8n^4r,$$

by $\Theta \le b^{4r^2}/\deg$ and $\rho \le b^{7n}$. Here $E+12 \le 6rs$: as $s \ge 20n+20$, $6rs-4r^2 = r(2s-20n-20) \ge s^2$, and with $r,k \le 2s$ and $s \ge 10^{10}n^4$, $7\cdot 10^9 n^5 + 168nk + 6\cdot 10^8 n^4 r + 12 \le 7\cdot 10^9 n^5 + 336ns + 12\cdot 10^8 n^4 s + 12 \le s^2$. So this part is at most $b^{-12}/\deg$.

Step 5. By (DN0)(a), $\kappa_0 = 1$, Steps 3 and 4, and with $x = b^{-12}$, $(1+x)^2 + x = 1 + (3+x)b^{-2}b^{-10} \le 1 + b^{-10}$ (as $b \ge 2$),

$$|c_F \mathcal{F}_0(p,q;W^\flat,1)| \le c_F \int_G \mathrm{FN}_0\, d\theta \le \frac{(1+b^{-12})^2 + b^{-12}}{\deg} \le \frac{1+b^{-10}}{\deg}.$$

□

Lemma 5.5.35. *Let ρ, b, s be real-place sizes, consider the data at $\mathbb{C}$ with box radius ρ, and let $(W, W^\flat)$ be a cut pair for b, f whose type is an irreducible unitary representation of $\mathrm{U}(k)$ on $\mathbb{C}^I$, I nonempty. For $T > 0$ let $B(T) = \{x \in G : |(x-1)_{ac}| \le T^{99n} \text{ for all } a, c\}$. Let $L \ge \rho$, $j \le \varrho$, p, q heads of length j for Γ_L and $t \ge 1$. Then*

$$c_F \,\mathrm{hdn}_j(p,q) \int_{G\setminus B(tL^{11n})} \mathrm{FM}_j(p,q,W^\flat,x)\, d\theta(x) \le \Theta\, L^{10^9 n^4} \rho^{24k}\, \frac{t^{10^8 n^4}}{t^s}.$$

Proof. Put $T = tL^{11n}$; $B(T)$ is closed, hence Borel. Let $\phi = \mathbf{1}_{G\setminus B(T)} \mathrm{FM}_j(p,q,W^\flat,\cdot)$, so $\phi \le \mathrm{FM}_j(p,q,W^\flat,\cdot)$. Let V be as in Lemma 5.5.4; by (i) there, $V \ge 1$ and $\mathrm{FML}_j(V;1)$. Suppose $t > 1$ and let $x \in G$. If $x \in B(T)$, $\phi(x) = 0$. Otherwise some entry of $x-1$ has modulus above T^{99n}, with $T \ge L^{11n}$ and $T^{-1} = t^{-1}L^{-11n} \le t^{-1}/2$ (as $L \ge \rho \ge 8$). By Lemma 5.5.9, $W^\flat$ is cut for the data at $\mathbb{C}$, so Lemma 5.5.25, (ii), with $\tau = t^{-1}$ shows that $(j,p,q,W^\flat)$ has a deep vector after the head of depth at least $t^2 = \sigma^{-1}$ at x, $\sigma = t^{-2}$. By (DN3)(c) of Theorem 5.4.1 (for Γ_L; $x \in \mathcal{P}_{\mathrm{pos}}$, p a head of length j, $V \ge 1$),

$$\phi(x)^2 = \mathrm{FM}_j(p,q,W^\flat,x)^2 \le \sigma^s V^2 \,\mathrm{WSP}_j(p,q,W^\flat,x) = (V/t^s)^2 \,\mathrm{WSP}_j(p,q,W^\flat,x).$$

So ϕ satisfies the hypotheses of Lemma 5.5.4, (ii), which gives

$$\#I\, c_F \,\mathrm{hdn}_j(p,q) \int \phi\, d\theta \le \Theta L^{10^9 n^4} \rho^{24k} t^{10^8 n^4}/t^s;$$

as $\#I \ge 1$ the claim follows. □

Lemma 5.5.36. *Let F be a large lattice datum over a finite place A with level unit M, ϱ pairs, $n = \varrho + 1$ and middle coordinates x_i, $i \in \{1,\dots,k\}$; let Z be an intermediate group, $j \le \varrho$, and $r \ge 0$ an integer with $r + 10^8 n^4 \le k$. Let α be a unit of A with $\alpha \in \mathcal{O}_A$ and $\bar\alpha = \alpha$, Γ the gauge of scale α and $\Lambda = 2/|\alpha|$ its loss. Let p, q be heads of length j for Γ, and if $j \ne 0$ let M^{-1} have size 1 for Γ. Let $f\colon G \to [0,\infty]$ satisfy $f \le \mathrm{FM}_j(p,q,W_Z,\cdot)$, and let $e \in [0,\infty]$ be such that, if $e < 1$, then $f(x)^2 \le (e^s)^2 \,\mathrm{WSP}_j(p,q,W_Z,x)$ for every $x \in \mathcal{P}_{\mathrm{pos}}$ (with WSP_j formed for Γ). Then*

$$\mathrm{hdn}_j(p,q) \int_G f\, d\theta \le \min(1,e)^r \,\mathrm{mr}(Z)\, \Lambda^{10^9 n^4}.$$

Proof. Write $W = W_Z$. As $\alpha \in \mathcal{O}_A$, $\Lambda = 2/|\alpha| \ge 2$.

Step 1: the rank condition. From the shape, $s = k - 7m - 3n - 13 \ge k - 66n - 6$, as $m \le 9n-1$; with $k \ge r + 10^8 n^4$ this gives $s \ge r + 9\cdot 10^7 n^4$, since $10^7 n^4 \ge 66n + 6$. In particular $s \ge 1$.

Step 2: support and shells. If $f(x) \neq 0$, then $\mathrm{FM}_j(p,q,W,x) \neq 0$, so x is a position by Lemma 5.5.40. By Lemma 5.5.37 the sets S_t ($t \geq 0$) of that lemma cover the positions, hence the support of f, and each S_t is a shell of height 2^t for (Γ, j, p, q, W).

Step 3: the floor mass. The sizes of scale 1 form a gauge of loss 2 with $A_{(11)} = \mathcal{O}_A$ of volume 1 (Lemma 5.2.51), so by Lemma 5.4.3 (ii) the bound $\mathrm{FML}_j(V;1)$ holds with $V = 2^{1287n} \geq 1$ (this bound reads no gauge).

Step 4: the shell sum. If $e = 0$, the hypothesis gives $f(x)^2 \leq 0$ on the positions ($s \geq 1$), so $f = 0$ by Step 2 and the claim holds. Let $e > 0$. If $e < 1$, then for every position x, $f(x)^2 \leq (e^s)^2 \mathrm{WSP}_j(p,q,W,x) \leq (Ve^s)^2 \mathrm{WSP}_j(p,q,W,x)$. The group measure θ is a Haar measure of G (Lemma 5.5.41 (1)). So (DN2) of Theorem 5.4.1, for the gauge Γ, the heads p, q of length j, the cut function W, the shells S_t of Step 2, $\phi = f \leq \mathrm{FM}_j(p,q,W,\cdot)$, V, e and the integer r with $9 \cdot 10^7 n^4 + r \leq s$ (Step 1), gives

$$\mathrm{hdn}_j(p,q) \int_G f \, d\theta \leq V \min(1,e)^r \, \Lambda^{4 \cdot 10^8 n^4} \, \mathrm{SN}(W).$$

Step 5. By Lemma 5.5.39, $\mathrm{SN}(W) \leq \mathrm{mr}(Z)$, and $V = 2^{1287n} \leq \Lambda^{1287n}$. As $1287n + 4 \cdot 10^8 n^4 \leq 10^9 n^4$ and $\Lambda \geq 1$,

$$\mathrm{hdn}_j(p,q) \int_G f \, d\theta \leq \min(1,e)^r \, \mathrm{mr}(Z) \, \Lambda^{10^9 n^4}.$$

□

Lemma 5.5.37. *Let F be a large lattice datum over a finite place A with level unit M, ϱ pairs and $n = \varrho + 1$, and let Z be an intermediate group. For $x \in G$ the set $I(x) = \{\varepsilon \in \mathcal{O}_A : \varepsilon M^{-1}(x-1)$ has entries in $\mathcal{O}_A\}$ is an ideal of finite index of $\mathcal{O}_A$; for $t \geq 0$ let S_t be the set of positions x for which t is the least integer $t' \geq 0$ with $[\mathcal{O}_A : I(x)] \leq 2^{t'}$. Let $\alpha \in \mathcal{O}_A$ be a unit with $\bar{\alpha} = \alpha$, Γ the gauge of scale α, $j \leq \varrho$, p, q heads of length j for Γ, and if $j \neq 0$ let M^{-1} have size 1 for Γ. Then the sets S_t ($t \geq 0$) cover $\mathcal{P}_{\mathrm{pos}}$, and each S_t is a shell of height 2^t for (Γ, j, p, q, W_Z).*

Proof. Let $\Lambda = 2/|\alpha| \geq 2$ be the loss of Γ. Componentwise, $I(x)$ is the set of ε with $v_i(\varepsilon_i) \geq \max(0, -\min v_i(\mathrm{entries}_i))$, so $I(x) = c_x \mathcal{O}_A$ for a unit $c_x \in \mathcal{O}_A$, of index $|c_x|^{-1} < \infty$. Hence each position lies in exactly one S_t. Fix t and put $\Delta = 2^t$; (SH1) holds. For $x \in S_t$, $[\mathcal{O}_A : I(x)] \leq \Delta \leq 2[\mathcal{O}_A : I(x)]$ (if $t \geq 1$, minimality gives $[\mathcal{O}_A : I(x)] > 2^{t-1}$; if $t = 0$, $\Delta = 1$).

(SH2). As $\zeta_{\mathcal{O}_A}(m) \geq 1$ and $\zeta_{\mathcal{O}_A}(m)^n \leq 2$, $\zeta_{\mathcal{O}_A}(m) \leq 2$; each ideal I with $[\mathcal{O}_A : I] \leq \Delta$ contributes at least Δ^{-m} to $\zeta_{\mathcal{O}_A}(m)$, so there are at most $2\Delta^m \leq 2\Delta^{9n}$ of them. For each such $I = c\mathcal{O}_A$ ($c \in \mathcal{O}_A$ a unit) let Γ_I be the gauge of scale $\bar{c}c$ (Lemma 5.2.51), of loss $2/|\bar{c}c| = 2[\mathcal{O}_A : I]^2 \leq \Lambda\Delta^2$. If $x \in S_t$ and $I = I(x)$, then $c \in I(x)$, so $cM^{-1}(x-1)$ is integral and $\bar{c}c\, M^{-1}(x-1)$ is integral: x is a position of size Γ_I. So S_t is covered by the positions of size Γ_I for the at most $2\Delta^{9n}$ ideals I of index at most Δ.

(SH3). Let $x \in S_t$ and $\sigma \in [0,1)$ with $\Lambda^{1200n^2} \leq \sigma^{400n^2}\Delta$. The unit $c = c_x$ satisfies the hypothesis of Lemma 5.5.38 (iii): if $\varepsilon \in \mathcal{O}_A$ and $\varepsilon M^{-1}(x-1)$ is integral, then $\varepsilon \in c\mathcal{O}_A$. Moreover, with $\Delta \leq 2|c|^{-1}$ and $\Lambda^{210n^2} \geq 2$,

$$|c| \, \Lambda^{990n^2} \leq \tfrac{1}{2}|c| \, \Lambda^{1200n^2} \leq \tfrac{1}{2}|c| \, \sigma^{400n^2}\Delta \leq \sigma^{400n^2}.$$

So by Lemma 5.5.38 (iii), (j, p, q, W_Z) has a deep vector after the head of depth at least $(\sigma^2)^{-1}$ at x. □

Lemma 5.5.38. *Let F be a large lattice datum over a finite place A with level unit M, ϱ pairs and $n = \varrho + 1$, and let Z be an intermediate group.*

(i) *For $x, y \in G$ and $a < \varrho$ there is a unit c of A with $\mathfrak{d}_a(x,y) = c\mathcal{O}_A$. If x is near at $a+1$, then $c \in \mathcal{O}_A$ and $\mathrm{dp}_a(x,y) = [\mathcal{O}_A : c\mathcal{O}_A] = |c|^{-1}$.*

(ii) *If $x, z \in G$ and $\langle W_Z(x), W_Z(z)\rangle \neq 0$, there are $u, v \in \mathsf{U}$ with $ux \in \mathrm{K}(1)$ and $(ux)^{-1}vz \in \mathrm{K}(M)$; in particular x and z are near at every $l \leq \varrho$.*

(iii) *Let $\alpha \in \mathcal{O}_A$ be a unit with $\bar{\alpha} = \alpha$, Γ the gauge of scale α, $\Lambda = 2/|\alpha|$, $j \leq \varrho$, p, q heads of length j for Γ, and if $j \neq 0$ let M^{-1} have size 1 for Γ. Let $x \in G$ and let $c \in \mathcal{O}_A$ be a unit such that $c^{-1}\varepsilon \in \mathcal{O}_A$ for every $\varepsilon \in \mathcal{O}_A$ for which $\varepsilon M^{-1}(x-1)$ has entries in $\mathcal{O}_A$. If $\sigma \in [0,1)$ and $|c|\,\Lambda^{990n^2} \leq \sigma^{400n^2}$, then (j,p,q,W_Z) has a deep vector after the head of depth at least σ^{-2} at x.*

(iv) *Let $e \in [0,\infty]$ satisfy $|c| \leq e$ for every unit $c \in \mathcal{O}_A$ with $c^{-1} \notin \mathcal{O}_A$. If $x \in G \setminus \mathrm{K}(M)$, then $(0,1,1,W_Z)$ has a deep vector after the head of depth at least e^{-1} at x.*

Proof. **(i).** Let L be the set of the pairs $(\omega, d) \in V \times \mathrm{span}\{e_b : b < a\}$ with $h(e_b, \omega) = 0$ for $b < a$, $x^{-1}\omega$ integral and $M^{-1}(x^{-1}\omega - (xy)^{-1}(\omega + d))$ integral. As h is linear in its second argument, L is an $\mathcal{O}_A$-module, and $\mathfrak{d}_a(x,y)$ is its image under the $\mathcal{O}_A$-linear map $(\omega, d) \mapsto h(e_a, \omega)$; so $\mathfrak{d}_a(x,y)$ is an $\mathcal{O}_A$-submodule of A. Choose units $c_0 \in \mathcal{O}_A$ and c_1 such that $c_0 x^{-1}$ and $c_0 M^{-1}(x^{-1} - (xy)^{-1})$ are integral and the entries of x lie in $c_1\mathcal{O}_A$. For $t \in c_0\mathcal{O}_A$ the pair $(te_a^*, 0)$ lies in L, so $c_0\mathcal{O}_A \subseteq \mathfrak{d}_a(x,y)$; for $(\omega, d) \in L$, $\omega = x(x^{-1}\omega)$ has entries in $c_1\mathcal{O}_A$, so $\mathfrak{d}_a(x,y) \subseteq c_1\mathcal{O}_A$. An $\mathcal{O}_A$-submodule N of A is the product of the $\mathcal{O}_{K_i}$-submodules $\epsilon_i N$ (ϵ_i the idempotents), and an $\mathcal{O}_{K_i}$-submodule of K_i between two nonzero fractional ideals is a fractional ideal, the discrete valuation ring $\mathcal{O}_{K_i}$ having totally ordered ideals. Hence $\mathfrak{d}_a(x,y) = c\mathcal{O}_A$ for a unit c.

Let x be near at $a+1$: $\xi x \in K = \mathrm{K}(1)$ for some $\xi \in R^{(a+1)}$. For $(\omega, d) \in L$, $\xi\omega = (\xi x)(x^{-1}\omega)$ is integral, so $h(e_a, \xi\omega) \in \mathcal{O}_A$. As ξ is an isometry, $h(e_a, \xi\omega) = h(\xi^{-1}e_a, \omega)$, and $\xi^{-1} \in R^{(a+1)}$ gives $\xi^{-1}e_a - e_a \in \mathrm{span}\{e_b : b < a\}$, orthogonal to ω; so $h(e_a, \omega) = h(e_a, \xi\omega) \in \mathcal{O}_A$. Thus $c\mathcal{O}_A \subseteq \mathcal{O}_A$, $c \in \mathcal{O}_A$, and $\mathrm{dp}_a(x,y) = \mathrm{vol}(c\mathcal{O}_A)^{-1} = |c|^{-1} = [c] = [\mathcal{O}_A : c\mathcal{O}_A]$.

(ii). $\langle W_Z(x), W_Z(z)\rangle = \sum_c \overline{\chi_M(xc^{-1})}\chi_M(zc^{-1})$ over the chosen representatives c of $\mathrm{Cos}(Z)$, so for some c both $\chi_M(xc^{-1})$ and $\chi_M(zc^{-1})$ are nonzero: there are $u', v' \in \mathsf{U}$ with $k_1 = u'^{-1}xc^{-1}$ and $k_2 = v'^{-1}zc^{-1}$ in $\mathrm{K}(M)$. Put $u = u'^{-1}$, $v = v'^{-1}$. Then $ux = k_1 c \in \mathrm{K}(1)$, as $c \in Z \leq \mathrm{K}(1)$ and $\mathrm{K}(M) \leq \mathrm{K}(1)$, and $(ux)^{-1}vz = c^{-1}k_1^{-1}k_2 c \in \mathrm{K}(M)$, as $\mathrm{K}(1)$ normalises $\mathrm{K}(M)$ (Lemma 5.5.44 (a)). Then $vz = ux \cdot (ux)^{-1}vz \in \mathrm{K}(1)$. For $l \leq \varrho$ and $b < l$, an element of U maps e_b to e_b plus coordinate vectors of level below b, i.e. the e_c, $c < b$; so $\mathsf{U} \subseteq R^{(l)}$, and x, z are near at l.

(iii). Suppose y is active at x for (j,p,q,W_Z) and $\mathrm{dp}_a(y,x) < \sigma^{-2}$ for every a with $j \leq a < \varrho$; we derive a contradiction. By (AC1) and (ii), y is near at every $a+1 \leq \varrho$, so by (i) $\mathfrak{d}_a(y,x) = d_a\mathcal{O}_A$ with units $d_a \in \mathcal{O}_A$ and $|d_a| = \mathrm{dp}_a(y,x)^{-1} > \sigma^2$. Put $C = \prod_{j \leq a < \varrho} d_a\bar{d}_a$, a unit in $\mathcal{O}_A$ with $\bar{C} = C$, $Cd_a^{-1} \in \mathcal{O}_A$ and, by (PA2), $|C| = \prod_a |d_a|^2 \geq \sigma^{4(\varrho - j)} \geq \sigma^{4n}$. Let Γ' be the gauge of scale $\alpha' = \alpha^{10n}C$ (Lemma 5.2.51). Then the elements of size $10n$ for Γ have size 1 for Γ' ($\alpha^{10n}b \in \mathcal{O}_A$ implies $\alpha' b \in \mathcal{O}_A$), and d_a^{-1} has size 1 for Γ' ($\alpha' d_a^{-1} = \alpha^{10n}Cd_a^{-1}$). By (ii) applied to $\langle W_Z(y), W_Z(yx)\rangle \neq 0$ there are $u, v \in \mathsf{U}$ with $uy \in K$ and $(uy)^{-1}vyx \in \mathrm{K}(M)$, so $vyx \in K$ and $M^{-1}(vyx - uy) = uy \cdot M^{-1}((uy)^{-1}vyx - 1)$ is integral, hence of size 2 for Γ'. By (AC2), (AC3), $w_{j,p}^{-1}y$ and $w_{j,q}^{-1}yx$ are near at j. Hence (DN4) of Theorem 5.4.1, for the gauge Γ' with second gauge $S = \Gamma$, the point y and the element x, shows that $M^{-1}(x-1)$ has size $99n$ for Γ': $\varepsilon = \alpha'^{99n} \in \mathcal{O}_A$ has $\varepsilon M^{-1}(x-1)$ integral. By hypothesis $c^{-1}\alpha'^{99n} \in \mathcal{O}_A$, so $|\alpha|^{990n^2}|C|^{99n} \leq |c|$. With $|\alpha| = 2/\Lambda \geq \Lambda^{-1}$ and $|C| \geq \sigma^{4n}$,

$$\Lambda^{-990n^2}\sigma^{396n^2} \leq |c| \leq \sigma^{400n^2}\Lambda^{-990n^2}.$$

Here $\sigma > 0$, since $|c| > 0$; so $\sigma^{396n^2} \le \sigma^{400n^2}$ contradicts $0 < \sigma < 1$.

(iv). Let y be active at x for $(0, 1, 1, W_Z)$ and suppose $\mathrm{dp}_a(y, x) < e^{-1}$ for every $a < \varrho$. As in (iii), $\mathfrak{d}_a(y, x) = d_a \mathcal{O}_A$ with units $d_a \in \mathcal{O}_A$ and $|d_a| = \mathrm{dp}_a(y, x)^{-1} > e$; by the hypothesis on e, $d_a^{-1} \in \mathcal{O}_A$. Let Γ_1 be the gauge of scale 1 (all sizes mean integral). The constant sequence 1 is a head of length 0, and d_a^{-1} has size 1. With u, v from (ii) as in (iii), (DN4) of Theorem 5.4.1 with $j = 0$ and $\Gamma = S = \Gamma_1$ gives that $M^{-1}(x - 1)$ is integral. Then $x = 1 + M \cdot M^{-1}(x - 1)$ is integral, and so is $x^{-1} = J^{-1} \bar{x}^{\mathsf{T}} J$; hence $x \in \mathrm{K}(M)$, a contradiction. So every active point has $a < \varrho$ with $\mathrm{dp}_a(y, x) \ge e^{-1}$. □

Lemma 5.5.39. *Let F be a large lattice datum over a finite place A and Z an intermediate group. Then $\mathrm{SN}(W_Z) \le \mathrm{mr}(Z)$.*

Proof. Write $W = W_Z$, $\eta = \eta_0 * \cdots * \eta_{\varrho+1}$, and recall $0 < \mathrm{fl} < \infty$. By Lemma 5.2.59 (iv) with $j = \varrho + 2$, $\theta = \eta * \theta_{\ge 0}$. By Lemma 5.5.41 (1) θ is a Haar measure of G; as G is unimodular, θ is also right invariant.

Step 1: $\mathrm{fl}\,\|W(y)\|^2 = \eta(\{u : uy \in Z\}) \le 1$ for $y \in G$. By Lemma 5.5.44 (c), Z is the finite disjoint union of the cosets $\mathrm{K}(M)c$, c the chosen representatives. For each c, Lemma 5.5.43 (level M, $\theta' = \delta_1$, $w = yc^{-1}$) gives $\eta(\{u : uyc^{-1} \in \mathrm{K}(M)\}) = \mathrm{fl}$ if $u' yc^{-1} \in \mathrm{K}(M)$ for some $u' \in \mathsf{U}$, and 0 otherwise; the first case is exactly $\chi_M(yc^{-1}) \ne 0$, i.e. $|\chi_M(yc^{-1})|^2 = 1$. Summing over c gives the identity. Since $Z \le \mathrm{K}(1)$, $\eta(\{u : uy \in Z\}) \le \eta(\{u : uy \in \mathrm{K}(1)\}) \le 1$ by Lemma 5.5.43 at level 1 (with $\theta' = \delta_1$, $w = y$).

Step 2: $\mathrm{fl} \int \|W\|^2 \, d\theta_{\ge 0} = \theta(Z)$. By Step 1 and Tonelli,

$$\mathrm{fl} \int \|W(x)\|^2 \, d\theta_{\ge 0}(x) = \int \eta(\{u : ux \in Z\}) \, d\theta_{\ge 0}(x) = (\eta * \theta_{\ge 0})(Z) = \theta(Z).$$

Step 3: the inner integral. Let $x \in G$ with $W(x) \ne 0$. By Lemma 5.5.44 (c) there is $u_0 \in \mathsf{U}$ with $u_0 x \in \mathrm{K}(1)$; put $w = (u_0 x)^{-1}$ and $S = \{y : vyw \in \mathrm{K}(M) \text{ for some } v \in \mathsf{U}\}$, an open set. If $\langle W(x), W(y) \rangle \ne 0$, then by Lemma 5.5.38 (ii) there are $u, v \in \mathsf{U}$ with $ux \in \mathrm{K}(1)$ and $k = (ux)^{-1} vy \in \mathrm{K}(M)$, and

$$u_0 u^{-1} v \, y \, w = u_0 u^{-1} (ux) k \, (u_0 x)^{-1} = (u_0 x) \, k \, (u_0 x)^{-1} \in \mathrm{K}(M),$$

as $\mathrm{K}(1)$ normalises $\mathrm{K}(M)$ (Lemma 5.5.44 (a)); so $y \in S$. By Cauchy–Schwarz and Step 1, $|\langle W(x), W(y) \rangle|^2 \le \|W(x)\|^2 \|W(y)\|^2 \le \|W(x)\|^2 \mathrm{fl}^{-1}$. By Lemma 5.5.43 (level M, $\theta' = \theta_{\ge 0}$, the element w) and right invariance,

$$\mathrm{fl}\, \theta_{\ge 0}(S) = (\eta * \theta_{\ge 0})(\{y : yw \in \mathrm{K}(M)\}) = \theta(\mathrm{K}(M) w^{-1}) = \theta(\mathrm{K}(M)).$$

Hence $\int |\langle W(x), W(y) \rangle|^2 \, d\theta_{\ge 0}(y) \le \|W(x)\|^2 \mathrm{fl}^{-1} \theta_{\ge 0}(S) = \|W(x)\|^2 \mathrm{fl}^{-2} \theta(\mathrm{K}(M))$; for $W(x) = 0$ both sides vanish.

Step 4. Integrating Step 3 over x and using Step 2,

$$\mathrm{SN}(W) = \mathrm{fl}\, \mathrm{PN}(W) \le \mathrm{fl} \cdot \mathrm{fl}^{-2} \theta(\mathrm{K}(M)) \int \|W\|^2 \, d\theta_{\ge 0} = \theta(Z) \, \theta(\mathrm{K}(M)) \, \mathrm{fl}^{-2} = \mathrm{mr}(Z).$$

□

Lemma 5.5.40. *Let F be a large lattice datum over a finite place A with level unit M and ϱ pairs, Z an intermediate group, $j \le \varrho$, $K \ge 0$ an integer, and p, q sequences of units of A. For a sequence r of units put $\mathrm{hw}_{K,j}(r) = \prod_{l<j} [r_0 \cdots r_l]^{K/2}$ if $r_0 \cdots r_l \in \mathcal{O}_A$ for every $l < j$, and $\mathrm{hw}_{K,j}(r) = 0$ otherwise. Suppose $\mathrm{FM}_j(p, q, W_Z, x) \ne 0$ for some $x \in G$. Then x is a position, $p_0 \cdots p_{l-1}$ and $q_0 \cdots q_{l-1}$ lie in $\mathcal{O}_A$ for every $l \le j$, and there is a unit α of A with $\alpha \in \mathcal{O}_A$ and $\bar{\alpha} = \alpha$ such that, for the gauge Γ of scale α and its loss $\Lambda = 2/|\alpha|$:*

(i) p and q are heads of length j for Γ;

(ii) if $j \neq 0$, M^{-1} has size 1; and $p_i^{-1} - q_i^{-1}$ has size 1 for every $i < j$;

(iii) $\Lambda^E \leq (2/|M|)^K \operatorname{hw}_{K,j}(p) \operatorname{hw}_{K,j}(q)$ for every integer $E \geq 0$ with $4E \leq K$.

Proof. Write $W = W_Z$, $w_p = w_{j,p}$, $w_q = w_{j,q}$ and $d = w_q^{-1} w_p$.

Step 1: support of the measures, and d commutes with $G^{(j)}$. The floor measure θ_l is concentrated on $G^{(l)}$: for $l < \varrho$ it is an image of a measure under $(c, v, \beta) \mapsto w_l(c)^{-1} n_l(v, \beta)$, and both factors fix the coordinate vectors e_b, e_b^* ($b < l$) of floor below l; for $l = \varrho$ by (PE2); for $l = \varrho + 1$ by (PE3); for $l \geq \varrho + 2$ it is δ_1. As the $G^{(l)}$ decrease in l, $\theta_{\geq j+1}$ is concentrated on $G^{(j+1)} \subseteq G^{(j)}$. The element d maps $e_a \mapsto p_a q_a^{-1} e_a$ and $e_a^* \mapsto \bar{p}_a^{-1} \bar{q}_a e_a^*$ for $a < j$ and fixes the other coordinate vectors. An element $\sigma \in G^{(j)}$ fixes the e_a, e_a^* ($a < j$) and hence preserves their orthogonal complement, the span of the coordinate vectors of floor at least j, on which d is the identity; so $d\sigma = \sigma d$.

Step 2: four near points. Let $x \in G$ with $\mathrm{FM}_j(p, q, W, x) \neq 0$. The integrand of FM_j is nonnegative, so there is $g \in G$ with $\mathrm{piv}_j(\mathcal{K}_W)(g, gx) \neq 0$ and $\mathrm{piv}_j(\mathcal{K}_W^d)(w_p^{-1} g, w_p^{-1} gx) \neq 0$. If $\mathrm{piv}_j(\mathcal{K})(g', y') = \int |\mathcal{K}_{j+1}(\xi g', \xi y')| \, d\theta_j(\xi) \neq 0$, then by Step 1 there is $\xi \in G^{(j)}$ with $\mathcal{K}_{j+1}(\xi g', \xi y') = \int \mathcal{K}(x' \xi g', x' \xi y') \, d\theta_{\geq j+1}(x') \neq 0$, and then $x' \in G^{(j)}$ with $\mathcal{K}(x' \xi g', x' \xi y') \neq 0$. So there are $\sigma, \sigma' \in G^{(j)}$ with

$$\langle W(\sigma g), W(\sigma g x) \rangle \neq 0, \qquad \langle W(\sigma' w_p^{-1} g), W(d \sigma' w_p^{-1} g x) \rangle \neq 0,$$

and $d\sigma' w_p^{-1} = \sigma' w_q^{-1}$ by Step 1. Hence W does not vanish at σg, $\sigma g x$, $\sigma' w_p^{-1} g$ and $\sigma' w_q^{-1} g x$. As W is cut (Lemma 5.5.44 (c)), for each of these points $z = \rho z_0$ ($\rho \in \{\sigma, \sigma'\}$) there is $u \in \mathsf{U}$ with $u \rho z_0 \in K$. Now $u\rho \in R^{(j)}$: for $b < j$, $\rho e_b = e_b$ and $u e_b - e_b$ is a combination of coordinate vectors of level below b, i.e. of the e_a, $a < b$, which is the defining condition of $R^{(j)}$ on the columns e_b, $b < j$. So g, gx, $w_p^{-1} g$ and $w_q^{-1} gx$ are near at j. Moreover, by Lemma 5.5.38 (ii) for $\langle W(\sigma g), W(\sigma g x) \rangle \neq 0$ there are $u_1, v_1 \in \mathsf{U}$ with $u_1 \sigma g \in \mathrm{K}(1)$ and $k_1 = (u_1 \sigma g)^{-1} v_1 \sigma g x \in \mathrm{K}(M)$; with $y_1 = \sigma g$ this reads $x = y_1^{-1} (v_1^{-1} u_1) y_1 \, k_1$, so x is a position (Definition 5.2.2).

Step 3: the head products are integral. The sizes of scale 1 ($A_{(i)} = \mathcal{O}_A$ for all i) form a gauge of F (Lemma 5.2.51 with $\alpha = 1$). By Lemma 5.4.13 (ii) for this gauge, applied to $\gamma = w_p$, $y = g$ and to $\gamma = w_q$, $y = gx$, and by Lemma 5.4.13 (iii),

$$p_0 \cdots p_{l-1} = D_l(w_p) \in \mathcal{O}_A, \qquad q_0 \cdots q_{l-1} = D_l(w_q) \in \mathcal{O}_A \qquad (0 \leq l \leq j),$$

as $j \leq \min(j, n)$. This is the first assertion.

Step 4: the scale. Put $P_l = p_0 \cdots p_{l-1}$, $P_l' = q_0 \cdots q_{l-1}$, $\beta = \prod_{l=1}^{j} P_l$, $\gamma = \prod_{l=1}^{j} P_l'$ and $\alpha = M \, \overline{\beta\gamma} \, \beta\gamma$. These are units in $\mathcal{O}_A$ (Step 3, and $\mathcal{O}_A$ is stable under the involution), and $\bar{\alpha} = \alpha$ as $\bar{M} = M$. Let Γ be the gauge of scale α (Lemma 5.2.51); size 1 means $\alpha x \in \mathcal{O}_A$.

(i) For $1 \leq l \leq j$, $\alpha P_l^{-1} = M \, \overline{\beta\gamma} \, \gamma \prod_{l' \neq l} P_{l'} \in \mathcal{O}_A$, so P_l^{-1} has size 1; $P_0 = 1$ has size 1. Hence p is a head of length j, and likewise q.

(ii) $\alpha M^{-1} = \overline{\beta\gamma} \, \beta\gamma \in \mathcal{O}_A$, so M^{-1} has size 1. For $i < j$, $p_i^{-1} = P_i P_{i+1}^{-1}$, so $\alpha p_i^{-1} = (\alpha P_{i+1}^{-1}) P_i \in \mathcal{O}_A$ by (i) and Step 3; likewise $\alpha q_i^{-1} \in \mathcal{O}_A$, and $\alpha(p_i^{-1} - q_i^{-1}) \in \mathcal{O}_A$.

(iii) By (PA2) and multiplicativity, $|\alpha| = |M| \, (|\beta||\gamma|)^2$, and $|\beta|, |\gamma|, |M| \leq 1$. Since all P_l are integral, $\operatorname{hw}_{K,j}(p) = \prod_{l<j} [P_{l+1}]^{K/2} = [\beta]^{K/2} = |\beta|^{-K/2}$, and $\operatorname{hw}_{K,j}(q) = |\gamma|^{-K/2}$. For $4E \leq K$, using $2/|M| \geq 1$ and $(|\beta||\gamma|)^{-1} \geq 1$,

$$\Lambda^E = \Big(\frac{2}{|M|}\Big)^E (|\beta||\gamma|)^{-2E} \leq \Big(\frac{2}{|M|}\Big)^K (|\beta||\gamma|)^{-K/2} = \Big(\frac{2}{|M|}\Big)^K \operatorname{hw}_{K,j}(p) \operatorname{hw}_{K,j}(q).$$

□

Lemma 5.5.41. *Let F be a large lattice datum over a finite place A, with group measure θ on the unitary group G.*

1. *θ is a Haar measure on G.*
2. *$\theta(\mathrm{K}(1)) \le 1 + 2^{-n}$.*

Proof. Let the shape be $(\varrho, \iota, m+1)$, so that $k+1 = |\iota| + 2m + 4 \ge 2m+4$.

Step 1: floor masses. Let $0 \le b \le \varrho$ and let S_b be the set of coordinate indices of floor at least b; identify V_b with A^{S_b}, so that vol_{V_b} is the Haar measure of A^{S_b} and $h(v,v) = \bar{v}^{\mathsf{T}} J_b v$ with J_b the $S_b \times S_b$ block of J. Since J pairs e_a with e_a^* (both of floor a) and the middle indices among themselves, J is block diagonal for S_b and its complement, so J_b and $J_b^{-1} = (J^{-1})_b$ are integral and hermitian. By Lemma 5.2.52 (ii), $(\mathrm{qv}_b(\beta))_\beta$ is a disintegration of vol_{V_b} along h, and $|S_b| \ge k+1 \ge 2m+1$; so Lemma 5.2.54 with $\mu = m$, $g = 1$, $x_0 = 0$ and $\beta = \delta h(f_b, f_b)$ gives $\mathrm{qv}_b(\mathcal{O}_A^{S_b}) \le \zeta_{\mathcal{O}_A}(m)$. The compact set $C_b = \{P_b(\kappa\kappa'^{-1} f_b) : \kappa, \kappa' \in K\}$ of Lemma 5.2.60 consists of integral vectors of V_b, as $\kappa\kappa'^{-1} \in \mathrm{K}(1)$ has integral entries. Hence, by Lemma 5.2.60 (iii), $\mathrm{FML}_b(\zeta_{\mathcal{O}_A}(m); 1)$ for every $b \le \varrho$.

Step 2: (2). By Lemma 5.2.59 (iv) with $j = \varrho+2$, $\theta = \eta_{<\varrho+2} * \theta_{\ge 0}$ with $\eta_{<\varrho+2} = \eta_0 * \cdots * \eta_{\varrho+1}$. By Lemma 5.5.43 at level 1 (factor 1), with $\theta' = \theta_{\ge 0}$ and $w = 1$,

$$\theta(\mathrm{K}(1)) = \theta_{\ge 0}\big(\{y \in G : uy \in \mathrm{K}(1) \text{ for some } u \in \mathsf{U}\}\big).$$

For $l = \varrho + 2$ the defining condition of $R^{(l)}$ is that of U, so $R^{(\varrho+2)} = \mathsf{U}$ and this is the set of the y near at $\varrho + 2$. By Step 1 and Lemma 5.2.60 (iv) with $l = 0$, $c = 1$,

$$\theta(\mathrm{K}(1)) \le \zeta_{\mathcal{O}_A}(m)^{\varrho+1} = \zeta_{\mathcal{O}_A}(m)^n \le 1 + 2^{-n}.$$

Step 3: (1). $\mathrm{K}(1)$ is an open subgroup of G (integral matrices form an open set and inversion is continuous) and compact (a closed set of integral matrices). By Lemma 5.2.61 (iii), (iv), θ is right invariant and nonzero. A compact subset of G is covered by finitely many right cosets $\mathrm{K}(1)g$, each of θ-measure $\theta(\mathrm{K}(1)) < \infty$ by Step 2 and right invariance; so θ is finite on compact sets. As G is a second countable locally compact group, θ is then a right Haar measure, and by uniqueness $\theta = d\,\nu$ for the left Haar measure ν that is right invariant (unimodularity) and some $d \in (0, \infty)$, $d \ne 0$ because $\theta \ne 0$. Hence θ is a Haar measure of G. □

Lemma 5.5.42. *Let F be a large lattice datum over a finite place A, with level unit M and group measure* vol*, and let Z be an intermediate group. Let $Q \ge 2$ be a real number with $|c| \le Q^{-1}$ for every unit c of A such that $c \in \mathcal{O}_A$ and $c^{-1} \notin \mathcal{O}_A$, and let $t \ge 0$ be an integer with $t + 2 \cdot 10^9 n^4 \le k$. Then*

$$\mathrm{mi}(Z) \le \mathrm{vol}(Z)\,\mathrm{vol}(\mathrm{K}(M))\,\big(\mathrm{vol}(Z) + Q^{-t}\big).$$

Proof. Write $W = W_Z$ and, for $x \in G$, $I(x) = \int_Z \chi_M(yxy^{-1})\,d\mathrm{vol}(y)$, so that $\mathrm{mi}(Z) = \int_G |I|^2\,d\mathrm{vol}$. As $\mathrm{K}(M)$ is open (Lemma 5.5.44 (a)),

$$\mathrm{mi}(Z) = \int_{\mathrm{K}(M)} |I|^2\,d\mathrm{vol} + \int_{G \setminus \mathrm{K}(M)} |I|^2\,d\mathrm{vol},$$

and the first term is $\mathrm{vol}(Z)^2 \mathrm{vol}(\mathrm{K}(M))$ by Lemma 5.5.44 (e).

The integrand off $\mathrm{K}(M)$**.** By Lemma 5.2.90, $|I(x)| = |\mathcal{K}_{W,0}(1,x)|\,\mathrm{fl}$. If $x' \mapsto \mathcal{K}_W(x', x'x)$ is $\theta_{\geq 0}$-integrable, then, as $\theta_{\geq 0} = \theta_{\geq 1} * \theta_0$ (Lemma 5.2.59 (i)), Fubini's theorem for the s-finite measures $\theta_{\geq 1}$, θ_0 gives, for θ_0-almost every ξ, the integrability of $x'' \mapsto \mathcal{K}_W(x''\xi, x''\xi x)$, whose integral is $\mathcal{K}_{W,1}(\xi, \xi x)$, and

$$\mathcal{K}_{W,0}(1,x) = \int_G \mathcal{K}_{W,1}(\xi,\xi x)\,d\theta_0(\xi), \qquad |\mathcal{K}_{W,0}(1,x)| \leq \int_G |\mathcal{K}_{W,1}(\xi,\xi x)|\,d\theta_0(\xi) = \mathrm{piv}_0(\mathcal{K}_W)(1,x);$$

otherwise $\mathcal{K}_{W,0}(1,x) = 0$. For $j = 0$ and $p = q = 1$ one has $w_{0,1} = 1$, $d = 1$ and $\mathfrak{m}_0 = \delta_1$, so $\mathrm{FM}_0(1,1,W,x) = \mathrm{piv}_0(\mathcal{K}_W)(1,x)^2$. Hence $|I(x)|^2 \leq \mathrm{fl}^2\,\mathrm{FM}_0(1,1,W,x)$ for every x.

Conclusion. By Lemma 5.5.45,

$$\int_{G\backslash \mathrm{K}(M)} |I|^2\,d\mathrm{vol} \leq \mathrm{fl}^2 \int_{G\backslash \mathrm{K}(M)} \mathrm{FM}_0(1,1,W,x)\,d\mathrm{vol}(x) \leq \mathrm{fl}^2 Q^{-t}\,\mathrm{mr}(Z) = Q^{-t}\,\mathrm{vol}(Z)\,\mathrm{vol}(\mathrm{K}(M)).$$

Adding the two terms, $\mathrm{mi}(Z) \leq \mathrm{vol}(Z)\,\mathrm{vol}(\mathrm{K}(M))\,(\mathrm{vol}(Z) + Q^{-t})$. □

Lemma 5.5.43. *Let F be a large lattice datum over a finite place A, with level unit M, and put $\eta_{<\varrho+2} = \eta_0 * \eta_1 * \cdots * \eta_{\varrho+1}$. Let $c \in \mathcal{O}_A$ be a unit with $\bar{c} = c$. Then for every s-finite Borel measure θ' on G and every $w \in G$,*

$$(\eta_{<\varrho+2} * \theta')(\{y \in G : yw \in \mathrm{K}(c)\}) = \Big(\prod_{l<\varrho} \mathrm{fm}(l,c)\Big) \cdot \theta'(\{y \in G : uyw \in \mathrm{K}(c) \text{ for some } u \in \mathsf{U}\}).$$

For $c = M$ the factor is fl*; for $c = 1$ it is* 1*.*

Proof. Fix c. $\mathrm{K}(c)$ is an open subgroup of G: for $x, y \in \mathrm{K}(c)$, $c^{-1}(xy-1) = c^{-1}(x-1)y + c^{-1}(y-1)$ and $c^{-1}(x^{-1}-1) = -x^{-1}c^{-1}(x-1)$ are integral, and integrality of matrices is an open condition. For a subgroup S of G and $w \in G$ the set $\{y : uyw \in \mathrm{K}(c) \text{ for some } u \in S\} = S\,\mathrm{K}(c)w^{-1}$ is open.

Step 1: unfolding. Let $j \geq 0$, θ' s-finite and $w \in G$. Then

$$(\eta_{<j} * \theta')(\{y : yw \in \mathrm{K}(c)\}) = \eta_{<j}(\mathrm{K}(c)) \cdot \theta'(\{y : uyw \in \mathrm{K}(c) \text{ for some } u \in N_{<j}\}). \tag{1}$$

By Tonelli the left side is $\int \eta_{<j}(\{x : xyw \in \mathrm{K}(c)\})\,d\theta'(y)$. If $uyw \in \mathrm{K}(c)$ for some $u \in N_{<j}$, then $\{x : xyw \in \mathrm{K}(c)\} = \{x : xu^{-1} \in \mathrm{K}(c)\}$, whose $\eta_{<j}$-measure is $(\eta_{<j} * \delta_{u^{-1}})(\mathrm{K}(c)) = \eta_{<j}(\mathrm{K}(c))$ by Lemma 5.4.14 (iii). Otherwise $xyw \notin \mathrm{K}(c)$ for every $x \in N_{<j}$, and $\eta_{<j}$ is carried by $N_{<j}$ (Lemma 5.4.14 (ii)), so the integrand vanishes. This gives (1).

Step 2: one floor. Let $l < \varrho$ and let $S \leq N_{<l}$ be a subgroup. For $v \in V_{l+1}$ and $\beta \in A^-$, some $u \in S$ has $un_l(v,\beta) \in \mathrm{K}(c)$ if and only if $c^{-1}v$ and $c^{-1}\beta$ are integral. Suppose $u \in S$ and $un \in \mathrm{K}(c)$, $n = n_l(v,\beta)$. As $u \in N_{<l}$, u maps e_l^*, e_l and every vector of V_{l+1} to itself plus a combination of the e_a, $a < l$ (their modified level is ϱ, and only indices of modified level below ϱ, the e_a with $a < l$, can occur). Hence $une_l^* - e_l^*$ equals $v + (\beta - \tau h(v,v))e_l$ plus a combination of the e_a, $a < l$, and integrality of $c^{-1}(un-1)$ gives $c^{-1}v$ and $c^{-1}(\beta - \tau h(v,v))$ integral. With $v_0 = c^{-1}v$, $h(v,v) = c^2 h(v_0,v_0)$ ($\bar{c} = c$), and $c^{-1}\tau h(v,v) = \tau c\,h(v_0,v_0) \in \mathcal{O}_A$ as J is integral; so $c^{-1}\beta$ is integral. Conversely, if $v = cv_0$ and $\beta = c\beta_0$ with v_0, β_0 integral, then $\beta - \tau h(v,v) = c\,t$ with $t = \beta_0 - \tau c h(v_0,v_0) \in \mathcal{O}_A$, and

$$(n-1)w = c\big(w_l^* v_0 + (tw_l^* - h(v_0,w))e_l\big),$$

where $w \mapsto w_l^* = h(e_l, w)$ and $w \mapsto h(v_0, w)$ have integral matrices. So $c^{-1}(n-1)$ is integral, $n = 1 + c \cdot c^{-1}(n-1)$ is integral, $n^{-1} = J^{-1}\bar{n}^{\mathsf{T}}J$ is integral, and $u = 1$ works. By the definition

of η_l, (PA3) and Lemma 5.2.55 (the skew ball of c^{-1}),

$$\begin{aligned}\eta_l(\{y : uy \in \mathrm{K}(c) \text{ for some } u \in S\}) &= \mathrm{vol}(\{v \in V_{l+1} : c^{-1}v \text{ integral}\}) \cdot \nu(\{\beta : c^{-1}\beta \in \mathcal{O}_A\}) \\ &= |c|^{\dim V_{l+1}}|c|^{1/2} = \mathrm{fm}(l,c),\end{aligned}$$

as $\dim V_{l+1} = k + 1 + 2(\varrho - l - 1) = d_l$.

Step 3: the congruence subgroup. For $j \le \varrho$, $\eta_{<j}(\mathrm{K}(c)) = \prod_{l<j} \mathrm{fm}(l,c)$, by induction on j: for $j = 0$, $\eta_{<0} = \delta_1$; for $j < \varrho$, $\eta_{<j+1} = \eta_{<j} * \eta_j$, and (1) with $\theta' = \eta_j$, $w = 1$, followed by Step 2 with $S = N_{<j}$, gives $\eta_{<j+1}(\mathrm{K}(c)) = \eta_{<j}(\mathrm{K}(c))\,\mathrm{fm}(j,c)$.

Step 4. As $\eta_l = \delta_1$ for $l \ge \varrho$, $\eta_{<\varrho+2} = \eta_{<\varrho}$; and $N_{<\varrho} = \mathsf{U}$, since for $l = \varrho$ the modified level of every index is its level. By (1) with $j = \varrho$ and Step 3,

$$(\eta_{<\varrho+2} * \theta')(\{y : yw \in \mathrm{K}(c)\}) = \Big(\prod_{l<\varrho} \mathrm{fm}(l,c)\Big)\theta'(\{y : uyw \in \mathrm{K}(c) \text{ for some } u \in \mathsf{U}\}).$$

For $c = M$ the factor is fl, and for $c = 1$ it is 1, as $|1| = 1$. □

Lemma 5.5.44. *Let F be a large lattice datum over a finite place A, with level unit M and group measure* vol.

(a) For every unit c of A, $\mathrm{K}(c)$ is a compact open subgroup of G, and $y\mathrm{K}(c)y^{-1} = \mathrm{K}(c)$ for $y \in \mathrm{K}(1)$; $\mathrm{K}(M) \le \mathrm{K}(1)$.

(b) The value $\chi_M(y)$ does not depend on the chosen u; $|\chi_M| \le 1$; $\chi_M(uy) = \psi(M^{-1}\sigma(u))\chi_M(y)$ for $u \in \mathsf{U}$; $\chi_M(yk) = \chi_M(y)$ for $k \in \mathrm{K}(M)$; $\chi_M = 1$ on $\mathrm{K}(M)$; and χ_M is continuous.

(c) For an intermediate group Z, the set $\mathrm{Cos}(Z)$ is finite and nonempty; W_Z is continuous and bounded, $W_Z(uy) = \psi(M^{-1}\sigma(u))W_Z(y)$ for $u \in \mathsf{U}$, and $W_Z(y) \ne 0$ implies $uy \in \mathrm{K}(1)$ for some $u \in \mathsf{U}$. In particular W_Z is cut.

(d) $0 < \mathrm{vol}(Z) < \infty$, $0 < \mathrm{vol}(\mathrm{K}(M)) < \infty$ and $0 < \mathrm{mr}(Z) < \infty$.

(e)

$$\int_{\mathrm{K}(M)} \Big| \int_Z \chi_M(yxy^{-1})\, d\mathrm{vol}(y) \Big|^2 d\mathrm{vol}(x) = \mathrm{vol}(Z)^2\, \mathrm{vol}(\mathrm{K}(M));$$

in particular $\mathrm{mi}(Z) \ge \mathrm{vol}(Z)^2\mathrm{vol}(\mathrm{K}(M))$.

Proof. **(a).** For $x \in G$ we have $x^{-1} = J^{-1}\bar{x}^{\mathsf{T}}J$, so x^{-1} is integral as soon as x is; hence $\mathrm{K}(1) = G \cap \mathrm{M}_r(\mathcal{O}_A)$ and $\mathrm{K}(c) = \mathrm{K}(1) \cap (1 + c\mathrm{M}_r(\mathcal{O}_A))$. Since $\mathcal{O}_A$ is compact and open in A and multiplication by the unit c is a homeomorphism of A, the sets $\mathrm{M}_r(\mathcal{O}_A)$ and $1 + c\mathrm{M}_r(\mathcal{O}_A)$ are compact and open in $\mathrm{M}_r(A)$; as G is closed in $\mathrm{M}_r(A)$, $\mathrm{K}(1)$ and $\mathrm{K}(c)$ are compact and open in G. If $x, y \in \mathrm{K}(c)$, then xy and x^{-1} lie in $\mathrm{K}(1)$, $c^{-1}(xy - 1) = c^{-1}(x - 1)y + c^{-1}(y - 1)$ and $c^{-1}(x^{-1} - 1) = -x^{-1}c^{-1}(x - 1)$ are integral; so $\mathrm{K}(c)$ is a subgroup. For $y \in \mathrm{K}(1)$ and $x \in \mathrm{K}(c)$, $yxy^{-1} \in \mathrm{K}(1)$ and $c^{-1}(yxy^{-1} - 1) = y\,c^{-1}(x - 1)\,y^{-1}$ is integral, so $y\mathrm{K}(c)y^{-1} \subseteq \mathrm{K}(c)$, and equality follows by applying this to y^{-1}. By definition $\mathrm{K}(M) \subseteq \mathrm{K}(1)$.

The simple-root sum. U acts trivially on the successive quotients of the flag $F_a = \langle e_0, \dots, e_a\rangle$ $(a < \varrho)$ and on $F_{\varrho-1}^{\perp}/F_{\varrho-1}$, which contains the image of z. Hence for $u \in \mathsf{U}$, $ue_{a+1} \in e_{a+1} + u_{a,a+1}e_a + F_{a-1}$ $(a + 1 < \varrho)$ and $uz \in z + u_{\varrho-1,z}e_{\varrho-1} + F_{\varrho-2}$, and $\sigma(u)$ is the sum of these coefficients $u_{a,a+1}$ and $u_{\varrho-1,z}$, which are matrix entries of u. Applying u to $ve_{a+1} \in e_{a+1} + v_{a,a+1}e_a + F_{a-1}$ gives $(uv)_{a,a+1} = u_{a,a+1} + v_{a,a+1}$, and likewise at z; so

$\sigma(uv) = \sigma(u) + \sigma(v)$ on U. If moreover $v \in \mathrm{K}(M)$, the off-diagonal entries of v are those of $v - 1$, so $M^{-1}\sigma(v) \in \mathcal{O}_A$, and $\psi(M^{-1}\sigma(v)) = 1$ by self-duality of $\mathcal{O}_A$ (take $r = 1$).

(b). Let $u, u' \in \mathsf{U}$ with $u^{-1}y, u'^{-1}y \in \mathrm{K}(M)$. Then $v = u^{-1}u' = (u^{-1}y)(u'^{-1}y)^{-1} \in \mathsf{U} \cap \mathrm{K}(M)$, $\sigma(u') = \sigma(u) + \sigma(v)$ and $\psi(M^{-1}\sigma(u')) = \psi(M^{-1}\sigma(u))\psi(M^{-1}\sigma(v)) = \psi(M^{-1}\sigma(u))$; so the value is independent of u. $|\chi_M| \leq 1$ as ψ is unitary. If $y \in v\mathrm{K}(M)$ with $v \in \mathsf{U}$, then $uy \in (uv)\mathrm{K}(M)$ and $\chi_M(uy) = \psi(M^{-1}\sigma(u))\psi(M^{-1}\sigma(v))$; if $y \notin \mathsf{U}\mathrm{K}(M)$ then $uy \notin \mathsf{U}\mathrm{K}(M)$ and both sides vanish. Right translation by $k \in \mathrm{K}(M)$ preserves $\mathsf{U}\mathrm{K}(M)$ and the admissible u, so $\chi_M(yk) = \chi_M(y)$; for $y \in \mathrm{K}(M)$ one may take $u = 1$, so $\chi_M(y) = \psi(0) = 1$. Being invariant under right translation by the open subgroup $\mathrm{K}(M)$, χ_M is locally constant, hence continuous.

(c). Z is a union of cosets of the open group $\mathrm{K}(M)$, hence open and closed in G; as $Z \subseteq \mathrm{K}(1)$, Z is compact, and the disjoint open cover of Z by its right cosets of $\mathrm{K}(M)$ is finite. It contains $\mathrm{K}(M)$, so $\mathrm{Cos}(Z) \neq \emptyset$. Each coordinate $y \mapsto \chi_M(yc^{-1})$ of W_Z is continuous with absolute value at most 1 by (b), so W_Z is continuous and $\|W_Z\| \leq (\#\mathrm{Cos}(Z))^{1/2}$; and $\chi_M(uyc^{-1}) = \psi(M^{-1}\sigma(u))\chi_M(yc^{-1})$ gives the transformation law. If $W_Z(y) \neq 0$, some $\chi_M(yc^{-1}) \neq 0$ with $c \in Z$, so $yc^{-1} \in v\mathrm{K}(M)$ for some $v \in \mathsf{U}$, and $v^{-1}y \in \mathrm{K}(M)c \subseteq \mathrm{K}(1)$. These are the properties of a cut function of frequency M^{-1} supported in $\mathsf{U}\mathrm{K}(1)$, the box being $\mathrm{K}(1)$.

(d). By Lemma 5.5.41 (1), vol is a Haar measure on G, so it is positive on the nonempty open sets $Z, \mathrm{K}(M)$ and finite on these compact sets. Since M is a unit of A, $|M| = \mathrm{vol}(M\mathcal{O}_A) \in (0, \infty)$ (a Haar measure of the compact open set $M\mathcal{O}_A$), so $\mathrm{fm}(l)$ and fl lie in $(0, \infty)$, and so does $\mathrm{mr}(Z) = \mathrm{vol}(Z)\mathrm{vol}(\mathrm{K}(M))\mathrm{fl}^{-2}$.

(e). Let $x \in \mathrm{K}(M)$. For $y \in Z \subseteq \mathrm{K}(1)$, $yxy^{-1} \in \mathrm{K}(M)$ by (a), so $\chi_M(yxy^{-1}) = 1$ by (b), and $\int_Z \chi_M(yxy^{-1})\, d\mathrm{vol}(y) = \mathrm{vol}(Z)$, finite by (d). Hence the integrand equals the constant $\mathrm{vol}(Z)^2$ on $\mathrm{K}(M)$, and its integral over $\mathrm{K}(M)$ is $\mathrm{vol}(Z)^2\mathrm{vol}(\mathrm{K}(M))$. The integrand of $\mathrm{mi}(Z)$ is nonnegative (and measurable: $x \mapsto \int_Z \chi_M(yxy^{-1})\, d\mathrm{vol}(y)$ is continuous by dominated convergence, χ_M being continuous and bounded and $\mathrm{vol}(Z)$ finite), so $\mathrm{mi}(Z)$ is at least its integral over $\mathrm{K}(M)$. □

Lemma 5.5.45. *Let F be a large lattice datum over a finite place A with level unit M, and let Z be an intermediate group. Let $Q \geq 2$ be a real number with $|c| \leq Q^{-1}$ for every unit $c \in \mathcal{O}_A$ with $c^{-1} \notin \mathcal{O}_A$, and let $t \geq 0$ be an integer with $t + 2 \cdot 10^9 n^4 \leq k$. Then*

$$\int_{G \backslash \mathrm{K}(M)} \mathrm{FM}_0(1, 1, W_Z, x)\, d\theta(x) \leq Q^{-t}\, \mathrm{mr}(Z).$$

Proof. Write $W = W_Z$, $N = 10^9 n^4$, and let Γ_1 be the gauge of scale 1 (Lemma 5.2.51 with $\alpha = 1$), of loss 2; the constant sequence 1 is a head of length 0 for Γ_1. Let $f = \mathbf{1}_{G \backslash \mathrm{K}(M)}\, \mathrm{FM}_0(1, 1, W, \cdot)$ and $e = Q^{-1} < 1$.

The gain. Let x be a position. If $x \in \mathrm{K}(M)$, $f(x) = 0$. If $x \notin \mathrm{K}(M)$, then by Lemma 5.5.38 (iv) (with $e = Q^{-1}$, admissible by the hypothesis on Q) $(0, 1, 1, W)$ has a deep vector after the head of depth at least Q at x, and (DN0)(d) of Theorem 5.4.1, for Γ_1 and $\sigma = Q^{-1}$, gives $f(x)^2 \leq (e^s)^2\, \mathrm{WSP}_0(1, 1, W, x)$, with WSP_0 formed for Γ_1.

The shell sum. Apply Lemma 5.5.36 with $\alpha = 1$, $j = 0$, $p = q = 1$, the integer $r = t + N$ (so $r + 10^8 n^4 \leq t + 2 \cdot 10^9 n^4 \leq k$), the function $f \leq \mathrm{FM}_0(1, 1, W, \cdot)$ and e:

$$\begin{aligned} \int_{G \backslash \mathrm{K}(M)} \mathrm{FM}_0(1, 1, W, x)\, d\theta(x) &= \int_G f\, d\theta \leq Q^{-(t+N)}\, \mathrm{mr}(Z)\, 2^N \\ &= Q^{-t}\, (2Q^{-1})^N\, \mathrm{mr}(Z) \leq Q^{-t}\, \mathrm{mr}(Z), \end{aligned}$$

as $Q \geq 2$. □

5.6 Filling data, unfolding and moments near the Bessel subgroup

This section connects the abstract filling datum, for which the two moments of Proposition 5.6.48 are proved, to the data. The data give an admissible filling datum that fits Y_0, whose classes and Bessel coordinates are coordinates on the source forms, $\mathcal{S} \cong \mathbb{C}^{\mathcal{C}\times I}$ (Proposition 5.6.4). The scaled Poincaré map has amplitude blocks (Proposition 5.6.16), which carries the second moment through Theorem 5.1.13. Near the Bessel subgroup the moments are those of the Bessel translates (Lemma 5.6.33, Proposition 5.6.41, Proposition 5.6.42), which together with the rarity of bad types carries the near-isometry. The classes are those of the definite group $\mathrm{U}(X_0)$, so that the coordinates are values of algebraic modular forms at class representatives [110] (Lemma 5.6.14), and the class number is controlled by a mass of hermitian lattices [97, 303]. The arrangement follows [259]; for another reading see [223].

Definition 5.6.1. Let $G_f = G_f(\{1,\dots,r\})$ and $\mathrm{K}_f = \mathbf{K}_0(\{1,\dots,r\})$ be the groups of (VP1) of Definition 3.2.2 (those of (FT1) of Definition 3.2.4 for $N = r$); $G_f = \mathrm{U}_r(\mathbb{A}_{E^+,f})$ for the unitary group U_r over E^+ of the standard form $\sum_i \bar{z}_i z'_i$ on E^r. For a chosen place w and $g \in G_f$ write $g_w \in \mathrm{GL}_r(E_w)$ for its component at w.

(HO1) Framed spaces. Here framed spaces are those of rank r of (FT1) of Definition 3.2.4, that is, framed hermitian spaces with signs $s\colon \{1,\dots,r\} \to \{\pm 1\}$ in the sense of (VP3) of Definition 3.2.2: signs the same at every $v \in \Phi$, a frame of $V \otimes_{E,v} \mathbb{C}$ with form $\mathrm{diag}(s)$ at each $v \in \Phi$, and the finite frame x_V. Through x_V, $\mathrm{U}(V)(\mathbb{A}_{E^+,f}) = G_f$ as in (VP3); the integral elements are K_f, the principal level is $\mathrm{K}_M = \{k \in \mathrm{K}_f :\ k_w \equiv 1 \bmod w \text{ for every chosen place } w\}$, and $\bar{G} = \mathrm{K}_f/\mathrm{K}_M$ by reduction at the chosen places. For a framed space, $\mathcal{A}_M(V)$ and the action of $\bar{G}$ of Definition 3.2.16 are taken in these frames: the archimedean type at v is the discrete series with Harish-Chandra parameter $\varsigma(v)$ relative to the signs s, and G_f acts on $\mathcal{A}_M(V)$-type forms by right translation ρ_V. The spaces $Y_0, W_1, \dots, W_m$ are framed spaces with the signs and frames fixed in Definition 3.2.13, Definition 3.2.15 and below; for W_j the signs are $+1$ at the first $k+j-1$ indices and -1 at the others.

(HO2) Gap. For a framed space V with signs s, ς **has gap Δ for** V if for every $v \in \Phi$ the entries $\varsigma(v)_1, \dots, \varsigma(v)_r$ are pairwise distinct and $|\varsigma(v)_i - \varsigma(v)_j| \geq \Delta$ whenever $s(i) \neq s(j)$.

(HO3) Hecke operators away from M. Let

$$G^M = \{g \in G_f :\ g_w = 1 \text{ for every chosen place } w\}.$$

The **spherical algebra away from** M is the set $\mathcal{H}^M$ of finitely supported $a = \sum_g a_g\, g \in \mathbb{C}[G^M]$ such that for every representation π of G_f on a complex vector space X and every $x \in X$ fixed by K_M, the vector $\pi(a)x = \sum_g a_g \pi(g)x$ is fixed by K_M. For $a \in \mathcal{H}^M$ and a framed space V let $T_a(V) = \rho_V(a)|_{\mathcal{A}_M(V)}$; the family $T_a = (T_a(V))_V$ is the **spherical Hecke operator away from** M given by a.

(HO4) Eigenparts. A finite place w of E is **split off the level** if $w \neq \bar{w}$ and neither w nor $\bar{w}$ is a chosen place; then G_f has the factor $\mathrm{GL}_r(E_w)$ at the place of E^+ below w, and for $1 \leq i \leq r$ let $a_{w,i} \in \mathcal{H}^M$ be the sum of the coset representatives of $\mathrm{GL}_r(\mathcal{O}_w)\,\mathrm{diag}(\varpi_w 1_i, 1_{r-i})\,\mathrm{GL}_r(\mathcal{O}_w)/\mathrm{GL}_r(\mathcal{O}_w)$, placed at w (and 1 elsewhere), with ϖ_w a uniformiser, and $T_{w,i} = T_{a_{w,i}}$. An **eigensystem** is a function q from the pairs (w,i) to $\mathbb{C}$; its **eigenpart** on a framed space V is

$$\mathcal{A}_M(V)[q] = \{y \in \mathcal{A}_M(V) :\ T_{w,i}(V)\, y = q(w,i)\, y \text{ for all } (w,i)\}.$$

The frame of Y_0 is its fixed frame, (XF1) of Definition 3.2.15, the frames of W_j those of its standard lattice $\mathcal{O}_E^r$ in (HP4), and the entries of $\varsigma(v)$ are attached to the basis by the numbering num of (HP2) of Definition 3.2.13, as in Definition 3.2.16.

Definition 5.6.2. Let E be a CM field, let Φ be its set of infinite places (all complex, $f = \#\Phi = [E^+ : \mathbb{Q}]$), let $N \geq 1$ and $s \in \{\pm 1\}^N$.

(EK1) Real group and size. $G_\infty = \prod_{v\in\Phi} \mathrm{U}(\operatorname{diag} s)$ is the real group of the signs s as in (FT1) of Definition 3.2.4, $\mathrm{U}(\operatorname{diag} s) = \{g \in \mathrm{GL}_N(\mathbb{C}) : \ \bar{g}^{\mathsf{T}} \operatorname{diag}(s)\, g = \operatorname{diag}(s)\}$, with a fixed Haar measure dg_v on each factor and the product measure dx on G_∞. The **archimedean size** of $x = (x_v)_v \in G_\infty$ is

$$\|x\|_\infty = \prod_{v\in\Phi} \|x_v\|,$$

$\|x_v\|$ the operator norm of x_v for the standard hermitian norm on $\mathbb{C}^N$.

(EK2) Real convolution. For a group $G = G_\infty \times G'$ (G' any group), a function α on G_∞ and a function F on G, the **real convolution** is

$$R(\alpha)F(g) = \int_{G_\infty} \alpha(h)\, F\big(g\,(h,1)\big)\, dh$$

at every g at which the integral converges absolutely.

(EK3) Idempotent kernel. A function $e\colon G_\infty \to \mathbb{C}$ is an **idempotent kernel** if it is continuous and integrable, $e(x^{-1}) = \overline{e(x)}$ for all x, $e(1) \neq 0$, and

$$\int_{G_\infty} e(z)\, e(x^{-1}z^{-1}y)\, dz = \frac{e(x^{-1})\, e(y)}{e(1)} \qquad (x, y \in G_\infty).$$

(EK4) Cusp kernel. For an integer $B \geq 0$, an idempotent kernel e is a **cusp kernel of order** B if

(a) $x \mapsto |e(x)|\, \|x\|_\infty^{B+1}$ is integrable on G_∞;

(b) $\sup_{x\in G_\infty} |e(x)|\, \|x\|_\infty^{B} < \infty$;

(c) there is a continuous compactly supported $\beta\colon G_\infty \to \mathbb{C}$ with $\int_{G_\infty} \beta(x)\, e(x^{-1}y)\, dx = e(y)$ for all $y \in G_\infty$.

Definition 5.6.3. Let D be admissible data and let V be a hermitian space over E of rank $n \geq 1$ with a frame and signs $s \in \{\pm 1\}^n$, the same at every $v \in \Phi$; with its frame V is a framed hermitian space with signs s on $\{1, \dots, n\}$ in the sense of (VP3) of Definition 3.2.2, that is, a framed space of rank n with signs s in the sense of (FT1) of Definition 3.2.4. Through the frame $\mathrm{U}(V)(\mathbb{A})$ is written as $\mathrm{G}(s) = G_\infty \times G_f$, $G_\infty = G_\infty(s)$, as in (VP3), and $u \in V$ has the coordinate vector $c(u) = c_V(u) \in \mathbb{A}_E^n$ of (VP3); $\|\cdot\|$ is the euclidean norm on $\prod_{v\in\Phi} \mathbb{C}^n$ of (VP4), and $\|g\|$ is the operator norm of a complex matrix g.

(DZ1) Gap. For $s \in \{\pm 1\}^n$ and $h, A \in \mathbb{N}$, an integer vector $l \in \mathbb{Z}^n$ has gap h, or large gap A, for s in the sense of (WM8) of Definition 3.2.19, on the index set $\{1, \dots, n\}$: its entries are distinct and congruent to $n-1$ modulo 2, and $|l_i - l_j| \geq 2h$ whenever $s_i \neq s_j$, respectively with $h = 500n^3 + A$. Then $\frac{1}{2}l$ is a Harish-Chandra parameter for $\mathrm{U}(\operatorname{diag} s)$.

(DZ2) Forms of a given type and level. For a family $\lambda = (\lambda(v))_{v\in\Phi}$ of Harish-Chandra parameters for $\mathrm{U}(\operatorname{diag} s)$, forms of type λ on V means the forms of parameter λ of (FT3) of Definition 3.2.4: the forms of type e_λ in the sense of (FT2), e_λ the type kernel built from Flensted-Jensen's functions $c_{\lambda(v)}$ of Definition 3.2.3 and the formal degrees $d_{\lambda(v)}$. For $0 \neq a \in \mathcal{O}_E$ the **principal level** $\mathrm{K}(a) \subseteq G_f$ consists of the elements whose matrix in the frame and whose inverse have entries in $\hat{\mathcal{O}}_E$ and which are congruent to 1 modulo a; a form **has level** $\mathrm{K}(a)$ if it is right invariant under $\mathrm{K}(a)$.

(DZ4) Cusp constant. $c_{\mathrm{cs}}(n) = 8fn\,\mathrm{rd}_E + 48n^6f^2q_1d + 1$.

(DZ5) Deep-level sizes. Integers $A, L, a_{\mathrm{p}} \geq 0$ and reals $\beta, R > 0$ satisfy the **deep-level size conditions for rank** n if

$$8(n+1)^2 < A, \quad a_{\mathrm{p}} < A, \quad 2f(n+1)+2 \leq A,$$
$$M\,c_{\mathrm{cs}}(n)^2 \leq \beta, \quad 2000(n+L+1)^6 \leq \beta, \quad f \leq \beta,$$
$$\beta^{20A(f+1)} \leq R, \qquad R^{3n+2} \leq M^{b/2}.$$

The last condition says $R^{3n+2} \leq |\pi_M^b|_v$ at every $v \in \Phi$, since $|\pi_M|_v^2 = \pi_M\bar{\pi}_M = M$. The **cusp threshold** is $\theta_{\mathrm{cs}} = \left(R^{-2}/(M\,c_{\mathrm{cs}}(n)^2)\right)^n$.

Proposition 5.6.4. *Let D be admissible data. There is a filling datum $\mathfrak{S}_D$ over E of shape (ϱ, k), $\varrho = (r-k-1)/2$, with level N, $\pi = \pi_N$ (as an idele), primes p_0, p_1, chosen places those of Definition 2.3.4 above p_0 and p_1, size b, source level the level K_0 of the source forms, and Bessel systems those of Proposition 5.7.3, such that:*

(i) $\mathfrak{S}_D$ is admissible, and E carries Weil data;

(ii) the level group of $\mathfrak{S}_D$ is $\bar{G}$, and $\mathfrak{S}_D$ fits Y_0: the space V of $\mathfrak{S}_D$ is Y_0 (the same basis vectors and hermitian form), so $\mathrm{U}(Y_0) = G$ with the same rational points; with $\iota\colon \mathrm{U}(X_0) \to G$ the inclusion, u_0 the central idele of (HP5) of Definition 3.2.13 and $\eta = \iota(u_0^{-1}1_{X_0}) \in G(\mathbb{A}_f)$, the element η commutes with $\iota(\mathrm{U}(X_0)(\mathbb{A}_f))$, the finite frame of Y_0 is $u_0\,\eta\,B\,k$ with B the lattice basis of (FG3) and $k \in \mathrm{GL}_r(\hat{\mathcal{O}})$ reducing to 1 at every chosen place, the principal level of Y_0 (as a group of operators on $Y_0 \otimes \mathbb{A}_f$) is $\mathrm{K}_M = \eta\,\mathrm{K}(M)\,\eta^{-1}$, and every $g \in G(\mathbb{A}_f)$ integral in the finite frame of Y_0 satisfies $\eta^{-1}g\eta \in K$, with $\mathrm{red}(\eta^{-1}g\eta)$ the family of reductions of g at the chosen places;

(iii) there is a linear isomorphism $\mathcal{S} \cong \mathbb{C}^{\mathcal{C}\times I}$ (the isomorphism θ: translation by the central idele u_0, evaluation at the class representatives, and a $\mathrm{U}(k)$-equivariant change of coordinates on the Weyl module), so $\dim \mathcal{S} = D_X = \#\mathcal{C}\cdot\#I \geq 1$;

(iv) $\#\bar{H}\cdot\#\mathcal{C}\cdot\prod_v \mathrm{fr}_v\,\#I_v^2/\deg_v \leq (1+b^{-9})\mathcal{D}_{\mathrm{main}}$ for the frame masses fr_v, and $\#I \leq b^{2kf}$.

Proof. The datum. Let $\mathfrak{S}_D$ have shape (ϱ, k), with $X_0 = E^k$ through the basis of X_0, numbered $x_1, \ldots, x_k$ (so $r = 2\varrho + k + 1$ and $n = \varrho + 1$), the CM field $E = E_s$, the prime N and the idele $\pi = \pi_N$, the primes p_0, p_1 with S_i the chosen places of D above p_i, the generator ϖ of the different of Lemma 3.2.35, the size b, the source level $K_X = K_0$, and at each $v \in \Phi$ the Bessel system $\mathfrak{B}_v$ of Proposition 5.7.3 (for (ϱ, k, b, f)). We check the clauses of Definition 5.1.9.

(FG1) and (FG3) are constructions. (FG2): N is prime by (AD6), $N \geq 5$ and $p_i < N$ by (S1) of Lemma 5.2.12; $\pi \in \hat{\mathcal{O}}$ and $\pi\bar{\pi} = N$ by (AD6). By Lemma 5.2.11 (a), (b), at a finite place w either π is a unit at w, or $w \in T_N$, in which case $q_w = N$ and $\bar{w} \notin T_N$, so π is a unit at $\bar{w}$; this

is the level clause of (FG2). By Lemma 5.2.11 (c), (d), $\#S_i = f$, $q_w = p_i$ for $w \in S_i$, $S_0 \cap S_1 = \emptyset$, no conjugate of a chosen place is chosen, and $M \mid x$ in $\hat{\mathcal{O}}$ if and only if $x \equiv \bar{x} \equiv 0$ modulo every chosen w. By Lemma 3.2.35 (i), $\bar{\varpi} = -\varpi$ and ϖ generates the different. $b \geq 2000$ by (S4) of Lemma 5.2.12. (FG4): K_0 is the subgroup of the compact group K^{st} of Lemma 5.6.5 cut out by congruences at the finitely many places of T_N and the chosen places, hence open and compact; it meets every conjugate of $\mathrm{U}(X_0)(E^+)$ trivially by Lemma 5.6.6. (FG5) is Proposition 5.7.3. (FG6): $b^{10} \leq p_i \leq b^{11}$ is (AD5); the other size fields are those of (S1) of Lemma 5.2.12, using $[E : \mathbb{Q}] = 2f$ (so $(\varrho+1)[E:\mathbb{Q}] \leq 2^{2\varrho}$ is $2f(\varrho+1) \leq 2^{2\varrho}$) and $2^{-\varrho} \leq b^{-6}$ from $b^{10} \leq 2^{\varrho}$.

Weil data. $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ divides $q_1(s) = q_0 \cdot 3\, l_0(s) \cdot 2^s$, because $2^4 \cdot 3 \cdot 5 \cdot 7 \mid q_0$ and $3 \mid 3\, l_0(s)$. So $E = K(\zeta_{q_1})$ contains a primitive 5040-th root of unity.

Admissibility, (i). (FA1): $K_X = K_0 = \{h : \iota(h) \in K(M)\}$ by Lemma 5.6.5 (b). (FA2): let $s = (s_w) \in \bar{G}$. For a chosen w, π and N are units at w (Lemma 5.2.11), so the lattice of (FG3) at w is the standard lattice $\mathcal{O}_{E_w}^r$, whose Gram matrix (hyperbolic planes, 1 on X_0, $-N$ at z) lies in $\mathrm{GL}_r(\mathcal{O}_{E_w})$. Let u be the place of E^+ below w; it splits ($\bar{w} \neq w$), $g \mapsto g_w$ identifies $G(E_u^+)$ with $\mathrm{GL}_r(E_w)$, and, the lattice being self-dual, the factor K_u of K with $\mathrm{GL}_r(\mathcal{O}_{E_w})$, which reduces onto $\mathrm{GL}_r(k_w)$. Distinct chosen places lie above distinct places of E^+ (Lemma 5.2.11 (c)). Choosing at each such u a lift of s_w in K_u and 1 at the other places gives $g \in K$ with $\mathrm{red}(g) = s$. (FA3) and (FA4): ϖ is purely imaginary, so $|\mathrm{rs}_v| = |\mathrm{Re}(Mv(\varpi)/2\pi i)| = M|\varpi|_v/2\pi$; by (S2) of Lemma 5.2.12, $|\varpi|_v \leq b$ and $1 \leq M|\varpi|_v/2\pi \leq b^{23}$. (FA5): by (S2), $\log N \leq 30(2\varrho+2)\log b = 60n \log b$. (FA6): the quantities $D_X = \#\mathcal{C} \cdot \#I$, $\mathcal{D}_{\mathrm{id}}$, $\mathcal{D}_{\mathrm{main}}$ of $\mathfrak{S}_D$ are those of Lemma 5.6.9, whose (iii) is (FA6). (FA7) is Lemma 5.6.8 (a). (FA8): by Lemma 5.6.5 (a), (c), the group $\{h : \iota(h) \in K\} = K^\pi$ has index $(N^k-1)^f$ in the standard level K^{st}, and $N^k - 1 \geq N^k/2 \geq N^k/b$. (FA9) is Lemma 5.6.12 (ii), whose quantities are those of $\mathfrak{S}_D$.

(ii). The level group of $\mathfrak{S}_D$ is $\prod_{w \in S_0 \cup S_1} \mathrm{GL}_r(k_w) = \bar{G}$. By (HP2), Y_0 has the basis $e_1, \dots, e_\varrho$, the basis of X_0, z, $e_\varrho^*, \dots, e_1^*$ (in the order of the string $v_1, \dots, v_{2\varrho+1} = e_1, \dots, e_\varrho, z, e_\varrho^*, \dots, e_1^*$) with the Gram matrix of (FG1); so $\mathrm{U}(Y_0) = G$ with the same rational points. As $s_0 = 2\varrho + 1$ is odd, the graded basis L_0 of (HP5) keeps the basis vectors of X_0 other than x_k and replaces $x_k, v_1, \dots, v_{2\varrho+1}$ by $\bar{\pi}^{-1}x_k + u_0 N^{-1} z$ and $u_0 v_1, \dots, u_0 v_{2\varrho+1}$, where $u_0 \bar{u}_0 = 1$. Let B be the lattice basis of (FG3) and $\eta = \iota(u_0^{-1} 1_{X_0}) \in \iota(\mathrm{U}(X_0)(\mathbb{A}_f))$, central there. Applying the scalar u_0 after η to B: the string vectors become $u_0 v_i$; x_j ($j < k$) becomes $u_0 u_0^{-1} x_j = x_j$; and $x_k/\bar{\pi} + z/N$ becomes $u_0(u_0^{-1}x_k/\bar{\pi} + z/N) = \bar{\pi}^{-1}x_k + u_0 N^{-1} z$. So $L_0 = u_0\, \eta\, B$, and the frame of Y_0 is $F = u_0 \eta B k$ with $k \in \mathrm{GL}_r(\hat{\mathcal{O}})$ reducing to 1 at every chosen place (HP5). Now $\mathcal{A}_M(Y_0)$ consists of the forms right invariant under $K_M = \{g : F^{-1} g F \in \mathrm{GL}_r(\hat{\mathcal{O}}),\ F^{-1} g F \equiv 1 \text{ at the chosen places}\}$. Since u_0 is a central scalar and conjugation by k preserves integrality and the congruence $\equiv 1$ at the chosen places, $g \in K_M$ if and only if $B^{-1}\eta^{-1} g \eta B$ is integral and $\equiv 1$ there, that is, $\eta^{-1} g \eta \in K(M)$; so $K_M = \eta K(M) \eta^{-1}$. Hence $\varphi \mapsto \varphi(\,\cdot\,\eta^{-1})$ is a linear isomorphism from $\mathcal{A}_M(Y_0)$ onto the space of forms on $G(\mathbb{A})$ of archimedean type ς (read in the frame of Y_0 at the infinite places, which is not changed) that are right invariant under $K(M)$, the forms of principal level of $\mathfrak{S}_D$: right translation by a finite element preserves left invariance, boundedness, continuity and the archimedean type, and $\varphi(gm\eta^{-1}) = \varphi(g\eta^{-1} \cdot \eta m \eta^{-1})$ with $\eta m \eta^{-1} \in K_M$ for $m \in K(M)$. It intertwines the actions of $\bar{G}$: if g is integral in the frame with reductions s, then $\eta^{-1} g \eta \in K$ has reduction s (again because $k \equiv 1$ at the chosen places), and $\varphi(h\eta^{-1}g) = \varphi(h(\eta^{-1}g\eta)\eta^{-1})$.

(iii) is Lemma 5.6.14 (b), with $D_X = \#\mathcal{C} \cdot \#I$. *(iv)* is Lemma 5.6.12 (i), and $\#I \leq b^{2kf}$ is (FA7). □

Lemma 5.6.5. *Let D be admissible data, and for $h \in \mathrm{U}(X_0)(\mathbb{A}_f)$ write $\iota(h) \in G(\mathbb{A}_f)$ for the element acting as h on X_0 and trivially on the hyperbolic planes and on z. Let $K^{\mathrm{st}} = \mathrm{U}(X_0)(\mathbb{A}_f) \cap \mathrm{GL}_k(\hat{\mathcal{O}})$ and let $K^\pi \subseteq K^{\mathrm{st}}$ be the subgroup of the h whose last row is $\equiv (0, \dots, 0, 1)$ (mod π_N). Then:*

(a) $\iota(h) \in K$ *if and only if* $h \in K^\pi$*;*

(b) $\iota(h) \in K(M)$ *if and only if* $h \in K_0$*;*

(c) $[K^{\mathrm{st}} : K^\pi] = (N^k - 1)^f$ *and* $[K^\pi : K_0] = \#\bar{H}$*, where* $\bar{H} = \mathrm{red}(\iota(\mathrm{U}(X_0)(\mathbb{A}_f)) \cap K)$*.*

Proof. Write $\pi = \pi_N$. Let $e_1, \dots, e_k$ be the basis of X_0, so that the lattice basis of (FG3) is the basis of V with e_k replaced by $e'_k = e_k/\bar{\pi} + z/N$, and L is its $\widehat{\mathcal{O}}$-span; K is the stabiliser of L in $G(\mathbb{A}_f)$. By Lemma 5.2.11 (b), π is a uniformiser at the f places of T_N and a unit elsewhere, $\bar{\pi}$ is a uniformiser exactly at the places of $\overline{T_N}$, $T_N \cap \overline{T_N} = \emptyset$, and $\pi\bar{\pi} = N$; hence $\pi\widehat{\mathcal{O}} + \bar{\pi}\widehat{\mathcal{O}} = \widehat{\mathcal{O}}$ and $\pi\widehat{\mathcal{O}} \cap \bar{\pi}\widehat{\mathcal{O}} = N\widehat{\mathcal{O}}$, and $\bar{\pi}/N = \pi^{-1}$.

(a). Let $h = (h_{ij}) \in \mathrm{U}(X_0)(\mathbb{A}_f)$; then $h^{-1} = h^*$, $(h^*)_{ij} = \overline{h_{ji}}$. The basis vectors outside $X_0 \oplus Ez$ and z are fixed by $\iota(h)$. For $j < k$, using $e_k = \bar{\pi}e'_k - \pi^{-1}z$,

$$\iota(h)e_j = \sum_{i<k} h_{ij}e_i + h_{kj}\bar{\pi}\, e'_k - h_{kj}\pi^{-1}z, \qquad \iota(h)e'_k = \sum_{i<k} \frac{h_{ik}}{\bar{\pi}}e_i + h_{kk}e'_k + \frac{1 - h_{kk}}{N}z.$$

As $\pi, \bar{\pi} \in \widehat{\mathcal{O}}$, $\iota(h)L \subseteq L$ holds if and only if

$$\text{(R)} \qquad h_{ij} \in \widehat{\mathcal{O}}\ (i,j<k), \quad h_{kj} \in \pi\widehat{\mathcal{O}}\ (j<k), \quad h_{ik} \in \bar{\pi}\widehat{\mathcal{O}}\ (i<k), \quad h_{kk} - 1 \in N\widehat{\mathcal{O}}.$$

Applying complex conjugation, condition (R) for $h^{-1} = h^*$ is again (R) for h. So $\iota(h) \in K$, i.e. $\iota(h)L \subseteq L$ and $\iota(h)^{-1}L \subseteq L$, if and only if (R) holds. If (R) holds, h and h^{-1} are integral and the last row of h is $\equiv (0, \dots, 0, 1) \pmod{\pi}$ because $N\widehat{\mathcal{O}} \subseteq \pi\widehat{\mathcal{O}}$; so $h \in K^\pi$. Conversely let $h \in K^\pi$. The entries of $hh^* = 1$ in column k read $\sum_j h_{ij}\overline{h_{kj}} = \delta_{ik}$. Conjugating $h_{kj} \equiv \delta_{kj} \pmod{\pi}$ gives $\overline{h_{kj}} \equiv \delta_{kj} \pmod{\bar{\pi}}$, and the h_{ij} are integral, so $h_{ik} \equiv \delta_{ik} \pmod{\bar{\pi}}$. Thus $h_{ik} \in \bar{\pi}\widehat{\mathcal{O}}$ for $i < k$, and $h_{kk} - 1 \in \pi\widehat{\mathcal{O}} \cap \bar{\pi}\widehat{\mathcal{O}} = N\widehat{\mathcal{O}}$; with $h_{kj} \in \pi\widehat{\mathcal{O}}$ $(j < k)$ this is (R).

(b). $K(M)$ is the set of $g \in K$ whose matrix in the lattice basis reduces to 1 at every chosen place w. A chosen place lies above p_0 or $p_1 \neq N$, so π and N are units at w, and the matrix B_w of the lattice basis in the standard basis lies in $\mathrm{GL}_r(\widehat{\mathcal{O}}_{E,w})$. The matrix of g in the lattice basis at w is $B_w^{-1}g_wB_w$; conjugation by an element of $\mathrm{GL}_r(\widehat{\mathcal{O}}_{E,w})$ preserves the congruence $\equiv 1 \pmod{w}$, so it reduces to 1 if and only if $g_w \equiv 1 \pmod{w}$. For $g = \iota(h)$ with $h \in K^\pi$ this says $h_w \equiv 1 \pmod{w}$. By (a), $\iota(h) \in K(M)$ if and only if $h \in K^\pi$ and $h_w \equiv 1 \pmod{w}$ at every chosen place w, which is the definition of K_0.

(c). The conditions defining K^π inside K^{st} concern only the components at the places of T_N, where π is not a unit. Let $w \in T_N$ and u the place of E^+ below it. By Lemma 5.2.11 (b), $\bar{w} \neq w$, so u splits in E, $h \mapsto h_w$ is an isomorphism $\mathrm{U}(X_0)(E_u^+) \to \mathrm{GL}_k(E_w)$ (the component at $\bar{w}$ is determined by $h_{\bar{w}} = \overline{(h_w^{\mathsf{T}})^{-1}}$, the bar applied entrywise through $E_w \cong E_{\bar{w}}$, so it is integral if and only if h_w is), and it carries the factor of K^{st} at u onto $\mathrm{GL}_k(\mathcal{O}_{E_w})$; distinct places of T_N lie above distinct places of E^+, again because $\overline{T_N} \cap T_N = \emptyset$. So $[K^{\mathrm{st}} : K^\pi] = \prod_{w \in T_N}[\mathrm{GL}_k(\mathcal{O}_{E_w}) : \Gamma_w]$, where Γ_w is the subgroup of the g whose last row is $\equiv (0, \dots, 0, 1) \pmod{w}$. As $q_w = N$ (Lemma 5.2.11 (a)), Γ_w is the preimage under the surjective reduction $\mathrm{GL}_k(\mathcal{O}_{E_w}) \to \mathrm{GL}_k(\mathbb{F}_N)$ of the stabiliser of the row vector $(0, \dots, 0, 1)$ for the right action $x \mapsto xg$; this action is transitive on the $N^k - 1$ nonzero row vectors, so the index is $N^k - 1$. With $\#T_N = f$ this gives $[K^{\mathrm{st}} : K^\pi] = (N^k - 1)^f$. Finally, by (a) $\iota(\mathrm{U}(X_0)(\mathbb{A}_f)) \cap K = \iota(K^\pi)$, so $\mathrm{red} \circ \iota \colon K^\pi \to \bar{G}$ has image $\bar{H}$, and by (b) its kernel is K_0; hence $[K^\pi : K_0] = \#\bar{H}$. □

Lemma 5.6.6. *Let D be admissible data and $\mathcal{C} = \mathrm{U}(X_0)(E^+)\backslash\mathrm{U}(X_0)(\mathbb{A}_f)/K_0$. Then $\mathcal{C}$ is finite, $\mathrm{U}(X_0)(E^+) \cap gK_0g^{-1} = \{1\}$ for every $g \in \mathrm{U}(X_0)(\mathbb{A}_f)$, and*

$$\#\mathcal{C} = \#\bar{H} \cdot (N^k - 1)^f \cdot \mathrm{mass}(I_k).$$

Proof. Put $\Gamma = \mathrm{U}(X_0)(E^+)$, $G_f = \mathrm{U}(X_0)(\mathbb{A}_f)$ and let $K^{\mathrm{st}} \supseteq K^\pi \supseteq K_0$ be as in Lemma 5.6.5; K^{st} is the stabiliser of $\widehat{\mathcal{O}}_E^k$ in G_f.

Freeness. K_0 is an open subgroup of K^{st}, and each element of K_0 reduces to 1 at every chosen place w, hence acts trivially on $\mathcal{O}_{E,w}^k / w\mathcal{O}_{E,w}^k$. By Lemma 4.6.2 (c), applied to the definite space X_0 with the lattice $\mathcal{O}_E^k$, $\Gamma \cap gK_0g^{-1} = \{1\}$ for every $g \in G_f$.

The count. By (2) of Lemma 3.2.9, E satisfies (Um). The space X_0 is E^k with $h(x,y) = \sum_i \bar{x}_i y_i$ and the lattice $\mathcal{O}_E^k$, and K_0 is an open subgroup of K^{st} with $\Gamma \cap gK_0g^{-1} = \{1\}$ for every $g \in G_f$ (Freeness). So Lemma 5.6.7 with $n = k$ and the level K_0 shows that $\mathcal{C}$ is finite and $\#\mathcal{C} = [K^{\mathrm{st}} : K_0] \cdot \mathrm{mass}(I_k)$.

The index. By Lemma 5.6.5 (c), $[K^{\mathrm{st}} : K_0] = [K^{\mathrm{st}} : K^\pi]\,[K^\pi : K_0] = (N^k - 1)^f \cdot \#\bar{H}$, which gives the formula. □

Lemma 5.6.7. *Let E be a CM field with (Um), E^+ its maximal totally real subfield, $n \geq 1$, $V = E^n$ with $h(x,y) = \sum_i \bar{x}_i y_i$ and $L = \mathcal{O}_E^n$. Put $\Gamma = \mathrm{U}(V)(E^+)$, $G_f = \mathrm{U}(V)(\mathbb{A}_{E^+,f})$ and let K^{st} be the stabiliser of $\widehat{L}$ in G_f. Let $\mathrm{K} \subseteq \mathrm{K}^{\mathrm{st}}$ be an open subgroup with $\Gamma \cap g\mathrm{K}g^{-1} = \{1\}$ for every $g \in G_f$. Then $\Gamma\backslash G_f/\mathrm{K}$ is finite and*

$$\#(\Gamma\backslash G_f/\mathrm{K}) = [\mathrm{K}^{\mathrm{st}} : \mathrm{K}] \cdot \mathrm{mass}(I_n).$$

Proof. **Step 1: classes of K^{st} and lattices of the genus.** For $g \in G_f$ put $L_g = V \cap g\widehat{L}$. Since $g_{\mathfrak{p}} \in \mathrm{U}(V)(E_{\mathfrak{p}}^+)$ fixes $\widehat{L}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$, L_g is a lattice in V with completions $(\widehat{L_g})_{\mathfrak{p}} = g_{\mathfrak{p}}\widehat{L}_{\mathfrak{p}}$ (a lattice is the intersection of V with its completions, and every family of local lattices that agrees with $\widehat{L}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ arises so); hence L_g lies in the genus of L. Conversely, if M' lies in the genus of L, choose isometries $g_{\mathfrak{p}}$ with $g_{\mathfrak{p}}\widehat{L}_{\mathfrak{p}} = \widehat{M}'_{\mathfrak{p}}$, and $g_{\mathfrak{p}} = 1$ at the almost all $\mathfrak{p}$ where $\widehat{M}'_{\mathfrak{p}} = \widehat{L}_{\mathfrak{p}}$; then $g = (g_{\mathfrak{p}}) \in G_f$ and $M' = L_g$. For $g, g' \in G_f$ and $\gamma \in \Gamma$, $\gamma L_{g'} = L_{\gamma g'}$, and $L_{\gamma g'} = L_g$ if and only if $\gamma g'\widehat{L} = g\widehat{L}$, that is $g \in \gamma g'\mathrm{K}^{\mathrm{st}}$. So $g \mapsto L_g$ induces a bijection from $\Gamma\backslash G_f/\mathrm{K}^{\mathrm{st}}$ onto the set of $\mathrm{U}(V)$-orbits of lattices in the genus of L (here $\mathrm{U}(V) = \Gamma$), and $\Gamma_g := \Gamma \cap g\mathrm{K}^{\mathrm{st}}g^{-1}$ is the stabiliser $\mathrm{Aut}(L_g)$, a finite group. The genus of L consists of finitely many orbits (finiteness of the class number, [33]), so $\Gamma\backslash G_f/\mathrm{K}^{\mathrm{st}}$ is finite.

Step 2: the fibres. K is open in the compact group K^{st}, so $[\mathrm{K}^{\mathrm{st}} : \mathrm{K}] < \infty$. The fibre of the projection $\Gamma\backslash G_f/\mathrm{K} \to \Gamma\backslash G_f/\mathrm{K}^{\mathrm{st}}$ over $\Gamma g\mathrm{K}^{\mathrm{st}}$ consists of the double cosets $\Gamma gk\mathrm{K}$, $k \in \mathrm{K}^{\mathrm{st}}$. Two of them coincide, $\Gamma gk'\mathrm{K} = \Gamma gk\mathrm{K}$, if and only if $gk'\mathrm{K} = \gamma gk\mathrm{K}$ for some $\gamma \in \Gamma$, and such a γ lies in $g\mathrm{K}^{\mathrm{st}}g^{-1}$, hence in Γ_g. So the fibre is the orbit set of Γ_g acting by left multiplication on $g\mathrm{K}^{\mathrm{st}}/\mathrm{K}$, a set of $[\mathrm{K}^{\mathrm{st}} : \mathrm{K}]$ elements. The action is free: $\gamma gk\mathrm{K} = gk\mathrm{K}$ means $\gamma \in \Gamma \cap (gk)\mathrm{K}(gk)^{-1} = \{1\}$. Hence the fibre has $[\mathrm{K}^{\mathrm{st}} : \mathrm{K}]/\#\Gamma_g$ elements, $\Gamma\backslash G_f/\mathrm{K}$ is finite, and

$$\#(\Gamma\backslash G_f/\mathrm{K}) = [\mathrm{K}^{\mathrm{st}} : \mathrm{K}] \sum_{\Gamma g\mathrm{K}^{\mathrm{st}}} \frac{1}{\#\Gamma_g}.$$

Step 3: the mass. By Step 1 the sum runs over representatives L_g of the $\mathrm{U}(V)$-orbits in the genus of L, with $\#\Gamma_g = |\mathrm{Aut}(L_g)|$; by Definition 3.2.5 it is $\mathrm{mass}(I_n)$. □

Lemma 5.6.8. *Let D be admissible data, $I = \prod_{v \in \Phi} I_v$ and $\deg = \prod_{v \in \Phi} \deg_v$. Then:*

(a) $\#I \leq b^{2kf}$ and $\#I \cdot ((k-1)!)^f \leq b^{5kf/4}$;

(b) with the sums over $v \in \Phi$,

$$\sum_v m(k-1)\log\kappa(v) \;\le\; \log\deg + f\sum_{l=k-1}^{r-1}\log l! \;\le\; \sum_v mr\Big(\log\kappa(v)+\frac{1}{x}\Big).$$

Proof. Sizes. By (AD2), $k \le r \le 3x^5$ and $m \le 2\sqrt{x}$; by (AD4), $\kappa(v) \ge xb/x_1$; $a = xb$. By Proposition 2.3.5, $x \ge (\log b)^{100}$ is large, $x_1 \ge x^{10^6}$ and $b \ge x_1^{10^6}$. Hence

$$(2m+1)r^8 \le (4\sqrt{x}+1)3^8x^{40} \le x^{42}, \qquad (k+a)k \le (3x^5+xb)\,3x^5 \le 6x^6b \le b^{5/4}, \qquad k+a \le b^2,$$

and $(2m+1)r^8/\kappa(v) \le x^{42}x_1/(xb) \le x_1^2/(xb) \le 1/x$.

(a). By Proposition 5.7.3 (ii), $\#I_v$ is the dimension of the irreducible representation of $\mathrm{U}(k)$ of the column count of v; this Schur module of GL_k is formed from the columns of height $t < k$, and $\mathrm{cols}_v(t) = 0$ for $1 < t < k$ (Definition 2.3.4), so it is that of the one-row weight λ_v: the representation $\mathrm{Sym}^{d_v}(\mathbb{C}^k)$ with $d_v = a$ for $v \in \Phi_a$ and $d_v = 0$ otherwise. So $\#I_v = \binom{k-1+d_v}{d_v} \le \binom{k+d_v}{k} \le (k+a)^k$. Multiplying over the f places and using $(k-1)! \le k^k$,

$$\#I \le (k+a)^{kf} \le b^{2kf}, \qquad \#I\cdot((k-1)!)^f \le \big((k+a)k\big)^{kf} \le b^{5kf/4}.$$

The parameter. Fix v and number the entries of $\varsigma = \varsigma(v)$ from 0 to $r-1$: $\varsigma_i = k/2-1-i$ for $i < k-1$ and $\varsigma_l = \Delta/2 - \kappa_{l+2-k}(v)$ for $k-1 \le l < r$, where $1 \le l+2-k \le m$ ($r-(k-1) = m$). Recall $\kappa_j(v) = \kappa(v) + (2j+1)\Delta/2$. For $i < l$:

- if $l < k-1$: $\varsigma_i - \varsigma_l = l - i$;
- if $i < k-1 \le l$: $\varsigma_i - \varsigma_l = k/2-1-i+\kappa(v)+2(l+2-k)\Delta/2$; as $|k/2-1-i| \le k \le r \le \Delta/2$, this lies between $\kappa(v)$ and $\kappa_{l+2-k}(v)$;
- if $k-1 \le i < l$: $\varsigma_i - \varsigma_l = 2(l-i)\Delta/2$, which lies between 1 and $\kappa_{l+2-k}(v)$, since $2(l-i) \le 2(l+2-k)$.

In all cases $1 \le \varsigma_i - \varsigma_l \le \kappa_{l+2-k}(v)$ when $l \ge k-1$.

(b). By Proposition 5.7.3 (i), $\deg_v = \mathrm{d}(\varsigma(v))$. As $\prod_{0\le i<l<r}(l-i) = \prod_{l<r} l!$ and, for $l < k-1$, $\prod_{i<l}(\varsigma_i - \varsigma_l) = l!$,

$$\log\deg_v = \sum_{l=k-1}^{r-1}\Big(\sum_{i<l}\log(\varsigma_i-\varsigma_l) - \log l!\Big).$$

All logarithms are ≥ 0. Lower bound: for each of the m values of $l \ge k-1$ keep the $k-1$ terms $i < k-1$, each $\ge \log\kappa(v)$; so $\log\deg_v + \sum_{l=k-1}^{r-1}\log l! \ge m(k-1)\log\kappa(v)$. Upper bound: each of the $l < r$ terms is at most $\log\kappa_{l+2-k}(v)$, and

$$\log\kappa_j(v) = \log\kappa(v) + \log\Big(1+\frac{(2j+1)\Delta/2}{\kappa(v)}\Big) \le \log\kappa(v) + \frac{(2m+1)r^8}{\kappa(v)} \le \log\kappa(v)+\frac{1}{x} \qquad (j \le m),$$

so the inner sum is at most $r(\log\kappa(v)+1/x)$ and $\log\deg_v + \sum_{l=k-1}^{r-1}\log l! \le mr(\log\kappa(v)+1/x)$. Summing over the f places v gives (b). □

Lemma 5.6.9. *Let D be admissible data, $D_X = \#\mathcal{C}\cdot\#I$, $\mathcal{D}_{\mathrm{id}} = \#\bar{R}_0\cdot D_X$ and $\mathcal{D}_{\mathrm{main}} = D_X^2/(\deg\cdot\mathrm{mass}(I_r))$. Then*

(i) $\dfrac{\mathcal{D}_{\mathrm{main}}}{\mathcal{D}_{\mathrm{id}}} = \dfrac{\#I\,(N^k-1)^f\,\mathrm{mass}(I_k)}{\deg\cdot\mathrm{mass}(I_r)\cdot M^{f\varrho(2k+2\varrho+1)}}$;

(ii) $\Big|\log\dfrac{\mathcal{D}_{\mathrm{main}}}{\mathcal{D}_{\mathrm{id}}} - fk\Big(\log M - \dfrac{15}{2}\log b\Big) - \log\big(\#I\,((k-1)!)^f\big)\Big| \le \dfrac{fk\log b}{4}$;

(iii) $b^{11kf}\,\mathcal{D}_{\mathrm{id}} \le \mathcal{D}_{\mathrm{main}} \le (M/b^6)^{kf}\,\mathcal{D}_{\mathrm{id}}$.

Proof. (i). $\mathcal{D}_{\mathrm{main}}/\mathcal{D}_{\mathrm{id}} = D_X/(\deg\cdot\mathrm{mass}(I_r)\cdot\#\bar R_0)$. By Lemma 5.6.6, $D_X = \#\bar H(N^k-1)^f\mathrm{mass}(I_k)\cdot\#I$, and by Lemma 5.2.67, $\#\bar R_0 = \#\bar H\cdot M^{f\varrho(2k+2\varrho+1)}$; $\#\bar H$ cancels.

Logarithm of the masses. By (S3) of Lemma 5.2.12, $8f \le 2^\varrho$ and $4\varrho+12 \le k$, so Lemma 5.6.10 applies with $u=\varrho$ to every j with $k \le j < r$. Summing over these $r-k = 2\varrho+1$ values,

$$\log\mathrm{mass}(I_r) - \log\mathrm{mass}(I_k) = f\sum_{i=k}^{r-1}\Big((i+\tfrac12)\log\mathrm{rd}_E + \log i! - (i+1)\log(2\pi)\Big) + e_1,$$
$$|e_1| \le (r-k)2^{1-\varrho}.$$

With $\sum_{i=k}^{r-1}(i+\frac12) = (2\varrho+1)(k+\varrho+\frac12)$ and $\sum_{i=k}^{r-1}(i+1) = (2\varrho+1)(k+\varrho+1)$.

Logarithm of the degree. By Lemma 5.6.8 (b),

$$\log\deg = -f\sum_{l=k-1}^{r-1}\log l! + (k-1)m\sum_v\log\kappa(v) + \theta'$$

with $0 \le \theta' \le m\big(m\sum_v\log\kappa(v) + fr/x\big)$, since $r-(k-1) = m$; note $\kappa(v)\ge 1$.

The level. By (AD6), $\log N = \lambda + e_3$ with $0 \le e_3 \le \sqrt{\log b}$ and $\lambda = \frac{m}{f}\sum_v\log\kappa(v) + (m-1)\log\frac{M\mathrm{rd}_E}{2\pi} - \frac{15}{2}\log b$. Also $f\log(N^k-1) = fk\log N - e_2$ with $0 \le e_2 = f\log\frac{N^k}{N^k-1} \le 2f/N$, since $N^k - 1 \ge N/2$ gives $\log\frac{N^k}{N^k-1} \le \frac{1}{N^k-1} \le \frac{2}{N}$.

Assembly. Insert these into the logarithm of (i), $\log\#I + f\log(N^k-1) + \log\mathrm{mass}(I_k) - \log\deg - \log\mathrm{mass}(I_r) - f\varrho(2k+2\varrho+1)\log M$. The factorials give $f\sum_{l=k-1}^{r-1}\log l! - f\sum_{i=k}^{r-1}\log i! = f\log(k-1)!$. Collecting the coefficients of $\log M$, $\log\mathrm{rd}_E$ and $\log 2\pi$, with $m-1 = 2\varrho+1$:

$$fk(m-1) - f\varrho(2k+2\varrho+1) = fk - f\varrho(2\varrho+1), \qquad fk(m-1) - f(2\varrho+1)(k+\varrho+\tfrac12) = -f(2\varrho+1)(\varrho+\tfrac12),$$

and $-fk(m-1) + f(2\varrho+1)(k+\varrho+1) = f(2\varrho+1)(\varrho+1)$. Hence, with

$$A = f(m-1)\Big(\varrho\log\frac{M\mathrm{rd}_E}{2\pi} + \log\frac{\sqrt{\mathrm{rd}_E}}{2\pi}\Big), \qquad B = m\sum_v\log\kappa(v),$$

we get

$$\log\frac{\mathcal{D}_{\mathrm{main}}}{\mathcal{D}_{\mathrm{id}}} - fk\Big(\log M - \frac{15}{2}\log b\Big) - \log\big(\#I((k-1)!)^f\big) = fk\,e_3 + (B-A) - \theta' - e_1 - e_2,$$

the κ-terms being $km\sum_v\log\kappa(v) - (k-1)m\sum_v\log\kappa(v) - \theta' = B - \theta'$. By (S3) of Lemma 5.2.12, $|e_1| + e_2 \le (r-k)2^{1-\varrho} + 2f/N \le \frac{2b^{-9}}{b^{-9}+2} \le \log(1+b^{-9})$, the middle step being the first clause of (S3) (with $2f/N \le 4f/N$) and the last the inequality $\log(1+t) \ge 2t/(t+2)$ for $t \ge 0$. So the right side lies between $-(|A-B| + \log(1+b^{-9}) + m(m\sum_v\log\kappa(v) + fr/x))$ and $fk\sqrt{\log b} + |A-B|$, and both bounds are at most $fk\log b/4$ in absolute value by the error clause of (S2) of Lemma 5.2.12. This is (ii).

(iii). By Lemma 5.6.8 (a) and $\#I \geq 1$, $0 \leq \log(\#I((k-1)!)^f) \leq \frac{5}{4} fk \log b$. By (AD5), $M \geq b^{20}$, so $\log M \geq 19 \log b$. From (ii),

$$\log \frac{\mathcal{D}_{\mathrm{main}}}{\mathcal{D}_{\mathrm{id}}} \geq fk\Big(19 - \frac{15}{2} - \frac{1}{4}\Big) \log b \geq 11 fk \log b,$$
$$\log \frac{\mathcal{D}_{\mathrm{main}}}{\mathcal{D}_{\mathrm{id}}} \leq fk\Big(\log M - \frac{15}{2} \log b + \frac{5}{4} \log b + \frac{1}{4} \log b\Big) = fk \log \frac{M}{b^6}.$$

□

Lemma 5.6.10. *Let P be a parameter pair, $s \geq 0$, $E = E_s$, and let $j, u \geq 0$ be integers with $8f \leq 2^u$ and $j \geq 4u + 12$. Then*

$$\Big|\log \mathrm{mass}(I_{j+1}) - \log \mathrm{mass}(I_j) - f\Big((j + \tfrac{1}{2}) \log \mathrm{rd}_E + \log j! - (j+1) \log(2\pi)\Big)\Big| \leq \frac{2}{2^u}.$$

Proof. By Lemma 3.2.9 (2), $E = E_s$ satisfies (Um), and E/F is unramified at every finite place, $F = E^+$. Here $[E : \mathbb{Q}] = 2f$ and $\mathrm{rd}_E = |d_E|^{1/(2f)}$, so the number $v = |d_E|^{1/(2f)}/2$ of Proposition 5.6.11 is $\mathrm{rd}_E/2$. Put $m = j + 1 \geq 4u + 13 \geq 3$. By Proposition 5.6.11 with $M = m$,

$$\begin{aligned} \log \frac{\mathrm{mass}(I_{j+1})}{\mathrm{mass}(I_j)} &= f\Big(\log j! + (j+1) \log \frac{\mathrm{rd}_E}{2\pi}\Big) - \frac{f}{2} \log \mathrm{rd}_E + \log L_E(m) \\ &= f\Big((j + \tfrac{1}{2}) \log \mathrm{rd}_E + \log j! - (j+1) \log(2\pi)\Big) + \log L_E(m), \end{aligned}$$

so it suffices to show $|\log L_E(m)| \leq 2/2^u$. Group the factors of $L_E(m)$ by the maximal ideals $\mathfrak{p}$ of $\mathcal{O}_F$, with $q = |\mathcal{O}_F/\mathfrak{p}|$ (the product converges absolutely). As $\mathfrak{p}$ is unramified in E, either it is inert, with one $\mathfrak{P} = \bar{\mathfrak{P}}$ of norm q^2 and factor $(1 - (-1)^m q^{-m})^{-1}$, or it splits into $\mathfrak{P} \neq \bar{\mathfrak{P}}$ of norm q with joint factor $(1 - q^{-m})^{-1}$. In both cases the factor lies between $(1 + q^{-m})^{-1}$ and $(1 - q^{-m})^{-1}$, so its logarithm has absolute value at most $-\log(1 - q^{-m})$. For a rational prime ℓ, the $\mathfrak{p} \mid \ell$ have $q = \ell^{a_\mathfrak{p}}$ with $\sum_{\mathfrak{p}|\ell} a_\mathfrak{p} \leq f$, and $-\log(1 - t^a) \leq -a \log(1-t)$ for $0 < t < 1$, $a \geq 1$ (as $(1-t)^a + t^a \leq 1$). Hence

$$|\log L_E(m)| \leq f \sum_\ell -\log(1 - \ell^{-m}) = f \log \zeta(m) \leq f(\zeta(m) - 1) \leq f\Big(2^{-m} + \int_2^\infty t^{-m}\, dt\Big) \leq 2f\, 2^{-m},$$

using $\log x \leq x - 1$ and $2^{1-m}/(m-1) \leq 2^{-m}$ for $m \geq 3$. With $f \leq 2^{u-3}$ and $m \geq 4u + 13$ this is at most $2^{u-2} 2^{-4u-13} \leq 2/2^u$. □

Proposition 5.6.11. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field with (Um), $f = [E : \mathbb{Q}]/2$, d_E its discriminant and $v = |d_E|^{1/(2f)}/2$. For an integer $M \geq 3$ and a maximal ideal $\mathfrak{P}$ of $\mathcal{O}_E$ with norm $N\mathfrak{P}$ put*

$$\epsilon_M(\mathfrak{P}) = \begin{cases} \big(1 - (-1)^M (N\mathfrak{P})^{-M/2}\big)^{-1} & \text{if } \bar{\mathfrak{P}} = \mathfrak{P}, \\ \big(1 - (N\mathfrak{P})^{-M}\big)^{-1/2} & \text{if } \bar{\mathfrak{P}} \neq \mathfrak{P}, \end{cases} \qquad L_E(M) = \prod_{\mathfrak{P}} \epsilon_M(\mathfrak{P}),$$

the product over all maximal ideals, absolutely convergent. Then for every integer $M \geq 3$

$$\frac{\mathrm{mass}(I_M)}{\mathrm{mass}(I_{M-1})} = \Big((M-1)! \Big(\frac{v}{\pi}\Big)^M\Big)^f (2v)^{-f/2}\, L_E(M).$$

Proof. Let F be the maximal totally real subfield of E, $d=[F:\mathbb{Q}]=f$, $d_F>0$ the discriminant of F (positive as F is totally real), χ the quadratic character of E/F and $q_{\mathfrak{p}}=|\mathcal{O}_F/\mathfrak{p}|$. By (Um), E/F is unramified at every finite place.

The ratio of two explicit masses. For $j\geq 1$ the lattice $\mathcal{O}_E^j$ with $h(x,y)=\sum_i \bar{x}_i y_i$ is unimodular: its Gram matrix in the standard basis is the identity, so $x\in(\mathcal{O}_E^j)^\vee$ if and only if all coordinates of x lie in $\mathcal{O}_E$. By the explicit mass formula for unimodular lattices of Gan–Yu in the form of Terakado–Yu [303] (Theorem 2.3.4 and its formula (2.6); hypotheses: E CM, E/F unramified at every finite place, the space definite and the lattice unimodular),

$$\operatorname{mass}(I_j)=2\,d_F^{j^2/2}\prod_{i=1}^{j}\Big(\frac{(i-1)!}{(2\pi)^i}\Big)^d L_F(i,\chi^i),$$

and these numbers are positive. For $M\geq 3$ the factors with $i\leq M-1$ cancel in the quotient, and $M^2-(M-1)^2=2M-1$:

$$\frac{\operatorname{mass}(I_M)}{\operatorname{mass}(I_{M-1})}=d_F^{M-\frac12}\Big(\frac{(M-1)!}{(2\pi)^M}\Big)^f L_F(M,\chi^M).$$

The discriminant. By the transitivity of discriminants, $|d_E|=d_F^{[E:F]}N_{F/\mathbb{Q}}(\mathfrak{d}_{E/F})$, and $\mathfrak{d}_{E/F}=\mathcal{O}_F$ because E/F is unramified at every finite place; so $|d_E|=d_F^2$ and $v=|d_E|^{1/(2f)}/2=d_F^{1/f}/2$. Hence

$$\Big((M-1)!\Big(\frac{v}{\pi}\Big)^M\Big)^f(2v)^{-f/2}=\big((M-1)!\big)^f\frac{d_F^M}{(2\pi)^{Mf}}\,d_F^{-1/2}=d_F^{M-\frac12}\Big(\frac{(M-1)!}{(2\pi)^M}\Big)^f.$$

The Euler product. Group the maximal ideals $\mathfrak{P}$ of $\mathcal{O}_E$ by $\mathfrak{p}=\mathfrak{P}\cap\mathcal{O}_F$; this is allowed since $M\geq 3$ makes both products absolutely convergent ($N\mathfrak{P}\geq q_{\mathfrak{p}}$ and at most two $\mathfrak{P}$ lie over $\mathfrak{p}$). As $\mathfrak{p}$ is unramified in E, either $\mathfrak{p}$ is inert, with the single prime $\mathfrak{P}=\mathfrak{p}\mathcal{O}_E=\bar{\mathfrak{P}}$ of norm $q_{\mathfrak{p}}^2$ and $\chi(\mathfrak{p})=-1$, so that

$$\epsilon_M(\mathfrak{P})=\big(1-(-1)^M q_{\mathfrak{p}}^{-M}\big)^{-1}=\big(1-\chi^M(\mathfrak{p})q_{\mathfrak{p}}^{-M}\big)^{-1}$$

($\chi^M(\mathfrak{p})=(-1)^M$ in both parities); or $\mathfrak{p}$ splits as $\mathfrak{P}\bar{\mathfrak{P}}$ with $\mathfrak{P}\neq\bar{\mathfrak{P}}$, both of norm $q_{\mathfrak{p}}$, $\chi(\mathfrak{p})=1$, and $\epsilon_M(\mathfrak{P})\epsilon_M(\bar{\mathfrak{P}})=(1-q_{\mathfrak{p}}^{-M})^{-1}=(1-\chi^M(\mathfrak{p})q_{\mathfrak{p}}^{-M})^{-1}$. Hence $L_E(M)=\prod_{\mathfrak{p}}(1-\chi^M(\mathfrak{p})q_{\mathfrak{p}}^{-M})^{-1}=L_F(M,\chi^M)$.

The three displays give the assertion. □

Lemma 5.6.12. *Let D be admissible data. Then*

(i) $\#\bar{H}\cdot\#\mathcal{C}\cdot\displaystyle\prod_{v\in\Phi}\frac{\mathrm{fr}_v\,(\#I_v)^2}{\deg_v}\leq(1+b^{-9})\,\mathcal{D}_{\mathrm{main}}$;

(ii) $c_S^\sharp\leq(1+b^{-9})\,\mathcal{D}_{\mathrm{main}}\,\deg c_G^f\,\mathrm{vol}(\hat{\mathcal{O}})^{-n}\,(\nu\Lambda_{\mathrm{fr}}/\#I)^2$.

Proof. Write $m_j=\operatorname{mass}(I_j)$ and $F=\prod_v\mathrm{fr}_v$; note $\prod_v\mathrm{fr}_v(\#I_v)^2/\deg_v=F\,(\#I)^2/\deg$.

(i). By Lemma 5.6.6, $\#\mathcal{C}=\#\bar{H}(N^k-1)^f m_k$, and $\mathcal{D}_{\mathrm{main}}=(\#\mathcal{C})^2(\#I)^2/(\deg\cdot m_r)$. Dividing both sides of (i) by the positive number $\#\mathcal{C}\,(\#I)^2/\deg$, (i) is equivalent to $\#\bar{H}\,F\,m_r\leq(1+b^{-9})\#\mathcal{C}$, that is, to $F\,m_r\leq(1+b^{-9})(N^k-1)^f m_k$. By Lemma 5.6.13, $F\,m_r\leq(1+b^{-9})N^{kf}m_k\big((N^k-1)/N^k\big)^{2f}$, and $N^{kf}\big((N^k-1)/N^k\big)^{2f}=(N^k-1)^f\big((N^k-1)/N^k\big)^f\leq(N^k-1)^f$.

An identity of constants. By Lemma 3.2.35 (iii), $\prod_v \mathrm{sph}_{i+1}(v) = (2\pi)^{(i+1)f}\mathrm{rd}_E^{-f(i+1/2)}(i!)^{-f}$, so $F = N^{kf}\prod_{i=k}^{r-1}(2\pi)^{(i+1)f}\mathrm{rd}_E^{-f(i+1/2)}(i!)^{-f}$. By the definitions of c_G and $\mathrm{vol}\,\mathrm{U}(j)$, $c_G = \frac{2\pi^k}{(k-1)!}\prod_{i=k}^{r-1}\frac{i!}{2\pi^{i+1}}$. Hence, using $\sum_{i=k}^{r-1} i = (2\varrho+1)(k+\varrho)$ and $\sum_{i=k}^{r-1}(i+\frac{1}{2}) = (2\varrho+1)(k+\varrho+\frac{1}{2})$,

$$F\,c_G^f = N^{kf}\Big(\frac{2\pi^k}{(k-1)!}\Big)^f 2^{f(2\varrho+1)(k+\varrho)}\,\mathrm{rd}_E^{-f(2\varrho+1)(k+\varrho+1/2)}.$$

As $|\mathrm{disc}\,E| = \mathrm{rd}_E^{2f}$, $c_S = \big(\pi^k/((k-1)!N^k)\big)^f \mathrm{rd}_E^{f/2}(2/\mathrm{rd}_E)^{fk}$. Comparing the exponents of 2, rd_E, N, π and $(k-1)!$,

$$F\,c_G^f = c_S\,N^{2kf}\,\mathrm{vol}(\hat{\mathcal{O}})^{n+2d}\,|\mathrm{disc}\,E|^{-\varrho/2}.$$

(ii). By the definition of ν, $\nu\Lambda_{\mathrm{fr}} = M^{(2d+\varrho)f}N^{-kf}\mathrm{vol}(\hat{\mathcal{O}})^{-d}|\mathrm{disc}\,E|^{\varrho/4}$ ($\Lambda_{\mathrm{fr}} > 0$ since $\mathrm{rs}_v \neq 0$: $|\mathrm{rs}_v| = M|\varpi|_v/2\pi > 0$), and $2d+\varrho = \varrho(2k+2\varrho+1)$, so $M^{(2d+\varrho)f} = \#\bar{\mathsf{U}}$ by Lemma 5.2.67 (a). Hence, by the identity,

$$c_G^f\,\mathrm{vol}(\hat{\mathcal{O}})^{-n}(\nu\Lambda_{\mathrm{fr}})^2 = (\#\bar{\mathsf{U}})^2N^{-2kf}\mathrm{vol}(\hat{\mathcal{O}})^{-n-2d}|\mathrm{disc}\,E|^{\varrho/2}c_G^f = \frac{(\#\bar{\mathsf{U}})^2c_S}{F},$$

and the right side of (ii) is $(1+b^{-9})\mathcal{D}_{\mathrm{main}}\deg(\#\bar{\mathsf{U}})^2c_S/((\#I)^2F)$. By Lemma 5.2.67 (b), $c_S^\sharp = \#\bar{R}_0\cdot\#\bar{\mathsf{U}}\cdot\#\mathcal{C}\cdot c_S = \#\bar{H}(\#\bar{\mathsf{U}})^2\#\mathcal{C}\,c_S$. Dividing by the positive number $(\#\bar{\mathsf{U}})^2c_S$ and multiplying by $F(\#I)^2/\deg$, (ii) becomes $\#\bar{H}\,\#\mathcal{C}\,F(\#I)^2/\deg \le (1+b^{-9})\mathcal{D}_{\mathrm{main}}$, which is (i). □

Lemma 5.6.13. *Let D be admissible data. For $v \in \Phi$ and $l \ge 1$ put*

$$\mathrm{sph}_l(v) = \frac{(2\pi)^l}{|\varpi|_v^{l-1/2}\,(l-1)!}, \qquad \mathrm{fr}_v = N^k\prod_{i=k}^{r-1}\mathrm{sph}_{i+1}(v)$$

*(the **sphere masses** and the **frame masses**). Then*

$$\Big(\prod_{v\in\Phi}\mathrm{fr}_v\Big)\,\mathrm{mass}(I_r)\Big(\frac{N^k}{N^k-1}\Big)^{2f} \le (1+b^{-9})\,N^{kf}\,\mathrm{mass}(I_k).$$

Proof. One rank. Fix i with $k \le i < r$. By Lemma 3.2.35 (iii), $\prod_v|\varpi|_v = \mathrm{rd}_E^f$, so

$$\log\prod_{v\in\Phi}\mathrm{sph}_{i+1}(v) = -f\Big((i+\tfrac{1}{2})\log\mathrm{rd}_E + \log i! - (i+1)\log(2\pi)\Big).$$

By (S3) of Lemma 5.2.12, $8f \le 2^\varrho$ and $4\varrho+12 \le k \le i$. Hence Lemma 5.6.10, with $j = i$ and $u = \varrho$, gives

$$\Big|\log\prod_v\mathrm{sph}_{i+1}(v) - \big(\log\mathrm{mass}(I_i) - \log\mathrm{mass}(I_{i+1})\big)\Big| \le \frac{2}{2^\varrho}.$$

Telescoping. Summing over the $r-k$ values of i, the masses telescope:

$$\log\prod_{i=k}^{r-1}\prod_v\mathrm{sph}_{i+1}(v) + \log\mathrm{mass}(I_r) - \log\mathrm{mass}(I_k) \le (r-k)\,2^{1-\varrho}.$$

Since $N \ge 2$ and $k \ge 1$, $N^k - 1 \ge N/2$, so $\log\frac{N^k}{N^k-1} \le \frac{N^k}{N^k-1} - 1 = \frac{1}{N^k-1} \le \frac{2}{N}$. As $\prod_v\mathrm{fr}_v = N^{kf}\prod_i\prod_v\mathrm{sph}_{i+1}(v)$, the logarithm of the left side of the claim minus $\log\big(N^{kf}\mathrm{mass}(I_k)\big)$ is at most

$$(r-k)\,2^{1-\varrho} + 2f\cdot\frac{2}{N} \le \frac{2b^{-9}}{b^{-9}+2} \le \log(1+b^{-9}),$$

by the first clause of (S3) of Lemma 5.2.12 and the elementary inequality $\log(1+t) \geq 2t/(t+2)$ for $t \geq 0$ (both sides vanish at $t = 0$, and the derivatives are $1/(1+t) \geq 4/(t+2)^2$). Exponentiating gives the claim. □

Lemma 5.6.14. *Let D be admissible data and fix representatives h_c of the classes $c \in \mathcal{C}$.*

(a) For each $v \in \Phi$ there is a linear isomorphism $A_v \colon W_{\lambda_v} \to \mathbb{C}^{I_v}$ with $A_v(\rho_{\lambda_v}(h^{\mathsf{T}})x) = A_v(x)\,\rho_v(h)$ for all $h \in \mathrm{U}(k)$ and $x \in W_{\lambda_v}$; put $A = \bigotimes_v A_v \colon W_\lambda \to \mathbb{C}^I$.

(b) The map $\theta \colon \mathcal{S} \to \mathbb{C}^{\mathcal{C}\times I}$, $\theta(\varphi)_{(c,\cdot)} = A\big(\varphi(h_c u_0)\big)$, where u_0 acts on X_0 as a scalar, is a linear isomorphism. In particular $\dim \mathcal{S} = \#\mathcal{C} \cdot \#I \geq 1$.

Proof. (a). By Proposition 5.7.3 (ii), ρ_v is isomorphic to the transpose of the Schur functor of the column count of v; the Schur module of GL_k is formed from the columns of height $t < k$, and $\mathrm{cols}_v(t) = 0$ for $1 < t < k$ (Definition 2.3.4), so its diagram is one row of length $\mathrm{cols}_v(1)$, the one-row weight λ_v (as in Lemma 5.6.8): there is an invertible matrix B_v, with rows indexed by the Weyl coordinates and columns by I_v, such that $B_v\rho_v(h) = \rho_{\lambda_v}(h^{\mathsf{T}})^{\mathsf{T}}B_v$ for all $h \in \mathrm{U}(k)$. Put $A_v(x) = x^{\mathsf{T}}B_v$ for a coordinate column vector x. Then $A_v(\rho_{\lambda_v}(h^{\mathsf{T}})x) = x^{\mathsf{T}}\rho_{\lambda_v}(h^{\mathsf{T}})^{\mathsf{T}}B_v = x^{\mathsf{T}}B_v\rho_v(h) = A_v(x)\rho_v(h)$, and A_v is bijective because B_v is invertible.

Translation by u_0. The idele $u_0 = (a/\bar{a})^{\lceil m/2\rceil}$ satisfies $u_0\bar{u}_0 = 1$, so the scalar matrix $u_0 \cdot 1_k$ lies in $\mathrm{U}(X_0)(\mathbb{A}_f)$ and is central there. Hence $R(u_0)\varphi = \varphi(\,\cdot\, u_0)$ maps $\mathcal{S}$ to itself: for $\gamma \in \mathrm{U}(X_0)(E^+)$ and $\omega \in K_0$, $\varphi(\gamma g\omega u_0) = \varphi(\gamma g u_0 \omega) = \gamma \cdot \varphi(g u_0)$. It is invertible, with inverse $R(u_0^{-1})$.

Evaluation at the classes. Put $\Gamma = \mathrm{U}(X_0)(E^+)$ and $G_f = \mathrm{U}(X_0)(\mathbb{A}_f)$. The map $\mathrm{ev} \colon \mathcal{S} \to W_\lambda^{\mathcal{C}}$, $\varphi \mapsto (\varphi(h_c))_c$, is linear. It is injective: every $g \in G_f$ is $\gamma h_c \omega$ with $\gamma \in \Gamma$, $\omega \in K_0$, and then $\varphi(g) = \gamma \cdot \varphi(h_c)$. It is surjective: given $(w_c)_c$, define $\varphi(\gamma h_c \omega) = \gamma \cdot w_c$; if $\gamma h_c \omega = \gamma' h_c \omega'$ then $\gamma'^{-1}\gamma = h_c\omega'\omega^{-1}h_c^{-1} \in \Gamma \cap h_c K_0 h_c^{-1} = \{1\}$ by Lemma 5.6.6, so φ is well defined, and it satisfies $\varphi(\gamma'' g \omega'') = \gamma'' \cdot \varphi(g)$ by construction.

(b). $\theta = (A^{\mathcal{C}}) \circ \mathrm{ev} \circ R(u_0)$ followed by the identification $(\mathbb{C}^I)^{\mathcal{C}} = \mathbb{C}^{\mathcal{C}\times I}$, a composition of linear isomorphisms. So $\dim \mathcal{S} = \#\mathcal{C} \cdot \#I$; $\mathcal{C}$ is nonempty and finite by Lemma 5.6.6, and each I_v is nonempty (a Bessel system has a nonempty index set), so $\dim \mathcal{S} \geq 1$. □

Definition 5.6.15. Let D be admissible data, $\mathfrak{S}$ a filling datum over E whose space V is Y_0 (the same basis vectors and hermitian form, so that $\mathrm{U}(Y_0) = G$ with the same rational points), with its classes $\mathcal{C}$, representatives h_c, Bessel coordinates I, Bessel functions W^{full} and the cutoff datum fixed with it, and let $\eta \in G(\mathbb{A}_f)$; the objects below depend on $(\mathfrak{S}, \eta)$, which is not written. Regard $\mathrm{U}(Y_0)(\mathbb{A}) = G(\mathbb{A}) = G(F_\infty) \times G(\mathbb{A}_f)$ as a group of operators, $G(F_\infty) = \prod_{v\in\Phi} \mathrm{U}(Y_0 \otimes_{E,v} \mathbb{C})$, matrices being taken in the basis of Y_0. Let s be the signs of Y_0, read on the basis $e_1, \dots, e_\varrho$, the basis of X_0, z, $e_1^*, \dots, e_\varrho^*$, and let p be a sign frame for s. For $v \in \Phi$ let B_v be the diagonal matrix of the basis scales bs_v and P_v the matrix of the frame of Y_0 at v.

Class coordinates. The **class coordinates** of $x \in G(\mathbb{A})$ for p are

$$\mathbf{y}_p(x) = \big(B_v^{-1} x_v P_v p^{-1}\big)_{v\in\Phi}. \tag{5.96}$$

Class-cut Bessel functions. For $a = (c, i) \in \mathcal{C} \times I$ the **class-cut Bessel function** is the function on $G(\mathbb{A})$

$$f_{a,p}(x) = \beta_c(\mathbf{y}_p(x))\, W_i^{\mathrm{full}}(\mathbf{y}_p(x))\, \mathbf{1}\big[x_f \in \eta^{-1} h_c \mathrm{K}_M\big]$$

when $\mathbf{y}_p(x) \in G_\infty$, and $f_{a,p}(x) = 0$ otherwise.

Class Poincaré series. The **class Poincaré series** of a is $\Pi_{a,p}(x) = \sum_{\gamma\in G(E)} f_{a,p}(\gamma x)$ for those $x \in G(\mathbb{A})$ at which the sum converges absolutely, and $\Pi_{a,p}(x) = 0$ at the other x.

Where p is fixed by the statement, it is not written: $\mathbf{y}$, f_a, Π_a.

Let D be admissible data. From here on, the class coordinates $\mathbf{y}_p$, the class-cut Bessel functions $f_{a,p}$ and the class Poincaré series $\Pi_{a,p}$ of D are those of the pair $(\mathfrak{S}_D, \eta)$, where $\mathfrak{S}_D$ is the filling datum and $\eta = \iota(u_0^{-1} 1_{X_0}) \in G(\mathbb{A}_f)$ the element of Proposition 5.6.4, (ii); its space is Y_0, as that statement says.

The scaled Poincaré map has amplitude blocks (Proposition 5.6.16), by the classical unfolding of a Poincaré series [234]. The Poincaré map represents Bessel periods [96, 153, 2], and in the coordinates of Proposition 5.6.4 it is described by class Poincaré series, sums over the rational points of a Bessel function at the infinite places (Lemma 5.6.18). The Petersson product of a translate of one such series with another unfolds into a sum of amplitudes over terms: the rational points that are integral between the two class representatives and reduce to the given level element, each weighted by an integral over G_∞ of one Bessel function against a translate of the other. From here on the second moment is a statement about sums of amplitudes over terms. For the convergence of Poincaré series on semisimple groups see [120, 38]; the same unfolding underlies the Rankin–Selberg method [148].

Proposition 5.6.16. *Let D be admissible data, and let Π_a ($a \in \mathcal{C} \times I$) be the class Poincaré series of Lemma 5.6.18, formed with the sign frame p of the Bessel systems of $\mathfrak{S}_D$, and suppose that $\mathcal{P}\varphi = \sum_{a \in \mathcal{C}\times I} \theta(\varphi)_a \, \Pi_a$ for every $\varphi \in \mathcal{S}$. Fix a Haar measure dg_f on G_f and put $V_M = \mathrm{vol}(\mathrm{K}_M)$; let dg_∞ be the Haar measure on $G_\infty^{Y_0}$ whose image under $g \mapsto \mathbf{y}((P_v g_v P_v^{-1})_v)$ is dy, and let the Petersson product on $\mathcal{A}_M(Y_0)$ be formed with the quotient of $dg_\infty \otimes dg_f$ by the counting measure on Γ_{Y_0}. Let ρ be the action of $\bar{G}$ on $\mathcal{A}_M(Y_0)$, which is unitary, and $c_{\mathcal{P}} = (V_M \nu)^{-1/2}$. Then:*

(i) (Petersson unfolding) for $k \in \mathrm{K}_f$ with reductions $s \in \bar{G}$ at the chosen places and $a = (c, i)$, $a' = (c', i') \in \mathcal{C} \times I$, the series defining $A^{\mathrm{full}}_{G(E)}(c, s, c')$ converges absolutely and

$$\langle \rho(k)\Pi_{a'}, \Pi_a \rangle = V_M \, A^{\mathrm{full}}_{G(E)}(c, s, c')_{i,i'};$$

(ii) the scaled Poincaré map $L = c_{\mathcal{P}}\, \mathcal{P} \circ \theta^{-1} \colon \mathbb{C}^{\mathcal{C}\times I} \to \mathcal{A}_M(Y_0)$ has amplitude blocks for ρ (TA6): $\langle \rho(s) L e_y, L e_x \rangle = \nu^{-1} A^{\mathrm{full}}_{G(E)}(c_x, s, c_y)_{i_x, i_y}$ for all $s \in \bar{G}$ and $x = (c_x, i_x)$, $y = (c_y, i_y)$; equivalently the (c, c') block of $L^* \rho(s) L$ is $\nu^{-1} \overline{A^{\mathrm{full}}_{G(E)}(c, s, c')}$.

Proof. The measure dg_∞ exists and is unique: $g \mapsto \mathbf{y}((P_v g_v P_v^{-1})_v) = t \cdot (p g_v p^{-1})_v$ with $t = (B_v^{-1} P_v p^{-1})_v \in G_\infty$ is a group isomorphism $g \mapsto (p g_v p^{-1})_v$ followed by a left translation, so it carries Haar measures to Haar measures. Work on $G(\mathbb{A}) = G(F_\infty) \times G(\mathbb{A}_f)$ (operators), with $\Gamma = G(E)$ and the Haar measure $\mu = \mu_\infty \otimes \mu_f$ that corresponds through the frames to $dg_\infty \otimes dg_f$; then $\mu_f(\mathrm{K}_M) = V_M$ (conjugation by the finite frame carries the principal level of (HO1) onto the operator group K_M), and $\int_{G(F_\infty)} \phi(\mathbf{y}(x))\, d\mu_\infty(x) = \int_{G_\infty} \phi \, dy$ for $\phi \geq 0$ measurable, by the choice of dg_∞. Γ is a discrete subgroup of the second countable locally compact group $G(\mathbb{A})$ (as E is discrete in $\mathbb{A}_E$), so by Lemma 5.6.20 it has a Borel fundamental set F_Γ, and the Petersson product is $\langle F, F' \rangle = \int_{F_\Gamma} \overline{F} F' \, d\mu$.

Step 1: the finite condition. Let k be as in (i), $\tilde{k} = \eta^{-1} k \eta$ (operators), so $\tilde{k} \in K$ and $\mathrm{red}(\tilde{k}) = s$ by Proposition 5.6.4 (ii). Let $x_f \in \Omega_c$ and $\gamma \in \Gamma$. Then $\gamma^{-1} x_f k \in \Omega_{c'}$ if and only if γ is a term over (c, s, c'). Indeed, $\mathrm{K}_M = \eta \mathrm{K}(M) \eta^{-1}$ and η commutes with $h_c \in \iota(\mathrm{U}(X_0)(\mathbb{A}_f))$ (Proposition 5.6.4 (ii)), so $\Omega_c = \eta^{-1} h_c \eta \mathrm{K}(M) \eta^{-1} = h_c \mathrm{K}(M) \eta^{-1}$ and $x_f = h_c m \eta^{-1}$ with $m \in \mathrm{K}(M)$, and with $t = h_c^{-1} \gamma h_{c'}$,

$$\gamma^{-1} x_f k \in h_{c'} \mathrm{K}(M) \eta^{-1} \iff t^{-1} m \tilde{k} \in \mathrm{K}(M) \iff t \in m \tilde{k}\, \mathrm{K}(M) = \{t' \in K : \mathrm{red}(t') = s\},$$

the last equality because $\mathrm{K}(M)$ is the kernel of $\mathrm{red} \colon K \to \bar{G}$ and $\mathrm{red}(m\tilde{k}) = s$.

Step 2: the series at the class point. Let $x \in G(F_\infty)$. Since $B_v^{-1}\gamma^{-1}B_v = (\gamma^{-1})_v$ by (5.5), $\mathbf{y}(\gamma^{-1}x) = \gamma^{-1}\mathbf{y}(x)$, and right translation by k does not change the archimedean part. As $\eta^{-1}h_c \in \Omega_c$, Step 1 says that the finite factor $\mathbf{1}[\gamma^{-1}\eta^{-1}h_c k \in \Omega_{c'}]$ of $f_{a'}(\gamma^{-1}(x,\eta^{-1}h_c)k)$ is 1 if γ is a term over (c,s,c') and 0 otherwise. Summing over $\gamma^{-1} \in \Gamma$ (the series converges absolutely by Lemma 5.6.18 (i)),

$$\Pi_{a'}((x,\eta^{-1}h_c)k) = \sum_{\gamma \text{ over } (c,s,c')} \beta_{c'}(\gamma^{-1}\mathbf{y}(x))\, W_{i'}^{\mathrm{full}}(\gamma^{-1}\mathbf{y}(x)).$$

Step 3: folding, proof of (i). The function $z(x) = \overline{\Pi_{a'}(xk)} = \overline{(\rho(k)\Pi_{a'})(x)}$ on $G(\mathbb{A})$ is bounded and Borel by Lemma 5.6.18 (i), left Γ-invariant, and right K_M-invariant: K_M is the kernel of the reduction $\mathrm{K}_f \to \bar{G}$ ((HO1)), hence normalised by k, and for $k' \in \mathrm{K}_M$, $\Pi_{a'}(xk'k) = \Pi_{a'}(xk\,(k^{-1}k'k)) = \Pi_{a'}(xk)$ by the right K_M-invariance of $\Pi_{a'}$ (Lemma 5.6.18 (ii)). By Lemma 5.6.17 (ii) for z and a, Step 2, the transport of μ_∞ to dy and the reality of the cutoffs,

$$\langle \rho(k)\Pi_{a'}, \Pi_a\rangle = \int_{F_\Gamma} z\,\Pi_a\, d\mu = V_M \int_{G_\infty} \sum_{\gamma \text{ over } (c,s,c')} \beta_c(y)\beta_{c'}(\gamma^{-1}y)\, W_i^{\mathrm{full}}(y)\, \overline{W_{i'}^{\mathrm{full}}(\gamma^{-1}y)}\, dy.$$

The integral over G_∞ of the absolute value of the term of γ is at most the amplitude majorant $M_\gamma(c,c')$ of (TA2) (an entry of $W(y) \otimes \overline{W(\gamma^{-1}y)}$ is at most its Hilbert–Schmidt norm), and $\sum_{\gamma \text{ over } (c,s,c')} M_\gamma(c,c') < \infty$ by Lemma 5.2.29 (ii) (with $T = G(E)$, $F_v = W_v$). So the series $\sum_\gamma A_\gamma(c,c')$ over the terms over (c,s,c') converges absolutely, $A^{\mathrm{full}}_{G(E)}(c,s,c')$ is defined, and by Fubini's theorem for the product of counting measure and dy the right side is $V_M \sum_\gamma A_\gamma(c,c')_{i,i'} = V_M\, A^{\mathrm{full}}_{G(E)}(c,s,c')_{i,i'}$.

Step 4: proof of (ii). By Proposition 5.8.4 (with $V = V' = Y_0$, using its (iii)) $\mathcal{A}_M(Y_0)$ is finite-dimensional; the Petersson product is positive definite on it, since a continuous F with $\int_{F_\Gamma} |F|^2 d\mu = 0$ vanishes μ-almost everywhere on $\Gamma F_\Gamma = G(\mathbb{A})$, hence everywhere; and ρ is unitary: for $k \in G(\mathbb{A}_f)$, $F_\Gamma k$ is again a Borel fundamental set and μ is right invariant ($G(\mathbb{A})$ is unimodular), so $\langle \rho(k)F, \rho(k)F'\rangle = \int_{F_\Gamma k} \overline{F}F'\, d\mu$; and for two Borel fundamental sets F_1, F_2 and an integrable Γ-invariant φ on them, $F_1 = \bigsqcup_{\gamma\in\Gamma}(F_1 \cap \gamma F_2)$ and $F_2 = \bigsqcup_\gamma(\gamma^{-1}F_1 \cap F_2)$ (left multiplication is free), so $\int_{F_1} \varphi\, d\mu = \sum_\gamma \int_{\gamma^{-1}F_1 \cap F_2} \varphi\, d\mu = \int_{F_2} \varphi\, d\mu$ by left invariance of μ. Since θ is a linear isomorphism (Proposition 5.6.4 (iii)), the hypothesis on $\mathcal{P}$ gives $\mathcal{P}\theta^{-1}e_a = \sum_{b\in\mathcal{C}\times I}(e_a)_b\,\Pi_b = \Pi_a$ for $a \in \mathcal{C}\times I$, so $Le_a = c_{\mathcal{P}}\Pi_a$, where $V_M > 0$ and $\nu > 0$ ($|\mathrm{rs}_v| \geq 1$ by (FA4), so $\Lambda_{\mathrm{fr}} > 0$). For $s \in \bar{G}$ choose $k \in \mathrm{K}_f$ with reductions s (reduction $\mathrm{K}_f \to \bar{G}$ is onto by (HO1)); then $\rho(s) = \rho(k)$ and by (i)

$$\langle \rho(s)Le_y, Le_x\rangle = c_{\mathcal{P}}^2\,\langle \rho(k)\Pi_y, \Pi_x\rangle = \frac{V_M}{V_M\nu} A^{\mathrm{full}}_{G(E)}(c_x, s, c_y)_{i_x,i_y},$$

which is (TA6). Finally $(L^*\rho(s)L)_{x,y} = \langle Le_x, \rho(s)Le_y\rangle = \overline{\langle \rho(s)Le_y, Le_x\rangle}$ gives the block form. □

Lemma 5.6.17. *Let D be admissible data, $G = \mathrm{U}(Y_0)$, $G(\mathbb{A}) = G(F_\infty) \times G(\mathbb{A}_f)$ as a group of operators, $\Gamma = G(E)$, and let p be a matrix with the properties required of it in Lemma 5.6.18, with f_a, Π_a formed for it. Let $\mu = \mu_\infty \otimes \mu_f$ be a left Haar measure of $G(\mathbb{A})$ with μ_∞, μ_f left Haar measures of $G(F_\infty)$, $G(\mathbb{A}_f)$, put $V_M = \mu_f(\mathrm{K}_M)$, and let $F_\Gamma \subseteq G(\mathbb{A})$ be a Borel set meeting every orbit Γx in exactly one point. Let $z\colon G(\mathbb{A}) \to \mathbb{C}$ be a bounded Borel function with $z(\gamma x) = z(x)$ and $z(xk) = z(x)$ for $\gamma \in \Gamma$, $k \in \mathrm{K}_M$, $x \in G(\mathbb{A})$. Then for every $a = (c,i) \in \mathcal{C}\times I$:*

(i) *f_a is integrable on $G(\mathbb{A})$, $z\,\Pi_a$ is integrable on F_Γ, and the function*

$$x \mapsto \beta_c(\mathbf{y}(x))\, W_i^{\mathrm{full}}(\mathbf{y}(x))\, z(x, \eta^{-1} h_c)$$

is integrable on $G(F_\infty)$, where $(x, \eta^{-1}h_c)$ is the element with archimedean component x and finite component $\eta^{-1}h_c$;

(ii)

$$\int_{F_\Gamma} z\,\Pi_a\, d\mu = \int_{G(\mathbb{A})} z\, f_a\, d\mu = V_M \int_{G(F_\infty)} \beta_c(\mathbf{y}(x))\, W_i^{\mathrm{full}}(\mathbf{y}(x))\, z(x, \eta^{-1}h_c)\, d\mu_\infty(x).$$

Proof. Write $\Omega_c = \eta^{-1} h_c \mathrm{K}_M$, an open subset of $G(\mathbb{A}_f)$, so that

$$f_a(x) = \beta_c(\mathbf{y}(x))\, W_i^{\mathrm{full}}(\mathbf{y}(x))\, \mathbf{1}_{\Omega_c}(x_f)$$

for $x = (x_\infty, x_f)$, with $\mathbf{y}(x)$ depending only on x_∞ (Definition 5.6.15); f_a is Borel, as β_c is Borel, W_i^{full} and $\mathbf{y}$ are continuous and Ω_c is open. By Lemma 5.6.18 (i) there is a constant C with $\sum_{\gamma\in\Gamma} |f_a(\gamma x)| \le C$ for every $x \in G(\mathbb{A})$; in particular $\Pi_a(x) = \sum_{\gamma\in\Gamma} f_a(\gamma x)$ converges absolutely at every x, and Π_a is Borel as a pointwise limit of finite sums of Borel functions (Γ is countable).

Step 1: the fundamental set. Γ is discrete in $G(\mathbb{A})$, because E is discrete in $\mathbb{A}_E$, and it acts freely by left multiplication; as F_Γ meets every orbit in exactly one point, $G(\mathbb{A}) = \bigsqcup_{\gamma\in\Gamma} \gamma F_\Gamma$. By left invariance of μ, $\mu(F_\Gamma)$ is the volume of $\Gamma\backslash G(\mathbb{A})$ for the quotient of μ by counting measure, which is finite by [37], for the hermitian space Y_0 over the CM extension E/E^+. Hence, for every Borel $\phi \ge 0$ on $G(\mathbb{A})$, by left invariance and monotone convergence,

$$\int_{G(\mathbb{A})} \phi\, d\mu = \sum_{\gamma\in\Gamma} \int_{F_\Gamma} \phi(\gamma x)\, d\mu(x) = \int_{F_\Gamma} \sum_{\gamma\in\Gamma} \phi(\gamma x)\, d\mu(x). \tag{5.97}$$

Step 2: integrability. With $\phi = |f_a|$, (5.97) gives $\int_{G(\mathbb{A})} |f_a|\, d\mu \le C\mu(F_\Gamma) < \infty$. Let $\|z\|_\infty$ be a bound of $|z|$. Then $|z\Pi_a| \le \|z\|_\infty C$ on F_Γ, which has finite measure, so $z\Pi_a$ is integrable on F_Γ; and $|z f_a| \le \|z\|_\infty |f_a|$ is integrable on $G(\mathbb{A})$.

Step 3: unfolding. By the invariance of z, $z(x)\Pi_a(x) = \sum_\gamma z(\gamma x) f_a(\gamma x)$ for every x, and $\sum_\gamma |z(\gamma x) f_a(\gamma x)| \le \|z\|_\infty C$. By the dominated convergence theorem on F_Γ (the partial sums are dominated by the constant $\|z\|_\infty C$), then left invariance and the disjoint decomposition of Step 1 (that is, (5.97) applied to the positive and negative parts of the real and imaginary parts of the integrable function $z f_a$),

$$\int_{F_\Gamma} z\,\Pi_a\, d\mu = \sum_{\gamma\in\Gamma} \int_{F_\Gamma} z(\gamma x) f_a(\gamma x)\, d\mu(x) = \sum_{\gamma\in\Gamma} \int_{\gamma F_\Gamma} z\, f_a\, d\mu = \int_{G(\mathbb{A})} z\, f_a\, d\mu,$$

the last series converging absolutely by (5.97) for $\phi = |z f_a|$.

Step 4: the finite variable. If $x_f \in \Omega_c$, then $x_f = \eta^{-1} h_c k$ with $k \in \mathrm{K}_M$, so $z(x_\infty, x_f) = z((x_\infty, \eta^{-1}h_c)k) = z(x_\infty, \eta^{-1}h_c)$. Hence $z(x) f_a(x) = A(x_\infty)\, \mathbf{1}_{\Omega_c}(x_f)$ with

$$A(x_\infty) = \beta_c(\mathbf{y}(x_\infty))\, W_i^{\mathrm{full}}(\mathbf{y}(x_\infty))\, z(x_\infty, \eta^{-1}h_c),$$

a Borel function on $G(F_\infty)$ (a section of the Borel function z is Borel). By left invariance $\mu_f(\Omega_c) = \mu_f(\mathrm{K}_M) = V_M$, and $0 < V_M < \infty$ as K_M is open and compact. By Tonelli's theorem for $\mu_\infty \otimes \mu_f$, $\int |z f_a|\, d\mu = V_M \int_{G(F_\infty)} |A|\, d\mu_\infty$, which is finite by Step 2, so A is integrable; and by Fubini's theorem $\int_{G(\mathbb{A})} z f_a\, d\mu = V_M \int_{G(F_\infty)} A\, d\mu_\infty$. With Step 3 this proves (ii), and (i) is Steps 2 and 4. □

Lemma 5.6.18. *Let D be admissible data and $\mathfrak{S}_D$ the filling datum of Proposition 5.6.4, with its fixed cutoff datum. Let s be the signs of Y_0 and p a sign frame for s such that for every $v \in \Phi$ and $y \in G_1$*

$$\int_{\mathrm{U}(\mathrm{diag}\, s)} d_{\varsigma(v)}\, \overline{c_{\varsigma(v)}(g)}\, W_v(y\, pgp^{-1})\, dg = W_v(y), \tag{5.98}$$

the formal degree being taken for the Haar measure dg. Write $\mathbf{y} = \mathbf{y}_p$, $f_a = f_{a,p}$ and $\Pi_a = \Pi_{a,p}$ for $a \in \mathcal{C} \times I$. Then $\mathbf{y}(x) \in G_\infty$ for every $x \in G(\mathbb{A})$, so that for $a = (c, i)$

$$f_a(x) = \beta_c(\mathbf{y}(x))\, W_i^{\mathrm{full}}(\mathbf{y}(x))\, \mathbf{1}[x_f \in \eta^{-1} h_c \mathrm{K}_M] \qquad (x \in G(\mathbb{A})),$$

and:

(i) for every $x \in G(\mathbb{A})$,

$$\sum_{\gamma \in G(E)} |f_a(\gamma x)| \le 2\, b^{12n^3 rf} \prod_{v \in \Phi} a_v;$$

in particular $\Pi_a(x) = \sum_{\gamma \in G(E)} f_a(\gamma x)$, the series converging absolutely, for every x;

(ii) the class Poincaré series Π_a, read on $G_\infty^{Y_0} \times G_f$ through the frames of Y_0 ($G_\infty^{Y_0} = \prod_v \mathrm{U}(\mathrm{diag}\, s)$), lies in $\mathcal{A}_M(Y_0)$.

Proof. By Proposition 5.6.4 (i), $\mathfrak{S}_D$ is admissible and E carries Weil data. Write $\Omega_c = \eta^{-1} h_c \mathrm{K}_M \subseteq G(\mathbb{A}_f)$, an open set.

Step 1: properties of $\mathbf{y}$ *and* Ω_c. Let H_v be the Gram matrix of $Y_0 \otimes_{E,v} \mathbb{C}$. By (RD1), $B_v^* H_v B_v = J_1$; by the frame condition of (VP3) of Definition 3.2.2 (as in (FT1)), $P_v^* H_v P_v = \mathrm{diag}(s)$; and $p^{-*}\mathrm{diag}(s) p^{-1} = J_1$. Hence for $x_v \in \mathrm{U}(H_v)$ the matrix $z = B_v^{-1} x_v P_v p^{-1}$ satisfies $z^* J_1 z = p^{-*} P_v^* x_v^* H_v x_v P_v p^{-1} = J_1$, so $\mathbf{y}(x) \in G_\infty$; and $\mathbf{y}$ is a homeomorphism $G(F_\infty) \to G_\infty$, being a product of left and right multiplications by fixed invertible matrices. Moreover:

(a) $\mathbf{y}(\gamma x) = \gamma_\infty \mathbf{y}(x)$ for $\gamma \in G(E)$, since $B_v^{-1} \gamma x_v P_v p^{-1} = (B_v^{-1} \gamma B_v)(B_v^{-1} x_v P_v p^{-1})$ and $B_v^{-1} \gamma B_v = \gamma_v$ by (5.5);

(b) $\mathbf{y}(x\,(P_v g_v P_v^{-1})_v) = \big(\mathbf{y}(x)_v\, pg_v p^{-1}\big)_v$ for $(g_v) \in \prod_v \mathrm{U}(\mathrm{diag}\, s)$;

(c) if $\gamma, \delta \in G(E)$ and $\gamma x_f, \delta x_f \in \Omega_c$, then $\delta\gamma^{-1}$ is a term over $(c, 1, c)$: writing $\gamma x_f = \eta^{-1} h_c k_1$, $\delta x_f = \eta^{-1} h_c k_2$ with $k_2 k_1^{-1} = \eta m \eta^{-1}$, $m \in \mathrm{K}(M)$ (Proposition 5.6.4 (ii): $\mathrm{K}_M = \eta \mathrm{K}(M) \eta^{-1}$), and since η commutes with h_c, which lies in $\iota(\mathrm{U}(X_0)(\mathbb{A}_f))$, we get $\delta\gamma^{-1} = \eta^{-1} h_c \eta m \eta^{-1} h_c^{-1} \eta = h_c m h_c^{-1}$, so $h_c^{-1}(\delta\gamma^{-1}) h_c = m \in \mathrm{K}(M) \subseteq K$ has reduction 1;

(d) $u\,\Omega_c = \Omega_c$ for $u \in \mathsf{U}_c(M)$: as above $\Omega_c = h_c \mathrm{K}(M) \eta^{-1}$, and $u h_c \mathrm{K}(M) = h_c (h_c^{-1} u h_c) \mathrm{K}(M) = h_c \mathrm{K}(M)$;

(e) $\Omega_c k = \Omega_c$ for $k \in \mathrm{K}_M$.

Since $\mathbf{y}(x) \in G_\infty$ for every x, the case $\mathbf{y}_p(x) \notin G_\infty$ of the definition of $f_{a,p}$ does not occur, and f_a is given by the displayed formula of the statement. The function f_a is Borel: β_c is Borel, W_i^{full} and $\mathbf{y}$ are continuous, Ω_c is open.

Step 2: proof of (i). Fix x. If no $\delta \in G(E)$ has $\delta x_f \in \Omega_c$, every term vanishes. Otherwise fix such a δ and put $y_0 = \mathbf{y}(\delta x)$. If $f_a(\gamma x) \neq 0$ then $\gamma x_f \in \Omega_c$, so $\tau = \delta\gamma^{-1}$ is a term over $(c, 1, c)$ by (c), and $\gamma = \tau^{-1}\delta$; the map $\gamma \mapsto \tau$ is injective. By (a), $\mathbf{y}(\gamma x) = \tau^{-1} y_0$, and $|W_i^{\mathrm{full}}(z)| = \prod_v |W_{v,i_v}(z_v)| \le \prod_v \|W_v(z_v)\|$. Hence

$$\sum_\gamma |f_a(\gamma x)| \le \sum_{\tau \text{ over } (c,1,c)} \beta_c(\tau^{-1} y_0) \prod_v \|W_v((\tau^{-1} y_0)_v)\|,$$

which is at most $2b^{12n^3rf}\prod_v a_v$ by Lemma 5.2.29 (i) with $s=1$, $T=G(E)$ and $F_v=W_v$ (the right side is the part $d=c$ of its sum). So the series $\sum_\gamma f_a(\gamma x)$ converges absolutely at every x, and by the definition of $\Pi_{a,p}$ its value is $\Pi_a(x)$.

Step 3: coset sums. Let $u\in \mathsf{U}_c(M)$ and $x\in G(\mathbb{A})$. By (a), (d) and Lemma 5.2.42,

$$f_a(ux)=\beta_c(u\cdot \mathbf{y}(x))\,W_i^{\mathrm{full}}(\mathbf{y}(x))\,\mathbf{1}[x_f\in\Omega_c].$$

By (RD6) the weights $a(u)=\beta_c^0(u\cdot z)$ ($z\in G_\infty$) have sum 1 over every right coset $\mathsf{U}_c^0(M)u'$, so by Lemma 5.2.21 and (RD5) $\sum_{u\in\mathsf{U}_c(M)}\beta_c(u\cdot z)=\sum_u a(u)\mathbf{1}_{\mathrm{red}}(u\cdot z)=1$. Let $D\subseteq G(E)$ contain one element of every right coset $\mathsf{U}_c(M)\delta$. By (i) the series $\Pi_a(x)$ converges absolutely, so it may be summed coset by coset:

$$\Pi_a(x)=\sum_{\delta\in D}F_\delta(x),\qquad F_\delta(x)=\sum_{u\in\mathsf{U}_c(M)}f_a(u\delta x)=W_i^{\mathrm{full}}(\mathbf{y}(\delta x))\,\mathbf{1}[\delta x_f\in\Omega_c],\tag{5.99}$$

and $\sum_{\delta\in D}|F_\delta(x)|\le\sum_\gamma|f_a(\gamma x)|$ is bounded by the constant C of (i).

Step 4: the type kernel is integrable. By Proposition 5.8.4 (iii), ς has gap Δ for Y_0 in the sense of (HO2). The entries of $\varsigma(v)$ are integers, so $l=2\varsigma(v)$ has even entries, which are congruent to $r-1$ modulo 2 because r is odd (Definition 2.3.4: $r=3^{e'+1}$); they are distinct, and $|l_i-l_j|\ge 2\Delta$ whenever $s_i\ne s_j$. As $\Delta=2r^8\ge 500r^3$, l has gap $500r^3$ for s in the sense of (DZ1). By Lemma 5.6.21 with $N=r$, $B=0$, the type kernel $e=e_\varsigma$, $e(g)=\prod_v e_v(g_v)$ with $e_v=d_{\varsigma(v)}\overline{c_{\varsigma(v)}}$, is a cusp kernel of order 0, in particular continuous and integrable with $e(1)\ne 0$. Then each e_v is continuous with $e_v(1)\ne 0$, so $0<\int|e_v|$, and $\prod_v\int|e_v|=\int|e|<\infty$ gives $e_v\in L^1$.

Step 5: proof of (ii). Read through the frames (the coordinate map of (VP3) of Definition 3.2.2), x corresponds to (g_∞,g_f) with $x_v=P_vg_vP_v^{-1}$, $x_f=Fg_fF^{-1}$ (F the finite frame), Γ_{Y_0} to $G(E)$, right translation by $(g,1)$ to $x\mapsto x(P_vg_vP_v^{-1})_v$, and K_M of (HO1) to the operator group K_M. *Invariance and bound.* For $\gamma'\in G(E)$, $\Pi_a(\gamma' x)=\sum_\gamma f_a(\gamma\gamma' x)=\Pi_a(x)$ by reindexing the absolutely convergent series; $f_a(xk)=f_a(x)$ for $k\in\mathrm{K}_M$ by (e), so Π_a is right K_M-invariant; $|\Pi_a|\le C$ by (i); and Π_a is Borel as a pointwise limit of Borel partial sums. *Type.* For $\delta\in D$ and $g=(g_v)_v$, by (b) the finite part is unchanged and

$$F_\delta(x(P_vg_vP_v^{-1})_v)=\mathbf{1}[\delta x_f\in\Omega_c]\prod_v W_{v,i_v}(\mathbf{y}(\delta x)_v\,pg_vp^{-1}).$$

The functions W_v are bounded and each e_v is integrable, so by Fubini's theorem and (5.98) at each place, $\int_{G_\infty^{Y_0}}e(g)F_\delta(x(P_vg_vP_v^{-1})_v)\,dg=F_\delta(x)$. As $\sum_\delta|F_\delta|\le C$ everywhere (Step 3) and e is integrable, dominated convergence and (5.99) give

$$\int_{G_\infty^{Y_0}}e(g)\,\Pi_a(x(P_vg_vP_v^{-1})_v)\,dg=\sum_{\delta\in D}F_\delta(x)=\Pi_a(x).$$

Continuity. In frame coordinates this reads

$$\Pi_a(h,g_f)=\int e(g)\Pi_a(hg,g_f)\,dg=\int e(h^{-1}g')\,\Pi_a(g',g_f)\,dg'$$

for $h\in G_\infty^{Y_0}$, by left invariance of dg. Since $e\in L^1$, $h\mapsto e(h^{-1}\cdot)$ is continuous into $L^1(G_\infty^{Y_0})$ (continuity of translation in L^1 of a locally compact group), and $|\Pi_a|\le C$, so $h\mapsto\Pi_a(h,g_f)$ is continuous for each g_f. As Π_a is right invariant under the open subgroup K_M, it is constant in g_f on the open sets $g_f\mathrm{K}_M$, hence continuous on $G_\infty^{Y_0}\times G_f$. Thus Π_a is continuous, bounded, left Γ_{Y_0}-invariant, right K_M-invariant and of type e_ς ((FT2) with e_ς integrable): $\Pi_a\in\mathcal{A}_M(Y_0)$. □

Lemma 5.6.19. *Let $\mathfrak{S}$ be an admissible filling datum. Then:*

(i) a residue domain exists;

(ii) for every class c the map $u \mapsto u_\infty$ is an injective homomorphism of $\mathsf{U}_c^0(M)$ onto a discrete subgroup of G_∞, and there is a Borel set $F_c \subseteq G_\infty$ such that $\beta_c^0 = \mathbf{1}_{F_c}$ satisfies $\sum_{u \in \mathsf{U}_c^0(M)} \beta_c^0(u \cdot y) = 1$ for every $y \in G_\infty$;

(iii) in particular $\mathfrak{S}$ has a cutoff datum.

Proof. Write $\mathbf{v}(x) = (v(x))_{v \in \Phi} \in \mathbb{C}^\Phi$ for $x \in E$, and give $\mathbb{C}^\Phi$ the real inner product $\langle a, a' \rangle = \mathrm{Re} \sum_{v \in \Phi} \bar{a}_v a'_v$, of real dimension $2f$. Since Φ contains exactly one embedding from each pair of complex conjugate embeddings of the CM field E, $\mathbf{v}$ extends to an isomorphism $E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}^\Phi$, so $\Gamma = \mathbf{v}(\mathcal{O}_E)$ is a lattice in $\mathbb{C}^\Phi$; moreover $v(\bar{x}) = \overline{v(x)}$, hence for $x, \eta \in E$

$$\langle \mathbf{v}(x), \mathbf{v}(\eta) \rangle = \mathrm{Re} \sum_{v \in \Phi} v(\bar{x}\eta) = \tfrac{1}{2} \mathrm{Tr}_{E/\mathbb{Q}}(\bar{x}\eta). \tag{5.100}$$

Step 1: covering radius of Γ. Let $\mathcal{O}_E^\vee = \{\eta \in E : \mathrm{Tr}_{E/\mathbb{Q}}(x\eta) \in \mathbb{Z} \text{ for all } x \in \mathcal{O}_E\}$, the inverse different; it is stable under complex conjugation, as $\mathcal{O}_E$ is. By (5.100) (extended by linearity to $E \otimes \mathbb{R}$; a point of $E \otimes \mathbb{R}$ pairing integrally with the $\mathbb{Q}$-basis $\mathcal{O}_E$ of E lies in E) the dual lattice is $\Gamma^* = \mathbf{v}(2\mathcal{O}_E^\vee)$. Let $0 \neq \eta \in \mathcal{O}_E^\vee$. By (FG2) ϖ generates the different $(\mathcal{O}_E^\vee)^{-1}$, so $\varpi\eta$ is a nonzero element of $\mathcal{O}_E$ and $\prod_{v \in \Phi} |v(\varpi\eta)|^2 = |N_{E/\mathbb{Q}}(\varpi\eta)| \geq 1$. Hence $|v(\varpi\eta)| \geq 1$ for some v, and by (FA3) $|v(\eta)| \geq 1/|\varpi|_v \geq 1/b$. So every nonzero $u = \mathbf{v}(2\eta) \in \Gamma^*$ has $\|u\| \geq 2|v(\eta)| \geq 2/b$. Banaszczyk's transference theorem [12] gives: for every $t \in \mathbb{C}^\Phi$ there is $x \in \mathcal{O}_E$ with $\|t - \mathbf{v}(x)\| \leq 2f/(2 \cdot 2/b) = fb/2$, in particular $|t_v - v(x)| \leq fb/2$ for every v.

Step 2: (i). Let $B = \{w \in \mathbb{C}^\Phi : |w_v| \leq b^{26} \ (v \in \Phi)\}$, a compact set, and $L = \mathbf{v}(M\mathcal{O}_E)$, a lattice acting on $\mathbb{C}^\Phi$ by translation. For $w \in \mathbb{C}^\Phi$ apply Step 1 to $t = -w/M$: some $x \in \mathcal{O}_E$ has $|v(x) + w_v/M| \leq fb/2$, so by (FG6) ($M \leq b^{23}$, $f \leq b/1000$)

$$|v(Mx) + w_v| \leq \tfrac{Mfb}{2} \leq \tfrac{b^{25}}{2000} \leq b^{26} \qquad (v \in \Phi),$$

that is, $(w + L) \cap B \neq \emptyset$. As L is discrete and B compact, $(w + L) \cap B$ is finite. Identify $\mathbb{C}^\Phi$ with $\mathbb{R}^{2f}$ by real and imaginary parts in a fixed order and let $\leq_{\mathrm{lex}}$ be the lexicographic order; the set $\{(a, a') : a \leq_{\mathrm{lex}} a'\}$ is a finite Boolean combination of the sets $\{a_i < a'_i\}$, $\{a_i = a'_i\}$, hence Borel. Put

$$D_{\mathrm{res}} = \bigcap_{l \in L} \{w \in B : \ w + l \notin B \text{ or } w \leq_{\mathrm{lex}} w + l\},$$

a countable intersection of Borel sets. For each w, the finite nonempty set $(w + L) \cap B$ has a unique $\leq_{\mathrm{lex}}$-least element, and the points of $D_{\mathrm{res}} \cap (w + L)$ are exactly the least elements of $(w + L) \cap B$; so D_{res} meets $w + L$ in exactly one point. Since $x \mapsto \mathbf{v}(Mx)$ is injective, there is exactly one $x \in \mathcal{O}_E$ with $(v(Mx) + w_v)_v \in D_{\mathrm{res}}$, and $D_{\mathrm{res}} \subseteq B$. So D_{res} is a residue domain (RD3).

Step 3: (ii). By (RD1) $\gamma \mapsto \gamma_\infty$ is a homomorphism, and it is injective since every v is injective and the basis scales are nonzero. Let $u \in \mathsf{U}_c^0(M)$. Then $h_c^{-1} u h_c \in \mathrm{K}(M) \subseteq K$, so u preserves the $\hat{\mathcal{O}}$-lattice $h_c L_0$ of (FG3), hence the $\mathcal{O}_E$-lattice $\Lambda_c = V \cap h_c L_0$ of $V = E^r$. Choose nonzero integers d_1, d_2 with $d_1 \mathcal{O}_E^r \subseteq \Lambda_c \subseteq d_2^{-1} \mathcal{O}_E^r$; then $u e_j \in d_1^{-1} u \Lambda_c = d_1^{-1} \Lambda_c \subseteq (d_1 d_2)^{-1} \mathcal{O}_E^r$ for every standard basis vector e_j, so all matrix entries of u lie in $(d_1 d_2)^{-1} \mathcal{O}_E$. By (5.5) the entries of u_v are $v(u_{ij})$ times nonzero constants depending only on (v, i, j). If u_∞ lies in a compact set $C \subseteq G_\infty$, every $|v(u_{ij})|$ is bounded by a constant depending on C, and since $\mathbf{v}((d_1 d_2)^{-1} \mathcal{O}_E)$ is discrete, only finitely many u qualify. So the image Γ_c of $\mathsf{U}_c^0(M)$ meets every compact set in a finite set, and is a discrete subgroup of $G_\infty = \prod_v G_1$, a closed subgroup of

$\prod_v \mathrm{GL}_r(\mathbb{C})$ whose topology has a countable base. By Lemma 5.6.20 there is a Borel $F_c \subseteq G_\infty$ meeting every orbit $\Gamma_c y$ in exactly one point. For $y \in G_\infty$, $u \mapsto u_\infty y$ is injective on $\mathsf{U}_c^0(M)$, so $\sum_{u \in \mathsf{U}_c^0(M)} \mathbf{1}_{F_c}(u_\infty y) = \#(F_c \cap \Gamma_c y) = 1$; and $\mathbf{1}_{F_c}$ is measurable with values in $[0,1]$.

Step 4: (iii). $G(E)$ is countable; choosing one element of every double coset $\mathsf{U}_c^0(M)\gamma\,\mathsf{U}_{c'}^0(M)$ gives sets $R_{c,c'}$ of representatives. With the residue domain of (i) and the β_c^0 of (ii),

$$(D_{\mathrm{res}}, (\beta_c^0)_c, (R_{c,c'})_{c,c'})$$

is a cutoff datum (RD6). □

Lemma 5.6.20. *Let G be a topological group whose topology has a countable base, and let $\Gamma \leq G$ be a discrete subgroup. Then there is a Borel set $F \subseteq G$ that meets every orbit Γx ($x \in G$) in exactly one point.*

Proof. Since Γ is discrete, there is an open neighbourhood W of 1 in G with $W \cap \Gamma = \{1\}$. By continuity of $(x,y) \mapsto xy^{-1}$ at $(1,1)$ there is an open neighbourhood V of 1 with $VV^{-1} \subseteq W$. The open sets Vg ($g \in G$) cover G; as G has a countable base, it is Lindelöf, so there are $g_1, g_2, \cdots \in G$ (a finite or infinite sequence) with $G = \bigcup_i Vg_i$.

Each orbit meets each Vg_i in at most one point. If $\gamma x = vg_i$ and $\gamma' x = v'g_i$ with $\gamma, \gamma' \in \Gamma$, $v, v' \in V$, then $\gamma'\gamma^{-1} = v'g_i x^{-1} \cdot x g_i^{-1} v^{-1} = v'v^{-1} \in VV^{-1} \cap \Gamma \subseteq W \cap \Gamma = \{1\}$, so $\gamma x = \gamma' x$.

The set. Put

$$F_i = Vg_i \setminus \bigcup_{j<i} \Gamma V g_j, \qquad F = \bigcup_i F_i.$$

Each $\Gamma V g_j = \bigcup_{\gamma \in \Gamma} \gamma V g_j$ is open, so every F_i is Borel, and so is the countable union F.

F meets every orbit exactly once. Let $O = \Gamma x$ and let i be the least index with $O \cap Vg_i \neq \emptyset$ (it exists since the Vg_i cover G); by the previous paragraph $O \cap Vg_i = \{p\}$ for one point p. For $j < i$ we have $O \cap \Gamma V g_j = \emptyset$, because O is Γ-stable and $O \cap Vg_j = \emptyset$; hence $p \in F_i$, and $F_j \cap O \subseteq Vg_j \cap O = \emptyset$. For $j > i$ we have $O = \Gamma p \subseteq \Gamma V g_i$, so $F_j \cap O = \emptyset$. Finally $F_i \cap O \subseteq Vg_i \cap O = \{p\}$. Thus $F \cap O = \{p\}$. □

Lemma 5.6.21. *Let E be a CM field with set of infinite places Φ, let $N \geq 1$, $s \in \{\pm 1\}^N$ and $B \geq 0$ an integer, and let $\lambda = (\lambda_v)_{v \in \Phi}$ be such that $2\lambda_v \in \mathbb{Z}^N$ has gap $500N^3 + B$ for s for every $v \in \Phi$. With the formal degrees taken for the Haar measures dg_v of $G_\infty = \prod_{v \in \Phi} \mathrm{U}(\mathrm{diag}\, s)$, the function*

$$e_\lambda(x) = \prod_{v \in \Phi} d_{\lambda_v}\, \overline{c_{\lambda_v}(x_v)} \qquad (x \in G_\infty)$$

is a cusp kernel of order B on G_∞.

Proof. Write $e_v(g) = d_{\lambda_v} \overline{c_{\lambda_v}(g)}$ on $G_v = \mathrm{U}(\mathrm{diag}\, s)$, so $e_\lambda(x) = \prod_v e_v(x_v)$, and put $h = 500N^3 + B$. A vector with gap h has gap h' for every $h' \leq h$, by the definition (DZ1).

Local facts. By Lemma 5.6.22 (gap $500N^3 \leq h$), each e_v is continuous and integrable on G_v, $e_v(g^{-1}) = \overline{e_v(g)}$, $e_v(1) \neq 0$, $\int e_v(z) e_v(x^{-1}z^{-1}y)\, dz = e_v(x^{-1}) e_v(y)/e_v(1)$, and there is a continuous compactly supported β_v on G_v with $\int \beta_v(x) e_v(x^{-1}g)\, dx = e_v(g)$. By Lemma 5.7.38(1) (gap h), $|c_{\lambda_v}(g)| \leq 1$ and $|c_{\lambda_v}(g)| \leq (2/\|g\|)^c$ for every integer $0 \leq c \leq h$; and $d_{\lambda_v} > 0$, as the inverse of the positive finite number $\int |c_{\lambda_v}|^2$.

e_λ is an idempotent kernel. It is continuous as a finite product of continuous functions of the coordinates, and integrable on the product G_∞ by Tonelli's theorem, $|e_\lambda(x)| = \prod_v |e_v(x_v)|$ being a product of integrable functions of distinct coordinates. Since $x^{-1} = (x_v^{-1})_v$, $e_\lambda(x^{-1}) = \prod_v \overline{e_v(x_v)} = \overline{e_\lambda(x)}$, and $e_\lambda(1) = \prod_v e_v(1) \neq 0$. For $x, y \in G_\infty$ the function $z \mapsto e_\lambda(z) e_\lambda(x^{-1}z^{-1}y)$

is the product over v of $z_v \mapsto e_v(z_v)e_v(x_v^{-1}z_v^{-1}y_v)$, each integrable on G_v (integrable times bounded, $|e_v| \le d_{\lambda_v}$); by Fubini's theorem and the local identity,

$$\int_{G_\infty} e_\lambda(z)e_\lambda(x^{-1}z^{-1}y)\,dz = \prod_v \frac{e_v(x_v^{-1})e_v(y_v)}{e_v(1)} = \frac{e_\lambda(x^{-1})e_\lambda(y)}{e_\lambda(1)}.$$

Decay (EK4)(b). With $c = B \le h$: $|e_\lambda(x)|\,\|x\|_\infty^B = \prod_v d_{\lambda_v}|c_{\lambda_v}(x_v)|\,\|x_v\|^B \le \prod_v d_{\lambda_v} 2^B$.

Moment (EK4)(a). Put $c' = h - B - 1 = 500N^3 - 1$, an integer with $c' > 8N^2$ (as $500N^3 \ge 500N^2 > 8N^2 + 1$). With $c = h$,

$$d_{\lambda_v}|c_{\lambda_v}(g)|\,\|g\|^{B+1} \le d_{\lambda_v}\Big(\frac{2}{\|g\|}\Big)^h \|g\|^{B+1} = 2^{B+1} d_{\lambda_v}\Big(\frac{2}{\|g\|}\Big)^{c'},$$

and $g \mapsto d_{\lambda_v}(2/\|g\|)^{c'}$ is integrable on G_v by Lemma 5.7.38(3), applied with large gap B (that is, gap $500N^3+B$) and $L = \max_i |2\lambda_{v,i}|$. So $|e_\lambda(x)|\,\|x\|_\infty^{B+1} = \prod_v d_{\lambda_v}|c_{\lambda_v}(x_v)|\,\|x_v\|^{B+1}$ is dominated by a product of nonnegative integrable functions of distinct coordinates, hence integrable by Tonelli's theorem.

Bump (EK4)(c). $\beta(x) = \prod_v \beta_v(x_v)$ is continuous with compact support $\prod_v \operatorname{supp}\beta_v$, and for $y \in G_\infty$, by Fubini's theorem (the integrand is bounded with compact support),

$$\int_{G_\infty} \beta(x)e_\lambda(x^{-1}y)\,dx = \prod_v \int_{G_v} \beta_v(x_v)e_v(x_v^{-1}y_v)\,dx_v = \prod_v e_v(y_v) = e_\lambda(y).$$

□

Lemma 5.6.22. *Let $n \ge 1$, $s \in \{\pm 1\}^n$, and let $l \in \mathbb{Z}^n$ have gap $500n^3$ for s; put $\lambda = \frac{1}{2}l$. Fix a Haar measure dg on $\mathrm{U}(\operatorname{diag} s)$, let d_λ be the formal degree for it, and put $e_\lambda(g) = d_\lambda\,\overline{c_\lambda(g)}$. Then:*

(i) e_λ is continuous and integrable, $e_\lambda(g^{-1}) = \overline{e_\lambda(g)}$, $e_\lambda(1) \ne 0$, and for all $x, y \in \mathrm{U}(\operatorname{diag} s)$

$$\int_{\mathrm{U}(\operatorname{diag} s)} e_\lambda(z)\,e_\lambda(x^{-1}z^{-1}y)\,dz = \frac{e_\lambda(x^{-1})\,e_\lambda(y)}{e_\lambda(1)};$$

(ii) there is a continuous compactly supported function β on $\mathrm{U}(\operatorname{diag} s)$ with $\int \beta(x)\,e_\lambda(x^{-1}g)\,dx = e_\lambda(g)$ for all g.

Proof. Write $G = \mathrm{U}(\operatorname{diag} s)$ and $c = c_\lambda$. Since $n \ge 1$, $500n^3 > 8n^2$, so Lemma 5.6.23 applies to l and dg; its $d_\lambda = (\int_G |c|^2 dg)^{-1}$ is the formal degree for dg, a positive real number. By Lemma 5.7.40, $c(g^{-1}) = \overline{c(g)}$.

(i). By Lemma 5.6.23 (i), c is continuous and integrable, hence so is $e_\lambda = d_\lambda \bar{c}$. As d_λ is real, $e_\lambda(g^{-1}) = d_\lambda \overline{c(g^{-1})} = d_\lambda c(g) = \overline{e_\lambda(g)}$, and $e_\lambda(1) = d_\lambda c(1) = d_\lambda \ne 0$, since $c(1) = \|m_{\mu_l}\|^{-2} \int_{K_s} |m_{\mu_l}|^2 dk = 1$ by (3.2), as $\mathrm{bd}_s(1) = 1$ and $m_w(1) = 1$. For $x, y \in G$, using $\overline{c(u)} = c(u^{-1})$ for $u = x^{-1}z^{-1}y$ and Lemma 5.6.23 (ii) with (x, a) replaced by (y^{-1}, x),

$$\int_G e_\lambda(z)\,e_\lambda(x^{-1}z^{-1}y)\,dz = d_\lambda \cdot d_\lambda \int_G c(y^{-1}zx)\,\overline{c(z)}\,dz = d_\lambda\,c(y^{-1})\,c(x) = d_\lambda\,\overline{c(y)}\,\overline{c(x^{-1})},$$

and the right side equals $e_\lambda(x^{-1})e_\lambda(y)/e_\lambda(1) = d_\lambda\overline{c(x^{-1})} \cdot d_\lambda\overline{c(y)}/d_\lambda$.

(ii). Let β be the smooth compactly supported function of Lemma 5.6.23 (iii), so

$$\int_G \beta(x)c(hx)\,dx = c(h) \quad \text{for all } h;$$

it is continuous. For $g \in G$, $e_\lambda(x^{-1}g) = d_\lambda \overline{c(x^{-1}g)} = d_\lambda c(g^{-1}x)$, so

$$\int_G \beta(x)\, e_\lambda(x^{-1}g)\, dx = d_\lambda \int_G \beta(x)\, c(g^{-1}x)\, dx = d_\lambda\, c(g^{-1}) = d_\lambda \overline{c(g)} = e_\lambda(g).$$

□

Lemma 5.6.23. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $h > 8(\#n)^2$ an integer, $l \in \mathbb{Z}^n$ a vector with gap h for ε, $\lambda = \frac{1}{2}l$, and dg a Haar measure on $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$.*

(i) c_λ is continuous, integrable and square integrable on G, and $d_\lambda = \left(\int_G |c_\lambda|^2 dg\right)^{-1}$ is a positive real number.

(ii) For all $x, a \in G$,

$$d_\lambda \int_G c_\lambda(xya)\, \overline{c_\lambda(y)}\, dy = c_\lambda(x)\, c_\lambda(a).$$

(iii) There is a smooth compactly supported function $\beta\colon G \to \mathbb{C}$ with $\int_G \beta(y)\, c_\lambda(gy)\, dy = c_\lambda(g)$ for every $g \in G$.

Proof. Write $c = c_\lambda$, $\mu = \mu_l$, $K = K_\varepsilon$ with its Haar probability measure dk, $J = \mathrm{diag}(\varepsilon)$, $\mathfrak{u}(J) = \{Y : Y^*J + JY = 0\}$, and $\|\cdot\|$ for the operator norm; m_μ and $\|m_\mu\|^2 > 0$ are as in Definition 3.2.3.

Unimodularity. The real bilinear form $B(X,Y) = \mathrm{Re}\,\mathrm{tr}(XY)$ on $\mathfrak{u}(J)$ is invariant under $\mathrm{Ad}(g)X = gXg^{-1}$ ($g \in G$) and nondegenerate: for $0 \neq X \in \mathfrak{u}(J)$ also $X^* \in \mathfrak{u}(J)$ and $B(X, X^*) = \mathrm{tr}(XX^*) > 0$.

Hence $|\det \mathrm{Ad}(g)| = 1$, the modular function of G is 1, and dg is invariant under left and right translations and under $g \mapsto g^{-1}$.

(i). By Lemma 5.6.24 (ii), c is locally the restriction of a holomorphic function, in particular continuous (indeed smooth) on G. By Lemma 5.7.40, $|c| \leq 1$ and $|c(g)| \leq (2/\|g\|)^h$; as $h > 8(\#n)^2$, the right side is integrable by Lemma 5.6.32 (ii) (with $S = G$, the right-invariant measure dg, finite on compact sets, the proper homomorphism $j = \mathrm{id}$ and $y = 1$). So c is integrable, and $|c|^2 \leq |c|$ is integrable. Since $\mathrm{bd}_\varepsilon(1) = 1$ and $m_w(1) = 1$, (3.2) gives $c(1) = \|m_\mu\|^{-2} \int_K |m_\mu|^2 dk = 1$; by continuity $|c| \geq \frac{1}{2}$ on a neighbourhood of 1, which has positive Haar measure. Hence $0 < \int_G |c|^2 < \infty$.

A translate identity. Let $w\colon G \to \mathbb{C}$ be integrable with

$$\int_K \overline{m_\mu(k)}\, w(k^{-1}g)\, dk = \|m_\mu\|^2\, w(g) \qquad (g \in G). \tag{5.101}$$

Then

$$\int_G w(g)\, c(xga)\, dg = c(x) \int_G w(g)\, c(ga)\, dg$$

for all $x, a \in G$. Indeed, the function $(k,g) \mapsto \overline{m_\mu(k)}\, w(g)\, c(xkga)$ is integrable on $K \times G$, being bounded by $\sup_K |m_\mu| \cdot |w(g)|$. Integrating first over k and using Lemma 5.6.31 (i) with $b = ga$, its integral is $\|m_\mu\|^2 c(x) \int_G w(g) c(ga)\, dg$. Integrating first over g, after the substitution $g \mapsto k^{-1}g$ (left invariance) the inner integral is $\int_G w(k^{-1}g) c(xga)\, dg$, and by Fubini's theorem and (5.101) the double integral is $\|m_\mu\|^2 \int_G w(g) c(xga)\, dg$. Divide by $\|m_\mu\|^2$.

(ii). Take $w(g) = c(g^{-1}) = \overline{c(g)}$ (Lemma 5.7.40), integrable by (i). It satisfies (5.101): $w(k^{-1}g) = c(g^{-1}k)$ and Lemma 5.6.31 (i) with $x = g^{-1}$, $b = 1$ gives $\int_K \overline{m_\mu(k)}\, c(g^{-1}k)\, dk = \|m_\mu\|^2 c(g^{-1}) c(1) = \|m_\mu\|^2 w(g)$. Hence

$$\int_G \overline{c(g)}\, c(xga)\, dg = c(x) \int_G \overline{c(g)}\, c(ga)\, dg.$$

The substitution $g \mapsto a^{-1}g^{-1}$, which preserves dg, carries $\overline{c(g)}\,c(ag) = c(g^{-1})\,c(ag)$ to $c(ga)\,c(g^{-1})$; so $\int_G \overline{c(g)}c(ga)\,dg = \int_G \overline{c(g)}c(ag)\,dg$, and the displayed identity with (x,a) replaced by $(a,1)$ evaluates the latter as $c(a)\int_G |c|^2 = c(a)/d_\lambda$. Together, $d_\lambda \int_G c(xga)\,\overline{c(g)}\,dg = c(x)c(a)$.

(iii). Let $\psi\colon \mathbb{R} \to [0,1]$ be smooth with $\psi = 1$ on $[-1,1]$ and $\psi = 0$ outside $[-2,2]$, and $\chi(g) = \psi(\|gg^* - 1\|_{\mathrm{HS}}^2)$. Then χ is smooth on G, $\chi(1) = 1$, $\chi(gk) = \chi(g)$ for $k \in K$, and χ has compact support: $\chi(g) \neq 0$ forces $\|g\|^2 = \|gg^*\| \leq 1 + \sqrt{2}$, and $\{g \in G : \|g\| \leq R\}$ is compact as G is closed in $\mathrm{M}_n(\mathbb{C})$. Put $w(g) = \chi(g^{-1})\,c(g^{-1})$, a continuous compactly supported function. It satisfies (5.101): $w(k^{-1}g) = \chi(g^{-1}k)c(g^{-1}k) = \chi(g^{-1})c(g^{-1}k)$, and Lemma 5.6.31 (i) (with $x = g^{-1}$, $b = 1$) gives $\int_K \overline{m_\mu(k)}c(g^{-1}k)\,dk = \|m_\mu\|^2 c(g^{-1})$. The translate identity with $a = 1$ gives $\int_G w(y)c(gy)\,dy = \kappa_0\,c(g)$, where

$$\kappa_0 = \int_G w(y)c(y)\,dy = \int_G \chi(y^{-1})\,|c(y)|^2\,dy > 0,$$

because $c(y^{-1}) = \overline{c(y)}$ and the integrand is continuous, nonnegative and equal to 1 at $y = 1$. Then $\beta = w/\kappa_0$ is smooth (as χ, c and inversion are smooth on G) and compactly supported, and $\int_G \beta(y)c(gy)\,dy = c(g)$. □

Lemma 5.6.24. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $J = \mathrm{diag}(\varepsilon)$, $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$, $\mathfrak{u}(J) = \{Y \in \mathrm{M}_n(\mathbb{C}) : Y^*J + JY = 0\}$, and let E_{ab} $(a,b \in n)$ be the matrix units. Call a function $f\colon G \to \mathbb{C}$* locally holomorphic *if every $g \in G$ has an open neighbourhood W in $\mathrm{GL}_n(\mathbb{C})$ with a holomorphic $F\colon W \to \mathbb{C}$ such that $f = F$ on $W \cap G$; for such (W,F) and $X \in \mathrm{M}_n(\mathbb{C})$ put $(Xf)(g) = F'(g)[gX]$ $(g \in W \cap G)$, the complex derivative of F at g in the direction gX.*

- *(i) For locally holomorphic f and $X \in \mathrm{M}_n(\mathbb{C})$ the function Xf does not depend on the choice of the pairs (W,F) and is locally holomorphic; $(Yf)(g) = \frac{d}{dt}|_{t=0} f(ge^{tY})$ for $Y \in \mathfrak{u}(J)$; $X \mapsto Xf$ is complex linear; and $X(X'f) - X'(Xf) = (XX' - X'X)f$ for all $X, X' \in \mathrm{M}_n(\mathbb{C})$.*
- *(ii) Let $l \in \mathbb{Z}^n$ have gap 0 for ε, $\lambda = \frac{1}{2}l$ and $\mu = \mu_l$. Then c_λ is locally holomorphic: for $g_0 \in G$ and $a = \mathrm{bd}_\varepsilon(g_0)$ one may take $W = \{y \in \mathrm{GL}_n(\mathbb{C}) : a^{-1}y$ accretive$\}$ and $F(y) = \mathrm{FJ}_l(a, a^{-1}y)$.*
- *(iii) In the situation of (ii), $E_{aa}c_\lambda = \mu_a\,c_\lambda$ for every $a \in n$, and $E_{ab}c_\lambda = 0$ whenever $a \neq b$, $\varepsilon_a = \varepsilon_b$ and $l_a > l_b$.*
- *(iv) In the situation of (ii),*

$$\sum_{a,b \in n} E_{ab}(E_{ba}c_\lambda) = \kappa_l\,c_\lambda, \qquad \kappa_l = \frac{1}{4}\sum_{a \in n}(l_a^2 - (2\rho_l(a))^2).$$

Proof. For $X \in \mathrm{M}_n(\mathbb{C})$ and a holomorphic function F on an open set $W \subseteq \mathrm{GL}_n(\mathbb{C})$ write $(D_XF)(y) = F'(y)[yX] = \frac{d}{ds}|_{s=0} F(y(1+sX))$ $(y \in W)$; D_XF is holomorphic on W and $X \mapsto D_XF$ is complex linear. Write $K = K_\varepsilon$ with its Haar probability measure dk.

(i). Let (W,F) be a pair for f and $g \in W \cap G$. For $Y \in \mathfrak{u}(J)$ the curve $t \mapsto ge^{tY}$ lies in G, in W for small $|t|$, and has derivative gY at 0; so $F'(g)[gY] = \frac{d}{dt}|_{t=0} F(ge^{tY}) = \frac{d}{dt}|_{t=0} f(ge^{tY})$, which depends only on f. Every $X \in \mathrm{M}_n(\mathbb{C})$ is $X = Y_1 + iY_2$ with

$$Y_1 = \tfrac{1}{2}(X - JX^*J), \qquad Y_2 = \tfrac{1}{2i}(X + JX^*J) \in \mathfrak{u}(J),$$

and $F'(g)$ is complex linear, so $F'(g)[gX] = (Y_1f)(g) + i(Y_2f)(g)$ depends only on f. On $W \cap G$ the function Xf is the restriction of the holomorphic D_XF, so Xf is locally holomorphic,

and complex linearity in X is clear. For the commutator, differentiate $y \mapsto F'(y)[yX']$ in the direction yX: $(D_X D_{X'}F)(y) = F''(y)[yX, yX'] + F'(y)[yXX']$; as $F''(y)$ is symmetric, $D_X D_{X'}F - D_{X'}D_X F = D_{XX'-X'X}F$. Restricting to $W \cap G$, where $X'f = D_{X'}F$ and hence $X(X'f) = D_X D_{X'}F$, gives the last claim.

(ii). By Lemma 5.6.25 (i), $a = \mathrm{bd}_\varepsilon(g_0)$ is invertible and $a^{-1}g_0$ is accretive; accretivity ($z+z^*$ positive definite) is an open condition, so W is an open neighbourhood of g_0. For $y \in W$ and $k \in K$ the matrix $k^*a^{-1}yk$ is accretive (Lemma 5.6.25 (ii)), so its determinant and principal minors do not vanish (Lemma 5.6.25 (ii)), and

$$\Phi_k(y) = \frac{m_\mu(ak)\,\overline{m_\mu(k)}}{m_{-l^\rho}(k^*a^{-1}yk)}$$

is continuous in $(y,k) \in W \times K$ and, for fixed k, holomorphic in y: by (3.1), $m_{-l^\rho}(Z)$ is an integer power of $\det Z$ times a product of principal minors of Z, and these are polynomials in the entries of y. Hence $F = \|m_\mu\|^{-2}\int_K \Phi_k\,dk$ is holomorphic on W (differentiation under the integral sign over the compact K, the y-derivatives of Φ_k being continuous on $W \times K$ by Cauchy's integral formula). For $y \in W \cap G$, $F(y) = c_\lambda(y)$ by Lemma 5.6.26 (i), a being block-diagonal with $a^{-1}y$ accretive.

(iii). Let $g_0 \in G$, $a' = \mathrm{bd}_\varepsilon(g_0^{-1})$, $W' = \{y \in \mathrm{GL}_n(\mathbb{C}) : a'^{-1}y^{-1}$ accretive$\}$ and $\tilde F(y) = \mathrm{FJ}_{-l}(a', a'^{-1}y^{-1})$. The vector $-l$ has gap 0, so, as in (ii) composed with the holomorphic map $y \mapsto y^{-1}$, W' is an open neighbourhood of g_0 and $\tilde F$ is holomorphic on it. For $y \in W' \cap G$: $\tilde F(y) = c_{-l/2}(y^{-1})$ by Lemma 5.6.26 (i), $= \overline{c_{l/2}(y^{-1})}$ by Lemma 5.6.29 (ii), $= c_{l/2}(y)$ by Lemma 5.7.40. So $(W', \tilde F)$ is a pair for c_λ at g_0.

Let B_l be as in Lemma 5.6.27 (invertible x with $x_{ij} = 0$ unless $\varepsilon_i = \varepsilon_j$ and $l_i \geq l_j$), and let $x \in \mathrm{B}_l$ and $y \in W'$ with $yx \in W'$. We show $\tilde F(yx) = x^\mu \tilde F(y)$, $x^\mu = \prod_i x_{ii}^{\mu_i}$. Put $g = (yx)^{-1} = x^{-1}y^{-1}$. The matrices a' and $x^{-1}a'$ are block-diagonal and invertible, $a'^{-1}g = a'^{-1}(yx)^{-1}$ is accretive as $yx \in W'$, and $(x^{-1}a')^{-1}g = a'^{-1}y^{-1}$ is accretive as $y \in W'$; so Lemma 5.6.26 (i) for $-l$ gives $\tilde F(yx) = \mathrm{FJ}_{-l}(a', a'^{-1}g) = \mathrm{FJ}_{-l}(x^{-1}a', a'^{-1}y^{-1})$. By Lemma 5.6.27 (iii), $\mu_{-l} = -\mu$ and $(-l)^\rho = -l^\rho$, so in the integrand (3.2) of $\mathrm{FJ}_{-l}(x^{-1}a', \cdot)$ only the factor $m_{-\mu}(x^{-1}a'k)$ involves x, and

$$m_{-\mu}(x^{-1}a'k) = m_\mu(k^{-1}a'^{-1}x) = x^\mu\, m_\mu(k^{-1}a'^{-1}) = x^\mu\, m_{-\mu}(a'k)$$

by Lemma 5.6.27 (i) (twice) and (ii) (with the left factor $1 \in \mathrm{B}_l$; μ satisfies its hypothesis by (iii) there). Hence $\tilde F(yx) = x^\mu\,\mathrm{FJ}_{-l}(a', a'^{-1}y^{-1}) = x^\mu \tilde F(y)$.

Now let $y \in W' \cap G$. If $a \neq b$, $\varepsilon_a = \varepsilon_b$ and $l_a > l_b$, then $x_t = 1 + tE_{ab} \in \mathrm{B}_l$ with $x_t^\mu = 1$ for every $t \in \mathbb{C}$, and $yx_t \in W'$ for small $|t|$; so $\tilde F(y(1+tE_{ab})) = \tilde F(y)$ and $(E_{ab}c_\lambda)(y) = (D_{E_{ab}}\tilde F)(y) = 0$. For $a = b$, $x_t = 1 + tE_{aa} \in \mathrm{B}_l$ for $t \neq -1$ with $x_t^\mu = (1+t)^{\mu_a}$, so $(E_{aa}c_\lambda)(y) = \frac{d}{dt}|_{t=0}(1+t)^{\mu_a}\tilde F(y) = \mu_a\, c_\lambda(y)$. As g_0 was arbitrary, (iii) holds on G.

(iv). Let g_0, a, W, F and Φ_k be as in (ii), and $\Omega = \sum_{a,b} D_{E_{ab}}D_{E_{ba}}$. The y-derivatives of order at most 2 of $\Phi_k(y)$ are continuous on $W \times K$, so $\Omega F = \|m_\mu\|^{-2}\int_K \Omega\Phi_k\,dk$. We show $\Omega\Phi_k = \kappa_l\Phi_k$ on W for each $k \in K$.

Change of basis. Put $X_{ab} = kE_{ab}k^{-1}$. Then $\sum_{a,b} X_{ab} \otimes X_{ba} = \sum_{c,d} E_{cd} \otimes E_{dc}$ in $\mathrm{M}_n(\mathbb{C}) \otimes \mathrm{M}_n(\mathbb{C})$,

and since $(X, X') \mapsto D_X D_{X'}$ is bilinear, $\Omega = \sum_{a,b} D_{X_{ab}}D_{X_{ba}}$.

Weights of Φ_k. Let B' be the set of invertible $t \in \mathrm{M}_n(\mathbb{C})$ with $t_{ij} = 0$ unless $l_i \leq l_j$. Lemma 5.6.27 (ii), applied to the signs $(1, \ldots, 1)$ and the vector $-l$ (for which the group of that lemma is B') and to the weight $-l^\rho$, which is admissible because $l_i^\rho \leq l_j^\rho$ whenever $l_i < l_j$ (Lemma 5.6.27 (iii)), gives $m_{-l^\rho}(Zt) = t^{-l^\rho} m_{-l^\rho}(Z)$ for invertible Z and $t \in \mathrm{B}'$. If y and $yktk^{-1}$ lie in W, then $k^*a^{-1}(yktk^{-1})k = (k^*a^{-1}yk)\,t$, so $\Phi_k(yktk^{-1}) = t^{l^\rho}\Phi_k(y)$. Taking $t = 1 + sE_{ij}$

with $i \neq j$ and $l_i < l_j$ (so $t^{l^\rho} = 1$), and $t = 1 + sE_{ii}$ (so $t^{l^\rho} = (1+s)^{l_i^\rho}$), and noting $yktk^{-1} = y(1 + sX_{ij})$, we get

$$D_{X_{ij}}\Phi_k = 0 \quad (i \neq j,\ l_i < l_j), \qquad D_{X_{ii}}\Phi_k = l_i^\rho\,\Phi_k.$$

Evaluation. By the commutator identity of (i) for the D's and $X_{ab}X_{ba} - X_{ba}X_{ab} = X_{aa} - X_{bb}$: for $l_a < l_b$, $D_{X_{ab}}D_{X_{ba}}\Phi_k = D_{X_{ba}}D_{X_{ab}}\Phi_k + D_{X_{aa}-X_{bb}}\Phi_k = (l_a^\rho - l_b^\rho)\Phi_k$; for $l_a > l_b$, $D_{X_{ba}}\Phi_k = 0$, so the term vanishes; for $a = b$ it is $(l_a^\rho)^2\Phi_k$. The entries of l are distinct, so

$$\Omega\Phi_k = \Big(\sum_a (l_a^\rho)^2 + \sum_{l_a<l_b} (l_a^\rho - l_b^\rho)\Big)\Phi_k = \sum_a l_a^\rho(l_a^\rho - 2\rho_l(a))\,\Phi_k,$$

since $\sum_{l_a<l_b}(l_a^\rho - l_b^\rho) = \sum_a l_a^\rho(\#\{b : l_b > l_a\} - \#\{b : l_b < l_a\}) = -\sum_a l_a^\rho\, 2\rho_l(a)$. The entries of l are congruent to $\#n - 1$ modulo 2, so $2l^\rho = l + 2\rho_l$ (Lemma 5.6.27 (iii)) and $l_a^\rho(l_a^\rho - 2\rho_l(a)) = \frac{1}{4}(l_a + 2\rho_l(a))(l_a - 2\rho_l(a))$; the sum is κ_l. Hence $\Omega F = \kappa_l F$ on W, and by (i), $\sum_{a,b} E_{ab}(E_{ba}c_\lambda) = (\Omega F)|_{W\cap G} = \kappa_l\, c_\lambda$ on $W \cap G$. As g_0 was arbitrary, (iv) holds on G. □

Lemma 5.6.25. *Let n be a nonempty finite set and $\varepsilon \in \{\pm 1\}^n$.*

1. *For every $g \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ the matrix $\mathrm{bd}_\varepsilon(g)$ is invertible and $\mathrm{bd}_\varepsilon(g)^{-1}g$ is accretive.*
2. *If z is accretive and k is invertible, then k^*zk is accretive, and $\Delta_P(y) \neq 0$ for every accretive y and every $P \subseteq n$.*

Proof. (ii) For $x \neq 0$, $x^*(k^*zk + k^*z^*k)x = (kx)^*(z + z^*)(kx) > 0$ as $kx \neq 0$. If y is accretive and $P \neq \emptyset$, let y_P be its submatrix on $P \times P$; for $0 \neq x \in \mathbb{C}^P$, extended by 0 to $\tilde{x} \in \mathbb{C}^n$, $x^*(y_P + y_P^*)x = \tilde{x}^*(y + y^*)\tilde{x} > 0$. If $y_P x = 0$ then $x^*(y_P + y_P^*)x = 2\,\mathrm{Re}\, x^*y_P x = 0$, so $x = 0$; hence y_P is invertible and $\Delta_P(y) \neq 0$. For $P = \emptyset$, $\Delta_\emptyset = 1$.

(i) Number n so that the indices with $\varepsilon_i = 1$ come first and write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $J = \mathrm{diag}(\varepsilon) = \mathrm{diag}(1, -1)$ in the corresponding blocks; if one block is empty, $\mathrm{bd}_\varepsilon(g) = g$ and $\mathrm{bd}_\varepsilon(g)^{-1}g = 1$, and there is nothing to prove. From $g^*Jg = J$ we get $g^{-1} = Jg^*J$, hence $gJg^* = J$, whose blocks read

$$AA^* - BB^* = 1, \qquad DD^* - CC^* = 1, \qquad AC^* = BD^*.$$

So $AA^* \geq 1$ and $DD^* \geq 1$, A and D are invertible, and $\mathrm{bd}_\varepsilon(g) = \mathrm{diag}(A, D)$ is invertible. Put $X = A^{-1}B$. Then $z = \mathrm{bd}_\varepsilon(g)^{-1}g = \begin{pmatrix} 1 & X \\ D^{-1}C & 1 \end{pmatrix}$, and $C^* = A^{-1}BD^*$ gives $(D^{-1}C)^* = C^*D^{-*} = X$, so

$$\tfrac{1}{2}(z + z^*) = \begin{pmatrix} 1 & X \\ X^* & 1 \end{pmatrix}.$$

Moreover $XX^* = A^{-1}(AA^* - 1)A^{-*} = 1 - (A^*A)^{-1}$, and $(A^*A)^{-1}$ is positive definite, so $\|X\|_{\mathrm{op}} < 1$. For $(u, v) \neq 0$, $|2\,\mathrm{Re}\, u^*Xv| \leq 2\|X\|_{\mathrm{op}}\|u\|\|v\| \leq \|X\|_{\mathrm{op}}(\|u\|^2 + \|v\|^2) < \|u\|^2 + \|v\|^2$, hence $(u, v)^*\frac{1}{2}(z + z^*)(u, v) = \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\, u^*Xv > 0$, and z is accretive. □

Lemma 5.6.26. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$ and $l \in \mathbb{Z}^n$ a vector with gap 0 for ε; write $\mu = \mu_l$. Call a complex $n \times n$ matrix x block-diagonal if $x_{ij} = 0$ whenever $\varepsilon_i \neq \varepsilon_j$.*

1. *If g is a complex $n \times n$ matrix and a, a' are block-diagonal matrices such that $a^{-1}g$ and $a'^{-1}g$ are accretive, then $\mathrm{FJ}_l(a, a^{-1}g) = \mathrm{FJ}_l(a', a'^{-1}g)$. In particular $c_{l/2}(g) = \mathrm{FJ}_l(a, a^{-1}g)$ for every $g \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ and every block-diagonal a with $a^{-1}g$ accretive.*

2. *If $g = pzq^*$ with p, q invertible block-diagonal and z accretive, then $a = pq^{-1}$ is block-diagonal with $a^{-1}g$ accretive, and*

$$\mathrm{FJ}_l(a, a^{-1}g) = \frac{1}{\|m_\mu\|^2} \int_{K_\varepsilon} \frac{m_\mu(pk)\,\overline{m_\mu(qk)}}{m_{-l^\rho}(k^* z k)}\, dk.$$

3. *Let b, c be invertible block-diagonal matrices with $b_{ij} = c_{ij} = 0$ whenever $l_i < l_j$. If y and $c^* y b$ are accretive, then*

$$\mathrm{FJ}_l(1, c^* y b) = \overline{c^\mu}\, b^\mu\, \mathrm{FJ}_l(1, y), \qquad b^\mu = \prod_{i \in n} b_{ii}^{\mu_i}.$$

Proof. Write $K = K_\varepsilon$, let $K_\mathbb{C}$ be the group of invertible block-diagonal matrices and $\mathrm{B}_l \subseteq K_\mathbb{C}$ the subgroup of the b with $b_{ij} = 0$ whenever $l_i < l_j$, as in Lemma 5.6.27 and Lemma 5.6.28. Gap 0 means: distinct entries, all congruent to $\#n - 1$ modulo 2. By Lemma 5.6.27 (iii) the weights μ and l^ρ both satisfy the hypothesis of (ii) there ($w_i \geq w_j$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$). The inverse of an accretive y is accretive, since $y^{-1} + y^{-*} = y^{-1}(y + y^*)y^{-*}$; an accretive matrix is invertible, and $k^* y k$ is accretive for invertible k (Lemma 5.6.25, (ii)).

Step 1: the integral I. Fix an invertible g. For invertible p, r with $rg^{-1}p$ accretive put

$$I(p, r) = \int_K \frac{m_\mu(pk)\, m_\mu(k^* r)}{m_{l^\rho}(k^* r g^{-1} p k)}\, dk.$$

The denominator is a product of powers of the determinant and of principal minors of the accretive matrix $k^*(rg^{-1}p)k$, all nonzero by Lemma 5.6.25 (ii); the integrand is continuous on the compact group K. If a is block-diagonal and $a^{-1}g$ is accretive, then $g^{-1}a = (a^{-1}g)^{-1}$ is accretive, and for $k \in K$ (so $k^{-1} = k^*$), Lemma 5.6.27 (i) gives $\overline{m_\mu(k)} = m_\mu(k^*)$ and $m_{-l^\rho}(k^* a^{-1} g k) = m_{l^\rho}(k^* g^{-1} a k)$. Comparing with (3.2),

$$\mathrm{FJ}_l(a, a^{-1}g) = \|m_\mu\|^{-2}\, I(a, 1). \tag{5.102}$$

Step 2: invariance. Let $rg^{-1}p$ be accretive and $x \in K_\mathbb{C}$. Then $(x^* r)g^{-1}(px) = x^*(rg^{-1}p)x$ is accretive and $I(px, x^* r) = I(p, r)$. Indeed, put

$$F(y) = \frac{m_\mu(py)\, m_\mu(y^* r)}{m_{l^\rho}(y^* r g^{-1} p y)} \qquad (y \in K_\mathbb{C});$$

F is continuous (principal minors are polynomials, the denominator does not vanish as above), and $F(xk)$ is the integrand of $I(px, x^* r)$ at k, $F(k)$ that of $I(p, r)$. For $b \in \mathrm{B}_l$, Lemma 5.6.27 (ii) (with the second factor 1 or the first factor 1) gives $m_\mu(pyb) = b^\mu m_\mu(py)$, $m_\mu(b^* y^* r) = \overline{b^\mu}\, m_\mu(y^* r)$ and $m_{l^\rho}(b^* Y b) = \overline{b^{l^\rho}}\, b^{l^\rho} m_{l^\rho}(Y)$ for $Y = y^* r g^{-1} p y$. As $\mu = l^\rho - 2\rho_l^{\mathrm{c}}$ by definition,

$$F(yb) = \frac{|b^\mu|^2}{|b^{l^\rho}|^2} F(y) = \left|b^{-2\rho_l^{\mathrm{c}}}\right|^2 F(y),$$

and Lemma 5.6.28 gives $\int_K F(xk)\, dk = \int_K F(k)\, dk$.

Step 3: the first variable alone. Let p be block-diagonal with $g^{-1}p$ accretive and Y block-diagonal. There is $\delta > 0$ such that for $|s|, |u| < \delta$ the matrices $1 + sY$, $1 + uY^*$ are invertible and $(1 + uY^*)g^{-1}p(1 + sY)$ is accretive (accretive matrices form an open set). On this bidisc put $\psi(s, u) = I\big(p(1 + sY), 1 + uY^*\big)$. The integrand is a continuous function of (s, u, k) and, for fixed k, a quotient of polynomials in (s, u) whose denominator (powers of determinants of invertible

matrices and principal minors of accretive matrices) does not vanish; hence ψ is holomorphic (continuity of parameter integrals over a compact space, Morera's theorem in each variable with Fubini, and Osgood's lemma). For complex t with $|t| < \delta$, $x = 1 + tY \in K_{\mathbb{C}}$ and $x^* = 1 + \bar{t}Y^*$, so Step 2 gives $\psi(t, \bar{t}) = \psi(0,0)$. Expand $\psi(s,u) - \psi(0,0) = \sum_{\alpha,\beta \geq 0} e_{\alpha\beta} s^\alpha u^\beta$, absolutely convergent for $|s|, |u| < \delta'$ with some $0 < \delta' \leq \delta$. With $t = \tau e^{i\theta}$, $0 \leq \tau < \delta'$,

$$0 = \sum_{\alpha,\beta} e_{\alpha\beta} \tau^{\alpha+\beta} e^{i(\alpha-\beta)\theta} \quad \text{for all } \theta,$$

so for each integer m the Fourier coefficient $\sum_{\beta \geq \max(0,-m)} e_{\beta+m,\beta}\, \tau^{2\beta+m}$ vanishes for all $0 \leq \tau < \delta'$; the exponents $2\beta + m$ are distinct, so every $e_{\beta+m,\beta} = 0$. Hence $\psi \equiv \psi(0,0)$ on the smaller bidisc, and in particular

$$I(p + sp\,Y, 1) = I(p, 1) \qquad (|s| < \delta'). \tag{5.103}$$

Step 4: proof of (i). The matrices a, a', g are invertible, and $g^{-1}a$, $g^{-1}a'$ are accretive. For $t \in [0,1]$ put $p_t = (1-t)a + ta'$: it is block-diagonal and $g^{-1}p_t = (1-t)g^{-1}a + tg^{-1}a'$ is accretive, a convex combination of accretive matrices; so p_t is invertible. Let $\phi(t) = I(p_t, 1)$. At $t_0 \in [0,1]$ apply (5.103) with $p = p_{t_0}$ and $Y = p_{t_0}^{-1}(a' - a)$, which is block-diagonal: $p_{t_0} + sp_{t_0}Y = p_{t_0+s}$, so ϕ is constant on a neighbourhood of t_0 in $[0,1]$. As $[0,1]$ is connected, $\phi(0) = \phi(1)$, that is $I(a,1) = I(a',1)$, and (i) follows from (5.102). For $g \in \mathrm{U}(\operatorname{diag}\varepsilon)$ the block-diagonal matrix $a_0 = \mathrm{bd}_\varepsilon(g)$ is invertible with $a_0^{-1}g$ accretive (Lemma 5.6.25, (i)), and $c_{l/2}(g) = \mathrm{FJ}_l(a_0, a_0^{-1}g)$ by definition; the second sentence follows.

Step 5: proof of (ii). The matrix $a = pq^{-1}$ is block-diagonal and $a^{-1}g = qzq^* = (q^*)^* z\, q^*$ is accretive. By (5.102) and Step 2 (applied to $(p,r) = (a,1)$, for which $g^{-1}a$ is accretive, and $x = q$),

$$\|m_\mu\|^2\, \mathrm{FJ}_l(a, a^{-1}g) = I(a,1) = I(aq, q^*) = I(p, q^*).$$

Here $q^*g^{-1}p = q^*q^{-*}z^{-1}p^{-1}p = z^{-1}$, $m_\mu(k^*q^*) = m_\mu((qk)^*) = \overline{m_\mu(qk)}$ and $m_{l^\rho}(k^*z^{-1}k) = m_{l^\rho}((k^*zk)^{-1}) = m_{-l^\rho}(k^*zk)$ by Lemma 5.6.27 (i); this is the asserted integral.

Step 6: proof of (iii). The hypotheses say $b, c \in \mathrm{B}_l$. The factor 1 is admissible for y and for c^*yb. Apply (ii) and (i) to $c^*yb = c^*\, y\, (b^*)^*$ (so $p = c^*$, $q = b^*$, $z = y$) and to $y = 1 \cdot y \cdot 1^*$:

$$\mathrm{FJ}_l(1, c^*yb) = \frac{1}{\|m_\mu\|^2} \int_K \frac{m_\mu(c^*k)\, \overline{m_\mu(b^*k)}}{m_{-l^\rho}(k^*yk)}\, dk, \qquad \mathrm{FJ}_l(1,y) = \frac{1}{\|m_\mu\|^2} \int_K \frac{m_\mu(k)\, \overline{m_\mu(k)}}{m_{-l^\rho}(k^*yk)}\, dk.$$

By Lemma 5.6.27 (ii), $m_\mu(c^*k) = \overline{c^\mu}\, m_\mu(k)$ and $m_\mu(b^*k) = \overline{b^\mu}\, m_\mu(k)$, which gives (iii). □

Lemma 5.6.27. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$ and $l \in \mathbb{Z}^n$ with distinct entries. Let B_l be the set of invertible complex $n \times n$ matrices b with $b_{ij} = 0$ unless $\varepsilon_i = \varepsilon_j$ and $l_i \geq l_j$, and for $w \in \mathbb{Z}^n$ and $b \in \mathrm{B}_l$ put $b^w = \prod_{i \in n} b_{ii}^{w_i}$.*

1. *For every invertible complex $n \times n$ matrix z and every $w \in \mathbb{Z}^n$, $m_w(z^*) = \overline{m_w(z)}$ and $m_{-w}(z) = m_w(z^{-1})$.*

2. *Let $w \in \mathbb{Z}^n$ satisfy $w_i \geq w_j$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$. Then every $b \in \mathrm{B}_l$ has nonzero diagonal entries, and*

$$m_w(c^*z\, b) = \overline{c^w}\, b^w\, m_w(z)$$

for every invertible complex $n \times n$ matrix z and all $b, c \in \mathrm{B}_l$.

3. *$2\rho_l(i) \geq 2\rho_l(j) + 2$ and $l_i^\rho \geq l_j^\rho$ whenever $l_i > l_j$. If moreover every entry of l is congruent to $\#n - 1$ modulo 2, then $2l^\rho = l + 2\rho_l$, $(-l)^\rho = -l^\rho$, $\mu_{-l} = -\mu_l$, and $\mu_l(i) \geq \mu_l(j)$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$.*

Proof. Throughout, z_{PQ} denotes the submatrix of a matrix z with rows in P and columns in Q, so that $\Delta_P(z) = \det z_{PP}$.

(i). $(z^*)_{PP} = (z_{PP})^*$, so $\Delta_P(z^*) = \overline{\Delta_P(z)}$ for every $P \subseteq n$ (including $P = n$, the determinant), and $m_w(z^*) = \overline{m_w(z)}$ by (3.1). For the second identity we use Jacobi's complementary minor formula $\Delta_P(z^{-1}) = \Delta_{n\smallsetminus P}(z)/\det z$ for invertible z and $P \subseteq n$ ([133], 0.8.4; for $P = \emptyset$ and $P = n$ it reads $1 = 1$). Put $w_- = \min_i w_i$ and $w_+ = \max_i w_i$. The minimum and maximum of $-w$ are $-w_+$ and $-w_-$, so by (3.1)

$$m_{-w}(z) = (\det z)^{-w_+} \prod_{-w_+ < t \le -w_-} \Delta_{\{i:\, -w_i \ge t\}}(z).$$

Substitute $t = 1 - s$: the integers t with $-w_+ < t \le -w_-$ are those with $w_- < s \le w_+$, and $\{i : -w_i \ge 1 - s\} = \{i : w_i < s\} = n \smallsetminus \{i : w_i \ge s\}$. By Jacobi's formula each factor is $\det z \cdot \Delta_{\{i:\, w_i \ge s\}}(z^{-1})$, and there are $w_+ - w_-$ factors, so

$$m_{-w}(z) = (\det z)^{-w_-} \prod_{w_- < s \le w_+} \Delta_{\{i:\, w_i \ge s\}}(z^{-1}) = (\det z^{-1})^{w_-} \prod_{w_- < s \le w_+} \Delta_{\{i:\, w_i \ge s\}}(z^{-1}) = m_w(z^{-1}).$$

(ii). Order n by decreasing l (possible as the entries are distinct). In this order every $b \in \mathrm{B}_l$ is upper triangular, since $b_{kj} \ne 0$ forces $l_k \ge l_j$; hence $\det b = \prod_i b_{ii}$, and the diagonal entries are nonzero because b is invertible. Call $P \subseteq n$ *closed* if $j \in P$, $\varepsilon_k = \varepsilon_j$ and $l_k > l_j$ imply $k \in P$. Every set $P_t = \{i : w_i \ge t\}$ is closed: if $j \in P_t$, $\varepsilon_k = \varepsilon_j$ and $l_k > l_j$, then $w_k \ge w_j \ge t$ by the hypothesis on w. The set n is closed as well.

Let P be closed, y any matrix and $b, c \in \mathrm{B}_l$. For $i, j \in P$, $(yb)_{ij} = \sum_k y_{ik} b_{kj}$, and $b_{kj} \ne 0$ forces $k = j$ or ($\varepsilon_k = \varepsilon_j$ and $l_k > l_j$), hence $k \in P$; so $(yb)_{PP} = y_{PP} b_{PP}$. Likewise $(c^* y)_{ij} = \sum_k \overline{c_{ki}}\, y_{kj}$, and $c_{ki} \ne 0$ forces $k \in P$; so $(c^* y)_{PP} = (c_{PP})^* y_{PP}$. The matrices b_{PP}, c_{PP} are upper triangular in the order by decreasing l, so

$$\Delta_P(c^* z b) = \Big(\prod_{i \in P} \overline{c_{ii}}\, b_{ii}\Big) \Delta_P(z) \qquad (P \text{ closed}).$$

Put $d_i = \overline{c_{ii}}\, b_{ii} \ne 0$. Applying this to $P = n$ and to the sets P_t in (3.1),

$$m_w(c^* z b) = m_w(z) \prod_{i \in n} d_i^{e_i}, \qquad e_i = w_- + \#\{t \in \mathbb{Z} :\, w_- < t \le w_+,\, t \le w_i\} = w_- + (w_i - w_-) = w_i,$$

and $\prod_i d_i^{w_i} = \overline{c^w}\, b^w$.

(iii). Let $l_i > l_j$. Every k with $l_k < l_j$ satisfies $l_k < l_i$, and j itself satisfies $l_j < l_i$; every k with $l_k > l_i$ satisfies $l_k > l_j$, and i itself satisfies $l_i > l_j$. Hence $2\rho_l(i) \ge 2\rho_l(j) + 2$, so $l_i + 2\rho_l(i) > l_j + 2\rho_l(j)$, and taking integer parts of halves, $l_i^\rho \ge l_j^\rho$.

Now assume the parity condition. For each i, $\#\{k : l_k < l_i\} + \#\{k : l_k > l_i\} = \#n - 1$ because the entries are distinct, so $2\rho_l(i) \equiv \#n - 1 \equiv l_i \pmod 2$; thus $l_i + 2\rho_l(i)$ is even and $2l_i^\rho = l_i + 2\rho_l(i)$. Replacing l by $-l$ exchanges the two counts in $2\rho_l$ and in $2\rho_l^{\mathrm{c}}$, so $2\rho_{-l} = -2\rho_l$ and $2\rho_{-l}^{\mathrm{c}} = -2\rho_l^{\mathrm{c}}$; as $-l$ satisfies the same parity condition, $2(-l)^\rho = -l - 2\rho_l = -2l^\rho$, and $\mu_{-l} = (-l)^\rho - 2\rho_{-l}^{\mathrm{c}} = -\mu_l$.

Finally let $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$. Write $\sigma(x) \in \{-1, 0, 1\}$ for the sign of an integer x, and $\tau_k = 1$ if $\varepsilon_k \ne \varepsilon_i$, $\tau_k = -1$ if $\varepsilon_k = \varepsilon_i$ (the same for j). By definition $2\rho_l(i) = \sum_k \sigma(l_i - l_k)$ and $2\rho_l^{\mathrm{c}}(i) = \sum_{k:\, \varepsilon_k = \varepsilon_i} \sigma(l_i - l_k)$, hence

$$2\mu_l(i) = l_i + 2\rho_l(i) - 2 \cdot 2\rho_l^{\mathrm{c}}(i) = l_i + \sum_k \tau_k\, \sigma(l_i - l_k),$$

and the same with j (with the same τ_k, since $\varepsilon_i = \varepsilon_j$). Therefore

$$2\mu_l(i) - 2\mu_l(j) = (l_i - l_j) + \sum_k \tau_k(\sigma(l_i - l_k) - \sigma(l_j - l_k)).$$

The bracket lies in $\{0, 1, 2\}$; it vanishes unless $l_j \le l_k \le l_i$, equals 1 for $k \in \{i, j\}$ (where $\tau_k = -1$), and equals 2 when $l_j < l_k < l_i$. The entries of l are distinct and congruent modulo 2, so the number N of k with $l_j < l_k < l_i$ is at most $\frac{1}{2}(l_i - l_j) - 1$. Hence the sum is at least $-2 - 2N \ge -(l_i - l_j)$, and $2\mu_l(i) - 2\mu_l(j) \ge 0$. □

Lemma 5.6.28. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$ and $l \in \mathbb{Z}^n$ with distinct entries. Let $K_{\mathbb{C}}$ be the group of invertible complex $n \times n$ matrices x with $x_{ij} = 0$ whenever $\varepsilon_i \ne \varepsilon_j$, let $\mathrm{B}_l \subseteq K_{\mathbb{C}}$ be the subgroup of the b with $b_{ij} = 0$ unless $\varepsilon_i = \varepsilon_j$ and $l_i \ge l_j$, and for $w \in \mathbb{Z}^n$ put $b^w = \prod_i b_{ii}^{w_i}$. Let $F\colon K_{\mathbb{C}} \to \mathbb{C}$ be continuous with*

$$F(yb) = \left|b^{-2\rho_l^c}\right|^2 F(y) \qquad (y \in K_{\mathbb{C}},\ b \in \mathrm{B}_l).$$

Then $\int_{K_\varepsilon} F(xk)\,dk = \int_{K_\varepsilon} F(k)\,dk$ for every $x \in K_{\mathbb{C}}$.

Proof. Write $n_\pm = \{i : \varepsilon_i = \pm 1\}$, so that $K_{\mathbb{C}} = \mathrm{GL}(\mathbb{C}^{n_+}) \times \mathrm{GL}(\mathbb{C}^{n_-})$ and $K_\varepsilon = \mathrm{U}(n_+) \times \mathrm{U}(n_-)$, the blocks being the restrictions to $n_\pm$ (a factor with empty index set is the trivial group). The diagonal entries of every $b \in \mathrm{B}_l$ are nonzero (Lemma 5.6.27, (ii)), so b^w is defined.

Block scalars are invisible. For nonzero complex numbers c_+, c_- let s_c be the diagonal matrix with entry $c_\pm$ at the indices in $n_\pm$; it is central in $K_{\mathbb{C}}$ and lies in B_l. For each sign e, $\sum_{i \in n_e} 2\rho_l^c(i) = \sum_{i,k \in n_e} \sigma(l_i - l_k) = 0$, where σ is the sign function, because the summand is antisymmetric in (i, k). Hence $|s_c^{-2\rho_l^c}| = |c_+|^{-\sum_{n_+} 2\rho_l^c} |c_-|^{-\sum_{n_-} 2\rho_l^c} = 1$ and

$$F(ys_c) = F(y) \qquad (y \in K_{\mathbb{C}}). \tag{5.104}$$

Reduction to the special linear group. Let G be the subgroup of $K_{\mathbb{C}}$ of the x whose blocks have determinant 1, and $K' = G \cap K_\varepsilon = \mathrm{SU}(n_+) \times \mathrm{SU}(n_-)$. Every $x \in K_{\mathbb{C}}$ is $x's_c$ with $x' \in G$: choose c_e with $c_e^{\#n_e} = \det x_e$ for each nonempty block x_e of x. Likewise every $k \in K_\varepsilon$ is $k's_u$ with $k' \in K'$ and $|u_\pm| = 1$. The map $K' \times \mathrm{U}(1)^2 \to K_\varepsilon$, $(k', u) \mapsto k's_u$, is a continuous surjective homomorphism (s_u is central), so the image of the product of the Haar probability measures is a left-invariant probability measure on K_ε, that is, its Haar probability measure. With (5.104) and the centrality of s_c this gives

$$\int_{K_\varepsilon} F(xk)\,dk = \int_{K'} \int_{\mathrm{U}(1)^2} F(x'k's_cs_u)\,du\,dk' = \int_{K'} F(x'k')\,dk',$$

and for $x = 1$, $\int_{K_\varepsilon} F(k)\,dk = \int_{K'} F(k')\,dk'$. It remains to show $\int_{K'} F(x'k')\,dk' = \int_{K'} F(k')\,dk'$ for $x' \in G$. Factors of G with one index are trivial and may be omitted; if G is trivial there is nothing to prove.

Iwasawa decomposition of G. G is a product of groups $\mathrm{SL}_m(\mathbb{C})$, $m \ge 2$, hence a connected semisimple Lie group with finite centre, and K' is a maximal compact subgroup. Let $A \subseteq G$ be the group of diagonal matrices with positive entries and $N \subseteq G$ the group of the y with $y_{ii} = 1$ and $y_{ij} = 0$ for $i \ne j$ unless $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$. Ordering each block by decreasing l, AN is the group of upper triangular matrices of G with positive diagonal, and $G = K'AN$ is the Iwasawa decomposition (Gram–Schmidt orthonormalisation of the columns, block by block). The Lie algebra $\mathfrak{a}$ of A consists of the real diagonal matrices H of trace zero in each block; the restricted

roots are $H \mapsto H_i - H_j$ ($i \neq j$ in one block), each with root space $\mathbb{C}E_{ij}$ of real dimension 2, and those occurring in the Lie algebra of N are the ones with $l_i > l_j$. Hence

$$\begin{aligned} 2\rho(H) &= 2 \sum_{\substack{\varepsilon_i = \varepsilon_j \\ l_i > l_j}} (H_i - H_j) \\ &= 2 \sum_i H_i (\#\{j : \varepsilon_j = \varepsilon_i,\ l_j < l_i\} - \#\{j : \varepsilon_j = \varepsilon_i,\ l_j > l_i\}) = 2 \sum_i 2\rho_l^{\mathrm{c}}(i) H_i, \end{aligned}$$

so $e^{-2\rho(H)} = \prod_i a_i^{-2 \cdot 2\rho_l^{\mathrm{c}}(i)}$ for $a = \exp H = \operatorname{diag}(a_i)$. The centraliser M of A in K' consists of diagonal unitary matrices.

Conclusion. For $g \in G$ write $g = \kappa(g) \exp(H(g))\, n(g)$ as in Theorem A.5.22. The matrix $\exp(H(g))\, n(g)$ lies in B_l and has diagonal $a = \exp H(g)$, so the transformation rule of F gives

$$F(g) = \left|a^{-2\rho_l^{\mathrm{c}}}\right|^2 F(\kappa(g)) = e^{-2\rho(H(g))} F(\kappa(g)).$$

Let f be the restriction of F to K'; it is continuous, and for $m \in M \subseteq \mathrm{B}_l$, $f(km) = |m^{-2\rho_l^{\mathrm{c}}}|^2 f(k) = f(k)$ since $|m_{ii}| = 1$. By Theorem A.5.22, for $x' \in G$,

$$\int_{K'} F(x'k')\, dk' = \int_{K'} f(\kappa(x'k'))\, e^{-2\rho(H(x'k'))}\, dk' = \int_{K'} f(k')\, dk' = \int_{K'} F(k')\, dk'.$$

□

Lemma 5.6.29. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $H \geq 0$ an integer and $l \in \mathbb{Z}^n$ a vector with gap H for ε. Then for every $g \in \mathrm{U}(\operatorname{diag}\varepsilon)$:*

1. $|c_{l/2}(g)|^2\, |\det \mathrm{bd}_\varepsilon(g)|^c \leq 1$ *for every integer* $0 \leq c \leq H$;
2. $c_{-l/2}(g) = \overline{c_{l/2}(g)}$.

Proof. Write $K = K_\varepsilon$ with Haar probability measure du and $\mu = \mu_l$. A vector with gap H has gap 0, so Lemma 5.6.26 and Lemma 5.6.27 (iii) apply to l and to $-l$. We use the facts of Lemma 5.6.30 on $\mathrm{U}(\operatorname{diag}\varepsilon)$: for x in it, $|\det x| = 1$, $|\Delta_{n \smallsetminus P}(x)| = |\Delta_P(x)|$ and $m_w(x^{-1}) = m_{-w}(x) = \overline{m_w(x)}$ (part (i)); every g in it is $g = kz$ with $k \in K$ and z a positive definite element (part (ii)); and $\mathrm{bd}_\varepsilon(xyx') = x\, \mathrm{bd}_\varepsilon(y)\, x'$ for block-diagonal x, x' (part (iii)), so $\mathrm{bd}_\varepsilon(kz) = k\, \mathrm{bd}_\varepsilon(z)$ and $\mathrm{bd}_\varepsilon(u^* z u) = u^* \mathrm{bd}_\varepsilon(z) u$ for $u \in K$.

The polar formula. Let $g = kz$ as above. The splitting $g = k\, z\, 1^*$ ($p = k$, $q = 1$ invertible block-diagonal, z accretive) and Lemma 5.6.26 (i), (ii) give

$$c_{l/2}(g) = \frac{1}{\|m_\mu\|^2} \int_K \frac{m_\mu(ku)\, \overline{m_\mu(u)}}{m_{-l^\rho}(u^* z u)}\, du. \tag{5.105}$$

(ii). By Lemma 5.6.27 (iii), $\mu_{-l} = -\mu$ and $(-l)^\rho = -l^\rho$, so (5.105) for $-l$ has integrand $m_{-\mu}(ku)\overline{m_{-\mu}(u)}/m_{l^\rho}(u^* z u)$ and normalisation $\|m_{-\mu}\|^2$. The matrices ku, u, $u^* z u$ lie in $\mathrm{U}(\operatorname{diag}\varepsilon)$, so by $m_{-w}(x) = \overline{m_w(x)}$ there, this integrand is the complex conjugate of that for l, and $\|m_{-\mu}\|^2 = \|m_\mu\|^2$. Hence $c_{-l/2}(g) = \overline{c_{l/2}(g)}$.

A determinant bound. Let $y \in \mathrm{U}(\operatorname{diag}\varepsilon)$ be positive definite, $a = \mathrm{bd}_\varepsilon(y)$ and $0 \leq c \leq H$ an integer. We show

$$|\det a|^c \leq |m_{-l^\rho}(y)|^2. \tag{5.106}$$

By Lemma 5.6.30 (ii), $a = \frac{1}{2}(y + y^{-1})$ is positive definite. Put $\nu = l^\rho$ and $v = \nu + c$ (add c to each entry); from (3.1), $m_v(x) = (\det x)^c m_\nu(x)$. Choose an integer B with all entries of ν and v in $[-B, B]$; then (3.1) reads $m_w(x) = (\det x)^{-B} \prod_{-B<t\le B} \Delta_{\{w\ge t\}}(x)$ for $w \in \{\nu, v\}$, where $\{w \ge t\} = \{i : w_i \ge t\}$. Fix t and put $Q = \{\nu \ge t\}$, $P = \{v \ge t\}$, $R = P \smallsetminus Q = \{i : t - c \le \nu_i \le t - 1\}$ and $S = \{v < t\} = n \smallsetminus P$. All indices in R have the same sign: if $\varepsilon_i \ne \varepsilon_j$ and, say, $l_i > l_j$, then $l_i - l_j \ge 2H$ by the gap and $2\rho_l(i) \ge 2\rho_l(j) + 2$, so $2\nu_i - 2\nu_j = l_i - l_j + 2\rho_l(i) - 2\rho_l(j) \ge 2H + 2$, both by Lemma 5.6.27 (iii), whereas $|\nu_i - \nu_j| \le c - 1 \le H - 1$ for $i, j \in R$. Hence a and y have the same $R \times R$ block. Fischer's inequality $\Delta_{X\sqcup Y}(A) \le \Delta_X(A)\Delta_Y(A)$ for positive definite A ([133], 7.8.5), applied to a with $P = Q \sqcup R$, then the complement identity for y ($n \smallsetminus R = Q \sqcup S$), then Fischer for y, gives

$$\Delta_P(a) \le \Delta_Q(a)\,\Delta_R(y) = \Delta_Q(a)\,\Delta_{Q\sqcup S}(y) \le \Delta_Q(a)\,\Delta_Q(y)\,\Delta_S(y),$$

all minors being positive. By Jacobi's formula and $|\det y| = 1$, $\Delta_S(y) = |\Delta_P(y^{-1})|$. Taking the product over $-B < t \le B$ and the factors $(\det\cdot)^{-B}$ (with $|\det y| = |\det y^{-1}| = 1$),

$$|m_v(a)| \le |m_\nu(a)|\,|m_\nu(y)|\,|m_v(y^{-1})| = |m_\nu(a)|\,|m_\nu(y)|\,|m_\nu(y^{-1})|,$$

and $|m_v(a)| = (\det a)^c |m_\nu(a)|$ with $m_\nu(a) \ne 0$. Dividing, and using $|m_\nu(y^{-1})| = |m_\nu(y)| = |m_{-\nu}(y)|$ by Lemma 5.6.30 (i), gives (5.106).

(i). Let $g = kz$ as above and $D = |\det \mathrm{bd}_\varepsilon(g)|^c = |\det \mathrm{bd}_\varepsilon(z)|^c > 0$ ($\mathrm{bd}_\varepsilon(g)$ is invertible by Lemma 5.6.25). For $u \in K$, $y = u^* z u$ is a positive definite element of $\mathrm{U}(\mathrm{diag}\,\varepsilon)$ with $\mathrm{bd}_\varepsilon(y) = u^* \mathrm{bd}_\varepsilon(z) u$, so (5.106) gives $|m_{-l^\rho}(u^* z u)| \ge D^{1/2}$. In (5.105), $2|m_\mu(ku)\,m_\mu(u)| \le |m_\mu(ku)|^2 + |m_\mu(u)|^2$, and $\int_K |m_\mu(ku)|^2 du = \int_K |m_\mu(u)|^2 du = \|m_\mu\|^2$ by the invariance of Haar measure. Therefore

$$|c_{l/2}(g)| \le \frac{1}{\|m_\mu\|^2} \cdot \frac{1}{D^{1/2}} \cdot \frac{\|m_\mu\|^2 + \|m_\mu\|^2}{2} = D^{-1/2},$$

that is $|c_{l/2}(g)|^2\,|\det \mathrm{bd}_\varepsilon(g)|^c \le 1$. □

Lemma 5.6.30. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$ and $J = \mathrm{diag}(\varepsilon)$.*

1. *Every $x \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ satisfies $x^{-1} = Jx^*J$, $\|x^{-1}\| = \|x\| \ge 1$ and $|\det x| = 1$, and for all $P \subseteq n$ and $w \in \mathbb{Z}^n$*

$$\Delta_P(x^{-1}) = \overline{\Delta_P(x)}, \qquad |\Delta_{n\smallsetminus P}(x)| = |\Delta_P(x)|, \qquad m_w(x^{-1}) = m_{-w}(x) = \overline{m_w(x)}.$$

2. *Every $g \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ is a product $g = kz$ with $k \in K_\varepsilon$ and $z \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ positive definite. For such z, $z^{-1} = JzJ$ and $\mathrm{bd}_\varepsilon(z) = \frac{1}{2}(z + z^{-1})$ is positive definite; and $\|g\| = \|z\|$.*

3. *If x, x' are matrices with $x_{ij} = x'_{ij} = 0$ whenever $\varepsilon_i \ne \varepsilon_j$, then $\mathrm{bd}_\varepsilon(xyx') = x\,\mathrm{bd}_\varepsilon(y)\,x'$ for every matrix y.*

Proof. (i). From $x^*Jx = J$ and $J^2 = 1$, $(Jx^*J)x = 1$, so $x^{-1} = Jx^*J$; as J is unitary, $\|x^{-1}\| = \|x^*\| = \|x\|$, and $1 \le \|x\|\,\|x^{-1}\| = \|x\|^2$. Taking determinants in $x^*Jx = J$ gives $|\det x|^2 = 1$. For $P \subseteq n$, $(x^{-1})_{PP} = J_{PP}(x_{PP})^* J_{PP}$ with J_{PP} diagonal of entries ± 1, so $\Delta_P(x^{-1}) = (\det J_{PP})^2 \overline{\Delta_P(x)} = \overline{\Delta_P(x)}$ (for $P = \emptyset$ both sides are 1). Jacobi's complementary minor formula $\Delta_P(x^{-1}) = \Delta_{n\smallsetminus P}(x)/\det x$ ([133], 0.8.4) and $|\det x| = 1$ give $|\Delta_{n\smallsetminus P}(x)| = |\Delta_P(x)|$. By (3.1), $m_w(x^{-1})$ is built from the $\Delta_P(x^{-1})$ and $\det x^{-1}$ exactly as $m_w(x)$ from the $\Delta_P(x)$ and $\det x$, so $m_w(x^{-1}) = \overline{m_w(x)}$; and $m_{-w}(x) = m_w(x^{-1})$ by Lemma 5.6.27 (i).

(ii). Let $z = (g^*g)^{1/2}$ be the positive definite square root. The hermitian matrix g^*g lies in $\mathrm{U}(\mathrm{diag}\,\varepsilon)$, so by (i) $J(g^*g)J = (g^*g)^{-1}$. Conjugation by the unitary involution J commutes with the continuous functional calculus of hermitian matrices, hence $JzJ = \left((g^*g)^{-1}\right)^{1/2} = z^{-1}$, and $z^*Jz = zJz = J\,(JzJ)\,z = J$: $z \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$. Then $k = gz^{-1} \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ and $k^*k = z^{-1}g^*gz^{-1} = 1$; from $k^*Jk = J$ and $k^* = k^{-1}$ we get $Jk = kJ$, so $k_{ij} = 0$ whenever $\varepsilon_i \neq \varepsilon_j$, and $k \in K_\varepsilon$. For any positive definite $z \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$, (i) gives $z^{-1} = Jz^*J = JzJ$, and $\mathrm{bd}_\varepsilon(z) = \frac{1}{2}(z + JzJ)$ (the entries with $\varepsilon_i = \varepsilon_j$ are kept, the others cancel), which is $\frac{1}{2}(z + z^{-1})$, positive definite as a sum of positive definite matrices. Finally $\|g\| = \|kz\| = \|z\|$ as k is unitary.

(iii). $(xyx')_{ij} = \sum_{a,b} x_{ia}y_{ab}x'_{bj}$, and a nonzero term has $\varepsilon_a = \varepsilon_i$ and $\varepsilon_b = \varepsilon_j$. If $\varepsilon_i = \varepsilon_j$, then $\varepsilon_a = \varepsilon_b$ in every nonzero term, so $(xyx')_{ij} = (x\,\mathrm{bd}_\varepsilon(y)\,x')_{ij}$. If $\varepsilon_i \neq \varepsilon_j$, then $(\mathrm{bd}_\varepsilon(xyx'))_{ij} = 0$, and every term of $(x\,\mathrm{bd}_\varepsilon(y)\,x')_{ij}$ would need $\varepsilon_i = \varepsilon_a = \varepsilon_b = \varepsilon_j$, so it vanishes as well. □

Lemma 5.6.31. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $l \in \mathbb{Z}^n$ a vector with gap 0 for ε, $\lambda = \frac{1}{2}l$ and $\mu = \mu_l$; for $o, \beta \in \mathbb{Z}^n$ put $o \cdot \beta = \sum_{i\in n} o_i\beta_i$.*

(i) For all $x, b \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$

$$\int_{K_\varepsilon} \overline{m_\mu(k)}\; c_\lambda(xkb)\,dk = \|m_\mu\|^2\, c_\lambda(x)\, c_\lambda(b).$$

(ii) Let $\nu \in \mathbb{Z}^n$ satisfy $\nu_i \geq \nu_j$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$, and suppose that some $o \in \mathbb{Z}^n$ with $o_i < o_j$ whenever $l_i > l_j$ has $o \cdot (\nu - \mu) > 0$. Then for all $x, b \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$

$$\int_{K_\varepsilon} \overline{m_\nu(k)}\; c_\lambda(xkb)\,dk = 0.$$

Proof. Write $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$, $J = \mathrm{diag}(\varepsilon)$, $K = K_\varepsilon$, $c = c_\lambda$, $\|\cdot\|$ for the operator norm, $\Delta_{P,Q}(y)$ for the minor of y with rows P and columns Q ($\Delta_P = \Delta_{P,P}$), and $t^w = \prod_i t_{ii}^{w_i}$. Fix $\nu \in \mathbb{Z}^n$ with $\nu_i \geq \nu_j$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$, and $o \in \mathbb{Z}^n$ with $o_i < o_j$ whenever $l_i > l_j$, such that $N = o \cdot (\nu - \mu) \geq 0$ and $N = 0$ only if $\nu = \mu$. We show, for all $x, b \in G$,

$$\int_K \overline{m_\nu(k)}\, c(xkb)\,dk = \begin{cases} \|m_\mu\|^2\, c(x)\, c(b) & \text{if } \nu = \mu, \\ 0 & \text{if } \nu \neq \mu. \end{cases} \tag{5.107}$$

For (i) take $\nu = \mu$ and $o = -l$: μ satisfies the condition on ν by Lemma 5.6.27 (iii), and $N = 0$. For (ii), $N > 0$ forces $\nu \neq \mu$. As the entries of l are distinct, o is injective: for $i \neq j$ with $l_i > l_j$, $o_i < o_j$. For $Q \subseteq n$ put $o(Q) = \sum_{i\in Q} o_i$.

Let B_l be as in Lemma 5.6.27, and let B' be the group of invertible t with $t_{ij} = 0$ unless $l_i \leq l_j$; Lemma 5.6.27 (ii), applied to the signs $(1, \dots, 1)$ and the vector $-l$ (for which its group is B') and to the weight $-l^\rho$ (admissible, since $l_i^\rho \leq l_j^\rho$ when $l_i < l_j$ by Lemma 5.6.27 (iii)), gives

$$m_{-l^\rho}(c^*Zt) = \overline{c^{-l^\rho}}\; t^{-l^\rho}\; m_{-l^\rho}(Z) \qquad (Z \text{ invertible},\ c, t \in \mathrm{B}'). \tag{5.108}$$

For $s \in \mathbb{C}^\times$ put $\tau_s = \mathrm{diag}(s^{o_i})_{i\in n}$; for $|s| = 1$, $\tau_s \in K$ and $\tau_s^* = \tau_s^{-1}$. For $x, b \in G$ put $T_x(b) = \int_K \overline{m_\nu(k)}\, c(xkb)\,dk$.

Step 1: Gauss factorisation. Number $n = \{i_1, \dots, i_n\}$ so that $l_{i_1} < \cdots < l_{i_n}$, and let $\mathcal{N}$ be the open set of matrices y whose leading principal minors $\Delta_{\{i_1,\dots,i_m\}}(y)$ $(1 \leq m \leq n)$ do not vanish; by Lemma 5.6.25 (ii) it contains every accretive matrix. Gaussian elimination in the order $i_1, \dots, i_n$ writes every $y \in \mathcal{N}$ uniquely as $y = L(y)U(y)$ with $L(y)$ unipotent and $L(y)_{ij} = 0$ unless $i = j$ or $l_i > l_j$, and $U(y) \in \mathrm{B}'$; the entries of $L(y)$ and $U(y)$ are rational functions of

the entries of y whose denominators are leading principal minors, so they are continuous on $\mathcal{N}$, and $L(1) = U(1) = 1$. For $s \in \mathbb{C}$ let $A_y(s)$ be the matrix with entries $L(y)_{ij} s^{o_j - o_i}$; wherever $i \neq j$ and $L(y)_{ij} \neq 0$ we have $l_i > l_j$, so $o_j - o_i$ is a positive integer; thus A_y is a polynomial in s, $A_y(0) = 1$, and $A_y(s) = \tau_s^{-1} L(y) \tau_s$ for $s \neq 0$. Since $L(y)^* \in \mathrm{B}'$ has diagonal 1, (5.108) gives $m_{-l^\rho}(L(y)Z) = m_{-l^\rho}(Z)$ for invertible Z, and therefore

$$m_{-l^\rho}(y) = U(y)^{-l^\rho}, \qquad m_{-l^\rho}(w\,\tau_s^{-1} y \tau_s) = m_{-l^\rho}(y)\, m_{-l^\rho}(w\, A_y(s)) \tag{5.109}$$

for $y \in \mathcal{N}$, $|s| = 1$ and every w with $wA_y(s)$ invertible: indeed $\tau_s^{-1} y \tau_s = A_y(s)\, U_s$ with $U_s = \tau_s^{-1} U(y) \tau_s \in \mathrm{B}'$, whose diagonal is that of $U(y)$, and (5.108) applies.

Step 2: m_ν along a torus. For $j, k \in K$ there is a polynomial $P_{k,j}$ with $P_{k,j}(0) = m_\nu(k)\, m_\nu(j^{-1})$ and

$$m_\nu(k \tau_s j^{-1}) = s^{o \cdot \nu} P_{k,j}(s) \qquad (s \in \mathbb{C}^\times).$$

For $P \subseteq n$, the Cauchy–Binet formula and the diagonality of τ_s give

$$\Delta_P(k \tau_s j^{-1}) = \sum_Q \Delta_{P,Q}(k)\, s^{o(Q)}\, \Delta_{Q,P}(j^{-1}),$$

the sum over $Q \subseteq n$ with $\#Q = \#P$. As k is block-diagonal for the sign blocks $\beta = \{i : \varepsilon_i = \pm 1\}$, $\Delta_{P,Q}(k) = 0$ unless $\#(Q \cap \beta) = \#(P \cap \beta)$ for both blocks. Now let $P = P_t = \{i : \nu_i \geq t\}$. By the condition on ν, $P_t \cap \beta$ consists of the $\#(P_t \cap \beta)$ indices of β with the largest l_i, that is, with the smallest o_i; as o is injective, every other subset of β of the same size has a strictly larger sum of the o_i. Hence every $Q \neq P_t$ with nonzero term has $o(Q) > o(P_t)$, and $\Delta_{P_t}(k \tau_s j^{-1}) = s^{o(P_t)} (\Delta_{P_t}(k) \Delta_{P_t}(j^{-1}) + s\, R_t(s))$ with a polynomial R_t; also $\det(k \tau_s j^{-1}) = \det(k) \det(j^{-1})\, s^{o(n)}$. Inserting this in (3.1) gives the claim, with

$$P_{k,j}(s) = (\det k \det j^{-1})^{\min_i \nu_i} \prod_{\min \nu < t \leq \max \nu} (\Delta_{P_t}(k) \Delta_{P_t}(j^{-1}) + s R_t(s)),$$

the exponent being $o(n) \min_i \nu_i + \sum_{\min \nu < t \leq \max \nu} o(P_t) = \sum_i o_i \nu_i$.

Step 3: a double integral, for b near 1. Fix $x \in G$, put $a = \mathrm{bd}_\varepsilon(x)$ and $z = a^{-1} x$, accretive by Lemma 5.6.25 (i). The set $\{j^* z j : j \in K\}$ is a compact set of accretive matrices and accretivity is an open condition, so there is $\delta \in (0, 1)$ such that wA is accretive whenever $w = j^* z j$ $(j \in K)$ and $\|A - 1\| \leq \delta$. By continuity of L at 1 there is $\delta' \in (0, \delta]$ such that every y with $\|y - 1\| \leq \delta'$ lies in $\mathcal{N}$ and satisfies $\|A_y(s) - 1\| \leq \delta$ for $|s| \leq 2$ (the entries of $A_y(s) - 1$ are $L(y)_{ij} s^{o_j - o_i}$, $i \neq j$). Let $\mathcal{U} = \{b \in G : \|b - 1\| \leq \delta'\}$, a neighbourhood of 1; for $b \in \mathcal{U}$ and $k \in K$, $y = k^* b k$ has $\|y - 1\| = \|b - 1\| \leq \delta'$, and b is accretive since $b + b^* \geq 2 - 2\|b - 1\| > 0$.

Let $b \in \mathcal{U}$ and $r \in K$. Then ar is block-diagonal and invertible and $(ar)^{-1}(xrb) = (r^* z r)\, b$ is accretive, so by Lemma 5.6.26 (i) and (3.2)

$$\|m_\mu\|^2 c(xrb) = \int_K \frac{m_\mu(ark)\, \overline{m_\mu(k)}}{m_{-l^\rho}(k^* r^* z r b k)}\, dk.$$

With $j = rk$ we have $k^* r^* z r b k = (j^* z j)(k^* b k)$, and $\overline{m_\nu(r)} = m_\nu(r^{-1}) = m_\nu(k j^{-1})$ by Lemma 5.6.30 (i). Put

$$H(j, k) = \frac{m_\nu(k j^{-1})\, m_\mu(aj)\, \overline{m_\mu(k)}}{m_{-l^\rho}((j^* z j)(k^* b k))} \qquad (j, k \in K),$$

continuous on $K \times K$, the argument of the denominator being accretive. Then

$$\overline{m_\nu(r)}\, \|m_\mu\|^2 c(xrb) = \int_K H(rk, k)\, dk;$$

integrating over r, using Fubini's theorem and, for fixed k, the invariance of dr under $r \mapsto rk$,

$$\|m_\mu\|^2\, T_x(b) = \int_K \int_K H(j,k)\, dj\, dk.$$

Step 4: averaging over the circle. For $|s| = 1$ the map $(j,k) \mapsto (j, k\tau_s)$ preserves $dj\, dk$, so, averaging over $s = e^{i\theta}$ and using Fubini's theorem, $\iint H = \iint M$ with $M(j,k) = \frac{1}{2\pi}\int_0^{2\pi} H(j, k\tau_{e^{i\theta}})\, d\theta$. Fix j, k, put $w = j^* z j$, $y = k^* b k$, and let $|s| = 1$. Then $(k\tau_s)^* b (k\tau_s) = \tau_s^{-1} y \tau_s$, $wA_y(s)$ is accretive, and (5.109) gives $m_{-l^\rho}(w\,\tau_s^{-1} y \tau_s) = m_{-l^\rho}(y)\, m_{-l^\rho}(wA_y(s))$. As $\tau_s \in \mathrm{B}_l$ is diagonal, $m_\mu(k\tau_s) = \tau_s^\mu m_\mu(k) = s^{o\cdot\mu} m_\mu(k)$ by Lemma 5.6.27 (ii) (for the weight μ, admissible by Lemma 5.6.27 (iii)), so $\overline{m_\mu(k\tau_s)} = s^{-o\cdot\mu}\, \overline{m_\mu(k)}$. With Step 2,

$$H(j, k\tau_s) = s^N P_{k,j}(s)\, q(s), \qquad q(s) = \frac{m_\mu(aj)\, \overline{m_\mu(k)}}{m_{-l^\rho}(y)\, m_{-l^\rho}\big(wA_y(s)\big)}.$$

For $|s| < 2$ the matrix $wA_y(s)$ is accretive, so its determinant and principal minors do not vanish, and q is holomorphic on $\{|s| < 2\}$. By the mean value property of the holomorphic function $s^N P_{k,j} q$, $M(j,k)$ is its value at $s = 0$. If $\nu \neq \mu$, then $N > 0$ and $M(j,k) = 0$. If $\nu = \mu$, then $N = 0$, and since $A_y(0) = 1$ and $m_\mu(j^{-1}) = \overline{m_\mu(j)}$ (Lemma 5.6.30 (i)),

$$M(j,k) = P_{k,j}(0)\, q(0) = \frac{m_\mu(aj)\, \overline{m_\mu(j)}}{m_{-l^\rho}(j^* z j)} \cdot \frac{|m_\mu(k)|^2}{m_{-l^\rho}(k^* b k)}.$$

Step 5: (5.107) *for* $b \in \mathcal{U}$. If $\nu \neq \mu$, Steps 3 and 4 give $\|m_\mu\|^2 T_x(b) = 0$; note $\|m_\mu\|^2 > 0$. If $\nu = \mu$: by (3.2), $\int_K \frac{m_\mu(aj)\overline{m_\mu(j)}}{m_{-l^\rho}(j^* z j)} dj = \|m_\mu\|^2 \mathrm{FJ}_l(a, a^{-1}x) = \|m_\mu\|^2 c(x)$, and $\int_K \frac{|m_\mu(k)|^2}{m_{-l^\rho}(k^* b k)} dk = \|m_\mu\|^2 \mathrm{FJ}_l(1, b) = \|m_\mu\|^2 c(b)$ by Lemma 5.6.26 (i) (1 is block-diagonal and b accretive). Hence $\|m_\mu\|^2 T_x(b) = \|m_\mu\|^4 c(x) c(b)$, that is $T_x(b) = \|m_\mu\|^2 c(x) c(b)$.

Step 6: all b*, by analytic continuation.* Let $b \in G$. By Lemma 5.6.30 (ii), $b = k_0 z_0$ with $k_0 \in K$ and $z_0 \in G$ positive definite, $z_0^{-1} = J z_0 J$. Let Z_0 be the hermitian logarithm of z_0; as J is a unitary involution, $\log(J z_0 J) = J Z_0 J$, so $J Z_0 J = -Z_0$, whence $Z_0^* J + J Z_0 = 0$. The block-diagonal unitary k_0 is e^{X_0} with X_0 block-diagonal and skew-hermitian (a logarithm in each block), so X_0 commutes with J and $X_0^* J + J X_0 = 0$. Thus $\gamma(t) = e^{tX_0} e^{tZ_0}$ is a curve in G with real-analytic entries, $\gamma(0) = 1$, $\gamma(1) = b$.

Let $\phi(t)$ be the right side of (5.107) at $\gamma(t)$: 0 if $\nu \neq \mu$, and $\|m_\mu\|^2 c(x) c(\gamma(t))$ if $\nu = \mu$; it is real-analytic in t, since c is locally the restriction of a holomorphic function (Lemma 5.6.24 (ii)). So is $t \mapsto T_x(\gamma(t))$: let $y_0 \in G$. For $r_0 \in K$ put $a_{r_0} = \mathrm{bd}_\varepsilon(x r_0 y_0)$; since $a_{r_0}^{-1} x r_0 y_0$ is accretive (Lemma 5.6.25 (i)), there are open sets $N_{r_0} \ni r_0$ in K and $V_{r_0} \ni y_0$ in $\mathrm{GL}_n(\mathbb{C})$ with $a_{r_0}^{-1} x r y$ accretive for $(r, y) \in N_{r_0} \times V_{r_0}$. Choose $r_1, \dots, r_m$ with $K = \bigcup_i N_{r_i}$, put $V = \bigcap_i V_{r_i}$, and let (χ_i) be a continuous partition of unity on K with $\operatorname{supp}\chi_i \subseteq N_{r_i}$. For $y \in V \cap G$, Lemma 5.6.26 (i) gives

$$T_x(y) = \sum_{i=1}^m \int_K \chi_i(r)\, \overline{m_\nu(r)}\, \mathrm{FJ}_l(a_{r_i}, a_{r_i}^{-1} x r y)\, dr,$$

and each term extends to a holomorphic function of $y \in V$: its integrand is continuous on $\operatorname{supp}\chi_i \times V$ and holomorphic in y (by (3.2), $\mathrm{FJ}_l(a, z)$ is the K-integral of a function continuous in (k, z) and holomorphic in z on accretive z, the denominator being a product of powers of principal minors of the accretive matrix $k^* z k$), so by Morera's theorem and Fubini's theorem in each variable the integral is holomorphic. Hence T_x is locally the restriction of a holomorphic function and $T_x \circ \gamma$ is real-analytic. The two real-analytic functions $T_x \circ \gamma$ and ϕ agree for t near 0, where $\gamma(t) \in \mathcal{U}$ (Step 5), hence on $[0,1]$; at $t = 1$ this is (5.107) for b. □

Lemma 5.6.32. *Let ι be a nonempty finite set of N elements, $\zeta \in \{\pm 1\}^\iota$, S a group with a σ-algebra for which left and right translations are measurable, ν a right-invariant measure on S, and $j\colon S \to \mathrm{U}(\operatorname{diag}\zeta)$ a homomorphism.*

1. *Let $y \in \mathrm{U}(\operatorname{diag}\zeta)$ and $D, R \geq 1$, $K \geq 3DR^2$ be reals. Then*

$$\nu\{s : \|j(s)y\| \leq R\} \leq K^{2N^2}\, \nu\{s : \|j(s) - 1\| \leq D^{-1}\}.$$

2. *Suppose moreover that S carries a topology whose Borel σ-algebra is the given one, that ν is finite on compact sets and that j is proper. Then $\nu\{s : \|j(s) - 1\| \leq R\} < \infty$ for every real R, and for every $y \in \mathrm{U}(\operatorname{diag}\zeta)$ and every integer $c > 8N^2$ the function $s \mapsto (2/\|j(s)y\|)^c$ is ν-integrable with*

$$\int_S \Big(\frac{2}{\|j(s)y\|}\Big)^c d\nu(s) \leq 16^c\, \nu\{s : \|j(s) - 1\| \leq 1\}.$$

Proof. For $g \in \mathrm{U}(\operatorname{diag}\zeta)$ and $Z = \operatorname{diag}(\zeta)$ we have $g^{-1} = Zg^*Z$, so $\|g^{-1}\| = \|g\|$, and $1 = \|Z\| = \|g^*Zg\| \leq \|g\|^2$, so $\|g\| \geq 1$. Where a set below is not known to be measurable, ν is applied as an outer measure; right translations are measurable bijections with measurable inverses, so the outer measure is right invariant as well.

(i) Put $f(s) = j(s)y$, $\mathcal{A} = \{s : \|f(s)\| \leq R\}$, $\mathcal{B} = \{s : \|j(s) - 1\| \leq D^{-1}\}$ and $\delta = (DR)^{-1}$. Call a subset $F \subseteq \mathcal{A}$ separated if $\|f(s) - f(s')\| \geq \delta$ for distinct $s, s' \in F$. For a separated F the closed balls of radius $\delta/2$ (operator norm) about the points $f(s)$, $s \in F$, have disjoint interiors and lie in the ball of radius $R + \delta/2$ about 0 in the real vector space of complex $N \times N$ matrices, of dimension $2N^2$; comparing Lebesgue measures, which scale by r^{2N^2} under dilation by r,

$$\#F \leq \Big(\frac{R + \delta/2}{\delta/2}\Big)^{2N^2} = (1 + 2DR^2)^{2N^2} \leq (3DR^2)^{2N^2} \leq K^{2N^2},$$

as $DR^2 \geq 1$. Take a separated F of maximal cardinality. For $s \in \mathcal{A} \setminus F$, $F \cup \{s\}$ is not separated, so there is $s' \in F$ with $\|f(s) - f(s')\| < \delta$; for $s \in F$ take $s' = s$. Then

$$\|j(ss'^{-1}) - 1\| = \|(f(s) - f(s'))f(s')^{-1}\| \leq \delta\, \|f(s')\| \leq \delta R = D^{-1},$$

since $f(s')^{-1} \in \mathrm{U}(\operatorname{diag}\zeta)$ has the norm of $f(s')$. So $s \in \mathcal{B}s'$, and $\mathcal{A} \subseteq \bigcup_{s' \in F} \mathcal{B}s'$. By right invariance $\nu(\mathcal{A}) \leq \#F\, \nu(\mathcal{B}) \leq K^{2N^2}\nu(\mathcal{B})$.

(ii) The set $\{s : \|j(s) - 1\| \leq R\}$ is the preimage under the proper map j of the intersection of the closed group $\mathrm{U}(\operatorname{diag}\zeta)$ with a closed ball of matrices, a compact set; so it is compact and has finite measure. Let $y \in \mathrm{U}(\operatorname{diag}\zeta)$, $c > 8N^2$, $F(s) = \|j(s)y\| \geq 1$, a continuous function, and $U_1 = \{s : \|j(s) - 1\| \leq 1\}$. For $0 < u < 1$ apply (i) with $D = 1$, $R = u^{-1/c} \geq 1$, $K = 3R^2$:

$$\nu\{s : F(s)^{-c} > u\} \leq \nu\{s : F(s) \leq u^{-1/c}\} \leq 3^{2N^2} u^{-4N^2/c}\, \nu(U_1),$$

while $\{F^{-c} > u\} = \emptyset$ for $u \geq 1$. By the layer-cake formula and $4N^2/c < \frac{1}{2}$,

$$\int_S F^{-c}\, d\nu = \int_0^1 \nu\{F^{-c} > u\}\, du \leq 3^{2N^2}\nu(U_1) \int_0^1 u^{-4N^2/c}\, du \leq 2 \cdot 3^{2N^2}\nu(U_1) < \infty.$$

So $(2/F)^c = 2^c F^{-c}$ is ν-integrable and, as $2N^2 + 1 \leq c$, $\int_S (2/F)^c\, d\nu \leq 2^c \cdot 2 \cdot 3^{2N^2}\nu(U_1) \leq 6^c\nu(U_1) \leq 16^c\nu(U_1)$. □

This group identifies the Bessel-period map at a suitable point with a combination of class Poincaré series, and computes the moments of a map with amplitude blocks near the Bessel subgroup. The first task is Lemma 5.6.33: the Bessel period of a source form, an integral over the compact quotient of the Bessel group, is represented by a unique form, and at a suitable point and linear form this form is the combination of the class Poincaré series with the source coordinates as coefficients. Its proof unfolds the Bessel period class by class into integrals over the real Bessel group (Proposition 5.6.35). The compactness rests on that of the adelic quotients of a unipotent group and of a definite unitary group [37], and the Bessel average of Flensted-Jensen's function is reproduced by the discrete-series kernel (Lemma 5.6.37). For Bessel periods of unitary groups see [96], for the real groups $\mathrm{U}(p,q)$ [125], and for the uniqueness of Bessel models at the real places [153].

After the lemma the parameters of the normalised Poincaré map are fixed for the rest of the paper: the Poincaré map is the Bessel-period map at a Bessel pair, the seesaw constant is one that Proposition 4.2.25 supplies for this map, and the divisibility errors are a family that Proposition 4.2.54 supplies. These parameters are used in two places only: in the proof of the main inequality (Section 5.9) and in the comparison operator of Section 6 (Proposition 6.6.13).

The second task is Proposition 5.6.42. On the level elements that agree with the Bessel subgroup at the chosen places above one prime, the Gram blocks of a map with amplitude blocks are the Bessel translates, up to a second moment of at most $b^{-12}\mathcal{D}_{\mathrm{id}}$. The traces of the Bessel translates are read through the reduced character, whose conjugation mean has square sum at most 2 (Lemma 5.6.43). The proposition also gives the passage from the terms of nonzero discriminant to all terms and the bound for the scalar cosets that Theorem 5.1.13 uses. The near-isometry Proposition 5.6.41, printed before it, rests on its later clauses: for central idempotents of small defect at the chosen places above p_0 and p_1, the good part of the map is a near-isometry with error $5b^{-4}$. With the rarity of bad types this is part (i) of Proposition 5.6.48.

Lemma 5.6.33. *Let D be admissible data.*

(i) For $g \in G(F_\infty)$, a linear form t on W_λ, $\varphi \in \mathcal{S}$ and $y \in \mathcal{A}_M(Y_0)$, the function $r \mapsto \mathcal{W}_{t,\varphi}(r)\,\overline{y(r\,(g,1))}$ on $R(\mathbb{A})$ is continuous, left invariant under $R(E)$ and integrable over $[R]$. There is a unique $\mathcal{P}_{g,t}\varphi \in \mathcal{A}_M(Y_0)$ with

$$\langle y, \mathcal{P}_{g,t}\varphi\rangle = \mathrm{BP}_{g,t}(\varphi, y) \qquad \text{for every } y \in \mathcal{A}_M(Y_0),$$

and $\mathcal{P}_{g,t}\colon \mathcal{S} \to \mathcal{A}_M(Y_0)$ is linear.

*(ii) Let p be the matrix of Proposition 5.7.3 (it has the properties required in Lemma 5.6.18), and Π_a ($a \in \mathcal{C}\times I$) the class Poincaré series of Lemma 5.6.18 (ii) for this p. There are $g_\infty \in G(F_\infty)$ and a linear form t_0 on W_λ (a **Bessel pair**) such that for every $\varphi \in \mathcal{S}$*

$$\mathcal{P}_{g_\infty,t_0}\,\varphi = \sum_{a\in\mathcal{C}\times I} \theta(\varphi)_a\,\Pi_a.$$

Proof. (i). A form $y \in \mathcal{A}_M(Y_0)$ is continuous and bounded and left invariant under the image of $G(E)$ ((FT2) of Definition 3.2.4); read on $G(\mathbb{A})$ through the frames, $x \mapsto y(x)$ is continuous on $G(\mathbb{A})$ and left $G(E)$-invariant, so $r \mapsto \overline{y(r(g,1))}$ is continuous on $R(\mathbb{A})$ and left $R(E)$-invariant. By Lemma 5.6.34 (iv) so is $\mathcal{W}_{t,\varphi}$, hence the product, which is therefore integrable over $[R]$ by Lemma 5.6.34 (iii). Consequently $\mathrm{BP}_{g,t}(\varphi, y)$ is linear in φ ($\mathcal{W}_{t,\varphi}$ is linear in φ) and conjugate-linear in y. By Proposition 5.8.4 with $V = V' = Y_0$ (its hypothesis is (iii) there), $\mathcal{A}_M(Y_0)$ is finite-dimensional. The Petersson product is positive definite on it: if $\langle F, F\rangle = 0$, the continuous

$|F|^2$ vanishes almost everywhere on a Borel fundamental set of the image of $G(E)$, hence almost everywhere by invariance, hence everywhere, Haar measure charging every nonempty open set. So $w \mapsto \langle \cdot, w\rangle$ is an injective linear map of $\mathcal{A}_M(Y_0)$ into the space of conjugate-linear forms on $\mathcal{A}_M(Y_0)$, which has the same finite dimension; it is bijective, and $\mathcal{P}_{g,t}\varphi$ exists and is unique. Linearity in φ follows from uniqueness and the linearity of $\mathrm{BP}_{g,t}$ in φ.

(ii). Let g_∞ and t_0 be as in Proposition 5.6.35, and $\varphi \in \mathcal{S}$. The form $\Sigma = \sum_a \theta(\varphi)_a \Pi_a$ lies in $\mathcal{A}_M(Y_0)$ by Lemma 5.6.18 (ii), and for every $y \in \mathcal{A}_M(Y_0)$

$$\langle y, \Sigma\rangle = \mathrm{BP}_{g_\infty,t_0}(\varphi, y) = \langle y, \mathcal{P}_{g_\infty,t_0}\varphi\rangle$$

by Proposition 5.6.35 and (i). Taking $y = \Sigma - \mathcal{P}_{g_\infty,t_0}\varphi$ and using positive definiteness gives $\Sigma = \mathcal{P}_{g_\infty,t_0}\varphi$. □

Let D be admissible data. From here on the following three choices are fixed.

- A Bessel pair (g_∞, t_0) as in Lemma 5.6.33 (ii) is fixed, and the **Poincaré map** is $\mathcal{P} = \mathcal{P}_{g_\infty,t_0} \colon \mathcal{S} \to \mathcal{A}_M(Y_0)$ of Lemma 5.6.33 (i): the linear map with $\langle y, \mathcal{P}\varphi\rangle = \mathrm{BP}_{g_\infty,t_0}(\varphi, y)$ for every $\varphi \in \mathcal{S}$ and every $y \in \mathcal{A}_M(Y_0)$.
- The **seesaw constant** $c_{\mathrm{ss}} \in \mathbb{C}^\times$ is a scalar as in Proposition 4.2.25, for this Poincaré map, fixed once.
- The **divisibility errors** $\mathrm{err} = (\mathrm{err}_\ell)_\ell = (\epsilon_\ell)_\ell$ are a family of nonnegative real numbers as in Proposition 4.2.54, indexed by the rational primes, fixed once.

The gain exponents g_ℓ, the inverse gain factor c_B, the normalised Poincaré map $\mathfrak{p} \colon \mathcal{S}^{\mathrm{ram}} \to \mathcal{T}$ and its translates $s\mathfrak{p}$ ($s \in \bar{G}$) of D are those of Definition 3.2.37 for the parameters $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$. These parameters have the properties assumed of them in the results stated for parameters: the family err satisfies the error conditions of Definition 3.2.37 and has the uniform local property of Proposition 4.2.54, by that proposition; for every ordinary twist ϑ the coefficient function Ψ_ϑ of Proposition 4.2.25 is admissible for ϑ and satisfies the seesaw identity $\Theta_\vartheta(\mathcal{P}\varphi) = c_{\mathrm{ss}} B_{\vartheta,\Psi_\vartheta}(\varphi)$ for every $\varphi \in \mathcal{S}^{\mathrm{ram}}$, by conditions 1–4 there; and $\mathcal{P}\varphi = \sum_{a \in \mathcal{C}\times I} \theta(\varphi)_a \, \Pi_a$ for every $\varphi \in \mathcal{S}$, with the coordinates θ and the class Poincaré series Π_a of Lemma 5.6.33 (ii), by that lemma.

Lemma 5.6.34. *Let D be admissible data, and put $\mathsf{U}(E) = \mathsf{U}(\mathbb{A}) \cap G(E)$.*

(i) $R(\mathbb{A})$ is a closed subgroup of $G(\mathbb{A})$ containing $\mathsf{U}(\mathbb{A})$ and $\iota(\mathrm{U}(X_0)(\mathbb{A}_{E^+}))$. For $r \in R(\mathbb{A})$, $a(r) \in \mathrm{U}(X_0)(\mathbb{A}_{E^+})$; $r \mapsto a(r)$ is a continuous homomorphism with kernel $\mathsf{U}(\mathbb{A})$ and $a(\iota(h)) = h$; hence $u(r) \in \mathsf{U}(\mathbb{A})$ and $r = u(r)\,\iota(a(r))$.

(ii) On $\mathsf{U}(\mathbb{A})$, ψ_{U} is a continuous homomorphism to the unit circle, trivial on $\mathsf{U}(E)$, and $\psi_{\mathsf{U}}(\iota(h) u\, \iota(h)^{-1}) = \psi_{\mathsf{U}}(u)$ for $h \in \mathrm{U}(X_0)(\mathbb{A}_{E^+})$, $u \in \mathsf{U}(\mathbb{A})$.

(iii) $R(E)$ is a discrete subgroup of the locally compact group $R(\mathbb{A})$, the quotient $R(E)\backslash R(\mathbb{A})$ is compact, and every continuous left $R(E)$-invariant function on $R(\mathbb{A})$ is bounded and integrable over $[R]$, for every left Haar measure dr.

(iv) For $\varphi \in \mathcal{S}$ and a linear form t on W_λ, $\mathcal{W}_{t,\varphi}$ is continuous and $\mathcal{W}_{t,\varphi}(\gamma r) = \mathcal{W}_{t,\varphi}(r)$ for $\gamma \in R(E)$, $r \in R(\mathbb{A})$.

Proof. Matrices are taken in the basis of Y_0, over the ring $\mathbb{A}_E$. For an even integer d let P_d be the $\mathbb{A}_E$-span of the basis vectors of grade $\geq d$. Grades are even, so a basis vector of grade $> g$ has grade $\geq g+2$. The form pairs v_i with $v_{2\varrho+2-i}$ (grades g and $-g$), is the identity on X_0 and has $\langle v_{\varrho+1}, v_{\varrho+1}\rangle = -N$; hence $P_2 = \langle v_1, \dots, v_\varrho\rangle$ is isotropic, $P_2^\perp = P_0 = P_2 \oplus X_0 \oplus \langle v_{\varrho+1}\rangle$, and P_0/P_2 with the induced form has the orthogonal basis X_0, $v_{\varrho+1}$, of norms 1 and $-N \in \mathbb{A}_E^\times$.

(i). By definition, $g \in G(\mathbb{A})$ lies in $R(\mathbb{A})$ if and only if $(g-1)v_k \in P_{\mathrm{gr}(v_k)+2}$ for every string vector v_k of grade ≥ 0 (that is $k \leq \varrho+1$). Such g maps P_d into itself for every even $d \geq 2$ (P_d is spanned by string vectors of grade $\geq d$), hence onto itself (g is injective and P_d is free of finite rank, so $g|_{P_d}$ is invertible by the determinant), and $P_0 = P_2^\perp$ onto itself as g is unitary. If $g, g' \in R(\mathbb{A})$ then $(gg'-1)v_k = g(g'-1)v_k + (g-1)v_k \in P_{\mathrm{gr}(v_k)+2}$ and $(g^{-1}-1)v_k = -g^{-1}(g-1)v_k \in P_{\mathrm{gr}(v_k)+2}$; the conditions are closed. So $R(\mathbb{A})$ is a closed subgroup; it contains $\mathsf{U}(\mathbb{A})$ (whose conditions include these) and $\iota(\mathrm{U}(X_0)(\mathbb{A}_{E^+}))$ (which fixes every v_i). Let $r \in R(\mathbb{A})$. It acts on P_0/P_2 by a unitary map $\bar r$ fixing $\bar v_{\varrho+1}$, so $\bar r$ preserves $\bar v_{\varrho+1}^\perp = \bar X_0$: for $x \in X_0$, $rx \equiv a(r)x \pmod{P_2}$, where $a(r)$ is the X_0-block of r, and $a(r)$ is unitary on X_0. As $r \mapsto \bar r$ is multiplicative, so is a; it is continuous (entries), and $a(\iota(h)) = h$. If $a(r) = 1$, then $r \in \mathsf{U}(\mathbb{A})$: the conditions $r_{ik} = \delta_{ik}$ for $\mathrm{gr}(i) \leq \mathrm{gr}(k)$ hold for string k of grade ≥ 0 by definition; for $k = x \in X_0$ because $rx \in x + P_2$; and for $k = v_j$ of grade $g_j < 0$ by unitarity: the coefficient of a basis vector i with $\mathrm{gr}(i) \leq g_j$ in rv_j is $\langle i^\vee, rv_j\rangle = \langle r^{-1}i^\vee, v_j\rangle$, where $i^\vee$ is the basis vector paired with i (up to the factor $-1/N$ only for $i = v_{\varrho+1}$, which has grade $0 > g_j$ and does not occur), of grade $-\mathrm{gr}(i) \geq -g_j \geq 2$; so $r^{-1}i^\vee \in i^\vee + P_{-\mathrm{gr}(i)+2}$ ($r^{-1} \in R(\mathbb{A})$), and $P_{-\mathrm{gr}(i)+2}$ is orthogonal to v_j because v_j pairs only with the vector of grade $-g_j < -\mathrm{gr}(i)+2$; hence the coefficient is $\langle i^\vee, v_j\rangle = \delta_{i,v_j}$. Conversely $\mathsf{U}(\mathbb{A}) \subseteq \ker a$, as its elements fix X_0 modulo P_2. So $\ker a = \mathsf{U}(\mathbb{A})$, and $u(r) = r\,\iota(a(r))^{-1} \in \ker a$.

(ii). Let $\mathfrak{e}$ be as in Definition 3.2.36: its entry (i,k) vanishes unless $\mathrm{gr}(i) = \mathrm{gr}(k) - 2$, so $\mathrm{tr}\,\mathfrak{e} = 0$. For $u \in \mathsf{U}(\mathbb{A})$ the entry (i,k) of $u-1$ vanishes unless $\mathrm{gr}(i) \geq \mathrm{gr}(k)+2$. For $u, u' \in \mathsf{U}(\mathbb{A})$ the entries (i,k) of $\mathfrak{e}(u-1)(u'-1)$ vanish unless $\mathrm{gr}(i) \geq \mathrm{gr}(k)+2$, so its trace is 0, and

$$\mathrm{tr}(\mathfrak{e}uu') = \mathrm{tr}(\mathfrak{e}u) + \mathrm{tr}(\mathfrak{e}u') - \mathrm{tr}(\mathfrak{e}) + \mathrm{tr}\big(\mathfrak{e}(u-1)(u'-1)\big) = \mathrm{tr}(\mathfrak{e}u) + \mathrm{tr}(\mathfrak{e}u').$$

As ψ_E is a continuous character of $\mathbb{A}_E$ trivial on E, ψ_{U} is a continuous homomorphism on $\mathsf{U}(\mathbb{A})$ with values of modulus one, and for $u \in \mathsf{U}(E)$, $\mathrm{tr}(\mathfrak{e}u)/(2M\varpi) \in E$, so $\psi_{\mathsf{U}}(u) = 1$. Finally $\iota(h)^{-1}\mathfrak{e}\,\iota(h) = \mathfrak{e}$: $\mathfrak{e}$ vanishes on X_0, which $\iota(h)$ preserves, and maps string vectors to multiples of string vectors, which $\iota(h)$ fixes; so $\mathrm{tr}(\mathfrak{e}\,\iota(h)u\iota(h)^{-1}) = \mathrm{tr}(\mathfrak{e}u)$.

(iii). $G(E)$ is discrete in $G(\mathbb{A})$, as E is discrete in $\mathbb{A}_E$, so $R(E)$ is discrete in the closed subgroup $R(\mathbb{A})$, which is locally compact. The group U is the unipotent radical of the stabiliser of the isotropic flag $P_{2\varrho} \subset \cdots \subset P_2$, a unipotent group over E^+; by Theorem A.4.3 there is a compact $C_{\mathsf{U}} \subseteq \mathsf{U}(\mathbb{A})$ with $\mathsf{U}(\mathbb{A}) = \mathsf{U}(E)C_{\mathsf{U}}$. The form of X_0 is definite at every embedding, so by Theorem A.4.5 there is a compact $C_H \subseteq \mathrm{U}(X_0)(\mathbb{A}_{E^+})$ with $\mathrm{U}(X_0)(\mathbb{A}_{E^+}) = \mathrm{U}(X_0)(E^+)C_H$. Let $r \in R(\mathbb{A})$; by (i), $r = u\,\iota(h)$ with $u \in \mathsf{U}(\mathbb{A})$, and $h = \delta c$ with δ rational, $c \in C_H$. Then $\iota(\delta)^{-1}u\,\iota(\delta) \in \mathsf{U}(\mathbb{A})$ ($\mathsf{U}(\mathbb{A}) = \ker a$ is normal) equals $u'c'$ with $u' \in \mathsf{U}(E)$, $c' \in C_{\mathsf{U}}$, and $r = \iota(\delta)u'{\cdot}c'\iota(c) \in R(E){\cdot}C$ with $C = C_{\mathsf{U}}\,\iota(C_H)$ compact. So $R(E)\backslash R(\mathbb{A})$ is compact. Let F be a Borel fundamental set for $R(E)$ in $R(\mathbb{A})$ (Lemma 5.6.20); as every orbit meets C, $1 \leq \sum_{\gamma\in R(E)} \mathbf{1}_C(\gamma x)$ for $x \in F$, and by left invariance of dr and disjointness of the γF,

$$dr(F) \leq \sum_{\gamma\in R(E)} dr(F \cap \gamma^{-1}C) = \sum_{\gamma\in R(E)} dr(\gamma F \cap C) \leq dr(C) < \infty.$$

A continuous left $R(E)$-invariant function takes on $R(\mathbb{A}) = R(E)C$ only the values it takes on C, so it is bounded, and it is integrable over $[R]$ (that is, on F).

(iv). Let $\gamma \in R(E)$. Its Levi part $a(\gamma)$ is the X_0-block of a rational matrix, so $a(\gamma) \in \mathrm{U}(X_0)(E^+)$, $\iota(a(\gamma)) \in G(E)$ and $u(\gamma) \in \mathsf{U}(E)$; the components of $a(\gamma)$ at $v \in \Phi$ are $F_v(a(\gamma))$.

By (i), $a(\gamma r) = a(\gamma)a(r)$ and $u(\gamma r) = \gamma r\,\iota(a(r))^{-1}\iota(a(\gamma))^{-1} = u(\gamma)\cdot\iota(a(\gamma))u(r)\iota(a(\gamma))^{-1}$, so $\psi_{\mathsf{U}}(u(\gamma r)) = \psi_{\mathsf{U}}(u(r))$ by (ii). The transformation law of source forms gives $\varphi(a(\gamma)_f a(r)_f) = \rho_\lambda((\overline{F_v(a(\gamma))})_v)\varphi(a(r)_f)$, and $\rho_\lambda((\overline{a(\gamma r)_v})_v)^{-1} = \rho_\lambda((\overline{a(r)_v})_v)^{-1}\rho_\lambda((\overline{F_v(a(\gamma))})_v)^{-1}$; inserting both in (3.5) gives $\mathcal{W}_{t,\varphi}(\gamma r) = \mathcal{W}_{t,\varphi}(r)$. Continuity: a, u, ψ_{U} and ρ_λ are continuous, and φ is locally constant, being right invariant under the open subgroup K_0. □

Proposition 5.6.35. *Let D be admissible data, let p be the matrix of Proposition 5.7.3 (it has the properties required in Lemma 5.6.18, by (iii) there), and let $\Pi_a \in \mathcal{A}_M(Y_0)$ ($a \in \mathcal{C}\times I$) be the class Poincaré series of Lemma 5.6.18 (ii) for this p. There are $g_\infty \in G(F_\infty)$ and a linear form t_0 on W_λ such that for every $\varphi \in \mathcal{S}$ and every $y \in \mathcal{A}_M(Y_0)$*

$$\Big\langle y,\ \sum_{a\in\mathcal{C}\times I} \theta(\varphi)_a\,\Pi_a \Big\rangle = \mathrm{BP}_{g_\infty,t_0}(\varphi, y).$$

Proof. Read everything on $G(\mathbb{A}) = G(F_\infty)\times G(\mathbb{A}_f)$ as a group of operators, $\Gamma = G(E)$. Let μ_∞ and μ_f be the Haar measures of $G(F_\infty)$ and $G(\mathbb{A}_f)$ that correspond through the frames of Y_0 to the measures of $G_\infty^{Y_0}$ and G_f fixed in (FT2), $\mu = \mu_\infty\otimes\mu_f$, $V_M = \mu_f(\mathrm{K}_M) > 0$, and F_Γ a Borel set meeting every orbit Γx in exactly one point (Lemma 5.6.20); the Petersson product is $\langle y, y'\rangle = \int_{F_\Gamma} \overline{y}\,y'\,d\mu$. Let $e(h) = e_\varsigma((P_v^{-1}h_vP_v)_v)$ be the type kernel read on $G(F_\infty)$. By (S6) of Lemma 5.2.12, $2\varsigma(v)$ has gap $500r^3 + r^7 \geq 500r^3$ for the signs of Y_0, so by Lemma 5.6.21 (with $B = 0$) e_ς is a cusp kernel, in particular continuous and integrable, and so is e for μ_∞. Read through the frames, (FT2) says that every $y \in \mathcal{A}_M(Y_0)$ is continuous, bounded, left Γ-invariant, right K_M-invariant and satisfies

$$\int_{G(F_\infty)} e(h)\,y(x\,(h,1))\,d\mu_\infty(h) = y(x) \qquad (x \in G(\mathbb{A})). \tag{5.110}$$

Fix a left Haar measure ds of $R(F_\infty)$; let g_∞ and t be as in Lemma 5.6.36 and κ as in Lemma 5.6.40 (iii) (for ds and the measure dr of Definition 3.2.36). Put $t_0 = (V_M/\kappa)\,t$. Fix $\varphi \in \mathcal{S}$ and $y \in \mathcal{A}_M(Y_0)$, and for $c \in \mathcal{C}$ put $W_c(s) = \mathcal{W}_{t,\varphi}(s, \hat{h}_c)$ ($s \in R(F_\infty)$) and $\Psi_c(x) = \sum_{i\in I}\theta(\varphi)_{(c,i)}W_i^{\mathrm{full}}(\mathbf{y}(x))$ ($x \in G(F_\infty)$).

Step 1: folding. The function $\overline{y}$ is bounded, Borel, left Γ-invariant and right K_M-invariant. By Lemma 5.6.17 for $z = \overline{y}$ and each $a = (c,i)$, and linearity of the Petersson product in the second argument,

$$\Big\langle y, \sum_a \theta(\varphi)_a\Pi_a\Big\rangle = \sum_a \theta(\varphi)_a \int_{F_\Gamma} \overline{y}\,\Pi_a\,d\mu = V_M\sum_{c\in\mathcal{C}}\int_{G(F_\infty)} \beta_c(\mathbf{y}(x))\,\Psi_c(x)\,\overline{y(x,\hat{h}_c)}\,d\mu_\infty(x),$$

each integral converging absolutely.

Step 2: the functions W_c. (a) W_c is continuous, by Lemma 5.6.34 (iv). (b) $W_c(us) = W_c(s)$ for $u \in \mathsf{U}_c(M)$, us the product of the archimedean component of u with s: in $R(\mathbb{A})$, $u\cdot(s,\hat{h}_c) = (us, \hat{h}_ck_u) = (us,\hat{h}_c)(1,k_u)$ with $k_u \in J_f'$ (Lemma 5.6.40 (i)), so $W_c(us) = \mathcal{W}_{t,\varphi}(u\cdot(s,\hat{h}_c)\cdot(1,k_u)^{-1}) = \mathcal{W}_{t,\varphi}(u\cdot(s,\hat{h}_c)) = W_c(s)$ by Lemma 5.6.38 (ii) and the left $R(E)$-invariance of Lemma 5.6.34 (iv). (c) $\mathcal{W}_{t,\varphi}$ is continuous, left $R(E)$-invariant (Lemma 5.6.34 (iv)), right J_f'-invariant (Lemma 5.6.38 (ii)) and integrable over $[R]$ (Lemma 5.6.34 (iii)); so by Lemma 5.6.40 (iii), $s\mapsto\beta_c(\mathbf{y}(s))W_c(s)$ is integrable on $R(F_\infty)$.

Step 3: the Bessel formula. By Lemma 5.6.36, for every $x \in G(F_\infty)$,

$$\Psi_c(x)\,\overline{y(x,\hat{h}_c)} = \int_{R(F_\infty)}\Phi_c(s,x)\,ds, \qquad \Phi_c(s,x) = W_c(s)\,\overline{e(g_\infty^{-1}s^{-1}x)}\,\overline{y(x,\hat{h}_c)}.$$

Step 4: moving the quotient. (a) $\Phi_c(us,ux)=\Phi_c(s,x)$ for $u\in \mathsf{U}_c(M)$ (archimedean components acting on the left): $W_c(us)=W_c(s)$ by Step 2 (b), $(us)^{-1}ux=s^{-1}x$, and $u\cdot(x,\hat h_c)=(ux,\hat h_c)(1,k_u)$ with $k_u\in J_f'\subseteq \mathrm{K}_M$, so $y(ux,\hat h_c)=y(u\cdot(x,\hat h_c)(1,k_u)^{-1})=y(x,\hat h_c)$ by right K_M- and left Γ-invariance. (b) A swap: for every Borel $\Theta\ge 0$ on $R(F_\infty)\times G(F_\infty)$ with $\Theta(us,ux)=\Theta(s,x)$ for $u\in \mathsf{U}_c(M)$,

$$\iint \beta_c(\mathbf{y}(x))\,\Theta(s,x)\,ds\,d\mu_\infty(x)=\iint \beta_c(\mathbf{y}(s))\,\Theta(s,x)\,ds\,d\mu_\infty(x).$$

Indeed, by Lemma 5.6.40 (ii) for s (an element of $G(F_\infty)$) and Tonelli's theorem, the left side is $\sum_u\iint \beta_c(\mathbf{y}(us))\beta_c(\mathbf{y}(x))\Theta(s,x)$; the substitution $(s,x)\mapsto(u^{-1}s,u^{-1}x)$, which preserves $ds\otimes\mu_\infty$ (left Haar measures), and the invariance of Θ turn the term of u into $\iint \beta_c(\mathbf{y}(s))\beta_c(\mathbf{y}(u^{-1}x))\Theta(s,x)$; summing over u ($u\mapsto u^{-1}$ permutes the group $\mathsf{U}_c(M)$) and using Lemma 5.6.40 (ii) for x gives the right side. (c) For $\Theta=|\Phi_c|$ the right side is finite: substituting $x=sg_\infty h$ (left invariance of μ_∞),

$$\iint \beta_c(\mathbf{y}(s))\,|\Phi_c(s,x)|\le \|y\|_\infty\,\|e\|_{L^1(\mu_\infty)}\int_{R(F_\infty)}\beta_c(\mathbf{y}(s))\,|W_c(s)|\,ds<\infty$$

by Step 2 (c). So both sides of (b) are finite for $|\Phi_c|$, hence for the positive and negative parts of $\operatorname{Re}\Phi_c$ and $\operatorname{Im}\Phi_c$, which are invariant by (a); subtracting, (b) holds for $\Theta=\Phi_c$, both sides being absolutely convergent. (d) The two sides. By Step 3 and Fubini's theorem the left side of (b) for Φ_c is $\int_{G(F_\infty)}\beta_c(\mathbf{y}(x))\Psi_c(x)\overline{y(x,\hat h_c)}\,d\mu_\infty(x)$. For fixed s, substituting $x=sg_\infty h$ and using (5.110) at the point $(sg_\infty,\hat h_c)$,

$$\int_{G(F_\infty)}\overline{e(g_\infty^{-1}s^{-1}x)}\,\overline{y(x,\hat h_c)}\,d\mu_\infty(x)=\overline{\int_{G(F_\infty)}e(h)\,y((sg_\infty,\hat h_c)(h,1))\,d\mu_\infty(h)}=\overline{y(sg_\infty,\hat h_c)},$$

so the right side of (b) for Φ_c is $\int_{R(F_\infty)}\beta_c(\mathbf{y}(s))\,W_c(s)\,\overline{y(sg_\infty,\hat h_c)}\,ds$. With Step 1,

$$\Big\langle y,\sum_a\theta(\varphi)_a\Pi_a\Big\rangle=V_M\sum_{c\in\mathcal{C}}\int_{R(F_\infty)}\beta_c(\mathbf{y}(s))\,W_c(s)\,\overline{y(sg_\infty,\hat h_c)}\,ds.$$

Step 5: the period, class by class. Let $F(r)=\mathcal{W}_{t,\varphi}(r)\,\overline{y(r\,(g_\infty,1))}$ on $R(\mathbb{A})$. It is continuous and left $R(E)$-invariant (Step 2 (c) and the left Γ-invariance of y), right J_f'-invariant ($r(1,k)(g_\infty,1)=r(g_\infty,1)(1,k)$, $k\in\mathrm{K}_M$, and Step 2 (c)), and integrable over $[R]$ (Lemma 5.6.34 (iii)). As $(s,\hat h_c)(g_\infty,1)=(sg_\infty,\hat h_c)$, $F(s,\hat h_c)=W_c(s)\overline{y(sg_\infty,\hat h_c)}$, and Lemma 5.6.40 (iii) gives

$$\mathrm{BP}_{g_\infty,t}(\varphi,y)=\int_{[R]}F\,dr=\kappa\sum_{c\in\mathcal{C}}\int_{R(F_\infty)}\beta_c(\mathbf{y}(s))\,W_c(s)\,\overline{y(sg_\infty,\hat h_c)}\,ds.$$

Comparing with Step 4 and using that $\mathcal{W}_{t,\varphi}$ is linear in t, $\langle y,\sum_a\theta(\varphi)_a\Pi_a\rangle=(V_M/\kappa)\,\mathrm{BP}_{g_\infty,t}(\varphi,y)=\mathrm{BP}_{g_\infty,t_0}(\varphi,y)$. □

Lemma 5.6.36. *Let D be admissible data, p the matrix of Proposition 5.7.3, $\mathbf{y}=\mathbf{y}_p$, and let $R(F_\infty)$, $\hat h_c=\eta^{-1}h_c$ be as in Lemma 5.6.40. Fix a left Haar measure ds of $R(F_\infty)$, and read the type kernel on $G(F_\infty)$ through the frames: $e(g)=e_\varsigma((P_v^{-1}g_vP_v)_v)$. There are $g_\infty\in G(F_\infty)$ and a linear form t on W_λ such that for every $\varphi\in\mathcal{S}$, $c\in\mathcal{C}$ and $x\in G(F_\infty)$ the function $s\mapsto\overline{e(g_\infty^{-1}s^{-1}x)}\,\mathcal{W}_{t,\varphi}(s,\hat h_c)$ is integrable on $R(F_\infty)$ and*

$$\sum_{i\in I}\theta(\varphi)_{(c,i)}\,W_i^{\mathrm{full}}(\mathbf{y}(x))=\int_{R(F_\infty)}\overline{e(g_\infty^{-1}s^{-1}x)}\;\mathcal{W}_{t,\varphi}(s,\hat h_c)\,ds.$$

Proof. Write $G_\infty = \prod_v G_1$, $B_v = \mathrm{diag}(\mathrm{bs}_v)$ and H_v for the Gram matrix of $Y_0 \otimes_{E,v} \mathbb{C}$; then $B_v^* H_v B_v = J_1$ (RD1), $P_v^* H_v P_v = \mathrm{diag}(s)$ (frames), $p^* J_1 p = \mathrm{diag}(s)$, and $\mathbf{y}(x)_v = B_v^{-1} x_v P_v p^{-1}$. For $v \in \Phi$ let $\mathfrak{B}_v$ be the Bessel system of Proposition 5.7.3 at v, with index set I_v, representation ρ_v and full function W_v, and $l_v = 2\varsigma(v)$; write $c^{(v)}(g) = c_{\varsigma(v)}(p^{-1}gp)$ for $g \in G_1$.

Step 1: the local integrals. By Proposition 5.7.3 (iv) there are $c_v \in \mathbb{R}$, $q_v \in G_1$ and a unit vector $e_v \in \mathbb{C}^{I_v}$ with $W_v = c_v\, W^{(v)}$, where $W^{(v)}(y) = \int_R c^{(v)}(q_v^{-1} s' y)\, \mathrm{wave}(-\sigma(s'))\, \rho_v(h_{s'})^{-1} e_v\, ds'$ and $ds' = du\, dh$. By (S6) of Lemma 5.2.12, l_v has large gap r^7 for s, so by Lemma 5.6.37 (i) the integral converges absolutely for every $y \in G_1$. The measure ds' is invariant under left translations and under $s' \mapsto s'^{-1}$: du is a two-sided Haar measure of U, invariant under $u \mapsto u^{-1}$ (Lemma 5.2.34), conjugation by $h \in H$ preserves du (Lemma 5.5.30 (vi)) and dh is the Haar probability measure of the compact group H; now $u_1 \cdot uh = (u_1 u)h$, $h_1 \cdot uh = (h_1 u h_1^{-1})(h_1 h)$ and $(uh)^{-1} = (h^{-1}u^{-1}h)h^{-1}$, and for fixed h the maps $u \mapsto u_1 u$, $u \mapsto h_1 u h_1^{-1}$, $u \mapsto h^{-1}u^{-1}h$ preserve du, so by Tonelli's theorem the three maps preserve $du \otimes dh$. By Lemma 5.5.30 (i), (ii), $h_{s'^{-1}} = h_{s'}^{-1}$ and $\sigma(s'^{-1}) = -\sigma(s')$; substituting $s' \mapsto s'^{-1}$,

$$W^{(v)}(y) = \int_R F_{v,y}(s')\, ds', \qquad F_{v,y}(s') = c^{(v)}(q_v^{-1} s'^{-1} y)\, \mathrm{wave}(\sigma(s'))\, \rho_v(h_{s'}) e_v, \tag{5.111}$$

with $F_{v,y}$ integrable. In particular ds' is a left Haar measure of R.

Step 2: the real Bessel group. For $s \in R(F_\infty)$ put $s'_v = B_v^{-1} s_v B_v$. As $B_v^* H_v B_v = J_1$, $s'_v \in G_1$. The conditions defining $R(\mathbb{A})$ at the frame vectors f_b (the string vectors of grade ≥ 0: $gf_b - f_b$ lies in the span of the vectors of larger grade, $f_0, \dots, f_{b-1}$) are, by Lemma 5.5.30 (i), those defining $R \subseteq G_1$, and conjugation by the diagonal B_v preserves them; so $s \mapsto (s'_v)_v$ is an isomorphism of topological groups $R(F_\infty) \to \prod_v R$, with inverse $(r_v) \mapsto (B_v r_v B_v^{-1})_v$. Since $\mathrm{bs}_v = 1$ on X_0, $h_{s'_v} = a(s)_v$ (the X_0-block). By Lemma 5.5.30 (ii) and (5.5), with $\mathrm{bs}_v(f_{a+1})/\mathrm{bs}_v(f_a) = \mathrm{rs}_v^{-1}$,

$$\sigma(s'_v) = \sum_{a<\varrho} (s_v)_{f_a f_{a+1}}\, \mathrm{rs}_v^{-1} = \mathrm{rs}_v^{-1}\, \alpha(u(s))_v,$$

with α as in Lemma 5.6.38 (i): the pairs (f_a, f_{a+1}) are the pairs (v_j, v_{j+1}), $j \leq \varrho$, and $u(s) = s\, \iota(a(s))^{-1}$ has the same entries in the columns f_{a+1}, as $\iota(a(s))$ fixes the string vectors. As both ds and $\prod_v ds'_v$ are left Haar measures, there is $c_1 > 0$ with $\int_{\prod_v R} \Phi \prod_v ds'_v = c_1 \int_{R(F_\infty)} \Phi((s'_v)_v)\, ds$ for every integrable or nonnegative Borel Φ. Finally $\mathbf{y}(s^{-1}x)_v = B_v^{-1} s_v^{-1} B_v\, \mathbf{y}(x)_v = s'^{-1}_v \mathbf{y}(x)_v$.

Step 3: the unipotent character at real points. Let $s \in R(F_\infty)$. The finite components of $u(s)$ are 1, so $\alpha(u(s))$ has finite component 0, and by Lemma 5.6.38 (i) and (WD4), $\psi_{\mathsf{U}}(u(s)) = \prod_v \exp(2\pi i \cdot 2\,\mathrm{Re}(\alpha_v/(Mv(\varpi))))$, $\alpha_v = \alpha(u(s))_v$. Since $\bar{\varpi} = -\varpi$ (Lemma 3.2.35 (i)), $Mv(\varpi)/2\pi i$ is real, so $Mv(\varpi) = 2\pi i\, \mathrm{rs}_v$; with $\sigma_v = \sigma(s'_v) = \alpha_v/\mathrm{rs}_v$ we get $2\pi i \cdot 2\,\mathrm{Re}(\sigma_v/(2\pi i)) = 2i\,\mathrm{Im}\,\sigma_v = \sigma_v - \bar{\sigma}_v$, hence

$$\psi_{\mathsf{U}}(u(s)) = \prod_{v \in \Phi} \mathrm{wave}(\sigma(s'_v)).$$

Step 4: the kernel. Put $g_v = B_v q_v p P_v^{-1}$. Then

$$g_v^* H_v g_v = P_v^{-*} p^* q_v^* J_1 q_v p P_v^{-1} = P_v^{-*} \mathrm{diag}(s) P_v^{-1} = H_v,$$

so $g_\infty = (g_v)_v \in G(F_\infty)$ and $\mathbf{y}(g_\infty) = (q_v)_v$. For $z \in G(F_\infty)$, $q_v^{-1}\mathbf{y}(z)_v = p\, P_v^{-1}(g_\infty^{-1} z)_v P_v\, p^{-1}$, so with $d = \prod_v d_{\varsigma(v)}$ (a positive real: $d_{\varsigma(v)} = \mathrm{d}(\varsigma(v)) > 0$ by Proposition 5.7.3 (i), the degree of a Bessel system being positive)

$$\prod_{v \in \Phi} c^{(v)}(q_v^{-1}\mathbf{y}(z)_v) = \prod_v c_{\varsigma(v)}(P_v^{-1}(g_\infty^{-1} z)_v P_v) = d^{-1}\, \overline{e(g_\infty^{-1} z)}. \tag{5.112}$$

Step 5: the source function at a class point. By Proposition 5.6.4 (ii), $\eta = \iota(u_0^{-1}1_{X_0})$, so $\hat{h}_c = \iota(u_0 h_c)$ (writing h_c also for its preimage in $\mathrm{U}(X_0)(\mathbb{A}_f)$). For $s \in R(F_\infty)$, by Lemma 5.6.34 (i), $a(s,\hat{h}_c)$ has the archimedean components $a(s)_v$ and the finite component $u_0 h_c$, and $u(s,\hat{h}_c) = (s,\hat{h}_c)\,\iota(a(s,\hat{h}_c))^{-1} = u(s)$. So, with $\xi_c = \varphi(h_c u_0)$ (u_0 is central in $\mathrm{U}(X_0)(\mathbb{A}_f)$),

$$\mathcal{W}_{t,\varphi}(s,\hat{h}_c) = \psi_{\mathsf{U}}(u(s))\; t\Big(\rho_\lambda\big((\overline{a(s)_v})_v\big)^{-1}\xi_c\Big).$$

Step 6: assembling. Let $A = \bigotimes_v A_v$ be as in Lemma 5.6.14, so $\theta(\varphi)_{(c,\cdot)} = A(\xi_c)$, a row vector in $\mathbb{C}^I$, and for $h = (h_v) \in \prod_v \mathrm{U}(k)$, tensoring (a) there over v, $A(\xi)(\bigotimes_v \rho_v(h_v)) = A(\rho_\lambda((h_v^{\mathsf{T}})_v)\xi) = A\big(\rho_\lambda((\bar{h}_v)_v)^{-1}\xi\big)$, as $h_v^{\mathsf{T}} = \bar{h}_v^{-1}$ for unitary h_v. Define the linear form

$$t(\zeta) = c_1\, d^{-1}\Big(\prod_v c_v\Big)\, A(\zeta)\cdot\Big(\bigotimes_v e_v\Big) \qquad (\zeta \in W_\lambda).$$

Fix φ, c, x and put $y = \mathbf{y}(x)$. As $W_i^{\mathrm{full}}(y) = \prod_v W_{v,i_v}(y_v)$, (5.111) and Fubini's theorem for the finitely many integrable functions F_{v,y_v} on the product $\prod_v R$ give

$$\sum_i \theta(\varphi)_{(c,i)} W_i^{\mathrm{full}}(y) = \Big(\prod_v c_v\Big)\int_{\prod_v R}\Big(\prod_v c^{(v)}(q_v^{-1}s_v'^{-1}y_v)\,\mathrm{wave}(\sigma(s_v'))\Big)\\ A(\xi_c)\Big(\bigotimes_v \rho_v(h_{s_v'})\Big)\Big(\bigotimes_v e_v\Big)\prod_v ds_v',$$

the integrand being integrable (it is a finite sum of products of coordinates of the F_{v,y_v}). Transport the integral to $R(F_\infty)$ by Step 2 ($s_v'^{-1}y_v = \mathbf{y}(s^{-1}x)_v$, $h_{s_v'} = a(s)_v$) and insert (5.112) with $z = s^{-1}x$, Step 3 and the identity for A above: the right side equals

$$c_1\, d^{-1}\Big(\prod_v c_v\Big)\int_{R(F_\infty)} \overline{e(g_\infty^{-1}s^{-1}x)}\;\psi_{\mathsf{U}}(u(s))\; A\Big(\rho_\lambda\big((\overline{a(s)_v})_v\big)^{-1}\xi_c\Big)\cdot\Big(\bigotimes_v e_v\Big)\,ds\\ = \int_{R(F_\infty)} \overline{e(g_\infty^{-1}s^{-1}x)}\;\mathcal{W}_{t,\varphi}(s,\hat{h}_c)\,ds,$$

by the definition of t and Step 5. The integrand on $R(F_\infty)$ is integrable, as its transport to $\prod_v R$ is. □

Lemma 5.6.37. *Fix the setting of Definition 5.1.6, signs s on the r basis indices of V_1 and a sign frame p for s. Let $l \in \mathbb{Z}^r$ have large gap A for s ($A \geq 0$ an integer); write $c_l^{G_1}(g) = c_{l/2}(p^{-1}gp)$ for $g \in G_1$, and d_l for the formal degree $(\int |c_{l/2}|^2)^{-1}$ for the Haar measure on $\mathrm{U}(\mathrm{diag}\, s)$ that $g \mapsto p^{-1}gp$ carries from dy. Every $s' \in R$ is uniquely $s' = uh$ with $u \in \mathsf{U}$, $h \in H$; put $h_{s'} = h$ and $\sigma(s') = \sigma(u)$, and let $ds' = du\,dh$ on R, dh the Haar probability measure. Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be a continuous unitary representation on $\mathbb{C}^I$, $v \in \mathbb{C}^I$ and $x \in G_1$. Then:*

(i) for every $y \in G_1$ the integral

$$W_x(y) = \int_R c_l^{G_1}(x^{-1}s'y)\,\mathrm{wave}(-\sigma(s'))\,\rho(h_{s'})^{-1}v\,ds'$$

converges absolutely; W_x is a continuous Bessel function of type ρ, integrable and square integrable on $\mathsf{U}\backslash G_1$; and $\int_{G_1} d_l\, c_l^{G_1}(a^{-1})\,W_x(ya)\,da = W_x(y)$ for every $y \in G_1$;

(ii) if moreover $u_0 \in \mathbb{Z}^k$ and $\langle v, \rho(m)v\rangle = c_{u_0/2}(m)$ for every $m \in \mathrm{U}(k)$ (Flensted-Jensen's function for the signs $(1,\dots,1)$), the period equals the squared norm:

$$\int_R c_l^{G_1}(x^{-1}s'x)\,\overline{c_{u_0/2}(h_{s'})}\,e^{\overline{\sigma(s')}-\sigma(s')}\,ds' = \langle v, W_x(x)\rangle = d_l\int_{\mathsf{U}\backslash G_1}\|W_x\|^2.$$

Proof. Write $c = c_l^{G_1}$, $W = W_x$, $\|\cdot\|_{\mathrm{op}}$ for the operator norm and $\langle\cdot,\cdot\rangle$ for the standard inner product of $\mathbb{C}^I$, conjugate-linear in the first argument. The map $\iota(g) = p^{-1}gp$ is an isomorphism of topological groups $G_1 \to \mathrm{U}(\mathrm{diag}\, s)$ (as $p^* J_1 p = \mathrm{diag}(s)$ and p is unitary, $\|\iota(g)\|_{\mathrm{op}} = \|g\|_{\mathrm{op}}$), and dy goes to a Haar measure dg of $\mathrm{U}(\mathrm{diag}\, s)$. The large gap $500r^3 + A$ exceeds $8r^2$, so Lemma 5.6.23 applies to l and dg; transporting by ι: c is continuous and integrable on G_1, $d_l > 0$, and by Lemma 5.6.23 (ii) with first argument g and second 1, together with $\overline{c(a)} = c(a^{-1})$ (Lemma 5.7.40),

$$\int_{G_1} d_l\, c(a^{-1})\, c(ga)\, da = c(g) \qquad (g \in G_1). \tag{5.113}$$

By Lemma 5.7.40 with exponent $9r^2 \le 500r^3 + A$, $|c(g)| \le (2/\|g\|_{\mathrm{op}})^{9r^2}$.

The group R. By (i), (ii) and (v) of Lemma 5.5.30, R is a closed subgroup of G_1, $s' \mapsto h_{s'}$ is a homomorphism, $\sigma(s's'') = \sigma(s') + \sigma(s'')$ on R, and ds' (the measure ds there) is a right-invariant Borel measure on R, finite on compact sets.

Unfolding. For Borel $f \ge 0$ on G_1, (vi) of Lemma 5.5.30 gives

$$\int_{\mathsf{U}\backslash G_1} \int_R f(s'y)\, ds'\, dy = \int_{G_1} f(g)\, dg, \tag{5.114}$$

and applied to the positive and negative parts of the real and imaginary parts, the identity holds for every integrable Borel f.

(i). The norm of the integrand is $|c(x^{-1}s'y)|\,\|v\|$, since ρ is unitary and $|\mathrm{wave}| = 1$. Apply Lemma 5.6.32 (ii) to the group R with ds', the homomorphism $j(s') = \iota(x^{-1}s'x)$ into $\mathrm{U}(\mathrm{diag}\, s)$, proper since R is closed and ι and conjugation are homeomorphisms, and the point $\iota(x^{-1}y)$: as $j(s')\iota(x^{-1}y) = \iota(x^{-1}s'y)$,

$$\int_R |c(x^{-1}s'y)|\, ds' \le \int_R \Big(\frac{2}{\|\iota(x^{-1}s'y)\|_{\mathrm{op}}}\Big)^{9r^2} ds' \le 16^{9r^2}\,\nu_x, \qquad \nu_x = \int_R \mathbf{1}\big[\|j(s')-1\|_{\mathrm{op}} \le 1\big]\, ds' < \infty,$$

uniformly in y. So W is defined and bounded by $\|v\|\, 16^{9r^2}\nu_x$. Continuity: for y in a compact neighbourhood C of y_0, $\|\iota(x^{-1}s'y)\|_{\mathrm{op}} \ge \|\iota(x^{-1}s'y_0)\|_{\mathrm{op}}/M$ with $M = \sup_{y\in C}\|y^{-1}y_0\|_{\mathrm{op}}$, so the norms of the integrands are dominated by the integrable $\|v\| M^{9r^2}(2/\|\iota(x^{-1}s'y_0)\|_{\mathrm{op}})^{9r^2}$, and dominated convergence applies. Equivariance: for $s'' \in R$, substituting $s' \mapsto s's''^{-1}$ (right invariance) and using the additivity of σ and $h_{s's''^{-1}} = h_{s'}h_{s''}^{-1}$,

$$W(s''y) = \int_R c(x^{-1}s'y)\,\mathrm{wave}\big(\sigma(s'') - \sigma(s')\big)\,\rho(h_{s''})\rho(h_{s'})^{-1}v\, ds' = \mathrm{wave}(\sigma(s''))\,\rho(h_{s''})\,W(y),$$

as $\mathrm{wave}(a+b) = \mathrm{wave}(a)\mathrm{wave}(b)$; for $s'' = u \in \mathsf{U}$ and $s'' = h \in H$ these are the two conditions of a Bessel function of type ρ. Integrability over $\mathsf{U}\backslash G_1$: $\|W(y)\| \le \|v\| \int_R |c(x^{-1}s'y)|\, ds'$, and by (5.114) the integral of the right side over $\mathsf{U}\backslash G_1$ is $\int_{G_1} |c(x^{-1}g)|\, dg = \int_{G_1} |c| < \infty$; as W is bounded it is also square integrable. Reproduction: the function $(a, s') \mapsto d_l\, c(a^{-1})\, c(x^{-1}s'ya)$ is integrable on $G_1 \times R$, the integral of its norm in s' being at most $d_l|c(a^{-1})|\,\|v\|\, 16^{9r^2}\nu_x$; by Fubini's theorem and (5.113) with $g = x^{-1}s'y$,

$$\int_{G_1} d_l\, c(a^{-1})W(ya)\, da = \int_R \Big(\int_{G_1} d_l\, c(a^{-1})\, c(x^{-1}s'ya)\, da\Big)\mathrm{wave}(-\sigma(s'))\rho(h_{s'})^{-1}v\, ds' = W(y).$$

(ii). First equality: $\overline{c_{u_0/2}(h)} = \overline{\langle v, \rho(h)v\rangle} = \langle \rho(h)v, v\rangle = \langle v, \rho(h)^{-1}v\rangle$ and $e^{\overline{\sigma}-\sigma} = \mathrm{wave}(-\sigma)$, so the period is $\int_R \langle v, c(x^{-1}s'x)\,\mathrm{wave}(-\sigma(s'))\rho(h_{s'})^{-1}v\rangle\, ds' = \langle v, W(x)\rangle$. Second equality: put

$\Phi(g) = d_l\,\overline{c(x^{-1}g)}\,\langle v, W(g)\rangle$, integrable on G_1 since c is integrable and W bounded. For $y \in G_1$ and $s' \in R$, by the equivariance, $\overline{\mathrm{wave}(\sigma)} = \mathrm{wave}(-\sigma)$ and unitarity,

$$\begin{aligned}\Phi(s'y) &= d_l\,\overline{c(x^{-1}s'y)}\,\mathrm{wave}(\sigma(s'))\,\langle \rho(h_{s'})^{-1}v, W(y)\rangle \\ &= d_l\,\overline{\langle W(y),\, c(x^{-1}s'y)\,\mathrm{wave}(-\sigma(s'))\,\rho(h_{s'})^{-1}v\rangle},\end{aligned}$$

so $\int_R \Phi(s'y)\,\overline{ds'} = d_l\overline{\langle W(y), W(y)\rangle} = d_l\|W(y)\|^2$. By (5.114), the substitution $g = xa$ (left invariance), $\overline{c(a)} = c(a^{-1})$ and the reproduction in (i) at $y = x$,

$$d_l \int_{\mathsf{U}\backslash G_1} \|W\|^2 = \int_{G_1} \Phi = \int_{G_1} d_l\, c(a^{-1})\,\langle v, W(xa)\rangle\, da = \Big\langle v, \int_{G_1} d_l\, c(a^{-1}) W(xa)\, da\Big\rangle = \langle v, W(x)\rangle.$$

□

Lemma 5.6.38. *Let D be admissible data and ϖ the generator of the different of E of Lemma 3.2.35.*

(i) For $u \in \mathsf{U}(\mathbb{A})$ put $\alpha(u) = \sum_{j=1}^{\varrho} u_{v_j v_{j+1}} \in \mathbb{A}_E$, the sum of the entries of u in the rows v_j and the columns v_{j+1} (matrices in the basis of Y_0). Then $\psi_{\mathsf{U}}(u) = \psi_E(-\alpha(u)/(M\varpi))$.

(ii) Let J' be the group of the $k \in R(\mathbb{A})$ whose components at the infinite places are 1 and whose finite component lies in K_M. Then $\mathcal{W}_{t,\varphi}(rk) = \mathcal{W}_{t,\varphi}(r)$ for every $\varphi \in \mathcal{S}$, every linear form t on W_λ, every $r \in R(\mathbb{A})$ and every $k \in J'$.

Proof. Matrices are taken in the basis of Y_0; g_{ik} is the coefficient of the basis vector i in g applied to the basis vector k. By (HP2), (HP3), $v_i = e_i$ and $v_{2\varrho+2-i} = e_i^*$ for $1 \le i \le \varrho$, and $v_{\varrho+1} = z$. Write the hermitian form $\langle\cdot,\cdot\rangle$ linear in the second argument (its Gram matrix is real, so the other convention gives the same formulas). Let P_2 be the span of $e_1, \dots, e_\varrho$ and $P_0 = P_2 \oplus X_0 \oplus \langle z\rangle$ (over $\mathbb{A}_E$); $P_0 = P_2^{\perp}$.

Step 1: two facts on additive characters. (a) $\psi_E(\bar{x}) = \psi_E(x)$ for $x \in \mathbb{A}_E$: the archimedean factor of ψ_E in (WD4) depends on $\mathrm{Re}\, x_v$ only, and if $x_f - a \in \hat{\mathcal{O}}$ with $a \in E$ then $\bar{x}_f - \bar{a} \in \hat{\mathcal{O}}$ and $\mathrm{Tr}_{E/\mathbb{Q}}(\bar{a}) = \mathrm{Tr}_{E/\mathbb{Q}}(a)$. If $\bar{z} = -z$, then $\psi_E(-z) = \psi_E(\bar{z}) = \psi_E(z)$, so $\psi_E(z)^2 = \psi_E(z)\psi_E(-z) = 1$; as $z/2$ has the same property, $\psi_E(z) = \psi_E(z/2)^2 = 1$. Applied to $z = (x - \bar{x})/2$:

$$\psi_E\Big(\frac{x+\bar{x}}{2}\Big) = \psi_E(x) \qquad (x \in \mathbb{A}_E). \tag{5.115}$$

(b) $\psi_f(y) = 1$ for $y \in \varpi^{-1}\hat{\mathcal{O}}$, with ϖ the generator of the different of Lemma 3.2.35: as $\mathcal{O}_E$ is dense in $\hat{\mathcal{O}}$ and $\varpi\hat{\mathcal{O}}$ is open, $\hat{\mathcal{O}} = \mathcal{O}_E + \varpi\hat{\mathcal{O}}$, so $y = a + o$ with $a \in \varpi^{-1}\mathcal{O}_E$ and $o \in \hat{\mathcal{O}}$; $\varpi^{-1}\mathcal{O}_E$ is the inverse different (Lemma 3.2.35 (i)), so $\mathrm{Tr}_{E/\mathbb{Q}}(a) \in \mathbb{Z}$ and $\psi_f(y) = \exp(2\pi i\,\mathrm{Tr}_{E/\mathbb{Q}}(a)) = 1$.

Step 2: the unipotent character in the entries above the diagonal. Let $u \in \mathsf{U}(\mathbb{A})$ and write $u_{j,j+1}$ for $u_{v_j v_{j+1}}$. Then $u = 1 + n$ with $n_{ik} \ne 0$ only if $\mathrm{gr}(i) \ge \mathrm{gr}(k) + 2$, so n is nilpotent, n^2 has nonzero entries only where $\mathrm{gr}(i) \ge \mathrm{gr}(k) + 4$, and $u^{-1} = 1 - n + n^2 - \dots$ has $(u^{-1})_{j,j+1} = -u_{j,j+1}$. The coefficient of v_i in a vector w is $\langle v_i^\vee, w\rangle$ with $v_i^\vee = v_{2\varrho+2-i}$ for $i \ne \varrho+1$ and $v_{\varrho+1}^\vee = -N^{-1}v_{\varrho+1}$. As u is unitary, $\langle v_i^\vee, u^{-1}w\rangle = \langle uv_i^\vee, w\rangle$, so

$$-u_{j,j+1} = (u^{-1})_{j,j+1} = \langle uv_j^\vee, v_{j+1}\rangle = \overline{\langle v_{j+1}, uv_j^\vee\rangle}.$$

For $1 \le j < \varrho$, $v_j^\vee = v_{2\varrho+2-j}$ and $\langle v_{j+1}, w\rangle$ is the coefficient of $v_{2\varrho+1-j}$ in w; for $j = \varrho$, $v_\varrho^\vee = v_{\varrho+2}$ and $\langle v_{\varrho+1}, w\rangle = -N$ times the coefficient of $v_{\varrho+1}$. Hence

$$u_{2\varrho+1-j,2\varrho+2-j} = -\overline{u_{j,j+1}} \quad (1 \le j < \varrho), \qquad N\,u_{\varrho+1,\varrho+2} = \overline{u_{\varrho,\varrho+1}}. \tag{5.116}$$

The shift $\mathfrak{e}$ of Definition 3.2.36 sends $v_l \mapsto c_l v_{l+1}$ ($1 \le l \le 2\varrho$) with $c_l = 1$ for $l \ne \varrho+1$, $c_{\varrho+1} = -N$, and kills X_0 and $v_{2\varrho+1}$; so $\mathrm{tr}(\mathfrak{e}u) = \sum_{l=1}^{2\varrho} c_l u_{l,l+1}$, and by (5.116), with $\alpha(u) = \sum_{j=1}^{\varrho} u_{j,j+1}$,

$$\mathrm{tr}(\mathfrak{e}u) = \sum_{j=1}^{\varrho} u_{j,j+1} - \overline{u_{\varrho,\varrho+1}} - \sum_{j=1}^{\varrho-1} \overline{u_{j,j+1}} = \alpha(u) - \overline{\alpha(u)}.$$

Put $x = \alpha(u)/(M\varpi)$. Since M is real and $\bar\varpi = -\varpi$, $\bar x = -\overline{\alpha(u)}/(M\varpi)$, so $\mathrm{tr}(\mathfrak{e}u)/(2M\varpi) = (x+\bar x)/2$, and by (5.115)

$$\psi_{\mathsf{U}}(u) = \psi_E\big(-\alpha(u)/(M\varpi)\big) \qquad (u \in \mathsf{U}(\mathbb{A})), \tag{5.117}$$

which is (i). Steps 3 to 5 prove (ii).

Step 3: the Levi part of an element of J'. By Proposition 5.6.4 (ii) the filling datum $\mathfrak{S}_D$ has the space Y_0 (same basis and form), $\mathrm{K}_M = \eta\,\mathrm{K}(M)\,\eta^{-1}$ with $\eta = \iota(u_0^{-1}1_{X_0})$, and η commutes with $\iota(\mathrm{U}(X_0)(\mathbb{A}_f))$. Let $k \in J'$ and $k' = \eta^{-1}k\eta$. As $\eta \in R(\mathbb{A})$ (Lemma 5.6.34 (i)), $k' \in R(\mathbb{A}) \cap \mathrm{K}(M)$, and $a(k') = a(\eta)^{-1}a(k)a(\eta) = a(k)$, $a(\eta)$ being the scalar u_0^{-1}. By Lemma 5.6.39 for $t = k'$, $a(k) = a(k')$ has components 1 at the infinite places and finite component in K_0, and $u' = u(k') = k'\iota(a(k'))^{-1} \in \mathsf{U}(\mathbb{A}) \cap \mathrm{K}(M)$, its components at the infinite places are 1, and $u(k) = k\,\iota(a(k))^{-1} = \eta k'\eta^{-1}\iota(a(k'))^{-1} = \eta u'\eta^{-1}$.

Step 4: $\psi_{\mathsf{U}}(u(k)) = 1$. The lattice L_0 of (FG3) has the basis e_a, e_a^* ($1 \le a \le \varrho$), z, $x_1, \dots, x_{k-1}$ and $x'_k = \bar\pi^{-1}x_k + N^{-1}z$ ($\pi = \pi_N$). Let u'_L be the matrix of u' in this basis: it is integral and reduces to 1 at every chosen place, so its off-diagonal entries lie in $\hat{\mathcal{O}}$ and are $\equiv 0$ modulo every chosen w. The basis vectors e_a, e_a^*, z of Y_0 are basis vectors of L_0, and z also occurs in x'_k with coefficient N^{-1}, while e_a, e_a^* occur in no other basis vector of L_0. Reading coefficients,

$$u'_{e_j,e_{j+1}} = (u'_L)_{e_j,e_{j+1}}, \quad u'_{e_\varrho,z} = (u'_L)_{e_\varrho,z}, \quad u'_{e^*_{j+1},e^*_j} = (u'_L)_{e^*_{j+1},e^*_j}, \quad u'_{z,e^*_\varrho} = (u'_L)_{z,e^*_\varrho} + N^{-1}(u'_L)_{x'_k,e^*_\varrho}.$$

So $u'_{j,j+1} \in \hat{\mathcal{O}}$ and $u'_{j,j+1} \equiv 0$ modulo every chosen w ($1 \le j \le \varrho$). By (5.116), $\overline{u'_{j,j+1}} = -u'_{e^*_{j+1},e^*_j}$ for $j < \varrho$ and $\overline{u'_{\varrho,\varrho+1}} = Nu'_{z,e^*_\varrho} = N(u'_L)_{z,e^*_\varrho} + (u'_L)_{x'_k,e^*_\varrho}$, which are $\equiv 0$ modulo every chosen w as well. By (FG2) (M divides $x \in \hat{\mathcal{O}}$ if and only if $x \equiv \bar x \equiv 0$ modulo every chosen place), $\alpha(u') \in M\hat{\mathcal{O}}$; and the archimedean components of $\alpha(u')$ vanish. So $y = -\alpha(u')/(M\varpi)$ has archimedean components 0 and finite component in $\varpi^{-1}\hat{\mathcal{O}}$, and by (5.117), (WD4) and Step 1 (b), $\psi_{\mathsf{U}}(u') = \psi_f(y_f) = 1$. By Lemma 5.6.34 (ii), $\psi_{\mathsf{U}}(u(k)) = \psi_{\mathsf{U}}(\eta u'\eta^{-1}) = \psi_{\mathsf{U}}(u') = 1$.

Step 5: conclusion. Let $r \in R(\mathbb{A})$. By Lemma 5.6.34 (i), $a(rk) = a(r)a(k)$, where $a(k)$ has components 1 at the infinite places (as k has) and finite component in K_0 (Step 3); and

$$u(rk) = rk\,\iota(a(k))^{-1}\iota(a(r))^{-1} = r\,u(k)\,\iota(a(r))^{-1} = u(r)\cdot\iota(a(r))\,u(k)\,\iota(a(r))^{-1},$$

so $\psi_{\mathsf{U}}(u(rk)) = \psi_{\mathsf{U}}(u(r))\,\psi_{\mathsf{U}}(u(k)) = \psi_{\mathsf{U}}(u(r))$ by Lemma 5.6.34 (ii) and Step 4. The archimedean components of $a(rk)$ are those of $a(r)$, and $\varphi(a(rk)_f) = \varphi(a(r)_f a(k)_f) = \varphi(a(r)_f)$ because φ is right K_0-invariant. Inserting this in (3.5) gives $\mathcal{W}_{t,\varphi}(rk) = \mathcal{W}_{t,\varphi}(r)$. □

Lemma 5.6.39. *Let D be admissible data, and let $t \in R(\mathbb{A})$ have components 1 at the infinite places and finite component in $\mathrm{K}(M)$. Then $a(t)$ has components 1 at the infinite places, its finite component lies in K_0, and $\iota(a(t))$ and $u(t)$ lie in $\mathrm{K}(M)$.*

Proof. By Proposition 5.6.4 (ii) the space of $\mathfrak{S}_D$ is Y_0 with the same basis and form, and by Proposition 5.6.4 (i) $\mathfrak{S}_D$ is admissible, with source level K_0. The components of $a(t)$ at the

infinite places are the X_0-blocks of those of t, which are 1. Let P_2 be the span of $e_1, \dots, e_\varrho$ and $P_0 = P_2 \oplus X_0 \oplus \langle z \rangle$, over $\mathbb{A}_E$, so that $P_0 = P_2^\perp$.

t on P_0/P_2. By the definition of $R(\mathbb{A})$ (the entries $t_{ik} = \delta_{ik}$ for every string vector k of grade ≥ 0 and every i with $\mathrm{gr}(i) \leq \mathrm{gr}(k)$), t maps e_a into $e_a + \langle e_1, \dots, e_{a-1} \rangle$ and z into $z + P_2$. So t maps P_2 onto itself (its matrix there is unitriangular), hence, being unitary, $P_0 = P_2^\perp$ onto itself, and it induces on P_0/P_2 an isometry of the induced form, which has the orthogonal basis given by X_0 and z, with $\langle z, z \rangle = -N$ a unit of $\mathbb{A}_E$; this isometry fixes the class of z, hence preserves the class of its orthogonal complement X_0, on which it acts by the block $a(t)$. Therefore

$$t\lambda \in \iota(a(t))\lambda + P_2 \qquad (\lambda \in (X_0 \oplus Ez) \otimes \mathbb{A}_f).$$

The lattice. The basis of L_0 (FG3) consists of e_a, e_a^* $(1 \leq a \leq \varrho)$, z, $x_1, \dots, x_{k-1}$ and $x_k' = \bar{\pi}^{-1} x_k + N^{-1} z$, $\pi = \pi_N$. Hence $L_0 = S_e \oplus \Lambda \oplus S_{e^*}$, where S_e, S_{e^*} are the $\hat{\mathcal{O}}$-spans of the e_a and of the e_a^* and Λ is the $\hat{\mathcal{O}}$-span of $x_1, \dots, x_{k-1}, x_k', z$, a lattice in $(X_0 \oplus Ez) \otimes \mathbb{A}_f$; and $L_0 \cap P_0 = S_e \oplus \Lambda$, $L_0 \cap P_2 = S_e$. Let $\lambda \in \Lambda$. As $t \in K$, $t\lambda \in L_0 \cap P_0 = S_e \oplus \Lambda$, and by the display $\iota(a(t))\lambda \in (X_0 \oplus Ez) \otimes \mathbb{A}_f$ is the Λ-component of $t\lambda$; so $\iota(a(t))\Lambda \subseteq \Lambda$. The same holds for t^{-1}, which satisfies the hypotheses, and $a(t^{-1}) = a(t)^{-1}$ (Lemma 5.6.34 (i)); so $\iota(a(t))\Lambda = \Lambda$. For a chosen place w, t reduces to 1 at w, that is $(t-1)L_0 \subseteq \mathfrak{p}_w L_0$ at w; taking Λ-components, $(\iota(a(t)) - 1)\Lambda \subseteq \mathfrak{p}_w \Lambda$ at w. Since $\iota(a(t))$ fixes S_e and S_{e^*} pointwise, it maps L_0 onto itself and reduces to 1 at every chosen place: $\iota(a(t)) \in \mathrm{K}(M)$. By (FA1), the finite component of $a(t)$ lies in the source level $\mathrm{K}_X = \mathrm{K}_0$. Finally $u(t) = t\,\iota(a(t))^{-1}$ is a product of two elements of $\mathrm{K}(M)$. □

Lemma 5.6.40. *Let D be admissible data and p a sign frame for the signs of Y_0. Put $R(F_\infty) = \{s \in G(F_\infty) : (s,1) \in R(\mathbb{A})\}$, $R(\mathbb{A}_f) = \{g \in G(\mathbb{A}_f) : (1,g) \in R(\mathbb{A})\}$, $J_f' = R(\mathbb{A}_f) \cap \mathrm{K}_M$ and $\hat{h}_c = \eta^{-1} h_c$ $(c \in \mathcal{C})$, and fix a left Haar measure ds of $R(F_\infty)$. Let $\mathsf{U}_c(M)$ be as in (RD2) of Definition 5.1.11.*

(i) *$\mathsf{U}_c(M) \subseteq R(E)$, and for $u \in \mathsf{U}_c(M)$ the element $k_u = \hat{h}_c^{-1} u \hat{h}_c$ (u read by its finite component) lies in J_f'.*

(ii) *$\sum_{u \in \mathsf{U}_c(M)} \beta_c(\mathbf{y}(ux)) = 1$ for every $c \in \mathcal{C}$ and $x \in G(F_\infty)$, where $ux \in G(F_\infty)$ is the product of the archimedean component of u and x.*

(iii) *There is a real $\kappa > 0$ such that for every Borel $F\colon R(\mathbb{A}) \to \mathbb{C}$ that is left invariant under $R(E)$, satisfies $F(r\,(1,k)) = F(r)$ for $r \in R(\mathbb{A})$, $k \in J_f'$, and is integrable over $[R]$: for every $c \in \mathcal{C}$ the function $s \mapsto \beta_c(\mathbf{y}(s))\, F(s, \hat{h}_c)$ is integrable on $R(F_\infty)$, and*

$$\int_{[R]} F\, dr = \kappa \sum_{c \in \mathcal{C}} \int_{R(F_\infty)} \beta_c(\mathbf{y}(s))\, F(s, \hat{h}_c)\, ds.$$

Proof. By Proposition 5.6.4 (i), (ii), $\mathfrak{S}_D$ is admissible, E carries Weil data, the space of $\mathfrak{S}_D$ is Y_0 with the same basis and form, $\mathrm{K}_M = \eta\, \mathrm{K}(M)\, \eta^{-1}$, $\eta = \iota(u_0^{-1} 1_{X_0})$ commutes with $\iota(\mathrm{U}(X_0)(\mathbb{A}_f))$, and the source level is $\mathrm{K}_X = \mathrm{K}_0$. We write h_c also for the element of $\mathrm{U}(X_0)(\mathbb{A}_f)$ with ι-image h_c. Since η commutes with h_c,

$$\hat{h}_c^{-1} x \in \mathrm{K}_M \iff h_c^{-1} x\, \eta \in \mathrm{K}(M) \qquad (x \in G(\mathbb{A}_f)), \tag{5.118}$$

because $\eta^{-1} \hat{h}_c^{-1} = \eta^{-1} h_c^{-1} \eta = h_c^{-1}$. The conditions defining $R(\mathbb{A})$ are conditions on matrix entries, imposed componentwise, so $R(\mathbb{A}) = R(F_\infty) \times R(\mathbb{A}_f)$, a product of closed subgroups;

$\hat h_c \in \iota(\mathrm{U}(X_0)(\mathbb{A}_f)) \subseteq R(\mathbb{A}_f)$ by Lemma 5.6.34 (i). The group $\mathsf{U}(\mathbb{A})$ of Definition 3.2.36 (the g with $g_{ik} = \delta_{ik}$ whenever $\mathrm{gr}(i) \le \mathrm{gr}(k)$) consists of the isometries acting trivially on the successive quotients of the flag $\langle e_1\rangle \subset \cdots \subset \langle e_1,\dots,e_\varrho\rangle \subset \langle e_1,\dots,e_\varrho\rangle^\perp \subset \dots$, so $\mathsf{U}(\mathbb{A}) \cap G(E)$ is the group $\mathsf{U}(E)$ of (RD2), and $\mathsf{U}_c(M) \subseteq \mathsf{U}(E) \subseteq R(E)$.

Step 1: the measures. Let dr_f be a left Haar measure of $R(\mathbb{A}_f)$. Then $ds \otimes dr_f$ is a left Haar measure of $R(\mathbb{A})$, so $dr = \kappa_0\, ds \otimes dr_f$ for some $\kappa_0 > 0$. J'_f is an open compact subgroup of $R(\mathbb{A}_f)$ (K_M is open and compact in $G(\mathbb{A}_f)$), so $0 < dr_f(J'_f) < \infty$; put $\kappa = \kappa_0\, dr_f(J'_f)$.

Step 2: the class cutoffs sum to one along $\mathsf{U}_c(M)$. For $\gamma \in G(E)$ and $x \in G(\mathbb{A})$, $\mathbf{y}(\gamma x) = \gamma_\infty \mathbf{y}(x)$, since $B_v^{-1} v(\gamma) x_v P_v p^{-1} = (B_v^{-1} v(\gamma) B_v)(B_v^{-1} x_v P_v p^{-1})$ and $B_v^{-1} v(\gamma) B_v = \gamma_v$ by (5.5). By (i) of Lemma 5.2.36 for the admissible datum $\mathfrak{S}_D$ with Weil data,

$$\sum_{u \in \mathsf{U}_c(M)} \beta_c(u \cdot y) = 1 \qquad (y \in G_\infty). \tag{5.119}$$

With $y = \mathbf{y}(x)$ and $\mathbf{y}(ux) = u \cdot \mathbf{y}(x)$ ($x \in G(F_\infty)$) this is (ii).

Step 3: the class weights. For $c \in \mathcal{C}$ and $r = (s, g) \in R(\mathbb{A})$ put $w_c(r) = \beta_c(\mathbf{y}(s))\, \mathbf{1}[\hat h_c^{-1} g \in \mathrm{K}_M] \ge 0$, a Borel function. We show $\sum_{\gamma \in R(E)} \sum_c w_c(\gamma r) = 1$ for every r.

(a) If $u \in \mathsf{U}_c(M)$ and $\hat h_c^{-1} g \in \mathrm{K}_M$, then $\hat h_c^{-1} u g \in \mathrm{K}_M$. Indeed $m = h_c^{-1} u h_c \in \mathrm{K}(M)$ by (RD2), so $u\hat h_c = u h_c \eta^{-1} = h_c m \eta^{-1} = \hat h_c (\eta m \eta^{-1})$ with $\eta m \eta^{-1} \in \mathrm{K}_M$, and $\hat h_c^{-1} u g = (\eta m \eta^{-1})(\hat h_c^{-1} g)$. As u, $\hat h_c \in R$, $k_u = \hat h_c^{-1} u \hat h_c = \eta m \eta^{-1}$ lies in $R(\mathbb{A}_f) \cap \mathrm{K}_M = J'_f$; with $\mathsf{U}_c(M) \subseteq R(E)$ (shown above) this is (i).

(b) For every $r = (s, g)$ there are $\gamma \in R(E)$ and c with $\hat h_c^{-1} \gamma g \in \mathrm{K}_M$. Let $g' = g\eta \in R(\mathbb{A}_f)$. By (FG4), $a(1, g') = \delta h_c \kappa'$ with $\delta \in \mathrm{U}(X_0)(E^+)$, a class c and $\kappa' \in \mathrm{K}_X$; by Lemma 5.6.34 (i), $\iota(\delta)^{-1} g' = n\, \iota(h_c \kappa')$ with $n = u(\iota(\delta)^{-1} g') \in \mathsf{U}(\mathbb{A}_f)$ (finite components). The group $O = h_c \mathrm{K}(M) h_c^{-1} \cap \mathsf{U}(\mathbb{A}_f)$ is an open subgroup of $\mathsf{U}(\mathbb{A}_f)$, so by strong approximation for the unipotent group U over E^+ [233] $n = u_1 o$ with $u_1 \in \mathsf{U}(E)$, $o \in O$. For $\gamma = u_1^{-1} \iota(\delta)^{-1} \in R(E)$, $h_c^{-1} \gamma g \eta = h_c^{-1} o\, \iota(h_c \kappa') = (h_c^{-1} o h_c)\, \iota(\kappa') \in \mathrm{K}(M)$, as $\iota(\kappa') \in \mathrm{K}(M)$ by (FA1); by (5.118), $\hat h_c^{-1} \gamma g \in \mathrm{K}_M$.

(c) If $\hat h_d^{-1} g \in \mathrm{K}_M$, $\gamma \in R(E)$ and $\hat h_c^{-1} \gamma g \in \mathrm{K}_M$, then $c = d$ and $\gamma \in \mathsf{U}_c(M)$. Indeed $\hat h_c^{-1} \gamma \hat h_d = (\hat h_c^{-1} \gamma g)(\hat h_d^{-1} g)^{-1} \in \mathrm{K}_M$, so $t = \eta^{-1} \hat h_c^{-1} \gamma \hat h_d \eta = h_c^{-1} \gamma h_d \in \mathrm{K}(M)$ (finite components), and $(1, t) \in R(\mathbb{A})$. By Lemma 5.6.39, $a(1, t) = h_c^{-1} a(\gamma)_f h_d \in \mathrm{K}_0 = \mathrm{K}_X$, where $a(\gamma) \in \mathrm{U}(X_0)(E^+)$ is the X_0-block of the rational matrix γ. So $h_d \in a(\gamma)^{-1} h_c \mathrm{K}_X$, whence $c = d$, and $a(\gamma) \in \mathrm{U}(X_0)(E^+) \cap h_c \mathrm{K}_X h_c^{-1} = \{1\}$ by (FG4). Hence $\gamma \in \mathsf{U}(\mathbb{A}) \cap G(E) = \mathsf{U}(E)$ (Lemma 5.6.34 (i)) and $h_c^{-1} \gamma h_c = t \in \mathrm{K}(M)$: $\gamma \in \mathsf{U}_c(M)$.

Let $r \in R(\mathbb{A})$; by (b) choose $\gamma_0 \in R(E)$ and d with $\hat h_d^{-1} (\gamma_0 r)_f \in \mathrm{K}_M$, and put $r' = \gamma_0 r = (s', g')$. Reindexing $\gamma \mapsto \gamma\gamma_0$, $\sum_\gamma \sum_c w_c(\gamma r) = \sum_\gamma \sum_c w_c(\gamma r')$. By (c) only $c = d$ and $\gamma \in \mathsf{U}_d(M)$ contribute; by (a) and Step 2, $w_d(ur') = \beta_d(\mathbf{y}(us')) = \beta_d(u \cdot \mathbf{y}(s'))$ for $u \in \mathsf{U}_d(M)$; by (5.119) the sum is 1.

Step 4: unfolding. $R(E)$ is discrete in $R(\mathbb{A})$ (Lemma 5.6.34 (iii)) and $R(\mathbb{A})$ has a countable base, so by Lemma 5.6.20 there is a Borel set F_R meeting every orbit $R(E)r$ in exactly one point; $R(\mathbb{A}) = \bigsqcup_{\gamma \in R(E)} \gamma F_R$, and the integral over $[R]$ is the integral over F_R. Put $w = \sum_c w_c$. For Borel $\phi \ge 0$, left invariance and monotone convergence give $\int_{R(\mathbb{A})} \phi\, dr = \int_{F_R} \sum_\gamma \phi(\gamma x)\, dr(x)$. With $\phi = w|F|$ and Step 3, $\int_{R(\mathbb{A})} w|F|\, dr = \int_{F_R} |F|\, dr < \infty$; applying the same identity to the positive and negative parts of the real and imaginary parts of the integrable wF, and using $F(\gamma x) = F(x)$,

$$\int_{R(\mathbb{A})} w\, F\, dr = \int_{F_R} \sum_\gamma w(\gamma x) F(x)\, dr(x) = \int_{[R]} F\, dr.$$

Step 5: one class. Let $g \in R(\mathbb{A}_f)$. As $\hat{h}_c \in R(\mathbb{A}_f)$, $\hat{h}_c^{-1} g \in \mathrm{K}_M$ if and only if $g \in \hat{h}_c J_f'$, and then $F(s,g) = F((s,\hat{h}_c)(1,\hat{h}_c^{-1}g)) = F(s,\hat{h}_c)$. So $w_c(s,g)F(s,g) = \beta_c(\mathbf{y}(s))F(s,\hat{h}_c)\,\mathbf{1}_{\hat{h}_c J_f'}(g)$, with $dr_f(\hat{h}_c J_f') = dr_f(J_f')$. Since $w_c|F| \le w|F|$ is integrable, Tonelli's theorem for $dr = \kappa_0\, ds \otimes dr_f$ shows that $s \mapsto \beta_c(\mathbf{y}(s))F(s,\hat{h}_c)$ (a Borel function) is integrable on $R(F_\infty)$, and Fubini's theorem gives $\int_{R(\mathbb{A})} w_c F\, dr = \kappa \int_{R(F_\infty)} \beta_c(\mathbf{y}(s))F(s,\hat{h}_c)\, ds$. Summing over the finitely many classes and using Step 4 proves the identity (iii). □

Proposition 5.6.41. *Let $\mathfrak{S}$ be an admissible filling datum over a field E with Weil data, let ρ be a unitary representation of $\bar{G}$ on a finite-dimensional Hilbert space $\mathcal{H}$, and let $P\colon \mathbb{C}^{\mathcal{C}\times I} \to \mathcal{H}$ have amplitude blocks (TA6). For $i = 0, 1$ let g_i be a central idempotent of $\mathbb{C}[\bar{G}_i]$ with*

$$\operatorname{Re}(1 - g_i)(1) \le \tfrac{1}{2}\, b^{-8}.$$

Then $\rho(g_0 \otimes g_1)\, P$ is a near-isometry with error $5b^{-4}$: for every subspace $U \subseteq \mathbb{C}^{\mathcal{C}\times I}$,

$$\|\rho(g_0 \otimes g_1) P \operatorname{pr}_U\|_{\mathrm{HS}}^2 \;\ge\; \dim U - 5b^{-4} D_X.$$

Proof. We use the following facts from Proposition 5.6.42, for $i \in \{0,1\}$; here Ω_i' is the set of $s \in \bar{G}$ that agree with an element of $\bar{R}_0$ at every chosen place above p_{1-i} (the set Ω_{1-i} there):

(M1) $\sum_{s\in\Omega_i'} \|P_s - r_\psi(s)\|^2 \le b^{-12}\mathcal{D}_{\mathrm{id}}$, and $\|P^*P - 1\|^2 \le b^{-12} D_X$ (clause (iii));

(M2) for $r \in \bar{R}_0$ and $s \in \bar{G}$: $r_\psi(r)$ is unitary, $r_\psi(rs) = r_\psi(r) r_\psi(s)$ and $P_{rs} = r_\psi(r) P_s$ (clause (v));

(M3) $\operatorname{tr} r_\psi(\iota_i t) = D_X\, \psi_U(\iota_i t)$ for $t \in \bar{G}_i$ (clause (vi)), and $\sum_{s\in\bar{G}} |\psi_U^\natural(s)|^2 \le 2$ (clause (iv)).

Central idempotents. A central idempotent x of $\mathbb{C}[H]$, H a finite group, is a sum of the primitive central idempotents $\chi(1)\#H^{-1}\sum_h \overline{\chi(h)} h$, so $x(h^{-1}) = \overline{x(h)}$ and x is a class function. Hence for a unitary representation σ of H, $\sigma(x) = \sum_h x(h)\sigma(h)$ is self-adjoint and idempotent, an orthogonal projection; and comparing coefficients at 1 in $x^2 = x$ gives $\sum_h |x(h)|^2 = x(1)$, which is real.

Step 1: P is nearly an isometry. For a subspace U, $\|P\operatorname{pr}_U\|^2 - \dim U = \operatorname{tr}(\operatorname{pr}_U(P^*P - 1))$, so by Cauchy–Schwarz for the Hilbert–Schmidt inner product, $\|\operatorname{pr}_U\|^2 = \dim U \le D_X$ and (M1),

$$\|P \operatorname{pr}_U\|^2 \ge \dim U - \sqrt{D_X}\,\sqrt{b^{-12} D_X} = \dim U - b^{-6} D_X.$$

Step 2: one prime. Let x be a central idempotent of $\mathbb{C}[\bar{G}_i]$ with $\operatorname{Re} x(1) \le \frac{1}{2} b^{-8}$ and $E_x = \rho(\iota_i x)$, an orthogonal projection. We show $\|E_x P\|^2 \le 2b^{-4} D_X$. Since $E_x^* E_x = E_x$,

$$\|E_x P\|^2 = \operatorname{tr}(P^* E_x P) = \sum_{t\in\bar{G}_i} x(t) \operatorname{tr} r_\psi(\iota_i t) + \operatorname{tr} Y, \qquad Y = \sum_{t\in\bar{G}_i} x(t)\big(P_{\iota_i t} - r_\psi(\iota_i t)\big),$$

and we bound the real parts of the two terms.

The Bessel term. By (M3) it is $D_X \sum_t x(t)\psi_U(\iota_i t)$. For $g \in \bar{G}$, $g\,\iota_i(t) g^{-1} = \iota_i(g_i t g_i^{-1})$, and $t \mapsto g_i t g_i^{-1}$ is a bijection of $\bar{G}_i$ preserving x; averaging over g replaces ψ_U by $\psi_U^\natural$. By Cauchy–Schwarz, $\sum_t |x(t)|^2 = x(1) \le \frac{1}{2} b^{-8}$ and (M3), $\big|\sum_t x(t)\psi_U^\natural(\iota_i t)\big|^2 \le \frac{1}{2} b^{-8} \cdot 2$, so the Bessel term has real part at most $b^{-4} D_X$.

The remainder. Let $F\colon \bar G \to \mathrm{End}(\mathbb{C}^{\mathcal{C}\times I})$ be $F(s) = P_s - r_\psi(s)$ for $s \in \Omega_i'$ and $F(s) = 0$ otherwise. The right convolution $(R_xF)(s) = \sum_t x(t)F(s\,\iota_i t)$ is $\sigma(x)$ for the right regular representation σ of $\bar G_i$ on the Hilbert space of functions $\bar G \to \mathrm{End}$ with $\|F\|^2 = \sum_s \|F(s)\|^2$, which is unitary; so R_x is an orthogonal projection and $\sum_s \|(R_xF)(s)\|^2 \le \sum_s \|F(s)\|^2$. For $r \in \bar R_0$ and $t \in \bar G_i$, $r\,\iota_i(t)$ agrees with r in the factor $\bar G_{1-i}$, so lies in Ω_i', and by (M2) $F(r\,\iota_i t) = r_\psi(r)F(\iota_i t)$; hence $(R_xF)(r) = r_\psi(r)Y$ has norm $\|Y\|$. With (M1),

$$\#\bar R_0\,\|Y\|^2 \le \sum_{s\in\bar G} \|(R_xF)(s)\|^2 \le \sum_{s\in\Omega_i'} \|P_s - r_\psi(s)\|^2 \le b^{-12}\#\bar R_0\,D_X,$$

so $|\operatorname{tr} Y| = |\langle 1, Y\rangle_{\mathrm{HS}}| \le \sqrt{D_X}\,\|Y\| \le b^{-6}D_X \le b^{-4}D_X$. Adding, $\|E_xP\|^2 \le 2b^{-4}D_X$.

Step 3: both primes. Let $E_i = \rho(\iota_i g_i)$, commuting orthogonal projections with $\rho(g_0 \otimes g_1) = E_0E_1$. Each $1 - g_i$ is a central idempotent of $\mathbb{C}[\bar G_i]$ with $\mathrm{Re}(1-g_i)(1) \le \frac12 b^{-8}$ and $\rho(\iota_i(1-g_i)) = 1 - E_i$, so Step 2 gives $\|(1-E_i)P\|^2 \le 2b^{-4}D_X$. For every vector y, $(1-E_0E_1)y = (1-E_0)y + E_0(1-E_1)y$ is an orthogonal sum and $\|E_0(1-E_1)y\| \le \|(1-E_1)y\|$; applied to $y = Pe$ for the vectors e of an orthonormal basis,

$$\|(1-E_0E_1)P\|^2 \le \|(1-E_0)P\|^2 + \|(1-E_1)P\|^2 \le 4b^{-4}D_X.$$

Step 4: conclusion. As E_0E_1 is an orthogonal projection, $\|P\mathrm{pr}_U\|^2 = \|E_0E_1P\mathrm{pr}_U\|^2 + \|(1-E_0E_1)P\mathrm{pr}_U\|^2$, and $\|(1-E_0E_1)P\mathrm{pr}_U\| \le \|(1-E_0E_1)P\|$. By Steps 1 and 3 and $b \ge 1$,

$$\|\rho(g_0\otimes g_1)P\,\mathrm{pr}_U\|^2 \ge \dim U - b^{-6}D_X - 4b^{-4}D_X \ge \dim U - 5b^{-4}D_X.$$

□

Proposition 5.6.42. *Let $\mathfrak{S}$ be admissible with Weil data. For $i \in \{0,1\}$ let Ω_i be the set of $s \in \bar G$ that agree with an element of $\bar R_0$ at every chosen place above p_i.*

(i) If $\Omega \subseteq \bar G$ avoids the scalar cosets (no $s \in \Omega$ is ζt with $t \in \bar R_0$ at every chosen place, for an integral $\zeta \ne 1$ with $\zeta\bar\zeta = 1$), $C \ge 0$ and $\mathrm{LM}(\{\mathsf{g} \ne 0\}, \Omega) \le C\nu^2$, then $\sqrt{\mathrm{SM}(A^{\mathrm{full}}_{G(E)\backslash R(E)}, \Omega)} \le \sqrt{C} + b^{-15}\sqrt{\mathcal{D}_{\mathrm{id}}}$.

(ii) $\mathrm{SM}(A^{\mathrm{full}}_{G(E)\backslash R(E)}, \Omega_i) \le b^{-12}\mathcal{D}_{\mathrm{id}}$.

(iii) If P has amplitude blocks for a representation ρ of $\bar G$, then $\sum_{s\in\Omega_i} \|P^*\rho(s)P - r_\psi(s)\|^2_{\mathrm{HS}} \le b^{-12}\mathcal{D}_{\mathrm{id}}$; in particular $\|P^*P - 1\|^2_{\mathrm{HS}} \le b^{-12}D_X$ and $\|P^*P\|^2_{\mathrm{HS}} \le 2D_X$.

(iv) $\sum_{s\in\bar G} |\psi^\natural_U(s)|^2 \le 2$.

(v) Equivariance. For $t \in \bar R_0$ and $s \in \bar G$, $r_\psi(t)$ is unitary and $r_\psi(ts) = r_\psi(t)r_\psi(s)$; if P has amplitude blocks for a unitary ρ, then moreover $P^*\rho(ts)P = r_\psi(t)\,P^*\rho(s)P$.

(vi) Trace. Write $\bar G = \bar G_0 \times \bar G_1$, $\bar G_i = \prod_{w\in S_i} \mathrm{GL}_r(k_w)$, and $\iota_i\colon \bar G_i \to \bar G$ for the inclusion of the factor i. If $s \in \bar G$ has trivial component in $\bar G_0$ or in $\bar G_1$, then $\operatorname{tr} r_\psi(s) = D_X\,\psi_U(s)$; in particular $\operatorname{tr} r_\psi(\iota_i t) = D_X\,\psi_U(\iota_i t)$ for $t \in \bar G_i$.

ar cosets and fixers. Call s in a scalar coset *if $s = \zeta t$ at every chosen place for some integral ζ with $\zeta\bar\zeta = 1$ and $t \in \bar R_0$. If $s \notin \bar R_0$ then $A^{\mathrm{full}}_{G(E)}(c,s,c') = A^{\mathrm{full}}_{G(E)\backslash R(E)}(c,s,c')$ for all c, c'. Open elements are in no scalar coset. If $\Theta\colon \bar G \to [0, 2D_X]$, then $\sum_{s \text{ in a scalar coset}} \Theta(s) \le 2M^f\mathcal{D}_{\mathrm{id}}$. Every $s \in \mathrm{Fix}(S_0, d)$ has meet norm at least p_0^f.*

Proof. Throughout, D_{id} stands for $\mathcal{D}_{\mathrm{id}} = \#\bar{R}_0\,\#\mathcal{C}\,\#I$; as $1 \in \bar{R}_0$ and $I \neq \emptyset$, $\#\mathcal{C} \leq \mathcal{D}_{\mathrm{id}}$. By (FG6), $k \geq 1000$ and $f \geq 1$, so $kf \geq 40$.

Step 0: blocks split along $R(E)$, and the entries of $P^*\rho(s)P$ and $r_\psi(s)$. For every $T \subseteq G(E)$ and every block (c,s,c') the amplitudes of the terms in T over (c,s,c') form an absolutely summable family, since their norms are bounded by the amplitude majorants, which are summable by Lemma 5.2.29 (ii). Hence

$$A_{G(E)}(c,s,c') = A_{R(E)}(c,s,c') + A_{G(E)\setminus R(E)}(c,s,c') \tag{5.120}$$

for the full and the cut amplitude sums. If P has amplitude blocks (TA6) for a representation ρ, then for $x = (c_x, i_x)$, $y = (c_y, i_y)$,

$$(P^*\rho(s)P)_{xy} = \langle Pe_x, \rho(s)Pe_y\rangle = \overline{\langle \rho(s)Pe_y, Pe_x\rangle} = \nu^{-1}\,\overline{A^{\mathrm{full}}_{G(E)}(c_x,s,c_y)_{i_x i_y}},$$

and $r_\psi(s)_{xy} = \nu^{-1}\overline{A^{\mathrm{full}}_{R(E)}(c_x,s,c_y)_{i_x i_y}}$ by (TA7). Terms compose and invert: if $h_c^{-1}\gamma h_{c'} = \kappa \in K$ with $\mathrm{red}(\kappa) = s$ and $h_{c'}^{-1}\delta h_{c''} = \kappa' \in K$ with $\mathrm{red}(\kappa') = t$, then $h_c^{-1}\gamma\delta h_{c''} = \kappa\kappa'$ and $h_{c'}^{-1}\gamma^{-1}h_c = \kappa^{-1}$, so $\gamma\delta$ is a term over (c, st, c'') and γ^{-1} one over (c', s^{-1}, c), red being a homomorphism. If $\mathcal{M}, \mathcal{N}$ are action matrices of $\gamma, \delta \in R(E)$, then $\mathcal{M}\mathcal{N}$ is one of $\gamma\delta$, and if $\mathcal{N}$ is unitary, $\mathcal{N}^* = \mathcal{N}^{-1}$ is one of δ^{-1} (apply $W(y) = W(\delta\cdot(\delta^{-1}\cdot y)) = \mathcal{N}W(\delta^{-1}\cdot y)$).

Step 1: elements of $\bar{R}_0$ fix the first frame vector. Let $t = \mathrm{red}(\kappa)$ with $\kappa \in R(\mathbb{A}_f) \cap K$. The vector $f_0 = e_1$ is a vector of the lattice basis (FG3) (only x_k is changed there), and $\kappa e_1 = e_1$: U acts trivially on the line $\langle e_1\rangle$, the first step of the flag it stabilises, and $H = \mathrm{U}(X_0)$ fixes e_1. As κ is an isometry, $h(e_1, \kappa x) = h(\kappa^{-1}e_1, x) = h(e_1, x)$, so the row of κ that gives the coordinate along e_1^* in the lattice basis is that of the identity. Reducing at a chosen place w, t_w fixes e_1, and so does the inverse adjoint of t_w (it fixes e_1 exactly when the e_1^*-row of t_w is that of the identity). Consequently, if $s \in \bar{G}$ and $s_w = t_w$, then s meets at $(w,0)$ and at $(w,1)$ (TA11), with $x = y$ the first coordinate vector.

Step 2: a root of unity that is 1 at the places of S_i is 1. Let $\zeta \in \mathcal{O}_E$, $\zeta\bar{\zeta} = 1$, with $\zeta \equiv 1$ modulo every $w \in S_i$. For every embedding $\sigma\colon E \to \mathbb{C}$, $\sigma(\bar{\zeta}) = \overline{\sigma(\zeta)}$ (E is CM), so $|\sigma(\zeta)|^2 = \sigma(\zeta\bar{\zeta}) = 1$. If $\zeta \neq 1$, then $\zeta - 1$ is a nonzero element of the product of the f distinct primes $w \in S_i$, each of norm p_i, so $p_i^f \leq |\mathrm{N}_{E/\mathbb{Q}}(\zeta - 1)| = \prod_\sigma |\sigma(\zeta) - 1| \leq 2^{2f}$, against $p_i \geq b^{10} > 4$ (FG6). Hence $\zeta = 1$.

Proof of (v). By Step 0 and Lemma 5.6.45 (ii), $A^{\mathrm{full}}_{R(E)}(c_y, s, c_x) = A^{\mathrm{full}}_{R(E)}(c_x, s^{-1}, c_y)^*$, whence $(r_\psi(s)^*)_{xy} = \overline{r_\psi(s)_{yx}} = r_\psi(s^{-1})_{xy}$: $r_\psi(s)^* = r_\psi(s^{-1})$ for all s. Let $t \in \bar{R}_0$ and $d \in \mathcal{C}$. By Lemma 5.6.45 (iii) (for d and $t^{-1} \in \bar{R}_0$) and (i) there are d' and a term $\delta \in R(E)$ over (d, t^{-1}, d') with a unitary action matrix $\mathcal{N}$. By Lemma 5.6.45 (ii), $A^{\mathrm{full}}_{R(E)}(d, t^{-1}, e) = \nu\,\delta_{ed'}\mathcal{N}$, so $A^{\mathrm{full}}_{R(E)}(e, t, d) = \nu\,\delta_{ed'}\mathcal{N}^*$ and

$$r_\psi(t)\,e_{(d,j)} = \sum_{l\in I} \mathcal{N}_{jl}\,e_{(d',l)} \qquad (j \in I). \tag{5.121}$$

For $t' \in \bar{R}_0$ choose likewise δ' over (d, t'^{-1}, d') with $\mathcal{N}'$, and δ over (d', t^{-1}, d'') with $\mathcal{N}$. Then $\delta'\delta \in R(E)$ is a term over $(d, (tt')^{-1}, d'')$ with action matrix $\mathcal{N}'\mathcal{N}$ (Step 0), so (5.121) gives $r_\psi(tt')e_{(d,j)} = \sum_l (\mathcal{N}'\mathcal{N})_{jl}e_{(d'',l)} = r_\psi(t)r_\psi(t')e_{(d,j)}$. With $\delta = 1$ over $(d, 1, d)$ and $\mathcal{N} = 1$, $r_\psi(1) = 1$; hence $r_\psi(t)^*r_\psi(t) = r_\psi(t^{-1})r_\psi(t) = 1$ and $r_\psi(t)$ is unitary. If $s \notin \bar{R}_0$, then $ts \notin \bar{R}_0$, no term of $R(E)$ lies over a block at s or at ts (Lemma 5.6.45 (ii)), and $r_\psi(ts) = 0 = r_\psi(t)r_\psi(s)$; if $s \in \bar{R}_0$ this was shown. So $r_\psi(ts) = r_\psi(t)r_\psi(s)$.

Now let P have amplitude blocks for a representation ρ (not necessarily unitary), and write $P_s = P^*\rho(s)P$. *Right equivariance: $P_{st} = P_s r_\psi(t)$ for $s \in \bar{G}$, $t \in \bar{R}_0$.* With $\delta, \mathcal{N}$ for d and t^{-1} as

above, Lemma 5.6.45 (iv) for the stable set $T = G(E)$ gives $A^{\mathrm{full}}_{G(E)}(c, st, d) = A^{\mathrm{full}}_{G(E)}(c, s, d')\mathcal{N}^*$, and by Step 0 and (5.121)

$$(P_s r_\psi(t))_{x,(d,j)} = \sum_l (P_s)_{x,(d',l)}\mathcal{N}_{jl} = \nu^{-1}\,\overline{(A^{\mathrm{full}}_{G(E)}(c_x, s, d')\mathcal{N}^*)_{i_x j}} = (P_{st})_{x,(d,j)}.$$

Left equivariance for unitary ρ. Let $t \in \bar{R}_0$, $\phi \in \mathbb{C}^{\mathcal{C}\times I}$ and $\Delta = \rho(t)P\phi - Pr_\psi(t)\phi$. For $s \in \bar{G}$ and ϕ', unitarity and right equivariance give $\langle \rho(s)P\phi', \Delta\rangle = \langle \phi', (P_{s^{-1}t} - P_{s^{-1}} r_\psi(t))\phi\rangle = 0$. Taking $(s, \phi') = (t, \phi)$ and $(1, r_\psi(t)\phi)$, $\langle \Delta, \Delta\rangle = 0$, so $\rho(t)P = Pr_\psi(t)$. Hence $P^*\rho(t) = (\rho(t^{-1})P)^* = (Pr_\psi(t^{-1}))^* = r_\psi(t)P^*$, and $P^*\rho(ts)P = r_\psi(t)\, P^*\rho(s)P$.

Proof of (vi). Let s have component 1 in $\bar{G}_j$, that is at every $w \in S_j$; so has s^{-1}. By Step 0, (v) and Lemma 5.6.45 (ii), (v),

$$\operatorname{tr} r_\psi(s) = \nu^{-1}\sum_{c\in\mathcal{C}}\sum_{i\in I}\overline{A^{\mathrm{full}}_{R(E)}(c, s, c)_{ii}} = \nu^{-1}\sum_{c,i} A^{\mathrm{full}}_{R(E)}(c, s^{-1}, c)_{ii} = \#\mathcal{C}\,\#I\,\psi_U(s) = D_X\,\psi_U(s).$$

For $t \in \bar{G}_i$, $\iota_i t$ has component 1 in $\bar{G}_{1-i}$.

Proof of (vii). If $s \notin \bar{R}_0$, no term of $R(E)$ lies over (c, s, c') (Lemma 5.6.45 (ii)), so $A^{\mathrm{full}}_{R(E)}(c, s, c') = 0$ and (5.120) gives the first claim. Let s be in a scalar coset, $s_w = \zeta t_w$ at every chosen place with $t \in \bar{R}_0$. By Step 1, $s_w f_0 = \zeta t_w e_1 = \zeta e_1$, so the leading minor of order 1 of $Z_w(s)$ is $h_w(e_1, \zeta e_1) = \zeta\, h(e_1, e_1) = 0$ at every $w \in S_0$ (e_1 is isotropic, and $\varrho \geq 1$); so s is not open (TA10). The element s is determined by t and by the residues of ζ at the $2f$ chosen places; by (FG2) there are at most $\prod_w \#k_w = p_0^f p_1^f = M^f$ families of residues, so at most $M^f \#\bar{R}_0$ elements lie in a scalar coset, and $\sum_{s \text{ in a scalar coset}} \Theta(s) \leq 2D_X\, M^f \#\bar{R}_0 = 2M^f \mathcal{D}_{\mathrm{id}}$. Finally let $s \in \mathrm{Fix}(S_0, d)$, $d = (d_w)_{w\in S_0}$, $d_w = (\epsilon_w, v_w)$. At each $w \in S_0$ the element g of (TA10) fixes $f \cdot v_w$, so $f \cdot v_w = g\, f \cdot v_w$ with $v_w \neq 0$: s meets at (w, ϵ_w). These are $\#S_0 = f$ distinct pairs, each contributing p_0, and the other factors of the meet norm are ≥ 1; so the meet norm of s is at least p_0^f.

Proof of (i). *Off $R(E)$ the blocks are those of Γ^*.* By Lemma 5.6.46, $A^{\mathrm{full}}_{G(E)\setminus R(E)}(c, s, c') = A^{\mathrm{full}}_{\Gamma^*}(c, s, c')$ for $s \in \Omega$. *Minkowski.* $\nu\sqrt{\mathrm{SM}(A, \Omega)}$ is the norm of the family $(A(c, s, c'))_{c,c'\in\mathcal{C},\, s\in\Omega}$ in the Hilbert space $\bigoplus \mathbb{C}^{I\times I}$ with Hilbert–Schmidt norms, so by the triangle inequality

$$\sqrt{\mathrm{SM}(A^{\mathrm{full}}_{\Gamma^*}, \Omega)} \leq \sqrt{\mathrm{SM}(A^{\mathrm{full}}_{\Gamma^*} - A^{\mathrm{cut}}_{\Gamma^*}, \Omega)} + \sqrt{\mathrm{SM}(A^{\mathrm{cut}}_{\Gamma^*}, \Omega)}.$$

The cost of the cut. Put $\eta = \nu/(2b^{k^3 f})$. For fixed (c, s), $\sum_{c'} \|A^{\mathrm{full}}_{\Gamma^*}(c, s, c') - A^{\mathrm{cut}}_{\Gamma^*}(c, s, c')\|^2 \leq \big(\sum_{c'} \|\cdots\|\big)^2 \leq \eta^2$ by Lemma 5.3.62 (ii). Summing over $c \in \mathcal{C}$ and $s \in \bar{G}$, and using $\#\bar{G} \leq \prod_w \#\mathrm{GL}_r(k_w) \leq M^{fr^2} \leq b^{23fr^2}$ (FG2), (FG6),

$$\mathrm{SM}(A^{\mathrm{full}}_{\Gamma^*} - A^{\mathrm{cut}}_{\Gamma^*}, \Omega) \leq \nu^{-2}\#\mathcal{C}\,\#\bar{G}\,\eta^2 \leq \tfrac{1}{4}\, b^{23fr^2 - 2k^3 f}\,\mathcal{D}_{\mathrm{id}} \leq b^{-40}\mathcal{D}_{\mathrm{id}},$$

The cut part. By Theorem 5.3.2 (iii) with $w = C$, $\mathrm{SM}(A^{\mathrm{cut}}_{\Gamma^*}, \Omega) \leq C + \mathcal{D}_{\mathrm{id}}/b^{kf} \leq C + b^{-40}\mathcal{D}_{\mathrm{id}}$. *Together,* with $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ and $2 \leq b^5$,

$$\sqrt{\mathrm{SM}(A^{\mathrm{full}}_{G(E)\setminus R(E)}, \Omega)} \leq b^{-20}\sqrt{\mathcal{D}_{\mathrm{id}}} + \sqrt{C} + b^{-20}\sqrt{\mathcal{D}_{\mathrm{id}}} \leq \sqrt{C} + b^{-15}\sqrt{\mathcal{D}_{\mathrm{id}}}.$$

Proof of (ii). *Ω_i avoids the scalar cosets.* Let $s \in \Omega_i$, $s_w = t_w$ for $w \in S_i$ with $t \in \bar{R}_0$, and suppose $s_w = \zeta t'_w$ at every chosen place with $\zeta \in \mathcal{O}_E$, $\zeta\bar{\zeta} = 1$, $t' \in \bar{R}_0$. For $w \in S_i$, Step 1 gives $e_1 = t_w e_1 = s_w e_1 = \zeta t'_w e_1 = \zeta e_1$ modulo w, so $\zeta \equiv 1$ modulo w, and $\zeta = 1$ by Step 2. *Meet*

norm. By Step 1 every $s \in \Omega_i$ meets at the $2f$ pairs (w, ϵ), $w \in S_i$, $\epsilon \in \{0,1\}$, so its meet norm is at least $q = p_i^{2f}$. *Links.* By Theorem 5.3.2 (iv), (FA6), and $M = p_i p_{1-i} \le p_i\, b^{11} \le b\, p_i^2$ (FG6),

$$\mathrm{LM}(\{\mathfrak{g} \neq 0\}, \Omega_i) \le \Big(\frac{b^{2f}}{p_i^{2f}}\Big)^k \Big(\frac{M}{b^6}\Big)^{kf} \mathcal{D}_{\mathrm{id}}\, \nu^2 \le \Big(\frac{b^2}{p_i^2} \cdot \frac{b\, p_i^2}{b^6}\Big)^{kf} \mathcal{D}_{\mathrm{id}}\, \nu^2 = b^{-3kf} \mathcal{D}_{\mathrm{id}}\, \nu^2 \le b^{-30} \mathcal{D}_{\mathrm{id}}\, \nu^2.$$

Conclusion. By (i) with $C = b^{-30}\mathcal{D}_{\mathrm{id}}$, $\sqrt{\mathrm{SM}(A^{\mathrm{full}}_{G(E)\backslash R(E)}, \Omega_i)} \le 2b^{-15}\sqrt{\mathcal{D}_{\mathrm{id}}}$, so $\mathrm{SM} \le 4b^{-30}\mathcal{D}_{\mathrm{id}} \le b^{-12}\mathcal{D}_{\mathrm{id}}$.

Proof of (iii). By Step 0 and (5.120), $(P_s - r_\psi(s))_{xy} = \nu^{-1}\overline{A^{\mathrm{full}}_{G(E)\backslash R(E)}(c_x, s, c_y)_{i_x i_y}}$, so $\|P_s - r_\psi(s)\|^2 = \nu^{-2}\sum_{c,c'} \|A^{\mathrm{full}}_{G(E)\backslash R(E)}(c, s, c')\|^2$, and summing over $s \in \Omega_i$ gives $\mathrm{SM}(A^{\mathrm{full}}_{G(E)\backslash R(E)}, \Omega_i) \le b^{-12}\mathcal{D}_{\mathrm{id}}$ by (ii). For $t \in \bar{R}_0$, right equivariance (proof of (v)) and $r_\psi(1) = 1$ give $P_t - r_\psi(t) = (P^*P - 1)r_\psi(t)$, whose norm is $\|P^*P - 1\|$ as $r_\psi(t)$ is unitary. Since $\bar{R}_0 \subseteq \Omega_i$, $\#\bar{R}_0\, \|P^*P - 1\|^2 \le b^{-12}\mathcal{D}_{\mathrm{id}} = b^{-12}\#\bar{R}_0 D_X$, so $\|P^*P - 1\|^2 \le b^{-12} D_X$. Then $\|P^*P\| \le \|P^*P - 1\| + \|1\| \le (b^{-6} + 1)\sqrt{D_X}$, and $\|P^*P\|^2 \le (1 + b^{-6})^2 D_X \le 2D_X$.

Proof of (iv). This is Lemma 5.6.43. □

Lemma 5.6.43. *Let $\mathfrak{S}$ be an admissible filling datum over a field E that carries Weil data. Then*

$$\sum_{s \in \bar{G}} \big|\psi^{\natural}_U(s)\big|^2 \le 2.$$

Proof. **Step 1: the set T.** Let T be the set of the chosen places $w \in S_0 \cup S_1$ and their conjugates $\bar{w}$; it is finite and stable under conjugation. By (FG2) the $2f$ chosen places are distinct, $\bar{w} \notin S_0 \cup S_1$ for chosen w, and conjugation is injective, so $\#T = 4f$ and, for $i = 0, 1$, T contains $2f$ distinct places above p_i, each of residue cardinality $q_w = p_i$. As $\sum_{w|p_i} e_w f_w = [E : \mathbb{Q}] = 2f$, these are all the places above p_0 and p_1; so $M = p_0 p_1$ is a unit at every finite place outside T, i.e. M^{-1} is a T-integer. By Lemma 5.2.82 there is a filling datum (P, F) at T; F is a large lattice datum with level unit M (the image of M in E_T). Write θ for its group measure, G_T for its unitary group, $K_1 = \mathrm{K}_T(1)$ and $K_M = \mathrm{K}_T(M)$, and χ_M for its extended Bessel character. Put

$$a(x) = \int_{K_1} \chi_M(yxy^{-1})\, d\theta(y) \qquad (x \in G_T),$$

well defined as χ_M is continuous with $|\chi_M| \le 1$ and $0 < \theta(K_1) < \infty$ (Lemma 5.5.44 (b), (d), with the intermediate group K_1).

Step 2: lifting and reduction at T. (a) The map $m \mapsto m_T$ sends K onto K_1: m and m^{-1} are integral at every place, so $m_T \in K_1$; conversely, for $y \in K_1$ the element $m \in \mathrm{GL}_r(\mathbb{A}_f)$ with components y_w at $w \in T$ and 1 elsewhere (in the lattice basis) is unitary (T is stable under conjugation), integral with integral inverse, so $m \in K$ and $m_T = y$. (b) For $y \in K_1$ let $\bar{y} \in \bar{G}$ have the components $y_w \bmod w$ at the chosen places w ($w \in T$); then $\bar{y} = \mathrm{red}(m)$ for every $m \in K$ with $m_T = y$, and $y \mapsto \bar{y}$ is a homomorphism $K_1 \to \bar{G}$, surjective by (a) and (FA2). Its kernel K_1' is open (it contains K_M, as $M \in \mathfrak{p}_w$ at each chosen w), and its fibres are the cosets gK_1'.

Step 3: $a(l_T) = \theta(K_1)\,\psi^{\natural}_U(\mathrm{red}(l))$ for $l \in K$. Let $s = \mathrm{red}(l)$ and $y \in K_1$; choose $m \in K$ with $m_T = y$ (Step 2). Then $y l_T y^{-1} = (mlm^{-1})_T$ with $mlm^{-1} \in K$ and $\mathrm{red}(mlm^{-1}) = \bar{y}s\bar{y}^{-1}$, so by Lemma 5.6.44

$$\chi_M(y l_T y^{-1}) = \psi_U(\bar{y}\, s\, \bar{y}^{-1}).$$

The integrand is constant on the fibres of $y \mapsto \bar{y}$. By Lemma 5.5.41 (1), θ is a Haar measure, so the $\#\bar{G}$ fibres $gK_1' \subseteq K_1$, translates of K_1', have equal measure $\theta(K_1)/\#\bar{G}$. Hence

$$a(l_T) = \frac{\theta(K_1)}{\#\bar{G}} \sum_{g \in \bar{G}} \psi_U(gsg^{-1}) = \theta(K_1)\, \psi_U^\natural(s).$$

Step 4: a on cosets of K_M. For $x \in G_T$, $k \in K_M$ and $y \in K_1$, $yxky^{-1} = (yxy^{-1})(yky^{-1})$ with $yky^{-1} \in K_M$ (Lemma 5.5.44 (a)), and χ_M is right K_M-invariant (Lemma 5.5.44 (b)); so $a(xk) = a(x)$. For each $s \in \bar{G}$ choose $l_s \in K$ with $\mathrm{red}(l_s) = s$ (FA2), and put $C_s = (l_s)_T K_M$. If $C_s \cap C_{s'} \neq \emptyset$ then $(l_s)_T^{-1}(l_{s'})_T \in K_M$, so $s = s'$ by Lemma 5.2.46 (ii): the C_s are pairwise disjoint, and $a = a((l_s)_T)$ on C_s.

Step 5: the mass integral. By Step 4, and since $\theta(C_s) = \theta(K_M)$ (translation invariance),

$$\theta(K_M) \sum_{s \in \bar{G}} |a((l_s)_T)|^2 = \sum_s \int_{C_s} |a|^2 d\theta \leq \int_{G_T} |a|^2 d\theta = \mathrm{mi}(K_1).$$

Apply Lemma 5.5.42 to F, $Z = K_1$, $Q = 2$, $t = n$. Its hypotheses hold: if c is a unit of E_T in $\mathcal{O}_T$ with $c^{-1} \notin \mathcal{O}_T$, some component c_w lies in $\mathfrak{p}_w$, so $|c| = \prod_{w \in T} |c_w|_w \leq q_w^{-1} \leq \min(p_0, p_1)^{-1} \leq b^{-10} \leq \frac{1}{2}$ by (FG6); and $n + 2 \cdot 10^9 n^4 \leq 10^{10} n^4 \leq n^6 \leq n^7 \leq k$ by (FG6). So $\mathrm{mi}(K_1) \leq \theta(K_1)\, \theta(K_M)\, (\theta(K_1) + 2^{-n})$. The Haar measure of E_T with $\mathcal{O}_T$ of mass 1 is the product of the local ones with $\mathcal{O}_w$ of mass 1, so $\mathcal{O}_T \setminus \mathcal{O}_T^\times = \bigcup_{w \in T} \{x \in \mathcal{O}_T : x_w \in \mathfrak{p}_w\}$ has measure at most $\sum_{w \in T} q_w^{-1} \leq 4f\, b^{-10} \leq \frac{1}{2}$ ($f \leq b/1000$ by (FG6)); by Lemma 5.2.87, $\theta(K_1) \geq 2^{-n}$. Hence $\mathrm{mi}(K_1) \leq 2\, \theta(K_1)^2 \theta(K_M)$.

Step 6: conclusion. By Step 3, $|a((l_s)_T)| = \theta(K_1) |\psi_U^\natural(s)|$. With Step 5,

$$\theta(K_M)\, \theta(K_1)^2 \sum_{s \in \bar{G}} |\psi_U^\natural(s)|^2 \leq 2\, \theta(K_1)^2\, \theta(K_M),$$

and $0 < \theta(K_M), \theta(K_1) < \infty$ (Lemma 5.5.44 (d)); dividing gives the claim. □

Lemma 5.6.44. *Let $\mathfrak{S}$ be a filling datum, T a finite set of finite places of E stable under complex conjugation such that M^{-1} is a T-integer, (P, F) a filling datum at T, and χ_M the extended Bessel character of F. Then*

$$\chi_M(m_T) = \psi_U(\mathrm{red}(m)) \qquad \textit{for every } m \in K.$$

Proof. Write U_T for the Bessel unipotent group of F over E_T; for $u \in \mathsf{U}(\mathbb{A}_f)$, $u_T \in \mathsf{U}_T$. The entries of u just above the diagonal of the frame block lie in the rows $e_1, \ldots, e_\varrho$; the coefficient of e_a of a vector is the same in the standard basis and in the lattice basis (they differ only in x_k, $x' = x_k/\bar{\pi} + z/N$, z), so $\sigma(u_T) = \sigma(u)_T$, and $\sigma(u) \in \hat{\mathcal{O}}$ if $u \in K$.

Step 1: ψ_f is trivial on $\varpi^{-1}\hat{\mathcal{O}}$. Let $x \in \varpi^{-1}\hat{\mathcal{O}}$ and $a \in E$ with $x - a \in \hat{\mathcal{O}}$. As $\varpi \in \mathcal{O}_E$, $a \in \varpi^{-1}\hat{\mathcal{O}} \cap E = \varpi^{-1}\mathcal{O}_E$, the inverse different of $E/\mathbb{Q}$ (ϖ generates the different), so $\mathrm{Tr}_{E/\mathbb{Q}}(a) \in \mathbb{Z}$ and $\psi_f(x) = 1$.

Step 2: the phase. Let $u \in \mathsf{U}(\mathbb{A}_f) \cap K$ and $x = M^{-1}\sigma(u) \in \mathbb{A}_f$. Let $\tilde{x} \in \mathbb{A}_f$ have the components of x at T and 0 elsewhere. Then $x - \tilde{x}$ vanishes at T, and at $w \notin T$ its component $M^{-1}\sigma(u)_w$ lies in $\mathcal{O}_w$, since M^{-1} is a T-integer and $\sigma(u)$ is integral. By additivity of ψ_f and Step 1,

$$\psi(u) = \psi_f(-x/\varpi) = \psi_f(-\tilde{x}/\varpi)\, \psi_f\big(-(x - \tilde{x})/\varpi\big) = \psi_T(x_T) = \psi_T\big(M^{-1}\sigma(u_T)\big).$$

Step 3: $\mathrm{red}(m) \in \bar{\mathsf{U}}$. Then $\mathrm{red}(m) = \mathrm{red}(u)$ for some $u \in \mathsf{U}(\mathbb{A}_f) \cap K$, and by Lemma 5.2.46 (ii) $u_T^{-1} m_T \in \mathrm{K}_T(M)$ with $u_T \in \mathsf{U}_T$. By the definition of χ_M and Step 2, $\chi_M(m_T) = \psi_T(M^{-1}\sigma(u_T)) = \psi(u) = \psi_U(\mathrm{red}(m))$, the last equality by (TA8).

Step 4: $\mathrm{red}(m) \notin \bar{\mathsf{U}}$. Then $\psi_U(\mathrm{red}(m)) = 0$. Suppose $\chi_M(m_T) \neq 0$: there is $u \in \mathsf{U}_T$ with $u^{-1} m_T \in \mathrm{K}_T(M)$. As m and m^{-1} have integral entries, $m_T \in \mathrm{K}_T(1)$, and so $u = m_T (u^{-1} m_T)^{-1} \in \mathrm{K}_T(1)$. Let $\tilde{u} \in G(\mathbb{A}_f)$ have, in the lattice basis, the components of u at the places of T and 1 at the other places. It is unitary, because the unitarity of u over E_T is the unitarity of its components at the places of T (T is stable under conjugation) and 1 is unitary; it lies in U at every place; and $\tilde{u}$, $\tilde{u}^{-1}$ have integral entries at every place. So $\tilde{u} \in \mathsf{U}(\mathbb{A}_f) \cap K$ and $\tilde{u}_T = u$, whence $\tilde{u}_T^{-1} m_T \in \mathrm{K}_T(M)$ and $\mathrm{red}(m) = \mathrm{red}(\tilde{u}) \in \bar{\mathsf{U}}$ by Lemma 5.2.46 (ii), a contradiction. Hence $\chi_M(m_T) = 0 = \psi_U(\mathrm{red}(m))$. □

Lemma 5.6.45. *Let $\mathfrak{S}$ be admissible with Weil data. For $\gamma \in R(E)$ call a matrix $\mathcal{M} \in \mathbb{C}^{I \times I}$ an* action matrix *of γ if $W(\gamma \cdot y) = \mathcal{M}\, W(y)$ for every $y \in G_\infty$.*

(i) Every $\gamma \in R(E)$ has a unitary action matrix.

(ii) Let $c, c' \in \mathcal{C}$ and $s \in \bar{G}$. If some $\gamma \in R(E)$ is a term over (c, s, c'), then $s \in \bar{R}_0$, and for every action matrix $\mathcal{M}$ of γ and every $d \in \mathcal{C}$

$$A^{\mathrm{full}}_{R(E)}(c, s, d) = \begin{cases} \nu\, \mathcal{M}, & d = c', \\ 0, & d \neq c'. \end{cases}$$

Moreover $A^{\mathrm{full}}_{R(E)}(c', s^{-1}, c) = A^{\mathrm{full}}_{R(E)}(c, s, c')^*$ for all $c, c' \in \mathcal{C}$ and $s \in \bar{G}$.

(iii) For every $c \in \mathcal{C}$ and $t \in \bar{R}_0$ there are $c' \in \mathcal{C}$ and a term $\gamma \in R(E)$ over (c, t, c').

(iv) Let $T \subseteq G(E)$ be stable, let $\delta \in R(E)$ be a term over (d, t, d') with an action matrix $\mathcal{N}$, and suppose that $\gamma \in T$ if and only if $\gamma\delta^{-1} \in T$, for every $\gamma \in G(E)$. Then for all $c \in \mathcal{C}$ and $s \in \bar{G}$

$$A^{\mathrm{full}}_T(c, st^{-1}, d) = A^{\mathrm{full}}_T(c, s, d')\, \mathcal{N}^*.$$

(v) Let $j \in \{0, 1\}$ and let $s \in \bar{G}$ have component 1 at every chosen place in S_j. Then $A^{\mathrm{full}}_{R(E)}(c, s^{-1}, c) = \nu\, \psi_U(s)\, 1_I$ for every $c \in \mathcal{C}$.

Proof. Write $X = \langle e_1, \dots, e_\varrho \rangle$, $X^* = \langle e_1^*, \dots, e_\varrho^* \rangle$ and $V_{\mathrm{m}} = X_0 \oplus Ez$, so that $V = X \oplus V_{\mathrm{m}} \oplus X^*$ and $X^\perp = X \oplus V_{\mathrm{m}}$; the lattice basis consists of the e_a, the e_a^* and a basis of $L_{\mathrm{m}} = L_0 \cap V_{\mathrm{m}}(\mathbb{A}_f)$ (the x_i, $i < k$, $x_k/\bar{\pi} + z/N$ and z), and matrices of elements of $G(\mathbb{A}_f)$ are taken in it, with blocks indexed by X, V_{m}, X^*. For $y \in G_\infty$ write $W(y) \otimes \overline{W(y)}$ for the matrix with entries $W(y)_i \overline{W(y)_j}$; for a matrix $\mathcal{M}$ and vectors p, q, $p \otimes \overline{\mathcal{M}q} = (p \otimes \bar{q})\, \mathcal{M}^*$ and $\mathcal{M}p \otimes \bar{q} = \mathcal{M}(p \otimes \bar{q})$.

Step 1: the Bessel subgroup. Let $A \in \{E, \mathbb{A}_f\}$, $u \in \mathsf{U}(A)$, $h \in H(A)$. (a) h fixes every e_a, e_a^* and z, so it stabilises the flag and acts trivially on its graded pieces; hence $h^{-1}uh$ stabilises the flag, acts trivially on the graded pieces, and acts on $X^\perp/X$ as the conjugate of the identity by the action of h, that is trivially; so $h^{-1}uh \in \mathsf{U}(A)$. (b) If $h \in \mathsf{U}(A)$, then for $x \in X_0 \subseteq X^\perp$, $hx \in (x + X) \cap X_0 = \{x\}$, so $h = 1$. By (a) and (b), $R(A)$ is a group and every element of it is uh for a unique pair (u, h). (c) For $x \in V_{\mathrm{m}} \subseteq X^\perp$, $hx \in V_{\mathrm{m}}$ and $uhx \in hx + X$, as u acts trivially on $X^\perp/X$. Hence, for $A = \mathbb{A}_f$, the $(V_{\mathrm{m}}, V_{\mathrm{m}})$-block of the matrix of uh is the matrix of $h|_{V_{\mathrm{m}}}$, while the matrix of h is the identity on the blocks X, X^* and has no other off-diagonal blocks. As $(uh)^{-1} = (h^{-1}u^{-1}h)h^{-1}$ is of the same form, we obtain: (d) if $uh \in K$, then h and h^{-1} have integral matrices, so $h \in K$; (e) if $uh \in K$ and uh reduces to 1 at a chosen place w,

then so does h, the entries of $h-1$ being 0 or entries of $uh-1$. In particular $uh \in \mathrm{K}(M)$ implies $h \in \mathrm{K}(M)$ and $u = (uh)h^{-1} \in \mathrm{K}(M)$.

Step 2: terms, conjugation, cutoffs. As K is a group and red a homomorphism: if γ is a term over (c, s, c') and δ one over (c', t, d), then $\gamma\delta$ is a term over (c, st, d) and γ^{-1} one over (c', s^{-1}, c). Let $\gamma \in R(E)$ be a term over (c, s, c'). Since $h_c, h_{c'} \in H(\mathbb{A}_f)$, $h_c^{-1}\gamma h_{c'} \in R(\mathbb{A}_f) \cap K$, so $s \in \bar{R}_0$. Moreover $\gamma\, \mathsf{U}_{c'}(M)\, \gamma^{-1} = \mathsf{U}_c(M)$: for $v \in \mathsf{U}_{c'}(M)$, $\gamma v \gamma^{-1} \in \mathsf{U}(E)$ by Step 1 (a), and $h_c^{-1}\gamma v\gamma^{-1}h_c = k(h_{c'}^{-1}vh_{c'})k^{-1} \in \mathrm{K}(M)$ with $k = h_c^{-1}\gamma h_{c'} \in K$, as $\mathrm{K}(M)$ is normal in K; the same with γ^{-1}, a term over (c', s^{-1}, c), gives the reverse inclusion. Finally, for every c and $y \in G_\infty$,

$$\sum_{v \in \mathsf{U}_c(M)} \beta_c(v \cdot y) = \sum_{v \in \mathsf{U}_c(M)} \beta_c(v^{-1} \cdot y) = 1 : \tag{5.122}$$

the first sum is (i) of Lemma 5.2.36, and the second is the first reindexed by $v \mapsto v^{-1}$, a bijection of the group $\mathsf{U}_c(M)$.

Step 3: the terms of $R(E)$ over $(c, 1, c')$ are $\mathsf{U}_c(M)$ if $c = c'$, and there are none if $c \neq c'$. Every $v \in \mathsf{U}_c(M)$ is such a term with $c' = c$. Conversely let $\gamma = u\alpha \in R(E)$ ($u \in \mathsf{U}(E)$, $\alpha \in H(E)$) with $h_c^{-1}\gamma h_{c'} \in \mathrm{K}(M)$. Then $h_c^{-1}\gamma h_{c'} = (h_c^{-1}uh_c)(h_c^{-1}\alpha h_{c'})$ with $h_c^{-1}uh_c \in \mathsf{U}(\mathbb{A}_f)$ (Step 1 (a)) and $h_c^{-1}\alpha h_{c'} \in H(\mathbb{A}_f)$, so by Step 1 (e) $h_c^{-1}\alpha h_{c'} \in \mathrm{K}(M) \cap \mathrm{U}(X_0)(\mathbb{A}_f) = \mathrm{K}_X$ (FA1). Thus $h_{c'} \in \alpha^{-1}h_c\mathrm{K}_X$ with $\alpha \in \mathrm{U}(X_0)(E^+)$, so $h_{c'}$ and h_c lie in the same double coset and $c = c'$. Then $h_c^{-1}\alpha h_c \in \mathrm{K}_X \cap h_c^{-1}\mathrm{U}(X_0)(E^+)h_c = \{1\}$ by (FG4), so $\alpha = 1$, $\gamma = u$, and $h_c^{-1}uh_c \in \mathrm{K}(M)$: $\gamma \in \mathsf{U}_c(M)$.

Step 4: proof of (i). Let $\gamma = u\alpha$ as in Step 3, with $u \in \mathsf{U}(E)$, $\alpha \in H(E)$. At $v \in \Phi$ the basis scales are 1 on X_0 and α fixes the other basis vectors, so by (5.5) α_v is the element of $H \subseteq G_1$ acting on $X_0 = \mathbb{C}^k$ by the matrix $v(\alpha)$; it is unitary, because α preserves the unit form of X_0 and $v(\bar{a}) = \overline{v(a)}$ for $a \in E$ (E is a CM field). As W_v is a Bessel function of type ρ_v, $W_v(\alpha_v y_v) = \rho_v(v(\alpha))W_v(y_v)$, so $W(\alpha \cdot y) = \mathcal{N}_\alpha W(y)$ with the Kronecker product $\mathcal{N}_\alpha = \bigotimes_{v \in \Phi} \rho_v(v(\alpha))$, a unitary matrix on $\mathbb{C}^I$. By Lemma 5.2.43, $W(\gamma \cdot y) = W(u \cdot (\alpha \cdot y)) = \psi(u)^{-1}\mathcal{N}_\alpha W(y)$. Since $|\psi(u)| = 1$ (Lemma 5.2.44 (i)), $\mathcal{M}_\gamma = \psi(u)^{-1}\mathcal{N}_\alpha$ is a unitary action matrix of γ.

Step 5: the identity integral. We show: for every $c \in \mathcal{C}$, $\int_{G_\infty} \beta_c(y)\|W(y)\|^2 dy = \nu\, \#I$, and

$$\int_{G_\infty} \beta_c(y)\, W(y) \otimes \overline{W(y)}\, dy = \nu\, 1_I. \tag{5.123}$$

For each v fix a Borel $\chi_v \colon G_1 \to [0, \infty]$ with $\int_{\mathsf{U}} \chi_v(ux)\, du = 1$ for all x (Definition 5.1.6), and put $\chi(y) = \prod_v \chi_v(y_v)$; by Tonelli $\int_{\mathsf{U}_\infty} \chi(uy)\, du = 1$. Since $|\mathrm{wave}(\cdot)| = 1$, each $\|W_v\|^2$ and each $W_{v,i}\overline{W_{v,j}}$ is left U-invariant, so $\phi_{ij}(y) = W(y)_i\overline{W(y)_j}$ and $\|W\|^2$ are left U_∞-invariant. Let $\phi \in \{\|W\|^2, \phi_{ij}\}$ and $F(u, y) = \beta_c(u^{-1}y)\chi(y)\phi(y)$ on $\mathsf{U}_\infty \times G_\infty$, a Borel function. By Tonelli, the invariance of du under inversion (Lemma 5.2.34 (iv)) and Lemma 5.2.33,

$$\begin{aligned}\iint |F|\, du\, dy &= \int_{G_\infty} \chi|\phi| \int_{\mathsf{U}_\infty} \beta_c(uy)\, du\, dy = \nu \int_{G_\infty} \chi|\phi| \\ &\leq \nu \int_{G_\infty} \chi\|W\|^2 = \nu \prod_v \int_{\mathsf{U}\backslash G_1} \|W_v\|^2 = \nu\, \#I\end{aligned}$$

by (BS1). So Fubini's theorem applies to F; with the substitution $y \mapsto uy$ in the inner integral (left invariance of the Cayley measures, Lemma 5.2.41) and $\phi(uy) = \phi(y)$,

$$\int_{G_\infty} \chi\phi\, \nu\, dy = \int_{\mathsf{U}_\infty} \int_{G_\infty} \beta_c(u^{-1}y)\chi(y)\phi(y)\, dy\, du = \int_{\mathsf{U}_\infty} \int_{G_\infty} \beta_c(y)\chi(uy)\phi(y)\, dy\, du = \int_{G_\infty} \beta_c\phi\, dy,$$

the same computation with $|\phi|$ showing that $\beta_c\phi$ is integrable. For $\phi = \|W\|^2$ this gives the first claim. For $\phi = \phi_{ij}$, $\int \chi\phi_{ij} = \prod_v \int_{G_1} \chi_v W_{v,i_v}\overline{W_{v,j_v}} = \prod_v \int_{\mathsf{U}\backslash G_1} W_{v,i_v}\overline{W_{v,j_v}}$ by Fubini and Lemma 5.5.29 (i) (complex form; the integrands are dominated by $\|W_v\|^2$). As

ρ_v is irreducible and $\int_{\mathsf{U}\backslash G_1}\|W_v\| < \infty$ by (BS1), Lemma 5.5.28 gives $\int_{\mathsf{U}\backslash G_1} W_{v,i_v}\overline{W_{v,j_v}} = \delta_{i_v j_v}\#I_v^{-1}\int_{\mathsf{U}\backslash G_1}\|W_v\|^2 = \delta_{i_v j_v}$. Hence $\int \beta_c\phi_{ij} = \nu\delta_{ij}$, which is (5.123).

Step 6: every action matrix is unitary, and $W(\gamma^{-1}\cdot y) = \mathcal{M}^*W(y)$. Let $\gamma \in R(E)$ and let $\mathcal{M}$ be an action matrix of γ. With $\mathcal{M}_\gamma$ of Step 4, $(\mathcal{M}-\mathcal{M}_\gamma)W(y) = 0$ for all y; multiplying by $\beta_c(y)\overline{W(y)}$, integrating (Step 5) and using (5.123), $0 = (\mathcal{M}-\mathcal{M}_\gamma)\int \beta_c\, W\otimes\overline{W}\,dy = \nu(\mathcal{M}-\mathcal{M}_\gamma)$. So $\mathcal{M} = \mathcal{M}_\gamma$ is unitary, and $W(y) = W(\gamma\cdot(\gamma^{-1}\cdot y)) = \mathcal{M}W(\gamma^{-1}\cdot y)$ gives $W(\gamma^{-1}\cdot y) = \mathcal{M}^*W(y)$; in particular $\mathcal{M}^*$ is the action matrix of γ^{-1}.

Step 7: proof of (ii). Let $\gamma \in R(E)$ be a term over (c,s,c') with action matrix $\mathcal{M}$; $s \in \bar{R}_0$ by Step 2. For $d \in \mathcal{C}$, by Step 2 the map $\delta \mapsto \gamma\delta$ is a bijection from the terms of $R(E)$ over $(c',1,d)$ onto those over (c,s,d) (with inverse $\gamma' \mapsto \gamma^{-1}\gamma'$). By Step 3 there are none if $d \neq c'$, so $A^{\mathrm{full}}_{R(E)}(c,s,d) = 0$; and for $d = c'$ the terms are the γv, $v \in \mathsf{U}_{c'}(M)$. For such v, $W(v^{-1}\cdot z) = W(z)$ (Lemma 5.2.42), so by Step 6

$$A_{\gamma v}(c,c') = \int_{G_\infty} \beta_c(y)\,\beta_{c'}(v^{-1}\cdot\gamma^{-1}y)\,W(y)\otimes\overline{\mathcal{M}^*W(y)}\,dy.$$

The integrands have norms $\beta_c(y)\beta_{c'}(v^{-1}\cdot\gamma^{-1}y)\|W(y)\|^2$ ($\mathcal{M}$ unitary), whose integrals add up over v, by (5.122) at $\gamma^{-1}y$ and Tonelli, to $\int\beta_c\|W\|^2 = \nu\#I < \infty$ (Step 5). So the sum over v may be taken inside the integral, and by (5.122) and (5.123)

$$A^{\mathrm{full}}_{R(E)}(c,s,c') = \int_{G_\infty}\beta_c(y)\,(W(y)\otimes\overline{W(y)})\mathcal{M}\,dy = \nu\,\mathcal{M}.$$

For the adjoint relation let $c,c' \in \mathcal{C}$, $s \in \bar{G}$. If some $\gamma \in R(E)$ is a term over (c,s,c'), with action matrix $\mathcal{M}$, then $\gamma^{-1} \in R(E)$ is a term over (c',s^{-1},c) with action matrix $\mathcal{M}^*$ (Steps 2 and 6), and by what was just shown $A^{\mathrm{full}}_{R(E)}(c',s^{-1},c) = \nu\mathcal{M}^* = A^{\mathrm{full}}_{R(E)}(c,s,c')^*$. Otherwise there is no term of $R(E)$ over (c',s^{-1},c) either (its inverse would be one over (c,s,c')), and both sums are empty, hence 0.

Step 8: proof of (iii). Let $c \in \mathcal{C}$ and $t \in \bar{R}_0$, $t = \mathrm{red}(m)$ with $m \in R(\mathbb{A}_f)\cap K$. By Step 1, $h_c m = uh$ with $u \in \mathsf{U}(\mathbb{A}_f)$ and $h \in H(\mathbb{A}_f)$, and by the class decomposition (FG4), $h = \alpha h_{c'}a$ with $c' \in \mathcal{C}$, $\alpha \in \mathrm{U}(X_0)(E^+) = H(E)$ and $a \in \mathrm{K}_X \subseteq \mathrm{K}(M)$ (FA1). The group $\mathsf{U}(\mathbb{A}_f)\cap h\mathrm{K}(M)h^{-1}$ is open in $\mathsf{U}(\mathbb{A}_f)$, as $\mathrm{K}(M)$ is open in $G(\mathbb{A}_f)$. By strong approximation for the unipotent group U over E^+ (Theorem A.4.7, S the archimedean places), $u = u_1k^{-1}$ with $u_1 \in \mathsf{U}(E)$ and $k \in \mathsf{U}(\mathbb{A}_f)\cap h\mathrm{K}(M)h^{-1}$. Put $\gamma = u_1\alpha \in R(E)$. Then

$$\gamma h_{c'} = uk\alpha h_{c'} = uk\,h\,a^{-1} = uh\,(h^{-1}kh)\,a^{-1} = h_c m\,k'', \qquad k'' = (h^{-1}kh)a^{-1} \in \mathrm{K}(M),$$

so $h_c^{-1}\gamma h_{c'} = mk'' \in K$ has reduction $\mathrm{red}(m) = t$: γ is a term over (c,t,c').

Step 9: proof of (iv). Let T, δ, $\mathcal{N}$ be as in (iv), $c \in \mathcal{C}$, $s \in \bar{G}$. By Step 2 and the hypothesis on T, $\gamma' \mapsto \gamma'\delta^{-1}$ is a bijection from the terms of T over (c,s,d') onto those of T over (c,st^{-1},d), with inverse $\gamma \mapsto \gamma\delta$. Since $W(\delta\cdot z) = \mathcal{N}W(z)$, for a term γ' of T over (c,s,d')

$$W(y)\otimes\overline{W(\delta\gamma'^{-1}y)} = (W(y)\otimes\overline{W(\gamma'^{-1}y)})\mathcal{N}^*.$$

Put $\varphi(\gamma',y) = \beta_c(y)\beta_d(\delta\cdot\gamma'^{-1}y)$. For a term $\gamma' \in T$ over (c,s,d') and $(v,v') \in \mathsf{U}_c(M)\times\mathsf{U}_{d'}(M)$, $\varphi(v\gamma'v'^{-1}, v\cdot y) = \beta_c(v\cdot y)\,\beta_d((\delta v'\delta^{-1})\cdot(\delta\cdot\gamma'^{-1}y))$; as $v' \mapsto \delta v'\delta^{-1}$ is a bijection of $\mathsf{U}_{d'}(M)$ onto $\mathsf{U}_d(M)$ (Step 2), (5.122) gives $\sum_{(v,v')}\varphi(v\gamma'v'^{-1}, v\cdot y) = 1$. By Lemma 5.2.40 ($T$ is stable), each $y \mapsto \varphi(\gamma',y)W(y)\otimes\overline{W(\gamma'^{-1}y)}$ is integrable and $A^{\mathrm{full}}_T(c,s,d') = \sum_{\gamma'}\int\varphi(\gamma',y)W(y)\otimes\overline{W(\gamma'^{-1}y)}\,dy$. By the display, the integrand of the amplitude $A_{\gamma'\delta^{-1}}(c,d)$ is this integrand times

$\mathcal{N}^*$, so $A_{\gamma'\delta^{-1}}(c,d) = (\int \varphi(\gamma',y)W(y)\otimes\overline{W(\gamma'^{-1}y)}\,dy)\mathcal{N}^*$. The series $A_T^{\mathrm{full}}(c,st^{-1},d)$ converges absolutely (Lemma 5.2.29 (ii)), so it may be reindexed by the bijection, and

$$A_T^{\mathrm{full}}(c,st^{-1},d) = \sum_{\gamma'} A_{\gamma'\delta^{-1}}(c,d) = A_T^{\mathrm{full}}(c,s,d')\,\mathcal{N}^*.$$

Step 10: proof of (v). Let j, s be as in (v) and $c\in\mathcal{C}$. First let $\gamma = u\alpha \in R(E)$ ($u\in\mathsf{U}(E)$, $\alpha\in H(E)$) be a term over (c,s^{-1},c). Then $m = h_c^{-1}\gamma h_c = (h_c^{-1}uh_c)h$ with $h = h_c^{-1}\alpha h_c \in H(\mathbb{A}_f)$ and $m\in K$ reducing to s^{-1}, which is 1 at every $w\in S_j$. By Step 1 (d), (e), $h\in K$ and h reduces to 1 at every $w\in S_j$. Let $t = \mathrm{tr}_{X_0}(\alpha) - k \in E$. Since α acts trivially on $X_0^\perp$ (of dimension $r-k$) and the trace is invariant under conjugation and change of basis, $t = \mathrm{tr}(h) - r = \mathrm{tr}(h-1)$, computed on the matrix of h in the lattice basis. Its entries are integral, so $t\in E\cap\hat{\mathcal{O}} = \mathcal{O}_E$, and its diagonal entries lie in the maximal ideal at every $w\in S_j$, so $t\in\mathfrak{p}_w$ for the f distinct places $w\in S_j$, each of norm p_j. For every embedding $\sigma\colon E\to\mathbb{C}$ the matrix $\sigma(\alpha)$ is unitary (Step 4), so $|\sigma(\mathrm{tr}_{X_0}\alpha)|\le k$ and $|\sigma(t)|\le 2k$. If $t\neq 0$, then $p_j^f \le |N_{E/\mathbb{Q}}(t)| = \prod_\sigma|\sigma(t)| \le (2k)^{2f}$, so $p_j\le 4k^2$; but $k\le f\le b/1000$ and $p_j\ge b^{10}$ (FG6), so $4k^2 < b^{10}\le p_j$. Hence $t=0$: the unitary matrix $\sigma(\alpha)$ has trace k, so its k eigenvalues, of absolute value 1 with sum k, all equal 1, and $\sigma(\alpha)$, being diagonalisable, is 1. As σ is injective, $\alpha = 1$ and $\gamma = u\in\mathsf{U}(E)$. By Step 4 its action matrix is $\psi(u)^{-1}1_I$, and by Lemma 5.2.44 (v), (iv), $s^{-1}\in\bar{\mathsf{U}}$ and $\psi(u) = \psi_U(s^{-1}) = \psi_U(s)^{-1}$, so the action matrix is $\psi_U(s)1_I$ and by (ii) $A^{\mathrm{full}}_{R(E)}(c,s^{-1},c) = \nu\,\psi_U(s)1_I$. It remains to see that such a term exists when $s\in\bar{\mathsf{U}}$, and that $\psi_U(s)=0$ otherwise. If $s\notin\bar{\mathsf{U}}$, then $\psi_U(s)=0$ (TA8), and by the first part there is no term of $R(E)$ over (c,s^{-1},c) (it would give $s^{-1}\in\bar{\mathsf{U}}$, a subgroup by Lemma 5.2.44 (iv)); the sum is empty and both sides vanish. If $s\in\bar{\mathsf{U}}$, let $g\in\mathsf{U}(\mathbb{A}_f)\cap K$ reduce to s^{-1}. By Step 1 (a), $h_cgh_c^{-1}\in\mathsf{U}(\mathbb{A}_f)$, and $\mathsf{U}(\mathbb{A}_f)\cap h_c\mathrm{K}(M)h_c^{-1}$ is an open subgroup, so by Theorem A.4.7 there is $u_1\in\mathsf{U}(E)$ with $u_1\in h_cgh_c^{-1}\,h_c\mathrm{K}(M)h_c^{-1}$, that is $h_c^{-1}u_1h_c\in g\,\mathrm{K}(M)\subseteq K$, of reduction s^{-1}. So u_1 is a term of $R(E)$ over (c,s^{-1},c), and the first part applies. □

Lemma 5.6.46. *Let $\mathfrak{S}$ be admissible with Weil data, and let $\Omega\subseteq\bar{G}$ contain no s of the following form: there are $\zeta\in\mathcal{O}_E$ with $\zeta\bar{\zeta}=1$ and $\zeta\neq 1$, and $t\in\bar{R}_0$, such that $s_w = \zeta\,t_w$ at every chosen place w (with ζ read in the residue field k_w). Then for all $s\in\Omega$, $c,c'\in\mathcal{C}$, and for the full as well as the cut amplitude sums,*

$$A_{G(E)\backslash R(E)}(c,s,c') = A_{\Gamma^*}(c,s,c').$$

Proof. Fix $s\in\Omega$, $c,c'\in\mathcal{C}$, and the side (full or cut). Put $T_0 = (G(E)\setminus R(E))\setminus\Gamma^*$, let T_1 be the set of $\gamma\in T_0$ without agreeing characters and $T_2 = T_0\setminus T_1$.

Step 1: splitting. By Lemma 5.3.18 (iii), $\Gamma^*\subseteq G(E)\setminus R(E)$, so $G(E)\setminus R(E)$ is the disjoint union of Γ^*, T_1 and T_2. The norm of each amplitude is at most its amplitude majorant, and the majorants of all terms over (c,s,c') have a finite sum (Lemma 5.2.29 (ii)); so the series of amplitudes over the terms of $G(E)\setminus R(E)$ converges absolutely and splits along the partition:

$$A_{G(E)\backslash R(E)}(c,s,c') = A_{\Gamma^*}(c,s,c') + A_{T_1}(c,s,c') + A_{T_2}(c,s,c').$$

Step 2: $A_{T_1}(c,s,c')=0$. T_1 is obtained from $R(E)$, Γ^* and the set of elements with agreeing characters by complements and intersections, so it is stable by Lemma 5.3.18 (ii); no element of it has agreeing characters. By Lemma 5.6.47, $A_{T_1}(c,s,c')=0$.

Step 3: no element of T_2 is a term over (c,s,c'). Suppose $\gamma\in T_2$ is one, and let $k = h_c^{-1}\gamma h_{c'}\in K$, with $\mathrm{red}(k) = s$. Since γ has agreeing characters and $\gamma\notin\Gamma^*$, $\mathsf{g}(\gamma)=0$ and

$\gamma X = X$. For $1 \le j \le \varrho$, $D_j(\gamma)$ is the determinant of $(h(e_{a+1}, \gamma e_{b+1}))_{a,b<j}$, which vanishes because $\gamma e_{b+1} \in X$ and X is isotropic; so $D_j(\gamma) = 0$ for $1 \le j \le \varrho$. Let j be the least integer in $\{0, \dots, \varrho+1\}$ such that some $\zeta \in E$ with $\zeta\bar{\zeta} = 1$ has $(\zeta, 1)$ occurring at every $a \ge j$. By Lemma 5.3.31 (ii), $j \le \varrho$ and γ satisfies the conditions of cell rank j, among them $D_j(\gamma) \ne 0$. Hence $j = 0$: there is ζ with $\zeta\bar{\zeta} = 1$ such that for every $a \in \{0, \dots, \varrho\}$ some $x, y \in E^{\varrho+1}$ satisfy $\sum_b x_b f_b = \gamma \sum_b y_b f_b$, $x_a = \zeta$, $y_a = 1$ and $x_b = y_b = 0$ for $b > a$. By induction on a, $\gamma f_a \in \zeta f_a + P_a$ with $P_a = \langle f_b : b < a \rangle$: the relation at a gives $\gamma f_a = \sum_{b \le a} x_b f_b - \sum_{b<a} y_b \gamma f_b$, and $\gamma f_b \in P_a$ for $b < a$ by induction. So $\gamma' = \bar{\zeta}\gamma \in G(E)$ satisfies $\gamma' f_a \in f_a + P_a$ for all a. Hence γ' stabilises each $F_b = \langle e_1, \dots, e_b \rangle$, acts trivially on F_b/F_{b-1} $(1 \le b \le \varrho)$, stabilises $X^\perp$, and induces on $X^\perp/X$ an isometry ι of the induced form fixing the class of z. Identify $X^\perp/X$ isometrically with $X_0 \oplus Ez \subseteq X^\perp$; ι fixes z, so it preserves $X_0 = z^\perp \cap (X_0 \oplus Ez)$, and its restriction defines $h_0 \in H(E)$. Then $g = \gamma' h_0^{-1}$ stabilises the flag and acts trivially on every F_b/F_{b-1} and on $X^\perp/X$; such elements form the unipotent radical of the stabiliser of the flag (its kernel to the Levi quotient $\prod_b \mathrm{GL}(F_b/F_{b-1}) \times \mathrm{U}(X^\perp/X)$), so $g \in \mathsf{U}(E)$ and $\gamma' = g h_0 \in R(E)$. At $a = 0$ the relation reads $\gamma e_1 = \zeta e_1$. As h_c and $h_{c'}$ fix e_1, $k e_1 = \zeta e_1$; e_1 is a vector of the lattice basis and k is integral, so $\zeta \in E \cap \hat{\mathcal{O}} = \mathcal{O}_E$. Then $k' = h_c^{-1}\gamma' h_{c'} = \bar{\zeta} k$ and its inverse ζk^{-1} are integral, so $k' \in K$; and $k' \in R(\mathbb{A}_f)$, as $h_c, h_{c'} \in H(\mathbb{A}_f)$ and $\gamma' \in R(E)$. So $t = \mathrm{red}(k') \in \bar{R}_0$, and $k = \zeta k'$ gives $s_w = \zeta\, t_w$ at every chosen place w, with ζ read in k_w. As $s \in \Omega$, the hypothesis forces $\zeta = 1$, so $\gamma = \gamma' \in R(E)$, contradicting $\gamma \in T_2 \subseteq G(E) \setminus R(E)$.

Step 4: conclusion. By Step 3 the sum $A_{T_2}(c, s, c')$ is empty, hence 0; with Steps 1 and 2, $A_{G(E)\setminus R(E)}(c, s, c') = A_{\Gamma^*}(c, s, c')$. □

Lemma 5.6.47. *Let $\mathfrak{S}$ be admissible with Weil data, and let $T \subseteq G(E)$ be stable and such that no element of T has agreeing characters. Then $A_T(c, s, c') = 0$ for all $c, c' \in \mathcal{C}$ and $s \in \bar{G}$, for the full and for the cut amplitude sums.*

Proof. Fix $W \in \{W^{\mathrm{full}}, W^{\mathrm{cut}}\}$, c, c' and s, and write $K(\gamma, y) = W(y) \otimes \overline{W(\gamma^{-1} y)}$, a continuous function of y (the functions of a Bessel system are continuous). Let X be the set of terms of T over (c, s, c').

Step 1: the action on G_∞. As $\gamma \mapsto \gamma_\infty$ is a homomorphism (RD1), $G(E)$ acts on G_∞ by $g \cdot y = g_\infty y$. Each map $y \mapsto g \cdot y$ is continuous, and $\int_{G_\infty} H(g \cdot y)\, dy = \int_{G_\infty} H\, dy$ for every Borel $H \colon G_\infty \to [0, \infty]$: the map is the product over $v \in \Phi$ of the left translations by $g_v \in G_1$, each of which preserves the Cayley measure (Lemma 5.2.41), and Tonelli's theorem applies to the product measure. For a subgroup $\Lambda \subseteq G(E)$ (countable, as $G(E)$ is), a Λ-*cutoff* is a Borel $\chi \colon G_\infty \to [0, \infty]$ with $\sum_{g \in \Lambda} \chi(g \cdot y) = 1$ for all y.

Step 2: equal indices. Let $\gamma \in G(E)$, $\Lambda = \mathsf{U}_c(M) \cap \gamma \mathsf{U}_{c'}(M)\gamma^{-1}$, and $u \in \mathsf{U}(E) \cap \gamma \mathsf{U}(E)\gamma^{-1}$. We show that $\Lambda \cap u^{-1}\Lambda u$ has the same finite index in Λ and in $u^{-1}\Lambda u$. The intersection $N = \mathsf{U} \cap \gamma \mathsf{U} \gamma^{-1}$ is a closed subgroup of the unipotent group U over E^+ (γ is rational), hence a unipotent linear algebraic group over E^+, with $N(E^+) = \mathsf{U}(E) \cap \gamma \mathsf{U}(E)\gamma^{-1} \ni u$ and $N_f = N(\mathbb{A}_{E^+,f}) = \mathsf{U}(\mathbb{A}_f) \cap \gamma \mathsf{U}(\mathbb{A}_f)\gamma^{-1}$, a closed subgroup of $G(\mathbb{A}_f)$. Put $O = h_c \mathrm{K}(M) h_c^{-1} \cap \gamma h_{c'} \mathrm{K}(M) h_{c'}^{-1} \gamma^{-1}$, a compact open subgroup of $G(\mathbb{A}_f)$; then $\Lambda = N(E^+) \cap O$ and $u^{-1}\Lambda u = N(E^+) \cap u^{-1} O u$. Let $B = N_f \cap O$ and $B' = B \cap u^{-1} B u$, compact open subgroups of N_f; then $\Lambda \cap u^{-1}\Lambda u = N(E^+) \cap B'$. For compact open subgroups $C' \subseteq C$ of N_f, the map $(N(E^+) \cap C)/(N(E^+) \cap C') \to C/C'$ induced by inclusion is injective, and surjective since every coset gC' is open in N_f and $N(E^+)$ is dense in N_f by strong approximation (Theorem A.4.7, S the archimedean places of E^+); and $[C : C'] = \mu(C)/\mu(C') < \infty$ for a Haar measure μ of N_f. Hence the two indices are $\mu(B)/\mu(B')$ and $\mu(u^{-1}Bu)/\mu(B')$. By left invariance, $\mu(u^{-1}Bu) = \mu(Bu) = \Delta(u)\mu(B)$, with the modular function Δ of N_f, a continuous homomorphism to $\mathbb{R}_{>0}$. As $u - 1$ is nilpotent, $u^l = \sum_{i<r} \binom{l}{i} (u-1)^i$ for $l \in \mathbb{Z}$, so all powers of u have entries in $d_0^{-1}\hat{\mathcal{O}}$ for a fixed

integer $d_0 \geq 1$ (in the lattice basis); the closure of $\langle u \rangle$ is compact, its image under Δ is a compact subgroup of $\mathbb{R}_{>0}$, so $\Delta(u) = 1$ and the indices agree.

Step 3: invariance and twist of the kernel. Let $\gamma \in X$ and $\Lambda_\gamma = \mathsf{U}_c(M) \cap \gamma \mathsf{U}_{c'}(M)\gamma^{-1}$. For $g = \gamma v \gamma^{-1} \in \Lambda_\gamma$ ($v \in \mathsf{U}_{c'}(M)$), $W(g \cdot y) = W(y)$ and $W(\gamma^{-1}(g \cdot y)) = W(v \cdot \gamma^{-1} y) = W(\gamma^{-1} y)$ by Lemma 5.2.42, so $K(\gamma, g \cdot y) = K(\gamma, y)$. As γ has no agreeing characters, there are $u, u' \in \mathsf{U}(E)$ with $u\gamma = \gamma u'$ and $\psi(u) \neq \psi(u')$; then $u = \gamma u' \gamma^{-1} \in \mathsf{U}(E) \cap \gamma \mathsf{U}(E) \gamma^{-1}$, and by Lemma 5.2.43 and $\gamma^{-1} u = u' \gamma^{-1}$,

$$K(\gamma, u \cdot y) = \psi(u)^{-1} W(y) \otimes \overline{\psi(u')^{-1} W(\gamma^{-1} y)} = c_0 \, K(\gamma, y), \qquad c_0 = \psi(u)^{-1} \psi(u') \neq 1,$$

since $|\psi(u')| = 1$ (Lemma 5.2.44 (i)).

Step 4: a weight adapted to the stabilisers. By (i) of Lemma 5.2.36, $\sum_{v \in \mathsf{U}_d(M)} \beta_d(v \cdot z) = 1$ for every class d and $z \in G_\infty$. As T is stable and $\mathrm{K}(M)$ is the kernel of red, X is invariant under left multiplication by $\mathsf{U}_c(M)$ and right multiplication by $\mathsf{U}_{c'}(M)$. Let $Q = \{\gamma \mathsf{U}_{c'}(M) : \gamma \in X\}$, on which $\mathsf{U}_c(M)$ acts by left multiplication, and choose $Q_0 \subseteq Q$ meeting every orbit in exactly one element. For $x = \gamma \mathsf{U}_{c'}(M) \in Q$ the stabiliser of x in $\mathsf{U}_c(M)$ is Λ_γ; choose representatives P_x of the cosets $\Lambda_\gamma \backslash \mathsf{U}_c(M)$ and put $\chi_x(y) = \sum_{p \in P_x} \beta_c(p \cdot y)$, a Λ_γ-cutoff, since $\sum_{g \in \Lambda_\gamma} \sum_p \beta_c(gp \cdot y) = \sum_{v \in \mathsf{U}_c(M)} \beta_c(v \cdot y) = 1$. Define $\varphi(\gamma, y) = \beta_{c'}(\gamma_\infty^{-1}) \, \chi_{\gamma \mathsf{U}_{c'}(M)}(y)$ if $\gamma \in X$ and $\gamma \mathsf{U}_{c'}(M) \in Q_0$, and $\varphi(\gamma, y) = 0$ otherwise; it is Borel in y. Let $\gamma \in X$, $x = \gamma \mathsf{U}_{c'}(M)$, $y \in G_\infty$. For $(v, v') \in \mathsf{U}_c(M) \times \mathsf{U}_{c'}(M)$, $(v \gamma v'^{-1})_\infty^{-1} = v'_\infty (v\gamma)_\infty^{-1}$ and $v \gamma v'^{-1} \mathsf{U}_{c'}(M) = vx$; summing first over v' (the identity above at $z = (v\gamma)_\infty^{-1}$),

$$\sum_{(v, v')} \varphi(v \gamma v'^{-1}, v \cdot y) = \sum_{v \in \mathsf{U}_c(M),\ vx \in Q_0} \chi_{vx}(v \cdot y) = \sum_{g \in \Lambda_{\gamma_0}} \chi_{x_0}(g a \cdot y) = 1,$$

where $x_0 = \gamma_0 \mathsf{U}_{c'}(M)$ is the element of Q_0 in the orbit of x, $a \in \mathsf{U}_c(M)$ satisfies $ax = x_0$, and $\{v : vx = x_0\} = \Lambda_{\gamma_0} a$.

Step 5: conclusion. By Lemma 5.2.40 (T is stable), each $y \mapsto \varphi(\gamma, y) K(\gamma, y)$ ($\gamma \in X$) is integrable and $A_T(c, s, c') = \sum_{\gamma \in X} \int \varphi(\gamma, y) K(\gamma, y) \, dy$. Let $\gamma \in X$ with $\varphi(\gamma, \cdot) \neq 0$; then $x = \gamma \mathsf{U}_{c'}(M) \in Q_0$ and $\beta_{c'}(\gamma_\infty^{-1}) > 0$, so $\int \chi_x \|K(\gamma, \cdot)\| < \infty$. Apply (iii) of Lemma 5.2.45 to the action of $G(E)$ on G_∞ of Step 1, $\Lambda = \Lambda_\gamma$, the Λ_γ-cutoff χ_x (in the roles of χ and χ'), $F = K(\gamma, \cdot)$ (Borel and Λ_γ-invariant by Step 3), and u, c_0 of Step 3; the indices agree by Step 2. Hence $\int \chi_x K(\gamma, y) \, dy = 0$, so $\int \varphi(\gamma, y) K(\gamma, y) \, dy = 0$ for every $\gamma \in X$, and $A_T(c, s, c') = 0$. □

The section closes with the two moments of a Poincaré map of class-series form (Proposition 5.6.48). For a map with $\mathcal{P}\varphi = \sum_a \theta(\varphi)_a \, \Pi_a$ there are an inner product on $\mathcal{S}$ and a scaling $L = c\mathcal{P}$ which, after e_0, is a near-isometry with error $5b^{-4} D_X$, and whose translated Gram operators satisfy the averaged second moment bound. These are exactly the hypotheses of Theorem B and of Proposition 5.1.4, and by Lemma 5.6.33 (ii) the Bessel-period map at a Bessel pair has this form. The proof assembles the results of Section 5: part (i) comes from the rarity of bad types and the near-isometry of the good part, and part (ii) from the averaged second moment, in which the main moment becomes the stated constant. Two facts produce that constant: the degree of the Bessel system at each infinite place is the Weyl product of the target parameter, which is the formal degree in Harish-Chandra's theorem [119] (Theorem 5.7.41), and the mass of the filling datum is that of the standard hermitian lattice.

Proposition 5.6.48. *Let D be admissible data, put $D_X = \dim \mathcal{S}$, and suppose that $\mathcal{P}\varphi = \sum_{a \in \mathcal{C} \times I} \theta(\varphi)_a \, \Pi_a$ for every $\varphi \in \mathcal{S}$. Then $D_X \geq 1$, and there are an inner product on $\mathcal{S}$ and a real $c > 0$ such that $L = c\mathcal{P} \colon \mathcal{S} \to \mathcal{A}_M(Y_0)$ satisfies:*

(i) $\|e_0 L \operatorname{pr}_U\|_{\mathrm{HS}}^2 \geq \dim U - 5b^{-4} D_X$ *for every subspace $U \subseteq \mathcal{S}$;*

(ii) $\displaystyle\sum_{s\in\bar G}\|L^{*}sL\|_{\mathrm{HS}}^{2}\le\Big(1+\frac{9}{b^{4}}\Big)\frac{D_X^{2}}{\operatorname{mass}(I_r)\prod_{v\in\Phi}\mathrm{d}(\varsigma(v))}.$

Proof. Let $\mathfrak{S}_D$ be the admissible filling datum of Proposition 5.6.4; E carries Weil data and the level group of $\mathfrak{S}_D$ is $\bar G$ (Proposition 5.6.4 (i), (ii)). By Proposition 5.6.4 (iii) there is a linear isomorphism $\theta\colon\mathcal{S}\to\mathbb{C}^{\mathcal{C}\times I}$, so $D_X=\#\mathcal{C}\cdot\#I\ge 1$ since both sets are nonempty. Give $\mathcal{S}$ the inner product transported by θ from the standard one, so that θ is unitary and Hilbert–Schmidt norms of maps out of $\mathcal{S}$ are computed in the coordinates. Let $c=c_{\mathcal{P}}=(V_M\nu)^{-1/2}>0$ be the constant of Proposition 5.6.16 and $L=c\,\mathcal{P}$. By Proposition 5.6.16 (ii), whose hypothesis on $\mathcal{P}$ is the one assumed here, the map $L\theta^{-1}=c_{\mathcal{P}}\,\mathcal{P}\circ\theta^{-1}\colon\mathbb{C}^{\mathcal{C}\times I}\to\mathcal{A}_M(Y_0)$ (the map written L in Proposition 5.6.16 (ii); here L is defined on $\mathcal{S}$) has amplitude blocks (TA6) for the action ρ of $\bar G$, which is unitary.

(i) Let g_0,g_1 be the central idempotents of Proposition 5.2.10 (iii). They satisfy $\operatorname{Re}\operatorname{coeff}_1(1-g_i)\le b^{-8}/2$, and $\rho(g_0\otimes g_1)=e_0$ on $\mathcal{A}_M(Y_0)$. By Proposition 5.6.41, applied to $P=L\theta^{-1}$, the map $e_0L\theta^{-1}$ is a near-isometry with error $5b^{-4}$. For a subspace $U\subseteq\mathcal{S}$, $\theta(U)$ has the same dimension and $\mathrm{pr}_{\theta(U)}=\theta\,\mathrm{pr}_U\theta^{-1}$ because θ is unitary; hence $\|e_0L\,\mathrm{pr}_U\|_{\mathrm{HS}}=\|e_0L\theta^{-1}\mathrm{pr}_{\theta(U)}\|_{\mathrm{HS}}$ and (i) follows.

(ii) By Theorem 5.1.13, applied to $P=L\theta^{-1}$, $\sum_s\|(L\theta^{-1})^{*}\rho(s)L\theta^{-1}\|_{\mathrm{HS}}^{2}\le(1+9b^{-4})\mathcal{D}_{\mathrm{main}}$, and the left side equals $\sum_s\|L^{*}sL\|_{\mathrm{HS}}^{2}$ by unitarity of θ. For $\mathfrak{S}_D$, $\mathcal{D}_{\mathrm{main}}=D_X^{2}/(\deg\cdot\operatorname{mass}(E,r))$ with $\deg=\prod_v\deg_v=\prod_v\mathrm{d}(\varsigma(v))$ by Proposition 5.7.3 (i), and $\operatorname{mass}(E,r)=\operatorname{mass}(I_r)$, the mass of the standard rank-r genus. Here $\operatorname{mass}(E,r)$ of Definition 5.1.9 is, by definition, the mass of the genus of the standard lattice $\mathcal{O}_E^{r}$ with the unit form, that is $\operatorname{mass}(I_r)$ of Definition 3.2.5: the classes $\mathrm{U}_r(E^{+})gK_0$ in $\mathrm{U}_r(\mathbb{A}_{E^{+},f})$, K_0 the stabiliser of $\widehat{\mathcal{O}}_E^{r}$, are the isometry classes in the genus, with $\mathrm{U}_r(E^{+})\cap gK_0g^{-1}$ the automorphism group of the lattice $g\widehat{\mathcal{O}}_E^{r}\cap E^{r}$; this is the same quantity that enters Proposition 5.8.54. □

5.7 Discrete series, Laplace's method and Bessel systems

This section supplies the archimedean ingredients of Section 5. Its main statement, Proposition 5.7.3, gives a Bessel system of discrete-series type at every infinite place: its degree is the Weyl product of the target parameter, it is reproduced by the discrete-series kernel built from Flensted-Jensen's function c_λ [81], and it is a multiple of a Bessel average at a point of controlled Bessel length. The degree turns the main moment into the constant of Proposition 5.6.48; the reproducing property is what the class Poincaré series and the Bessel point require; and the local bounds that Theorem 5.5.3 provides for these systems enter the archimedean factors of the sieve.

The heart of the section is the formal degree. For the Cayley measure, $\big(\int|c_\lambda|^{2}\big)^{-1}$ is the Weyl product when the parameter has a large gap (Lemma 5.7.4). This rests on Harish-Chandra's identities for the orbital integrals of c_λ on the compact torus (Proposition 5.7.5), proved by writing these integrals as antisymmetric Laurent polynomials and comparing top coefficients, with the multiplicities of the K-types; in our reading these multiplicities are the parts of Blattner's formula that the argument needs (compare Theorem A.5.13). The discrete series of the real unitary groups are Harish-Chandra's [119]; for their lowest K-types see [250], for accounts [251, 163, 315], and for characters of the maximal compact subgroup, Weyl modules and Schur functors [92, 3, 91]. The section continues with Laplace's method (Theorem 5.7.21), which gives the lower bound for a period that the construction needs, and closes with the properties of the Bessel averages (Proposition 5.7.30) and Harish-Chandra's formula for the formal degree (Theorem 5.7.41). The arrangement follows [259].

Definition 5.7.1. Let E be a CM field with maximal totally real subfield E^+, $N \geq 1$, and $\mathbb{A}_{E,f}$ the finite adeles of E; $G_f = \{g \in \mathrm{GL}_N(\mathbb{A}_{E,f}) : \bar{g}^{\mathsf{T}} g = 1\}$ as in (FT1).

Rational isometries as matrices. Let U be a framed space of rank N. Fix a basis of U over E and write the finite frame x_U and the elements of $\mathrm{U}(U)(E^+)$ as matrices in it, so $x_U \in \mathrm{GL}_N(\mathbb{A}_{E,f})$ and $\gamma \in \mathrm{GL}_N(E)$. The **characteristic polynomial** $\mathrm{charpoly}(\gamma) \in E[X]$ of γ is that of this matrix. Elements of $\mathrm{U}(U)(E^+)$ are identified with their images in $\Gamma_U \subseteq G_\infty \times G_f$.

Orbital integrals. For functions $k\colon G_\infty \to \mathbb{C}$ and $\phi\colon G_f \to \mathbb{C}$ write $(k \otimes \phi)(g) = k(g_\infty)\,\phi(g_f)$ for $g = (g_\infty, g_f) \in G_\infty \times G_f$. For $\gamma \in \mathrm{U}(U)(E^+)$ let $Z_\gamma \subseteq \Gamma_U$ be the image of its centraliser in $\mathrm{U}(U)(E^+)$. The **orbital integral** of $k \otimes \phi$ at γ is

$$I_\gamma(k \otimes \phi) = \int_{Z_\gamma \backslash (G_\infty \times G_f)} (k \otimes \phi)(x^{-1}\gamma x)\,dx, \tag{5.124}$$

for the quotient of the measure of $G_\infty \times G_f$ of (FT2) by counting measure on Z_γ.

Definition 5.7.2. A **Laplace datum** consists of nonempty finite sets $\mathfrak{n}$, $\mathfrak{m}$, signs $\varepsilon \in \{\pm 1\}^{\mathfrak{n}}$ and $\eta \in \{\pm 1\}^{\mathfrak{m}}$, integer vectors $l \in \mathbb{Z}^{\mathfrak{n}}$ and $k \in \mathbb{Z}^{\mathfrak{m}}$ (with lowest K-types μ_l for the signs ε and μ_k for the signs η), a group S, a homomorphism $j = (j_1, j_2)\colon S \to \mathrm{U}(\mathrm{diag}\,\varepsilon) \times \mathrm{U}(\mathrm{diag}\,\eta)$, a function $\varphi\colon S \to \mathbb{C}$, a finite set X, an integer $M \geq 0$, a bijection $e\colon X \sqcup \{0, \dots, M-1\} \to \mathfrak{n}$, a map $\tau\colon X \to \mathfrak{m}$, reals $B \geq 1$ and $\beta \geq 10^{12}$ and integers $L, h \geq 0$, such that (LA1)–(LA4) below hold. Here ι_e and ι_τ are the $\mathfrak{n} \times X$ and $\mathfrak{m} \times X$ matrices whose column x is the standard basis vector at $e(x)$, respectively at $\tau(x)$.

(LA1) Steep. $\varepsilon_{e(x)} = 1$ and $\mu_l(e(x)) = 0$ for $x \in X$; $\varepsilon_{e(i)} = (-1)^i$ for $0 \leq i < M$; and $i \mapsto \mu_l(e(i))$ is strictly decreasing on $\{0, \dots, M-1\}$ with values in $[-3B, -B]$.

(LA2) Phase. For all $s, t \in S$:

$$\varphi(s) = \mathrm{tr}\big(\mathrm{diag}(\mu_l)(j_1(s) - 1)\big) - \mathrm{tr}\big(\mathrm{diag}(\mu_k)(j_2(s) - 1)\big), \qquad \varphi(st) = \varphi(s) + \varphi(t),$$

$\mathrm{Re}\,\varphi(s) = 0$, and $\mathrm{tr}(j_1(s) - 1) = \mathrm{tr}(j_2(s) - 1)$.

(LA3) Second factor and diagonal copy. $\eta_{\tau(x)} = 1$ and $\mu_k(\tau(x)) = 0$ for $x \in X$; $\mu_k(i) \geq 0$ for every $i \in \mathfrak{m}$, and $\mu_k(i) \leq 3B$ for every $i \in \mathfrak{m}$ such that $\eta_{i'} \neq \eta_i$ for some $i' \in \mathfrak{m}$; and for every unitary $X \times X$ matrix u there is $s \in S$ with $j_1(s) - 1 = \iota_e(u - 1)\iota_e^*$ and $j_2(s) - 1 = \iota_\tau(u - 1)\iota_\tau^*$.

(LA4) Gap and size. l has large gap h for ε and k has large gap h for η; $|l_a| \leq L$ and $|k_b| \leq L$ for all $a \in \mathfrak{n}$, $b \in \mathfrak{m}$; $\#\mathfrak{n} + \#\mathfrak{m} + L \leq \beta$ and $10^{30}\big((\#\mathfrak{n} + \#\mathfrak{m})^2 \log \beta\big)^3 \leq h$.

For $\mathfrak{n}$, ε, l, X, M, e and B as above (without the other data), l is **steep** for (ε, e, B) if (LA1) holds.

Proposition 5.7.3. *Let D be admissible data, $\varrho = (r - k - 1)/2$, and $s \in \{\pm 1\}^r$ the signs of Y_0. There is a unitary $r \times r$ matrix p with $p^* J_1 p = \mathrm{diag}(s)$ such that for every $v \in \Phi$ there is a Bessel system $\mathfrak{B}_v$ for (ϱ, k, b, f), with representation ρ and full function W, such that:*

(i) its degree is $d(\varsigma(v))$;

(ii) there is an invertible matrix B_v with $B_v\,\rho(h) = S_{w_v}(h^{\mathsf{T}})^{\mathsf{T}} B_v$ for every unitary $k \times k$ matrix h, where S_{w_v} is the Schur matrix (polynomial representation of $\mathrm{GL}_k(\mathbb{C})$ in the standard basis of its Weyl module) of the Young diagram with $\mathrm{cols}_v(t)$ columns of height t for $1 \leq t < k$ and no columns of height $t \geq k$;

(iii) for every Haar measure dg on $\mathrm{U}(\operatorname{diag} s)$, with $d_{\varsigma(v)}$ the formal degree for it, and every $y \in G_1$,

$$\int_{\mathrm{U}(\operatorname{diag} s)} d_{\varsigma(v)}\, \overline{c_{\varsigma(v)}(g)}\, W(y\, pgp^{-1})\, dg = W(y);$$

(iv) there are $c \in \mathbb{R}$, a point $x \in G_1$ with $\ell(x) \le b^{5n}$ and a unit vector $e \in \mathbb{C}^I$ such that $W = c\, W_x$, where W_x is the Bessel average of Proposition 5.7.30 formed with the signs s, the matrix p, $l = 2\varsigma(v)$, the representation ρ and the vector e; and $W^\flat$ is the cut of W.

Proof. Throughout, $n = \varrho + 1$, a is the long order of D, and the r basis vectors of Y_0 (numbered as in Definition 3.2.16) and of V_1 (in the order $e_1, \dots, e_\varrho$, the basis of X_0, z, $e_1^*, \dots, e_\varrho^*$) are both identified with $\{0, \dots, r-1\}$; signs and parameters are read through this identification. Since $k = r + 1 - m$ with $m \ge 2$ lifts, $k \le r$, and $\varrho = (r - k - 1)/2 = m/2 - 1$.

Step 1: the sign frame. By Lemma 5.7.20 (for $z = \mathrm{I}$) there is a bijection e_0 from $X \sqcup \{0, \dots, 2\varrho + 1\}$, $X = \{0, \dots, k-2\}$, to the indices of Y_0 with $s(e_0(x)) = 1$ and $s(e_0(i)) = (-1)^i$; so s has exactly $\varrho + 1$ entries -1. The hermitian involution J_1 has the eigenvalue -1 with multiplicity $\varrho + 1$: on each hyperbolic plane $\langle e_b, e_b^* \rangle$ its matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has eigenvalues ± 1, it is 1 on X_0 and -1 on z. By the spectral theorem $J_1 = q \operatorname{diag}(s')\, q^*$ with q unitary and s' a sign vector with $\varrho + 1$ entries -1; for a permutation matrix w with $w^{\mathsf{T}} \operatorname{diag}(s') w = \operatorname{diag}(s)$, the matrix $p = qw$ is unitary with $p^* J_1 p = \operatorname{diag}(s)$. Fix it. Then $\iota(g) = p^{-1} g p$ is an isomorphism of topological groups $G_1 \to \mathrm{U}(\operatorname{diag} s)$ that preserves operator norms.

Step 2: the measures. The map ι carries the Cayley measure dy of G_1 to the Cayley measure dg of $\mathrm{U}(\operatorname{diag} s)$. Indeed, with S as in Definition 3.2.1, let $y \mapsto y'$ be the real-linear bijection of $\mathrm{M}_r(\mathbb{R})$ with $S(y') = p^* S(y) p$. As $p^{-1} J_1 = \operatorname{diag}(s) p^*$, ι carries $(1 + \frac{1}{2} J_1 S(y))(1 - \frac{1}{2} J_1 S(y))^{-1}$ to $(1 + \frac{1}{2}\operatorname{diag}(s) S(y'))(1 - \frac{1}{2}\operatorname{diag}(s) S(y'))^{-1}$, and $\det(1 - \frac{1}{2} J_1 S(y)) = \det(1 - \frac{1}{2}\operatorname{diag}(s) S(y'))$. The coordinates $\operatorname{Im} x_{aa}$, $\operatorname{Re} x_{ab}$, $\operatorname{Im} x_{ab}$ $(a < b)$ of skew-hermitian matrices are orthogonal for the euclidean structure $\operatorname{Re}\operatorname{tr}(X^* Y)$, with fixed lengths, so Lebesgue measure in them is a constant multiple of the euclidean volume, which $X \mapsto p^* X p$ preserves; as S carries Lebesgue measure of $\mathrm{M}_r(\mathbb{R})$ to Lebesgue measure in these coordinates, $y \mapsto y'$ preserves Lebesgue measure. So the two Cayley measures, images of the same density under corresponding charts, correspond under ι. Consequently, for a parameter l, the formal degree d_l of Proposition 5.7.30 (for the measure carried from dy) is the formal degree for the Cayley measure of $\mathrm{U}(\operatorname{diag} s)$.

Fix $v \in \Phi$ for the rest of the proof.

Step 3: the parameter. Put $l = 2\varsigma(v) \in \mathbb{Z}^r$, so $c_l^{G_1}(g) = c_{\varsigma(v)}(\iota(g))$. By (S6), l has large gap r^7 for s and $|l_i| \le 3a$. Since $r \ge 100$ (S4), $r^7 \ge 10 r^5$, so l also has large gap $10 r^5$, and by Lemma 5.7.4 and Step 2, $d_l = \mathrm{d}(\varsigma(v))$; in particular $d_l > 0$.

Step 4: the representation on the values. Let $x_1, \dots, x_k$ be the standard basis of X_0 and w the weight with $w(h) = \operatorname{cols}_v(h)$ for $1 \le h < k$; its highest weight is $\lambda_w(0) = \operatorname{cols}_v(1)$ and $\lambda_w(p) = 0$ for $p \ge 1$, as $\operatorname{cols}_v(h) = 0$ for $1 < h < k$ (Definition 2.3.4). Let $u(i) = 2\lambda_w(i-1) + k + 1 - 2i$ $(1 \le i \le k)$; it is a doubled Harish-Chandra parameter of $(w, 0)$. By Lemma 5.7.17, u has every gap for the signs $(1, \dots, 1)$, its lowest K-type is $\mu_u(i_0) = \operatorname{cols}_v(1)$ at the index $i_0 = 1$ and $\mu_u = 0$ elsewhere, and there are a continuous unitary representation $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$, $I = I_w$, an invertible matrix B_v and a unit vector e with $B_v \rho(h) = S_w(h^{\mathsf{T}})^{\mathsf{T}} B_v$, $\langle e, \rho(h) e \rangle = c_{u/2}(h)$, and only scalars commuting with ρ; as ρ is unitary and finite dimensional, it is irreducible. Since $\operatorname{cols}_v(1) \le a$ and $k \le a$ (S4), $|u(i)| \le 2a + k - 1 \le 3a$.

Step 5: a stationary point. Apply Proposition 5.7.30 (a) with the signs s, l, the set X and the bijection e_0 of Step 1, a bijection $\tau\colon X \sqcup \{*\} \to \{1, \dots, k\}$ with $\tau(*) = i_0$, $B = \kappa_0(v)$, the integer called a there equal to $\operatorname{cols}_v(1)$, $h = r^7$, $L = 3a$ and $\beta = b^3$. Its hypotheses hold:

steepness of $l = 2\varsigma(v)$ for $(s, e_0, \kappa_0(v))$ with $M = 2\varrho + 2 = 2n$ is Lemma 5.7.20; the lowest K-type of u is as required by Step 4; $3\kappa_0(v) + \mathrm{cols}_v(1) \le 3\kappa_0(v) + a \le b^2$ (S6); $8nr \le b$ (S4); l and u have large gap r^7 (Steps 3, 4); $|l_i|, |u_i| \le 3a$; $10^{12} \le b^3$ (S6), $r + k + 3a \le b^3$ and $10^{30}((r+k)^2 \log b^3)^3 \le r^7$ (S4). So there is $x \in G_1$ with $\ell(x) \le b^{5n}$ and $V(x) \le b^{60(r+k)^2}\,|\Pi|$, Π the period of Proposition 5.7.30 (a) for u. As $\langle e, \rho(h)e\rangle = c_{u/2}(h)$, Proposition 5.7.30 (b) with $P = b^{60(r+k)^2}$ gives $V(x) \le b^{60(r+k)^2}\, d_l \int_{\mathsf{U}\backslash G_1} \|W_x\|^2$, where W_x is the Bessel average for ρ, e and x.

Step 6: the cut pair and Schur's bound. Apply Proposition 5.7.30 (c) with $A = r^7$, $L = 3a$, $m = 60(r+k)^2$, f and b: $\|e\| = 1$; the volume bound and $\ell(x) \le b^{5n}$ are Step 5; $b \ge 2000$, $r \ge 100$ and $r + 3a + 1 \le b^4$ are (S4); $m + 99r^2 \le 240r^2 + 99r^2 \le 400r^2$ as $k \le r$; and $3r^3 f \le r^7$ is (S6). So there are $c \in \mathbb{R}$ and a continuous $W = c\,W_x$ with $\int_{G_1} d_l\, c_l^{G_1}(a'^{-1}) W(ya')\, da' = W(y)$ for all y, such that $(W, W^\flat)$, $W^\flat = \chi(\ell(\cdot))W$, is a cut pair for b, f. Then $W^\flat$ is continuous: χ is continuous, and so is ℓ, because $\ell(y') \le \ell(y)\,\|y^{-1}y'\|_{\mathrm{op}}$ for all y, y' (as $\|sy'\|_{\mathrm{op}} + 1 \le (\|sy\|_{\mathrm{op}} + 1)\|y^{-1}y'\|_{\mathrm{op}}$, elements of G_1 having operator norm at least 1). By (BS1), W is integrable over $\mathsf{U}\backslash G_1$ and $\int_{\mathsf{U}\backslash G_1} \|W\|^2 = \#I$. For $P \ge b^5$, Proposition 5.7.30 (d) with $A = r^7$, $L = 3a$ and $B = P$ applies, since $(r+1)^4(3a + r + 1 + 2n^2) \le b^5 \le P$ (S4): W has Schur's bound $(P, d_l^{-1}\#I)$.

Step 7: local bounds. Apply Theorem 5.5.3 to the cut pair $(W, W^\flat)$ of the irreducible type ρ, with $\deg = d_l$. Its hypotheses: $b \ge 2$; $8r \le 8nr \le b \le b^{6n+2}$; $k = s_0 + 3n + 6$ with $s_0 = k - 3n - 6 > 10^9 n^3$ (S4); $n^7 \le k$ (S1); $10^8 \le \varrho$ (S6); $4000n \le f$ (S4). The coefficient bound: by Lemma 5.7.19 (ii) for $F = W$ (a Bessel function of type ρ, integrable over $\mathsf{U}\backslash G_1$ and reproduced by $d_l c_l^{G_1}$, l of large gap r^7),

$$\int_{G_1} \Big| \int_{\mathsf{U}\backslash G_1} \langle W(y), W(yx)\rangle\, dy \Big|^2 dx = d_l^{-1} \Big(\int_{\mathsf{U}\backslash G_1} \|W\|^2 \Big)^2 = \frac{(\#I)^2}{d_l}.$$

The constant: $d_l = \mathrm{d}(\varsigma(v)) = \prod_{i<j} \frac{|l_i - l_j|/2}{j-i} \le (3a)^{r^2} \le b^{2r^2}$, as $|l_i - l_j|/2 \le 3a \le b^2$ (S4); and with Θ of Definition 5.1.6, $\mathrm{vol}\,\mathrm{U}(k)/\mathrm{vol}\,\mathrm{U}(r) = \prod_{i=k}^{r-1} \frac{i!}{2\pi^{i+1}} \le r^{r^2} \le b^{r^2}$ ($i! \le r^r$, $2\pi^{i+1} \ge 1$, $r \le b$) and $\pi^{(2\varrho+1)k} \le b^{r^2}$ ($\pi \le b$, $(2\varrho+1)k \le r^2$, (S4)); so $d_l\Theta \le b^{4r^2}$. The ball volume: by Lemma 5.7.38 (3) for $\mathrm{U}(\mathrm{diag}\, s)$ with the measure carried from dy, the large gap r^7, $L = 3a$ and $c = 9r^2 > 8r^2$, and as ι preserves operator norms,

$$\int_{G_1} d_l \Big(\frac{2}{\|g\|_{\mathrm{op}}} \Big)^{9r^2} dg \le 8\big(2000(r + 3a + 1)^6\big)^{2r^2} 2^{9r^2} \le 8\big(2000\, b^{24}\big)^{2r^2} 2^{9r^2} \le b^{50r^2+1} 2^{9r^2},$$

using $r+3a+1 \le b^4$ and $8 \le 2000 \le b$. For $T \ge 1$ and $\|g\|_{\mathrm{op}} \le T$ we have $1 \le (T/2)^{9r^2}(2/\|g\|_{\mathrm{op}})^{9r^2}$, hence $d_l\,\mathrm{vol}\{g \in G_1 : \|g\|_{\mathrm{op}} \le T\} \le T^{9r^2} b^{50r^2+1} \le T^{9r^2} b^{99r^2}$. So Theorem 5.5.3 gives forms J that are local bounds for $W^\flat$ with degree d_l.

Step 8: the Bessel system. I is finite and nonempty, ρ is irreducible and unitary, $(W, W^\flat)$ are continuous Bessel functions of type ρ forming a cut pair, $\deg = d_l > 0$, $8r \le b$, W has Schur's bound $(P, \deg^{-1}\#I)$ for every $P \ge b^5$, and J are local bounds for $W^\flat$ with degree deg: this is a Bessel system $\mathfrak{B}_v$ for (ϱ, k, b, f). (i) is Step 3. (ii) is Step 4, the Young diagram of w having $\mathrm{cols}_v(t)$ columns of height t for $1 \le t < k$ and none of height $t \ge k$. (iii): let dg' be a Haar measure of $\mathrm{U}(\mathrm{diag}\, s)$ and d' the formal degree for it; then $d'dg' = d_l\, dg$, the formal degree scaling inversely to the measure. Substituting $a' = \iota^{-1}(g) = pgp^{-1}$ in the reproduction of Step 6 and using $c_{\varsigma(v)}(g^{-1}) = \overline{c_{\varsigma(v)}(g)}$ (Lemma 5.7.40),

$$W(y) = \int_{\mathrm{U}(\mathrm{diag}\, s)} d_l\, \overline{c_{\varsigma(v)}(g)}\, W(ypgp^{-1})\, dg = \int_{\mathrm{U}(\mathrm{diag}\, s)} d'\, \overline{c_{\varsigma(v)}(g)}\, W(ypgp^{-1})\, dg'.$$

(iv): $W = c\,W_x$ with c real, $\ell(x) \le b^{5n}$ and W_x the Bessel average for the signs s, p, $l = 2\varsigma(v)$, ρ and the unit vector e (Steps 1 and 3 to 6), and $W^\flat$ is its cut (Step 6). □

Lemma 5.7.4. *Let $n = \{1, \dots, N\}$ ($N \geq 1$), $\varepsilon \in \{\pm 1\}^n$, and let $l \in \mathbb{Z}^n$ have large gap $10N^5$ for ε; put $\lambda = \frac{1}{2}l$. Let dg be the Cayley measure of $\mathrm{U}(\operatorname{diag}\varepsilon)$. Then*

$$\Big(\int_{\mathrm{U}(\operatorname{diag}\varepsilon)} |c_\lambda|^2\, dg\Big)^{-1} = \mathrm{d}(\lambda) = \prod_{1\leq a<b\leq N} \frac{|\lambda_a - \lambda_b|}{b-a}.$$

Proof. Write d_λ for the left side and dk for the Haar probability measure of $\mathrm{U}(n)$. For $a \in n$ let $p(a) = \#\{a' : l_{a'} > l_a\}$; as l is injective, p is a bijection $n \to \{0, \dots, N-1\}$, and we write a_p for the index with $p(a_p) = p$.

The lowest K-type for the compact form. Let Λ be the lowest K-type of l for the signs $(1, \dots, 1)$ (Definition 3.2.3). All signs agree, so $2\rho_l^{\mathrm{c}} = 2\rho_l$ with $2\rho_l(a) = (N-1-p(a)) - p(a)$. As l has a gap, $l_a \equiv N-1 \pmod 2$, so $l_a + 2\rho_l(a)$ is even and $l^\rho(a) = \lambda_a + \frac{N-1}{2} - p(a)$; hence

$$\Lambda(a) = l^\rho(a) - 2\rho_l(a) = \lambda_a - \tfrac{N-1}{2} + p(a).$$

Put $L_p = \Lambda(a_p)$. The entries of l are distinct and of the same parity, so $\lambda_{a_p} - \lambda_{a_{p+1}} \geq 1$ and $L_p \geq L_{p+1}$.

A Weyl module with coefficient m_Λ. Let $w(h) = L_{h-1} - L_h \geq 0$ for $1 \leq h < N$, a weight for the rank N, with highest weight $\lambda_w(p) = \sum_{p<h<N} w(h) = L_p - L_{N-1}$, and put $u(a) = 2\lambda_w(p(a)) + N - 1 - 2p(a)$. The numbers $2\lambda_w(q) + N - 1 - 2q$ are strictly decreasing in q, so u is injective and is a doubled Harish-Chandra parameter of $(w, 0)$ ((WM6) of Definition 3.2.19), and $\#\{a' : u(a') > u(a)\} = p(a)$. By Lemma 5.7.17 (for $k = N$, the rows indexed by $n = \{1, \dots, N\}$), $\mu_u(a) = \lambda_w(p(a)) = \Lambda(a) - c_0$ with $c_0 = L_{N-1}$, and there are an irreducible unitary representation ρ of $\mathrm{U}(n)$ on $\mathbb{C}^{I_w}$ and a unit vector e with $m_{\mu_u}(h) = \langle e, \rho(h)e\rangle$ for $h \in \mathrm{U}(n)$, where $\#I_w = d_N(\lambda_w)$. By Schur's orthogonality relations for the compact group $\mathrm{U}(n)$, $\int |\langle e, \rho(k)e\rangle|^2 dk = 1/\#I_w$. By (3.1), $\min_a \mu_u(a) = \lambda_w(N-1) = 0$, and $\{a : \Lambda(a) \geq t\} = \{a : \mu_u(a) \geq t - c_0\}$, so $m_\Lambda(h) = (\det h)^{c_0} m_{\mu_u}(h)$; as $|\det k| = 1$ on $\mathrm{U}(n)$,

$$\int_{\mathrm{U}(n)} |m_\Lambda(k)|^2\, dk = \frac{1}{d_N(\lambda_w)}.$$

Conclusion. By Proposition 5.7.5 (i), $d_\lambda \int_{\mathrm{U}(n)} |m_\Lambda|^2 dk = 1$, so $d_\lambda = d_N(\lambda_w)$. Finally, as $\lambda_w(p) - \lambda_w(q) + q - p = L_p - L_q + q - p = \lambda_{a_p} - \lambda_{a_q} > 0$ for $p < q$,

$$d_N(\lambda_w) = \prod_{0\leq p<q<N} \frac{\lambda_{a_p} - \lambda_{a_q}}{q-p} = \frac{\prod_{1\leq a<b\leq N} |\lambda_a - \lambda_b|}{\prod_{1\leq a<b\leq N} (b-a)} = \mathrm{d}(\lambda),$$

because both products of differences run over all unordered pairs of indices, and $\prod_{0\leq p<q<N}(q-p) = \prod_{j=1}^{N-1} j! = \prod_{1\leq a<b\leq N}(b-a)$. □

Proposition 5.7.5. *Let n be a nonempty finite set, $N = \#n$, $\varepsilon \in \{\pm 1\}^n$, and let $l \in \mathbb{Z}^n$ have large gap $10N^5$ for ε; put $\lambda = \frac{1}{2}l$. Let dg be the Cayley measure of $\mathrm{U}(\operatorname{diag}\varepsilon)$ and $d_\lambda = \big(\int_{\mathrm{U}(\operatorname{diag}\varepsilon)} |c_\lambda|^2 dg\big)^{-1}$.*

(i) Let Λ be the lowest K-type of l for the signs $(1, \dots, 1)$, and dk the Haar probability measure of the group $\mathrm{U}(n)$ of unitary $n \times n$ matrices. Then

$$d_\lambda \int_{\mathrm{U}(n)} |m_\Lambda(k)|^2\, dk = 1.$$

(ii) Let $\kappa_a = \frac{1}{2}(l_a + N - 1)$ $(a \in n)$, *and for* $x \in \mathbb{C}^n$ *put*

$$V_{\mathrm{c}}(x) = \prod_{\substack{\varepsilon_a = \varepsilon_b \\ l_b < l_a}} (x_a - x_b), \qquad D(x) = \prod_{\substack{\varepsilon_a \neq \varepsilon_b \\ l_a < l_b}} (x_a - x_b), \qquad A(x) = \sum_{\substack{w \in S_n \\ \varepsilon \circ w = \varepsilon}} \operatorname{sgn}(w) \prod_{a \in n} x_a^{\kappa_{w(a)}},$$

the products over ordered pairs (a, b). *If* $x_a \in \mathbb{C}$, $|x_a| = 1$, *are pairwise distinct, then* $g \mapsto c_\lambda(g^{-1} \operatorname{diag}(x)\, g)$ *is integrable on* $\mathrm{U}(\operatorname{diag} \varepsilon)$ *and*

$$V_{\mathrm{c}}(x)\, D(x)\, d_\lambda \int_{\mathrm{U}(\operatorname{diag}\varepsilon)} c_\lambda(g^{-1}\operatorname{diag}(x) g)\, dg = A(x).$$

Proof. For $x \in \mathbb{C}^n$ with $|x_a| = 1$ write $t_x = \operatorname{diag}(x)$ and, when the entries are pairwise distinct, $\Theta(x) = d_\lambda \int c_\lambda(g^{-1} t_x g)\, dg$. By Lemma 5.6.23 (i) (the gap h exceeds $8N^2$) d_λ is a positive real number. Let $\delta_{\mathrm{c}}(a) = \#\{b : \varepsilon_b = \varepsilon_a,\ l_b < l_a\}$, let p, q be the numbers of indices with $\varepsilon = 1$, $\varepsilon = -1$, and number n as $a_0, \dots, a_{N-1}$ by decreasing l, so $o(a_j) = j$.

Step 1: three identities. As $l_a \equiv N - 1 \pmod 2$ and $\#\{b : l_b < l_a\} = N - 1 - o(a)$, $2\rho_l(a) = N - 1 - 2o(a)$ and $l^\rho = \kappa - o$. (α) For the signs $(1, \dots, 1)$, $2\rho_l^{\mathrm{c}} = 2\rho_l$, so $\Lambda_{\mathrm{c}} = l^\rho - 2\rho_l$, that is $\Lambda_{\mathrm{c}}(a_j) = \kappa(a_j) - (N-1) + j$; it is nonincreasing in j (κ is strictly increasing along l, Lemma 5.7.18 (a)), and $\Lambda_{\mathrm{c}}(a_j) + N - 1 - j = \kappa(a_j)$. (β) For the signs ε, $2\rho_l^{\mathrm{c}}(a) = \delta_{\mathrm{c}}(a) - \#\{b : \varepsilon_b = \varepsilon_a,\ l_b > l_a\}$, so $\mu_l + \delta_{\mathrm{c}} = \kappa - o + \#\{b : \varepsilon_b = \varepsilon_a,\ l_b > l_a\} = \kappa - \delta$, and $\mu_l + \delta_{\mathrm{c}} + (N'+1)\delta = \mathrm{top}$. (γ) $V = V_{\mathrm{c}} \prod_{\varepsilon_a \neq \varepsilon_b,\, l_b < l_a} (X_a - X_b) = (-1)^{pq} V_{\mathrm{c}} D$, each of the pq unordered pairs of opposite signs giving one factor of D with the opposite orientation.

Step 2: the two Laurent polynomials. By Proposition 5.7.6, $g \mapsto c_\lambda(g^{-1} t_x g)$ is integrable at every x with distinct entries, and there is a W_K-antisymmetric $P \in \mathcal{R}$, a cone-series multiple of $D^{N'+1}$ at $\mu_l + \delta_{\mathrm{c}}$, with $P(x) = V_{\mathrm{c}}(x) D(x)^{N'+1} \Theta(x)$ at these x. By (β) and Lemma 5.7.18 (b), $P_{\mathrm{top}} = 1$ and every exponent of P has a W_K-conjugate of level at most $o \cdot \mathrm{top}$. Put $\mathrm{num}^\eta = D^{N'} A_\kappa^\eta$ $(\eta \in \{\varepsilon, \mathbf{1}\})$. By Lemma 5.7.18 (c), then (b) with $a = \kappa$, $k = N'$, num^ε is W_K-antisymmetric, $\mathrm{num}^\varepsilon_{\mathrm{top}} = 1$, and every exponent of num^ε has a W_K-conjugate of level at most $o \cdot \mathrm{top}$.

Step 3: comparison on the accretive torus. Let σ and Off be as in Proposition 5.7.10, and let x have $|x_a| = 1$, distinct entries and Jt_x accretive. By (a) there, $\Theta(x) = \frac{d_\lambda}{\sigma}\big(\|m_{\Lambda_{\mathrm{c}}}\|^2 \chi(t_x) - \mathrm{Off}(Jt_x)\big)$. Weyl's character formula [92], in the numbering $a_0, \dots, a_{N-1}$ with the highest weight Λ_{c} (nonincreasing by (α)), gives

$$\chi(t_x) = \frac{\det\big(x_{a_k}^{\Lambda_{\mathrm{c}}(a_j) + N - 1 - j}\big)_{j,k}}{\det\big(x_{a_k}^{N-1-j}\big)_{j,k}} = \frac{\det\big(x_b^{\kappa_a}\big)_{a,b}}{\prod_{j<k}(x_{a_j} - x_{a_k})} = \frac{A_\kappa^{\mathbf{1}}(x)}{V(x)},$$

by (α), by reindexing rows and columns by the same bijection, and by the Leibniz formula $\det(x_b^{\kappa_a}) = \sum_{v \in S_n} \operatorname{sgn}(v) \prod_a x_a^{\kappa_{v(a)}}$; here $V(x) = \prod_{j<k}(x_{a_j} - x_{a_k}) \neq 0$. With (γ), $V_{\mathrm{c}}(x) D(x) \chi(t_x) = (-1)^{pq} A_\kappa^{\mathbf{1}}(x)$. Hence, with

$$s = (-1)^{pq}\, \frac{d_\lambda \|m_{\Lambda_{\mathrm{c}}}\|^2}{\sigma}, \qquad Q = -(-1)^{pq} \frac{d_\lambda}{\sigma}\, V\, D^{N'} \in \mathcal{R},$$

and $V_{\mathrm{c}} D^{N'+1} = (-1)^{pq} V D^{N'}$,

$$(P - s\, \mathrm{num}^{\mathbf{1}})(x) = V_{\mathrm{c}}(x) D(x)^{N'+1} \Theta(x) - s\, D(x)^{N'} A_\kappa^{\mathbf{1}}(x) = Q(x)\, \mathrm{Off}(Jt_x).$$

Step 4: low levels. Let $o \cdot \gamma \leq o \cdot \mathrm{top}$. The exponents β of Q are exponents of $V D^{N'}$, so by Lemma 5.7.18 (d), $o \cdot \gamma \leq o \cdot \mathrm{top} < o \cdot (\kappa - o) + o \cdot \beta + h$. By Step 3 and Proposition 5.7.10

(b), $(P - s\,\mathrm{num}^{\mathbf{1}})_\gamma = 0$, and by Lemma 5.7.18 (d), $\mathrm{num}^{\mathbf{1}}_\gamma = \mathrm{num}^\varepsilon_\gamma$. So $P_\gamma = s\,\mathrm{num}^\varepsilon_\gamma$ whenever $o\cdot\gamma \le o\cdot\mathrm{top}$.

Step 5: identification. By Steps 2 and 4 and Lemma 5.7.18 (e) (with $m = o\cdot\mathrm{top}$, $c = s$), $P = s\,\mathrm{num}^\varepsilon$. Comparing the coefficients at top (Step 2), $1 = s\cdot 1$, so $s = 1$.

Step 6: conclusion. (i): $1 = |s| = d_\lambda \|m_{\Lambda_\mathrm{c}}\|^2/|\sigma| = d_\lambda \int_{\mathrm{U}(n)} |m_\Lambda(k)|^2 dk$, as $d_\lambda > 0$, $|\sigma| = 1$ and $\Lambda = \Lambda_\mathrm{c}$. (ii): integrability is Step 2; and for x with $|x_a| = 1$ and distinct entries, $P = \mathrm{num}^\varepsilon$ gives $V_\mathrm{c}(x)D(x)^{N'+1}\Theta(x) = D(x)^{N'}A^\varepsilon_\kappa(x)$, where $D(x)\neq 0$ (distinct entries) and

$$A^\varepsilon_\kappa(x) = \sum_{w\in S_n,\,\varepsilon\circ w=\varepsilon} \mathrm{sgn}(w)\prod_a x_a^{\kappa_{w(a)}} = A(x);$$

dividing by $D(x)^{N'}$ gives

$$V_\mathrm{c}(x)D(x)\,d_\lambda \int c_\lambda(g^{-1}\mathrm{diag}(x)g)\,dg = A(x).$$

□

Proposition 5.7.6. *Let n be a nonempty finite set, $N = \#n$, $\varepsilon\in\{\pm1\}^n$, and let $l\in\mathbb{Z}^n$ have large gap $10N^5$ for ε; put $\lambda = \frac12 l$, let dg be the Cayley measure of $\mathrm{U}(\mathrm{diag}\,\varepsilon)$ and $d_\lambda = \left(\int |c_\lambda|^2 dg\right)^{-1}$. For $x\in\mathbb{C}^n$ with $|x_a| = 1$ let $t_x = \mathrm{diag}(x)\in\mathrm{U}(\mathrm{diag}\,\varepsilon)$. Let the Laurent polynomial ring $\mathcal{R}$, coefficients P_γ, W_K, D, $N' = 4N^2$, W_K-antisymmetry and cone-series multiples be as in Lemma 5.7.18 for these n,ε,l, and put $V_\mathrm{c} = \prod_{\varepsilon_a=\varepsilon_b,\ l_b<l_a}(X_a - X_b)$ and $\delta_\mathrm{c}(a) = \#\{b : \varepsilon_b = \varepsilon_a,\ l_b < l_a\}$. Then for every x with $|x_a| = 1$ and pairwise distinct entries the function $g\mapsto c_\lambda(g^{-1}t_x g)$ is integrable on $\mathrm{U}(\mathrm{diag}\,\varepsilon)$, and there is $P\in\mathcal{R}$ which is W_K-antisymmetric, is a cone-series multiple of $D^{N'+1}$ at $\mu_l + \delta_\mathrm{c}$, and satisfies*

$$P(x) = V_\mathrm{c}(x)\,D(x)^{N'+1}\,d_\lambda \int_{\mathrm{U}(\mathrm{diag}\,\varepsilon)} c_\lambda(g^{-1}t_x g)\,dg$$

for every such x.

Proof. Write $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$, $K = K_\varepsilon$, $Y = D^{N'+1}\in\mathcal{R}$, and for x with $|x_a| = 1$ and distinct entries $\Theta(x) = d_\lambda\int_G c_\lambda(g^{-1}t_x g)\,dg$. The vector l is injective, and μ_l is K-dominant (Lemma 5.6.27 (iii)). Every factor of D is $\pm(X_a - X_b)$ with $\varepsilon_a = 1$, $\varepsilon_b = -1$, each such pair once, and W_K permutes the indices of each sign; so $D(X\circ w) = D(X)$ and $Y_{\beta\circ w} = Y_\beta$ for $w\in W_K$, and Y has integer coefficients.

Step 1: integrability and the class function. If x has distinct entries, $D(x)\neq 0$, and by Lemma 5.7.8 (iii) $g\mapsto c_\lambda(g^{-1}t_x g)$ is integrable. Let d, Φ and h be as in Lemma 5.7.8: h is a continuous class function on K with $h(t_x) = Y(x)\Theta(x)$ at these x.

Step 2: the coefficients of h are integers. Let ν be K-dominant and $a_\nu = \int_K \overline{\chi^K_\nu}\,h\,dk$. For $g\in G$ let $\psi_g(k) = c_\lambda(g^{-1}kg)$, continuous on K. Since $|d(k)\psi_g(k)|\le\Phi(g)$ with Φ integrable and χ^K_ν is bounded, Fubini and Lemma 5.7.9 (c) (for $D = Y$, $f = \psi_g$) give

$$a_\nu = d_\lambda\int_G\int_K \overline{\chi^K_\nu}\,d\,\psi_g\,dk\,dg = d_\lambda\int_G\sum_\beta Y_\beta\,s_{e(\beta)}\,R_{\nu_{e(\beta)}}(g)\,dg = \sum_\beta Y_\beta\,s_{e(\beta)}\,m(\nu_{e(\beta)}),$$

$e(\beta) = \nu + \delta_\mathrm{c} - \beta$, using that each $R_{\nu'}$ is integrable with $d_\lambda\int_G R_{\nu'} = m(\nu')$ (Proposition 5.7.7). Hence $a_\nu\in\mathbb{Z}$.

Step 3: finitely many coefficients. By the theorem of the highest weight for $\mathrm{U}(n_+)\times\mathrm{U}(n_-)$, the χ^K_ν (ν K-dominant) are the characters of a set of representatives of the irreducible continuous

representations of K; by Theorem A.5.21 (i) they are orthonormal in $L^2(K)$, so by Bessel's inequality $\sum_\nu |a_\nu|^2 \le \int_K |h|^2 dk < \infty$. As the a_ν are integers, the set F of ν with $a_\nu \ne 0$ is finite.

Step 4: completeness. The continuous class function $f = h - \sum_{\nu\in F} a_\nu \chi^K_\nu$ satisfies $\int_K f\,\overline{\chi^K_\nu}\,dk = 0$ for every K-dominant ν, hence for every irreducible character; by Theorem A.5.21 (ii) it vanishes almost everywhere, hence everywhere (it is continuous and dk charges every nonempty open set). So $h = \sum_{\nu\in F} a_\nu \chi^K_\nu$.

Step 5: the Laurent polynomial. Put $P = \sum_{\nu\in F} a_\nu A^\varepsilon_{\nu+\delta_c} \in \mathcal{R}$. Each alternant is W_K-antisymmetric ($(A^\varepsilon_\mu)_{\gamma\circ w} = \mathrm{sgn}(w)(A^\varepsilon_\mu)_\gamma$), hence so is P. By Lemma 5.7.9 (a), $P(x) = V_c(x)h(t_x)$ for every x on the torus, and by Step 1, $P(x) = V_c(x)D(x)^{N'+1}\Theta(x)$ when x has distinct entries.

Step 6: P is a cone-series multiple of $D^{N'+1}$ at $\mu_l+\delta_c$. Let $t(\gamma) = s_\gamma\, m(\nu_\gamma)$. By Lemma 5.7.9 (b), $A^\varepsilon_{e\circ w} = \mathrm{sgn}(w)A^\varepsilon_e$ for $w \in W_K$ forces $s_{e\circ w} = \mathrm{sgn}(w)s_e$ and, when $s_e \ne 0$, $\nu_{e\circ w} = \nu_e$ (alternants $A^\varepsilon_{\mu+\delta_c}$ of distinct K-dominant μ have disjoint sets of exponents); so $t(e\circ w) = \mathrm{sgn}(w)t(e)$. Hence $Q_\gamma = \sum_\beta Y_\beta\, t(\gamma-\beta)$ (a finite sum) is W_K-antisymmetric in γ, by the substitution $\beta \mapsto \beta\circ w$ and the invariance of Y. Two W_K-antisymmetric functions on $\mathbb{Z}^n$ that agree at every $\nu+\delta_c$ with ν K-dominant agree everywhere: by (b), every γ with $s_\gamma \ne 0$ is $(\nu_\gamma+\delta_c)\circ w$ for some $w \in W_K$, and at a γ fixed by an odd element of W_K both vanish. At $\gamma = \nu+\delta_c$, $s_\gamma = 1$ and $\nu_\gamma = \nu$, so $P_\gamma = a_\nu$ by (b), and $Q_\gamma = \sum_\beta Y_\beta s_{e(\beta)} m(\nu_{e(\beta)}) = a_\nu$ by Step 2. So $P_\gamma = \sum_\beta Y_\beta\, t(\gamma-\beta)$ for all γ. Further $t(\mu_l+\delta_c) = 1\cdot m(\mu_l) = 1$. If $t(\gamma) \ne 0$, then $s_\gamma \ne 0$ and $m(\nu_\gamma) \ne 0$, so $\nu_\gamma - \mu_l \in C_l$ (Proposition 5.7.7), and $\mu = \nu_\gamma + \delta_c$ satisfies $(A^\varepsilon_\mu)_\gamma = s_\gamma \ne 0$ by (b) and $\mu - (\mu_l+\delta_c) \in C_l$. This is the cone condition of Lemma 5.7.18. □

Proposition 5.7.7. *Let n be a nonempty finite set, $N = \#n$, $\varepsilon \in \{\pm1\}^n$, and let $l \in \mathbb{Z}^n$ have large gap $10N^5$ for ε; put $\lambda = \frac12 l$, let dg be the Cayley measure of $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$, $d_\lambda = \left(\int_G |c_\lambda|^2 dg\right)^{-1}$, and dk the Haar probability measure of $K = K_\varepsilon$. Call $\nu \in \mathbb{Z}^n$ K-*dominant *if $\nu(i) \ge \nu(j)$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$, and for such ν let χ^K_ν be the character of the irreducible continuous representation of K of highest weight ν for the diagonal torus and the Borel subgroup of the invertible matrices b with $b_{ij} = 0$ unless $\varepsilon_i = \varepsilon_j$ and $l_i \ge l_j$. Let $C_l = \{\beta \in \mathbb{Z}^n : o'\cdot\beta \le 0$ for every $o' \in \mathbb{Z}^n$ with $o'(a) < o'(b)$ whenever $l_b < l_a\}$. Then for every K-dominant ν the function*

$$R_\nu(g) = \int_K c_\lambda(g^{-1}kg)\,\overline{\chi^K_\nu(k)}\,dk$$

is integrable on G, and $m(\nu) = d_\lambda \int_G R_\nu(g)\,dg$ is a nonnegative integer, with $m(\mu_l) = 1$, and $m(\nu) = 0$ unless $\nu - \mu_l \in C_l$.

Proof. Write $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$, $K = K_\varepsilon$, $c = c_\lambda$, $d = d_\lambda$, $\mu = \mu_l$, and $\|m_w\|^2 = \int_K |m_w|^2 dk$ for $w \in \mathbb{Z}^n$, a positive number since m_w is continuous on K with $m_w(1) = 1$. The large gap $10N^5$ is the gap $500N^3 + 10N^5 > 8N^2$, hence also gap 0. The Cayley measure dg is a Haar measure on G, invariant under left and right translations (Lemma 5.2.41). By Lemma 5.6.23 (i), c is continuous and square integrable and $d > 0$; by Lemma 5.7.40, $|c| \le 1$ and $c(g^{-1}) = \overline{c(g)}$; and $c(1) = 1$ by (3.2), as $\mathrm{bd}_\varepsilon(1) = 1$ and $m_w(1) = 1$. The lowest K-type μ is K-dominant by Lemma 5.6.27 (iii). Fix a K-dominant ν.

Step 1: the character as a conjugation mean. We show

$$\int_K m_\nu(y^{-1}ky)\,dy = \|m_\nu\|^2\,\chi^K_\nu(k) \qquad (k \in K). \tag{5.125}$$

Let $n_\pm = \{a : \varepsilon_a = \pm1\}$, so that $K = \mathrm{U}(n_+)\times\mathrm{U}(n_-)$, $k = (k_+, k_-)$, and dk is the product of the Haar probability measures; for $w \in \mathbb{Z}^n$ let $w_\pm$ be its restriction to $n_\pm$. For block-diagonal invertible g and $P \subseteq n$, $\Delta_P(g) = \Delta_{P\cap n_+}(g_+)\Delta_{P\cap n_-}(g_-)$; in (3.1) the factors of block $n_\pm$ with $t \le \min w_\pm$ are $\det g_\pm$ and those with $t > \max w_\pm$ are 1, so $m_w(g) = m_{w_+}(g_+)\,m_{w_-}(g_-)$ (a factor

for an empty block being 1). By the theorem of the highest weight for the connected compact group K, the irreducible representation of highest weight ν (for the diagonal torus and the Borel subgroup of the statement) is the outer tensor product of the irreducible representations of $\mathrm{U}(n_\pm)$ of highest weights $\nu_\pm$ for the Borel subgroups of invertible $b_\pm$ with $(b_\pm)_{ij}=0$ when $l_i<l_j$; so $\chi^K_\nu(k)=\chi_{\nu_+}(k_+)\chi_{\nu_-}(k_-)$, where $\chi_{\nu_\pm}$ is the character of Lemma 5.7.16 for $n_\pm$ and the numbering of $n_\pm$ by decreasing l (along which $\nu_\pm$ decreases, by K-dominance; for an empty block put $\chi=1$). By Fubini's theorem $\|m_\nu\|^2=\|m_{\nu_+}\|^2\|m_{\nu_-}\|^2$, and Lemma 5.7.16 (iv) in each nonempty block gives (5.125).

Consequently, for $g\in G$: the integrand of $R_\nu(g)$ is continuous on K, and inserting the complex conjugate of (5.125), using Fubini's theorem on $K\times K$ (continuous integrand) and the invariance of dk under $k\mapsto yky^{-1}$,

$$R_\nu(g)=\frac{1}{\|m_\nu\|^2}\int_K\int_K \overline{m_\nu(k)}\; c(g^{-1}y\,k\,y^{-1}g)\,dk\,dy. \tag{5.126}$$

Step 2: the lowest K-type. For $\nu=\mu$, Lemma 5.6.31 (i) with $x=g^{-1}y$, $b=y^{-1}g$ turns (5.126) into $R_\mu(g)=\int_K c(g^{-1}y)\,c(y^{-1}g)\,dy=\int_K|c(y^{-1}g)|^2dy\ge 0$. By Tonelli's theorem and left invariance, $\int_G R_\mu\,dg=\int_K\int_G|c(y^{-1}g)|^2dg\,dy=\int_G|c|^2dg=d^{-1}$. So R_μ is integrable and $d\int_G R_\mu\,dg=1$.

Step 3: off the cone. If $\nu-\mu\notin C_l$, there is $o\in\mathbb{Z}^n$ with $o(a)<o(b)$ whenever $l_b<l_a$ and $o\cdot(\nu-\mu)>0$. By Lemma 5.6.31 (ii) the inner integral in (5.126) vanishes for every y, so $R_\nu=0$: R_ν is integrable with integral 0.

Step 4: the representation generated by c. On $L^2(G)=L^2(G,dg)$ let $\langle u,w\rangle=\int_G\overline{u}\,w\,dg$ and $(R_xf)(g)=f(gx)$. By right invariance each R_x is unitary, $R_xR_{x'}=R_{xx'}$, and $x\mapsto R_xf$ is continuous (continuity of translation in L^2 of a locally compact group). Let V be the closed span of the R_xc, $x\in G$; it is $R(G)$-stable. For $x,y\in G$, the substitution $g\mapsto gx^{-1}$ and Lemma 5.6.23 (ii) with $(x,a)=(1,x^{-1}y)$ give

$$\langle R_xc,R_yc\rangle=\int_G\overline{c(g)}\,c(gx^{-1}y)\,dg=d^{-1}c(1)c(x^{-1}y)=d^{-1}c(x^{-1}y). \tag{5.127}$$

We show

$$\int_G|\langle R_gc,u\rangle|^2\,dg=d^{-2}\|u\|^2\qquad(u\in V). \tag{5.128}$$

Let first $u=\sum_i\alpha_iR_{x_i}c$, a finite sum. By (5.127),

$$\langle R_gc,u\rangle=d^{-1}\sum_i\alpha_ic(g^{-1}x_i)\quad\text{and}\quad\overline{c(g^{-1}x_i)}\,c(g^{-1}x_j)=c(x_i^{-1}g)\,\overline{c(x_j^{-1}g)};$$

the substitution $g=x_jy$ and Lemma 5.6.23 (ii) with $(x,a)=(x_i^{-1}x_j,1)$ give

$$\int_G c(x_i^{-1}g)\overline{c(x_j^{-1}g)}\,dg=d^{-1}c(x_i^{-1}x_j)=\langle R_{x_i}c,R_{x_j}c\rangle.$$

Summing with the weights $\overline{\alpha_i}\alpha_j$ gives (5.128) for u. For general $u\in V$ choose u_m in the span with $u_m\to u$. By the case just shown, applied to $u_m-u_{m'}$, the functions $F_m(g)=\langle R_gc,u_m\rangle$ form a Cauchy sequence in $L^2(G)$; they converge pointwise to $F(g)=\langle R_gc,u\rangle$, since $|F_m(g)-F(g)|\le\|c\|\,\|u_m-u\|$. A subsequence of an L^2-convergent sequence converges almost everywhere, so the L^2 limit is F and $\|F\|_2=\lim\|F_m\|_2=d^{-1}\|u\|$.

Step 5: V is irreducible. Define the bounded operator Q on $L^2(G)$ by

$$\langle w,Qf\rangle=\|m_\mu\|^{-2}\int_K\overline{m_\mu(k)}\,\langle w,R_kf\rangle\,dk$$

(the integrand is continuous in k and bounded by $\sup_K |m_\mu| \, \|w\| \, \|f\|$; Riesz's theorem). Let $b \in G$ and $w \in L^2(G)$. The function $(k,g) \mapsto \overline{m_\mu(k)}\,\overline{w(g)}\,c(gkb)$ is integrable on $K \times G$, because $\int_K \int_G |w(g)|\,|c(gkb)|\,dg\,dk \le \|w\|\,\|c\|$ (Cauchy–Schwarz and right invariance) and m_μ is bounded on K. So Fubini's theorem and Lemma 5.6.31 (i) with $x = g$ give

$$\langle w, QR_b c\rangle = \|m_\mu\|^{-2} \int_G \overline{w(g)} \int_K \overline{m_\mu(k)}\, c(gkb)\, dk\, dg = c(b)\,\langle w, c\rangle,$$

that is, $QR_b c = c(b)\,c$. By linearity and continuity $Q(V) \subseteq \mathbb{C}c$, and $Qc = c(1)c = c$. Now let A be a bounded operator on V with $AR_x u = R_x A u$ for $x \in G$, $u \in V$, and let A^* be its adjoint on V. For $u, w \in V$,

$$\begin{aligned}\langle w, QAu\rangle &= \frac{1}{\|m_\mu\|^2} \int_K \overline{m_\mu(k)}\,\langle w, AR_k u\rangle\, dk \\ &= \frac{1}{\|m_\mu\|^2} \int_K \overline{m_\mu(k)}\,\langle A^* w, R_k u\rangle\, dk = \langle A^* w, Qu\rangle = \langle w, AQu\rangle,\end{aligned}$$

and QAu, AQu lie in V, so $QA = AQ$ on V. Hence $Ac = AQc = QAc \in \mathbb{C}c$, say $Ac = \alpha c$; then $AR_x c = R_x A c = \alpha R_x c$ for every x, and $A = \alpha$ on V by linearity and continuity. If $V' \subseteq V$ is a closed $R(G)$-stable subspace, then so is its orthogonal complement in V (the R_x are unitary and preserve V), so the orthogonal projection of V onto V' commutes with every R_x and is 0 or 1. As $c \ne 0$, $V \ne 0$; so R restricted to V is an irreducible unitary representation of G.

Step 6: $m(\nu)$ is a nonnegative integer. Let τ_ν be the irreducible representation of K with character χ_ν^K, $d_\nu = \chi_\nu^K(1)$ its dimension, and V_ν the closed span of the K-subrepresentations of V isomorphic to τ_ν. By Theorem A.5.17 (1) (with $j = \varepsilon$) and Step 5, V_ν is finite-dimensional. A finite-dimensional space spanned by irreducible K-stable subspaces, all isomorphic to τ_ν, is the direct sum of some of them, so $\dim V_\nu = r_\nu d_\nu$ with $r_\nu \in \mathbb{N}$. By the Peter–Weyl theory for the unitary representation $k \mapsto R_k|_V$ of the compact group K ([163], Chapter I), the operator E on V defined by $\langle w, Eu\rangle = d_\nu \int_K \overline{\chi_\nu^K(k)}\,\langle w, R_k u\rangle\, dk$ is the orthogonal projection of V onto V_ν. Let $e_1, \dots, e_D$ be an orthonormal basis of V_ν, $D = r_\nu d_\nu$. For $g \in G$ and $k \in K$, (5.127) with $(x,y) = (g, kg)$ gives $c(g^{-1}kg) = d\,\langle R_g c, R_k R_g c\rangle$, and $R_g c \in V$; hence

$$R_\nu(g) = d \int_K \overline{\chi_\nu^K(k)}\,\langle R_g c, R_k R_g c\rangle\, dk = \frac{d}{d_\nu}\,\langle R_g c, E R_g c\rangle = \frac{d}{d_\nu}\,\|E R_g c\|^2 = \frac{d}{d_\nu} \sum_{i=1}^{D} |\langle R_g c, e_i\rangle|^2.$$

This is a nonnegative continuous function of g, and by (5.128), $\int_G R_\nu\, dg = \frac{d}{d_\nu}\, D\, d^{-2} = r_\nu / d$. So R_ν is integrable and $m(\nu) = d \int_G R_\nu\, dg = r_\nu$ is a nonnegative integer. With Step 2, $m(\mu) = 1$; with Step 3, $m(\nu) = 0$ unless $\nu - \mu \in C_l$. □

Lemma 5.7.8. *Let n be a nonempty finite set, $N = \#n$, $\varepsilon \in \{\pm 1\}^n$, let $l \in \mathbb{Z}^n$ have large gap $10N^5$ for ε, $\lambda = \frac{1}{2} l$, $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$ with its Cayley measure dg, $d_\lambda = (\int_G |c_\lambda|^2 dg)^{-1}$, $K = K_\varepsilon$, $N' = 4N^2$, $\|\cdot\|$ the operator norm, and*

$$D(x) = \prod_{\substack{\varepsilon_a \ne \varepsilon_b \\ l_a < l_b}} (x_a - x_b) \qquad (x \in \mathbb{C}^n),$$

the product over ordered pairs; for $x \in \mathbb{C}^n$ with $|x_a| = 1$ let $t_x = \mathrm{diag}(x)$. Then:

(i) there is a continuous function d on K with $d(yky^{-1}) = d(k)$ and $d(yt_x y^{-1}) = D(x)^{N'+1}$ for all $y, k \in K$ and all x with $|x_a| = 1$;

(ii) there is a dg-integrable function Φ on G with $|d(k)\, c_\lambda(g^{-1}kg)| \le \Phi(g)$ for all $k \in K$, $g \in G$;

(iii) the function $h(k) = d_\lambda \int_G d(k)\, c_\lambda(g^{-1}kg)\, dg$ *is continuous on* K *with* $h(yky^{-1}) = h(k)$, *and for every* x *with* $|x_a| = 1$ *and* $D(x) \neq 0$ *the function* $g \mapsto c_\lambda(g^{-1}t_x g)$ *is* dg*-integrable and*

$$h(yt_xy^{-1}) = D(x)^{N'+1}\, d_\lambda \int_G c_\lambda(g^{-1}t_x g)\, dg \qquad (y \in K).$$

Proof. Let $n_\pm = \{a : \varepsilon_a = \pm 1\}$, $p = \#n_+$, $q = \#n_-$; write matrices in blocks for $n = n_+ \sqcup n_-$, so that $K = \mathrm{U}(n_+) \times \mathrm{U}(n_-)$ consists of the block-diagonal unitary $k = (k_+, k_-)$. Every factor of D is $\pm(x_a - x_b)$ with $a \in n_+$, $b \in n_-$, and every such pair occurs once, so $D = \sigma_D\, D_+$ with $D_+(x) = \prod_{a\in n_+,\, b\in n_-}(x_a - x_b)$ and a fixed sign σ_D. The large gap gives gap $h_0 = 500N^3 + 10N^5$, and $pq(N'+1) \leq N^2(4N^2+1) \leq h_0$.

(i). Put $R(k) = \det(k_+ \otimes 1 - 1 \otimes k_-)$ on $\mathbb{C}^{n_+} \otimes \mathbb{C}^{n_-}$ (for $p = 0$ or $q = 0$, $R = 1$), a polynomial in the entries of k, and $d = (\sigma_D R)^{N'+1}$. For $y \in K$, $(yky^{-1})_+ \otimes 1 - 1 \otimes (yky^{-1})_-$ is the conjugate of $k_+ \otimes 1 - 1 \otimes k_-$ by $y_+ \otimes y_-$, so d is a class function; at $k = t_x$ the operator is diagonal with entries $x_a - x_b$, so $d(t_x) = (\sigma_D D_+(x))^{N'+1} = D(x)^{N'+1}$.

A bound for the noncompact root product. Let $u \in G$, $|x_a| = 1$ and $g' = u^{-1}t_x u$. Then

$$\Big(\frac{\|u\|}{2}\Big)^2 |D(x)| \leq (2\|g'\|)^{pq}. \tag{5.129}$$

If $p = 0$ or $q = 0$, then G is a unitary group, $\|u\| = \|g'\| = 1$ and $D = 1$, and (5.129) is clear; let $p, q \geq 1$ and write $u = \begin{pmatrix} A & B \\ C & E \end{pmatrix}$. From $u^*Ju = J$, $A^*A = 1 + C^*C$ and $E^*E = 1 + B^*B$; from $uJu^* = J$ (as $u^{-1} = Ju^*J$), $EE^* = 1 + CC^*$. So all singular values of A are ≥ 1, $\|A\|^2 = 1 + \|C\|^2 = \|E\|^2$ and $\|B\|^2 = \|E\|^2 - 1$; hence $\|u\|^2 \leq \|AA^* + BB^*\| + \|CC^* + EE^*\| \leq \|A\|^2 + \|B\|^2 + \|C\|^2 + \|E\|^2 \leq 4\|A\|^2$, and $(\|u\|/2)^2 \leq \|A\|^2 \leq |\det A|^2$. Let $T = \prod_{b\in n_-}(t_x - x_b)$, diagonal with $T_{aa} = 0$ for $a \in n_-$ and $T_{aa} = \prod_{b\in n_-}(x_a - x_b)$ for $a \in n_+$, and $M = u^{-1}Tu = \prod_{b\in n_-}(g' - x_b)$. As T vanishes on n_-, the $n_+ \times n_+$ block of M is $(u^{-1})_{++}T_{++}A$, and $(u^{-1})_{++} = (Ju^*J)_{++} = A^*$; so $\Delta_{n_+}(M) = |\det A|^2 D_+(x)$. On the other hand $\|g'\| \geq 1$ (it lies in G; apply the above to g'), so $\|M\| \leq (\|g'\| + 1)^q \leq (2\|g'\|)^q$, and a principal minor of size p is at most $\|M\|^p$ in modulus. This gives (5.129).

(ii). Let $k \in K$, $g \in G$. By the spectral theorem in each block, $k = yt_xy^{-1}$ with $y \in K$ and $|x_a| = 1$; put $u = y^{-1}g \in G$, so $\|u\| = \|g\|$ and $g^{-1}kg = u^{-1}t_x u = g'$. By Lemma 5.7.40 with $c = pq(N'+1) \leq h_0$ and (5.129),

$$\begin{aligned} |d(k)\, c_\lambda(g^{-1}kg)| = |D(x)|^{N'+1}|c_\lambda(g')| &\leq \Big(\frac{(2\|g'\|)^{pq}\, 4}{\|g\|^2}\Big)^{N'+1}\Big(\frac{2}{\|g'\|}\Big)^{pq(N'+1)} \\ &= 4^{pq(N'+1)}\Big(\frac{2}{\|g\|}\Big)^{2N'+2} = \Phi(g). \end{aligned}$$

Φ is integrable by Lemma 5.7.38 (3) with $c = 2N'+2 = 8N^2+2 > 8N^2$ (dg is a Haar measure by Lemma 5.2.41, and $d_\lambda > 0$).

(iii). c_λ is continuous (Lemma 5.6.23 (i)), so $(k, g) \mapsto d(k)c_\lambda(g^{-1}kg)$ is continuous and, by (ii) and dominated convergence, h is continuous. For $y \in K$, the substitution $g \mapsto yg$ (left invariance, Lemma 5.2.41) and $d(yky^{-1}) = d(k)$ give $h(yky^{-1}) = h(k)$. If $D(x) \neq 0$, then $|c_\lambda(g^{-1}t_x g)| \leq \Phi(g)/|D(x)|^{N'+1}$, so the orbital integral converges absolutely, and $h(yt_xy^{-1}) = h(t_x) = D(x)^{N'+1} d_\lambda \int_G c_\lambda(g^{-1}t_x g)\, dg$. □

Lemma 5.7.9. *Let* n *be a nonempty finite set,* $\varepsilon \in \{\pm 1\}^n$, $l \in \mathbb{Z}^n$ *injective,* $K = K_\varepsilon$, *and for* $x \in \mathbb{C}^n$ *with* $|x_a| = 1$ *let* $t_x = \mathrm{diag}(x)$. *Put* $\delta_{\mathrm{c}}(a) = \#\{b : \varepsilon_b = \varepsilon_a,\ l_b < l_a\}$ *and* $V_{\mathrm{c}} = \prod_{\varepsilon_a = \varepsilon_b,\ l_b < l_a}(X_a - X_b)$ *(ordered pairs).*

(a) *(Weyl) For K-dominant ν and every x with $|x_a| = 1$, $V_{\mathrm{c}}(x)\,\chi^K_\nu(t_x) = A^\varepsilon_{\nu+\delta_{\mathrm{c}}}(x)$.*

(b) *(Straightening) For every $e \in \mathbb{Z}^n$ there are $s_e \in \{0, 1, -1\}$ and a K-dominant ν_e with $A^\varepsilon_e = s_e\, A^\varepsilon_{\nu_e+\delta_{\mathrm{c}}}$; and for K-dominant μ and $\gamma \in \mathbb{Z}^n$, the coefficient of X^γ in $A^\varepsilon_{\mu+\delta_{\mathrm{c}}}$ is s_γ if $s_\gamma \neq 0$ and $\nu_\gamma = \mu$, and 0 otherwise.*

(c) *(Brauer's rule) Let $D \in \mathcal{R}$ satisfy $D_{\beta\circ w} = D_\beta$ for $w \in W_K$, and let d be a continuous function on K with $d(yt_xy^{-1}) = D(x)$ for $y \in K$ and $|x_a| = 1$. Then for every continuous f on K and every K-dominant ν,*

$$\int_K \overline{\chi^K_\nu(k)}\, d(k) f(k)\, dk = \sum_\beta D_\beta\, s_{\nu+\delta_{\mathrm{c}}-\beta} \int_K \overline{\chi^K_{\nu_{\nu+\delta_{\mathrm{c}}-\beta}}(k)}\, f(k)\, dk.$$

Proof. Let $n_\pm = \{a : \varepsilon_a = \pm 1\}$, so $K = \mathrm{U}(n_+) \times \mathrm{U}(n_-)$ and $W_K = S(n_+) \times S(n_-)$; every element of K is yt_xy^{-1} with $y \in K$ and $|x_a| = 1$ (spectral theorem in each block). Call x regular if $x_a \neq x_b$ whenever $a \neq b$ and $\varepsilon_a = \varepsilon_b$; the regular points are dense in the torus. For $\mu \in \mathbb{Z}^n$, $A^\varepsilon_\mu = A^+_\mu A^-_\mu$ with $A^\pm_\mu = \sum_{v\in S(n_\pm)} \mathrm{sgn}(v) \prod_{a\in n_\pm} X_a^{\mu(v(a))}$, as sgn is multiplicative on W_K.

(a). The irreducible representation of K of highest weight ν is the outer tensor product of the irreducible representations of $\mathrm{U}(n_\pm)$ of highest weights $\nu|_{n_\pm}$ (for the diagonal tori and the upper triangular Borel subgroups in the numberings by decreasing l), so $\chi^K_\nu(t_x) = \chi^+(t_{x,+})\chi^-(t_{x,-})$. Number n_+ as $a_0, \dots, a_{p-1}$ by decreasing l; then $\delta_{\mathrm{c}}(a_i) = p - 1 - i$ and $\nu(a_0) \geq \cdots \geq \nu(a_{p-1})$. For regular x, Theorem A.5.20 gives $\chi^+(t_{x,+}) = \det(x_{a_k}^{\nu(a_j)+p-1-j})_{j,k} / \det(x_{a_k}^{p-1-j})_{j,k}$; by the Leibniz formula (with $v(a_j) = a_{\pi(j)}$) the numerator is $A^+_{\nu+\delta_{\mathrm{c}}}(x)$, and the denominator is $\prod_{j<k}(x_{a_j} - x_{a_k}) = \prod_{a,b\in n_+,\, l_b<l_a}(x_a - x_b)$. The same holds for n_-, and V_{c} is the product of the two Vandermonde products. So (a) holds at regular x, and at all x by continuity of both sides.

(b). If $e(a) = e(b)$ for some $a \neq b$ with $\varepsilon_a = \varepsilon_b$, the transposition τ of a, b lies in W_K, is odd and fixes e, so $A^\varepsilon_e = \sum_w \mathrm{sgn}(\tau w) X^{e\circ\tau w} = -A^\varepsilon_e$; take $s_e = 0$ and any ν_e. Otherwise there is a unique $w_0 \in W_K$ such that $e' = e \circ w_0$ satisfies $e'(i) > e'(j)$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$. Put $\nu_e = e' - \delta_{\mathrm{c}}$ and $s_e = \mathrm{sgn}(w_0)$. If $\varepsilon_i = \varepsilon_j$, $l_i > l_j$ and no c of the same sign has $l_j < l_c < l_i$, then $\delta_{\mathrm{c}}(i) = \delta_{\mathrm{c}}(j) + 1$ and $e'(i) \geq e'(j) + 1$, so $\nu_e(i) \geq \nu_e(j)$; by transitivity ν_e is K-dominant. Substituting $w = w_0w'$, $A^\varepsilon_e = \sum_{w'} \mathrm{sgn}(w_0w') X^{e'\circ w'} = s_e A^\varepsilon_{\nu_e+\delta_{\mathrm{c}}}$. For the second claim: $\mu + \delta_{\mathrm{c}}$ is strictly decreasing along decreasing l in each block, so the monomials $X^{(\mu+\delta_{\mathrm{c}})\circ w}$, $w \in W_K$, are distinct, and X^γ occurs iff $\gamma = (\mu + \delta_{\mathrm{c}}) \circ w$ for some w, with coefficient $\mathrm{sgn}(w)$; this holds iff γ has distinct entries in each block, $w_0 = w^{-1}$ for γ, i.e. $\nu_\gamma = \mu$, and then $\mathrm{sgn}(w) = s_\gamma$.

(c). For $|x_a| = 1$, $\overline{D(x)} = \sum_\beta \overline{D_\beta} x^{-\beta}$. By the invariance of the coefficients of D, reindexing $\beta \mapsto \beta \circ w$ in each term,

$$\overline{D(x)}\, A^\varepsilon_{\nu+\delta_{\mathrm{c}}}(x) = \sum_w \mathrm{sgn}(w) \sum_\beta \overline{D_\beta}\, x^{(\nu+\delta_{\mathrm{c}}-\beta)\circ w} = \sum_\beta \overline{D_\beta}\, A^\varepsilon_{\nu+\delta_{\mathrm{c}}-\beta}(x) = \sum_\beta \overline{D_\beta}\, s_e\, A^\varepsilon_{\nu_e+\delta_{\mathrm{c}}}(x),$$

with $e = \nu + \delta_{\mathrm{c}} - \beta$, by (b). By (a), at regular x (where $V_{\mathrm{c}}(x) \neq 0$) this says $\overline{d}\,\chi^K_\nu = \sum_\beta \overline{D_\beta} s_e \chi^K_{\nu_e}$ at t_x; by continuity at every t_x, and since both sides are class functions on K, on all of K. Multiplying the complex conjugate of this identity by f and integrating gives (c) (s_e is real). □

Proposition 5.7.10. *Let n be a nonempty finite set, $N = \#n$, $\varepsilon \in \{\pm 1\}^n$, and let $l \in \mathbb{Z}^n$ have large gap $10N^5$ for ε; put $\lambda = \frac{1}{2}l$, $J = \mathrm{diag}(\varepsilon)$, let dg be the Cayley measure of $\mathrm{U}(\mathrm{diag}\,\varepsilon)$, and for $x \in \mathbb{C}^n$ with $|x_a| = 1$ let $t_x = \mathrm{diag}(x)$. Let Λ_{c} be the lowest K-type of l for the signs $(1, \dots, 1)$, $\|m_{\Lambda_{\mathrm{c}}}\|^2 = \int_{\mathrm{U}(n)} |m_{\Lambda_{\mathrm{c}}}(k)|^2 dk$ (dk the Haar probability measure), and χ the character of the irreducible representation of $\mathrm{U}(n)$ whose highest weight, in the numbering of n by decreasing*

l, is Λ_{c}. Let $\mathcal{R}$, the coefficients P_γ, exponents, the level $o\cdot\gamma$, κ and $h = 500N^3 + 10N^5$ be as in Lemma 5.7.18. There are $\sigma \in \mathbb{C}$ with $|\sigma| = 1$ and a function Off *on the accretive $n\times n$ matrices such that:*

(a) for every x with $|x_a| = 1$, pairwise distinct entries and Jt_x accretive,

$$\sigma \int_{\mathrm{U}(\operatorname{diag}\varepsilon)} c_\lambda(g^{-1}t_x g)\,dg = \|m_{\Lambda_{\mathrm{c}}}\|^2\,\chi(t_x) - \mathrm{Off}(Jt_x);$$

(b) if $P, Q \in \mathcal{R}$ satisfy $P(x) = Q(x)\,\mathrm{Off}(Jt_x)$ for every such x, and $\gamma \in \mathbb{Z}^n$ satisfies $o\cdot\gamma < o\cdot(\kappa - o) + o\cdot\beta + h$ for every exponent β of Q, then $P_\gamma = 0$.

Proof. Write $G = \mathrm{U}(\operatorname{diag}\varepsilon)$, $K = K_\varepsilon$, $\mu = \mu_l$ (for the signs ε), $\Lambda = \Lambda_{\mathrm{c}}$, $m = m_\Lambda$, $\|m\|^2 = \|m_\Lambda\|^2$, and let dk, du, dv, dy denote the Haar probability measure of $\mathrm{U}(n)$. The vector l has gap $h = 500N^3 + 10N^5$, in particular gap 0, so its entries are distinct and $\equiv N-1 \pmod 2$. For a diagonal matrix $\operatorname{diag}(d)$, $m_w(\operatorname{diag}(d)) = \prod_a d_a^{w_a}$ ((5.134) of Lemma 5.7.13). By Lemma 5.6.27 (iii), l^ρ is strictly increasing along l, $\mu(i) \ge \mu(j)$ whenever $\varepsilon_i = \varepsilon_j$ and $l_i > l_j$, and $\Lambda(i) \ge \Lambda(j)$ whenever $l_i > l_j$; and $\Lambda = l^\rho - 2\rho_l$ ($2\rho_l^{\mathrm{c}} = 2\rho_l$ for constant signs). Put

$$\sigma = m_{-\Lambda-\mu}(J) = \prod_a \varepsilon_a^{-\Lambda(a)-\mu(a)}, \qquad \text{Off the off-orbit integral of Lemma 5.7.13 for } (\varepsilon, l);$$

$|\sigma| = 1$, and neither depends on x.

Step 1: the density. Fix x with $|x_a| = 1$, distinct entries and $A = Jt_x$ accretive. For $y \in \mathrm{GL}_n(\mathbb{C})$, y^*Ay is accretive ($y^*(A + A^*)y$ is positive definite), so its principal minors are nonzero (Lemma 5.6.25 (ii)), and y^*Jy is invertible; so

$$F(y) = \frac{m_{-\Lambda}(y^*Jy)}{m_{-l^\rho}(y^*Ay)}$$

is continuous on $\mathrm{GL}_n(\mathbb{C})$. The weights $-\Lambda$ and $-l^\rho$ are nonincreasing along l, so Lemma 5.6.27 (ii), for the signs $(1,\dots,1)$ and the vector $-l$ (whose triangular group is B_l'), gives $m_w(b^*Zb) = |b^w|^2 m_w(Z)$ for $b \in \mathrm{B}_l'$ and $w \in \{-\Lambda, -l^\rho\}$. With $(yb)^*M(yb) = b^*(y^*My)b$,

$$F(yb) = \left|b^{l^\rho-\Lambda}\right|^2 F(y) = \left|b^{2\rho_l}\right|^2 F(y) \qquad (b \in \mathrm{B}_l').$$

By Lemma 5.7.11, F is integrable on G and, by the definition of Off,

$$\int_G F(g)\,dg = \int_{\mathrm{U}(n)\cap\Omega_{\varepsilon,l}} F(k)\,dk = \int_{\mathrm{U}(n)} F(k)\,dk - \mathrm{Off}(Jt_x). \tag{5.130}$$

Step 2: the orbital integral. Let $g \in G$. Then $g^{-1}t_x g \in G$, and since $g^{-1} = Jg^*J$, $J^{-1}(g^{-1}t_x g) = g^*Ag$ is accretive; J is block-diagonal. By Lemma 5.6.26 (i) and (3.2),

$$c_\lambda(g^{-1}t_x g) = \mathrm{FJ}_l(J, g^*Ag) = \frac{1}{\|m_\mu\|^2}\int_K \frac{m_\mu(Jk)\,\overline{m_\mu(k)}}{m_{-l^\rho}(k^*g^*Agk)}\,dk.$$

By Lemma 5.6.27 (ii) (with $c = J$, $b = 1$), $m_\mu(Jk) = J^\mu m_\mu(k)$, $J^\mu = \prod_a \varepsilon_a^{\mu(a)}$. For $k \in K$, $k^*g^*Jgk = k^*Jk = J$, so $F(gk) = m_{-\Lambda}(J)/m_{-l^\rho}((gk)^*A(gk))$ with $m_{-\Lambda}(J) = \prod_a \varepsilon_a^{\Lambda(a)}$. Hence, with $w(k) = |m_\mu(k)|^2/\|m_\mu\|^2$, a continuous function of integral 1 on K,

$$\sigma\, c_\lambda(g^{-1}t_x g) = \int_K w(k)\,F(gk)\,dk.$$

By Tonelli and the right invariance of dg under $K \subseteq G$ (Lemma 5.2.41), $\int_G \int_K w(k)|F(gk)|\,dk\,dg = \int_G |F|\,dg < \infty$; so $g \mapsto c_\lambda(g^{-1}t_x g)$ (continuous, as a parameter integral over K) is integrable on G, and by Fubini

$$\sigma \int_G c_\lambda(g^{-1}t_x g)\,dg = \int_K w(k) \int_G F(gk)\,dg\,dk = \int_G F(g)\,dg. \tag{5.131}$$

Step 3: the integral over the unitary group. Number n by decreasing l; this numbering has Λ nonincreasing, so Lemma 5.7.16 applies to $\nu = \Lambda$, with $\chi_\Lambda = \chi$. Let ρ, e be as in its (i) and $d = d_\Lambda$. For the signs $(1, \dots, 1)$, $K_{(1,\dots,1)} = \mathrm{U}(n)$, every matrix is block-diagonal, the lowest K-type of l is Λ, and Flensted-Jensen's function c^{c} of λ for these signs satisfies, for $h' \in \mathrm{U}(n)$ (where $\mathrm{bd}(h') = h'$ and $m_{-l^\rho}(k^*k) = 1$),

$$c^{\mathrm{c}}(h') = \frac{1}{\|m\|^2} \int_{\mathrm{U}(n)} m(h'k)\overline{m(k)}\,dk = d \int \langle \rho(h')^* e, \rho(k)e\rangle \overline{\langle e, \rho(k)e\rangle}\,dk = \langle \rho(h')^* e, e\rangle = m(h'),$$

by Lemma 5.7.16 (i), (ii) and Schur's orthogonality relations. Let $y \in \mathrm{U}(n)$, $a_y = y^{-1}Jy$ (unitary) and $z_y = y^{-1}Ay$ (accretive); then $a_y z_y = y^{-1}t_x y \in \mathrm{U}(n)$ and $a_y^{-1}(a_y z_y) = z_y$, so by Lemma 5.6.26 (i) for the signs $(1, \dots, 1)$,

$$m(y^{-1}t_x y) = c^{\mathrm{c}}(a_y z_y) = \mathrm{FJ}_l^{\mathrm{c}}(a_y, z_y) = \frac{1}{\|m\|^2} \int_{\mathrm{U}(n)} \frac{m(a_y k)\,\overline{m(k)}}{m_{-l^\rho}(k^* z_y k)}\,dk,$$

$\mathrm{FJ}_l^{\mathrm{c}}$ being Flensted-Jensen's integral for these signs. The integrand is continuous on $\mathrm{U}(n) \times \mathrm{U}(n)$ in (y, k). By Lemma 5.7.16 (iv) and the substitution $(y, k) = (vu, u^{-1})$, which preserves $dy\,dk$ (right translation in the first factor, then inversion in the second), under which $a_y k = u^{-1}a_v$, $\overline{m(u^{-1})} = m(u)$ (Lemma 5.6.27 (i)) and $k^* z_y k = v^* A v$,

$$\|m\|^2 \chi(t_x) = \int m(y^{-1}t_x y)\,dy = \frac{1}{\|m\|^2} \int_{\mathrm{U}(n)} \frac{\int_{\mathrm{U}(n)} m(u^{-1}a_v)\,m(u)\,du}{m_{-l^\rho}(v^*Av)}\,dv = \int_{\mathrm{U}(n)} \frac{m(a_v)}{m_{-l^\rho}(v^*Av)}\,dv,$$

by Lemma 5.7.16 (iii). Finally $m(a_v) = m_{-\Lambda}(a_v^{-1}) = m_{-\Lambda}(v^*Jv)$ by Lemma 5.6.27 (i) and $J^{-1} = J$; so $\|m\|^2 \chi(t_x) = \int_{\mathrm{U}(n)} F\,dv$.

Step 4: (a). By (5.131), (5.130) and Step 3, $\sigma \int_G c_\lambda(g^{-1}t_x g)\,dg = \|m_{\Lambda_{\mathrm{c}}}\|^2 \chi(t_x) - \mathrm{Off}(Jt_x)$.

Step 5: (b). The vector $o(a) = \#\{b : l_b > l_a\}$ satisfies $o(a) < o(b)$ whenever $l_b < l_a$, and $2\rho_l(a) = N - 1 - 2o(a)$, so $l^\rho = \frac{1}{2}(l + 2\rho_l) = \kappa - o$. Hence the hypothesis of (b) reads $o \cdot \gamma < o \cdot l^\rho + o \cdot \beta + h$ for every exponent β of Q, and l has gap h; Lemma 5.7.15 gives $P_\gamma = 0$. □

Lemma 5.7.11. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$ with its Cayley measure dg, $\mathrm{U}(n)$ the group of unitary $n \times n$ matrices with its Haar probability measure dk, and $l \in \mathbb{Z}^n$ injective. For $b \in \mathrm{B}_l'$ and $w \in \mathbb{Z}^n$ put $b^w = \prod_i b_{ii}^{w_i}$. Let $F\colon \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$ be continuous with*

$$F(yb) = \left|b^{2\rho_l}\right|^2 F(y) \qquad (y \in \mathrm{GL}_n(\mathbb{C}),\ b \in \mathrm{B}_l').$$

Then F restricted to G is dg-integrable, and

$$\int_G F(g)\,dg = \int_{\mathrm{U}(n) \cap \Omega_{\varepsilon,l}} F(k)\,dk.$$

Proof. Write $\Omega = \Omega_{\varepsilon,l}$, $N = \#n$, $J = \operatorname{diag}(\varepsilon)$, and let dt be the Haar probability measure of the compact group T. Call a function Φ on $\mathrm{GL}_n(\mathbb{C})$ a density if $\Phi(yb) = |b^{2\rho_l}|^2\Phi(y)$ for all y and $b \in \mathrm{B}'_l$. By Lemma 5.6.28, applied to the signs $(1,\dots,1)$ and the vector $-l$ (then the group $K_{\mathbb{C}}$ there is $\mathrm{GL}_n(\mathbb{C})$, K_ε is $\mathrm{U}(n)$, its triangular group is B'_l, and $-2\rho^{\mathrm{c}}_{-l} = 2\rho_l$),

$$\int_{\mathrm{U}(n)} \Phi(xk)\,dk = \int_{\mathrm{U}(n)} \Phi(k)\,dk \qquad (x \in \mathrm{GL}_n(\mathbb{C})) \tag{5.132}$$

for every continuous density Φ.

Step 1: an invariant functional. For $f \in C_c(G)$ let $F_f(y) = |\beta(y)^{2\rho_l}|^2 \int_{\mathrm{T}} f(\gamma(y)t)\,dt$ for $y \in \Omega$ and $F_f(y) = 0$ for $y \in \mathrm{GL}_n(\mathbb{C}) \smallsetminus \Omega$. It is a density: for $y \in \Omega$ and $b \in \mathrm{B}'_l$, Lemma 5.7.12 (i) gives $\gamma(yb) = \gamma(y)t_b$ and $|\beta(yb)^{2\rho_l}| = |\beta(y)^{2\rho_l}|\,|b^{2\rho_l}|$, and dt is invariant under t_b; if $y \notin \Omega$ then $yb \notin \Omega$. It is continuous: $C = \operatorname{supp}(f)\,\mathrm{T}$ is compact, $C\mathrm{B}'_l$ is closed in $\mathrm{GL}_n(\mathbb{C})$ (a product of a compact set and a closed subgroup) and contained in Ω, F_f vanishes outside it, and F_f is continuous on the open set Ω by Lemma 5.7.12 (ii); the open sets Ω and $\mathrm{GL}_n(\mathbb{C}) \smallsetminus C\mathrm{B}'_l$ cover $\mathrm{GL}_n(\mathbb{C})$. Moreover, for $g_0 \in G$, $\gamma(g_0^{-1}y) = g_0^{-1}\gamma(y)$ and $\beta(g_0^{-1}y) = \beta(y)$, so the function $f(g_0^{-1}\cdot)$ gives the density $F_f(g_0^{-1}\cdot)$. Put

$$\Lambda(f) = \int_{\mathrm{U}(n)} F_f(k)\,dk.$$

Λ is a positive linear functional on $C_c(G)$, and left invariant by (5.132) with $x = g_0^{-1}$. By Lemma 5.2.41 the Cayley measure is a Haar measure of the locally compact group G, so by the Riesz representation theorem and the uniqueness of Haar measure there is $c \geq 0$ with $\Lambda(f) = c\int_G f\,dg$ for all $f \in C_c(G)$. For $t \in \mathrm{T}$, right translation by t preserves $\mathrm{U}(n) \cap \Omega$ and dk, and $\gamma(kt) = \gamma(k)t$, $|\beta(kt)^{2\rho_l}| = |\beta(k)^{2\rho_l}|$; so by Fubini on $\mathrm{U}(n) \times \mathrm{T}$,

$$\Lambda(f) = \int_{\mathrm{U}(n)\cap\Omega} \bigl|\beta(k)^{2\rho_l}\bigr|^2 f(\gamma(k))\,dk. \tag{5.133}$$

Step 2: the unitary factor. For $g \in G$ let $g = \phi(g)b$ with $\phi(g)$ unitary and $b \in \mathrm{B}^+_l$ (Lemma 5.7.12 (iii)). Then $\phi(g) = gb^{-1} \in \mathrm{U}(n) \cap \Omega$ and $\gamma(\phi(g)) = g$; for $k \in \mathrm{U}(n) \cap \Omega$, $\gamma(k) = k\beta(k)^{-1}$, so $\phi(\gamma(k)) = k$. By Lemma 5.7.12 (ii) (also for the signs $(1,\dots,1)$), ϕ is a diffeomorphism of G onto the open subset $\mathrm{U}(n) \cap \Omega$ of $\mathrm{U}(n)$, with inverse γ. Let $\mathfrak{t}_l$ be the real space of the Y with $Y_{ij} = 0$ for $l_i > l_j$ and real diagonal. Differentiating $g = \phi(g)\,b(g)$ at $g = 1$ in the direction $X \in \mathfrak{u}(J) = \{X : X^*J + JX = 0\}$ gives $X = d\phi_1(X) + Z$ with $d\phi_1(X)$ skew-hermitian and $Z \in \mathfrak{t}_l$; a skew-hermitian matrix in $\mathfrak{t}_l$ is 0 (its entries below the pattern vanish, hence all off-diagonal entries, and its diagonal is imaginary and real), so this decomposition is unique: $d\phi_1(X) = K$ with $K_{ij} = X_{ij}$ for $l_i > l_j$ and $K_{ii} = X_{ii}$ (the diagonal of $X \in \mathfrak{u}(J)$ is imaginary).

Step 3: the differential in Cayley coordinates. Let $T_J(y) = JS(y)$ and $T_1(y) = S(y)$ for $y \in \mathrm{M}_n(\mathbb{R})$, the coordinates of Definition 3.2.1, and $L = T_1^{-1} \circ d\phi_1 \circ T_J$. For $X = JS(y)$ and $K = d\phi_1(X) = S(y')$: $\mathrm{i}y'_{ii} = K_{ii} = \varepsilon_i \mathrm{i} y_{ii}$, and for $l_i > l_j$, $S(y')_{ij} = K_{ij} = \varepsilon_i S(y)_{ij}$. Since $(y_{ij}, y_{ji}) \mapsto S(y)_{ij}$ is a real-linear bijection $\mathbb{R}^2 \to \mathbb{C}$, the map $(y_{ij}, y_{ji}) \mapsto (y'_{ij}, y'_{ji})$ is conjugate to multiplication by $\varepsilon_i \in \{\pm 1\}$ on $\mathbb{C} \cong \mathbb{R}^2$; so L is block diagonal with blocks of determinant ± 1, and $|\det L| = 1$.

Step 4: $c = 1$. Write $\operatorname{cay}(Y) = (1 + \frac{1}{2}Y)(1 - \frac{1}{2}Y)^{-1}$; its differential at 0 is the identity, and by (5.20) of Lemma 5.2.41, in the coordinates y,

$$\int_G \psi\,dg = \frac{1}{\operatorname{vol}\mathrm{U}(N)} \int \psi(\operatorname{cay}(T_J y))\,m_J(y)\,dy, \qquad m_J(y) = |\det(1 - \tfrac{1}{2}T_J(y))|^{-2N},$$

for Borel $\psi \ge 0$, the integral over the y with $\det(1 - \frac{1}{2}T_J(y)) \neq 0$; the same holds for the Cayley measure of $\mathrm{U}(n)$ with T_1, m_1, and that measure is dk, being a Haar measure (Lemma 5.2.41) of total mass 1 (Lemma 5.5.21). By the inverse function theorem there is an open neighbourhood W of 0 in $\mathrm{M}_n(\mathbb{R})$ on which $\Psi(y) = T_1^{-1}(\mathrm{cay}^{-1}(\phi(\mathrm{cay}(T_J y))))$ is defined, is a diffeomorphism onto an open set, and $y \mapsto \mathrm{cay}(T_J y)$ is a diffeomorphism onto an open neighbourhood U_0 of 1 in G; here $\Psi(0) = 0$ and $d\Psi_0 = L$ by the chain rule. Let $f \in C_c(G)$ be supported in U_0. In (5.133) the integrand vanishes unless $\gamma(k) \in U_0$, i.e. $k = \phi(\gamma(k)) \in \phi(U_0)$; substituting $k = \mathrm{cay}(T_1\Psi(y)) = \phi(\mathrm{cay}(T_J y))$, $\gamma(k) = \mathrm{cay}(T_J y)$,

$$\Lambda(f) = \frac{1}{\mathrm{vol}\,\mathrm{U}(N)} \int_W f(\mathrm{cay}(T_J y))\, h(y)\, dy, \qquad h(y) = \left|\beta(\phi(\mathrm{cay}(T_J y)))^{2\rho_l}\right|^2 m_1(\Psi(y))\, |\det d\Psi_y|,$$

while $c\int_G f\, dg = \frac{c}{\mathrm{vol}\,\mathrm{U}(N)} \int_W f(\mathrm{cay}(T_J y))\, m_J(y)\, dy$. As $f(\mathrm{cay}(T_J \cdot))$ runs over $C_c(W)$ and h, m_J are continuous on W, $h = c\, m_J$ on W. At $y = 0$: $\beta(\phi(1)) = \beta(1) = 1$, $m_1(0) = m_J(0) = 1$ and $|\det d\Psi_0| = |\det L| = 1$; so $c = 1$.

Step 5: conclusion. First let $F \ge 0$, and $f = F|_G$, continuous. For $y \in \Omega$, $F(y) = F(\gamma(y)\beta(y)) = |\beta(y)^{2\rho_l}|^2 f(\gamma(y))$. For $j \ge 1$ let $\chi_j(g) = \min\bigl(1, \max(0, j + 1 - \|g\|)\bigr)$ ($\|\cdot\|$ the operator norm); then $f_j = \chi_j f \in C_c(G)$ and $0 \le f_j \uparrow f$ pointwise. By Steps 1 and 4, (5.133) and monotone convergence on both sides,

$$\int_G f\, dg = \lim_j \int_G f_j\, dg = \lim_j \int_{\mathrm{U}(n)\cap\Omega} \left|\beta(k)^{2\rho_l}\right|^2 f_j(\gamma(k))\, dk = \int_{\mathrm{U}(n)\cap\Omega} F(k)\, dk \le \int_{\mathrm{U}(n)} F\, dk < \infty.$$

For general F, the four functions $\max(\pm\mathrm{Re}\, F, 0)$, $\max(\pm\mathrm{Im}\, F, 0)$ are continuous nonnegative densities (the factor $|b^{2\rho_l}|^2$ is positive); each is integrable on G with the asserted identity, and F is their combination with coefficients $1, -1, \mathrm{i}, -\mathrm{i}$. □

Lemma 5.7.12. *Let n be a nonempty finite set, $N = \#n$, $\varepsilon \in \{\pm 1\}^n$, $J = \mathrm{diag}(\varepsilon)$, $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$, and let $l \in \mathbb{Z}^n$ be injective. Let B_l' be the group of invertible complex $n \times n$ matrices b with $b_{ij} = 0$ whenever $l_i > l_j$, B_l^+ its subgroup of the b with positive real diagonal entries, T the group of diagonal matrices with entries of modulus 1, and*

$$\Omega_{\varepsilon,l} = \{gb :\ g \in G,\ b \in \mathrm{B}_l'\}.$$

(i) $G \cap \mathrm{B}_l' = \mathrm{T}$. *Every $y \in \Omega_{\varepsilon,l}$ is $y = \gamma(y)\beta(y)$ for exactly one pair $\gamma(y) \in G$, $\beta(y) \in \mathrm{B}_l^+$. For $y \in \Omega_{\varepsilon,l}$, $g \in G$ and $b \in \mathrm{B}_l'$: $gyb \in \Omega_{\varepsilon,l}$, $\gamma(gyb) = g\,\gamma(y)\,t_b$ with $t_b = \mathrm{diag}(b_{ii}/|b_{ii}|)$, and $\beta(gyb)_{ii} = \beta(y)_{ii}\,|b_{ii}|$ for every i.*

(ii) *$\Omega_{\varepsilon,l}$ is open in $\mathrm{M}_n(\mathbb{C})$, and $\gamma\colon \Omega_{\varepsilon,l} \to G$ and $\beta\colon \Omega_{\varepsilon,l} \to \mathrm{B}_l^+$ are smooth.*

(iii) *For $\varepsilon = (1, \dots, 1)$, $\Omega_{\varepsilon,l} = \mathrm{GL}_n(\mathbb{C})$: every invertible y is $y = kb$ with k unitary and $b \in \mathrm{B}_l^+$, both unique.*

(iv) *$K_\varepsilon \subseteq G \subseteq \Omega_{\varepsilon,l}$.*

Proof. Write $\Omega = \Omega_{\varepsilon,l}$. For $X \in \mathrm{M}_n(\mathbb{C})$, $X^*J + JX = 0$ means $X_{ij} = -\varepsilon_i\varepsilon_j\overline{X_{ji}}$ for all i, j; these X form the Lie algebra $\mathfrak{u}(J)$ of the closed subgroup G of $\mathrm{GL}_n(\mathbb{C})$, of real dimension N^2 ($X \mapsto JX$ is a real-linear bijection onto the skew-hermitian matrices). Let $\mathfrak{t}_l$ be the real vector space of the Y with $Y_{ij} = 0$ whenever $l_i > l_j$ and $Y_{ii} \in \mathbb{R}$; it has real dimension $N(N-1) + N = N^2$, and B_l^+ is an open subset of it. The diagonal of a product of elements of B_l' is the product of their diagonals, so B_l^+ is a group.

(i). Let $t \in G \cap \mathrm{B}_l'$. Then $t^{-1} \in \mathrm{B}_l'$, so $(t^{-1})_{ij} \neq 0$ implies $l_i \le l_j$; and $t^{-1} = Jt^*J$, so $(t^{-1})_{ij} = \varepsilon_i\varepsilon_j\overline{t_{ji}} \neq 0$ implies $l_j \le l_i$. As l is injective, t^{-1} and hence t is diagonal, and $t^*Jt = J$ gives $|t_{ii}| = 1$. Conversely $\mathrm{T} \subseteq G \cap \mathrm{B}_l'$. Let $y = gb \in \Omega$; with t_b as in the statement, $t_b \in \mathrm{T} \subseteq G$ and $b_+ = t_b^{-1}b \in \mathrm{B}_l^+$, so $y = (gt_b)b_+$ is a factorisation. If $g_1b_1 = g_2b_2$ with $g_i \in G$, $b_i \in \mathrm{B}_l^+$, then $g_2^{-1}g_1 = b_2b_1^{-1} \in G \cap \mathrm{B}_l^+$, a diagonal matrix with entries of modulus 1 and positive, so $g_1 = g_2$ and $b_1 = b_2$. Finally, for $y \in \Omega$, $g \in G$, $b \in \mathrm{B}_l'$ write $b = t_bb_+$ as above; then

$$gyb = \big(g\gamma(y)t_b\big)\big(t_b^{-1}\beta(y)t_b\, b_+\big),$$

where $g\gamma(y)t_b \in G$ and $t_b^{-1}\beta(y)t_bb_+ \in \mathrm{B}_l^+$ (conjugation by a diagonal matrix keeps the pattern of zeros and the diagonal); by uniqueness $\gamma(gyb) = g\gamma(y)t_b$, and the diagonal entries of $\beta(gyb)$ are $\beta(y)_{ii}|b_{ii}|$.

(ii). Let $\mu\colon G \times \mathrm{B}_l^+ \to \mathrm{M}_n(\mathbb{C})$, $\mu(g,b) = gb$, a smooth map between manifolds of real dimension $2N^2$, whose image is Ω by (i). The tangent spaces at (g,b) are $g\,\mathfrak{u}(J)$ and $\mathfrak{t}_l$, and

$$d\mu_{(g,b)}(gX,Y) = gXb + gY = g\big(X + Yb^{-1}\big)b \qquad (X \in \mathfrak{u}(J),\ Y \in \mathfrak{t}_l).$$

The map $Y \mapsto Yb^{-1}$ is a linear bijection of $\mathfrak{t}_l$ (the product has the zero pattern of B_l' and real diagonal entries Y_{ii}/b_{ii}; its inverse is $Y' \mapsto Y'b$). So $d\mu_{(g,b)}$ is bijective as soon as $\mathfrak{u}(J) \cap \mathfrak{t}_l = 0$, by dimensions. Let $X \in \mathfrak{u}(J) \cap \mathfrak{t}_l$: for $l_i > l_j$, $X_{ij} = 0$ and hence $X_{ji} = -\varepsilon_i\varepsilon_j\overline{X_{ij}} = 0$, so X is diagonal; $X_{ii} = -\overline{X_{ii}}$ is imaginary and real, so $X = 0$. By the inverse function theorem μ is a local diffeomorphism; it is injective by (i). Hence $\Omega = \mu(G \times \mathrm{B}_l^+)$ is open, μ is a diffeomorphism onto Ω, and $(\gamma,\beta) = \mu^{-1}$ is smooth.

(iii). Let $\varepsilon = (1,\dots,1)$, so G is the unitary group. List n as $c_1,\dots,c_N$ with $l(c_1) < \cdots < l(c_N)$; then $b \in \mathrm{B}_l'$ if and only if $b_{c_ic_j} = 0$ for $i > j$. For invertible y, the Gram–Schmidt process applied to the columns $y_{\bullet c_1},\dots,y_{\bullet c_N}$ gives orthonormal vectors $k_{\bullet c_1},\dots,k_{\bullet c_N}$ and numbers $b_{c_ic_j}$ $(i \le j)$, $b_{c_jc_j} > 0$, with $y_{\bullet c_j} = \sum_{i\le j} k_{\bullet c_i}b_{c_ic_j}$; that is $y = kb$ with k unitary and $b \in \mathrm{B}_l^+$. Uniqueness is (i).

(iv). $K_\varepsilon \subseteq G$ by definition, and $g = g \cdot 1$ with $1 \in \mathrm{B}_l'$. □

Lemma 5.7.13. *Let n be a nonempty finite set, $\varepsilon \in \{\pm1\}^n$, $J = \mathrm{diag}(\varepsilon)$, $h \in \mathbb{N}$, and let $l \in \mathbb{Z}^n$ have gap h for ε. Let Λ_{c} be the lowest K-type of l for the signs $(1,\dots,1)$, $\mathrm{U}(n)$ the group of unitary $n \times n$ matrices with its Haar probability measure dk, and for an accretive $n \times n$ matrix A put*

$$\mathrm{Off}(A) = \int_{\mathrm{U}(n)\smallsetminus\Omega_{\varepsilon,l}} \frac{m_{-\Lambda_{\mathrm{c}}}(k^*Jk)}{m_{-l^\rho}(k^*Ak)}\,dk.$$

(i) The integrand is continuous on $\mathrm{U}(n)$, and Off is a holomorphic function of the entries of A on the open set of accretive matrices; in particular it is continuous there.

(ii) Let $o \in \mathbb{Z}^n$ satisfy $o(a) < o(b)$ whenever $l_b < l_a$, and let $e \in \mathbb{Z}^n$. There are $C \ge 0$ and $s_0 \in \mathbb{N}$ such that for every integer $s \ge s_0$,

$$\Big|\mathrm{Off}\big(\mathrm{diag}(2^{-s\,o(a)-e(a)})_{a\in n}\big)\Big| \le C\,2^{-s\,(o\cdot l^\rho+h)}, \qquad o\cdot l^\rho = \sum_a o(a)\,l_a^\rho.$$

Proof. Write $\Omega = \Omega_{\varepsilon,l}$ and $\mu = l^\rho$. For a diagonal matrix $\mathrm{diag}(d)$ with nonzero entries, $\Delta_P(\mathrm{diag}(d)) = \prod_{a\in P} d_a$, so by (3.1)

$$m_w(\mathrm{diag}(d)) = \prod_a d_a^{w_- + \#\{t\colon w_- < t \le w_a\}} = \prod_a d_a^{w_a} \qquad (w \in \mathbb{Z}^n,\ w_- = \min w). \tag{5.134}$$

By Lemma 5.6.27 (iii) (the entries of l are distinct and $\equiv \#n - 1 \pmod 2$), $2\mu = l + 2\rho_l$ and $2\rho_l$ is strictly increasing along l; so μ is strictly increasing along l, and $|\mu_a - \mu_b| \geq \frac{1}{2}|l_a - l_b|$.

(i). For k unitary, k^*Jk is invertible and k^*Ak is accretive when A is, so all principal minors of k^*Ak are nonzero (Lemma 5.6.25 (ii)). By (3.1), $m_{-\mu}(Z)$ is a power of $\det Z$ times a product of principal minors of Z; hence $(A,k) \mapsto m_{-\Lambda_c}(k^*Jk)/m_{-\mu}(k^*Ak)$ is continuous on (accretive matrices)$\times$U(n) and, for fixed k, a rational function of the entries of A without poles on the accretive matrices. Ω is open (Lemma 5.7.12 (ii)), so $\mathrm{U}(n) \setminus \Omega$ is compact. A parameter integral over a compact set of a jointly continuous function that is holomorphic in each parameter is continuous and, by Morera's theorem in each variable with Fubini, holomorphic in each variable separately, hence holomorphic (Osgood's lemma).

A large entry outside the orbit. There is $r > 0$ such that every $k \in \mathrm{U}(n) \setminus \Omega$ has indices i, j with $\varepsilon_i \neq \varepsilon_j$ and

$$r\,|m_{-\Lambda_c}(k^*Jk)| < |k_{ij}|^{2h}. \tag{5.135}$$

Indeed, let $k \in \mathrm{U}(n) \setminus \Omega$. If $k_{ij} = 0$ whenever $\varepsilon_i \neq \varepsilon_j$, then k commutes with J, so $k^*Jk = J$ and $k \in K_\varepsilon \subseteq \Omega$ (Lemma 5.7.12 (iv)); so some $k_{ij} \neq 0$ with $\varepsilon_i \neq \varepsilon_j$. The function $(r', k') \mapsto |k'_{ij}|^{2h} - r'|m_{-\Lambda_c}(k'^*Jk')|$ is continuous and positive at $(0, k)$, hence positive on $[0, r_k) \times V_k$ for some $r_k > 0$ and some open neighbourhood V_k of k. Finitely many V_k cover the compact set $\mathrm{U}(n) \setminus \Omega$; take r half the least of the corresponding r_k.

(ii). Let $C' \in \mathbb{N}$ with $C' \geq 2\max_a |e(a)|$, let $s \geq s_0 = C'$, $f = -(s\,o + e)$, $D = \mathrm{diag}(2^{f_a})$, and $m = s - C' \geq 0$. We apply Lemma 5.7.14 with $w = -\mu$. If $w_c < w_a$, then $\mu_a < \mu_c$, so $l_a < l_c$ and $o(c) + 1 \leq o(a)$; hence $f_a \leq -s(o(c)+1) - e(a) = f_c - s + e(c) - e(a) \leq f_c - |m|$. For $k \in \mathrm{U}(n) \setminus \Omega$ take i, j as in (5.135); then $k_{ij} \neq 0$ and, as $\varepsilon_i \neq \varepsilon_j$, $|w_i - w_j| \geq \frac{1}{2}|l_i - l_j| \geq h$. By Lemma 5.7.14 and (5.134),

$$|m_{-\mu}(k^*Dk)| \geq |m_{-\mu}(D)|\big(|k_{ij}|^2 2^{s-C'}\big)^h = 2^{s\,o\cdot\mu + e\cdot\mu}\,|k_{ij}|^{2h}\,2^{(s-C')h},$$

and with (5.135) the integrand of $\mathrm{Off}(D)$ at k has modulus at most $r^{-1}2^{C'h - e\cdot\mu}\,2^{-s(o\cdot\mu+h)}$. As dk has mass 1, (ii) holds with $C = r^{-1}2^{C'h - e\cdot\mu}$. □

Lemma 5.7.14. *Let n be a nonempty finite set, $w, f \in \mathbb{Z}^n$, $m \in \mathbb{Z}$, $h \in \mathbb{N}$ and $i, j \in n$, such that*

$$f_a + |m| \leq f_c \quad \textit{whenever } w_c < w_a, \qquad h \leq |w_i - w_j|.$$

Let $D = \mathrm{diag}(2^{f_a})_{a\in n}$ and let u be a unitary $n \times n$ matrix. Then

$$|m_w(D)|\,\big(|u_{ij}|^2\,2^m\big)^h \leq |m_w(u^*Du)|.$$

Proof. For $f' \in \mathbb{Z}^n$ write $M_{f'} = u^*\mathrm{diag}(2^{f'_a})u$, a positive definite matrix with $\det M_{f'} = 2^{\sum_a f'_a}$, so its principal minors are positive. If $f' \leq f''$ entrywise then $M_{f''} - M_{f'}$ is positive semidefinite, hence so is every principal submatrix of it, and the determinant is monotone on positive definite matrices for this order:

$$\Delta_P(M_{f'}) \leq \Delta_P(M_{f''}) \qquad (P \subseteq n,\ f' \leq f''). \tag{5.136}$$

Reduction to $w_j \leq w_i$. The hypotheses hold for $(-w, -f)$ in place of (w, f): if $-w_c < -w_a$ then $w_a < w_c$, so $f_c + |m| \leq f_a$, that is $-f_a + |m| \leq -f_c$. By Lemma 5.6.27 (i), $m_{-w}(z^{-1}) = m_w(z)$, and $(u^*Du)^{-1} = u^*D^{-1}u$, $D^{-1} = \mathrm{diag}(2^{-f_a})$; so both sides of the assertion are unchanged. Exchanging w with $-w$ if necessary we may assume $w_j \leq w_i$, hence $w_j \leq w_i - h$.

Expansion. Let $w_- = \min w$, $w_+ = \max w$ and $P_t = \{a : w_a \geq t\}$. By (3.1) and $\det(u^*Du) = \det D$,

$$\frac{|m_w(u^*Du)|}{|m_w(D)|} = \prod_{w_- < t \leq w_+} \frac{\Delta_{P_t}(M_f)}{\Delta_{P_t}(M_f|_{u=1})}, \qquad \Delta_{P_t}(M_f|_{u=1}) = \Delta_{P_t}(D) = 2^{\sum_{a\in P_t} f_a}.$$

As $w_- \leq w_j \leq w_i - h$ and $w_i \leq w_+$, the interval $(w_i - h, w_i]$ contains exactly h integers, all in $(w_-, w_+]$. It therefore suffices to show, for each integer t with $w_- < t \leq w_+$,

$$\Delta_{P_t}(D)\, g_t \leq \Delta_{P_t}(M_f), \qquad g_t = \begin{cases} |u_{ij}|^2 2^m & \text{if } w_i - h < t \leq w_i, \\ 1 & \text{otherwise.} \end{cases} \tag{5.137}$$

Proof of (5.137). Fix t, let $P = P_t$ and $Q = n \smallsetminus P = \{a : w_a < t\}$, nonempty as $t > w_-$. Choose $c \in Q$ with $f_c = \min_Q f$ and put $g' = \min(f, f_c)$ entrywise. For $a \in P$, $w_c < t \leq w_a$, so $f_a + |m| \leq f_c$; hence $g' = f$ on P and $g' = f_c$ on Q. Let $q = \{j\}$ if $w_i - h < t \leq w_i$ and $q = \emptyset$ otherwise; in the first case $w_j \leq w_i - h < t$, so $q \subseteq Q$; put $p = Q \smallsetminus q$, so $n \smallsetminus q = p \sqcup P$. First, $g_t \leq 2^{f_c \# q}\, \Delta_q(M_{g'}^{-1})$: for $q = \emptyset$ both sides are 1; for $q = \{j\}$, $M_{g'}^{-1} = u^* \mathrm{diag}(2^{-g'_a}) u$ has (j,j) entry $\sum_a |u_{aj}|^2 2^{-g'_a} \geq |u_{ij}|^2 2^{-f_i}$ ($i \in P$ as $w_i \geq t$), and $f_c - f_i \geq |m| \geq m$. Second, by Jacobi's complementary minor formula ([133], 0.8.4), Fischer's inequality for the positive definite matrix $(M_{g'})_{n\smallsetminus q, n\smallsetminus q}$ and the partition $p \sqcup P$ ([133], 7.8.5), and (5.136) with $g' \leq f_c$ and $g' \leq f$,

$$\Delta_q(M_{g'}^{-1}) = \frac{\Delta_{n\smallsetminus q}(M_{g'})}{\det M_{g'}} \leq \frac{\Delta_p(M_{g'})\, \Delta_P(M_{g'})}{\det M_{g'}} \leq \frac{\Delta_p(2^{f_c}\cdot 1)\, \Delta_P(M_f)}{2^{\sum_a g'_a}} = \frac{2^{f_c \# p}\, \Delta_P(M_f)}{2^{\sum_P f_a + f_c \# Q}},$$

using $M_{f_c} = 2^{f_c}\cdot 1$ (u unitary). Since $\#p + \#q = \#Q$, the two estimates give $g_t \leq \Delta_P(M_f)/2^{\sum_P f_a}$, which is (5.137). □

Lemma 5.7.15. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $J = \mathrm{diag}(\varepsilon)$, $h \in \mathbb{N}$, let $l \in \mathbb{Z}^n$ have gap h for ε, and let* Off *be the off-orbit integral of Lemma 5.7.13 for (ε, l). In $\mathcal{R} = \mathbb{C}[X_a^{\pm 1} : a \in n]$ write P_γ for the coefficient of $X^\gamma = \prod_a X_a^{\gamma_a}$ and call the γ with $P_\gamma \neq 0$ the exponents of P. For $x \in \mathbb{C}^n$ with $|x_a| = 1$ let $t_x = \mathrm{diag}(x)$. Let $P, Q \in \mathcal{R}$ satisfy*

$$P(x) = Q(x)\, \mathrm{Off}(J t_x)$$

for every x with $|x_a| = 1$, pairwise distinct entries and Jt_x accretive. Let $o \in \mathbb{Z}^n$ satisfy $o(a) < o(b)$ whenever $l_b < l_a$, and let $\gamma \in \mathbb{Z}^n$ satisfy $o\cdot\gamma < o\cdot l^\rho + o\cdot\beta + h$ for every exponent β of Q ($o\cdot\gamma = \sum_a o(a)\gamma_a$). Then $P_\gamma = 0$.

Proof. Step 1: all accretive points of the torus. Let A be the set of $x \in (S^1)^n$ with Jt_x accretive, an open subset of the torus. The points of A with two equal entries lie in finitely many subtori $\{x_a = x_b\}$ ($a \neq b$), closed with empty interior; so the points with distinct entries are dense in A. Both sides of the identity are continuous on A (Off is continuous on the accretive matrices by Lemma 5.7.13 (i)), so $P(x) = Q(x)\mathrm{Off}(Jt_x)$ for all $x \in A$.

Step 2: continuation to the positive axis. Let $f \in \mathbb{Z}^n$. For $z \in \mathbb{C} \smallsetminus \{0\}$ put $D(z) = \mathrm{diag}(z^{f_a})$ and, for a Laurent polynomial R, $T_R(z) = \sum_\gamma R_\gamma\, \varepsilon^\gamma z^{f\cdot\gamma}$ with $\varepsilon^\gamma = \prod_a \varepsilon_a^{\gamma_a}$. For $|z| = 1$ the point $x(z) = (\varepsilon_a z^{f_a})_a$ lies on the torus, $Jt_{x(z)} = D(z)$ and $R(x(z)) = T_R(z)$. The set U of $z \neq 0$ with $D(z)$ accretive is open and contains $(0,\infty)$ ($D(z)$ is then positive definite); let U_1 be its connected component containing $(0,\infty)$. On U the function $z \mapsto T_P(z) - T_Q(z)\,\mathrm{Off}(D(z))$ is holomorphic (Lemma 5.7.13 (i)); by Step 1 it vanishes at the points $e^{\mathrm{i}\theta} \in U$ with θ small, which accumulate at $1 \in U_1$. By the identity theorem it vanishes on U_1:

$$T_P(z) = T_Q(z)\, \mathrm{Off}\big(\mathrm{diag}(z^{f_a})\big) \qquad (z > 0). \tag{5.138}$$

Step 3: the lowest level of P vanishes. Suppose $P_\gamma \neq 0$. Let β_0 be an exponent of P of least level $o\cdot\beta_0$; then $o\cdot\beta_0 \le o\cdot\gamma$. Put $L = o\cdot l^\rho + h$, fix $e\in\mathbb{Z}^n$, and define the polynomials (in t)

$$p(t) = \sum_{\gamma'} P_{\gamma'}\varepsilon^{\gamma'} 2^{-e\cdot\gamma'}\, t^{\,o\cdot\gamma' - o\cdot\beta_0}, \qquad q(t) = \sum_{\beta} Q_\beta \varepsilon^\beta 2^{-e\cdot\beta}\, t^{\,o\cdot\beta + L - o\cdot\beta_0},$$

over the exponents of P, resp. of Q. The exponents of t in p are ≥ 0 by the choice of β_0, and those in q are ≥ 1, since $o\cdot\beta_0 \le o\cdot\gamma < L + o\cdot\beta$; so $q(0)=0$. Take $f = s\,o + e$ and $z = \frac12$ in (5.138): then $T_P(\frac12) = 2^{-s\,o\cdot\beta_0} p(2^{-s})$ and $T_Q(\frac12) = 2^{-s\,o\cdot\beta_0} 2^{sL} q(2^{-s})$, so

$$p(2^{-s}) = 2^{sL} q(2^{-s})\,\mathrm{Off}\big(\mathrm{diag}(2^{-s\,o(a)-e(a)})\big) \qquad (s\in\mathbb{N}).$$

By Lemma 5.7.13 (ii), $|p(2^{-s})| \le C|q(2^{-s})|$ for $s\ge s_0$; letting $s\to\infty$, $p(0)=0$, that is

$$\sum_{\gamma'\in E_0} P_{\gamma'}\varepsilon^{\gamma'}\, 2^{-e\cdot\gamma'} = 0 \qquad (e\in\mathbb{Z}^n),$$

where E_0 is the set of exponents γ' of P with $o\cdot\gamma' = o\cdot\beta_0$. For distinct γ' the homomorphisms $e\mapsto 2^{-e\cdot\gamma'}$ from $\mathbb{Z}^n$ to $\mathbb{C}^\times$ are distinct (they differ at a unit vector), hence linearly independent (Dedekind–Artin). So $P_{\beta_0}\varepsilon^{\beta_0} = 0$, contradicting the choice of β_0. Hence $P_\gamma = 0$. □

Lemma 5.7.16. *Let n be a nonempty finite set, $N = \#n$, $\mathrm{U}(n)$ the group of unitary $n\times n$ matrices with its Haar probability measure dk, and $\nu\in\mathbb{Z}^n$. Let $a_0,\dots,a_{N-1}$ be a numbering of n with $\nu(a_0)\ge\nu(a_1)\ge\cdots\ge\nu(a_{N-1})$. Let χ_ν be the character of the irreducible continuous representation of $\mathrm{U}(n)$ of highest weight $(\nu(a_0),\dots,\nu(a_{N-1}))$ for the diagonal torus and the Borel subgroup of the invertible matrices b with $b_{a_i a_j}=0$ for $i>j$, let $d_\nu = \chi_\nu(1)$, and $\|m_\nu\|^2 = \int_{\mathrm{U}(n)} |m_\nu(k)|^2\,dk$. Then:*

(i) there are a continuous homomorphism ρ from $\mathrm{U}(n)$ to the unitary group of a finite-dimensional complex Hilbert space H, such that $\operatorname{tr}\rho(k) = \chi_\nu(k)$ for $k\in\mathrm{U}(n)$ and only the scalars commute with every $\rho(k)$, and a unit vector $e\in H$ with $m_\nu(k) = \langle e, \rho(k)e\rangle$ for every $k\in\mathrm{U}(n)$;

(ii) $\|m_\nu\|^2 = 1/d_\nu$;

(iii) $\int_{\mathrm{U}(n)} m_\nu(u^{-1}a)\,m_\nu(u)\,du = \|m_\nu\|^2\, m_\nu(a)$ for every $a\in\mathrm{U}(n)$;

(iv) $\int_{\mathrm{U}(n)} m_\nu(y^{-1}ty)\,dy = \|m_\nu\|^2\,\chi_\nu(t)$ for every $t\in\mathrm{U}(n)$.

Proof. Step 1: transport to the rows $1,\dots,N$. Let $c = \nu(a_{N-1}) = \min\nu$ and $\nu' = \nu - c$. In (3.1), $\min\nu' = 0$, $\max\nu' = \max\nu - c$ and $\{i:\nu'_i\ge t\} = \{i:\nu_i\ge t+c\}$, so

$$m_\nu(g) = (\det g)^c\, m_{\nu'}(g) \qquad (g \text{ invertible}). \tag{5.139}$$

For a matrix z indexed by n let z' be the $N\times N$ matrix with $z'_{i+1,j+1} = z_{a_i a_j}$ $(0\le i,j<N)$; then $z' = PzP^{-1}$ for the permutation matrix P with $Pe_{a_j} = e_{j+1}$, and $\Delta_S(z) = \Delta_{S'}(z')$ for $S\subseteq n$ and $S' = \{j+1 : a_j\in S\}$, the two submatrices differing by the same reordering of rows and columns. Put $k = N$, $w(h) = \nu'(a_{h-1}) - \nu'(a_h) \ge 0$ for $1\le h<N$; its highest weight is $\lambda(p) = \sum_{p<h<N} w(h) = \nu'(a_p)$, as $\nu'(a_{N-1}) = 0$. Let $u(j) = 2\lambda(j-1) + N + 1 - 2j$ $(1\le j\le N)$. As λ is nonincreasing, u is strictly decreasing, so it is injective and $\#\{j' : u(j') > u(j)\} = j-1$; each value $u(j) = 2(0+\lambda(p)) + N - 1 - 2p$ with $p = j-1$, so u is a doubled Harish-Chandra

parameter of $(w,0)$. By Lemma 5.7.17 (i), $\mu_u(j) = \lambda(j-1) = \nu'(a_{j-1})$; hence $\{j : \mu_u(j) \geq t\} = \{i : \nu'_i \geq t\}'$ for every t, and by (3.1)

$$m_{\mu_u}(z') = m_{\nu'}(z) \qquad (z \text{ invertible}). \tag{5.140}$$

Step 2: proof of (i). Let ρ_0, e be as in Lemma 5.7.17 (ii), on $H = \mathbb{C}^{I_w}$, and put $\rho(k) = (\det k)^c \rho_0(k')$ for $k \in \mathrm{U}(n)$. Since $k \mapsto k' = PkP^{-1}$ is an isomorphism of $\mathrm{U}(n)$ onto $\mathrm{U}(N)$ and $|\det k| = 1$, ρ is a continuous homomorphism into the unitary group of H, and a matrix commuting with every $\rho(k)$ commutes with every $\rho_0(h)$, so it is a scalar. By (5.139), (5.140) and Lemma 5.7.17 (ii), $\langle e, \rho(k)e\rangle = (\det k)^c m_{\mu_u}(k') = (\det k)^c m_{\nu'}(k) = m_\nu(k)$. By Lemma 5.7.17 (iii), $\operatorname{tr}\rho(k) = (\det k)^c \chi_\lambda(PkP^{-1})$. Conjugation by P carries the diagonal torus of $\mathrm{U}(N)$ to that of $\mathrm{U}(n)$ and the upper triangular Borel subgroup to the group of the b with $b_{a_i a_j} = 0$ for $i > j$ (as $(PbP^{-1})_{i+1,j+1} = b_{a_i a_j}$), and twisting by the character $\det^c$ adds c to every weight; so $k \mapsto (\det k)^c \chi_\lambda(PkP^{-1})$ is the character of the irreducible representation of $\mathrm{U}(n)$ of highest weight $(\lambda(0)+c, \dots, \lambda(N-1)+c) = (\nu(a_0), \dots, \nu(a_{N-1}))$, that is χ_ν. In particular $d_\nu = \dim H$.

Step 3: proof of (ii)–(iv). By (i), ρ is irreducible and unitary, so Schur's orthogonality relations for the compact group $\mathrm{U}(n)$ give

$$\int_{\mathrm{U}(n)} \langle x, \rho(k)y\rangle \, \overline{\langle x', \rho(k)y'\rangle}\, dk = d_\nu^{-1} \langle x, x'\rangle \langle y', y\rangle \qquad (x, y, x', y' \in H).$$

With $x = y = x' = y' = e$ this is (ii). For (iii), $m_\nu(u^{-1}a) = \langle e, \rho(u)^{-1}\rho(a)e\rangle = \langle \rho(u)e, \rho(a)e\rangle = \overline{\langle \rho(a)e, \rho(u)e\rangle}$, so with $x = y = y' = e$, $x' = \rho(a)e$,

$$\int m_\nu(u^{-1}a)\, m_\nu(u)\, du = \int \langle e, \rho(u)e\rangle \, \overline{\langle \rho(a)e, \rho(u)e\rangle}\, du = d_\nu^{-1} \langle e, \rho(a)e\rangle = \|m_\nu\|^2 m_\nu(a).$$

For (iv), let $T = \int_{\mathrm{U}(n)} \rho(y)^{-1}\rho(t)\rho(y)\, dy$, the integral of a continuous function with values in the operators on H. The substitution $y \mapsto yz$ shows $\rho(z)^{-1} T \rho(z) = T$ for $z \in \mathrm{U}(n)$, so $T = \tau \cdot 1$ by (i), and taking traces $\tau d_\nu = \operatorname{tr}\rho(t) = \chi_\nu(t)$. Hence $\int m_\nu(y^{-1}ty)\, dy = \langle e, Te\rangle = \chi_\nu(t)/d_\nu = \|m_\nu\|^2 \chi_\nu(t)$. □

Lemma 5.7.17. *Let $k \geq 1$, let w be a weight for the rank k (integers $w(h) \geq 0$ for $1 \leq h < k$), with highest weight $\lambda(p) = \sum_{p<h<k} w(h)$ $(0 \leq p < k)$, and let $\mathbb{V}_w$ be its Weyl module, the rows of the arrays of (WM1) being indexed by $\{1, \dots, k\}$. Let $u \in \mathbb{Z}^k$ be a doubled Harish-Chandra parameter of $(w,0)$, and for $1 \leq a \leq k$ put $p(a) = \#\{a' : u(a') > u(a)\}$. Write $\mathrm{U}(k)$ for the group of unitary $k \times k$ matrices and $\langle \cdot, \cdot \rangle$ for the standard inner product of $\mathbb{C}^{I_w}$ (conjugate-linear in the first argument).*

(i) $u(a) = 2\lambda(p(a)) + k - 1 - 2p(a)$ *for every $1 \leq a \leq k$; for the signs $(1, \dots, 1)$, u has gap h for every $h \in \mathbb{N}$, and its lowest K-type is $\mu_u(a) = \lambda(p(a))$ $(1 \leq a \leq k)$.*

(ii) There are a continuous homomorphism ρ from $\mathrm{U}(k)$ to the group of unitary $I_w \times I_w$ matrices, an invertible complex $I_w \times I_w$ matrix C and a unit vector $e \in \mathbb{C}^{I_w}$ such that

$$C\,\rho(h) = S_w(h^{\mathsf{T}})^{\mathsf{T}}\, C \quad \text{and} \quad \langle e, \rho(h)e\rangle = c_{u/2}(h) = m_{\mu_u}(h) \qquad (h \in \mathrm{U}(k)),$$

where $c_{u/2}$ is Flensted-Jensen's function for the signs $(1, \dots, 1)$, and every complex matrix that commutes with $\rho(h)$ for all $h \in \mathrm{U}(k)$ is a scalar multiple of the identity. Moreover $\#I_w = d_k(\lambda) = \prod_{0 \leq p < q < k} \frac{\lambda(p) - \lambda(q) + q - p}{q - p}$.

(iii) For $h \in \mathrm{U}(k)$, $\operatorname{tr}\rho(h) = \chi_\lambda(h)$, where χ_λ is the character of the irreducible continuous representation of $\mathrm{U}(k)$ of highest weight $(\lambda(0), \dots, \lambda(k-1))$ for the diagonal torus and the Borel subgroup of upper triangular matrices.

Proof. Minors are taken with rows in increasing order.

(i). By (WM6), u is injective and each value $u(a)$ is one of the k numbers $\ell_p = 2\lambda(p) + k - 1 - 2p$ $(0 \le p < k)$. As λ is nonincreasing, $\ell_0 > \ell_1 > \cdots > \ell_{k-1}$; so u is a bijection of $\{1,\dots,k\}$ onto $\{\ell_0,\dots,\ell_{k-1}\}$, and the index a with $u(a) = \ell_p$ has exactly p larger values, that is $p(a) = p$. Every $\ell_p \equiv k-1 \pmod 2$, and no two indices carry different signs, so u has gap h for every h ((WM8) of Definition 3.2.19). For the lowest K-type of Definition 3.2.3: all signs agree, so $2\rho_u^{\mathrm c} = 2\rho_u$, and $2\rho_u(a) = \#\{u(a') < u(a)\} - \#\{u(a') > u(a)\} = k - 1 - 2p(a)$. Hence $u^\rho(a) = \lfloor \frac12 (u(a) + 2\rho_u(a)) \rfloor = \lambda(p(a)) + k - 1 - 2p(a)$ and $\mu_u(a) = u^\rho(a) - 2\rho_u(a) = \lambda(p(a))$.

Properties of the Schur matrices. By Lemma 4.6.10 (a), (b): $S_w(zz') = S_w(z)S_w(z')$, $S_w(1) = 1$, $S_w(z)^{\mathsf T} G_w = G_w S_w(z^{\mathsf T})$ for all complex $k \times k$ matrices z, z', and $G_w = F_w^{\mathsf T} F_w$ is a real positive definite matrix. By Lemma 4.6.10 (c) with $e = 0$, where $\rho_{w,0}(h) = S_w(\bar h)$, we have $\langle S_w(\bar h)\xi, S_w(\bar h)\xi'\rangle_w = \langle \xi, \xi'\rangle_w$ for unitary h; as $\bar h$ runs over all unitary matrices when h does, $S_w(g)^* G_w S_w(g) = G_w$ for every unitary g. The entries of $S_w(z)$ are polynomials in the entries of z: by (WM2) the coefficients of $b_i(zy)$ in the basis $(b_j(y))$ are, for the matrix of indeterminates z over $\mathbb Z[z_{ab}]$, polynomials in the z_{ab}, and they specialise to $S_w(z)$.

The representation and the intertwiner. Let B be the positive definite square root of G_w (real symmetric, $B^2 = G_w$) and put $\rho(h) = B\, S_w(h)\, B^{-1}$ and $C = B$. Then ρ is a homomorphism, continuous because S_w has polynomial entries, and $\rho(h)$ is unitary for $h \in \mathrm U(k)$: $\rho(h)^*\rho(h) = B^{-1} S_w(h)^* G_w S_w(h) B^{-1} = B^{-1} G_w B^{-1} = 1$. The identity $S_w(z)^{\mathsf T} G_w = G_w S_w(z^{\mathsf T})$ with $z = h^{\mathsf T}$ reads $S_w(h^{\mathsf T})^{\mathsf T} G_w = G_w S_w(h)$, so

$$C\rho(h) = G_w S_w(h) B^{-1} = S_w(h^{\mathsf T})^{\mathsf T} G_w B^{-1} = S_w(h^{\mathsf T})^{\mathsf T} C.$$

Only scalars commute with ρ. Let M commute with every $\rho(h)$, and $M' = B^{-1} M B$; then $M' S_w(h) = S_w(h) M'$ for $h \in \mathrm U(k)$. Each entry of $M' S_w(g) - S_w(g) M'$ is a polynomial in the entries of $g \in \mathrm M_k(\mathbb C)$ that vanishes on $\mathrm U(k)$, so it vanishes identically by Lemma 4.6.13 (one factor, $n = k$). Hence M' commutes with $S_w(z)$ for every $z \in \mathrm{GL}_k(\mathbb C)$, and $M'^{\mathsf T}$ commutes with $\pi(g) = S_w(g^{\mathsf T})^{\mathsf T}$ for every $g \in \mathrm{GL}_k(\mathbb C)$. Let $\mathcal N$ be the complex span of the products of minors Δ_T; the b_i, a $\mathbb Z$-basis of the $\mathbb Z$-module ∇_w of integer polynomials, are linearly independent over $\mathbb Q$ and hence over $\mathbb C$, so $\xi \mapsto \sum_i \xi_i b_i$ is an isomorphism $\mathbb C^{I_w} \to \mathcal N$, and it carries $\pi(g)$ to $P \mapsto P(g^{\mathsf T}\cdot)$: $\sum_i \xi_i b_i(g^{\mathsf T} y) = \sum_{i,j} \xi_i S_w(g^{\mathsf T})_{ij} b_j(y)$. Only the columns $0 \le c < k-1$ of the arrays occur in the minors, so $\mathcal N$ is a space of polynomials in the entries of a $k \times k$ matrix $X = (y_{a,c})_{1\le a\le k, 0\le c<k}$. A minor Δ_A on h rows and the columns $0,\dots,h-1$ is homogeneous of degree 1 in each column $c < h$ and does not involve the others, so Δ_T is homogeneous of degree $\sum_{h>c} w(h) = \lambda(c)$ in column c; and adding a multiple of a column c' to a later column c changes no minor (if $c < h$ then $c' < h$ and the determinant is unchanged; if $c \ge h$ column c does not occur). So $\mathcal N$ is a nonzero subspace, stable under $P \mapsto P(g^{\mathsf T}\cdot)$, of the module $\nabla_{\mathbb C}(\lambda)$ of Theorem A.5.19 (λ is nonincreasing and $\lambda(k-1) = 0$). That module is irreducible, so $\mathcal N = \nabla_{\mathbb C}(\lambda)$, π is irreducible, and $\#I_w = \dim \nabla_{\mathbb C}(\lambda) = d_k(\lambda)$ by the dimension formula of Theorem A.5.19. By Schur's lemma $M'^{\mathsf T}$, hence M' and $M = BM'B^{-1}$, is a scalar.

The diagonal coefficient of the highest vector. Let y^u be the array of (WM6), $y^u_{a,c} = 1$ if $c = p(a)$ and 0 otherwise, $v^+ = \mathrm{ev}_w(y^u)$ and $e = Bv^+$. As B is real symmetric, $\langle e, \rho(h)e\rangle = \overline{v^+}^{\mathsf T} G_w S_w(h) v^+ = \langle v^+, S_w(h)v^+\rangle_w$. By the definition of S_w, $S_w(h)\,\mathrm{ev}_w(y) = \mathrm{ev}_w(hy)$, and by the definition of F_w, $F_w\,\mathrm{ev}_w(y) = (\Delta_T(y))_{T\in\mathcal C_w}$; since F_w is real,

$$\langle \mathrm{ev}_w(y), \mathrm{ev}_w(y')\rangle_w = \sum_{T\in\mathcal C_w} \overline{\Delta_T(y)}\,\Delta_T(y').$$

For $1 \le h < k$ put $P_h = \{a : p(a) < h\}$ and let a_c be the index with $p(a_c) = c$. The columns $0,\dots,h-1$ of y^u are the unit vectors at $a_0,\dots,a_{h-1}$; so for a set A of h rows, $\Delta_A(y^u) = 0$ unless

$A = P_h$, and $\Delta_{P_h}(y^u) = \sigma_h$, the sign of the permutation that lists $a_0, \dots, a_{h-1}$ in increasing order. Hence $\Delta_T(y^u) = 0$ for every increasing filling except the filling T^+ with $T^+(h,j) = P_h$ for all h, j, for which $\Delta_{T^+}(y^u) = \prod_h \sigma_h^{w(h)}$. Further $(hy^u)_{a,c} = h_{a\,a_c}$, so $\Delta_{P_{h'}}(hy^u)$ is the determinant of the submatrix of h with rows $P_{h'}$ and columns $a_0, \dots, a_{h'-1}$, which is $\sigma_{h'}\,\Delta_{P_{h'}}(h)$. Therefore

$$\langle e, \rho(h)e\rangle = \prod_{1\le h'<k} \sigma_{h'}^{2w(h')}\,\Delta_{P_{h'}}(h)^{w(h')} = \prod_{1\le h'<k} \Delta_{P_{h'}}(h)^{w(h')} = m_{\mu_u}(h).$$

For the last equality use (3.1) with $\mu_u(a) = \lambda(p(a))$ from (i): $\min \mu_u = \lambda(k-1) = 0$, and for $\lambda(h') < t \le \lambda(h'-1)$ $(1 \le h' < k)$ the set $\{a : \mu_u(a) \ge t\}$ is $P_{h'}$, a range of $\lambda(h'-1) - \lambda(h') = w(h')$ values of t. At $h = 1$ this gives $\|e\|^2 = m_{\mu_u}(1) = 1$.

Flensted-Jensen's function on $\mathrm{U}(k)$. For the signs $(1, \dots, 1)$, $K_\varepsilon = \mathrm{U}(k)$, $\mathrm{bd}(h) = h$ and $h^{-1}h = 1$ is accretive, so by (3.2), with $m_{-u^\rho}(k'^* k') = m_{-u^\rho}(1) = 1$,

$$c_{u/2}(h) = \frac{1}{\|m_{\mu_u}\|^2} \int_{\mathrm{U}(k)} m_{\mu_u}(hk')\,\overline{m_{\mu_u}(k')}\,dk'.$$

Here $m_{\mu_u}(g) = \langle e, \rho(g)e\rangle$ and $m_{\mu_u}(hk') = \langle \rho(h)^* e, \rho(k')e\rangle$. The representation ρ is unitary and irreducible of dimension $\delta = \#I_w$, so Schur's orthogonality relations for the compact group $\mathrm{U}(k)$ with its Haar probability measure give $\int \langle x, \rho(k')y\rangle\overline{\langle x', \rho(k')y'\rangle}\,dk' = \delta^{-1}\langle x, x'\rangle\langle y', y\rangle$. Hence $\|m_{\mu_u}\|^2 = \delta^{-1}$ and the integral is $\delta^{-1}\langle \rho(h)^* e, e\rangle = \delta^{-1}\langle e, \rho(h)e\rangle$; so $c_{u/2}(h) = \langle e, \rho(h)e\rangle = m_{\mu_u}(h)$.

(iii). By the paragraph on the commutant, $\xi \mapsto \sum_i \xi_i b_i$ is an isomorphism of $\mathbb{C}^{I_w}$ onto $\nabla_{\mathbb{C}}(\lambda)$ that carries $\pi(g) = S_w(g^{\mathsf{T}})^{\mathsf{T}}$ to $P \mapsto P(g^{\mathsf{T}}\cdot)$; by Theorem A.5.19 this is the irreducible rational $\mathrm{GL}_k(\mathbb{C})$-module of highest weight λ for the diagonal torus and the upper triangular Borel subgroup. Its restriction to $\mathrm{U}(k)$ is irreducible, because $\pi(h) = C\rho(h)C^{-1}$ and only scalars commute with ρ; so it is the irreducible continuous representation of $\mathrm{U}(k)$ of highest weight $(\lambda(0), \dots, \lambda(k-1))$, the torus weights of a rational module being those of its restriction. Hence $\chi_\lambda(h) = \operatorname{tr}\pi(h) = \operatorname{tr}(C\rho(h)C^{-1}) = \operatorname{tr}\rho(h)$. □

Lemma 5.7.18. *Let n be a nonempty finite set, $N = \#n$, $\varepsilon \in \{\pm 1\}^n$, and let $l \in \mathbb{Z}^n$ have gap $h = 500N^3 + 10N^5$ for ε. In the Laurent polynomial ring $\mathcal{R} = \mathbb{C}[X_a^{\pm 1} : a \in n]$ write P_γ for the coefficient of $X^\gamma = \prod_a X_a^{\gamma_a}$ $(\gamma \in \mathbb{Z}^n)$; the* exponents *of P are the γ with $P_\gamma \ne 0$. Let $W_K = \{w \in S_n : \varepsilon \circ w = \varepsilon\}$, $o(a) = \#\{b : l_b > l_a\}$, the* level *$o\cdot\gamma = \sum_a o(a)\gamma_a$, $\kappa = \frac{1}{2}(l + N - 1)$, $\delta(a) = \#\{b : \varepsilon_b \ne \varepsilon_a,\ l_a < l_b\}$, $N' = 4N^2$ and* top *$= \kappa + N'\delta$. Put*

$$D = \prod_{\substack{\varepsilon_a \ne \varepsilon_b \\ l_a < l_b}} (X_a - X_b), \qquad V = \prod_{l_b < l_a} (X_a - X_b), \qquad A_\mu^\eta = \sum_{w \in S_n,\ \eta\circ w = \eta} \operatorname{sgn}(w)\, X^{\mu\circ w} \quad (\eta \in \{\varepsilon, \mathbf{1}\}),$$

the products over ordered pairs, $\mathbf{1}$ the constant sign $+1$. The cone *of l is $C_l = \{\beta \in \mathbb{Z}^n : o'\cdot\beta \le 0$ for every $o' \in \mathbb{Z}^n$ with $o'(a) < o'(b)$ whenever $l_b < l_a\}$. For $a \in \mathbb{Z}^n$ and $k \in \mathbb{N}$, $P \in \mathcal{R}$ is a* cone-series multiple of D^k at a *if there is $t\colon \mathbb{Z}^n \to \mathbb{C}$ with $t(a) = 1$, $P_\gamma = \sum_\beta (D^k)_\beta\, t(\gamma - \beta)$ for all γ, and, whenever $t(\gamma) \ne 0$, some μ with $(A_\mu^\varepsilon)_\gamma \ne 0$ and $\mu - a \in C_l$. P is W_K-*antisymmetric *if $P_{\gamma\circ w} = \operatorname{sgn}(w) P_\gamma$ for $w \in W_K$ and all γ. Then:*

(a) $\kappa \in \mathbb{Z}^n$; κ and top *are strictly increasing along l ($l_j < l_i$ implies $\kappa_j < \kappa_i$ and $\mathrm{top}_j < \mathrm{top}_i$).*

(b) If P is a cone-series multiple of D^k at a and $a + k\delta =$ top, then $P_{\mathrm{top}} = 1$, and every exponent γ of P has some $w \in W_K$ with $o\cdot(\gamma\circ w) \le o\cdot$ top.

(c) $D^{N'} A_\kappa^\varepsilon$ is W_K-antisymmetric and a cone-series multiple of $D^{N'}$ at κ.

(d) If $o\cdot\gamma \le o\cdot\mathrm{top}$, then $(D^{N'}A^{\mathbf{1}}_{\kappa})_\gamma = (D^{N'}A^{\varepsilon}_{\kappa})_\gamma$; and every exponent β of $VD^{N'}$ satisfies $o\cdot\mathrm{top} < o\cdot(\kappa-o)+o\cdot\beta+h$.

(e) Let $P,Q\in\mathcal{R}$ be W_K-antisymmetric, $m\in\mathbb{Z}$ and $c\in\mathbb{C}$, such that every exponent γ of P or of Q has some $w\in W_K$ with $o\cdot(\gamma\circ w)\le m$, and $P_\gamma = cQ_\gamma$ whenever $o\cdot\gamma\le m$. Then $P=cQ$.

Proof. As l has a gap, it is injective and $l_a \equiv N-1 \pmod 2$, so $\kappa\in\mathbb{Z}^n$; o is a bijection $n\to\{0,\dots,N-1\}$ that strictly reverses the order of l ($l_b<l_a \Rightarrow o(a)<o(b)$), and $0\le\delta\le N$. If $\varepsilon_a\ne\varepsilon_b$, then $|\kappa_a-\kappa_b| = \frac12|l_a-l_b|\ge h$.

Step 1: rearrangement with a gain. Let $b\in\mathbb{Z}^n$ be injective and strictly increasing along l, so that o strictly reverses its order. Then for $w\in S_n$ and $i\in n$,

$$o\cdot b + |b_i - b_{w(i)}| \le o\cdot(b\circ w), \tag{5.141}$$

and in particular $o\cdot(b\circ w)\ge o\cdot b$ with equality only for $w=1$. Indeed, for $c\in\{1,-1\}$ the integer vector $o+c\,e_i$ still weakly reverses the order of b (if $b_j<b_{j'}$ then $o(j')\le o(j)-1$), so by the rearrangement inequality $(o+ce_i)\cdot b\le(o+ce_i)\cdot(b\circ w)$, that is $o\cdot b+cb_i\le o\cdot(b\circ w)+cb_{w(i)}$; take the better of $c=\pm1$. If $w\ne1$, some i has $b_{w(i)}\ne b_i$, and (5.141) is strict.

Step 2: proof of (a). κ is strictly increasing along l. Let $l_j<l_i$. If $\kappa_i-\kappa_j>N'N$, then $\mathrm{top}_i-\mathrm{top}_j\ge\kappa_i-\kappa_j-N'N>0$ as $0\le\delta\le N$. Otherwise $0<\kappa_i-\kappa_j\le 4N^3<h$, so every a with $l_j\le l_a\le l_i$ has $\varepsilon_a=\varepsilon_i=\varepsilon_j$ (else $|\kappa_a-\kappa_i|\ge h$ or $|\kappa_a-\kappa_j|\ge h$, while $\kappa_j\le\kappa_a\le\kappa_i$). Then every b counted in $\delta(j)$ ($\varepsilon_b\ne\varepsilon_j$, $l_b>l_j$) has $l_b>l_i$ and $\varepsilon_b\ne\varepsilon_i$, so $\delta(j)\le\delta(i)$ and $\mathrm{top}_j<\mathrm{top}_i$.

Step 3: the monomials of D^k. Each factor X_a-X_b of D has $l_a<l_b$, so $o(a)>o(b)$: the monomial X_a has the larger level. Expanding the product, X^δ (the product of the first terms) has coefficient 1 in D and every other monomial has strictly smaller level; the same holds for D^k and $X^{k\delta}$, and all exponents of D^k are ≥0. Moreover $D(X\circ w)=D(X)$ for $w\in W_K$: writing each factor as $\pm(X_a-X_b)$ with $\varepsilon_a=+1$, $\varepsilon_b=-1$, $D=\pm\prod_{\varepsilon_a=1,\varepsilon_b=-1}(X_a-X_b)$ with a fixed sign, and w permutes the indices of each sign; so $(D^k)_{\beta\circ w}=(D^k)_\beta$.

Step 4: proof of (b). Let t be as in the definition. Suppose $(D^k)_\beta\, t(\gamma-\beta)\ne0$. There is μ with $(A^\varepsilon_\mu)_{\gamma-\beta}\ne0$ and $\mu-a\in C_l$; so $\gamma-\beta=\mu\circ v$ for some $v\in W_K$, that is $\mu=(\gamma-\beta)\circ w$ with $w=v^{-1}\in W_K$, and $o\cdot(\mu-a)\le0$ gives $o\cdot(\gamma\circ w)\le o\cdot(\beta\circ w)+o\cdot a$. By Step 3, $\beta\circ w$ is an exponent of D^k, so $o\cdot(\beta\circ w)\le k\,o\cdot\delta$, with equality only if $\beta\circ w=k\delta$. Hence

$$o\cdot(\gamma\circ w)+[\beta\circ w\ne k\delta]\le o\cdot\mathrm{top}. \tag{5.142}$$

If γ is an exponent of P, some β has $(D^k)_\beta t(\gamma-\beta)\ne0$, and (5.142) gives the second claim. For $\gamma=\mathrm{top}$: if a term β is nonzero, (5.142) and Step 1 (for $b=\mathrm{top}$, strictly increasing along l by (a)) force $w=1$ and then $\beta=k\delta$. So $P_{\mathrm{top}}=(D^k)_{k\delta}\,t(\mathrm{top}-k\delta)=t(a)=1$.

Step 5: proof of (c). By Step 3 and $\mathrm{sgn}(wv)=\mathrm{sgn}(w)\mathrm{sgn}(v)$, $D^{N'}$ is W_K-invariant and A^ε_κ is W_K-antisymmetric, hence so is the product (the coefficient of $X^{\gamma\circ w}$ in a product FG with F invariant is $\sum_\beta F_{\beta\circ w}G_{(\gamma-\beta)\circ w}$). Take $t(\gamma)=(A^\varepsilon_\kappa)_\gamma$: then $(D^{N'}A^\varepsilon_\kappa)_\gamma=\sum_\beta(D^{N'})_\beta t(\gamma-\beta)$; $t(\kappa)=1$, because $\kappa\circ w=\kappa$ only for $w=1$ (κ is injective); and if $t(\gamma)\ne0$, $\mu=\kappa$ works, $0\in C_l$.

Step 6: proof of (d). Let $o\cdot\gamma\le o\cdot\mathrm{top}$. The coefficient $(D^{N'}A^{\mathbf{1}}_\kappa)_\gamma$ is $\sum_\beta(D^{N'})_\beta\sum_v\mathrm{sgn}(v)[\kappa\circ v=\gamma-\beta]$ over all $v\in S_n$, and the same with $v\in W_K$ for A^ε. Suppose $\kappa\circ v=\gamma-\beta$ with $(D^{N'})_\beta\ne0$ and $v\notin W_K$: some i has $\varepsilon_{v(i)}\ne\varepsilon_i$, and (5.141) for $b=\kappa$ gives $o\cdot\kappa+h\le o\cdot(\kappa\circ v)=o\cdot\gamma-o\cdot\beta\le o\cdot\gamma$ ($\beta\ge0$, $o\ge0$); but $o\cdot\gamma\le o\cdot\kappa+N'o\cdot\delta\le o\cdot\kappa+4N^2\cdot N^3<o\cdot\kappa+h$ (as $0\le o(a)<N$ and $0\le\delta(a)\le N$). So only $v\in W_K$ contribute, and the two coefficients agree. For the bound, $VD^{N'}$ is a polynomial, so $o\cdot\beta\ge0$ for its exponents, and $o\cdot\mathrm{top}-o\cdot(\kappa-o)=o\cdot o+N'\,o\cdot\delta\le N\cdot N^2+4N^2\cdot N^3<h$.

Step 7: proof of (e). Let γ be an exponent of P or of Q and $w \in W_K$ with $o \cdot (\gamma \circ w) \le m$. Then $\operatorname{sgn}(w) P_\gamma = P_{\gamma\circ w} = cQ_{\gamma\circ w} = c\operatorname{sgn}(w) Q_\gamma$, so $P_\gamma = cQ_\gamma$; if γ is an exponent of neither, both sides vanish. □

Lemma 5.7.19. *Fix the setting of Definition 5.1.6, signs ε on the r basis indices of V_1 and a sign frame p for ε. Let $l \in \mathbb{Z}^r$ have large gap A for ε ($A \ge 0$ an integer); write $\psi(g) = c_{l/2}(p^{-1}gp)$ ($g \in G_1$), d_l for the formal degree $(\int |c_{l/2}|^2)^{-1}$ for the Haar measure on $\mathrm{U}(\operatorname{diag}\varepsilon)$ carried from dy by $g \mapsto p^{-1}gp$, and $e = d_l\psi$. Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be a unitary representation on $\mathbb{C}^I$, with the standard inner product $\langle\cdot,\cdot\rangle$ of $\mathbb{C}^I$ (conjugate-linear in the first argument), and F a Bessel function of type ρ, integrable over $\mathsf{U}\backslash G_1$, with $\int_{G_1} e(a^{-1})F(ya)\,da = F(y)$ for all $y \in G_1$; put $\|F\|_2^2 = \int_{\mathsf{U}\backslash G_1} \|F\|^2$. Then:*

(i) for all $a, b \in G_1$ the following integral converges absolutely, and

$$\int_{\mathsf{U}\backslash G_1} \langle F(ya), F(yb)\rangle\, dy = \psi(a^{-1}b)\, \|F\|_2^2;$$

(ii) $\displaystyle\int_{G_1} \Big| \int_{\mathsf{U}\backslash G_1} \langle F(y), F(yg)\rangle\, dy \Big|^2 dg = d_l^{-1}\, \|F\|_2^4;$

(iii) let $X\colon G_1 \to \mathbb{C}$ be integrable with $\int_{G_1} X(a^{-1})\, e(ya)\, da = X(y)$ for all y, and put $V(y) = \int_{G_1} X(a^{-1})F(ya)\, da$. Then for every integrable ϕ on G_1

$$\int_{\mathsf{U}\backslash G_1} \Big\| \int_{G_1} \phi(y)\, V(\gamma y)\, dy \Big\|^2 d\gamma \le \Big(d_l^{-1} \int_{G_1} |X|^2 \Big) \Big(d_l^{-1}\, \|F\|_2^2 \Big) \int_{G_1} |\phi|^2.$$

Proof. Write $G = G_1$, $c = c_{l/2}$, $d = d_l$, $\iota(g) = p^{-1}gp$, $M_F = \sup_G \|F\| < \infty$ (F is bounded) and $\|F\|_1 = \int_{\mathsf{U}\backslash G} \|F\| < \infty$. Integrals over G without a named measure are for dy, and integrable means integrable for dy.

The group and the kernel. As p is unitary and $p^* J_1 p = \operatorname{diag}(\varepsilon)$, ι is an isomorphism of topological groups $G \to \mathrm{U}(\operatorname{diag}\varepsilon)$; let dg be the image of dy, the Haar measure of the statement. By Lemma 5.2.41, dy is invariant under $y \mapsto ayb$ for all $a, b \in G$. It is also invariant under $y \mapsto y^{-1}$: the image of dy under inversion is again invariant under left and right translations, hence equal to $\gamma_0\, dy$ for some $\gamma_0 > 0$ by the uniqueness of Haar measure, and applying inversion twice gives $\gamma_0^2 = 1$. The vector l has gap $500r^3 + A > 8r^2$ for ε ((WM8) of Definition 3.2.19), so Lemma 5.6.23 (i), (ii) and Lemma 5.7.40 apply to l and dg. Transported by ι they give: ψ is continuous and integrable, $|\psi| \le 1$, $\psi(g^{-1}) = \overline{\psi(g)}$, $d > 0$, $\int_G |\psi|^2 = d^{-1}$, and

$$d \int_G \psi(xya)\, \overline{\psi(y)}\, dy = \psi(x)\, \psi(a) \qquad (x, a \in G). \tag{5.143}$$

Hence e is continuous and integrable, $|e| \le d$, $e(g^{-1}) = \overline{e(g)}$, and

$$\int_G e(z)\, e(x^{-1}z^{-1}y)\, dz = \frac{e(x^{-1})\, e(y)}{d} \qquad (x, y \in G): \tag{5.144}$$

after the substitution $z \mapsto z^{-1}$ the left side is $d^2 \int_G \overline{\psi(z)}\, \psi(x^{-1}zy)\, dz = d\, \psi(x^{-1})\psi(y)$ by (5.143). Finally X is bounded, $|X(y)| \le \int_G |X(a^{-1})|\, |e(ya)|\, da \le d \int_G |X|$, hence square integrable.

A coefficient identity. Let E be $\mathbb{C}^I$ or $\mathbb{C}$ with its standard inner product, μ a σ-finite Borel measure on G and $f\colon G\to E$ a bounded Borel function with $\int\|f\|\,d\mu<\infty$ such that, for all $y,s,t\in G$,

$$\begin{aligned} f(y)&=\int_G e(t^{-1})f(yt)\,dt,\\ \int\|f(ys)\|\,d\mu(y)&=\int\|f\|\,d\mu,\\ \int\langle f(ys),f(yt)\rangle\,d\mu(y)&=\int\langle f(y),f(ys^{-1}t)\rangle\,d\mu(y). \end{aligned}\tag{5.145}$$

Claim: for all $a,b\in G$, $\int\langle f(ya),f(yb)\rangle\,d\mu(y)=\psi(a^{-1}b)\int\|f\|^2d\mu$. Put $M=\sup\|f\|$, $m=\int\|f\|\,d\mu$ and $k(g)=\int\langle f(y),f(yg)\rangle\,d\mu(y)$; the integrand is bounded by $M\|f(y)\|$, so $|k|\le Mm$, and k is Borel by Tonelli's theorem applied to the positive and negative parts of the real and imaginary parts of $(y,g)\mapsto\langle f(y),f(yg)\rangle$. By the third identity of (5.145), $\int\langle f(ys),f(yt)\rangle\,d\mu=k(s^{-1}t)$; with $(s,t)=(g,1)$ this gives $k(g^{-1})=\int\langle f(yg),f(y)\rangle\,d\mu=\overline{k(g)}$. With $(s,t)=(g^{-1},1)$, $k(g)=\int\langle f(yg^{-1}),f(y)\rangle\,d\mu(y)$; insert $f(y)=\int e(t^{-1})f(yt)\,dt$. The function $(y,t)\mapsto e(t^{-1})\langle f(yg^{-1}),f(yt)\rangle$ is integrable for $\mu\otimes dt$, as it is bounded by $M|e(t^{-1})|\,\|f(yg^{-1})\|$, whose double integral is $Mm\int|e|$ by the second identity of (5.145). By Fubini's theorem,

$$k(g)=\int_G e(t^{-1})\int\langle f(yg^{-1}),f(yt)\rangle\,d\mu(y)\,dt=\int_G e(t^{-1})\,k(gt)\,dt\qquad(g\in G).\tag{5.146}$$

Conjugating (5.146) at g^{-1} and using $\overline{e(t^{-1})}=e(t)$, $\overline{k(g^{-1}t)}=k(t^{-1}g)$ gives

$$k(g)=\int_G e(s)\,k(s^{-1}g)\,ds.$$

Insert (5.146) at $s^{-1}g$ and substitute $t\mapsto g^{-1}st$:

$$k(g)=\int_G e(s)\int_G e(t^{-1})\,k(s^{-1}gt)\,dt\,ds=\int_G e(s)\int_G e(t^{-1}s^{-1}g)\,k(t)\,dt\,ds.$$

The integrand is bounded by $Mm\,|e(s)|\,|e(t^{-1}s^{-1}g)|$, whose double integral is $Mm(\int|e|)^2$ by the invariance of dt; so Fubini's theorem, (5.144) with $(x,y)=(t,g)$ and (5.146) at $g=1$ give

$$k(g)=\int_G k(t)\int_G e(s)\,e(t^{-1}s^{-1}g)\,ds\,dt=\frac{e(g)}{d}\int_G e(t^{-1})\,k(t)\,dt=\psi(g)\,k(1).$$

As $k(1)=\int\|f\|^2d\mu$, the claim follows from $\int\langle f(ya),f(yb)\rangle\,d\mu=k(a^{-1}b)$.

Integrals over $\mathsf{U}\backslash G$. By Definition 5.1.6, $\int_{\mathsf{U}\backslash G}\Theta=\int_G\Theta\,\chi\,dy$ for left U-invariant Θ, where $\chi\ge0$ is a Borel cutoff with $\int_{\mathsf{U}}\chi(uy)\,du=1$ for every y. For left U-invariant Borel $\Theta\ge0$, every $g\in G$ and any two such cutoffs χ,χ',

$$\int_G\Theta(yg)\,\chi(y)\,dy=\int_G\Theta(y)\,\chi'(y)\,dy.\tag{5.147}$$

Indeed $\chi_g(y)=\chi(yg^{-1})$ is again a cutoff and $\int\Theta(yg)\chi(y)\,dy=\int\Theta(y)\chi_g(y)\,dy$ by right invariance; and for cutoffs χ_1,χ_2, by Tonelli's theorem on $\mathsf{U}\times G$, the substitution $y\mapsto u^{-1}y$ (left invariance of dy) and $\Theta(u^{-1}y)=\Theta(y)$,

$$\int_G\Theta\chi_2\,dy=\int_{\mathsf{U}}\int_G\chi_1(uy)\chi_2(y)\Theta(y)\,dy\,du=\int_{\mathsf{U}}\int_G\chi_1(y)\chi_2(u^{-1}y)\Theta(y)\,dy\,du=\int_G\chi_1\Theta\,dy,$$

since $\int_{\mathsf{U}}\chi_2(u^{-1}y)\,du=\int_{\mathsf{U}}\chi_2(uy)\,du=1$ by the inversion invariance of du (Lemma 5.2.34 (iv)). By linearity (5.147) holds for complex left U-invariant Θ with $\int|\Theta|\chi\,dy<\infty$. Fix χ and put $\nu=\chi\,dy$, a σ-finite measure (dy is σ-finite and ν is finite on $C\cap\{\chi\le m\}$ for compact C).

For Bessel functions F_1, F_2 of type ρ and $a, b \in G$ the functions $y \mapsto \langle F_1(ya), F_2(yb)\rangle$ and $y \mapsto \|F_1(ya)\|$ are left U-invariant, as $F_i(uya) = \mathrm{wave}(\sigma(u))F_i(ya)$ with $|\mathrm{wave}| = 1$. So by (5.147), $\int \|F(ys)\|\,d\nu = \|F\|_1$, $\int \|F(ys)\|^2 d\nu = \|F\|_2^2 \le M_F\|F\|_1$, and $\int\langle F(ys), F(yt)\rangle\,d\nu = \int\langle F(y), F(ys^{-1}t)\rangle\,d\nu$ (apply (5.147) with $g = s$ to $\Theta(y) = \langle F(y), F(ys^{-1}t)\rangle$, bounded by $M_F\|F(y)\|$). Together with the reproduction of F, these are the hypotheses (5.145) for $f = F$ and $\mu = \nu$.

(i) and (ii). The integrand of (i) is bounded by $M_F\|F(ya)\|$, which is ν-integrable; the claim for (F, ν) gives (i). For (ii), (i) with $(a, b) = (1, g)$ makes the inner integral $\psi(g)\|F\|_2^2$, and $\int_G |\psi|^2 = d^{-1}$.

Schur orthogonality for X. The function $Y(y) = \overline{X(y^{-1})}$ is bounded, Borel and integrable, and $\int e(t^{-1})Y(yt)\,dt = \overline{\int e(t)X(t^{-1}y^{-1})\,dt} = \overline{\int X(a^{-1})e(y^{-1}a)\,da} = Y(y)$ (conjugate, use $\overline{e(t^{-1})} = e(t)$, substitute $t = y^{-1}a$). With $\mu = dy$ the other two identities of (5.145) hold by right invariance. The claim for (Y, dy), followed by the substitution $y \mapsto y^{-1}$, gives

$$\int_G X(a^{-1}g)\,\overline{X(b^{-1}g)}\,dg = \psi(a^{-1}b)\int_G |X|^2 \qquad (a, b \in G). \tag{5.148}$$

The translates in $L^2(\nu)$. Let $\mathcal{H} = L^2(G, \nu; \mathbb{C}^I)$ with $\langle f, f'\rangle_{\mathcal{H}} = \int\langle f, f'\rangle\,d\nu$, and $w_a \in \mathcal{H}$ the class of $y \mapsto F(ya)$. By (i), $\langle w_a, w_b\rangle_{\mathcal{H}} = \psi(a^{-1}b)\|F\|_2^2$, so $\|w_a\|_{\mathcal{H}} = \|F\|_2$ and $\|w_a - w_b\|_{\mathcal{H}}^2 = 2\|F\|_2^2\,\mathrm{Re}(\psi(1) - \psi(a^{-1}b))$, which tends to 0 as $b \to a$. Thus $a \mapsto w_a$ is continuous and bounded, hence strongly measurable (G is second countable), and for integrable f the Bochner integral $\int f(a)w_a\,da \in \mathcal{H}$ exists, with $\langle u, \int f(a)w_a\,da\rangle_{\mathcal{H}} = \int f(a)\langle u, w_a\rangle_{\mathcal{H}}\,da$ for $u \in \mathcal{H}$. Put $v_g = \int X(a^{-1})w_{ga}\,da = \int X(a^{-1}g)\,w_a\,da$ $(g \in G)$. Then $\|v_g - v_{g'}\|_{\mathcal{H}} \le \|F\|_2\int |X(ag) - X(ag')|\,da$, which tends to 0 as $g' \to g$ by the continuity of right translation in $L^1(G)$; so $g \mapsto v_g$ is continuous, with $\|v_g\|_{\mathcal{H}} \le \|F\|_2\int |X|$. By (i) and the reproduction of X,

$$\langle w_b, v_g\rangle_{\mathcal{H}} = \|F\|_2^2\int_G X(a^{-1})\,\psi(b^{-1}ga)\,da = \frac{\|F\|_2^2}{d}\,X(b^{-1}g) \qquad (b, g \in G), \tag{5.149}$$

and $\langle u, v_g\rangle_{\mathcal{H}} = 0$ for every g if u is orthogonal to all w_a. In particular, as $\langle v_1, w_a\rangle_{\mathcal{H}} = \overline{\langle w_a, v_1\rangle_{\mathcal{H}}}$,

$$\|v_1\|_{\mathcal{H}}^2 = \int_G X(a^{-1})\,\langle v_1, w_a\rangle_{\mathcal{H}}\,da = \frac{\|F\|_2^2}{d}\int_G |X(a^{-1})|^2\,da = \frac{\|F\|_2^2}{d}\int_G |X|^2. \tag{5.150}$$

Orthogonality on the span. Let H_0 be the linear span of the w_a. For $u \in H_0$, by (5.149) the function $g \mapsto \langle u, v_g\rangle_{\mathcal{H}}$ is a finite linear combination of left translates of X, hence bounded and integrable, and for $u, u' \in H_0$

$$\int_G \langle u, v_g\rangle_{\mathcal{H}}\,\langle v_g, u'\rangle_{\mathcal{H}}\,dg = \frac{\|v_1\|_{\mathcal{H}}^2}{d}\,\langle u, u'\rangle_{\mathcal{H}}. \tag{5.151}$$

Both sides are conjugate-linear in u and linear in u', so it suffices to take $u = w_a$, $u' = w_b$: by (5.149) and (5.148) the left side is $(\|F\|_2^2/d)^2\int X(a^{-1}g)\overline{X(b^{-1}g)}\,dg = (\|F\|_2^2/d)^2\psi(a^{-1}b)\int |X|^2$, and by (5.150) and (i) so is the right side.

(iii). Let ϕ be integrable; if $\int |\phi|^2 = \infty$ there is nothing to prove. Put $\Omega = \int_G \phi(g)\,v_g\,dg \in \mathcal{H}$ and $B = d^{-1}\|v_1\|_{\mathcal{H}}^2\int |\phi|^2$. For $u \in H_0$, by the Cauchy–Schwarz inequality and (5.151) with $u' = u$,

$$|\langle u, \Omega\rangle_{\mathcal{H}}|^2 = \Big|\int_G \phi(g)\,\langle u, v_g\rangle_{\mathcal{H}}\,dg\Big|^2 \le \int_G |\phi|^2\int_G |\langle u, v_g\rangle_{\mathcal{H}}|^2\,dg = B\,\|u\|_{\mathcal{H}}^2.$$

Both sides are continuous in u, so the inequality holds on the closure $\overline{H_0}$. If u is orthogonal to H_0, then $\langle u, \Omega\rangle_{\mathcal{H}} = \int \phi(g)\langle u, v_g\rangle_{\mathcal{H}}\,dg = 0$; so $\Omega \in (H_0^{\perp})^{\perp} = \overline{H_0}$, and $u = \Omega$ gives $\|\Omega\|_{\mathcal{H}}^4 \le B\|\Omega\|_{\mathcal{H}}^2$, that is $\|\Omega\|_{\mathcal{H}}^2 \le B$.

It remains to compare Ω with the function on the left of (iii). Put $Z(s) = \int_G \phi(y)X(sy)\,dy$; by Tonelli's theorem and right invariance $\int|Z| \le \int|\phi|\int|X|$. For $\gamma \in G$, the substitution $a = y^{-1}t$ and Fubini's theorem (the integrand $\phi(y)X(t^{-1}y)F(\gamma t)$ has norm with double integral at most $M_F\int|\phi|\int|X|$) give

$$\Phi(\gamma) := \int_G \phi(y)\,V(\gamma y)\,dy = \int_G \phi(y)\int_G X(t^{-1}y)\,F(\gamma t)\,dt\,dy = \int_G Z(t^{-1})\,F(\gamma t)\,dt.$$

So Φ is bounded by $M_F\int|Z|$, Borel by Tonelli's theorem, a Bessel function of type ρ (the left transformations pass through the integral), and $\int\|\Phi\|\,d\nu \le \int|Z(t^{-1})|\int\|F(\gamma t)\|\,d\nu(\gamma)\,dt = \|F\|_1\int|Z|$; hence its class $\tilde\Phi$ lies in $\mathcal{H}$. In $\mathcal{H}$, $\int Z(t^{-1})w_t\,dt = \Omega$: for $u \in \mathcal{H}$, by the substitution $a = y^{-1}t$ and Fubini's theorem ($|\langle u, w_t\rangle_{\mathcal{H}}| \le \|u\|_{\mathcal{H}}\|F\|_2$),

$$\begin{aligned}\langle u,\Omega\rangle_{\mathcal{H}} &= \int_G \phi(y)\int_G X(a^{-1})\langle u, w_{ya}\rangle_{\mathcal{H}}\,da\,dy = \int_G\Big(\int_G \phi(y)X(t^{-1}y)\,dy\Big)\langle u, w_t\rangle_{\mathcal{H}}\,dt\\ &= \Big\langle u, \int_G Z(t^{-1})\,w_t\,dt\Big\rangle_{\mathcal{H}}.\end{aligned}$$

By Fubini's theorem for $\nu\otimes dt$ (the integrand $\langle\Phi(\gamma), Z(t^{-1})F(\gamma t)\rangle$ is bounded by

$$\sup\|\Phi\|\,|Z(t^{-1})|\,\|F(\gamma t)\|,$$

of double integral $\sup\|\Phi\|\,\|F\|_1\int|Z|$),

$$\|\tilde\Phi\|_{\mathcal{H}}^2 = \int\Big\langle\Phi(\gamma), \int_G Z(t^{-1})F(\gamma t)\,dt\Big\rangle\,d\nu(\gamma) = \int_G Z(t^{-1})\,\langle\tilde\Phi, w_t\rangle_{\mathcal{H}}\,dt = \langle\tilde\Phi,\Omega\rangle_{\mathcal{H}} \le \|\tilde\Phi\|_{\mathcal{H}}\|\Omega\|_{\mathcal{H}}.$$

Hence $\|\tilde\Phi\|_{\mathcal{H}} \le \|\Omega\|_{\mathcal{H}}$. As $\|\Phi\|^2$ is left U-invariant, the left side of (iii) is $\|\tilde\Phi\|_{\mathcal{H}}^2 \le \|\Omega\|_{\mathcal{H}}^2 \le B$, and by (5.150) $B = (d^{-1}\int|X|^2)(d^{-1}\|F\|_2^2)\int|\phi|^2$. □

Lemma 5.7.20. *Let D be admissible data, $\varrho = m/2-1$ and $z \in \{\mathrm{I},\mathrm{II}\}$. Let $\mathfrak{n}_z = \{0,\dots,r-1+[z]\}$ be the index set of the node $(z,0,0)$ (the basis of Y_0, and for $z=\mathrm{II}$ the basis of $\tilde Y_0^0 = Y_0\oplus\langle 1\rangle$), let ε_z be its signs, and for $v\in\Phi$ let $\lambda_z(v) = \frac12 p_{z,0,0}(v)$, that is $\lambda_z(v)(i) = \varsigma_i(v) + \frac12[z]$; thus $\lambda_{\mathrm{I}}(v) = \varsigma(v)$ is the Harish-Chandra parameter of Y_0 and $\lambda_{\mathrm{II}}(v)$ is minus that of $\tilde Y_0^0$. Let $X = \{0,\dots,k-2\}$ and let $e\colon X\sqcup\{0,\dots,2\varrho+1+[z]\}\to\mathfrak{n}_z$ be given by $e(x) = x$ for $x\in X$ and $e(i) = k-1+i$ on the second summand. Then e is a bijection, and for every $v\in\Phi$ the doubled parameter $l = p_{z,0,0}(v) = 2\lambda_z(v)$ is steep for $(\varepsilon_z, e, \kappa_0(v))$: with $\mu_z(v) = \mu_l$ the lowest K-type of l for the signs ε_z,*

$$\varepsilon_z(e(x)) = 1,\quad \mu_z(v)(e(x)) = 0 \qquad (x\in X), \qquad\qquad \varepsilon_z(e(i)) = (-1)^i \qquad (0\le i\le 2\varrho+1+[z]),$$

and $i\mapsto\mu_z(v)(e(i))$ is strictly decreasing with values in $[-3\kappa_0(v), -\kappa_0(v)]$.

Proof. Fix z and $v\in\Phi$ and write $N = r+[z]$, $\lambda_i = \lambda_z(v)(i)$, $\varepsilon_i = \varepsilon_z(i)$, $\mu = \mu_z(v)$, $\kappa = \kappa(v)$ and $\kappa_j = \kappa_j(v)$; note $\kappa_{j+1}-\kappa_j = \Delta$ and $\kappa_0 = \kappa + \frac{\Delta}{2}$.

Sizes. We use three inequalities:

$$\text{(a) } m+1 < r, \qquad \text{(b) } N \le r+1 < \Delta, \qquad \text{(c) } \Delta^2 \le \kappa. \tag{5.152}$$

By Lemma 2.3.6, $\log x \ge 10^{14}C_P \ge 10^{14}\cdot 18480$, $x^{10^6} \le x_1$ and $x_1^{10^6} \le b$. By (AD2), $m \le 2\sqrt{x} < x^5 - 1 \le r-1$, which is (a), and (b) holds since $\Delta = 2r^8$. For (c): by (AD2), $\Delta^2 = 4r^{16} \le 4\cdot 3^{16}x^{80} \le x^{81} \le x_1$, and by (AD4), $\kappa\in\{b,\ xb/x_1\}$, so $\kappa \ge b/x_1 \ge x_1 \ge \Delta^2$, because $x\ge 1$ and $x_1^2 \le b$.

The bijection and the signs. Since $r = k-1+m$ and $m = 2\varrho+2$, $\#\mathfrak{n}_z = N = (k-1)+(2\varrho+2+[z])$, and e maps X onto $\{0,\dots,k-2\}$ and the second summand onto $\{k-1,\dots,N-1\}$; so e is a bijection. At the node $(z,0,0)$ there are no blocks, and by (GP5) with $b=0$ the sign of the base index i is $(-1)^{e_i}$ with $e_i = \max(i+1-k,0)$. Hence $\varepsilon_i = 1$ for $i \le k-1$ and $\varepsilon_{k-1+a} = (-1)^a$ for $a \ge 0$; in particular $\varepsilon(e(x)) = 1$ for $x \in X$ and $\varepsilon(e(i)) = (-1)^i$.

The entries. By (GP5),

$$\lambda_i = \tfrac{k}{2} - 1 - i + \tfrac{1}{2}[z] \quad (0 \le i \le k-2), \qquad \lambda_{k-1+a} = \tfrac{\Delta}{2} - \kappa_{a+1} + \tfrac{1}{2}[z] \quad (0 \le a \le m-1+[z]).$$

The sequence (λ_i) is strictly decreasing: it falls by 1 along the first range, by Δ along the second, and $\lambda_{k-1} - \lambda_{k-2} = -\kappa - \Delta - 1 + \frac{k}{2} < 0$ since $k < \Delta$. So λ is regular; its entries lie in $\mathbb{Z} + \frac{N-1}{2}$ because r is odd (AD2); it is a Harish-Chandra parameter for $\mathrm{U}(\operatorname{diag}\varepsilon)$.

The lowest K-type. Let $l = 2\lambda = p_{z,0,0}(v)$. As (λ_i) is strictly decreasing, the definitions of Definition 3.2.3 give $2\rho_l(i) = \#\{i' : i' > i\} - \#\{i' : i' < i\} = N-1-2i$ and $2\rho_l^{\mathrm{c}}(i) = \sum \operatorname{sgn}(i'-i)$ over the $i' \ne i$ with $\varepsilon_{i'} = \varepsilon_i$. The entries of λ lie in $\mathbb{Z} + \frac{N-1}{2}$, so $l_i + 2\rho_l(i) = 2\lambda_i + N - 1 - 2i$ is even and $l_i^\rho = \lambda_i + \rho_l(i)$; hence $2\mu(i) = 2\lambda_i + 2\rho_l(i) - 4\rho_l^{\mathrm{c}}(i)$. In $2\rho_l(i) - 4\rho_l^{\mathrm{c}}(i)$ an index $i' \ne i$ contributes $-\operatorname{sgn}(i'-i)$ if $\varepsilon_{i'} = \varepsilon_i$ and $\operatorname{sgn}(i'-i)$ otherwise, that is $-\varepsilon_i\varepsilon_{i'}\operatorname{sgn}(i'-i)$. Therefore

$$2\mu(i) = 2\lambda_i - L_i, \qquad L_i = \sum_{i' \ne i} \varepsilon_i\varepsilon_{i'} \operatorname{sgn}(i'-i), \qquad |L_i| \le N-1. \tag{5.153}$$

The near entries. Let $0 \le i \le k-2$, so $\varepsilon_i = 1$. The indices $i' < i$ have sign $+1$ and contribute $-i$ to L_i; the indices $i < i' \le k-1$ have sign $+1$ and contribute $k-1-i$; the indices $i' = k-1+a$, $1 \le a \le m-1+[z]$, contribute $\sum_{a=1}^{m-1+[z]}(-1)^a$, which is -1 for $[z]=0$ and 0 for $[z]=1$ (m is even), that is $[z]-1$. So $L_i = k-2-2i+[z] = 2\lambda_i$, and (5.153) gives $\mu(e(x)) = \mu(x) = 0$ for $x \in X$.

The far entries. Let $0 \le a \le m-1+[z]$ and $i = e(a) = k-1+a$. By (5.153),

$$2\mu(e(a)) = \Delta - 2\kappa_{a+1} + [z] - L_{k-1+a}.$$

Strict decrease: $2\mu(e(a+1)) - 2\mu(e(a)) = -2\Delta + L_{k-1+a} - L_{k+a} \le -2\Delta + 2(N-1) < 0$ by (5.152)(b). Upper bound: $[z] - L_i \le N$, so $2\mu(e(a)) \le \Delta - 2\kappa_1 + N = -2\kappa - 2\Delta + N \le -2\kappa - \Delta = -2\kappa_0$, by (b). Lower bound: $[z] - L_i \ge -N$ and $a+1 \le m+1$, so

$$2\mu(e(a)) \ge \Delta - 2\kappa_{m+1} - N = -2\kappa - 2(m+1)\Delta - N \ge -2\kappa - 3\Delta^2 \ge -6\kappa - 3\Delta = -6\kappa_0,$$

where $2(m+1)\Delta + N \le 2\Delta^2 + \Delta \le 3\Delta^2$ by (a), (b), and $3\Delta^2 \le 3\kappa \le 4\kappa + 3\Delta$ by (c). Hence $\mu(e(a)) \in [-3\kappa_0, -\kappa_0]$ for every a, which completes the proof. □

The construction of a Bessel system needs a lower bound for a period: an integral, over a group S mapped into two real unitary groups, of a coefficient of one discrete series against the conjugate of a coefficient of another, twisted by a unitary phase. Theorem 5.7.21 gives it under four conditions on the parameters, the phase and the embedding. It is used in the stationary point of the Bessel averages (Proposition 5.7.30 (a)) and again in Section 6, for the auxiliary forms (Lemma 6.4.109, Lemma 6.4.98). The method is Laplace's [66] (see [132] for the related method of stationary phase). After the phase is removed, Flensted-Jensen's function near the identity has a Gaussian shape (Lemma 5.7.24); in our reading the integrand is then nonnegative up to a small error near the identity and decays away from it. The Gaussian shape rests on an expansion to second order of the principal minors in Flensted-Jensen's formula (Proposition 5.7.25, Proposition 5.7.26).

Theorem 5.7.21. *Let $(\mathfrak{n},\mathfrak{m},\varepsilon,\eta,l,k,S,j,\varphi,X,M,e,\tau,B,\beta,L,h)$ be a Laplace datum, $j=(j_1,j_2)$. Suppose that S carries a topology and its Borel σ-algebra, for which left and right translations are measurable, that ν is a right-invariant measure on S that is finite on compact sets, and that j_1 is proper and j_2 continuous. Then $\theta(s)=c_{l/2}(j_1(s))\,\overline{c_{k/2}(j_2(s))}\,e^{-\varphi(s)}$ is ν-integrable, the set $\{s\in S:\|j_1(s)\|_{\mathrm{op}}\le\beta+1,\ \|j_2(s)\|_{\mathrm{op}}\le\beta+1\}$ has finite measure, and*

$$\nu\{s\in S:\|j_1(s)\|_{\mathrm{op}}\le\beta+1,\ \|j_2(s)\|_{\mathrm{op}}\le\beta+1\}\le\beta^{20(\#\mathfrak{n}+\#\mathfrak{m})^2}\Big|\int_S\theta\,d\nu\Big|.$$

Proof. Put $N=(\#\mathfrak{n}+\#\mathfrak{m})^2\ge1$ and let $J\colon S\to\mathrm{U}(\mathrm{diag}(\varepsilon\oplus\eta))$, $J(s)=\mathrm{diag}(j_1(s),j_2(s))$, the block sum, a homomorphism into the unitary group of the $(\mathfrak{n}\sqcup\mathfrak{m})$-indexed signs $\varepsilon\oplus\eta$. For block-diagonal matrices $\|J(s)\|=\max(\|j_1(s)\|,\|j_2(s)\|)$ and $\|J(s)-1\|=\max(\|j_1(s)-1\|,\|j_2(s)-1\|)$. Let G be as in Lemma 5.7.22, so $G(s)=(2/\|J(s)\|)^{9N}$.

Step 1: J is proper. j_1 is proper, hence continuous, and j_2 is continuous, so J is continuous. If $C\subseteq\mathrm{U}(\mathrm{diag}(\varepsilon\oplus\eta))$ is compact, its image C_1 under the continuous map $g\mapsto$ (upper left $\mathfrak{n}\times\mathfrak{n}$ block) is compact, and $J^{-1}(C)$ is a closed subset (closed as J is continuous and C is closed in the Hausdorff group) of the compact set $j_1^{-1}(C_1\cap\mathrm{U}(\mathrm{diag}\,\varepsilon))$; so $J^{-1}(C)$ is compact.

Step 2: integrability. The function $c_{l/2}$ is continuous on $\mathrm{U}(\mathrm{diag}\,\varepsilon)$: by Lemma 5.6.25, in (3.2) of Definition 3.2.3 with $a=\mathrm{bd}_\varepsilon(g)$, $z=a^{-1}g$, the integrand is a continuous function of $(g,k)\in\mathrm{U}(\mathrm{diag}\,\varepsilon)\times K_\varepsilon$ (its denominators do not vanish), and an integral over the compact K_ε of a jointly continuous function is continuous in the parameter. Likewise for $c_{k/2}$, and φ is continuous by the formula of (LA2). So θ is continuous, hence measurable. By Lemma 5.6.32 (ii) for the proper J, $y=1$ and $c=9N>8N$, G is ν-integrable and

$$\int_S G\,d\nu\le16^{9N}\,\nu(U_1),\qquad U_1=\{s:\|J(s)-1\|\le1\},\tag{5.154}$$

and by Lemma 5.7.22 (i), $|\theta|\le G$, so θ is ν-integrable.

Step 3: covering. Let $\mathcal{L}=\{s:\|j_1(s)\|\le\beta+1,\ \|j_2(s)\|\le\beta+1\}=\{s:\|J(s)\|\le\beta+1\}$ and $\mathcal{S}=\{s:\|J(s)-1\|\le(200\beta^5)^{-1}\}$. Apply Lemma 5.6.32 (i) to J (index set of $\#\mathfrak{n}+\#\mathfrak{m}$ elements) with $y=1$, $D=200\beta^5$, $R=\beta+1$, $K=\beta^8$: indeed $3DR^2=600\beta^5(\beta+1)^2\le1200\beta^7\le\beta^8$ as $\beta\ge1200$. So

$$\nu(\mathcal{L})\le\beta^{16N}\nu(\mathcal{S}),$$

and $\nu(\mathcal{S})<\infty$ by Lemma 5.6.32 (ii); hence $\nu(\mathcal{L})<\infty$. Also $U_1\subseteq\mathcal{L}$, since $\|J(s)\|\le1+\|J(s)-1\|\le2\le\beta+1$.

Step 4: integration of the pointwise bounds. By Lemma 5.7.22 (ii), $|\theta|\ge\frac14$ on $\mathcal{S}$, and by (i), $|\theta|\le2\,\mathrm{Re}\,\theta+6\beta^{-50N}G$ everywhere. Hence, with (5.154),

$$\tfrac14\nu(\mathcal{S})\le\int_S|\theta|\,d\nu\le2\,\mathrm{Re}\int_S\theta\,d\nu+6\beta^{-50N}\int_S G\,d\nu\le2I+6\beta^{-50N}16^{9N}\nu(\mathcal{L}),\qquad I=\Big|\int_S\theta\,d\nu\Big|.$$

Step 5: conclusion. Write $V=\nu(\mathcal{L})<\infty$. Steps 3 and 4 give $V\le\beta^{16N}(8I+24\beta^{-50N}16^{9N}V)$. Since $16\le\beta$ and $48\le\beta$, $48\cdot16^{9N}\beta^{16N}\le\beta\cdot\beta^{9N}\beta^{16N}\le\beta^{50N}$, so $24\beta^{16N}\beta^{-50N}16^{9N}\le\frac12$ and $V\le8\beta^{16N}I+\frac12V$, i.e. $V\le16\beta^{16N}I\le\beta^{20N}I$, as $16\le\beta^{4N}$. □

Lemma 5.7.22. *Let $(\mathfrak{n},\mathfrak{m},\varepsilon,\eta,l,k,S,j,\varphi,X,M,e,\tau,B,\beta,L,h)$ be a Laplace datum, $j=(j_1,j_2)$. Put $N=(\#\mathfrak{n}+\#\mathfrak{m})^2$,*

$$\theta(s)=c_{l/2}(j_1(s))\,\overline{c_{k/2}(j_2(s))}\,e^{-\varphi(s)}\quad\text{and}\quad G(s)=\big(2/\max(\|j_1(s)\|,\|j_2(s)\|)\big)^{9N}.$$

Then for every $s\in S$:

1. $|\theta(s)| \le G(s)$ *and* $|\theta(s)| \le 2\operatorname{Re}\theta(s) + 6\beta^{-50N}G(s)$;

2. *if* $\max(\|j_1(s)-1\|, \|j_2(s)-1\|) \le (200\beta^5)^{-1}$, *then* $|\theta(s)| \ge \frac{1}{4}$.

Proof. Throughout, $s \in S$, $g_1 = j_1(s)$, $g_2 = j_2(s)$, $x_i = g_i - 1$, and $y(g) = g - \mathrm{bd}(g)$ is the noncompact part (for the signs of the group of g). By (LA2), $\operatorname{Re}\varphi = 0$, so

$$|\theta(s)| = |c_{l/2}(g_1)|\,|c_{k/2}(g_2)|. \tag{5.155}$$

Since $\beta \ge 10^{12} > e$, $\log\beta \ge 1$; put $T = N\log\beta \ge 1$. By (LA4), $h \ge 10^{30}T^3 \ge 18N$. The vectors l, k have gap $500(\#\mathfrak{n})^3 + h \ge h$, respectively $500(\#\mathfrak{m})^3 + h \ge h$, and so does $-k$ (gaps are invariant under $l \mapsto -l$).

Step 1: decay. By Lemma 5.7.38 (1), $|c_{l/2}| \le 1$, $|c_{k/2}| \le 1$ and $|c_{l/2}(g)| \le (2/\|g\|)^c$, $|c_{k/2}(g)| \le (2/\|g\|)^c$ for every integer $0 \le c \le h$. Bounding by (5.155) the factor at the component of larger norm by the decay and the other by 1, with $c = 9N$ and $c = 18N$:

$$|\theta(s)| \le G(s), \qquad |\theta(s)| \le G(s)^2. \tag{5.156}$$

This is the first claim of (i).

Step 2: far from the compact subgroups. Put $\rho = 10^{-13}/T$ and suppose $\|y(g_1)\|_{\mathrm{HS}}^2 + \|y(g_2)\|_{\mathrm{HS}}^2 \ge \rho^2$. For $g \in \mathrm{U}(\operatorname{diag}\sigma)$, $J = \operatorname{diag}(\sigma)$ and $g' = JgJ$ we have $\mathrm{bd}(g) = \frac{1}{2}(g+g')$, $y(g) = \frac{1}{2}(g-g')$ and $g^*g' = g^*JgJ = 1 = g'^*g$, hence $\mathrm{bd}(g)^*\mathrm{bd}(g) - y(g)^*y(g) = \frac{1}{2}(g^*g' + g'^*g) = 1$. With $\sigma_i \ge 0$ the eigenvalues of y^*y,

$$|\det \mathrm{bd}(g)|^2 = \det(1+y^*y) = \prod_i (1+\sigma_i) \ge 1 + \sum_i \sigma_i = 1 + \|y(g)\|_{\mathrm{HS}}^2.$$

By Lemma 5.6.29 (i) with $c = h$, $|c_{l/2}(g_1)|^4 \le |\det \mathrm{bd}(g_1)|^{-2h} \le (1+\|y(g_1)\|_{\mathrm{HS}}^2)^{-h}$, and likewise for $c_{k/2}(g_2)$. By (5.155),

$$|\theta(s)|^4 \le \big((1+\|y(g_1)\|_{\mathrm{HS}}^2)(1+\|y(g_2)\|_{\mathrm{HS}}^2)\big)^{-h} \le (1+\rho^2)^{-h}.$$

As $\log(1+u) \ge 2u/(u+2)$ for $u \ge 0$, $\rho \le 10^{-13}$ and $\rho^2T^2 = 10^{-26}$,

$$h\log(1+\rho^2) \ge 10^{30}T^3\frac{2\rho^2}{\rho^2+2} = \frac{2\cdot 10^4\,T}{\rho^2+2} \ge 400\,T,$$

so $|\theta(s)| \le e^{-100T}$, and therefore $|\theta(s)| \le 2\operatorname{Re}\theta(s) + 3e^{-100T}$, as $\operatorname{Re}\theta \ge -|\theta|$.

Step 3: the phase at the logarithms. For pairs $Z = (Z_1, Z_2)$ of $\mathfrak{n}\times\mathfrak{n}$ and $\mathfrak{m}\times\mathfrak{m}$ matrices (added and multiplied componentwise) put $\Psi(Z) = \operatorname{tr}(\operatorname{diag}(\mu_l)Z_1) - \operatorname{tr}(\operatorname{diag}(\mu_k)Z_2)$, a linear form, and $x(t) = (j_1(t)-1, j_2(t)-1)$, so that $\Psi(x(t)) = \varphi(t)$ by (LA2). Since $x(s)(1+x(t)) = x(st) - x(t)$, (LA2) gives $\Psi(x(s)(1+x(t))) = \varphi(st) - \varphi(t) = \varphi(s)$, and by induction on i, writing $x = x(s)$,

$$\Psi\big(x^i x(1+x(t))\big) = 0^i\varphi(s) \quad (t \in S,\ i \ge 0),$$

because $x^{i+1}x(1+x(t)) = x^i x(1+x(st)) - x^i x(1+x(t))$. With $t = 1$: $\Psi(x^{i+1}) = 0^i\varphi(s)$. If $\|x_1\|_{\mathrm{HS}}, \|x_2\|_{\mathrm{HS}} \le \frac{1}{2}$, then $(1+rx)^{-1}x = \sum_{i\ge0}(-r)^i x^{i+1}$ converges absolutely for $0 \le r \le 1$, and integrating over r (lg as in Lemma 5.7.24, applied componentwise),

$$\Psi\big(\lg(x_1), \lg(x_2)\big) = \int_0^1 \sum_{i\ge0}(-r)^i 0^i\varphi(s)\,dr = \varphi(s). \tag{5.157}$$

Step 4: near the compact subgroups. Suppose $\|y(g_1)\|_{\mathrm{HS}}^2+\|y(g_2)\|_{\mathrm{HS}}^2\le\rho^2$. As $\rho\le\frac16$, Lemma 5.7.23 gives $t\in S$ with $\theta(t)=\theta(s)$ and $\|j_i(t)-1\|_{\mathrm{HS}}\le\delta:=4\rho$, $\delta T=4\cdot10^{-13}\le10^{-12}$; replace s by t, so that $\|x_i\|_{\mathrm{HS}}\le\delta\le\frac12$. By Lemma 5.6.29 (ii), $\overline{c_{k/2}(g_2)}=c_{-k/2}(g_2)$. From the definition of the lowest K-type (Definition 3.2.3), $2\rho_{-k}=-2\rho_k$ and $2\rho^{\mathrm c}_{-k}=-2\rho^{\mathrm c}_k$, and $k_i+2\rho_k(i)$ is even (the entries of k are distinct and $\equiv\#\mathfrak m-1 \bmod 2$, and $2\rho_k(i)\equiv\#\mathfrak m-1 \bmod 2$), so $\mu_{-k}=-\mu_k$. By (5.157),

$$\theta(s)=R_1R_2,\qquad R_1=\frac{c_{l/2}(g_1)}{\exp\mathrm{tr}(\mathrm{diag}(\mu_l)\lg(x_1))},\quad R_2=\frac{c_{-k/2}(g_2)}{\exp\mathrm{tr}(\mathrm{diag}(\mu_{-k})\lg(x_2))}.$$

Apply Lemma 5.7.24 to (ε,l,g_1) and to $(\eta,-k,g_2)$, with $A=h$, L, δ, T: $1\le T$; $(\#\mathfrak n)^2\le N\le T$; $(\#\mathfrak n)^2\log(\#\mathfrak n+L+1)\le N\log\beta=T$ since $\#\mathfrak n+L+1\le\beta$ by (LA4); the same for $\mathfrak m$; $\delta T\le10^{-12}$. So $|R_i-\varkappa_i|\le\varkappa_i/10+E$ with $\varkappa_i\in[0,1]$ and $E=e^{-100T}\in[0,1]$. Writing $R_1R_2-\varkappa_1\varkappa_2=\varkappa_1(R_2-\varkappa_2)+(R_1-\varkappa_1)\varkappa_2+(R_1-\varkappa_1)(R_2-\varkappa_2)$,

$$|\theta(s)-p|\le\varkappa_1(\tfrac{\varkappa_2}{10}+E)+(\tfrac{\varkappa_1}{10}+E)\varkappa_2+(\tfrac{\varkappa_1}{10}+E)(\tfrac{\varkappa_2}{10}+E)\le0.21\,p+4E,\qquad p=\varkappa_1\varkappa_2\in[0,1].$$

Hence $|\theta(s)|\le p+|\theta(s)-p|$ and $\mathrm{Re}\,\theta(s)\ge p-|\theta(s)-p|$ give $|\theta(s)|-2\,\mathrm{Re}\,\theta(s)\le3|\theta(s)-p|-p\le12E$.

Step 5: the cone with a decaying error. By Steps 2 and 4, $d(s):=|\theta(s)|-2\,\mathrm{Re}\,\theta(s)\le12E$ for every s, and $E=\beta^{-100N}=(\beta^{-50N})^2$. Also $d(s)\le3|\theta(s)|\le3G(s)^2$ by (5.156). If $d(s)>6\beta^{-50N}G(s)$ (≥0), then $d(s)^2>36\beta^{-100N}G(s)^2\ge12\beta^{-100N}\cdot3G(s)^2\ge d(s)\cdot d(s)$, a contradiction. So $|\theta(s)|\le2\,\mathrm{Re}\,\theta(s)+6\beta^{-50N}G(s)$, which completes (i).

Step 6: near the identity. By (3.2) of Definition 3.2.3,

$$c_{l/2}(1)=\mathrm{FJ}_l(1,1)=\|m_{\mu_l}\|^{-2}\int_{K_\varepsilon}|m_{\mu_l}(k)|^2/m_{-l^\rho}(1)\,dk=1,$$

as every principal minor of 1 is 1; likewise $c_{k/2}(1)=1$. By Lemma 5.7.38 (2) with $x=1$ (large gap h, entries bounded by L), $|c_{l/2}(g_1)-1|\le100(\#\mathfrak n+L+1)^5\|g_1-1\|\le100\beta^5(200\beta^5)^{-1}=\frac12$ under the hypothesis of (ii), as $\#\mathfrak n+L+1\le\beta$. So $|c_{l/2}(g_1)|\ge\frac12$, likewise $|c_{k/2}(g_2)|\ge\frac12$, and (5.155) gives (ii). □

Lemma 5.7.23. *Let $\mathfrak n,\mathfrak m,\varepsilon,\eta,l,k,S,j=(j_1,j_2),\varphi,X,M,e,\tau,B$ be data as in Definition 5.7.2 (without β, L, h), with $B\ge1$, and suppose that (LA1), (LA2) and (LA3) of Definition 5.7.2 hold. Put $\theta(s)=c_{l/2}(j_1(s))\,\overline{c_{k/2}(j_2(s))}\,e^{-\varphi(s)}$, and for g in $\mathrm U(\mathrm{diag}\,\varepsilon)$ or $\mathrm U(\mathrm{diag}\,\eta)$ let $y(g)=g-\mathrm{bd}(g)$ be its noncompact part (for the signs ε, respectively η). Let $0<r\le\frac16$ and $s\in S$ with*

$$\|y(j_1(s))\|_{\mathrm{HS}}^2+\|y(j_2(s))\|_{\mathrm{HS}}^2\le r^2.$$

Then there is $t\in S$ with $\theta(t)=\theta(s)$, $\|j_1(t)-1\|_{\mathrm{HS}}\le4r$ and $\|j_2(t)-1\|_{\mathrm{HS}}\le4r$.

Proof. All norms are Hilbert–Schmidt norms. For g in a unitary group $\mathrm U(\mathrm{diag}\,\sigma)$ ($\sigma=\varepsilon$ or η) write $x(g)=g-1$ and $J=\mathrm{diag}(\sigma)$.

Step 1: the row identity. For $g\in\mathrm U(\mathrm{diag}\,\sigma)$ and every index i,

$$2\,\mathrm{Re}\,x(g)_{ii}=2\sum_j|y(g)_{ij}|^2-\sum_j|x(g)_{ij}|^2.\tag{5.158}$$

Indeed $g^*Jg=J$ gives $g^{-1}=Jg^*J$, so $(gJg^*J)_{ii}=1$, i.e. $\sum_j\sigma_i\sigma_j|g_{ij}|^2=1$, i.e. $\sum_j|g_{ij}|^2-2\sum_j|y(g)_{ij}|^2=1$; and $\sum_j|x(g)_{ij}|^2=\sum_j|g_{ij}|^2-2\,\mathrm{Re}\,g_{ii}+1$, $\mathrm{Re}\,x(g)_{ii}=\mathrm{Re}\,g_{ii}-1$. Summing (5.158) over i:

$$\|x(g)\|^2=2\|y(g)\|^2-2\,\mathrm{Re}\,\mathrm{tr}\,x(g).\tag{5.159}$$

Step 2: a weighted form. Let v, w be real vectors and a, C reals with $av_i^2 \le w_i$ for all i and $w_i \le C$ for every i such that $\sigma_{i'} \ne \sigma_i$ for some i'. Then

$$a\|\mathrm{diag}(v)x(g)\|^2 + 2\,\mathrm{Re}\,\mathrm{tr}\big(\mathrm{diag}(w)x(g)\big) \le 2C\|y(g)\|^2. \tag{5.160}$$

Multiply (5.158) by w_i and sum: the left side of (5.160) is

$$\sum_i (av_i^2 - w_i) \sum_j |x(g)_{ij}|^2 + \sum_i 2w_i \sum_j |y(g)_{ij}|^2.$$

The first sum is ≤ 0; in the second, $y(g)_{ij} \ne 0$ only if $\sigma_i \ne \sigma_j$, and then $w_i \le C$.

Step 3: rows and columns off X. Let P be the diagonal $\mathfrak{n} \times \mathfrak{n}$ matrix with entry 1 at the indices $e(i)$, $0 \le i < M$, and 0 at the indices $e(x)$, $x \in X$. For $s \in S$ write $g_1 = j_1(s)$, $g_2 = j_2(s)$, $Y(s) = \|y(g_1)\|^2 + \|y(g_2)\|^2$. Apply (5.160) to g_1 with $v =$ diag entries of P, $w = -\mu_l$, $a = B$, $C = 3B$: by (LA1), $-\mu_l = 0 \ge B \cdot 0$ on $e(X)$ and $B \le -\mu_l \le 3B$ off $e(X)$. Apply it to g_2 with $v = 0$, $w = \mu_k$, $a = 0$, $C = 3B$, legitimate by (LA3). This gives

$$B\|Px(g_1)\|^2 - 2\,\mathrm{Re}\,\mathrm{tr}\big(\mathrm{diag}(\mu_l)x(g_1)\big) \le 6B\|y(g_1)\|^2, \qquad 2\,\mathrm{Re}\,\mathrm{tr}\big(\mathrm{diag}(\mu_k)x(g_2)\big) \le 6B\|y(g_2)\|^2.$$

By (LA2) the difference of the two traces is $\varphi(s)$, which has real part 0. Adding and dividing by $B > 0$:

$$\|P\,x(j_1(s))\|^2 \le 6\,Y(s) \qquad (s \in S). \tag{5.161}$$

For the columns apply (5.161) to s^{-1}: $j_1(s^{-1}) = g_1^{-1} = Jg_1^*J$ ($J = \mathrm{diag}(\varepsilon)$), so $x(j_1(s^{-1})) = Jx(g_1)^*J$ and $\|PJx(g_1)^*J\| = \|x(g_1)P\|$ (J is diagonal unitary and commutes with P); likewise $y(g^{-1}) = Jy(g)^*J$ has the norm of $y(g)$, so $Y(s^{-1}) = Y(s)$. Hence $\|x(j_1(s))P\|^2 \le 6Y(s)$.

Step 4: the block on X. Let a be the $X \times X$ matrix $a_{xx'} = (g_1)_{e(x)e(x')}$, $g_1 = j_1(s)$, and $W = Px(g_1)$. Since $\varepsilon = 1$ on $e(X)$, the $(e(x), e(x'))$ entry of $g_1^*Jg_1 = J$ reads $(a^*a)_{xx'} + \sum_{i \notin e(X)} \varepsilon_i \overline{(g_1)_{i,e(x)}} (g_1)_{i,e(x')} = \delta_{xx'}$, and $(g_1)_{i,e(x)} = W_{i,e(x)}$ for $i \notin e(X)$. So $a^*a - 1$ is a submatrix of $-W^*JW$, and by (5.161) and the hypothesis

$$\|a^*a - 1\| \le \|W\|^2 \le 6r^2 \le r < 1.$$

Hence a^*a is invertible, a is invertible, and $a = k_1 z_1$ with k_1 unitary and $z_1 = (a^*a)^{1/2}$ positive definite. If $\sigma_i \ge 0$ are the eigenvalues of z_1, then $|\sigma_i - 1| \le |\sigma_i - 1|(\sigma_i + 1) = |\sigma_i^2 - 1|$, so $\|z_1 - 1\| \le \|z_1^2 - 1\| \le r$.

Step 5: translation by the diagonal copy. By (LA3) there is $s_0 \in S$ with $j_1(s_0) - 1 = \iota_e(k_1^* - 1)\iota_e^*$ and $j_2(s_0) - 1 = \iota_\tau(k_1^* - 1)\iota_\tau^*$; put $t = s_0 s$, $h_1 = j_1(s_0)$, $h_2 = j_2(s_0)$. The matrix $h_1 - 1$ is supported on $e(X) \times e(X)$, where $\varepsilon = 1$ and $\mu_l = 0$ (LA1); the matrix $h_2 - 1$ is supported on $\tau(X) \times \tau(X)$, where $\eta = 1$ and $\mu_k = 0$ (LA3). (a) $\varphi(t) = \varphi(s)$: by (LA2), $\varphi(s_0) = \mathrm{tr}(\mathrm{diag}(\mu_l)(h_1 - 1)) - \mathrm{tr}(\mathrm{diag}(\mu_k)(h_2 - 1)) = 0$, as the diagonals of $h_1 - 1$, $h_2 - 1$ live where the weights vanish, and $\varphi(t) = \varphi(s_0) + \varphi(s)$. (b) $c_{l/2}(h_1 g) = c_{l/2}(g)$ for every $g \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$, and likewise $c_{k/2}(h_2 g') = c_{k/2}(g')$. Let $h = h_1$, $\mu = \mu_l$. The off-diagonal entries of h lie in $e(X) \times e(X)$, where the signs agree, so h is block diagonal, $\mathrm{bd}_\varepsilon(hg) = h\,\mathrm{bd}_\varepsilon(g)$ and $\mathrm{bd}_\varepsilon(hg)^{-1}hg = \mathrm{bd}_\varepsilon(g)^{-1}g$. By (3.2) of Definition 3.2.3 it remains to show $m_\mu(hy) = m_\mu(y)$ for invertible y. If $X = \emptyset$ then $h = 1$. Otherwise $\min \mu \le 0 \le \max \mu$. For $t \in \mathbb{Z}$ and $Q = \{i : \mu_i \ge t\}$, $h_{ij} = 0$ for $i \in Q$, $j \notin Q$ (such $i \ne j$ have $\mu_i \ne \mu_j$, while h has off-diagonal entries only between indices of weight 0), so $\Delta_Q(hy) = \Delta_Q(h)\Delta_Q(y)$; and $\Delta_Q(h) = \det h$ if $t \le 0$ (then $Q \supseteq e(X)$) and $\Delta_Q(h) = 1$ if $t > 0$ (then $Q \cap e(X) = \emptyset$). By (3.1), $m_\mu(hy) = m_\mu(y) \cdot (\det h)^{\min \mu} (\det h)^{\#\{t : \min \mu < t \le 0\}} = m_\mu(y)$. The same argument applies to h_2, μ_k, η. (c) Hence $\theta(t) = \theta(s)$. (d) The noncompact parts keep their norms: h_1 is block diagonal and in $\mathrm{U}(\mathrm{diag}\,\varepsilon)$, hence commutes with J and is unitary;

$y(h_1 g) = h_1 g - h_1 \mathrm{bd}_\varepsilon(g) = h_1 y(g)$ has the norm of $y(g)$. Likewise for h_2. So $Y(t) = Y(s) \le r^2$. (e) The block of $j_1(t) = h_1 g_1$ on X: the rows $e(x)$ of h_1 are those of $\iota_e k_1^* \iota_e^*$, so this block is $k_1^* a = z_1$.

Step 6: the bounds at t. Every entry of $x(j_1(t))$ lies in a row off $e(X)$, or a column off $e(X)$, or in $e(X) \times e(X)$, so by Step 3 at t, Step 5 (d), (e) and Step 4,

$$\|x(j_1(t))\|^2 \le \|Px(j_1(t))\|^2 + \|x(j_1(t))P\|^2 + \|z_1 - 1\|^2 \le 6r^2 + 6r^2 + r^2 = 13r^2.$$

By (5.159) for $j_1(t)$ and $j_2(t)$ and $\operatorname{tr} x(j_1(t)) = \operatorname{tr} x(j_2(t))$ (LA2), $\|x(j_2(t))\|^2 = \|x(j_1(t))\|^2 - 2\|y(j_1(t))\|^2 + 2\|y(j_2(t))\|^2 \le 13r^2 + 2r^2 = 15r^2$. Since $15 < 16$, both norms are $\le 4r$. □

Lemma 5.7.24. *For a complex square matrix Y with $\|Y\|_{\mathrm{HS}} \le \frac{1}{2}$ put $\lg(Y) = \int_0^1 (1+tY)^{-1} Y\, dt$, the logarithm of $1+Y$. Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $l \in \mathbb{Z}^n$, $A, L \ge 0$ integers and δ, T reals. Suppose that l has large gap A for ε, $|l_i| \le L$ for every i, and*

$$1 \le T, \qquad (\#n)^2 \le T, \qquad (\#n)^2 \log(\#n + L + 1) \le T, \qquad \delta T \le 10^{-12}.$$

Then for every $g \in \mathrm{U}(\operatorname{diag}\varepsilon)$ with $\|g - 1\|_{\mathrm{HS}} \le \delta$ there is a real number $\varkappa \in [0,1]$ with

$$\Big| \frac{c_{l/2}(g)}{\exp \operatorname{tr}\big(\operatorname{diag}(\mu_l) \lg(g-1)\big)} - \varkappa \Big| \le \frac{\varkappa}{10} + e^{-100T}.$$

Proof. Write $N = \#n$, $X = g - 1$, $\mu = \mu_l$, $\Upsilon = l^\rho$, $c_\mu = \|m_\mu\|^2 = \int_{K_\varepsilon} |m_\mu|^2 dh$, $\|\cdot\|$ for the Hilbert–Schmidt norm, and gw_w as in Lemma 5.7.28. Since $T \ge 1$, $\|X\| \le \delta \le \delta T \le 10^{-12}$. Put

$$E = \operatorname{tr}\big(\operatorname{diag}(\mu) \lg(X)\big), \qquad q_0 = \tfrac{1}{4}\mathrm{gw}_\Upsilon(X) \ge 0, \qquad \varkappa = e^{-q_0} \in (0,1].$$

A vector with large gap A has gap 0.

The formula. Proposition 5.7.25 gives an accretive matrix y with $c_{l/2}(g) = \mathrm{FJ}_l(1, y)$, that is, by (3.2),

$$c_{l/2}(g) = \frac{1}{c_\mu} \int_{K_\varepsilon} \frac{|m_\mu(h)|^2}{m_{-\Upsilon}(h^* y h)}\, dh. \tag{5.162}$$

The denominator. Fix $h \in K_\varepsilon$. Let $w = -\Upsilon$, $w_\pm$ its largest and smallest entry, $P_\tau = \{i : w_i \ge \tau\}$ and p_τ the diagonal projection onto P_τ. If $i \in P_\tau$ and $i' \notin P_\tau$, then $\Upsilon_i < \Upsilon_{i'}$, hence $l_i < l_{i'}$ (if $l_i > l_{i'}$, then $\Upsilon_i \ge \Upsilon_{i'}$ by Lemma 5.6.27 (iii)). So Proposition 5.7.25 applies to every P_τ and gives numbers η_τ with

$$\Delta_{P_\tau}(h^* y h) = \exp\Big(\operatorname{tr}(p_\tau \lg(X)) + \tfrac{1}{4}\|[p_\tau, X]\|^2 + \eta_\tau\Big), \qquad |\eta_\tau| \le 2 \cdot 10^5\, \delta(\|[p_\tau, X]\|^2 + \|[p_\tau, h]\|^2);$$

applied to the set n itself, where both commutators vanish, it gives $\det(h^* y h) = \exp \operatorname{tr} \lg(X)$. Inserting this into (3.1) for the weight w and summing over $w_- < \tau \le w_+$ with Lemma 5.7.28 (i) (the traces add up to $\operatorname{tr}(\operatorname{diag}(w) \lg(X))$, the squares to $\mathrm{gw}_w(X) = \mathrm{gw}_\Upsilon(X)$),

$$m_{-\Upsilon}(h^* y h) = \exp\big(q_0 - E - r(h)\big), \qquad r(h) = \operatorname{tr}\big(\operatorname{diag}(\Upsilon - \mu) \lg(X)\big) - \sum_\tau \eta_\tau. \tag{5.163}$$

Bound for $r(h)$. By definition $\Upsilon - \mu = 2\rho_l^{\mathrm{c}}$, whose entries have absolute value at most N, and $|\lg(X)_{ii}| \le \|\lg(X)\| \le 2\delta$ by Lemma 5.7.27; so the trace term is at most $2N^2\delta$. By Lemma 5.7.28 (i), $\sum_\tau \|[p_\tau, X]\|^2 = 4q_0$ and $\sum_\tau \|[p_\tau, h]\|^2 = \mathrm{gw}_\Upsilon(h) \le \mathrm{gw}_\mu(h) + 2N\|h\|^2 = \mathrm{gw}_\mu(h) + 2N^2$, since $|\Upsilon_i - \Upsilon_{i'}| \le |\mu_i - \mu_{i'}| + 2N$ and $\|h\|^2 = N$ for unitary h. Hence, with $G_0(h) = q_0 + \frac{1}{2}\mathrm{gw}_\mu(h)$,

$$|r(h)| \le 2N^2\delta + 2 \cdot 10^5 \delta\big(4q_0 + \mathrm{gw}_\mu(h) + 2N^2\big) \le 10^6 \delta\big(G_0(h) + N^2\big). \tag{5.164}$$

The comparison. By (5.162), (5.163) and $\int |m_\mu(h)|^2 dh = c_\mu$ (the measure has mass one),

$$\frac{c_{l/2}(g)}{e^E} - \varkappa = \frac{1}{c_\mu}\int_{K_\varepsilon} |m_\mu(h)|^2 e^{-q_0}\big(e^{r(h)}-1\big)dh,$$
$$\frac{\varkappa}{10} + e^{-100T} = \frac{1}{c_\mu}\int_{K_\varepsilon} \Big(\frac{\varkappa}{10}|m_\mu(h)|^2 + e^{-100T}c_\mu\Big)dh.$$

So it suffices to show, for every $h \in K_\varepsilon$,

$$|m_\mu(h)|^2 e^{-q_0}\big|e^{r(h)}-1\big| \le \frac{\varkappa}{10}|m_\mu(h)|^2 + e^{-100T}c_\mu. \tag{5.165}$$

If $|r(h)| < \frac{1}{20}$, then $|e^{r(h)}-1| \le |r(h)|e^{|r(h)|} < \frac{1}{10}$ and (5.165) holds without the last term. Suppose $|r(h)| \ge \frac{1}{20}$. Since $10^6\delta \le 10^{-6}/T$ and $N^2 \le T$, (5.164) gives $\frac{1}{20} \le 10^{-6}(G_0(h)+T)/T$, that is $G_0(h) \ge (5\cdot 10^4 - 1)T$, and also $|r(h)| \le 10^{-6}(G_0(h)+T)$. By Lemma 5.7.28 (ii), $|m_\mu(h)|^2 e^{-q_0} \le e^{-G_0(h)}$, and $|e^r - 1| \le e^{|r|}$, so the left side of (5.165) is at most

$$\exp\big(-G_0(h)+|r(h)|\big) \le \exp\big(-(1-10^{-6})G_0(h) + 10^{-6}T\big) \le e^{-49998\,T}.$$

By Lemma 5.7.29, $c_\mu \ge \frac{1}{4}(30N(N+L))^{-2N^2}$, so, with $\log 4 < 2$, $\log 30 < 4$ and $\log(N(N+L)) \le 2\log(N+L+1)$,

$$\log c_\mu \ge -2 - 8N^2 - 4N^2\log(N+L+1) \ge -14T$$

by the three lower bounds for T. Hence $e^{-49998T} \le e^{-100T}c_\mu$, which proves (5.165); integrating it and dividing by c_μ gives the claim with $\varkappa = e^{-q_0}$. □

Proposition 5.7.25. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $l \in \mathbb{Z}^n$ a vector with gap 0 for ε, and $g \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ with $X = g - 1$ satisfying $\|X\| < \frac{1}{1000}$. For $\Pi \subseteq n$ let p_Π be the diagonal matrix with entry 1 at the indices in Π and 0 elsewhere. Then there is an accretive matrix y with $c_{l/2}(g) = \mathrm{FJ}_l(1,y)$ such that for every $\Pi \subseteq n$ with $l_i < l_{i'}$ for all $i \in \Pi$, $i' \notin \Pi$, and every $h \in K_\varepsilon$, there is a complex number η with*

$$\Delta_\Pi(h^* y h) = \exp\Big(\mathrm{tr}\big(p_\Pi \lg(X)\big) + \tfrac{1}{4}\big\|[p_\Pi, X]\big\|^2 + \eta\Big), \qquad |\eta| \le 200000\,\|X\|\big(\|[p_\Pi,X]\|^2 + \|[p_\Pi,h]\|^2\big).$$

Proof. Write $J = \mathrm{diag}(\varepsilon)$ and $\delta_0 = \|X\|$. For a matrix Z let $Z^\uparrow$ keep the entries $Z_{ii'}$ with $\varepsilon_i = \varepsilon_{i'}$ and $l_i < l_{i'}$ and replace the others by 0, and $Z^\downarrow$ the same with $l_i > l_{i'}$. Replacing entries by 0 does not increase $\|\cdot\|$, and $\|M\|_{\mathrm{op}} \le \|M\|$; we use $\|MN\| \le \|M\|\|N\|$ freely. Put

$$y = (1 - X^\uparrow)\, g\, (1 - X^\downarrow).$$

With $X_1 = (1-X^\uparrow)g - 1 = X - X^\uparrow - X^\uparrow X$ we have $y - 1 = X_1 - X^\downarrow - X_1 X^\downarrow$, and

$$\|X_1\| \le 2\delta_0 + \delta_0^2, \qquad \|y-1\| \le 3\delta_0 + 3\delta_0^2 + \delta_0^3 \le 4\delta_0 < \tfrac{1}{2}. \tag{5.166}$$

The formula. A matrix Z with $\|Z-1\|_{\mathrm{op}} < 1$ is accretive, since $Z + Z^* = 2 + (Z-1) + (Z-1)^*$. So g and y are accretive. By Lemma 5.6.26 (i) with the block-diagonal factor 1, $c_{l/2}(g) = \mathrm{FJ}_l(1,g)$. Let $b_1 = (1-X^\uparrow)^*$ and $b_2 = 1 - X^\downarrow$. Both are invertible, with diagonal entries 1 and with $(b_1)_{ij} = (b_2)_{ij} = 0$ whenever $\varepsilon_i \ne \varepsilon_j$ or $l_i < l_j$ (a nonzero off-diagonal entry $(b_1)_{ij} = -\overline{X^\uparrow_{ji}}$ needs $l_j < l_i$, and $(b_2)_{ij} = -X^\downarrow_{ij}$ needs $l_i > l_j$). Since $y = b_1^* g b_2$, Lemma 5.6.26 (iii) gives $\mathrm{FJ}_l(1,y) = \mathrm{FJ}_l(1,g) = c_{l/2}(g)$, the two characters being 1 at matrices with diagonal entries 1.

Minors of products. If $M_{ii'}=0$ for all $i\in\Pi$, $i'\notin\Pi$, or $N_{i'i}=0$ for all $i'\notin\Pi$, $i\in\Pi$, then $\Delta_\Pi(MN)=\Delta_\Pi(M)\Delta_\Pi(N)$, because $(MN)_{\Pi\Pi}=M_{\Pi\Pi}N_{\Pi\Pi}+M_{\Pi,n\smallsetminus\Pi}N_{n\smallsetminus\Pi,\Pi}$. Let $\Pi'\subseteq n$ satisfy $l_{i'}<l_i$ for all $i\in\Pi'$, $i'\notin\Pi'$. Then $(X^\uparrow)_{ii'}=0$ and $(X^\downarrow)_{i'i}=0$ for $i\in\Pi'$, $i'\notin\Pi'$, so the minor on Π' is multiplicative on the three factors of y; and $\Delta_{\Pi'}(1-X^\uparrow)=\Delta_{\Pi'}(1-X^\downarrow)=1$, the submatrices being triangular with diagonal 1 in the order of l. Hence

$$\Delta_{\Pi'}(y)=\Delta_{\Pi'}(g);\qquad\text{in particular}\quad\det y=\det g. \tag{5.167}$$

Setting up. Fix Π and h as in the statement; write $p=p_\Pi$, $A=[p,X]$, $\alpha_0=\|A\|$, $\kappa_0=\|[p,h]\|$. The map $Z\mapsto[p,Z]$ multiplies the entry $Z_{ii'}$ by 1, -1 or 0; so it commutes with $Z\mapsto Z^\uparrow, Z^\downarrow, \mathrm{bd}_\varepsilon(Z)$ and kills diagonal entries. Let $A_\mathrm{c}=\mathrm{bd}_\varepsilon(A)=A^\uparrow+A^\downarrow$ and $A_\mathrm{n}=A-A_\mathrm{c}$; they have norm at most α_0, $\|A_\mathrm{n}-A_\mathrm{c}\|=\alpha_0$ (disjoint supports), and $\mathrm{tr}(A_\mathrm{c}A_\mathrm{n})=0$, since a product of an entry of A_c at (i,j) and of A_n at (j,i) would need $\varepsilon_i=\varepsilon_j$ and $\varepsilon_i\neq\varepsilon_j$.

The commutator of y. With $G=A^\uparrow X+X^\uparrow A$ we have $[p,X_1]=A-A^\uparrow-G$, and then

$$[p,y-1]=A_\mathrm{n}-R,\qquad R=G+X_1A^\downarrow+(A-A^\uparrow-G)X^\downarrow, \tag{5.168}$$

with $\|R\|\le 2\delta_0\alpha_0+(2\delta_0+\delta_0^2)\alpha_0+(\alpha_0+2\delta_0\alpha_0)\delta_0\le 7\,\delta_0\alpha_0$.

Three expansions. Let $p'=1-p=p_{n\smallsetminus\Pi}$ and $q=hph^*$, an orthogonal projection with $\|p-q\|=\|[p,h]h^*\|=\kappa_0$. We apply Proposition 5.7.26 three times, with the data (p,q,X,Y,δ,c) there as follows; $4\delta_0<\frac12$ throughout.

1. $(p',p',X,-A,\delta_0,\alpha_0)$: $[p',X]=-A$. So $\det(1+p'X)=\exp(\mathrm{tr}(p'\lg(X))-\frac14\mathrm{tr}(A^2)+\eta_1)$ with $|\eta_1|\le 100\,\delta_0\alpha_0^2$.

2. $(p',p',y-1,-A_\mathrm{n},4\delta_0,4\alpha_0)$: by (5.166) and (5.168), $\|y-1\|\le 4\delta_0$ and $[p',y-1]+A_\mathrm{n}=R$ has norm at most $16\delta_0\alpha_0$. So $\det(1+p'(y-1))=\exp(\mathrm{tr}(p'\lg(y-1))-\frac14\mathrm{tr}(A_\mathrm{n}^2)+\eta_2)$ with $|\eta_2|\le 6400\,\delta_0\alpha_0^2$.

3. $(q,p,y-1,A_\mathrm{n},4\delta_0,4(\alpha_0+\kappa_0))$: h and p commute with J, so $[q,p]$ is block-diagonal and its trace against A_n, supported on the pairs of different signs, vanishes; and $[q,y-1]-A_\mathrm{n}=[q-p,y-1]-R$ has norm at most $2\kappa_0\cdot4\delta_0+7\delta_0\alpha_0\le16\delta_0(\alpha_0+\kappa_0)$. So $\det(1+q(y-1))=\exp(\mathrm{tr}(p\lg(y-1))-\frac14\mathrm{tr}(A_\mathrm{n}^2)+\eta_3)$ with $|\eta_3|\le 6400\,\delta_0(\alpha_0+\kappa_0)^2$.

The left sides are minors: $1+p'Z$ is block triangular with the identity block on Π, so $\det(1+p'X)=\Delta_{n\smallsetminus\Pi}(g)$ and $\det(1+p'(y-1))=\Delta_{n\smallsetminus\Pi}(y)$; and $\det(1+q(y-1))=\det\big(h(1+p(h^*yh-1))h^*\big)=\Delta_\Pi(h^*yh)$. The complement of Π satisfies the condition on Π' above, so the first two are equal by (5.167), and nonzero. By Lemma 5.7.27 and (5.167), $\exp\mathrm{tr}\lg(y-1)=\det y=\det g=\exp\mathrm{tr}\lg(X)$. Multiplying (iii) by (ii), dividing by (i) and using $p+p'=1$,

$$\Delta_\Pi(h^*yh)=\exp\Big(\mathrm{tr}(p\lg(X))+\tfrac14\mathrm{tr}(A^2)-\tfrac12\mathrm{tr}(A_\mathrm{n}^2)+\eta_2+\eta_3-\eta_1\Big). \tag{5.169}$$

The sign. Put $V=g^{-1}-1$ and $E=VA+AV+VAV$. From $g^*=Jg^{-1}J$, $X^*=JVJ$; and $[p,V]=[p,g^{-1}]=-g^{-1}Ag^{-1}=-(A+E)$, so $A^*=-[p,X^*]=-J[p,V]J=J(A+E)J$ (p commutes with J). Conjugation by J fixes A_c and changes the sign of A_n, so

$$\|A\|^2=\mathrm{tr}(A^*A)=\mathrm{tr}\big((A+E)(A_\mathrm{c}-A_\mathrm{n})\big)=\mathrm{tr}(A_\mathrm{c}^2)-\mathrm{tr}(A_\mathrm{n}^2)+\mathrm{tr}\big(E(A_\mathrm{c}-A_\mathrm{n})\big).$$

With $\mathrm{tr}(A^2)=\mathrm{tr}(A_\mathrm{c}^2)+\mathrm{tr}(A_\mathrm{n}^2)$ the exponent of (5.169) becomes $\mathrm{tr}(p\lg(X))+\frac14\|A\|^2+\eta$ with

$$\eta=\eta_2+\eta_3-\eta_1+\tfrac14\mathrm{tr}\big(E(A_\mathrm{n}-A_\mathrm{c})\big).$$

By Lemma 5.7.27, $\|V\| \le 2\delta_0$, so $\|E\| \le (4\delta_0 + 4\delta_0^2)\alpha_0 \le 5\delta_0\alpha_0$. Therefore

$$\begin{aligned}|\eta| &\le \delta_0(6400\alpha_0^2 + 6400(\alpha_0+\kappa_0)^2 + 100\alpha_0^2 + 2\alpha_0^2)\\ &\le 19302\,\delta_0(\alpha_0^2+\kappa_0^2) \le 200000\,\|X\|(\|[p,X]\|^2 + \|[p,h]\|^2).\end{aligned}$$

□

Proposition 5.7.26. *Let Ω be a finite set, p, q, X, Y complex $\Omega\times\Omega$ matrices and δ, c real numbers. Suppose that p and q are orthogonal projections ($p = p^2 = p^*$, $q = q^2 = q^*$), that $\delta \le \frac12$, and that*

$$\|X\| \le \delta, \qquad \operatorname{tr}([p,q]\,Y) = 0, \qquad \|[p,X] - Y\| \le \delta c, \qquad \|Y\| \le c, \qquad \|q-p\| \le c.$$

Then there is a complex number η with $|\eta| \le 100\,\delta c^2$ and

$$\det(1+pX) = \exp\Big(\operatorname{tr}\big(q\lg(X)\big) - \tfrac14\operatorname{tr}(Y^2) + \eta\Big).$$

Proof. The numbers δ and c are nonnegative, being bounds of norms. We use $|\operatorname{tr}(MN)| \le \|M\|\|N\|$, $\|MN\| \le \|M\|_{\mathrm{op}}\|N\|$, $\|N\|_{\mathrm{op}} \le \|N\|$ and $\|p\|_{\mathrm{op}}, \|q\|_{\mathrm{op}} \le 1$ throughout. Put $A = [p,X]$ and $D = q - p$; then $\|A\| \le \|Y\| + \|A - Y\| \le c + \delta c \le \frac32 c$, and $\|pX\| \le \|X\| \le \delta \le \frac12$.

Reduction to an integral. By Lemma 5.7.27, $\det(1+pX) = \exp\operatorname{tr}\lg(pX)$, and $\operatorname{tr}\lg(pX)$ and $\operatorname{tr}(q\lg(X))$ are integrals over $s\in[0,1]$. For $s\in[0,1]$ put $W = (1+sX)^{-1}$, $V = (1+spX)^{-1}$; by Lemma 5.7.27, $\|W\|_{\mathrm{op}}, \|V\|_{\mathrm{op}} \le 2$ and $\|W-1\|, \|V-1\| \le 2\delta$. Since $\frac14 = \int_0^1 \frac{s}{2}\,ds$, the assertion holds with

$$\eta = \int_0^1 \mathcal{J}(s)\,ds, \qquad \mathcal{J}(s) = \operatorname{tr}(VpX) - \operatorname{tr}(qWX) + \tfrac{s}{2}\operatorname{tr}(Y^2),$$

and it suffices to show $|\mathcal{J}(s)| \le 100\,\delta c^2$ for each s. As $\operatorname{tr}(A^2 - Y^2) = \operatorname{tr}((A-Y)(A+Y))$, we split $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$ with

$$\mathcal{J}_1 = \operatorname{tr}(VpX) - \operatorname{tr}(pWX) + \tfrac{s}{2}\operatorname{tr}(A^2), \qquad \mathcal{J}_2 = -\operatorname{tr}(DWX), \qquad \mathcal{J}_3 = -\tfrac{s}{2}\operatorname{tr}\big((A-Y)(A+Y)\big).$$

At once $|\mathcal{J}_3| \le \frac12\delta c(\frac32 c + c) = \frac54\delta c^2$.

The term $\mathcal{J}_1$. Direct computation gives $pAp = 0 = (1-p)A(1-p)$, $pA = pX(1-p)$, $Ap = -(1-p)Xp$, and $\operatorname{tr}(pA^2) = \operatorname{tr}(pA(1-p)A) = \operatorname{tr}((1-p)ApA) = \operatorname{tr}((1-p)A^2)$, so $\operatorname{tr}(pA^2) = \frac12\operatorname{tr}(A^2)$. From $p(1+sX) - (1+spX)p = s\,pA$, multiplying by V on the left and W on the right, $Vp - pW = s\,VpAW$; hence

$$\operatorname{tr}(VpX) - \operatorname{tr}(pWX) = s\operatorname{tr}(pA\,WXV) = s\operatorname{tr}\big(pA(1-p)\,WXV\,p\big),$$

using $pA = pA(1-p)$ and $\operatorname{tr}(pM) = \operatorname{tr}(pMp)$. Now $[p,W] = -W[p,W^{-1}]W = -sWAW$, so $(1-p)W = W(1-p) + sWAW$; and $(1-p)V^{-1} = 1-p$ gives $(1-p)V = 1-p$, so $(1-p)Vp = 0$ and $Vp = pVp$. Therefore

$$(1-p)WXVp = W(1-p)XpVp + sWAWXVp = -WAVp + sWAWXVp.$$

With $\frac{s}{2}\operatorname{tr}(A^2) = s\operatorname{tr}(pA\,Ap)$ and $WAV - A = (W-1)AV + A(V-1)$,

$$\mathcal{J}_1 = -s\operatorname{tr}\big(pA((W-1)AV + A(V-1))p\big) + s^2\operatorname{tr}\big(pA\cdot WAW\cdot XVp\big),$$

so $|\mathcal{J}_1| \le \|A\|^2(2\delta\cdot 2 + 2\delta) + \|A\|\cdot 2\cdot 2\|A\|\cdot\delta\cdot 2 = 14\,\delta\|A\|^2 \le \frac{63}{2}\delta c^2$.

The term $\mathcal{J}_2$. From $p^2=p$, $q^2=q$: $D^2 = p+q-pq-qp$ commutes with p ($pD^2 = D^2p = p-pqp$), $pDp = -pD^2$ and $(1-p)D(1-p) = (1-p)q(1-p) = (1-p)D^2$. Hence $D = D_0 + D_1$ with

$$D_0 = (1-2p)D^2, \qquad D_1 = pq(1-p) + (1-p)qp, \qquad \text{while} \qquad [p,q] = pq(1-p) - (1-p)qp.$$

For every matrix Z, $[p,Z] = pZ(1-p) - (1-p)Zp$, and multiplying out, $\operatorname{tr}([p,q][p,Z]) = -\operatorname{tr}(pq(1-p)Z) - \operatorname{tr}((1-p)qpZ) = -\operatorname{tr}(D_1 Z)$. For $Z = WX$, $[p,WX] = [p,W]X + W[p,X] = WA - sWAWX$, and with the hypothesis $\operatorname{tr}([p,q]Y) = 0$,

$$\mathcal{J}_2 = -\operatorname{tr}(D_0 WX) + \operatorname{tr}\Big([p,q]\big((A-Y) + (W-1)A - sWAWX\big)\Big).$$

Here $\|D_0\| = \|D^2\| \le \|D\|_{\mathrm{op}}\|D\| \le c^2$ because $1-2p$ is unitary; $\|WX\| \le 2\delta$; $\|[p,q]\| = \|D_1\| \le \|D\| \le c$, since $[p,q]$ and D_1 have the same entries up to sign in the blocks $p\cdot(1-p)$ and $(1-p)\cdot p$, which are the off-diagonal blocks of D; and the second factor has norm at most $\delta c + 2\delta\cdot\frac{3}{2}c + 2\cdot\frac{3}{2}c\cdot 2\cdot\delta = 10\,\delta c$. Hence $|\mathcal{J}_2| \le 2\delta c^2 + 10\delta c^2$.

Together $|\mathcal{J}(s)| \le (\frac{63}{2} + 12 + \frac{5}{4})\delta c^2 \le 100\,\delta c^2$. □

Lemma 5.7.27. *Let Y be a complex square matrix with $\|Y\|_{\mathrm{HS}} \le \frac{1}{2}$. Then for every $s \in [0,1]$ the matrix $1+sY$ is invertible, with*

$$\|(1+sY)^{-1}\|_{\mathrm{op}} \le 2, \qquad \|(1+sY)^{-1} - 1\|_{\mathrm{HS}} \le 2s\|Y\|_{\mathrm{HS}},$$

and the matrix $\lg(Y) = \int_0^1 (1+sY)^{-1}Y\,ds$ satisfies

$$\det(1+Y) = \exp\operatorname{tr}\lg(Y), \qquad \|\lg(Y)\|_{\mathrm{HS}} \le 2\|Y\|_{\mathrm{HS}}.$$

Proof. Since $\|sY\|_{\mathrm{op}} \le \|Y\|_{\mathrm{HS}} \le \frac{1}{2}$, the Neumann series $\sum_{k\ge 0}(-sY)^k$ converges to $(1+sY)^{-1}$ and $\|(1+sY)^{-1}\|_{\mathrm{op}} \le \sum_k 2^{-k} = 2$. From $(1+sY)^{-1} - 1 = -sY(1+sY)^{-1}$ and $\|MN\|_{\mathrm{HS}} \le \|M\|_{\mathrm{HS}}\|N\|_{\mathrm{op}}$ we get $\|(1+sY)^{-1}-1\|_{\mathrm{HS}} \le 2s\|Y\|_{\mathrm{HS}}$, and likewise $\|(1+sY)^{-1}Y\|_{\mathrm{HS}} \le 2\|Y\|_{\mathrm{HS}}$, which integrates to $\|\lg(Y)\|_{\mathrm{HS}} \le 2\|Y\|_{\mathrm{HS}}$. The function $f(s) = \det(1+sY)$ is a polynomial without zeros on $[0,1]$, and by Jacobi's formula for the derivative of a determinant, $f'(s) = f(s)\operatorname{tr}\big((1+sY)^{-1}Y\big)$. Hence $\frac{d}{ds}\big(f(s)\exp(-\int_0^s \operatorname{tr}((1+\sigma Y)^{-1}Y)\,d\sigma)\big) = 0$, and with $f(0) = 1$,

$$\det(1+Y) = f(1) = \exp\int_0^1 \operatorname{tr}\big((1+sY)^{-1}Y\big)ds = \exp\operatorname{tr}\lg(Y).$$

□

Lemma 5.7.28. *Let n be a nonempty finite set and $w \in \mathbb{Z}^n$; put $w_- = \min_i w_i$, $w_+ = \max_i w_i$, let p_t ($t \in \mathbb{Z}$) be the diagonal matrix with entry 1 at the indices i with $w_i \ge t$ and 0 elsewhere, and for a complex $n\times n$ matrix Z put*

$$\mathrm{gw}_w(Z) = \sum_{i,i'\in n} |w_i - w_{i'}|\,|Z_{ii'}|^2.$$

1. *For every complex $n\times n$ matrix Z,*

$$\sum_{w_- < t \le w_+} \big\|[p_t, Z]\big\|^2 = \mathrm{gw}_w(Z), \qquad w_-\operatorname{tr}(Z) + \sum_{w_- < t \le w_+} \operatorname{tr}(p_t Z) = \operatorname{tr}\big(\operatorname{diag}(w)Z\big).$$

2. *For every unitary $n\times n$ matrix h, $|m_w(h)|^2 \le \exp\big(-\frac{1}{2}\mathrm{gw}_w(h)\big)$.*

Proof. (i). The entry of $[p_t, Z]$ at (i,i') is $Z_{ii'}$ times the difference of the indicators of $w_i \ge t$ and $w_{i'} \ge t$. This difference is ± 1 for the integers t with $\min(w_i, w_{i'}) < t \le \max(w_i, w_{i'})$, which are $|w_i - w_{i'}|$ in number and lie in $(w_-, w_+]$, and 0 otherwise; summing $|[p_t, Z]_{ii'}|^2$ over t and (i,i') gives the first identity. The diagonal entry of $p_t Z$ at i is Z_{ii} for the $w_i - w_-$ integers $t \in (w_-, w_+]$ with $t \le w_i$, and 0 otherwise; so the left side of the second identity is $\sum_i (w_- + w_i - w_-) Z_{ii}$.

(ii). As $|\det h| = 1$, (3.1) gives $|m_w(h)|^2 = \prod_{w_- < t \le w_+} |\Delta_{P_t}(h)|^2$ with $P_t = \{i : w_i \ge t\}$, and by (i) it suffices to show $|\Delta_P(h)|^2 \le \exp(-\frac{1}{2}\|[p,h]\|^2)$ for every $P \subseteq n$ and $p = p_P$ the diagonal projection onto P. The matrix $(1-p) + ph$ has the rows of h at the indices in P and those of 1 elsewhere, so its determinant is $\Delta_P(h)$; by Sylvester's identity $\det(1 + MN) = \det(1 + NM)$ with $M = -p$, $N = p(1-h)$,

$$\Delta_P(h) = \det\bigl(1 - p(1-h)\bigr) = \det\bigl(1 - p(1-h)p\bigr) = \det Z, \qquad Z = (1-p) + php.$$

Now $Z^*Z = (1-p) + ph^*php$ is positive semidefinite, and for its eigenvalues $a \ge 0$ the inequality $a \le e^{a-1}$ gives $|\det Z|^2 = \det(Z^*Z) \le \exp(\operatorname{tr}(Z^*Z) - \#n) = \exp(\|php\|^2 - \operatorname{tr} p)$. As h is unitary, $\|ph\|^2 = \|hp\|^2 = \operatorname{tr} p$, and $[p,h] = ph(1-p) - (1-p)hp$ with the two terms supported on disjoint sets of entries, so

$$\|[p,h]\|^2 = \|ph(1-p)\|^2 + \|(1-p)hp\|^2 = (\operatorname{tr} p - \|php\|^2) + (\operatorname{tr} p - \|php\|^2),$$

and the exponent is $-\frac{1}{2}\|[p,h]\|^2$. □

Lemma 5.7.29. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $L \ge 0$ an integer and $l \in \mathbb{Z}^n$ a vector with gap 0 for ε and $|l_i| \le L$ for all i. Then $|\mu_l(i)| \le \frac{1}{2}(L + \#n)$ for all i, and*

$$\|m_{\mu_l}\|^2 \ge \tfrac{1}{4}\bigl(30\,\#n\,(\#n + L)\bigr)^{-2(\#n)^2}.$$

Proof. Write $N = \#n$, $\mu = \mu_l$, $K = K_\varepsilon$ with Haar probability measure ν, and $\|\cdot\|$ for the operator norm.

Size of μ. By Lemma 5.6.27 (iii), $2l^\rho = l + 2\rho_l$, so with σ the sign function,

$$2\mu(i) = l_i + 2\rho_l(i) - 2 \cdot 2\rho_l^{\mathrm{c}}(i) = l_i + \sum_{k:\,\varepsilon_k \ne \varepsilon_i} \sigma(l_i - l_k) - \sum_{k:\,\varepsilon_k = \varepsilon_i} \sigma(l_i - l_k),$$

and $|2\mu(i)| \le L + N - 1$. Put $M = \frac{1}{2}(L+N)$, so $|\mu(i)| \le M$.

m_μ near 1. Let $h \in K$ with $\|h - 1\| \le r$, where $r = 1/D$ and $D = 10N(N+L)$. By (3.1), $m_\mu(h)$ is a product of $|\mu_-|$ factors $(\det h)^{\pm 1}$ and $\mu_+ - \mu_-$ principal minors $\Delta_P(h)$ ($\mu_\pm$ the largest and smallest entry), at most $3M \le 2(L+N)$ factors z in all. For a minor, $\Delta_P(h) = \det(1+E)$ with $E = (h-1)_{PP}$, $\|E\| \le r$, so its eigenvalues e_j satisfy $|e_j| \le r$ and $|\Delta_P(h) - 1| = |\prod_j (1 + e_j) - 1| \le \prod_j (1 + |e_j|) - 1 \le e^{Nr} - 1$; the same holds for $\det h$, and $|(\det h)^{-1} - 1| = |\det h - 1|$ as $|\det h| = 1$. As $Nr \le \frac{1}{10}$, $e^{Nr} - 1 \le 2Nr$. With $|\prod_j z_j - 1| \le \prod_j (1 + |z_j - 1|) - 1$ (induction on the number of factors),

$$|m_\mu(h) - 1| \le \exp\bigl(2(L+N) \cdot 2Nr\bigr) - 1 = e^{0.4} - 1 < \tfrac{1}{2},$$

so $|m_\mu(h)| \ge \frac{1}{2}$ whenever $\|h - 1\| \le 1/D$.

Measure of a small ball. Apply Lemma 5.6.32 (i) to the group $S = K$ with the right-invariant measure ν, the inclusion $j\colon K \to \mathrm{U}(\operatorname{diag}\varepsilon)$, $y = 1$, $R = 1$, the given $D \ge 1$ and the constant $3D$: since every $h \in K$ is unitary, $\|h\| = 1$, and

$$1 = \nu\{h : \|h\| \le 1\} \le (3D)^{2N^2}\, \nu\{h : \|h - 1\| \le 1/D\}.$$

Hence $\|m_\mu\|^2 \ge \frac{1}{4}\,\nu\{h : \|h-1\| \le 1/D\} \ge \frac{1}{4}(3D)^{-2N^2} = \frac{1}{4}\bigl(30N(N+L)\bigr)^{-2N^2}$. □

The Bessel systems of the data are Bessel averages of Flensted-Jensen's function, and Proposition 5.7.30 proves the four properties of these averages that the construction uses: a stationary point of controlled Bessel length, a bound for the ball volume, a cut pair reproduced by the discrete-series kernel, and Schur's bound. The stationary point comes from Laplace's method applied to the Bessel subgroup and is found with Witt's extension theorem for hermitian spaces (Theorem A.6.7; for the quadratic case see [330]); the ball volume follows from the identity for the Bessel average (Lemma 5.6.37). Schur's bound needs control of derivatives: a bounded function reproduced by the kernel has derivatives given by convolution with derivative kernels, which are bounded through the filtration of the module generated by Flensted-Jensen's function (Lemma 5.7.35, Lemma 5.7.36); see [315, 165].

The section closes with Harish-Chandra's formula for the formal degree (Theorem 5.7.41): for every sign vector and every strictly decreasing parameter with the right integrality, the square-integrable representation exists, is unique up to equivalence, and has the Weyl product as formal degree for the Haar measure normalised by the compact dual. The proof combines Harish-Chandra's formula [119, 251] with Godement's Schur orthogonality and Weyl's dimension formula [92]. In the definite case the formula is Weyl's dimension formula, and this is how the trace comparison reads it (Proposition 5.8.54, Lemma 5.8.56).

Proposition 5.7.30. *Fix the setting of Definition 5.1.6, signs ε on the r basis indices of V_1 and a unitary matrix p with $p^{*}J_1p=\operatorname{diag}(\varepsilon)$. For $l\in\mathbb{Z}^r$ (a doubled parameter, $\lambda=\frac12 l$) write $c_l^{G_1}(g)=c_\lambda(p^{-1}gp)$ for $g\in G_1$, and d_l for the formal degree of c_λ for the Haar measure on $\mathrm{U}(\operatorname{diag}\varepsilon)$ that $g\mapsto p^{-1}gp$ carries from dy. Let ds be the measure $du\,dh$ on R and, for $s=uh\in R$, $h_s=h$. Let $\rho\colon\mathrm{U}(k)\to\mathrm{U}(I)$ be a continuous unitary representation on $\mathbb{C}^I$, $v\in\mathbb{C}^I$ and $x\in G_1$; the **Bessel average** is*

$$W_x(y)=\int_R c_l^{G_1}(x^{-1}sy)\,\mathrm{wave}\big(-\sigma(s)\big)\,\rho(h_s)^{-1}v\,ds\qquad(y\in G_1),$$

*and the **ball volume** is $V(x)=\mathrm{vol}_R\{s\in R:\|p^{-1}x^{-1}sxp-1\|_{\mathrm{op}}\le 1\}$.*

(a) Stationary point. Let X be a finite set, $e\colon X\sqcup\{0,\dots,2n-1\}\to\{1,\dots,r\}$ and $\tau\colon X\sqcup\{*\}\to\{1,\dots,k\}$ bijections, $B\ge1$, b,β reals and $a,L,h\ge0$ integers. Suppose that l is steep for (ε,e,B) (with $M=2n$); that $u\in\mathbb{Z}^k$ has, for the signs all $+1$, lowest K-type $\mu_u(\tau(*))=a$ and $\mu_u=0$ elsewhere; that $3B+a\le b^2$ and $8nr\le b$; that l (for ε) and u have large gap h; that $|l_i|\le L$ and $|u_i|\le L$ for all entries; and that $10^{12}\le\beta$, $r+k+L\le\beta$ and $10^{30}\big((r+k)^2\log\beta\big)^3\le h$. Then there is $x\in G_1$ with $\ell(x)\le b^{5n}$ and

$$V(x)\le\beta^{20(r+k)^2}\Big|\int_R c_l^{G_1}(x^{-1}sx)\,\overline{c_{u/2}(h_s)}\,e^{\overline{\sigma(s)}-\sigma(s)}\,ds\Big|.$$

(b) Ball volume. Suppose that l has large gap A for ε for some integer $A\ge0$, that $\langle v,\rho(m)v\rangle=c_{u/2}(m)$ for every $m\in\mathrm{U}(k)$, where $u\in\mathbb{Z}^k$ (signs all $+1$), and that

$$V(x)\le P\,\Big|\int_R c_l^{G_1}(x^{-1}sx)\,\overline{c_{u/2}(h_s)}\,e^{\overline{\sigma(s)}-\sigma(s)}\,ds\Big|$$

for a real $P\ge0$. Then $V(x)\le P\,d_l\int_{\mathrm{U}\backslash G_1}\|W_x\|^2$.

(c) Cut pair. Let $A,L,m\ge0$ and $f\ge1$ be integers and b real. Suppose that l has large gap A for ε and $|l_i|\le L$ for all i, that $\|v\|=1$, $V(x)\le b^m d_l\int_{\mathrm{U}\backslash G_1}\|W_x\|^2$, $\ell(x)\le b^{5n}$, $b\ge2000$, $r\ge100$, $m+99r^2\le400r^2$, $r+L+1\le b^4$ and $3r^3f\le A$. Then there are $c\in\mathbb{R}$ and a continuous F with $F=cW_x$, $\int_{G_1}d_l\,c_l^{G_1}(a^{-1})F(ya)\,da=F(y)$ for all y, and such that $(F,\chi(\ell(\cdot))F)$ is a cut pair (for b,f), where $\chi=1$ on $[0,\frac12 b^{6n+2}]$, $\chi=0$ on $[b^{6n+2},\infty)$, and χ is linear in between.

(d) Schur's bound. Let $A, L \geq 0$ be integers and B real. Suppose that l has large gap A for ε, $|l_i| \leq L$ for all i, and $(r+1)^4(L+r+1+2n^2) \leq B$. If F is a Bessel function of type ρ, integrable over $\mathsf{U}\backslash G_1$, with $\int_{G_1} d_l\, c_l^{G_1}(a^{-1})F(ya)\,da = F(y)$ for all y, then F has Schur's bound $(B,\ d_l^{-1}\int_{\mathsf{U}\backslash G_1}\|F\|^2)$.

Proof. The objects $c_l^{G_1}$, d_l, ds, h_s, $\sigma(s)$, W_x and $V(x)$ of the statement are those of Lemma 5.7.31, Lemma 5.6.37, Lemma 5.7.33 and Lemma 5.7.34 (with $u_0 = u$), and $V(x) = \int_R \mathbf{1}[\|p^{-1}x^{-1}sxp - 1\|_{\mathrm{op}} \leq 1]\,ds$.

(a) is Lemma 5.7.31, whose hypotheses are those of (a).

(b). The representation ρ is continuous and unitary, l has large gap A, and $\langle v, \rho(m)v\rangle = c_{u/2}(m)$ for $m \in \mathrm{U}(k)$; so Lemma 5.6.37 (ii) applies and shows that the period

$$\int_R c_l^{G_1}(x^{-1}sx)\overline{c_{u/2}(h_s)}e^{\overline{\sigma(s)}-\sigma(s)}ds$$

equals $d_l\int_{\mathsf{U}\backslash G_1}\|W_x\|^2$, a nonnegative real number. Hence its modulus is $d_l\int_{\mathsf{U}\backslash G_1}\|W_x\|^2$, and the hypothesis $V(x) \leq P\,|\text{period}|$ gives $V(x) \leq P\,d_l\int_{\mathsf{U}\backslash G_1}\|W_x\|^2$.

(c) is Lemma 5.7.33, whose hypotheses are those of (c) (ρ continuous, $\|v\| = 1$).

(d) is Lemma 5.7.34, whose hypotheses are those of (d). □

Lemma 5.7.31. *Fix the setting of Definition 5.1.6, signs ε on the r basis indices of V_1 and a sign frame p for ε. Every $s \in R$ is uniquely $s = uh$ with $u \in \mathsf{U}$, $h \in H$; put $h_s = h$, $\sigma(s) = \sigma(u)$, and let $ds = du\,dh$ on R, dh the Haar probability measure. Let X be a finite set, $e\colon X \sqcup \{0,\dots,2n-1\} \to \{1,\dots,r\}$ and $\tau\colon X \sqcup \{*\} \to \{1,\dots,k\}$ bijections, $B \geq 1$, b, β reals, $a, L, h \geq 0$ integers, $l \in \mathbb{Z}^r$ and $u_0 \in \mathbb{Z}^k$. Suppose that l is steep for (ε, e, B) (with $M = 2n$); that u_0 has, for the signs all $+1$, lowest K-type $\mu_{u_0}(\tau(*)) = a$ and $\mu_{u_0} = 0$ elsewhere; that $3B + a \leq b^2$ and $8nr \leq b$; that l (for ε) and u_0 (for the signs all $+1$) have large gap h; that $|l_i| \leq L$ and $|u_{0,i}| \leq L$ for all entries; and that $10^{12} \leq \beta$, $r + k + L \leq \beta$ and $10^{30}\big((r+k)^2\log\beta\big)^3 \leq h$. Then there is $x \in G_1$ with $\ell(x) \leq b^{5n}$ and*

$$\int_R \mathbf{1}[\|p^{-1}x^{-1}sxp - 1\|_{\mathrm{op}} \leq 1]\,ds \leq \beta^{20(r+k)^2}\Big|\int_R c_{l/2}(p^{-1}x^{-1}sxp)\,\overline{c_{u_0/2}(h_s)}\,e^{\overline{\sigma(s)}-\sigma(s)}\,ds\Big|.$$

Proof. Write b_i for the basis vector of V_1 at the index i, $p_i = pb_i$ for the column of p at i, $\Lambda = \mu_l$ (for ε) and $\Lambda' = \mu_{u_0}$ (for the signs all $+1$). The columns p_i are orthonormal, and $p_i^* J_1 p_{i'} = \varepsilon_i\delta_{ii'}$. The steepness hypothesis is (LA1) of Definition 5.7.2 for l, the signs ε, e, B and $M = 2n = 2\varrho+2$: $\varepsilon_{e(\xi)} = 1$ and $\Lambda(e(\xi)) = 0$ for $\xi \in X$; $\varepsilon_{e(m)} = (-1)^m$ for $0 \leq m \leq 2\varrho+1$; and $m \mapsto \Lambda(e(m))$ is strictly decreasing on $\{0,\dots,2\varrho+1\}$ with values in $[-3B, -B]$. All values of Λ are integers.

Nodes and Lagrange weights. Put $t_0 = 0$ and $t_{m+1} = \Lambda(e(m)) - a$ for $0 \leq m \leq 2\varrho+1$. These $2\varrho+3$ integers satisfy $t_0 > t_1 > \cdots > t_{2\varrho+2}$, since $\Lambda(e(m)) - a \leq -B < 0$, and $|t_{m+1}| \leq 3B + a \leq b^2$. For $0 \leq m \leq 2\varrho+2$ the product $\prod_{m'\neq m}(t_m - t_{m'})$ has exactly m negative factors and absolute value at least 1; so $\omega_m = \big|\prod_{m'\neq m}(t_m - t_{m'})\big|^{1/2} \geq 1$ and $\omega_m^2 = (-1)^m\prod_{m'\neq m}(t_m - t_{m'})$. By Lagrange interpolation at the $2\varrho+3$ nodes, for every polynomial π of degree at most $2\varrho+2$,

$$[\text{coefficient of } t^{2\varrho+2} \text{ in } \pi] = \sum_{m=0}^{2\varrho+2}\frac{\pi(t_m)}{\prod_{m'\neq m}(t_m - t_{m'})} = \sum_{m=0}^{2\varrho+2}(-1)^m\frac{\pi(t_m)}{\omega_m^2}. \tag{5.170}$$

Lagrange vectors. Define

$$w^{(0)} = \omega_0 \sum_{m=0}^{2\varrho+1} \frac{1}{\omega_{m+1}}\, p_{e(m)}, \qquad w^{(k)} = \sum_{m=0}^{2\varrho+1} \frac{t_{m+1}^k}{\omega_{m+1}}\, p_{e(m)} \quad (k \geq 1).$$

As $\varepsilon_{e(m)} = (-1)^m$, for $k, k' \geq 1$ with $K = k + k' \leq 2\varrho + 2$, (5.170) for $\pi = t^K$ (which vanishes at $t_0 = 0$) gives

$$w^{(k)*} J_1 w^{(k')} = \sum_{m=0}^{2\varrho+1} (-1)^m \frac{t_{m+1}^K}{\omega_{m+1}^2} = -\sum_{m=0}^{2\varrho+2} (-1)^m \frac{t_m^K}{\omega_m^2} = -\delta_{K,2\varrho+2}.$$

Likewise, (5.170) for $\pi = 1$ and for $\pi = t^{k''}$ gives $w^{(0)*} J_1 w^{(0)} = \omega_0^2(\omega_0^{-2} - 0) = 1$ and $w^{(0)*} J_1 w^{(k'')} = -\omega_0 \delta_{k'',2\varrho+2} = 0$ for $1 \leq k'' \leq 2\varrho + 1$. Summarising, for $k, k' \geq 1$, $k + k' \leq 2\varrho + 2$ and $1 \leq k'' \leq 2\varrho + 1$:

$$w^{(0)*} J_1 w^{(0)} = 1, \qquad w^{(0)*} J_1 w^{(k'')} = 0, \qquad w^{(k)*} J_1 w^{(k')} = \begin{cases} -1 & k + k' = 2\varrho + 2, \\ 0 & k + k' < 2\varrho + 2. \end{cases} \tag{5.171}$$

The vectors $p_{e(\xi)}$, $\xi \in X$, are J_1-orthonormal with squares $\varepsilon_{e(\xi)} = 1$ and J_1-orthogonal to every $w^{(k)}$, which is a combination of the $p_{e(m)}$. Since the p_i are orthonormal, $\|w^{(k)}\|^2 = \sum_m t_{m+1}^{2k} \omega_{m+1}^{-2} \leq (2\varrho + 2) b^{4k}$ for $k \geq 1$.

The point x. Compare the families

$$\Phi = \big((p_{e(\xi)})_{\xi \in X},\ w^{(0)}, w^{(1)}, \ldots, w^{(\varrho+1)}\big), \qquad \Psi = \big((e^0_{\tau(\xi)})_{\xi \in X},\ e^0_{\tau(*)},\ f_0, \ldots, f_\varrho\big).$$

By (5.171) with $k = a' + 1$, $k' = a'' + 1$ ($0 \leq a', a'' \leq \varrho$, so $k + k' \leq 2\varrho + 2$ with equality only for $a' = a'' = \varrho$) and $k'' \leq \varrho + 1 \leq 2\varrho + 1$, the Gram matrix of Φ for h is the identity on the first $\#X + 1$ vectors, -1 at $(w^{(\varrho+1)}, w^{(\varrho+1)})$ and 0 elsewhere; this is the Gram matrix of Ψ, since the e_i^0 are orthonormal for h and orthogonal to the f_c, and $h(f_c, f_{c'}) = -\delta_{c\varrho}\delta_{c'\varrho}$. The family Ψ consists of distinct basis vectors. The family Φ is independent: its matrix of J_1-products with the vectors $p_{e(\xi)}$ ($\xi \in X$), $w^{(0)}$, $w^{(2\varrho+1)}, w^{(2\varrho)}, \ldots, w^{(\varrho+1)}$ is triangular with diagonal entries ± 1, because by (5.171) $w^{(2\varrho+2-c)*} J_1 w^{(c')}$ is 0 for $1 \leq c' < c$ and -1 for $c' = c$, $w^{(2\varrho+2-c)*} J_1 w^{(0)} = 0$ ($1 \leq c \leq \varrho + 1$), and the $p_{e(\xi)}$ are orthogonal to all $w^{(k)}$. So the linear map from $\langle \Phi \rangle$ to V_1 sending Φ to Ψ termwise is injective and preserves h, and by Witt's theorem for the nondegenerate hermitian form h on $\mathbb{C}^r$ ([249], Chapter 7) there is $x \in G_1$ with

$$x p_{e(\xi)} = e^0_{\tau(\xi)}\ (\xi \in X), \qquad x w^{(0)} = e^0_{\tau(*)}, \qquad x w^{(c+1)} = f_c\ (0 \leq c \leq \varrho). \tag{5.172}$$

Its Bessel length. Put $w_c = w^{(c+1)}$ for $0 \leq c \leq 2\varrho$. Then $w_c = x^{-1} f_c$ for $c \leq \varrho$ by (5.172); $h(w_c, w_{c'})$ is -1 if $c + c' = 2\varrho$ and 0 if $c + c' < 2\varrho$ by (5.171); and $\|w_c\|^2 \leq (2\varrho + 2) b^{4(2\varrho+1)} \leq b^{8\varrho+6}$, as $2\varrho + 2 \leq 8nr \leq b$ and $b \geq 1$. By Lemma 5.7.32 with $M = b^{4\varrho+3} \geq 1$,

$$\ell(x) \leq 4(\varrho + 1)\, b^{4\varrho+3} \leq b^{4\varrho+4} \leq b^{5n},$$

since $4n \leq b$ and $4\varrho + 4 \leq 5\varrho + 5$.

The conjugated weight matrix. Put $q = xp$. Then $q^* J_1 q = p^* J_1 p = \operatorname{diag}(\varepsilon)$, so $q^{-1} = \operatorname{diag}(\varepsilon) q^* J_1$ and $g \mapsto q^{-1} g q$ maps G_1 onto $\mathrm{U}(\operatorname{diag} \varepsilon)$. Let $D = q \operatorname{diag}(\Lambda)\, q^{-1}$, so that

$$\operatorname{tr}(\operatorname{diag}(\Lambda)\, q^{-1} Y q) = \operatorname{tr}(DY) \quad \text{for every } r \times r \text{ matrix } Y.$$

The matrix $p\,\mathrm{diag}(\Lambda)p^{-1}$ multiplies p_i by $\Lambda(i)$, so it kills $p_{e(\xi)}$, and since $\Lambda(e(m)) = t_{m+1} + a$ it sends $w^{(k)}$ to $aw^{(k)} + w^{(k+1)}$ for $k \geq 1$ and $w^{(0)}$ to $aw^{(0)} + \omega_0 w^{(1)}$. Applying x and (5.172):

$$De^0_{\tau(\xi)} = 0, \qquad De^0_{\tau(*)} = a\, e^0_{\tau(*)} + \omega_0 f_0, \qquad Df_c = a\, f_c + f_{c+1} \quad (0 \leq c < \varrho). \tag{5.173}$$

Moreover $J_1 D = J_1 q\,\mathrm{diag}(\Lambda)\mathrm{diag}(\varepsilon)\, q^* J_1$ is hermitian, so $h(v, Dv') = h(Dv, v')$ for all v, v'.

$D - C$ is block upper triangular for the levels, that is, $(D-C)b_j$ lies in the span of the basis vectors of level at most that of b_j, for every basis vector b_j. Recall that the e^*_c-coordinate of a vector v is $h(e_c, v)$, and that $e_c = f_{c-1}$ for $1 \leq c \leq \varrho$, so $De_c = ae_c + f_c$ by (5.173). For $b_j = e_c$: $(D-C)e_c = ae_c + f_c - f_c = ae_c$. For $b_j = e^0_i$: $Ce^0_i = 0$ and, τ being a bijection, De^0_i lies in $\langle e^0_{\tau(*)}, f_0\rangle$, of levels $\varrho+1$ and 1. For $b_j = z$ (level $\varrho+1$): $h(e_c, Dz) = h(De_c, z) = h(ae_c + f_c, z) = -\delta_{c\varrho}$, so the e^*-part of Dz is $-e^*_\varrho = Cz$, and $(D-C)z$ has levels at most $\varrho+1$. For $b_j = e^*_d$ (level $2\varrho + 2 - d$): $h(e_c, De^*_d) = h(ae_c + f_c, e^*_d) = a\delta_{cd} + \delta_{c+1,d}$ (a is real), so the e^*-part of De^*_d is $ae^*_d + e^*_{d-1}$ (the second term absent for $d = 1$), while $Ce^*_d = e^*_{d-1}$ (and $Ce^*_1 = 0$); hence $(D-C)e^*_d$ is ae^*_d plus vectors of level at most $\varrho+1$. Finally, the block of D on X_0 is, by (5.173), the diagonal matrix with entry a at $\tau(*)$ and 0 elsewhere, which is $\mathrm{diag}(\Lambda')$ by the hypothesis on u_0; and C vanishes on X_0.

The data for Laplace's method. Take $S = R$ with the topology of G_1 and its Borel sets, $\nu = ds$, $\mathfrak{n}$ the r basis indices with the signs ε, $\mathfrak{m} = \{1, \dots, \dim X_0\}$ with the signs $\eta = (1, \dots, 1)$, the vectors l and $k = u_0$, $j = (j_1, j_2)$ with $j_1(s) = q^{-1}sq \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ and $j_2(s) = h_s \in H$, $\varphi(s) = \sigma(s) - \overline{\sigma(s)}$, the set X, $M = 2n$, the bijection e, the map $\tau|_X$, and B, β, L, h. By Lemma 5.5.30 (v), translations of R are continuous and ds is right invariant and finite on compact sets. By (i) of the same lemma, j_2 is a continuous homomorphism, and j_1 is a homomorphism which is the restriction to the closed subset R of the homeomorphism $g \mapsto q^{-1}gq$ of G_1 onto $\mathrm{U}(\mathrm{diag}\,\varepsilon)$, hence proper. We check that these data form a Laplace datum, that is (LA1)–(LA4) of Definition 5.7.2.

(LA1) is the steepness hypothesis.

(LA2) Let $s \in R$. By Lemma 5.5.30 (iii) for $Z = D - C$, (iv), and the description of $D - C$ on X_0 (where $h_s - 1$ is supported),

$$\begin{aligned}\mathrm{tr}\big(\mathrm{diag}(\Lambda)(j_1(s)-1)\big) &= \mathrm{tr}\big(D(s-1)\big)\\ &= \mathrm{tr}\big((D-C)(s-h_s)\big) + \mathrm{tr}\big((D-C)(h_s-1)\big) + \mathrm{tr}\big(C(s-1)\big)\\ &= 0 + \mathrm{tr}\big(\mathrm{diag}(\Lambda')(h_s-1)\big) + \sigma(s) - \overline{\sigma(s)},\end{aligned}$$

which is the first condition. By Lemma 5.5.30 (ii), φ is additive, and $\mathrm{Re}\,\varphi = 0$. By (iii) for $Z = 1$, $\mathrm{tr}(j_1(s) - 1) = \mathrm{tr}(s - 1) = \mathrm{tr}(h_s - 1) = \mathrm{tr}(j_2(s) - 1)$.

(LA3) $\eta_{\tau(\xi)} = 1$ and $\Lambda'(\tau(\xi)) = 0$ for $\xi \in X$; $\Lambda' \geq 0$ since $a \geq 0$; all signs η_i are equal, so the bound by $3B$ is not required. Let u be a unitary $X \times X$ matrix and $h_u = 1 + \iota_\tau(u - 1)\iota^*_\tau \in H$, regarded as an element of $H \subseteq R$; then $j_2(h_u) - 1 = \iota_\tau(u-1)\iota^*_\tau$. By (5.172), $q^{-1}e^0_{\tau(\xi)} = b_{e(\xi)}$; and since $J_1 e^0_i = e^0_i$ and $J_1 q = (q^{-1})^*\mathrm{diag}(\varepsilon)$, $(e^0_{\tau(\xi)})^* q = (e^0_{\tau(\xi)})^* J_1 q = (q^{-1}e^0_{\tau(\xi)})^*\mathrm{diag}(\varepsilon) = \varepsilon_{e(\xi)} b^*_{e(\xi)} = b^*_{e(\xi)}$. As $h_u - 1 = \sum_{\xi,\xi' \in X} (u-1)_{\xi\xi'}\, e^0_{\tau(\xi)}(e^0_{\tau(\xi')})^*$, we get $j_1(h_u) - 1 = q^{-1}(h_u - 1)q = \iota_e(u-1)\iota^*_e$.

(LA4) are the hypotheses on gaps, entries, β and h, with $\#\mathfrak{n} + \#\mathfrak{m} = r + k$.

Conclusion. By Theorem 5.7.21, $\theta(s) = c_{l/2}(p^{-1}x^{-1}sxp)\,\overline{c_{u_0/2}(h_s)}\, e^{\overline{\sigma(s)} - \sigma(s)}$ is ds-integrable and

$$ds\{s \in R : \|j_1(s)\| \leq \beta + 1,\ \|h_s\| \leq \beta + 1\} \leq \beta^{20(r+k)^2} \Big| \int_R \theta\, ds \Big|.$$

If $\|p^{-1}x^{-1}sxp-1\| \le 1$, then $\|j_1(s)\| \le 2 \le \beta+1$, and $\|h_s\| = 1 \le \beta+1$ as h_s is unitary. So the closed set $\{s \in R : \|p^{-1}x^{-1}sxp-1\| \le 1\}$ lies in the set on the left, which proves the inequality of the lemma; and $\ell(x) \le b^{5n}$ was shown above. □

Lemma 5.7.32. *Fix the setting of Definition 5.1.6. Let $y \in G_1$, $M \ge 1$ a real number and $w_0, \dots, w_{2\varrho} \in V_1$ with*

- *$w_a = y^{-1}f_a$ for $0 \le a \le \varrho$;*
- *$\|w_a\| \le M$ (Euclidean norm of $\mathbb{C}^r$) for $0 \le a \le 2\varrho$;*
- *$h(w_a, w_{a'}) = -1$ if $a + a' = 2\varrho$ and $h(w_a, w_{a'}) = 0$ if $a + a' < 2\varrho$.*

Then $\ell(y) \le 4(\varrho+1)M$.

Proof. The matrix J_1 is real symmetric with $J_1^2 = 1$, so it is unitary; it exchanges e_b and e_b^*, fixes the x_i and sends z to $-z$. Hence $|h(v,v')| = |\langle v, J_1 v'\rangle| \le \|v\|\,\|v'\|$.

Gram–Schmidt. The vectors $w_0, \dots, w_\varrho$ are the images under y^{-1} of the independent vectors $f_0, \dots, f_\varrho$, so they are independent. Let $o_0, \dots, o_\varrho$ be their Gram–Schmidt orthogonalisation for the Euclidean product, not normalised: o_a is the orthogonal projection of w_a to the orthogonal complement of $\langle w_b : b < a\rangle$. Then the o_a are pairwise orthogonal and nonzero, $\|o_a\| \le \|w_a\| \le M$, and $o_a = \sum_{b \le a} c_{ba} w_b$ with complex c_{ba} and $c_{aa} = 1$. For $0 \le a \le \varrho$ we have $2\varrho - a \le 2\varrho$, and $h(w_{2\varrho-a}, w_b)$ is 0 for $b < a$ and -1 for $b = a$; hence

$$-1 = h(w_{2\varrho-a}, o_a), \qquad 1 \le \|w_{2\varrho-a}\|\,\|o_a\| \le M\|o_a\|,$$

so $M^{-1} \le \|o_a\| \le M$.

An isometry with prescribed columns. For $a < \varrho$ put $v_a = \|o_a\|^{-2} o_a$, so that $\langle o_b, v_a\rangle = \delta_{ab}$ for $b \le \varrho$. Compare the families

$$\Phi = (o_0, \dots, o_\varrho,\ J_1 v_0, \dots, J_1 v_{\varrho-1}), \qquad \Psi = (f_0, \dots, f_\varrho,\ e_1^*, \dots, e_\varrho^*).$$

Their Gram matrices for h agree: $h(o_a, o_b) = \sum_{a' \le a,\, b' \le b} \overline{c_{a'a}} c_{b'b} h(f_{a'}, f_{b'})$, where $h(f_{a'}, f_{b'}) = -1$ if $a' = b' = \varrho$ and 0 otherwise, and $c_{\varrho a} = \delta_{\varrho a}$ for $a \le \varrho$; so $h(o_a, o_b) = -\delta_{a\varrho}\delta_{b\varrho} = h(f_a, f_b)$. Next $h(o_b, J_1 v_a) = \langle o_b, J_1^2 v_a\rangle = \delta_{ab} = h(f_b, e_{a+1}^*)$, since $f_b = e_{b+1}$ for $b < \varrho$ and $f_\varrho = z \perp e_{a+1}^*$. Last $h(J_1 v_a, J_1 v_b) = h(v_a, v_b) = \|o_a\|^{-2}\|o_b\|^{-2} h(o_a, o_b) = 0 = h(e_{a+1}^*, e_{b+1}^*)$ for $a, b < \varrho$. The family Ψ is part of the basis and its Gram matrix is invertible (it consists of the hyperbolic pairs (e_b, e_b^*) and $h(z,z) = -1$); so a relation $\sum_i \alpha_i \Phi_i = 0$ gives, after pairing with each Φ_j, the product of this invertible matrix with α, hence $\alpha = 0$. Thus Φ is independent, and the linear map from $\langle\Psi\rangle$ to V_1 sending Ψ_i to Φ_i is injective and preserves h. By Witt's theorem for the nondegenerate hermitian form h on $\mathbb{C}^r$ ([249], Chapter 7) it extends to an isometry A of (V_1, h); so $A \in G_1$, $Af_a = o_a$ $(a \le \varrho)$ and $Ae_{a+1}^* = J_1 v_a$ $(a < \varrho)$.

The norm of A^{-1}. Here $\|Af_a\| = \|o_a\| \le M$ for $0 \le a \le \varrho$, and $\|Ae_b^*\| = \|J_1 v_{b-1}\| = \|o_{b-1}\|^{-1} \le M$ for $1 \le b \le \varrho$, as J_1 is unitary. So Lemma 5.3.58, applied to $A^{-1} \in G_1$, gives $\|A^{-1}\| + 1 \le 4(\varrho+1)M$.

An element of R. Put $s = A^{-1}y^{-1} \in G_1$. For $b \le \varrho$, $s^{-1}f_b = yAf_b = yo_b = \sum_{b' \le b} c_{b'b}\, y w_{b'} = f_b + \sum_{b' < b} c_{b'b} f_{b'}$. By Lemma 5.5.30 (i), $s^{-1} \in R$, hence $s \in R$, and $sy = A^{-1}$. Therefore $\ell(y) \le \|sy\| + 1 = \|A^{-1}\| + 1 \le 4(\varrho+1)M$. □

Lemma 5.7.33. *Fix the setting of Definition 5.1.6, signs ε on the r basis indices of V_1 and a sign frame p for ε. Let $A, L, m \ge 0$ and $f \ge 1$ be integers and b real, and let $l \in \mathbb{Z}^r$ have large*

gap A for ε with $|l_i| \le L$ for all i; write $c_l^{G_1}(g) = c_{l/2}(p^{-1}gp)$ $(g \in G_1)$ and d_l for the formal degree $(\int |c_{l/2}|^2)^{-1}$ for the Haar measure on $\mathrm{U}(\operatorname{diag}\varepsilon)$ carried from dy by $g \mapsto p^{-1}gp$. Every $s \in R$ is uniquely $s = uh$ with $u \in \mathsf{U}$, $h \in H$; put $h_s = h$, $\sigma(s) = \sigma(u)$, $ds = du\,dh$ with dh the Haar probability measure. Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be a continuous unitary representation on $\mathbb{C}^I$, $v \in \mathbb{C}^I$ with $\|v\| = 1$, $x \in G_1$, and

$$W_x(y) = \int_R c_l^{G_1}(x^{-1}sy)\,\mathrm{wave}\big(-\sigma(s)\big)\,\rho(h_s)^{-1}v\,ds, \qquad V(x) = \int_R \mathbf{1}\big[\|p^{-1}x^{-1}sxp - 1\|_{\mathrm{op}} \le 1\big]\,ds.$$

Suppose that $V(x) \le b^m d_l \int_{\mathsf{U}\backslash G_1} \|W_x\|^2$, $\ell(x) \le b^{5n}$, $b \ge 2000$, $r \ge 100$, $m + 99r^2 \le 400r^2$, $r + L + 1 \le b^4$ and $3r^3 f \le A$. Then there are $c \in \mathbb{R}$ and a continuous $F\colon G_1 \to \mathbb{C}^I$ with $F = c\,W_x$, $\int_{G_1} d_l\, c_l^{G_1}(a^{-1})F(ya)\,da = F(y)$ for all $y \in G_1$, and such that $(F, \chi(\ell(\cdot))F)$ is a cut pair (for b, f), where $\chi = 1$ on $[0, \frac{1}{2}b^{6n+2}]$, $\chi = 0$ on $[b^{6n+2}, \infty)$, and χ is linear in between.

Proof. Write $\psi = c_l^{G_1}$ and $\iota(g) = p^{-1}gp$. As p is unitary with $p^*J_1p = \operatorname{diag}(\varepsilon)$, ι is an isomorphism of topological groups $G_1 \to \mathrm{U}(\operatorname{diag}\varepsilon)$ with $\|\iota(g)\| = \|g\|$ and $\|\iota(g) - 1\| = \|g - 1\|$; it carries the Cayley measure dy, which is left and right invariant (Lemma 5.2.41) and finite on compact sets, to a Haar measure dg of $\mathrm{U}(\operatorname{diag}\varepsilon)$, and d_l is the formal degree for dg. Every $g \in G_1$ has $\|g\| \ge 1$, since $g^{-1} = J_1g^*J_1$ has the same norm and $1 \le \|g\|\,\|g^{-1}\|$. The vector l has gap $500r^3 + A$. Put $N = 9r^2$.

Inputs on the coefficient. By Lemma 5.7.40, transported by ι,

$$|\psi(g)| \le \Big(\frac{2}{\|g\|}\Big)^j \qquad (g \in G_1,\ 0 \le j \le 500r^3 + A). \tag{5.174}$$

By Lemma 5.7.38 (3) with the integer $N > 8r^2$, transported by ι, $d_l > 0$ and, using $2000 \le b$, $r + L + 1 \le b^4$ and $8 \le b$,

$$d_l \int_{G_1} \Big(\frac{2}{\|g\|}\Big)^N dg \le 8\big(2000(r + L + 1)^6\big)^{2r^2} 2^N \le b \cdot b^{50r^2} \cdot 2^N. \tag{5.175}$$

The Bessel average. By Lemma 5.6.37 (i) (ρ continuous unitary, l of large gap A), $W = W_x$ is a continuous Bessel function of type ρ, integrable and square integrable over $\mathsf{U}\backslash G_1$, and $\int_{G_1} d_l\psi(a^{-1})W(ya)\,da = W(y)$. As ρ is unitary, $|\mathrm{wave}| = 1$ and $\|v\| = 1$,

$$\|W(y)\| \le \int_R |\psi(x^{-1}sy)|\,ds \qquad (y \in G_1). \tag{5.176}$$

The ball has positive finite volume. The map $j(s) = \iota(x^{-1}sx)$ is a homomorphism $R \to \mathrm{U}(\operatorname{diag}\varepsilon)$; it is the restriction to the closed subgroup R (Lemma 5.5.30 (i)) of a homeomorphism, hence proper, and $V(x) = ds\{s : \|j(s) - 1\| \le 1\}$. By Lemma 5.6.32 (ii), applicable by Lemma 5.5.30 (v), $V(x) < \infty$. The set $\{s \in R : \|x^{-1}sx - 1\| < 1\}$ is an open neighbourhood of 1 in R contained in the ball. Its preimage under the homeomorphism $(u, h) \mapsto uh$ of Lemma 5.5.30 (i) contains a product $O_1 \times O_2$ of nonempty open sets of U and H, and $du(O_1) > 0$, $dh(O_2) > 0$: countably many left translates of O_1 cover the second countable group U, whose measure is $du(\mathsf{U}) = \infty$ by Lemma 5.2.34 (iii) (applied to the whole coordinate space), and du is left invariant (Lemma 5.2.34 (iv)); likewise finitely many translates of O_2 cover H, of mass 1. Hence $V(x) > 0$.

Normalisation. Let $Q = \int_{\mathsf{U}\backslash G_1} \|W\|^2 < \infty$. By hypothesis $0 < V(x) \le b^m d_l Q$, so $Q > 0$. Put $c = (\#I/Q)^{1/2}$ and $F = cW$. Then F is a continuous Bessel function of type ρ, integrable over $\mathsf{U}\backslash G_1$, with $\int_{\mathsf{U}\backslash G_1} \|F\|^2 = \#I$, and $\int_{G_1} d_l\psi(a^{-1})F(ya)\,da = F(y)$ by linearity.

The Bessel length along R. For $y, y' \in G_1$ and $s' \in R$, $\|s'y'\| + 1 \le \|s'y\|\,\|y^{-1}y'\| + \|y^{-1}y'\|$ (as $\|y^{-1}y'\| \ge 1$); taking the infimum over s', $\ell(y') \le \|y^{-1}y'\|\,\ell(y)$. Hence ℓ is continuous (as $\|y^{-1}y'\| \to 1$ when $y' \to y$), and since $s' \mapsto s's$ is a bijection of R for $s \in R$, $\ell(sy) = \ell(y)$. For $s \in R$ and $y \in G_1$ this gives $\ell(y) = \ell(sy) \le \|x^{-1}sy\|\,\ell(x) \le \|x^{-1}sy\|\,b^{5n}$, so with $\lambda(y) = \max(1, \ell(y)/b^{6n})$:

$$\ell(y)/b^{6n} \le \|x^{-1}sy\|/b^n \le \|x^{-1}sy\|/2.$$

The decay majorant. Let $\Phi(y) = \int_R (2/\|x^{-1}sy\|)^N ds$, a Borel function of y by Tonelli's theorem, left R-invariant by the right invariance of ds. Let $j_0 = 3r^3 f \le A$ and $g = x^{-1}sy$. If $\lambda(y) = 1$, (5.174) with exponent N gives $\lambda(y)^{j_0}|\psi(g)| \le (2/\|g\|)^N$. Otherwise $\lambda(y) = \ell(y)/b^{6n} \le \|g\|/2$, and (5.174) with exponent $j_0 + N \le A + 500r^3$ gives $\lambda(y)^{j_0}|\psi(g)| \le (\|g\|/2)^{j_0}(2/\|g\|)^{j_0+N} = (2/\|g\|)^N$. Integrating over s and using (5.176),

$$\lambda(y)^{3r^3 f}\,\|F(y)\| \le c\,\Phi(y) \qquad (y \in G_1). \tag{5.177}$$

By Lemma 5.6.32 (ii) for j, the point $\iota(x^{-1}y)$ (so that $j(s)\iota(x^{-1}y) = \iota(x^{-1}sy)$) and the integer $N > 8r^2$: $\Phi(y) \le 16^N V(x)$. By the unfolding Lemma 5.5.30 (vi) and left invariance of dy, $\int_{\mathsf{U}\backslash G_1} \Phi = \int_{G_1} (2/\|x^{-1}g\|)^N dg = \int_{G_1} (2/\|g\|)^N dg$. Put $A_{\mathrm{m}} = c\,\Phi$ and $a = c\,16^N V(x) < \infty$. Then $A_{\mathrm{m}} \le a$, and by the hypothesis on $V(x)$, $c^2 Q = \#I$ and (5.175),

$$\begin{aligned} a \int_{\mathsf{U}\backslash G_1} A_{\mathrm{m}} &= c^2\,16^N V(x) \int_{G_1} \Big(\frac{2}{\|g\|}\Big)^N dg \le \#I\, b^m\, 16^N\, d_l \int_{G_1} \Big(\frac{2}{\|g\|}\Big)^N dg \\ &\le \#I\, b^{m+1+50r^2}\, 32^N \le b^{m+99r^2} \#I \le b^{400r^2} \#I, \end{aligned}$$

using $32^N \le b^N = b^{9r^2}$ and $1 + 59r^2 \le 99r^2$.

The cut. Let $F^\flat(y) = \chi(\ell(y))F(y)$. As $\chi \in [0,1]$ and ℓ is continuous and left R-invariant, $F^\flat$ is a bounded Borel Bessel function of type ρ with $\|F^\flat\| \le \|F\|$ and $\|F^\flat - F\| = (1-\chi(\ell))\|F\| \le \|F\|$. We check (BS1)–(BS3) of a cut pair for $(F, F^\flat)$.

(BS1) F is integrable over $\mathsf{U}\backslash G_1$, $\int \|F\|^2 = \#I$ and $\int \|F^\flat\|^2 \le \int \|F\|^2 = \#I$.

(BS3) A_{m} is Borel, left R-invariant, $A_{\mathrm{m}} \le a < \infty$, $\lambda^{3r^3 f}\|F\| \le A_{\mathrm{m}}$ by (5.177), $\lambda^{3r^3 f}\|F^\flat\| \le \lambda^{3r^3 f}\|F\| \le A_{\mathrm{m}}$, and $a \int A_{\mathrm{m}} \le b^{400r^2}\#I$.

(BS2) If $\ell(y) > b^{6n+2}$ then $\chi(\ell(y)) = 0$, so $F^\flat$ vanishes outside the cut; if $F^\flat(y) \ne F(y)$ then $\chi(\ell(y)) \ne 1$, so $\ell(y) > \frac{1}{2}b^{6n+2}$. At such y, $\lambda(y) \ge \ell(y)/b^{6n} > b^2/2 \ge b$, so $b^{3r^3 f}\|F^\flat(y) - F(y)\| \le \lambda(y)^{3r^3 f}\|F(y)\| \le A_{\mathrm{m}}(y) \le a$; this holds trivially where $F^\flat = F$. Multiplying the two bounds, $b^{6r^3 f}\|F^\flat - F\|^2 \le a\,A_{\mathrm{m}}$ on G_1, both sides left U-invariant, and integrating, $\int_{\mathsf{U}\backslash G_1} \|F^\flat - F\|^2 \le b^{400r^2 - 6r^3 f}\#I \le b^{-2r^3 f}\#I$, because $400r^2 \le 4r^3 \le 4r^3 f$ for $r \ge 100$, $f \ge 1$, and $b \ge 1$.

So $(F, \chi(\ell(\cdot))F)$ is a cut pair for b, f, with $F = cW_x$ continuous and reproduced, as claimed. □

Lemma 5.7.34. *Fix the setting of Definition 5.1.6, signs ε on the r basis indices of V_1 and a sign frame p for ε. Let $A, L \ge 0$ be integers, B real, and $l \in \mathbb{Z}^r$ with large gap A for ε and $|l_i| \le L$ for all i; write $c_l^{G_1}(g) = c_{l/2}(p^{-1}gp)$ $(g \in G_1)$ and d_l for the formal degree $(\int |c_{l/2}|^2)^{-1}$ for the Haar measure on $\mathrm{U}(\mathrm{diag}\,\varepsilon)$ carried from dy by $g \mapsto p^{-1}gp$. Suppose $(r+1)^4(L+r+1+2n^2) \le B$. Let $\rho\colon \mathrm{U}(k) \to \mathrm{U}(I)$ be a unitary representation on $\mathbb{C}^I$ and F a Bessel function of type ρ, integrable over $\mathsf{U}\backslash G_1$, with $\int_{G_1} d_l\, c_l^{G_1}(a^{-1})F(ya)\,da = F(y)$ for all $y \in G_1$. Then F has Schur's bound $(B,\ d_l^{-1}\int_{\mathsf{U}\backslash G_1}\|F\|^2)$.*

Proof. Write $\psi = c_l^{G_1}$, $e = d_l\psi$, $D = d_l^{-1}\int_{\mathsf{U}\backslash G_1}\|F\|^2$, and let X_w be the kernels of Lemma 5.7.35. The Bessel function F is bounded and Borel and $\int e(a^{-1})F(ya)\,da = F(y)$; so by Lemma 5.7.35 (iii) the functions $V_w(y) = \int_{G_1} X_w(a^{-1})F(ya)\,da$ are Bessel functions of type ρ, $V_\emptyset = F$, and $\frac{d}{dt}V_w(\gamma(t)) = \sum_{a,b} Z_{ab}V_{(a,b)w}(\gamma(t))$ whenever $\gamma'(t) = \gamma(t)Z$. Let $|w| \le 2n^2$ and ϕ be integrable on G_1. By Lemma 5.7.35 (i), X_w is integrable with $\int X_w(a^{-1})e(ya)\,da = X_w(y)$, so Lemma 5.7.19 (iii) applies to F and $X = X_w$:

$$\int_{\mathsf{U}\backslash G_1}\Big\|\int_{G_1}\phi(y)V_w(\gamma y)\,dy\Big\|^2 d\gamma \le \Big(d_l^{-1}\int_{G_1}|X_w|^2\Big)D\int_{G_1}|\phi|^2 \le B^{2|w|}D\int_{G_1}|\phi|^2,$$

by Lemma 5.7.35 (ii) and $(r+1)^4(L+r+1+|w|) \le (r+1)^4(L+r+1+2n^2) \le B$. So V_w satisfies Schur$(B^{2|w|}D)$ for $|w| \le 2n^2$, and F has Schur's bound (B, D). □

Lemma 5.7.35. *Fix the setting of Definition 5.1.6, signs ε on the r basis indices of V_1 and a sign frame p for ε. Let $A, L \ge 0$ be integers and $l \in \mathbb{Z}^r$ with large gap A for ε and $|l_i| \le L$ for all i; write $\psi(g) = c_{l/2}(p^{-1}gp)$, d_l for the formal degree for the Haar measure on $\mathrm{U}(\operatorname{diag}\varepsilon)$ carried from dy by $g \mapsto p^{-1}gp$, and $e = d_l\psi$. Words are finite sequences w of pairs of basis indices, $|w|$ is the length and $(a,b)w$ the word with the letter (a,b) prefixed. There are functions $X_w\colon G_1 \to \mathbb{C}$ such that:*

(i) $X_\emptyset = e$; every X_w is integrable and square integrable, and $\int_{G_1} X_w(a^{-1})\,e(ya)\,da = X_w(y)$ for all $y \in G_1$;

(ii) $d_l^{-1}\int_{G_1}|X_w|^2 \le \big((r+1)^4(L+r+1+|w|)\big)^{2|w|}$ for every word w;

(iii) if $F\colon G_1 \to \mathbb{C}^I$ is bounded and Borel with $\int_{G_1} e(a^{-1})F(ya)\,da = F(y)$ for all y, then the functions $V_w(y) = \int_{G_1} X_w(a^{-1})F(ya)\,da$ are bounded and continuous, $V_\emptyset = F$, and for every differentiable curve γ in G_1 with $\gamma'(t) = \gamma(t)Z$ at t,

$$\frac{d}{dt}V_w(\gamma(t)) = \sum_{a,b} Z_{ab}\,V_{(a,b)w}(\gamma(t));$$

if moreover F is a Bessel function of type ρ for a unitary representation ρ of $\mathrm{U}(k)$, so is every V_w.

Proof. Let $\iota(g) = p^{-1}gp$, an isomorphism of topological groups $G_1 \to G$ carrying dy to dg (a Haar measure: dy is left and right invariant by Lemma 5.2.41); write $c = c_{l/2}$, so $\psi = c \circ \iota$, and $d = d_l$. The gap $500r^3 + A$ exceeds $8r^2$. For a basis pair (a,b) let $\hat{E}_{ab} = p^{-1}E_{ab}p$, of operator norm 1 as p is unitary. By Lemma 5.7.36 (ii) there is a smooth compactly supported β on G with $c = f_\beta$. Define bumps $\beta_\emptyset = \beta$, $\beta_{(a,b)w} = (\beta_w)_{\hat{E}_{ab}}$, and put $X_w = d\,f_{\beta_w} \circ \iota$ and $b_w = \beta_w \circ \iota$, a smooth compactly supported function on G_1.

(ii). By Lemma 5.7.36 (ii), $f_{\beta_w} = \hat{E}_{a_1b_1}(\cdots(\hat{E}_{a_sb_s}c))$ for $w = ((a_1,b_1),\dots,(a_s,b_s))$. As $c \in M_0$, Lemma 5.7.37 applied s times (to steps $s-1,\dots,0$, with $\|\hat{E}_{ab}\| = 1$) gives $\|f_{\beta_w}\|_2 \le \prod_{j<s}(r+1)^4(L+r+1+j)\,\|c\|_2 \le \big((r+1)^4(L+r+1+s)\big)^s d^{-1/2}$, using $\|c\|_2^2 = d^{-1}$; so $d^{-1}\int_{G_1}|X_w|^2 = d\,\|f_{\beta_w}\|_2^2$ is at most the claimed bound.

(i). $X_\emptyset = d\,c \circ \iota = e$. Each f_φ is bounded and square integrable (Lemma 5.7.36 (i)) and integrable, since $\int_G |f_\varphi| \le \|\varphi\|_1\|c\|_1$ (c is integrable, Lemma 5.6.23 (i)). For the reproduction it suffices, transporting by ι, to show $d\int_G f_\varphi(a^{-1})c(ya)\,da = f_\varphi(y)$. By Fubini's theorem (the integrand is bounded by $|\varphi(z)||c(a^{-1}z)|$, integrable) this is $\int \varphi(z)\,d\int c(ya)c(a^{-1}z)\,da\,dz$, and

with $c(a^{-1}z) = \overline{c(z^{-1}a)}$ (Lemma 5.7.40), the substitution $a = zu$ and Lemma 5.6.23 (ii) with $x = yz$ and $a = 1$, $d\int c(yzu)\overline{c(u)}\,du = c(yz)c(1) = c(yz)$; so the integral is $f_\varphi(y)$.

(iii). Let F be as stated and $\|F\| \le C$. By the definition of f_{β_w} and the transport, $X_w(a^{-1}) = d\int_{G_1} b_w(z)\psi(a^{-1}z)\,dz$, so by Fubini's theorem (integrand bounded by $C|b_w(z)||\psi(a^{-1}z)|$) and the substitution $a = zt$,

$$V_w(y) = \int_{G_1} b_w(z)\Big(\int_{G_1} e(t^{-1})F(yzt)\,dt\Big)dz = \int_{G_1} b_w(z)F(yz)\,dz.$$

Hence (also directly from $X_\emptyset = e$) $V_\emptyset(y) = \int e(a^{-1})F(ya)\,da = F(y)$; $\|V_w\| \le C\|b_w\|_1$; and $V_w(y) = \int b_w(y^{-1}x)F(x)\,dx$ is continuous in y by dominated convergence (for y in a compact set the integrands are supported in a fixed compact set and bounded). For the derivative let $\gamma'(t) = \gamma(t)Z$, so $Z = \gamma(t)^{-1}\gamma'(t)$ lies in the Lie algebra of G_1 and $\gamma(t+s)^{-1} = (1 - sZ + o(s))\gamma(t)^{-1}$. By the chain rule, $\frac{d}{dt}b_w(\gamma(t)^{-1}x) = (b_w)_Z(\gamma(t)^{-1}x)$, where $(b_w)_Z(x) = \frac{d}{ds}|_0 b_w(e^{-sZ}x)$; for t in a compact interval these derivatives are bounded and supported in a fixed compact set, so differentiation under the integral gives $\frac{d}{dt}V_w(\gamma(t)) = \int (b_w)_Z(\gamma(t)^{-1}x)F(x)\,dx$. Since $\iota(e^{-sZ}x) = e^{-s\iota(Z)}\iota(x)$, $(b_w)_Z = (\beta_w)_{p^{-1}Zp}\circ\iota$, and by complex linearity of $X \mapsto (\beta_w)_X$ (Lemma 5.7.36 (i)) and $p^{-1}Zp = \sum_{a,b} Z_{ab}\hat{E}_{ab}$, $(b_w)_Z = \sum Z_{ab}\, b_{(a,b)w}$. Hence $\frac{d}{dt}V_w(\gamma(t)) = \sum_{a,b} Z_{ab}V_{(a,b)w}(\gamma(t))$. Finally, if F is a Bessel function of type ρ, then $V_w(uy) = \int X_w(a^{-1})\mathrm{wave}(\sigma(u))F(ya)\,da = \mathrm{wave}(\sigma(u))V_w(y)$ for $u \in \mathsf{U}$ and likewise $V_w(hy) = \rho(h)V_w(y)$ for $h \in H$ (the linear map $\rho(h)$ commutes with the integral); with boundedness and continuity, V_w is a Bessel function of type ρ. □

Lemma 5.7.36. *Let n be a nonempty finite set, $\varepsilon \in \{\pm1\}^n$, $J = \mathrm{diag}(\varepsilon)$, $G = \mathrm{U}(J)$ with a Haar measure dg, $\mathfrak{u}(J) = \{Y : Y^*J + JY = 0\}$, and E_{ab} the matrix units. Let $h > 8(\#n)^2$ be an integer, $l \in \mathbb{Z}^n$ with gap h for ε, $c = c_{l/2}$, $\mu = \mu_l$, $d = (\int_G |c|^2)^{-1}$, and $\langle u, v\rangle = \int_G \overline{u}\,v$ on $L^2(G)$. For a smooth compactly supported φ on G put $f_\varphi(g) = \int_G \varphi(y)\,c(gy)\,dy$; for $Y \in \mathfrak{u}(J)$ put $\varphi_Y(y) = \frac{d}{dt}|_{t=0}\varphi(e^{-tY}y)$, and for $X = Y_1 + iY_2$ with $Y_1, Y_2 \in \mathfrak{u}(J)$ (a unique decomposition) put $\varphi_X = \varphi_{Y_1} + i\varphi_{Y_2}$. Let $\mathcal{H}_c$ be the span of the functions $X_1(X_2(\cdots(X_s c)))$, $s \ge 0$, $X_j \in \mathrm{M}_n(\mathbb{C})$.*

- *(i) φ_Y and φ_X are smooth with support in that of φ, $X \mapsto \varphi_X$ is complex linear, $|f_\varphi| \le \|\varphi\|_1$, $\|f_\varphi\|_2 \le \|\varphi\|_1\|c\|_2$, and for $Y \in \mathfrak{u}(J)$: $\frac{d}{dt}|_{t=0} f_\varphi(ge^{tY}) = f_{\varphi_Y}(g)$ for every g, and $t^{-1}(f_\varphi(\cdot\, e^{tY}) - f_\varphi) \to f_{\varphi_Y}$ in $L^2(G)$ as $t \to 0$.*
- *(ii) There is a smooth compactly supported β with $c = f_\beta$; for every word, $X_1(X_2(\cdots(X_s c))) = f_\varphi$ with $\varphi = (\cdots((\beta_{X_s})_{X_{s-1}})\cdots)_{X_1}$. So $\mathcal{H}_c$ consists of bounded square integrable functions f_φ, is stable under every $X \in \mathrm{M}_n(\mathbb{C})$, and $X(X'u) - X'(Xu) = (XX' - X'X)u$ for $u \in \mathcal{H}_c$.*
- *(iii) $\langle E_{ab}u, v\rangle = \varepsilon_a\varepsilon_b\langle u, E_{ba}v\rangle$ for $u, v \in \mathcal{H}_c$ and $a, b \in n$.*
- *(iv) $\|c\|_2^2 = d^{-1}$ and $|u(g)| \le d^{1/2}\|u\|_2$ for every $u \in \mathcal{H}_c$ and $g \in G$.*
- *(v) $E_{aa}c = \mu_a c$; $E_{ab}c = 0$ if $a \ne b$, $\varepsilon_a = \varepsilon_b$ and $l_a > l_b$; and $\Omega u = \kappa_l u$ for every $u \in \mathcal{H}_c$, where $\Omega u = \sum_{a,b} E_{ab}(E_{ba}u)$ and $\kappa_l = \frac{1}{4}\sum_a (l_a^2 - (2\rho_l(a))^2) \ge -\frac{1}{4}(\#n)^3$.*

Proof. G is unimodular, so dg is invariant under left and right translations; write R_x for right translation, a unitary operator on $L^2(G)$. By Lemma 5.6.23 ($h > 8(\#n)^2$), c is continuous, integrable and square integrable, $d > 0$, Schur orthogonality (ii) holds, and there is a smooth compactly supported β with $f_\beta = c$ (part (iii) there). By Lemma 5.7.40, $|c| \le 1$.

(i). Smoothness and support are clear, and $Y \mapsto \varphi_Y$ is real linear on $\mathfrak{u}(J)$; as $\mathrm{M}_n(\mathbb{C}) = \mathfrak{u}(J) \oplus i\,\mathfrak{u}(J)$ ($X = Y_1 + iY_2$ with $Y_1 = \frac{1}{2}(X - JX^*J)$, $Y_2 = \frac{1}{2i}(X + JX^*J)$), $X \mapsto \varphi_X$ is complex linear.

$|f_\varphi| \le \|\varphi\|_1$ as $|c| \le 1$, and by Minkowski's integral inequality $\|f_\varphi\|_2 \le \int |\varphi(y)|\,\|R_y c\|_2\,dy = \|\varphi\|_1\|c\|_2$. For $Y \in \mathfrak{u}(J)$, left invariance gives $f_\varphi(ge^{tY}) = \int \varphi(e^{-tY}y)c(gy)\,dy = f_{\varphi(e^{-tY}\cdot)}(g)$. The functions $\psi_t = t^{-1}(\varphi(e^{-tY}\cdot) - \varphi) - \varphi_Y$, $0 < |t| \le 1$, are supported in the compact set $\{e^{-sY} : |s| \le 1\}\operatorname{supp}\varphi$ and tend to 0 uniformly (Taylor's formula; the second t-derivative of $\varphi(e^{-tY}y)$ is bounded there); so $\|\psi_t\|_1 \to 0$, and $|f_{\psi_t}| \le \|\psi_t\|_1$, $\|f_{\psi_t}\|_2 \le \|\psi_t\|_1\|c\|_2$ give both limits.

(ii). By induction on s: the function $u = X_2(\cdots(X_s c))$ equals $f_{\varphi'}$ with φ' the bump of the claim for the shorter word, and u is locally holomorphic (Lemma 5.6.24 (i), (ii)). For $X_1 = Y_1 + iY_2$, Lemma 5.6.24 (i) gives $X_1 u = Y_1 u + iY_2 u$ with $(Y_j u)(g) = \frac{d}{dt}|_0 u(ge^{tY_j}) = f_{\varphi'_{Y_j}}(g)$ by (i); so $X_1 u = f_{\varphi'_{X_1}}$. The rest follows from (i) and the commutator identity of Lemma 5.6.24 (i).

(iii). Let $u = f_\varphi$, $v = f_{\varphi'}$ in $\mathcal{H}_c$ and $Y \in \mathfrak{u}(J)$. The function $t \mapsto \langle R_{e^{tY}}u, R_{e^{tY}}v\rangle = \langle u, v\rangle$ is constant, and by the L^2 limit of (i) its derivative at 0 is $\langle Yu, v\rangle + \langle u, Yv\rangle$ (with $Yu = f_{\varphi_Y}$ by (ii)). So $\langle Yu, v\rangle = -\langle u, Yv\rangle$. For $X = Y_1 + iY_2$, as the product is conjugate-linear in the first argument, $\langle Xu, v\rangle = \langle u, (-Y_1 + iY_2)v\rangle = \langle u, (JX^*J)v\rangle$; for $X = E_{ab}$, $JE_{ba}J = \varepsilon_a\varepsilon_b E_{ba}$.

(iv). $\|c\|_2^2 = d^{-1}$ by definition. Let $u = f_\varphi \in \mathcal{H}_c$ and $g \in G$. By Fubini's theorem (the integrand is bounded by $|\varphi(y)||c(z)|$) and Lemma 5.6.23 (ii) with $x = 1$, $a = gy$, $c(1) = 1$: $\langle c, R_g u\rangle = \int \varphi(y) \int c(zgy)\overline{c(z)}\,dz\,dy = d^{-1}u(g)$. By Cauchy–Schwarz, $|u(g)| \le d\,\|c\|_2\|R_g u\|_2 = d^{1/2}\|u\|_2$.

(v). The first two statements are Lemma 5.6.24 (iii) (gap $h \ge 0$). By Lemma 5.6.24 (iv), $\Omega c = \kappa_l c$. The element $\sum_{a,b} E_{ab}E_{ba}$ is central in the universal enveloping algebra of $\mathfrak{gl}_n$: from $[E_{cd}, E_{ab}] = \delta_{da}E_{cb} - \delta_{bc}E_{ad}$,

$$\sum_{a,b}[E_{cd}, E_{ab}]E_{ba} = \sum_b E_{cb}E_{bd} - \sum_a E_{ad}E_{ca}, \qquad \sum_{a,b} E_{ab}[E_{cd}, E_{ba}] = \sum_a E_{ad}E_{ca} - \sum_b E_{cb}E_{bd},$$

so $\sum_{a,b}[E_{cd}, E_{ab}E_{ba}] = 0$. By the commutator identity of (ii), the operator Ω on $\mathcal{H}_c$ therefore commutes with every E_{cd}, hence with every word, and $\Omega(X_1\cdots X_s c) = X_1\cdots X_s\Omega c = \kappa_l X_1\cdots X_s c$. Finally $|2\rho_l(a)| \le \#n - 1$, so $\kappa_l \ge -\frac{1}{4}\#n(\#n-1)^2 \ge -\frac{1}{4}(\#n)^3$. □

Lemma 5.7.37. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $B, L \ge 0$ integers, and $l \in \mathbb{Z}^n$ with large gap B for ε and $|l_a| \le L$ for all a; let $\mathcal{H}_c$ be as in Lemma 5.7.36. Let $M_0 \subseteq \mathcal{H}_c$ be the span of the functions $E_{a_1b_1}(\cdots(E_{a_mb_m}c))$ with $m \ge 0$ and $\varepsilon_{a_j} = \varepsilon_{b_j}$ for all j, and $M_{s+1} = M_s + \sum_{a,b} E_{ab}M_s$. Then for every $s \ge 0$, $y \in M_s$ and $X \in \mathrm{M}_n(\mathbb{C})$, $Xy \in M_{s+1}$ and*

$$\|Xy\|_2 \le (\#n + 1)^4\,(L + \#n + 1 + s)\,\|X\|\,\|y\|_2,$$

with $\|X\|$ the operator norm.

Proof. Write $N_0 = \#n$, $\mu = \mu_l$, call (a, b) *compact* if $\varepsilon_a = \varepsilon_b$, *lowering* if moreover $l_a < l_b$, and put $\Omega_K = \sum_{(a,b)\text{ compact}} E_{ab}E_{ba}$, an operator on $\mathcal{H}_c$ by (ii). The gap $500N_0^3 + B > 8N_0^2$, so Lemma 5.7.36 applies. $Xy \in M_{s+1}$ since $Xy = \sum X_{ab}E_{ab}y$ by complex linearity.

Step 1: weights. From $[E_{aa}, E_{cd}] = (\delta_{ac} - \delta_{ad})E_{cd}$ and (v), for a word $w = ((a_1, b_1), \ldots, (a_m, b_m))$ and $wc = E_{a_1b_1}(\cdots(E_{a_mb_m}c))$ we get $E_{aa}(wc) = \nu_w(a)\,wc$ with $\nu_w = \mu + \sum_j(\delta_{a_j} - \delta_{b_j})$ (here δ_a is the indicator vector of a), an integer vector with $|\nu_w(a) - \mu_a| \le m$. By (iii) E_{aa} is symmetric on $\mathcal{H}_c$, so weight vectors of distinct weights are orthogonal. The entries of l are distinct integers in $[-L, L]$ of one parity, so $N_0 \le L + 1$; $2\rho_l(a) = (N_0 - 1) - 2\#\{j : l_j > l_a\}$ and l_a are both congruent to $N_0 - 1$ modulo 2, so $\mu_a = \frac{1}{2}(l_a + 2\rho_l(a)) - 2\rho_l^{\mathrm{c}}(a)$; with $|2\rho_l(a)|, |2\rho_l^{\mathrm{c}}(a)| \le N_0 - 1$, $|\mu_a| \le \frac{1}{2}(L + N_0 - 1) + N_0 - 1 \le L + N_0$.

Step 2: highest vectors. Call $z \in \mathcal{H}_c$ *K-highest of weight ν* if $E_{aa}z = \nu_a z$ for all a and $E_{ab}z = 0$ for compact (a,b) with $l_a > l_b$. For such z, as in the computation for c: for compact (a,b) with $l_a > l_b$, $E_{ab}E_{ba}z = E_{ba}E_{ab}z + (E_{aa} - E_{bb})z = (\nu_a - \nu_b)z$, and $E_{ba}z = 0$ when $l_a < l_b$; hence

$$\Omega_K z = c_K(\nu)z, \qquad c_K(\nu) = \sum_a \nu_a^2 + \sum_{\varepsilon_a=\varepsilon_b,\ l_a>l_b} (\nu_a - \nu_b) = \sum_a \nu_a(\nu_a + 2\rho_l^{\mathrm{c}}(a)).$$

If $W \neq 0$ is a finite-dimensional subspace of $\mathcal{H}_c$ stable under the E_{ab} with (a,b) compact, then W contains a nonzero K-highest vector: the E_{aa} commute and are symmetric on W (by (iii)), so W is the orthogonal sum of joint eigenspaces, with real weights; take a weight ν of W maximising $\sum_a l_a\nu_a$ and $z \neq 0$ of weight ν; for compact (a,b) with $l_a > l_b$, $E_{ab}z \in W$ has weight $\nu + \delta_a - \delta_b$ of larger pairing with l, so $E_{ab}z = 0$.

Step 3: a closure property. Let $T \subseteq \mathcal{H}_c$ be a subspace stable under the lowering E_{ab} and containing wc for all words w of length at most s (arbitrary letters). Then $M_s \subseteq T$. Indeed, for a set Σ of pairs of signs and a lowering-stable subspace U, the space $U^\Sigma = \{y : E_{cd}y \in U \text{ whenever } (\varepsilon_c,\varepsilon_d) \in \Sigma\}$ is lowering-stable: for lowering (a,b) and $y \in U^\Sigma$, $E_{cd}E_{ab}y = E_{ab}E_{cd}y + \delta_{da}E_{cb}y - \delta_{bc}E_{ad}y$, and $(\varepsilon_c,\varepsilon_b)$, $(\varepsilon_a,\varepsilon_d)$ equal $(\varepsilon_c,\varepsilon_d)$ when the deltas are nonzero (as $\varepsilon_a = \varepsilon_b$). For $s = 0$: let T_0 be the smallest lowering-stable subspace containing c. Then $c \in T_0^{\Sigma_0}$ for Σ_0 the diagonal pairs of signs (lowering letters map c into T_0, $E_{aa}c = \mu_a c$, raising compact letters kill c), and $T_0^{\Sigma_0}$ is lowering-stable, so $T_0 \subseteq T_0^{\Sigma_0}$: T_0 is stable under all compact E_{ab}, hence $M_0 \subseteq T_0 \subseteq T$. For $s+1$: T^{all} (Σ all pairs) is lowering-stable and contains wc for $|w| \leq s$, so by induction $M_s \subseteq T^{\mathrm{all}}$, i.e. $E_{ab}M_s \subseteq T$; with $M_s \subseteq T$ this gives $M_{s+1} \subseteq T$.

Step 4: M_s is finite-dimensional and stable under compact letters. Ω_K commutes with the compact E_{cd} (it is the sum of the Casimir elements of the two blocks $\{\varepsilon = \pm 1\}$, each central in its enveloping algebra by the identity of Lemma 5.7.36 (v) for that block), so $\Omega_K = c_K(\mu)$ on M_0, as c is K-highest of weight μ by (v). By Step 3 ($s = 0$, $T = T_0$), M_0 is spanned by the vectors wc with all letters lowering. For such w of length m with $wc \neq 0$, by (iii) and Step 1, $c_K(\mu)\|wc\|^2 = \langle wc, \Omega_K wc\rangle = \sum_{\text{compact}} \|E_{ba}wc\|^2 \geq \sum_a \nu_w(a)^2\|wc\|^2$, while $\sum_a l_a\nu_w(a) \leq \sum_a l_a\mu_a - m$ (each lowering letter changes the pairing by $l_a - l_b \leq -1$). Hence $0 \leq \sum_a (l_a + \nu_w(a))^2 \leq \sum_a l_a^2 + 2\sum_a l_a\mu_a - 2m + c_K(\mu)$, which bounds m; so M_0 is spanned by finitely many vectors, and M_{s+1} is a finite sum of images of M_s. Stability under compact E_{cd} by induction: $E_{cd}E_{ab}y = E_{ab}E_{cd}y + [E_{cd}, E_{ab}]y$, with $[E_{cd}, E_{ab}]$ a combination of matrix units.

Step 5: Ω_K on M_s. Put $N = L + N_0 + s$. By (iii), Ω_K is symmetric on $\mathcal{H}_c$ and maps M_s into itself (Step 4); let e be an eigenvalue of $\Omega_K|_{M_s}$ and $W \subseteq M_s$ its eigenspace, stable under compact letters (they commute with Ω_K). By Step 2, W contains a nonzero K-highest z of some weight ν, and $e = c_K(\nu)$. Some word w of length at most s has $\langle wc, z\rangle \neq 0$: otherwise the orthogonal complement of z, which is lowering-stable (by (iii) the adjoint of a lowering E_{ab} is E_{ba}, and $E_{ba}z = 0$), would contain $M_s \ni z$ by Step 3. Pairing $E_{aa}wc = \nu_w(a)wc$ with z and using the symmetry of E_{aa} gives $\nu = \nu_w$, so $|\nu_a| \leq |\mu_a| + s \leq N$ by Step 1 and $e = c_K(\nu) \leq \sum_a |\nu_a|(|\nu_a| + N_0) \leq N_0N(N + N_0) \leq 2N_0N^2$. As M_s has an orthonormal eigenbasis of Ω_K, $\langle y, \Omega_K y\rangle \leq 2N_0N^2\|y\|_2^2$ for $y \in M_s$.

Step 6: conclusion. For $y \in \mathcal{H}_c$, by (iii), $\|E_{ab}y\|^2 = \varepsilon_a\varepsilon_b\langle y, E_{ba}E_{ab}y\rangle$; summing over compact pairs gives $\langle y, \Omega_K y\rangle$ and over the others $-\langle y, (\Omega - \Omega_K)y\rangle$. So by (v), for $y \in M_s$,

$$\sum_{a,b} \|E_{ab}y\|^2 = 2\langle y, \Omega_K y\rangle - \kappa_l\|y\|^2 \leq (4N_0N^2 + \tfrac{1}{4}N_0^3)\|y\|^2 \leq 5N_0N^2\|y\|^2.$$

Since $|X_{ab}| \leq \|X\|$ and by Cauchy–Schwarz over the N_0^2 pairs, $\|Xy\| \leq \|X\| \sum_{a,b} \|E_{ab}y\| \leq \|X\|N_0(5N_0N^2)^{1/2}\|y\| \leq (N_0+1)^4(N+1)\|X\|\,\|y\|$. □

Lemma 5.7.38. *Let $n \geq 1$, $s \in \{\pm 1\}^n$, $l \in \mathbb{Z}^n$, $\lambda = \frac{1}{2}l$, let dg be a Haar measure on $\mathrm{U}(\operatorname{diag} s)$, d_λ the formal degree with respect to it, and $\|\cdot\|$ the operator norm. Let $h, B, L \in \mathbb{N}$.*

1. *If l has gap h for s, then $c_\lambda(g^{-1}) = \overline{c_\lambda(g)}$, $|c_\lambda(g)| \leq 1$, and $|c_\lambda(g)| \leq (2/\|g\|)^c$ for every integer $0 \leq c \leq h$ and every g.*
2. *If l has large gap B for s and $|l_i| \leq L$ for all i, then for all $x, g \in \mathrm{U}(\operatorname{diag} s)$*
$$|c_\lambda(xg) - c_\lambda(x)| \leq 100(n + L + 1)^5 \, \|g - 1\|.$$
3. *If l has large gap B for s, $|l_i| \leq L$ for all i, and $c > 8n^2$ is an integer, then $d_\lambda > 0$, $g \mapsto d_\lambda (2/\|g\|)^c$ is integrable, and*
$$\int_{\mathrm{U}(\operatorname{diag} s)} d_\lambda \Big(\frac{2}{\|g\|}\Big)^c dg \leq 8\big(2000(n + L + 1)^6\big)^{2n^2} 2^c.$$

Proof. Write $J = \operatorname{diag}(s)$, $G = \mathrm{U}(\operatorname{diag} s)$, $K = K_s$ (the unitary elements of G), and call a matrix block-diagonal if its (i, j) entry vanishes whenever $s_i \neq s_j$. A vector with large gap B has gap 0. By Lemma 5.6.30: every $x \in G$ satisfies $\|x^{-1}\| = \|x\| \geq 1$ and $m_w(x^{-1}) = \overline{m_w(x)}$ (part (i)); every $g \in G$ is $g = kz$ with $k \in K$ and $z \in G$ positive definite, $\|g\| = \|z\|$, $z^{-1} = JzJ$ and $\mathrm{bd}_s(z) = \frac{1}{2}(z + z^{-1})$ (part (ii)); and $\mathrm{bd}_s(g) = k \, \mathrm{bd}_s(z)$ (part (iii)), so $|\det \mathrm{bd}_s(g)| = |\det \mathrm{bd}_s(z)|$.

(1) is Lemma 5.7.40.

Differentiability. By Lemma 5.6.24 (ii), for $g_0 \in G$ and $a = \mathrm{bd}_s(g_0)$ there are an open neighbourhood W of g_0 in $\mathrm{GL}_n(\mathbb{C})$ and the holomorphic function $F(x) = \mathrm{FJ}_l(a, a^{-1}x)$ on W with $c_\lambda = F$ on $W \cap G$. Consequently, for a differentiable curve γ in G and $x \in G$, $t \mapsto c_\lambda(x\gamma(t))$ is differentiable with derivative $DF(y)[y\,X]$ at t, where $y = x\gamma(t)$ and $X = \gamma(t)^{-1}\gamma'(t)$; differentiating $\gamma^* J \gamma = J$ gives $X^*J + JX = 0$, and $DF(y)[yX] = \frac{d}{d\sigma}\big|_{\sigma=0} c_\lambda(ye^{\sigma X})$ because $\sigma \mapsto ye^{\sigma X}$ is a curve in G with derivative yX at 0.

(2). Let $x, g \in G$ and $A = g - 1$. If $\|A\| \geq 1$, then $|c_\lambda(xg) - c_\lambda(x)| \leq 2 \leq 100(n + L + 1)^5 \|A\|$ by (1). Let $\|A\| < 1$. For $t \in [0, 1]$ put $u_t = 1 - \frac{t-1}{2}A$; as $\|\frac{t-1}{2}A\| \leq \frac{1}{2}$, u_t is invertible with $\|u_t^{-1}\| \leq 2$ (Neumann series). Put $\gamma(t) = 1 + tAu_t^{-1}$. Then $\gamma(0) = 1$, $\gamma(1) = 1 + A = g$, and $\gamma(t) \in G$: indeed $\gamma(t)u_t = 1 + \frac{1+t}{2}A$, and with $\alpha = \frac{1+t}{2}$, $\beta = \frac{1-t}{2}$ (so $u_t = 1 + \beta A$, $\alpha + \beta = 1$, $\alpha - \beta = t$),

$$(1 + \alpha A)^* J (1 + \alpha A) - (1 + \beta A)^* J (1 + \beta A) = t(A^*J + JA + A^*JA) = t\,(g^*Jg - J) = 0,$$

so $u_t^* \gamma(t)^* J \gamma(t) u_t = u_t^* J u_t$ and $\gamma(t)^* J \gamma(t) = J$. Moreover $\frac{d}{dt} u_t^{-1} = \frac{1}{2} u_t^{-1} A u_t^{-1}$, so

$$\gamma'(t) = Au_t^{-1} + \tfrac{t}{2}\, Au_t^{-1}Au_t^{-1}, \qquad \|\gamma'(t)\| \leq 2\|A\| + 2\|A\|^2 \leq 4\|A\|, \qquad \|\gamma(t)\| \leq 1 + 2\|A\| \leq 3.$$

With $X_t = \gamma(t)^{-1}\gamma'(t)$, $\|X_t\| \leq \|\gamma(t)^{-1}\| \, \|\gamma'(t)\| = \|\gamma(t)\| \, \|\gamma'(t)\| \leq 12\|A\|$. By the differentiability paragraph and Lemma 5.7.39 (large gap B, entries bounded by L), $\phi(t) = c_\lambda(x\gamma(t))$ is differentiable on $[0, 1]$ with $|\phi'(t)| \leq (n+1)^4(n + L + 1) \cdot 12\|A\| \leq 12(n + L + 1)^5 \|A\|$. The mean value inequality gives $|c_\lambda(xg) - c_\lambda(x)| = |\phi(1) - \phi(0)| \leq 100(n + L + 1)^5 \|g - 1\|$.

(3). Let $c' > 8n^2$ be the integer of the statement (written c there) and put $\delta = (200(n + L + 1)^5)^{-1}$, $\kappa = 2000(n + L + 1)^6$. We apply Lemma 5.6.32 to the group $S = G$ with the Haar measure $\nu = dg$ (right invariant, as G is unimodular; finite on compact sets), j the identity (a proper homomorphism) and $y = 1$.

Square integrability and $d_\lambda > 0$. By (1) with $c = 500n^3 \leq 500n^3 + B$, $|c_\lambda(g)|^2 \leq (2/\|g\|)^{1000n^3}$ and $1000n^3 > 8n^2$, so $|c_\lambda|^2$ is integrable by Lemma 5.6.32 (ii). By (2) with $x = 1$ and $c_\lambda(1) = 1$,

$|c_\lambda(g)| \geq \frac{1}{2}$ whenever $\|g-1\| \leq \delta$; this set contains a neighbourhood of 1, of positive Haar measure. Hence $0 < \int_G |c_\lambda|^2 dg < \infty$ and $d_\lambda = (\int_G |c_\lambda|^2)^{-1} > 0$, and

$$\nu\{g : \|g-1\| \leq \delta\} \leq \int_G 4|c_\lambda|^2 \, dg = \frac{4}{d_\lambda}.$$

Balls. For $R \geq 1$, Lemma 5.6.32 (i) with $D = \delta^{-1}$ and the constant $\kappa R^2 \geq 3 \cdot 200(n+L+1)^5 R^2 = 3DR^2$ in the place of its K gives

$$\nu\{g : \|g\| \leq R\} \leq (\kappa R^2)^{2n^2} \cdot \frac{4}{d_\lambda}.$$

Integrability and the bound. The function $g \mapsto d_\lambda (2/\|g\|)^{c'}$ is integrable by Lemma 5.6.32 (ii). Since $\|g\| \geq 1$, the layer-cake formula gives

$$\int_G \|g\|^{-c'} dg = \int_0^1 \nu\{g : \|g\| < \tau^{-1/c'}\} \, d\tau \leq \frac{4\kappa^{2n^2}}{d_\lambda} \int_0^1 \tau^{-4n^2/c'} d\tau = \frac{4\kappa^{2n^2}}{d_\lambda} \cdot \frac{c'}{c'-4n^2} \leq \frac{8\kappa^{2n^2}}{d_\lambda},$$

as $c' > 8n^2$. Multiplying by $d_\lambda 2^{c'}$ gives $\int_G d_\lambda (2/\|g\|)^{c'} dg \leq 8\big(2000(n+L+1)^6\big)^{2n^2} 2^{c'}$. Replacing dg by a positive multiple divides d_λ by the same factor, so the left side does not depend on the choice of dg. □

Lemma 5.7.39. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $B, L \geq 0$ integers, and $l \in \mathbb{Z}^n$ a vector with large gap B for ε and $|l_i| \leq L$ for all i. Let $y \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ and let X be a complex $n \times n$ matrix with $X^* \mathrm{diag}(\varepsilon) + \mathrm{diag}(\varepsilon) X = 0$. Then $t \mapsto c_{l/2}(ye^{tX})$ is differentiable at $t = 0$ and*

$$\Big| \frac{d}{dt}\Big|_{t=0} c_{l/2}(ye^{tX}) \Big| \leq (\#n+1)^4 \, (\#n+L+1) \, \|X\|.$$

Proof. Write $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$ with a Haar measure, $c = c_{l/2}$, $d = (\int_G |c|^2)^{-1}$, and use the module $\mathcal{H}_c$, the smoothings f_φ and the bumps φ_X of Lemma 5.7.36; the gap $500(\#n)^3 + B$ exceeds $8(\#n)^2$. By Lemma 5.7.36 (ii), $c = f_\beta$ for a smooth compactly supported β, and $Xc = f_{\beta_X} \in \mathcal{H}_c$. By (i) of the same lemma, $t \mapsto f_\beta(ye^{tX})$ is differentiable at 0 with derivative $f_{\beta_X}(y) = (Xc)(y)$. As $c \in M_0$ (the filtration of Lemma 5.7.37), that lemma with $s = 0$ gives $\|Xc\|_2 \leq (\#n+1)^4 (L + \#n + 1) \|X\| \, \|c\|_2$, and by Lemma 5.7.36 (iv),

$$\begin{aligned} \Big| \frac{d}{dt}\Big|_{t=0} c(ye^{tX}) \Big| &= |(Xc)(y)| \leq d^{1/2} \|Xc\|_2 \\ &\leq (\#n+1)^4 (\#n+L+1) \, \|X\| \, d^{1/2} \|c\|_2 = (\#n+1)^4 (\#n+L+1) \, \|X\|. \end{aligned}$$

□

Lemma 5.7.40. *Let n be a nonempty finite set, $\varepsilon \in \{\pm 1\}^n$, $h \geq 0$ an integer and $l \in \mathbb{Z}^n$ a vector with gap h for ε. Then for every $g \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$*

$$c_{l/2}(g^{-1}) = \overline{c_{l/2}(g)}, \qquad |c_{l/2}(g)| \leq 1, \qquad |c_{l/2}(g)| \leq \Big(\frac{2}{\|g\|}\Big)^c \quad \textit{for every integer } 0 \leq c \leq h.$$

Proof. Write $G = \mathrm{U}(\mathrm{diag}\,\varepsilon)$, $J = \mathrm{diag}(\varepsilon)$, $K = K_\varepsilon$, $\mu = \mu_l$ and $c = c_{l/2}$ where no exponent is meant; call a matrix block-diagonal if its (i,j) entry vanishes whenever $\varepsilon_i \neq \varepsilon_j$. A vector with gap h has gap 0, so Lemma 5.6.26 and Lemma 5.6.29 apply. By Lemma 5.6.30: every $x \in G$ satisfies $\|x^{-1}\| = \|x\|$ and $m_w(x^{-1}) = \overline{m_w(x)}$ for $w \in \mathbb{Z}^n$ (part (i)); every $g \in G$ is $g = kz$ with $k \in K$ and $z \in G$ positive definite, $\|g\| = \|z\|$, $z^{-1} = JzJ$ and $\mathrm{bd}_\varepsilon(z) = \frac{1}{2}(z + z^{-1})$ (part (ii)); and $\mathrm{bd}_\varepsilon(g) = k\,\mathrm{bd}_\varepsilon(z)$ (part (iii)), so $|\det \mathrm{bd}_\varepsilon(g)| = |\det \mathrm{bd}_\varepsilon(z)|$ as $|\det k| = 1$.

Inversion. Let $g = kz$ as above. Then $g = k\,z\,1^*$ and $g^{-1} = 1 \cdot z^{-1} \cdot k^*$ are splittings as in Lemma 5.6.26 (ii), with block-diagonal invertible outer factors ($p = k$, $q = 1$, respectively $p = 1$, $q = k$) and positive definite, hence accretive, middle factors z and z^{-1}. By Lemma 5.6.26 (i), (ii),

$$c(g) = \frac{1}{\|m_\mu\|^2} \int_K \frac{m_\mu(ku)\,\overline{m_\mu(u)}}{m_{-l^\rho}(u^* z u)}\,du, \qquad c(g^{-1}) = \frac{1}{\|m_\mu\|^2} \int_K \frac{m_\mu(u)\,\overline{m_\mu(ku)}}{m_{-l^\rho}(u^* z^{-1} u)}\,du.$$

For $u \in K$ the matrix $u^* z u$ lies in G and $u^* z^{-1} u = (u^* z u)^{-1}$, so $m_{-l^\rho}(u^* z^{-1} u) = \overline{m_{-l^\rho}(u^* z u)}$ by Lemma 5.6.30 (i). Hence the second integrand is the complex conjugate of the first, and $c(g^{-1}) = \overline{c(g)}$.

Bound by one. Lemma 5.6.29 (i) with exponent 0 gives $|c(g)|^2 \leq 1$.

Decay. We show $(\|g\|/2)^2 \leq |\det \mathrm{bd}_\varepsilon(g)|$. Diagonalise the positive definite matrix z by a unitary matrix, with eigenvalues $x_1, \ldots, x_n > 0$; then z^{-1} has the eigenvalues x_i^{-1} in the same eigenbasis, so $\mathrm{bd}_\varepsilon(z) = \frac{1}{2}(z + z^{-1})$ has the eigenvalues $\frac{1}{2}(x_i + x_i^{-1}) \geq 1$, and $\det \mathrm{bd}_\varepsilon(z) = \prod_i \frac{1}{2}(x_i + x_i^{-1})$, each factor being at least 1. Moreover $\|z\| = \max_i x_i$. Since $z^{-1} = JzJ$ is unitarily conjugate to z, the eigenvalues of z^{-1}, which are the x_i^{-1}, are those of z with multiplicities; hence $\max_i x_i^{-1} = \max_i x_i = \|z\|$, that is $\min_i x_i = \|z\|^{-1}$. If the maximum and the minimum are attained at different indices, then

$$\det \mathrm{bd}_\varepsilon(z) \geq \tfrac{1}{2}(\|z\| + \|z\|^{-1}) \cdot \tfrac{1}{2}(\|z\|^{-1} + \|z\|) \geq (\|z\|/2)^2;$$

otherwise $\max_i x_i = \min_i x_i = \|z\|^{-1}$ forces $\|z\| = 1$ and $\det \mathrm{bd}_\varepsilon(z) \geq 1 \geq (\|z\|/2)^2$. With $\|g\| = \|z\|$ this proves the claim. Now Lemma 5.6.29 (i) gives, for every integer $0 \leq c \leq h$,

$$|c(g)|^2 \Big(\frac{\|g\|}{2}\Big)^{2c} \leq |c(g)|^2\,|\det \mathrm{bd}_\varepsilon(g)|^c \leq 1,$$

that is $|c(g)| \leq (2/\|g\|)^c$. □

Theorem 5.7.41. *Let $N \geq 1$ and let $\varepsilon \in \{+,-\}^N$ be a sign vector, with the groups J_ε, G_ε, K_ε of (Dp1) and, for $\lambda \in \mathbb{R}^N$, the Weyl product $\mathrm{d}(\lambda)$ of (Dp2) in Definition 5.1.1.*

Let ω be a left-invariant holomorphic differential form of top degree on $\mathrm{GL}_N(\mathbb{C})$ such that the measure $|\omega|$ induced on $\mathrm{U}(N)$ has total mass 1, and let $\mu_\varepsilon^{\mathrm{cd}}$ be the measure $|\omega|$ induced on G_ε (the compact-dual Haar measure*; it is a Haar measure on G_ε and does not depend on the choice of ω). Let $\lambda \in \mathbb{R}^N$ be strictly decreasing with $\lambda_a - \frac{N-1}{2} \in \mathbb{Z}$ for all a. Then:*

(i) there is a square-integrable representation of G_ε with Harish-Chandra parameter λ relative to ε, and any two are unitarily equivalent;

(ii) for every such representation σ on $\mathcal{H}$ and all $v, w, v', w' \in \mathcal{H}$ the function

$$g \mapsto \langle w, \sigma(g)v\rangle \overline{\langle w', \sigma(g)v'\rangle}$$

is $\mu_\varepsilon^{\mathrm{cd}}$-integrable and

$$\int_{G_\varepsilon} \langle w, \sigma(g)v\rangle\,\overline{\langle w', \sigma(g)v'\rangle}\,d\mu_\varepsilon^{\mathrm{cd}}(g) = \frac{\langle v', v\rangle\,\langle w, w'\rangle}{\mathrm{d}(\lambda)};$$

(iii) for every Haar measure dg on G_ε, with $c>0$ defined by $\mu_\varepsilon^{\mathrm{cd}} = c\,dg$ (so c depends only on ε and dg), the same function is dg-integrable and

$$\int_{G_\varepsilon} \langle w, \sigma(g)v\rangle\, \overline{\langle w', \sigma(g)v'\rangle}\, dg = \frac{\langle v', v\rangle\, \langle w, w'\rangle}{c\,\mathrm{d}(\lambda)}.$$

Proof. **Step 0: $\mu_\varepsilon^{\mathrm{cd}}$ is a well-defined Haar measure, and G_ε is unimodular.** The Lie algebra of G_ε is $\mathfrak{u}_\varepsilon = \{X : X^*J_\varepsilon + J_\varepsilon X = 0\}$; $X \mapsto J_\varepsilon X$ maps it bijectively onto the skew-hermitian matrices, so $\dim_{\mathbb{R}} \mathfrak{u}_\varepsilon = N^2$, and $\mathfrak{u}_\varepsilon \cap i\mathfrak{u}_\varepsilon = 0$ (if $X, iX \in \mathfrak{u}_\varepsilon$ then $J_\varepsilon X = 0$). Hence a real basis of $\mathfrak{u}_\varepsilon$ is a complex basis of $\mathfrak{gl}_N(\mathbb{C})$. A left-invariant holomorphic top-degree form is $\omega_g = \omega_e \circ \Lambda^{N^2}(dL_{g^{-1}})$ with $0 \neq \omega_e \in \Lambda^{N^2}_{\mathbb{C}}(\mathfrak{gl}_N(\mathbb{C}))^*$, unique up to a factor in $\mathbb{C}^\times$; its restriction to G_ε is therefore nowhere zero and invariant under left translation by G_ε, so $|\omega|$ induces a left Haar measure on G_ε (and on $\mathrm{U}(N)$, of finite positive mass since $\mathrm{U}(N)$ is compact). The normalisation fixes the factor up to absolute value 1, so $\mu_\varepsilon^{\mathrm{cd}}$ does not depend on ω. The real form $(X,Y) \mapsto \mathrm{tr}(XY)$ on $\mathfrak{u}_\varepsilon$ is real-valued ($\overline{\mathrm{tr}(XY)} = \mathrm{tr}(Y^*X^*) = \mathrm{tr}(J_\varepsilon Y J_\varepsilon J_\varepsilon X J_\varepsilon) = \mathrm{tr}(XY)$), nondegenerate (its complexification is the trace form of $\mathfrak{gl}_N(\mathbb{C})$) and Ad-invariant, so $|\det \mathrm{Ad}(g)| = 1$ for $g \in G_\varepsilon$ and G_ε is unimodular.

Step 1: decomposition of $\mu_\varepsilon^{\mathrm{cd}}$. Let $m\colon \mathrm{SL}_N(\mathbb{C}) \times \mathbb{C}^\times \to \mathrm{GL}_N(\mathbb{C})$, $m(x,z) = zx$. It is a surjective holomorphic homomorphism ($z^N = \det g$, $x = z^{-1}g$), with kernel $\{(\zeta^{-1}I, \zeta) : \zeta^N = 1\}$ of order N and $dm_e(X,t) = X + tI$ bijective. Let ω_1 be a left-invariant holomorphic top-degree form on $\mathrm{SL}_N(\mathbb{C})$ with $|\omega_1|$ of total mass 1 on $\mathrm{SU}(N)$, and $\omega_2 = (2\pi i)^{-1} z^{-1} dz$ on $\mathbb{C}^\times$, so that $|\omega_2|$ induces the probability Haar measure dz on $Z = \{\zeta I : |\zeta| = 1\} \cong \{|\zeta| = 1\}$. Let ω' be the left-invariant form on $\mathrm{GL}_N(\mathbb{C})$ with $\omega'_e \circ \Lambda^{N^2}(dm_e) = (\mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^*\omega_2)_e$; since m is a homomorphism, $m^*\omega'$ is left-invariant and agrees with $\mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^*\omega_2$ at e, so the two are equal. Let $H \in \{\mathrm{U}(N), G_\varepsilon\}$ and $H_1 = H \cap \mathrm{SL}_N(\mathbb{C})$. For $g \in H$, $|\det g| = 1$ (from $g^*J_\varepsilon g = J_\varepsilon$, resp. $g^*g = I$), so m maps $H_1 \times Z$ onto H; the fibre over $m(x,z)$ is $\{(\zeta^{-1}x, \zeta z) : \zeta^N = 1\} \subseteq H_1 \times Z$, of N elements; and dm_e maps $\mathrm{Lie}(H_1) \oplus i\mathbb{R}$ bijectively onto $\mathrm{Lie}(H)$. So $m\colon H_1 \times Z \to H$ is an N-sheeted covering, and the change of variables formula gives, for Borel $f \geq 0$ on H,

$$\int_{H_1}\int_Z f(zx)\, dz\, d|\omega_1|(x) = N \int_H f\, d|\omega'|,$$

the density induced by $m^*\omega' = \mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^*\omega_2$ on $H_1 \times Z$ being the product of those induced by ω_1 and ω_2. For $H = \mathrm{U}(N)$ and $f = 1$ this gives $N|\omega'|(\mathrm{U}(N)) = 1$; so $N\omega'$ is normalised as ω in the statement and, by Step 0, $\mu_\varepsilon^{\mathrm{cd}} = N|\omega'|$ on G_ε. With $G_1 = G_\varepsilon \cap \mathrm{SL}_N(\mathbb{C})$ and μ_1 the measure induced by $|\omega_1|$ on G_1,

$$\int_{G_\varepsilon} f\, d\mu_\varepsilon^{\mathrm{cd}} = \int_{G_1}\int_Z f(zx)\, dz\, d\mu_1(x) \tag{1}$$

for Borel $f \geq 0$, and by Fubini for every $\mu_\varepsilon^{\mathrm{cd}}$-integrable f.

Step 2: ε not constant. Then $N \geq 2$ and both signs occur. Statement (i) is Lemma 5.7.42 (c). Let σ on $\mathcal{H}$ be square-integrable with Harish-Chandra parameter λ relative to ε, i.e. a discrete series with parameter λ. By Lemma 5.7.42 (b), $\sigma(\zeta I) = \zeta^{c(\lambda)}\mathrm{id}$ and $\sigma|_{G_1}$ is unitarily equivalent to Harish-Chandra's discrete series π_{λ° of $G_1 \cong \mathrm{SU}(p,q)$, where $\lambda^\circ = \lambda - \frac{c(\lambda)}{N}(1,\dots,1)$. Apply Theorem A.5.12 with $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$, $G = \mathrm{SL}_N(\mathbb{C})$, $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_1)$ with Cartan involution $\theta(X) = J_\varepsilon X J_\varepsilon$ (that is, $X \mapsto -X^*$), $G_{\mathbb{R}} = G_1$ (connected and linear, Lemma 5.7.42), $T_{\mathbb{R}} = T_1$ the diagonal torus of G_1, $K_{\mathbb{R}} = K_\varepsilon \cap G_1$, $U_{\mathbb{R}} = \mathrm{SU}(N)$ (its Lie algebra $\mathfrak{su}(N)$ is a compact real form of $\mathfrak{sl}_N(\mathbb{C})$ and $J_\varepsilon\mathfrak{su}(N)J_\varepsilon = \mathfrak{su}(N)$), $\omega = \omega_1$, so $\mu_{\mathbb{R}} = \mu_1$; $\Phi = \{e_a - e_b : a \neq b\}$ on $\mathbb{R}^N_0$ as in Lemma 5.7.42, $\Phi^+ = \{e_a - e_b : a < b\}$, $\rho_a = \frac{N+1}{2} - a$, $\Lambda = P_1$ and $\nu = \lambda^\circ$, which lies in $\rho + P_1$ and is regular by Lemma 5.7.42 (a), and is dominant for Φ^+ because λ is strictly decreasing.

By Theorem A.5.12 (b) the character of π_{λ° is Θ_{λ°, so by (c) its formal degree with respect to μ_1 is $\prod_{a<b}(\lambda^\circ, e_a - e_b)/(\rho, e_a - e_b)$. The Killing form of $\mathfrak{sl}_N(\mathbb{C})$ is $2N\,\mathrm{tr}(XY)$, so the induced form on $\mathbb{R}_0^N$ is a positive multiple of the euclidean inner product, and the factor cancels in each quotient: the formal degree is $\prod_{a<b}(\lambda_a - \lambda_b)/(b-a) = \mathrm{d}(\lambda)$. A unitary equivalence U preserves inner products and matrix coefficients ($\langle w, \sigma(x)v\rangle = \langle Uw, \pi_{\lambda^\circ}(x)Uv\rangle$), so by Theorem A.5.18 for π_{λ° (irreducible, square-integrable by Theorem A.5.12 (a)), for all $v, w, v', w' \in \mathcal{H}$

$$\int_{G_1} \langle w, \sigma(x)v\rangle \overline{\langle w', \sigma(x)v'\rangle}\, d\mu_1(x) = \frac{\langle v', v\rangle\langle w, w'\rangle}{\mathrm{d}(\lambda)},$$

the integrand being μ_1-integrable. Let $F(g) = \langle w, \sigma(g)v\rangle\overline{\langle w', \sigma(g)v'\rangle}$. For $x \in G_1$, $|\zeta| = 1$, $\sigma(\zeta x) = \zeta^{c(\lambda)}\sigma(x)$ and $|\zeta^{c(\lambda)}| = 1$, so $F(\zeta x) = F(x)$. By (1) and Tonelli, $\int_{G_\varepsilon} |F|\, d\mu_\varepsilon^{\mathrm{cd}} = \int_{G_1} |F|\, d\mu_1 < \infty$, and by (1) and Fubini $\int_{G_\varepsilon} F\, d\mu_\varepsilon^{\mathrm{cd}} = \int_{G_1} F\, d\mu_1$. This is (ii).

Step 3: ε constant. Then $J_\varepsilon = \pm I$, $G_\varepsilon = K_\varepsilon = \mathrm{U}(N)$, $\mu_\varepsilon^{\mathrm{cd}}$ is the Haar probability measure, $\rho_n(\varepsilon) = 0$ and $\rho_c(\varepsilon)_a = \frac{N+1}{2} - a$. Put $\Lambda = \Lambda_\varepsilon(\lambda) = \lambda - \rho_c(\varepsilon)$. Then $\Lambda_a = (\lambda_a - \frac{N-1}{2}) + a - 1 \in \mathbb{Z}$ and $\Lambda_a - \Lambda_{a+1} = \lambda_a - \lambda_{a+1} - 1 \geq 0$, so Λ is a dominant weight; let V_Λ be the irreducible representation of $\mathrm{U}(N)$ with highest weight Λ, with an invariant inner product. Put $\Lambda' = \Lambda - \Lambda_N(1, \dots, 1)$, nonincreasing with $\Lambda'_N = 0$. By Theorem A.5.19 (with $K = \mathbb{C}$, $k = N$), $\nabla_{\mathbb{C}}(\Lambda') \otimes \det^{\Lambda_N}$ is an irreducible rational $\mathrm{GL}_N(\mathbb{C})$-module of highest weight Λ and dimension $d_N(\Lambda') = d_N(\Lambda)$ (d_N depends only on the differences $\Lambda_a - \Lambda_b$); as $\mathrm{U}(N)$ is Zariski dense in $\mathrm{GL}_N(\mathbb{C})$, its restriction to $\mathrm{U}(N)$ is irreducible of highest weight Λ, hence isomorphic to V_Λ (theorem of the highest weight). Hence

$$\dim V_\Lambda = \prod_{a<b} \frac{\Lambda_a - \Lambda_b + b - a}{b - a} = \prod_{a<b} \frac{\lambda_a - \lambda_b}{b - a} = \mathrm{d}(\lambda).$$

(i): V_Λ is finite-dimensional, so strongly continuous with bounded, hence square-integrable, matrix coefficients; its only K_ε-type is Λ, so the occurrence and minimality conditions hold. If σ on $\mathcal{H}$ is square-integrable with parameter λ, $\mathcal{H}$ contains a finite-dimensional, hence closed, invariant subspace on which $\mathrm{U}(N)$ acts by V_Λ; by irreducibility it is $\mathcal{H}$, and by Schur's lemma the two invariant inner products are proportional, so σ is unitarily equivalent to V_Λ. (ii): $\dim \mathcal{H} = D = \mathrm{d}(\lambda)$, F is continuous on the compact group, hence integrable, and Theorem A.5.18 gives a formal degree d_σ for $\mu_\varepsilon^{\mathrm{cd}}$. For a unit vector v and an orthonormal basis $e_1, \dots, e_D$ of $\mathcal{H}$,

$$\frac{D}{d_\sigma} = \sum_{i=1}^{D} \int_{G_\varepsilon} |\langle e_i, \sigma(g)v\rangle|^2\, d\mu_\varepsilon^{\mathrm{cd}}(g) = \int_{G_\varepsilon} \|\sigma(g)v\|^2\, d\mu_\varepsilon^{\mathrm{cd}}(g) = 1,$$

so $d_\sigma = \mathrm{d}(\lambda)$, which is (ii).

Step 4: (iii). If $\mu_\varepsilon^{\mathrm{cd}} = c\, dg$, then F is dg-integrable and $\int F\, dg = c^{-1} \int F\, d\mu_\varepsilon^{\mathrm{cd}}$; (ii) gives (iii). The constant c is determined by the two Haar measures $\mu_\varepsilon^{\mathrm{cd}}$ and dg alone. □

Lemma 5.7.42. *Let $n \geq 2$, $j \in \{\pm 1\}^n$ with $p(j), q(j) \geq 1$, and let λ be a Harish-Chandra parameter. Put $G = \mathrm{U}(J_j)$, $G_1 = \{g \in G : \det g = 1\}$, $T_1 = T \cap G_1$, $Z = \{\zeta I : |\zeta| = 1\}$,*

$$c(\lambda) = \sum_{a=1}^{n} \lambda_a, \qquad \lambda^\circ = \lambda - \frac{c(\lambda)}{n}(1, \dots, 1),$$

and identify the differentials of characters of T_1 with elements of $\mathbb{R}_0^n = \{x \in \mathbb{R}^n : \sum_a x_a = 0\}$, x corresponding to $\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \mapsto e^{i\sum_a x_a\theta_a}$. Then:

(a) $c(\lambda) \in \mathbb{Z}$, *and* λ° *is regular for the roots* $e_a - e_b$ *of* $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{t}_1)$ *and lies in* $\rho + P_1$, *where* P_1 *is the lattice of differentials of characters of* T_1 *and* $\rho_a = \frac{n+1}{2} - a$;

(b) every discrete series $(\pi, \mathcal{H})$ *of* G *with parameter* λ *satisfies* $\pi(\zeta I) = \zeta^{c(\lambda)} \operatorname{id}_{\mathcal{H}}$ *for* $|\zeta| = 1$, *and* $\pi|_{G_1}$ *is unitarily equivalent to Harish-Chandra's discrete series* π_{λ° *of* G_1;

(c) there is a discrete series of G *with parameter* λ, *and any two are unitarily equivalent.*

Proof. **Setting.** $G_1 \cong \mathrm{SU}(p(j), q(j))$ is a connected linear semisimple Lie group with finite centre, $K_1 = K_j \cap G_1$ is a maximal compact subgroup and T_1 a compact Cartan subgroup. The roots of $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{t}_1)$ are the $e_a - e_b$ $(a \neq b)$, with root space $\mathbb{C}E_{ab}$, compact if and only if $j_a = j_b$. Let $\Phi_c^+ = \{e_a - e_b : a < b,\ j_a = j_b\}$; its half-sum is $\rho_c(j)$, and the Weyl group W_c of K_1 is the group of permutations preserving j. We use the euclidean inner product on $\mathbb{R}^n$ and the orthogonal projection $\mathrm{pr}\colon \mathbb{R}^n \to \mathbb{R}^n_0$. The restriction to T_1 of the character of T of weight $\mu \in \mathbb{Z}^n$ has differential $\mathrm{pr}(\mu)$, and every character of the subtorus T_1 extends to T; so $P_1 = \mathrm{pr}(\mathbb{Z}^n)$, and a character of T_1 is determined by its differential. Note $\rho, \rho_c(j), \rho_n(j) \in \mathbb{R}^n_0$ and $\rho = \rho_c(j) + \rho_n(j)$. Put

$$\Lambda = \beta_j(\lambda) = \lambda + \rho_n(j) - \rho_c(j) = \lambda + \rho - 2\rho_c(j).$$

Proof of (a), and integrality of Λ. As $\lambda_a - \frac{n-1}{2} \in \mathbb{Z}$, $c(\lambda) \in \frac{n(n-1)}{2} + \mathbb{Z} = \mathbb{Z}$. $\lambda^\circ_a - \lambda^\circ_b = \lambda_a - \lambda_b \neq 0$ for $a \neq b$, so λ° is regular. Since $\lambda_a - \rho_a = (\lambda_a - \frac{n-1}{2}) + a - 1 \in \mathbb{Z}$, we have $\lambda - \rho \in \mathbb{Z}^n$ and $\lambda^\circ - \rho = \mathrm{pr}(\lambda - \rho) \in P_1$. Moreover $\lambda + \rho \in \mathbb{Z}^n$ (as $\lambda_a + \rho_a = (\lambda_a - \frac{n-1}{2}) + n - a$) and $2\rho_c(j) \in \mathbb{Z}^n$, so $\Lambda \in \mathbb{Z}^n$, $\mathrm{pr}(\Lambda) = \lambda^\circ + \rho_n(j) - \rho_c(j)$ and $\sum_a \Lambda_a = c(\lambda)$.

Step 1: the minimal K_1-type of $\pi_{\lambda'}$. Let $\lambda' \in \rho + P_1$ be regular and dominant for Φ_c^+, let $\Phi'^+ = \{e_a - e_b : \lambda'_a > \lambda'_b\}$ (so $\Phi'^+ \supseteq \Phi_c^+$ and $\Phi'^+ \cap \Phi_c = \Phi_c^+$), Φ'^+_n its noncompact part with half-sum ρ'_n, i.e.

$$(\rho'_n)_a = \tfrac{1}{2}\big(\#\{b : j_b \neq j_a,\ \lambda'_b < \lambda'_a\} - \#\{b : j_b \neq j_a,\ \lambda'_b > \lambda'_a\}\big),$$

and $\rho' = \rho_c(j) + \rho'_n$ the half-sum of Φ'^+. For a K_1-type of $\pi_{\lambda'}$ with highest weight ν (for Φ_c^+) put $\nu^\sharp = \nu + \rho_c(j)$, which lies in $\rho_c(j) + P_1$ and is regular and dominant for Φ_c^+. We claim: $\|\nu + 2\rho_c(j)\| \geq \|\lambda' + \rho'\|$, with equality if and only if $\nu^\sharp = \mu_0 := \lambda' + \rho'_n$, and the K_1-type with $\nu^\sharp = \mu_0$ occurs in $\pi_{\lambda'}$ with multiplicity one. By Blattner's formula (Theorem A.5.13, with the positive system Φ'^+ for which λ' is dominant) the multiplicity of ν is $\sum_{w \in W_c} \varepsilon(w) \mathcal{P}_n(w\nu^\sharp - \lambda' - \rho'_n) \geq 1$, so some $w \in W_c$ has $w\nu^\sharp = \lambda' + \rho'_n + S$ with S a sum of elements of Φ'^+_n. As $\nu^\sharp$ is dominant, $\nu^\sharp - w\nu^\sharp$ is a nonnegative combination of Φ_c^+, each element of which pairs positively with $\rho_c(j)$; with $\|w\nu^\sharp\| = \|\nu^\sharp\|$ this gives

$$\begin{aligned}\|\nu^\sharp + \rho_c(j)\|^2 &= \|w\nu^\sharp + \rho_c(j)\|^2 + 2\langle \nu^\sharp - w\nu^\sharp, \rho_c(j)\rangle \\ &\geq \|\lambda' + \rho' + S\|^2 = \|\lambda' + \rho'\|^2 + 2\langle \lambda' + \rho', S\rangle + \|S\|^2 \geq \|\lambda' + \rho'\|^2,\end{aligned}$$

since $\lambda' + \rho'$ pairs positively with every element of Φ'^+. Equality forces $S = 0$ and $\langle \nu^\sharp - w\nu^\sharp, \rho_c(j)\rangle = 0$, hence $w\nu^\sharp = \nu^\sharp$, $w = 1$ ($\nu^\sharp$ is regular) and $\nu^\sharp = \mu_0$. Conversely, $\mu_0 - \rho_c(j) - (\lambda' - \rho) = 2\rho + (\rho' - \rho) - 2\rho_c(j)$ lies in the root lattice, which is contained in P_1, so $\mu_0 \in \rho_c(j) + P_1$; and μ_0 is regular and dominant for Φ_c^+: for $a < b$ with $j_a = j_b$, $\lambda'_a - \lambda'_b$ is a positive integer (it is $\langle \lambda', e_a - e_b\rangle$, an integer as $\lambda' \in \rho + P_1$), and $(\rho'_n)_a \geq (\rho'_n)_b$ because $\lambda'_a > \lambda'_b$. Its multiplicity is $\sum_w \varepsilon(w) \mathcal{P}_n(w\mu_0 - \mu_0) = 1$: for $w \neq 1$, $w\mu_0 - \mu_0 = -C$ with $C \neq 0$ a nonnegative combination of Φ_c^+, so $\langle \lambda', -C\rangle < 0$, whereas every sum of elements of Φ'^+_n pairs nonnegatively with λ'; thus $\mathcal{P}_n(-C) = 0$.

Step 2: restriction of a discrete series with parameter λ. Let $(\pi,\mathcal{H})$ be a discrete series of G with parameter λ. By Schur's lemma $\pi(\zeta I)=\chi(\zeta)\mathrm{id}$ for a character χ of the central subgroup Z. A vector of T-weight μ is multiplied by $\zeta^{\sum_a \mu_a}$ under $\zeta I\in T$; hence every T-weight μ occurring in a K_j-type of π has $\sum_a \mu_a = c$ for one integer c with $\chi(\zeta)=\zeta^c$, and since Λ is the highest weight of a K_j-type of π, $c=\sum_a \Lambda_a = c(\lambda)$. This is the first claim of (b). As $G=G_1Z$ and Z acts by scalars, $\pi_1=\pi|_{G_1}$ is irreducible. The multiplication map $G_1\times Z\to G$ is a surjective homomorphism with finite kernel, so it carries a Haar measure of $G_1\times Z$ to a Haar measure of G; since $|\langle v,\pi(xz)u\rangle| = |\langle v,\pi(x)u\rangle|$ $(x\in G_1,\ z\in Z)$, the matrix coefficients of π_1 are square-integrable on G_1. By Harish-Chandra's classification (Theorem A.5.14) $\pi_1\cong\pi_{\lambda'}$ for some regular $\lambda'\in\rho+P_1$, which after conjugation by W_c is dominant for Φ_c^+. Since $K_j=K_1Z$ with Z scalar, each K_j-isotypic subspace of $\mathcal{H}$ of highest weight μ is K_1-isotypic of highest weight $\mathrm{pr}(\mu)$, so the K_1-types of π_1 are the $\mathrm{pr}(\mu)$, μ running over the K_j-types of π, all with $\sum_a\mu_a=c(\lambda)$, and

$$\|\mu+2\rho_c(j)\|^2 = \|\mathrm{pr}(\mu)+2\rho_c(j)\|^2 + \frac{c(\lambda)^2}{n}.$$

By the definition of a discrete series with parameter λ, Λ minimises $\|\mu+2\rho_c(j)\|$ over the K_j-types of π; hence $\mathrm{pr}(\Lambda)$ minimises $\|\nu+2\rho_c(j)\|$ over the K_1-types ν of $\pi_1\cong\pi_{\lambda'}$, and by the uniqueness in Step 1

$$\lambda^\circ+\rho_n(j)=\mathrm{pr}(\Lambda)+\rho_c(j)=\lambda'+\rho'_n. \tag{$*$}$$

Step 3: $(*)$ forces $\lambda'=\lambda^\circ$. Adding $\rho_c(j)$ to both sides of $(*)$ gives

$$v:=\lambda^\circ+\rho=\lambda'+\rho'.$$

For $a<b$, $v_a-v_b=(\lambda_a-\lambda_b)+(b-a)>0$; so v is regular and $\{\alpha:\langle v,\alpha\rangle>0\}=\Phi^+_{\mathrm{st}}:=\{e_a-e_b:a<b\}$. On the other hand ρ' is the half-sum of $\Phi'^+=\{e_a-e_b:\lambda'_a>\lambda'_b\}$, so $\rho'_a=\frac12(\#\{b:\lambda'_b<\lambda'_a\}-\#\{b:\lambda'_b>\lambda'_a\})$, and $\lambda'_a>\lambda'_b$ implies $\rho'_a-\rho'_b\geq 1$. Hence every $e_a-e_b\in\Phi'^+$ has $\langle v,e_a-e_b\rangle=(\lambda'_a-\lambda'_b)+(\rho'_a-\rho'_b)>0$, i.e. $\Phi'^+\subseteq\Phi^+_{\mathrm{st}}$. Both sets contain exactly one of $\pm\alpha$ for every root α, so $\Phi'^+=\Phi^+_{\mathrm{st}}$, $\rho'=\rho$ and $\rho'_n=\rho'-\rho_c(j)=\rho_n(j)$. Now $(*)$ gives $\lambda'=\lambda^\circ$, so $\pi|_{G_1}\cong\pi_{\lambda^\circ}$. This proves (b).

Step 4: existence. By (a), π_{λ° is defined; let it act on $\mathcal{H}_0$. The centre $G_1\cap Z=\{\zeta I:\zeta^n=1\}$ of G_1 acts by scalars $\chi_0(\zeta)$ (Schur's lemma). By Step 1 with $\lambda'=\lambda^\circ$ (strictly decreasing, so $\rho'_n=\rho_n(j)$ and $\mu_0-\rho_c(j)=\mathrm{pr}(\Lambda)$), π_{λ° contains the K_1-type of highest weight $\mathrm{pr}(\Lambda)$; its highest weight vector transforms under T_1 by the restriction of the character of T of weight Λ, and for $\zeta^n=1$ the element $\zeta I\in T_1$ multiplies it by $\zeta^{\sum_a\Lambda_a}=\zeta^{c(\lambda)}$. Hence $\chi_0(\zeta)=\zeta^{c(\lambda)}$, and

$$\pi(x\,\zeta I)=\zeta^{c(\lambda)}\pi_{\lambda^\circ}(x)\qquad(x\in G_1,\ |\zeta|=1)$$

is well defined: if $x\zeta I=x'\zeta' I$ then $x'^{-1}x=(\zeta'/\zeta)I\in G_1\cap Z$ and $\pi_{\lambda^\circ}(x)=(\zeta'/\zeta)^{c(\lambda)}\pi_{\lambda^\circ}(x')$. It is a unitary representation of G, strongly continuous because the multiplication map $G_1\times Z\to G$ is a surjective homomorphism of Lie groups with finite kernel, hence open, and irreducible because its restriction to G_1 is. Its matrix coefficients are square-integrable on G by the Haar measure argument of Step 2. Its K_j-types are the representations of $K_j=K_1Z$ obtained from the K_1-types ν of π_{λ° by letting ζI act by $\zeta^{c(\lambda)}$, with highest weight the unique μ with $\mathrm{pr}(\mu)=\nu$ and $\sum_a\mu_a=c(\lambda)$. For $\nu=\mathrm{pr}(\Lambda)$ this is $\mu=\Lambda$, so Λ occurs; and by the displayed identity of Step 2 and the minimality in Step 1 (where $\|\nu+2\rho_c(j)\|=\|\nu^\sharp+\rho_c(j)\|$ is minimal exactly at $\nu=\mathrm{pr}(\Lambda)$), every K_j-type μ satisfies $\|\mu+2\rho_c(j)\|\geq\|\Lambda+2\rho_c(j)\|$. So π is a discrete series of G with parameter λ.

Step 5: uniqueness. If π,π' are discrete series of G with parameter λ, by (b) there is a unitary U with $U\pi(x)=\pi'(x)U$ for $x\in G_1$, and $\pi(\zeta I)=\zeta^{c(\lambda)}=\pi'(\zeta I)$; as $G=G_1Z$, U intertwines π and π'. This proves (c). □

5.8 The trace formula, equal multiplicities and the target dimension

This section compares forms of the target rank on spaces of different signatures. It opens with framed spaces of rank r, whose unitary groups are identified with one group G_f at the finite places, with the spherical Hecke operators away from the level and their eigenparts (Definition 5.6.1; see [248, 49, 109]), and with the objects of the geometric side of the trace formula.

The main statement is Proposition 5.8.4: on two framed spaces on which the target parameter has the gap, every Hecke operator away from M has the same trace, characteristic polynomial and determinant on the good forms, every eigensystem has the same multiplicity, and the good forms on each W_j have dimension $d_{\mathcal{T}}$. Its input is the trace comparison Theorem 5.8.5, for Hecke functions cuspidal at one split place. The section also shows that a series of translations with coefficients independent of the space acts as one Hecke operator on all framed spaces (Proposition 5.8.55), and it bounds the target dimension by a mass, since on the definite space the forms are vector-valued functions on the finite adelic group [110] (Lemma 5.8.56, Proposition 5.8.54).

The comparison uses simple trace formulas, in the tradition from Selberg [258] through Arthur and Clozel [8, 7] to Labesse's comparison for unitary groups and discrete series [177, 58]. It has three steps. The simple trace formula (Theorem 5.8.6) uses a kernel at the infinite places made of coefficients of the lowest K-types scaled by formal degrees, which under the gap is a cusp kernel; in our reading such kernels play the part of pseudo-coefficients of discrete series [59]. The key step is the term-by-term matching (Theorem 5.8.30). Characteristic polynomials with a root off the unit circle contribute nothing; elliptic ones contribute a constant times the compact characters at the eigenvalues, by Harish-Chandra's formula for elliptic orbital integrals [119] (Lemma 5.8.49); and reducible ones contribute nothing for Hecke functions cuspidal at one place with a simple zero modulo another. The last step removes the simple-zero hypothesis by cutting the level at an auxiliary split place (Theorem 5.8.50). The comparison of unitary groups of different signatures is also part of the endoscopic classification [209, 156]; here only parameters with a large gap and test functions cuspidal at one place are needed, so that the non-elliptic terms vanish and the elliptic ones match directly.

Definition 5.8.1. Let F be a totally real number field of degree f, E/F a CM extension unramified at every finite place, $D_E = |d_E|$, V an E-vector space of dimension n with a nondegenerate hermitian form h (E-linear in the first variable), L an $\mathcal{O}_E$-lattice in V that is self-dual for h, and $G = \mathrm{U}(V,h)$. For every real place v of F fix a complex place w of E above v, put $V_v = V \otimes_{E,w} \mathbb{C}$ with the hermitian form h_v induced by h, and fix a **majorant** h_v^+: for an h_v-orthogonal decomposition $V_v = V_v^+ \oplus V_v^-$ with h_v positive definite on V_v^+ and negative definite on V_v^-, $h_v^+ = h_v$ on V_v^+, $h_v^+ = -h_v$ on V_v^-, and $V_v^+ \perp V_v^-$ for h_v^+.

Lattices and metrics at a point. For $x \in G(\mathbb{A}_F)$ let Λ_x be the $\mathcal{O}_E$-lattice of V with completions $x_v L_v$ at every finite place v, and for every real place v let

$$\langle y, z\rangle_{x,v} = h_v^+(x_v^{-1}y,\, x_v^{-1}z) \qquad (y, z \in V_v).$$

For a finitely generated $\mathcal{O}_E$-submodule $S \subseteq V$ of rank $r = \dim_E ES \geq 1$, the **lattice height** of S at x is

$$\mathrm{ht}_x(S) = 2^{fr} D_E^{-r/2}\, \mathrm{covol}_x(S),$$

where $\mathrm{covol}_x(S)$ is the covolume of S in $\prod_{v\ \mathrm{real}} (ES)_v$ (with $(ES)_v = ES \otimes_{E,w} \mathbb{C} \subseteq V_v$) for the Lebesgue measure of the euclidean inner product $\sum_v \mathrm{Re}\langle\cdot,\cdot\rangle_{x,v}$; and $\mathrm{ht}_x(0) = 1$. If $S' \subsetneq S$ is saturated in S ($S' = S \cap ES'$) of rank r', the **step size** of S over S' at x is

$$\mathrm{st}_x(S/S') = \big(\mathrm{ht}_x(S)/\mathrm{ht}_x(S')\big)^{1/(2f(r-r'))}.$$

Isotropic flags. For a composition $j = (j_1, \dots, j_a)$ of an integer $J \geq 1$ (positive integers with sum J), an **isotropic flag of type** j is a chain $W_\bullet = (0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_a)$ of E-subspaces of V with $\dim W_i - \dim W_{i-1} = j_i$ and $h(W_a, W_a) = 0$. Its **step-size vector** at x is

$$\mathrm{sv}_x(W_\bullet) = (t_1, \dots, t_a) \in (0,\infty)^a, \qquad t_i = \mathrm{st}_x\big((\Lambda_x \cap W_i)/(\Lambda_x \cap W_{i-1})\big).$$

Exponents and flag density. For such j put $c_i(j) = n - 2J + j_i + 2\sum_{k>i} j_k$ $(1 \leq i \leq a)$, $d(j) = 2nJ - 2J^2 - \sum_i j_i^2$, and define the **flag density** $\mathrm{fd}_j \colon (0,\infty)^a \to (0,\infty)$ by

$$\mathrm{fd}_j(t) = \prod_{i=1}^{a} 2f j_i \, t_i^{2f j_i c_i(j) - 1}. \tag{5.178}$$

Definition 5.8.2. Let E/F be a quadratic extension of number fields with nontrivial automorphism $e \mapsto \bar{e}$, and $r \geq 1$. For a commutative F-algebra k (in practice F or a completion F_v) write $E_k = E \otimes_F k$, with $e \otimes c \mapsto \bar{e} \otimes c$, and $V_k = V \otimes_F k$ for a hermitian space V over E, with the extended form.

For $P \in E[X]$ monic of degree r with $P(0) \neq 0$, its **conjugate reciprocal** is $P^{\#}(X) = \overline{P(0)}^{\,-1} X^r \bar{P}(X^{-1})$, $\bar{P}$ the polynomial with conjugated coefficients; P is **conjugate-reciprocal** if $P = P^{\#}$. $\mathcal{P}_r(E/F)$ is the set of monic irreducible conjugate-reciprocal $P \in E[X]$ of degree r.

For $P \in \mathcal{P}_r(E/F)$ let $L_P = E[X]/(P)$ (a field of degree r over E), α_P the class of X, and $z \mapsto z^*$ the F-algebra involution of L_P with $e^* = \bar{e}$ $(e \in E)$ and $\alpha_P^* = \alpha_P^{-1}$ (it exists because $\bar{P}(\alpha_P^{-1}) = 0$, as $P = P^{\#}$); $K_P = \{z \in L_P : z^* = z\}$. For k as above put $L_{P,k} = L_P \otimes_F k$, $K_{P,k} = K_P \otimes_F k$, extend $*$, and define:

- the **norm** $N(z) = zz^* \in K_{P,k}$ $(z \in L_{P,k})$, and the **norm-one torus** $T_P(k) = \{z \in L_{P,k}^\times : zz^* = 1\}$ (the k-points of an F-torus T_P);
- for $x \in K_{P,k}^\times$ the **trace form** h_x on the free E_k-module $L_{P,k}$ of rank r:

$$h_x(y, y') = \mathrm{Tr}_{L_{P,k}/E_k}(x\, y\, y'^*);$$

- the **class group** $C_P(k) = K_{P,k}^\times / N(L_{P,k}^\times)$, and, for a hermitian space V over E of rank r, the **local class set**

$$X_P(V,k) = \{[x] \in C_P(k) : (L_{P,k}, h_x) \text{ is isometric to } V_k\}.$$

For F-algebra maps $k \to k'$ (such as $F \to F_v$) the induced map $C_P(k) \to C_P(k')$ is written $c \mapsto c_{k'}$.

Definition 5.8.3. Let E/F be a CM extension, $r \geq 1$, V a hermitian space over E of rank r with an $\mathcal{O}_E$-lattice Λ, $G = \mathrm{U}(V)$, and $P \in \mathcal{P}_r(E/F)$. For a finite place v let $\mathcal{O}_{L,v}$ be the integral closure of $\mathcal{O}_{F,v}$ in L_{P,F_v} and $T_P(\mathcal{O}_v) = T_P(F_v) \cap \mathcal{O}_{L,v}^\times$, a compact open subgroup of $T_P(F_v)$.

(Oz1) Canonical embedding. Let k be F, a completion F_v, or $\mathbb{A}_F$, and let $\gamma \in G(k)$ have characteristic polynomial P (over E_k). Every $z \in L_{P,k} = E_k[X]/(P)$ is $z = q(\alpha_P)$ for some $q \in E_k[X]$, and $\iota_\gamma(z) = q(\gamma)$ is well defined because $P(\gamma) = 0$. The **canonical embedding** of γ is the restriction $\iota_\gamma \colon T_P(k) \to \mathrm{GL}_{E_k}(V_k)$; for $g \in G(k)$, $\iota_{g\gamma g^{-1}}(z) = g\, \iota_\gamma(z)\, g^{-1}$.

(Oz2) Torus measures. A **family of torus measures** for P is a family $(dt_v)_v$, indexed by all places v of F, of Haar measures dt_v on $T_P(F_v)$ with $dt_v(T_P(\mathcal{O}_v)) = 1$ for all but finitely many finite v. Its **product** dt is the restricted product measure on $T_P(\mathbb{A}_F)$ (restricted with respect to the $T_P(\mathcal{O}_v)$), and the **torus volume** is

$$\tau(P) = \mathrm{vol}\big(T_P(F)\backslash T_P(\mathbb{A}_F)\big) \in (0, \infty],$$

for the quotient of dt by the counting measure on the discrete subgroup $T_P(F)$.

(Oz3) Orbital integrals. Let v be a place of F, μ_v a Haar measure on $G(F_v)$, dt_v a Haar measure on $T_P(F_v)$, and $\gamma \in G(F_v)$ with characteristic polynomial P and centraliser G_γ such that $\iota_\gamma(T_P(F_v)) = G_\gamma$. For a function h on $G(F_v)$ the **orbital integral** is

$$O_\gamma(h) = \int_{G_\gamma\backslash G(F_v)} h(x^{-1}\gamma x)\, d\dot{x}, \qquad d\dot{x} = \mu_v/(\iota_\gamma)_* dt_v,$$

whenever the integral converges absolutely. It depends only on the $G(F_v)$-conjugacy class of γ (by (Oz1) and the invariance of μ_v under left translation).

(Oz4) Stable orbital integrals. With μ_v, dt_v as in (Oz3), let $\mathcal{C}_P(V, F_v)$ be the set of $G(F_v)$-conjugacy classes of elements of $G(F_v)$ with characteristic polynomial P. If $\mathcal{C}_P(V, F_v)$ is finite and every $O_{\gamma_C}(h)$ converges absolutely ($\gamma_C \in C$), the **stable orbital integral** of h along P is

$$SO_v(P, h) = \sum_{C \in \mathcal{C}_P(V, F_v)} O_{\gamma_C}(h)$$

(an empty sum is 0).

Proposition 5.8.4. *Let D be admissible data and let V, V' be framed spaces such that ς has gap Δ for V and for V'. Then $\mathcal{A}_M(V)$, $\mathcal{A}_M(V')$ are finite-dimensional, every T_a ($a \in \mathcal{H}^M$) preserves the good forms, and:*

(i) for every $a \in \mathcal{H}^M$, T_a has the same trace, the same characteristic polynomial and the same determinant on $\mathcal{A}^{\mathrm{good}}(V)$ as on $\mathcal{A}^{\mathrm{good}}(V')$;

(ii) for every eigensystem q,

$$\dim\big(\mathcal{A}^{\mathrm{good}}(V) \cap \mathcal{A}_M(V)[q]\big) = \dim\big(\mathcal{A}^{\mathrm{good}}(V') \cap \mathcal{A}_M(V')[q]\big);$$

(iii) ς has gap Δ for Y_0 and for W_j, $1 \le j \le m$, and $\dim \mathcal{A}^{\mathrm{good}}(W_j) = d_{\mathcal{T}}$ for $1 \le j \le m$.

Proof. Step 0: the field and the parameter. $E = K(\zeta_{q_1})$ with $q_1 = q_0 T(s)$, $18480 \mid q_0$ and $9 \mid T(s)$ ($l_0(s)$ is a power of 3 at least 81), so $5040 = 16 \cdot 9 \cdot 5 \cdot 7$ divides q_1 and E contains a primitive 5040-th root of unity; $[E^+ : \mathbb{Q}] = f = \varphi(q_1)$ is even (Lemma 2.3.7). The rank $r = 3^{e'+1} \ge x^5$ is odd and ≥ 2, so the entries of $\varsigma(v)$, which are integers, lie in $\frac{r-1}{2} + \mathbb{Z}$, and $\Delta = 2r^8 \ge 500r^3 + 10r^5$. Hence the gap hypothesis of Theorem 5.8.5 holds for $\lambda = \varsigma$ and V, V'. By Lemma 4.6.2(a) every chosen place w has $w \ne \bar{w}$ and $\bar{w}$ not chosen. Hence also the hypotheses of Theorem 5.8.6 hold for $U = V$ (and for $U = V'$), $N = r$, $\lambda = \varsigma$ and $K = \mathrm{K}_M$, and its clause (i) gives that $\mathcal{A}_M(V)$ and $\mathcal{A}_M(V')$ are finite-dimensional.

Step 1: Hecke operators preserve good forms. Let $\sigma\colon \bar{G} \to \mathrm{K}_f$ be a section of the reduction whose values have component 1 at every place of E^+ not below a chosen place. Then $\bar{g} \in \bar{G}$ acts on $\mathcal{A}_M(V)$ by $\rho_V(\sigma(\bar{g}))$. An element of G^M has component 1 at the places below the chosen

places, so it commutes with every $\sigma(\bar g)$ in G_f; hence $T_a(V)$ commutes with the action of $\bar G$, in particular with the projector e_0, and preserves $\mathcal{A}^{\mathrm{good}}(V)$.

Step 2: traces. As e_0 is a central idempotent of $\mathbb{C}[\bar G]$ with $e_0 = \sum_c e_c$, $e_c = \bigotimes_w e_{c_w}$, over the good families c,

$$\operatorname{tr}(T_a \mid \mathcal{A}^{\mathrm{good}}(V)) = \operatorname{tr}(T_a(V)\, e_0 \mid \mathcal{A}_M(V)) = \sum_c \operatorname{tr}(T_a(V)\, e_c \mid \mathcal{A}_M(V)),$$

and likewise for V'. Fix a good family c; it has a chosen place w with c_w primitive, hence irreducible. Put $\tilde e_c = \sum_{\bar g} e_c(\bar g)\,\sigma(\bar g) \in \mathbb{C}[\mathrm{K}_f]$ and $b = a\,\tilde e_c \in \mathbb{C}[G_f]$. As K_M is normal in K_f, b maps K_M-fixed vectors to K_M-fixed vectors in every representation, and $\rho_V(b) = T_a(V)e_c$ on $\mathcal{A}_M(V)$, the same for V'. Let $\phi(g) = \sum_h b_h \mathbf{1}_{\mathrm{K}_M}(h^{-1}g)$. Then $\phi(g) = \sum_h a_h \psi(h^{-1}g)$ with $\psi(y) = e_c(\bar y)$ for $y \in \mathrm{K}_f$ ($\bar y$ its reduction) and $\psi(y) = 0$ otherwise. Let T, t, $\tilde u$ be as in the definition of a function cuspidal at w, (FT4) of Definition 3.2.4, and $y \in G_f$. If $y \notin \mathrm{K}_f$, all $y\iota_w(\tilde u) \notin \mathrm{K}_f$. If $y \in \mathrm{K}_f$, the reduction of $y\iota_w(\tilde u)$ is $\bar y$ times the element with component ut at w and 1 at the other chosen places ($\bar w$ is not chosen), so

$$\sum_{u \in U_T} \psi(y\,\iota_w(\tilde u)) = \prod_{w' \neq w} e_{c_{w'}}(\bar y_{w'}) \cdot \sum_{u \in U_T} e_{c_w}(\bar y_w\, u\, t) = 0$$

by Theorem A.5.28. Applying this to $y = h^{-1}g$ gives the cuspidality of $\phi = \phi_b^{\mathrm{K}_M}$ at w. By Theorem 5.8.5 with $K = \mathrm{K}_M$, $\lambda = \varsigma$ and the element b in place of a, $\operatorname{tr}(T_a(V)e_c) = \operatorname{tr}(T_a(V')e_c)$. Summing over c gives equal traces.

Step 3: characteristic polynomials and determinants. $\mathcal{H}^M$ is a subalgebra containing 1 and $T_a^n = T_{a^n}$, so Step 2 gives $\operatorname{tr}(T_a^n)$ equal on both good spaces for all $n \geq 0$ ($n = 0$: equal dimensions). Over $\mathbb{C}$ the power sums of the eigenvalues determine the characteristic polynomial (Newton's identities), hence also the determinant.

Step 4: eigenparts. We show that $\mathcal{A}_M(V)$ is the direct sum of the eigenparts $\mathcal{A}_M(V)[q]$, finitely many of them nonzero. Give $\mathcal{A}_M(V)$ the Petersson inner product, for which every $\rho_V(g)$, $g \in G_f$, is unitary. Let w be split off the level and v the place of E^+ below it; then K_M has the full component $K_w = \mathrm{GL}_r(\mathcal{O}_w)$ at v, and for $d \in \mathrm{GL}_r(E_w)$ write $T_{[d]}$ for the operator $\rho_V(\sum_j g_j)$, g_j over $K_w d K_w / K_w$ placed at v. With the Haar measure of $\mathrm{GL}_r(E_w)$ giving K_w volume 1, $T_{[d]}x = \int_{K_w d K_w} \rho_V(h)x\, dh$ for $x \in \mathcal{A}_M(V)$; so the $T_{[d]}$ represent the convolution algebra $\mathcal{H}(\mathrm{GL}_r(E_w), K_w)$ of K_w-biinvariant compactly supported functions, which is commutative [109]. (a) Commutation: $T_{w,i}$ and $T_{w',i'}$ commute if w, w' lie over the same place of E^+ (both lie in the image of that commutative algebra; for $w' = \bar w$ through the identification of (HO4)), and otherwise because they translate at different components of G_f. (b) Adjoints: for $x, y \in \mathcal{A}_M(V)$, by unitarity and unimodularity, $\langle T_{[d]}x, y\rangle = \int_{K_w d K_w} \langle x, \rho_V(h^{-1})y\rangle dh = \langle x, T_{[d^{-1}]}y\rangle$. For $d = \operatorname{diag}(\varpi_w 1_i, 1_{r-i})$, $K_w d^{-1} K_w = \varpi_w^{-1} K_w \operatorname{diag}(\varpi_w 1_{r-i}, 1_i) K_w$ (a permutation matrix in K_w reorders the diagonal), so $T_{w,i}^* = T_{w,r}^{-1} T_{w,r-i}$ with $T_{w,0} = 1$; here $T_{w,r} = \rho_V(\varpi_w 1_r)$ is invertible, so $T_{w,r}^{-1}$ is a polynomial in $T_{w,r}$ by the Cayley–Hamilton theorem. Hence the algebra $\mathcal{B} \subseteq \operatorname{End} \mathcal{A}_M(V)$ generated by all $T_{w,i}$ is commutative and closed under adjoints; it is finite-dimensional, so it is generated by finitely many $T_{w,i}$, which are commuting normal operators. By the spectral theorem for commuting normal operators [133] (Theorem 2.5.5) they are simultaneously unitarily diagonalisable, so $\mathcal{A}_M(V)$ is the orthogonal direct sum of the joint eigenspaces of $\mathcal{B}$. Each joint eigenspace is $\mathcal{A}_M(V)[q]$ for the eigensystem q of its eigenvalues, eigenparts of distinct eigensystems meet in 0, and only finitely many are nonzero as $\dim \mathcal{A}_M(V) < \infty$. Since $\bar G$ commutes with the $T_{w,i}$ (Step 1), e_0 preserves each eigenpart and $\mathcal{A}^{\mathrm{good}}(V) = \bigoplus_q \mathcal{A}^{\mathrm{good}}(V) \cap \mathcal{A}_M(V)[q]$; the same for V'. Let F be the finite set of eigensystems occurring for V or V', and q an eigensystem. For

each $q' \in F \setminus \{q\}$ choose (w,i) with $q'(w,i) \neq q(w,i)$ and put

$$a = \prod_{q' \in F \setminus \{q\}} \frac{a_{w,i} - q'(w,i)}{q(w,i) - q'(w,i)} \in \mathcal{H}^M.$$

Then T_a is the identity on the q-part and 0 on the q'-parts, $q' \neq q$, on both spaces; its trace on good forms is the dimension in (ii), and Step 2 gives (ii).

Step 5: the spaces Y_0, W_j. Index the entries of $\varsigma(v)$ by $p = 0, \dots, r-1$ in the displayed order of Definition 3.2.16, matched with the basis of Y_0 by the numbering num of (HP2) of Definition 3.2.13: $\varsigma(v)_p$ with $p < k-1$ lies in $[1-\frac{k}{2}, \frac{k}{2}-1]$, and $\varsigma(v)_{k-2+i} = \frac{\Delta}{2} - \kappa_i(v) = -\kappa(v) - i\Delta$ for $1 \leq i \leq m$ (by (AD4), $\kappa_i = \kappa + (2i+1)\frac{\Delta}{2}$). The entries are pairwise distinct. Two entries $-\kappa - i\Delta$, $-\kappa - i'\Delta$ with $i \neq i'$ differ by at least Δ, and a small entry differs from $-\kappa - i\Delta$ by at least $\kappa + \Delta - \frac{k}{2} \geq \Delta$ ($\kappa \geq 1$, $k < r < \Delta$). So ς has gap Δ for any signs that are constant on the $k-1$ small indices. The signs of W_j are $+1$ on the first $k+j-1$ indices ((HP4) of Definition 3.2.13); the signs of Y_0 ((HP2) of Definition 3.2.13) are -1 exactly at the indices $k-2+i$ with $i \geq 2$ even (equivalently, at the indices p with $p-k$ non-negative and even), $m/2 = \varrho + 1$ of them. Both are constant on the small indices, which gives the gaps. Finally T_1 is the identity, so (i) for $a = 1$, $V = Y_0$, $V' = W_j$ gives $\dim \mathcal{A}^{\mathrm{good}}(W_j) = \dim \mathcal{A}^{\mathrm{good}}(Y_0) = d_{\mathcal{T}}$. □

Theorem 5.8.5. *Let E be a CM field with totally real subfield E^+ such that E contains a primitive 5040-th root of unity and $[E^+ : \mathbb{Q}]$ is even. Let $N \geq 2$, let $G_f = G_f(\{1,\dots,N\})$, $\mathrm{K}_f = \mathbf{K}_0(\{1,\dots,N\})$ be as in (VP1), and let V, V' be framed spaces of rank N over E whose signs do not depend on the infinite place: $s_v = s$ and $s'_v = s'$ for all v, with fixed $s, s' \colon \{1,\dots,N\} \to \{\pm 1\}$. Let $\lambda = (\lambda_v)_v$, indexed by the infinite places v of E^+, with $\lambda_v \in (\frac{N-1}{2} + \mathbb{Z})^N$ having pairwise distinct entries and*

$$|\lambda_{v,i} - \lambda_{v,j}| \;\geq\; 500N^3 + 10N^5 \quad \text{whenever } s_v(i) \neq s_v(j) \text{ or } s'_v(i) \neq s'_v(j).$$

For a framed space $U \in \{V, V'\}$ and an open subgroup $K \subseteq \mathrm{K}_f$ let $\mathcal{A}_K(U,\lambda)$ be the space of forms on $\mathrm{U}(U)$ of level K whose archimedean type at every v is the discrete series with Harish-Chandra parameter λ_v relative to the signs of U at v. Let $K \subseteq \mathrm{K}_f$ be open and let $a = \sum_g a_g\, g \in \mathbb{C}[G_f]$ be finitely supported such that $\pi(a)$ maps K-fixed vectors to K-fixed vectors in every representation π of G_f. Let w be a finite place of E with $w \neq \bar{w}$, and suppose that the function $\phi_a^K(g) = \sum_h a_h \mathbf{1}_K(h^{-1}g)$ on G_f is cuspidal at w. Then $\mathcal{A}_K(V,\lambda)$ and $\mathcal{A}_K(V',\lambda)$ are finite-dimensional, $\rho_V(a)$ and $\rho_{V'}(a)$ preserve them, and

$$\mathrm{tr}\big(\rho_V(a) \mid \mathcal{A}_K(V,\lambda)\big) = \mathrm{tr}\big(\rho_{V'}(a) \mid \mathcal{A}_K(V',\lambda)\big).$$

Proof. For $U \in \{V, V'\}$ let X_U be the space of forms on $\mathrm{U}(U)$ of type e_λ^U (of all levels), so that $X_U^K = \mathcal{A}_K(U,\lambda)$ for every open $K \subseteq \mathrm{K}_f$. Right translation by G_f commutes with right convolution by e_λ^U at the infinite places, so ρ_U is a representation of G_f on X_U; it is unitary for the Petersson inner product, because right translation preserves the invariant measure on $\Gamma_U \backslash G^U$, which has finite volume, and forms of type e_λ^U are bounded, hence square integrable. The hypothesis on λ says that λ satisfies the gap condition of Theorem 5.8.6 for the signs s and for the signs s'.

Step 1, finiteness and stability. By Theorem 5.8.6 (i), $\mathcal{A}_K(V,\lambda)$ and $\mathcal{A}_K(V',\lambda)$ are finite-dimensional. Since a maps K-fixed vectors to K-fixed vectors in every representation of G_f, $\rho_U(a)$ maps $X_U^K = \mathcal{A}_K(U,\lambda)$ into itself for $U = V, V'$.

Step 2, the auxiliary place. By Theorem 5.8.50 (CU1) there is a finite place $v \neq \bar{v}$ of E with $v \notin \{w, \bar{w}\}$, $\iota_v(\mathrm{GL}_N(\mathcal{O}_v)) \subseteq K$, and a a combination of elements integral at v. The group $K' = \{k \in K : \ k_v \in J_v\}$ is open in K_f, so by Theorem 5.8.6 (i) the spaces $X_V^{K'} = \mathcal{A}_{K'}(V, \lambda)$ and $X_{V'}^{K'} = \mathcal{A}_{K'}(V', \lambda)$ are finite-dimensional.

Step 3, equality of traces at the cut level. Let $l, m \geq 0$. By Theorem 5.8.50 (CU2), $\rho_U(b_{l,m})$ maps $\mathcal{A}_{K'}(U, \lambda)$ into itself, so Theorem 5.8.6 (iii), applied to U, the level K' and the element $b_{l,m}$, whose function at level K' is $\phi_{l,m}$, gives

$$\mu_f(K') \operatorname{tr}\big(\rho_U(b_{l,m}) \mid \mathcal{A}_{K'}(U, \lambda)\big) = \sum_{R \in E[X]} I_R^U(\phi_{l,m}) \qquad (U = V, V').$$

By Theorem 5.8.50 (CU3), $\phi_{l,m}$ is cuspidal at w, and every g with $\phi_{l,m}(g) \neq 0$ has g_v integral with 0 a simple root of the characteristic polynomial of its reduction modulo v. These are the hypotheses of (TM3) of Theorem 5.8.30 with $w' = v$; together with $\mu_{5040} \subseteq E$ and $[E^+ : \mathbb{Q}]$ even, the consequence stated there gives $I_R^V(\phi_{l,m}) = I_R^{V'}(\phi_{l,m})$ for every R. The two sums above are therefore equal, and since $\mu_f(K') > 0$ for the open subgroup K',

$$\operatorname{tr}\big(\rho_V(b_{l,m}) \mid X_V^{K'}\big) = \operatorname{tr}\big(\rho_{V'}(b_{l,m}) \mid X_{V'}^{K'}\big) \qquad \text{for all } l, m \geq 0.$$

Step 4, return to level K. The unitary representations ρ_V on X_V and $\rho_{V'}$ on $X_{V'}$, with finite-dimensional K'-fixed spaces (Step 2), the place v and the element a satisfy the hypotheses of Theorem 5.8.50 (CU2), and the traces of all $b_{l,m}$ agree by Step 3. Hence $\operatorname{tr}(\rho_V(a) \mid \mathcal{A}_K(V, \lambda)) = \operatorname{tr}(\rho_{V'}(a) \mid \mathcal{A}_K(V', \lambda))$. □

The simple trace formula is stated for one framed space. Its proof is reduction theory and decay: Mahler's criterion for framed spaces, uniform rapid decay of the forms reproduced by a cusp kernel, the kernel bound by the inverse first minimum, the uniform bound of the polynomial parts of the kernel, and the spectral expansion of the kernel. The finiteness of the volume of the adelic quotient (Theorem 4.2.18) rests on the theorem of Borel and Harish-Chandra (Theorem A.4.6; see [37] and [33]). For rapid decay of cusp forms see [120], [208] and [200].

Theorem 5.8.6. *Let E be a CM field with totally real subfield E^+ and a CM type fixing an embedding $\Phi_v \colon E \to \mathbb{C}$ for each infinite place v. Let $N \geq 2$, let U be a framed space of rank N over E whose signs do not depend on the infinite place, $s_v = s$ for all v with a fixed $s \colon \{1, \dots, N\} \to \{\pm 1\}$, and let $\lambda = (\lambda_v)_v$ with $\lambda_v \in (\frac{N-1}{2} + \mathbb{Z})^N$ having pairwise distinct entries and $|\lambda_{v,i} - \lambda_{v,j}| \geq 500N^3 + 10N^5$ whenever $s_v(i) \neq s_v(j)$. Put $G_\infty^U = G_\infty = \prod_v \mathrm{U}(\mathrm{diag}(s_v))$ and $G^U = G_\infty^U \times G_f$, with $\Gamma_U \subseteq G^U$ the image of $\mathrm{U}(U)(E^+)$ under the frames, as in (FT1). Let*

$$e_\lambda^U(g_\infty) = \prod_v d(\lambda_v, s_v)\, \overline{c_{\lambda_v, s_v}(g_v)},$$

where c_{λ_v, s_v} is Flensted-Jensen's function of the parameter λ_v for the signs s_v as in (FT3) (the matrix coefficient, at a vector of the lowest K-type, of the discrete series of $\mathrm{U}(\mathrm{diag}(s_v))$ with Harish-Chandra parameter λ_v, with $c_{\lambda_v, s_v}(1) = 1$), and $d(\lambda_v, s_v) = (\int |c_{\lambda_v, s_v}|^2)^{-1}$ its formal degree for the Haar measure fixed below. Let $K \subseteq \mathrm{K}_f$ be open, let $a \in \mathbb{C}[G_f]$ be finitely supported, and let $\phi(g) = \sum_h a_h \mathbf{1}_K(h^{-1}g)$. For a polynomial $R \in E[X]$ put

$$I_R^U(\phi) = \int_{\Gamma_U \backslash G^U} \sum_{\gamma \in \mathrm{U}(U)(E^+),\ \mathrm{charpoly}(\gamma) = R} (e_\lambda^U \otimes \phi)(x^{-1}\gamma x)\, dx,$$

with Haar measures fixed as follows: on each $\mathrm{U}(\mathrm{diag}(s_v))$ a fixed Haar measure, the same one as in $d(\lambda_v, s_v)$, on G_f a fixed Haar measure μ_f independent of U, and on $\Gamma_U \backslash G^U$ the quotient by the counting measure on Γ_U. Then:

1. *$\mathcal{A}_K(U,\lambda)$ is finite-dimensional;*

2. *for every R the inner sum converges absolutely for every x and is integrable on $\Gamma_U\backslash G^U$, and the family $(I_R^U(\phi))_{R\in E[X]}$ is absolutely summable;*

3. *if $\rho_U(a)$ maps $\mathcal{A}_K(U,\lambda)$ into itself, then*

$$\mu_f(K)\ \mathrm{tr}(\rho_U(a)\mid \mathcal{A}_K(U,\lambda)) = \sum_{R\in E[X]} I_R^U(\phi).$$

Proof. Write $G_\infty = G_\infty^U$, $\Gamma = \Gamma_U$, $\mathcal{A} = \mathcal{A}_K(U,\lambda)$, $e = e_\lambda^U$, and for $x \in G^U$ and $S \subseteq E[X]$

$$s_S(x) = \sum_{\gamma\in \mathrm{U}(U)(E^+),\ \mathrm{charpoly}(\gamma)\in S} (e\otimes\phi)(x^{-1}\gamma x),$$

so that $I_R^U(\phi) = \int_{\Gamma\backslash G^U} s_{\{R\}}$. The function ϕ is locally constant with compact support (a finite combination of indicators of compact open cosets hK).

Step 0: the kernel is a cusp kernel of order $B = 10N^5$. As $\lambda_v \in (\frac{N-1}{2}+\mathbb{Z})^N$ has pairwise distinct entries, $2\lambda_v \in \mathbb{Z}^N$ has distinct entries congruent to $N-1$ modulo 2, and $|2\lambda_{v,i} - 2\lambda_{v,j}| \geq 2(500N^3+B)$ when $s(i)\neq s(j)$: so $2\lambda_v$ has gap $500N^3+B$ for s in the sense of (DZ1) of Definition 5.6.3. By Lemma 5.6.21, e is a cusp kernel of order B on G_∞ (the formal degrees are taken for the fixed Haar measures). As $N\geq 2$, $B = 10N^5 \geq 13N^3 > 8N^2$. The space $\mathcal{A}$ consists of the continuous bounded functions on G^U that are left Γ-invariant, right K-invariant and satisfy $R(e)F = F$ (right K-invariance gives the open subgroup required in the type condition).

Step 1: (i). Let R_1 be the constant of Lemma 5.8.7 for e, K and $A = 1$, and $R_0 = \max(R_1, 1)$; let C be the compact set of Lemma 5.8.20 for $T = 2R_0$. By the continuity of left translation in $L^1(G_\infty)$ there is a neighbourhood W_∞ of 1 in G_∞ with $\int |e(u^{-1}h) - e(h)|\,dh \leq \frac{1}{2}$ for $u \in W_\infty$; put $W = W_\infty \times K$. For $F\in\mathcal{A}$ with $M = \sup|F|$, $g\in G^U$ and $w = (u,k)\in W$,

$$|F(gw) - F(g)| = |F(g(u,1)) - F(g)| = \left|\int (e(u^{-1}h) - e(h))F(g(h,1))\,dh\right| \leq \tfrac{1}{2}M, \qquad (1)$$

using right K-invariance, $F = R(e)F$ and the substitution $h\mapsto u^{-1}h$. The compact set C is covered by finitely many sets $a_1W,\dots,a_mW$ with $a_j\in C$. We claim that $F\mapsto (F(a_1),\dots,F(a_m))$ is injective on $\mathcal{A}$, so $\dim\mathcal{A}\leq m$. Let $F\in\mathcal{A}$ vanish at all a_j and suppose $M = \sup|F| > 0$. Choose g with $|F(g)| > M/2$. By Lemma 5.8.7, $|F(g)|\lambda_U(g)^{-1} \leq MR_0$ ($\lambda_U(g)$ the first minimum of (VP4) of Definition 3.2.2), so $\lambda_U(g)^{-1} < 2R_0$; by Lemma 5.8.20 some $\gamma\in\Gamma$ has $\gamma g\in C$, so $\gamma g \in a_jW$ for some j, and by (1) and Γ-invariance $|F(g)| = |F(\gamma g) - F(a_j)| \leq M/2$, a contradiction. So $F = 0$.

Step 2: (ii), the terms. By Proposition 5.8.22(ii) (with $k = e$, $j = 0$, $r = 1$, $g = x$, Z the γ with polynomial in S; (EK4)(b) and $B\geq 8N^2+1$), the series $s_S(x)$ converges absolutely for every x and every S. Each s_S is measurable (a countable sum of measurable functions) and left Γ-invariant: for $\delta\in\mathrm{U}(U)(E^+)$ the bijection $\gamma\mapsto\delta^{-1}\gamma\delta$ preserves characteristic polynomials and $(\delta x)^{-1}\gamma(\delta x) = x^{-1}(\delta^{-1}\gamma\delta)x$. By Proposition 5.8.23 there is D with $|s_S(x)|\leq D$ for all x and S. The quotient $\Gamma\backslash G^U$ has finite volume vol by Theorem 4.2.18 (through the frames G^U is the adelic unitary group of U and Γ its rational points). So every s_S is integrable on $\Gamma\backslash G^U$; in particular $I_R^U(\phi)$ is defined.

Step 3: (ii), summability. For a finite $F\subseteq E[X]$, absolute convergence gives $s_F = \sum_{R\in F} s_{\{R\}}$, hence $\int s_F = \sum_{R\in F} I_R^U(\phi)$. Given a finite F, let $F_1\subseteq F$ be the set of R with $\mathrm{Re}\, I_R^U(\phi)\geq 0$; then

$\sum_{R\in F_1} \operatorname{Re} I_R^U(\phi) = \operatorname{Re}\int s_{F_1} \le D\,\mathrm{vol}$, and likewise for the negative real parts and for the imaginary parts. So $\sum_{R\in F} |I_R^U(\phi)| \le 4D\,\mathrm{vol}$ for every finite F: the family is absolutely summable. To identify its sum, enumerate the countable set $E[X]$ as $R_1, R_2, \dots$ and put $F_n = \{R_1, \dots, R_n\}$. For each x, $s_{F_n}(x) \to s_{E[X]}(x)$ by absolute convergence of the series over γ, and $|s_{F_n}| \le D$; by dominated convergence on the finite measure space,

$$\sum_{R\in E[X]} I_R^U(\phi) = \lim_n \int s_{F_n} = \int_{\Gamma\backslash G^U} s_{E[X]}. \tag{2}$$

Step 4: (iii). Assume that $\rho_U(a)$ maps $\mathcal{A}$ into itself, and let (b_i) be an orthonormal basis of $\mathcal{A}$ for the Petersson product (finite by Step 1). By Proposition 5.8.28 (cusp kernel of order $B > 8N^2$, $K \subseteq \mathrm{K}_f$, $\mathcal{A}$ finite-dimensional) with $g = g' = x$,

$$s_{E[X]}(x) = \mu_f(K) \sum_i (\rho_U(a) b_i)(x)\, \overline{b_i(x)}.$$

Each product is integrable (bounded functions, finite volume), so integrating and using (2),

$$\sum_{R\in E[X]} I_R^U(\phi) = \mu_f(K) \sum_i \int_{\Gamma\backslash G^U} \overline{b_i}\, \rho_U(a) b_i = \mu_f(K) \sum_i \langle b_i, \rho_U(a) b_i\rangle = \mu_f(K)\, \mathrm{tr}\big(\rho_U(a) \mid \mathcal{A}\big).$$

□

Lemma 5.8.7. *Let E be a CM field, $N \ge 1$, $s \in \{\pm 1\}^N$, and U a framed space of rank N over E with signs $s_v = s$ at every infinite place v; let $G^U = G_\infty \times G_f$ with G_∞ the real group of s, and Γ_U the image of $\mathrm{U}(U)(E^+)$ under the frames. Let $B \ge 0$ and let e be a cusp kernel of order B on G_∞; let $K \subseteq G_f$ be an open subgroup and $A \ge 0$ an integer. There is $R \ge 0$ such that for every continuous bounded function F on G^U that is left Γ_U-invariant, right K-invariant and satisfies $R(e)F = F$, and every real c with $|F| \le c$ on G^U,*

$$|F(g)|\, \lambda_U(g)^{-A} \le c\, R \qquad (g \in G^U).$$

Proof. Put $J = K \cap \mathrm{K}_f$, an open subgroup of G_f. Let $\delta_0 > 0$ be the constant of Proposition 5.8.8 for the cusp kernel e and the exponent $j = NA$, and $\kappa_J > 0$ that of Proposition 5.8.13(ii) for J; put $\delta_1 = \min(\delta_0, 1)$ and

$$R = \big(\kappa_J/\delta_1^N\big)^A.$$

Let F and c be as in the statement; then $c \ge |F(1)| \ge 0$. Let $g \in G^U$. The first minimum $\lambda_U(g)$ is positive (the nonzero vectors of the lattice $L_{g,\infty}$ of Lemma 5.8.15(iii) have norm bounded below), so there is $\delta > 0$ with $\delta^N = \kappa_J \lambda_U(g)$. We show

$$|F(g)| \le (\delta/\delta_1)^{NA}\, c. \tag{1}$$

If $\delta \le 1$, Proposition 5.8.13(ii) gives a nonzero isotropic subspace $A_0 \subseteq U$ with $N_{A_0}(\mathbb{A}_{E^+})\, g \subseteq N_{A_0}(E^+)\, g\, Z_\delta\, J$. The function F is continuous, bounded by c, satisfies $R(e)F = F$, is right J-invariant ($J \subseteq K$) and left invariant under $N_{A_0}(E^+) \subseteq \Gamma_U$; so Proposition 5.8.8, with $T = c$, the subgroup J and $j = NA$, gives $|F(g)| \le (\delta/\delta_0)^{NA} c \le (\delta/\delta_1)^{NA} c$. If $\delta > 1$, then $\delta/\delta_1 > 1$ and (1) follows from $|F(g)| \le c$.

By (1),

$$|F(g)|\lambda_U(g)^{-A} \le c\Big(\frac{\delta^N}{\delta_1^N}\Big)^A \lambda_U(g)^{-A} = c\Big(\frac{\kappa_J}{\delta_1^N}\Big)^A = c\, R.$$

The constant R depends only on e, A, N, E and K. □

Proposition 5.8.8. *Let E be a CM field, $N \geq 1$, $s \in \{\pm 1\}^N$, $B \geq 0$, e a cusp kernel of order B on G_∞, and $j \geq 0$ an integer. There is $\delta_0 > 0$, depending only on e and j, with the following property. Let U be a framed space of rank N over E with signs $s_v = s$ for all v, $G^U = G_\infty \times G_f$, $A \subseteq U$ a nonzero isotropic subspace, $J \subseteq G_f$ a subgroup, $\delta \geq 0$ and $g \in G^U$ with $N_A(\mathbb{A}_{E^+})\, g \subseteq N_A(E^+)\, g\, Z_\delta\, J$. Let F be a continuous function on G^U with $|F| \leq T$, $R(e)F = F$, F right J-invariant and $F(\nu x) = F(x)$ for $\nu \in N_A(E^+)$. Then*

$$|F(g)| \leq \Big(\frac{\delta}{\delta_0}\Big)^j T.$$

Proof. Write $\|z\| = \|z\|_\infty$ for $z \in G_\infty$. For $z_v \in \mathrm{U}(\mathrm{diag}\, s)$, $z_v^{-1} = \mathrm{diag}(s)\bar{z}_v^{\mathsf{T}}\mathrm{diag}(s)$, so $\|z_v^{-1}\| = \|z_v\| \geq 1$; hence $\|z\| \geq \|z_v\| \geq 1$ for every v, and $\|zw\| \leq \|z\|\,\|w\|$.

Step 1: a reproducing bump. Let β be as in (EK4)(c). For a bounded measurable f on G^U with $R(e)f = f$, Fubini's theorem ($|\beta(x)e(x^{-1}y)|\sup|f|$ is integrable) and the substitution $y = xy'$ give

$$f(g') = \int e(y) f(g'(y,1))\, dy = \int \beta(x) \int e(y') f(g'(xy',1))\, dy'\, dx = \int \beta(x) f(g'(x,1))\, dx.$$

Step 2: small oscillation. Let $\Omega = \{ux : \max_v \|u_v - 1\| \leq 1,\ x \in \mathrm{supp}\,\beta\}$, a compact set; let $c \geq 1$ bound $\|\cdot\|$ on Ω, and choose $0 < \delta_0 \leq 1$ with $\int |\beta(u^{-1}x) - \beta(x)|\, dx \leq \frac{1}{2}c^{-2j}$ whenever $\max_v \|u_v - 1\| \leq \delta_0$ (continuity of left translation in L^1). Then: if f is bounded, continuous, $R(e)f = f$, $g' \in G^U$, $C' \geq 0$ and $|f(g'(w,1))| \leq C'\|w\|^{2j}$ for all $w \in G_\infty$, then $|f(g'(u,1)) - f(g')| \leq C'/2$ for $\max_v \|u_v - 1\| \leq \delta_0$. Indeed, by Step 1 and left invariance, $f(g'(u,1)) - f(g') = \int (\beta(u^{-1}x) - \beta(x)) f(g'(x,1))\, dx$; where the integrand is nonzero, $x \in \mathrm{supp}\,\beta$ or $u^{-1}x \in \mathrm{supp}\,\beta$, so $x \in \Omega$ and $|f(g'(x,1))| \leq C'c^{2j}$; so the difference is at most $C'c^{2j} \cdot \frac{1}{2}c^{-2j}$.

Step 3: halving. Put $b = (\delta/\delta_0)^j T$ and let $\mathcal{C}$ be the set of $C \geq 0$ with $|F(g(z,1))| \leq C\|z\|^{2j}$ for all $z \in G_\infty$; $T \in \mathcal{C}$. We show: if $C \in \mathcal{C}$ then $\max(C/2, b) \in \mathcal{C}$. Let $z \in G_\infty$.

If $\|z\|^2\delta \leq \delta_0$: for $x \in Z_\delta$ write $g(xz,1) = g(z,1)(u,1)$ with $u = z^{-1}xz$; then $\|u_v - 1\| = \|z_v^{-1}(x_v - 1)z_v\| \leq \|z\|^2\delta \leq \delta_0$. Step 2 at $g' = g(z,1)$ with $C' = C\|z\|^{2j}$ applies, since $|F(g(z,1)(w,1))| = |F(g(zw,1))| \leq C\|zw\|^{2j} \leq C'\|w\|^{2j}$; so $|F(g(xz,1)) - F(g(z,1))| \leq \frac{C}{2}\|z\|^{2j}$ for all $x \in Z_\delta$. By Lemma 5.8.9(ii) (with $y = z$, $D = \frac{C}{2}\|z\|^{2j}$), $|F(g(z,1))| \leq \frac{C}{2}\|z\|^{2j}$.

If $\|z\|^2\delta > \delta_0$: then $(\delta/\delta_0)^j\|z\|^{2j} \geq 1$ and $|F(g(z,1))| \leq T \leq b\|z\|^{2j}$.

Iterating from $C_0 = T$, $C_{k+1} = \max(C_k/2, b)$ gives $C_k \leq \max(2^{-k}T, b)$ in $\mathcal{C}$ for all k; at $z = 1$, $|F(g)| \leq \max(2^{-k}T, b)$ for every k, hence $|F(g)| \leq b$. The constant δ_0 was chosen from e and j only. □

Lemma 5.8.9. *Let E be a CM field, $N \geq 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs $s_v = s$ for all v, $G^U = G_\infty \times G_f$ and Γ_U the image of $\mathrm{U}(U)(E^+)$ under the frames. Let $A \subseteq U$ be a nonzero isotropic subspace, N_A the unipotent radical of its stabiliser in $\mathrm{U}(U)$ (a unipotent group over E^+), with $N_A(\mathbb{A}_{E^+}) \subseteq G^U$ through the frames and $N_A(E^+) \subseteq \Gamma_U$. Let $B \geq 0$, e a cusp kernel of order B on G_∞, and F a continuous bounded function on G^U with $R(e)F = F$ and $F(\nu x) = F(x)$ for $\nu \in N_A(E^+)$, $x \in G^U$. Let dn be the invariant probability measure on the compact quotient $N_A(E^+)\backslash N_A(\mathbb{A}_{E^+})$.*

(i) For every $g \in G^U$, $\int F(ng)\, dn = 0$.

(ii) Let $J \subseteq G_f$ be a subgroup such that F is right J-invariant, $\delta \geq 0$, and $g \in G^U$ with

$$N_A(\mathbb{A}_{E^+})\, g \subseteq N_A(E^+)\, g\, Z_\delta\, J, \qquad Z_\delta = \{z \in G_\infty : \max_v \|z_v - 1\| \leq \delta\}$$

(G_∞ and G_f embedded in G^U as $z \mapsto (z,1)$, $k \mapsto (1,k)$). If $y \in G_\infty$ and $D \geq 0$ satisfy $|F(g(zy,1)) - F(g(y,1))| \leq D$ for all $z \in Z_\delta$, then $|F(g(y,1))| \leq D$.

Proof. By Theorem A.4.3, applied to the unipotent group N_A over E^+, the subgroup $N_A(E^+)$ is discrete and cocompact in the unimodular group $N_A(\mathbb{A}_{E^+})$; so the quotient carries an invariant probability measure dn, invariant also under right translation by $N_A(\mathbb{A}_{E^+})$. Let $\Sigma = \sup|F|$. Fix an E-basis of U with Gram matrix J_U and frames X, so $c(u) = X^{-1}u$, G^U preserves the form D_s ($\mathrm{diag}(s)$ at each infinite place, 1 at the finite places), and let $Y \in M_N(E)$ be a nonzero matrix whose columns span A. An element $h \in G^U$ lies in $N_A(\mathbb{A}_{E^+})$ if and only if $hX^{-1}Y = X^{-1}Y$ and $\bar{w}^{\mathsf{T}} D_s h u = \bar{w}^{\mathsf{T}} D_s u$ for all adelic vectors u, w orthogonal (for D_s) to the columns of $X^{-1}Y$.

(i). Fix g and put $A' = g^{-1}X^{-1}Y$ and $A_v = A'_v \in M_N(\mathbb{C})$ at the infinite place v; $A_v \neq 0$ ($Y \neq 0$ and $g^{-1}X^{-1}$ is invertible) and $\bar{A}_v^{\mathsf{T}} \mathrm{diag}(s) A_v = \Phi_v(\bar{Y}^{\mathsf{T}} J_U Y) = 0$, as A is isotropic and g preserves D_s. Let $R_v \subseteq \mathrm{U}(\mathrm{diag}\, s)$ be as in Lemma 5.8.10 for A_v. For $r \in \prod_v R_v$ the element $n_r = g(r,1)g^{-1}$ lies in $N_A(\mathbb{A}_{E^+})$: $n_r X^{-1}Y = g(r,1)A' = gA' = X^{-1}Y$, since $r_v A_v = A_v$ and the finite component of $(r,1)$ is 1; and the orthogonality condition for n_r is that of $(r,1)$ for A', transported by g, which preserves D_s. Define, for $x \in G_\infty$,

$$\Psi(x) = \int F\big(ng(x,1)\big)\, dn,$$

well defined since $n \mapsto F(ng(x,1))$ is continuous, bounded and left $N_A(E^+)$-invariant. Ψ is bounded by Σ and continuous by dominated convergence (finite measure, F continuous and bounded). For $r \in \prod_v R_v$, $\Psi(rx) = \int F(nn_r g(x,1))\, dn = \Psi(x)$ by right invariance of dn. Since $F = R(e)F$ and $|e(x)|\Sigma$ is integrable on $G_\infty \times (N_A(E^+)\backslash N_A(\mathbb{A}_{E^+}))$, Fubini's theorem gives

$$\Psi(1) = \iint e(x) F\big(ng(x,1)\big)\, dx\, dn = \int_{G_\infty} e(x)\Psi(x)\, dx = 0$$

by Lemma 5.8.10 with $y = 1$. As $\Psi(1) = \int F(ng)\, dn$, this is (i).

(ii). Let $n \in N_A(\mathbb{A}_{E^+})$. By hypothesis $ng = \nu g(z,1)(1,k)$ with $\nu \in N_A(E^+)$, $z \in Z_\delta$, $k \in J$. As $(z,1)(1,k)(y,1) = (zy,1)(1,k)$, the invariances of F give $F(ng(y,1)) = F(g(zy,1))$, so $|F(ng(y,1)) - F(g(y,1))| \leq D$ for every n. Integrating against the probability measure dn and using (i) at the point $g(y,1)$,

$$|F(g(y,1))| = \Big|\int \big(F(g(y,1)) - F(ng(y,1))\big)\, dn\Big| \leq D.$$

□

Lemma 5.8.10. *Let E be a CM field with set of infinite places Φ, $N \geq 1$, $s \in \{\pm 1\}^N$, $B \geq 0$ and e a cusp kernel of order B on $G_\infty = \prod_{v\in\Phi} \mathrm{U}(\mathrm{diag}\, s)$. For each $v \in \Phi$ let $A_v \in M_N(\mathbb{C})$ be nonzero with $\bar{A}_v^{\mathsf{T}} \mathrm{diag}(s) A_v = 0$, and let R_v be the group of $r \in \mathrm{U}(\mathrm{diag}\, s)$ with $rA_v = A_v$ and $\bar{w}^{\mathsf{T}} \mathrm{diag}(s)\, ru = \bar{w}^{\mathsf{T}} \mathrm{diag}(s)\, u$ for all $u, w \in \mathbb{C}^N$ with $\bar{A}_v^{\mathsf{T}} \mathrm{diag}(s) u = \bar{A}_v^{\mathsf{T}} \mathrm{diag}(s) w = 0$ (the unipotent radical of the stabiliser of the column span of A_v). Let Ψ be a bounded continuous function on G_∞ with $\Psi(rx) = \Psi(x)$ for all $x \in G_\infty$ and $r \in \prod_v R_v$. Then*

$$\int_{G_\infty} e(x)\, \Psi(yx)\, dx = 0 \qquad (y \in G_\infty).$$

Proof. Write $J = \mathrm{diag}(s)$, $\|\cdot\|$ for operator norms, and $A_v^\perp = \{u \in \mathbb{C}^N : A_v^* J u = 0\}$ ($P^* = \bar{P}^{\mathsf{T}}$). For $z \in \mathrm{U}(J)$, $z^{-1} = Jz^*J$, so $\|z^{-1}\| = \|z\|$ and $1 \le \|z\|^2$; hence $\|x\|_\infty \ge \|x_v\| \ge 1$ for $x \in G_\infty$ and every v. We apply Lemma 5.8.11 to $G = G_\infty$ (a second countable locally compact group; the product dx of Haar measures is a left Haar measure) with the following data.

The kernel. e is integrable and, by (EK3) with $x = 1$, $\int e(z)e(z^{-1}y)\,dz = e(1)e(y)/e(1) = e(y)$.

The group D. Let D be the set of $(r, c) \in G_\infty \times \mathbb{C}^\times$ such that for every v: $r_v A_v = cA_v$, and $w^* J r_v u = w^* J u$ for all $u, w \in A_v^\perp$. If $(r, c) \in D$, then r_v maps $A_v^\perp$ onto itself: from $r_v^* J r_v = J$ and $r_v^{-1} A_v = c^{-1} A_v$, $A_v^* J r_v u = (r_v^{-1} A_v)^* J u = \bar{c}^{-1} A_v^* J u = 0$ for $u \in A_v^\perp$, and r_v is injective on the finite-dimensional space $A_v^\perp$. Hence D is a subgroup: for $(r, c), (r', c') \in D$, $r_v r'_v A_v = cc' A_v$ and $w^* J r_v r'_v u = w^* J r'_v u = w^* J u$ for $u, w \in A_v^\perp$ (as $r'_v u \in A_v^\perp$); and $r_v^{-1} A_v = c^{-1} A_v$, while for $u, w \in A_v^\perp$ the vector $u' = r_v^{-1} u$ lies in $A_v^\perp$, so $w^* J u = w^* J r_v u' = w^* J u' = w^* J r_v^{-1} u$; so $(r^{-1}, c^{-1}) \in D$. Moreover $(r, 1) \in D$ if and only if $r \in \prod_v R_v$, so Ψ satisfies the invariance hypothesis of Lemma 5.8.11.

The height. Fix $v_0 \in \Phi$ and put $H(x) = \log \|x_{v_0}^{-1} A_{v_0}\|$, defined and continuous as $A_{v_0} \neq 0$. For $(r, c) \in D$, $(rx)_{v_0}^{-1} A_{v_0} = x_{v_0}^{-1} r_{v_0}^{-1} A_{v_0} = c^{-1} x_{v_0}^{-1} A_{v_0}$, so $H(rx) = H(x) - \log|c|$. For $y, g \in G_\infty$: $\|(yg)_{v_0}^{-1} A_{v_0}\| \le \|g_{v_0}^{-1}\|\, \|y_{v_0}^{-1} A_{v_0}\|$ and $\|y_{v_0}^{-1} A_{v_0}\| \le \|g_{v_0}\|\, \|(yg)_{v_0}^{-1} A_{v_0}\|$, so with $\|g_{v_0}^{-1}\| = \|g_{v_0}\| \ge 1$,

$$|H(yg) - H(y)| \le \log \|g_{v_0}\| \le \|g_{v_0}\| \le \ell(g) := \|g\|_\infty.$$

ℓ is continuous, and $|e|\ell \le |e|\, \|\cdot\|_\infty^{B+1}$ is integrable by (EK4)(a).

Dilations near 1. Let W be a neighbourhood of 1 in G_∞; it contains $\{b : \|b_v - 1\| \le \delta \text{ for all } v\}$ for some $\delta > 0$. Put $c = 1 + \delta$, so $\log|c| > 0$. Let $x \in G_\infty$. For each v, Lemma 5.8.12 (with A_v, x_v, δ) gives $b_v \in \mathrm{U}(J)$ with $\|b_v - 1\| \le \delta$ such that $x_v b_v x_v^{-1}$ maps A_v to cA_v and satisfies $w^* J x_v b_v x_v^{-1} u = w^* J u$ for $u, w \in A_v^\perp$. Then $b = (b_v)_v \in W$ and $(xbx^{-1}, c) \in D$.

By Lemma 5.8.11 with $f = \Psi$, $\int e(x)\Psi(yx)\,dx = 0$ for every $y \in G_\infty$. □

Lemma 5.8.11. *Let G be a second countable locally compact group with a left Haar measure dg, and let $e \in L^1(G)$ satisfy $\int_G e(x)\, e(x^{-1}g)\,dx = e(g)$ for all $g \in G$. Let $H\colon G \to \mathbb{R}$ be continuous and $\ell\colon G \to [0, \infty)$ measurable with $|e|\,\ell$ integrable and*

$$|H(yg) - H(y)| \le \ell(g) \qquad (y, g \in G).$$

Let $D \subseteq G \times \mathbb{C}^\times$ be a subgroup with $H(rg) = H(g) - \log|c|$ for all $(r, c) \in D$, $g \in G$, and assume that for every neighbourhood W of 1 in G there is $c \in \mathbb{C}^\times$ with $\log|c| \neq 0$ such that for every $x \in G$ some $b \in W$ has $(xbx^{-1}, c) \in D$. Then for every bounded continuous $f\colon G \to \mathbb{C}$ with $f(rg) = f(g)$ whenever $(r, 1) \in D$ and $g \in G$,

$$\int_G e(g)\, f(xg)\,dg = 0 \qquad (x \in G).$$

Proof. Write C_b for the bounded continuous functions on G with the sup norm $\|\cdot\|$, and $\|e\|_1 = \int |e|$.

The operator T. For $\varphi \in C_b$ put $T\varphi(y) = \int e(g)\varphi(yg)\,dg$. Then $|T\varphi| \le \|e\|_1 \|\varphi\|$, and $T\varphi$ is continuous by dominated convergence (G is metrizable, φ continuous and bounded, e integrable). T is idempotent: by Fubini's theorem ($\iint |e(x)||e(x^{-1}g)|\,dg\,dx = \|e\|_1^2$ by left invariance) and the substitution $g \mapsto x^{-1}g$,

$$T(T\varphi)(y) = \int e(x) \int e(x^{-1}g)\varphi(yg)\,dg\,dx = \int \Big(\int e(x) e(x^{-1}g)\,dx \Big) \varphi(yg)\,dg = T\varphi(y).$$

Call φ *invariant* if $\varphi(rg) = \varphi(g)$ whenever $(r,1) \in D$. If φ is invariant, so is $T\varphi$ ($T\varphi(rg) = \int e(h)\varphi(rgh)\,dh$). Let $V = \{\varphi \in C_b : \ T\varphi = \varphi,\ \varphi \text{ invariant}\}$, a normed space; for invariant $\varphi \in C_b$ we have $T\varphi \in V$. As $Tf \in V$ and $Tf(x)$ is the integral of the statement, it suffices to show $V = 0$.

A translation close to 1. Left translation is continuous on $L^1(G)$, so $W = \{b \in G : \ \int |e(g) - e(b^{-1}g)|\,dg < \frac{1}{2}\}$ is a neighbourhood of 1. Let z be as in the hypothesis for W, $\tau = \log|z| \neq 0$, and for every $y \in G$ choose $b(y) \in W$ with $(yb(y)y^{-1}, z) \in D$; put $a = b(1)$, so $(a,z) \in D$. For $\varphi \in V$ let $U\varphi(y) = \varphi(ay)$. Then $U\varphi \in V$: $T(U\varphi)(y) = T\varphi(ay)$, and for $(r,1) \in D$ also $(ara^{-1},1) = (a,z)(r,1)(a,z)^{-1} \in D$, so $\varphi(arg) = \varphi(ara^{-1}\,ag) = \varphi(ag)$. We claim $\|\varphi - U\varphi\| \leq \frac{1}{2}\|\varphi\|$. Let $y \in G$ and $\beta = b(y)$. Since $(y\beta y^{-1}a^{-1},1) = (y\beta y^{-1},z)(a,z)^{-1} \in D$, invariance gives $\varphi(ay) = \varphi(y\beta y^{-1}a^{-1}\cdot ay) = \varphi(y\beta)$; with $T\varphi = \varphi$ and the substitution $g \mapsto \beta^{-1}g$,

$$\varphi(y) - \varphi(ay) = T\varphi(y) - T\varphi(y\beta) = \int (e(g) - e(\beta^{-1}g))\varphi(yg)\,dg,$$

of modulus at most $\frac{1}{2}\|\varphi\|$, as $\beta \in W$. So $\|1 - U\| \leq \frac{1}{2}$ in the algebra of bounded operators on V.

A twisted operator. Choose an integer $n \geq 1$ with $n > \frac{\pi}{|\tau|}\int |e|\ell$, put $\theta = -\pi/(\tau n)$ and $\chi(g) = \exp(i\theta H(g))$, a continuous function of modulus 1; χ is invariant, since $H(rg) = H(g)$ for $(r,1) \in D$. For $\varphi \in V$ the product $\chi\varphi$ is invariant, so $S\varphi = T(\chi\varphi) \in V$; S is a bounded operator on V. As $H(ag) = H(g) - \tau$, $\chi(ag) = \omega\chi(g)$ with $\omega = e^{i\pi/n}$, hence

$$U(S\varphi)(y) = \int e(g)\chi(ayg)\varphi(ayg)\,dg = \omega\int e(g)\chi(yg)\,(U\varphi)(yg)\,dg = \omega\,S(U\varphi)(y),$$

so $US = \omega SU$ and $US^n = \omega^n S^n U = -S^n U$.

$S^n = 0$. Let $X = S^n$, so $UX = -XU$. Then $(1-U)X + X(1-U) = 2X - UX - XU = 2X$, and $2\|X\| \leq \|1-U\|\,\|X\| + \|X\|\,\|1-U\| \leq \|X\|$, so $X = 0$.

S is injective. Let $\varphi \in V$ with $S\varphi = 0$, and $y \in G$. Then $T(\chi\varphi - \chi(y)\varphi)(y) = 0 - \chi(y)T\varphi(y) = -\chi(y)\varphi(y)$, so

$$|\varphi(y)| = \Big|\int e(g)\big(\chi(yg) - \chi(y)\big)\varphi(yg)\,dg\Big| \leq |\theta|\Big(\int |e|\ell\Big)\|\varphi\|,$$

using $|e^{it} - e^{is}| \leq |t-s|$ and $|H(yg) - H(y)| \leq \ell(g)$. Here $q = |\theta|\int |e|\ell = \frac{\pi}{|\tau|n}\int |e|\ell < 1$, so $\|\varphi\| \leq q\|\varphi\|$ and $\varphi = 0$.

Conclusion. For $\psi \in V$, $S^n\psi = 0$ and S is injective, so $\psi = 0$. Hence $V = 0$, $Tf = 0$, and $\int e(g)f(xg)\,dg = Tf(x) = 0$. □

Lemma 5.8.12. *Let $N \geq 1$, $s \in \{\pm 1\}^N$, $J = \mathrm{diag}(s)$ and $\mathrm{U}(J) = \{y \in \mathrm{GL}_N(\mathbb{C}) : y^*Jy = J\}$ ($y^* = \bar{y}^{\mathsf{T}}$), and write $\|\cdot\|$ for the operator norm. Let $A \in \mathrm{M}_N(\mathbb{C})$ with $A^*JA = 0$, $x \in \mathrm{U}(J)$ and $c > 0$. Then there is $y \in \mathrm{U}(J)$ with $\|y - 1\| \leq c$ such that $g = xyx^{-1}$ satisfies $gA = (1+c)A$ and $w^*Jgv = w^*Jv$ for all $v, w \in \mathbb{C}^N$ with $A^*Jv = A^*Jw = 0$.*

Proof. Note $J^* = J$, $J^2 = 1$, and $x^{-1} = Jx^*J$. Let $A' = x^{-1}A$ and $L \subseteq \mathbb{C}^N$ its column space; L is isotropic for J, since $A'^*JA' = A^*(x^{-1})^*Jx^{-1}A = A^*JA = 0$. Let p be the orthogonal projection (for the standard hermitian product) onto L, $p = PP^*$ for a matrix P whose columns are an orthonormal basis of L, and $q = JpJ$, the orthogonal projection onto JL. Isotropy gives $P^*JP = 0$, hence

$$pJp = 0, \qquad pq = pJpJ = 0, \qquad qp = (pq)^* = 0, \qquad pJu = 0 \text{ whenever } P^*Ju = 0. \tag{1}$$

Put $t = (1+c)^{-1} \in (0,1)$ and $y = 1 + c(p - tq)$.

$y \in \mathrm{U}(J)$. As p, q are self-adjoint, $y^* = y$. Using $qJ = Jp$, $Jq = pJ$, $p^2 = p$ and (1): $pJq = pJ$, $qJp = Jp$, $qJq = JpJpJ = 0$, so $(p - tq)J(p - tq) = -t(pJ + Jp)$, and

$$y^*Jy = J + c(pJ + Jp) - ct(qJ + Jq) + c^2(p - tq)J(p - tq) = J + (c - ct - c^2t)(pJ + Jp) = J,$$

since $ct + c^2t = c$. So y is invertible and lies in $\mathrm{U}(J)$.

$\|y - 1\| \le c$. By (1) the ranges of p and q are orthogonal, and $p - tq$ is 1 on the range of p, $-t$ on that of q and 0 on the orthogonal of both; so $\|p - tq\| \le \max(1, t) = 1$ and $\|y - 1\| = c\|p - tq\| \le c$.

The dilation. $pA' = A'$, and $qA' = Jp(JA') = 0$ by (1), since $P^*JA' = 0$ (L is isotropic and contains the columns of A'). So $yA' = (1 + c)A'$ and $gA = xyA' = (1 + c)A$.

Trivial on $A^\perp$ modulo A. Let v, w with $A^*Jv = A^*Jw = 0$ and $v' = x^{-1}v$, $w' = x^{-1}w$. Then $A'^*Jv' = A^*Jv = 0$, so $P^*Jv' = 0$ (the columns of P are combinations of those of A'), and likewise for w'. As $w^*Jx = w'^*x^*Jx = w'^*J$,

$$w^*Jgv - w^*Jv = c\, w'^*J(p - tq)v' = c((pJw')^*v' - t\, w'^*(pJv')) = 0$$

by (1), using $Jq = pJ$ and $(Jp)^* = pJ$. □

Proposition 5.8.13. *Let E be a CM field with maximal totally real subfield E^+, $f = [E^+ : \mathbb{Q}]$, root discriminant $\mathrm{rd}_E = |d_E|^{1/(2f)}$ and different $\mathfrak{D}_E$; let $N \ge 1$. Say that a framed space U of rank N over E with hermitian form h has a* cusp frame of size δ at $g \in G^U = G_\infty \times G_f$ covered by $J \subseteq G_f$ *if there are a nonzero isotropic subspace $A \subseteq U$ and reals $0 \le a \le b$ with $ab \le \delta$ such that A is spanned by vectors $u \in L_g$ with $|u|_g \le a$, $A^\perp$ is spanned by vectors $w \in L_g$ with $|w|_g \le b$, and $N_A(\mathbb{A}_{E^+})\, g \subseteq N_A(E^+)\, g\, Z_\delta\, J$.*

(i) Let $N_D \ge 1$ be a rational integer in $\mathfrak{D}_E$, $\kappa = 8fN\,\mathrm{rd}_E + 48N^6f^2N_D + 1$, $M \ge 1$ an integer and $0 < \delta \le 1$. If U is a framed space of rank N over E and $g \in G^U$ satisfies $\lambda_U(g) < (\delta/(M\kappa^2))^N$, then U has a cusp frame of size δ at g covered by $\mathrm{K}(M)$.

(ii) For every open subgroup $J \subseteq G_f$ there is $\kappa_J > 0$, depending only on E, N and J, with the following property. Let U be a framed space of rank N over E, $g \in G^U$ and $\delta > 0$ with $\delta^N = \kappa_J\lambda_U(g)$. If $\delta \le 1$, then U has a cusp frame of size δ at g covered by J.

Proof. (i). We have $\kappa \ge 1$ and $\kappa^2 \ge \kappa \ge fN\,\mathrm{rd}_E$. Put $\theta = \delta/(M\kappa^2)$; then $0 \le \theta\kappa = \delta/(M\kappa) \le 1$ and $\lambda_U(g) < \theta^N$. By Lemma 5.8.14 there are a nonzero isotropic A and $0 \le a \le b$ with A, $A^\perp$ spanned by vectors of L_g of lengths at most a, b, and $ab \le \theta\kappa = \delta/(M\kappa)$. As $\kappa \ge 48N^6f^2N_D \ge 6N^2fN_D$, $ab \le \delta/(6MN^2fN_D)$, so Lemma 5.8.17 gives $N_A(\mathbb{A}_{E^+})g \subseteq N_A(E^+)gZ_\delta\mathrm{K}(M)$; and $ab \le \delta/(M\kappa) \le \delta$.

(ii). The principal levels $\mathrm{K}(M)$, $M \ge 1$, form a basis of neighbourhoods of 1 in G_f (the sets $1 + M\,\mathrm{M}_N(\widehat{\mathcal{O}}_E)$ form one in $\mathrm{GL}_N(\mathbb{A}_{E,f})$, and G_f carries the subspace topology), so there is $M \ge 1$ with $\mathrm{K}(M) \subseteq J$. Let $N_D = |d_E|$, which lies in $\mathfrak{D}_E$ (the norm of an ideal lies in the ideal, and $\mathrm{N}(\mathfrak{D}_E) = |d_E|$), let κ be as in (i) and put $c = M\kappa^2$ and $\kappa_J = 2c^N$; these depend only on E, N, J. Let U, g, δ be as in (ii) with $\delta \le 1$. The first minimum is positive (the nonzero points of the lattice $L_{g,\infty}$ of Lemma 5.8.15(iii) have norm bounded below), so $\lambda_U(g) = \delta^N/(2c^N) < (\delta/c)^N$, and (i) gives a cusp frame of size δ at g covered by $\mathrm{K}(M) \subseteq J$, hence by J. □

Lemma 5.8.14. *Let E be a CM field with maximal totally real subfield E^+, $f = [E^+ : \mathbb{Q}]$ and root discriminant $\mathrm{rd}_E = |d_E|^{1/(2f)}$; let $N \ge 1$ and U a framed space of rank N over E with hermitian form h, and $G^U = G_\infty \times G_f$. Let κ, θ be real numbers with $\kappa^2 \ge fN\,\mathrm{rd}_E$ and*

$0 \le \theta\kappa \le 1$, *and let* $g \in G^U$ *with* $\lambda_U(g) < \theta^N$. *Then there are a nonzero isotropic subspace* $A \subseteq U$ *and real numbers* $0 \le a \le b$ *with* $ab \le \theta\kappa$ *such that* A *is spanned over* E *by vectors* $u \in L_g$ *with* $|u|_g \le a$, *and* $A^\perp = \{w \in U : h(u,w) = 0 \text{ for all } u \in A\}$ *is spanned over* E *by vectors* $w \in L_g$ *with* $|w|_g \le b$.

Proof. As $\kappa^2 \ge fN \operatorname{rd}_E \ge 1$, $\kappa \ge 1$, so $0 \le \theta \le 1$ and $\theta^N \le 1$; since $\lambda_U(g) = \min(1, \inf\{|u|_g : 0 \ne u \in L_g\}) < \theta^N \le 1$ (the set L_g being that of Lemma 5.8.15), there is $\xi \in L_g$, $\xi \ne 0$, with $|\xi|_g < \theta^N$. Let $y_1, \dots, y_N$ be the basis of Lemma 5.8.16 and $\alpha_j = |y_j|_g$, so $\alpha_1 \le \cdots \le \alpha_N$ and $\prod_j \alpha_j \le (fN \operatorname{rd}_E)^{N/2} \le \kappa^N$.

The longest vector is long: $|\xi|_g \, \alpha_N \ge 1$. Otherwise $|\xi|_g |y_j|_g \le |\xi|_g \alpha_N < 1$ for every j, so $h(\xi, y_j) = 0$ by Lemma 5.8.15(ii); as the y_j span U and h is nondegenerate, $\xi = 0$, a contradiction. In particular all $\alpha_j > 0$ (each $y_j \ne 0$, and $|u|_g > 0$ for $u \ne 0$ because $u \mapsto (g^{-1}c(u))_\infty$ is injective).

The other lengths have a small product. Hence

$$\prod_{j=1}^{N-1} \alpha_j = \frac{\prod_{j=1}^{N} \alpha_j}{\alpha_N} \le \kappa^N |\xi|_g < (\theta\kappa)^N.$$

Put $c = \theta\kappa \in [0,1]$. The left side is positive, so $c > 0$; and $N \ge 2$, since for $N = 1$ the empty product 1 would be $< c \le 1$. Let $k = N - 1 \ge 1$ and $P = \prod_{j=1}^{k} \alpha_j$; then $P^2 < c^{2k+2} \le c^k$ as $0 < c \le 1$.

Pairing from both ends. For $1 \le j \le k$ put $j' = N - j$, so $j \mapsto j'$ is an involution of $\{1, \dots, k\}$ and $\prod_{j=1}^{k} \alpha_j \alpha_{j'} = P^2 < c^k$. Hence $\alpha_j \alpha_{j'} < c$ for some j (otherwise the product of the k factors, each $\ge c > 0$, would be $\ge c^k$). Replacing j by j' if necessary, there is $i \in \{1, \dots, k\}$ with $\alpha_i \le \alpha_{i'}$ and $\alpha_i \alpha_{i'} < c$, where $i' = N - i$.

The frame. Let A be the E-span of $y_1, \dots, y_i$ and B that of $y_1, \dots, y_{i'}$; $\dim A = i \ge 1$ and $\dim B = N - i$. For $p \le i$ and $q \le i'$ we have $\alpha_p \alpha_q \le \alpha_i \alpha_{i'} < c \le 1$, and for $p, q \le i$ we have $\alpha_p \alpha_q \le \alpha_i^2 \le \alpha_i \alpha_{i'} < 1$; by Lemma 5.8.15(ii), $h(y_p, y_q) = 0$ in both cases. So A is isotropic and $B \subseteq A^\perp$; as h is nondegenerate, $\dim A^\perp = N - i = \dim B$, so $B = A^\perp$. With $a = \alpha_i$ and $b = \alpha_{i'}$: A is spanned by $y_1, \dots, y_i \in L_g$ of lengths $\le a$, $A^\perp$ by $y_1, \dots, y_{i'} \in L_g$ of lengths $\le b$, $0 \le a \le b$ and $ab < \theta\kappa$. □

Lemma 5.8.15. *Let* E *be a CM field with maximal totally real subfield* E^+, $f = [E^+ : \mathbb{Q}]$, *ring of integers* $\mathcal{O}_E$ *and discriminant* d_E, *and let* Φ *be the set of infinite places of* E^+. *Let* $N \ge 1$ *and let* U *be a framed space of rank* N *over* E *with hermitian form* h *(conjugate-linear in the first argument) and* $G^U = G_\infty \times G_f$. *Give* $\prod_{v \in \Phi} \mathbb{C}^N$ *the real inner product* $\langle a, b \rangle = \operatorname{Re} \sum_{v \in \Phi} \sum_{i=1}^{N} \bar{a}_{v,i} b_{v,i}$, *its norm* $\|\cdot\|$ *and the Lebesgue measure of* $\langle\,,\rangle$. *For* $g \in G^U$ *let* $L_g = L_g(U)$ *and* $|u|_g$ *be the lattice and the length at* g *of (VP4), that is,*

$$L_g = \{u \in U : \ (g^{-1}c(u))_f \in \widehat{\mathcal{O}}_E^N\}, \qquad |u|_g = \|(g^{-1}c(u))_\infty\|,$$

and put $L_{g,\infty} = \{(g^{-1}c(u))_\infty : \ u \in L_g\}$. *Then for every* $g \in G^U$:

(i) L_g *is a finitely generated* $\mathcal{O}_E$*-submodule of* U *that spans* U *over* E, *and* $L_g = \{w \in U : h(u,w) \in \mathcal{O}_E \text{ for all } u \in L_g\}$;

(ii) if $u, w \in L_g$ *and* $|u|_g \, |w|_g < 1$, *then* $h(u,w) = 0$;

(iii) $L_{g,\infty}$ *is a lattice in the real vector space* $\prod_{v \in \Phi} \mathbb{C}^N$ *of dimension* $2fN$, *and its covolume is* $2^{-fN} |d_E|^{N/2}$;

(iv) the dual lattice $L^*_{g,\infty} = \{y : \langle y, z\rangle \in \mathbb{Z} \text{ for all } z \in L_{g,\infty}\}$ *is* $\{\sigma((g^{-1}c(w))_\infty) : \ w \in 2\mathfrak{D}_E^{-1}L_g\}$*, where* $\mathfrak{D}_E$ *is the different of* E *and* σ *multiplies the coordinate* (v,i) *by the sign* s_i*.*

Proof. Fix an E-basis of U, so that $U = E^N$ and $h(u,w) = \bar{u}^{\mathsf{T}} J w$ with an invertible hermitian $J \in \mathrm{M}_N(E)$, and let $x \in \mathrm{GL}_N(\mathbb{A}_E)$ be the matrix of the frames, so that $c(u) = x^{-1}u$ and $\bar{x}^{\mathsf{T}} J x = D_s$, the adelic matrix equal to $\mathrm{diag}(s)$ at every $v \in \Phi$ and to 1 at the finite places. For $u \in E^N$ put $\alpha(u) = g^{-1}x^{-1}u \in \mathbb{A}_E^N$. As g and g^{-1} preserve D_s,

$$h(u,w) = \overline{\alpha(u)}^{\mathsf{T}} D_s\, \alpha(w) \qquad (u, w \in E^N), \tag{1}$$

an identity in $\mathbb{A}_E$, in which E is embedded diagonally. At $v \in \Phi$, with $\Phi_v \colon E \to \mathbb{C}$ the chosen embedding, (1) reads $\Phi_v(h(u,w)) = \sum_i s_i \overline{\alpha(u)_{v,i}}\, \alpha(w)_{v,i}$; at the finite places it reads $h(u,w) = \overline{\alpha(u)_f}^{\mathsf{T}} \alpha(w)_f$.

(i). We have $L_g = E^N \cap (xg)_f \widehat{\mathcal{O}}_E^N$. The set $(xg)_f\widehat{\mathcal{O}}_E^N$ is a compact open $\widehat{\mathcal{O}}_E$-submodule of $\mathbb{A}_{E,f}^N$, so there are nonzero integers m, m' with $m'\widehat{\mathcal{O}}_E^N \subseteq (xg)_f\widehat{\mathcal{O}}_E^N \subseteq m^{-1}\widehat{\mathcal{O}}_E^N$; as $E \cap \widehat{\mathcal{O}}_E = \mathcal{O}_E$, this gives $m'\mathcal{O}_E^N \subseteq L_g \subseteq m^{-1}\mathcal{O}_E^N$. So L_g is an $\mathcal{O}_E$-module ($\widehat{\mathcal{O}}_E$ is stable under $\mathcal{O}_E$), finitely generated ($\mathcal{O}_E$ is noetherian) and spans E^N. For $u, w \in L_g$ the finite form of (1) gives $h(u,w) \in \widehat{\mathcal{O}}_E \cap E = \mathcal{O}_E$. Conversely let $w \in E^N$ with $h(u,w) \in \mathcal{O}_E$ for all $u \in L_g$. The set $\{\alpha(u)_f : u \in L_g\} = \widehat{\mathcal{O}}_E^N \cap (g^{-1}x^{-1})_f E^N$ is dense in $\widehat{\mathcal{O}}_E^N$, since E^N is dense in $\mathbb{A}_{E,f}^N$ and $\widehat{\mathcal{O}}_E^N$ is open. So $\bar{a}^{\mathsf{T}}\alpha(w)_f \in \widehat{\mathcal{O}}_E$ for all a in a dense subset of $\widehat{\mathcal{O}}_E^N$, hence, $\widehat{\mathcal{O}}_E$ being closed, for all $a \in \widehat{\mathcal{O}}_E^N$; taking for a the unit vectors, $\alpha(w)_f \in \widehat{\mathcal{O}}_E^N$, that is, $w \in L_g$.

(ii). Let $t = h(u,w)$; by (i), $t \in \mathcal{O}_E$. For $v \in \Phi$, by (1) and the Cauchy–Schwarz inequality, $|\Phi_v(t)| \le \|\alpha(u)_v\|\,\|\alpha(w)_v\| \le |u|_g |w|_g < 1$, and the conjugate embedding gives the same modulus. Every embedding of E into $\mathbb{C}$ is Φ_v or its conjugate for some $v \in \Phi$, so $|\mathrm{N}_{E/\mathbb{Q}}(t)| = \prod_{v\in\Phi} |\Phi_v(t)|^2 < 1$; as $\mathrm{N}_{E/\mathbb{Q}}(t) \in \mathbb{Z}$, it is 0, and $t = 0$.

(iii). Let $\beta \colon E^N \otimes_{\mathbb{Q}} \mathbb{R} \to \prod_{v\in\Phi} \mathbb{C}^N$ be the $\mathbb{R}$-linear extension of $u \mapsto \alpha(u)_\infty$. It is the composite of the Minkowski isomorphism $E^N \otimes_{\mathbb{Q}} \mathbb{R} \to \prod_{v\in\Phi} \mathbb{C}^N$, $u \mapsto (\Phi_v(u))_v$ (all infinite places of E are complex, one above each $v \in \Phi$), with the $\mathbb{R}$-linear automorphism $(g^{-1}x^{-1})_\infty$; so β is an isomorphism, and $L_{g,\infty} = \beta(L_g)$ is a lattice, since L_g lies between the lattices $m'\mathcal{O}_E^N$ and $m^{-1}\mathcal{O}_E^N$ of $E^N \otimes \mathbb{R}$. The map σ of (iv) is an isometry of $\prod_v \mathbb{C}^N$. Since $\mathrm{Tr}_{E/\mathbb{Q}}(t) = 2\sum_{v\in\Phi} \mathrm{Re}\,\Phi_v(t)$, (1) gives

$$\mathrm{Tr}_{E/\mathbb{Q}}\, h(u,w) = 2\,\langle \beta(u), \sigma\beta(w)\rangle \qquad (u, w \in E^N). \tag{2}$$

The dual lattice. The form $B(u,w) = \frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}} h(u,w)$ is $\mathbb{Q}$-bilinear and nondegenerate on E^N (J is invertible and the trace form of $E/\mathbb{Q}$ is nondegenerate). Let $u_1, \dots, u_{2fN} \in L_g$ be a $\mathbb{Z}$-basis of L_g (a $\mathbb{Q}$-basis of E^N) and $w_1, \dots, w_{2fN} \in E^N$ the dual basis for B. By (2), $\langle \beta(u_k), \sigma\beta(w)\rangle = B(u_k, w)$ for $w \in E^N \otimes \mathbb{R}$ ($\mathbb{R}$-linear extension). Every $y \in \prod_v \mathbb{C}^N$ is $\sigma\beta(w)$ for one $w = \sum_k B(u_k,w) w_k \in E^N \otimes \mathbb{R}$, and y lies in the dual lattice $L^*_{g,\infty} = \{y : \langle y, z\rangle \in \mathbb{Z} \text{ for } z \in L_{g,\infty}\}$ if and only if all $B(u_k, w) \in \mathbb{Z}$. Hence $L^*_{g,\infty} = \sigma\beta(M)$ with

$$M = \{w \in E^N : \ \mathrm{Tr}_{E/\mathbb{Q}} h(u,w) \in 2\mathbb{Z} \text{ for all } u \in L_g\}.$$

$M = 2\mathfrak{D}_E^{-1}L_g$, with $\mathfrak{D}_E$ the different of E. As L_g is an $\mathcal{O}_E$-module and $h(tu, w) = \bar{t}\, h(u,w)$, $w \in M$ if and only if $\mathrm{Tr}_{E/\mathbb{Q}}(t\, h(u, \frac{w}{2})) \in \mathbb{Z}$ for all $t \in \mathcal{O}_E$ and $u \in L_g$, that is, $h(u, \frac{w}{2}) \in \mathfrak{D}_E^{-1}$ for all $u \in L_g$. We show $\{w' \in E^N : h(L_g, w') \subseteq \mathfrak{D}_E^{-1}\} = \mathfrak{D}_E^{-1}L_g$. The inclusion $\supseteq$ holds since $h(u, tw'') = t\, h(u, w'') \in \mathfrak{D}_E^{-1}$ for $t \in \mathfrak{D}_E^{-1}$, $u, w'' \in L_g$. Conversely, if $h(L_g, w') \subseteq \mathfrak{D}_E^{-1}$, then $h(L_g, tw') \subseteq \mathcal{O}_E$ for $t \in \mathfrak{D}_E$, so $tw' \in L_g$ by (i); writing $1 = \sum_k a_k t_k$ with $a_k \in \mathfrak{D}_E^{-1}$, $t_k \in \mathfrak{D}_E$, we get $w' = \sum_k a_k (t_k w') \in \mathfrak{D}_E^{-1}L_g$.

Covolumes. Write $\mathrm{cv}(X)$ for the covolume of a lattice X. The $\mathcal{O}_E$-module L_g is torsion free of rank N, so at every prime $\mathfrak{p}$ of $\mathcal{O}_E$ its localisation is free of rank N over the discrete valuation ring $\mathcal{O}_{E,\mathfrak{p}}$; the finite module $\mathfrak{D}_E^{-1}L_g/L_g$ is the direct sum of its localisations $\mathfrak{p}^{-e_\mathfrak{p}}L_{g,\mathfrak{p}}/L_{g,\mathfrak{p}} \cong (\mathcal{O}_{E,\mathfrak{p}}/\mathfrak{p}^{e_\mathfrak{p}})^N$, $e_\mathfrak{p} = v_\mathfrak{p}(\mathfrak{D}_E)$, so $[\mathfrak{D}_E^{-1}L_g : L_g] = \mathrm{N}(\mathfrak{D}_E)^N = |d_E|^N$ (Dedekind: the norm of the different is $|d_E|$). Hence, σ preserving volume and the real dimension being $2fN$,

$$\mathrm{cv}(L_{g,\infty}^*) = \mathrm{cv}(\beta(2\mathfrak{D}_E^{-1}L_g)) = 2^{2fN}\,|d_E|^{-N}\,\mathrm{cv}(L_{g,\infty}).$$

As $\mathrm{cv}(L_{g,\infty}^*)\,\mathrm{cv}(L_{g,\infty}) = 1$, we get $\mathrm{cv}(L_{g,\infty})^2 = 2^{-2fN}|d_E|^N$, which is (iii). The two displays $L_{g,\infty}^* = \sigma\beta(M)$ and $M = 2\mathfrak{D}_E^{-1}L_g$ are (iv). □

Lemma 5.8.16. *Let E be a CM field with maximal totally real subfield E^+, $f = [E^+ : \mathbb{Q}]$, discriminant d_E and root discriminant $\mathrm{rd}_E = |d_E|^{1/(2f)}$; let $N \geq 1$, U a framed space of rank N over E and $G^U = G_\infty \times G_f$. For every $g \in G^U$ there is a basis $y_1, \dots, y_N$ of U over E with $y_j \in L_g$, $|y_1|_g \leq |y_2|_g \leq \cdots \leq |y_N|_g$ and*

$$\prod_{j=1}^{N} |y_j|_g \leq (fN\,\mathrm{rd}_E)^{N/2}.$$

Proof. Identify $U = E^N$ by an E-basis and let $x \in \mathrm{GL}_N(\mathbb{A}_E)$ be the matrix of the frames in this basis, so that $c(u) = x^{-1}u$. Put $a = x_f g_f \in \mathrm{GL}_N(\mathbb{A}_f)$ and $r_v = x_v g_v \in \mathrm{GL}_N(\mathbb{C})$ for $v \in \Phi$. In the notation of Lemma 5.3.83 (with $s = N$),

$$\Lambda = \{u \in E^N :\ g_f^{-1}x_f^{-1}u \in \widehat{\mathcal{O}}_E^N\} = L_g,$$
$$\|u\|_r = \Big(\sum_{v\in\Phi} \|g_v^{-1}x_v^{-1}v(u)\|^2\Big)^{1/2} = \|(g^{-1}c(u))_\infty\| = |u|_g,$$

and $j(\Lambda) = L_{g,\infty}$, for the same euclidean structure on $\prod_v \mathbb{C}^N$.

The determinant factor is 1. By Lemma 5.8.15 (iii), $L_{g,\infty}$ has covolume $2^{-fN}|d_E|^{N/2}$; by (i) of Lemma 5.3.83, it has covolume $2^{-fN}|d_E|^{N/2}[\det a]\prod_v |\det r_v|^{-2}$. Hence $\prod_v |\det r_v|^2\,[\det a]^{-1} = 1$.

The basis. By (ii) of Lemma 5.3.83 there is a basis $y_1, \dots, y_N$ of U over E with $y_j \in \Lambda = L_g$ and $\prod_j |y_j|_g^{2f} = \prod_j \|y_j\|_r^{2f} \leq (fN\,\mathrm{rd}_E)^{fN}$, that is $\prod_j |y_j|_g \leq (fN\,\mathrm{rd}_E)^{N/2}$. Renumbering the y_j by increasing length changes neither the product nor the other properties. □

Lemma 5.8.17. *Let E be a CM field with maximal totally real subfield E^+, $f = [E^+ : \mathbb{Q}]$ and different $\mathfrak{D}_E$, and let $N_D \geq 1$ be a rational integer with $N_D \in \mathfrak{D}_E$. Let $N \geq 1$, U a framed space of rank N over E with hermitian form h, $G^U = G_\infty \times G_f$, and for an integer $M \geq 1$ let $\mathrm{K}(M) = \{k \in \mathrm{K}_f :\ k - 1 \in M\,\mathrm{M}_N(\widehat{\mathcal{O}}_E)\}$ be the principal level M. Let $0 < \delta \leq 1$, $g \in G^U$, $A \subseteq U$ a nonzero isotropic subspace and $0 \leq a \leq b$ real numbers such that A is spanned by vectors $u \in L_g$ with $|u|_g \leq a$, $A^\perp$ is spanned by vectors $w \in L_g$ with $|w|_g \leq b$, and*

$$ab \leq \frac{\delta}{6MN^2fN_D}.$$

Then

$$N_A(\mathbb{A}_{E^+})\,g \subseteq N_A(E^+)\,g\,Z_\delta\,\mathrm{K}(M).$$

Proof. Coordinates. Fix an E-basis $U = E^N$, with $h(u,w) = u^*Jw$ for an invertible hermitian $J \in \mathrm{M}_N(E)$ ($P^* = \bar{P}^{\mathsf{T}}$), and let $x \in \mathrm{GL}_N(\mathbb{A}_E)$ be the matrix of the frames: $c(u) = x^{-1}u$, $x^*Jx = D_s$, the adelic matrix equal to $\mathrm{diag}(s)$ at the infinite places and to 1 at the finite places, and $G^U = \{k \in \mathrm{GL}_N(\mathbb{A}_E) : k^*D_sk = D_s\}$. Put $p = xg$; then $p^*Jp = g^*D_sg = D_s$, that is, $J = (p^{-1})^*D_sp^{-1}$. Choose a basis of A among the given spanning vectors and let $X \in \mathrm{M}_N(E)$ have these vectors as its first columns and 0 as its other columns; likewise $Y \in \mathrm{M}_N(E)$ with a basis of $A^\perp$ chosen among the given spanning vectors. Then $X^*JX = 0$, $X^*JY = 0$, the columns of X and Y lie in L_g, and the columns of $\tilde{X} = p^{-1}X$ and $\tilde{Y} = p^{-1}Y$ are the vectors $g^{-1}c(u)$ of these columns. Let $\mathcal{N}$ and ρ be as in Lemma 5.8.18 for J, X. The unipotent radical of the stabiliser of the isotropic subspace A consists of the isometries that fix A pointwise and act trivially on $A^\perp/A$; for $R = E$ or $R = \mathbb{A}_E$ the second condition on u reads $w^*J(u-1)v = 0$ for $v, w \in R^N$ orthogonal to A, because $A \otimes_E R = \{z : \ w^*Jz = 0 \text{ for all } w \in A^\perp \otimes_E R\}$ ($A = (A^\perp)^\perp$ and R is free over E), and orthogonality to A is orthogonality to the columns of X. Since $G^U = x^{-1}\,\mathrm{U}(J)(\mathbb{A}_E)\,x$ through the frames, $k \mapsto xkx^{-1}$ maps $N_A(\mathbb{A}_{E^+})$ onto $\mathcal{N}(\mathbb{A}_E)$ and $N_A(E^+)$ onto $\mathcal{N}(E)$.

A splitting. There are $e, b' \in \mathrm{M}_N(E)$ with $Ye + b'X^*J = 1$: the kernel of $v \mapsto X^*Jv$ is $A^\perp$, the column space of Y; for a complement C of $A^\perp$ in E^N this map is injective on C, so there is b' with $b'X^*Jv = v$ for $v \in C$, and there is e with Ye the projection onto $A^\perp$ along C; then $v = Yev + b'X^*J(v - Yev) = Yev + b'X^*Jv$. Put $\beta_0 = b'/(2M)$, so $Ye + 2(M\beta_0)X^*J = 1$, and apply Lemma 5.8.18 with $b = M\beta_0$.

Reduction. Let $k \in N_A(\mathbb{A}_{E^+})$ and $n = xkx^{-1} \in \mathcal{N}(\mathbb{A}_E)$. By Lemma 5.8.18(iii), $n = \rho(Yc')$ with $c' = e(n - 1 - MX\beta_0^*Jn)M\beta_0 = Mc$, $c = e(n - 1 - MX\beta_0^*Jn)\beta_0 \in \mathrm{M}_N(\mathbb{A}_E)$. Applying Lemma 5.8.19 to each entry, choose $r(c) \in \mathrm{M}_N(E)$ such that $\Delta c = c - r(c)$ has finite part in $\mathrm{M}_N(\widehat{\mathcal{O}}_E)$ and all entries of modulus $\le \frac{1}{2}fN_D$ at every $v \in \Phi$. Put $c_2 = -M\,r(c)^*Y^*JYc \in \mathrm{M}_N(\mathbb{A}_E)$, choose $r(c_2)$ in the same way, $\Delta c_2 = c_2 - r(c_2)$, and let

$$Z_0 = M\big(Xr(c_2) - Yr(c)\big) \in \mathrm{M}_N(E), \qquad X^*JZ_0 = 0.$$

By Lemma 5.8.18(i) for $R = E$, $\nu_0 = \rho(Z_0) \in \mathcal{N}(E)$. As $Z_0^*J(MYc) = M^2\big(r(c_2)^*X^*JYc - r(c)^*Y^*JYc\big) = Mc_2$, Lemma 5.8.18(ii) gives

$$\nu_0 n = \rho\big(Z_0 + MYc - X(Mc_2)\big) = \rho(V), \qquad V = M\big(Y\Delta c - X\Delta c_2\big).$$

Conjugation by p. Let $T = VX^*$ and $\tilde{t} = (\tilde{Y}\Delta c - \tilde{X}\Delta c_2)\tilde{X}^*$, so $p^{-1}V = M(\tilde{Y}\Delta c - \tilde{X}\Delta c_2)$. From $J = (p^{-1})^*D_sp^{-1}$: $p^{-1}TJp = M\tilde{t}D_s$, $p^{-1}T^*Jp = \tilde{X}(p^{-1}V)^*D_s = M\tilde{t}^*D_s$ and $p^{-1}T^*JTJp = \tilde{X}(p^{-1}V)^*D_s(p^{-1}V)\tilde{X}^*D_s = M^2\tilde{t}^*D_s\tilde{t}D_s$. Hence

$$\kappa_0 := p^{-1}\nu_0 np = 1 + 2(M\tilde{t} - M\tilde{t}^* - M^2\tilde{t}^*D_s\tilde{t})D_s,$$

and $\kappa_0 = g^{-1}(x^{-1}\nu_0 x)\,k\,g \in G^U$.

Finite part. The columns of $\tilde{X}_f$, $\tilde{Y}_f$ lie in $\widehat{\mathcal{O}}_E^N$ (columns of X, Y lie in L_g), $\Delta c_f, (\Delta c_2)_f \in \mathrm{M}_N(\widehat{\mathcal{O}}_E)$ and $(D_s)_f = 1$, so $\tilde{t}_f \in \mathrm{M}_N(\widehat{\mathcal{O}}_E)$ and $(\kappa_0)_f - 1 \in M\,\mathrm{M}_N(\widehat{\mathcal{O}}_E)$. As $(\kappa_0)_f \in G_f$ is integral, $(\kappa_0)_f \in \mathrm{K}_f$, hence $(\kappa_0)_f \in \mathrm{K}(M)$.

Infinite part. Let $v \in \Phi$ and write $\|\cdot\|$ for the operator norm. A column of $\tilde{X}_v$ is $(g^{-1}c(u))_v$ with $\|(g^{-1}c(u))_v\| \le |u|_g \le a$, so $\|\tilde{X}_v\| \le \sqrt{N}\,a$ (Frobenius norm); likewise $\|\tilde{Y}_v\| \le \sqrt{N}\,b$, and $\|\Delta c_v\|, \|(\Delta c_2)_v\| \le N \cdot \frac{1}{2}fN_D$. With $a \le b$,

$$\|\tilde{t}_v\| \le (\sqrt{N}b + \sqrt{N}a)\tfrac{1}{2}NfN_D \cdot \sqrt{N}a \le N^2fN_D\,ab \le \frac{\delta}{6M}.$$

As $\|\mathrm{diag}(s)\| = 1$ and $\delta \le 1$, $\|(\kappa_0)_v - 1\| \le 2\big(\frac{\delta}{6} + \frac{\delta}{6} + \frac{\delta^2}{36}\big) = \frac{2\delta}{3} + \frac{\delta^2}{18} \le \delta$. So $(\kappa_0)_\infty \in Z_\delta$.

Conclusion. With $\nu = x^{-1}\nu_0 x \in N_A(E^+)$ we have $g^{-1}\nu kg = \kappa_0 \in Z_\delta\mathrm{K}(M)$ (G_∞ and G_f embedded as $z \mapsto (z,1)$, $k \mapsto (1,k)$), so $kg = \nu^{-1}g\kappa_0 \in N_A(E^+)\,g\,Z_\delta\,\mathrm{K}(M)$. □

Lemma 5.8.18. *Let E be a CM field and R a commutative E-algebra with an involution $r \mapsto \bar{r}$ extending complex conjugation of E (we use $R = E$ and $R = \mathbb{A}_E$); for a matrix P over R put $P^* = \bar{P}^{\mathsf{T}}$. Let $N \geq 1$, $J \in \mathrm{GL}_N(E)$ with $J^* = J$, and $X, Y, e, b \in \mathrm{M}_N(E)$ with*

$$X^*JX = 0, \qquad X^*JY = 0, \qquad Ye + 2bX^*J = 1.$$

Let $\mathcal{N}(R)$ be the set of $u \in \mathrm{M}_N(R)$ with $u^*Ju = J$, $uX = X$ and $w^*Juv = w^*Jv$ for all $v, w \in R^N$ with $X^*Jv = X^*Jw = 0$. For $W \in \mathrm{M}_N(R)$ with $X^*JW = 0$ put $T = WX^*$ and

$$\rho(W) = 1 + 2\,(T - T^* - T^*JT)\,J.$$

Then:

(i) $\rho(W) \in \mathcal{N}(R)$;

(ii) if also $X^*JZ = 0$, *then* $\rho(W)\rho(Z) = \rho(W + Z - X(W^*JZ))$;

(iii) every $u \in \mathcal{N}(R)$ *equals* $\rho(Yc)$ *with* $c = e\,(u - 1 - Xb^*Ju)\,b \in \mathrm{M}_N(R)$.

Proof. Throughout, P^* is additive, $(PQ)^* = Q^*P^*$ (R is commutative) and $J^* = J$. From $X^*JX = 0$ and $X^*JW = 0$ we get, by applying $*$, $X^*JX = 0$ and $W^*JX = 0$.

Notation for (i) and (ii). For W with $X^*JW = 0$ put $T_W = WX^*$ and $S_W = (T_W - T_W^*)J$. For two such W, Z:

$$\begin{aligned} T_W J T_Z &= W(X^*JZ)X^* = 0, \qquad & T_W^* J T_Z^* &= X(W^*JX)Z^* = 0, \\ T_W J T_Z^* &= W(X^*JX)Z^* = 0, \qquad & T_W^* J T_Z &= X(W^*JZ)X^*, \end{aligned}$$

so

$$\begin{aligned} &S_W S_Z = -X(W^*JZ)X^*J, \qquad S_W X = W(X^*JX) - X(W^*JX) = 0, \\ &X^*JS_W = \big(X^*JW\,X^* - X^*JX\,W^*\big)J = 0. \end{aligned} \tag{1}$$

By (1) every product of three of the matrices $S_W, S_Z, \ldots$ vanishes (the middle factor is killed on the left by X^*J or on the right by X), and $S_W^2 = -T_W^*JT_WJ$; hence

$$\rho(W) = 1 + 2S_W + 2S_W^2. \tag{2}$$

(i). We have $S_W^* = J(T_W^* - T_W)$, so $S_W^*J = -JS_W$ and $(S_W^*)^2J = -S_W^*JS_W = JS_W^2$. By (2), $\rho(W)^*J = J(1 - 2S_W + 2S_W^2)$, and as $S_W^3 = S_W^4 = 0$,

$$\rho(W)^*J\rho(W) = J(1 - 2S_W + 2S_W^2)(1 + 2S_W + 2S_W^2) = J(1 - 4S_W^2 + 4S_W^2) = J.$$

By (1) and (2), $\rho(W)X = X$. Let $v, w \in R^N$ with $X^*Jv = X^*Jw = 0$, hence $w^*JX = 0$. Then $S_Wv = W(X^*Jv) - X(W^*Jv) = -X(W^*Jv)$, so $w^*JS_Wv = -(w^*JX)(W^*Jv) = 0$ and $S_W^2v = -S_WX(W^*Jv) = 0$; hence $w^*J\rho(W)v = w^*Jv$. So $\rho(W) \in \mathcal{N}(R)$.

(ii). Put $V = W + Z - X(W^*JZ)$; then $X^*JV = 0$, and as $(W^*JZ)^* = Z^*JW$,

$$S_V = S_W + S_Z + C, \qquad C = -X\big(W^*JZ - Z^*JW\big)X^*J.$$

By (1), $CS = SC = 0$ for $S \in \{S_W, S_Z\}$ and $C^2 = 0$, so $S_V^2 = (S_W + S_Z)^2$. Expanding with (2) and dropping the products of three or four S's,

$$\rho(W)\rho(Z) - \rho(V) = 4S_WS_Z - 2S_WS_Z - 2S_ZS_W - 2C = 2(S_WS_Z - S_ZS_W) - 2C = 0$$

by the first identity of (1).

(iii). Let $u \in \mathcal{N}(R)$ and $n = u - 1$. Unitarity reads

$$n^*J + Jn + n^*Jn = 0. \tag{3}$$

From $u^*Ju = J$ and $uX = X$: $X^*Ju = (uX)^*Ju = X^*J$, so $X^*Jn = 0$ and, applying $*$, $n^*JX = 0$. The columns of Y satisfy $X^*JY = 0$, so the last condition on u gives $Y^*JnY = 0$. Applying $*$ to $Ye + 2bX^*J = 1$ gives $1 = e^*Y^* + 2JXb^*$; multiplying on the right by JnY,

$$JnY = e^*(Y^*JnY) + 2JX(b^*JnY) = 2JX\gamma, \qquad \text{so} \qquad nY = 2X\gamma, \quad \gamma = b^*JnY. \tag{4}$$

The parameter. With $c = e(n - Xb^*Ju)b$ and $Ye = 1 - 2bX^*J$,

$$W := Yc = (1 - 2bX^*J)(n - Xb^*Ju)b = nb - X\beta, \qquad \beta = b^*Jub,$$

since $X^*Jn = 0$ and $X^*JX = 0$. Also $X^*JW = X^*JYc = 0$. Put $T = WX^*$. *The three terms of* $\rho(W) - 1$. Using $2bX^*J = 1 - Ye$ and (4),

$$2TJ = 2nbX^*J - 2X\beta X^*J = n - nYe - 2X\beta X^*J = n - 2X\gamma e - 2X\beta X^*J.$$

Next $T^* = XW^* = X(b^*n^* - \beta^*X^*)$, so $2T^*J = 2Xb^*n^*J - 2X\beta^*X^*J$. Finally $W^*JW = (b^*n^* - \beta^*X^*)J(nb - X\beta) = b^*n^*Jnb$, the other three terms containing $n^*JX = 0$, $X^*Jn = 0$ or $X^*JX = 0$; so $2T^*JTJ = 2X(b^*n^*Jnb)X^*J$. Hence $\rho(W) - 1 = n + \mathcal{E}$ with

$$\mathcal{E} = -2X\gamma e - 2X(\beta - \beta^*)X^*J - 2Xb^*n^*J - 2X(b^*n^*Jnb)X^*J.$$

$\mathcal{E} = 0$. By (3), $-2Xb^*n^*J = 2Xb^*Jn + 2Xb^*n^*Jn$. Multiplying $n = n(Ye + 2bX^*J)$ on the left by b^*J and by b^*n^*J, and using (4) and $n^*JX = 0$,

$$b^*Jn = \gamma e + 2(b^*Jnb)X^*J, \qquad b^*n^*Jn = 2b^*(n^*JX)\gamma e + 2(b^*n^*Jnb)X^*J = 2(b^*n^*Jnb)X^*J.$$

Substituting,

$$\mathcal{E} = 2X\big(-(\beta - \beta^*) + 2b^*Jnb + b^*n^*Jnb\big)X^*J.$$

As $\beta = b^*Jb + b^*Jnb$ and $\beta^* = b^*u^*Jb = b^*Jb + b^*n^*Jb$, the bracket is $b^*(Jn + n^*J + n^*Jn)b = 0$ by (3). So $\rho(Yc) = 1 + n = u$. □

Lemma 5.8.19. *Let E be a CM field with maximal totally real subfield E^+, $f = [E^+ : \mathbb{Q}]$, Φ the set of infinite places of E^+ with a chosen embedding $\Phi_v \colon E \to \mathbb{C}$ above each $v \in \Phi$, so that $E_\infty = \prod_{v\in\Phi} \mathbb{C}$, and $\mathfrak{D}_E$ the different of E. Let $N_D \geq 1$ be a rational integer with $N_D \in \mathfrak{D}_E$. Then for every $c = (c_\infty, c_f) \in \mathbb{A}_E$ there is $a \in E$ with*

$$c_f - a \in \widehat{\mathcal{O}}_E \qquad \text{and} \qquad |c_v - \Phi_v(a)| \leq \tfrac{1}{2} f N_D \quad \text{for every } v \in \Phi,$$

where $c_v \in \mathbb{C}$ is the component of c_∞ at v.

Proof. The finite part. E is dense in $\mathbb{A}_{E,f}$ (approximation theorem) and $\widehat{\mathcal{O}}_E$ is open, so there is $r \in E$ with $c_f - r \in \widehat{\mathcal{O}}_E$.

The lattice of integers and its dual. Give $\prod_{v\in\Phi} \mathbb{C}$ the real inner product $\langle y, z\rangle = \operatorname{Re} \sum_v \bar{y}_v z_v$, of dimension $2f$, and let $\phi \colon E \to \prod_v \mathbb{C}$, $\phi(t) = (\Phi_v(t))_v$. Every embedding of E is some Φ_v or its conjugate, so $\langle \phi(t), \phi(t')\rangle = \frac{1}{2}\operatorname{Tr}_{E/\mathbb{Q}}(\bar{t}t')$. The image $\Gamma = \phi(\mathcal{O}_E)$ is a lattice (Minkowski). Let $t_1, \ldots, t_{2f}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_E$ and $t'_1, \ldots, t'_{2f} \in E$ the dual basis for the nondegenerate $\mathbb{Q}$-bilinear

form $\frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(\bar{t}t')$; then $\phi(t'_1),\dots,\phi(t'_{2f})$ is the dual basis of $\phi(t_1),\dots,\phi(t_{2f})$, so the dual lattice is

$$\Gamma^* = \phi(M), \qquad M = \{t' \in E :\ \mathrm{Tr}_{E/\mathbb{Q}}(\bar{t}t') \in 2\mathbb{Z} \text{ for all } t \in \mathcal{O}_E\} = 2\mathfrak{D}_E^{-1},$$

as $\bar{t}$ runs through $\mathcal{O}_E$ with t and $\mathfrak{D}_E^{-1} = \{y \in E : \mathrm{Tr}_{E/\mathbb{Q}}(y\mathcal{O}_E) \subseteq \mathbb{Z}\}$.

The dual minimum. Let $0 \neq t' \in M$ and $y = \frac{1}{2}N_D t'$. As $N_D \in \mathfrak{D}_E$, $y \in N_D\mathfrak{D}_E^{-1} \subseteq \mathfrak{D}_E\mathfrak{D}_E^{-1} = \mathcal{O}_E$, and $y \neq 0$, so $|\mathrm{N}_{E/\mathbb{Q}}(y)| = \prod_v |\Phi_v(y)|^2 \geq 1$ and $|\Phi_v(y)| \geq 1$ for some v. Hence $\|\phi(t')\| = \frac{2}{N_D}\|\phi(y)\| \geq \frac{2}{N_D}$.

Rounding. By Theorem A.6.1, with $d = 2f$ and $\lambda = 2/N_D$, the point $z = (c_v - \Phi_v(r))_v$ has some $t \in \mathcal{O}_E$ with $\|z - \phi(t)\| \leq \frac{2f}{2\cdot 2/N_D} = \frac{1}{2}fN_D$. Put $a = r + t$. Then $c_f - a = (c_f - r) - t \in \widehat{\mathcal{O}}_E$ and $|c_v - \Phi_v(a)| = |z_v - \Phi_v(t)| \leq \|z - \phi(t)\| \leq \frac{1}{2}fN_D$ for every v. □

Lemma 5.8.20. *Let E be a CM field, $N \geq 1$, U a framed space of rank N over E, $G^U = G^U_\infty \times G_f$ and $\Gamma_U \subseteq G^U$ the image of $\mathrm{U}(U)(E^+)$ under the frames. For every real T there is a compact set $C \subseteq G^U$ such that for every $g \in G^U$ with $\lambda_U(g)^{-1} \leq T$ some $\gamma \in \Gamma_U$ has $\gamma g \in C$.*

Proof. Fix an E-basis of U, so $U = E^N$ with Gram matrix J; the frames form $x \in \mathrm{GL}_N(\mathbb{A}_E)$ with $\bar{x}^{\mathsf{T}}Jx = D_s$ ($\mathrm{diag}(s_v)$ at the infinite place v, 1 at the finite places), $c(u) = x^{-1}u$, $G^U = \{h \in \mathrm{GL}_N(\mathbb{A}_E) : \bar{h}^{\mathsf{T}}D_s h = D_s\}$ and $\Gamma_U = \{x^{-1}\gamma x :\ \gamma \in \mathrm{GL}_N(E),\ \bar{\gamma}^{\mathsf{T}}J\gamma = J\}$. The set G^U is closed in $M_N(\mathbb{A}_E)$, and its topology is the subspace topology, since inversion $h \mapsto D_s\bar{h}^{\mathsf{T}}D_s$ is continuous for it. Let $c = fN|d_E|$, as in Lemma 5.8.21, and assume $T \geq 0$ (otherwise there is no g).

Step 1: a bounded rational basis. Let g satisfy $\lambda_U(g)^{-1} \leq T$. By Lemma 5.8.21 there is $Y \in \mathrm{GL}_N(E)$ whose columns u_j lie in L_g with $|u_j|_g \leq cT$. The adelic matrix $M = g^{-1}x^{-1}Y$ has columns $g^{-1}c(u_j)$, so M_f has entries in $\widehat{\mathcal{O}}_E$ and every entry of M at every infinite place has modulus at most cT. Thus M lies in the compact set

$$\mathrm{Bx} = \{M \in M_N(\mathbb{A}_E) :\ M_f \in M_N(\widehat{\mathcal{O}}_E),\ |M_{v,ij}| \leq cT \text{ for all } v,i,j\}.$$

As g preserves D_s, $\bar{Y}^{\mathsf{T}}JY = \bar{M}^{\mathsf{T}}\bar{g}^{\mathsf{T}}D_s g M = \bar{M}^{\mathsf{T}}D_s M$.

Step 2: finitely many Gram matrices. The set $Q = \{\bar{M}^{\mathsf{T}}D_s M :\ M \in \mathrm{Bx}\}$ is compact, and $M_N(E)$ is discrete and closed in $M_N(\mathbb{A}_E)$ (E is a discrete subgroup of $\mathbb{A}_E$); so $Q \cap M_N(E)$ is a finite set. For each of its elements T' that is the Gram matrix $\bar{Y}^{\mathsf{T}}JY$ of a basis Y arising in Step 1 choose one such basis $Y_{T'}$; let $F \subseteq \mathrm{GL}_N(\mathbb{A}_E)$ be the finite set of the matrices $Y_{T'}^{-1}x$.

Step 3: the compact set. Let $C = \{h \in G^U :\ h^{-1} \in \mathrm{Bx} \cdot F\}$. As $\mathrm{Bx} \cdot F$ is a finite union of continuous images of Bx, it is compact, $G^U \cap \mathrm{Bx} \cdot F$ is compact, and C, its image under inversion, is compact. For g, Y, M as in Step 1, let $T' = \bar{Y}^{\mathsf{T}}JY$ and $\gamma_0 = Y_{T'}Y^{-1} \in \mathrm{GL}_N(E)$. Then $\bar{\gamma}_0^{\mathsf{T}}J\gamma_0 = \bar{Y}^{-\mathsf{T}}T'Y^{-1} = J$, so $\gamma = x^{-1}\gamma_0 x \in \Gamma_U$, and

$$(\gamma g)^{-1} = g^{-1}x^{-1}\gamma_0^{-1}x = g^{-1}x^{-1}Y\,Y_{T'}^{-1}x = M\,(Y_{T'}^{-1}x) \in \mathrm{Bx} \cdot F.$$

Hence $\gamma g \in C$. □

Lemma 5.8.21. *Let E be a CM field with totally real subfield E^+, $f = [E^+ : \mathbb{Q}]$ and discriminant d_E, let $N \geq 1$ and let U be a framed space of rank N over E, with real group G^U_∞ and $G^U = G^U_\infty \times G_f$. For $g \in G^U$ let*

$$L_g = \{u \in U :\ (g^{-1}c(u))_f \in \widehat{\mathcal{O}}_E^N\}, \qquad |u|_g = \|(g^{-1}c(u))_\infty\|,$$

$c(u) \in \mathbb{A}_E^N$ the coordinate vector of u and $\|\cdot\|$ the Euclidean norm on $\prod_v \mathbb{C}^N$. Then for every $g \in G^U$ there is a basis $u_1,\dots,u_N$ of U over E with $u_j \in L_g$ and

$$|u_j|_g \leq f\,N\,|d_E|\ \lambda_U(g)^{-1} \qquad (1 \leq j \leq N).$$

Proof. Let $L_{g,\infty} = \{(g^{-1}c(u))_\infty : u \in L_g\}$, $\langle\,,\rangle$ and σ be as in Lemma 5.8.15, and let β be the $\mathbb{R}$-linear extension of $u \mapsto (g^{-1}c(u))_\infty$ to $U \otimes_{\mathbb{Q}} \mathbb{R}$, so $|u|_g = \|\beta(u)\|$. Put $n = 2fN$ and $d = |d_E|$. By Lemma 5.8.15(iii), $L_{g,\infty}$ is a lattice of rank n in $\prod_v \mathbb{C}^N$, so $\lambda_1(L_{g,\infty}) = \min\{|u|_g : 0 \neq u \in L_g\} > 0$ and $\lambda_U(g) = \min(1, \lambda_1(L_{g,\infty}))$.

Step 1: $\lambda_1(L^*_{g,\infty}) \geq 2\lambda_1(L_{g,\infty})/d$. By Lemma 5.8.15(iv) a nonzero vector of $L^*_{g,\infty}$ is $\sigma\beta(w)$ with $0 \neq w \in 2\mathfrak{D}_E^{-1}L_g$, $\mathfrak{D}_E$ the different of E. The rational integer $d = \mathrm{N}(\mathfrak{D}_E)$ lies in $\mathfrak{D}_E$, so $\frac{d}{2}w \in d\mathfrak{D}_E^{-1}L_g \subseteq L_g$, and $\frac{d}{2}w \neq 0$. Hence $\lambda_1(L_{g,\infty}) \leq |\frac{d}{2}w|_g = \frac{d}{2}\|\beta(w)\| = \frac{d}{2}\|\sigma\beta(w)\|$.

Step 2: transference. By Banaszczyk's transference theorem [12], $\lambda_n(L_{g,\infty})\lambda_1(L^*_{g,\infty}) \leq n$; with Step 1,

$$\lambda_n(L_{g,\infty}) \leq \frac{n\,d}{2\lambda_1(L_{g,\infty})} = \frac{fN|d_E|}{\lambda_1(L_{g,\infty})} \leq \frac{fN|d_E|}{\lambda_U(g)}.$$

So there are $u'_1, \ldots, u'_n \in L_g$ with $|u'_k|_g \leq fN|d_E|\lambda_U(g)^{-1}$ and $\beta(u'_1), \ldots, \beta(u'_n)$ linearly independent over $\mathbb{R}$. Their E-span W is U: otherwise $\beta(W \otimes_{\mathbb{Q}} \mathbb{R})$ would contain them and have real dimension $2f \dim_E W < n$. Choose $u_1, \ldots, u_N$ among the u'_k forming an E-basis. □

Proposition 5.8.22. *Let E be a CM field, $f = [E^+ : \mathbb{Q}]$, $N \geq 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs $s_v = s$ for all v, $G^U = G_\infty \times G_f$ and Γ_U the image of $\mathrm{U}(U)(E^+)$ under the frames. Let $\varphi\colon G_f \to \mathbb{C}$ be locally constant with compact support.*

(i) There is $C \geq 0$ such that for all $g, x \in G^U$

$$\sum_{\gamma \in \Gamma_U} |\varphi((g^{-1}\gamma x)_f)|\, \|(g^{-1}\gamma x)_\infty\|_\infty^{-(8N^2+1)} \leq C\, \lambda_U(x)^{-4fN^2}.$$

(ii) Let $j \geq 0$ and $B \geq 8N^2 + 1 + j$ be integers and $k\colon G_\infty \to \mathbb{C}$ with $\sup_y |k(y)|\, \|y\|_\infty^B < \infty$. There is $C' \geq 0$ such that for all $g, x \in G^U$, every $Z \subseteq \Gamma_U$ and every real $r \geq 0$ with $r\,\|(g^{-1}\gamma x)_\infty\|_\infty \geq 1$ for all $\gamma \in Z$ with $\varphi((g^{-1}\gamma x)_f) \neq 0$,

$$\sum_{\gamma \in Z} |k((g^{-1}\gamma x)_\infty)\, \varphi((g^{-1}\gamma x)_f)| \leq C'\, r^j \lambda_U(x)^{-4fN^2}.$$

In particular ($j = 0$, $r = 1$) for $B > 8N^2$ these sums converge absolutely, with a bound uniform in g and Z.

Proof. Write $A = 8N^2+1$, $\mathrm{ht}(x) = \lambda_U(x)^{-1} \geq 1$ (the height $\mathrm{ht}_U(x)$ of (VP4) of Definition 3.2.2), $G_v = \mathrm{U}(\mathrm{diag}\, s)$, μ_v its Haar measure, and $m(y) = |\varphi(y_f)|\, \|y_\infty\|_\infty^{-A}$ for $y \in G^U$. For $z \in G_v$, $z^{-1} = \mathrm{diag}(s)\bar{z}^{\mathsf{T}}\mathrm{diag}(s)$, so $\|z^{-1}\| = \|z\|$, and $1 = \|zz^{-1}\| \leq \|z\|^2$; hence $\|y\|_\infty \geq 1$. Fix an E-basis of U with Gram matrix J and frames $X \in \mathrm{GL}_N(\mathbb{A}_E)$, so $c(u) = X^{-1}u$ and $\gamma \in \mathrm{U}(U)(E^+)$ goes to $X^{-1}\gamma X \in \Gamma_U$; then $c(\gamma u) = (X^{-1}\gamma X)\, c(u)$. Use L_x, $|u|_x$ as in Lemma 5.8.21 and $c_0 = fN|d_E|$.

Step 1: a level for φ. The support S of φ is compact and open. For $y \in S$ choose an open compact subgroup $J_y \subseteq \mathrm{K}_f$ with φ constant on yJ_y, and an open subgroup J'_y with $J'_yJ'_y \subseteq J_y$; finitely many $y_iJ'_{y_i}$ cover S, and $J = \bigcap_i J'_{y_i}$ is an open compact subgroup of K_f. If $y \in S$ and $u \in J$, then $y \in y_iJ'_{y_i}$ for some i and $y, yu \in y_iJ_{y_i}$, so $\varphi(yu) = \varphi(y)$; if $y \notin S$ and $yu \in S$, the same applied to yu and u^{-1} gives $y \in S$. So $\varphi(yu) = \varphi(y)$ for all $y \in G_f$, $u \in J$.

Step 2: nearly central rational points are trivial. Let $x \in G^U$ and $\gamma \in \Gamma_U$ (from $\gamma_0 \in \mathrm{U}(U)(E^+)$) with $t = x^{-1}\gamma x$, $t_f \in \mathrm{K}_f$ and $\|t_v - 1\| \leq ((c_0+1)\mathrm{ht}(x)^2)^{-1}$ for all v. By Lemma 5.8.21

there is a basis $u_1,\dots,u_N$ of U in L_x with $|u_j|_x \le c_0\mathrm{ht}(x)$. The vector $w=\gamma_0 u_j-u_j$ has $x^{-1}c(w)=(t-1)\,x^{-1}c(u_j)$; its finite part lies in $\widehat{\mathcal{O}}_E^N$ (as t_f is integral), so $w\in L_x$, and

$$|w|_x \le \max_v \|t_v-1\|\,|u_j|_x \le \frac{c_0\,\mathrm{ht}(x)}{(c_0+1)\mathrm{ht}(x)^2} < \lambda_U(x).$$

As every nonzero vector of L_x has length at least $\lambda_U(x)$ ((VP4) of Definition 3.2.2), $w=0$. So γ_0 fixes a basis, $\gamma_0=1$ and $\gamma=1$.

Step 3: disjoint boxes. For $0<\delta\le 1$ let $\mathrm{Bx}_\delta=\{z\in G^U:\ \|z_v-1\|\le\delta \text{ for all } v,\ z_f\in J\}$ and put $\delta(x)=(4(c_0+1)\mathrm{ht}(x)^2)^{-1}\le\frac14$. If $g^{-1}\gamma xz=g^{-1}\gamma' xz'$ with $\gamma,\gamma'\in\Gamma_U$ and $z,z'\in\mathrm{Bx}_{\delta(x)}$, then $x^{-1}\gamma'^{-1}\gamma x=z'z^{-1}$; here $(z'z^{-1})_f\in J\subseteq \mathrm{K}_f$ and, as $\|z_v^{-1}\|=\|z_v\|\le 1+\delta(x)\le 2$,

$$\|z'_v z_v^{-1}-1\|\le\|z'_v-z_v\|\,\|z_v^{-1}\|\le 2\delta(x)\cdot 2=((c_0+1)\mathrm{ht}(x)^2)^{-1}.$$

By Step 2, $\gamma=\gamma'$. So the sets $g^{-1}\gamma x\,\mathrm{Bx}_{\delta(x)}$, $\gamma\in\Gamma_U$, are pairwise disjoint.

Step 4: the majorant on a box and its integral. For $y\in G^U$ and $z\in\mathrm{Bx}_\delta$: $\|(yz)_v\|\le\|y_v\|\,\|z_v\|\le 2\|y_v\|$ and $\varphi((yz)_f)=\varphi(y_f)$ (Step 1), so $m(y)\le 2^{fA}m(yz)$. Next, $\int_{G^U} m=\prod_v\int_{G_v}\|z\|^{-A}d\mu_v\cdot\int_{G_f}|\varphi|$; the last factor is finite (φ is bounded with compact support), and so is each $\int_{G_v}\|z\|^{-A}$: with $h=500N^3$, the vector l with $l_i=N-1+2hi$ has gap h for s, so by Lemma 5.7.38(3) (large gap 0, $c=A>8N^2$) $z\mapsto d_{l/2}(2/\|z\|)^A$ is integrable with $d_{l/2}>0$.

Step 5: volume of a box. Let $\Omega=\{z\in G_v:\|z\|\le 2\}$, a neighbourhood of 1, so $\mu_v(\Omega)>0$. For $D\ge 1$ choose a maximal subset $\{z_i\}\subseteq\Omega$ with $\|z_i-z_{i'}\|>1/(2D)$ for $i\ne i'$. The balls of radius $1/(4D)$ about the z_i in $M_N(\mathbb{C})\cong\mathbb{R}^{2N^2}$ (operator norm) are disjoint and lie in the ball of radius 3 about 0, so comparing Lebesgue volumes there are at most $(12D)^{2N^2}$ of them. For $z\in\Omega$ some z_i has $\|z-z_i\|\le 1/(2D)$, and $w=z_i^{-1}z$ has $\|w-1\|\le\|z_i^{-1}\|\,\|z-z_i\|\le 1/D$. So Ω is covered by $(12D)^{2N^2}$ left translates of $\{w:\|w-1\|\le 1/D\}$, and by left invariance $\mu(\mathrm{Bx}_{1/D})\ge v_0(12D)^{-2N^2 f}$ with $v_0=\prod_v\mu_v(\Omega)\cdot\mu_f(J)>0$.

Step 6: (i). With $\delta=\delta(x)$, Steps 3 and 4 and left invariance of the Haar measure give

$$\mu(\mathrm{Bx}_\delta)\sum_{\gamma\in\Gamma_U} m(g^{-1}\gamma x)\le 2^{fA}\sum_\gamma\int_{\mathrm{Bx}_\delta} m(g^{-1}\gamma xz)\,dz=2^{fA}\sum_\gamma\int_{g^{-1}\gamma x\mathrm{Bx}_\delta} m\le 2^{fA}\int_{G^U} m.$$

By Step 5 with $D=\delta^{-1}=4(c_0+1)\mathrm{ht}(x)^2$, the sum is at most $2^{fA}v_0^{-1}\big(48(c_0+1)\big)^{2N^2 f}\int m\cdot\mathrm{ht}(x)^{4fN^2}$; this is (i).

Step 7: (ii). Let $C_k=\sup|k|\,\|\cdot\|_\infty^B$. For $\gamma\in Z$ with $\varphi((g^{-1}\gamma x)_f)\ne 0$ and $y=(g^{-1}\gamma x)_\infty$, $\|y\|_\infty\ge 1$ and $r\|y\|_\infty\ge 1$ give $\|y\|_\infty^{-B}\le\|y\|_\infty^{-A}\|y\|_\infty^{-j}\le r^j\|y\|_\infty^{-A}$. So each term is at most $C_k r^j m(g^{-1}\gamma x)$, and (i) gives (ii) with $C'=C_kC$. □

Proposition 5.8.23. *Let E be a CM field, $N\ge 1$, $s\in\{\pm1\}^N$, U a framed space of rank N over E with signs $s_v=s$ for all v, $G^U=G_\infty\times G_f$, and Γ_U the image of $\mathrm{U}(U)(E^+)$ under the frames; for $\gamma\in\Gamma_U$ let $\mathrm{charpoly}(\gamma)\in E[X]$ be the characteristic polynomial of the corresponding rational isometry of U. Let $B\ge 13N^3$, let e be a cusp kernel of order B on G_∞, and let $\varphi\colon G_f\to\mathbb{C}$ be locally constant with compact support. There is $D\ge 0$ such that for every $x\in G^U$ and every set $S\subseteq E[X]$*

$$\Big|\sum_{\gamma\in\Gamma_U,\ \mathrm{charpoly}(\gamma)\in S} e\big((x^{-1}\gamma x)_\infty\big)\,\varphi\big((x^{-1}\gamma x)_f\big)\Big|\le D,$$

the sum being absolutely convergent.

Proof. For $\gamma \in \Gamma_U$ write γ_0 for the rational isometry with image γ, so $\mathrm{charpoly}(\gamma) = \mathrm{charpoly}(\gamma_0)$, and $t_\gamma(x) = e((x^{-1}\gamma x)_\infty)\varphi((x^{-1}\gamma x)_f)$.

A left level of φ. Let S_φ be the support of φ, compact and open. For $y \in S_\varphi$ choose an open compact subgroup $J_y \subseteq G_f$ with φ constant on $J_y y$ and an open subgroup J'_y with $J'_y J'_y \subseteq J_y$; finitely many $J'_{y_i} y_i$ cover S_φ; put $J = \bigcap_i J'_{y_i}$, an open subgroup. If $y \in S_\varphi$ and $u \in J$, then $y \in J'_{y_i} y_i$ for some i, so $y, u^{-1}y \in J_{y_i} y_i$ and $\varphi(u^{-1}y) = \varphi(y)$. If $y \notin S_\varphi$ and $u^{-1}y \in S_\varphi$, the same applied to $u^{-1}y$ and u^{-1} gives $y \in S_\varphi$, which is absurd. So $\varphi(u^{-1}y) = \varphi(y)$ for all $u \in J$, $y \in G_f$.

Constants. As $\|y\|_\infty \geq 1$ on G_∞, Proposition 5.8.22(ii) with $j = 0$, $r = 1$, $k = e$ (bounded against $\|\cdot\|_\infty^B$ by (EK4)(b), $B \geq 8N^2 + 1$) gives C_0 with $\sum_{\gamma \in Z} |t_\gamma(x)| \leq C_0 \lambda_U(x)^{-4fN^2}$ for all $x \in G^U$ and $Z \subseteq \Gamma_U$ ($\lambda_U(x)$ the first minimum of (VP4) of Definition 3.2.2); in particular the sum in the statement converges absolutely. Let C_1 be the constant of Lemma 5.8.24 for $k = e$ and φ, C_2 that of Lemma 5.8.26 for e, φ and J, κ_J that of Proposition 5.8.13(ii) for J, and

$$D = \max(C_0,\ C_1 + C_2)\, \kappa_J^{4fN^2}.$$

The bound. Let $x \in G^U$ and $S \subseteq E[X]$, and $Z_S = \{\gamma : \mathrm{charpoly}(\gamma_0) \in S\}$. The first minimum is positive (the nonzero vectors of the lattice $L_{x,\infty}$ of Lemma 5.8.15(iii) have norm bounded below), so there is $\delta > 0$ with $\delta^N = \kappa_J \lambda_U(x)$.

If $\delta > 1$, then $\lambda_U(x)^{-1} = \kappa_J \delta^{-N} < \kappa_J$, and $|\sum_{Z_S} t_\gamma(x)| \leq C_0 \lambda_U(x)^{-4fN^2} \leq C_0 \kappa_J^{4fN^2} \leq D$.

If $\delta \leq 1$, Proposition 5.8.13(ii) gives a nonzero isotropic $A \subseteq U$ and $0 \leq a \leq b$ with $ab \leq \delta$, A and $A^\perp$ spanned by vectors of L_x of lengths at most a and b, and $N_A(\mathbb{A}_{E^+})x \subseteq N_A(E^+)x Z_\delta J$. Split Z_S into $Z_1 = \{\gamma \in Z_S : \gamma_0 A = A\}$ and $Z_2 = Z_S \setminus Z_1$; by absolute convergence $\sum_{Z_S} t_\gamma(x) = \sum_{Z_1} t_\gamma(x) + \sum_{Z_2} t_\gamma(x)$. By Lemma 5.8.26, $|\sum_{Z_1} t_\gamma(x)| \leq C_2(\delta^N \lambda_U(x)^{-1})^{4fN^2} = C_2 \kappa_J^{4fN^2}$, and by Lemma 5.8.24, applied to Z_2 and the frame (A, a, b), $|\sum_{Z_2} t_\gamma(x)| \leq C_1 \kappa_J^{4fN^2}$. Hence $|\sum_{Z_S} t_\gamma(x)| \leq (C_1 + C_2)\kappa_J^{4fN^2} \leq D$. □

Lemma 5.8.24. *Let E be a CM field, $f = [E^+ : \mathbb{Q}]$, $N \geq 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs $s_v = s$ for all v and hermitian form h, $G^U = G_\infty \times G_f$, and for $\gamma \in \Gamma_U$ let $\gamma_0 \in \mathrm{U}(U)(E^+)$ be the rational isometry whose image is γ. Let $B \geq 13N^3$ be an integer, $k\colon G_\infty \to \mathbb{C}$ with $\sup_y |k(y)|\,\|y\|_\infty^B < \infty$, and $\varphi\colon G_f \to \mathbb{C}$ locally constant with compact support. There is $C \geq 0$ with the following property. Let $x \in G^U$, $\delta \geq 0$, $A \subseteq U$ a nonzero isotropic subspace and $0 \leq a \leq b$ with $ab \leq \delta$ such that A is spanned by vectors $u \in L_x$ with $|u|_x \leq a$ and $A^\perp$ by vectors $w \in L_x$ with $|w|_x \leq b$, and let $Z \subseteq \Gamma_U$ be a set with $\gamma_0 A \neq A$ for all $\gamma \in Z$. Then*

$$\sum_{\gamma \in Z} \left|k((x^{-1}\gamma x)_\infty)\, \varphi((x^{-1}\gamma x)_f)\right| \leq C\, (\delta^N \lambda_U(x)^{-1})^{4fN^2}.$$

Proof. Let Ω be the support of φ, a compact subset of G_f, and c_Ω the constant of Lemma 5.8.25 for Ω. Put $j = 4N^3$; then $8N^2 + 1 + j \leq 13N^3 \leq B$ (as $9N^3 \geq 9N^2 \geq 8N^2 + 1$), and let C' be the constant of Proposition 5.8.22(ii) for j, B, k, φ. Put $C = C' c_\Omega^{4fN^3}$.

Let x, δ, A, a, b, Z be as in the statement and $r = (c_\Omega \delta)^f$. Let $\gamma \in Z$ with $\varphi((x^{-1}\gamma x)_f) \neq 0$, so $(x^{-1}\gamma x)_f \in \Omega$. As γ_0 is injective and $\gamma_0 A \neq A$, $\gamma_0 A \not\subseteq A$, so some vector y of the given spanning set of A has $\gamma_0 y \notin A$. Since h is nondegenerate, $A = (A^\perp)^\perp$, so $h(w, \gamma_0 y) \neq 0$ for some $w \in A^\perp$; writing w as a combination of the given spanning vectors of $A^\perp$ (h is conjugate-linear in its first argument), some spanning vector y' of $A^\perp$ has $h(y', \gamma_0 y) \neq 0$. Here $y, y' \in L_x$ and $|y|_x |y'|_x \leq ab \leq \delta$, so Lemma 5.8.25 gives $r\,\|(x^{-1}\gamma x)_\infty\|_\infty \geq 1$.

So the hypothesis of Proposition 5.8.22(ii) holds for $g = x$, Z and r, and

$$\sum_{\gamma\in Z} \bigl|k((x^{-1}\gamma x)_\infty)\varphi((x^{-1}\gamma x)_f)\bigr| \le C' r^j \lambda_U(x)^{-4fN^2} = C' c_\Omega^{4fN^3} \delta^{4fN^3} \lambda_U(x)^{-4fN^2}$$
$$= C\bigl(\delta^N \lambda_U(x)^{-1}\bigr)^{4fN^2}.$$

□

Lemma 5.8.25. *Let E be a CM field, $f = [E^+ : \mathbb{Q}]$, $N \ge 1$, U a framed space of rank N over E with hermitian form h, $G^U = G_\infty \times G_f$, and for $\gamma \in \Gamma_U$ let $\gamma_0 \in \mathrm{U}(U)(E^+)$ be the rational isometry whose image is γ. Let $\Omega \subseteq G_f$ be compact. There is $c_\Omega \ge 0$ such that for all $x \in G^U$, $\gamma \in \Gamma_U$, $y, y' \in L_x$ and $\delta \ge 0$ with*

$$|y|_x\,|y'|_x \le \delta, \qquad (x^{-1}\gamma x)_f \in \Omega, \qquad h(y', \gamma_0 y) \ne 0,$$

one has $(c_\Omega\delta)^f\,\|(x^{-1}\gamma x)_\infty\|_\infty \ge 1$.

Proof. Fix an E-basis $U = E^N$ with Gram matrix J and let $P \in \mathrm{GL}_N(\mathbb{A}_E)$ be the matrix of the frames, so $c(u) = P^{-1}u$ and $\gamma = P^{-1}\gamma_0 P$. For $t = x^{-1}\gamma x$ and $u \in U$,

$$x^{-1}c(\gamma_0 u) = x^{-1}P^{-1}\gamma_0 P\,P^{-1}u = t\,x^{-1}c(u). \tag{1}$$

At $v \in \Phi$ the frame is a basis in which the form is $\mathrm{diag}(s)$, and x_v preserves $\mathrm{diag}(s)$; so $\Phi_v(h(u,w)) = \overline{(x^{-1}c(u))_v}^{\mathsf{T}}\mathrm{diag}(s)\,(x^{-1}c(w))_v$ and, by the Cauchy–Schwarz inequality,

$$|\Phi_v(h(u,w))| \le \|(x^{-1}c(u))_v\|\,\|(x^{-1}c(w))_v\| \qquad (u, w \in U). \tag{2}$$

The constant. The finite adeles are the increasing union of the open sets $(m!)^{-1}\widehat{\mathcal{O}}_E$, $m \ge 1$, so the compact set of entries of the matrices in Ω lies in one of them: there is an integer $c_\Omega \ge 1$ with $c_\Omega t_f \in \mathrm{M}_N(\widehat{\mathcal{O}}_E)$ for all $t_f \in \Omega$.

The estimate. Let x, γ, y, y', δ be as in the statement and $t = x^{-1}\gamma x$. By (1), $(x^{-1}c(c_\Omega\gamma_0 y))_f = c_\Omega t_f\,(x^{-1}c(y))_f \in \widehat{\mathcal{O}}_E^N$, so $c_\Omega\gamma_0 y \in L_x$, and by Lemma 5.8.15(i) the element $z = h(y', c_\Omega\gamma_0 y) = c_\Omega h(y', \gamma_0 y)$ lies in $\mathcal{O}_E$; it is nonzero. By (2) and (1), for every $v \in \Phi$,

$$|\Phi_v(z)| \le c_\Omega\|(x^{-1}c(y'))_v\|\,\|t_v\|\,\|(x^{-1}c(y))_v\| \le c_\Omega|y'|_x\,|y|_x\,\|t_v\| \le c_\Omega\delta\,\|t_v\|.$$

Every embedding of E is some Φ_v or its conjugate, so $1 \le |\mathrm{N}_{E/\mathbb{Q}}(z)| = \prod_v |\Phi_v(z)|^2$, hence $1 \le \prod_v |\Phi_v(z)| \le (c_\Omega\delta)^f \prod_v \|t_v\| = (c_\Omega\delta)^f \|t_\infty\|_\infty$. □

Lemma 5.8.26. *Let E be a CM field, $f = [E^+ : \mathbb{Q}]$, $N \ge 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs $s_v = s$ for all v, $G^U = G_\infty \times G_f$, and for $\gamma \in \Gamma_U$ let $\gamma_0 \in \mathrm{U}(U)(E^+)$ be the rational isometry whose image is γ. Let $B \ge 13N^3$ be an integer, e a cusp kernel of order B on G_∞, $\varphi\colon G_f \to \mathbb{C}$ locally constant with compact support, and $J \subseteq G_f$ an open subgroup with $\varphi(u^{-1}y) = \varphi(y)$ for all $u \in J$, $y \in G_f$. There is $C \ge 0$ with the following property. Let $x \in G^U$, $\delta \ge 0$ and $A \subseteq U$ a nonzero isotropic subspace with $N_A(\mathbb{A}_{E^+})\,x \subseteq N_A(E^+)\,x\,Z_\delta\,J$, and let $S \subseteq E[X]$. Then the sum below converges absolutely and*

$$\Bigl|\sum_{\substack{\gamma\in\Gamma_U:\ \gamma_0 A = A,\\ \mathrm{charpoly}(\gamma_0)\in S}} e\bigl((x^{-1}\gamma x)_\infty\bigr)\,\varphi\bigl((x^{-1}\gamma x)_f\bigr)\Bigr| \le C\,\bigl(\delta^N\lambda_U(x)^{-1}\bigr)^{4fN^2}.$$

Proof. Constants. Since $\|y\|_\infty \geq 1$ on G_∞ ($\|z_v^{-1}\| = \|z_v\|$ for $z_v \in \mathrm{U}(\operatorname{diag} s)$, so $1 \leq \|z_v\|^2$), the hypothesis of Proposition 5.8.22(ii) with $j = 0$ and $r = 1$ holds for every set of rational points; as $B \geq 8N^2 + 1$ and $\sup |e| \, \|\cdot\|_\infty^B < \infty$ by (EK4)(b), there is $C_0 \geq 0$ with

$$\sum_{\gamma \in Z} |e((g^{-1}\gamma x)_\infty)\varphi((g^{-1}\gamma x)_f)| \leq C_0 \lambda_U(x)^{-4fN^2} \qquad (g, x \in G^U,\ Z \subseteq \Gamma_U). \tag{1}$$

Let $\delta_0 > 0$ be the constant of Proposition 5.8.8 for e and $j' = 4fN^3$, and $C = C_0 \delta_0^{-4fN^3}$.

The partial kernel. Fix x, δ, A, S, let $Z = \{\gamma \in \Gamma_U : \gamma_0 A = A,\ \mathrm{charpoly}(\gamma_0) \in S\}$ and, for $g \in G^U$,

$$F(g) = \sum_{\gamma \in Z} t_\gamma(g), \qquad t_\gamma(g) = e\big((g^{-1}\gamma x)_\infty\big)\varphi\big((g^{-1}\gamma x)_f\big).$$

By (1) the sum converges absolutely and $|F| \leq T := C_0 \lambda_U(x)^{-4fN^2}$ on G^U; the sum in the statement is $F(x)$.

F is right J-invariant. For $u \in J$, $(g(1,u))^{-1}\gamma x = (1, u^{-1})\, g^{-1}\gamma x$, and $\varphi(u^{-1}y) = \varphi(y)$; so $t_\gamma(g(1,u)) = t_\gamma(g)$.

$R(e)F = F$. By (EK3) with $x = 1$, $\int e(z)e(z^{-1}y)\, dz = e(y)$ for $y \in G_\infty$. As $(g(h,1))^{-1}\gamma x = (h^{-1}, 1)\, g^{-1}\gamma x$, this gives $\int e(h)\, t_\gamma(g(h,1))\, dh = t_\gamma(g)$ for each γ. By (1),

$$\sum_\gamma \int |e(h)|\, |t_\gamma(g(h,1))|\, dh \leq T \int |e| < \infty,$$

so by Fubini's theorem

$$R(e)F(g) = \sum_\gamma \int e(h) t_\gamma(g(h,1))\, dh = F(g).$$

F is continuous. Let $g, g' \in G^U$ with $g'_f \in g_f J$, an open condition on g'. By the two previous steps and the left invariance of the Haar measure,

$$F(g') = \int e(h) F((g'_\infty h, g'_f))dh = \int e(g'^{-1}_\infty h)\, F((h, g_f))dh,$$

and the same for g; so $|F(g') - F(g)| \leq T \int |e(g'^{-1}_\infty h) - e(g_\infty^{-1}h)|\, dh$, which tends to 0 as $g'_\infty \to g_\infty$ (continuity of left translation in $L^1(G_\infty)$).

F is left invariant under $N_A(E^+)$. Let $\nu \in N_A(E^+) \subseteq \Gamma_U$; ν_0 fixes A pointwise and induces the identity on $A^\perp/A$, and so does ν_0^{-1}. The bijection $\gamma \mapsto \nu^{-1}\gamma$ of Γ_U maps Z onto Z: $\nu_0^{-1}\gamma_0$ keeps A if and only if γ_0 does, and then $\mathrm{charpoly}(\nu_0^{-1}\gamma_0) = \mathrm{charpoly}(\gamma_0)$ by Lemma 5.8.27(ii). Since $t_\gamma(\nu g) = t_{\nu^{-1}\gamma}(g)$ and the sum converges absolutely, $F(\nu g) = F(g)$.

Decay. Proposition 5.8.8, applied to F, T, the subgroup J, δ, A, the point x and $j' = 4fN^3$, gives

$$|F(x)| \leq (\delta/\delta_0)^{4fN^3} T = C_0 \delta_0^{-4fN^3} \delta^{4fN^3} \lambda_U(x)^{-4fN^2} = C\big(\delta^N \lambda_U(x)^{-1}\big)^{4fN^2}.$$

□

Lemma 5.8.27. *Let E be a CM field with maximal totally real subfield E^+, $N \geq 1$, U a nondegenerate hermitian space over E of dimension N with form h, $A \subseteq U$ a nonzero isotropic subspace and $A^\perp = \{w \in U : h(a,w) = 0$ for all $a \in A\}$. Let $\gamma_0 \in \mathrm{U}(U)(E^+)$ with $\gamma_0 A = A$. Then:*

(i) the characteristic polynomial of γ_0 in $E[X]$ is not irreducible;

(ii) if $\nu \in \mathrm{U}(U)(E^+)$ fixes A pointwise and induces the identity on $A^\perp/A$, then $\nu\gamma_0$ and γ_0 have the same characteristic polynomial.

Proof. Let $r = \dim A \geq 1$. As h is nondegenerate, $\dim A^\perp = N - r$, and $A \subseteq A^\perp$ since A is isotropic; so $r \leq N - r$, hence $1 \leq r < N$. For an endomorphism ϕ of U that maps each space of a flag $0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_m = U$ into itself, the characteristic polynomial of ϕ is the product of those of the endomorphisms induced on U_i/U_{i-1} (block triangular matrix in an adapted basis).

(i). By the remark, applied to $0 \subseteq A \subseteq U$, the characteristic polynomial of γ_0 is the product of those of $\gamma_0|_A$ and of the map induced on U/A, of degrees $r \geq 1$ and $N - r \geq 1$; so it is not irreducible.

(ii). The isometry γ_0 keeps $A^\perp$: for $w \in A^\perp$ and $a \in A$, $h(a, \gamma_0 w) = h(\gamma_0^{-1}a, w) = 0$, as $\gamma_0^{-1}A = A$. The isometry ν keeps A and $A^\perp$ in the same way, and induces the identity on $U/A^\perp$: for $u \in U$ and $a \in A$, $h(a, \nu u) = h(\nu a, \nu u) = h(a, u)$, so $\nu u - u \in A^\perp$. Hence γ_0, ν and $\nu\gamma_0$ map each space of the flag $0 \subseteq A \subseteq A^\perp \subseteq U$ into itself, and as ν induces the identity on A, on $A^\perp/A$ and on $U/A^\perp$, the maps induced by $\nu\gamma_0$ and by γ_0 on these three quotients coincide. By the remark, $\nu\gamma_0$ and γ_0 have the same characteristic polynomial. □

Proposition 5.8.28. *Let E be a CM field, $N \geq 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs $s_v = s$ for all v, $G^U = G_\infty \times G_f$ and Γ_U the image of $\mathrm{U}(U)(E^+)$ under the frames. Fix a Haar measure μ_f on G_f, the product Haar measure on G^U, and on $\Gamma_U\backslash G^U$ the quotient by the counting measure on Γ_U. Let $B > 8N^2$, let e be a cusp kernel of order B on G_∞, and let $K \subseteq \mathrm{K}_f$ be an open subgroup. Assume that the space $\mathcal{A}_K(U, e)$ of forms of type e right invariant under K (FT2) is finite-dimensional, and let $(b_i)_{i\in I}$ be an orthonormal basis for the Petersson product $\langle F, F'\rangle = \int_{\Gamma_U\backslash G^U} \overline{F}F'$ of the measures just fixed. For finitely supported $a = \sum_h a_h h \in \mathbb{C}[G_f]$ put $\phi(y) = \sum_h a_h \mathbf{1}_K(h^{-1}y)$ and $(\rho_U(a)F)(g) = \sum_h a_h F(g(1,h))$. Then for all $g, g' \in G^U$ the series below converges absolutely and*

$$\sum_{\gamma\in\Gamma_U} e((g^{-1}\gamma g')_\infty)\,\phi((g^{-1}\gamma g')_f) = \mu_f(K)\sum_{i\in I}(\rho_U(a)b_i)(g)\,\overline{b_i(g')}.$$

Proof. By Theorem 4.2.18 (through the frames G^U is the adelic unitary group of U and Γ_U its group of rational points) the quotient $\Gamma_U\backslash G^U$ has finite volume, so the inner product is defined on bounded functions; it is positive definite on continuous functions, as a nonzero continuous Γ_U-invariant function is nonzero on the image of a nonempty open set, which has positive measure. For $x \in G^U$ and $\gamma \in \Gamma_U$ write $\kappa_x(\gamma, y) = e((y^{-1}\gamma x)_\infty)\mathbf{1}_K((y^{-1}\gamma x)_f)$.

Step 1: absolute convergence. ϕ is locally constant with compact support, and e has $\sup|e|\,\|\cdot\|_\infty^B < \infty$ with $B \geq 8N^2 + 1$ (EK4)(b). By Proposition 5.8.22(ii) ($j = 0$, $r = 1$, $Z = \Gamma_U$) the left side converges absolutely, and likewise, for every x, $\sum_\gamma |\kappa_x(\gamma, y)| \leq C'\lambda_U(x)^{-4fN^2}$ for all y ($\lambda_U(x)$ the first minimum of (VP4) of Definition 3.2.2), with C' independent of x and y (the test function $\mathbf{1}_K$).

Step 2: the kernel of the level is a form. Fix x and put $P_x(y) = \sum_{\gamma\in\Gamma_U} \kappa_x(\gamma, y)$. By Step 1, P_x is bounded, and it is measurable as a countable sum of measurable functions. For $\delta \in \Gamma_U$, $\kappa_x(\gamma, \delta y) = \kappa_x(\delta^{-1}\gamma, y)$, so P_x is left Γ_U-invariant; for $k \in K$, $(yk)^{-1}\gamma x = k^{-1}(y^{-1}\gamma x)$ has the same real part and finite part in $K(y^{-1}\gamma x)_f$, so P_x is right K-invariant. Moreover $R(e)P_x = P_x$: for $h \in G_\infty$, $(y(h,1))^{-1}\gamma x$ has real part $h^{-1}w_\gamma$, $w_\gamma = (y^{-1}\gamma x)_\infty$, and unchanged finite part, and

$$\int \sum_\gamma |e(h)|\,|\kappa_x(\gamma, y(h,1))|\,dh \leq C'\lambda_U(x)^{-4fN^2}\int |e| < \infty,$$

so summation and integration may be exchanged, and $\int e(h)e(h^{-1}w)\,dh = e(w)$ by (EK3) with $x = 1$. By Lemma 5.8.29, $P_x = R(e)P_x$ is continuous; so $P_x \in \mathcal{A}_K(U, e)$. As $e(z^{-1}) = \overline{e(z)}$ and $\mathbf{1}_K(z^{-1}) = \mathbf{1}_K(z)$, replacing γ by γ^{-1} gives

$$\sum_\gamma e((x^{-1}\gamma g')_\infty)\mathbf{1}_K((x^{-1}\gamma g')_f) = \overline{P_x(g')} \qquad (g' \in G^U). \tag{1}$$

Step 3: $\langle P_x, v\rangle = \mu_f(K)\,v(x)$ for $v \in \mathcal{A}_K(U, e)$. By (1) with $g' = y$, $\overline{P_x(y)}\,v(y) = \sum_\gamma f(\gamma y)$ with $f(y) = e((x^{-1}y)_\infty)\mathbf{1}_K((x^{-1}y)_f)v(y)$, using $v(\gamma y) = v(y)$. The function f is integrable on G^U ($|f| \le \sup|v|\,|e((x^{-1}y)_\infty)|\mathbf{1}_K((x^{-1}y)_f)$), so by the defining property of the quotient measure and left invariance,

$$\langle P_x, v\rangle = \int_{G^U} f = \int_{G_\infty}\int_K e(h)\,v(x(h,k))\,dk\,dh = \mu_f(K)\int_{G_\infty} e(h)v(x(h,1))\,dh = \mu_f(K)\,v(x),$$

since $v(x(h,k)) = v(x(h,1)(1,k)) = v(x(h,1))$ and $R(e)v = v$.

Step 4: one group element. Let $h \in G_f$ and $\phi_h(y) = \mathbf{1}_K(h^{-1}y)$. As $(g(1,h))^{-1}\gamma g' = (1,h^{-1})\,g^{-1}\gamma g'$, (1) with $x = g(1,h)$ gives $\sum_\gamma e((g^{-1}\gamma g')_\infty)\phi_h((g^{-1}\gamma g')_f) = \overline{P_{g(1,h)}(g')}$. Expanding $P = P_{g(1,h)} = \sum_i \langle b_i, P\rangle b_i$ and using Step 3,

$$\overline{P(g')} = \sum_i \langle P, b_i\rangle\,\overline{b_i(g')} = \mu_f(K)\sum_i b_i(g(1,h))\,\overline{b_i(g')} = \mu_f(K)\sum_i (\rho_U(h)b_i)(g)\,\overline{b_i(g')}.$$

Step 5: linearity. $\phi = \sum_h a_h\phi_h$ is a finite sum, each series converges absolutely (Step 1), and both sides are linear in a; Step 4 gives the claim. □

Lemma 5.8.29. *Let E be a CM field, $N \ge 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs $s_v = s$ for all v, $G^U = G_\infty \times G_f$ and Γ_U the image of $\mathrm{U}(U)(E^+)$ under the frames. Let e be an idempotent kernel on G_∞, $K \subseteq G_f$ an open subgroup, and F a bounded measurable function on G^U that is left Γ_U-invariant and right K-invariant. Then $\Phi = R(e)F$ is defined everywhere, continuous, bounded by $\sup|F|\int|e|$, left Γ_U-invariant and right K-invariant, and $R(e)\Phi = \Phi$.*

Proof. Let $M = \sup|F|$. For $g \in G^U$ the function $h \mapsto e(h)F(g(h,1))$ is measurable and bounded in modulus by $M|e(h)|$, so $\Phi(g) = \int e(h)F(g(h,1))\,dh$ is defined and $|\Phi(g)| \le M\int|e|$. For $\gamma \in \Gamma_U$, $\Phi(\gamma g) = \Phi(g)$ since $F(\gamma g(h,1)) = F(g(h,1))$; for $k \in K$, $g(1,k)(h,1) = g(h,1)(1,k)$ (the real and finite components commute), so $\Phi(g(1,k)) = \Phi(g)$.

Continuity. Let $g \in G^U$. For $u \in G_\infty$, the substitution $h \mapsto u^{-1}h$ (left invariance of the Haar measure) gives $\Phi(g(u,1)) = \int e(h)F(g(uh,1))\,dh = \int e(u^{-1}h)F(g(h,1))\,dh$, hence

$$|\Phi(g(u,1)) - \Phi(g)| \le M\int_{G_\infty}|e(u^{-1}h) - e(h)|\,dh,$$

which tends to 0 as $u \to 1$, by the continuity of left translation in $L^1(G_\infty)$. For (u,k) with $k \in K$, $\Phi(g(u,k)) = \Phi(g(u,1)(1,k)) = \Phi(g(u,1))$. As $G_\infty \times K$ is a neighbourhood of 1 in G^U, Φ is continuous at g.

Reproduction. For $g \in G^U$, the function $(h,z) \mapsto e(h)e(h^{-1}z)F(g(z,1))$ on $G_\infty \times G_\infty$ is integrable, with $\iint|e(h)|\,|e(h^{-1}z)|\,M\,dz\,dh = M(\int|e|)^2$. By Fubini's theorem and the substitution $z = hh'$,

$$R(e)\Phi(g) = \int e(h)\int e(h')F(g(hh',1))\,dh'\,dh = \int\Big(\int e(h)e(h^{-1}z)\,dh\Big)F(g(z,1))\,dz.$$

The identity (EK3) with $x = 1$ gives $\int e(h)e(h^{-1}z)\,dh = e(1)e(z)/e(1) = e(z)$, so $R(e)\Phi(g) = \Phi(g)$. □

The matching has three clauses, one for each kind of characteristic polynomial. The elliptic clause rests on a description of the elliptic conjugacy classes of a framed space as a product of finite classes, independent of the space, and real classes at each infinite place; this is a local-global statement for hermitian forms [180, 147]. The level cut that follows is linear algebra of Hecke operators at one place, based on the relation $T^+T^- = q^{N-2}(1+(q^N-1)e)$ between the raising operator of a line and its adjoint (Proposition 5.8.52). The group closes with the three statements on the good forms and the definite space announced at the opening of the section.

Theorem 5.8.30. *Let E be a CM field with totally real subfield E^+ and a CM type $(\Phi_v)_v$, let $N \geq 2$, let $K \subseteq \mathrm{K}_f$ be open, let $a \in \mathbb{C}[G_f]$ be finitely supported, and let $\phi(g) = \sum_h a_h \mathbf{1}_K(h^{-1}g)$. Let U be a framed space of rank N over E whose signs do not depend on the infinite place, $s_v = s$ for all v with a fixed $s\colon \{1,\dots,N\} \to \{\pm 1\}$, and let $\lambda = (\lambda_v)_v$, $\lambda_v \in (\frac{N-1}{2} + \mathbb{Z})^N$, have pairwise distinct entries and $|\lambda_{v,i} - \lambda_{v,j}| \geq 500N^3 + 10N^5$ whenever $s_v(i) \neq s_v(j)$. Let $R \in E[X]$.*

(TM1) If R is irreducible and some root of $\Phi_v(R)$, for some infinite place v, has absolute value $\neq 1$, then $I_R^U(\phi) = 0$.

(TM2) Suppose that E contains a primitive 5040-th root of unity and $[E^+ : \mathbb{Q}]$ is even. If R is irreducible and all roots of $\Phi_v(R)$ lie on the unit circle for every v, let $z_v \in \mathrm{U}(N)$ be the diagonal matrix of these roots and χ_{λ_v} the character of the irreducible representation of the compact group $\mathrm{U}(N)$ with Harish-Chandra parameter λ_v. There is $c_R(\phi) \in \mathbb{C}$, depending only on R and ϕ (not on U, s or λ), such that

$$I_R^U(\phi) = c_R(\phi) \prod_v \overline{\chi_{\lambda_v}(z_v)}.$$

(TM3) Suppose that ϕ is cuspidal at a finite place w of E with $w \neq \bar{w}$, and that w' is a finite place of E such that for every $g \in G_f$ with $\phi(g) \neq 0$ the component $g_{w'} \in \mathrm{GL}_N(E_{w'})$ has entries in $\mathcal{O}_{w'}$ and 0 is a simple root of the characteristic polynomial of its reduction modulo w'. If R is not irreducible, then $I_R^U(\phi) = 0$.

Consequently, under the hypotheses of (TM2) and (TM3), for two framed spaces V, V' of rank N with signs s, s', each independent of the infinite place, such that λ satisfies the gap condition for both, $I_R^V(\phi) = I_R^{V'}(\phi)$ for every $R \in E[X]$.

Proof. (TM3) is Proposition 5.8.31.

Normalisations. Replacing the Haar measure dg_v of $\mathrm{U}(\mathrm{diag}(s_v))$ by $\tau\, dg_v$ ($\tau > 0$) divides $d(\lambda_v, s_v)$ by τ and multiplies the quotient measure by τ, so $I_R^U(\phi)$ does not change; a change of μ_f multiplies all $I_R^U(\phi)$ by one positive constant. As (TM1) and (TM3) assert vanishing, for them we take the measures of (FT2) of Definition 3.2.4. Put $e = e_\lambda^U$ and $B = 10N^5$. Since $N \geq 2$, $B \geq 13N^3$. For every v, $2\lambda_v$ has distinct entries congruent to $N-1$ modulo 2 and $|\lambda_{v,i} - \lambda_{v,j}| \geq 500N^3 + B$ whenever $s(i) \neq s(j)$, so by Lemma 5.6.21 e is a cusp kernel of order B on G_∞^U. The subgroup K is open in the compact group K_f, hence compact, so ϕ is locally constant with compact support.

Proof of (TM1). Let R be irreducible and let some root of $\Phi_v(R)$ have absolute value $\neq 1$. Let X_R be the set of $\gamma \in \mathrm{U}(U)(E^+)$ with $\mathrm{charpoly}(\gamma) = R$ and C_R a set of representatives of its conjugacy classes. The function e is continuous with $\sup |e| \, \| \cdot \|_\infty^B < \infty$ by (EK4)(b), so Proposition 5.8.36(ii) with $k = e$ gives

$$I_R^U(\phi) = \sum_{\gamma \in C_R} I_\gamma(e \otimes \phi),$$

the orbital integrals of Definition 5.7.1. Every $\gamma \in C_R$ has the irreducible characteristic polynomial R, a root of whose image under Φ_v lies off the unit circle; so $I_\gamma(e \otimes \phi) = 0$ by Proposition 5.8.38. Hence $I_R^U(\phi) = 0$.

Proof of (TM2). The hypothesis on roots of unity in E is not used. Let R be irreducible with all roots of $\Phi_v(R)$ on the unit circle for every v. If R is not monic of degree N, no $\gamma \in \mathrm{U}(U)(E^+)$ has characteristic polynomial R, so $I_R^U(\phi) = 0$ for every U, and $c_R(\phi) = 0$ works. Let R be monic of degree N; for a framed space U let $X_R(U)$, $\mathcal{C}(U)$, $\mathcal{C}_f(U)$ and $\mathcal{C}_v(s)$ be as in Proposition 5.8.44, and $\mathcal{C}_f$ the set $\mathcal{C}_f(U)$, which by Proposition 5.8.44(i) does not depend on U. For $C \in \mathcal{C}_f$ choose $y_C \in C$; it is the finite component of some $\gamma \in X_R(U)$ for some framed space U, so $Z(y_C)$ is discrete, $I_{y_C}^{\mathrm{fin}}(|\phi|) < \infty$, and $I_{y_C}^{\mathrm{fin}}(\phi)$ does not depend on the choice of y_C (Proposition 5.8.42 (i), (iii) with $k = \|\cdot\|_\infty^{-B}$, (iv)); here I^{fin} is formed with μ_f. Put

$$c_R(\phi) = \sum_{C \in \mathcal{C}_f} I_{y_C}^{\mathrm{fin}}(\phi),$$

which involves only R, ϕ and μ_f; its absolute convergence is shown below.

Fix U, s, λ as in the statement and $e = e_\lambda^U$. The identities of Proposition 5.8.36(ii) and Proposition 5.8.42(ii) are homogeneous under the changes of measure of the first paragraph (replacing dg_v by $\tau\, dg_v$ replaces $d(\lambda_v, s_v)$ by $d(\lambda_v, s_v)/\tau$ and leaves both sides unchanged, and replacing μ_f by $\tau'\mu_f$ multiplies both sides by τ'), so they hold for the measures of the statement. By Proposition 5.8.36(ii) with $k = e$,

$$I_R^U(\phi) = \sum_{[\gamma] \in \mathcal{C}(U)} I_\gamma(e \otimes \phi), \qquad \sum_{[\gamma] \in \mathcal{C}(U)} |I_\gamma(e \otimes \phi)| < \infty.$$

For $\gamma \in X_R(U)$, Proposition 5.8.42(ii), (iii) give $I_\gamma(e \otimes \phi) = I_\gamma^\infty(e)\, I_{\gamma_f}^{\mathrm{fin}}(\phi)$. For an infinite place v and $t \in \mathrm{U}(\operatorname{diag} s)$ with characteristic polynomial $\Phi_v(R)$ put

$$A_v(t) = \int_{\mathrm{U}(\operatorname{diag} s)} d(\lambda_v, s)\, \overline{c_{\lambda_v, s}(h^{-1}th)}\, dh.$$

By Lemma 5.8.43(iv) the class of t contains some $\operatorname{diag}(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(N)})$, ζ_i the roots of $\Phi_v(R)$, so by Lemma 5.8.49 (applicable: $2\lambda_v$ has gap $500N^3 + 10N^5$ for s, the ζ_i are distinct of absolute value 1) the integral converges absolutely; by invariance of dh under left translation it depends only on the class $[t] \in \mathcal{C}_v(s)$. As e is the product of these kernels over the places, Fubini's theorem gives $I_\gamma^\infty(e) = \prod_v A_v([\gamma_v])$. With Proposition 5.8.44(ii) and Proposition 5.8.42(iv), the classes $[\gamma]$ correspond to the pairs $(C, (\omega_v)_v) \in \mathcal{C}_f \times \prod_v \mathcal{C}_v(s)$, and

$$I_\gamma(e \otimes \phi) = I_{y_C}^{\mathrm{fin}}(\phi) \prod_v A_v(\omega_v).$$

Absolute convergence of $c_R(\phi)$. Let $k_0 = \|\cdot\|_\infty^{-B}$. By Proposition 5.8.36(ii) for (k_0, ϕ) and the Tonelli identity in the proof of Proposition 5.8.42(ii), $\sum_{[\gamma] \in \mathcal{C}(U)} I_\gamma^\infty(k_0)\, I_{\gamma_f}^{\mathrm{fin}}(|\phi|) < \infty$. Fix $(\omega_v^0)_v \in \prod_v \mathcal{C}_v(s)$ (nonempty sets, Proposition 5.8.44(iii)). For the classes $[\gamma]$ corresponding to $(C, (\omega_v^0)_v)$, $C \in \mathcal{C}_f$, the element γ_∞ runs through one conjugacy class of G_∞^U, so $I_\gamma^\infty(k_0)$ is a constant κ, and $\kappa > 0$ (a positive continuous integrand). Hence $\sum_{C \in \mathcal{C}_f} I_{y_C}^{\mathrm{fin}}(|\phi|) < \infty$.

Summation. The sets $\mathcal{C}_v(s)$ are finite and the series over $\mathcal{C}_f$ converges absolutely, so

$$I_R^U(\phi) = \sum_{C \in \mathcal{C}_f} \sum_{(\omega_v) \in \prod_v \mathcal{C}_v(s)} I_{y_C}^{\mathrm{fin}}(\phi) \prod_v A_v(\omega_v) = c_R(\phi) \prod_v S_v, \qquad S_v = \sum_{\omega \in \mathcal{C}_v(s)} A_v(\omega).$$

Let $p = \#\{i : s(i) = 1\}$, $q = N - p$. By Lemma 5.8.43(iv) each class $\omega \in \mathcal{C}_v(s)$ contains $\mathrm{diag}(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(N)})$ for exactly $p!\,q!$ permutations σ, and every such diagonal matrix lies in one of these classes; so by Lemma 5.8.49

$$S_v = \frac{1}{p!\,q!} \sum_{\sigma \in S_N} A_v(\mathrm{diag}(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(N)})) = (-1)^{pq}\, \overline{\chi_{\lambda_v}(z_v)}.$$

The signs s are the same at all $[E^+ : \mathbb{Q}]$ infinite places, an even number, so $\prod_v (-1)^{pq} = 1$ and $I_R^U(\phi) = c_R(\phi) \prod_v \overline{\chi_{\lambda_v}(z_v)}$.

Proof of the consequence. Let $R \in E[X]$. If R is irreducible with a root of absolute value $\neq 1$ under some Φ_v, then $I_R^V(\phi) = 0 = I_R^{V'}(\phi)$ by (TM1), applied to $U = V$ and $U = V'$. If R is irreducible with all roots on the unit circle, then by (TM2), with the same $c_R(\phi)$ for both spaces, $I_R^V(\phi) = c_R(\phi) \prod_v \overline{\chi_{\lambda_v}(z_v)} = I_R^{V'}(\phi)$; the right side involves neither the space nor its signs. If R is not irreducible, both terms vanish by (TM3). □

Proposition 5.8.31. *Let E be a CM field with totally real subfield E^+ and a CM type $(\Phi_v)_v$, let $N \geq 2$, and let U be a framed space of rank N over E whose signs do not depend on the infinite place, $s_v = s$ for all v. Let $\lambda = (\lambda_v)_v$, $\lambda_v \in (\frac{N-1}{2} + \mathbb{Z})^N$, have pairwise distinct entries and $|\lambda_{v,i} - \lambda_{v,j}| \geq 500N^3 + 10N^5$ whenever $s(i) \neq s(j)$. Let $K \subseteq \mathrm{K}_f$ be open, $a \in \mathbb{C}[G_f]$ finitely supported and $\phi(g) = \sum_h a_h \mathbf{1}_K(h^{-1}g)$. Suppose that ϕ is cuspidal at a finite place w of E with $w \neq \bar{w}$, and that w' is a finite place of E with $w' \neq \bar{w}'$ such that for every $g \in G_f$ with $\phi(g) \neq 0$ the component $g_{w'} \in \mathrm{GL}_N(E_{w'})$ has entries in $\mathcal{O}_{w'}$ and 0 is a simple root of the characteristic polynomial of its reduction modulo w'. If $R \in E[X]$ is not irreducible, then $I_R^U(\phi) = 0$.*

Proof. Step 1: I_R^U is a linear form on all test functions. Every locally constant compactly supported f on G_f is right invariant under an open subgroup $K' \subseteq \mathrm{K}_f$ and is a finite combination of indicators of cosets hK', that is $f(g) = \sum_h b_h \mathbf{1}_{K'}(h^{-1}g)$ with b finitely supported. By Theorem 5.8.6(ii), applied to K' and b, the sum over γ in $I_R^U(f)$ converges absolutely for every x and is integrable on $\Gamma_U \backslash G^U$. So $I_R^U(f)$ is defined for every such f, and $f \mapsto I_R^U(f)$ is linear.

Step 2: invariance under conjugation. Let $g \in G_f$ and $f^g(z) = f(g^{-1}zg)$. The function $x \mapsto \sum_{\mathrm{charpoly}(\gamma)=R} (e_\lambda^U \otimes f)(x^{-1}\gamma x)$ is left invariant under Γ_U (replacing x by δx permutes the γ by $\gamma \mapsto \delta^{-1}\gamma\delta$, which keeps the characteristic polynomial), and its value for f^g at x is its value for f at $x(1, g)$. The quotient measure on $\Gamma_U \backslash G^U$ is invariant under right translation by G^U, so $I_R^U(f^g) = I_R^U(f)$.

Step 3: support. For $\gamma \in \mathrm{U}(U)(E^+)$ and $x \in G^U$, the image of γ in G_f is conjugate in $\mathrm{GL}_N(\mathbb{A}_{E,f})$, through the finite frame, to the matrix of γ; hence for every finite place u of E at which G_f has the factor $\mathrm{GL}_N(E_u)$, the component $(x^{-1}\gamma x)_{f,u}$ has as characteristic polynomial the image of $\mathrm{charpoly}(\gamma)$ in $E_u[X]$. So if f vanishes at all z with $\mathrm{charpoly}(z_w) = R$, every term of $I_R^U(f)$ vanishes and $I_R^U(f) = 0$.

Step 4: R is a product of two coprime factors. Suppose $I_R^U(\phi) \neq 0$. Then some γ with $\mathrm{charpoly}(\gamma) = R$ and some x give $\phi(g) \neq 0$ for $g = (x^{-1}\gamma x)_f$; so R is monic of degree N, and by Step 3 at w' and the hypothesis, R has coefficients in $\mathcal{O}_{w'}$ and 0 is a simple root of its reduction $\bar{R}$ modulo w'. R is not a proper power: if $R = Q^j$ with $Q \in E[X]$ and $j \geq 2$, we may take Q monic (divide by its leading coefficient, a j-th root of 1); the roots of Q are roots of the monic integral polynomial R, hence integral over $\mathcal{O}_{w'}$, so the coefficients of Q, symmetric functions of them, lie in $\mathcal{O}_{w'}$; then $\bar{R} = \bar{Q}^j$ and $\bar{Q}(0) = 0$, so 0 has multiplicity at least $j \geq 2$ in $\bar{R}$, a contradiction. Let P be a monic irreducible factor of R and $R = P^i C$ with $i \geq 1$ and $P \nmid C$. If $C = 1$, then $R = P$ is irreducible or $R = P^i$ with $i \geq 2$ is a proper power, both excluded. So $A = P^i$ and C are monic of positive degree and coprime, and $R = AC$; their images in $E_w[X]$ are still coprime (a Bezout identity $uA + vC = 1$ maps to $E_w[X]$).

Step 5: conclusion. By Steps 1–3 the linear form $D = I_R^U$ satisfies (i) and (ii) of Proposition 5.8.32 for the images of A and C in $E_w[X]$ (for (i) take $g = \iota_w(g_w)$ in Step 2). Since ϕ is cuspidal at w, Proposition 5.8.32 gives $I_R^U(\phi) = 0$, contradicting the assumption. Hence $I_R^U(\phi) = 0$. □

Proposition 5.8.32. *Let E be a CM field, $N \geq 2$, w a finite place of E with $w \neq \bar{w}$, and $A, C \in E_w[X]$ monic, coprime and of positive degrees with $\deg A + \deg C = N$; let $\Omega = \{z \in G_f : \mathrm{charpoly}(z_w) = AC\}$. Let D be a linear form on the space of locally constant compactly supported functions on G_f such that*

1. *$D\big(f(\iota_w(g)^{-1} \cdot \iota_w(g))\big) = D(f)$ for all f and all $g \in \mathrm{GL}_N(E_w)$, and*
2. *$D(f) = 0$ for every f that vanishes on Ω.*

Let $K \subseteq \mathrm{K}_f$ be open, $b \in \mathbb{C}[G_f]$ finitely supported, and $\phi(g) = \sum_h b_h \mathbf{1}_K(h^{-1}g)$. If ϕ is cuspidal at w, then $D(\phi) = 0$.

Proof. Put $F = E_w$, $G_w = \mathrm{GL}_N(F)$, $K_w = \mathrm{GL}_N(\mathcal{O}_w)$, $a = \deg A$, $c = \deg C$, and let X, M, $g \cdot z = \iota_w(g) z \iota_w(g)^{-1}$ and $\Psi \mapsto \Psi^\flat$ be as in Lemma 5.8.33. Let $N_a \subseteq G_w$ be the group of matrices $n_Z = \begin{pmatrix} 1_a & Z \\ 0 & 1_c \end{pmatrix}$, $Z \in \mathrm{M}_{a\times c}(F)$, with the Haar measure dn that is additive Haar measure in Z, and $P = MN_a$.

Step 1: a form on Ω. Ω is closed in G_f (Lemma 5.8.33(i)), so every locally constant compactly supported f_0 on Ω extends to one on G_f: cover its support by finitely many disjoint sets $V_i \cap \Omega$, $V_i \subseteq G_f$ compact open and pairwise disjoint, on which f_0 is constant, and take $f = \sum_i f_0(V_i \cap \Omega)\mathbf{1}_{V_i}$. By (ii), $D_\Omega(f_0) = D(f)$ does not depend on the extension, and by (i), since Ω is stable under G_w, $D_\Omega(f_0(g^{-1} \cdot \,\cdot)) = D_\Omega(f_0)$.

Step 2: a form on $G_w \times X$. For a locally constant compactly supported Ψ on $G_w \times X$ put $D'(\Psi) = D_\Omega(\Psi^\flat)$ (Lemma 5.8.33(ii)). By the equivariance in Lemma 5.8.33(ii) and Step 1, $D'(\Psi(g_0^{-1}\cdot, \cdot)) = D_\Omega(\Psi^\flat(g_0^{-1} \cdot \,\cdot)) = D'(\Psi)$. X is a closed subset of G_f, hence locally compact, Hausdorff and totally disconnected; so Lemma 5.8.34 applies with $H = G_w$ and $Y = X$: $D'(\Psi) = 0$ whenever $\int_{G_w} \Psi(g, y)\, dg = 0$ for every $y \in X$.

Step 3: the function Ψ. Put $\Psi(g, y) = \mathbf{1}_{K_w N_a}(g)\, \phi(g \cdot y)$. The set $K_w N_a$ is open (a union of translates of K_w) and closed (compact times closed), so Ψ is locally constant. For the support, let $y \in X$ with $y_w = \mathrm{diag}(y_1, y_2)$, and $n = n_Z$. Since products of two matrices of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ vanish,

$$n_Z y_w n_Z^{-1} = \begin{pmatrix} y_1 & L_y(Z) \\ 0 & y_2 \end{pmatrix}, \qquad L_y(Z) = Zy_2 - y_1 Z. \tag{5.179}$$

L_y is a linear automorphism of $\mathrm{M}_{a\times c}(F)$: $Zy_2 = y_1 Z$ forces $Z = 0$ because y_1, y_2 have no common eigenvalue (their characteristic polynomials A, C are coprime). If $\Psi(g, y) \neq 0$ write $g = k n_Z$ with $k \in K_w$; then $z' = n_Z \cdot y$ lies in the compact set $C' = \bigcup_{k \in K_w} k^{-1} \cdot \mathrm{supp}\, \phi$, and by (5.179) z'_w is block upper triangular, y equals z' away from the place of E^+ below w, y_w is the block diagonal of z'_w, and $Z = L_y^{-1}$ applied to the corner of z'_w. These are continuous functions of z' on the compact set of block upper triangular elements of C', so (g, y) lies in a compact set: Ψ has compact support.

Step 4: the image of Ψ is a multiple of ϕ. Let $z = g \cdot y \in \Omega$ and write $g = kmn$ with $k \in K_w$, $m \in M$, $n \in N_a$ (Iwasawa decomposition $G_w = K_w P$). For $h \in M$, $gh^{-1} = k(mh^{-1})(hnh^{-1})$ with $hnh^{-1} \in N_a$, so $gh^{-1} \in K_w N_a$ iff $mh^{-1} \in K_w N_a \cap M$. This intersection is $K_w \cap M$: if

$mh^{-1} = k'n'$, then $k' \in K_w \cap P$, and an element of $K_w \cap P$ has integral invertible diagonal blocks, so $K_w \cap P = (K_w \cap M)(K_w \cap N_a)$ and $mh^{-1} \in K_w \cap M$. As $\Psi(gh^{-1}, h \cdot y) = \mathbf{1}_{K_w N_a}(gh^{-1})\phi(z)$,

$$\Psi^\flat(z) = \phi(z) \int_M \mathbf{1}_{(K_w \cap M)m}(h)\, dh = c_0\, \phi(z), \qquad c_0 = \mathrm{vol}(K_w \cap M) > 0.$$

Step 5: the fibre integrals vanish. Let $y \in X$. By the integration formula of the Iwasawa decomposition, for suitable Haar measures $\int_{G_w} f(g)\, dg = \int_{K_w} \int_M \int_{N_a} f(kmn)\, dn\, dm\, dk$ for compactly supported continuous f. For $f(g) = \Psi(g, y)$ the integrand vanishes unless $m \in K_w \cap M$ (Step 4 with $h = 1$), and then $km \in K_w$; substituting $k \mapsto km^{-1}$,

$$\int_{G_w} \Psi(g, y)\, dg = c_0 \int_{K_w} \int_{N_a} \phi((kn) \cdot y)\, dn\, dk.$$

Now $n \cdot y = y\, \iota_w(\theta_y(n))$ with $\theta_y(n) = y_w^{-1} n y_w n^{-1} \in N_a$, and by (5.179) $\theta_y(n_Z) = n_{Z'}$ with $Z' = y_1^{-1} L_y(Z)$, a linear automorphism of $\mathrm{M}_{a \times c}(F)$. So

$$\int_{N_a} \phi((kn) \cdot y)\, dn = |\det(y_1^{-1} L_y)|^{-1} \int_{N_a} \phi\big(\iota_w(k)\, y\, \iota_w(n)\, \iota_w(k)^{-1}\big)\, dn = 0$$

by Lemma 5.8.35 with $j = a$ and $x = y$.

Conclusion. By Steps 2, 4 and 5, $c_0 D(\phi) = D_\Omega(c_0 \phi|_\Omega) = D_\Omega(\Psi^\flat) = D'(\Psi) = 0$, so $D(\phi) = 0$. □

Lemma 5.8.33. *Let E be a CM field, $N \geq 2$, and w a finite place of E with $w \neq \bar{w}$; put $F = E_w$ and $G_w = \mathrm{GL}_N(F)$. Let $A, C \in F[X]$ be monic and coprime, of degrees $a, c \geq 1$ with $a + c = N$. Let*

$$\Omega = \{z \in G_f :\ \mathrm{charpoly}(z_w) = AC\},$$
$$X = \{y \in \Omega :\ y_w = \mathrm{diag}(y_1, y_2),\ \mathrm{charpoly}(y_1) = A,\ \mathrm{charpoly}(y_2) = C\},$$

with $y_1 \in \mathrm{GL}_a(F)$, $y_2 \in \mathrm{GL}_c(F)$, and let $M = \mathrm{GL}_a(F) \times \mathrm{GL}_c(F) \subseteq G_w$ (block-diagonal), with a Haar measure dh. Write $g \cdot z = \iota_w(g)\, z\, \iota_w(g)^{-1}$ for $g \in G_w$, $z \in G_f$. Then:

1. *Ω and X are closed in G_f; $\Omega = G_w \cdot X$; and if $g \cdot y = g' \cdot y'$ with $y, y' \in X$, then $h = g'^{-1} g \in M$ and $y' = h \cdot y$.*
2. *For every locally constant compactly supported Ψ on $G_w \times X$ the formula*

$$\Psi^\flat(g \cdot y) = \int_M \Psi(gh^{-1},\, h \cdot y)\, dh \qquad (g \in G_w,\ y \in X)$$

defines a locally constant compactly supported function $\Psi^\flat$ on Ω, and for $g_0 \in G_w$ the function $(g, y) \mapsto \Psi(g_0^{-1} g, y)$ has image $z \mapsto \Psi^\flat(g_0^{-1} \cdot z)$.

Proof. Choose $u, v \in F[X]$ with $uA + vC = 1$. For a matrix $Z \in \mathrm{M}_N(F)$ with $\mathrm{charpoly}(Z) = AC$ put $e(Z) = (vC)(Z)$. By Cayley–Hamilton $(AC)(Z) = 0$, so $e(Z)(1 - e(Z)) = (uv\, AC)(Z) = 0$: $e(Z)$ is an idempotent. Its image is $\ker A(Z)$: $A(Z)e(Z) = (vAC)(Z) = 0$, and if $A(Z)x = 0$ then $x = (uA + vC)(Z)x = e(Z)x$. Likewise $\ker e(Z) = \mathrm{im}(1 - e(Z)) = \ker C(Z)$. So $F^N = \ker A(Z) \oplus \ker C(Z)$, both summands Z-stable. The restriction Z_1 of Z to $\ker A(Z)$ is killed by A, so every root of its characteristic polynomial is a root of A; likewise for the restriction Z_2 to $\ker C(Z)$ and C. Since $\mathrm{charpoly}(Z_1)\mathrm{charpoly}(Z_2) = AC$ and A, C have no common root, $\mathrm{charpoly}(Z_1) = A$ and $\mathrm{charpoly}(Z_2) = C$; in particular $\dim \ker A(Z) = a$. For $y \in X$, $A(y_2)$

is invertible ($uA(y_2) = 1 - vC(y_2) = 1$), so $\ker A(y_w) = F^a \oplus 0$, $\ker C(y_w) = 0 \oplus F^c$, and $e(y_w) = \varepsilon := \operatorname{diag}(1_a, 0_c)$. Also $e(gZg^{-1}) = g\,e(Z)\,g^{-1}$.

(i). The coefficients of $\operatorname{charpoly}(z_w)$ and the blocks of z_w depend continuously on z, so Ω and X are closed. Let $z \in \Omega$ and choose $g \in G_w$ mapping $F^a \oplus 0$ onto $\ker A(z_w)$ and $0 \oplus F^c$ onto $\ker C(z_w)$; then $g^{-1} z_w g$ preserves both coordinate subspaces, and its blocks are conjugate to Z_1, Z_2 for $Z = z_w$, with characteristic polynomials A, C. So $y = g^{-1} \cdot z \in X$ (its components away from the place of E^+ below w are those of z). If $g \cdot y = g' \cdot y'$ with $y, y' \in X$, then $h = g'^{-1} g$ satisfies $h y_w h^{-1} = y'_w$, hence maps $\ker A(y_w) = F^a \oplus 0$ onto $\ker A(y'_w) = F^a \oplus 0$ and $0 \oplus F^c$ onto itself; so $h \in M$, and $y' = h \cdot y$.

Local sections. Let $z_0 = g_0 \cdot y_0 \in \Omega$ with $y_0 \in X$. The map $z \mapsto e(z_w)$ is continuous on Ω and $e(z_{0,w}) = g_0 \varepsilon g_0^{-1}$. Put $x(z) = e(z_w)e(z_{0,w}) + (1 - e(z_w))(1 - e(z_{0,w}))$; it is continuous with $x(z_0) = 1$, so it is invertible on an open neighbourhood U of z_0 in Ω, and $x(z)e(z_{0,w}) = e(z_w)e(z_{0,w}) = e(z_w)x(z)$. For $z \in U$ put $g(z) = x(z) g_0$; then $g(z) \varepsilon g(z)^{-1} = e(z_w)$, so $g(z)^{-1} z_w g(z)$ commutes with ε (as z_w commutes with the polynomial $e(z_w)$ in z_w) and is block diagonal with blocks conjugate to the restrictions of z_w to $\ker A(z_w)$, $\ker C(z_w)$. Hence $y(z) = g(z)^{-1} \cdot z \in X$, and $g \colon U \to G_w$, $y \colon U \to X$ are continuous with $z = g(z) \cdot y(z)$.

(ii), well defined. Let S be the compact support of Ψ and S_1 its projection to G_w. For fixed (g, y) the integrand vanishes unless $gh^{-1} \in S_1$, i.e. $h \in M \cap S_1^{-1} g$, a compact set; the integrand is locally constant, so the integral is finite. If $g \cdot y = g' \cdot y'$, then by (i) $g' = g h_0^{-1}$ and $y' = h_0 \cdot y$ with $h_0 \in M$, and the substitution $h \mapsto h h_0^{-1}$ ($M = \mathrm{GL}_a(F) \times \mathrm{GL}_c(F)$ is unimodular) gives $\int_M \Psi(g' h^{-1}, h \cdot y')\, dh = \int_M \Psi(gh^{-1}, h \cdot y)\, dh$. So $\Psi^\flat$ is well defined on $\Omega = G_w \cdot X$.

(ii), locally constant with compact support. Let $z_0 \in \Omega$, and U, $g(z)$, $y(z)$ as above; shrink U to a compact neighbourhood U_0 of z_0 in Ω. For $z \in U_0$ the integrand $(z, h) \mapsto \Psi(g(z) h^{-1}, h \cdot y(z))$ vanishes unless h lies in the compact set $Q = M \cap S_1^{-1} g(U_0)$, and it is locally constant on $U_0 \times Q$, being a locally constant function composed with a continuous map. Cover $\{z_0\} \times Q$ by finitely many open boxes on which it is constant; on the intersection U_1 of their first factors, $\Psi^\flat(z) = \int_Q \Psi(g(z) h^{-1}, h \cdot y(z))\, dh = \Psi^\flat(z_0)$. The support of $\Psi^\flat$ lies in the image of S under the continuous map $(g, y) \mapsto g \cdot y$, which is compact.

(ii), equivariance. For $z = g \cdot y$, $g_0^{-1} \cdot z = (g_0^{-1} g) \cdot y$, so $\int_M \Psi(g_0^{-1} g h^{-1}, h \cdot y)\, dh = \Psi^\flat((g_0^{-1} g) \cdot y) = \Psi^\flat(g_0^{-1} \cdot z)$. □

Lemma 5.8.34. *Let H be a locally compact totally disconnected group with a left Haar measure μ, and Y a locally compact totally disconnected Hausdorff space. For such a space Z let $C_c^\infty(Z)$ be the space of locally constant complex functions of compact support on Z. Let $D \colon C_c^\infty(H \times Y) \to \mathbb{C}$ be a linear form with*

$$D\big(\Psi(h_0^{-1} \cdot, \cdot)\big) = D(\Psi) \qquad (h_0 \in H,\ \Psi \in C_c^\infty(H \times Y)).$$

Then there is a linear form T on $C_c^\infty(Y)$ with $D(\Psi) = T\big(y \mapsto \int_H \Psi(h, y)\, d\mu(h)\big)$ for every Ψ. In particular $D(\Psi) = 0$ whenever $\int_H \Psi(h, y)\, d\mu(h) = 0$ for every $y \in Y$.

Proof. Product decomposition. Let $\Psi \in C_c^\infty(H \times Y)$. Compact open sets form a basis of both spaces, so every point of the compact support of Ψ has a neighbourhood $U \times V$, with $U \subseteq H$ and $V \subseteq Y$ compact open, on which Ψ is constant; finitely many such boxes $U_1 \times V_1, \dots, U_m \times V_m$ cover the support. Let α run over the nonempty sets $\bigcap_{i \in S} U_i \cap \bigcap_{i \notin S} (H \smallsetminus U_i)$, $S \neq \emptyset$, and β over the analogous sets for the V_i; they are compact open, and the products $\alpha \times \beta$ partition $(\bigcup U_i) \times (\bigcup V_i)$. If a product $\alpha \times \beta$ meets a box $U_i \times V_i$, then $\alpha \subseteq U_i$ and $\beta \subseteq V_i$, so $\alpha \times \beta \subseteq U_i \times V_i$ and Ψ is constant on it; otherwise $\alpha \times \beta$ misses the support. Hence $\Psi = \sum_j c_j\, \mathbf{1}_{\alpha_j} \otimes \mathbf{1}_{\beta_j}$, a finite sum of products $f \otimes \psi$ with $f \in C_c^\infty(H)$, $\psi \in C_c^\infty(Y)$.

Invariant forms on H. Let L be a linear form on $C_c^\infty(H)$ with $L(f(h_0^{-1}\cdot)) = L(f)$, and fix a compact open subgroup K_0. Every $f \in C_c^\infty(H)$ is right invariant under some compact open subgroup $K_1 \subseteq K_0$ (local constancy and compact support), so $f = \sum_j c_j \mathbf{1}_{g_j K_1}$ with finitely many disjoint cosets. By invariance $L(\mathbf{1}_{gK_1}) = L(\mathbf{1}_{K_1})$, and $\mathbf{1}_{K_0}$ is the sum of the $[K_0 : K_1]$ indicators $\mathbf{1}_{kK_1}$, so $L(\mathbf{1}_{K_1}) = L(\mathbf{1}_{K_0})/[K_0 : K_1] = L(\mathbf{1}_{K_0})\mu(K_1)/\mu(K_0)$. Hence $L(f) = \sum_j c_j L(\mathbf{1}_{K_0})\mu(K_1)/\mu(K_0) = \big(L(\mathbf{1}_{K_0})/\mu(K_0)\big)\int_H f\,d\mu$.

Conclusion. For $\psi \in C_c^\infty(Y)$ the form $f \mapsto D(f\otimes\psi)$ is invariant in this sense, so $D(f\otimes\psi) = T(\psi)\int_H f\,d\mu$ with $T(\psi) = D(\mathbf{1}_{K_0}\otimes\psi)/\mu(K_0)$, which is linear in ψ. For $\Psi = \sum_i f_i \otimes \psi_i$,

$$D(\Psi) = \sum_i T(\psi_i)\int_H f_i\,d\mu = T\Big(\sum_i \Big(\int_H f_i\,d\mu\Big)\psi_i\Big) = T\Big(y \mapsto \int_H \Psi(h,y)\,d\mu(h)\Big).$$

□

Lemma 5.8.35. *Let E be a CM field, $N \geq 2$, $K \subseteq \mathrm{K}_f$ open, $a \in \mathbb{C}[G_f]$ finitely supported, $\phi(g) = \sum_h a_h \mathbf{1}_K(h^{-1}g)$, and let w be a finite place of E with $w \neq \bar{w}$ such that ϕ is cuspidal at w. For $1 \leq j \leq N-1$ let $N_j \subseteq \mathrm{GL}_N(E_w)$ be the group of matrices $\begin{pmatrix} 1_j & Z \\ 0 & 1_{N-j}\end{pmatrix}$, with a Haar measure dn. Then for all $k \in \mathrm{GL}_N(\mathcal{O}_w)$ and $x \in G_f$ the function $n \mapsto \phi(\iota_w(k)\,x\,\iota_w(n)\,\iota_w(k)^{-1})$ on N_j is locally constant with compact support, and*

$$\int_{N_j} \phi\big(\iota_w(k)\,x\,\iota_w(n)\,\iota_w(k)^{-1}\big)\,dn = 0.$$

Proof. Write $\bar{m}$ for the reduction modulo w of $m \in \mathrm{GL}_N(\mathcal{O}_w)$, k_w for the residue field, and $K_w(1)$ for the kernel of $m \mapsto \bar{m}$. Take $T = \{1,\dots,j\}$; then $U_T = \{1 + \epsilon Z(1-\epsilon)\} \subseteq \mathrm{GL}_N(k_w)$ is the group of matrices $\begin{pmatrix} 1_j & Z \\ 0 & 1\end{pmatrix}$ over k_w, and reduction maps $N_j(\mathcal{O}_w) = N_j \cap \mathrm{GL}_N(\mathcal{O}_w)$ onto U_T with kernel $N_j \cap K_w(1)$.

Right invariance under $\iota_w(K_w(1))$. Let $g \in G_f$, $t \in \mathrm{GL}_N(k_w)$, and take two families of lifts of the elements ut ($u \in U_T$) that differ only at one u_0, with lifts m, m' there. Subtracting the two vanishing sums of the cuspidality condition gives $\phi(g\,\iota_w(m)) = \phi(g\,\iota_w(m'))$. Every element of $\mathrm{GL}_N(k_w)$ is of the form $u_0 t$ (take $u_0 = 1$), so $\phi(g\,\iota_w(m))$ depends only on $\bar{m}$; with $m = 1$ and $m' \in K_w(1)$ this gives $\phi(g\,\iota_w(m')) = \phi(g)$.

Integrals over $N_j(\mathcal{O}_w)$. Fix k and put $\phi_k(z) = \phi(\iota_w(k) z \iota_w(k)^{-1})$. As $K_w(1)$ is normal in $\mathrm{GL}_N(\mathcal{O}_w)$, ϕ_k is right invariant under $\iota_w(K_w(1))$. For $u \in U_T$ choose $\tilde{n}_u \in N_j(\mathcal{O}_w)$ reducing to u. Then for every $x' \in G_f$

$$\int_{N_j(\mathcal{O}_w)} \phi_k(x'\iota_w(n))\,dn = \mathrm{vol}(N_j \cap K_w(1)) \sum_{u \in U_T} \phi\big(\iota_w(k) x'\,\iota_w(\tilde{n}_u k^{-1})\big) = 0,$$

because $\tilde{n}_u k^{-1} \in \mathrm{GL}_N(\mathcal{O}_w)$ is a lift of $u\bar{k}^{-1}$, so the sum is the cuspidality sum for $g = \iota_w(k)x'$ and $t = \bar{k}^{-1}$.

Integral over N_j. The function $n \mapsto \phi_k(x\iota_w(n))$ is locally constant, and it vanishes unless $\iota_w(n)$ lies in the compact set $x^{-1}\iota_w(k)^{-1}\,\mathrm{supp}(\phi)\iota_w(k)$; since ι_w is a closed embedding and N_j is closed, its support is compact. So it meets only finitely many cosets $n_i N_j(\mathcal{O}_w)$ of the open subgroup $N_j(\mathcal{O}_w)$, and

$$\int_{N_j} \phi_k(x\iota_w(n))\,dn = \sum_i \int_{N_j(\mathcal{O}_w)} \phi_k\big(x\iota_w(n_i)\iota_w(n)\big)\,dn = 0$$

by the previous step with $x' = x\iota_w(n_i)$. □

Proposition 5.8.36. *Let E be a CM field with maximal totally real subfield E^+, $N \geq 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs s, and $G^U = G_\infty \times G_f$. Let $B \geq 13N^3$ be an integer, $k\colon G_\infty \to \mathbb{C}$ measurable with $\sup_{y \in G_\infty} |k(y)|\, \|y\|_\infty^B < \infty$, $\phi\colon G_f \to \mathbb{C}$ locally constant with compact support, and $R \in E[X]$ irreducible. Let X_R be the set of $\gamma \in \mathrm{U}(U)(E^+)$ with* $\mathrm{charpoly}(\gamma) = R$*, and $C_R \subseteq X_R$ a set of representatives of the $\mathrm{U}(U)(E^+)$-conjugacy classes contained in X_R. Then:*

(i) for every $\gamma \in X_R$ the function $x \mapsto (k \otimes \phi)(x^{-1}\gamma x)$ is integrable on $Z_\gamma \backslash G^U$;

(ii) $\sum_{\gamma \in C_R} \int_{Z_\gamma \backslash G^U} |(k \otimes \phi)(x^{-1}\gamma x)|\, dx < \infty$, the series $\sum_{\gamma \in X_R} (k \otimes \phi)(x^{-1}\gamma x)$ converges absolutely for every $x \in G^U$, its sum is integrable on $\Gamma_U \backslash G^U$, and

$$\int_{\Gamma_U \backslash G^U} \sum_{\gamma \in X_R} (k \otimes \phi)(x^{-1}\gamma x)\, dx = \sum_{\gamma \in C_R} I_\gamma(k \otimes \phi).$$

Proof. Write $G = G^U$, $\Gamma_0 = \mathrm{U}(U)(E^+)$, identified with Γ_U through the frames, and $f_\gamma(x) = (k \otimes \phi)(x^{-1}\gamma x)$ for $\gamma \in \Gamma_0$, $x \in G$. These functions are measurable (k is measurable, ϕ is locally constant). For $\delta \in \Gamma_0$, $f_\gamma(\delta x) = f_{\delta^{-1}\gamma\delta}(x)$, and f_γ is left invariant under Z_γ.

Quotient measures. G is a second countable locally compact group, and $\Gamma_0 \subseteq \mathrm{GL}_N(E)$ is countable; every subgroup Δ of Γ_U is a countable discrete subgroup of G, and the quotient measure on $\Delta \backslash G$ is the measure with $\int_{\Delta\backslash G} \sum_{\delta \in \Delta} f(\delta x)\, dx = \int_G f$ for measurable $f \geq 0$ on G. Let $Z \subseteq \Delta$ be subgroups of Γ_U and $F \geq 0$ measurable on G and left Z-invariant. Then

$$\int_{Z\backslash G} F(x)\, dx = \int_{\Delta \backslash G} \sum_{\delta \in Z\backslash\Delta} F(\delta x)\, dx. \tag{5.180}$$

Indeed, a countable discrete subgroup of a second countable locally compact group has a Borel fundamental domain D (a Borel set meeting every orbit Zx in exactly one point). For $f = F\mathbf{1}_D$ we have $\sum_{\zeta \in Z} f(\zeta x) = F(x)$ for all x, hence $\int_{Z\backslash G} F = \int_G f = \int_{\Delta\backslash G} \sum_{\delta\in\Delta} f(\delta x)\, dx$, and grouping Δ into the cosets $Z\delta$ turns the inner sum into $\sum_{\delta \in Z\backslash\Delta} \sum_{\zeta \in Z} f(\zeta\delta x) = \sum_{\delta \in Z\backslash\Delta} F(\delta x)$. By linearity, (5.180) also holds for every complex left Z-invariant measurable F with $\int_{Z\backslash G} |F| < \infty$, the inner series then converging absolutely for almost every x.

A uniform bound. Every $\gamma \in X_R$ has irreducible characteristic polynomial, so Lemma 5.8.37 (applicable: $B \geq 13N^3$, $\sup |k|\, \|\cdot\|_\infty^B < \infty$, ϕ locally constant with compact support) gives $C \geq 0$ with

$$S(x) := \sum_{\gamma \in X_R} |f_\gamma(x)| \leq C \qquad (x \in G).$$

Conjugation by $\delta \in \Gamma_0$ preserves the characteristic polynomial and so permutes X_R; with $f_\gamma(\delta x) = f_{\delta^{-1}\gamma\delta}(x)$ this shows $S(\delta x) = S(x)$. By Theorem 4.2.18, applied to $\mathrm{U}(U)$ over E/E^+ (through the frames $G = \mathrm{U}(U)(\mathbb{A}_{E^+})$ and Γ_U is the image of $\mathrm{U}(U)(E^+)$), $\Gamma_U \backslash G$ has finite volume; so $\int_{\Gamma_U\backslash G} S \leq C \,\mathrm{vol}(\Gamma_U \backslash G) < \infty$.

Orbits. For $\gamma \in C_R$ the map $Z_\gamma \delta \mapsto \delta^{-1}\gamma\delta$ is a bijection from $Z_\gamma \backslash \Gamma_0$ onto the conjugacy class of γ ($\delta^{-1}\gamma\delta = \delta'^{-1}\gamma\delta'$ if and only if $\delta'\delta^{-1} \in Z_\gamma$), and X_R is the disjoint union of these classes. Hence

$$S(x) = \sum_{\gamma \in C_R} \sum_{\delta \in Z_\gamma \backslash \Gamma_0} |f_\gamma(\delta x)|.$$

Integrating over $\Gamma_U \backslash G$, exchanging the integral with the sum over C_R (Tonelli) and applying (5.180) with $Z = Z_\gamma$, $\Delta = \Gamma_U$, $F = |f_\gamma|$ gives

$$\sum_{\gamma \in C_R} \int_{Z_\gamma \backslash G} |f_\gamma| = \int_{\Gamma_U \backslash G} S < \infty.$$

This is the first assertion of (ii), and it gives (i) for $\gamma \in C_R$. For an arbitrary $\gamma \in X_R$ write $\gamma = \delta^{-1}\gamma'\delta$ with $\gamma' \in C_R$, $\delta \in \Gamma_0$; then $Z_\gamma = \delta^{-1}Z_{\gamma'}\delta$, $f_\gamma(x) = f_{\gamma'}(\delta x)$, and $x \mapsto \delta x$ induces a bijection $Z_\gamma\backslash G \to Z_{\gamma'}\backslash G$ which preserves the quotient measures (the measure of G is left invariant, and $\zeta \mapsto \delta\zeta\delta^{-1}$ carries counting measure on Z_γ to that on $Z_{\gamma'}$); so f_γ is integrable on $Z_\gamma\backslash G$. This proves (i).

The identity. As $S \le C$, the series $\sum_{\gamma\in X_R} f_\gamma(x)$ converges absolutely for every x; its sum is measurable, left Γ_U-invariant and bounded by S, hence integrable on $\Gamma_U\backslash G$. Since $\int_{\Gamma_U\backslash G}\sum_{\gamma\in C_R}\sum_{\delta\in Z_\gamma\backslash\Gamma_0}|f_\gamma(\delta x)|\,dx = \int S < \infty$, the integral may be exchanged with the double series (Fubini for the counting measure on the countable index set), and (5.180) for the integrable f_γ gives

$$\int_{\Gamma_U\backslash G}\sum_{\gamma\in X_R} f_\gamma = \sum_{\gamma\in C_R}\int_{\Gamma_U\backslash G}\sum_{\delta\in Z_\gamma\backslash\Gamma_0} f_\gamma(\delta x)\,dx = \sum_{\gamma\in C_R}\int_{Z_\gamma\backslash G} f_\gamma = \sum_{\gamma\in C_R} I_\gamma(k\otimes\phi),$$

the last integrals being absolutely convergent by (i), so that they are the orbital integrals (5.124). □

Lemma 5.8.37. *Let E be a CM field, $N \ge 1$, $s \in \{\pm1\}^N$, U a framed space of rank N over E with signs $s_v = s$ for all v, $G^U = G_\infty \times G_f$, and for $\gamma \in \Gamma_U$ let $\gamma_0 \in \mathrm{U}(U)(E^+)$ be the rational isometry whose image is γ. Let $B \ge 13N^3$ be an integer, $k\colon G_\infty \to \mathbb{C}$ with $\sup_y |k(y)|\,\|y\|_\infty^B < \infty$, and $\varphi\colon G_f \to \mathbb{C}$ locally constant with compact support. There is $C \ge 0$ such that for every $x \in G^U$*

$$\sum_{\substack{\gamma\in\Gamma_U:\ \mathrm{charpoly}(\gamma_0)\\ \textit{irreducible in } E[X]}} |k((x^{-1}\gamma x)_\infty)\,\varphi((x^{-1}\gamma x)_f)| \le C.$$

Proof. Let $f = [E^+ : \mathbb{Q}]$ and Z the set of $\gamma \in \Gamma_U$ with $\mathrm{charpoly}(\gamma_0)$ irreducible. As $\|y\|_\infty \ge 1$ on G_∞ and $B \ge 8N^2 + 1$, Proposition 5.8.22(ii) with $j = 0$, $r = 1$ gives C_0 with

$$\sum_{\gamma\in Z} |k((x^{-1}\gamma x)_\infty)\varphi((x^{-1}\gamma x)_f)| \le C_0\lambda_U(x)^{-4fN^2}$$

for all x ($\lambda_U(x)$ the first minimum of (VP4) of Definition 3.2.2). Let C_1 be the constant of Lemma 5.8.24 for k, φ, and κ that of Proposition 5.8.13(ii) for the open subgroup G_f; put $C = \max(C_0, C_1)\kappa^{4fN^2}$.

Let $x \in G^U$. The first minimum is positive (the nonzero vectors of the lattice $L_{x,\infty}$ of Lemma 5.8.15(iii) have norm bounded below), so there is $\delta > 0$ with $\delta^N = \kappa\lambda_U(x)$. If $\delta > 1$, then $\lambda_U(x)^{-1} < \kappa$ and the sum is at most $C_0\kappa^{4fN^2} \le C$. If $\delta \le 1$, Proposition 5.8.13(ii) gives a nonzero isotropic $A \subseteq U$ and $0 \le a \le b$ with $ab \le \delta$ such that A and $A^\perp$ are spanned by vectors of L_x of lengths at most a and b. By Lemma 5.8.27(i), no $\gamma \in Z$ has $\gamma_0 A = A$. So Lemma 5.8.24 applies to Z and bounds the sum by $C_1(\delta^N\lambda_U(x)^{-1})^{4fN^2} = C_1\kappa^{4fN^2} \le C$. □

Proposition 5.8.38. *Let E be a CM field with maximal totally real subfield E^+ and, for each infinite place v of E^+, the chosen embedding $\Phi_v\colon E \to \mathbb{C}$ above v. Let $N \ge 1$, $s \in \{\pm1\}^N$, U a framed space of rank N over E with signs s, $G^U = G_\infty \times G_f$, $B \ge 13N^3$ an integer, e a cusp kernel of order B on G_∞, and $\phi\colon G_f \to \mathbb{C}$ locally constant with compact support. Let $\gamma \in \mathrm{U}(U)(E^+)$ have irreducible characteristic polynomial $R \in E[X]$, and suppose that for some infinite place v some root of $\Phi_v(R)$ (the polynomial with coefficients mapped by Φ_v) has absolute value $\ne 1$. Then*

$$I_\gamma(e\otimes\phi) = 0.$$

Proof. For $t \in \mathrm{U}(U)(E^+)$ write t_∞ for the real component of its image in Γ_U and $t_v \in \mathrm{U}(\operatorname{diag} s)$ for the component of t_∞ at v: it is the matrix of $t \otimes 1$ on $U \otimes_{E,\Phi_v} \mathbb{C}$ in the frame at v. So $t \mapsto t_v$ is a homomorphism, and $A = \gamma_v$ has characteristic polynomial $\Phi_v(R)$. As R is irreducible over a field of characteristic 0 it is separable, so $\Phi_v(R)$ has N distinct roots $\zeta_1, \dots, \zeta_N$, and $|\zeta_i| \neq 1$ for some i. Hence there is $k \in \mathrm{GL}_N(\mathbb{C})$ with $kAk^{-1} = \operatorname{diag}(\zeta_1, \dots, \zeta_N)$. Let $q \in \mathbb{C}^{1\times N}$ be the i-th row of k, a nonzero row vector, and $|\cdot|$ the standard norm of row vectors; for a matrix b, $|wb| \le |w|\,\|b\|$, $\|b\|$ the operator norm (that of b^{T} is the same).

Rows of the diagonaliser. $qA = \zeta_i q$, the i-th row of $\operatorname{diag}(\zeta)k = kA$. If $t \in \mathrm{GL}_N(\mathbb{C})$ commutes with A, then $M = ktk^{-1}$ commutes with $\operatorname{diag}(\zeta)$; comparing entries, $\zeta_a M_{ab} = M_{ab}\zeta_b$, so $M_{ab} = 0$ for $a \neq b$ and M is diagonal and invertible. Hence $qt = M_{ii}\, q$ with $M_{ii} \neq 0$.

The function H. Put $H(y) = \log|q\, y_v|$ for $y \in G_\infty$; it is well defined (y_v is invertible and $q \neq 0$) and continuous. For $a, b \in G_\infty$, $|qa_v b_v| \le |qa_v|\,\|b_v\|$ and $|qa_v| \le |qa_v b_v|\,\|b_v^{-1}\|$, and $\|b_v^{-1}\| = \|b_v\|$ because $b_v^{-1} = \operatorname{diag}(s)\bar{b}_v^{\mathsf{T}}\operatorname{diag}(s)$; as $\|b_v\| \ge 1$ and $\log r \le r$,

$$|H(ab) - H(a)| \le \log\|b_v\| \le \|b_v\| \le \|b\|_\infty.$$

Moreover $H(\gamma_\infty y) = \log|qAy_v| = \log|\zeta_i| + H(y)$, and for $t \in \mathrm{U}(U)(E^+)$ commuting with γ, t_v commutes with A, so $H(t_\infty y) = \log|M_{ii}| + H(y)$ with $M = kt_v k^{-1}$.

Conclusion. Lemma 5.8.39 applies with $\ell = \log|\zeta_i|$ and gives $\log|\zeta_i| \cdot I_\gamma(e \otimes \phi) = 0$. Since $|\zeta_i| \neq 1$, $\log|\zeta_i| \neq 0$, so $I_\gamma(e \otimes \phi) = 0$. □

Lemma 5.8.39. *Let E be a CM field with maximal totally real subfield E^+, $N \ge 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs s, $G^U = G_\infty \times G_f$, $B \ge 13N^3$ an integer, e a cusp kernel of order B on G_∞, $\phi\colon G_f \to \mathbb{C}$ locally constant with compact support, and $\gamma \in \mathrm{U}(U)(E^+)$ with irreducible characteristic polynomial; for $t \in \mathrm{U}(U)(E^+)$ write $t_\infty \in G_\infty$ for the real component of its image in Γ_U. Let $H\colon G_\infty \to \mathbb{R}$ be measurable with*

$$|H(ab) - H(a)| \le \|b\|_\infty \qquad (a, b \in G_\infty),$$

let $\ell \in \mathbb{R}$ with $H(\gamma_\infty y) = \ell + H(y)$ for all $y \in G_\infty$, and suppose that for every $t \in \mathrm{U}(U)(E^+)$ commuting with γ there is $r_t \in \mathbb{R}$ with $H(t_\infty y) = r_t + H(y)$ for all y. Then

$$\ell \cdot I_\gamma(e \otimes \phi) = 0.$$

Proof. Let m, $c(x) = x^{-1}\gamma x$, the rotation ρ and the kernel $P = P_\gamma$ (for e, ϕ) be as in Lemma 5.8.41 and Lemma 5.8.40; then $c(x)_\infty = x_\infty^{-1}\gamma_\infty x_\infty$. On triples $(x, y, u) \in G^U \times G_\infty \times G_\infty$ put

$$a_1 = H(x_\infty y) - H(x_\infty), \qquad a_2 = H(x_\infty yu) - H(x_\infty y), \qquad a_3 = \ell + H(x_\infty) - H(x_\infty yu),$$

measurable functions with $a_1 + a_2 + a_3 = \ell$. For $\zeta \in Z_\gamma$, $(\zeta x)_\infty = \zeta_\infty x_\infty$ and $H(\zeta_\infty w) = r_\zeta + H(w)$, so each a_i is left invariant under Z_γ in x.

Integrability. By the Lipschitz bound, $|a_1| \le \|y\|_\infty$ and $|a_2| \le \|u\|_\infty$; as $\|\cdot\|_\infty \ge 1$ on G_∞, the weights 1, a_1, a_2 are bounded by $\|y\|_\infty\|u\|_\infty$. So P, a_1P, a_2P are m-integrable by Lemma 5.8.40, and so is $a_3P = \ell P - a_1P - a_2P$.

The rotation moves the weights. Let $(x', y', u') = \rho(x, y, u)$. Then $x'_\infty = x_\infty y$, $y' = u$ and $x_\infty yuu' = x_\infty c(x)_\infty = \gamma_\infty x_\infty$, so $x'_\infty y'u' = \gamma_\infty x_\infty$ and

$$a_1 \circ \rho = H(x_\infty yu) - H(x_\infty y) = a_2, \qquad a_2 \circ \rho = H(\gamma_\infty x_\infty) - H(x_\infty yu) = a_3,$$

using $H(\gamma_\infty x_\infty) = \ell + H(x_\infty)$. Since $P \circ \rho = P$ (Lemma 5.8.41(b)), Lemma 5.8.41(a) applied to $f = a_1 P$ (with $f \circ \rho = a_2 P$) and to $f = a_2 P$ (with $f \circ \rho = a_3 P$) gives

$$\int a_2 P \, dm = \int a_1 P \, dm, \qquad \int a_3 P \, dm = \int a_2 P \, dm.$$

Idempotence kills a_2. Let $b(x,w) = H(x_\infty w) - H(x_\infty)$, measurable and left Z_γ-invariant in x. Then $b(x, yu) = a_1 + a_2$ and $b(x,y) = a_1$, both integrable against P; Lemma 5.8.41(c) gives $\int (a_1 + a_2) P \, dm = \int a_1 P \, dm$, so $\int a_2 P \, dm = 0$, and hence $\int a_1 P \, dm = \int a_3 P \, dm = 0$.

Conclusion. By Lemma 5.8.41(b) (P is m-integrable) $\int P \, dm = I_\gamma(e \otimes \phi)$, so

$$\ell \, I_\gamma(e \otimes \phi) = \int (a_1 + a_2 + a_3) P \, dm = 0.$$

□

Lemma 5.8.40. *Let E be a CM field with maximal totally real subfield E^+, $N \geq 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs s, $G^U = G_\infty \times G_f$, $B \geq 13N^3$ an integer, e a cusp kernel of order B on G_∞, $\phi \colon G_f \to \mathbb{C}$ locally constant with compact support, and $\gamma \in \mathrm{U}(U)(E^+)$ with irreducible characteristic polynomial. For $x \in G^U$ and $y, u \in G_\infty$ put $c = x^{-1}\gamma x$ and*

$$P_\gamma(x,y,u) = e(y)\, e(u)\, e(u^{-1}y^{-1}c_\infty)\, \phi(c_f).$$

Let $a \colon G^U \times G_\infty \times G_\infty \to \mathbb{C}$ be measurable, left invariant under Z_γ in the first variable, with $|a(x,y,u)| \leq \|y\|_\infty \|u\|_\infty$. Then aP_γ is integrable on $(Z_\gamma \backslash G^U) \times G_\infty \times G_\infty$ for the product of the quotient measure and the measures of G_∞.

Proof. Write $\|\cdot\| = \|\cdot\|_\infty$. For $z \in G_\infty$ every component $z_v \in \mathrm{U}(\mathrm{diag}\, s)$ satisfies $z_v^{-1} = \mathrm{diag}(s) \bar{z}_v^{\mathsf{T}} \mathrm{diag}(s)$, so $\|z_v\| = \|z_v^{-1}\| \geq 1$; hence $\|z\| \geq 1$, and $\|zw\| \leq \|z\|\,\|w\|$. The function P_γ is left invariant under Z_γ in x ($x^{-1}\gamma x$ is), and aP_γ is measurable (e is continuous, ϕ locally constant).

Let $C_e = \sup_z |e(z)|\,\|z\|^B$, finite by (EK4)(b), and $M(y) = |e(y)|\,\|y\|^{B+1}$, integrable on G_∞ by (EK4)(a). Fix (x,y,u), put $c = x^{-1}\gamma x$ and $z = u^{-1}y^{-1}c_\infty$. Then $c_\infty = yuz$, so $\|c_\infty\| \leq \|y\|\,\|u\|\,\|z\|$ and

$$|e(z)| \leq C_e \|z\|^{-B} \leq C_e \left(\|y\|\,\|u\|\right)^B \|c_\infty\|^{-B}.$$

Hence

$$|a(x,y,u) P_\gamma(x,y,u)| \leq \|y\|\,\|u\|\,|e(y)|\,|e(u)|\,|e(z)|\,|\phi(c_f)| \leq C_e\, M(y)\, M(u)\, |(k_0 \otimes \phi)(x^{-1}\gamma x)|,$$

with $k_0(w) = \|w\|^{-B}$ on G_∞. The function k_0 is continuous and $\sup |k_0|\,\|\cdot\|^B = 1$, so by Proposition 5.8.36(i) the function $x \mapsto (k_0 \otimes \phi)(x^{-1}\gamma x)$ is integrable on $Z_\gamma \backslash G^U$. By Tonelli's theorem the right side, a product of integrable functions of x, of y and of u, is integrable on $(Z_\gamma \backslash G^U) \times G_\infty \times G_\infty$, and therefore so is aP_γ. □

Lemma 5.8.41. *Let E be a CM field with maximal totally real subfield E^+, $N \geq 1$, $s \in \{\pm 1\}^N$, U a framed space of rank N over E with signs s, $G^U = G_\infty \times G_f$, and $\gamma \in \mathrm{U}(U)(E^+)$. Let m be the product of the quotient measure on $Z_\gamma \backslash G^U$ and the measures of G_∞ on $(Z_\gamma \backslash G^U) \times G_\infty \times G_\infty$. For $x \in G^U$ write $c(x) = x^{-1}\gamma x$, and define*

$$\rho(x,y,u) = \left(x\,(y,1),\ u,\ u^{-1}y^{-1}c(x)_\infty\right) \qquad (x \in G^U,\ y,u \in G_\infty).$$

(a) *Let f be measurable on $G^U \times G_\infty \times G_\infty$, left invariant under Z_γ in the first variable, such that f and $f \circ \rho$ are m-integrable. Then $\int f \circ \rho \, dm = \int f \, dm$.*

Let moreover e be an idempotent kernel on G_∞, $\phi\colon G_f \to \mathbb{C}$ measurable and bounded, and

$$P_\gamma(x,y,u) = e(y)\, e(u)\, e\big(u^{-1}y^{-1}c(x)_\infty\big)\, \phi\big(c(x)_f\big).$$

(b) *$P_\gamma \circ \rho = P_\gamma$; and if P_γ is m-integrable, then $x \mapsto (e \otimes \phi)(x^{-1}\gamma x)$ is integrable on $Z_\gamma\backslash G^U$ and $\int P_\gamma \, dm = I_\gamma(e \otimes \phi)$.*

(c) *Let $b\colon G^U \times G_\infty \to \mathbb{C}$ be measurable and left invariant under Z_γ in the first variable, such that $(x,y,u) \mapsto b(x,yu)P_\gamma(x,y,u)$ and $(x,y,u) \mapsto b(x,y)P_\gamma(x,y,u)$ are m-integrable. Then*

$$\int b(x,yu)\, P_\gamma(x,y,u)\, dm = \int b(x,y)\, P_\gamma(x,y,u)\, dm.$$

Proof. Write ν for the quotient measure on $Z_\gamma\backslash G^U$ and $\mu = dy\, du$ on $G_\infty \times G_\infty$. For $\zeta \in Z_\gamma$, $c(\zeta x) = c(x)$; so $\rho(\zeta x, y, u) = (\zeta x(y,1), u, u^{-1}y^{-1}c(x)_\infty)$, and ρ, $f \circ \rho$, P_γ are compatible with the left action of Z_γ in the first variable. The groups G_∞ and G^U are unimodular (products of reductive groups over local fields, restricted at the finite places); hence the Haar measure of G_∞ is invariant under inversion and under left and right translations, and ν is invariant under right translation by elements of G^U (for measurable $f_0 \ge 0$ on G^U and $g \in G^U$, $\int_{Z_\gamma\backslash G^U} \sum_\zeta f_0(\zeta x g)\, dx = \int f_0(xg)\, dx = \int f_0$).

(a) For $x \in G^U$ let $T_x(y,u) = (u,\ u^{-1}c(x)_\infty y^{-1})$. Then T_x preserves μ: it is $(y,u) \mapsto (c(x)_\infty y^{-1}, u)$, which preserves μ by invariance of dy under inversion and left translation, followed by $(w,u) \mapsto (u, u^{-1}w)$, which preserves μ because $\int\int g(u,u^{-1}w)\, dw\, du = \int\int g(u,w')\, dw'\, du$ by left invariance of dw. Since $c(x(y,1))_\infty = y^{-1}c(x)_\infty y$,

$$\rho(x,y,u) = \big(x(y,1),\ T_{x(y,1)}(y,u)\big).$$

By Fubini's theorem for the integrable $f \circ \rho$,

$$\int f \circ \rho \, dm = \int \Big(\int f(x(y,1), T_{x(y,1)}(y,u))\, d\nu(x) \Big) d\mu(y,u).$$

For fixed (y,u) the function $F(x) = f(x, T_x(y,u))$ is left Z_γ-invariant, and the inner integral is $\int F(x(y,1))\, d\nu(x) = \int F\, d\nu$ by the right invariance of ν. The map $\Theta(x,y,u) = (x, T_x(y,u))$ is a measurable bijection of $(Z_\gamma\backslash G^U) \times G_\infty \times G_\infty$ preserving m (for each x, T_x preserves μ; Tonelli), so $f \circ \Theta$ is m-integrable with $\int f \circ \Theta \, dm = \int f \, dm$, and by Fubini $\int\int f(x, T_x(y,u))\, d\nu(x)\, d\mu = \int f \circ \Theta \, dm$. Together, $\int f \circ \rho \, dm = \int f \, dm$.

(b) Let $(x',y',u') = \rho(x,y,u)$. Then $c(x') = (y,1)^{-1}c(x)(y,1)$, so $c(x')_f = c(x)_f$ and $u'^{-1}y'^{-1}c(x')_\infty = \big(c(x)_\infty^{-1}yu\big)u^{-1}y^{-1}c(x)_\infty y = y$. Hence

$$P_\gamma(\rho(x,y,u)) = e(u)\, e\big(u^{-1}y^{-1}c(x)_\infty\big)\, e(y)\, \phi\big(c(x)_f\big) = P_\gamma(x,y,u).$$

By (EK3) with $x = 1$, $\int e(z)e(z^{-1}w)\, dz = e(w)$ for every $w \in G_\infty$; below it is used only at points w where the integrand is integrable, which Fubini's theorem supplies. Suppose P_γ is m-integrable. By Fubini, for ν-almost every x the function $(y,u) \mapsto P_\gamma(x,y,u)$ is μ-integrable, and for such x and almost every y the function $u \mapsto e(u)e(u^{-1}y^{-1}c(x)_\infty)$ is integrable; by (EK3) its integral is $e(y^{-1}c(x)_\infty)$. Then $y \mapsto e(y)e(y^{-1}c(x)_\infty)$ is integrable with integral $e(c(x)_\infty)$, again by (EK3). So for ν-almost every x,

$$\int P_\gamma(x,y,u)\, d\mu(y,u) = e(c(x)_\infty)\, \phi\big(c(x)_f\big) = (e \otimes \phi)(x^{-1}\gamma x),$$

and the left side is a ν-integrable function of x with integral $\int P_\gamma\,dm$. Hence $x \mapsto (e\otimes\phi)(x^{-1}\gamma x)$ is integrable on $Z_\gamma\backslash G^U$ and $\int P_\gamma\,dm = I_\gamma(e\otimes\phi)$.

(c) By Fubini, for ν-almost every x both

$$(y,u)\mapsto b(x,yu)P_\gamma(x,y,u) \quad\text{and}\quad (y,u)\mapsto b(x,y)P_\gamma(x,y,u)$$

are μ-integrable; fix such an x and write $c=c(x)$. In the first, substitute $u=y^{-1}w$ for fixed y (left invariance of du); the resulting function

$$(y,w)\mapsto b(x,w)e(y)e(y^{-1}w)e(w^{-1}c_\infty)\phi(c_f)$$

is the image of an integrable function under a μ-preserving bijection, hence integrable, and Fubini in the order w outside gives

$$\begin{aligned}\int b(x,yu)P_\gamma(x,y,u)\,d\mu &= \int b(x,w)\,e(w^{-1}c_\infty)\,\phi(c_f)\Big(\int e(y)e(y^{-1}w)\,dy\Big)dw\\ &= \int b(x,w)\,e(w)\,e(w^{-1}c_\infty)\,\phi(c_f)\,dw,\end{aligned}$$

the inner integral being $e(w)$ by (EK3) for almost every w. In the second, integrating first over u gives, by (EK3), $\int e(u)e(u^{-1}y^{-1}c_\infty)\,du = e(y^{-1}c_\infty)$ for almost every y, so

$$\int b(x,y)P_\gamma(x,y,u)\,d\mu = \int b(x,y)\,e(y)\,e(y^{-1}c_\infty)\,\phi(c_f)\,dy.$$

The two right sides agree. Integrating over x (Fubini for the two m-integrable functions) gives (c). □

Proposition 5.8.42. *Let E be a CM field with maximal totally real subfield E^+ and chosen embeddings $\Phi_v\colon E\to\mathbb{C}$ at the infinite places v of E^+. Let $N\ge 1$, $s\in\{\pm1\}^N$, U a framed space of rank N over E with signs s, and $G^U = G_\infty\times G_f$. Let $R\in E[X]$ be irreducible such that every root of $\Phi_v(R)$ has absolute value 1 for every v, and let $\gamma\in \mathrm{U}(U)(E^+)$ have characteristic polynomial R; write $\gamma_\infty\in G_\infty$ and $\gamma_f\in G_f$ for the components of its image in Γ_U. For $y\in G_f$ let $E[y]=\{r(y): r\in E[X]\}\subseteq \mathrm{M}_N(\mathbb{A}_{E,f})$ and $Z(y)=E[y]\cap G_f$. For functions k on G_∞ and ϕ on G_f put*

$$I_\gamma^\infty(k)=\int_{G_\infty} k(h^{-1}\gamma_\infty h)\,dh, \qquad I_y^{\mathrm{fin}}(\phi)=\int_{Z(y)\backslash G_f}\phi(v^{-1}yv)\,dv,$$

when $Z(y)$ is discrete and the integrals converge absolutely.

(i) $t\mapsto t_f$ (the finite component of the image in Γ_U) is an isomorphism of Z_γ onto $Z(\gamma_f)$, and $Z(\gamma_f)$ is a discrete subgroup of G_f.

(ii) If k, ϕ are measurable with $I_\gamma^\infty(|k|)<\infty$ and $I_{\gamma_f}^{\mathrm{fin}}(|\phi|)<\infty$, then $x\mapsto(k\otimes\phi)(x^{-1}\gamma x)$ is integrable on $Z_\gamma\backslash G^U$ and $I_\gamma(k\otimes\phi)=I_\gamma^\infty(k)\,I_{\gamma_f}^{\mathrm{fin}}(\phi)$.

(iii) If $B\ge 13N^3$ is an integer, k is measurable with $\sup|k|\,\|\cdot\|_\infty^B<\infty$ and ϕ is locally constant with compact support, then $I_\gamma^\infty(|k|)<\infty$ and $I_{\gamma_f}^{\mathrm{fin}}(|\phi|)<\infty$.

(iv) For $g\in G_f$ and $y=g^{-1}\gamma_f g$, $Z(y)=g^{-1}Z(\gamma_f)g$, and $I_y^{\mathrm{fin}}(|\phi|)=I_{\gamma_f}^{\mathrm{fin}}(|\phi|)$ and $I_y^{\mathrm{fin}}(\phi)=I_{\gamma_f}^{\mathrm{fin}}(\phi)$ for every measurable ϕ for which the latter integral converges absolutely.

Proof. Write $G = G^U$, $Z = Z_\gamma$ and $Z^f = Z(\gamma_f)$. In a basis of U the image of $t \in \mathrm{U}(U)(E^+)$ in G_f is $t_f = x_U^{-1} t x_U$, x_U the finite frame as a matrix (Definition 5.7.1); so $r(t)_f = r(t_f)$ for $r \in E[X]$ whenever $r(t)$ is unitary, and $t \mapsto t_f$ is injective ($\mathrm{M}_N(E) \to \mathrm{M}_N(\mathbb{A}_{E,f})$ is injective).

(i) *The centraliser.* The characteristic polynomial R of γ is irreducible, so it is the minimal polynomial of γ, $L = E[X]/(R)$ is a field, and U is an L-vector space (X acting by γ) of dimension $N/\deg R = 1$. An E-linear map commuting with γ is L-linear, hence multiplication by an element of L, that is $r(\gamma)$ for some $r \in E[X]$. So Z_γ is the image of $E[\gamma] \cap \mathrm{U}(U)(E^+)$, and its finite components lie in $E[\gamma_f] \cap G_f = Z^f$.

Surjectivity. The map $L \to \mathrm{M}_N(\mathbb{A}_{E,f})$, $r \mapsto r(\gamma_f)$, is a well defined E-algebra map ($R(\gamma_f) = x_U^{-1} R(\gamma) x_U = 0$ by Cayley–Hamilton), injective as L is a field. The class α of X is invertible in L; for $r \in E[X]$ let $r^\natural \in E[X]$ represent $\bar{r}(\alpha^{-1})$, where $\bar{r}$ has conjugated coefficients. For the standard form, $g^* = \bar{g}^{\mathsf{T}}$ and $\gamma_f^* = \gamma_f^{-1}$, so $r(\gamma_f)^* = \bar{r}(\gamma_f^{-1}) = r^\natural(\gamma_f)$; likewise, for the adjoint with respect to the form of U (conjugate-linear in scalars), $r(\gamma)^* = \bar{r}(\gamma^{-1}) = r^\natural(\gamma)$. Hence, using the injectivity of $r \mapsto r(\gamma_f)$ and of $r \mapsto r(\gamma)$ on L,

$$r(\gamma_f) \in G_f \iff r^\natural(\alpha) r(\alpha) = 1 \text{ in } L \iff r(\gamma) \in \mathrm{U}(U)(E^+).$$

So every element $r(\gamma_f)$ of Z^f is t_f for $t = r(\gamma) \in Z_\gamma$; with injectivity, $t \mapsto t_f$ is an isomorphism $Z_\gamma \to Z^f$.

Discreteness. At an infinite place v, $\gamma_v \in \mathrm{U}(\operatorname{diag} s)$ has characteristic polynomial $\Phi_v(R)$, whose roots are distinct (R is separable) and of absolute value 1. For $t \in Z_\gamma$, t_v commutes with γ_v, so by Lemma 5.8.43(i) t_v lies in the set C_v of elements of $\mathrm{U}(\operatorname{diag} s)$ acting on each eigenline of γ_v by a scalar of absolute value 1; in an eigenbasis these are the diagonal matrices with entries in the unit circle, so C_v is compact. Let $W \subseteq G_f$ be a compact open neighbourhood of 1. As Γ_U is discrete in G, $Z \cap (\prod_v C_v \times W)$ is a discrete subset of a compact set, hence finite; by injectivity $Z^f \cap W$ is finite. A subgroup of a Hausdorff group that meets some neighbourhood of 1 in a finite set is discrete; so Z^f is discrete.

(ii) Z^f is a countable discrete subgroup of the second countable group G_f, so it has a Borel fundamental domain D_f for its action by left multiplication. Then $D = G_\infty \times D_f$ is a Borel fundamental domain for Z acting on G: for $x = (h, v)$, $\zeta x \in D$ if and only if $\zeta_f v \in D_f$, and by (i) exactly one $\zeta \in Z$ has this property. For a measurable left Z-invariant $F \geq 0$ on G, $f = F\mathbf{1}_D$ satisfies $\sum_{\zeta \in Z} f(\zeta x) = F(x)$, so $\int_{Z\backslash G} F = \int_D F$; likewise $\int_{Z^f\backslash G_f} F_f = \int_{D_f} F_f$ for left Z^f-invariant $F_f \geq 0$. For $x = (h, v)$, $x^{-1}\gamma x = (h^{-1}\gamma_\infty h,\, v^{-1}\gamma_f v)$, and $v \mapsto \phi(v^{-1}\gamma_f v)$ is left Z^f-invariant (Z^f commutes with γ_f). By Tonelli,

$$\int_{Z\backslash G} \left|(k \otimes \phi)(x^{-1}\gamma x)\right| dx = \int_{G_\infty} |k(h^{-1}\gamma_\infty h)|\, dh \int_{D_f} |\phi(v^{-1}\gamma_f v)|\, dv = I_\gamma^\infty(|k|)\, I_{\gamma_f}^{\mathrm{fin}}(|\phi|) < \infty,$$

and then Fubini's theorem gives $I_\gamma(k \otimes \phi) = \int_D k(h^{-1}\gamma_\infty h)\phi(v^{-1}\gamma_f v) = I_\gamma^\infty(k)\, I_{\gamma_f}^{\mathrm{fin}}(\phi)$.

(iii) Let $k_0(w) = \|w\|_\infty^{-B}$, continuous with $\sup k_0 \| \cdot \|_\infty^B = 1$, and $C_k = \sup |k| \,\| \cdot \|_\infty^B$, so $|k| \leq C_k k_0$. Let $W = \gamma_f \mathrm{K}_f$, compact and open, and $\phi_0 = \mathbf{1}_W$. By Proposition 5.8.36(i) for (k_0, ϕ_0) and the Tonelli identity of (ii) (valid for nonnegative functions without any finiteness assumption), $I_\gamma^\infty(k_0)\, I_{\gamma_f}^{\mathrm{fin}}(\phi_0) < \infty$. We show $I_{\gamma_f}^{\mathrm{fin}}(\phi_0) > 0$. The set $S = \{v \in G_f : v^{-1}\gamma_f v \in W\}$ is open, contains 1 and is left Z^f-invariant. By (i) there is an open neighbourhood $O \subseteq S$ of 1 with $OO^{-1} \cap Z^f = \{1\}$; then for every v at most one $\zeta \in Z^f$ has $\zeta v \in O$, and if one has, $v \in \zeta^{-1} S = S$. So $\sum_{\zeta \in Z^f} \mathbf{1}_O(\zeta v) \leq \mathbf{1}_S(v)$ and

$$I_{\gamma_f}^{\mathrm{fin}}(\phi_0) = \int_{Z^f\backslash G_f} \mathbf{1}_S \;\geq \int_{Z^f\backslash G_f} \sum_\zeta \mathbf{1}_O(\zeta v)\, dv = \mathrm{vol}(O) > 0.$$

Hence $I^\infty_\gamma(k_0) < \infty$ and $I^\infty_\gamma(|k|) \le C_k I^\infty_\gamma(k_0) < \infty$. Finally Proposition 5.8.36(i) for (k_0, ϕ) and the same Tonelli identity give $I^\infty_\gamma(k_0)\, I^{\mathrm{fin}}_{\gamma_f}(|\phi|) < \infty$, and $I^\infty_\gamma(k_0) > 0$ (a positive continuous integrand), so $I^{\mathrm{fin}}_{\gamma_f}(|\phi|) < \infty$.

(iv) $E[g^{-1}\gamma_f g] = g^{-1}E[\gamma_f]g$ and $g^{-1}G_f g = G_f$, so $Z(y) = g^{-1}Z^f g$, again discrete. The map $w \mapsto g^{-1}w$ induces a bijection $Z^f\backslash G_f \to Z(y)\backslash G_f$ ($Z(y)g^{-1}w = g^{-1}Z^f w$) which preserves the quotient measures (left translation preserves the measure of G_f, and conjugation by g carries counting measure on Z^f to that on $Z(y)$), and $(g^{-1}w)^{-1}y(g^{-1}w) = w^{-1}\gamma_f w$. Both identities follow. □

Lemma 5.8.43. *Let $N \ge 1$, $s \in \{\pm 1\}^N$, $p = \#\{i : s(i) = 1\}$, $q = N - p$, let $h(u,u') = \bar{u}^{\mathsf{T}}\mathrm{diag}(s)\,u'$ on $\mathbb{C}^N$ and $G = \mathrm{U}(\mathrm{diag}\, s) = \{g \in \mathrm{GL}_N(\mathbb{C}) : \bar{g}^{\mathsf{T}}\mathrm{diag}(s)\, g = \mathrm{diag}(s)\}$. Let $\zeta_1, \dots, \zeta_N \in \mathbb{C}$ be distinct with $|\zeta_i| = 1$, and $Q = \prod_i (X - \zeta_i)$. For $\sigma \in S_N$ put $t_\sigma = \mathrm{diag}(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(N)})$.*

(i) Let $t \in G$ have characteristic polynomial Q. Its ζ_i-eigenspace is a line ℓ_i; $h(u,u') = 0$ for $u \in \ell_i$, $u' \in \ell_j$, $i \ne j$; and $h(u,u)$ is real and nonzero for $u \in \ell_i \smallsetminus 0$, with a sign $\varepsilon_t(i) \in \{\pm 1\}$ independent of u, and $\#\{i : \varepsilon_t(i) = 1\} = p$. Every $g \in G$ commuting with t acts on each ℓ_i by a scalar of absolute value 1.

(ii) Elements $t, t' \in G$ with characteristic polynomial Q are conjugate in G if and only if $\varepsilon_t = \varepsilon_{t'}$.

(iii) For $\sigma \in S_N$, $t_\sigma \in G$ has characteristic polynomial Q and $\varepsilon_{t_\sigma} = s \circ \sigma^{-1}$.

(iv) Every G-conjugacy class C of elements of G with characteristic polynomial Q contains t_σ for exactly $p!\,q!$ permutations $\sigma \in S_N$.

Proof. (i) As the ζ_i are distinct roots of the characteristic polynomial, t is diagonalisable with one-dimensional eigenspaces ℓ_i. For $u \in \ell_i$, $u' \in \ell_j$, $h(u,u') = h(tu, tu') = \bar{\zeta}_i\zeta_j\, h(u,u')$, and $\bar{\zeta}_i\zeta_j = \zeta_j/\zeta_i \ne 1$ for $i \ne j$ (as $|\zeta_i| = 1$); so $h(u,u') = 0$. Choose $f_i \in \ell_i \smallsetminus 0$. Then (f_i) is a basis in which the Gram matrix of h is diagonal with entries $h(f_i, f_i)$; these are real (h is hermitian) and nonzero (h is nondegenerate), and $h(cf_i, cf_i) = |c|^2 h(f_i, f_i)$, so the sign $\varepsilon_t(i)$ does not depend on the choice. By Sylvester's law of inertia the number of positive entries is the number p of positive entries of $\mathrm{diag}(s)$. If $g \in G$ commutes with t, it preserves each eigenspace, $gf_i = \mu_i f_i$, and $h(f_i, f_i) = h(gf_i, gf_i) = |\mu_i|^2 h(f_i, f_i)$ gives $|\mu_i| = 1$.

(ii) If $t' = gtg^{-1}$ with $g \in G$, then $g\ell_i$ is the ζ_i-eigenline of t' and $h(gu, gu) = h(u,u)$, so $\varepsilon_{t'} = \varepsilon_t$. Conversely let $\varepsilon_t = \varepsilon_{t'} = \varepsilon$. Choose eigenvectors f_i of t and f'_i of t' for ζ_i, scaled so that $h(f_i, f_i) = h(f'_i, f'_i) = \varepsilon(i)$. The linear map g with $gf_i = f'_i$ carries a basis with Gram matrix $\mathrm{diag}(\varepsilon)$ to a basis with the same Gram matrix, so it preserves h, that is $g \in G$; and $gtg^{-1}f'_i = \zeta_i f'_i = t' f'_i$ for all i, so $gtg^{-1} = t'$.

(iii) t_σ is diagonal with entries of absolute value 1, so $\bar{t}_\sigma^{\mathsf{T}}\mathrm{diag}(s)t_\sigma = \mathrm{diag}(s)$ and $t_\sigma \in G$, with characteristic polynomial Q. The standard basis vector e_k is an eigenvector for $\zeta_{\sigma(k)}$, so $\ell_i = \mathbb{C}e_{\sigma^{-1}(i)}$ and $\varepsilon_{t_\sigma}(i) = \mathrm{sign}\, h(e_{\sigma^{-1}(i)}, e_{\sigma^{-1}(i)}) = s(\sigma^{-1}(i))$.

(iv) Let C be such a class and $\varepsilon = \varepsilon_t$ for $t \in C$; by (ii), $C = \{t' \in G : \mathrm{charpoly}(t') = Q,\ \varepsilon_{t'} = \varepsilon\}$, and by (iii), $t_\sigma \in C$ if and only if $s \circ \sigma^{-1} = \varepsilon$. By (i), ε and s take the value 1 equally often, so there is σ_0 with $s \circ \sigma_0^{-1} = \varepsilon$. For $\sigma = \sigma_0\tau$, $s \circ \sigma^{-1} = (s \circ \tau^{-1}) \circ \sigma_0^{-1}$, which equals ε if and only if $s \circ \tau^{-1} = s$, that is, τ permutes $\{i : s(i) = 1\}$ and $\{i : s(i) = -1\}$ separately. There are $p!\,q!$ such τ. □

Proposition 5.8.44. *Let E be a CM field with maximal totally real subfield E^+ and chosen embeddings $\Phi_v\colon E \to \mathbb{C}$ at the infinite places v of E^+, let $N \geq 1$, and let $R \in E[X]$ be monic and irreducible of degree N such that every root of $\Phi_v(R)$ has absolute value 1 for every v. For a framed space U of rank N over E, with signs s, let $X_R(U)$ be the set of $\gamma \in \mathrm{U}(U)(E^+)$ with characteristic polynomial R, $\mathcal{C}(U)$ the set of $\mathrm{U}(U)(E^+)$-conjugacy classes contained in $X_R(U)$, and $\mathcal{C}_f(U)$ the set of G_f-conjugacy classes of the finite components $\gamma_f \in G_f$ of the images of the $\gamma \in X_R(U)$ in Γ_U. For an infinite place v let $\mathcal{C}_v(s)$ be the set of $\mathrm{U}(\operatorname{diag} s)$-conjugacy classes of elements of $\mathrm{U}(\operatorname{diag} s)$ with characteristic polynomial $\Phi_v(R)$. Then:*

(i) $\mathcal{C}_f(U)$ is the same set of G_f-conjugacy classes for all framed spaces U of rank N over E (of any signs);

(ii) $[\gamma] \mapsto ([\gamma_f], ([\gamma_v])_v)$, with $\gamma_v \in \mathrm{U}(\operatorname{diag} s)$ the components of the real part of the image of γ, is a bijection $\mathcal{C}(U) \to \mathcal{C}_f(U) \times \prod_v \mathcal{C}_v(s)$;

(iii) every $\mathcal{C}_v(s)$ is finite and nonempty.

Proof. Put $F = E^+$; E/F is a CM extension and all infinite places of F are real. The unitary group of a hermitian form and its isometry classes are unchanged if the form h is replaced by $(u, u') \mapsto h(u', u)$, so the convention of (FT1) (conjugate-linear in the first argument) may be exchanged for the one used for unitary dual pairs and global theta lifts over a CM extension.

R is conjugate-reciprocal. The roots of $\Phi_v(R)$ have absolute value 1, so $R(0) \neq 0$. The polynomial $R^\# = \overline{R(0)}^{-1} X^N \bar{R}(X^{-1})$ is monic of degree N, and since complex conjugation on E is induced by complex conjugation under Φ_v, the roots of $\Phi_v(R^\#)$ are the $1/\bar{\zeta} = \zeta$ for the roots ζ of $\Phi_v(R)$; as these are N distinct numbers (R is separable), $\Phi_v(R^\#) = \Phi_v(R)$, hence $R^\# = R$. So $P := R \in \mathcal{P}_N(E/F)$ in the notation of Definition 5.8.2.

Step 1: local classes. Let $V_0 = E^N$ with the standard form, $\Lambda = \mathcal{O}_E^N$, and for a finite place v let K_v be the stabiliser of Λ_v in $\mathrm{U}(V_0)(F_v)$; then $G_f = \prod_v{}' \mathrm{U}(V_0)(F_v)$ (restricted with respect to the K_v) and $\mathrm{K}_f = \prod_v K_v$. For a finite place v, the component at v of the finite frame x_U is an isometry $V_{0,v} \to U_v = U \otimes_F F_v$; hence $X_P(U, F_v) = X_P(V_0, F_v)$, and for $\gamma \in X_R(U)$ the component $(\gamma_f)_v = x_U^{-1} \gamma x_U \in \mathrm{U}(V_0)(F_v)$ satisfies $c_{F_v}((\gamma_f)_v) = c_{F_v}(\gamma)$ (compose the isometry φ of Lemma 5.8.46(b) for γ with x_U^{-1}). For a real place v, $E \otimes_F F_v = \mathbb{C}$ through Φ_v, and the frame at v identifies $\mathrm{U}(U)(F_v)$ with $\mathrm{U}(\operatorname{diag} s)$, $\gamma \mapsto \gamma_v$, preserving characteristic polynomials up to Φ_v; so by Lemma 5.8.46(c) the map $[\gamma_v] \mapsto c_{F_v}(\gamma)$ is a bijection $\mathcal{C}_v(s) \to X_P(U, F_v)$, and $\mathcal{C}_v(s)$ is finite by Lemma 5.8.46(e). It is nonempty: $\operatorname{diag}(\zeta_1, \ldots, \zeta_N)$, the ζ_i the roots of $\Phi_v(R)$, lies in $\mathrm{U}(\operatorname{diag} s)$ since $|\zeta_i| = 1$. This proves (iii).

Step 2: global classes. By Proposition 5.8.45 (H1)–(H3), $[\gamma] \mapsto (c_{F_v}(\gamma))_v$ is a bijection of $\mathcal{C}(U)$ onto the set $\mathcal{F}(U)$ of families $(c_v)_v$ over all places of F with $c_v \in X_P(U, F_v)$ and $c_v = 1$ for all but finitely many v. By Step 1, $\mathcal{F}(U) = \mathcal{F}_f \times \prod_{v|\infty} X_P(U, F_v)$, where $\mathcal{F}_f$ is the set of families over the finite places with $c_v \in X_P(V_0, F_v)$, almost all equal to 1; $\mathcal{F}_f$ does not depend on U.

Step 3: finite classes. Let $y, y' \in G_f$ have characteristic polynomial P. We claim: y and y' are conjugate in G_f if and only if $c_{F_v}(y_v) = c_{F_v}(y'_v)$ for every finite v. If $y' = g^{-1} y g$ with $g \in G_f$, the classes agree by Lemma 5.8.46(b). Conversely, by Lemma 5.8.46(c) there is $g_v \in \mathrm{U}(V_0)(F_v)$ with $g_v^{-1} y_v g_v = y'_v$ for every finite v. For all but finitely many v, $y_v, y'_v \in K_v$ (as $y, y' \in G_f$) and the conclusion of Lemma 5.8.47(a) for (V_0, Λ) holds: $\{x : x^{-1} y_v x \in K_v\} = T_{y_v} K_v$, T_{y_v} the centraliser of y_v. For such v, $g_v = tk$ with $t \in T_{y_v}$, $k \in K_v$, and $k^{-1} y_v k = y'_v$; replacing g_v by k, the family $(g_v)_v$ lies in G_f and conjugates y to y'.

By Step 1 and the claim, $[y] \mapsto (c_{F_v}(y_v))_{v \text{ finite}}$ is an injective map from the set of G_f-classes of elements of G_f with characteristic polynomial P to the families of local classes, and on $\mathcal{C}_f(U)$

it takes values in $\mathcal{F}_f$. It maps $\mathcal{C}_f(U)$ onto $\mathcal{F}_f$: a family in $\mathcal{F}_f$, completed by any classes at the real places (nonempty sets by (iii)), lies in $\mathcal{F}(U)$ and is the family of some $\gamma \in X_R(U)$ (Step 2), whose finite component has the given finite classes (Step 1). So $\mathcal{C}_f(U)$ is the preimage of $\mathcal{F}_f$ under one injective map that does not depend on U, which proves (i).

Step 4: the bijection. The map of (ii) is the composite of the bijection $\mathcal{C}(U) \to \mathcal{F}_f \times \prod_{v|\infty} X_P(U, F_v)$ of Step 2 with the inverses of the bijections $\mathcal{C}_f(U) \to \mathcal{F}_f$ (Step 3) and $\mathcal{C}_v(s) \to X_P(U, F_v)$ (Step 1): indeed the finite part of the family of γ is the image of $[\gamma_f]$, and its component at a real v is the image of $[\gamma_v]$. So it is a bijection. □

Proposition 5.8.45. *Let E/F be a CM extension, $r \geq 1$, $P \in \mathcal{P}_r(E/F)$ and V a hermitian space over E of rank r. For $\gamma \in \mathrm{U}(V)(F)$ with characteristic polynomial P and a place v of F (finite or real) write $c_v(\gamma) = c_{F_v}(\gamma) \in X_P(V, F_v)$. Then:*

(H1) $c_v(\gamma) = (c_F(\gamma))_{F_v}$ for every v, and $c_v(\gamma) = 1$ for all but finitely many v;

(H2) two elements $\gamma, \gamma' \in \mathrm{U}(V)(F)$ with characteristic polynomial P are conjugate in $\mathrm{U}(V)(F)$ if and only if $c_v(\gamma) = c_v(\gamma')$ for every place v;

(H3) for every family $(c_v)_v$ with $c_v \in X_P(V, F_v)$ for every place v and $c_v = 1$ for all but finitely many v, there is $\gamma \in \mathrm{U}(V)(F)$ with characteristic polynomial P and $c_v(\gamma) = c_v$ for every v.

Proof. Write $L = L_P$, $K = K_P$, $\alpha = \alpha_P$. Since $E \subseteq L$, $E \not\subseteq K$ and $K \cap E = F$, while $[L : K] = 2 = [E : F]$, we have $L = KE \cong K \otimes_F E$; with $E = F(\sqrt{d})$, $d \in F$ totally negative (E/F is CM), $L = K(\sqrt{d})$, $*$ is the nontrivial automorphism of L/K and $N = N_{L/K}$. For a place v of F, $K_{P,F_v} = \prod_{w|v} K_w$ and $L_{P,F_v} = \prod_{w|v} L \otimes_K K_w$ over the places w of K above v. Fix bases and write $\det V \in F^\times$ for the determinant of a Gram matrix of V (well defined up to $N_{E/F}(E^\times)$), and similarly over F_v.

A determinant formula. For $x \in K_{P,k}^\times$ ($k = F$ or F_v), in the E_k-basis $1, \alpha, \dots, \alpha^{r-1}$ of $L_{P,k}$ the Gram matrix of h_x is $G_x = (\mathrm{Tr}(x\alpha^i\alpha^{-j}))_{ij}$, and $h_x(y, y') = h_1(xy, y')$, so $\det G_x = N_{L/E}(x) \det G_1$. As $L = K \otimes_F E$, $N_{L/E}(x) = N_{K/F}(x)$ for $x \in K$. Put $d_P = \det G_1 \in F^\times$ (G_1 is hermitian, so its determinant is fixed by conjugation). Thus

$$\det G_x = N_{K/F}(x)\, d_P. \tag{1}$$

If $[x] \in X_P(V, F_v)$, then $\det G_x \in \det V \cdot N(E_v^\times)$, so with η_v the quadratic character of $F_v^\times$ attached to E_v/F_v:

$$\eta_v(N_{K_{P,F_v}/F_v}(x)) = \eta_v(\det V\, d_P^{-1}). \tag{2}$$

(H1). By Lemma 5.8.46(b) for $k = F$ there are $x \in K^\times$ and an isometry $\varphi\colon (L, h_x) \to V$ with $\varphi m_\alpha = \gamma\varphi$; their base changes to F_v have the same properties, so $c_v(\gamma) = [x]_{F_v}$. For all but finitely many v, x is a unit at every $w \mid v$ and L/K is unramified at every $w \mid v$; then x is a norm from $L \otimes_K K_w$ (units are norms in an unramified quadratic extension, everything is a norm in the split algebra), so $[x]_{F_v} = 1$.

(H2). Conjugate elements have equal classes by Lemma 5.8.46(b) at $k = F_v$. Conversely let $c_v(\gamma) = c_v(\gamma')$ for all v, with global (x, φ), (x', φ') as in (H1). Then $u = x/x' \in K^\times$ lies in $N(L_w^\times)$ for every place w of K, $L_w = L \otimes_K K_w$. At a real place w (embedding σ of K), $\sigma(d) < 0$, $L_w \cong \mathbb{C}$ and $\sigma(u) > 0$, so $\sigma(u) = a^2 - \sigma(d)b^2$ with $a, b \in \mathbb{R}$; at a finite w, $u = a^2 - db^2$ with $a, b \in K_w$ (if L_w is a field this is $u \in N(L_w^\times)$, and if d is a square in K_w the form $a^2 - db^2$ represents every element). The Hasse norm theorem A.2.11 for the field K and $\delta = d$ gives

$u = a^2 - db^2 = N(z)$, $z = a + b\sqrt{d} \in L^\times$. So $x = x'N(z)$, i.e. $c_F(\gamma) = [x] = [x'] = c_F(\gamma')$ in $C_P(F)$, and γ, γ' are conjugate in $\mathrm{U}(V)(F)$ by the injectivity in Lemma 5.8.46(c) for $k = F$.

(H3), a global element of K. Choose $x_v \in K_{P,F_v}^\times$ representing c_v, with $x_v = 1$ whenever $c_v = 1$; write $x_{v,w}$ for its components. The family $(x_{v,w})_w$ is an idele of K (almost all components are 1). For the quadratic extension L/K with local characters ω_w of $K_w^\times$ (trivial exactly on $N(L_w^\times)$), the norm compatibility of local class field theory for $L_w = K_w \otimes_{F_v} E_v$ gives $\prod_{w|v} \omega_w(x_{v,w}) = \eta_v(N_{K_{P,F_v}/F_v}(x_v))$, which equals $\eta_v(\det V\, d_P^{-1})$ by (2). Hence

$$\prod_w \omega_w(x_{v,w}) = \prod_v \eta_v(\det V\, d_P^{-1}) = 1$$

by the product formula for the Hecke character η of E/F on $F^\times$ (Hilbert reciprocity). By global class field theory for L/K (the kernel of $\prod_w \omega_w$ on the ideles of K is $K^\times N(\mathbb{I}_L)$), there are $x \in K^\times$ and an idele b of L with $x_{v,w} = xN(b_w)$ for all w; so $[x]_{F_v} = c_v$ for every v.

(H3), a global isometry. For every v, (L_{P,F_v}, h_x) is isometric to V_{F_v}, since $[x]_{F_v} = c_v \in X_P(V, F_v)$ and $h_{xN(z)} \cong h_x$ through m_z. In particular, at every complex embedding τ of E (above a real place of F) the signatures of h_x and of V agree, and $q = \det G_x / \det V \in F^\times$ is a norm from E_v at every finite v and positive at every real v. By Theorem A.2.11 for F and $\delta = d$, $q = N_{E/F}(z_0)$ with $z_0 \in E^\times$, i.e. $\det G_x = z_0 \bar{z}_0 \det V$. Landherr's theorem A.2.12 (E is a CM field) gives an isometry $\varphi\colon (L, h_x) \to V$. Then $\gamma = \varphi m_\alpha \varphi^{-1} \in \mathrm{U}(V)(F)$ has characteristic polynomial P (Lemma 5.8.46(a)) and $c_F(\gamma) = [x]$, so $c_v(\gamma) = [x]_{F_v} = c_v$ by (H1). □

Lemma 5.8.46. *Let E/F be a quadratic extension of number fields, $r \geq 1$, $P \in \mathcal{P}_r(E/F)$, V a hermitian space over E of rank r, and k either F or a completion of F. Write $L = L_{P,k}$, $\alpha = \alpha_P$.*

(a) For $x \in K_{P,k}^\times$, h_x is a nondegenerate hermitian form on the E_k-module L, and multiplication m_α by α is an isometry of h_x with characteristic polynomial P.

(b) For every $\gamma \in \mathrm{U}(V)(k)$ with characteristic polynomial P there are $x \in K_{P,k}^\times$ and an isometry $\varphi\colon (L, h_x) \to V_k$ with $\varphi \circ m_\alpha = \gamma \circ \varphi$. The class $c_k(\gamma) = [x] \in X_P(V, k)$ depends only on the $\mathrm{U}(V)(k)$-conjugacy class of γ.

(c) $\gamma \mapsto c_k(\gamma)$ induces a bijection from the set of $\mathrm{U}(V)(k)$-conjugacy classes of elements with characteristic polynomial P onto $X_P(V, k)$.

(d) For γ, x, φ as in (b), $z \mapsto \varphi \circ m_z \circ \varphi^{-1}$ is an isomorphism of $T_P(k)$ onto the centraliser of γ in $\mathrm{U}(V)(k)$.

(e) If k is a completion of F, then $C_P(k)$, hence $X_P(V, k)$, is finite.

Proof. Write $\mathrm{Tr} = \mathrm{Tr}_{L/E_k}$. Since P is irreducible over a field of characteristic 0, L_P/E is separable, so the trace form $(y, y') \mapsto \mathrm{Tr}(yy')$ on L is nondegenerate (base change of a nondegenerate form). The involution $*$ is a ring automorphism of L which acts on E_k as $e \mapsto \bar{e}$; computing the trace of m_{z^*} in a basis (b_i^*) when (b_i) is an E_k-basis gives $\mathrm{Tr}(z^*) = \overline{\mathrm{Tr}(z)}$.

(a) h_x is E_k-linear in y and conjugate-linear in y' ($(ey')^* = \bar{e}\, y'^*$). Hermitian: $\overline{h_x(y', y)} = \mathrm{Tr}((xy'y^*)^*) = \mathrm{Tr}(x^*y'^*y) = h_x(y, y')$ as $x^* = x$. Nondegenerate: if $h_x(y, y') = 0$ for all y' then $\mathrm{Tr}(xyw) = 0$ for all $w \in L$ (y'^* runs over L), so $xy = 0$ and $y = 0$, x being a unit. Isometry: $h_x(\alpha y, \alpha y') = \mathrm{Tr}(x\, y\, y'^*\, \alpha\alpha^*) = h_x(y, y')$ since $\alpha^* = \alpha^{-1}$. The characteristic polynomial of m_α on $L = E_k[X]/(P)$ is P (basis $1, \alpha, \ldots, \alpha^{r-1}$, companion matrix).

A lemma on invariant forms. Let h be a nondegenerate hermitian form on L with $h(\alpha y, \alpha y') = h(y, y')$. Then $h = h_x$ for a unique $x \in K_{P,k}^\times$. Indeed $h(\alpha y, y') = h(y, \alpha^{-1} y') = h(y, \alpha^* y')$, and $h(ey, y') = h(y, \bar{e} y')$ for $e \in E_k$; as L is generated as a ring by E_k and $\alpha^{\pm 1}$ ($\alpha^{-1} \in E_k[\alpha]$ because $P(0) \neq 0$), $h(zy, y') = h(y, z^* y')$ for all $z \in L$. The map $w \mapsto h(w, 1)$ is E_k-linear, so by nondegeneracy of the trace form there is a unique $x \in L$ with $h(w, 1) = \mathrm{Tr}(xw)$; then $h(y, y') = h(y'^* y, 1) = \mathrm{Tr}(xyy'^*)$. Comparing $h(y', y) = \overline{h(y, y')}$ gives $\mathrm{Tr}((x - x^*) y' y^*) = 0$ for all y, y', so $x = x^*$; nondegeneracy of h forces $x \in L^\times$. Uniqueness: $h_x = h_{x'}$ gives $\mathrm{Tr}((x - x')w) = 0$ for all w.

(b) Write $E_k = \prod_j E_j$ with fields E_j (one factor, or two if $k \supseteq E$ splits E), $V_k = \prod_j V_j$ and P_j the image of P in $E_j[X]$. On each V_j, γ is E_j-linear with characteristic polynomial P_j, which is separable (image of a separable polynomial); a separable characteristic polynomial equals the minimal polynomial, so $V_j \cong E_j[X]/(P_j)$ as $E_j[X]$-modules (X acting by γ). Hence V_k is a free module of rank one over $E_k[X]/(P) = L$ with α acting by γ; for a generator v_0, $\varphi_0(y) = y \cdot v_0$ is an E_k-linear bijection $L \to V_k$ with $\varphi_0 m_\alpha = \gamma \varphi_0$. The pull-back $h(y, y') = \langle \varphi_0 y, \varphi_0 y' \rangle_V$ is nondegenerate hermitian and m_α-invariant because γ is unitary, so $h = h_x$ with $x \in K_{P,k}^\times$ by the lemma, and $\varphi = \varphi_0$ is an isometry $(L, h_x) \to V_k$. In particular $[x] \in X_P(V, k)$.

Conjugacy and norms. Let (γ, x, φ) and (γ', x', φ') be as in (b). For $g \in \mathrm{U}(V)(k)$ with $\gamma' = g\gamma g^{-1}$, the map $\psi = \varphi'^{-1} g \varphi$ is an E_k-linear bijection $L \to L$ commuting with m_α, hence L-linear, i.e. $\psi = m_z$ with $z \in L^\times$; it is an isometry $h_x \to h_{x'}$, so $\mathrm{Tr}(x' z z^* y y'^*) = \mathrm{Tr}(xyy'^*)$ for all y, y', i.e. $x = x' N(z)$. Conversely, if $x = x' N(z)$ with $z \in L^\times$, then $g = \varphi' m_z \varphi^{-1}$ is an isometry of V_k, so $g \in \mathrm{U}(V)(k)$, and $g\gamma g^{-1} = \gamma'$. Taking $\gamma' = \gamma$ shows that $[x]$ does not depend on the choice of (x, φ), and in general that $c_k(\gamma) = c_k(\gamma')$ if and only if γ, γ' are conjugate. This proves (b) and the injectivity in (c).

(c) If $[x] \in X_P(V, k)$, choose an isometry $\varphi\colon (L, h_x) \to V_k$; $\gamma = \varphi m_\alpha \varphi^{-1}$ is unitary by (a), has characteristic polynomial P, and $c_k(\gamma) = [x]$.

(d) With $\gamma' = \gamma$, $x' = x$, $\varphi' = \varphi$, the computation above shows that g centralises γ if and only if $g = \varphi m_z \varphi^{-1}$ with $N(z) = 1$, and $z \mapsto \varphi m_z \varphi^{-1}$ is an injective homomorphism.

(e) Let $k = F_v$. Since $*$ is a nontrivial involution of L_P (nontrivial on E), $[L_P : K_P] = 2$. Then $K_{P,k} = \prod_w K_{P,w}$ over the places w of K_P above v, and $L_{P,k} = \prod_w L_P \otimes_{K_P} K_{P,w}$, each factor a quadratic field extension of $K_{P,w}$ or $K_{P,w} \times K_{P,w}$ with $*$ the conjugation, resp. the swap; N is the componentwise norm. In the split case the norm is surjective; for a quadratic extension of local fields the norm subgroup has index 2 (local class field theory, [224], Chapter V; for $\mathbb{C}/\mathbb{R}$ the norms are the positive reals, of index 2). So $|C_P(k)| \leq 2^{\#\{w|v\}}$. □

Lemma 5.8.47. *Let E/F be a CM extension, $r \geq 1$, $P \in \mathcal{P}_r(E/F)$, V a hermitian space over E of rank r with an $\mathcal{O}_E$-lattice Λ, and $G = \mathrm{U}(V)$; for a finite place v of F let $\Lambda_v = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$ and let $K_v \subseteq G(F_v)$ be its stabiliser. Then:*

(a) for all but finitely many finite places v of F: if $\gamma \in K_v$ has characteristic polynomial P, then $c_{F_v}(\gamma) = 1$ and $\{x \in G(F_v) : x^{-1}\gamma x \in K_v\} = T_\gamma K_v$, where T_γ is the centraliser of γ; consequently, for Haar measures μ_v on $G(F_v)$ and dt_v on T_γ,

$$\int_{T_\gamma \backslash G(F_v)} \mathbf{1}_{K_v}(x^{-1}\gamma x)\, d\dot{x} = \frac{\mu_v(K_v)}{dt_v(T_\gamma \cap K_v)};$$

(b) $T_P(F)\backslash T_P(\mathbb{A}_F)$ is compact;

(c) for all but finitely many finite places v of F: there is $\gamma \in K_v$ with characteristic polynomial P, and every such γ satisfies $\iota_\gamma^{-1}(K_v) = T_P(\mathcal{O}_v)$.

Proof. Write $L = L_P$, $K = K_P$, $\alpha = \alpha_P$; Since $*$ is nontrivial on $E \subseteq L$, $E \not\subseteq K$, $K \cap E = F$ and $[L : K] = 2 = [E : F]$, so $L = KE \cong K \otimes_F E$. For a finite place v let $\mathcal{O}_{L,v}$, $\mathcal{O}_{K,v}$, $\mathcal{O}_{E,v}$ be the integral closures of $\mathcal{O}_{F,v}$ in L_{P,F_v}, K_{P,F_v}, E_v, and $\Lambda_v^\vee = \{y : \langle y, \Lambda_v\rangle \subseteq \mathcal{O}_{E,v}\}$.

Excluded places. Let Σ be the finite set of finite v at which one of the following fails: E_v/F_v is unramified; the coefficients of P are in $\mathcal{O}_{E,v}$, $P(0) \in \mathcal{O}_{E,v}^\times$ and the discriminant of P is in $\mathcal{O}_{E,v}^\times$; $\Lambda_v^\vee = \Lambda_v$. (Each condition fails at finitely many v; for the last, $\Lambda^\vee$ and Λ are two $\mathcal{O}_E$-lattices of V, hence equal at almost all places.) Let $v \notin \Sigma$. Then $\mathcal{O}_{E,v}[X]/(P)$ is finite étale over $\mathcal{O}_{E,v}$, hence equal to $\mathcal{O}_{L,v}$, with $\alpha \in \mathcal{O}_{L,v}^\times$; it is stable under $*$ ($\alpha^* = \alpha^{-1}$); its trace dual for $\mathrm{Tr}_{L/E}$ is itself; and $\mathcal{O}_{L,v}$ is a finite product of discrete valuation rings, each factor of $L_{P,F_v} = \prod_w L \otimes_K K_w$ being unramified over K_w (since $L \otimes_K K_w = K_w \otimes_{F_v} E_v$ and E_v/F_v is unramified).

(a), the class. Let $\gamma \in K_v$ have characteristic polynomial P. Then Λ_v is stable under γ and γ^{-1}, hence a module over $\mathcal{O}_{E,v}[\gamma] \cong \mathcal{O}_{L,v}$ ($\alpha \mapsto \gamma$, the isomorphism of Lemma 5.8.46(b)), finitely generated and torsion free of rank one over each factor; as $\mathcal{O}_{L,v}$ is a product of principal ideal domains, $\Lambda_v = \mathcal{O}_{L,v} w_0$ for some w_0. With $\varphi(y) = y \cdot w_0$, Lemma 5.8.46(b) gives $x \in K_{P,F_v}^\times$ with $\varphi\colon (L_{P,F_v}, h_x) \to V_{F_v}$ an isometry, $\varphi(\mathcal{O}_{L,v}) = \Lambda_v$. The dual of $\mathcal{O}_{L,v}$ for h_x is $\{y : \mathrm{Tr}(xy\mathcal{O}_{L,v}) \subseteq \mathcal{O}_{E,v}\} = x^{-1}\mathcal{O}_{L,v}$ (trace self-duality and $\mathcal{O}_{L,v}^* = \mathcal{O}_{L,v}$); self-duality of Λ_v gives $x \in \mathcal{O}_{L,v}^\times \cap K_{P,F_v} = \mathcal{O}_{K,v}^\times$. A unit is a norm from an unramified quadratic extension of local fields and from a split algebra, so $c_{F_v}(\gamma) = [x] = 1$.

(a), the orbit. Clearly $T_\gamma K_v \subseteq \{x : x^{-1}\gamma x \in K_v\}$. Conversely, if $x^{-1}\gamma x \in K_v$, then γ stabilises the self-dual lattice $x\Lambda_v$, which as above equals $\mathcal{O}_{L,v} w$ with $w = \varphi(z)$, $z \in L_{P,F_v}^\times$; it is self-dual for the form $h_{xN(z)}$ read through $y \mapsto y \cdot w$, so $N(z) \in \mathcal{O}_{K,v}^\times$. In each factor L_w of L_{P,F_v}: if L_w is an unramified field over K_w, $N(z_w)$ a unit forces $z_w \in \mathcal{O}_{L_w}^\times$; if $L_w = K_w \times K_w$ with $*$ the swap, $z_w = (z_1, z_2)$ with $z_1 z_2$ a unit, and $z_w = (z_1, z_1^{-1}) \cdot (1, z_1 z_2)$ with (z_1, z_1^{-1}) of norm 1 and $(1, z_1 z_2)$ a unit. So $z = tu$ with $N(t) = 1$ and $u \in \mathcal{O}_{L,v}^\times$, and $x\Lambda_v = \mathcal{O}_{L,v}\varphi(tu) = t \cdot \Lambda_v$ where t acts as $\varphi m_t \varphi^{-1} \in T_\gamma$ (Lemma 5.8.46(d)). Hence $(\varphi m_t \varphi^{-1})^{-1} x$ stabilises Λ_v, i.e. $x \in T_\gamma K_v$. The integral is then $\int_{T_\gamma \backslash T_\gamma K_v} d\dot{x} = \mu_v(K_v)/dt_v(T_\gamma \cap K_v)$ by the quotient integral formula.

(c) Let $v \notin \Sigma$. If $\gamma \in K_v$ has characteristic polynomial P, take w_0 and $\varphi(y) = y \cdot w_0$ as in the proof of the class in (a), so $\varphi(\mathcal{O}_{L,v}) = \Lambda_v$; for $z = q(\alpha) \in L_{P,F_v}$, $\varphi(zy) = q(\gamma)\varphi(y)$, i.e. $\iota_\gamma(z) = \varphi m_z \varphi^{-1}$ on $T_P(F_v)$. Hence $\iota_\gamma(z) \in K_v$ if and only if $z\mathcal{O}_{L,v} = \mathcal{O}_{L,v}$, i.e. $z \in \mathcal{O}_{L,v}^\times$; so $\iota_\gamma^{-1}(K_v) = T_P(\mathcal{O}_v)$. Existence: the computation of duals in (a) with $x = 1$ shows that $\mathcal{O}_{L,v}$ is self-dual for h_1, and Λ_v is self-dual in V_{F_v}. The Gram matrices of $\mathcal{O}_{E,v}$-bases of these two lattices lie in $\mathrm{GL}_r(\mathcal{O}_{E,v})$, so both determinants are units of $\mathcal{O}_{F,v}$, and units are norms from E_v (E_v/F_v unramified, or split). Nondegenerate hermitian spaces over E_v/F_v of the same rank whose determinants agree modulo norms are isometric (Jacobowitz [147]; in the split case every such space of rank r is isometric to the standard one), so there is an isometry $\psi\colon (L_{P,F_v}, h_1) \to V_{F_v}$. Then $\psi(\mathcal{O}_{L,v})$ is a self-dual lattice of V_{F_v}, and Lemma 5.8.48 gives $g \in G(F_v)$ with $g\psi(\mathcal{O}_{L,v}) = \Lambda_v$. With $\varphi = g\psi$, the element $\gamma = \varphi m_\alpha \varphi^{-1}$ lies in $G(F_v)$ and has characteristic polynomial P (Lemma 5.8.46(a)), and $\gamma\Lambda_v = \varphi(\alpha\mathcal{O}_{L,v}) = \Lambda_v$ since $\alpha \in \mathcal{O}_{L,v}^\times$; so $\gamma \in K_v$.

(b) T_P is the Weil restriction from K to F of the norm-one torus of the quadratic field extension L/K, which has no nontrivial K-rational character; so T_P has no nontrivial F-rational character. It contains no nontrivial F-split torus: if $S \subseteq T_P$ were one, choose $\chi \in X^*(T_P)$ whose restriction to S is nontrivial (restriction $X^*(T_P) \to X^*(S)$ is surjective for a subtorus); the absolute Galois group of F acts on $X^*(T_P)$ through a finite quotient Γ, trivially on $X^*(S)$, and restriction is Γ-equivariant, so $\sum_{g\in\Gamma} g\chi$ is an F-rational character of T_P whose restriction to S is $|\Gamma|\,\chi|_S \neq 0$ ($X^*(S)$ is torsion free), a contradiction. As T_P is a torus, hence connected reductive, the compactness criterion of Borel–Harish-Chandra and Mostow–Tamagawa [37], [216] shows that $T_P(F)\backslash T_P(\mathbb{A}_F)$ is compact. □

Lemma 5.8.48. *Let F be a non-archimedean local field with valuation ring $\mathcal{O}$, and let K be either the unramified quadratic field extension of F or $F \times F$, with its non-trivial F-involution c and maximal order R (R is the valuation ring of K, resp. $\mathcal{O} \times \mathcal{O}$). Let W be a free K-module of finite rank with a non-degenerate form $k\colon W \times W \to K$, K-linear in the first variable, with $k(y,x) = c(k(x,y))$. Call an R-lattice $L \subseteq W$ (a finitely generated R-submodule spanning W over K) **self-dual** if $L = \{x \in W : \ k(x,L) \subseteq R\}$. Then for any two self-dual R-lattices L, L' of W there is a K-linear $g\colon W \to W$ with $k(gx,gy) = k(x,y)$ for all x, y and $gL = L'$.*

Proof. K **a field: a self-dual lattice has an orthonormal basis.** Here R is a discrete valuation ring with uniformiser ϖ (the extension is unramified), every finitely generated torsion-free R-module is free, and $\mathrm{Tr}\colon R \to \mathcal{O}$ and $N\colon R^\times \to \mathcal{O}^\times$ are surjective (unramified extensions of local fields; Serre, *Local fields*, Ch. V, §2). Self-duality of L means $k(L,L) \subseteq R$ and that $x \mapsto k(\cdot, x)$ maps L onto $\mathrm{Hom}_R(L,R)$ (every R-linear $L \to R$ extends to a K-linear form on W, which is $k(\cdot,x)$ for a unique $x \in W$, and then $k(L,x) \subseteq R$ gives $x \in L$). We show by induction on $\mathrm{rk}_R L$ that L has a basis $e_1, \dots, e_r$ with $k(e_i, e_j) = \delta_{ij}$. Let $x \in L \smallsetminus \varpi L$. Then $k(x,\cdot)\colon L \to R$ is onto: otherwise its image lies in ϖR, so the R-linear map $k(\cdot,x) = c \circ k(x,\cdot)$ is ϖ times an element of $\mathrm{Hom}_R(L,R)$, which by self-duality is $k(\cdot,x')$ with $x' \in L$; as $k(\cdot,\varpi x') = \varpi\, k(\cdot,x')$, non-degeneracy gives $x = \varpi x' \in \varpi L$. Choose $y \in L$ with $k(x,y) = 1$. We find $z \in L$ with $u = k(z,z) \in \mathcal{O}^\times$ ($k(z,z)$ is c-invariant, so lies in $R \cap F = \mathcal{O}$): take $z = x$ or $z = y$ if $k(x,x)$ or $k(y,y)$ is a unit; otherwise choose $a \in R$ with $\mathrm{Tr}(a) = 1$ and $z = x + ay$, for which

$$k(z,z) = k(x,x) + \mathrm{Tr}\big(\bar{a}\, k(x,y)\big) + N(a)\, k(y,y) \equiv \mathrm{Tr}(\bar{a}) = 1 \pmod{\varpi}.$$

Then $L = Rz \oplus L_1$ with $L_1 = \{w \in L : \ k(w,z) = 0\}$, via $w = k(w,z)u^{-1}z + (w - k(w,z)u^{-1}z)$; the Gram matrix of L in a basis adapted to this sum is block diagonal with the unit u in one block, so its other block is invertible over R and L_1 is self-dual in $z^\perp$. Choose $t \in R^\times$ with $N(t) = u$ and put $e_1 = t^{-1}z$, so $k(e_1,e_1) = 1$, and apply the induction hypothesis to L_1. Given orthonormal bases (e_i) of L and (e_i') of L', the K-linear g with $ge_i = e_i'$ preserves k and maps L onto L'.

$K = F \times F$**.** Let ϵ_1, ϵ_2 be the idempotents of K (interchanged by c) and $W_i = \epsilon_i W$, so $W = W_1 \oplus W_2$. Writing $k = \epsilon_1 k_1 + \epsilon_2 k_2$ with F-valued k_i, linearity in the first variable and semilinearity in the second give $k_1(x,y) = b(x_1, y_2)$ for an F-bilinear $b\colon W_1 \times W_2 \to F$ ($x = x_1 + x_2$, $x_i \in W_i$), and hermitian symmetry gives $k_2(x,y) = b(y_1, x_2)$; non-degeneracy of k makes b perfect. A K-linear $g = (g_1, g_2)$, $g_i \in \mathrm{GL}(W_i)$, preserves k if and only if $g_2 = (g_1^*)^{-1}$, with g_1^* the adjoint for b. Every R-lattice is $L = L_1 \oplus L_2$ with $L_i = \epsilon_i L$ an $\mathcal{O}$-lattice of W_i, and $\{x : \ k(x,L) \subseteq R\} = \{x_1 : \ b(x_1, L_2) \subseteq \mathcal{O}\} \oplus \{x_2 : \ b(L_1, x_2) \subseteq \mathcal{O}\}$; hence L is self-dual if and only if $L_2 = L_1^\perp := \{x_2 : \ b(L_1, x_2) \subseteq \mathcal{O}\}$ (the condition on L_1 then follows from perfectness of b). For self-dual $L = L_1 \oplus L_1^\perp$ and $L' = L_1' \oplus L_1'^\perp$ choose $g_1 \in \mathrm{GL}(W_1)$ with $g_1 L_1 = L_1'$ (both are free $\mathcal{O}$-modules of rank $\dim W_1$); then $(g_1^*)^{-1} L_1^\perp = (g_1 L_1)^\perp = L_1'^\perp$, and $g = (g_1, (g_1^*)^{-1})$ preserves k and maps L onto L'. □

Lemma 5.8.49. *Let $N \geq 1$, $s \in \{\pm 1\}^N$, $p = \#\{i : s(i) = 1\}$, $q = N - p$, and $G = \mathrm{U}(\mathrm{diag}\, s)$ with a Haar measure dh. Let $\lambda \in (\frac{1}{2}\mathbb{Z})^N$ be such that 2λ has gap $500N^3 + 10N^5$ for s, let d_λ be the formal degree for dh, and $e_\lambda = d_\lambda \overline{c_\lambda}$ on G. Let χ_λ be the character of the irreducible representation of the compact group $\mathrm{U}(N)$ with Harish-Chandra parameter λ. Let $z_1, \dots, z_N \in \mathbb{C}$ be distinct with $|z_i| = 1$, and for $\sigma \in S_N$ let $t_\sigma = \mathrm{diag}(z_{\sigma(1)}, \dots, z_{\sigma(N)})$. Then for every σ the function $h \mapsto e_\lambda(h^{-1} t_\sigma h)$ is integrable on G, and*

$$(-1)^{pq} \sum_{\sigma \in S_N} \int_G e_\lambda(h^{-1} t_\sigma h)\, dh = p!\, q!\, \overline{\chi_\lambda(\mathrm{diag}(z_1, \dots, z_N))}.$$

Proof. Index the basis by $n = \{1, \dots, N\}$, put $l = 2\lambda$ (large gap $10N^5$ for s, (WM8) of Definition 3.2.19), $x = (z_1, \dots, z_N)$, and for $a \in n$ let $p(a) = \#\{a' : l_{a'} > l_a\}$ and a_p the index with $p(a_p) = p$ (l is injective). Let dg be the Cayley measure of G and, for $y \in \mathbb{C}^n$ with distinct entries of modulus one, $\Theta(y) = d_\lambda^{\mathrm{Cay}} \int_G c_\lambda(g^{-1}\mathrm{diag}(y)g)\, dg$, with d_λ^{Cay} the formal degree for dg. If $dh = t\, dg$, then $d_\lambda = d_\lambda^{\mathrm{Cay}}/t$, so $d_\lambda \int_G c_\lambda(h^{-1}\mathrm{diag}(y)h)\, dh = \Theta(y)$, and integrability does not depend on the Haar measure. For $\tau \in S_N$ write $x \circ \tau = (z_{\tau(1)}, \dots, z_{\tau(N)})$, so $t_\tau = \mathrm{diag}(x \circ \tau)$; its entries are distinct.

The identity for one permutation. By Proposition 5.7.5 (ii) (with $\varepsilon = s$), for every τ the function $h \mapsto c_\lambda(h^{-1}t_\tau h)$ is integrable, hence so is $h \mapsto e_\lambda(h^{-1}t_\tau h) = d_\lambda \overline{c_\lambda(h^{-1}t_\tau h)}$, and

$$V_{\mathrm{c}}(x \circ \tau)\, D(x \circ \tau)\, \Theta(x \circ \tau) = A(x \circ \tau),$$

with V_{c}, D, κ and A as there. Let $V(y) = \prod_{(a,b):\, l_b < l_a} (y_a - y_b)$. Each factor of D is $y_a - y_b = -(y_b - y_a)$ for a pair with $s_a \neq s_b$ and $l_a < l_b$, and these pairs correspond to the pq unordered pairs of indices with different signs; so $D(y) = (-1)^{pq} \prod_{s_a \neq s_b,\ l_b < l_a} (y_a - y_b)$ and $V_{\mathrm{c}}(y)D(y) = (-1)^{pq}V(y)$. In the order $a_0, \dots, a_{N-1}$, $V(y) = \prod_{p<q}(y_{a_p} - y_{a_q}) = \det\bigl(y_{a_q}^{N-1-p}\bigr)_{p,q}$ (Vandermonde). Replacing y by $x \circ \tau$ permutes the columns of this matrix by a permutation conjugate to τ, so $V(x \circ \tau) = \mathrm{sgn}(\tau)V(x)$. Hence

$$(-1)^{pq}\, \mathrm{sgn}(\tau)\, V(x)\, \Theta(x \circ \tau) = A(x \circ \tau) \qquad (\tau \in S_N). \tag{5.181}$$

Summation. Let $W = \{w \in S_N : s \circ w = s\}$, so $\#W = p!\, q!$. For $w \in W$, by the Leibniz formula and a permutation of rows,

$$\sum_{\tau \in S_N} \mathrm{sgn}(\tau) \prod_a x_{\tau(a)}^{\kappa_{w(a)}} = \det\bigl(x_b^{\kappa_{w(a)}}\bigr)_{a,b} = \mathrm{sgn}(w)\, \Delta(x), \qquad \Delta(x) = \det\bigl(x_b^{\kappa_a}\bigr)_{a,b}.$$

Multiplying (5.181) by $\mathrm{sgn}(\tau)$ and summing over τ gives

$$(-1)^{pq}\, V(x) \sum_{\tau \in S_N} \Theta(x \circ \tau) = \sum_{w \in W} \mathrm{sgn}(w)^2 \Delta(x) = p!\, q!\, \Delta(x).$$

Weyl's character formula. Reordering rows and columns by the same permutation, $\Delta(x) = \det\bigl(x_{a_q}^{\kappa_{a_p}}\bigr)_{p,q}$, and $\kappa_{a_p} = \lambda_{a_p} + \frac{N-1}{2} = \mu_p + N - 1 - p$ with $\mu_p = \lambda_{a_p} - \frac{N-1}{2} + p$. The μ_p are integers ($\lambda_a \in \frac{N-1}{2} + \mathbb{Z}$) and nonincreasing ($\lambda_{a_p} - \lambda_{a_{p+1}} \geq 1$, the entries of l being distinct of equal parity); μ is the highest weight, for the diagonal torus and the order $a_0, \dots, a_{N-1}$ of the basis, of the irreducible representation of $\mathrm{U}(N)$ with Harish-Chandra parameter λ. Weyl's character formula for $\mathrm{U}(N)$ [91] gives $\chi_\lambda(\mathrm{diag}(x)) = \det\bigl(x_{a_q}^{\mu_p+N-1-p}\bigr) / \det\bigl(x_{a_q}^{N-1-p}\bigr) = \Delta(x)/V(x)$, where $V(x) \neq 0$ as the z_i are distinct. Therefore $(-1)^{pq} \sum_\tau \Theta(x \circ \tau) = p!\, q!\, \chi_\lambda(\mathrm{diag}(x))$.

Conclusion. As d_λ is real, $\int_G e_\lambda(h^{-1}t_\tau h)\, dh = \overline{d_\lambda \int_G c_\lambda(h^{-1}t_\tau h)\, dh} = \overline{\Theta(x \circ \tau)}$, and conjugating the last identity gives

$$(-1)^{pq} \sum_{\sigma \in S_N} \int_G e_\lambda(h^{-1}t_\sigma h)\, dh = p!\, q!\, \overline{\chi_\lambda(\mathrm{diag}(z_1, \dots, z_N))}.$$

□

Theorem 5.8.50. *Let E be a CM field containing a primitive 5040-th root of unity, $N \geq 1$, and $G_f = G_f(\{1, \dots, N\})$, $\mathrm{K}_f = \mathbf{K}_0(\{1, \dots, N\})$ as in (VP1). For a finite place v of E with $v \neq \bar{v}$ let $\iota_v \colon \mathrm{GL}_N(E_v) \to G_f$ be the embedding at the place of E^+ below v (component 1 elsewhere), and for $g \in G_f$ let $g_v \in \mathrm{GL}_N(E_v)$ be its component and $g^{(v)} = g\, \iota_v(g_v)^{-1}$; extend $g \mapsto g^{(v)}$*

linearly to $a \mapsto a^{(v)}$ on $\mathbb{C}[G_f]$. Fix $i \in \{1,\dots,N\}$, let $J_v \subseteq \mathrm{GL}_N(\mathcal{O}_v)$ be the stabiliser of the line of the i-th basis vector modulo v, $t_v = \mathrm{diag}(1,\dots,\varpi_v,\dots,1)$ with a uniformiser ϖ_v at position i, and $T_v^+ \in \mathbb{C}[\mathrm{GL}_N(E_v)]$ the sum of a set of representatives of $J_v t_v J_v / J_v$. For an open subgroup $K \subseteq \mathrm{K}_f$ put $K' = \{k \in K :\ k_v \in J_v\}$, and for $a \in \mathbb{C}[G_f]$ and $l,m \geq 0$ put

$$b_{l,m} = \iota_v(T_v^+)^{m+1}\,(a^{(v)})^{l+1} \in \mathbb{C}[G_f].$$

(CU1) For every open $K \subseteq \mathrm{K}_f$, every finitely supported $a \in \mathbb{C}[G_f]$ and every finite place w of E there is a finite place v of E with $v \neq \bar v$ and $v \notin \{w, \bar w\}$, such that $\iota_v(\mathrm{GL}_N(\mathcal{O}_v)) \subseteq K$ and a is a linear combination of elements g with $g_v \in \mathrm{GL}_N(\mathcal{O}_v)$.

(CU2) Let $K \subseteq \mathrm{K}_f$ be open, let $a \in \mathbb{C}[G_f]$ be finitely supported and map K-fixed vectors to K-fixed vectors in every representation of G_f, and let v be as in (CU1) for K and a. Let ρ, ρ' be unitary representations of G_f on complex inner product spaces X, X' such that $X^{K'}$ and $X'^{K'}$ are finite-dimensional. Then each $b_{l,m}$ maps $X^{K'}$ and $X'^{K'}$ into themselves, and if

$$\mathrm{tr}(\rho(b_{l,m}) \mid X^{K'}) = \mathrm{tr}(\rho'(b_{l,m}) \mid X'^{K'}) \qquad \text{for all } l,m \geq 0,$$

then $\mathrm{tr}(\rho(a) \mid X^K) = \mathrm{tr}(\rho'(a) \mid X'^K)$.

(CU3) In the situation of (CU2), let $\phi(g) = \sum_h a_h \mathbf{1}_K(h^{-1}g)$ be cuspidal at a finite place w with $w \neq \bar w$ and $v \notin \{w, \bar w\}$. Then for all $l,m \geq 0$ the function $\phi_{l,m}(g) = \sum_h (b_{l,m})_h \mathbf{1}_{K'}(h^{-1}g)$ is cuspidal at w, and for every g with $\phi_{l,m}(g) \neq 0$ the component g_v has entries in $\mathcal{O}_v$ and 0 is a simple root of the characteristic polynomial of its reduction modulo v.

Proof. Throughout, for a finite place v of E with $v \neq \bar v$ let u be the place of E^+ below v. Then $E \otimes_{E^+} E_u^+ = E_v \times E_{\bar v}$, the factor of G_f at u is identified with $\mathrm{GL}_N(E_v)$ by the component at v, and ι_v is the inclusion of this factor; the factor of K_f at u is $\iota_v(\mathrm{GL}_N(\mathcal{O}_v))$. For an open subgroup $L \subseteq G_f$ and $c \in \mathbb{C}[G_f]$ write $\phi_c^L(g) = \sum_h c_h \mathbf{1}_L(h^{-1}g)$; then $\phi_{xc}^L(g) = \phi_c^L(x^{-1}g)$ for $x \in G_f$, and $c \mapsto \phi_c^L$ is linear.

Proof of (CU1). Infinitely many rational primes split completely in E (Chebotarev's density theorem), and a place v of E above such a prime has $E_v = \mathbb{Q}_p$ and $v \neq \bar v$, because the place of E^+ below it has two places of E above it. The remaining conditions exclude finitely many v. First, $v \in \{w, \bar w\}$ excludes at most two. Second, K is open in the compact group $K_f = \prod_u K_{f,u}$, so it contains $\prod_{u \notin S} K_{f,u}$ for a finite set S of places of E^+, and then $\iota_v(\mathrm{GL}_N(\mathcal{O}_v)) = K_{f,u} \subseteq K$ whenever $u \notin S$. Third, a has finitely many elements in its support, and each element g of the restricted product G_f has $g_v \in \mathrm{GL}_N(\mathcal{O}_v)$ for all but finitely many v. Any split v outside these finitely many places satisfies (CU1).

Reductions for (CU2) and (CU3). Fix K, a and v as in (CU2), so that $v \neq \bar v$, $\iota_v(\mathrm{GL}_N(\mathcal{O}_v)) \subseteq K$ and every g in the support of a has $g_v \in \mathrm{GL}_N(\mathcal{O}_v)$. Let $G^{(u)} \subseteq G_f$ be the subgroup of elements with component 1 at u. Every element of $G^{(u)}$ commutes with every element of $\iota_v(\mathrm{GL}_N(E_v))$, and $g^{(v)} \in G^{(u)}$. Hence $g \mapsto g^{(v)}$ is a homomorphism: $(gh)^{(v)} = g\,h\,\iota_v(h_v)^{-1}\iota_v(g_v)^{-1} = g\,\iota_v(g_v)^{-1}\,h\,\iota_v(h_v)^{-1} = g^{(v)}h^{(v)}$, using that $h^{(v)}$ commutes with $\iota_v(g_v)^{-1}$. For $k \in K$ we have $\iota_v(k_v) \in K_{f,u} \subseteq K$, so $k^{(v)} \in K$, and $k^{(v)}$ has component $1 \in J_v$ at v, so $k^{(v)} \in K'$; also $\iota_v(J_v) \subseteq K'$. Since $k = k^{(v)}\iota_v(k_v)$, this gives, for a representation ρ of G_f on X and $y \in X$:

$$y \in X^{K'} \iff y \text{ is fixed by } \rho(k^{(v)})\ (k \in K) \text{ and by } \rho(\iota_v(j))\ (j \in J_v), \tag{5.182}$$

and the same with $(K, \mathrm{GL}_N(\mathcal{O}_v))$ in place of (K', J_v). Let $\rho^{(v)}(g) = \rho(g^{(v)})$, a representation of G_f on X by the homomorphism property; the first condition in (5.182) says that y is K-fixed for $\rho^{(v)}$.

The operators at the cut level. Put $A = \rho(a^{(v)}) = \rho^{(v)}(a)$, and let T^+, T^-, e be the operators of Proposition 5.8.52 for the local field E_v, the index i and the representation $\rho \circ \iota_v$ of $\mathrm{GL}_N(E_v)$, acting on the vectors fixed by $\rho(\iota_v(J_v))$; thus T^+ is the action of $\iota_v(T_v^+)$ there. By the hypothesis on a, applied to $\rho^{(v)}$, the operator A maps the K-fixed vectors of $\rho^{(v)}$ to themselves; it commutes with $\rho(\iota_v(\mathrm{GL}_N(E_v)))$, so it maps the $\rho(\iota_v(J_v))$-fixed vectors to themselves. By Proposition 5.8.52(ii), $T^\pm$ and e preserve the $\rho(\iota_v(J_v))$-fixed vectors, and as combinations of elements of $\iota_v(\mathrm{GL}_N(E_v))$ they commute with $\rho^{(v)}(G_f)$, so they preserve the K-fixed vectors of $\rho^{(v)}$. By (5.182), A, $T^\pm$ and e preserve $X^{K'}$, and $\rho(b_{l,m})$ acts on $X^{K'}$ as $(T^+)^{m+1}A^{l+1} = A^{l+1}(T^+)^{m+1}$ (the factors commute); in particular $b_{l,m}$ maps $X^{K'}$ into itself. The same holds for ρ'.

Proof of (CU2). Restrict all operators to $X^{K'}$. By Proposition 5.8.52(iii), (iv) (the representation $\rho \circ \iota_v$ is unitary), $\langle T^+x, y\rangle = \langle x, T^-y\rangle$ and $T^+T^- = d(1 + ce)$ with $d = q_v^{N-2} > 0$ and $c = q_v^N - 1 > 0$, q_v the order of the residue field of v. By Proposition 5.8.52(ii), e maps $X^{K'}$ into the vectors fixed by $\rho(\iota_v(\mathrm{GL}_N(\mathcal{O}_v)))$; it also preserves the K-fixed vectors of $\rho^{(v)}$, so by (5.182) for K it maps $X^{K'}$ into X^K, and it is the identity on X^K, which is fixed by $\iota_v(\mathrm{GL}_N(\mathcal{O}_v)) \subseteq K$. So e is an idempotent of $X^{K'}$ with image X^K. Finally A commutes with $T^\pm$, since $\rho^{(v)}(G_f)$ commutes with $\rho(\iota_v(\mathrm{GL}_N(E_v)))$. Thus A, T^+, T^-, e on $X^{K'}$, and the corresponding operators on $X'^{K'}$, satisfy the hypotheses of Lemma 5.8.51 with the same d and c. The traces in that lemma are those of $\rho(b_{l,m})$ and $\rho'(b_{l,m})$ on $X^{K'}$ and $X'^{K'}$, equal by hypothesis; hence $\mathrm{tr}(Ae \mid X^{K'}) = \mathrm{tr}(A'e' \mid X'^{K'})$.

It remains to identify $\mathrm{tr}(Ae \mid X^{K'})$ with $\mathrm{tr}(\rho(a) \mid X^K)$. For $y \in X^K$ and g in the support of a, $\iota_v(g_v) \in K$, so $\rho(g)y = \rho(g^{(v)})\rho(\iota_v(g_v))y = \rho(g^{(v)})y$; hence $\rho(a)y = Ay$, and A maps X^K into itself. Let $\iota\colon X^K \to X^{K'}$ be the inclusion and $p\colon X^{K'} \to X^K$ the map e, so that $e = \iota \circ p$ and $p \circ \iota = \mathrm{id}$. Then $A\iota = \iota\, \rho(a)|_{X^K}$ and

$$\mathrm{tr}(Ae \mid X^{K'}) = \mathrm{tr}(\iota\, \rho(a)|_{X^K}\, p) = \mathrm{tr}(\rho(a)|_{X^K}\, p\, \iota) = \mathrm{tr}(\rho(a) \mid X^K).$$

The same holds for ρ', which proves (CU2).

Proof of (CU3). First, for $c \in \mathbb{C}[G_f]$ a combination of elements h with $h_v \in \mathrm{GL}_N(\mathcal{O}_v)$,

$$\phi^{K'}_{c^{(v)}}(g) = \mathbf{1}_{J_v}(g_v)\, \phi^K_c(g) \qquad (g \in G_f). \tag{5.183}$$

By linearity it suffices to take $c = h$. Then $(h^{(v)})^{-1}g = \iota_v(h_v)h^{-1}g$ has component $h_v h_v^{-1} g_v = g_v$ at v, and since $\iota_v(h_v) \in K$ it lies in K iff $h^{-1}g \in K$; so it lies in K' iff $h^{-1}g \in K$ and $g_v \in J_v$. The element a^{l+1} is a combination of products of elements of the support of a, which are integral at v, and $(a^{l+1})^{(v)} = (a^{(v)})^{l+1}$; so (5.183) gives $\phi^{K'}_{(a^{(v)})^{l+1}}(g) = \mathbf{1}_{J_v}(g_v)\phi^K_{a^{l+1}}(g)$, and $\phi_{l,m} = \phi^{K'}_{b_{l,m}}$ is the combination $\sum_x \beta_x\, \phi^{K'}_{(a^{(v)})^{l+1}}(x^{-1}\,\cdot\,)$, with $\iota_v(T_v^+)^{m+1} = \sum_x \beta_x x$.

Cuspidality at w. The defining condition, $\sum_{u' \in U_T} \psi(g\, \iota_w(\tilde{u}')) = 0$ for all g, T, t and lifts, is linear in ψ and is preserved by left translations $\psi \mapsto \psi(x^{-1}\,\cdot\,)$, since it is required for every g. Now $\phi^K_{a^{l+1}} = \phi^K_{a^l . a} = \sum_x (a^l)_x\, \phi(x^{-1}\,\cdot\,)$ is cuspidal at w. Since $v \notin \{w, \bar{w}\}$, the places of E^+ below v and w differ, so $(g\, \iota_w(\tilde{u}'))_v = g_v$ and the factor $\mathbf{1}_{J_v}(g_v)$ is constant on each sum of the condition; hence $\phi^{K'}_{(a^{(v)})^{l+1}}$ is cuspidal at w, and so is its combination of left translates $\phi_{l,m}$.

Support at v. By Lemma 5.8.53 for E_v (with $m + 1 \geq 1$), the support of $(T_v^+)^{m+1}$ consists of elements $x' \in \mathrm{GL}_N(E_v)$ such that $x'j$ is integral with 0 a simple root of the characteristic polynomial of its reduction for every $j \in J_v$. The support of $(a^{(v)})^{l+1}$ lies in $G^{(u)}$. If $\phi_{l,m}(g) \neq 0$, there are x' in the first support and h in the second with $(\iota_v(x')h)^{-1}g \in K'$; its component at v is $x'^{-1}g_v$ (as $h_v = 1$), so $g_v = x'j$ with $j \in J_v$, and g_v has the stated property. □

Lemma 5.8.51. *Let $d, c > 0$ be real numbers. Let X be a finite-dimensional complex inner product space with linear operators A, T^+, T^-, e such that $\langle T^+x, y\rangle = \langle x, T^-y\rangle$ for all $x, y \in X$,*

$e^2 = e$,

$$T^+T^- = d\,(1 + c\,e),$$

and A commutes with T^+ and with T^-. Let X', A', T'^+, T'^-, e' satisfy the same conditions with the same d, c. If

$$\operatorname{tr}\big(A^{l+1}(T^+)^{m+1} \mid X\big) = \operatorname{tr}\big(A'^{\,l+1}(T'^+)^{m+1} \mid X'\big) \qquad \text{for all integers } l, m \geq 0,$$

then $\operatorname{tr}(Ae \mid X) = \operatorname{tr}(A'e' \mid X')$.

Proof. For $\mu \in \mathbb{C}$ let $X_\mu = \ker(A - \mu)^{\dim X}$ be the generalised eigenspace of A, so that $X = \bigoplus_\mu X_\mu$ with finitely many nonzero summands, and define X'_μ in the same way for A'. The relation $T^+T^- = d(1+ce)$ gives $e = c^{-1}(d^{-1}T^+T^- - 1)$, so e commutes with A. Every operator commuting with A preserves each X_μ; write B_μ for the restriction of such an operator B to X_μ. The restrictions $T^\pm_\mu$, e_μ satisfy the same three relations on X_μ with its induced inner product; in particular $T^-_\mu = (T^+_\mu)^*$, the adjoint in X_μ. The same holds on X'.

Traces split over the generalised eigenspaces. Let B commute with A and let $l \geq 0$. On X_μ the operator $A_\mu - \mu$ is nilpotent and $A_\mu^l - \mu^l = (A_\mu - \mu)P(A_\mu)$ for a polynomial P, so $(A_\mu^l - \mu^l)B_\mu$ is a product of commuting operators one of which is nilpotent; it is nilpotent and has trace 0. Hence

$$\operatorname{tr}(A^l B \mid X) = \sum_\mu \mu^l \operatorname{tr}(B_\mu \mid X_\mu). \tag{5.184}$$

The traces of the powers of T^+ on each X_μ, $\mu \neq 0$, agree. Let S be the finite set of nonzero μ with $X_\mu \neq 0$ or $X'_\mu \neq 0$. Fix $m \geq 0$ and put $\delta(\mu) = \operatorname{tr}((T^+_\mu)^{m+1}) - \operatorname{tr}((T'^+_\mu)^{m+1})$. By (5.184) with $B = (T^+)^{m+1}$ and exponent $l+1$ (the term $\mu = 0$ vanishes) and the hypothesis, $\sum_{\mu\in S} \mu^{l+1}\delta(\mu) = 0$ for $l = 0, \dots, |S| - 1$. The matrix $(\mu^{l+1})_{l,\mu}$ is a Vandermonde matrix times $\operatorname{diag}(\mu)_{\mu\in S}$, invertible since the $\mu \in S$ are distinct and nonzero. So $\delta(\mu) = 0$ for every $\mu \neq 0$ and every $m \geq 0$.

Determinants. Let $\mu \in S$, $n_\mu = \dim X_\mu$ and $r_\mu = \operatorname{rank} e_\mu$. Since e_μ is idempotent, $X_\mu = \operatorname{im} e_\mu \oplus \ker e_\mu$ and $1 + ce_\mu$ is multiplication by $1 + c$ on the first summand and the identity on the second. Therefore

$$|\det T^+_\mu|^2 = \det\big(T^+_\mu (T^+_\mu)^*\big) = \det\big(d(1 + ce_\mu)\big) = d^{\,n_\mu}(1 + c)^{r_\mu} > 0,$$

so T^+_μ is invertible; likewise $|\det T'^+_\mu|^2 = d^{\,n'_\mu}(1 + c)^{r'_\mu}$ and T'^+_μ is invertible.

The characteristic polynomials of T^+_μ and T'^+_μ agree. For an eigenvalue ν let $a(\nu)$, $a'(\nu)$ be its algebraic multiplicities for T^+_μ and T'^+_μ; all eigenvalues of both are nonzero. Then $\operatorname{tr}((T^+_\mu)^j) = \sum_\nu a(\nu)\nu^j$ and similarly for T'^+_μ, and these agree for every $j \geq 1$ by the previous step. The same Vandermonde argument, over the finite set of nonzero ν that are eigenvalues of either operator, gives $a(\nu) = a'(\nu)$ for all ν. Hence the two characteristic polynomials are equal; comparing degrees gives $n_\mu = n'_\mu$, and comparing constant terms gives $\det T^+_\mu = \det T'^+_\mu$.

Ranks. By the two previous steps $d^{\,n_\mu}(1 + c)^{r_\mu} = d^{\,n_\mu}(1 + c)^{r'_\mu}$, and $1 + c > 1$, so $r_\mu = r'_\mu$ for every $\mu \neq 0$.

Conclusion. By (5.184) with $l = 1$ and $B = e$, and since the trace of an idempotent is its rank,

$$\operatorname{tr}(Ae \mid X) = \sum_\mu \mu \operatorname{tr}(e_\mu) = \sum_{\mu\neq 0} \mu\, r_\mu = \sum_{\mu\neq 0} \mu\, r'_\mu = \operatorname{tr}(A'e' \mid X').$$

□

Proposition 5.8.52. *Let F be a non-archimedean local field with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field k of order q, let $N \geq 1$ and $1 \leq i \leq N$. Let $J \subseteq \mathrm{GL}_N(\mathcal{O})$ be the group of elements whose reduction modulo ϖ maps the line $ke_i \subseteq k^N$ to itself, and let $t = \mathrm{diag}(t_1, \ldots, t_N)$ with $t_i = \varpi$ and $t_j = 1$ for $j \neq i$. Then:*

1. *each of JtJ and $Jt^{-1}J$ is the disjoint union of exactly q^{N-1} left cosets gJ.*

Let ρ be a representation of $\mathrm{GL}_N(F)$ on a complex vector space X, and for $y \in X^J$ put

$$T^+ y = \sum_{gJ \subseteq JtJ} \rho(g) y, \qquad T^- y = \sum_{gJ \subseteq Jt^{-1}J} \rho(g) y, \qquad e\, y = \frac{1}{[\mathrm{GL}_N(\mathcal{O}) : J]} \sum_{kJ \subseteq \mathrm{GL}_N(\mathcal{O})} \rho(k) y,$$

the sums running over the left cosets of J in the indicated sets (each term depends only on the coset). Then:

2. *T^+, T^- and e map X^J into X^J; e maps X^J onto $X^{\mathrm{GL}_N(\mathcal{O})}$ and is the identity on $X^{\mathrm{GL}_N(\mathcal{O})}$;*
3. *for every $y \in X^J$,*
$$T^+ T^- y = q^{N-2}(y + (q^N - 1)\, e\, y);$$
4. *if X carries an inner product with $\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle$ for all g, then $\langle T^+ x, y \rangle = \langle x, T^- y \rangle$ for all $x, y \in X^J$.*

Proof. Write $\bar{g} \in \mathrm{GL}_N(k)$ for the reduction of $g \in \mathrm{GL}_N(\mathcal{O})$; reduction is surjective onto $\mathrm{GL}_N(k)$, and its kernel lies in every group considered below. Let $\ell_0 = ke_i$ and $H_0 = \{x \in k^N : x_i = 0\}$, so $\ell_0 \not\subseteq H_0$. Let $J^\flat \subseteq \mathrm{GL}_N(\mathcal{O})$ be the group of elements whose reduction maps H_0 to itself. An element $g \in \mathrm{GL}_N(\mathcal{O})$ lies in J iff $g_{ai} \in \varpi\mathcal{O}$ for $a \neq i$, and in $J^\flat$ iff $g_{ib} \in \varpi\mathcal{O}$ for $b \neq i$. For a subgroup $J_1 \subseteq \mathrm{GL}_N(\mathcal{O})$ containing the kernel of reduction, $gJ_1 = g'J_1$ for $g, g' \in \mathrm{GL}_N(\mathcal{O})$ iff $\bar{g}^{-1}\bar{g}' \in \bar{J}_1$.

Conjugation by t. $(tgt^{-1})_{ab} = t_a g_{ab} t_b^{-1}$: conjugation by t multiplies row i by ϖ and divides column i by ϖ. For $g \in J$ the entries g_{ai}/ϖ $(a \neq i)$ are integral, so tgt^{-1} is integral with the same unit determinant and with row i off the diagonal in $\varpi\mathcal{O}$; thus $tJt^{-1} \subseteq J^\flat$. The same computation with t^{-1} gives $t^{-1}J^\flat t \subseteq J$. Hence $tJt^{-1} = J^\flat$.

Cosets in JtJ. The map $j \mapsto jtJ$ induces a bijection $J/(J \cap tJt^{-1}) \to JtJ/J$, since $jtJ = j'tJ$ iff $j^{-1}j' \in tJt^{-1}$. Here $J \cap tJt^{-1} = J \cap J^\flat$, and $J/(J \cap J^\flat) \cong \bar{J}/(\bar{J} \cap \bar{J}^\flat)$ is in bijection, by $j \mapsto \bar{j}H_0$, with the orbit of H_0 under $\bar{J}$, the stabiliser of ℓ_0. That orbit is the set of hyperplanes $H \subseteq k^N$ with $\ell_0 \not\subseteq H$: such an H is $\ker \xi$ for a unique linear form ξ with $\xi(e_i) = 1$, and the element $g \in \mathrm{GL}_N(k)$ with $ge_i = e_i$ and $ge_j = e_j - \xi(e_j)e_i$ for $j \neq i$ lies in $\bar{J}$ and maps H_0 into, hence onto, H, since $\xi(ge_j) = 0$; conversely every element of $\bar{J}$ maps H_0 to a hyperplane not containing ℓ_0. There are q^{N-1} such forms. So

$$JtJ = \bigsqcup_{H \not\supseteq \ell_0} j_H t J, \qquad j_H \in J,\ \bar{j}_H H_0 = H, \tag{5.185}$$

with q^{N-1} cosets. Since $\mathrm{GL}_N(F)$ is unimodular and J is compact open, the number of left cosets of J in a double coset equals the number of right cosets, and inversion maps the right cosets in JtJ onto the left cosets in $Jt^{-1}J$; this proves (i).

Cosets in $Jt^{-1}J$. By $tJt^{-1} = J^\flat$ we have $Jt^{-1}J = t^{-1}(J^\flat J)$, and $hJ \mapsto \bar{h}\ell_0$ is a bijection from $J^\flat J/J \cong J^\flat/(J^\flat \cap J)$ onto the orbit of ℓ_0 under $\bar{J}^\flat$, the stabiliser of H_0. That orbit is the set of lines $\ell \not\subseteq H_0$: such a line is kz for a unique z with $z_i = 1$, and the element h with $he_i = z$

and $he_j = e_j$ for $j \neq i$ lies in $\bar{J}^\flat$ and maps ℓ_0 onto ℓ; conversely $\bar{J}^\flat$ maps ℓ_0 to lines not contained in H_0. There are q^{N-1} such lines. So

$$Jt^{-1}J = \bigsqcup_{\ell \not\subseteq H_0} t^{-1} h_\ell J, \qquad h_\ell \in J^\flat,\ \bar{h}_\ell \ell_0 = \ell. \tag{5.186}$$

Proof of (ii). For $j \in J$ left multiplication by j permutes the left cosets of J in JtJ, in $Jt^{-1}J$ and in $\mathrm{GL}_N(\mathcal{O})$, so $\rho(j)T^\pm y = T^\pm y$ and $\rho(j)ey = ey$; the same argument for $j \in \mathrm{GL}_N(\mathcal{O})$ shows that ey is fixed by $\mathrm{GL}_N(\mathcal{O})$. If y is fixed by $\mathrm{GL}_N(\mathcal{O})$, every term of ey is y, so $ey = y$.

Proof of (iii). Let $y \in X^J$. For a line $\ell \subseteq k^N$ put $y[\ell] = \rho(g)y$ for any $g \in \mathrm{GL}_N(\mathcal{O})$ with $\bar{g}\ell_0 = \ell$; this is well defined because two such g differ on the right by an element of J, and $\rho(h)y[\ell] = y[\bar{h}\ell]$ for $h \in \mathrm{GL}_N(\mathcal{O})$, $y[\ell_0] = y$. By (5.186), $\rho(t)T^- y = \sum_{\ell \not\subseteq H_0} y[\ell]$. By (ii) $T^- y \in X^J$, so by (5.185)

$$T^+T^- y = \sum_{H \not\supseteq \ell_0} \rho(j_H)\rho(t)T^- y = \sum_{H \not\supseteq \ell_0} \sum_{\ell \not\subseteq H_0} y[\bar{j}_H \ell] = \sum_{H \not\supseteq \ell_0} \sum_{\ell \not\subseteq H} y[\ell],$$

since $\bar{j}_H$ maps the lines not contained in H_0 bijectively onto those not contained in H. Exchange the sums. The line ℓ_0 lies in no H of the outer sum and is counted q^{N-1} times. A line $\ell = kz \neq \ell_0$ is counted once for each form ξ with $\xi(e_i) = 1$ and $\xi(z) \neq 0$; since e_i, z are linearly independent, q^{N-2} of the q^{N-1} forms with $\xi(e_i) = 1$ have $\xi(z) = 0$, so ℓ is counted $q^{N-1} - q^{N-2}$ times. The number of lines is $[\mathrm{GL}_N(\mathcal{O}) : J] = (q^N - 1)/(q-1)$, and $\sum_\ell y[\ell] = \sum_{kJ \subseteq \mathrm{GL}_N(\mathcal{O})} \rho(k)y = \frac{q^N-1}{q-1}\, ey$. Hence

$$T^+T^- y = q^{N-1}y + q^{N-2}(q-1)\Big(\frac{q^N-1}{q-1}\, ey - y\Big) = q^{N-2}\big(y + (q^N - 1)\, ey\big).$$

Proof of (iv). Let $x, y \in X^J$ and $g = jtj'$ with $j, j' \in J$. Then

$$\langle \rho(g)x, y\rangle = \langle \rho(t)\rho(j')x, \rho(j)^{-1}y\rangle = \langle \rho(t)x, y\rangle$$

by unitarity and the invariance of x and y. Summing over the q^{N-1} cosets of (i), $\langle T^+x, y\rangle = q^{N-1}\langle \rho(t)x, y\rangle$. In the same way, every $g \in Jt^{-1}J$ gives $\langle x, \rho(g)y\rangle = \langle x, \rho(t^{-1})y\rangle$, so

$$\langle x, T^- y\rangle = q^{N-1}\langle x, \rho(t)^{-1}y\rangle = q^{N-1}\langle \rho(t)x, y\rangle.$$

□

Lemma 5.8.53. *Let F be a non-archimedean local field with ring of integers $\mathcal{O}$ and uniformiser ϖ, let $N \geq 1$, $1 \leq i \leq N$, let $J \subseteq \mathrm{GL}_N(\mathcal{O})$ be the stabiliser of the line of e_i modulo ϖ, $t = \mathrm{diag}(1, \ldots, \varpi, \ldots, 1)$ with ϖ at position i, and let $T^+ \in \mathbb{C}[\mathrm{GL}_N(F)]$ be the sum of a set of representatives of the left cosets of J in JtJ. Then for every $m \geq 1$ the element $(T^+)^m$ is a linear combination of elements $x \in \mathrm{GL}_N(F)$ with the following property: for every $j \in J$ the matrix xj has entries in $\mathcal{O}$, and 0 is a simple root of the characteristic polynomial of its reduction modulo ϖ.*

Proof. Let k be the residue field, $\bar{g}$ the reduction of a matrix g with entries in $\mathcal{O}$, and $\ell_0 = ke_i$. Let D' be the set of $g \in \mathrm{GL}_N(F)$ with entries in $\mathcal{O}$ such that $\bar{g}\ell_0 \subseteq \ell_0$ and the map induced by $\bar{g}$ on k^N/ℓ_0 is invertible, and let $D \subseteq D'$ be the subset of those with $\bar{g}e_i = 0$.

Closure properties. Reduction is multiplicative on matrices with entries in $\mathcal{O}$, and the induced maps on k^N/ℓ_0 multiply; hence $D'D' \subseteq D'$. If $g \in D$ and $h \in D'$ then $\overline{gh}\, e_i = \bar{g}(\bar{h}e_i) \in \bar{g}\ell_0 = 0$ and $\overline{hg}\, e_i = \bar{h}(0) = 0$, so $DD' \subseteq D$ and $D'D \subseteq D$; also $DD \subseteq D$.

The generators. $J \subseteq D'$: an element of J is a unit of $\mathrm{M}_N(\mathcal{O})$, its reduction is invertible and preserves ℓ_0, so the induced map on the quotient is invertible. $t \in D$: $\bar{t}$ is diagonal with entry 0 at i and 1 elsewhere, so $\bar{t}e_i = 0$ and $\bar{t}$ induces the identity on k^N/ℓ_0. Hence $JtJ \subseteq D'DD' \subseteq D$, so T^+ is a sum of elements of D, and for $m \geq 1$ the element $(T^+)^m$ is a linear combination of products of m elements of D, which lie in D.

Elements of D. Let $x \in D$ and $j \in J$; then $xj \in DD' \subseteq D$. For $g \in D$, in a basis of k^N beginning with e_i the matrix $\bar{g}$ has the block form $\begin{pmatrix} 0 & * \\ 0 & g' \end{pmatrix}$ with g' the invertible map induced on k^N/ℓ_0. So the characteristic polynomial of $\bar{g}$ is $X \cdot \det(X - g')$, and $\det(0 - g') = \pm \det g' \neq 0$: the root 0 is simple. Applied to $g = xj$ this is the claim. □

Proposition 5.8.54. *Let D be admissible data. Then E satisfies (Um) of Definition 3.2.5,*

$$\dim_{\mathbb{C}} \mathcal{A}_M(W_m) = \#\bar{G} \cdot \mathrm{mass}(I_r) \cdot \prod_{v \in \Phi} \mathrm{d}(\varsigma(v)), \qquad \text{and} \qquad d_{\mathcal{T}} \;\leq\; \#\bar{G} \cdot \mathrm{mass}(I_r) \cdot \prod_{v \in \Phi} \mathrm{d}(\varsigma(v)).$$

Proof. Step 0: (Um). $E = E_s$ satisfies (Um) by (2) of Lemma 3.2.9.

Step 1: the space and its forms. W_m has all signs $+1$ ($k + m - 1 = r$), so it is the standard definite space E^r with lattice $L = \mathcal{O}_E^r$ and the identity frame; $G = \mathrm{U}(W_m) = \mathrm{U}_r$, $\Gamma = G(E^+)$, $G_f = G_f(\{1, \ldots, r\})$ and $\mathrm{K}_f = \mathbf{K}_0(\{1, \ldots, r\})$ (the stabiliser of $\widehat{L}$) as in (VP1) of Definition 3.2.2, and K_M as in (HO1). At $v \in \Phi$ the group is $\mathrm{U}(r)$, compact. The entries of $\varsigma(v)$ are strictly decreasing integers ($\varsigma(v)_{k-2+i} = -\kappa(v) - i\Delta$ for $1 \leq i \leq m$, below the entries $\frac{k}{2} - 1 > \cdots > 1 - \frac{k}{2}$) and r is odd, so Theorem 5.7.41(i) gives a unique irreducible representation τ_v of $\mathrm{U}(r)$ with Harish-Chandra parameter $\varsigma(v)$. For the compact group, with μ^{cd} the Haar probability measure, Schur's orthogonality gives $\int |\langle w, \tau_v(g)v' \rangle|^2 \, dg = \|v'\|^2 \|w\|^2 / \dim \tau_v$, and Theorem 5.7.41(ii) gives the same with $\mathrm{d}(\varsigma(v))$; so $\dim \tau_v = \mathrm{d}(\varsigma(v))$. Let $\tau = \bigotimes_v \tau_v$ on W_τ, $\dim W_\tau = \prod_v \mathrm{d}(\varsigma(v))$. By Lemma 5.8.56(ii), $\mathcal{A}_M(W_m)$ is isomorphic to the space of $\varphi \colon G_f \to W_\tau^\vee$ with $\varphi(\gamma_f g k) = \tau^\vee(\gamma_\infty)\varphi(g)$ for $\gamma \in \Gamma$, $k \in \mathrm{K}_M$, where $\tau^\vee$ is the contragredient of τ and $\dim W_\tau^\vee = \dim W_\tau$.

Step 2: free classes. $\Gamma \backslash G_f / \mathrm{K}_M$ is finite (G_f/K_M is covered by finitely many Γ-orbits since $\Gamma \backslash G_f / \mathrm{K}_f$ is finite and $[\mathrm{K}_f : \mathrm{K}_M] < \infty$). A function in $\mathcal{A}_M(W_m)$ is determined by its values at representatives g_x of the double cosets, and the value at g_x is an arbitrary vector fixed by $\tau^\vee(\gamma_\infty)$ for the $\gamma \in \Gamma$ with $\gamma_f \in g_x \mathrm{K}_M g_x^{-1}$. By Lemma 4.6.2(c) (with any chosen place w; K_M acts trivially on L_w / wL_w) these groups are trivial, so

$$\dim \mathcal{A}_M(W_m) = \#(\Gamma \backslash G_f / \mathrm{K}_M) \cdot \prod_{v \in \Phi} \mathrm{d}(\varsigma(v)).$$

Step 3: counting classes. By Step 1, W_m is E^r with $h(x, y) = \sum_i \bar{x}_i y_i$ and the lattice $L = \mathcal{O}_E^r$, and K_f is the stabiliser of $\widehat{L}$ in G_f; K_M is an open subgroup of K_f, and $\Gamma \cap g\mathrm{K}_M g^{-1} = \{1\}$ for every $g \in G_f$ by Lemma 4.6.2(c), as in Step 2. By Step 0 and Lemma 5.6.7 with $n = r$ and the level K_M,

$$\#(\Gamma \backslash G_f / \mathrm{K}_M) = [\mathrm{K}_f : \mathrm{K}_M] \cdot \mathrm{mass}(I_r).$$

Finally $\mathrm{K}_f / \mathrm{K}_M \cong \bar{G}$: by Lemma 4.6.2(a) each chosen place w lies over a split place of E^+ and distinct chosen places lie over distinct places, where K_f has the factor $\mathrm{GL}_r(\mathcal{O}_w)$, whose reduction onto $\mathrm{GL}_r(k_w)$ is surjective; K_M is the kernel of the product of these reductions. This proves the formula for $\dim \mathcal{A}_M(W_m)$.

Step 4. By Proposition 5.8.4(iii) with $j = m$, $d_{\mathcal{T}} = \dim \mathcal{A}^{\mathrm{good}}(W_m) \leq \dim \mathcal{A}_M(W_m)$.

□

Proposition 5.8.55. *Let D be admissible data and V, V' framed spaces such that ς has gap Δ for both. Let $(\gamma_i)_{i\in I}$ be a countable family in G^M and $(c_i)_{i\in I}$ complex numbers, and for $U \in \{V, V'\}$ let R_U be a linear operator on $\mathcal{A}_M(U)$ such that for all $z, y \in \mathcal{A}_M(U)$ the family $\left(c_i\langle z, \rho_U(\gamma_i)y\rangle\right)_{i\in I}$ is summable with sum $\langle z, R_U y\rangle$. Then there is $a \in \mathcal{H}^M$ with $R_V = T_a(V)$ and $R_{V'} = T_a(V')$. In particular R_U preserves $\mathcal{A}^{\mathrm{good}}(U)$, and the restrictions of R_V and $R_{V'}$ to the good forms have the same characteristic polynomial and the same determinant.*

Proof. Step 1: one translation. Let $\gamma \in G^M$. Put $H' = K_f \cap G^M$, an open compact subgroup contained in K_M, and $H = H' \cap \gamma H'\gamma^{-1}$, of finite index in H'. Let

$$p_\gamma = \frac{1}{[H' : H]} \sum_{k\in H'/H} k\gamma \ \in \mathbb{C}[G^M],$$

which does not depend on the representatives k when applied to H'-fixed vectors (if $h \in H$ then $\gamma^{-1}h\gamma \in H'$). It lies in $\mathcal{H}^M$: let x be fixed by K_M in a representation π. Left multiplication by H' permutes the classes kH, so $\pi(p_\gamma)x$ is H'-fixed. $K_M = K_M^c H'$, where K_M^c is the subgroup of elements of K_M with component 1 at the places not below chosen places; elements of K_M^c commute with G^M and fix x, so they fix $\pi(p_\gamma)x$. Since ρ_U is unitary and $z \in \mathcal{A}_M(U)$ is fixed by $H' \subseteq K_M$,

$$\langle z, \rho_U(p_\gamma)y\rangle = \frac{1}{[H' : H]} \sum_k \langle \rho_U(k)^{-1}z, \rho_U(\gamma)y\rangle = \langle z, \rho_U(\gamma)y\rangle \qquad (z, y \in \mathcal{A}_M(U)),$$

for every framed space U.

Step 2: the series. The spaces $\mathcal{A}_M(V)$, $\mathcal{A}_M(V')$ are finite-dimensional (Proposition 5.8.4). For $a \in \mathcal{H}^M$ let $\Psi(a)$ be the pair of sesquilinear forms $(z, y) \mapsto \langle z, T_a(V)y\rangle$ on $\mathcal{A}_M(V)$ and $(z, y) \mapsto \langle z, T_a(V')y\rangle$ on $\mathcal{A}_M(V')$. Ψ is linear and factors through the finite-dimensional space $\operatorname{End}\mathcal{A}_M(V) \times \operatorname{End}\mathcal{A}_M(V')$, so its image $\mathcal{Z}$ is finite-dimensional, hence closed in the (finite-dimensional) space of pairs of forms. By Step 1 the i-th pair of terms is $\Psi(c_i p_{\gamma_i}) \in \mathcal{Z}$; the pair of sums $\left(\langle z, R_V y\rangle, \langle z, R_{V'} y\rangle\right)$ is a limit of finite partial sums, so it lies in $\mathcal{Z}$: it equals $\Psi(a)$ for some $a \in \mathcal{H}^M$. As the Petersson product is nondegenerate, $R_V = T_a(V)$ and $R_{V'} = T_a(V')$.

Step 3. By Proposition 5.8.4, T_a preserves the good forms and has the same characteristic polynomial and determinant on $\mathcal{A}^{\mathrm{good}}(V)$ and on $\mathcal{A}^{\mathrm{good}}(V')$.

Application. □

Lemma 5.8.56. *Let V be a framed space of rank r all of whose signs are $+1$. Through its frames, $\mathrm{U}(V)(\mathbb{A}_{E^+}) = G_\infty \times G_f$ with $G_\infty = \prod_{v\in\Phi} \mathrm{U}(r)$ compact; let $\Gamma \subseteq G_\infty \times G_f$ be the image of $\mathrm{U}(V)(E^+)$, with elements $\gamma = (\gamma_\infty, \gamma_f)$.*

1. *Let π be an irreducible unitary representation of G_∞ on W, $\pi^\vee$ its contragredient on the dual space $W^\vee$, $u \in W$ a unit vector, and $e(h) = \dim W \cdot \overline{\langle u, \pi(h)u\rangle}$. A function $F\colon G_\infty \times G_f \to \mathbb{C}$ is bounded, continuous, left invariant under Γ, right invariant under an open subgroup of G_f, and satisfies $\int_{G_\infty} e(h)F(xh, g)\,dh = F(x, g)$ for all (x, g), if and only if*

$$F(x, g) = \varphi(g)(\pi(x)u) \qquad (x \in G_\infty,\ g \in G_f)$$

for a function $\varphi\colon G_f \to W^\vee$ that is right invariant under an open subgroup of G_f and satisfies $\varphi(\gamma_f g) = \pi^\vee(\gamma_\infty)\varphi(g)$ for all $\gamma \in \Gamma$. The function φ is unique; for an open subgroup $K \subseteq G_f$, F is right invariant under K if and only if φ is; and $\varphi \mapsto F$ is linear and commutes with right translation by G_f.

2. *Suppose that for every $v \in \Phi$ the entries of $\varsigma(v)$ are strictly decreasing and lie in $\frac{r-1}{2}+\mathbb{Z}$. Let π_v be the irreducible representation of $\mathrm{U}(r)$ with Harish-Chandra parameter $\varsigma(v)$ and $\pi = \bigotimes_v \pi_v$ on $W = \bigotimes_v W_v$. Then $\varphi \mapsto F$ of (i) restricts to a linear isomorphism, commuting with right translation by G_f, from the space of functions $\varphi\colon G_f \to W^\vee$ with*

$$\varphi(\gamma_f g k) = \pi^\vee(\gamma_\infty)\varphi(g) \qquad (\gamma \in \Gamma,\ g \in G_f,\ k \in \mathrm{K}_M)$$

onto $\mathcal{A}_M(V)$; and $\dim W = \prod_v \dim \pi_v$.

Proof. Proof of (i), necessity. Let F have the listed properties and fix g. Since $\overline{\langle u, \pi(h)u\rangle} = \langle \pi(h)u, u\rangle = \langle u, \pi(h^{-1})u\rangle$, the substitution $y = xh$ (Haar measure is invariant) gives

$$\begin{aligned} F(x,g) &= \dim W \int_{G_\infty} \langle u, \pi(y^{-1}x)u\rangle F(y,g)\,dy \\ &= \dim W \int_{G_\infty} \langle \pi(y)u, \pi(x)u\rangle F(y,g)\,dy = \varphi(g)(\pi(x)u), \end{aligned}$$

where $\varphi(g) \in W^\vee$ is the linear form $z \mapsto \dim W \int_{G_\infty} F(y,g)\langle \pi(y)u, z\rangle\,dy$ (the integrand is continuous on a compact group). If F is right invariant under an open K, then $\varphi(gk)(\pi(x)u) = \varphi(g)(\pi(x)u)$ for all x; since π is irreducible and $u \neq 0$, the vectors $\pi(x)u$ span W, so $\varphi(gk) = \varphi(g)$. Left invariance under $\gamma \in \Gamma$ gives $\varphi(\gamma_f g)(\pi(\gamma_\infty)\pi(x)u) = \varphi(g)(\pi(x)u)$ for all x, hence $\varphi(\gamma_f g) \circ \pi(\gamma_\infty) = \varphi(g)$, that is $\varphi(\gamma_f g) = \pi^\vee(\gamma_\infty)\varphi(g)$. Uniqueness of φ follows from the same spanning property.

Proof of (i), sufficiency. Let φ be given, right invariant under an open subgroup L, and $F(x,g) = \varphi(g)(\pi(x)u)$. Then φ is constant on the open cosets gL, so F is continuous; F is right invariant under L and, by the transformation law, left invariant under Γ. Give $W^\vee$ the norm dual to that of W, for which $\pi^\vee$ is unitary. Then $\|\varphi(\gamma_f g)\| = \|\varphi(g)\|$ and $\|\varphi(gl)\| = \|\varphi(g)\|$ for $l \in L$, so $\|\varphi\|$ is a function on $\Gamma_f \backslash G_f / L$, Γ_f the projection of Γ. This set is finite: it is the double coset space $\mathrm{U}(V)(E^+)\backslash \mathrm{U}(V)(\mathbb{A}_{E^+})/(G_\infty \times L)$, finite by Borel's finiteness of class numbers [33]. Hence $|F(x,g)| \leq \|\varphi(g)\|$ is bounded. For the convolution identity fix (x,g) and let $w \in W$ represent the linear form $\varphi(g) \circ \pi(x)$, that is $\varphi(g)(\pi(x)z) = \langle w, z\rangle$. Schur's orthogonality relations for the compact group G_∞ give

$$\begin{aligned} \int_{G_\infty} e(h)F(xh,g)\,dh &= \dim W \int_{G_\infty} \langle w, \pi(h)u\rangle\, \overline{\langle u, \pi(h)u\rangle}\,dh \\ &= \langle u,u\rangle\,\langle w,u\rangle = \varphi(g)(\pi(x)u) = F(x,g). \end{aligned}$$

Linearity of $\varphi \mapsto F$ is clear, and $F(x,gh) = \varphi(gh)(\pi(x)u)$, so right translation of F by $h \in G_f$ corresponds to right translation of φ; the statement about K-invariance was shown in the first part.

Proof of (ii). By Theorem 5.7.41(i), applied to the sign vector $(+,\dots,+)$, for which the group is the compact group $\mathrm{U}(r)$, there is up to equivalence exactly one square-integrable representation π_v with Harish-Chandra parameter $\varsigma(v)$; a square-integrable irreducible unitary representation of a compact group is finite-dimensional. We show that $c_{\varsigma(v)}(h) = \langle u_v, \pi_v(h)u_v\rangle$ for a unit vector $u_v \in W_v$. Write $\lambda = \varsigma(v)$, $l = 2\lambda$, $K = \mathrm{U}(r)$ with its Haar probability measure, and $\mu = \mu_l$, $m = m_\mu$ as in Definition 3.2.3. For the signs $(+,\dots,+)$, $\mathrm{bd}(g) = g$, $\mathrm{bd}(g)^{-1}g = 1$ and $m_{-l^\rho}(k^*k) = m_{-l^\rho}(1) = 1$, so (3.2) gives

$$c_\lambda(g) = \|m\|^{-2} \int_K m(gk)\,\overline{m(k)}\,dk \qquad (g \in K).$$

As λ is strictly decreasing with entries in $\frac{r-1}{2}+\mathbb{Z}$, $2\rho_l(i)=r+1-2i$ and $l_i+2\rho_l(i)$ is even, so $\mu(i)=\lambda_i-\frac{r+1}{2}+i$: $\mu=\lambda-\rho$ is integral and weakly decreasing, and it is the highest weight $\Lambda_\varepsilon(\lambda)$ of (Dp3) of Definition 5.1.1 for $\varepsilon=(+,\dots,+)$ (all roots compact). (a) *m is a highest weight vector for right translation.* If b is invertible and $b_{ij}=0$ whenever $j\in P$, $i\notin P$, the $P\times P$ block of gb is $g_{PP}b_{PP}$, so $\Delta_P(gb)=\Delta_P(g)\Delta_P(b)$. The sets $\{i:\mu_i\ge t\}$ are initial segments of $\{1,\dots,r\}$, so every invertible upper triangular b has this property for all of them, and (3.1) gives $m(gb)=m(g)\prod_i b_{ii}^{\mu_i}$. In particular $m(kt)=t^\mu m(k)$ for diagonal $t\in K$, and $m(g(1+xE_{ij}))=m(g)$ for $i<j$, $x\in\mathbb{C}$. (b) *The right translates of m.* On K, m is a polynomial in the entries of k and $\bar k$ ($\det k^{-1}=\overline{\det k}$), so the functions $R(h)m$, $R(h)f(k)=f(kh)$, span a finite-dimensional space $V\subseteq C(K)$, on which R is a continuous unitary representation for the inner product of $L^2(K)$. The function m and its right translates are restrictions of holomorphic functions on $\mathrm{GL}_r(\mathbb{C})$ (polynomials in the entries and $\det^{-1}$), so the complexified action of $X\in\mathfrak{gl}_r(\mathbb{C})$ on V is $f\mapsto\frac{d}{dx}\big|_{x=0}f(k\exp(xX))$; by (a), E_{ij} ($i<j$) kills m and the diagonal matrices act on m by the weight μ. The $U(\mathfrak{gl}_r(\mathbb{C}))$-module generated by m is spanned by products of the E_{ij} with $i>j$ applied to m, so its weight-μ space is $\mathbb{C}m$; it is a direct sum of irreducible submodules (complete reducibility), the components of m in the summands are of weight μ, hence multiples of m, so m lies in one summand, which therefore is the whole module: it is irreducible. As K is connected, this module is V. So (R,V) is irreducible with highest weight μ; by the theorem of the highest weight it is isomorphic to π_v, which by (Dp3) is irreducible and contains, hence equals, the K-type of highest weight $\Lambda_\varepsilon(\lambda)=\mu$. (c) *$c_\lambda$ is a coefficient.* The linear form $f\mapsto f(1)$ on the finite-dimensional Hilbert space V is $f\mapsto\langle v_0,f\rangle$ for some $v_0\in V$, and $v_0\ne0$ as $m(1)=1$; then $f(k)=(R(k)f)(1)=\langle v_0,R(k)f\rangle$, in particular $m(gk)=\langle R(g^{-1})v_0,R(k)m\rangle$. Schur's orthogonality relations for the irreducible unitary (R,V) of dimension n, $\int_K\langle a,R(k)b\rangle\overline{\langle c,R(k)d\rangle}\,dk=n^{-1}\langle a,c\rangle\langle d,b\rangle$, give $\int_K m(gk)\overline{m(k)}\,dk=n^{-1}\|m\|^2\langle R(g^{-1})v_0,v_0\rangle$ and $\|m\|^2=n^{-1}\|m\|^2\|v_0\|^2$. Hence $c_\lambda(g)=\|v_0\|^{-2}\langle v_0,R(g)v_0\rangle$. Transporting $v_0/\|v_0\|$ by an isomorphism $V\to W_v$ that is unitary (invariant inner products on an irreducible representation are proportional) gives the unit vector u_v. By Schur's orthogonality relations, $\int_{\mathrm{U}(r)}|c_{\varsigma(v)}|^2\,dh=1/\dim W_v$ for the probability measure, so the formal degree is $d_{\varsigma(v)}=\dim W_v$; the convolution condition of Definition 3.2.16 does not depend on the Haar measure, since $d_{\varsigma(v)}\,dh$ does not. Hence, with $u=\bigotimes_v u_v$,

$$e_\varsigma(h)=\prod_v\dim W_v\,\overline{\langle u_v,\pi_v(h_v)u_v\rangle}=\dim W\cdot\overline{\langle u,\pi(h)u\rangle},$$

and $\pi=\bigotimes_v\pi_v$ is an irreducible unitary representation of the product group G_∞. So e_ς is the kernel e of (i), and $\mathcal{A}_M(V)$, the space of functions of archimedean type ς that are right invariant under K_M, corresponds by (i) with $K=\mathrm{K}_M$ to the stated space of functions φ. □

5.9 The main inequality

This section proves the main inequality Theorem 5.9.1, $m\cdot\mathrm{rem}\le f\,a\,(\log b)^5$ for every datum, the one result of a later section that Section 2 uses. It is printed after the last of the results that provide its parameters, and Theorem C is cited forward from Section 6. The proof first checks that the parameters fixed in Section 5.6, the Poincaré map at a Bessel pair with a seesaw constant and a family of divisibility errors, have the properties that Theorems A and B assume: the errors by Proposition 4.2.54, the seesaw identity by Proposition 4.2.25, and the two moments by Lemma 5.6.33 and Proposition 5.6.48. The rest is the comparison of two bounds for the covolume of one family of target forms, sketched in the Introduction: the main amount cancels, and the factor m, the number of lifts, survives.

Theorem 5.9.1. *For admissible data D over a parameter pair P,*

$$m \cdot \mathrm{rem} \;\le\; f\,a\,(\log b)^5.$$

Proof. Put

$$\ell = f\,a\,(\log b)^4, \qquad h = h_{\mathrm{main}} - m\,\mathrm{rem} + \ell, \qquad \iota_0 = 10\log(1+2b^2)\,u_{\mathrm{err}}.$$

Sizes. By Proposition 2.3.5, $\log\log b \ge 10^{12} C_P$, and $C_P \ge q_0 \ge 18480$ since all summands of C_P in Definition 2.3.3 other than q_0 are nonnegative. So $\log b \ge e^{10^{12}}$; in particular $b \ge 3$, $4/b^2 \le \frac{1}{2}$, $\log(1+2b^2) \le \log(b^3) = 3\log b$ and $\log b \ge 128$. Also $f, a, m > 0$.

The parameters. Theorems A and B are applied below to the parameters $(\mathcal{P}, c_{\mathrm{ss}}, \mathrm{err})$ fixed above (the Poincaré map, the seesaw constant and the divisibility errors), and these have every property those theorems assume. By Proposition 4.2.54, the family err is nonnegative, satisfies the error conditions of Definition 3.2.37 and has the uniform local property. By Proposition 4.2.25, applied to $\mathcal{P} = \mathcal{P}_{g_\infty, t_0}$ and its scalar c_{ss}, for every ordinary twist ϑ the datum Ψ_ϑ there is admissible for ϑ (conditions 1–3) and satisfies $\Theta_\vartheta(\mathcal{P}\varphi) = c_{\mathrm{ss}} B_{\vartheta,\Psi_\vartheta}(\varphi)$ for every $\varphi \in \mathcal{S}^{\mathrm{ram}}$ (condition 4): this is the seesaw identity assumed in Theorem 4.3.1 and Theorem 4.4.1. By Lemma 5.6.33 (ii), $\mathcal{P}\varphi = \sum_{a \in \mathcal{C}\times I} \theta(\varphi)_a\, \Pi_a$ for every $\varphi \in \mathcal{S}$, the class Poincaré series being formed with the matrix of Proposition 5.7.3, the sign frame of the Bessel systems of the filling datum: this is the hypothesis of Proposition 5.6.48. By that proposition, $\mathcal{S} \ne 0$ and there are an inner product on $\mathcal{S}$ and a real $c_0 > 0$ for which $c_0\mathcal{P}$ satisfies the two moment bounds (i) and (ii) there; these are the bounds (i) and (ii) assumed in Theorem 5.1.2, word for word (with c_0 for the constant c there), so $\mathcal{P}$ satisfies the hypothesis of Theorem 5.1.2.

Enough translates (Theorem B). By Theorem 5.1.2, $d_{\mathcal{T}} \ge 1$ and there is $t \subseteq \bar{G}$ with $\#t \cdot \dim \mathcal{S}^{\mathrm{ram}} \le 5 d_{\mathcal{T}} \log(1+2b^2)$ and $\dim W \ge (1 - 4/b^2) d_{\mathcal{T}}$, where $W = \sum_{s\in t} s\mathfrak{p}(\mathcal{S}^{\mathrm{ram}})$.

Short forms (Theorem A(a)). Let Sh be the set of $\varphi \in \mathcal{S}^{\mathrm{ram}}$ integral at every prime identification with $\|s\mathfrak{p}\varphi\|_\sigma^2 \le e^h$ for all $s \in \bar{G}$ and all automorphisms σ of $\mathbb{C}$. By Theorem 4.3.1, Sh spans $\mathcal{S}^{\mathrm{ram}}$.

The family. As each $s\mathfrak{p}$ is linear, the set $I = \bigcup_{s\in t} s\mathfrak{p}(\mathrm{Sh})$ spans W. Choose a basis $y_1, \dots, y_n$ of W inside I. Then y is linearly independent, $(1 - 4/b^2) d_{\mathcal{T}} \le n \le d_{\mathcal{T}}$, and $y_i = s_i \mathfrak{p} \varphi_i$ with $s_i \in t$, $\varphi_i \in \mathrm{Sh}$.

Covolume. $\|y_i\|_\sigma^2 \le e^h$ for all i, σ, so by Lemma 3.1.7 the family y has covolume at most nh.

Index (Theorem A(b)). Each φ_i is integral at every prime identification, so by Theorem 4.4.1 the family y has index at most $\#t \cdot \dim \mathcal{S}^{\mathrm{ram}} \cdot 2u_{\mathrm{err}} \le 5 d_{\mathcal{T}} \log(1+2b^2) \cdot 2u_{\mathrm{err}} = d_{\mathcal{T}}\,\iota_0$ (index bounds can be enlarged, as $u_{\mathrm{err}} \ge 0$).

Lower bound (Theorem C). By Theorem 6.1.4 with $c = nh$ and $\delta = d_{\mathcal{T}}\iota_0$,

$$d_{\mathcal{T}}(h_{\mathrm{main}} - 3\ell) \le n(h_{\mathrm{main}} + \ell) - n\,m\,\mathrm{rem} + 2 d_{\mathcal{T}} \iota_0. \tag{5.187}$$

Comparison: the main amount cancels. By Lemma 2.4.8, $h_{\mathrm{main}} + \ell \ge 0$, so $n(h_{\mathrm{main}} + \ell) \le d_{\mathcal{T}}(h_{\mathrm{main}} + \ell)$ as $n \le d_{\mathcal{T}}$. Inserting this in (5.187), the terms $d_{\mathcal{T}} h_{\mathrm{main}}$ cancel and

$$n\,m\,\mathrm{rem} \le d_{\mathcal{T}}\, e_0, \qquad e_0 = 4\ell + 2\iota_0 \ge 0.$$

Division. Since $4/b^2 \le \frac{1}{2}$, $n \ge d_{\mathcal{T}}/2 > 0$. If $m\,\mathrm{rem} \le 0$ the claim holds, as $fa(\log b)^5 > 0$. Otherwise $m\,\mathrm{rem} \le (d_{\mathcal{T}}/n) e_0 \le 2e_0$, and

$$2e_0 = 8fa(\log b)^4 + 40\log(1+2b^2)\, fa(\log b)^3 \le (8 + 120)\, fa(\log b)^4 \le fa(\log b)^5,$$

by $\log(1+2b^2) \le 3\log b$ and $\log b \ge 128$. □

6 Integral forms are long: the lower bound (Theorem C)

This section proves Theorem C (Theorem 6.1.4), the lower half of the comparison: an almost full family of target forms with index at most δ and covolume at most c satisfies $d_{\mathcal{T}}(h_{\mathrm{main}} - 3\eta) \leq c + 2\delta$. The proof (Section 6.1) combines the product formula of Section 3 (Theorem 3.1.5) with four inputs. Auxiliary forms, integral after one scaling for all twists and short at the trivial twist, complete the family to a basis (Lemma 6.4.5). The determinant $Z(\vartheta)$ of the chain factors into those of the m single lifts, up to chain ratios of Hecke L-values (Proposition 6.5.1). Along a regular family of twists, the height of the ratios $Z_j(1)/Z_j(\vartheta_\iota)$ of one lift is bounded by the main amount and the denominator budget (Theorem 6.6.6). Regular families exist by Section 7 (Theorem 7.3.1).

The inputs are statements about theta lifts between unitary groups, with the splittings of Kudla [172], and one identity underlies them: the exchange identity (Theorem 6.3.2). It compares the theta integral from a space W to a space V with an integral, over the unitary group of a smaller space V' with $\operatorname{rk} V = \operatorname{rk} V' + 2e$, of a theta kernel from V' to W, at all finite data matched modulo a modulus $\mathfrak{m}$, with a constant whose squared modulus is a fixed multiple of a product of partial Hecke L-values of the character [128, 294], the multiple not depending on $\mathfrak{m}$ or the character. Its ingredients are classical: Rallis's towers of theta lifts [236], the unfolding of Fourier coefficients of theta integrals [237, 171], the seesaw principle, and the analysis of theta integrals outside Weil's convergent range familiar from the regularised Siegel–Weil formula [323, 176, 139]. The identity is not in the literature in this form, because the lifted objects are characters with ramified data and the constant must be uniform in the character; it is proved in full. In our reading it is an identity of Rallis–Waldspurger type, a period of theta lifts whose square is an L-value, as in the Rallis inner product formula and in the formulas for toric and Gross–Prasad periods [313, 111, 141] (see [96]).

After Theorem C and its notions and the background on the theta correspondence in towers (Section 6.2), the section proves the exchange identity with the one-square identity and the explicit matching of class sums (Section 6.3); that auxiliary forms span, by transferring the chain of lifts across two grids of hermitian spaces to definite spaces (Section 6.4); the norm of the chain, by the Rallis inner product formula (Section 6.5); the cost of one lift, with a comparison operator (Section 6.6); and the bounds for Hecke values along a regular family, with the test forms of the denominator conditions (Section 6.7). The cost of one lift uses Theorem B, for the Poincaré map fixed in Section 5, and the equality of multiplicities (Proposition 5.8.4); from Section 4 the section uses the good projection, the Weil representation and the theta functions with complex multiplication and their jets. For another reading of this part of the argument see [223].

6.1 Theorem C

The section opens with the three definitions that the proof of Theorem C reads besides those of Section 3: auxiliary target forms (Definition 6.1.1), whose lifts become integral for all twists after a scaling and whose conjugate squared norms are at most $\exp(2x_1^2 b^2)$; the volumes $Z_j(\vartheta)$ of the single lifts (Definition 6.1.2); and regular families of twists (Definition 6.1.3), which have small ratios of Hecke L-values and bounded denominators. In our reading the denominators measure congruences between lifts and other forms, as in Ribet's method [241]. The proof also reads the critical pairs and the lift of a character of U(1) to a definite group, defined in Section 4.2 (Definition 4.2.4, Definition 4.2.5); for theta lifts of characters of U(1) see [121], and their squared norms are Hecke L-values (Proposition 6.7.13).

The proof applies the product formula to the target lattice, with one twist of a regular

family at each prime identification. The given family is completed to a basis by auxiliary forms (Lemma 6.4.5); each costs $2x_1^2b^2 + b$, and there are at most $4d_{\mathcal{T}}/b^2$ of them, so that their total cost is at most $d_{\mathcal{T}}b$. The ratio of volumes factors into the ratios of the m single lifts and a power of a product of chain ratios (Proposition 6.5.1). The single lifts are bounded by the cost of one lift (Theorem 6.6.6), whose sum over the m lifts produces h_{main} and the error; the chain ratios and the scaling of the auxiliary forms are of small height along a regular family (Lemma 6.1.5); and the scales of the datum absorb the rest (Lemma 6.1.6). The ratios of Hecke L-values are positive and equal to 1 at the trivial twist (Lemma 6.1.7), by Hecke's theory and the non-vanishing on the line $\operatorname{Re} s = 1$ [128, 294, 182]. For the arithmetic of the values of Hecke L-functions of CM fields at integers see [274, 159]; only their non-vanishing and the heights of their ratios are used here.

Definition 6.1.1. Let γ be a function from the m-fold twists to $\mathbb{C}$. A target form $y \in \mathcal{T}$ is **auxiliary for** γ if

(AF1) $\gamma(\vartheta)\,\Theta_\vartheta y \in \Lambda_\vartheta(\iota)$ for every prime identification $\iota \mid \ell$ and every m-fold twist ϑ with $\ell \nmid \mathrm{N}Q(\vartheta)$;

(AF2) $\|y\|_\sigma^2 \le \exp(2x_1^2b^2)$ for every field automorphism σ of $\mathbb{C}$, the conjugate norm being that of the target lattice (the squared norm of the conjugate $\sigma(\Theta_1 y)$).

Definition 6.1.2. For an m-fold twist ϑ and $1 \le j \le m$ the **volume of the j-th lift** is

$$Z_j(\vartheta) = \det\!\big(\Theta_{j,\vartheta^j}^{*}\,\Theta_{j,\vartheta^j}\big),$$

the Gram determinant of the images under Θ_{j,ϑ^j} of an orthonormal basis of $\mathcal{A}^{\mathrm{good}}(W_j)$; note the power ϑ^j. The volume of the chain is the chain determinant $Z(\vartheta)$ of the target lattice.

Definition 6.1.3. Integral forms. For a descent pair O, with spaces $W = W(O)$ and $V = \mathrm{I}_n$, let $\mathcal{A}(V) = \mathrm{Form}(O)$, a finite-dimensional space of algebraic forms on $\mathrm{U}(V)$. For a field embedding $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$, a form Ξ is **integral at** η if for every $g \in G_f(n)$ all coordinates of $\prod_\tau \det(g_{\ell,\tau})^{-e_O(\tau)}\, S_{\kappa_O}\big((g_{\ell,\tau}^{-1})_\tau\big)\,\eta(\Xi(g))$ have absolute value at most 1, where η is applied coordinatewise, S_{κ_O} is the Schur matrix of (WM2) of Definition 3.2.19, and $g_{\ell,\tau} \in \mathrm{GL}_n(\mathbb{C}_\ell)$ is the image of the component of g at the place of E induced by $x \mapsto \eta(\tau(\bar{x}))$ (the ℓ-adic gauge at η); a form is integral at a prime identification $\iota \mid \ell$ if it is integral at the embedding $\mathbb{C} \xrightarrow{\iota} \overline{\mathbb{Q}}_\ell \subset \mathbb{C}_\ell$, and **integral above** ℓ if it is integral at every embedding $\mathbb{C} \to \mathbb{C}_\ell$.

False lifts. Let e be an idempotent of the group algebra of $\prod_{w \in M_O} \mathrm{GL}_{\mathrm{rk}W}(k_w)$, acting on $\mathcal{A}(V)$ through (DP5), and $Z' \subseteq \mathcal{A}(V)$ a subspace. For a finite place w of E with $w \ne \bar{w}$ and $w, \bar{w}$ not among the places of O, the **false lifts** at w are

$$\mathcal{F}_w(e, Z') = e\,\mathcal{A}(V) \;\cap\; \ker H_w(\lambda_w) \;\cap\; Z',$$

where $H_w(\lambda_w)$ is the Hecke polynomial operator of O at w evaluated at the **test value** $\lambda_w = X_*(w)$, the lift Hecke root. A form $y \in \mathcal{A}(V)$ is **orthogonal to the false lifts** for (e, Z') if $y \perp \mathcal{F}_w(e, Z')$ for every such w.

Test forms. A form $\Xi \in \mathcal{A}(V)$ is a **test form for** ℓ if it is integral above ℓ and orthogonal to the false lifts for $(e_{X-1}, \ker O^*)$, where $e_{X-1} = \bigotimes_{w \in M_O} e_{X-1}$ is the tensor product of the label idempotents of the polynomial $X - 1$ over the residue fields k_w (zero unless $\mathrm{rk}W = 1$; for $\mathrm{rk}W = 1$ it is the projector onto the vectors fixed by $\prod_w k_w^\times$), and $O^* = \mathrm{Th}_O$ is the normalised descent.

L-value lifts. For a main pair $p = (\nu, z)$ and a twist ϑ, $O_{p,\vartheta} = O_{X(p,\vartheta)}$ is the descent pair of the L-value data $X(p,\vartheta)$ (rank $2z$, character $\nu\vartheta$), and $\lambda_{p,\vartheta} = \lambda_{X(p,\vartheta)} \in \mathcal{A}(V)$ its lift; $\mathrm{pr}_{p,\vartheta}$ is the orthogonal projection to the line $\mathbb{C}\lambda_{p,\vartheta}$.

Denominators. Let $\iota \mid \ell$, ϑ a twist, $1 \le j \le m$ and $\beta \in \mathbb{R}$; let pr be the orthogonal projection of $\mathcal{A}_{j,\vartheta}(V^{\mathrm{def}})$ onto $\mathcal{L} = \mathcal{L}_{j,\vartheta}$. The twist ϑ has **denominators at most** β **at** ι **for** j if

(DL1) Clearing. there is $B \in \mathrm{End}_{\mathbb{C}}(\mathcal{L})$ with $\|\iota(\det B)\| \ge \exp(-(\dim \mathcal{L})\, r\, \beta)$ such that $B(\mathrm{pr}\, y)$ is integral at ι for every y that is integral at ι and orthogonal to the false lifts for $(e_0, \mathcal{L}^{\perp})$;

(DL2) Main pairs. for every main pair p for j and all test forms Ξ, Ξ' for ℓ of $O_{p,\vartheta}$,

$$\|\iota\langle \mathrm{pr}_{p,\vartheta}\Xi, \mathrm{pr}_{p,\vartheta}\Xi'\rangle\| \le e^{\beta}.$$

For a family $\theta = (\theta_\iota)$ of m-fold twists, one for each prime identification, and $1 \le j \le m$, a **denominator bound** for (θ, j) is a family of reals (γ_ℓ), one for each prime, with

(DB1) $\sum_{\ell \in T} \gamma_\ell \le \beta_{\mathrm{den}}$ for every finite set T of primes;

(DB2) for every $\iota \mid \ell$ the twist $\theta_\iota^{\,j}$ has denominators at most γ_ℓ at ι for j.

Small ratios and regular families. A family $\theta = (\theta_\iota)$ of m-fold twists, one for each prime identification, has **small ratios** if

(SR1) $\ell \nmid \mathrm{N}Q(\theta_\iota)$ for every $\iota \mid \ell$;

(SR2) for every $1 \le j \le m$, every ramified critical pair p and every $s \in \{\pm 1\}$, the family $\iota \mapsto r_p(\theta_\iota^{\,j}, s)^{-1}$ has height at most x_1.

It is **regular** if it has small ratios and

(RW) for every $1 \le j \le m$ there is a denominator bound for (θ, j).

Theorem 6.1.4. *Let D be admissible data over a parameter pair P, and let $y_1, \dots, y_n \in \mathcal{T}$ be linearly independent target forms with $n \ge (1 - 4/b^2)\, d_{\mathcal{T}}$. Let $\delta, c \in \mathbb{R}$, and suppose that the family $(y_1, \dots, y_n)$ has index at most δ and covolume at most c in the target lattice. Then*

$$d_{\mathcal{T}}\Big(h_{\mathrm{main}} - 3\, f\, a\, (\log b)^4\Big) \;\le\; c + 2\delta.$$

Proof. Write $d = d_{\mathcal{T}}$ and $L = \log b$. The proof applies Theorem 3.1.5 to the target lattice, completing y to a basis by auxiliary forms and taking one twist of a regular family for each prime identification.

Step 1: twists. By Theorem 7.3.1 there is a regular family $\theta = (\theta_\iota)$ of m-fold twists. By (SR1), $\ell \nmid \mathrm{N}Q(\theta_\iota)$ for $\iota \mid \ell$, so θ_ι is allowed at ℓ in the target lattice (TL2).

Step 2: completion by auxiliary forms. Let $e \in \mathbb{Z}$ and γ, with $\gamma(\vartheta) \ne 0$ and $|\gamma(\vartheta)| = \mathrm{N}Q(\vartheta)^{e/2} \prod_{0 \le i,j < m} \tau_{ij}(\vartheta)$, be as in Lemma 6.4.5. Since $\mathcal{T}$ is spanned by the forms auxiliary for γ and $y_1, \dots, y_n$ are linearly independent, the exchange lemma gives auxiliary forms $z_1, \dots, z_{n'}$ such that $y_1, \dots, y_n, z_1, \dots, z_{n'}$ is a basis of $\mathcal{T}$; so $n + n' = d$ and, by $n \ge (1 - 4/b^2)d$, $n' \le 4d/b^2$.

Step 3: the hypotheses of Theorem 3.1.5. The target lattice is a twisted lattice on the finite-dimensional inner product space $\mathcal{T}$ by Lemma 3.2.32. Apply Theorem 3.1.5 to it, with the family y, the completion z, the twists $\xi_\iota = \theta_\iota$ (Step 1), the function γ and the constants

$$b_{\mathrm{sh}} = 2x_1^2 b^2, \qquad b_{\mathrm{den}} = b, \qquad b_{\mathrm{vol}} = d(b + faL^4 - h_{\mathrm{main}}).$$

- Θ_ϑ is injective for every m-fold twist ϑ, by Proposition 6.5.1(iii).
- Hypothesis (1) is the assumption of the theorem.
- Hypothesis (2): for $\iota \mid \ell$ and ϑ allowed at ℓ, i.e. $\ell \nmid \mathrm{N}Q(\vartheta)$, $\gamma(\vartheta)\Theta_\vartheta z_j \in \Lambda_\vartheta(\iota)$ by (AF1).
- Hypothesis (3): $\|z_j\|_\sigma^2 \le \exp(2x_1^2 b^2)$ for every automorphism σ by (AF2).
- Hypothesis (4), the denominators: as $\gamma(\theta_\iota) \neq 0$,

$$|\gamma(\theta_\iota)|^{-2} = \big((\mathrm{N}Q(\theta_\iota)^{e/2} \prod_{0\le i,j<m} \tau_{ij}(\theta_\iota)\big)^{-1}\big)^2.$$

  By Lemma 6.1.5(c) the inner family has height at most $m^2 t x_1$ (θ has small ratios), so by Lemma 3.1.8(e), (a) and Lemma 6.1.6(i) the family $\iota \mapsto |\gamma(\theta_\iota)|^{-2}$ has height at most $2m^2 t x_1 \le b$.
- Hypothesis (4), the volumes: by (Vo) and (TL1) the volume at ϑ is $Z(\vartheta)$. Let $e' \in \mathbb{Z}$ be the integer of Proposition 6.5.1. By its parts (iii) and (iv) all volumes are positive and, for every m-fold twist ϑ,

$$\frac{Z(1)}{Z(\vartheta)} = \Big((\mathrm{N}Q(\vartheta)^{e'/2} \textstyle\prod_{1\le i<j\le m} q_{ij}(\vartheta)\big)^{-1}\Big)^d \prod_{j=1}^{m} \frac{Z_j(1)}{Z_j(\vartheta)},$$

  the bracket being nonzero. By Lemma 6.1.5(c) and Lemma 6.1.6(i) the family

$$\iota \mapsto \big(\mathrm{N}Q(\theta_\iota)^{e'/2} \prod_{i<j} q_{ij}(\theta_\iota)\big)^{-1}$$

  has height at most $m^2 t x_1 \le b$, so its d-th power has height at most db (Lemma 3.1.8(e)). By Theorem 6.6.6, for $1 \le j \le m$ the family $\iota \mapsto Z_j(1)/Z_j(\theta_\iota)$ has height at most $dk(L\beta_{\mathrm{den}} - h_{\mathrm{dir}})$. By Lemma 3.1.8(e) the family $\iota \mapsto Z(1)/Z(\theta_\iota)$ has height at most

$$db + m\,dk\,(L\beta_{\mathrm{den}} - h_{\mathrm{dir}}) = d\big(b + faL^4 - h_{\mathrm{main}}\big) = b_{\mathrm{vol}},$$

  using $mkL\beta_{\mathrm{den}} = faL^4$ (Lemma 6.1.6(vi)) and $mkh_{\mathrm{dir}} = h_{\mathrm{main}}$.

Hence Theorem 3.1.5 gives

$$0 \le c + 2\delta + n'\big(2x_1^2 b^2 + b\big) + d\big(b + faL^4 - h_{\mathrm{main}}\big).$$

Step 4: the completion cost and the conclusion. By Step 2 and Lemma 6.1.6(iii),

$$n'\big(2x_1^2 b^2 + b\big) \le \frac{4d}{b^2}\big(2x_1^2 b^2 + b\big) = d \cdot 4\big(2x_1^2 + b^{-1}\big) \le db,$$

and $db \le d\,faL^4$ by Lemma 6.1.6(iv). Substituting both bounds for the two terms db,

$$0 \le c + 2\delta + 3\,d\,faL^4 - d\,h_{\mathrm{main}}, \qquad \text{that is,} \qquad d\big(h_{\mathrm{main}} - 3faL^4\big) \le c + 2\delta.$$

□

Lemma 6.1.5. *Let D be admissible data.*

(a) For $0 \le i < j \le m$ the chain ratio q_{ij}, and for $0 \le i, j < m$ the turn ratio τ_{ij}, is a ramified ratio. Every ramified ratio takes values in $\mathbb{R}_{\ge 0}$.

(b) Let $\theta = (\theta_\iota)$ be a family of m-fold twists with small ratios, $e \in \mathbb{Z}$, A a finite set and $(F_a)_{a\in A}$ ramified ratios. Then the family

$$\iota \longmapsto \Big(\mathrm{N}Q(\theta_\iota)^{e/2} \prod_{a\in A} F_a(\theta_\iota)\Big)^{-1}$$

has height at most $\#A \cdot t \cdot x_1$.

(c) In particular, for θ and e as in (b), the families

$$\iota \mapsto \Big(\mathrm{N}Q(\theta_\iota)^{e/2} \prod_{1\le i<j\le m} q_{ij}(\theta_\iota)\Big)^{-1}, \qquad \iota \mapsto \Big(\mathrm{N}Q(\theta_\iota)^{e/2} \prod_{0\le i,j<m} \tau_{ij}(\theta_\iota)\Big)^{-1}$$

have height at most $m^2 t\, x_1$.

Proof. **(a)** Write $\eta = \upsilon^{-\Delta}\chi_M$; by (LF2), (LF3) of Definition 3.2.26 it has weight $-\Delta$ and tame part ω, and $\mu_j^0 = \mu^\flat \eta^j$ has weight $-\kappa_j$ and tame part ω^j (weights add and tame parts multiply under products of characters). Recall $t = 5r$, $\Delta = 2r^8$ and $\kappa_j = \kappa + (2j+1)\frac{\Delta}{2}$ with $\kappa(v) \in \{b, a/x_1\}$, $a/x_1 = xb/x_1 \le b$.

Chain ratios. Let $0 \le i < j \le m$, $n = j - i \in [1, m]$, and $\nu = \mu_j^0/\mu_i^0 = \eta^n$, of weight $-n\Delta$ and tame part ω^n. Then (ν, t) is a critical pair: $1 \le t \le t$, and $t = 5r \le \Delta \le n\Delta \le m\Delta \le 4b$. It is ramified, since $\omega_w^n = 1$ off the chosen places and $\omega_w^n \ne 1$ at the chosen places by Lemma 6.7.3 ($1 \le n \le m$). By definition $q_{ij}(\vartheta) = F_{\nu,t}(\vartheta^n)$, so q_{ij} is a ramified ratio. *Turn ratios.* Let $0 \le i, j < m$ and $\nu = \mu_{i+1}^0 \mu_{j+1}^0$, of weight $-(\kappa_{i+1} + \kappa_{j+1})$ and tame part ω^{i+j+2}. Then (ν, t) is a critical pair, since $t \le \Delta \le \kappa_{i+1}(v) + \kappa_{j+1}(v) \le 4b$, and it is ramified by Lemma 6.7.3, as $2 \le i + j + 2 \le 2m$. With $n = j + 1 \in [1, m]$, $\tau_{ij}(\vartheta) = F_{\nu,t}(\vartheta^n)$. Finally, a ramified ratio is a finite product of the ratios $r_{(\nu,z')}$, which are quotients of absolute values, hence ≥ 0.

(b) Fix $a \in A$ and write $F_a(\vartheta) = F_{\nu,z}(\vartheta^n)$ with (ν, z) a ramified critical pair and $1 \le n \le m$. For $1 \le z' \le z$ the lower point (ν, z') is a critical pair with the same tame part, hence ramified, and by (SR2) (with $j = n$, $s = 1$) the family $\iota \mapsto r_{(\nu,z')}(\theta_\iota^n)^{-1}$ has height at most x_1. Under the convention $u^{-1} = 0$ for $u = 0$, inversion is multiplicative on $\mathbb{R}$, so

$$F_a(\theta_\iota)^{-1} = \prod_{z'=1}^{z} r_{(\nu,z')}(\theta_\iota^n)^{-1},$$

and by Lemma 3.1.8(e), (a) the family $\iota \mapsto F_a(\theta_\iota)^{-1}$ has height at most $z\, x_1 \le t\, x_1$ (as $z \le t$ and $x_1 > 0$). By Lemma 3.1.8(e) again, $\iota \mapsto \prod_{a\in A} F_a(\theta_\iota)^{-1}$ has height at most $\#A\, t\, x_1$. By (SR1), $N_\iota = \mathrm{N}Q(\theta_\iota)$ is a positive integer prime to ℓ for $\iota \mid \ell$, so by Lemma 3.1.8(b) with exponent $-e/2$ the family $\iota \mapsto N_\iota^{-e/2} \prod_a F_a(\theta_\iota)^{-1} = \big(N_\iota^{e/2} \prod_a F_a(\theta_\iota)\big)^{-1}$ has height at most $\#A\, t\, x_1$.

(c) Apply (b) with (a) to $A = \{(i, j) : 1 \le i < j \le m\}$, of size $m(m-1)/2 \le m^2$, and to $A = \{(i, j) : 0 \le i, j < m\}$, of size m^2, and use Lemma 3.1.8(a) to raise the bound to $m^2 t\, x_1$. □

Lemma 6.1.6. *For admissible data D, with $A = f\, m\, b\, (\log b)^2$:*

(i) $2m^2 t\, x_1 \le b$;

(ii) $x^{99} + x^{202} b^2 \le 2x_1^2 b^2$;

(iii) $4(2x_1^2 + b^{-1}) \le b$;

(iv) $b \le f\, a\, (\log b)^4$;

(v) $3A + r\,\beta_{\mathrm{den}} + 10\,m\,x_1 + k\,(6\beta_{\mathrm{den}} - h_{\mathrm{dir}}) \le k\,(\log b \cdot \beta_{\mathrm{den}} - h_{\mathrm{dir}})$;

(vi) $k \log b \cdot \beta_{\mathrm{den}} = f\,a\,(\log b)^4/m$.

Proof. **Sizes.** $C_P \ge 6$ by its definition in Definition 2.3.3 (it contains the summand $2d+6$ and the other summands are nonnegative), so by Proposition 2.3.5 $\log\log b \ge 6 \cdot 10^{12}$, hence $L := \log b \ge 13$, $x \ge (\log b)^{100} \ge 10^9$, $x^{10^6} \le x_1$ and $x_1^{10^6} \le b$. By (AD2), $m \le 2\sqrt{x}$ and $x^5 \le r \le 3x^5$, so $m^2 \le 4x$, $t \le 15x^5$, $m \le r/2$ and $k = r + 1 - m \ge r/2$. Finally $f \ge 1$, as E has an infinite place.

(i) $2m^2 t\,x_1 \le 2 \cdot 4x \cdot 15x^5 \cdot x_1 = 120x^6 x_1 \le x_1^2 \le b$.

(ii) $x^{202} \le x_1 \le x_1^2$ and $x^{99} \le x_1^2 \le x_1^2 b^2$, so the left side is at most $x_1^2 b^2 + x_1^2 b^2$.

(iii) $b^{-1} \le 1 \le x_1^2/4$, so $4(2x_1^2 + b^{-1}) \le 9x_1^2 \le b$.

(iv) $f\,a\,(\log b)^4 \ge a = xb \ge b$, since $f \ge 1$, $\log b \ge 1$ and $x \ge 1$.

(vi) $\beta_{\mathrm{den}} = faL^3/(mk)$, so $kL\beta_{\mathrm{den}} = faL^4/m$.

(v) The term $-kh_{\mathrm{dir}}$ appears on both sides; it suffices to show $3A + r\beta_{\mathrm{den}} + 10mx_1 + 6k\beta_{\mathrm{den}} \le kL\beta_{\mathrm{den}}$. Put $u = k\beta_{\mathrm{den}} = faL^3/m$. Since $r \le 2k$, $r\beta_{\mathrm{den}} \le 2u$, and it suffices that $3A + 10mx_1 \le u(L-8)$. Now

$$u(L-8) = \frac{f\,x\,b\,L^3(L-8)}{m} \ge \frac{f\,b\,L^2\sqrt{x}\,L(L-8)}{2} \ge 7\,f\,b\,L^2\sqrt{x},$$

using $m \le 2\sqrt{x}$ and $L(L-8) \ge 65$. On the other side $3A = 3fmbL^2 \le 6fbL^2\sqrt{x}$ and $10mx_1 \le 20\sqrt{x}\,x_1 \le fbL^2\sqrt{x}$, since $b \ge x_1^{10^6} \ge 20x_1$. Adding gives the claim. □

Lemma 6.1.7. *Let D be admissible data.*

(a) Let ν be an automorphic character of $\mathrm{U}(1)$ of weight e and tame part ψ with $e(v_0) \ne 0$ for some $v_0 \in \Phi$, and let S be a finite set of finite places of E with $\psi_w = 1$ for every $w \notin S$. Then $L^S(s,\nu) \ne 0$ for every real $s \ge 1$.

(b) For every critical pair p, every tame character ϑ and every sign $s \in \{\pm 1\}$, $r_p(\vartheta, s) > 0$, and $r_p(1,s) = 1$.

(c) Every ramified ratio F satisfies $F(\vartheta) > 0$ for every m-fold twist ϑ, and $F(1) = 1$. In particular $q_{ij}(1) = 1$ for $0 \le i < j \le m$ and $\tau_{ij}(1) = 1$ for $0 \le i, j < m$.

Proof. **(a) The base change is a unitary Hecke character, not of the form $|\cdot|_{\mathbb{A}}^{i\tau}$.** Let $\chi(x) = \nu(x/\bar{x})$ for $x \in \mathbb{A}_E^\times$. Since $x/\bar{x} \in \mathrm{U}(1)(\mathbb{A})$, $x/\bar{x} \in \mathrm{U}(1)(E)$ for $x \in E^\times$, and ν is continuous with values of modulus 1, χ is a unitary Hecke character of E, and $L^S(s,\nu)$ is the partial L-function of χ without the Euler factors at $S \cup \bar{S}$.

χ is unramified at every finite place w with $w, \bar{w} \notin S$: for $y \in \mathcal{O}_w^\times$, viewed as the idele with component y at w and 1 elsewhere, $y/\bar{y}$ has integral finite components, components 1 at the infinite places and nontrivial reductions only at w and $\bar{w}$, so by the definition of weight and tame part $\chi(y) = \psi_w(y \bmod w)\,\psi_{\bar{w}}(\bar{y}^{-1} \bmod \bar{w}) = 1$ (one factor if $w = \bar{w}$).

At v_0 (a complex place, on which complex conjugation of E acts as $z \mapsto \bar{z}$), the same definition gives, for $z \in E_{v_0}^\times = \mathbb{C}^\times$,

$$\chi_{v_0}(z) = (z/\bar{z})^{e(v_0)} = (z/|z|)^{2e(v_0)},$$

which is nontrivial on the unit circle because $e(v_0) \ne 0$. The component at v_0 of $|\cdot|_{\mathbb{A}}^{i\tau}$ is $z \mapsto |z|^{2i\tau}$, trivial on the unit circle. So $\chi \ne |\cdot|_{\mathbb{A}}^{i\tau}$ for every $\tau \in \mathbb{R}$.

(a) Nonvanishing. By Theorem A.1.12(3) (Hecke [128]; non-vanishing on $\operatorname{Re} s = 1$, [182]), the L-function $L(s,\chi)$ is holomorphic and has no zero in $\operatorname{Re} s \geq 1$. By the previous paragraph every place where χ ramifies lies in $S \cup \bar{S}$, where the local factor of $L(s,\chi)$ is 1, so

$$L^S(s,\nu) = L(s,\chi) \prod_w (1 - \chi(\varpi_w)\,\mathrm{N}w^{-s}),$$

the product over the finitely many $w \in S \cup \bar{S}$ at which χ is unramified, ϖ_w a uniformiser. For real $s \geq 1$, $|\chi(\varpi_w)\mathrm{N}w^{-s}| \leq \mathrm{N}w^{-1} \leq 1/2$, so every factor is nonzero, and $L^S(s,\nu) \neq 0$.

(b) Let $p = (\nu, z)$ with ν of weight $-c$, tame part trivial off the chosen places, and $1 \leq z \leq c(v)$ for every $v \in \Phi$. For a tame character ϑ of weight 0 with tame part trivial off its modulus Q, the character $(\nu\vartheta^s)^{-1}$ has weight c (weights add under products and change sign under inversion, by the definition of weight) and tame part trivial off $S_M \cup Q$; the same holds for ν^{-1} with S_M. As $c(v) \geq z \geq 1 \neq 0$ and $z \geq 1$, part (a) gives

$$L^{S_M \cup Q}(z, (\nu\vartheta^s)^{-1}) \neq 0, \qquad L^{S_M}(z, \nu^{-1}) \neq 0,$$

so $r_p(\vartheta, s)$ is a quotient of two positive numbers, hence > 0. For $\vartheta = 1$ the modulus is empty and $1^s = 1$, so numerator and denominator coincide and $r_p(1,s) = 1$.

(c) Let $F(\vartheta) = F_{\nu,z}(\vartheta^n) = \prod_{z'=1}^{z} r_{(\nu,z')}(\vartheta^n)$, with (ν, z) a ramified critical pair and $1 \leq n \leq m$; each (ν, z') is a critical pair (a lower point), and ϑ^n is a tame character. By (b) every factor is > 0, so $F(\vartheta) > 0$. At $\vartheta = 1$, $1^n = 1$, so every factor is $r_{(\nu,z')}(1) = 1$ and $F(1) = 1$. The chain and turn ratios are ramified ratios by Lemma 6.1.5(a), so $q_{ij}(1) = \tau_{ij}(1) = 1$. □

6.2 Theta correspondence: mixed models, towers and unfolding in stages

This section collects the facts about theta lifts between unitary groups on which the exchange identity rests: the Weil representation in a mixed model, attached to a splitting of a hermitian space along an isotropic frame; the tower property, which computes a constant term of a lift along a new hyperbolic plane; and three principles of integration over adelic quotients. The theta correspondence is classical [322, 134, 136] (for unitary dual pairs and their splittings see [172, 121], for the local theory [206, 98]). Here, however, the lifted objects are characters and ramified data rather than cusp forms, and the constants must be uniform in the tame character, so every statement is proved, from the Weil representation of a unitary pair, Tate's Fourier analysis [294], Witt's theorem, strong approximation and Hecke's theory of L-functions [128]. The section adds two definitions: the line terms along a hyperbolic plane with their cone integrals (Definition 6.2.1), and, during the unfolding, the specification of the exchange data (Definition 6.2.23). The organisation follows [259]; for another reading see [223].

The idea throughout is unfolding. Along a splitting a theta integral separates into theta functions of a smaller pair, and an integral over the larger unitary group unfolds, orbit by orbit, into integrals over stabilisers. The main instance is Rallis's tower property [236], in the form Proposition 6.2.2: for a split tame character $\mathfrak{c}$ and the plane data of the exchange, with $W' = W \oplus \mathbb{H}$, $V' = V \oplus \mathbb{H}$ and $e = \operatorname{rk} V - \operatorname{rk} W/2$, the plane constant term of the large side is γ times the large side of the smaller pair at a cone integral, where $|\gamma| = C\, L^{\sharp}_{\mathfrak{c}}(z_e)$ with C independent of $\mathfrak{c}$ and $z_e = e + 1$ or $-e$; only values at points ≥ 1 occur. It differs from Rallis's statement for cusp forms in that the lifted object is a character, the data are ramified at Σ, and the orbit of isotropic vectors survives as a Tate integral. For constant terms in the regularised Siegel–Weil formula see [176, 139], and for constant terms of automorphic forms [208]. The proof has two halves, an unfolding and the evaluation of a Tate integral, given in the two groups that follow. The norms of the lifts are computed later by the Rallis inner product formula (Section 6.5), for cusp forms the formula of [238, 187].

Definition 6.2.1. Throughout, W', W, V', V are spaces with points with signs s' on n', s on n, t' on r', t on r; $A = (k_A, l_A, i_A)$ is a base-adapted splitting of W' over W and $X = (k_X, l_X, i_X)$ a base-adapted splitting of V' over V, both along the one-element set $\{o\}$. Vectors of $\mathrm{Hom}(W', V')$ are $r' \times n'$ matrices.

(LN1) Splittings induced on the target. For a splitting $D = (k, l, i)$ of a space with a point Y over Y_0 along p and a space with a point T with signs on n_T, the **splitting induced on the target** is the splitting of $\mathrm{Hom}(T, Y)$ over $\mathrm{Hom}(T, Y_0)$ along $p \times n_T$ with frames $k \otimes J_T^{\mathsf{T}}$, $l \otimes 1$ and embedding $i \otimes 1$. Its zero term is written $Z^{\mathrm{tg}}_{D,T} \colon \mathcal{M}_{T,Y} \to \mathcal{M}_{T,Y_0}$, and the zero term along the splitting of $\mathrm{Hom}(Y, T)$ induced by D ((MX1); zero terms as in (MX4) of Definition 4.1.18) is written $Z^{\mathrm{sc}}_{D,T} \colon \mathcal{M}_{Y,T} \to \mathcal{M}_{Y_0,T}$.

(LN2) Line terms. For $a \in \mathbb{A}_E$ the **line terms** at a are the maps $\mathrm{Ln}^{\pm}_{A,X}(a) \colon \mathcal{M}_{W',V'} \to \mathcal{M}_{W,V}$,

$$\mathrm{Ln}^{+}_{A,X}(a) f = Z^{\mathrm{tg}}_{X,W}(f[a\, k_{X,\mathbb{A}}]), \qquad \mathrm{Ln}^{-}_{A,X}(a) f = Z^{\mathrm{sc}}_{A,V}\Big(Z^{\mathrm{tg}}_{X,W'}\big(\rho(a\, l_{X,\mathbb{A}}\, \mathrm{pr}_l) f\big)\Big),$$

where $f[a\, k_{X,\mathbb{A}}] \in \mathcal{M}_{W,V'}$ is the slice of f at the adelic $r' \times 1$ matrix $a\, k_{X,\mathbb{A}}$ along the splitting of $\mathrm{Hom}(W', V')$ induced by A ((MX1); slices as in (MX4) of Definition 4.1.18), pr_l is the coordinate of A, and the adelic $r' \times n'$ matrix $a\, l_{X,\mathbb{A}} \mathrm{pr}_l$ is read as a vector of $\mathrm{Hom}(W', V')$.

(LN3) The plane splitting, the line splittings and real line terms. Put $q = (n \sqcup \{o\}) \sqcup r$ and let $\{*\}$ be a one-element set. The **plane splitting** $D^{\mathrm{pl}}_{A,X}$ is the splitting of $\mathrm{Hom}(W', V')$ over $\mathbb{H}_{\{*\}} \oplus \mathrm{Hom}(W, V)$ along q with frames

$$k = \Big(k_X \otimes \begin{pmatrix} i_A^* J_{W'} \\ k_A^* J_{W'} \end{pmatrix}^{\mathsf{T}} \;\Big|\; (i_X J_V^{-1}) \otimes (k_A^* J_{W'})^{\mathsf{T}}\Big), \qquad l = \Big(l_X \otimes \begin{pmatrix} J_W^{-1} i_A^* J_{W'} \\ l_A^* J_{W'} \end{pmatrix}^{\mathsf{T}} \;\Big|\; i_X \otimes (l_A^* J_{W'})^{\mathsf{T}}\Big)$$

(column blocks indexed by $n \sqcup \{o\}$ and by r) and embedding

$$i = \Big(k_X \otimes (l_A^* J_{W'})^{\mathsf{T}} \;\Big|\; l_X \otimes (k_A^* J_{W'})^{\mathsf{T}} \;\Big|\; i_X \otimes (J_W^{-1} i_A^* J_{W'})^{\mathsf{T}}\Big)$$

(column blocks indexed by the first and the second copy of $\{*\}$ in $\mathbb{H}_{\{*\}}$ and by $r \times n$). Let R^+ be the tautological splitting of $\mathbb{H}_{\{*\}} \oplus \mathrm{Hom}(W, V)$ over $\mathrm{Hom}(W, V)$ along $\{*\}$ and R^- its opposite splitting. The **line splittings** are the composites $D^{\pm}_{A,X} = D^{\mathrm{pl}}_{A,X} \circ R^{\pm}$, splittings of $\mathrm{Hom}(W', V')$ over $\mathrm{Hom}(W, V)$ along $q \sqcup \{*\}$. The **line basis** $m_{A,X} \in \mathrm{GL}_{q \sqcup \{*\}}(\mathbb{A}_E)$ is block diagonal with the blocks $\begin{pmatrix} \bar{x}_W & 0 \\ 0 & 1 \end{pmatrix}$ on $n \sqcup \{o\}$, $(x_V^{-1})^*$ on r and 1 on $\{*\}$. For $a \in \mathbb{C}^{\Phi}$ let $b_a \in \mathbb{C}^{\Phi \times (q \sqcup \{*\})}$ have the entries a_w at $(w, *)$ ($w \in \Phi$) and 0 elsewhere. The **real line terms** at a are the maps $\mathcal{F}_{W',V'} \to \mathcal{F}_{W,V}$

$$\mathrm{Ln}^{\pm,\infty}_{A,X}(a) v = \mathrm{rsl}_{D^{\pm}_{A,X}, m_{A,X}}(b_a; v),$$

the real slice of (MX7) for the space with a point $\mathrm{Hom}(W', V')$, the splitting $D^{\pm}_{A,X}$ and the basis $m_{A,X}$ (the Fock models of (MX7) for $\mathrm{Hom}(W', V')$ and $\mathrm{Hom}(W, V)$ are $\mathcal{F}_{W',V'}$ and $\mathcal{F}_{W,V}$).

(LN4) Cone integrals. Let $\mathrm{K}_{\max}(t')$ be the compact subgroup of the $u \in \mathrm{G}(t')$ with $u_w^* u_w = 1$ at every $w \in \Phi$ and $u_f \in \mathbf{K}_0(r')$, with its Haar probability measure du, and let $(\mathbb{C}^{\times})^{\Phi}$

carry the measure $d^\times a = \prod_{w\in\Phi} |a_w|^{-2}\, d\lambda(a_w)$, λ the Lebesgue measure of $\mathbb{C} = \mathbb{R}^2$. Let $\mu\colon \Phi \to \mathbb{Z}$ and $\gamma_\mu(w) = \lfloor \#n'/2 \rfloor - \mu(w)$. For $z \in \mathbb{R}$ and $v \in \mathcal{F}_{W',V'}$ the **cone integral** $(+)$ and the **dual cone integral** $(-)$ are the Bochner integrals in $\mathcal{F}_{W,V}$

$$\mathrm{Cone}^{\pm}_{A,X,\mu}(z;v) = \int_{\mathrm{K}_{\max}(t')\times(\mathbb{C}^\times)^\Phi} \prod_{w\in\Phi} \Big(\frac{a_w}{|a_w|}\Big)^{2\mu(w)} |a_w|^{2z}\, \mathrm{Ln}^{\pm,\infty}_{A,X}(a)\big(\Omega_{\gamma_\mu,0}(1,u_\infty)v\big)\, du\, d^\times a,$$

defined when the integrand is Bochner integrable; here $u_\infty = (u_w)_{w\in\Phi} \in G_\infty(t')$ and $\Omega_{\gamma_\mu,0}$ is the real Weil representation of the pair (W',V') with weights $(\gamma_\mu, 0)$.

(LN5) The Tate test vector and the Gamma integral. For $m\colon \Phi \to \mathbb{N}$ the **Tate test vector** of W is the Fock vector $\tau_{W,m} \in \mathcal{F}_{W,W}$ ((WR5), index set $\Phi \times n \times n$) of the polynomial $\prod_{w\in\Phi}\prod_{i\in n} X^{m(w)}_{(w,i,i)}$, and for $x \in \mathbb{R}$ the **Gamma integral** is

$$\Gamma^\infty_m(x) = \int_{(\mathbb{C}^\times)^\Phi} \exp\Big(-\sum_{w\in\Phi} |a_w|^2\Big) \prod_{w\in\Phi} |a_w|^{2m(w)+2x}\, d^\times a \in [0,\infty].$$

(LN6) Line characters and the Tate integrand. Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, with base change $\tilde\nu$ (both as in Definition 3.2.22), and υ the splitting character ((WD3) of Definition 3.2.12). The **line characters** of $\mathfrak{c}$ are the automorphic characters of $\mathrm{U}(1)$

$$\alpha_W = \nu^{-1}\upsilon^{\lfloor \#n/2 \rfloor}, \qquad \alpha_{W'} = \nu^{-1}\upsilon^{\lfloor \#n'/2 \rfloor}.$$

For $f \in \mathcal{M}_{W',V'}$, $u \in \mathbb{A}_E^\times$, $g \in \mathrm{G}(s)$ and real x the **Tate integrand** is the function of $a \in \mathbb{A}_E^\times$, $h \in \mathrm{G}(t)$ and $\kappa \in \mathrm{K}_{\max}(t')$

$$Z_x(f;u,g)(a,h,\kappa) = \tilde\nu(au)\,|au|^x \sum_{\xi\in E^\times} \theta^{W,V}_{\alpha_W,\alpha_W}\Big(\mathrm{Ln}^+_{A,X}(\xi au)\big(\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f\big)\Big)(g,h),$$

$|\cdot|$ the module of $\mathbb{A}_E^\times$, $\theta^{W,V}_{\alpha,\beta}$ the theta kernel of (WC4) of Definition 4.1.15, $\omega_{\alpha,\beta}$ as in (WR4) of Definition 3.2.18, and the sum taken when it converges absolutely (0 otherwise).

Proposition 6.2.2. *Let W', W, V', V, A, X be as above, let b be an index set with bijections $\sigma\colon b\sqcup r \to n$ and $\sigma'\colon b\sqcup r' \to n'$, assume that $\#n$ is even, put $e = \#r - \#n/2$, and let $\mu\colon \Phi \to \mathbb{Z}$. Put $z_e = e+1$ and $\pm = +$ if $e \geq 0$, and $z_e = -e$ and $\pm = -$ if $e < 0$; so $z_e \geq 1$. Then there is $C > 0$ such that for every split tame character $\mathfrak{c} = (\nu,\Sigma)$ of weight μ there is $\gamma \in \mathbb{C}$ with*

$$|\gamma| = C\, L^\sharp_{\mathfrak{c}}(z_e)$$

and the following property: for every Schwartz vector $v' \in \mathcal{F}_{W',V'}$, every $y \in \mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$, and all $u \in \mathbb{A}_E^\times$, $g \in \mathrm{G}(s)$ with $u = 1$ or $\#r = \#n$, the cone integral $\mathrm{Cone}^{\pm}_{A,X,\mu}(z_e;v')$ is defined and

$$\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v',y)(u,g) = \gamma\, \theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\mathrm{Cone}^{\pm}_{A,X,\mu}(z_e;v'),\Phi^{\mathrm{st}}_{\sigma,y})(g).$$

Proof. Fix a Haar measure on $\mathbb{A}_E^\times$, and let Y, the characters α_W, $\alpha_{W'}$ and the functions $Z_x(f;u,g)$ on Y be those of Lemma 6.2.3 and Lemma 6.2.36 (the two lemmas use the same formula). Let $c_0 \neq 0$ be the constant of Lemma 6.2.3, and $C_+, C_- > 0$ and the idele α_0 those of Lemma 6.2.36; none depends on the character. Put

$$C = |c_0|\, C_\pm\, |\alpha_0|^{z_e} > 0.$$

Let $\mathfrak{c}=(\nu,\Sigma)$ be a split tame character of weight μ, let $\epsilon_\pm$ be the numbers of Lemma 6.2.36 for $\mathfrak{c}$, put $Q=\prod_{w\in\Sigma}\mathrm{N}w$ and

$$\gamma = c_0\,\epsilon_\pm\,|\alpha_0|^{z_e}\,Q^{z_e-1/2}\,L^\Sigma(z_e,\tilde\nu).$$

By (XD7), $L^\sharp_{\mathfrak{c}}(z_e)=|L^\Sigma(z_e,\tilde\nu)|\,Q^{z_e-1/2}$, so $|\gamma|=|c_0|\,C_\pm\,|\alpha_0|^{z_e}\,Q^{z_e-1/2}|L^\Sigma(z_e,\tilde\nu)|=C\,L^\sharp_{\mathfrak{c}}(z_e)$.

Let v', y, u, g be as in the statement and $f=\varphi^{\mathrm{pl}}_{A,\sigma'}(v',y)$. By Lemma 6.2.36, $Z_x(f;u,g)$ is integrable on Y for every real x, in particular for $x=e+1$, and the cone integrals $\mathrm{Cone}^\pm_{A,X,\mu}(z_e;v')$ are defined as $z_e\geq 1$. By Lemma 6.2.3 (whose hypotheses are those of the statement, y being integral),

$$\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v',y)(u,g)=c_0\int_Y Z_{e+1}(f;u,g).$$

If $e\geq 0$, then $e+1=z_e$ and the first formula of Lemma 6.2.36 at $x=z_e$ gives $\int_Y Z_{z_e}(f;u,g)=\epsilon_+|\alpha_0|^{z_e}Q^{z_e-1/2}L^\Sigma(z_e,\tilde\nu)\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\mathrm{Cone}^+_{A,X,\mu}(z_e;v'),\Phi^{\mathrm{st}}_{\sigma,y})(g)$. If $e<0$, then $e+1=1-z_e$ and the second formula at $x=z_e$ gives the same with ϵ_- and Cone^-. In both cases multiplying by c_0 gives the assertion. □

The first half of the proof unfolds (Lemma 6.2.3). The constant term along an isotropic line is a sum over the rational vectors; the zero vector contributes nothing at the plane datum, and the nonzero isotropic lines form one orbit by Witt's theorem, so their terms unfold to a Tate integral along the new line, whose integrand is a sum of theta kernels of the smaller pair at line terms. The unfolding is legitimate because the kernels decay. In Weil's convergent range every theta integral converges [323]; for the pairs of the exchange this need not hold, and the group isolates the target-bounded data, which include the local data of the exchange (Lemma 6.2.26). The analytic input is the decay estimate Lemma 6.2.27: for data supported at a split place on surjective (injective) matrices, the kernel decays faster than any power of the height of the target (source) point. Its proof combines a uniform moderate-growth bound, reduction with heights, short bases by Banaszczyk's transference theorem [12], short isotropic vectors and an estimate for functionals invariant along rank-one maps; estimates of this kind are familiar from the regularised Siegel–Weil formula [176, 139].

The mixed model, standard in the local theory [206, 172, 136], enters through zero terms and slices: a splitting of V along an isotropic frame gives each vector a zero term and slices, which are vectors of the model of the complement, and the zero term intertwines the stabiliser of the frame with the smaller pair up to a determinant character. Locally constant, compactly supported functions lie in the span of the Schrödinger functions (Lemma 6.2.22; for Schwartz–Bruhat functions see [322, 294]), so the local data of the exchange are defined for all of them. Right after this, Definition 6.2.23 specifies the exchange data: from now on, for every space with a point V, the functional θ_V is the canonical theta functional of V (Definition 4.1.15), and every theta kernel and theta integral of the exchange data is formed with it.

Lemma 6.2.3. *Let W',W,V',V, A, X be as above, let b be an index set with a bijection $\sigma'\colon b\sqcup r'\to n'$, assume that $\#n$ is even, and put $e=\#r-\#n/2$. Fix a Haar measure $d^\times a$ on $\mathbb{A}_E^\times$ and give $E^\times\backslash\mathbb{A}_E^\times\times[\mathrm{U}(V)]\times\mathrm{K}_{\max}(t')$ the product of its quotient by counting measure on $E^\times$, the quotient measure of $[\mathrm{U}(V)]$ and the Haar probability measure. For a split tame character $\mathfrak{c}=(\nu,\Sigma)$ put $\alpha_W=\nu^{-1}\upsilon^{\#n/2}$ and $\alpha_{W'}=\nu^{-1}\upsilon^{\#n'/2}$, and for $f\in\mathcal{M}_{W',V'}$, $u\in\mathbb{A}_E^\times$, $g\in\mathrm{G}(s)$ and real x put*

$$Z_x(f;u,g)(a,h,\kappa)=\tilde\nu(au)\,|au|^x\sum_{\xi\in E^\times}\theta^{W,V}_{\alpha_W,\alpha_W}\Big(\mathrm{Ln}^+_{A,X}(\xi au)(\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f)\Big)(g,h)$$

for $a\in\mathbb{A}_E^\times$, $h\in\mathrm{G}(t)$, $\kappa\in\mathrm{K}_{\max}(t')$. Then there is a real number $c_0\neq 0$, depending only on A, X and $d^\times a$, with the following property. Let $\mu\colon\Phi\to\mathbb{Z}$, let $\mathfrak{c}=(\nu,\Sigma)$ be a split tame character

of weight μ, let $v \in \mathcal{F}_{W',V'}$ be a Schwartz vector, $y \in \mathrm{M}_{b\times n}(E_\Sigma)$, $g \in \mathrm{G}(s)$ and $u \in \mathbb{A}_E^\times$ with $u = 1$ or $\#r = \#n$, and $f = \varphi^{\mathrm{pl}}_{A,\sigma'}(v,y)$. Then the series defining $Z_{e+1}(f;u,g)(a,h,\kappa)$ converges absolutely, $Z_{e+1}(f;u,g)$ is a function on $E^\times\backslash\mathbb{A}_E^\times \times [\mathrm{U}(V)] \times \mathrm{K}_{\max}(t')$, and, if it is integrable there,

$$\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v,y)(u,g) = c_0 \int Z_{e+1}(f;u,g)(a,h,\kappa)\, d(a,h,\kappa).$$

Proof. Let $c_0 \neq 0$ be the constant of Lemma 6.2.7 for A, X and $d^\times a$. Fix μ, $\mathfrak{c} = (\nu,\Sigma)$, v, y, g, u as in the statement, $f = \varphi^{\mathrm{pl}}_{A,\sigma'}(v,y) = v \otimes [\Phi']_\Sigma$ with $\Phi' = \Phi^{\mathrm{st}}_{\sigma',y(\mathrm{pr}_i)_\Sigma}$, and $\alpha' = \alpha_{W'} = \nu^{-1}\upsilon^{\#n'/2} = \alpha_W\upsilon$ ($\#n' = \#n + 2$ is even).

Step 1: the constant term as a theta integral. By (XD9), $\theta^{\mathrm{lg}}_{\mathfrak{c}}(W',V';v,\Phi') = I^{W',V'}_{\alpha',\alpha'}(f)$, so by (XD10)

$$\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v,y)(u,g) = \int I^{W',V'}_{\alpha',\alpha'}(f)\big(\mathrm{rad}^{\mathrm{ln}}_A(q_1,q_2)\,\mathrm{lev}_A(u,g)\big)\, dq.$$

The datum f is target-bounded by Lemma 6.2.26: v is a Schwartz vector, and Φ' is locally constant with compact support and vanishes at every $r' \times n'$ matrix x without a right inverse (if $\Phi'(x) \neq 0$, the $n' \times n'$ matrix with the rows of $y(\mathrm{pr}_i)_\Sigma$ and of x is invertible, so the rows of x are linearly independent).

Step 2: the sum over the rational vectors. Put $F(h) = \omega_{\alpha',\alpha'}(\mathrm{lev}_A(u,g),h)f$ for $h \in \mathrm{G}(t')$. By Lemma 6.2.4 (target V', the point $\mathrm{lev}_A(u,g)$), the series $\sum_{x\in E^{r'}} \mathrm{ct}_A(x;F(h))$ converge absolutely and

$$\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v,y)(u,g) = \int_{[\mathrm{U}(V')]} \sum_{x\in E^{r'}} \mathrm{ct}_A\big(x;F(h)\big)\, dh.$$

By (2) of Lemma 6.2.18, $\mathrm{ct}_A(0;F(h)) = 0$ for every h.

Step 3: the unfolded integrand is Z_{e+1}. Let $a' \in \mathbb{A}_E^\times$, $h \in \mathrm{G}(t)$, $\kappa \in \mathrm{K}_{\max}(t')$, put $f_\kappa = \omega_{\alpha',\alpha'}(1,\kappa)f$ and $\lambda(q) = \theta^{W,V}_{\alpha_W,\alpha_W}\big(\mathrm{Ln}^+_{A,X}(q)f_\kappa\big)(g,h)$ for $q \in \mathbb{A}_E$. Since $\omega_{\alpha',\alpha'}$ is a representation, $F(\mathrm{lev}^{\mathrm{ln}}_X(a',h)\kappa) = \omega_{\alpha',\alpha'}(\mathrm{lev}_A(u,g),\mathrm{lev}^{\mathrm{ln}}_X(a',h))f_\kappa$, and (2) of Lemma 6.2.9 for the character α_W (with $c = a'$, $x = a'u$) and the definition of theta kernels give, for $\xi \in E$,

$$\theta_{\mathrm{Hom}(W,V)}\Big(\mathrm{Ln}^+_{A,X}(\xi)F\big(\mathrm{lev}^{\mathrm{ln}}_X(a',h)\kappa\big)\Big) = \frac{|u|^{(\#r+2)/2}|a'|^{-\#n/2}}{\alpha_W[\#r](u)\,\alpha_W[\#n](a')}\,\lambda(\xi a'u).$$

Here $\tilde\alpha_W = \tilde\nu^{-1}\upsilon^{\#n}$, so $\alpha_W[\#n] = \tilde\nu^{-1}$ and $\alpha_W[\#r] = \tilde\nu^{-1}\upsilon^{\#n-\#r}$; with $e+1 = \#r - \#n/2 + 1$,

$$|a'|^{\#r+1}\,\frac{|u|^{(\#r+2)/2}|a'|^{-\#n/2}}{\alpha_W[\#r](u)\,\alpha_W[\#n](a')} = \tilde\nu(a'u)\,|a'u|^{e+1}\cdot\upsilon(u)^{\#r-\#n}|u|^{(\#n-\#r)/2} = \tilde\nu(a'u)\,|a'u|^{e+1},$$

because $u = 1$ or $\#r = \#n$. By (2) of Lemma 6.2.24 applied to $F(\mathrm{lev}^{\mathrm{ln}}_X(a',h)\kappa)$, the series over $\xi \in E$ of the left side converges absolutely; hence $\sum_{\xi\in E^\times}\lambda(\xi a'u)$ converges absolutely, which is the absolute convergence of the series defining $Z_{e+1}(f;u,g)(a',h,\kappa)$, and the function Θ of Lemma 6.2.7 for $a = b = \alpha'$, $g' = \mathrm{lev}_A(u,g)$ is

$$\Theta(a',h,\kappa) = \tilde\nu(a'u)\,|a'u|^{e+1}\sum_{\xi\in E^\times}\lambda(\xi a'u) = Z_{e+1}(f;u,g)(a',h,\kappa).$$

It is a function on Y: replacing a' by $\zeta a'$ with $\zeta \in E^\times$ permutes the ξ, and $\tilde\nu(\zeta) = \nu(\zeta/\bar\zeta) = 1$, $|\zeta| = 1$; replacing h by γh with $\gamma \in \Gamma_V$ does not change λ, by Lemma 6.2.5.

Step 4: conclusion. If $Z_{e+1}(f;u,g)$ is integrable on Y, the hypotheses of Lemma 6.2.7 hold for $a=b=\alpha'$, f, $g'=\mathrm{lev}_A(u,g)$ (Steps 2 and 3), and with Step 2

$$\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v,y)(u,g)=c_0\int_Y Z_{e+1}(f;u,g)(a,h,\kappa)\,d(a,h,\kappa).$$

□

Lemma 6.2.4. *Let W', W be spaces with points with signs s' on n', s on n, A a splitting of W' over W along a one-element set, V a space with a point with signs t on r, a,b automorphic characters of $\mathrm{U}(1)$, $f\in\mathcal{M}_{W',V}$ target-bounded and $g\in\mathrm{G}(s')$. Then for every $h\in\mathrm{G}(t)$ the series $\sum_{x\in E^r}\mathrm{ct}_A(x;\omega_{a,b}(g,h)f)$ (x read as an $r\times 1$ matrix) converges absolutely, its sum is a bounded measurable function on $[\mathrm{U}(V)]$, and*

$$\int I^{W',V}_{a,b}(f)(\mathrm{rad}^{\mathrm{ln}}_A(q_1,q_2)\,g)\,dq=\int_{[\mathrm{U}(V)]}\sum_{x\in E^r}\mathrm{ct}_A(x;\omega_{a,b}(g,h)f)\,dh.$$

Proof. Write $A=(k,l,i)$ with adelic frames $k_{\mathbb{A}},l_{\mathbb{A}},i_{\mathbb{A}}$ and coordinates $\mathrm{pr}_k,\mathrm{pr}_l,\mathrm{pr}_i$, so that pr_l (resp. pr_i, pr_k) is 1 on $l_{\mathbb{A}}$ (resp. $i_{\mathbb{A}}$, $k_{\mathbb{A}}$) and 0 on the other two frames. Let $\varepsilon_{s'},\varepsilon_s,\varepsilon_t$ be the adelic sign matrices of W',W,V, $\theta_0=\theta_{\mathrm{Hom}(W,V)}$, and for $h\in\mathrm{G}(t)$ put $\varphi_h=\omega_{a,b}(g,h)f$. For $\xi\in E^r$ put $x_\xi=x_V^{-1}\xi$ and $h_V(\xi)=\xi^*J_V\xi$; it lies in F (J_V is hermitian), and $x_\xi^*\varepsilon_t x_\xi=h_V(\xi)$ because $x_V^*J_Vx_V=\varepsilon_t$.

Step 1: radical elements. For an adelic $n\times 1$ matrix c and an adele τ let $r(c,\tau)=\mathrm{rad}_A(c,\tau)$ and $N=r(c,\tau)-1=-i_{\mathbb{A}}c\,\mathrm{pr}_l+k_{\mathbb{A}}c^*\varepsilon_s\mathrm{pr}_i-k_{\mathbb{A}}Y\mathrm{pr}_l$ with $Y=\frac12(\jmath^{-1}(\tau+\bar\tau)+c^*\varepsilon_sc)$. (a) N kills $k_{\mathbb{A}}$, maps $i_{\mathbb{A}}$ into the span of $k_{\mathbb{A}}$ and everything into the span of $i_{\mathbb{A}},k_{\mathbb{A}}$, so it is nilpotent and $\det r(c,\tau)=1$. Moreover $N^2=-k_{\mathbb{A}}c^*\varepsilon_sc\,\mathrm{pr}_l$ and $N^3=0$, so $r(c,\tau)^{-1}=1-N+N^2$ and

$$r(c,\tau)k_{\mathbb{A}}=k_{\mathbb{A}},\qquad \mathrm{pr}_i\,r(c,\tau)\,i_{\mathbb{A}}=1,\qquad r(c,\tau)^{-1}l_{\mathbb{A}}=l_{\mathbb{A}}+i_{\mathbb{A}}c+k_{\mathbb{A}}(Y-c^*\varepsilon_sc).$$

(b) As pr_l kills the image of N, $r(0,\tau)r(c,\tau')=r(c,\tau+\tau')$; hence, for adeles e,

$$\mathrm{rad}^{\mathrm{ln}}_A(q_1,q_2+e)=\mathrm{rad}^{\mathrm{ln}}_A(0,e)\,\mathrm{rad}^{\mathrm{ln}}_A(q_1,q_2).$$

(c) For $e\in E$, $x_{W'}\mathrm{rad}^{\mathrm{ln}}_A(0,e)x_{W'}^{-1}=1-k\,\jmath^{-1}(e+\bar e)\,k^*J_{W'}$ (as $k_{\mathbb{A}}=x_{W'}^{-1}k$ and $\mathrm{pr}_l=k_{\mathbb{A}}^*\varepsilon_{s'}=k^*J_{W'}x_{W'}$) has entries in E; since $\mathrm{rad}^{\mathrm{ln}}_A(0,e)\in\mathrm{G}(s')$, it lies in $\Gamma_{W'}$.

Step 2: the term of ξ at a radical point. Let $\varphi\in\mathcal{M}_{W',V}$, $\xi\in E^r$, $q=(q_1,q_2)$, $c=x_W^{-1}q_1$. By (MX6) with trivial characters, $\mathrm{term}_\xi(\varphi)(\mathrm{rad}^{\mathrm{ln}}_A(q),1)=\theta_0\big((\omega_{1,1}(r,1)\varphi)[x_\xi]\big)$ with $r=r(c,2q_2)$. Apply (2) of Lemma 6.2.45 to the splitting A along o, the space V, $g:=r$, $h:=1$, $\alpha=\upsilon^{-1}$, $\beta=1$: by Step 1 (a) its hypotheses hold with $m=1$, $g_0=1$, its two coordinates are $\mathrm{pr}_ir^{-1}l_{\mathbb{A}}=c$ and $\mathrm{pr}_kr^{-1}l_{\mathbb{A}}=\jmath^{-1}(q_2+\bar q_2)-\frac12c^*\varepsilon_sc$, and $\chi(\det m)=|\det m|=1$. Hence

$$\mathrm{term}_\xi(\varphi)(\mathrm{rad}^{\mathrm{ln}}_A(q),1)=\psi_E\big(\tfrac12(q_2+\bar q_2)h_V(\xi)\big)\,\psi_E\big(-\tfrac14\jmath\,h_V(\xi)\,c^*\varepsilon_sc\big)\,\theta_0\big(\rho(x_\xi c^*\varepsilon_s)\,\varphi[x_\xi]\big).\quad(6.1)$$

Consequences. (i) $|\mathrm{term}_\xi(\varphi)(\mathrm{rad}^{\mathrm{ln}}_A(q),1)|\le\|\theta_0\|\,\|\varphi[x_\xi]\|$, as $|\psi_E|=1$ and each $\rho(z)$ preserves the norm of $L^1(\mu)$ ($|\beta|=1$, μ translation invariant). (ii) $\mathrm{term}_\xi(\varphi)(\mathrm{rad}^{\mathrm{ln}}_A(q_1,q_2),1)=\chi_\xi(q_2)\,\mathrm{term}_\xi(\varphi)(\mathrm{rad}^{\mathrm{ln}}_A(q_1,0),1)$ with $\chi_\xi(q_2)=\psi_E(\frac12(q_2+\bar q_2)h_V(\xi))$. (iii) If $h_V(\xi)=0$, the function $q_1\mapsto\mathrm{term}_\xi(\varphi)(\mathrm{rad}^{\mathrm{ln}}_A(q_1,0),1)=\theta_0(\rho(z)\varphi[x_\xi])$, $z=x_\xi c^*\varepsilon_s$, is continuous ($z\mapsto\rho(z)F$ is continuous, translation being continuous in $L^1(\mu)$ and β continuous of modulus 1) and invariant under $q_1\mapsto q_1+\gamma$, $\gamma\in\mathrm{M}_{n\times1}(E)$. Indeed this adds to z the vector $u=x_\xi\gamma^*(x_W^{-1})^*\varepsilon_s=x_V^{-1}(\xi\gamma^*J_W)x_W$, a rational vector of $\mathrm{Hom}(W,V)$. The bicharacter of $\mathrm{Hom}(W,V)$ (signs $t(i)s(j)$)

is $\beta(z,u)=\psi_E(\frac{1}{2}\operatorname{tr}(u^*\varepsilon_t z\varepsilon_s))$ in matrix form, and $u^*\varepsilon_t z\varepsilon_s = x_W^* J_W \gamma\, h_V(\xi)\, c^* = 0$; by (WD7), $\rho(u)\rho(z)=\beta(z,u)^{-1}\rho(u+z)=\rho(u+z)$, and $\theta_0(\rho(u)F)=\theta_0(F)$ by (WD9), θ_0 being a theta functional by Lemma 4.1.17.

Step 3: theta expansion and a summable bound. Let $h\in \mathrm{G}(t)$, $q=(q_1,q_2)$, $r=\mathrm{rad}_A^{\mathrm{ln}}(q)$. Since $\omega_{a,b}$ is a representation ((2) of Lemma 4.1.1) and $\det r = 1$ (Step 1 (a)), $\omega_{a,b}(rg,h)=\omega_{1,1}(r,1)\,\omega_{a,b}(g,h)$. By the theta expansion along the splitting of $\mathrm{Hom}(W',V)$ over $\mathrm{Hom}(W,V)$ induced by A ((5) of Lemma 6.2.44) and (MX6),

$$\vartheta(q,h) := \theta_{a,b}^{W',V}(f)(rg,h) = \theta_{\mathrm{Hom}(W',V)}\big(\omega_{1,1}(r,1)\varphi_h\big) = \sum_{\xi\in E^r} \mathrm{term}_\xi(\varphi_h)(r,1),$$

absolutely convergent, and by Step 2 (i) the ξ-th term is bounded by $\|\theta_0\|\|\varphi_h[x_\xi]\|$, independently of q, where $\sum_\xi \|\varphi_h[x_\xi]\|<\infty$ by (5) of Lemma 6.2.44. By (MX10) and Step 2 (i), $|\mathrm{ct}_A(\xi;\varphi_h)|\le \|\theta_0\|\|\varphi_h[x_\xi]\|$, so $\sum_\xi \mathrm{ct}_A(\xi;\varphi_h)$ converges absolutely.

Step 4: fundamental domains and boundedness of the kernel. Let $P\subset\prod_{v\in\Phi}E_v$ be a half-open fundamental parallelepiped of $\mathcal{O}_E$; since $\mathbb{A}_{E,f}=E+\widehat{\mathcal{O}}_E$ and $E\cap(\prod_v E_v\times\widehat{\mathcal{O}}_E)=\mathcal{O}_E$, $Q_2=P\times\widehat{\mathcal{O}}_E$ is a relatively compact Borel fundamental domain of E in $\mathbb{A}_E$, and $Q_1=Q_2^n$ one of $\mathrm{M}_{n\times 1}(E)$ in $\mathrm{M}_{n\times 1}(\mathbb{A}_E)$. With the Haar measures giving them mass 1, the integral over a quotient of a periodic function is its integral over Q_1, resp. Q_2. Let $R\subset\mathrm{G}(t)$ be a Borel set of representatives of $\Gamma_V\backslash\mathrm{G}(t)$ (Γ_V is discrete and $\mathrm{G}(t)$ second countable); it has finite measure by Theorem 4.2.18, and the integral over $[\mathrm{U}(V)]$ of a left Γ_V-invariant function is its integral over R. Let $Q\subset\mathrm{M}_{n\times1}(\mathbb{A}_E)\times\mathbb{A}_E$ be relatively compact. The closure of $\{\mathrm{rad}_A^{\mathrm{ln}}(q)g : q\in Q\}$ is compact, so by target-boundedness ($|\theta_{a,b}^{W',V}(f)(x,h)|\le C_0\,\mathrm{ht}_{W'}(x)^{B_0}$) and (3) of Lemma 6.2.6, ϑ is bounded on $Q\times\mathrm{G}(t)$. It is continuous in (q,h), as $(x,F)\mapsto\omega(x)F$ is continuous ((2) of Lemma 4.1.1) and $\theta_{\mathrm{Hom}(W',V)}$ is bounded; so on $Q\times R$ it is bounded, measurable and integrable, and Fubini's theorem applies in any order. By Lemma 6.2.5, $\vartheta(q,\gamma h)=\vartheta(q,h)$ for $\gamma\in\Gamma_V$, so $I_{a,b}^{W',V}(f)(\mathrm{rad}_A^{\mathrm{ln}}(q)g)=\int_R\vartheta(q,h)\,dh$, and by Step 1 (b), (c) and Lemma 6.2.5, $q_2\mapsto\vartheta(q_1,q_2,h)$ and $q_2\mapsto I_{a,b}^{W',V}(f)(\mathrm{rad}_A^{\mathrm{ln}}(q_1,q_2)g)$ are E-periodic.

Step 5: the inner integral. Fix q_1. By Step 4 (with $Q=\{q_1\}\times Q_2$) and Fubini,

$$G(q_1) := \int_{\mathbb{A}_E/E} I_{a,b}^{W',V}(f)\big(\mathrm{rad}_A^{\mathrm{ln}}(q_1,q_2)g\big)\,dq_2 = \int_R\int_{Q_2}\vartheta(q_1,q_2,h)\,dq_2\,dh.$$

For fixed h, dominated convergence (Step 3) and Step 2 (ii) give

$$\int_{Q_2}\vartheta(q_1,q_2,h)\,dq_2 = \sum_\xi \mathrm{term}_\xi(\varphi_h)(\mathrm{rad}_A^{\mathrm{ln}}(q_1,0),1)\int_{Q_2}\chi_\xi.$$

The function χ_ξ is a continuous character of $\mathbb{A}_E$, trivial on E (there $\frac{1}{2}(q_2+\bar{q}_2)h_V(\xi)\in E$, and ψ_E is trivial on E), so $\int_{Q_2}\chi_\xi$ is its mean over the compact group $\mathbb{A}_E/E$: 1 if $\chi_\xi=1$, 0 otherwise. If $h_V(\xi)=0$, $\chi_\xi=1$. If $h_V(\xi)\neq 0$, pick $v\in\Phi$; $h_V(\xi)_v$ is real and nonzero, and for the adele q_2 with component $t\in\mathbb{R}$ at v and 0 elsewhere (WD4) gives $\chi_\xi(q_2)=\exp(-4\pi i\,t\,h_V(\xi)_v)$, which is $-i$ for $t=(8h_V(\xi)_v)^{-1}$. Hence

$$\int_{Q_2}\vartheta(q_1,q_2,h)\,dq_2 = \sigma(q_1,h) := \sum_{\xi\in E^r,\ h_V(\xi)=0}\mathrm{term}_\xi(\varphi_h)\big(\mathrm{rad}_A^{\mathrm{ln}}(q_1,0),1\big),$$

and $G(q_1)=\int_R\sigma(q_1,h)\,dh$. By Step 2 (iii) each term of σ is $\mathrm{M}_{n\times1}(E)$-periodic in q_1, so $\sigma(\cdot,h)$ and G are periodic.

Step 6: the outer integral. By periodicity, the left side of the assertion is $\int_{Q_1}G(q_1)\,dq_1=\int_{Q_1}\int_R\sigma(q_1,h)\,dh\,dq_1$. Here $\sigma=\int_{Q_2}\vartheta\,dq_2$ is measurable and bounded on $Q_1\times R$ (Step 4 with

$Q = Q_1 \times Q_2$), so Fubini gives $\int_R \int_{Q_1} \sigma(q_1, h)\, dq_1\, dh$. For fixed h, dominated convergence (Step 3) and (MX10) (for isotropic ξ the integral over the quotient of the periodic term is the integral over Q_1; for the others $\mathrm{ct}_A = 0$) give

$$\int_{Q_1} \sigma(q_1, h)\, dq_1 = \sum_{\xi \in E^r} \mathrm{ct}_A(\xi; \varphi_h) = \int_{Q_1 \times Q_2} \vartheta(q, h)\, dq.$$

The right side is measurable in h (Fubini), bounded (Step 4), and left Γ_V-invariant (Lemma 6.2.5); so $h \mapsto \sum_\xi \mathrm{ct}_A(\xi; \omega_{a,b}(g,h)f)$ is a bounded measurable function on $[\mathrm{U}(V)]$, and its integral over R, which is the left side of the assertion, is its integral over $[\mathrm{U}(V)]$. □

Lemma 6.2.5. *Let S, T be spaces with points with signs s on n and s' on n', and α, β automorphic characters of* $\mathrm{U}(1)$. *For $\gamma \in \Gamma_S$, $\gamma' \in \Gamma_T$ and $\varphi \in \mathcal{M}_{S,T}$,*

$$\theta_{\mathrm{Hom}(S,T)}(\omega_{\alpha,\beta}(\gamma, \gamma')\varphi) = \theta_{\mathrm{Hom}(S,T)}(\varphi);$$

consequently $\theta^{S,T}_{\alpha,\beta}(\varphi)(\gamma g, \gamma' h) = \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)$ for all $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$.

Proof. Write $\mathcal{M} = \mathcal{M}_{S,T}$, $\theta = \theta_{\mathrm{Hom}(S,T)}$ and $\omega = \omega_{\alpha,\beta}$; let $\mathcal{M}^\infty \subset \mathcal{M}$ be the vectors smooth for ρ (those φ for which $z \mapsto \rho(z)\varphi$ is smooth in the archimedean and locally constant in the finite components of z).

Smooth vectors are preserved and dense. For $q = (g, h)$, (2) of Lemma 4.1.1 gives

$$\rho(z)\omega(q)\varphi = \omega(q)\rho(P^{-1}z)\varphi,$$

P the pair map of q, a continuous linear automorphism of $\mathrm{Hom}(S,T)(\mathbb{A}_E)$ preserving the archimedean and finite components; so $\omega(q)\mathcal{M}^\infty \subset \mathcal{M}^\infty$. Density is Gårding's argument: the Heisenberg operators satisfy

$$\rho(z')\rho(z) = \beta(z', z)\rho(z' + z)$$

with β the bicharacter of the model (a direct computation from (WD6), (WD7)); for κ the product of a smooth compactly supported function of the archimedean components and the normalised indicator of a compact open subgroup of the finite components, $\rho(\kappa)\varphi = \int \kappa(z)\rho(z)\varphi\, dz$ satisfies

$$\rho(z')\rho(\kappa)\varphi = \int \kappa(z - z')\beta(z', z)\rho(z)\varphi\, dz,$$

which is smooth in the archimedean part of z' and, as $\beta(z', z) = 1$ for z in the compact support of κ and z' in a small enough compact open subgroup of the finite part, locally constant in its finite part; and $\rho(\kappa_j)\varphi \to \varphi$ for an approximate identity (κ_j) of this form, by the strong continuity of ρ.

Smooth vectors. By Lemma 4.1.17, applied to the fixed line functionals and transporting elements of (WC1)–(WC3), the canonical functional θ is bounded, rational-Heisenberg invariant and normalized. It is therefore a theta functional in the sense of (WD9), as required for the local theta input of Lemma 4.1.2. Let Φ, $\omega_{\psi_F,\chi}$ and $c \neq 0$ be supplied by the action and theta-distribution comparison of Lemma 4.1.2, so that $\Phi \circ \omega(q) = \omega_{\psi_F,\chi}(q) \circ \Phi$ on $\mathcal{M}^\infty$ and $\theta = c\,\theta_{\mathrm{dist}} \circ \Phi$ there, θ_{dist} the theta distribution; here $\mathrm{G}(s)$, $\mathrm{G}(s')$ are identified with $\mathrm{U}(S)(\mathbb{A}_F)$, $\mathrm{U}(T)(\mathbb{A}_F)$ through the points ((VP2) of Definition 3.2.2), under which Γ_S, Γ_T are the groups of rational points. For $\varphi \in \mathcal{M}^\infty$ and $q = (\gamma, \gamma')$ rational, (ii) of Theorem A.5.4 gives

$$\theta(\omega(q)\varphi) = c\,\theta_{\mathrm{dist}}(\omega_{\psi_F,\chi}(q)\Phi\varphi) = c\,\theta_{\mathrm{dist}}(\Phi\varphi) = \theta(\varphi).$$

All vectors. Both $\varphi \mapsto \theta(\omega(q)\varphi)$ and θ are continuous on $\mathcal{M}$ (θ is bounded and $\omega(q)$ is a bounded operator), and they agree on the dense subspace $\mathcal{M}^\infty$; hence they agree on $\mathcal{M}$. Finally $\theta^{S,T}_{\alpha,\beta}(\varphi)(\gamma g, \gamma' h) = \theta(\omega(\gamma, \gamma')\omega(g,h)\varphi) = \theta(\omega(g,h)\varphi)$, as ω is a representation. □

Lemma 6.2.6. *Let V be a space with a point with signs s on n.*

1. $\mathrm{ht}_V(\gamma g) = \mathrm{ht}_V(g)$ *for all* $\gamma \in \Gamma_V$ *and* $g \in \mathrm{G}(s)$.
2. *For every compact set* $K \subset \mathrm{G}(s)$ *there is* $c_K \geq 1$ *with* $\mathrm{ht}_V(gk) \leq c_K \,\mathrm{ht}_V(g)$ *for all* $g \in \mathrm{G}(s)$ *and* $k \in K$.
3. *For every compact set* $K \subset \mathrm{G}(s)$ *the height* ht_V *is bounded on* $\Gamma_V K$.

Proof. For $g \in \mathrm{G}(s)$ let $L(g) = \{u \in E^n : ((x_V g)^{-1}u)_f \in \widehat{\mathcal{O}}_E^n\}$ be the lattice of V_g and, for $u \in E^n$, $|u|_g = \|((x_V g)^{-1}u)_\infty\|$, the euclidean norm on $\mathbb{C}^{\Phi\times n}$; so $\lambda_V(g)$ is the infimum of 1 and of $|u|_g$ over $u \in L(g) \setminus \{0\}$, and $\mathrm{ht}_V(g) = \lambda_V(g)^{-1}$.

(1). Let $\gamma \in \Gamma_V$, so $\delta = x_V \gamma x_V^{-1} \in \mathrm{U}(V)(F) \subset \mathrm{GL}_n(E)$ and $x_V \gamma g = \delta x_V g$. For $u \in E^n$, $(x_V \gamma g)^{-1}u = (x_V g)^{-1}\delta^{-1}u$. Since $u \mapsto \delta^{-1}u$ is a bijection of $E^n \setminus \{0\}$ onto itself, it maps $L(\gamma g) \setminus \{0\}$ bijectively onto $L(g) \setminus \{0\}$ with $|u|_{\gamma g} = |\delta^{-1}u|_g$. Hence $\lambda_V(\gamma g) = \lambda_V(g)$.

(2). Let K be compact. The infinite component $k \mapsto (k_v)_{v\in\Phi}$ is continuous, so $M = \sup_{k\in K} \max_{v\in\Phi} \|k_v\|$ (operator norms) is finite. The finite component $k \mapsto k_f$ is continuous, so the entries of the k_f, $k \in K$, form a compact subset C of $\mathbb{A}_{E,f}$. The open sets $\prod_{w\in R} E_w \times \prod_{w\notin R} \mathcal{O}_{E,w}$, R finite, cover $\mathbb{A}_{E,f}$ and increase with R, so C lies in one of them, and its projection to each E_w, $w \in R$, is compact, hence contained in $\varpi_w^{-m_w}\mathcal{O}_{E,w}$ for some $m_w \geq 0$. Let N be the product of the $p_w^{m_w}$, p_w the residue characteristic of w, over $w \in R$; then $Nc \in \widehat{\mathcal{O}}_E$ for every $c \in C$, so $k_f\widehat{\mathcal{O}}_E^n \subset N^{-1}\widehat{\mathcal{O}}_E^n$ for every $k \in K$. Put $c_K = \max(1, MN)$.

Let $g \in \mathrm{G}(s)$, $k \in K$ and $u \in L(gk) \setminus \{0\}$. At the finite places $(x_V g)^{-1}u = k_f\,(x_V gk)^{-1}u \in k_f\widehat{\mathcal{O}}_E^n \subset N^{-1}\widehat{\mathcal{O}}_E^n$, so $Nu \in L(g) \setminus \{0\}$. At the infinite places $(x_V g)^{-1}u = k_\infty (x_V gk)^{-1}u$, and k_∞ is block diagonal over Φ with blocks of norm at most M, so $|u|_g \leq M|u|_{gk}$. Therefore

$$|u|_{gk} \geq M^{-1}|u|_g = (MN)^{-1}|Nu|_g \geq (MN)^{-1}\lambda_V(g) \geq c_K^{-1}\lambda_V(g).$$

Taking the infimum over u, and using $1 \geq \lambda_V(g) \geq c_K^{-1}\lambda_V(g)$, gives $\lambda_V(gk) \geq c_K^{-1}\lambda_V(g)$, that is $\mathrm{ht}_V(gk) \leq c_K\,\mathrm{ht}_V(g)$.

(3). For $\gamma \in \Gamma_V$ and $k \in K$, (1) and (2) with $g = 1$ give $\mathrm{ht}_V(\gamma k) = \mathrm{ht}_V(k) \leq c_K\,\mathrm{ht}_V(1)$, and $\mathrm{ht}_V(1)$ is finite by (VP4) of Definition 3.2.2. □

Lemma 6.2.7. *Let W', W, V', V, A, X be as above, fix a Haar measure $d^\times a$ on $\mathbb{A}_E^\times$, and give $Y = E^\times\backslash\mathbb{A}_E^\times \times [\mathrm{U}(V)] \times \mathrm{K}_{\max}(t')$ the product of the quotient of $d^\times a$ by counting measure on $E^\times$, the quotient measure of $[\mathrm{U}(V)]$ and the Haar probability measure. There is a real number $c_0 \neq 0$, depending only on A, X and $d^\times a$, with the following property. Let a, b be automorphic characters of $\mathrm{U}(1)$, $f \in \mathcal{M}_{W',V'}$, $g \in \mathrm{G}(s')$, and $F(h) = \omega_{a,b}(g,h)f$ for $h \in \mathrm{G}(t')$. Assume that $\sum_{x\in E^{r'}} \mathrm{ct}_A(x; F(h))$ converges absolutely for every h, that $\mathrm{ct}_A(0; F(h)) = 0$ for every h, and that the function*

$$\Theta(a', h, \kappa) = |a'|^{\#r+1} \sum_{\xi\in E^\times} \theta_{\mathrm{Hom}(W,V)}\Big(\mathrm{Ln}_{A,X}^+(\xi)\, F\big(\mathrm{lev}_X^{\mathrm{ln}}(a', h)\,\kappa\big)\Big)$$

(the series converging absolutely) is a function on Y that is integrable on Y. Then

$$\int_{[\mathrm{U}(V')]} \sum_{x\in E^{r'}} \mathrm{ct}_A(x; F(h))\, dh = c_0 \int_Y \Theta.$$

Proof. Write $G' = \mathrm{G}(t')$, $\Gamma = \Gamma_{V'}$, $\Gamma_P = \Gamma_{P,X}$, $x = x_{V'}$, $\theta_0 = \theta_{\mathrm{Hom}(W,V)}$, and for a line (one-dimensional subspace) $\ell \subset E^{r'}$ and $h \in G'$ put $S_\ell(h) = \sum_{y \in \ell \setminus \{0\}} \mathrm{ct}_A(y; F(h))$ and $S(h) = \sum_{y \in E^{r'}} \mathrm{ct}_A(y; F(h))$; all these series converge absolutely by hypothesis. Let $\mathcal{L}$ be the set of isotropic lines ($h_{V'}$ vanishes on them) and $\ell_0 = Ek_X$, which lies in $\mathcal{L}$ by (MX1).

Step 1: grouping by lines. By (MX10), $\mathrm{ct}_A(y; \cdot) = 0$ unless y is isotropic, and $\mathrm{ct}_A(0; F(h)) = 0$ by hypothesis. Every nonzero isotropic y lies in exactly one line, which is isotropic, so rearranging the absolutely convergent series gives $S(h) = \sum_{\ell \in \mathcal{L}} S_\ell(h)$, with $\sum_\ell |S_\ell(h)| < \infty$.

Step 2: equivariance. Let $\gamma \in \Gamma$, $\delta = x\gamma x^{-1} \in \mathrm{U}(V')(F)$, $y \in E^{r'}$ and $h \in G'$; we show $\mathrm{ct}_A(\delta y; F(\gamma h)) = \mathrm{ct}_A(y; F(h))$. Both vanish unless y is isotropic, as δ preserves $h_{V'}$. Since $\det \gamma = \det \delta$ has norm 1 and lies in E, $a(\det \gamma) = 1$, so $F(\gamma h) = \omega_{a,b}(1, \gamma)F(h) = \omega_{1,1}(1, \gamma)F(h)$ ((2) of Lemma 4.1.1), and $\omega_{1,1}(r, 1)\omega_{1,1}(1, \gamma) = \omega_{1,1}(1, \gamma)\omega_{1,1}(r, 1)$ for every $r \in \mathrm{G}(s')$. Moreover $x^{-1}\delta y = \gamma\,(x^{-1}y)$. By (2) of Lemma 6.2.45 for $g = 1$ (so $m = 1$, $g_0 = 1$ and both coordinates vanish), with the characters $(\upsilon^{-1}, 1)$, $(\omega_{1,1}(1, \gamma)G)[\gamma z] = \omega_{\upsilon^{-1},1}(1, \gamma)(G[z])$ for $G \in \mathcal{M}_{W',V'}$ and adelic z; and $\theta_0 \circ \omega_{\upsilon^{-1},1}(1, \gamma) = \theta_0$ by Lemma 6.2.5. Applying this to $G = \omega_{1,1}(r, 1)F(h)$, $z = x^{-1}y$, $r = \mathrm{rad}_A^{\mathrm{ln}}(q_1, 0)$ and integrating over q_1 in (MX10) gives the claim. Summing over the nonzero vectors of ℓ, $S_{\delta\ell}(\gamma h) = S_\ell(h)$.

Step 3: one orbit, and the unfolded sum. If $\ell = Ey \in \mathcal{L}$, the map $\sigma\colon \ell \to E^{r'}$, $y \mapsto k_X$, is injective and isometric ($h_{V'}(y, y) = 0 = h_{V'}(k_X, k_X)$), so by Witt's theorem (Theorem A.6.7) there is $\delta \in \mathrm{U}(V')(F)$ with $\delta y = k_X$, i.e. $\delta\ell = \ell_0$. The stabiliser of ℓ_0 corresponds to Γ_P by its definition. Let $R \subset \Gamma$ be a set of representatives of $\Gamma_P \backslash \Gamma$; then $\gamma \mapsto (x\gamma x^{-1})^{-1}\ell_0$ is a bijection $R \to \mathcal{L}$, and Step 2 (with $\ell = (x\gamma x^{-1})^{-1}\ell_0$) gives $S_{(x\gamma x^{-1})^{-1}\ell_0}(h) = \Phi(\gamma h)$, $\Phi := S_{\ell_0}$. Hence

$$S(h) = \sum_{\gamma \in R} \Phi(\gamma h), \qquad \sum_{\gamma \in R} |\Phi(\gamma h)| < \infty \qquad (h \in G').$$

Taking $\gamma \in \Gamma_P$ in Step 2 shows that Φ is left Γ_P-invariant.

Step 4: Φ on the line. By (1) of Lemma 6.2.8, $\Phi(h) = \sum_{\xi \in E^\times} \theta_0(\mathrm{Ln}^+_{A,X}(\xi)F(h))$. For $n = \mathrm{rad}_X(c, d) \in N_X$, $F(nh) = \omega_{a,b}(1, n)F(h)$, and by (2) of Lemma 6.2.8 each term is unchanged; so Φ is left N_X-invariant, and $\Theta(a', h, \kappa) = |a'|^{\#r+1}\Phi(\mathrm{lev}_X^{\mathrm{ln}}(a', h)\kappa)$. Each $h \mapsto \mathrm{ct}_A(y; F(h))$ is continuous: its integrand $(q_1, h) \mapsto \theta_0((\omega_{1,1}(\mathrm{rad}_A^{\mathrm{ln}}(q_1, 0)g, h)f)[x^{-1}y])$ is continuous ((2) of Lemma 4.1.1; the Hom slice map is bounded, being a zero term, (7) of Lemma 4.1.1, after an isometric Heisenberg operator) and so bounded on compact sets, and the integral may be taken over a relatively compact fundamental domain; dominated convergence applies. So S, the S_ℓ and Φ are Borel.

Step 5: integration. Φ is Borel and left invariant under N_X and Γ_P, and $\Theta = \Theta_\Phi$ is integrable on Y by hypothesis; by Lemma 6.2.15, Φ is integrable on $\Gamma_P \backslash G'$ and $\int_{\Gamma_P \backslash G'} \Phi = c_1 \int_Y \Theta$, with $c_1 > 0$ depending only on X and $d^\times a$. Let D be a Borel fundamental domain of Γ in G'; then $D_P = \bigsqcup_{\gamma \in R} \gamma D$ is one of Γ_P (as $G' = \bigsqcup_{\pi \in \Gamma_P} \bigsqcup_{\gamma \in R} \pi\gamma D$), and by the left invariance of the measure, for every left Γ_P-invariant Borel $f \geq 0$, $\int_{D_P} f = \int_D \sum_{\gamma \in R} f(\gamma h)\, dh$. With $f = |\Phi|$ this shows that $\sum_{\gamma \in R} |\Phi(\gamma \cdot)|$ is integrable on D, so by dominated convergence and Step 3

$$\int_{[\mathrm{U}(V')]} S = \int_D \sum_{\gamma \in R} \Phi(\gamma h)\, dh = \sum_{\gamma \in R} \int_{\gamma D} \Phi = \int_{D_P} \Phi = \int_{\Gamma_P \backslash G'} \Phi = c_1 \int_Y \Theta.$$

This is the assertion with $c_0 = c_1$. □

Lemma 6.2.8. *Let W', W, V', V, A, X be as above, let $k_X \in \mathrm{M}_{r' \times 1}(E)$ be the rational frame k of X, and $f \in \mathcal{M}_{W',V'}$.*

1. *For every $\xi \in E^\times$, $\mathrm{ct}_A(\xi k_X; f) = \theta_{\mathrm{Hom}(W,V)}(\mathrm{Ln}^+_{A,X}(\xi)f)$.*

2. For all automorphic characters a, b of $\mathrm{U}(1)$, $c \in \mathrm{M}_{r\times 1}(\mathbb{A}_E)$, $d \in \mathbb{A}_E$ and $\xi \in \mathbb{A}_E$, with $\mathrm{rad}_X(c,d)$ the radical element of X,

$$\mathrm{Ln}^+_{A,X}(\xi)\Big(\omega_{a,b}(1,\mathrm{rad}_X(c,d))\,f\Big) = \mathrm{Ln}^+_{A,X}(\xi)f.$$

Proof. For a splitting (k,l,i) and $c' \in \mathrm{M}_{n_0\times p}(\mathbb{A}_E)$, (MX2) gives $\mathrm{rad}(c',0)k_{\mathbb{A}} = k_{\mathbb{A}}$, $\mathrm{rad}(c',0)i_{\mathbb{A}}w = i_{\mathbb{A}}w + k_{\mathbb{A}}c'^{*}\varepsilon w$ and $\mathrm{rad}(c',0)l_{\mathbb{A}} = l_{\mathbb{A}} - i_{\mathbb{A}}c' - \frac{1}{2}k_{\mathbb{A}}c'^{*}\varepsilon c'$; multiplying out, $\mathrm{rad}(c',0)\mathrm{rad}(-c',0) = 1$, so $\mathrm{pr}_i\,\mathrm{rad}(c',0)^{-1}l_{\mathbb{A}} = c'$. Likewise every $\mathrm{rad}(c',d')$ fixes $k_{\mathbb{A}}$ and has $\mathrm{pr}_i\,\mathrm{rad}(c',d')\,i_{\mathbb{A}} = 1$.

Proof of (1). Put $x = \xi k_X$; then $x^*J_{V'}x = |\xi|^2k_X^*J_{V'}k_X = 0$, and $x_{V'}^{-1}x = \xi k_{X,\mathbb{A}}$ is isotropic for $\varepsilon_{t'}$. Let $q \in \mathrm{M}_{n\times 1}(\mathbb{A}_E)$, $c' = x_W^{-1}q$ and $\eta = \mathrm{rad}^{\mathrm{ln}}_A(q,0) = \mathrm{rad}_A(c',0)$. By the first paragraph $\eta k_{A,\mathbb{A}} = k_{A,\mathbb{A}}$, $\mathrm{pr}_i\eta i_{A,\mathbb{A}} = 1$ and $\mathrm{pr}_i\eta^{-1}l_{A,\mathbb{A}} = c'$. So (2) of Lemma 6.2.45 (source W' split by A, target V', $m=1$, $g_0=1$, $h=1$, characters υ^{-1} and 1), whose phase is 1 because $\xi k_{X,\mathbb{A}}$ is isotropic, gives

$$(\omega_{1,1}(\eta,1)f)[\xi k_{X,\mathbb{A}}] = \rho(\xi k_{X,\mathbb{A}}c'^{*}\varepsilon_s)\,G, \qquad G = f[\xi k_{X,\mathbb{A}}] \in \mathcal{M}_{W,V'}.$$

Hence, with $G^\theta(z) = \theta_{\mathrm{Hom}(W,V')}(\rho(z)G)$, the integrand of (MX10) at q is $G^\theta(k_{X,\mathbb{A}}(x_W^{-1}\bar\xi q)^*\varepsilon_s)$. On the other side, the adelic frame k of the splitting induced on the target by X has the columns $x_{V'}^{-1}k_Xe_j^{\mathsf{T}}J_Wx_W = k_{X,\mathbb{A}}(x_W^{-1}e_j)^*\varepsilon_s$ ($j \in n$), so it sends $b \in \mathbb{A}_E^n$ to $k_{X,\mathbb{A}}(x_W^{-1}\bar b)^*\varepsilon_s$, and (MX4) at the point 0 gives

$$\theta_{\mathrm{Hom}(W,V)}(Z^{\mathrm{tg}}_{X,W}G) = \int_{\mathbb{A}_E^n/E^n} G^\theta(k_{X,\mathbb{A}}(x_W^{-1}\bar b)^*\varepsilon_s)\,db.$$

The map $q \mapsto b = \xi\bar q$ is an automorphism of the topological group $\mathbb{A}_E^n$ carrying E^n onto E^n ($\xi \neq 0$), so it preserves the invariant probability measure of the quotient, and $\bar b = \bar\xi q$. Therefore $\mathrm{ct}_A(\xi k_X; f) = \theta_{\mathrm{Hom}(W,V)}(Z^{\mathrm{tg}}_{X,W}(f[\xi k_{X,\mathbb{A}}]))$, which is $\theta_{\mathrm{Hom}(W,V)}(\mathrm{Ln}^+_{A,X}(\xi)f)$ by (LN2).

Proof of (2). Let $\varrho = \mathrm{rad}_X(c,d)$; it fixes $k_{X,\mathbb{A}}$, so $\varrho^{-1}k_{X,\mathbb{A}} = k_{X,\mathbb{A}}$, and $\mathrm{pr}_i\varrho i_{X,\mathbb{A}} = 1$. By (2) of Lemma 6.2.45 with the source point 1 ($m=1$, $g_0=1$, $W=B=0$; characters $a\upsilon^{-1}$ and b), $(\omega_{a,b}(1,\varrho)f)[\xi k_{X,\mathbb{A}}] = \omega_{a\upsilon^{-1},b}(1,\varrho)(f[\xi k_{X,\mathbb{A}}])$. By (1) of Lemma 6.2.9 (target V' split by X, $m=1$, $h_0=1$, source point 1; characters $a\upsilon^{-1}$ and $b\upsilon^{-1}$), $Z^{\mathrm{tg}}_{X,W}(\omega_{a\upsilon^{-1},b}(1,\varrho)G) = Z^{\mathrm{tg}}_{X,W}(G)$ for every $G \in \mathcal{M}_{W,V'}$. With (LN2) this is the assertion. □

Lemma 6.2.9. *1. Let $D = (k,l,i)$ be a splitting of a space with a point Y (signs σ) over Y_0 (signs σ_0) along p, and T a space with a point with signs τ on n_T. Let $h \in \mathrm{G}(\sigma)$, $m \in \mathrm{GL}_p(\mathbb{A}_E)$ and $h_0 \in \mathrm{G}(\sigma_0)$ with $h\,k_{\mathbb{A}} = k_{\mathbb{A}}m$ and $\mathrm{pr}_i\,h\,i_{\mathbb{A}} = h_0$, let $g \in \mathrm{G}(\tau)$ and α,β automorphic characters of $\mathrm{U}(1)$. Then for $F \in \mathcal{M}_{T,Y}$*

$$Z^{\mathrm{tg}}_{D,T}(\omega_{\alpha,\beta\upsilon^{\#p}}(g,h)F) = \alpha[\#n_T](\det m)\,|\det m|^{\#n_T/2}\,\omega_{\alpha,\beta}(g,h_0)\,Z^{\mathrm{tg}}_{D,T}(F).$$

2. Let W', W, V', V, A, X be as above, a an automorphic character of $\mathrm{U}(1)$, $u, c \in \mathbb{A}_E^\times$, $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(t)$, $\xi \in \mathbb{A}_E$ and $f \in \mathcal{M}_{W',V'}$. Put $\gamma = \mathrm{lev}_A(u,g)$, $\eta = \mathrm{lev}^{\mathrm{ln}}_X(c,h)$ and $x = cu$. Then

$$\mathrm{Ln}^+_{A,X}(\xi)(\omega_{av,av}(\gamma,\eta)f) = \frac{|u|^{(\#r+2)/2}\,|c|^{-\#n/2}}{a[\#r](u)\,a[\#n](c)}\,\omega_{a,a}(g,h)\,\mathrm{Ln}^+_{A,X}(\xi x)f,$$

$$\mathrm{Ln}^-_{A,X}(\xi)(\omega_{av,av}(\gamma,\eta)f) = \frac{|u|^{\#r/2}\,|c|^{-(\#n+2)/2}}{a[\#r](u)\,a[\#n](c)}\,\omega_{a,a}(g,h)\,\mathrm{Ln}^-_{A,X}(\xi\bar x^{-1})f.$$

Proof. For adelic vectors y of T (resp. of Y, of W') and z write $[y,z] = y\,z^*\varepsilon$, an adelic matrix read as a vector of the relevant Hom space, ε the sign matrix of the source. The pair map sends (g,h) to $P_{g,h}\colon X \mapsto hXg^{-1}$, and $\varepsilon g^{-1} = g^*\varepsilon$ gives

$$P_{g,h}[y,z] = [hy, gz]. \tag{$*$}$$

For a splitting $D = (k,l,i)$ and an element h with $hk_{\mathbb{A}} = k_{\mathbb{A}}m$, $\mathrm{pr}_i h i_{\mathbb{A}} = h_0$, the matrix of h in the adelic basis $(k_{\mathbb{A}}, i_{\mathbb{A}}, l_{\mathbb{A}})$ is block upper triangular with diagonal blocks m, h_0, $(m^*)^{-1}$ (the last because h preserves the form, $l^*_{\mathbb{A}}\varepsilon k_{\mathbb{A}} = 1$ and $k^*_{\mathbb{A}}\varepsilon k_{\mathbb{A}} = k^*_{\mathbb{A}}\varepsilon i_{\mathbb{A}} = 0$), so

$$\det h = \det m\,\det h_0\,\overline{\det m}^{\,-1}. \tag{$**$}$$

Proof of (1). The adelic frames of the splitting of $\mathrm{Hom}(T,Y)$ induced on the target are, by (LN1) and the point $x_Y \otimes (x_T^{-1})^{\mathsf{T}}$ (which reads M as $x_Y^{-1}Mx_T$, and $e_j^{\mathsf{T}}J_T x_T = (x_T^{-1}e_j)^*\varepsilon_\tau$): the columns $[k_{\mathbb{A}}e_b, d_j]$ for k, and the embedding $[w,z] \mapsto [i_{\mathbb{A}}w, z]$, where $d_j = x_T^{-1}e_j$ ($b \in p$, $j \in n_T$). Writing $y = l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b$, the vectors $[l_{\mathbb{A}}a, z]$, $[k_{\mathbb{A}}b, z]$ lie in the spans of the frames l and k of the induced splitting, so its coordinate $\mathrm{pr}_i^{\mathrm{tg}}$ sends $[y,z]$ to $[\mathrm{pr}_i y, z]$. Let $P = P_{g,h}$ and $P_0 = P_{g,h_0}$. By $(*)$, $P[k_{\mathbb{A}}e_b, d_j] = [k_{\mathbb{A}}me_b, gd_j]$; as $[\,\cdot\,,\cdot\,]$ is linear in the first and conjugate-linear in the second argument, P maps the frame k of the induced splitting to itself times $M = m \otimes \overline{g'}$, where $gd_j = \sum_{j'} d_{j'}g'_{j'j}$, $g' = x_T g x_T^{-1}$; so $\det M = (\det m)^{\#n_T}\,\overline{\det g}^{\#p}$. By $(*)$ and the formula for $\mathrm{pr}_i^{\mathrm{tg}}$, $\mathrm{pr}_i^{\mathrm{tg}} P i_{\mathbb{A}}^{\mathrm{tg}}[w,z] = [h_0 w, gz] = P_0[w,z]$. So Lemma 6.2.10, applied to the space with a point $\mathrm{Hom}(T,Y)$, the induced splitting and P, gives

$$Z^{\mathrm{tg}}_{D,T}(\omega(P)F) = \upsilon(\det M)^{-1}|\det M|^{1/2}\,\omega(P_0)\,Z^{\mathrm{tg}}_{D,T}(F),$$

with $\upsilon(\det M)^{-1} = \upsilon(\det m)^{-\#n_T}\upsilon(\det g)^{\#p}$ (as $\upsilon(\bar y) = \upsilon(y)^{-1}$) and $|\det M| = |\det m|^{\#n_T}$ ($|\det g| = 1$). By (WR4) and $(**)$, $\omega_{\alpha,\beta\upsilon^{\#p}}(g,h) = \alpha(\det h)(\beta\upsilon^{\#p})(\det g)^{-1}\omega(P)$ with $\alpha(\det h) = \tilde\alpha(\det m)\,\alpha(\det h_0)$. Collecting the scalars,

$$\begin{aligned}&\tilde\alpha(\det m)\,\upsilon(\det m)^{-\#n_T}|\det m|^{\#n_T/2}\cdot\alpha(\det h_0)\beta(\det g)^{-1}\\ &\quad= \alpha[\#n_T](\det m)\,|\det m|^{\#n_T/2}\cdot\alpha(\det h_0)\beta(\det g)^{-1},\end{aligned}$$

which is the assertion since $\omega_{\alpha,\beta}(g,h_0) = \alpha(\det h_0)\beta(\det g)^{-1}\omega(P_0)$.

Proof of (2). By (MX2), $\gamma k_{A,\mathbb{A}} = k_{A,\mathbb{A}}u$, $\gamma l_{A,\mathbb{A}} = l_{A,\mathbb{A}}\bar u^{-1}$, $\gamma i_{A,\mathbb{A}} = i_{A,\mathbb{A}}g$, hence $\mathrm{pr}_k\gamma = u\,\mathrm{pr}_k$, $\mathrm{pr}_l\gamma = \bar u^{-1}\mathrm{pr}_l$, $\mathrm{pr}_i\gamma i_{A,\mathbb{A}} = g$ (coordinates of A); and $\eta = \mathrm{lev}_X(c^{-1},h)$ has $\eta k_{X,\mathbb{A}} = k_{X,\mathbb{A}}c^{-1}$, $\eta l_{X,\mathbb{A}} = l_{X,\mathbb{A}}\bar c$, $\mathrm{pr}_i\eta i_{X,\mathbb{A}} = h$.

Step 1: the translation. By (2) of Lemma 4.1.1, $\omega(P)^{-1}\rho(z)\omega(P) = \rho(P^{-1}z)$, and $\omega_{av,av}(\gamma,\eta)$ is a scalar times $\omega(P_{\gamma,\eta})$; so $\rho(z)\,\omega_{av,av}(\gamma,\eta) = \omega_{av,av}(\gamma,\eta)\,\rho(\eta^{-1}z\gamma)$. For $z = \xi k_{X,\mathbb{A}}\mathrm{pr}_k$ this is $\eta^{-1}z\gamma = \xi cu\,k_{X,\mathbb{A}}\mathrm{pr}_k = \xi x\,k_{X,\mathbb{A}}\mathrm{pr}_k$, and for $z = \xi l_{X,\mathbb{A}}\mathrm{pr}_l$ it is $\xi\bar c^{-1}\bar u^{-1}l_{X,\mathbb{A}}\mathrm{pr}_l = \xi\bar x^{-1}l_{X,\mathbb{A}}\mathrm{pr}_l$.

Step 2: the sign $+$. By (LN2) and (MX4), $\mathrm{Ln}^+_{A,X}(\xi) = Z^{\mathrm{tg}}_{X,W}\circ Z_S\circ\rho(\xi k_{X,\mathbb{A}}\mathrm{pr}_k)$, S the splitting of $\mathrm{Hom}(W',V')$ induced by A ($T = V'$). By Step 1 it suffices to move $\omega_{av,av}(\gamma,\eta)$ past Z_S and $Z^{\mathrm{tg}}_{X,W}$. The zero-term law of (2) of Lemma 6.2.45 at $x = 0$ (source W' split by A, $p = \{o\}$, $m = u$, $g_0 = g$, target V' of rank $\#r' = \#r + 2$; characters $\alpha = a$, $\beta = av$) gives

$$Z_S(\omega_{av,av}(\gamma,\eta)F) = (av)[\#r+2](u)^{-1}|u|^{(\#r+2)/2}\,\omega_{a,av}(g,\eta)Z_S(F),$$

and (1) (target V' split by X, $m = c^{-1}$, $h_0 = h$, source W of rank $\#n$; $\alpha = \beta = a$) gives

$$Z^{\mathrm{tg}}_{X,W}(\omega_{a,av}(g,\eta)F') = a[\#n](c)^{-1}|c|^{-\#n/2}\,\omega_{a,a}(g,h)Z^{\mathrm{tg}}_{X,W}(F').$$

Since $\widetilde{v}(y) = v(y/\bar{y}) = v(y)^2$, $(av)[\#r+2] = \tilde{a}\, v^2 v^{-\#r-2} = a[\#r]$, and the product of the two factors is the one stated.

Step 3: the sign $-$. By (LN2), $\mathrm{Ln}^-_{A,X}(\xi) = Z^{\mathrm{sc}}_{A,V} \circ Z^{\mathrm{tg}}_{X,W'} \circ \rho(\xi l_{X,\mathbb{A}} \mathrm{pr}_l)$. By (1) (target V' split by X, $m = c^{-1}$, source W' of rank $\#n+2$; $\alpha = av$, $\beta = a$),

$$Z^{\mathrm{tg}}_{X,W'}(\omega_{av,av}(\gamma,\eta)F) = (av)[\#n+2](c)^{-1}|c|^{-(\#n+2)/2}\, \omega_{av,a}(\gamma,h) Z^{\mathrm{tg}}_{X,W'}(F),$$

and by (2) of Lemma 6.2.45 at $x = 0$ (source W' split by A, $m = u$, target V of rank $\#r$; $\alpha = \beta = a$)

$$Z^{\mathrm{sc}}_{A,V}(\omega_{av,a}(\gamma,h)F') = a[\#r](u)^{-1}|u|^{\#r/2}\, \omega_{a,a}(g,h) Z^{\mathrm{sc}}_{A,V}(F').$$

As $(av)[\#n+2] = a[\#n]$, the product is the stated factor; with Step 1 this proves the second formula. □

Lemma 6.2.10. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p an index set, and (k,l,i) a splitting of V over V_0 along p with zero term Z. Let $g \in \mathrm{G}(s)$, $m \in \mathrm{GL}_p(\mathbb{A}_E)$ and $g_0 \in \mathrm{G}(s_0)$ satisfy $g\, k_{\mathbb{A}} = k_{\mathbb{A}} m$ and $\mathrm{pr}_i\, g\, i_{\mathbb{A}} = g_0$. Then for every $f \in \mathcal{M}_s$*

$$Z(\omega_s(g)f) = v(\det m)^{-1}\, |\det m|^{1/2}\, \omega_{s_0}(g_0)\, Z(f).$$

Proof. Throughout, an element g *in the stabiliser* means $g \in \mathrm{G}(s)$ with $gk_{\mathbb{A}} = k_{\mathbb{A}} m$, $\mathrm{pr}_i g i_{\mathbb{A}} = g_0$ for some $m \in \mathrm{GL}_p(\mathbb{A}_E)$, $g_0 \in \mathrm{G}(s_0)$.

Step 1: radical elements. (a) For g in the stabiliser, $\varepsilon_s g = g^{-*}\varepsilon_s$ gives $\mathrm{pr}_l g = k^*_{\mathbb{A}}\varepsilon_s g = (g^{-1}k_{\mathbb{A}})^*\varepsilon_s = m^{-*}\mathrm{pr}_l$; so $\mathrm{pr}_l g i_{\mathbb{A}} = 0$ and $g i_{\mathbb{A}} w = i_{\mathbb{A}} g_0 w + k_{\mathbb{A}}(\mathrm{pr}_k g i_{\mathbb{A}} w)$. (b) $r = g\,\mathrm{lev}(m, g_0)^{-1}$ satisfies $rk_{\mathbb{A}} = k_{\mathbb{A}}$ and $\mathrm{pr}_i r i_{\mathbb{A}} = g_0 g_0^{-1} = 1$. (c) Every $r \in \mathrm{G}(s)$ with $rk_{\mathbb{A}} = k_{\mathbb{A}}$ and $\mathrm{pr}_i r i_{\mathbb{A}} = 1$ is a radical element. Write $r(l_{\mathbb{A}}a) = l_{\mathbb{A}}A_1 a + i_{\mathbb{A}}A_2 a + k_{\mathbb{A}}A_3 a$ and $r(i_{\mathbb{A}}w) = l_{\mathbb{A}}B_1 w + i_{\mathbb{A}}w + k_{\mathbb{A}}B_3 w$. As r preserves $z^*\varepsilon_s z'$, pairing with $r(k_{\mathbb{A}}y) = k_{\mathbb{A}}y$ gives $A_1 = 1$, $B_1 = 0$; pairing $r(i_{\mathbb{A}}w)$ with $r(l_{\mathbb{A}}a)$ gives $B_3 = -A_2^*\varepsilon_{s_0}$; pairing $r(l_{\mathbb{A}}a)$ with $r(l_{\mathbb{A}}a')$ gives $A_3 + A_3^* + A_2^*\varepsilon_{s_0}A_2 = 0$. Put $c = -A_2$ and $b = -\jmath(A_3 + \frac{1}{2}c^*\varepsilon_{s_0}c)$; then b is hermitian ($\bar{\jmath} = -\jmath$ and the last relation), $\frac{1}{2}(\jmath^{-1}(b+b^*) + c^*\varepsilon_{s_0}c) = -A_3$, and by (MX2)

$$\mathrm{rad}(c,b)(l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}y) = l_{\mathbb{A}}a + i_{\mathbb{A}}(w - ca) + k_{\mathbb{A}}(y + c^*\varepsilon_{s_0}w - \tfrac{1}{2}(\jmath^{-1}(b+b^*) + c^*\varepsilon_{s_0}c)a)$$

agrees with r; so $r = \mathrm{rad}(c,b)$. Conversely the display shows that $\mathrm{rad}(c,b)$ is in the stabiliser with $m = 1$, $g_0 = 1$, and that it depends on b only through $b + b^*$. (d) From the display, by direct composition,

$$\begin{aligned} \mathrm{rad}(c_1,b_1)\mathrm{rad}(c_2,b_2) &= \mathrm{rad}\big(c_1 + c_2,\ b_1 + b_2 + \jmath c_1^*\varepsilon_{s_0}c_2\big), \\ \mathrm{lev}(t,1)\,\mathrm{rad}(c,b)\,\mathrm{lev}(t,1)^{-1} &= \mathrm{rad}(ct^*, tbt^*). \end{aligned} \tag{6.2}$$

Step 2: a scalar for every element of the stabiliser. Let g be in the stabiliser and $\Pi = \omega_{s_0}(g_0)^{-1} Z \omega_s(g)$, a bounded linear map $\mathcal{M}_s \to \mathcal{M}_{s_0}$. By (1) of Lemma 6.2.44 for Z (at $x = 0$, $a = 0$), $Z\rho(k_{\mathbb{A}}y) = Z$ and $Z\rho(i_{\mathbb{A}}w + k_{\mathbb{A}}y) = \rho(w)Z$. Hence $\Pi\rho(k_{\mathbb{A}}y) = \omega_{s_0}(g_0)^{-1} Z\rho(k_{\mathbb{A}}my)\omega_s(g) = \Pi$, and by Step 1 (a)

$$\Pi\rho(i_{\mathbb{A}}w) = \omega_{s_0}(g_0)^{-1} Z\rho(i_{\mathbb{A}}g_0 w + k_{\mathbb{A}}\mathrm{pr}_k g i_{\mathbb{A}} w)\omega_s(g) = \omega_{s_0}(g_0)^{-1}\rho(g_0 w) Z\omega_s(g) = \rho(w)\Pi.$$

By (1) of Lemma 6.2.44, $\Pi_x = \Pi\rho(l_{\mathbb{A}}x)$ obeys the mixed law in the hypothesis of (2) of Lemma 6.2.46. Let g_D be as in (1) of Lemma 6.2.46 and $u = \omega_s(g_D)v^0$. By (2) there, $\Pi_x u = C'|e_p(x)|^2 v^0$ for some C'; by Lemma 6.2.11, $u[x] = C|e_p(x)|^2 v^0$ with $C \neq 0$. With

$\gamma = C'/C$ we get $\Pi_x u = \gamma\, u[x]$ for all x. For $z = l_{\mathbb{A}} a + i_{\mathbb{A}} w + k_{\mathbb{A}} y$ the mixed law for Π and for Z gives

$$\Pi\rho(z)u = \psi_E(\tfrac{1}{2}\jmath\, y^* a)\rho(w)\Pi_a u = \gamma\,\psi_E(\tfrac{1}{2}\jmath\, y^* a)\rho(w)u[a] = \gamma\, Z\rho(z)u.$$

For $f \in \mathcal{M}_s$ put $h = \omega_s(g_D)^{-1} f$; by the superposition (1) of Lemma 6.2.12 and $\omega_s(g_D)\rho(z) = \rho(g_D z)\omega_s(g_D)$, $f = \int h(z)\rho(g_D z)u\, d\mu_n(z)$, so $\Pi f = \gamma Z f$ as Π, Z are bounded. Thus there is $c(g) \in \mathbb{C}$ with

$$Z\,\omega_s(g) = c(g)\,\omega_{s_0}(g_0)\,Z. \tag{6.3}$$

Since $Zu = u[0] = Cv^0 \neq 0$, $c(g)$ is unique; and $c(gg') = c(g)c(g')$ for g, g' in the stabiliser (apply (6.3) twice; gg' acts by mm' and $g_0 g_0'$), so $c(g)c(g^{-1}) = c(1) = 1$ and $c(g) \neq 0$.

Step 3: radical elements act by 1. Let $r = \mathrm{rad}(c, b)$ and $L = \mathrm{lev}(2\cdot 1_p, 1)$. With $b' = b/3$, (6.2) gives $L\,\mathrm{rad}(c, b')L^{-1} = \mathrm{rad}(2c, 4b')$ and $\mathrm{rad}(c, b')\, r = \mathrm{rad}(2c, b' + b + \jmath c^* \varepsilon_{s_0} c)$; both have hermitian part $\frac{4}{3}(b + b^*)$ in the second argument, so they are equal. Multiplicativity of c gives $c(\mathrm{rad}(c, b')) = c(L)c(\mathrm{rad}(c, b'))c(L)^{-1} = c(\mathrm{rad}(c, b'))\, c(r)$, hence $c(r) = 1$: $Z\omega_s(r) = Z$.

Step 4: the Levi block form at the adapted point. Let $L = \mathrm{lev}(m, g_0)$, $x_{\mathbb{H}}$ the point of $\mathbb{H}_p$, and $g_1 = x_{\mathbb{H}}^{-1}\mathrm{diag}(m^{-*}, m)x_{\mathbb{H}}$. As $L(l_{\mathbb{A}}\ k_{\mathbb{A}}\ i_{\mathbb{A}}) = (l_{\mathbb{A}} m^{-*}\ k_{\mathbb{A}} m\ i_{\mathbb{A}} g_0)$, the matrix F_D of Lemma 6.2.46 satisfies $LF_D = F_D\,\mathrm{diag}(g_1, g_0)$. With the permutation matrix P sending the basis vector j to $\tau(j)$, $g_D P = F_D$ (by (1) of Lemma 6.2.46), so

$$g_D^{-1} L g_D = P\,\mathrm{diag}(g_1, g_0)\,P^{-1},$$

and $\mathrm{diag}(g_1, g_0)$ preserves the sign matrix of $s^{\mathbb{H}} \sqcup s_0$ (as $P^*\varepsilon_s P$ is that matrix); so $g_1 \in \mathrm{G}(s^{\mathbb{H}})$. Its multiplier: at $v \in \Phi$, conjugation by P maps the block of sign -1 for $s^{\mathbb{H}} \sqcup s_0$ onto that for s, and the block of $\mathrm{diag}(g_1, g_0)_v$ of sign -1 is $\mathrm{diag}((g_{1,v})_-, (g_{0,v})_-)$; at the finite places $\Delta(P\,X\,P^{-1}) = \Delta(X)$ and $\Delta(X_1 \oplus X_2) = \Delta(X_1)\Delta(X_2)$ by Lemma 4.1.13, P being integral. Hence

$$\Delta_s(g_D^{-1} L g_D) = \Delta_{s^{\mathbb{H}}}(g_1)\,\Delta_{s_0}(g_0). \tag{6.4}$$

Step 5: the phase of the hyperbolic multiplier. We show $\Delta_{s^{\mathbb{H}}}(g_1)\, \upsilon(\det m) > 0$. At $v \in \Phi$, $x_{\mathbb{H},v} = 2^{-1/2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $g_{1,v} = \frac{1}{2}\begin{pmatrix} A+B & B-A \\ B-A & A+B \end{pmatrix}$ with $A = m_v^{-*}$, $B = m_v$; its block of sign -1 (the second copy of p) is $\frac{1}{2}(m_v^{-*} + m_v)$, and

$$\overline{\det \tfrac{1}{2}(m_v^{-*} + m_v)} \cdot \frac{\det m_v}{|\det m_v|} = \frac{\det(\frac{1}{2}(m_v^{-1} + m_v^*)\, m_v)}{|\det m_v|} = \frac{\det \frac{1}{2}(1 + m_v^* m_v)}{|\det m_v|} > 0.$$

At the finite places $x_{\mathbb{H},f} = \begin{pmatrix} 1 & -c \\ c_0 & c\bar{c}_0 \end{pmatrix}$ has integral entries and determinant $c(c_0 + \bar{c}_0) = c$, a unit, so $x_{\mathbb{H},f} \in \mathrm{GL}_{2p}(\widehat{\mathcal{O}}_E)$, and the last display of Lemma 4.1.13 (with $A = m_f$, $k = x_{\mathbb{H},f}$) gives $\Delta(g_{1,f})\, \upsilon(\det m_f) = |\Delta(m_f)|^2 |\det m_f|^{-1/2} > 0$, $\det m_f$ read with components 1 at Φ. Splitting the idele $\det m$ into its archimedean part (finite components 1) and $\det m_f$, (WD3) gives $\upsilon(\det m) = \prod_v \frac{\det m_v}{|\det m_v|}\, \upsilon(\det m_f)$; multiplying the factors proves the claim.

Step 6: the Levi scalar. For $f, f' \in \mathcal{M}_s$ put $B(f, f') = \int_{\mathbb{A}_E^p} \langle f[x], f'[x]\rangle\, dx$. Every vector a of $\mathcal{M}_{s_0}$ satisfies $|a(y)| \leq \|a\|$ (reproducing identity (WD7), $|\beta_{s_0}| = 1$, $|e_{n_0}| \leq 1$), so $|\langle a, a'\rangle| \leq \|a\|\,\|a'\|$; moreover $\|f[x]\| \leq \|Z\|\,\|f\|$, and by (1), (2) of Lemma 6.2.44, $\int \|f'[x]\| dx = \int_{\mathbb{A}_E^p/E^p} \sum_\xi \|(\rho(l_{\mathbb{A}}x)f')[\xi]\|\, dx \leq M\|f'\|$. With the continuity in (4) there, B is a bounded sesquilinear form. It is Heisenberg invariant: for $z = l_{\mathbb{A}} a + i_{\mathbb{A}} w + k_{\mathbb{A}} y$, $(\rho(z)f)[x] = \psi_E(\cdot)\rho(w)f[x + a]$ with a phase of modulus 1, $\rho(w)$ preserves the inner product, and dx is translation invariant. By Lemma 6.2.14, $B = K\langle\cdot,\cdot\rangle$ with $K = B(v^0, v^0)$.

Let $u = \omega_s(g_D)v^0$, so $u[x] = C\varphi(x)v^0$ with $\varphi(x) = |e_p(x)|^2$ (continuous, $\varphi \geq 0$, $\varphi(0) = 1$, integrable), and $v = \omega_s(L)u$. Since $L^{-1}l_{\mathbb{A}} = l_{\mathbb{A}}m^*$, (6.3) gives $v[x] = Z\omega_s(L)\rho(l_{\mathbb{A}}m^*x)u = c(L)\,C\,\varphi(m^*x)\,\omega_{s_0}(g_0)v^0$. Put $I = \int \varphi^2 > 0$ and $J = \int \varphi(x)\varphi(m^*x)\,dx \geq 0$; the substitution $x \mapsto m^*x$ multiplies Haar measure by $|\det m^*| = |\det m|$, so $\int \varphi(m^*x)^2 dx = |\det m|^{-1}I$. Recall $\langle v^0, v^0\rangle = 1$ and $\langle v^0, \omega(h)v^0\rangle = \overline{\Delta(h)}^{-1}$ (Lemma 6.2.13), and that ω preserves inner products. Then:

- $B(u,u) = |C|^2 I = K$;
- $B(v,v) = |c(L)|^2|C|^2|\det m|^{-1}I = K$, so $|c(L)|^2 = |\det m|$;
- $B(u,v) = |C|^2 c(L)\,J\,\overline{\Delta_{s_0}(g_0)}^{-1}$ equals

$$K\langle u, v\rangle = K\langle v^0, \omega_s(g_D^{-1}Lg_D)v^0\rangle = |C|^2 I\,\overline{\Delta_{s^{\mathbb{H}}}(g_1)}^{-1}\overline{\Delta_{s_0}(g_0)}^{-1}$$

  by (6.4).

Hence $c(L)\,J\,\overline{\Delta_{s^{\mathbb{H}}}(g_1)} = I > 0$, so $J > 0$ and $c(L)\overline{\Delta_{s^{\mathbb{H}}}(g_1)} > 0$. By Step 5, $\overline{\Delta_{s^{\mathbb{H}}}(g_1)}$ is a positive multiple of $\overline{\upsilon(\det m)}^{-1} = \upsilon(\det m)$, so $c(L)\upsilon(\det m) > 0$, and with $|c(L)| = |\det m|^{1/2}$ we get $c(L) = \upsilon(\det m)^{-1}|\det m|^{1/2}$.

Step 7: conclusion. For g as in the statement, $r = gL^{-1}$ with $L = \mathrm{lev}(m, g_0)$ is a radical element (Step 1), so by Steps 3 and 6,

$$Z\omega_s(g) = Z\omega_s(r)\omega_s(L) = Z\omega_s(L) = \upsilon(\det m)^{-1}|\det m|^{1/2}\,\omega_{s_0}(g_0)Z.$$

□

Lemma 6.2.11. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set, (k,l,i) a splitting of V over V_0 along p, $m \in \mathrm{GL}_p(\mathbb{A}_E)$ and $g \in \mathrm{G}(s)$ such that the splitting is adapted at g in the basis m ((MX5) of Definition 3.2.21). Then there is $C_g \in \mathbb{C}$ with $C_g \neq 0$ and $(\omega_s(g)v^0)[x] = C_g|e_p(m^*x)|^2v^0$ for all $x \in \mathbb{A}_E^p$.*

Proof. **The constant exists.** By (1) of Lemma 6.2.44 for $\Pi = Z$, the slice maps $\Pi_x\colon f \mapsto f[x]$ satisfy the hypothesis of (2) of Lemma 6.2.46, which gives $C_g \in \mathbb{C}$ with $(\omega_s(g)v^0)[x] = C_g|e_p(m^*x)|^2v^0$ for all x. Put $u = \omega_s(g)v^0$ and suppose $C_g = 0$, so that $u[x] = 0$ for every $x \in \mathbb{A}_E^p$.

The zero term vanishes. For $z \in \mathbb{A}_E^n$ write $gz = l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b$. By the mixed law ((1) of Lemma 6.2.44)

$$Z(\rho(gz)u) = (\rho(gz)u)[0] = \psi_E(\tfrac{1}{2}\jmath\, b^*a)\rho(w)u[a] = 0.$$

Let $f \in \mathcal{M}_s$ and $h = \omega_s(g^{-1})f$. By the superposition (1) of Lemma 6.2.12, $h = \int h(z)\rho(z)v^0\,d\mu_n(z)$ as a Bochner integral in $\mathcal{M}_s$. Applying the bounded operator $\omega_s(g)$ and $\omega_s(g)\rho(z) = \rho(gz)\omega_s(g)$ ((2) of Lemma 4.1.1),

$$f = \int h(z)\rho(gz)u\,d\mu_n(z),$$

and as Z is bounded and linear,

$$Z(f) = \int h(z)Z(\rho(gz)u)\,d\mu_n(z) = 0.$$

Contradiction. Hence $f[\xi] = Z(\rho(l_{\mathbb{A}}\xi)f) = 0$ for all f and $\xi \in E^p$, and the theta expansion ((3) of Lemma 6.2.44) gives $\theta_V(f) = 0$ for every $f \in \mathcal{M}_s$. But θ_V is a theta functional (Lemma 4.1.17), so $\int_{W_n/V(E)} |\theta_V(\rho(x)v^0)|^2 dx = 1$ (WD9), which is impossible if $\theta_V = 0$. Therefore $C_g \neq 0$. □

Lemma 6.2.12. *Let s be signs on an index set n.*

1. *(Superposition.) For $h \in \mathcal{M}_s$ the map $z \mapsto h(z)\rho(z)v^0$ from $(\mathbb{A}_E^n, \mu_n)$ to $\mathcal{M}_s$ is Bochner integrable, and $h = \int h(z)\,\rho(z)v^0\,d\mu_n(z)$.*

2. *(Twisted self-convolution.) For all $x, y \in \mathbb{A}_E^n$,*

$$\int_{\mathbb{A}_E^n} e_n(z)\,\beta_s(z,y)\,\beta_s(z+y,x)\,e_n(x-z-y)\,d\mu_n(z) = e_n(y)\,e_n(x).$$

3. *(Vacuum projection.) For $f \in \mathcal{M}_s$ the map $z \mapsto e_n(z)\rho(z)f$ is Bochner integrable, and*

$$\int_{\mathbb{A}_E^n} e_n(z)\,\rho(z)f\,d\mu_n(z) = \langle v^0, f\rangle\, v^0.$$

4. *(Partial projections.) Let Σ be a set of places of E, $A_\Sigma = \{z \in \mathbb{A}_E^n : z_w = 0 \text{ for } w \notin \Sigma\}$, and μ_Σ the Haar measure on A_Σ that is the product of the factors of μ_n at the places of Σ (WD6). For $f \in \mathcal{M}_s$ put $P_\Sigma f = \int_{A_\Sigma} e_n(z)\rho(z)f\,d\mu_\Sigma(z)$. Then P_Σ is a bounded operator, $\langle P_\Sigma f, f'\rangle = \langle f, P_\Sigma f'\rangle$, P_Σ commutes with $\rho(y)$ for every y whose components at the places of Σ are 0, and $P_\Sigma\rho(y)v^0 = e_n(y)v^0$ for $y \in A_\Sigma$; in particular $P_\Sigma v^0 = v^0$. For Σ the set of all places, P_Σ is the operator of (3).*

Proof. **Step 0: evaluation and the Heisenberg law.** For $h \in \mathcal{M}_s$ and every x, $|h(x)| \le \|h\|$, by the reproducing identity (WD7), $|\beta_s| = 1$ and $|e_n| \le 1$; so $h \mapsto h(x)$ is a bounded functional and an element of $\mathcal{M}_s$ is determined by its class in $L^1(\mu_n)$. As β_s is additive in each variable and $\beta_s(y,z) = \beta_s(z,y)^{-1}$ ($\psi_E(\bar t) = \psi_E(t)$, $\bar\jmath = -\jmath$), $(\rho(z)\rho(y)h)(x) = \beta_s(z,x)\beta_s(y,x-z)h(x-z-y) = \beta_s(z,y)\,(\rho(z+y)h)(x)$.

Step 1: (1). The map $z \mapsto h(z)\rho(z)v^0$ is strongly measurable (strong continuity) with norm $|h(z)|\,\|v^0\|$, hence Bochner integrable. By Step 0 the value of its integral at x is $\int h(z)\beta_s(z,x)e_n(x-z)\,d\mu_n(z) = h(x)$, by (WD7).

Step 2: (2), reduction. Write $x = (x_\infty, x_f)$. In the coordinates $x_{v,i} = \kappa_v u_{v,i}$ at $v \in \Phi$ and $x_f = \alpha\xi$ at the finite places, μ_n is the product of $\pi^{-1}du_{v,i}$ over (v,i) and the Haar measure $d\xi$ with $\widehat{\mathcal{O}}_E^n$ of mass 1 (WD6), and, as $\jmath_v\kappa_v^2 = i/(2\pi)$, $\pi\,\mathrm{Im}(\jmath_v)\kappa_v^2 = \frac12$ and $(\alpha\eta)^*(\alpha\xi) = \alpha\bar\alpha\,\eta^*\xi$,

$$e_n(x) = \prod_{v,i} e^{-|u_{v,i}|^2/2}\cdot\prod_i \psi_0(\tfrac{c_0}{2}\bar\xi_i\xi_i)\mathbf{1}_{\widehat{\mathcal{O}}_E}(\xi_i), \qquad \beta_s(x,y) = \prod_{v,i} b_{s_i}(u_{v,i},u'_{v,i})\cdot\prod_i \psi_0(\tfrac12\bar\eta_i\xi_i),$$

for y with coordinates u', η, where $b_\sigma(t,t') = \exp(\sigma(\bar t't - \bar t t')/2)$ (the archimedean factor of ψ_E is $\exp(-2\pi i\,2\,\mathrm{Re}(\cdot))$). So the integral in (2) is a product of one-dimensional integrals, and it suffices to prove (R) and (F) below.

Step 3: (R) archimedean, $\sigma = \pm1$: $\pi^{-1}\int_{\mathbb{C}} e(t)\,b_\sigma(t,t')\,b_\sigma(t+t',u)\,e(u-t-t')\,dt = e(t')e(u)$ with $e(t) = e^{-|t|^2/2}$. For $\sigma = 1$, expanding $|u-t'-t|^2$, the exponent of the integrand is

$$-|t|^2 + \bar u t - \bar t t' - \tfrac12|u-t'|^2 + \tfrac12(\bar u t' - \bar t' u).$$

By the Gaussian integral $\pi^{-1}\int_{\mathbb{C}}\exp(-|t|^2 + at + c\bar t)\,dt = e^{ac}$ ($a, c \in \mathbb{C}$; both sides are entire in (a,c) and agree for $c = \bar a$ by the substitution $t \mapsto t + c$), with $a = \bar u$, $c = -t'$, the integral is $\exp(-\bar u t' - \frac12|u-t'|^2 + \frac12(\bar u t' - \bar t' u)) = \exp(-\frac12(|u|^2 + |t'|^2))$. For $\sigma = -1$ substitute $t \mapsto \bar t$: $b_{-1}(t,t') = b_1(\bar t, \bar t')$ and $e(\bar t) = e(t)$, so the integral is the case $\sigma = 1$ at $(\bar u, \bar t')$, again $e(t')e(u)$.

Step 4: (F) finite: $\int_{\mathbb{A}_{E,f}} \varepsilon(\zeta)\,\psi_0(\frac12\bar\eta\zeta)\,\psi_0(\frac12\bar\xi(\zeta+\eta))\,\varepsilon(\xi-\zeta-\eta)\,d\zeta = \varepsilon(\eta)\varepsilon(\xi)$ with $\varepsilon(\zeta) = \psi_0(\frac{c_0}{2}\bar\zeta\zeta)\mathbf{1}_{\widehat{\mathcal{O}}_E}(\zeta)$. We use $\psi_0(\bar t) = \psi_0(t)^{-1}$ (as $\psi_{E,f}(\bar t) = \psi_{E,f}(t)$ and $\overline{\jmath\alpha\bar\alpha} = -\jmath\alpha\bar\alpha$), $\psi_0(\widehat{\mathcal{O}}_E) = 1$, $c_0 \in \mathcal{O}_E$ and $c_0 + \bar c_0 = 1$ (WD1); in particular $\psi_0(\frac12(X-\bar X)) = \psi_0(\frac12 X)^2 = \psi_0(X) = 1$ for $X \in \widehat{\mathcal{O}}_E$. The integrand vanishes unless $\zeta \in \widehat{\mathcal{O}}_E$ and $\xi-\eta-\zeta \in \widehat{\mathcal{O}}_E$, so both sides vanish unless $d = \xi-\eta \in \widehat{\mathcal{O}}_E$ (for the right side: see below). Let $d \in \widehat{\mathcal{O}}_E$ and $\zeta \in \widehat{\mathcal{O}}_E$. Expanding, and replacing $\psi_0(-\frac{c_0}{2}\bar\zeta d)$ by $\psi_0(\frac{\bar c_0}{2}\zeta\bar d)$ and $\psi_0(c_0\bar\zeta\zeta)$ by 1, the integrand is

$$\psi_0(\tfrac{c_0}{2}\bar d d + \tfrac12\bar\xi\eta)\,\psi_0\Big(\tfrac12\zeta(\bar\eta+\bar\xi-c_0\bar d+\bar c_0\bar d)\Big) = \psi_0(\tfrac{c_0}{2}\bar d d + \tfrac12\bar\xi\eta)\,\psi_0(\zeta(\bar c_0\bar\xi + c_0\bar\eta)),$$

since $\bar\eta+\bar\xi-(c_0-\bar c_0)(\bar\xi-\bar\eta) = (1-c_0+\bar c_0)\bar\xi + (1+c_0-\bar c_0)\bar\eta = 2\bar c_0\bar\xi + 2c_0\bar\eta$. By self-duality the mean of $\psi_0(\zeta t)$ over $\zeta \in \widehat{\mathcal{O}}_E$ is $\mathbf{1}_{\widehat{\mathcal{O}}_E}(t)$, and $\bar c_0\bar\xi + c_0\bar\eta = \bar\eta + \bar c_0\bar d$ is integral iff η is. So the left side is $\psi_0(\frac{c_0}{2}\bar d d + \frac12\bar\xi\eta)\mathbf{1}[\xi,\eta \in \widehat{\mathcal{O}}_E]$, and for ξ,η integral its ratio to $\varepsilon(\eta)\varepsilon(\xi)$ is $\psi_0(\frac12(\bar c_0\bar\xi\eta - c_0\bar\eta\xi)) = \psi_0(\frac12(X-\bar X)) = 1$ with $X = \bar c_0\bar\xi\eta$. If $d \notin \widehat{\mathcal{O}}_E$ the right side vanishes too, as then ξ and η are not both integral.

Step 4b: the same at a set of places. Let Σ be a set of places. The identities of Steps 3 and 4 are local: (R) concerns one coordinate at one infinite place, and Step 4 holds verbatim with $\mathbb{A}_{E,f}$ and $\widehat{\mathcal{O}}_E$ replaced by the restricted product of the E_w and by $\prod_w \mathcal{O}_{E,w}$ over the finite $w \in \Sigma$, because ψ_0 is the product of its local components and $\prod_{w\in\Sigma}\mathcal{O}_{E,w}$ is its own annihilator (the annihilator of $\widehat{\mathcal{O}}_E = \prod_w \mathcal{O}_{E,w}$ is the product of the local annihilators). For $y \in A_\Sigma$ the integrand of $(P_\Sigma\rho(y)v^0)(x) = \int_{A_\Sigma} e_n(z)\beta_s(z,y)\beta_s(z+y,x)e_n(x-z-y)\,d\mu_\Sigma(z)$ is, by Step 0, a product over the places: at the places of Σ these local identities give the factor e_n of y times that of x, and at the places outside Σ the factor is that of $e_n(x)$, as z and y vanish there. Hence $P_\Sigma\rho(y)v^0 = e_n(y)v^0$ (the factors of $e_n(y)$ outside Σ being $e_n(0)$-factors, equal to 1). Boundedness: $\|P_\Sigma\| \le \int_{A_\Sigma}|e_n|\,d\mu_\Sigma < \infty$. Symmetry: $\rho(z)$ preserves the inner product and $\rho(z)^{-1} = \rho(-z)$, e_n is real and even and μ_Σ is invariant under $z \mapsto -z$, so $\langle P_\Sigma f, f'\rangle = \int e_n(z)\langle f, \rho(-z)f'\rangle\,d\mu_\Sigma(z) = \langle f, P_\Sigma f'\rangle$. Commutation: if y vanishes at the places of Σ and $z \in A_\Sigma$, then $\beta_s(y,z) = \beta_s(z,y) = 1$, so $\rho(y)\rho(z) = \rho(z)\rho(y)$ by Step 0.

Step 5: (3). The map $z \mapsto e_n(z)\rho(z)f$ is Bochner integrable as in Step 1, and $Pf = \int e_n(z)\rho(z)f\,d\mu_n(z)$ is bounded linear in f with $\|P\| \le \|e_n\|_1$. For $f = \rho(y)v^0$, Step 0 and (2) give, at every x, $(P\rho(y)v^0)(x) = \int e_n(z)\beta_s(z,y)\beta_s(z+y,x)e_n(x-z-y)\,d\mu_n(z) = e_n(y)e_n(x)$, so $P\rho(y)v^0 = e_n(y)v^0$. For general f, (1) and boundedness of P give $Pf = \int f(y)e_n(y)\,d\mu_n(y)\,v^0 = \langle v^0, f\rangle v^0$. □

Lemma 6.2.13. *Let s be signs on an index set n.*

1. *For every $f \in \mathcal{M}_s$, $\langle v^0, f\rangle = f(0)$. In particular $\langle v^0, v^0\rangle = 1$.*
2. *For every $g \in \mathrm{G}(s)$, $\Delta_s(g) \neq 0$,*

$$|\Delta_s(g)|^2 = \prod_{v\in\Phi} |\det(g_v)_-|^2 \cdot |\det u|^{-1} \qquad \textit{and} \qquad \langle v^0, \omega_s(g)v^0\rangle = \overline{\Delta_s(g)}^{-1},$$

where $u \in \mathrm{GL}_n(\mathbb{A}_{E,f})$ is any element with $u\widehat{\mathcal{O}}_E^n = \widehat{\mathcal{O}}_E^n \cap g_f^{-1}\widehat{\mathcal{O}}_E^n$.

Proof. **Step 1: the vacuum is real and even.** Write $x = (x_\infty, x_f) \in \mathbb{A}_E^n$ and $e_n(x) = e_\infty(x_\infty)e_f(x_f)$ with the two factors of (WD5). The factor e_∞ is real and even. If $x_f \notin \alpha\widehat{\mathcal{O}}_E^n$ then $e_f(x_f) = e_f(-x_f) = 0$. If $x_f = \alpha z$ with $z \in \widehat{\mathcal{O}}_E^n$, then $\frac{c_0\jmath}{2}\sum_i \bar x_i x_i = \jmath\alpha\bar\alpha\,\frac{c_0}{2}z^*z$, so

$$e_f(\alpha z) = \psi_0(\tfrac{c_0}{2}z^*z)\mathbf{1}[z \in \widehat{\mathcal{O}}_E^n] \qquad (z \in \mathbb{A}_{E,f}^n). \tag{6.5}$$

This is even in z, and $e_f(\alpha z)\,\overline{e_f(\alpha z)}^{-1} = \psi_0(c_0 z^* z) = 1$ for $z \in \widehat{\mathcal{O}}_E^n$, because $c_0 \in \mathcal{O}_E$ (WD1), so $c_0 z^* z \in \widehat{\mathcal{O}}_E$, and ψ_0 is trivial on $\widehat{\mathcal{O}}_E$. Hence $\overline{e_n(x)} = e_n(x) = e_n(-x)$ for all x.

Step 2: (1). By (WD7) at the point $x = 0$, with $\beta_s(y,0) = \psi_E(0) = 1$, and by Step 1,

$$f(0) = \int f(y)\,\beta_s(y,0)\,e_n(-y)\,d\mu_n(y) = \int \overline{e_n(y)}\,f(y)\,d\mu_n(y) = \langle v^0, f\rangle.$$

For $f = v^0$ this gives $\langle v^0, v^0\rangle = e_n(0) = 1$.

Step 3: the vacuum coefficient as an integral. Let $g \in \mathrm{G}(s)$. By (1) applied to $\omega_s(g)v^0 \in \mathcal{M}_s$, by (WR3) at $x = 0$ and by Step 1,

$$\begin{aligned}\langle v^0, \omega_s(g)v^0\rangle = (\omega_s(g)v^0)(0) &= \Delta_s(g)\int e_n(y)\,\beta_s(gy,0)\,e_n(-gy)\,d\mu_n(y)\\ &= \Delta_s(g)\int e_n(y)\,e_n(gy)\,d\mu_n(y).\end{aligned}$$

The measure μ_n is the product of the image μ_∞ of $\pi^{-fn}du$ under $u \mapsto (\kappa_v u_{v,i})$ and the image μ_f of dz under $z \mapsto \alpha z$ (WD6); the integrand is the product of an archimedean and a finite factor, each bounded by a Gaussian, respectively an indicator of a compact set; so by Fubini the integral is the product $I_\infty I_f$ of the two integrals below.

Step 4: the archimedean integral. Since $\pi\,\mathrm{Im}(\jmath_v)\kappa_v^2 = \frac{1}{2}$, we have $e_\infty((\kappa_v u_v)_v) = \prod_v \exp(-\frac{1}{2}u_v^* u_v)$, and $(g\,(\kappa_v u_v))_v = \kappa_v g_v u_v$. By the Gaussian integral $\int_{\mathbb{C}^n}\exp(-u^*Au)\,du = \pi^{\#n}\det(A)^{-1}$ for positive definite hermitian A,

$$I_\infty = \prod_{v\in\Phi}\pi^{-\#n}\int_{\mathbb{C}^n}\exp(-u^*\tfrac{1}{2}(1+g_v^*g_v)u)\,du = \prod_{v\in\Phi}\det\big(\tfrac{1}{2}(1+g_v^*g_v)\big)^{-1}.$$

Fix v and order the indices with the signs $+1$ first: $g_v = \begin{pmatrix} A & B\\ C & D\end{pmatrix}$, $D = (g_v)_-$. The relation $g_v^*\mathrm{diag}(s)g_v = \mathrm{diag}(s)$ reads $A^*A = 1 + C^*C$, $D^*D = 1 + B^*B$, $A^*B = C^*D$. Hence

$$\tfrac{1}{2}(1+g_v^*g_v) = \begin{pmatrix} A^*A & C^*D\\ D^*C & D^*D\end{pmatrix},$$

$D^*D \geq 1$ is invertible, so D is invertible, and the Schur complement of D^*D is $A^*A - C^*D(D^*D)^{-1}D^*C = A^*A - C^*C = 1$. Therefore $\det\frac{1}{2}(1+g_v^*g_v) = \det(D^*D) = |\det(g_v)_-|^2$ (an empty block has determinant 1), and $I_\infty = \prod_v |\det(g_v)_-|^{-2}$.

Step 5: the finite integral. By (6.5), $g_f\alpha z = \alpha g_f z$ and $(g_f z)^*(g_f z) = z^*z$ (as $g_f^*g_f = 1$),

$$e_f(\alpha z)\,e_f(\alpha g_f z) = \psi_0(c_0 z^* z)\,\mathbf{1}[z \in \widehat{\mathcal{O}}_E^n \cap g_f^{-1}\widehat{\mathcal{O}}_E^n] = \mathbf{1}[z \in u\widehat{\mathcal{O}}_E^n],$$

the phase being 1 as in Step 1. The Haar measure of $u\widehat{\mathcal{O}}_E^n$ is $|\det u|$ times that of $\widehat{\mathcal{O}}_E^n$, which is 1; so $I_f = |\det u|$.

Step 6: (2). The determinants $\det(g_v)_-$ are nonzero (Step 4), $|v| = 1$ and $\det u$ is an idele, so by (WR3) $\Delta_s(g) \neq 0$ and $|\Delta_s(g)|^2 = \prod_v |\det(g_v)_-|^2\,|\det u|^{-1} = (I_\infty I_f)^{-1}$. By Steps 3–5, $\langle v^0, \omega_s(g)v^0\rangle = \Delta_s(g)\,|\Delta_s(g)|^{-2} = \overline{\Delta_s(g)}^{-1}$. □

Lemma 6.2.14. *Let s be signs on an index set n, and let $B\colon \mathcal{M}_s \times \mathcal{M}_s \to \mathbb{C}$ be conjugate-linear in the first and linear in the second argument, with $|B(f,f')| \leq C\,\|f\|\,\|f'\|$ for some C and all f, f', and with*

$$B(\rho(z)f, \rho(z)f') = B(f,f') \qquad (z \in \mathbb{A}_E^n,\ f, f' \in \mathcal{M}_s).$$

Then $B(f,f') = B(v^0,v^0)\,\langle f, f'\rangle$ for all $f, f' \in \mathcal{M}_s$.

Proof. **Step 1: moving Heisenberg operators across B.** For $a, b \in \mathcal{M}_s$ and $z \in \mathbb{A}_E^n$, invariance under $\rho(-z)$ and $\rho(-z)\rho(z) = 1$ give $B(\rho(z)a, b) = B(a, \rho(-z)b)$; the same holds for $\langle\cdot,\cdot\rangle$, which is invariant under ρ ($|\beta_s| = 1$ and μ_n is translation invariant).

Step 2: the projection is symmetric for B. Let $Pf = \int e_n(y)\rho(y)f\, d\mu_n(y)$, so that $Pf = \langle v^0, f\rangle v^0$ by (3) of Lemma 6.2.12. Since B is bounded, $f' \mapsto B(f, f')$ is a bounded linear functional and $f \mapsto B(f, f')$ a bounded conjugate-linear one, so both commute with Bochner integrals (the latter with conjugated scalars). By Step 1, e_n real and even, and the invariance of μ_n under $y \mapsto -y$,

$$\begin{aligned} B(f, Pf') &= \int e_n(y) B(f, \rho(y)f')\, d\mu_n(y) = \int e_n(y) B(\rho(-y)f, f')\, d\mu_n(y) \\ &= B\Big(\int e_n(-y)\rho(-y)f\, d\mu_n(y), f'\Big) = B(Pf, f'). \end{aligned}$$

Step 3: the value at the vacuum. As $Pv^0 = \langle v^0, v^0\rangle v^0 = v^0$, Step 2 gives for every f

$$B(f, v^0) = B(f, Pv^0) = B(Pf, v^0) = B(\langle v^0, f\rangle v^0, v^0) = \langle f, v^0\rangle\, B(v^0, v^0).$$

Step 4: conclusion. By (1) of Lemma 6.2.12, $f' = \int f'(z)\rho(z)v^0\, d\mu_n(z)$. By boundedness, Step 1 and Step 3,

$$\begin{aligned} B(f, f') &= \int f'(z) B(\rho(-z)f, v^0)\, d\mu_n(z) = B(v^0, v^0) \int f'(z)\langle \rho(-z)f, v^0\rangle\, d\mu_n(z) \\ &= B(v^0, v^0) \int f'(z)\langle f, \rho(z)v^0\rangle\, d\mu_n(z), \end{aligned}$$

and the last integral is $\langle f, \int f'(z)\rho(z)v^0\, d\mu_n(z)\rangle = \langle f, f'\rangle$, the inner product being bounded on $\mathcal{M}_s$ ($|g(x)| \le \|g\|$ for $g \in \mathcal{M}_s$, so $|\langle g, g'\rangle| \le \|g\|\,\|g'\|$). □

Lemma 6.2.15. *Let V', V be spaces with points with signs t' on r' and t on r, X a base-adapted splitting of V' over V along a one-element set, and $d^\times a$ a Haar measure on $\mathbb{A}_E^\times$. Let $Y = E^\times\backslash\mathbb{A}_E^\times \times [\mathrm{U}(V)] \times \mathrm{K}_{\max}(t')$ carry the product of the quotient of $d^\times a$ by counting measure on $E^\times$, the quotient measure of $[\mathrm{U}(V)]$ and the Haar probability measure. There is $c_1 > 0$, depending only on X and $d^\times a$, with the following property. Let $\varphi\colon \mathrm{G}(t') \to \mathbb{C}$ be Borel and left invariant under N_X and under $\Gamma_{P,X}$, and*

$$\Theta_\varphi(a, h, \kappa) = |a|^{\#r+1}\, \varphi\big(\mathrm{lev}_X^{\ln}(a, h)\, \kappa\big) \qquad (a \in \mathbb{A}_E^\times,\ h \in \mathrm{G}(t),\ \kappa \in \mathrm{K}_{\max}(t')).$$

Then Θ_φ is left invariant under $E^\times \times \Gamma_V$ in (a, h), so it is a function on Y. If $\varphi \ge 0$, then $\int_{\Gamma_{P,X}\backslash \mathrm{G}(t')} \varphi = c_1 \int_Y \Theta_\varphi$ in $[0, \infty]$. If Θ_φ is integrable on Y, then φ is integrable on $\Gamma_{P,X}\backslash\mathrm{G}(t')$ and $\int_{\Gamma_{P,X}\backslash \mathrm{G}(t')} \varphi = c_1 \int_Y \Theta_\varphi$.

Proof. Write $G' = \mathrm{G}(t')$ with its measure μ (a left Haar measure), $K = \mathrm{K}_{\max}(t')$ with $d\kappa$, $N = N_X$ with a Haar measure dn, $\Gamma_P = \Gamma_{P,X}$, $M = \mathbb{A}_E^\times \times \mathrm{G}(t)$ with $dm = d^\times a\, dh$ (a Haar measure), $L(m) = \mathrm{lev}_X^{\ln}(a, h)$ for $m = (a, h)$, $w(m) = |a|^{\#r+1}$, $M_F = E^\times \times \Gamma_V$ and $N_F = N \cap \Gamma_{V'}$. By (1) of Lemma 6.2.16, L is a continuous injective homomorphism normalising N, and by (3) of Lemma 6.2.16 applied to $\mathrm{lev}_X(a^{-1}, h)$,

$$\int_N f\big(L(m)\, n\, L(m)^{-1}\big)\, dn = w(m) \int_N f(n)\, dn \qquad (m \in M,\ f \ge 0 \text{ Borel}). \tag{6.6}$$

By (4) of Lemma 6.2.16 (with $\xi \mapsto \xi^{-1}$), $(\delta, \nu) \mapsto L(\delta)\nu$ is a bijection $M_F \times N_F \to \Gamma_P$; in particular $L(\delta) \in \Gamma_P$, and $w(\delta) = 1$ for $\delta \in M_F$ by the product formula. Hence $\Theta_\varphi((\xi, \gamma_0)(a, h), \kappa) = w(\xi a, \gamma_0 h)\varphi(L(\xi, \gamma_0)L(a, h)\kappa) = \Theta_\varphi(a, h, \kappa)$.

Step 1: the Iwasawa measure is a multiple of Haar measure. Let $\Pi\colon M\times N\times K\to G'$, $\Pi(m,n,\kappa)=L(m)n\kappa$, and $\Lambda=\Pi_*(dm\,dn\,d\kappa)$. By (6) and (2) of Lemma 6.2.16, $G'=P_XK$ and $P_X=N\,L(M)=L(M)\,N$, so Π is onto. It is proper: if $L(m)n\kappa\in C$, C compact, then $L(m)n\in CK\cap P_X$, a compact set (P_X is closed), and by (2) of Lemma 6.2.16 the components m, n of $p=L(m)n=(L(m)nL(m)^{-1})L(m)$ are continuous functions of $p\in P_X$; so $\Pi^{-1}(C)$ lies in a compact set and $\Lambda(C)<\infty$. Thus Λ is a nonzero Borel measure on the second countable locally compact space G', finite on compact sets, hence a Radon measure. It is invariant under the action $(p,k)\cdot g=pgk^{-1}$ of $H=P_X\times K$: right translation by K preserves $d\kappa$; left translation by $n_0\in N$ sends $L(m)n\kappa$ to $L(m)(L(m)^{-1}n_0L(m))n\kappa$ and dn is left invariant; left translation by $L(m_0)$ sends it to $L(m_0m)n\kappa$ and dm is left invariant; and $P_X=N\,L(M)$. The measure μ is also H-invariant: it is left invariant, and right invariant under the compact group K, on which the modular function of G' is trivial. The action of H on G' is transitive, with the compact stabiliser $\{(\kappa_0,\kappa_0):\kappa_0\in P_X\cap K\}$ of 1, and the orbit map from the quotient of H by it to G' is a continuous bijection between homogeneous spaces of the σ-compact group H, hence a homeomorphism. An H-invariant Radon measure on the quotient of H by a compact subgroup is unique up to a positive factor (Weil's theorem on invariant measures on homogeneous spaces; see Folland, *A Course in Abstract Harmonic Analysis*, Theorem 2.49). Hence there is $d\in(0,\infty)$ with

$$\int_{G'} f\,d\mu=d^{-1}\int_M\int_N\int_K f(L(m)n\kappa)\,d\kappa\,dn\,dm\qquad(f\ge0\text{ Borel}).\tag{6.7}$$

Step 2: fundamental domains. $\Gamma_P\subset\Gamma_{V'}$ is countable and discrete, and so are $N_F\subset N$ and $M_F\subset M$; let $D\subset G'$, $D_N\subset N$ and $D_M=D_1\times D_2\subset M$ be Borel fundamental domains for their left actions (D_1 for $E^\times$ in $\mathbb{A}_E^\times$, D_2 for Γ_V in $\mathrm{G}(t)$). For $f\ge0$ Borel on N, $\int_N f\,dn=\int_{D_N}\sum_{\nu\in N_F}f(\nu n)\,dn$, and likewise on M. Let C be the compact set of (5) of Lemma 6.2.16; then $\sum_\nu\mathbf{1}_C(\nu n)\ge1$ for every n, so $v_N:=dn(D_N)\le\int_{D_N}\sum_\nu\mathbf{1}_C(\nu n)\,dn=dn(C)<\infty$, and $v_N>0$ since the countably many νD_N cover N.

Step 3: the identity for $\varphi\ge0$. By the definition of the quotient measure and (6.7),

$$\int_{\Gamma_P\backslash G'}\varphi=\int_{G'}\varphi\mathbf{1}_D\,d\mu=d^{-1}\int_M\int_N\int_K\varphi(L(m)n\kappa)\mathbf{1}_D\big(L(m)n\kappa\big)\,d\kappa\,dn\,dm.$$

Since $L(m)n=(L(m)nL(m)^{-1})L(m)$ and φ is left N-invariant, $\varphi(L(m)n\kappa)=\varphi(L(m)\kappa)$. By (6.6) with $f(n)=\mathbf{1}_D(nL(m)\kappa)$, $\int_N\mathbf{1}_D(L(m)n\kappa)\,dn=w(m)A(L(m)\kappa)$ with $A(g)=\int_N\mathbf{1}_D(ng)\,dn$. By Tonelli,

$$\int_{\Gamma_P\backslash G'}\varphi=d^{-1}\int_K\int_M H_\kappa(m)\,A\big(L(m)\kappa\big)\,dm\,d\kappa,\qquad H_\kappa(m)=w(m)\varphi\big(L(m)\kappa\big).$$

For $g\in G'$ and $\delta\in M_F$, (6.6) with $w(\delta)=1$ gives $A(L(\delta)g)=\int_N\mathbf{1}_D(L(\delta)ng)\,dn$, so by Tonelli and the bijection $M_F\times N_F\to\Gamma_P$,

$$\sum_{\delta\in M_F}A(L(\delta)g)=\int_{D_N}\sum_{\nu\in N_F}\sum_{\delta\in M_F}\mathbf{1}_D\big(L(\delta)\nu\,ng\big)\,dn=\int_{D_N}\sum_{\gamma\in\Gamma_P}\mathbf{1}_D(\gamma ng)\,dn=v_N.$$

As H_κ is left M_F-invariant (shown above for Θ_φ),

$$\int_M H_\kappa(m)A(L(m)\kappa)\,dm=\int_{D_M}H_\kappa(m)\sum_{\delta\in M_F}A\big(L(\delta)L(m)\kappa\big)\,dm=v_N\int_{D_M}H_\kappa(m)\,dm.$$

Hence $\int_{\Gamma_P\backslash G'}\varphi=c_1\int_K\int_{D_M}\Theta_\varphi(m,\kappa)\,dm\,d\kappa=c_1\int_Y\Theta_\varphi$ with $c_1=v_N/d$. Replacing dn by $t\,dn$ multiplies v_N and d by t, so c_1 depends only on X, $d^\times a$ and the fixed measures of $\mathrm{G}(t')$ and $\mathrm{G}(t)$.

Step 4: integrable φ. $|\varphi|$ is Borel, ≥ 0 and left invariant under N and Γ_P, and $\Theta_{|\varphi|} = |\Theta_\varphi|$; by Step 3, φ is integrable on $\Gamma_P\backslash G'$ when Θ_φ is integrable on Y. Apply Step 3 to the positive and negative parts of $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$, which have the same invariance and finite integrals, and use linearity. □

Lemma 6.2.16. *Let V', V be spaces with points with signs t' on r' and t on r, and $X = (k, l, i)$ a base-adapted splitting of V' over V along a one-element set, with adelic frames $k_\mathbb{A}, l_\mathbb{A}, i_\mathbb{A}$, coordinates $\mathrm{pr}_k, \mathrm{pr}_l, \mathrm{pr}_i$, Levi elements $\mathrm{lev} = \mathrm{lev}_X$ and radical elements $\mathrm{rad} = \mathrm{rad}_X$. Put*

$$N_X = \{\mathrm{rad}(c, b) : c \in \mathrm{M}_{r\times 1}(\mathbb{A}_E),\ b \in \mathbb{A}_E\}, \qquad P_X = \{g \in \mathrm{G}(t') : gk_\mathbb{A} \in k_\mathbb{A}\mathbb{A}_E^\times\},$$

and let $\Gamma_{P,X}$ be the set of $\gamma \in \Gamma_{V'}$ with $x_{V'}\gamma x_{V'}^{-1}k \in kE^\times$.

1. *N_X is the set of $g \in \mathrm{G}(t')$ with $gk_\mathbb{A} = k_\mathbb{A}$ and $\mathrm{pr}_i\, g\, i_\mathbb{A} = 1$; it is a closed subgroup of $\mathrm{G}(t')$. The map $(m, h) \mapsto \mathrm{lev}(m, h)$ is a continuous injective homomorphism $\mathbb{A}_E^\times \times \mathrm{G}(t) \to \mathrm{G}(t')$ whose image normalises N_X, and $\mathrm{lev}(m, h) \in N_X$ only for $(m, h) = (1, 1)$.*

2. *P_X is a closed subgroup of $\mathrm{G}(t')$, and every $g \in P_X$ is $g = n\,\mathrm{lev}(m, h)$ with unique $n \in N_X$, $m \in \mathbb{A}_E^\times$, $h \in \mathrm{G}(t)$, namely $m = \mathrm{pr}_k\, g\, k_\mathbb{A}$ and $h = \mathrm{pr}_i\, g\, i_\mathbb{A}$; so n, m, h depend continuously on g.*

3. *N_X is unimodular, and every Haar measure dn of N_X satisfies, for all (m, h) and Borel $f \geq 0$ on N_X,*

$$\int_{N_X} f\big(\mathrm{lev}(m, h)\, n\, \mathrm{lev}(m, h)^{-1}\big)\, dn = |m|^{-(\#r+1)} \int_{N_X} f(n)\, dn.$$

4. *$\Gamma_{P,X} = P_X \cap \Gamma_{V'}$, and the map $((\xi, \gamma_0), \nu) \mapsto \mathrm{lev}(\xi, \gamma_0)\,\nu$ is a bijection from $(E^\times \times \Gamma_V) \times (N_X \cap \Gamma_{V'})$ onto $\Gamma_{P,X}$.*

5. *There is a compact set $C \subset N_X$ with $N_X = (N_X \cap \Gamma_{V'})\, C$.*

6. *$\mathrm{G}(t') = P_X\, \mathrm{K}_{\max}(t')$.*

Proof. Write $x = x_{V'}$, $\varepsilon = \varepsilon_{t'}$, $p_0 = \#r$, and $\mathcal{F} = (l_\mathbb{A} \mid k_\mathbb{A} \mid i_\mathbb{A})$. From (MX1) and $x^*J_{V'}x = \varepsilon$, $x_V^*J_Vx_V = \varepsilon_t$: $k_\mathbb{A}^*\varepsilon k_\mathbb{A} = l_\mathbb{A}^*\varepsilon l_\mathbb{A} = 0$, $l_\mathbb{A}^*\varepsilon k_\mathbb{A} = 1$, $i_\mathbb{A}^*\varepsilon i_\mathbb{A} = \varepsilon_t$, $i_\mathbb{A}^*\varepsilon k_\mathbb{A} = i_\mathbb{A}^*\varepsilon l_\mathbb{A} = 0$, and $\mathrm{pr}_k = l_\mathbb{A}^*\varepsilon$, $\mathrm{pr}_l = k_\mathbb{A}^*\varepsilon$, $\mathrm{pr}_i = \varepsilon_t i_\mathbb{A}^*\varepsilon$. So $\mathcal{F}^*\varepsilon\mathcal{F}$ is invertible, $\mathcal{F}$ is invertible, and $k_\mathbb{A}\mathrm{pr}_k + l_\mathbb{A}\mathrm{pr}_l + i_\mathbb{A}\mathrm{pr}_i = 1$. For $g \in \mathrm{G}(t')$, $\varepsilon g = (g^*)^{-1}\varepsilon$, so $\mathrm{pr}_l\, g = (g^{-1}k_\mathbb{A})^*\varepsilon$.

Step 1: radical elements. Let $N = \mathrm{rad}(c, b) - 1 = -i_\mathbb{A}c\,\mathrm{pr}_l + k_\mathbb{A}c^*\varepsilon_t\mathrm{pr}_i - k_\mathbb{A}Y\mathrm{pr}_l$, $Y = \frac{1}{2}(\jmath^{-1}(b+\bar{b})+c^*\varepsilon_t c)$. Then $\mathrm{rad}(c, b)k_\mathbb{A} = k_\mathbb{A}$, $\mathrm{rad}(c, b)i_\mathbb{A} = i_\mathbb{A}+k_\mathbb{A}c^*\varepsilon_t$, $\mathrm{rad}(c, b)l_\mathbb{A} = l_\mathbb{A}-i_\mathbb{A}c-k_\mathbb{A}Y$. As pr_l kills the image of N, $\mathrm{rad}(0, b)\mathrm{rad}(c, b') = \mathrm{rad}(c, b+b')$; and with $z = c_1^*\varepsilon_t c_2$ (so $c_2^*\varepsilon_t c_1 = \bar{z}$), multiplying out gives $\mathrm{rad}(c_1, 0)\mathrm{rad}(c_2, 0) = \mathrm{rad}(c_1+c_2, \frac{1}{2}\jmath(z-\bar{z}))$. Hence, as b enters only through $b + \bar{b} \in \mathbb{A}_F$,

$$\begin{gathered}\Psi(c_1, \beta_1)\Psi(c_2, \beta_2) = \Psi\big(c_1 + c_2,\ \beta_1 + \beta_2 + \jmath(z - \bar{z})\big),\\ \Psi(c, \beta) := \mathrm{rad}(c, \beta/2) \quad (c \in \mathrm{M}_{r\times 1}(\mathbb{A}_E),\ \beta \in \mathbb{A}_F),\end{gathered} \tag{6.8}$$

with $\jmath(z - \bar{z}) \in \mathbb{A}_F$. Ψ maps onto N_X, and it is injective with continuous inverse: $c = -\mathrm{pr}_i\, n\, l_\mathbb{A}$ and $\jmath^{-1}\beta = -2\,\mathrm{pr}_k\, n\, l_\mathbb{A} - c^*\varepsilon_t c$.

Step 2: (1). Let N' be the set of $g \in \mathrm{G}(t')$ with $gk_\mathbb{A} = k_\mathbb{A}$, $\mathrm{pr}_i g i_\mathbb{A} = 1$. For $g \in N'$, $\mathrm{pr}_l g i_\mathbb{A} = k_\mathbb{A}^*\varepsilon i_\mathbb{A} = 0$, so $gi_\mathbb{A} = i_\mathbb{A}+k_\mathbb{A}y$ with $y = \mathrm{pr}_k g i_\mathbb{A}$; from this N' is closed under products and inverses. By Step 1, $N_X \subset N'$. Conversely let $q \in N'$, $qi_\mathbb{A} = i_\mathbb{A}+k_\mathbb{A}y$, $ql_\mathbb{A} = l_\mathbb{A}+i_\mathbb{A}c'+k_\mathbb{A}z'$ (here $\mathrm{pr}_l ql_\mathbb{A} = k_\mathbb{A}^*\varepsilon l_\mathbb{A} = 1$). Since q preserves the form, $(qi_\mathbb{A})^*\varepsilon(ql_\mathbb{A}) = 0$ gives $\varepsilon_t c' + y^* = 0$ and $(ql_\mathbb{A})^*\varepsilon(ql_\mathbb{A}) = 0$

gives $z'+\bar z'=-c'^{*}\varepsilon_t c'$. With $c=-c'$ and $b=-\jmath(z'+\frac12 c^{*}\varepsilon_t c)$ one has $y=c^{*}\varepsilon_t$ and $-Y=z'$, so q and $\mathrm{rad}(c,b)$ agree on the columns of the invertible $\mathcal F$; $q=\mathrm{rad}(c,b)$. So $N_X=N'$, closed. The Levi elements satisfy $\mathrm{lev}(m,h)k_{\mathbb A}=k_{\mathbb A}m$, $\mathrm{lev}(m,h)i_{\mathbb A}=i_{\mathbb A}h$, $\mathrm{lev}(m,h)l_{\mathbb A}=l_{\mathbb A}\bar m^{-1}$, whence lev is a continuous homomorphism, $\mathrm{pr}_k\mathrm{lev}(m,h)k_{\mathbb A}=m$, $\mathrm{pr}_i\mathrm{lev}(m,h)i_{\mathbb A}=h$ (injectivity), and $\mathrm{lev}(m,h)\in N'$ forces $m=1$, $h=1$. Finally, using $h^{*}\varepsilon_t=\varepsilon_t h^{-1}$, a direct computation on the three terms of N gives

$$\mathrm{lev}(m,h)\,\Psi(c,\beta)\,\mathrm{lev}(m,h)^{-1}=\Psi(hc\bar m,\ m\bar m\beta). \tag{6.9}$$

Step 3: (2). $g\in \mathrm{G}(t')$ lies in P_X iff $\mathrm{pr}_l gk_{\mathbb A}=0$ and $\mathrm{pr}_i gk_{\mathbb A}=0$: then $gk_{\mathbb A}=k_{\mathbb A}m$, $m=\mathrm{pr}_k gk_{\mathbb A}$, and m is a unit since $(gl_{\mathbb A})^{*}\varepsilon k_{\mathbb A}m=l_{\mathbb A}^{*}\varepsilon k_{\mathbb A}=1$. So P_X is closed, and a subgroup. For $g\in P_X$, $\mathrm{pr}_l gi_{\mathbb A}=(k_{\mathbb A}m^{-1})^{*}\varepsilon i_{\mathbb A}=0$, so $gi_{\mathbb A}=i_{\mathbb A}h+k_{\mathbb A}y$ with $h=\mathrm{pr}_i gi_{\mathbb A}$, and $\varepsilon_t=(gi_{\mathbb A})^{*}\varepsilon(gi_{\mathbb A})=h^{*}\varepsilon_t h$: $h\in \mathrm{G}(t)$. Then $n=g\,\mathrm{lev}(m,h)^{-1}$ fixes $k_{\mathbb A}$ and $ni_{\mathbb A}=i_{\mathbb A}+k_{\mathbb A}yh^{-1}$, so $n\in N_X$ by (1). If $n\,\mathrm{lev}(m,h)=n'\mathrm{lev}(m',h')$, then $\mathrm{lev}(m'^{-1}m,h'^{-1}h)\in N_X$, so $(m,h)=(m',h')$ and $n=n'$.

Step 4: (3). Let dc, $d\beta$ be Haar measures of $\mathrm{M}_{r\times 1}(\mathbb A_E)=\mathbb A_E^{p_0}$ and $\mathbb A_F$, and dn the image of $dc\,d\beta$ under the homeomorphism Ψ. By (6.8), a left (or right) translation is, in the coordinates, a translation in c followed, for each c, by a translation in β; it preserves $dc\,d\beta$ by Fubini. So dn is a left and right Haar measure, N_X is unimodular, and every Haar measure is a multiple of dn. By (6.9) conjugation by $\mathrm{lev}(m,h)$ is, in the coordinates, $(c,\beta)\mapsto(hc\bar m,m\bar m\beta)$. The first map is given by the matrix $h\bar m\in \mathrm{GL}_{p_0}(\mathbb A_E)$ and multiplies dc by $|\det h|\,|\bar m|^{p_0}=|m|^{p_0}$, since $\det h\overline{\det h}=1$ and conjugation preserves the module; the second multiplies $d\beta$ by $|m\bar m|_{\mathbb A_F}=|m|$. Hence $\int f(\mathrm{lev}\,n\,\mathrm{lev}^{-1})\,dn=|m|^{-(p_0+1)}\int f\,dn$.

Step 5: (4). Let $\gamma\in\Gamma_{V'}$, $\delta=x\gamma x^{-1}\in \mathrm{U}(V')(F)$. Then $\gamma k_{\mathbb A}=x^{-1}\delta k$, and as δk and $k\neq 0$ are rational, $\delta k\in k\mathbb A_E^{\times}$ iff $\delta k\in kE^{\times}$; so $\Gamma_{P,X}=P_X\cap\Gamma_{V'}$. Using $xk_{\mathbb A}=k$, $\mathrm{pr}_k x^{-1}=l^{*}J_{V'}$, $\mathrm{pr}_l x^{-1}=k^{*}J_{V'}$, $xi_{\mathbb A}=ix_V$, $\mathrm{pr}_i x^{-1}=x_V^{-1}J_V^{-1}i^{*}J_{V'}$ (from $\varepsilon_t x_V^{*}=x_V^{-1}J_V^{-1}$), one gets for $\xi\in E^{\times}$, $\gamma_0\in\Gamma_V$

$$x\,\mathrm{lev}(\xi,\gamma_0)\,x^{-1}=k\xi l^{*}J_{V'}+i\,(x_V\gamma_0x_V^{-1})\,J_V^{-1}i^{*}J_{V'}+l\bar\xi^{-1}k^{*}J_{V'}\in \mathrm{GL}_{r'}(E),$$

which lies in $\mathrm{U}(V')(F)$ because $\mathrm{lev}(\xi,\gamma_0)\in \mathrm{G}(t')$; so $\mathrm{lev}(\xi,\gamma_0)\in\Gamma_{P,X}$, and the map lands in $\Gamma_{P,X}$. Conversely, for $\gamma\in\Gamma_{P,X}$, (2) gives $\gamma=n\,\mathrm{lev}(\xi,h)$ with $\xi=\mathrm{pr}_k\gamma k_{\mathbb A}\in E^{\times}$ and $h=\mathrm{pr}_i\gamma i_{\mathbb A}=x_V^{-1}(J_V^{-1}i^{*}J_{V'}\delta i)x_V$, so $x_Vhx_V^{-1}$ has entries in E and, h lying in $\mathrm{G}(t)$, $h\in\Gamma_V$; then $\nu=\mathrm{lev}(\xi,h)^{-1}\gamma=\mathrm{lev}(\xi,h)^{-1}n\,\mathrm{lev}(\xi,h)\in N_X\cap\Gamma_{V'}$. Injectivity: $\mathrm{lev}(\xi,\gamma_0)\nu=\mathrm{lev}(\xi',\gamma_0')\nu'$ gives $\mathrm{lev}(\xi'^{-1}\xi,\gamma_0'^{-1}\gamma_0)\in N_X$, so $(\xi,\gamma_0)=(\xi',\gamma_0')$ by (1).

Step 6: (5). For $u\in E^{p_0}$ and $\beta_0\in F$, $\Psi(x_V^{-1}u,\beta_0)\in\Gamma_{V'}$: conjugated by x, the three terms of N become $-iuk^{*}J_{V'}$, $ku^{*}i^{*}J_{V'}$ and $-k\frac12(\jmath^{-1}\beta_0+u^{*}J_Vu)k^{*}J_{V'}$, all rational. As $\mathbb A_E/E$ and $\mathbb A_F/F$ are compact, there are compact $C_0\subset\mathbb A_E^{p_0}$ and $C_F\subset\mathbb A_F$ with $\mathbb A_E^{p_0}=E^{p_0}+C_0$ and $\mathbb A_F=F+C_F$. Let $n=\Psi(c,\beta)$; choose u with $c-x_V^{-1}u\in C_1:=x_V^{-1}C_0$. By (6.8), $\Psi(c',0)^{-1}=\Psi(-c',0)$ (z is then real) and $\Psi(-x_V^{-1}u,0)\,n=\Psi(c-x_V^{-1}u,\beta+\varrho)$ for some $\varrho\in\mathbb A_F$; choose $\beta_0\in F$ with $\beta+\varrho-\beta_0\in C_F$; then $\Psi(0,-\beta_0)\Psi(-x_V^{-1}u,0)\,n\in\Psi(C_1\times C_F)=:C$, compact.

Step 7: (6). Let v be a place of F; write G_v, K_v for the components at v of $\mathrm{G}(t')$ and $\mathrm{K}_{\max}(t')$, and P_v for the stabiliser in G_v of the line spanned by $(k_{\mathbb A})_v$. For $v\in\Phi$, $G_v=\mathrm{U}(\mathrm{diag}\,t')$, the basis $e_1=(k_{\mathbb A})_v$, $f_1=(l_{\mathbb A})_v$, $u_c=$ the columns of $(i_{\mathbb A})_v$ has the relations of (i) of Lemma 6.2.17 for the form $z^{*}\mathrm{diag}(t')y$ (first paragraph), it is orthonormal for the standard form (base-adapted), so $K_v=\{u\in G_v:u^{*}u=1\}$ is the stabiliser of the majorant, and the signature (p,q) has $\min(p,q)\geq 1$ (the span of e_1,f_1 has signature $(1,1)$); (i) with $J=1$ gives $G_v=P_vK_v$. For finite v, $E\otimes F_v$ is an unramified field or $F_v\times F_v$ (Lemma 3.2.9), G_v is the unitary group of $z^{*}y$ on $(E\otimes F_v)^{r'}$, $K_v=G_v\cap\mathrm{GL}_{r'}(\mathcal O_{E,v})$ is the stabiliser of the self-dual lattice $\mathcal O_{E,v}^{r'}$, and the line of $(k_{\mathbb A})_v$ is free of rank one and isotropic ($a(k_{\mathbb A})_v=0$ implies $a=a\,(l_{\mathbb A})_v^{*}(k_{\mathbb A})_v=0$); (ii)

gives $G_v = P_v K_v$. Let $g \in \mathrm{G}(t')$ and R the finite set of finite v with $g_v \notin \mathrm{GL}_{r'}(\mathcal{O}_{E,v})$. Write $g_v = p_v \kappa_v$ for $v \in \Phi \cup R$, and put $p_v = 1$, $\kappa_v = g_v \in K_v$ otherwise. Then $p = (p_v) \in \mathrm{G}(t')$, $\kappa = (\kappa_v) \in \mathrm{K}_{\max}(t')$, $g = p\kappa$, and $p_v (k_{\mathbb{A}})_v = (k_{\mathbb{A}})_v m_v$ with $m_v \in (E \otimes F_v)^\times$, $m_v = 1$ for $v \notin \Phi \cup R$; so $m \in \mathbb{A}_E^\times$ and $p \in P_X$. □

Lemma 6.2.17. *All symbols introduced here are local to this statement and to the results that cite it. Let $j = (j_1, \dots, j_a)$ be a composition of $J \geq 1$.*

(i) Real place. *Let $V_v = \mathbb{C}^n$ carry a hermitian form h_v of signature (p, q) with $J \leq \min(p, q)$, and let $(e_1, \dots, e_J, u_1, \dots, u_{n-2J}, f_1, \dots, f_J)$ be a basis with $h_v(e_s, f_t) = \delta_{st}$, $h_v(e_s, e_t) = h_v(f_s, f_t) = 0$, $h_v(e_s, u_c) = h_v(f_s, u_c) = 0$, $h_v(u_c, u_d) = \pm\delta_{cd}$. Let h_v^+ be the positive definite hermitian form for which this basis is orthonormal (it is a majorant of h_v), $G_v = \mathrm{U}(V_v, h_v)$, K_v the stabiliser of h_v^+ in G_v, W_i the span of $e_1, \dots, e_{j_1 + \cdots + j_i}$, and P_v the stabiliser of $W_\bullet$ in G_v. Let dg be the Haar measure of G_v and $d_l p$ the left Haar measure of P_v of density 1 at 1 with respect to the Lebesgue measures of $\langle Y, Z \rangle = \mathrm{Re}\,\mathrm{tr}(Y Z^*)$ (Z^* the h_v^+-adjoint) on their Lie algebras, and dk the Haar probability measure of K_v. Then $G_v = P_v K_v$ and for every Borel $\varphi \colon G_v \to [0, \infty]$*

$$\int_{G_v} \varphi \, dg = c_{\mathrm{Iw}} \int_{P_v} \int_{K_v} \varphi(pk) \, dk \, d_l p, \qquad c_{\mathrm{Iw}} = \frac{2^{-d(j)/2} \operatorname{vol} \mathrm{U}(p) \operatorname{vol} \mathrm{U}(q)}{\operatorname{vol} \mathrm{U}(p - J) \operatorname{vol} \mathrm{U}(q - J) \prod_{i=1}^{a} 2^{j_i^2/2} \operatorname{vol} \mathrm{U}(j_i)}.$$

(ii) Finite place. *Let F_v be a nonarchimedean local field, E_v an unramified quadratic extension of F_v or $F_v \times F_v$, V_v a free E_v-module of rank n with a nondegenerate hermitian form h_v, L_v a self-dual lattice, $G_v = \mathrm{U}(V_v, h_v)$, K_v the stabiliser of L_v, $W_\bullet$ an isotropic flag of type j in V_v (free submodules, $\operatorname{rank} W_i - \operatorname{rank} W_{i-1} = j_i$, $h_v(W_a, W_a) = 0$), and P_v its stabiliser. Let dg be the Haar measure of G_v with $\int_{K_v} dg = 1$, $d_l p$ the left Haar measure of P_v with $\int_{P_v \cap K_v} d_l p = 1$, and dk the Haar probability measure of K_v. Then $G_v = P_v K_v$ and $\int_{G_v} \varphi \, dg = \int_{P_v} \int_{K_v} \varphi(pk) \, dk \, d_l p$ for every Borel $\varphi \geq 0$.*

Proof. **A general principle.** Let G be a unimodular locally compact group, $K \subseteq G$ a compact subgroup and $P \subseteq G$ a closed subgroup with $G = PK$; put $K_P = P \cap K$ (compact). By Weil's integration formula there is a G-invariant Radon measure dy on G/K with $\int_G \varphi \, dg = \int_{G/K} \int_K \varphi(gk) \, dk \, dy(gK)$, dk the Haar probability measure; likewise $\int_P \psi \, d_l p = \int_{P/K_P} \int_{K_P} \psi(pl) \, dl \, dz(pK_P)$ for a P-invariant Radon measure dz on P/K_P (dl the Haar probability measure; the modular function of P is trivial on the compact K_P). The map $\iota \colon P/K_P \to G/K$, $pK_P \mapsto pK$, is a P-equivariant homeomorphism ($G = PK$, $P \cap K = K_P$, and $P/K_P \to G/K$ is a continuous bijection of homogeneous spaces of the σ-compact group P, hence open). A P-invariant Radon measure on P/K_P is unique up to a positive factor, so $\iota^* dy = c \, dz$ for some $c > 0$. For $\varphi \geq 0$ put $\Psi(p) = \int_K \varphi(pk) \, dk$, which is right K_P-invariant; then

$$\int_G \varphi \, dg = \int_{G/K} \Psi \, dy = c \int_{P/K_P} \Psi \, dz = c \int_P \Psi \, d_l p = c \int_P \int_K \varphi(pk) \, dk \, d_l p. \tag{$*$}$$

(ii). K_v is a special (indeed hyperspecial) maximal compact subgroup of G_v, so $G_v = P_v K_v$ for every parabolic subgroup P_v by the Iwasawa decomposition of Bruhat–Tits ([309], 3.3.2); P_v, the stabiliser of an isotropic flag, is parabolic. Apply $(*)$ to $\varphi = \mathbf{1}_{K_v}$: the left side is 1 and the right side is $c \int_{P_v} \int_{K_v} \mathbf{1}_{K_v}(pk) \, dk \, d_l p = c \int_{P_v \cap K_v} d_l p = c$. So $c = 1$.

(i): the Lie algebras. Order the basis as $(E_1, \dots, E_a, U, F_a, \dots, F_1)$, where E_i, F_i are the spans of the e_s, f_s with $j_1 + \cdots + j_{i-1} < s \leq j_1 + \cdots + j_i$ and U the span of the u_c. The h_v-orthogonal of W_i is $E_1 \oplus \cdots \oplus E_a \oplus U \oplus F_a \oplus \cdots \oplus F_{i+1}$, so P_v is the group of block upper triangular elements of G_v. Let H be the matrix of h_v in this basis; it is the matrix

of the involution σ with $h_v(y,z) = h_v^+(\sigma y, z)$, $H = H^* = H^{-1}$, and $\sigma(e_s) = f_s$, $\sigma(f_s) = e_s$, $\sigma(u_c) = \pm u_c$. The Lie algebra is $\mathfrak{g} = \{X : X^*H + HX = 0\}$; it is stable under $\theta(X) = -X^*$ (if $X^*H = -HX$ then $HX^*H = -X$, i.e. $XH + HX^* = 0$) and under the projections of $\mathfrak{gl}_n$ onto the strictly block upper, block diagonal and strictly block lower parts (the condition is $X = -HX^*H$, and $X \mapsto -HX^*H$ respects the three parts since H reverses the order of the blocks). Hence $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \bar{\mathfrak{n}}$ with $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{m}$ the Lie algebra of P_v, $\mathfrak{m}$ the block diagonal elements, $\theta(\mathfrak{n}) = \bar{\mathfrak{n}}$, and the three parts are pairwise orthogonal for $\langle \cdot, \cdot \rangle$ (disjoint supports of matrix entries). Further $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, $\mathfrak{k} = \{\theta X = X\}$ the Lie algebra of K_v, $\mathfrak{s} = \{\theta X = -X\}$, orthogonal, and $\mathfrak{m} = \mathfrak{k}_M \oplus \mathfrak{s}_M$ with $\mathfrak{k}_M = \mathfrak{m} \cap \mathfrak{k}$, $\mathfrak{s}_M = \mathfrak{m} \cap \mathfrak{s}$. Counting dimensions, $\dim \mathfrak{m} = 2\sum_i j_i^2 + (n-2J)^2$ and $\dim \mathfrak{n} = (n^2 - \dim \mathfrak{m})/2 = d(j)$.

(i): the compact groups. K_v commutes with σ, so it is $\mathrm{U}(V_v^+) \times \mathrm{U}(V_v^-)$ for the ± 1 eigenspaces of σ, of dimensions p, q, and its Lie algebra carries the Frobenius metric on each factor: in the metric of $\langle \cdot, \cdot \rangle$ it has volume $\mathrm{vol}\,\mathrm{U}(p)\,\mathrm{vol}\,\mathrm{U}(q)$. An element $k \in K_v \cap P_v$ preserves $W_{i-1} \subseteq W_i$ and is h_v^+-unitary, so it preserves E_i, hence $\sigma(E_i) = F_i$, hence U; so $K_M := K_v \cap P_v$ consists of the block diagonal elements $(k_1, \dots, k_a, k_0, k_a', \dots, k_1')$ with $k_i \in \mathrm{U}(j_i)$, $k_i' = (k_i^*)^{-1} = k_i$ in the f-basis, and $k_0 \in \mathrm{U}(p-J) \times \mathrm{U}(q-J)$. The element $X \in \mathfrak{u}(j_i)$ occurs twice, so its norm is $\sqrt{2}$ times its Frobenius norm; hence in the metric of $\langle \cdot, \cdot \rangle$, $\mathrm{vol}(K_M) = \mathrm{vol}\,\mathrm{U}(p-J)\,\mathrm{vol}\,\mathrm{U}(q-J) \prod_i 2^{j_i^2/2} \mathrm{vol}\,\mathrm{U}(j_i)$.

(i): the constant. $G_v = P_v K_v$ by the Iwasawa decomposition of the real reductive group G_v ([163], Chapter V): the stabiliser $P_{\min}$ of a maximal isotropic flag refining $W_\bullet$ is a minimal parabolic subgroup contained in P_v, the Iwasawa decomposition gives $G_v = K_v P_{\min}$ for the maximal compact subgroup K_v, and taking inverses $G_v = P_{\min} K_v \subseteq P_v K_v$. Give G_v the left-invariant Riemannian metric equal to $\langle \cdot, \cdot \rangle$ at 1; as $\mathrm{Ad}(K_v)$ preserves $\langle \cdot, \cdot \rangle$ it is also right K_v-invariant, and its volume measure is dg. Give G_v/K_v the G_v-invariant metric equal to $\langle \cdot, \cdot \rangle|_{\mathfrak{s}}$ at the base point. Then $G_v \to G_v/K_v$ is a Riemannian submersion whose fibres gK_v are isometric (by left translation) to K_v with its bi-invariant metric, and the coarea formula gives $(*)$'s first identity with dy the Riemannian measure of G_v/K_v times $\mathrm{vol}(K_v)$. In the same way ($\mathrm{Ad}(K_M)$ preserves $\mathfrak{p}$ and the inner product) dz is $\mathrm{vol}(K_M)$ times the Riemannian measure of P_v/K_M for the metric $\langle \cdot, \cdot \rangle|_{\mathfrak{p} \ominus \mathfrak{k}_M}$. Both Riemannian measures are P_v-invariant, so the ratio of ι^* of the first to the second is the constant $|\det d\iota|$ at the base point. There $d\iota$ is the restriction to $\mathfrak{p} \ominus \mathfrak{k}_M = \mathfrak{n} \oplus \mathfrak{s}_M$ of the orthogonal projection onto $\mathfrak{s}$: it is the identity on $\mathfrak{s}_M$, and $Y \in \mathfrak{n}$ goes to $(Y - \theta Y)/2$ (as $Y = \frac{1}{2}(Y + \theta Y) + \frac{1}{2}(Y - \theta Y)$ with the parts in $\mathfrak{k}$, $\mathfrak{s}$). For an orthonormal basis $Y_1, \dots, Y_d$ of $\mathfrak{n}$,

$$\left\langle \tfrac{Y_s - \theta Y_s}{2}, \tfrac{Y_t - \theta Y_t}{2} \right\rangle = \tfrac{1}{4}\big(\langle Y_s, Y_t \rangle + \langle \theta Y_s, \theta Y_t \rangle\big) = \tfrac{1}{2}\delta_{st},$$

since $\mathfrak{n} \perp \bar{\mathfrak{n}}$ and θ is an isometry, and these images are orthogonal to $\mathfrak{s}_M \subseteq \mathfrak{m}$. Hence $|\det d\iota| = 2^{-d(j)/2}$, and

$$\iota^* dy = \mathrm{vol}(K_v) \cdot 2^{-d(j)/2} \cdot \frac{dz}{\mathrm{vol}(K_M)},$$

so $c = 2^{-d(j)/2} \mathrm{vol}(K_v)/\mathrm{vol}(K_M) = c_{\mathrm{Iw}}$ by the two volumes above, and $(*)$ is the claim. □

Lemma 6.2.18. *Let W', W, V' and A be as above, $\mathfrak{c} = (\nu, \Sigma)$ a split tame character, b an index set with a bijection $\sigma' \colon b \sqcup r' \to n'$, $v \in \mathcal{F}_{W',V'}$, $y \in \mathrm{M}_{b \times n}(E_\Sigma)$, and Z_S the zero term along the splitting S of $\mathrm{Hom}(W', V')$ over $\mathrm{Hom}(W, V')$ induced by A. Then:*

1. *$Z_S(\varphi^{\mathrm{pl}}_{A,\sigma'}(v,y)) = 0$;*
2. *for all automorphic characters a, b of $\mathrm{U}(1)$, $u \in \mathbb{A}_E^\times$, $g \in \mathrm{G}(s)$ and $h \in \mathrm{G}(t')$,*

$$\mathrm{ct}_A\Big(0;\ \omega_{a,b}\big(\mathrm{lev}_A(u,g), h\big)\varphi^{\mathrm{pl}}_{A,\sigma'}(v,y)\Big) = 0.$$

Proof. The function $\Phi = \Phi^{\mathrm{st}}_{\sigma',y(\mathrm{pr}_i)_\Sigma}$ is locally constant with compact support (it vanishes unless the stacked matrix is integral, and is then a function of its reduction modulo the uniformisers), so the plane datum $f = v \otimes [\Phi]_\Sigma$ is defined by Lemma 6.2.22. For adelic vectors y' of V' and z of W' write $\langle y', z\rangle = y'z^*\varepsilon_{s'}$, and let k_A, l_A, i_A be the adelic frames of A.

Step 1: S is adapted. By (MX4), the rational frames of S have the columns $J_{V'}^{-1} e_m\, k^* J_{W'}$, $e_m\, l^* J_{W'}$ ($m \in r'$, k, l the rational frames of A), and its embedding sends M_0 to $M_0 J_W^{-1} i^* J_{W'}$. In adelic coordinates, using $x^{-1}J^{-1} = \varepsilon x^*$ and $J_{W'}x_{W'} = (x_{W'}^{-1})^*\varepsilon_{s'}$, they are $\langle \varepsilon_{t'} x_{V'}^* e_m, k_A\rangle$, $\langle x_{V'}^{-1} e_m, l_A\rangle$, and the embedding sends the adelic unit matrix $e_m e_j^{\mathsf{T}}$ of $\mathrm{Hom}(W, V')$ to $\varepsilon_s(j)\langle e_m, i_A e_j\rangle$. Let $m_S = (\varepsilon_{t'} x_{V'}^*)^{-1} \in \mathrm{GL}_{r'}(\mathbb{A}_E)$ (the index set $r' \times \{o\}$ of S read as r'). Then $k_{\mathbb{A}} m_S$ has the columns $\langle e_m, k_A\rangle$ and $l_{\mathbb{A}} m_S^{-*} = l_{\mathbb{A}} x_{V'} \varepsilon_{t'}$ the columns $\varepsilon_{t'}(m)\langle e_m, l_A\rangle$. As in (MX5), these columns and the $\langle e_m, i_A e_j\rangle$ are, up to signs, the vectorised products of the orthonormal (at ∞) and integral (at the finite places) columns e_m and the columns of $(l_A \mid k_A \mid i_A)$, A being base-adapted; so S is adapted at 1 in the basis m_S.

Step 2: the standard datum vanishes on the frame at $x = 0$. At the finite places $\langle y', z\rangle = y'z^*$, and with $d_K = (\mathrm{pr}_k)_\Sigma = (l_A^*)_\Sigma$, $d_L = (\mathrm{pr}_l)_\Sigma = (k_A^*)_\Sigma$, $d_I = (\mathrm{pr}_i)_\Sigma = (i_A^*)_\Sigma$, the frame $\mathcal{F}_\Sigma$ of S in the basis m_S sends $(0, z, Y)$ ($z \in E_\Sigma^{r'}$, $Y \in \mathrm{M}_{r'\times n}(E_\Sigma)$) to $z\, d_L + Y\, d_I$. The matrix of (XD10) for Φ at this point has rows $y\, d_I$ and $z\, d_L + Y\, d_I$; it equals, up to a permutation of rows, $\begin{pmatrix} 0 & y & 0 \\ 0 & Y & z \end{pmatrix} R$ with $R = (d_K; d_I; d_L)$, whose first column is zero. Its determinant is 0, not a unit, so $\Phi(\mathcal{F}_\Sigma(0, z, Y)) = 0$ for all z, Y.

Step 3: proof of (1). Apply Lemma 6.2.19 to S, the basis m_S, the point $x = 0$, Φ, $\Psi = 0$ and $f_i = 1$: its hypothesis holds by Step 2, and the slice at $m_S^{-*}(\alpha_0 \cdot 0) = 0$ is $Z_S(\rho(0)f) = Z_S(f)$. So $Z_S(f)$ is a multiple of $\mathrm{rsl}_{S,m_S}(0; v) \otimes [0]_\Sigma = 0$ ($\Psi \mapsto \Psi \otimes_\Sigma v^0$ is linear).

Step 4: proof of (2). Let $F = \omega_{a,b}(\mathrm{lev}_A(u, g), h)f$. By (MX2), $\gamma = \mathrm{lev}_A(u, g)$ satisfies $\gamma k_A = k_A u$ and $\mathrm{pr}_i \gamma i_A = g$, so (2) of Lemma 6.2.45 at $x = 0$ (characters $a\upsilon^{-1}$ and b) gives $Z_S(F) = b[\#r'](u)^{-1}|u|^{\#r'/2}\omega_{a\upsilon^{-1},b}(g, h)Z_S(f) = 0$ by (1). For $q \in \mathrm{M}_{n\times 1}(\mathbb{A}_E)$ the element $\eta = \mathrm{rad}_A^{\mathrm{ln}}(q, 0) = \mathrm{rad}_A(x_W^{-1}q, 0)$ satisfies $\eta k_A = k_A$ and $\mathrm{pr}_i \eta i_A = 1$ ((MX2): $\mathrm{pr}_l k_A = \mathrm{pr}_i k_A = 0$, $\mathrm{pr}_l i_A = 0$, $\mathrm{pr}_i i_A = 1$), so (2) of Lemma 6.2.45 at $x = 0$ with the characters υ^{-1} and 1 gives $Z_S(\omega_{1,1}(\eta, 1)F) = Z_S(F) = 0$. The theta term of (MX6) at $x = 0$ is $\theta_{\mathrm{Hom}(W,V')}((\omega_{1,1}(\eta, 1)F)[0]) = \theta_{\mathrm{Hom}(W,V')}(Z_S(\omega_{1,1}(\eta, 1)F)) = 0$, so the integrand of (MX10) vanishes and $\mathrm{ct}_A(0; F) = 0$. □

Lemma 6.2.19. *Let W', T', W, T be spaces with points with signs on n', r', n, r, and D a splitting of $\mathrm{Hom}(W', T')$ over $\mathrm{Hom}(W, T)$ along a nonempty index set p which is adapted at 1 in the basis $m \in \mathrm{GL}_p(\mathbb{A}_E)$. Let $\mathcal{F} = (l_{\mathbb{A}} m^{-*} \mid k_{\mathbb{A}} m \mid i_{\mathbb{A}})$ be its frame, with column blocks indexed by p, p and $r \times n$, and let Σ be a nonempty finite set of finite places of E containing the conjugate of none of its members. Let $\mathcal{F}_\Sigma$ be the matrix of the components at Σ of the finite part of $\mathcal{F}$, $\psi_\Sigma(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$ for $t \in E_\Sigma$ (read as a finite adele), dt the Haar probability measure on $\mathcal{O}_\Sigma$, and $\alpha_0 \in \mathbb{A}_E^\times$ the idele with components κ_v at $v \in \Phi$ and finite part α. Let $v \in \mathcal{F}_{W',T'}$, let $x \in \mathbb{A}_E^p$ have infinite part $x_\infty \in \mathbb{C}^{\Phi\times p}$ and finite part x_f, let Φ and Ψ be functions on the $r' \times n'$, resp. $r \times n$, matrices over E_Σ with $\Phi^\sharp, \Psi^\sharp \in \mathcal{T}_\Sigma$, and let $f_i \colon E_\Sigma \to \mathbb{C}$ ($i \in p$) be continuous on $\mathcal{O}_\Sigma$, such that*

$$\Phi(\mathcal{F}_\Sigma(x_{f,\Sigma}, z, Y)) = \prod_{i\in p} \mathbf{1}_{\mathcal{O}_\Sigma}(z_i)\, f_i(z_i) \cdot \Psi(Y) \qquad (z \in E_\Sigma^p,\ Y \in \mathrm{M}_{r\times n}(E_\Sigma)),$$

where $x_{f,\Sigma}$ is the vector of the components of x_f at Σ and the value of $\mathcal{F}_\Sigma$ is read as an $r' \times n'$

matrix. Then

$$(v\otimes[\Phi]_\Sigma)\big[m^{-*}(\alpha_0 x)\big] = c\,\mathrm{rsl}_{D,m}(x_\infty;v)\otimes[\Psi]_\Sigma,$$
$$c = \mathbf{1}\big[x_f \text{ integral outside } \Sigma\cup\bar\Sigma\big]\,\mathrm{vf}(D)\prod_{i\in p}\int_{\mathcal{O}_\Sigma}\psi_\Sigma\big((\bar x_{f,i})_\Sigma\, t\big)f_i(t)\,dt,$$

where $(\bar x_{f,i})_\Sigma$ is the vector of the components at Σ of the conjugate of the adele $x_{f,i}$.

Proof. Write $V=\mathrm{Hom}(W',T')$, $V_0=\mathrm{Hom}(W,T)$, Z for the zero term and $h[x']$ for the slices of D; $\mathcal{F}_f$ is the finite part of $\mathcal{F}$. Put $\psi_0(t)=\psi_{E,f}(\jmath\alpha\bar\alpha t)$ for finite adeles t (so $\psi_\Sigma=\psi_0$ on E_Σ). For a finite adelic vector u and a set S of places, u_S keeps the components at S and sets the others to 0; $u^*u'=\sum_j \bar u_j u'_j$ (a finite adele). Recall that $\Sigma\cap\bar\Sigma=\emptyset$.

Step 1: three facts on the finite frame and on ψ_0. (a) At the finite places $\varepsilon_s=1$, so $x^*_{V,f}J_V x_{V,f}=1$ and $(x^{-1}_{V,f}u)^*(x^{-1}_{V,f}u')=u^*J_V u'$; with the relations of (MX1) and $x^*_{V_0,f}J_{V_0}x_{V_0,f}=1$ this gives $\mathcal{F}_f^*\mathcal{F}_f=Q_0$, the matrix of the form $(a,b,w;a',b',w')\mapsto a^*b'+b^*a'+w^*w'$. Since D is adapted at 1 in the basis m, $\mathcal{F}_f$ has entries in $\widehat{\mathcal{O}}_E$, and so has $\mathcal{F}_f^{-1}=Q_0\mathcal{F}_f^*$. Hence, placewise, $u\mapsto\mathcal{F}_f u$ is a bijection of the vectors supported on a set S of finite places that preserves integrality at every place, and for $u=(a,b,w)$, $u'=(a',b',w')$,

$$(\mathcal{F}_f u)^*(\mathcal{F}_f u') = a^*b'+b^*a'+w^*w'. \tag{1}$$

(b) $\psi_0(t+\bar t)=1$ for every finite adele t: with $u=\jmath\alpha\bar\alpha t$ we have $\jmath\alpha\bar\alpha\bar t=-\bar u$ ($\bar\jmath=-\jmath$), and $\psi_{E,f}(\bar u)=\psi_{E,f}(u)$ because $\widehat{\mathcal{O}}_E$ is stable under conjugation and $\mathrm{Tr}_{E/\mathbb{Q}}(\bar a)=\mathrm{Tr}_{E/\mathbb{Q}}(a)$. (c) By Lemma 4.1.3, $\widehat{\mathcal{O}}_E^{\,p}$ is its own annihilator under $(u,u')\mapsto\psi_0(u^*u')$. Hence for a finite adelic vector c, $\int_{\mathcal{O}_\Sigma^p}\psi_0(c^*t)\,dt$ is 1 if c is integral at $\bar\Sigma$ and 0 otherwise: c^*t for t supported on Σ only involves $c_{\bar\Sigma}$, the character $t\mapsto\psi_0(c^*_{\bar\Sigma}t)$ is trivial on $\mathcal{O}_\Sigma^p$ exactly when it is trivial on $\widehat{\mathcal{O}}_E^{\,p}$ (the pairing with $c_{\bar\Sigma}$ only sees the components at Σ), that is when $c_{\bar\Sigma}$ is integral, and the integral of a nontrivial character over a compact group is 0.

Step 2: the real part. Since $\alpha_0 x=\mathrm{rv}(x_\infty)+\mathrm{fv}(x_f)$, the point $m^{-*}(\alpha_0 x)$ is $m^{-*}\mathrm{rv}(x_\infty)+y_f$ with $y_f=\alpha m^{-*}x_f$, whose infinite components are 0. Let $h=\Phi^\sharp\otimes_\Sigma v^0$. By (3) of Lemma 6.2.20 (applicable: $p\neq\emptyset$, D adapted at 1 in the basis m),

$$(v\otimes[\Phi]_\Sigma)\big[m^{-*}(\alpha_0x)\big]=\mathrm{rsl}_{D,m}(x_\infty;v)\otimes(v_1\otimes h)[y_f].$$

By (3) of Lemma 4.1.3, h is a finite combination of vectors $\rho(\mathrm{fv}(y))v^0$, and $v_1\otimes\rho(\mathrm{fv}(y))v^0=\rho(\mathrm{fv}(y))(v_1\otimes v^0)=\rho(\mathrm{fv}(y))v^0$ by (3), (4) of Lemma 6.2.21; so $v_1\otimes h=h$. It remains to show

$$h[y_f]=c\;\Psi^\sharp\otimes_\Sigma v^0, \tag{2}$$

because then the right side above is $c\,\mathrm{rsl}_{D,m}(x_\infty;v)\otimes(\Psi^\sharp\otimes_\Sigma v^0)=c\,\mathrm{rsl}_{D,m}(x_\infty;v)\otimes[\Psi]_\Sigma$.

Step 3: the slice of one finite translate of the vacuum. Let y be a finite adelic vector of V supported on $\Sigma\cup\bar\Sigma$ and $(a,b,w)=\mathcal{F}_f^{-1}y$, so $y=l_{\mathbb{A}}m^{-*}a+k_{\mathbb{A}}mb+i_{\mathbb{A}}w$ at the finite places and a,b,w are supported on $\Sigma\cup\bar\Sigma$. Then $\mathrm{fv}(y)=l_{\mathbb{A}}A+k_{\mathbb{A}}B+i_{\mathbb{A}}\mathrm{fv}(w)$ with $A=\alpha m^{-*}a$, $B=\alpha mb$. The mixed law ((1) of Lemma 6.2.44 for $\Pi=Z$) at $y_f=\alpha m^{-*}x_f$ gives, since $B^*(2y_f+A)=\alpha\bar\alpha\,b^*(2x_f+a)$ and $\psi_E=\psi_{E,f}$ on finite adeles,

$$(\rho(\mathrm{fv}(y))v^0)[y_f]=\psi_0\big(\tfrac12 b^*(2x_f+a)\big)\,\rho(\mathrm{fv}(w))\,v^0\big[\alpha m^{-*}(x_f+a)\big].$$

By Lemma 6.2.11 at $g=1$, $v^0[x']=C_1|e_p(m^*x')|^2v^0$, and $C_1=\mathrm{vf}(D)$ by (4) of Lemma 6.2.20 (take $x'=0$). At a finite adelic point $|e_p(\alpha u)|^2=\mathbf{1}[u\in\widehat{\mathcal{O}}_E^{\,p}]$ by (WD5). Hence

$$(\rho(\mathrm{fv}(y))v^0)[y_f]=\mathrm{vf}(D)\,\mathbf{1}\big[x_f+a\in\widehat{\mathcal{O}}_E^{\,p}\big]\,\psi_0\big(b^*x_f+\tfrac12 b^*a\big)\,\rho(\mathrm{fv}(w))v^0. \tag{3}$$

Step 4: the partial Fourier integral of one translate. For a function Θ on $r' \times n'$ matrices over E_Σ with $\Theta^\sharp \in \mathcal{T}_\Sigma$ (a smooth compactly supported function, (1) of Lemma 4.1.3) put

$$\Theta_{x_f}(Y) = \int_{E_\Sigma^p} \psi_0(x_f^* t)\, \Theta\big(\mathcal{F}_\Sigma(x_{f,\Sigma}, t, Y)\big)\, dt \qquad (Y \in \mathrm{M}_{r\times n}(E_\Sigma)),$$

dt the Haar measure of E_Σ^p giving $\mathcal{O}_\Sigma^p$ mass 1; $\Theta \mapsto \Theta_{x_f}$ is linear, and $\psi_0(x_f^* t) = \prod_i \psi_\Sigma((\bar{x}_{f,i})_\Sigma t_i)$. Let y, a, b, w be as in Step 3 and $\Theta^\sharp = \tau_y$, that is $\Theta(X) = \psi_0(\frac{1}{2} y^*(2X + y_\Sigma))\, \mathbf{1}[X + y_\Sigma \text{ integral}]$ for X supported on Σ. For $X = \mathcal{F}_\Sigma(x_{f,\Sigma}, t, Y)$ we have $X + y_\Sigma = \mathcal{F}_f((x_f + a)_\Sigma, t + b_\Sigma, Y + w_\Sigma)$, which by Step 1 (a) is integral exactly when its three coordinates are, and by (1)

$$y^*(2X + y_\Sigma) = a^*(2t + b_\Sigma) + b^*(2x_{f,\Sigma} + a_\Sigma) + w^*(2Y + w_\Sigma).$$

The terms in t of the integrand are $\psi_0((x_f + a)^* t)\, \mathbf{1}[t + b_\Sigma \in \mathcal{O}_\Sigma^p]$; substituting $t = t' - b_\Sigma$ and applying Step 1 (c) to $c = x_f + a$,

$$\Theta_{x_f}(Y) = \mathbf{1}\big[(x_f + a)_{\Sigma\cup\bar{\Sigma}} \text{ integral}\big]\, \psi_0(-x_f^* b_\Sigma - \tfrac{1}{2} a^* b_\Sigma + b^* x_{f,\Sigma} + \tfrac{1}{2} b^* a_\Sigma)\, \tau_w(Y),$$

where $\tau_w(Y) = \psi_0(w^* Y + \frac{1}{2} w^* w_\Sigma)\mathbf{1}[Y + w_\Sigma \text{ integral}]$ is the function τ_w of (WR6) for V_0, so that $(\Theta_{x_f})^\sharp \otimes_\Sigma v^0$ is the same scalar times $\tau_w \otimes_\Sigma v^0 = \rho(\mathrm{fv}(w)) P^\Sigma v^0 = \rho(\mathrm{fv}(w)) v^0$ ((1), (3) of Lemma 4.1.3).

Step 5: comparison. Since a is supported on $\Sigma \cup \bar{\Sigma}$,

$$\mathbf{1}[x_f + a \in \widehat{\mathcal{O}}_E^p] = \mathbf{1}[x_f \text{ integral outside } \Sigma \cup \bar{\Sigma}] \cdot \mathbf{1}[(x_f + a)_{\Sigma\cup\bar{\Sigma}} \text{ integral}].$$

The two phases, that of (3) and that of Step 4, differ by ψ_0 of

$$b^*(x_f - x_{f,\Sigma}) + x_f^* b_\Sigma + \tfrac{1}{2}(b^*(a - a_\Sigma) + a^* b_\Sigma) = (z + \bar{z}) + \tfrac{1}{2}(z' + \bar{z}'), \qquad z = x_{f,\bar{\Sigma}}^* b_\Sigma,\ z' = a_{\bar{\Sigma}}^* b_\Sigma,$$

because b is supported on $\Sigma \cup \bar{\Sigma}$ (so $b^*(x_f - x_{f,\Sigma}) = b_{\bar{\Sigma}}^* x_{f,\bar{\Sigma}} = \bar{z}$, and likewise for a), and by Step 1 (b) this is 1. Comparing (3) with Step 4,

$$(\tau_y \otimes_\Sigma v^0)[y_f] = (\rho(\mathrm{fv}(y)) v^0)[y_f] = \mathrm{vf}(D)\, \mathbf{1}[x_f \text{ integral outside } \Sigma\cup\bar{\Sigma}]\, (\Theta_{x_f})^\sharp \otimes_\Sigma v^0 \qquad (\Theta^\sharp = \tau_y),$$

the first equality by (WR6) with $P^\Sigma v^0 = v^0$ and the independence of the section ((1) of Lemma 4.1.3).

Step 6: proof of (2). Write $\Phi^\sharp = \sum_y c_y \tau_y$ (finitely many y supported on $\Sigma \cup \bar{\Sigma}$), so that $\Phi = \sum_y c_y \Theta_y$ with $\Theta_y^\sharp = \tau_y$. The zero term is linear and $\Theta \mapsto \Theta^\sharp \otimes_\Sigma v^0$ and $\Theta \mapsto \Theta_{x_f}$ are linear, so Step 5 gives $h[y_f] = \mathrm{vf}(D)\, \mathbf{1}[x_f \text{ integral outside } \Sigma \cup \bar{\Sigma}]\, (\Phi_{x_f})^\sharp \otimes_\Sigma v^0$. By the hypothesis on Φ, for $t \in E_\Sigma^p$ the integrand of $\Phi_{x_f}(Y)$ is $\prod_i \psi_\Sigma((\bar{x}_{f,i})_\Sigma t_i) \mathbf{1}_{\mathcal{O}_\Sigma}(t_i) f_i(t_i) \cdot \Psi(Y)$, so by Fubini $\Phi_{x_f} = \prod_i \int_{\mathcal{O}_\Sigma} \psi_\Sigma((\bar{x}_{f,i})_\Sigma t) f_i(t)\, dt \cdot \Psi$. This is (2). □

Lemma 6.2.20. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set, $D = (k, l, i)$ a splitting of V over V_0 along p with zero term Z, and $m \in \mathrm{GL}_p(\mathbb{A}_E)$ such that D is adapted at 1 in the basis m. Let $f \in \mathcal{M}_s$.*

1. *For $x \in \mathbb{A}_E^p$ with components 0 at the finite places, $(v_1 \otimes f)[x] = |e_p(m^* x)|^2\, (v_1 \otimes f)[0]$, and $v_1 \otimes ((v_1 \otimes f)[0]) = (v_1 \otimes f)[0]$ (the outer datum formed in $\mathcal{M}_{s_0}$).*
2. *For $v \in \mathcal{F}_s$, $Z(v \otimes f) = Z_{D,m}^\infty(v) \otimes Z(v_1 \otimes f)$.*
3. *For $v \in \mathcal{F}_s$, $b \in \mathbb{C}^{\Phi\times p}$ and $y \in \mathbb{A}_E^p$ with components 0 at the infinite places, $(v \otimes f)[m^{-*}\mathrm{rv}(b) + y] = \mathrm{rsl}_{D,m}(b; v) \otimes (v_1 \otimes f)[y]$.*

4. $Z(v^0) = \mathrm{vf}(D)\, v^0$ *and* $\mathrm{vf}(D) \neq 0$.

Proof. **Step 1: the real adapted frame.** Let $G_\infty = (l_{\mathbb{A}} m^{-*}\ k_{\mathbb{A}} m\ i_{\mathbb{A}})_\infty$, block diagonal over Φ; by adaptedness at 1 each block is square with orthonormal columns, hence unitary. Write $y \in \mathbb{C}^{\Phi\times n}$ as $y = G_\infty(a; b; w)$ with $a, b \in \mathbb{C}^{\Phi\times p}$, $w \in \mathbb{C}^{\Phi\times n_0}$. Then $\mathrm{rv}(y) = l_{\mathbb{A}} m^{-*}\mathrm{rv}(a) + k_{\mathbb{A}} m\,\mathrm{rv}(b) + i_{\mathbb{A}}\mathrm{rv}(w)$ (κ_w is a scalar), ν is invariant under G_∞, and $v_1(y) = v_1(a)v_1(b)v_1(w)$. For real vectors, $\psi_E(\frac{1}{2}\jmath\,\mathrm{rv}(b)^*\mathrm{rv}(c)) = \exp((b^*c - c^*b)/2)$ (all signs +1).

Step 2: (1). Let x be as in (1) and $X \in \mathbb{C}^{\Phi\times p}$ with $m^*x = \mathrm{rv}(X)$. By Step 1 and the mixed law with $A = m^{-*}\mathrm{rv}(a)$, $B = m\,\mathrm{rv}(b)$, $W = \mathrm{rv}(w)$, for which $B^*(2x + A) = \mathrm{rv}(b)^*\mathrm{rv}(2X + a)$,

$$\begin{aligned}(v_1 \otimes f)[x] &= \int v_1(y)\,(\rho(\mathrm{rv}(y))f)[x]\,d\nu(y)\\ &= \iint v_1(a)v_1(b)\,e^{(b^*(2X+a)-(2X+a)^*b)/2}\,P_0\Big(f\big[m^{-*}\mathrm{rv}(X+a)\big]\Big)\,d\nu(a)\,d\nu(b),\end{aligned}$$

where $P_0 h = v_1 \otimes h = \int v_1(w)\rho(\mathrm{rv}(w))h\,d\nu(w)$ on $\mathcal{M}_{s_0}$; Fubini applies, the integrand being bounded by $v_1(y)\|Z\|\|f\|$. The Fourier transform of the Gaussian gives $\int v_1(b)e^{(b^*c-c^*b)/2}d\nu(b) = 2^N e^{-|c|^2/2}$, $N = \#\Phi\,\#p$, and $|a|^2 + |2X + a|^2 = 2|X|^2 + 2|X + a|^2$; substituting $a \mapsto a - X$,

$$(v_1 \otimes f)[x] = 2^N e^{-|X|^2} K, \qquad K = \int e^{-|a|^2} P_0(f[m^{-*}\mathrm{rv}(a)])\,d\nu(a).$$

At $x = 0$ this is $2^N K$. Since P_0 is bounded and $P_0P_0 = P_0$ ((5) of Lemma 6.2.21 for s_0), $P_0K = K$. As $e^{-|X|^2} = |e_p(m^*x)|^2$ for x with finite components 0 ((WD5), $2\pi\,\mathrm{Im}(\jmath_w)\kappa_w^2 = 1$), (1) follows.

Step 3: (2). Let $F = v_1 \otimes f$; then $v \otimes f = v \otimes F$. For $y \in \mathbb{C}^{\Phi\times n}$ put $A = \mathrm{pr}_l\mathrm{rv}(y)$, $W = \mathrm{pr}_i\mathrm{rv}(y) = \mathrm{rv}(w(y))$, $B = \mathrm{pr}_k\mathrm{rv}(y)$, with $a(y)$, $w(y)$, $\chi(y)$ as in (MX7); then $m^*A = \mathrm{rv}(a(y))$ and $\psi_E(\frac{1}{2}\jmath B^*A) = \chi(y)$. By the mixed law at $x = 0$ and (1),

$$Z(v \otimes F) = \int v(y)\,\chi(y)\,\rho(W)F[A]\,d\nu(y) = \int v(y)\,\chi(y)\,e^{-a(y)^*a(y)}\,\rho(\mathrm{rv}(w(y)))F[0]\,d\nu(y).$$

On the other side, $h \mapsto h \otimes F[0]$ is bounded on $L^1(\nu_{\Phi\times n_0})$, which contains $\mathcal{F}_{s_0}$ with the same norm, so it commutes with the Bochner integral defining $Z^\infty_{D,m}(v)$, and

$$\begin{aligned}Z^\infty_{D,m}(v) \otimes F[0] &= \int v(y)\chi(y)e^{-a(y)^*a(y)}\,(\rho^\infty(w(y))v_1) \otimes F[0]\,d\nu(y)\\ &= \int v(y)\chi(y)e^{-a(y)^*a(y)}\,\rho(\mathrm{rv}(w(y)))\,(v_1 \otimes F[0])\,d\nu(y),\end{aligned}$$

and $v_1 \otimes F[0] = F[0]$ by (1). Since $F[0] = Z(v_1 \otimes f)$, (2) follows.

Step 4: (3). The vectors $l_{\mathbb{A}} m^{-*}\mathrm{rv}(b) = \mathrm{rv}((l_{\mathbb{A}} m^{-*})_\infty b)$ and $l_{\mathbb{A}} y$ have disjoint supports, so $\rho(l_{\mathbb{A}}(m^{-*}\mathrm{rv}(b) + y)) = \rho(\mathrm{rv}((l_{\mathbb{A}} m^{-*})_\infty b))\rho(l_{\mathbb{A}} y)$, and by the calculus of data

$$\rho\big(l_{\mathbb{A}}(m^{-*}\mathrm{rv}(b) + y)\big)(v \otimes f) = \big(\rho^\infty((l_{\mathbb{A}} m^{-*})_\infty b)v\big) \otimes \rho(l_{\mathbb{A}} y)f.$$

Applying Z and (2) to the Fock vector $\rho^\infty((l_{\mathbb{A}} m^{-*})_\infty b)v$ gives $\mathrm{rsl}_{D,m}(b; v) \otimes Z(v_1 \otimes \rho(l_{\mathbb{A}} y)f)$, and $v_1 \otimes \rho(l_{\mathbb{A}} y)f = \rho(l_{\mathbb{A}} y)(v_1 \otimes f)$, whose zero term is $(v_1 \otimes f)[y]$.

Step 5: (4). The splitting is adapted at $1 \in \mathrm{G}(s)$ in the basis m, so Lemma 6.2.11 gives $C \neq 0$ with $(v^0)[x] = C|e_p(m^*x)|^2 v^0$; at $x = 0$, $Z(v^0) = Cv^0$, and by (MX8) $\mathrm{vf}(D) = \langle v^0, Cv^0\rangle / \langle v^0, v^0\rangle = C$. □

Lemma 6.2.21. *Let s be signs on an index set n. For $w \in L^1(\nu)$ and $f \in \mathcal{M}_s$ put $w \otimes f = \int w(t)\,\rho(\mathrm{rv}(t))f\,d\nu(t)$.*

1. *This is a Bochner integral in $\mathcal{M}_s$, $\|w \otimes f\| \le \|w\|_{L^1(\nu)}\|f\|$, and for $w \in \mathcal{F}_\varepsilon$ it is the datum of (WR6).*
2. *For $t_0 \in \mathbb{C}^\iota$, $(\rho^\infty(t_0)w) \otimes f = \rho(\mathrm{rv}(t_0))(w \otimes f)$.*
3. *If $z \in \mathbb{A}_E^n$ has components 0 at the infinite places, then $\rho(\mathrm{rv}(t))\rho(z) = \rho(z)\rho(\mathrm{rv}(t)) = \rho(\mathrm{rv}(t)+z)$; consequently $w \otimes (Xf) = X(w \otimes f)$ for $X = \rho(z)$, for $X = \rho_f(y)$ ($y \in \mathbb{A}_{E,f}^n$) and for $X = P^A$.*
4. *$\int v_1(t)\,\beta_\varepsilon^\infty(t,u)\,v_1(u-t)\,d\nu(t) = v_1(u)$ for every $u \in \mathbb{C}^\iota$; so $v_1 \in \mathcal{F}_\varepsilon$. Moreover $v_1 \otimes v^0 = v^0$.*
5. *For $v \in \mathcal{F}_\varepsilon$ and $f \in \mathcal{M}_s$, $v \otimes (v_1 \otimes f) = v \otimes f$; in particular $v_1 \otimes (v_1 \otimes f) = v_1 \otimes f$.*

Proof. **Step 0: rules for ρ.** $\rho(z)\rho(z') = \beta_s(z,z')\rho(z+z')$, as β_s is additive in each variable and $\beta_s(z',z) = \beta_s(z,z')^{-1}$. If z has components only at the infinite and z' only at the finite places, $\beta_s(z,z') = \beta_s(z',z) = 1$. In normalized coordinates the archimedean factor of $\beta_s(\mathrm{rv}(t),\mathrm{rv}(t'))$ is $\prod_{(v,i)} \exp(s_i(\bar{t}'_{v,i}t_{v,i} - \bar{t}_{v,i}t'_{v,i})/2) = \beta_\varepsilon^\infty(t,t')$, because $\jmath_v\kappa_v^2 = i/(2\pi)$ and the archimedean factor of ψ_E is $\exp(-2\pi i\,2\,\mathrm{Re}(\cdot))$.

(1) The integrand is strongly measurable (strong continuity) with norm $|w(t)|\,\|f\|$; (WR6) is the same integral.

(2) Substitute $t = t' + t_0$ (ν is translation invariant). By additivity and $\beta_\varepsilon^\infty(t_0,t_0) = 1$, $\beta_\varepsilon^\infty(t_0,t'+t_0) = \beta_\varepsilon^\infty(t_0,t')$, and by Step 0, $\rho(\mathrm{rv}(t')+\mathrm{rv}(t_0)) = \beta_\varepsilon^\infty(t_0,t')^{-1}\rho(\mathrm{rv}(t_0))\rho(\mathrm{rv}(t'))$. So $(\rho^\infty(t_0)w) \otimes f = \int \beta_\varepsilon^\infty(t_0,t)w(t-t_0)\rho(\mathrm{rv}(t))f\,d\nu(t) = \rho(\mathrm{rv}(t_0))\int w(t')\rho(\mathrm{rv}(t'))f\,d\nu(t')$.

(3) The first part is Step 0. Hence $\rho(\mathrm{rv}(t))$ commutes with $\rho(z)$, with $\rho_f(y)$, a scalar multiple of $\rho(\mathrm{fv}(y))$ (WD8), and with P^A, a Bochner mean of such operators (WR6); a bounded operator commuting with every $\rho(\mathrm{rv}(t))$ commutes with $w \otimes \cdot$.

(4) The integrand is a product over the coordinates $c = (v,i)$. For a coordinate of sign $+1$, expanding $|u-t|^2$, the exponent of $e^{-|t|^2/2}\exp((\bar{u}t - \bar{t}u)/2)e^{-|u-t|^2/2}$ is $-|t|^2 + \bar{u}t - \frac{1}{2}|u|^2$, and the Gaussian integral $\pi^{-1}\int_{\mathbb{C}} \exp(-|t|^2 + at)\,dt = 1$ ($a \in \mathbb{C}$; entire in a, and 1 for $a = 0$ by rotation invariance, the integral of $\exp(-|t|^2)t^k$ vanishing for $k \ge 1$) gives $e^{-|u|^2/2}$. For sign -1 substitute $t \mapsto \bar{t}$. This is the reproducing identity of (WR5), and v_1 is integrable. For $v_1 \otimes v^0$: its value at x (normalized archimedean coordinates u) is $\int v_1(t)\beta_s(\mathrm{rv}(t),x)e_n(x-\mathrm{rv}(t))\,d\nu(t)$; by Step 0 and the product form of e_n (WD5) this is $e_f(x_f)\int v_1(t)\beta_\varepsilon^\infty(t,u)v_1(u-t)\,d\nu(t) = e_f(x_f)v_1(u) = e_n(x)$.

(5) By Fubini (all functions integrable, the operators isometric) and Step 0,

$$\begin{aligned} v \otimes (v_1 \otimes f) &= \iint v(t)v_1(t')\beta_\varepsilon^\infty(t,t')\rho(\mathrm{rv}(t+t'))f\,d\nu\,d\nu \\ &= \int \Big(\int v(t)\beta_\varepsilon^\infty(t,\tau)v_1(\tau-t)\,d\nu(t)\Big)\rho(\mathrm{rv}(\tau))f\,d\nu(\tau), \end{aligned}$$

substituting $t' = \tau - t$ and using $\beta_\varepsilon^\infty(t,\tau-t) = \beta_\varepsilon^\infty(t,\tau)$. The inner integral is $v(\tau)$ by the reproducing identity of $v \in \mathcal{F}_\varepsilon$ (WR5). By (4) this applies to $v = v_1$. □

Lemma 6.2.22. *Let A be a finite set of finite places of E containing the conjugate of none of its members, N a finite index set and $E_A = \prod_{w\in A} E_w$. For every locally constant, compactly supported $\Phi\colon E_A^N \to \mathbb{C}$, the function $x \mapsto \Phi(x_A)$ of finite adelic vectors $x \in \mathbb{A}_{E,f}^N$ lies in $\mathcal{T}_A$. In particular, for a split tame character (ν,Σ), spaces S, T and a locally constant, compactly supported function Φ on $n_T \times n_S$ matrices over E_Σ, $\Phi^\sharp \in \mathcal{T}_\Sigma$ and $\Phi^\flat \in \mathcal{T}_{\bar{\Sigma}}$, so the local datum $v \otimes [\Phi]_\Sigma$ and the conjugate local datum $v \otimes [\Phi]_{\bar{\Sigma}}$ are defined for every $v \in \mathcal{F}_{S,T}$.*

Proof. For $z \in E_A^N$ read as the finite adelic vector with components z_w at $w \in A$ and 0 elsewhere, and $\beta \in E_A^N$, put $e_\beta(z) = \psi_{E,f}(\beta^{\mathsf{T}} z)$. Let $\mathcal{O}_A = \prod_{w \in A} \mathcal{O}_{E,w}$.

Step 1: the functions τ_y. Let y be supported at $A \cup \bar{A}$, $y = a + b$ with a supported at A and b at $\bar{A}$. Then $(2x + y)_A = 2x_A + a$, and the components of $\bar{y}$ at A are $\bar{b}$ (complex conjugation carries the components at $\bar{A}$ to A), while those of $\bar{a}$ sit at $\bar{A}$ and meet the zero components of $(2x + y)_A$. Hence

$$\tau_y(x) = \psi_{E,f}\big(\tfrac{1}{2}\jmath\alpha\bar{\alpha}\,\bar{b}^{\mathsf{T}} a\big)\; e_\beta(x_A)\; \mathbf{1}\big[x_A + a \in \mathcal{O}_A^N\big], \qquad \beta = \jmath\alpha\bar{\alpha}\,\bar{b} \in E_A^N.$$

The components of $\jmath\alpha\bar{\alpha}$ at A are nonzero, so as b runs over the vectors supported at $\bar{A}$, β runs over all of E_A^N; the first factor is a nonzero constant. So $\mathcal{T}_A$ contains every function $x \mapsto e_\beta(x_A)\mathbf{1}_{c+\mathcal{O}_A^N}(x_A)$, $\beta, c \in E_A^N$.

Step 2: indicators of small cosets. Let $k \geq 0$ and $L_k = \prod_{w \in A} \varpi_w^k \mathcal{O}_{E,w}^N$, ϖ_w uniformisers. The local components of the nontrivial character $\psi_{E,f}$ of $\mathbb{A}_{E,f}/(\text{the image of } E)$ are nontrivial, so $\beta \mapsto e_\beta$ identifies E_A^N with its group of characters (self-duality of local fields, Tate's thesis), and every character of the compact open subgroup $\mathcal{O}_A^N$ extends to E_A^N (Pontryagin duality); hence every character χ of the finite group $\mathcal{O}_A^N / L_k$ is $z \mapsto e_\beta(z)$ on $\mathcal{O}_A^N$ for some $\beta = \beta_\chi$. By orthogonality of characters, for $c \in E_A^N$ and $z \in E_A^N$,

$$\mathbf{1}_{c+L_k}(z) = \mathbf{1}_{c+\mathcal{O}_A^N}(z)\,\mathbf{1}_{L_k}(z-c) = \frac{1}{[\mathcal{O}_A^N : L_k]} \sum_\chi e_{\beta_\chi}(-c)\, e_{\beta_\chi}(z)\, \mathbf{1}_{c+\mathcal{O}_A^N}(z),$$

so $x \mapsto \mathbf{1}_{c+L_k}(x_A)$ lies in $\mathcal{T}_A$ by Step 1.

Step 3: Schwartz-Bruhat functions. Φ is locally constant with compact support, so for some $k \geq 0$ it is constant on the cosets of the open subgroup L_k, and its support meets finitely many of them; thus $\Phi = \sum_c \Phi(c)\mathbf{1}_{c+L_k}$ (finite sum over representatives) and $x \mapsto \Phi(x_A)$ lies in $\mathcal{T}_A$ by Step 2.

Local data. With $N = n_T \times n_S$ and $A = \Sigma$ this gives $\Phi^\sharp \in \mathcal{T}_\Sigma$. Since $(\bar{y})_\Sigma$ is the complex conjugate of the matrix of components of y at $\bar{\Sigma}$, $\Phi^\flat(y) = \Phi'(y_{\bar{\Sigma}})$ with $\Phi'(z) = \Phi(\bar{z})$ locally constant with compact support on matrices over $E_{\bar{\Sigma}}$, and $\bar{\Sigma}$ contains the conjugate of none of its members; so $\Phi^\flat \in \mathcal{T}_{\bar{\Sigma}}$. □

Definition 6.2.23. Let $E = E_s$ be the field of a parameter pair P and an index $s \geq 0$, as in Definition 3.2.22.

(XA1) The functionals of the exchange data. From now on, for every space with a point V, the functional θ_V of (XD2) of Definition 3.2.22 is the canonical theta functional of V ((WC3) of Definition 4.1.15), which is a bounded linear functional on the theta model of the signs of V, as (XD2) requires. Theta kernels, theta integrals, target-bounded data, equal-rank ratios, the large and small sides, plane constant terms and the row-column exchange identity of Definition 3.2.22, and every object defined from them, are formed with these functionals; in particular the theta kernel $\theta_{\alpha,\beta}^{S,T}$ of (XD2) is the theta kernel of (WC4) of Definition 4.1.15.

Lemma 6.2.24. *Let W', W, V', V, A, X be as above. Let $Z^{\mathrm{pl}} \colon \mathcal{M}_{W',V'} \to \mathcal{M}_{s_{\mathbb{H}}}$ be the zero term along $D_{A,X}^{\mathrm{pl}}$, $s_{\mathbb{H}}$ the signs of $\mathbb{H}_{\{*\}} \oplus \mathrm{Hom}(W,V)$, and for $F \in \mathcal{M}_{s_{\mathbb{H}}}$ and $a \in \mathbb{A}_E$ let $F[a]_\pm \in \mathcal{M}_{W,V}$ be the slice of F at a along $R^\pm$. Then for every $f \in \mathcal{M}_{W',V'}$:*

1. $\mathrm{Ln}_{A,X}^+(a)f = (Z^{\mathrm{pl}}f)[a]_+$ *and* $\mathrm{Ln}_{A,X}^-(a)f = (Z^{\mathrm{pl}}f)[a]_-$ *for every $a \in \mathbb{A}_E$;*

2. *the series* $\sum_{\xi\in E}\theta_{\mathrm{Hom}(W,V)}(\mathrm{Ln}^{+}_{A,X}(\xi)f)$ *and* $\sum_{\xi\in E}\theta_{\mathrm{Hom}(W,V)}(\mathrm{Ln}^{-}_{A,X}(\xi)f)$ *converge absolutely, and both sums equal* $\theta_{\mathbb{H}_{\{*\}}\oplus\mathrm{Hom}(W,V)}(Z^{\mathrm{pl}}f)$.

Proof. For $y\in E^{r'}$ and $z\in E^{n'}$ write $[y,z]=y\,z^{*}J_{W'}\in\mathrm{Hom}(W',V')$, and for $z\in E^{n}$ write $[y,z]_{W}=y\,z^{*}J_{W}\in\mathrm{Hom}(W,V')$. By the computation in (LN3), the form of $\mathrm{Hom}(W',V')$ satisfies $h([y,z],[y',z'])=h_{V'}(y,y')\,h_{W'}(z',z)$. The point of $\mathrm{Hom}(W',V')$ reads a rational vector M as the adelic matrix $x_{V'}^{-1}Mx_{W'}$; since $x_{W'}^{*}J_{W'}x_{W'}=\varepsilon_{s'}$, the coordinates of A are $\mathrm{pr}_k=l_{A,\mathbb{A}}^{*}\varepsilon_{s'}=l_A^{*}J_{W'}x_{W'}$ and $\mathrm{pr}_l=k_A^{*}J_{W'}x_{W'}$. Hence

$$x_{V'}^{-1}[k_X,l_A]x_{W'}=k_{X,\mathbb{A}}\,\mathrm{pr}_k,\qquad x_{V'}^{-1}[l_X,k_A]x_{W'}=l_{X,\mathbb{A}}\,\mathrm{pr}_l. \tag{$*$}$$

Step 1: the frames. Reading the Kronecker products of (LN3), (LN1) and (MX4) column by column:

- $D^{\mathrm{pl}}_{A,X}$ has the frame k with columns $[k_X,i_Ae_j]$ $(j\in n)$, $[k_X,k_A]$, $[i_XJ_V^{-1}e_m,k_A]$ $(m\in r)$, and the embedding i^{pl} with columns $[k_X,l_A]$, $[l_X,k_A]$ (the two copies of $*$) and $[i_Xe_m,i_AJ_W^{-1}e_j]$ $((m,j)\in r\times n)$. The tautological R^{+} has l, k the first and second copy of $*$ and i the block of $\mathrm{Hom}(W,V)$; R^{-} exchanges l and k. By (MX5), $D^{\mathrm{pl}}\circ R^{\pm}$ has the embedding with columns $[i_Xe_m,i_AJ_W^{-1}e_j]$ and the frame k consisting of the columns of k of D^{pl} together with $[l_X,k_A]$ (for $+$), resp. $[k_X,l_A]$ (for $-$); and $i^{\mathrm{pl}}l^{R^{+}}=[k_X,l_A]$, $i^{\mathrm{pl}}l^{R^{-}}=[l_X,k_A]$.
- The splitting S of $\mathrm{Hom}(W',V')$ over $\mathrm{Hom}(W,V')$ induced by A ((MX4), $T=V'$) has k with columns $[J_{V'}^{-1}e_{m'},k_A]$ $(m'\in r')$ and embedding $M\mapsto MJ_W^{-1}i_A^{*}J_{W'}$, which sends $[y,z]_W$ to $[y,i_Az]$. The splitting S^{tg} of $\mathrm{Hom}(W,V')$ over $\mathrm{Hom}(W,V)$ induced on the target by X ((LN1), $T=W$) has k with columns $[k_X,e_j]_W$ and embedding with columns $[i_Xe_m,J_W^{-1}e_j]_W$. So $S\circ S^{\mathrm{tg}}$ has the frame k with columns $[J_{V'}^{-1}e_{m'},k_A]$, $[k_X,i_Ae_j]$ and the embedding with columns $[i_Xe_m,i_AJ_W^{-1}e_j]$.
- The splitting S' of $\mathrm{Hom}(W',V')$ over $\mathrm{Hom}(W',V)$ induced on the target by X $(T=W')$ has k with columns $[k_X,e_{j'}]$ $(j'\in n')$ and embedding $M\mapsto i_XM$, sending $[y,z]$ to $[i_Xy,z]$; the splitting S'^{sc} of $\mathrm{Hom}(W',V)$ over $\mathrm{Hom}(W,V)$ induced by A $(T=V)$ has k with columns $[J_V^{-1}e_m,k_A]$ and embedding with columns $[e_m,i_AJ_W^{-1}e_j]$. So $S'\circ S'^{\mathrm{sc}}$ has the frame k with columns $[k_X,e_{j'}]$, $[i_XJ_V^{-1}e_m,k_A]$ and the embedding with columns $[i_Xe_m,i_AJ_W^{-1}e_j]$.

Step 2: the zero terms agree. Let U^{+} be the span of the $[y,k_A]$ $(y\in E^{r'})$ and the $[k_X,z]$ with $z\in i_AE^{n}$, and U^{-} the span of the $[k_X,z]$ $(z\in E^{n'})$ and the $[i_Xy,k_A]$ $(y\in E^{r})$. By the product formula for h and $h_{W'}(k_A,k_A)=0$, $h_{W'}(i_Az,k_A)=0$, $h_{V'}(k_X,k_X)=0$, $h_{V'}(k_X,i_Xy)=0$, both are totally isotropic. Every column of the frames k of $D^{\mathrm{pl}}\circ R^{+}$ and of $S\circ S^{\mathrm{tg}}$ lies in U^{+}, and every column of those of $D^{\mathrm{pl}}\circ R^{-}$ and of $S'\circ S'^{\mathrm{sc}}$ lies in U^{-}; so in each pair $k^{*}Jk'=0$, and the embeddings agree by Step 1. By (2) of Lemma 6.2.25 and then (1) of Lemma 6.2.25,

$$Z_{D^{\mathrm{pl}}\circ R^{+}}=Z_{S\circ S^{\mathrm{tg}}}=Z_{S^{\mathrm{tg}}}\circ Z_S,\qquad Z_{D^{\mathrm{pl}}\circ R^{-}}=Z_{S'\circ S'^{\mathrm{sc}}}=Z_{S'^{\mathrm{sc}}}\circ Z_{S'},$$

and also $Z_{D^{\mathrm{pl}}\circ R^{\pm}}=Z_{R^{\pm}}\circ Z^{\mathrm{pl}}$.

Step 3: proof of (1). The adelic frames multiply as $i^{\mathrm{pl}}_{\mathbb{A}}l^{R^{\pm}}_{\mathbb{A}}=x^{-1}i^{\mathrm{pl}}l^{R^{\pm}}$, x the point of $\mathrm{Hom}(W',V')$, so by Step 1 and $(*)$, $i^{\mathrm{pl}}_{\mathbb{A}}l^{R^{+}}_{\mathbb{A}}a=a\,k_{X,\mathbb{A}}\mathrm{pr}_k$ and $i^{\mathrm{pl}}_{\mathbb{A}}l^{R^{-}}_{\mathbb{A}}a=a\,l_{X,\mathbb{A}}\mathrm{pr}_l$. By (1) of Lemma 6.2.44, $Z^{\mathrm{pl}}\rho(i^{\mathrm{pl}}_{\mathbb{A}}w)=\rho(w)Z^{\mathrm{pl}}$. Hence, with Step 2,

$$\begin{aligned}(Z^{\mathrm{pl}}f)[a]_{+}&=Z_{R^{+}}\big(\rho(l^{R^{+}}_{\mathbb{A}}a)Z^{\mathrm{pl}}f\big)=Z_{R^{+}}Z^{\mathrm{pl}}\big(\rho(a\,k_{X,\mathbb{A}}\mathrm{pr}_k)f\big)\\&=Z_{S^{\mathrm{tg}}}\Big(Z_S\big(\rho(a\,k_{X,\mathbb{A}}\mathrm{pr}_k)f\big)\Big)=Z^{\mathrm{tg}}_{X,W}\big(f[a\,k_{X,\mathbb{A}}]\big),\end{aligned}$$

the last step by the definition of the Hom slice in (MX4) for S; this is $\mathrm{Ln}^+_{A,X}(a)f$ by (LN2). In the same way $(Z^{\mathrm{pl}}f)[a]_- = Z_{S'^{\mathrm{sc}}}(Z_{S'}(\rho(a\,l_{X,\mathbb{A}}\mathrm{pr}_l)f)) = Z^{\mathrm{sc}}_{A,V}(Z^{\mathrm{tg}}_{X,W'}(\rho(a\,l_{X,\mathbb{A}}\mathrm{pr}_l)f)) = \mathrm{Ln}^-_{A,X}(a)f$.

Step 4: proof of (2). Apply (3) of Lemma 6.2.44 to $F = Z^{\mathrm{pl}}f$ and to the splittings R^+ and R^- of $\mathbb{H}_{\{*\}} \oplus \mathrm{Hom}(W,V)$ over $\mathrm{Hom}(W,V)$ along the nonempty set $\{*\}$: $\theta_{\mathbb{H}_{\{*\}}\oplus\mathrm{Hom}(W,V)}(F) = \sum_{\xi\in E}\theta_{\mathrm{Hom}(W,V)}(F[\xi]_\pm)$ for both signs, the series converging absolutely. By (1), $F[\xi]_\pm = \mathrm{Ln}^\pm_{A,X}(\xi)f$. □

Lemma 6.2.25. *Let V, V_0, V_1 be spaces with points.*

1. *Let D be a splitting of V over V_0 along p and D' a splitting of V_0 over V_1 along q, with zero terms Z_D, $Z_{D'}$. Then the zero term along the composite $D\circ D'$ is $Z_{D\circ D'} = Z_{D'}\circ Z_D$.*
2. *Let $D = (k,l,i)$ and $D' = (k',l',i')$ be splittings of V over V_0 along p and q with $i' = i$ and $k^*J_Vk' = 0$. Then $k' = kc$ for a matrix $c \in \mathrm{M}_{p\times q}(E)$ with a two-sided inverse, and $Z_{D'} = Z_D$.*

Proof. For $f \in \mathcal{M}_s$ write $f^\theta(x) = \theta_V(\rho(x)f)$. The function f^θ is continuous on $\mathbb{A}^n_E$: θ_V is bounded, and $z \mapsto \rho(z)f$ is continuous into $\mathcal{M}_s$, because $\|\rho(z)f - \rho(z')f\|_1 \le \int |\beta_s(z,x+z) - \beta_s(z',x+z)|\,|f(x)|\,d\mu_n(x) + \int |f(x-z)-f(x-z')|\,d\mu_n(x)$ (μ_n is a Haar measure of $\mathbb{A}^n_E$), the first term tends to 0 as $z' \to z$ by dominated convergence (β_s is continuous of modulus 1) and the second by continuity of translation in $L^1(\mu_n)$. By (7) of Lemma 4.1.1 the integrand of (MX4) is a function on the compact group $\mathbb{A}^p_E/E^p$, and the zero term is the unique vector with the property (MX4).

Proof of (1). Write $D = (k,l,i)$, $D' = (k',l',i')$. By (MX5) the composite has frames $(k \mid ik')$ and embedding ii', so its adelic frames are $(k_\mathbb{A} \mid x_V^{-1}ik') = (k_\mathbb{A} \mid i_\mathbb{A}k'_\mathbb{A})$ and $x_V^{-1}ii'x_{V_1} = i_\mathbb{A}i'_\mathbb{A}$, using $x_V^{-1}ik' = (x_V^{-1}ix_{V_0})(x_{V_0}^{-1}k')$. Let $f \in \mathcal{M}_s$ and $y \in \mathbb{A}^{n_1}_E$. The function

$$F(b,b') = f^\theta(i_\mathbb{A}i'_\mathbb{A}y + i_\mathbb{A}k'_\mathbb{A}b' + k_\mathbb{A}b) \qquad (b \in \mathbb{A}^p_E,\ b' \in \mathbb{A}^q_E)$$

is the integrand of (MX4) for $D\circ D'$ at y, hence continuous and a function on $\mathbb{A}^p_E/E^p \times \mathbb{A}^q_E/E^q$, whose invariant probability measure is the product of those of the factors. By (MX4) for D at the point $i'_\mathbb{A}y + k'_\mathbb{A}b'$ and then for D' at y,

$$\theta_{V_1}(\rho(y)Z_{D'}(Z_Df)) = \int_{\mathbb{A}^q_E/E^q}\theta_{V_0}(\rho(i'_\mathbb{A}y + k'_\mathbb{A}b')Z_Df)\,db' = \int_{\mathbb{A}^q_E/E^q}\int_{\mathbb{A}^p_E/E^p} F(b,b')\,db\,db',$$

which by Fubini (a continuous function on a compact group) is $\int F$ over the product, the right side of (MX4) for $D\circ D'$ at y. So $Z_{D'}(Z_Df)$ has the defining property of the zero term of f along $D\circ D'$, and equals it by uniqueness.

Proof of (2). By (MX1), $l^*J_Vk = 1$, $k^*J_Vk = 0$, $i^*J_Vk = i^*J_Vl = 0$, $i^*J_Vi = J_{V_0}$ and $E^n = lE^p \oplus kE^p \oplus iE^{n_0}$. Write $k' = la + kc + iw$ with $a \in \mathrm{M}_{p\times q}(E)$, $c \in \mathrm{M}_{p\times q}(E)$, $w \in \mathrm{M}_{n_0\times q}(E)$. Then $0 = k^*J_Vk' = (l^*J_Vk)^*a = a$ and $0 = i'^*J_Vk' = i^*J_Vk' = J_{V_0}w$, so $a = 0$, $w = 0$ and $k' = kc$. As $k'^*J_Vk = (k^*J_Vk')^* = 0$ and $i = i'$, the same argument gives $k = k'c'$ with $c' \in \mathrm{M}_{q\times p}(E)$. Then $k = kcc'$ and $k' = k'c'c$; since $l^*J_Vk = 1$ and $l'^*J_Vk' = 1$, the columns of k and of k' are linearly independent, so $cc' = 1$ and $c'c = 1$. Hence $k'_\mathbb{A} = x_V^{-1}kc = k_\mathbb{A}c$ and $i'_\mathbb{A} = i_\mathbb{A}$. The map $b \mapsto cb$ is an isomorphism of topological groups $\mathbb{A}^q_E \to \mathbb{A}^p_E$ carrying E^q onto E^p; it induces an isomorphism of the compact quotients, which carries the invariant probability measure to the invariant probability measure (uniqueness of Haar measure). Therefore, for $f \in \mathcal{M}_s$ and $y \in \mathbb{A}^{n_0}_E$,

$$\int_{\mathbb{A}^q_E/E^q} f^\theta(i'_\mathbb{A}y + k'_\mathbb{A}b)\,db = \int_{\mathbb{A}^q_E/E^q} f^\theta(i_\mathbb{A}y + k_\mathbb{A}cb)\,db = \int_{\mathbb{A}^p_E/E^p} f^\theta(i_\mathbb{A}y + k_\mathbb{A}b'')\,db''.$$

So $Z_D(f)$ has the defining property of the zero term of f along D', and $Z_{D'}(f) = Z_D(f)$ by uniqueness. □

Lemma 6.2.26. *Let S, T be spaces with index sets n_S, n_T, let A be a nonempty finite set of finite places of E containing the conjugate of none of its members, $E_A = \prod_{w\in A} E_w$, let v be a Schwartz vector for (S,T), and let φ be a locally constant, compactly supported function on $\mathrm{M}_{n_T\times n_S}(E_A)$ vanishing at every matrix without a right inverse. Put $\varphi^\sharp(y) = \varphi(y_A)$ for finite adelic $n_T \times n_S$ matrices y (an element of $\mathcal{T}_A$ by Lemma 6.2.22). Then the datum $v \otimes (\varphi^\sharp \otimes_A v^0) \in \mathcal{M}_{S,T}$ is target-bounded.*

Proof. By Lemma 6.2.22, $\varphi^\sharp \in \mathcal{T}_A$. A matrix over $E_A = \prod_{w\in A} E_w$ has a right inverse if and only if each of its components x_w ($w \in A$) has one. Fix $w \in A$ (A is nonempty). If $\varphi^\sharp(\eta) = \varphi(\eta_A) \neq 0$ for a finite adelic $n_T \times n_S$ matrix η, then η_A has a right inverse, hence so has η_w. So (1) of Lemma 6.2.27 applies to S, T, A, $\Psi = \varphi^\sharp$ and w; with $N = 0$ it gives $B, M \in \mathbb{N}$ and $C \geq 0$ such that

$$|\theta^{S,T}_{\alpha,\beta}(v \otimes (\varphi^\sharp \otimes_A v^0))(g,h)| \leq C \,\mathrm{mom}_M(v)\,\mathrm{ht}_S(g)^B$$

for all automorphic characters α, β of $\mathrm{U}(1)$ and all $g \in \mathrm{U}(S)(\mathbb{A}_F)$, $h \in \mathrm{U}(T)(\mathbb{A}_F)$. As v is a Schwartz vector, $\mathrm{mom}_M(v) < \infty$, and the datum is target-bounded with the exponent B and the constant $C\,\mathrm{mom}_M(v)$. □

Lemma 6.2.27. *Let S, T be spaces with points with signs s on n and s' on n', A a finite set of finite places of E containing the conjugate of none of its members, $\Psi \in \mathcal{T}_A$ a function of finite adelic $n' \times n$ matrices, and $w \in A$. Write $\Theta(v)(g,h) = \theta^{S,T}_{\alpha,\beta}(v \otimes (\Psi \otimes_A v^0))(g,h)$.*

1. *If η_w has a right inverse in $\mathrm{M}_{n\times n'}(E_w)$ for every η with $\Psi(\eta) \neq 0$, then for every $N \in \mathbb{N}$ there are $B, M \in \mathbb{N}$ and $C \geq 0$ such that*

   $$|\Theta(v)(g,h)|\,\mathrm{ht}_T(h)^N \leq C\,\mathrm{mom}_M(v)\,\mathrm{ht}_S(g)^B$$

   for all automorphic characters α, β of $\mathrm{U}(1)$, all $v \in \mathcal{F}_{S,T}$ with $\mathrm{mom}_M(v) < \infty$, and all $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$.

2. *If η_w has a left inverse in $\mathrm{M}_{n\times n'}(E_w)$ for every η with $\Psi(\eta) \neq 0$, then for every N there are B, M, C with $|\Theta(v)(g,h)|\,\mathrm{ht}_S(g)^N \leq C\,\mathrm{mom}_M(v)\,\mathrm{ht}_T(h)^B$ for the same α, β, v, g, h.*

Proof. Let $\Theta_0 = \theta_{\mathrm{Hom}(S,T)}$ and $f_0 = \Psi \otimes_A v^0$. Since $|\alpha(\det h)\beta(\det g)^{-1}| = 1$, $|\Theta(v)(g,h)| = |\Theta_0(\omega_{1,1}(g,h)(v \otimes f_0))|$, so we work with $\alpha = \beta = 1$ and the functional $\ell = \Theta_0$; the constants below do not depend on α, β.

Inputs. (a) By Lemma 6.2.28 there are $Y_0 \geq 0$, $A_0 \in \mathbb{N}$ with

$$|\Theta_0(\omega_{1,1}(g,h)f)| \leq Y_0\|f\|(\mathrm{ht}_S(g)\mathrm{ht}_T(h))^{A_0}$$

for all g, h, f. (b) By (VP3) of Definition 3.2.2 S and T are framed hermitian spaces and $\lambda_S(g) = \mathrm{ht}_S(g)^{-1}$, $\lambda_T(h) = \mathrm{ht}_T(h)^{-1}$ are the first minima of (VP4) for both readings; by (VP4) their lattices at g, resp. h, are those of S_g, T_h and their lengths are the euclidean norms of the real components; so by Lemma 5.8.21, for $c_S = f\,\#n\,|d_E|$ every g has $c \in \mathrm{GL}_n(E)$ whose columns lie in the lattice of S_g with lengths at most $c_S\mathrm{ht}_S(g)$; likewise $c_T = f\,\#n'\,|d_E|$ for T. (c) By Lemma 6.2.30, if $\mathrm{ht}_T(h) > 2$ there is a nonzero $d \in E^{n'}$ in the lattice of T_h with $d^* J_T d = 0$ and length at most $2/\mathrm{ht}_T(h)$; likewise for S. (d) By Lemma 4.1.17, Θ_0 is invariant under $\rho(\gamma)$ for every rational vector $\gamma = x_T^{-1} X x_S$ ($X \in \mathrm{M}_{n'\times n}(E)$) of $\mathrm{Hom}(S,T)$; in particular under $\rho(z)$, $z \in Z_{d,c}(E)$, for all d, c (notation of Lemma 6.2.31).

(1). Let g, h with $H = \mathrm{ht}_T(h) > 2$, c as in (b) for g and d as in (c) for h (p a one-element set, $q = n$). We check that (d, c) is a short pair for (g, h, Θ_0, x, y) with $x = 2/H$, $y = c_S \mathrm{ht}_S(g)$. Conditions (i) and (ii) hold by (b), (c); (iii) by (d). For (iv), first alternative, let $\Psi(\eta) \neq 0$, $d' = (x_T h)^{-1} d$, $c' = (x_S g)^{-1} c$. The component of d'^*_f at w is the row $a = \overline{d'_{\bar{w}}}^{\mathsf{T}}$, which is nonzero because $d \neq 0$ and $((x_T h)^{-1})_{\bar{w}}$ is invertible; if R is a right inverse of η_w, then $a\eta_w \neq 0$ (else $a = a\eta_w R = 0$); and c'_w is invertible since c is. Hence $(d'^*_f \eta c'_f)_w = a\eta_w c'_w \neq 0$. Moreover $xy = 2c_S \mathrm{ht}_S(g)/H$. Now apply (2) of Lemma 6.2.31 with $\ell = \Theta_0$, N, Y_0, A_0, $Z_0 = c_S$, $r = \mathrm{ht}_S(g) \geq 1$ and $H = \mathrm{ht}_T(h) > 0$: its growth hypothesis is (a), and for $H > 2$ the short pair just built has $xy \leq 2Z_0 r/H$. It gives C', independent of v, g, h, with

$$|\Theta(v)(g,h)| \, \mathrm{ht}_T(h)^N \leq C' \, \mathrm{mom}_{N+A_0}(v) \, \mathrm{ht}_S(g)^{N+2A_0},$$

which is (1) with $M = N + A_0$, $B = N + 2A_0$, $C = C'$.

(2). Exchange the roles: for $H = \mathrm{ht}_S(g) > 2$ take $c \in E^n$ as in (c) for S at g (q a one-element set, $c^* J_S c = 0$, length at most $2/H$) and $d \in \mathrm{GL}_{n'}(E)$ as in (b) for T at h ($p = n'$, lengths at most $c_T \mathrm{ht}_T(h)$). Condition (iv): the component a of d'^*_f at w is the invertible matrix $\overline{d'_{\bar{w}}}^{\mathsf{T}}$, $c'_w \neq 0$ since $c \neq 0$, and $\eta_w c'_w \neq 0$ because η_w has a left inverse; so $a\eta_w c'_w \neq 0$. Then (2) of Lemma 6.2.31 with $Z_0 = c_T$, $r = \mathrm{ht}_T(h)$, $H = \mathrm{ht}_S(g)$ (and (a), which is symmetric in the two heights) gives (2) with the same M, B. □

Lemma 6.2.28. *Let S, T be spaces with points with signs s on n and s' on n'. There are $Y \geq 0$ and $A \in \mathbb{N}$ such that:*

1. *for all automorphic characters α, β of $\mathrm{U}(1)$, all $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$ and $\varphi \in \mathcal{M}_{S,T}$,*

$$|\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)| \leq Y \, \|\varphi\| \, (\mathrm{ht}_S(g) \, \mathrm{ht}_T(h))^A;$$

2. *for every $g \in \mathrm{G}(s)$ there is $\gamma \in \Gamma_S$ such that $\|\omega_{\alpha,\beta}(\gamma g, 1)\varphi\| \leq Y \, \mathrm{ht}_S(g)^A \, \|\varphi\|$ for all automorphic characters α, β of $\mathrm{U}(1)$ and all $\varphi \in \mathcal{M}_{S,T}$.*

Proof. **Reduction to $\alpha = \beta = 1$.** By (WC4) and (WR4),

$$\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h) = \alpha(\det h)\beta(\det g)^{-1} \theta^{S,T}_{1,1}(\varphi)(g,h),$$

and automorphic characters of $\mathrm{U}(1)$ take values of modulus 1. So it suffices to treat $\omega = \omega_{1,1}$. By (2) of Lemma 4.1.1, ω is a representation of $\mathrm{G}(s) \times \mathrm{G}(s')$ by bounded operators of $\mathcal{M}_{S,T}$ and $(k, \varphi) \mapsto \omega(k)\varphi$ is continuous (the pair map of (VP5) of Definition 3.2.2 is a continuous homomorphism).

Operators on a compact set. Let $\Omega_S \subseteq \mathrm{G}(s)$ and $\Omega_T \subseteq \mathrm{G}(s')$ be the compact sets of Lemma 6.2.29 for S and T. The set $B_S = \{(x, 1) : x \in G_\infty(s), \ \|x_v\| \leq 2 \ (v \in \Phi)\}$ is compact: $\mathrm{U}(\mathrm{diag}\, s)$ is closed in $\mathrm{M}_n(\mathbb{C})$, and on it $\|x^{-1}\| = \|\mathrm{diag}(s) x^* \mathrm{diag}(s)\| = \|x\|$, so $\{x \in \mathrm{U}(\mathrm{diag}\, s) : \|x\| \leq 2\}$ is a compact subset of $\mathrm{GL}_n(\mathbb{C})$. Likewise B_T. Let $K = (\Omega_S \cup B_S \cup \{1\}) \times (\Omega_T \cup B_T \cup \{1\})$, a compact set. For each φ the map $k \mapsto \omega(k)\varphi$ is continuous, hence bounded on K; since $\mathcal{M}_{S,T}$ is a Banach space ((1) of Lemma 4.1.10), the uniform boundedness principle gives $C_0 = \max(1, \sup_{k \in K} \|\omega(k)\|) < \infty$. Choose $A \in \mathbb{N}$ with $2^A \geq C_0$.

The bound. Let $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$. By Lemma 6.2.29, $g = \gamma z (x_1 \cdots x_k, 1)$ and $h = \delta z' (y_1 \cdots y_l, 1)$ with $\gamma \in \Gamma_S$, $\delta \in \Gamma_T$, $z \in \Omega_S$, $z' \in \Omega_T$, all $(x_i, 1) \in B_S$, $(y_j, 1) \in B_T$, $2^k \leq$

$\mathrm{ht}_S(g)$ and $2^l \le \mathrm{ht}_T(h)$. By Lemma 6.2.5, $\Theta(\omega(g,h)\varphi) = \Theta(\omega(\gamma^{-1}g, \delta^{-1}h)\varphi)$, and since ω is a representation of the product group,

$$\omega(\gamma^{-1}g, \delta^{-1}h) = \omega(z,1) \prod_{i=1}^{k} \omega((x_i,1),1) \cdot \omega(1,z') \prod_{j=1}^{l} \omega(1,(y_j,1)),$$

a product of $k+l+2$ operators $\omega(\kappa)$ with $\kappa \in K$. Hence

$$|\Theta(\omega(g,h)\varphi)| \le \|\Theta\|\, 2^{A(k+l+2)}\, \|\varphi\| \le 4^A \|\Theta\| \left(\mathrm{ht}_S(g)\, \mathrm{ht}_T(h)\right)^A \|\varphi\|,$$

so (1) holds with $Y = 4^A \max(1, \|\Theta\|)$. For (2), the same decomposition of g gives $\omega(\gamma^{-1}g, 1) = \omega(z,1) \prod_{i=1}^{k} \omega((x_i,1),1)$, a product of $k+1$ operators $\omega(\kappa)$ with $\kappa \in K$, so $\|\omega(\gamma^{-1}g,1)\varphi\| \le 2^{A(k+1)}\|\varphi\| \le 2^A \mathrm{ht}_S(g)^A \|\varphi\| \le Y\, \mathrm{ht}_S(g)^A \|\varphi\|$; the element $\gamma^{-1} \in \Gamma_S$ works for all α, β, since $\omega_{\alpha,\beta}(\gamma^{-1}g, 1)$ is $\beta(\det(\gamma^{-1}g))^{-1}$, of modulus 1, times $\omega(\gamma^{-1}g,1)$ (WR4). □

Lemma 6.2.29. *Let V be a space with a point with signs s on n.*

1. *There is a compact set $\Omega \subseteq \mathrm{G}(s)$ such that every $g \in \mathrm{G}(s)$ can be written*

$$g = \gamma\, z\, (x_1 x_2 \cdots x_k, 1) \qquad (\gamma \in \Gamma_V,\ z \in \Omega,\ k \in \mathbb{N},\ x_1, \ldots, x_k \in G_\infty(s))$$

   with $\|x_{i,v}\| \le 2$ for all i and all $v \in \Phi$, and $2^k \le \mathrm{ht}_V(g)$.

2. *By (VP3) of Definition 3.2.2, V is the framed hermitian space (E^n, h_V) with signs s on n, its frame at $v \in \Phi$ being the columns of $x_{V,v}$ and its isometry $z \mapsto x_{V,f} z$ (for $n \ne \emptyset$, a framed space of rank $\#n$ in the sense of (FT1) of Definition 3.2.4); its coordinate map is $c_V(u) = x_V^{-1} u$ and its rational points are Γ_V; and by (VP4) of Definition 3.2.2, for $g \in \mathrm{G}(s)$, $L_g(V)$ is the lattice of V_g, $|u|_g = \|((x_V g)^{-1}u)_\infty\|$, and the first minimum $\lambda_V(g) = \mathrm{ht}_V(g)^{-1}$ is the same for both readings.*

Proof. Write $D = \mathrm{diag}(s)$ and, for $g \in \mathrm{G}(s)$ and u in the lattice $L(g)$ of V_g, $w_g(u) = ((x_V g)^{-1}u)_\infty \in \mathbb{C}^{\Phi \times n}$, with components $w_g(u)_v \in \mathbb{C}^n$, and $N_g(u) = \|w_g(u)\|$; so $\lambda_V(g) = \min(1, \inf\{N_g(u) : 0 \ne u \in L(g)\})$.

Step 1: one halving step. We show: for every $g \in \mathrm{G}(s)$ there is $y \in G_\infty(s)$ with $\|y_v\| \le 2$ for all $v \in \Phi$ and $\mathrm{ht}_V(g(y,1)^{-1}) \le \max(\mathrm{ht}_V(g)/2,\, 2)$.

(a) The short vectors span isotropic subspaces. Let $u, u' \in L(g)$ with $N_g(u) < 1$, $N_g(u') < 1$, and $q = u'^* J_V u \in E$. Since $(x_V g)^* J_V (x_V g) = \varepsilon_s$, $q = w^* \varepsilon_s w'$ componentwise with $w = (x_V g)^{-1}u'$, $w' = (x_V g)^{-1}u$. At the finite places $\varepsilon_s = 1$ and w_f, w'_f are integral, so $q \in \mathcal{O}_E$. At $v \in \Phi$, by the Cauchy–Schwarz inequality, $|q_v| = |w_g(u')_v^* D w_g(u)_v| \le N_g(u') N_g(u) < 1$, and $|\bar{q}_v| = |q_v|$. Hence $|N_{E/\mathbb{Q}}(q)| < 1$ and $q = 0$. So for each $v \in \Phi$ the span $W_v \subseteq \mathbb{C}^n$ of the vectors $w_g(u)_v$, $u \in L(g)$, $N_g(u) < 1$, satisfies $a^* D b = 0$ for $a, b \in W_v$.

(b) A dilation of an isotropic subspace. Let $W \subseteq \mathbb{C}^n$ satisfy $a^* D b = 0$ for $a, b \in W$, and put $W' = DW$. For $a, b \in W$, $\langle a, Db\rangle = a^* D b = 0$ and $\langle Da, Db\rangle = a^* b$, $\langle \cdot, \cdot \rangle$ the standard inner product; so $W \perp W'$ and $W \cap W' = 0$. Let $R = (W + W')^\perp$. For $r \in R$ and $a \in W$, $\langle Dr, a\rangle = \langle r, Da\rangle = 0$ and $\langle Dr, Da\rangle = \langle r, a\rangle = 0$, so $DR = R$. Let y act by 2 on W, by $\frac{1}{2}$ on W' and by 1 on R. These three subspaces are mutually orthogonal, so y is self-adjoint with $\|y\| \le 2$ and $\|y^{-1}\| \le 2$. Moreover $Dy = y^{-1}D$: on $a \in W$ both sides give $2Da$; on $Da \in W'$ both give $\frac{1}{2}a$ (as $D^2 = 1$); on R both are D. Hence $y^* D y = y D y = y\, y^{-1} D = D$, i.e. $y \in \mathrm{U}(\mathrm{diag}\, s)$.

(c) Apply (b) to W_v at each $v \in \Phi$, obtaining $y = (y_v) \in G_\infty(s)$, and put $g' = g(y,1)^{-1}$. Then $(x_V g')^{-1} = (y,1)(x_V g)^{-1}$ has the same finite component as $(x_V g)^{-1}$, so $L(g') = L(g)$, and

$w_{g'}(u)_v = y_v w_g(u)_v$. For $0 \neq u \in L(g)$: if $N_g(u) < 1$, then $w_g(u)_v \in W_v$ for every v, so $N_{g'}(u) = 2N_g(u) \geq 2\lambda_V(g)$ (as $\lambda_V(g) \leq N_g(u)$); if $N_g(u) \geq 1$, then $N_g(u) \leq \max_v \|y_v^{-1}\| \, N_{g'}(u) \leq 2N_{g'}(u)$, so $N_{g'}(u) \geq \frac{1}{2}$. Therefore $\lambda_V(g') \geq \min(2\lambda_V(g), \frac{1}{2})$, which is the claimed bound for $\mathrm{ht}_V(g') = \lambda_V(g')^{-1}$.

Step 2: proof of (2), and bounded height. By (VP3) of Definition 3.2.2 V is a framed hermitian space and $\lambda_V(g)$ is the first minimum of (VP4) for both readings ($g \in \mathrm{G}(s)$); with the coordinate map c_V and the lattice $L_g(V)$ of V_g of (VP4), $|u|_g = N_g(u)$. This is (2). By Lemma 5.8.20 for this framed space with $T = 2$ there is a compact $\Omega \subseteq \mathrm{G}(s)$ such that every g with $\mathrm{ht}_V(g) \leq 2$ has some $\gamma_0 \in \Gamma_V$ with $\gamma_0 g \in \Omega$.

Step 3: proof of (1), by induction. We show by induction on $k \in \mathbb{N}$: every g with $\mathrm{ht}_V(g) \leq 2^{k+1}$ is $\gamma z(x_1 \cdots x_k, 1)$ with $\gamma \in \Gamma_V$, $z \in \Omega$ and $x_i \in G_\infty(s)$, $\|x_{i,v}\| \leq 2$. For $k = 0$, Step 2 gives $g = \gamma_0^{-1}(\gamma_0 g)$. If $\mathrm{ht}_V(g) \leq 2^{k+2}$, Step 1 gives y with $\mathrm{ht}_V(g(y,1)^{-1}) \leq \max(2^{k+1}, 2) = 2^{k+1}$; by the induction hypothesis $g(y,1)^{-1} = \gamma z(x_1 \cdots x_k, 1)$, so $g = \gamma z(x_1 \cdots x_k y, 1)$, with $x_{k+1} = y$. Finally, for any g choose $k \in \mathbb{N}$ with $2^k \leq \mathrm{ht}_V(g) < 2^{k+1}$ (possible as $1 \leq \mathrm{ht}_V(g) < \infty$) and apply the claim. □

Lemma 6.2.30. *Let S be a space with a point x_S and signs s on n, and $g \in \mathrm{G}(s)$ with $\mathrm{ht}_S(g) > 2$. Then there is a nonzero u in the lattice of S_g with $u^* J_S u = 0$ and $\|((x_S g)^{-1}u)_\infty\| \leq 2/\mathrm{ht}_S(g)$.*

Proof. Put $L = \{u \in E^n : ((x_S g)^{-1}u)_f \in \widehat{\mathcal{O}}_E^n\}$, the lattice of S_g, and $N(u) = \|((x_S g)^{-1}u)_\infty\|$. The map $u \mapsto ((x_S g)^{-1}u)_\infty$ embeds L as a discrete subgroup of $\mathbb{C}^{\Phi \times n}$ (L is a lattice of E^n and $(x_S g)_\infty$ is invertible), so only finitely many $u \in L$ have $N(u) \leq 1$. As $\lambda_S(g) = \mathrm{ht}_S(g)^{-1} < \frac{1}{2} < 1$, the infimum defining $\lambda_S(g)$ is a minimum over these finitely many nonzero u: there is $u \in L \setminus \{0\}$ with $N(u) = \lambda_S(g) \leq 2/\mathrm{ht}_S(g)$.

It remains to see that $q = u^* J_S u$ vanishes. It lies in F (J_S is hermitian). Put $w = (x_S g)^{-1}u$; since $(x_S g)^* J_S (x_S g) = \varepsilon_s$ ($x_S^* J_S x_S = \varepsilon_s$ and $g^* \varepsilon_s g = \varepsilon_s$), $q = w^* \varepsilon_s w$ componentwise. At a finite place $\varepsilon_s = 1$ and w_f is integral, so q is integral at every finite place: $q \in \mathcal{O}_F$. At a real place σ of F, below the place $v \in \Phi$ of E, $|\sigma(q)| = |\sum_i s_i |w_{i,v}|^2| \leq \|w_\infty\|^2 = N(u)^2 < \frac{1}{4}$. Hence $q \in \mathcal{O}_F$ has $|N_{F/\mathbb{Q}}(q)| < 1$, so $q = 0$. □

Lemma 6.2.31. *Let S, T be spaces with points with signs s on n and s' on n', α, β automorphic characters of $\mathrm{U}(1)$, A a finite set of finite places of E containing the conjugate of none of its members, $\Psi \in \mathcal{T}_A$ (a function of finite adelic $n' \times n$ matrices), $f_0 = \Psi \otimes_A v^0 \in \mathcal{M}_{S,T}$, and p, q finite index sets. For $d \in \mathrm{M}_{n' \times p}(E)$, $c \in \mathrm{M}_{n \times q}(E)$ and a ring $R \supseteq E$ put $Z_{d,c}(R) = \{x_T^{-1} \, dY \, c^* J_S \, x_S : Y \in \mathrm{M}_{p \times q}(R)\}$; $Z_{d,c}(\mathbb{A}_E)/Z_{d,c}(E)$ is a compact group with Haar probability measure dz. Call (d,c) a **short pair** for (g,h,ℓ,x,y), where $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$, ℓ is a bounded linear functional on $\mathcal{M}_{S,T}$ and $x, y \geq 0$, if:*

(i) the columns of d lie in the lattice of T_h and have lengths at most x, and the columns of c lie in the lattice of S_g and have lengths at most y;

(ii) $d^* J_T d = 0$ or $c^* J_S c = 0$;

(iii) $\ell(\rho(z)f) = \ell(f)$ for all $z \in Z_{d,c}(E)$ and $f \in \mathcal{M}_{S,T}$;

(iv) either, with $d' = (x_T h)^{-1} d$ and $c' = (x_S g)^{-1} c$, every finite adelic $n' \times n$ matrix η with $\Psi(\eta) \neq 0$ has some $w \in A$ with $(d_f'^* \, \eta \, c_f')_w \neq 0$; or $\int_{Z_{d,c}(\mathbb{A}_E)/Z_{d,c}(E)} \ell(\rho(z)f) \, dz = 0$ for every $f \in \mathcal{M}_{S,T}$.

Let $N \in \mathbb{N}$.

1. *There is $C \geq 0$ such that for all g, h, ℓ, x, y and every short pair (d, c) for them, and every $v \in \mathcal{F}_{S,T}$ with $\mathrm{mom}_N(v) < \infty$,*

$$\left|\ell(\omega_{\alpha,\beta}(g,h)(v \otimes f_0))\right| \leq C \,\|\ell \circ \omega_{\alpha,\beta}(g,h)\|\,(xy)^N \,\mathrm{mom}_N(v).$$

2. *Let $Y_0, Z_0 \geq 0$ and $A_0 \in \mathbb{N}$. There is $C' \geq 0$ with the following property. Let $v \in \mathcal{F}_{S,T}$ with $\mathrm{mom}_{N+A_0}(v) < \infty$, g, h, ℓ as above and $r \geq 1$, $H > 0$ such that $|\ell(\omega_{\alpha,\beta}(g,h)f)| \leq Y_0 \|f\|(rH)^{A_0}$ for all $f \in \mathcal{M}_{S,T}$ and, if $H > 2$, there is a short pair for (g, h, ℓ, x, y) with $xy \leq 2Z_0 r/H$. Then*

$$\left|\ell(\omega_{\alpha,\beta}(g,h)(v \otimes f_0))\right| H^N \leq C' \,\mathrm{mom}_{N+A_0}(v)\, r^{N+2A_0}.$$

Proof. Let $V = \mathrm{Hom}(S,T)$, signs $s' \otimes s$ on $n' \times n$, point x_V with $x_V^{-1} X = x_T^{-1} X x_S$.

Step 1: the rank-one maps. Let g, h, ℓ, x, y, d, c be as in (1), k the image of (g,h) under the pair map, d_i, c_j the columns, and $X_{ij} = d_i c_j^* J_S \in \mathrm{M}_{n' \times n}(E)$. Since $(x_S g)^* J_S (x_S g) = \varepsilon_s$, $c_j^* J_S x_S g = c_j'^* \varepsilon_s$, so the coordinates of X_{ij} at k are

$$(x_V k)^{-1} X_{ij} = (x_T h)^{-1} X_{ij} (x_S g) = d_i'\, c_j'^* \varepsilon_s.$$

Its finite part $d'_{i,f} c'^*_{j,f}$ is integral by (i), and

$$\sum_{v \in \Phi} \|d'_{i,v} c'^*_{j,v} \varepsilon_s\|^2 = \sum_v \|d'_{i,v}\|^2 \|c'_{j,v}\|^2 \leq \|d'_{i,\infty}\|^2 \|c'_{j,\infty}\|^2 \leq (xy)^2$$

(Frobenius norms). So the X_{ij} ($\#p \cdot \#q$ of them) lie in the lattice of V_k with lengths at most xy. They span $Z = \{dYc^* J_S : Y \in \mathrm{M}_{p\times q}(E)\}$, and for Y, Y' the form of V gives $\mathrm{tr}\big((dY'c^*J_S)^* J_T (dYc^*J_S) J_S^{-1}\big) = \mathrm{tr}\big(Y'^*(d^*J_T d) Y (c^* J_S c)\big) = 0$ by (ii); so Z is isotropic, $x_V^{-1} Z = Z_{d,c}(E)$ and the $\mathbb{A}_E$-span of $x_V^{-1} Z$ is $Z_{d,c}(\mathbb{A}_E)$. In the same way $k^{-1} Z_{d,c}(\mathbb{A}_E) = I$ with $I = \{d' Y c'^* \varepsilon_s : Y \in \mathrm{M}_{p\times q}(\mathbb{A}_E)\}$, an $\mathbb{A}_E$-submodule of $\mathbb{A}_E^{n'\times n}$.

Step 2: (1). Put $\chi = \alpha(\det h)\beta(\det g)^{-1}$ (modulus 1) and $\Lambda = \chi\ell$, so $\Lambda \circ \omega_{s'\otimes s}(k) = \ell \circ \omega_{\alpha,\beta}(g,h)$, and Λ is invariant under $\rho(x_V^{-1}u)$, $u \in Z$, by (iii). Let Λ_Z be its mean as in (1) of Lemma 6.2.32, and $F = \omega_{s'\otimes s}(k)(v \otimes f_0)$. By (2) of Lemma 6.2.32 (for V, A, Ψ, N, $r = \#p \cdot \#q$, $\delta = xy$; constant C),

$$|\Lambda(F) - \Lambda_Z(F)| \leq C \,\|\ell \circ \omega_{\alpha,\beta}(g,h)\|\,(xy)^N \,\mathrm{mom}_N(v),$$

and $|\ell(\omega_{\alpha,\beta}(g,h)(v \otimes f_0))| = |\Lambda(F)|$. It remains to show $\Lambda_Z(F) = 0$. In the second case of (iv), $\Lambda_Z = \chi \int \ell(\rho(z)\cdot)\,dz = 0$. In the first case let $\Lambda' = \Lambda_Z \circ \omega_{s'\otimes s}(k)$. For $z' \in I$, $kz' \in Z_{d,c}(\mathbb{A}_E)$ and $\omega(k)\rho(z') = \rho(kz')\omega(k)$ ((2) of Lemma 4.1.1), so $\Lambda'(\rho(z')f) = \Lambda_Z(\rho(kz')\omega(k)f) = \Lambda'(f)$ by (1) of Lemma 6.2.32. Let η satisfy $\Psi(\eta) \neq 0$ and $w \in A$ with $(d'^*_f \eta c'_f)_w \neq 0$; choose an entry (i,j) with $(d'^*_{i,f}\eta c'_{j,f})_w \neq 0$ and put $z = d'_i c'^*_j \varepsilon_s \in I$. Then $z_f^* \eta = \sum_{a,b} \overline{z_{f,ab}}\eta_{ab} = d'^*_{i,f}\eta c'_{j,f}$, whose component at w is not 0. So Lemma 6.2.35 (signs $s' \otimes s$ on $n' \times n$, the module I, the functional Λ') gives $\Lambda'(\Psi \otimes_A f) = 0$ for every f. Finally, by (3) of Lemma 6.2.21, $v \otimes f_0 = \sum_y c_y(\Psi)\, v \otimes \rho(\mathrm{fv}(y))P^A v^0 = \Psi \otimes_A (v \otimes v^0)$, so $\Lambda_Z(F) = \Lambda'(\Psi \otimes_A (v \otimes v^0)) = 0$.

Step 3: (2). Let C be the constant of (1) for $N + A_0$ in place of N, and $L = \ell \circ \omega_{\alpha,\beta}(g,h)$; then $\|L\| \leq Y_0(rH)^{A_0}$, and $\mathrm{mom}_0(v) = \|v\| \leq \mathrm{mom}_{N+A_0}(v)$. If $H > 2$, (1) applied to the short pair with $xy \leq 2Z_0 r/H$ gives

$$\begin{aligned} |L(v \otimes f_0)| H^N &\leq C\, Y_0 (rH)^{A_0} \Big(\frac{2Z_0 r}{H}\Big)^{N+A_0} H^N \mathrm{mom}_{N+A_0}(v) \\ &= C\, Y_0 (2Z_0)^{N+A_0} r^{N+2A_0}\, \mathrm{mom}_{N+A_0}(v). \end{aligned}$$

If $H \le 2$,

$$|L(v \otimes f_0)|H^N \le \|L\|\,\|f_0\|\,\|v\|\,2^N \le Y_0(2r)^{A_0}2^N\|f_0\|\,\mathrm{mom}_{N+A_0}(v)$$
$$\le 2^{N+A_0}Y_0\|f_0\|\,r^{N+2A_0}\mathrm{mom}_{N+A_0}(v),$$

as $r \ge 1$. So $C' = 2^{N+A_0}Y_0(C\,Z_0^{N+A_0} + \|f_0\|)$ works. □

Lemma 6.2.32. *Let V be a space with a point with signs s on n, $\varepsilon(v,i) = s_i$ on $\iota = \Phi \times n$, A a finite set of finite places of E containing the conjugate of none of its members, $\Psi \in \mathcal{T}_A$, $f_0 = \Psi \otimes_A v^0$, and $N, r \in \mathbb{N}$. For an E-subspace $Z \subseteq E^n$ with $u'^* J_V u = 0$ for all $u, u' \in Z$, let $Z(\mathbb{A}) \subseteq \mathbb{A}_E^n$ be the $\mathbb{A}_E$-span of $x_V^{-1}Z$, $Z(E) = x_V^{-1}Z$, and dz the Haar probability measure of the compact group $Z(\mathbb{A})/Z(E)$.*

1. *Let Z be such a subspace and Λ a bounded linear functional on $\mathcal{M}_s$ with $\Lambda(\rho(x_V^{-1}u)f) = \Lambda(f)$ for all $u \in Z$, $f \in \mathcal{M}_s$. Then for every f the function $z \mapsto \Lambda(\rho(z)f)$ on $Z(\mathbb{A})$ is continuous and $Z(E)$-periodic, $\Lambda_Z(f) = \int_{Z(\mathbb{A})/Z(E)} \Lambda(\rho(z)f)\,dz$ is a bounded linear functional, $\Lambda_Z(\rho(z)f) = \Lambda_Z(f)$ for $z \in Z(\mathbb{A})$, and $\|\Lambda_Z \circ \omega_s(g)\| \le \|\Lambda \circ \omega_s(g)\|$ for every $g \in \mathrm{G}(s)$.*

2. *There is $C \ge 0$ with the following property. Let $g \in \mathrm{G}(s)$, $\delta \ge 0$, and let $u_1, \dots, u_r$ be vectors of the lattice of V_g with $\|((x_V g)^{-1}u_i)_\infty\| \le \delta$ that span a subspace Z as above; let Λ and Λ_Z be as in (1). Then for every $v \in \mathcal{F}_\varepsilon$ with $\mathrm{mom}_N(v) < \infty$,*

$$\big|(\Lambda - \Lambda_Z)(\omega_s(g)(v \otimes f_0))\big| \le C\,\|\Lambda \circ \omega_s(g)\|\,\delta^N\,\mathrm{mom}_N(v).$$

Proof. **Step 1: the Heisenberg operators of $Z(\mathbb{A})$ commute.** Since $x_V^* J_V x_V = \varepsilon_s$, $(x_V^{-1})^* \varepsilon_s x_V^{-1} = J_V$. For $a = \sum_i c_i x_V^{-1}u_i$, $b = \sum_j c'_j x_V^{-1}u'_j$ ($c_i, c'_j \in \mathbb{A}_E$, $u_i, u'_j \in Z$), $\bar{b}^{\mathsf{T}}\varepsilon_s a = \sum_{i,j} \bar{c}'_j c_i\, u'^*_j J_V u_i = 0$, so $\beta_s(a,b) = 1$ (WD6) and $\rho(a)\rho(b) = \rho(a+b)$ on $Z(\mathbb{A})$. Also $Z(\mathbb{A})/Z(E)$ is compact: for a basis $u_1, \dots, u_m$ of Z, $(c_i) \mapsto \sum c_i x_V^{-1}u_i$ maps $\mathbb{A}_E^m$ onto $Z(\mathbb{A})$ and E^m into $Z(E)$, so the quotient is a continuous image of the compact $(\mathbb{A}_E/E)^m$; and $Z(E) \subseteq x_V^{-1}E^n$ is discrete.

Step 2: (1). For $u \in Z$, $\Lambda(\rho(z + x_V^{-1}u)f) = \Lambda(\rho(x_V^{-1}u)\rho(z)f) = \Lambda(\rho(z)f)$ by Step 1; continuity is the strong continuity of ρ. So Λ_Z is defined, linear, and $|\Lambda_Z(f)| \le \|\Lambda\|\|f\|$. For $z' \in Z(\mathbb{A})$, $\Lambda_Z(\rho(z')f) = \int \Lambda(\rho(z+z')f)\,dz = \Lambda_Z(f)$ by Step 1 and invariance of dz. Finally $\rho(z)\omega_s(g) = \omega_s(g)\rho(g^{-1}z)$, so $|\Lambda(\rho(z)\omega_s(g)f)| \le \|\Lambda \circ \omega_s(g)\|\,\|\rho(g^{-1}z)f\| = \|\Lambda \circ \omega_s(g)\|\|f\|$, and integrating gives $\|\Lambda_Z \circ \omega_s(g)\| \le \|\Lambda \circ \omega_s(g)\|$.

Step 3: an integer that fixes f_0. By (WR6) and $P^A v^0 = v^0$, $f_0 = \sum_k c_k \rho(\mathrm{fv}(y_k))v^0$ for finitely many $y_k \in \mathbb{A}_{E,f}^n$. Choose an even integer $M \ge 2$ with $\frac{M}{2}y_k \in \widehat{\mathcal{O}}_E^n$ for all k. For $y \in \widehat{\mathcal{O}}_E^n$, $\beta_s(\mathrm{fv}(My), \mathrm{fv}(y_k)) = \psi_0(\frac{M}{2}\bar{y}_k^{\mathsf{T}}y) = 1$ and $\beta_s(\mathrm{fv}(y_k), \mathrm{fv}(My)) = \psi_0(\frac{M}{2}\bar{y}^{\mathsf{T}}y_k) = 1$, so $\rho(\mathrm{fv}(My))$ commutes with $\rho(\mathrm{fv}(y_k))$; and $\rho(\mathrm{fv}(My))v^0 = \psi_0(\frac{c_0M^2}{2}\bar{y}^{\mathsf{T}}y)^{-1}\rho_f(My)v^0 = v^0$ ($M^2/2$ and c_0 integral). Hence $\rho(\mathrm{fv}(My))f_0 = f_0$ for all $y \in \widehat{\mathcal{O}}_E^n$.

Step 4: rational shifts act on $w \otimes f_0$ as short real translations. By compactness of $\mathbb{A}_E/E$ there is $R > 0$ such that every $c \in \mathbb{A}_E$ is $a + w$ with $a \in E$, $w_f \in \widehat{\mathcal{O}}_E$ and $|w_v| \le R$ for $v \in \Phi$; applied to c/M this writes $c = Ma' + Mw$ with $a' \in E$. Let g, δ, $u_1, \dots, u_r$, Z be as in (2), and $z \in Z(\mathbb{A})$; as the u_i span Z, $z = \sum_i c_i x_V^{-1}u_i$, and writing $c_i = Ma'_i + Mw_i$ we put $u = -M\sum_i a'_i u_i \in Z$, so that $g^{-1}(x_V^{-1}u + z) = Mp$ with $p = \sum_i w_i (x_V g)^{-1}u_i$. The finite part p_f is integral (the u_i lie in the lattice of V_g), and $\|p_\infty\| \le rR\delta$. The archimedean part Mp_∞ and the finite part Mp_f have disjoint supports, so $\beta_s(Mp_\infty, Mp_f) = 1$ and $\rho(Mp) = \rho(Mp_\infty)\rho(Mp_f)$. Now $Mp_f = \mathrm{fv}(M\alpha^{-1}p_f)$ with $\alpha^{-1}p_f$ integral (α^{-1} has integral components, (WD2)), and

$Mp_\infty = \mathrm{rv}(y)$ with $y_{v,i} = \kappa_v^{-1} M p_{v,i}$, $\|y\|_2 \le C_0\delta$, $C_0 = MrR \max_v \kappa_v^{-1}$. By Lemma 6.2.21 and Step 3, for every $w \in \mathcal{F}_\varepsilon$,

$$\rho(g^{-1}(x_V^{-1}u + z))(w \otimes f_0) = \rho(\mathrm{rv}(y))(w \otimes \rho(\mathrm{fv}(M\alpha^{-1}p_f))f_0) = (\rho^\infty(y)w) \otimes f_0. \tag{1}$$

Step 5: (2). Let $\Lambda' = \Lambda - \Lambda_Z$; it is invariant under $\rho(x_V^{-1}u)$, $u \in Z$ (by hypothesis and (1)), and $\int \Lambda'(\rho(z)F)\,dz = \Lambda_Z(F) - \Lambda_Z(F) = 0$ for every F, so

$$\Lambda'(F) = \int_{Z(\mathbb{A})/Z(E)} (\Lambda'(F) - \Lambda'(\rho(z)F))\,dz, \qquad |\Lambda'(F)| \le \sup_{z \in Z(\mathbb{A})} |\Lambda'(F) - \Lambda'(\rho(z)F)|. \tag{2}$$

Define $\ell(w) = \Lambda'(\omega_s(g)(w \otimes f_0))$ on $\mathcal{F}_\varepsilon$; it is bounded with $\|\ell\| \le \|\Lambda' \circ \omega_s(g)\|\,\|f_0\| \le 2\|f_0\|\,\|\Lambda \circ \omega_s(g)\|$ by (1). For $z \in Z(\mathbb{A})$ with u, y as in Step 4, Step 1, invariance and (1) give

$$\begin{aligned}\Lambda'(\rho(z)\omega_s(g)(w \otimes f_0)) &= \Lambda'(\rho(x_V^{-1}u + z)\omega_s(g)(w \otimes f_0)) \\ &= \Lambda'(\omega_s(g)\rho(g^{-1}(x_V^{-1}u + z))(w \otimes f_0)) = \ell(\rho^\infty(y)w).\end{aligned}$$

With (2) for $F = \omega_s(g)(w \otimes f_0)$: $|\ell(w)| \le \sup_{\|y\|_2 \le C_0\delta} |\ell(w) - \ell(\rho^\infty(y)w)|$ for every w. By Lemma 6.2.34 with $t = C_0\delta$ (its constant C_F for ι, ε, N), $|\ell(v)| \le C_F\|\ell\|(C_0\delta)^N \mathrm{mom}_N(v)$, which is (2) with $C = 2C_F C_0^N \|f_0\|$. □

Lemma 6.2.33. *Let s be signs on n, and A a finite set of finite places of E containing the conjugate of none of its members. For functions $\Psi, \Psi' \in \mathcal{T}_A$ put*

$$\langle \Psi, \Psi' \rangle_A = \int_{X_A} \overline{\Psi(x)}\,\Psi'(x)\,dx,$$

where X_A is the set of $x \in \mathbb{A}_{E,f}^n$ whose components at the finite places outside A are integral, and dx is the Haar measure of $\mathbb{A}_{E,f}^n$ giving $\widehat{\mathcal{O}}_E^n$ mass 1. Then:

1. *for $\Psi, \Psi' \in \mathcal{T}_A$ and $f, f' \in \mathcal{M}_s$, $\langle \Psi \otimes_A f, \Psi' \otimes_A f' \rangle = \langle \Psi, \Psi' \rangle_A \langle P^A f, P^A f' \rangle$;*
2. *$P^A v^0 = v^0$;*
3. *if A' is a finite set of finite places containing the conjugate of none of its members, with $(A' \cup \bar{A}') \cap (A \cup \bar{A}) = \emptyset$, then $P^{A'}(\Psi \otimes_A v^0) = \Psi \otimes_A v^0$ for every $\Psi \in \mathcal{T}_A$;*
4. *for $\Psi \in \mathcal{T}_A$, $v_1 \otimes (\Psi \otimes_A v^0) = \Psi \otimes_A v^0$, where v_1 is the real vacuum of (WR5) of Definition 3.2.18 for the signs s on $\Phi \times n$;*
5. *for $\Psi \in \mathcal{T}_A$ and every $y \in \widehat{\mathcal{O}}_E^n$ whose components at the places of $A \cup \bar{A}$ are 0, $\rho_f(y)(\Psi \otimes_A v^0) = \Psi \otimes_A v^0$ (for $A = \emptyset$, $\Psi = 1$: $\rho_f(y)v^0 = v^0$ for every $y \in \widehat{\mathcal{O}}_E^n$);*
6. *for Fock vectors v, v' (signs s on $\Phi \times n$) and $f, f' \in \mathcal{M}_s$, $\langle v \otimes f, v' \otimes f' \rangle = \langle v, v' \rangle \langle v_1 \otimes f, v_1 \otimes f' \rangle$, the first factor the inner product of the Fock model;*
7. *let Σ be a set of finite places of F, and call $f \in \mathcal{M}_s$ a **vacuum above** Σ if $\rho_f(y)f = f$ for every $y \in \widehat{\mathcal{O}}_E^n$ whose components at the finite places not above Σ are 0. If f, f' are vacua above Σ and $\gamma \in \mathrm{G}(s)$ has components 1 at the infinite places and at the finite places not above Σ, then $\langle f', \omega_s(\gamma)f \rangle = \langle v^0, \omega_s(\gamma)v^0 \rangle \langle f', f \rangle$.*

Proof. **Step 1: (1).** Each $\Psi \in \mathcal{T}_A$ depends only on the components at A (Lemma 4.1.3 identifies $\mathcal{T}_A$ with functions on $Q_A = \prod_{w\in A} E_w^n$). The Haar measure of $\mathbb{A}_{E,f}^n$ with $\widehat{\mathcal{O}}_E^n$ of mass 1 is the product of the local ones with $\mathcal{O}_{E,w}^n$ of mass 1, and $X_A = Q_A \times \prod_{w\notin A} \mathcal{O}_{E,w}^n$; so $\langle \Psi, \Psi'\rangle_A = \int_{Q_A} \bar{\Psi}\Psi'$, and (1) is (1) of Lemma 4.1.3.

Step 2: the finite Heisenberg operators of integral vectors. For $y \in \widehat{\mathcal{O}}_E^n$, (WD8) gives $\rho_f(y) = \psi_0(\frac{c_0}{2} y^* y)\rho(\mathrm{fv}(y)) = e_n(\alpha y)\rho(\alpha y)$ (WD5). For $y, y' \in \widehat{\mathcal{O}}_E^n$, $\beta_s(\mathrm{fv}(y), \mathrm{fv}(y')) = \psi_0(\frac{1}{2} y'^* y)$, so

$$\rho_f(y)\rho_f(y')\rho_f(y+y')^{-1} = \psi_0(\tfrac{1}{2} y'^* y - \tfrac{c_0}{2}(y^* y' + y'^* y)) = \psi_0(\tfrac{1}{2}(X - \bar{X})) = 1, \qquad X = \bar{c}_0\, y'^* y \in \widehat{\mathcal{O}}_E,$$

using $1 - c_0 = \bar{c}_0$, $\psi_0(\bar{t}) = \psi_0(t)^{-1}$ and $\psi_0(\widehat{\mathcal{O}}_E) = 1$; so $y \mapsto \rho_f(y)$ is a continuous homomorphism on $\widehat{\mathcal{O}}_E^n$. For a set S of finite places let O_S be the integral vectors with components 0 outside S. With $\Sigma = S$ in (4) of Lemma 6.2.12, $P_S = \int_{O_S} \rho_f(y)\, dy$ (Haar probability measure), so $\rho_f(y) P_S = P_S$ for $y \in O_S$ (translation invariance), and a vector f satisfies $\rho_f(y) f = f$ for all $y \in O_S$ if and only if $P_S f = f$. As $P_S v^0 = v^0$, $\rho_f(y) v^0 = v^0$ for $y \in O_S$; with S all finite places this is the case $A = \emptyset$ of (5). Moreover $P^A = P_{A\cup \bar{A}}$ by (WR6).

Step 3: (5), (2), (3). Let $\Psi \in \mathcal{T}_A$, so $\Psi \otimes_A v^0 = \sum_{y'} c_{y'}(\Psi)\rho(\mathrm{fv}(y'))P^A v^0$ with y' supported at $A \cup \bar{A}$. If $y \in \widehat{\mathcal{O}}_E^n$ has components 0 at $A \cup \bar{A}$, then $\rho_f(y)$ commutes with $\rho(\mathrm{fv}(y'))$ (disjoint supports) and with $P^A = P_{A\cup\bar{A}}$ ((4) of Lemma 6.2.12), so $\rho_f(y)(\Psi \otimes_A v^0) = \sum c_{y'} \rho(\mathrm{fv}(y'))P^A \rho_f(y) v^0 = \Psi \otimes_A v^0$ by Step 2: this is (5). (2) is $P_{A\cup\bar{A}} v^0 = v^0$. For (3), $P^{A'} = P_{A'\cup\bar{A}'}$ is the mean of the $\rho_f(y)$, $y \in O_{A'\cup\bar{A}'}$, and these y vanish at $A \cup \bar{A}$; by (5) each fixes $\Psi \otimes_A v^0$.

Step 4: (4). By (3) and (4) of Lemma 6.2.21, $v_1 \otimes (\Psi \otimes_A v^0) = \sum c_{y'} \rho(\mathrm{fv}(y'))P^A(v_1 \otimes v^0) = \Psi \otimes_A v^0$.

Step 5: (6). Let $P = P_\Phi$ be the partial projection at the infinite places; in normalized coordinates $e_n(\mathrm{rv}(t)) = v_1(t)$ and μ_Φ is the image of ν, so $Pf = v_1 \otimes f$, and $P^2 = P$ by (5) of Lemma 6.2.21. First, $P\rho(\mathrm{rv}(\tau))P = v_1(\tau)P$: for $z = z_\infty + z_f$ (archimedean and finite parts), (4) of Lemma 6.2.12 gives $P\rho(z)v^0 = \rho(z_f)P\rho(z_\infty)v^0 = e_n(z_\infty)\rho(z_f)v^0$, and then $P\rho(\mathrm{rv}(\tau))P\rho(z)v^0 = e_n(z_\infty)\rho(z_f)P\rho(\mathrm{rv}(\tau))v^0 = v_1(\tau)P\rho(z)v^0$; superposition and boundedness extend this to all vectors. Next, for $v \in \mathcal{F}_\varepsilon$ the reproducing identity (WR5) says $v = \int v(\tau)\rho^\infty(\tau) v_1\, d\nu(\tau)$ as a Bochner integral in $L^1(\nu)$ (strong continuity, Fubini), so by (1), (2), (5) of Lemma 6.2.21, $v \otimes f = v \otimes Pf = \int v(\tau)\rho(\mathrm{rv}(\tau))Pf\, d\nu(\tau)$. Hence, with $\rho(\mathrm{rv}(\tau))^{-1}\rho(\mathrm{rv}(\tau')) = \beta_\varepsilon^\infty(-\tau, \tau')\rho(\mathrm{rv}(\tau'-\tau))$ and $\langle Pf, X\rangle = \langle Pf, PX\rangle$ (P symmetric, $P^2 = P$),

$$\langle v \otimes f, v' \otimes f'\rangle = \iint \overline{v(\tau)} v'(\tau') \beta_\varepsilon^\infty(-\tau, \tau')\, v_1(\tau' - \tau)\, d\nu\, d\nu \cdot \langle Pf, Pf'\rangle.$$

Since $\overline{\beta_\varepsilon^\infty(-\tau, \tau')} = \beta_\varepsilon^\infty(\tau, \tau')$ and v_1 is real, the τ-integral is the conjugate of $\int v(\tau)\beta_\varepsilon^\infty(\tau, \tau') v_1(\tau' - \tau)\, d\nu(\tau) = v(\tau')$, by (WR5); so the double integral is $\langle v, v'\rangle$, and $\langle Pf, Pf'\rangle = \langle v_1 \otimes f, v_1 \otimes f'\rangle$.

Step 6: (7). Let S be the set of places of E above Σ, R its complement in the set of all places, $c = \langle v^0, \omega_s(\gamma) v^0\rangle$, and $B(h, h') = \langle P_S h, \omega_s(\gamma) P_S h'\rangle - c\langle P_S h, P_S h'\rangle$, a bounded sesquilinear form. For $y \in A_R$, $\gamma y = y$, so $\omega_s(\gamma)$ commutes with $\rho(y)$, hence with P_R, and $P_R \omega_s(\gamma) v^0 = \omega_s(\gamma) v^0$. For $z = z_S + z_R$, $z' = z'_S + z'_R$ (parts in A_S, A_R), Step 2 and (4) of Lemma 6.2.12 give $P_S \rho(z) v^0 = e_n(z_S)\rho(z_R)v^0$, and with $Z = z'_R - z_R \in A_R$,

$$\begin{aligned}\langle P_S\rho(z)v^0, \omega_s(\gamma) P_S \rho(z')v^0\rangle &= \overline{e_n(z_S)} e_n(z'_S)\beta_s(-z_R, z'_R)\langle \rho(-Z)v^0, P_R\omega_s(\gamma)v^0\rangle \\ &= \overline{e_n(z_S)} e_n(z'_S)\beta_s(-z_R, z'_R)\, e_n(Z)\, c,\end{aligned}$$

by the symmetry of P_R and $P_R\rho(-Z)v^0 = e_n(-Z)v^0 = e_n(Z)v^0$. The case $\gamma = 1$ gives the same expression with c replaced by $\langle v^0, v^0\rangle = 1$. So B vanishes on pairs of Heisenberg translates of

v^0, hence, by superposition in both arguments, everywhere. If f, f' are vacua above Σ, then $P_S f = f$, $P_S f' = f'$ by Step 2, and $B(f', f) = 0$ is (7). $\square$

Lemma 6.2.34. *Let ε be signs on a finite index set ι and $N \in \mathbb{N}$. There is $C \geq 0$ with the following property. Let $t \geq 0$ and let ℓ be a bounded linear functional on $\mathcal{F}_\varepsilon$ such that, if $t < 1$,*

$$|\ell(f)| \leq \sup_{y \in \mathbb{C}^\iota,\ \|y\|_2 \leq t} \big|\ell(f) - \ell(\rho^\infty(y) f)\big| \qquad \text{for every } f \in \mathcal{F}_\varepsilon.$$

Then $|\ell(v)| \leq C\,\|\ell\|\,t^N \operatorname{mom}_N(v)$ for every $v \in \mathcal{F}_\varepsilon$ with $\operatorname{mom}_N(v) < \infty$.

Proof. Write $D = \operatorname{diag}(\varepsilon)$, $\nu = \nu_\iota$, $\rho = \rho^\infty$, $\beta = \beta^\infty_\varepsilon$, $e = e^\infty$, and $\|\cdot\|$ for the norm of (XD3); let $c_\iota \geq 1$ with $1 + \|z\|_2 \leq c_\iota(1 + \|z\|)$ for all z (equivalence of norms on $\mathbb{C}^\iota$). Note $|\beta| = 1$, $\beta(x, y) = \exp(i \operatorname{Im} y^* D x)$ is additive in each variable, and $(\rho(z)v_1)(x) = \beta(z, x) e(x - z)$.

Superposition. For $v \in \mathcal{F}_\varepsilon$, the map $z \mapsto v(z)\rho(z)v_1$ is Bochner integrable in $\mathcal{F}_\varepsilon$ (continuous, with norm $|v(z)|\,\|v_1\|$), and its integral is v: evaluated at x (Fubini) it is $\int v(z)\beta(z, x)e(x - z)\,d\nu(z) = v(x)$, the reproducing identity of (WR5). Hence $\ell(v) = \int v(z)\,\ell(\rho(z)v_1)\,d\nu(z)$ and

$$|\ell(v)| \leq \int |v(z)|\,|\ell(\rho(z)v_1)|\,d\nu(z). \tag{1}$$

The case $t \geq 1$. Then $|\ell(v)| \leq \|\ell\|\,\|v\| = \|\ell\|\operatorname{mom}_0(v) \leq \|\ell\|\,t^N \operatorname{mom}_N(v)$.

The case $t < 1$; iteration. Put $\Delta_y = 1 - \rho(y)$. We show by induction on k that $|\ell(f)| \leq \sup|\ell(\Delta_{y_k} \cdots \Delta_{y_1} f)|$ for every f, the supremum over $y_1, \ldots, y_k$ with $\|y_i\|_2 \leq t$ (it is at most $2^k \|\ell\| \|f\|$, since $\rho(y)$ is an isometry). For $k = 0$ there is nothing to show. Given the claim for k, apply the hypothesis to $f' = \Delta_{y_k} \cdots \Delta_{y_1} f$: $|\ell(f')| \leq \sup_{\|y\|_2 \leq t} |\ell(\Delta_y f')|$; taking the supremum over $y_1, \ldots, y_k$ gives the claim for $k + 1$. With $k = N$ and $f = \rho(z)v_1$,

$$|\ell(\rho(z)v_1)| \leq \|\ell\| \sup_{\|y_i\|_2 \leq t} \big\|\Delta_{y_N} \cdots \Delta_{y_1} \rho(z) v_1\big\|. \tag{2}$$

An N-th difference of a coherent vector. Fix z and $y_1, \ldots, y_N$ with $\|y_i\|_2 \leq t \leq 1$. For $\sigma = (\sigma_1, \ldots, \sigma_N) \in [0, 1]^N$ let $G_\sigma = \rho(\sigma_N y_N) \cdots \rho(\sigma_1 y_1)\rho(z)v_1$ and $w_\sigma = \sum_j \sigma_j y_j$. Applying $(\rho(a)c)(x) = \beta(a, x)c(x - a)$ repeatedly,

$$G_\sigma(x) = \prod_{i=1}^{N} \beta\Big(\sigma_i y_i,\ x - \sum_{j>i} \sigma_j y_j\Big) \cdot \beta(z, x - w_\sigma)\, e(x - w_\sigma - z) = \exp Q(\sigma, x),$$

where, by additivity of β and $e(u) = \exp(-u^*u/2)$, $Q(\cdot, x)$ is a polynomial of degree at most 2 in σ. Expanding the product,

$$\Delta_{y_N} \cdots \Delta_{y_1} \rho(z) v_1 = \sum_{S \subseteq \{1, \ldots, N\}} (-1)^{\#S} G_{\mathbf{1}_S}$$

($\rho(0) = 1$), and by the fundamental theorem of calculus in each variable this function of x equals

$$(-1)^N \int_{[0,1]^N} \partial_{\sigma_1} \cdots \partial_{\sigma_N} \exp Q(\sigma, x)\,d\sigma.$$

By the formula for mixed derivatives of an exponential,

$$\partial_{\sigma_1} \cdots \partial_{\sigma_N} e^Q = e^Q \sum_\pi \prod_{B \in \pi} \partial_B Q,$$

the sum over the partitions π of $\{1,\dots,N\}$, ∂_B the mixed derivative in the σ_i, $i\in B$; $\partial_B Q=0$ if $\#B\ge 3$. Reading off Q (each term is $\pm\frac12$ or $\pm\frac{i}{2}$ times a^*Db or a^*b with a,b among x, z, the $\sigma_j y_j$, and $|a^*Db|\le\|a\|_2\|b\|_2$): $|\partial_{\sigma_i}Q|\le 3\|y_i\|_2(\|x\|_2+\|z\|_2+N)$ and $|\partial_{\sigma_i}\partial_{\sigma_j}Q|\le 2\|y_i\|_2\|y_j\|_2$ for $i\ne j$, on $[0,1]^N$. Hence, with b_N the number of partitions,

$$|\partial_{\sigma_1}\cdots\partial_{\sigma_N}e^{Q}|\le b_N(3t)^N(1+N+\|x\|_2+\|z\|_2)^N\,e(x-w_\sigma-z),$$

using $|e^Q|=|G_\sigma(x)|=e(x-w_\sigma-z)$. Integrating over x, substituting $x=x'+w_\sigma+z$ and using $\|w_\sigma\|_2\le N$ and $1+2N+2\|z\|_2+\|x'\|_2\le 2(1+N)(1+\|z\|_2)(1+\|x'\|_2)$,

$$\|\Delta_{y_N}\cdots\Delta_{y_1}\rho(z)v_1\|\le C_1\,t^N(1+\|z\|_2)^N,\qquad C_1=b_N6^N(1+N)^N\int(1+\|x'\|_2)^N e(x')\,d\nu(x')<\infty.$$

Conclusion. By (2), $|\ell(\rho(z)v_1)|\le C_1\|\ell\|t^N(1+\|z\|_2)^N\le C_1c_\iota^N\|\ell\|t^N(1+\|z\|)^N$, and (1) gives $|\ell(v)|\le C_1c_\iota^N\|\ell\|t^N\mathrm{mom}_N(v)$. So $C=\max(1,C_1c_\iota^N)$ works in both cases. □

Lemma 6.2.35. *Let s be signs on a finite index set m, A a finite set of finite places of E containing the conjugate of none of its members, $\Psi\in\mathcal{T}_A$, $I\subseteq\mathbb{A}_E^m$ an $\mathbb{A}_E$-submodule, and Λ a bounded linear functional on $\mathcal{M}_s$ with $\Lambda(\rho(z)f)=\Lambda(f)$ for all $z\in I$ and $f\in\mathcal{M}_s$. Suppose that for every $y\in\mathbb{A}_{E,f}^m$ with $\Psi(y)\ne 0$ there are $z\in I$ and $w\in A$ with $(z_f^*y)_w\ne 0$. Then $\Lambda(\Psi\otimes_A f)=0$ for every $f\in\mathcal{M}_s$.*

Proof. Fix f. Functions in $\mathcal{T}_A$ are read on Q_A.

Step 1: a character for each point of the support. Let $x\in Q_A$ with $\Psi(x)\ne 0$ (read as a finite adelic vector with components 0 off A). By hypothesis there are $z\in I$ and $w\in A$ with $b=(z_f^*x)_w=\sum_k\overline{z_{k,\bar w}}\,x_{k,w}\ne 0$ (on adeles, $\bar z_w=\overline{z_{\bar w}}$). The restriction of ψ_0 to E_w is a nontrivial continuous character ($\jmath\alpha\bar\alpha\ne 0$ and $\psi_{E,f}$ is nontrivial on E_w), so there is $t_0\in E_w$ with $\psi_0(t_0)\ne 1$; as $a\mapsto\overline{\alpha_{\bar w}^{-1}}\,a\,b$ is a bijection $E_{\bar w}\to E_w$, there is $a_x\in\mathbb{A}_E$ with components 0 except at $\bar w$ and $\overline{\alpha_{\bar w}^{-1}(a_x)_{\bar w}}\,b=t_0$. Put $z_x=a_xz\in I$, supported at $\bar w\in\bar A$, and $y_x=\alpha^{-1}(z_x)_f$, so $\mathrm{fv}(y_x)=z_x$. Since $(y_x)_A=0$,

$$(R_A(y_x)\Phi)(x')=\chi_x(x')\,\Phi(x'),\qquad \chi_x(x')=\psi_0(\bar y_x^{\mathsf T}x'_A)=\psi_0\Big(\overline{\alpha_{\bar w}^{-1}(a_x)_{\bar w}}\sum_k\overline{z_{k,\bar w}}\,x'_{k,w}\Big),$$

for $\Phi\in\mathcal{T}_A$; χ_x is a continuous character of Q_A with $\chi_x(x)=\psi_0(t_0)\ne 1$.

Step 2: a decomposition of Ψ. Let $U_0\subseteq Q_A$ be an open compact subgroup under which Ψ is invariant (Ψ is locally constant with compact support). For each x in the compact support K of Ψ let $U_x\subseteq U_0$ be an open compact subgroup on which χ_x is trivial (its kernel is open). Finitely many $x_1+U_{x_1},\dots,x_r+U_{x_r}$ cover K; let $U=\bigcap_j U_{x_j}$. Then $\Psi=\sum_c\Psi(c)\mathbf{1}_{c+U}$, the finite sum over the cosets $c+U$ meeting K (Ψ is constant on each, as $U\subseteq U_0$). Each such coset meets some $x_j+U_{x_j}$, hence lies in it, and there χ_{x_j} is constant, equal to $\chi_{x_j}(x_j)\ne 1$.

Step 3: each piece is killed. Let $c+U$ be one of these cosets, j as in Step 2, and $\Phi=\mathbf{1}_{c+U}\in\mathcal{T}_A$. Then $R_A(y_{x_j})\Phi=\chi_{x_j}(x_j)\Phi$, so by invariance of Λ under $\rho(z_{x_j})=\rho(\mathrm{fv}(y_{x_j}))$,

$$\Lambda(\Phi\otimes_A f)=\Lambda\big(\rho(\mathrm{fv}(y_{x_j}))(\Phi\otimes_A f)\big)=\chi_{x_j}(x_j)\,\Lambda(\Phi\otimes_A f),$$

and $\Lambda(\Phi\otimes_A f)=0$. By linearity of $\Psi\mapsto\Psi\otimes_A f$ and Step 2, $\Lambda(\Psi\otimes_A f)=0$. □

The second half evaluates the Tate integral (Lemma 6.2.36). Poisson's formula on the new line and the factorisation of the line values of standard data into a matrix Gauss sum, a finite factor and a kernel of the smaller pair reduce it to Tate's integral at the finite places, which

gives $|\alpha_0|^x Q^{x-1/2} L^\Sigma(x,\tilde\nu)$ as in [294, 128], times real cone integrals, which converge and are again Schwartz vectors.

The group continues with the mixed model in its global form. The Weil representation of a pair of spaces with points is the classical Weil representation of the unitary dual pair, split by a pair of characters of U(1) as in [172], and Lemma 6.2.44 gives the four properties that every unfolding uses. Real slices of Schwartz vectors decay rapidly, and every smooth compactly supported family of unitary translates of a Schwartz vector is a slice family (Lemma 6.2.43); in our reading this is what lets the exchange identity prescribe the real vectors of its data.

The section closes with three principles of integration over quotients by discrete subgroups, used repeatedly in the exchange identity: unfolding in stages (Lemma 6.2.47) and the averaging criterion (Lemma 6.2.50), both consequences of one unfolding formula (Lemma 6.2.48), an instance of Weil's formula for quotient measures [320]; and a constancy criterion for automorphic functions on a unitary group that splits off a hyperbolic plane (Lemma 6.2.49), proved by strong approximation [167, 233]. In our reading the averaging criterion is the measure-theoretic form of the vanishing of a period against a character that is nontrivial on the stabiliser.

Lemma 6.2.36. *Let W', W, V', V, A, X be as above, let b be an index set with bijections $\sigma\colon b \sqcup r \to n$ and $\sigma'\colon b \sqcup r' \to n'$, assume that $\#n$ is even, and let $\mu\colon \Phi \to \mathbb{Z}$. Fix a Haar measure on $\mathbb{A}_E^\times$ and give $Y = E^\times\backslash\mathbb{A}_E^\times \times [\mathrm{U}(V)] \times \mathrm{K}_{\max}(t')$ the measure of Lemma 6.2.3. For a split tame character $\mathfrak{c} = (\nu,\Sigma)$ let α_W, $\alpha_{W'}$ and, for $f \in \mathcal{M}_{W',V'}$, $u \in \mathbb{A}_E^\times$, $g \in \mathrm{G}(s)$ and real x, the function $Z_x(f;u,g)$ on Y be as in Lemma 6.2.3:*

$$Z_x(f;u,g)(a,h,\kappa) = \tilde\nu(au)\,|au|^x \sum_{\xi\in E^\times} \theta^{W,V}_{\alpha_W,\alpha_W}\Big(\mathrm{Ln}^+_{A,X}(\xi au)(\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f)\Big)(g,h),$$
$$\alpha_W = \nu^{-1}\upsilon^{\#n/2},\ \alpha_{W'} = \nu^{-1}\upsilon^{\#n'/2}.$$

Let $\alpha_0 \in \mathbb{A}_E^\times$ be the idele with components κ_v at $v \in \Phi$ and finite part the Weil idele α ((WD2), (WD6) of Definition 3.2.12). Then there are real numbers $C_+, C_- > 0$ such that for every split tame character $\mathfrak{c} = (\nu,\Sigma)$ of weight μ there are $\epsilon_+, \epsilon_- \in \mathbb{C}$ with $|\epsilon_\pm| = C_\pm$ and the following property. Let $Q = \prod_{w\in\Sigma} \mathrm{N}w$, let $v \in \mathcal{F}_{W',V'}$ be a Schwartz vector, $y \in \mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$, $f = \varphi^{\mathrm{pl}}_{A,\sigma'}(v,y)$, and $u \in \mathbb{A}_E^\times$, $g \in \mathrm{G}(s)$ with $u = 1$ or $\#r = \#n$. Then $Z_x(f;u,g)$ is integrable on Y for every real x and $x \mapsto \int_Y Z_x(f;u,g)$ is a real-analytic function on $\mathbb{R}$; for every $x \geq 1$ the cone integrals $\mathrm{Cone}^\pm_{A,X,\mu}(x;v)$ are defined, and

$$\int_Y Z_x(f;u,g) = \epsilon_+\,|\alpha_0|^x\,Q^{x-1/2}L^\Sigma(x,\tilde\nu)\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\mathrm{Cone}^+_{A,X,\mu}(x;v),\Phi^{\mathrm{st}}_{\sigma,y})(g),$$
$$\int_Y Z_{1-x}(f;u,g) = \epsilon_-\,|\alpha_0|^x\,Q^{x-1/2}L^\Sigma(x,\tilde\nu)\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\mathrm{Cone}^-_{A,X,\mu}(x;v),\Phi^{\mathrm{st}}_{\sigma,y})(g).$$

Proof. Let $l_\pm$ be the constants of Lemma 6.2.37, and let $\Lambda_{\mathfrak{c}}$ denote the finite factor of a split tame character $\mathfrak{c}$, defined by the same formula in Lemma 6.2.37 and Lemma 6.2.41. Write $\mathbb{A}_E^\times = (\mathbb{C}^\times)^\Phi \times \mathbb{A}_{E,f}^\times$; the fixed Haar measure of $\mathbb{A}_E^\times$ is $d^\times a \otimes \rho$ for a Haar measure ρ of $\mathbb{A}_{E,f}^\times$, by uniqueness of Haar measure. The idele α_0 is that of Lemma 6.2.37. Put

$$C_\pm = |l_\pm|\,\rho(\widehat{\mathcal{O}}_E^\times) > 0.$$

Fix a split tame character $\mathfrak{c} = (\nu,\Sigma)$ of weight μ, let γ and the finite idele c (with $|\gamma|^2 = Q^{-1}$, $|c| = Q^{-1}$) be those of Lemma 6.2.37 for $\mathfrak{c}$, put $\delta = c^{-1}\alpha_0$ and

$$\epsilon_\pm = l_\pm\,\gamma\,Q^{1/2}\,\tilde\nu(\delta)\,\rho(\widehat{\mathcal{O}}_E^\times).$$

As $|\tilde\nu(\delta)|=1$ and $|\gamma|Q^{1/2}=1$, $|\epsilon_\pm|=C_\pm$. Fix v, y, u, g as in the statement and $f=\varphi^{\mathrm{pl}}_{A,\sigma'}(v,y)$. For $h\in \mathrm{G}(t)$, $\kappa\in \mathrm{K}_{\max}(t')$ and an adele q let $\lambda^{\pm}_{h,\kappa}(q)$ be the line values of Lemma 6.2.40 for ν, f, g, h, κ, and for $a'\in\mathbb{C}^\Phi$ let $m^{\pm}(h,\kappa,a')=\theta^{W,V}_{\alpha_W,\alpha_W}(\mathrm{Ln}^{\pm,\infty}_{A,X}(a')(\Omega_{\gamma_\mu,0}(1,\kappa_\infty)v)\otimes[\Phi^{\mathrm{st}}_{\sigma,y}]_\Sigma)(g,h)$.

Step 1: factorisation. By Lemma 6.2.37, applied to the adele $q\alpha_0^{-1}$, for every adele q

$$\lambda^{\pm}_{h,\kappa}(q)=l_\pm\,\gamma\,\Lambda_{\mathfrak{c}}((q\alpha_0^{-1})_f c)\,m^{\pm}(h,\kappa,(q\alpha_0^{-1})_\infty).$$

At $q=0$ the finite factor is $\Lambda_{\mathfrak{c}}(0)=0$, as 0 is not a unit; so $\lambda^{+}_{h,\kappa}(0)=\lambda^{-}_{h,\kappa}(0)=0$.

Step 2: Poisson's formula. Let $P_+(q)=q$ and $P_-(q)=\bar q^{-1}$ on $\mathbb{A}_E^\times$, and

$$G^{\pm}_x(q)=\tilde\nu(q)|q|^x\lambda^{\pm}_{h,\kappa}(q).$$

Let $a\in\mathbb{A}_E^\times$ and $q=au$. By Step 1 and Lemma 6.2.40 ($\#n$ even), the series converge absolutely and

$$\sum_\xi \lambda^{+}_{h,\kappa}(\xi q)=|q|^{-1}\sum_\xi\lambda^{-}_{h,\kappa}(\xi\bar q^{-1}),$$

sums over $\xi\in E^\times$. Since $\tilde\nu(\xi)=1$, $|\xi|=1$, $\tilde\nu(\bar q^{-1})=\nu(q/\bar q)=\tilde\nu(q)$ and $|\bar q^{-1}|=|q|^{-1}$, this gives, for every real x,

$$Z_x(f;u,g)(a,h,\kappa)=\sum_{\xi\in E^\times}G^{+}_x\big(\xi P_+(au)\big),\qquad Z_{1-x}(f;u,g)(a,h,\kappa)=\sum_{\xi\in E^\times}G^{-}_x\big(\xi P_-(au)\big).$$

Step 3: product form. For $q'\in\mathbb{A}_E^\times$ the adele $q'\delta\alpha_0^{-1}=q'c^{-1}$ has infinite part q'_∞ and finite part q'_fc^{-1}, so Step 1 gives $\lambda^{\pm}_{h,\kappa}(q'\delta)=l_\pm\gamma\,\Lambda_{\mathfrak{c}}(q'_f)\,m^{\pm}(h,\kappa,q'_\infty)$. Since ν has weight μ, $\tilde\nu(a)=\prod_{w\in\Phi}(a_w/|a_w|)^{2\mu(w)}$ for $a\in(\mathbb{C}^\times)^\Phi$, and $|a|=\prod_{w\in\Phi}|a_w|^2$; hence

$$G^{\pm}_x(q'\delta)=l_\pm\gamma\,\tilde\nu(\delta)|\delta|^x\cdot k^{\pm}_x(h,\kappa,q'_\infty)\cdot\Lambda_{\mathfrak{c}}(q'_f)\tilde\nu(q'_f)|q'_f|^x,$$

with $k^{\pm}_x$ the cone kernel of Lemma 6.2.42 (for σ, y, g and the sign $\pm$).

Step 4: the case $x>1$. The map $a\mapsto q'(a)=P_\pm(au)\delta^{-1}$ is a homeomorphism of $\mathbb{A}_E^\times$ that preserves its Haar measure (translations do, and $q\mapsto\bar q^{-1}$ is a continuous automorphism of order two, so its modulus is 1) and permutes the cosets of $E^\times$ ($P_-(\xi a)=\bar\xi^{-1}P_-(a)$). Let $F^{\pm}_x(q',h,\kappa)=G^{\pm}_x(q'\delta)$. By Step 3, Lemma 6.2.42 ($x>0$) and Lemma 6.2.41 ($x>1$), $F^{\pm}_x$ is integrable on $\mathbb{A}_E^\times\times[\mathrm{U}(V)]\times\mathrm{K}_{\max}(t')$, the integral of $|F^{\pm}_x|$ being a product of two finite integrals by Tonelli. For such a function the quotient integral formula for the discrete subgroup $E^\times$ (counting measure; Tonelli for $|F^{\pm}_x|$, then Fubini) shows that $a\mapsto\sum_{\xi\in E^\times}F^{\pm}_x(\xi q'(a),h,\kappa)$ is integrable on Y with integral $\int F^{\pm}_x$. By Step 2 this function is $Z_x(f;u,g)$, resp. $Z_{1-x}(f;u,g)$, so these are integrable and, by Fubini, Lemma 6.2.42 and Lemma 6.2.41,

$$\int_Y Z_x(f;u,g),\ \int_Y Z_{1-x}(f;u,g)\ =\ l_\pm\gamma\,\tilde\nu(\delta)|\delta|^x\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\mathrm{Cone}^{\pm}_{A,X,\mu}(x;v),\Phi^{\mathrm{st}}_{\sigma,y})(g)\,\rho(\widehat{\mathcal{O}}_E^\times)L^\Sigma(x,\tilde\nu)$$

(sign $+$ for Z_x, sign $-$ for Z_{1-x}). As $|\delta|^x=|c|^{-x}|\alpha_0|^x=Q^x|\alpha_0|^x$, the factor is $\epsilon_\pm\,|\alpha_0|^x\,Q^{x-1/2}$: the two formulas hold for $x>1$.

Step 5: integrability for every real x. By definition

$$Z_x(f;u,g)(a,h,\kappa)=Z_0(f;u,g)(a,h,\kappa)\,|au|^x.$$

Given x, let $R=|x|+2>1$, so $1-R<x<R$. Then $|au|^x\le|au|^{1-R}+|au|^R$, so $|Z_{x'}(f;u,g)|\le|Z_{1-R}(f;u,g)|+|Z_R(f;u,g)|$ for all $x'\in[1-R,R]$, and the right side is integrable by Step 4.

Hence $Z_x(f;u,g)$ is integrable, and by dominated convergence $x' \mapsto \int_Y Z_{x'}(f;u,g)$ is continuous on $(1-R,R)$; so it is continuous on $\mathbb{R}$.

Step 6: the point $x = 1$. For $x \geq 1$ the cone integrals are defined by Lemma 6.2.42. Both sides of the two formulas are continuous on $[1,\infty)$: the left sides by Step 5; on the right, $|\alpha_0|^x$ and $Q^{x-1/2}$ are continuous, $L^\Sigma(x,\tilde\nu)$ is continuous by Lemma 6.2.41, and $x \mapsto \theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\mathrm{Cone}^{\pm}_{A,X,\mu}(x;v),\Phi^{\mathrm{st}}_{\sigma,y})(g)$ is the integral of $k^{\pm}_x$, continuous by Lemma 6.2.42. They agree on $(1,\infty)$ by Step 4, hence also at $x=1$.

Step 7: analyticity in x. Write $\ell(a) = \log|au|$, a function on Y because $|\xi| = 1$ for $\xi \in E^\times$. Fix $x_0 \in \mathbb{R}$ and let $|x - x_0| < 1$. By Step 5, $Z_x(f;u,g) = Z_{x_0}(f;u,g)\, e^{(x-x_0)\ell} = \sum_{m\geq 0} \frac{(x-x_0)^m}{m!} Z_{x_0}(f;u,g)\,\ell^m$, and

$$\sum_{m\geq 0} \frac{|x-x_0|^m}{m!} \big|Z_{x_0}(f;u,g)\big|\,|\ell|^m = \big|Z_{x_0}(f;u,g)\big|\, e^{|x-x_0||\ell|} \leq \big|Z_{x_0+1}(f;u,g)\big| + \big|Z_{x_0-1}(f;u,g)\big|,$$

which is integrable on Y by Step 5. By Tonelli each $Z_{x_0}(f;u,g)\ell^m$ is integrable and, by Fubini for the sum over m, $\int_Y Z_x(f;u,g) = \sum_{m\geq 0} c_m (x-x_0)^m$ with $c_m = \frac{1}{m!}\int_Y Z_{x_0}(f;u,g)\ell^m$, the series converging for $|x-x_0|<1$. So the function is given by a convergent power series near every point. □

Lemma 6.2.37. *Let W', W, V', V, A, X be as above, let b be an index set with bijections $\sigma\colon b \sqcup r \to n$ and $\sigma'\colon b \sqcup r' \to n'$, assume that $\#n$ is even, let $\mu\colon \Phi \to \mathbb{Z}$, and let $\alpha_0 \in \mathbb{A}_E^\times$ be the idele with components κ_v at $v \in \Phi$ and finite part α. For a split tame character $\mathfrak{c} = (\nu,\Sigma)$ put $\alpha_W = \nu^{-1}\upsilon^{\#n/2}$, $\alpha_{W'} = \nu^{-1}\upsilon^{\#n'/2}$, $Q = \prod_{w\in\Sigma} \mathrm{N}w$, and let $\Lambda_{\mathfrak{c}}(a) = \chi_{\mathfrak{c}}(a_\Sigma)^{-1}\chi_{\mathfrak{c}}((\bar a)_\Sigma)$ for finite adeles a integral at every finite place and units at every place of $\Sigma\cup\bar\Sigma$, and $\Lambda_{\mathfrak{c}}(a) = 0$ for all other finite adeles. Then there are $l_+, l_- \in \mathbb{C}^\times$ with the following property. For every split tame character $\mathfrak{c} = (\nu,\Sigma)$ of weight μ there are $\gamma \in \mathbb{C}$ with $|\gamma|^2 = Q^{-1}$ and a finite idele c with $|c| = Q^{-1}$ such that for every $v \in \mathcal{F}_{W',V'}$, every $y \in \mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$, all $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(t)$, $\kappa \in \mathrm{K}_{\max}(t')$, both signs $\pm$ and every adele x with infinite part $x_\infty \in \mathbb{C}^\Phi$ and finite part x_f,*

$$\begin{aligned}&\theta^{W,V}_{\alpha_W,\alpha_W}\Big(\mathrm{Ln}^{\pm}_{A,X}(x\alpha_0)(\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f)\Big)(g,h)\\ &\qquad = l_\pm\,\gamma\,\Lambda_{\mathfrak{c}}(x_f c)\,\theta^{W,V}_{\alpha_W,\alpha_W}\Big(\mathrm{Ln}^{\pm,\infty}_{A,X}(x_\infty)(\Omega_{\gamma_\mu,0}(1,\kappa_\infty)v)\otimes\big[\Phi^{\mathrm{st}}_{\sigma,y}\big]_\Sigma\Big)(g,h),\end{aligned}$$

where $f = \varphi^{\mathrm{pl}}_{A,\sigma'}(v,y)$ and κ_∞ is the infinite component of κ.

Proof. Fix a split tame character $\mathfrak{c} = (\nu,\Sigma)$ of weight μ with tame part ψ and semilocal character $\chi = \chi_{\mathfrak{c}}$. For $t \in \mathcal{O}_\Sigma^\times$, read as a finite adele, $t/\bar t$ has the components t_w at $w\in\Sigma$, $\bar t_w^{-1}$ at $\bar w$ and 1 elsewhere, and $\psi_{\bar w} = 1$ for $w \in \Sigma$; so the weight formula of Definition 3.1.2 gives $\chi(t) = \prod_{w\in\Sigma}\psi_w(t_w \bmod w)$. Let $\varpi \in \mathcal{O}_\Sigma$ have a uniformiser ϖ_w as component at each $w\in\Sigma$, $R = \mathcal{O}_\Sigma/\varpi\mathcal{O}_\Sigma = \prod_{w\in\Sigma} k_w$, so $\#R = Q$, and $\psi_\Sigma(t) = \psi_{E,f}(\jmath\alpha\bar\alpha t)$ on E_Σ, dt the Haar probability measure of $\mathcal{O}_\Sigma$.

Step 0: a Gauss sum. By Lemma 4.1.3, $\widehat{\mathcal{O}}_E$ is its own annihilator under $(u,v)\mapsto\psi_{E,f}(\jmath\alpha\bar\alpha\bar u v)$; taking $u = \bar d$ for $d \in E_\Sigma$ (read as an adele) and v integral, $t\mapsto\psi_\Sigma(dt)$ is trivial on $\mathcal{O}_\Sigma$ if and only if $d\in\mathcal{O}_\Sigma$. Hence $\psi_R(t \bmod \varpi) = \psi_\Sigma(\varpi^{-1}t)$ is a well defined primitive additive character of R. The character χ^{-1}, read on R through reduction and extended by 0 to the nonunits, is a multiplicative character of R, and it is primitive in the sense of Lemma 6.2.39: if $r\neq 0$, some component $r_w \neq 0$; choose $u\in k_w^\times$ with $\psi_w(u)\neq 1$ and $c\in R$ with $c_w = (u-1)/r_w$, $c_{w'} = 0$ otherwise; then $\chi^{-1}(1+cr) = \psi_w(u)^{-1} \notin \{0,1\}$. Let $G = \sum_{u\in R}\chi^{-1}(u)\psi_R(u)$, so $|G|^2 = Q$ by Lemma 6.2.39 (T a point), and $\gamma_0 = Q^{-1}G$. We claim, for $d\in E_\Sigma$,

$$\int_{\mathcal{O}_\Sigma} \chi^{-1}(t)\,\psi_\Sigma(dt)\,dt = \gamma_0\,\chi(\varpi d) \ \text{ if } \varpi d \in \mathcal{O}_\Sigma^\times, \qquad = 0 \ \text{ otherwise.} \tag{G}$$

Indeed $\mathcal{O}_\Sigma$ is the disjoint union of the cosets $u+\varpi\mathcal{O}_\Sigma$ ($u\in R$), each of measure Q^{-1}, and χ^{-1} is constant on each. So the integral is $\sum_u \chi^{-1}(u)\psi_\Sigma(du)\int_{\varpi\mathcal{O}_\Sigma}\psi_\Sigma(dp)\,dp$; the inner integral of a character of the compact group $\varpi\mathcal{O}_\Sigma$ is Q^{-1} if $\varpi d\in\mathcal{O}_\Sigma$ and 0 otherwise. If $e=\varpi d\in\mathcal{O}_\Sigma$, then $\psi_\Sigma(du)=\psi_R(eu)$ and Lemma 6.2.39 with $S=\chi^{-1}$ gives $\sum_u\chi^{-1}(u)\psi_R(eu)=G\,\chi(e)$ if e is a unit and 0 otherwise.

Step 1: the compact point. Let $\kappa=(\kappa_\infty,\kappa_f)\in \mathrm{K}_{\max}(t')$, $f=\varphi^{\mathrm{pl}}_{A,\sigma'}(v,y)$ and $\Phi'=\Phi^{\mathrm{st}}_{\sigma',y(\mathrm{pr}_i)_\Sigma}$. As y and $(\mathrm{pr}_i)_\Sigma$ are integral (base-adaptedness), Φ' vanishes off the integral matrices and on them depends only on the reduction modulo ϖ; so $\Phi'^\sharp=\Phi_0^\natural$ for the modulus $\prod_{w\in\Sigma}w$ and a function Φ_0 on $r'\times n'$ matrices over R. The character $\alpha_{W'}=\nu^{-1}\upsilon^{\#n'/2}$ has weight γ_μ (υ has weight 1). By (5) and (6) of Lemma 4.1.1 (source point 1),

$$\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f=\alpha_{W'}(\det\kappa_f)\,\big(\Omega_{\gamma_\mu,0}(1,\kappa_\infty)v\big)\otimes\big[\Phi'_\kappa\big]_\Sigma,\qquad \Phi'_\kappa(X)=\Phi'(\kappa_\Sigma^{-1}X),$$

$\kappa_\Sigma\in\mathrm{GL}_{r'}(\mathcal{O}_\Sigma)$ the components of κ_f at Σ. Replacing the rows X by $\kappa_\Sigma^{-1}X$ multiplies the matrix of (XD10) by an integral invertible block matrix of determinant $\det\kappa_\Sigma^{-1}$, so $\Phi'_\kappa=\chi(\det\kappa_\Sigma)\Phi'$. The finite idele $z=\det\kappa_f\in\mathrm{U}(1)(\mathbb{A})$ has infinite components 1 and integral finite components, so $\nu(z)=\prod_{w\in\Sigma}\psi_w(z_w \bmod w)=\chi(\det\kappa_\Sigma)$ and $\upsilon(z)=1$ ((WD3)); hence $\alpha_{W'}(\det\kappa_f)\chi(\det\kappa_\Sigma)=1$ and, with $v_\kappa=\Omega_{\gamma_\mu,0}(1,\kappa_\infty)v$,

$$\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f=v_\kappa\otimes[\Phi']_\Sigma.$$

Step 2: line terms as slices along $D^\pm$. Let $a\in\mathbb{A}_E$ and $F\in\mathcal{M}_{W',V'}$. By (1) of Lemma 6.2.24, (1) of Lemma 6.2.44 for the plane splitting and (1) of Lemma 6.2.25,

$$\mathrm{Ln}^\pm_{A,X}(a)F=Z_{R^\pm}\big(\rho(l_{\mathbb{A}}^{R^\pm}a)Z^{\mathrm{pl}}F\big)=Z_{D^\pm}\big(\rho(i_{\mathbb{A}}^{\mathrm{pl}}l_{\mathbb{A}}^{R^\pm}a)F\big),$$

and $i_{\mathbb{A}}^{\mathrm{pl}}l_{\mathbb{A}}^{R^\pm}a$ is the frame $l_{\mathbb{A}}$ of the composite $D^\pm_{A,X}$ ((MX5)) applied to $ae_*\in\mathbb{A}_E^{q\sqcup\{*\}}$. So $\mathrm{Ln}^\pm_{A,X}(a)F$ is the slice $F[ae_*]$ along $D^\pm_{A,X}$. Since $m=m_{A,X}$ is block diagonal with block 1 at $*$, $ae_*=m^{-*}(ae_*)$; for $a=x\alpha_0$, $ae_*=\alpha_0(xe_*)$.

Step 3: the slice. By (1) of Lemma 6.2.38, $D^\pm_{A,X}$ is adapted at 1 in the basis m. Let ϵ be the number of (2) of Lemma 6.2.38 for $\mathfrak{c}$, σ, σ', and put $f_i=1$ ($i\in q$) and $f_*(t)=\epsilon\,\chi^{-1}(x_{f,\Sigma})\chi^{-1}(t)$, which are locally constant. By (2) of Lemma 6.2.38 with $a=x_{f,\Sigma}$, the hypothesis of Lemma 6.2.19 holds for the point xe_*, Φ' and $\Psi=\Phi^{\mathrm{st}}_{\sigma,y}$; both are locally constant with compact support, so $\Phi'^\sharp,\Psi^\sharp\in\mathcal{T}_\Sigma$ by Lemma 6.2.22. The vector xe_* has finite components 0 off $*$, so the integrals for $i\in q$ equal 1, and by (G) with $d=(\bar x_f)_\Sigma$

$$\int_{\mathcal{O}_\Sigma}\psi_\Sigma((\bar x_f)_\Sigma t)f_*(t)\,dt=\epsilon\,\gamma_0\,\chi^{-1}(x_{f,\Sigma})\,\chi(\varpi(\bar x_f)_\Sigma)\,\mathbf{1}[\varpi(\bar x_f)_\Sigma\in\mathcal{O}_\Sigma^\times].$$

As $\mathrm{rsl}_{D^\pm,m}(b_{x_\infty};\cdot)=\mathrm{Ln}^{\pm,\infty}_{A,X}(x_\infty)$ by (LN3), Lemma 6.2.19 and Steps 1, 2 give

$$\mathrm{Ln}^\pm_{A,X}(x\alpha_0)\big(\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f\big)=\mathrm{vf}(D^\pm_{A,X})\,\epsilon\,\gamma_0\,\Lambda'(x_f)\,\mathrm{Ln}^{\pm,\infty}_{A,X}(x_\infty)v_\kappa\otimes\big[\Phi^{\mathrm{st}}_{\sigma,y}\big]_\Sigma,$$

with $\Lambda'(x_f)=\mathbf{1}[x_f$ integral outside $\Sigma\cup\bar\Sigma]\,\chi^{-1}(x_{f,\Sigma})\,\chi(\varpi(\bar x_f)_\Sigma)\,\mathbf{1}[\varpi(\bar x_f)_\Sigma\in\mathcal{O}_\Sigma^\times]$.

Step 4: the shift and the constants. Let c be the finite idele with component $\overline{\varpi_w}\in E_{\bar w}$ at $\bar w$ for $w\in\Sigma$ and 1 elsewhere; $|c|=\prod_{w\in\Sigma}\mathrm{N}\bar w^{-1}=Q^{-1}$. Then $(x_fc)_\Sigma=x_{f,\Sigma}$, $(\overline{x_fc})_\Sigma=\varpi(\bar x_f)_\Sigma$, and x_fc is integral outside $\Sigma\cup\bar\Sigma$ exactly when x_f is; it is a unit at Σ exactly when $x_{f,\Sigma}$ is and a unit at $\bar\Sigma$ exactly when $\varpi(\bar x_f)_\Sigma$ is. Comparing with the definition of $\Lambda_{\mathfrak{c}}$ (χ^{-1} vanishes off the units), $\Lambda'(x_f)=\Lambda_{\mathfrak{c}}(x_fc)$. Put $l_\pm=\mathrm{vf}(D^\pm_{A,X})$, nonzero by (4) of Lemma 6.2.20 and independent of $\mathfrak{c}$, and $\gamma=\epsilon\gamma_0$, so $|\gamma|^2=Q^{-1}$. Applying the linear functional $\theta^{W,V}_{\alpha_W,\alpha_W}(\cdot)(g,h)$ gives the assertion. □

Lemma 6.2.38. *Let W', W, V', V, A, X be as above.*

1. *For both signs, $D^{\pm}_{A,X}$ is adapted at 1 in the basis $m_{A,X}$.*
2. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character with semilocal character $\chi = \chi_{\mathfrak{c}}$, b an index set and $\sigma\colon b \sqcup r \to n$, $\sigma'\colon b \sqcup r' \to n'$ bijections. Let $\mathcal{F}^{\pm}_{\Sigma}$ be the matrix of the components at Σ of the finite part of the frame $(l_{\mathbb{A}} m^{-*}_{A,X} \mid k_{\mathbb{A}} m_{A,X} \mid i_{\mathbb{A}})$ of $D^{\pm}_{A,X}$ (column blocks indexed by $q \sqcup \{*\}$, $q \sqcup \{*\}$, $r \times n$; values read as $r' \times n'$ matrices), $(\mathrm{pr}_i)_{\Sigma} \in \mathrm{M}_{n \times n'}(E_{\Sigma})$ the components at Σ of the finite part of the coordinate pr_i of A, and $e_* \in E_{\Sigma}^{q \sqcup \{*\}}$ the vector with entry 1 at $*$ and 0 elsewhere. Then there is $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$, depending only on A, X, σ, σ' and $\mathfrak{c}$, such that for both signs, all $y \in \mathrm{M}_{b \times n}(E_{\Sigma})$, $a \in E_{\Sigma}$, $z \in E_{\Sigma}^{q \sqcup \{*\}}$ and $Y \in \mathrm{M}_{r \times n}(E_{\Sigma})$*

$$\Phi^{\mathrm{st}}_{\sigma',\, y(\mathrm{pr}_i)_{\Sigma}}\big(\mathcal{F}^{\pm}_{\Sigma}\,(a e_*, z, Y)\big) = \epsilon \prod_{i \in q} \mathbf{1}_{\mathcal{O}_{\Sigma}}(z_i) \cdot \mathbf{1}_{\mathcal{O}_{\Sigma}}(z_*)\, \chi^{-1}(a)\, \chi^{-1}(z_*) \cdot \Phi^{\mathrm{st}}_{\sigma, y}(Y).$$

Proof. For adelic vectors y of V' and z of W' write $\langle y, z\rangle = y\, z^* \varepsilon_{s'}$, an adelic $r' \times n'$ matrix; it is linear in y and conjugate-linear in z. For rational $y \in E^{r'}$, $z \in E^{n'}$ the rational vector $y z^* J_{W'}$ of $\mathrm{Hom}(W', V')$ has the adelic coordinate $\langle x^{-1}_{V'} y, x^{-1}_{W'} z\rangle$, because $J_{W'} x_{W'} = (x^{-1}_{W'})^* \varepsilon_{s'}$. Write k_X, l_X, i_X and k_A, l_A, i_A for the adelic frames of X and A; then $x^{-1}_{W'}(\text{rational } i_A) = i_A x^{-1}_{W}$ and likewise for X. Recall $J^{-1}(x^{-1})^* = x\varepsilon$ for every space with a point.

Step 1: the frame. By (LN3) the rational frames of $D^{\pm}_{A,X}$ consist of rank-one maps $y z^* J_{W'}$, and the line basis $m = m_{A,X}$ is block diagonal with blocks $\bar{x}_W$ (n), 1 (o), $(x^{-1}_V)^*$ (r), 1 $(*)$; on the l-block the frame is multiplied by m^{-*}, with blocks $(x^{-1}_W)^{\mathsf{T}}$, 1, x_V, 1. Using the conjugate-linearity in z, $J^{-1}_W (x^{-1}_W)^* = x_W \varepsilon_s$ and $J^{-1}_V (x^{-1}_V)^* = x_V \varepsilon_t$, the columns of the adelic frame $\mathcal{F}^{\pm} = (l_{\mathbb{A}} m^{-*} \mid k_{\mathbb{A}} m \mid i_{\mathbb{A}})$ of $D^{\pm}_{A,X}$ are, with $\varepsilon_s(j)$, $\varepsilon_t(m')$ the diagonal entries (signs at the infinite places, 1 at the finite places):

- l-block: $\varepsilon_s(j)\langle l_X, i_A e_j\rangle$ $(j \in n)$, $\langle l_X, l_A\rangle$ (o), $\langle i_X e_{m'}, l_A\rangle$ $(m' \in r)$, and at $*$: $\langle k_X, l_A\rangle$ for $+$, $\langle l_X, k_A\rangle$ for $-$;
- k-block: $\langle k_X, i_A e_j\rangle$, $\langle k_X, k_A\rangle$, $\varepsilon_t(m')\langle i_X e_{m'}, k_A\rangle$, and at $*$: $\langle l_X, k_A\rangle$ for $+$, $\langle k_X, l_A\rangle$ for $-$;
- i-block: $\varepsilon_s(j)\langle i_X e_{m'}, i_A e_j\rangle$ $((m', j) \in r \times n)$.

So, up to signs, the columns are the $\langle y, z\rangle$ with y running over the columns of $B_X = (l_X \mid k_X \mid i_X)$ and z over those of $B_A = (l_A \mid k_A \mid i_A)$, each pair once (there are $(\#r + 2)(\#n + 2)$ columns).

Step 2: proof of (1). As A and X are base-adapted, B_A and B_X have orthonormal columns at every infinite place and entries in $\widehat{\mathcal{O}}_E$ at the finite places. At an infinite place, the euclidean inner product of the vectorisations of $\langle y, z\rangle$ and $\langle y', z'\rangle$ is $(y'^* y)\,\overline{(z'^* z)}$ ($\varepsilon_{s'}$ is unitary), so the columns of $\mathcal{F}^{\pm}$ are orthonormal; at a finite place $\varepsilon_{s'} = 1$ and the entries $y_a \overline{z_b}$ are integral because complex conjugation preserves $\widehat{\mathcal{O}}_E$. This is adaptedness at 1 in the basis m ((MX5)).

Step 3: the frame at Σ. At the finite places $\langle y, z\rangle = y z^*$. Let K_X, L_X, I_X be the components at Σ of k_X, l_X, i_X, put $P_X = (K_X \mid I_X \mid L_X) \in \mathrm{M}_{r'}(\mathcal{O}_{\Sigma})$, and let $R = (\mathrm{pr}_k; \mathrm{pr}_i; \mathrm{pr}_l)_{\Sigma} \in \mathrm{M}_{n'}(\mathcal{O}_{\Sigma})$ be the rows of the components at Σ of the coordinates of A, so that $(\mathrm{pr}_k)_{\Sigma} = (l^*_A)_{\Sigma}$, $(\mathrm{pr}_l)_{\Sigma} = (k^*_A)_{\Sigma}$, $(\mathrm{pr}_i)_{\Sigma} = (i^*_A)_{\Sigma}$. The identities $\mathrm{pr}_{\bullet}\, \bullet_{\mathbb{A}} = 1$, $\mathrm{pr}_{\bullet}\, \bullet'_{\mathbb{A}} = 0$ and $k_{\mathbb{A}}\mathrm{pr}_k + i_{\mathbb{A}}\mathrm{pr}_i + l_{\mathbb{A}}\mathrm{pr}_l = 1$ of (MX1), for A and for X, hold componentwise; with base-adaptedness they give $R \in \mathrm{GL}_{n'}(\mathcal{O}_{\Sigma})$ and $P_X \in \mathrm{GL}_{r'}(\mathcal{O}_{\Sigma})$. By Step 1, for $a \in E_{\Sigma}$, $z \in E_{\Sigma}^{q \sqcup \{*\}}$, $Y \in \mathrm{M}_{r \times n}(E_{\Sigma})$, with $z_n = (z_j)_{j \in n}$ (a row), z_o, $z_r = (z_{m'})_{m' \in r}$ (a column) and z_*,

$$\mathcal{F}^{\pm}_{\Sigma}(a e_*, z, Y) = P_X\, N^{\pm}\, R, \qquad N^{+} = \begin{pmatrix} a & z_n & z_o \\ 0 & Y & z_r \\ 0 & 0 & z_* \end{pmatrix}, \qquad N^{-} = \begin{pmatrix} z_* & z_n & z_o \\ 0 & Y & z_r \\ 0 & 0 & a \end{pmatrix}$$

(rows of $N^{\pm}$ indexed by o, r, o; columns by o, n, o).

Step 4: proof of (2). Let $M = \mathcal{F}_{\Sigma}^{\pm}(ae_*, z, Y)$ and $\tilde{M} \in \mathrm{M}_{n'}(E_\Sigma)$ the matrix whose row $\sigma'(j)$ is row j of $y(\mathrm{pr}_i)_\Sigma$ ($j \in b$) and whose row $\sigma'(i)$ is row i of M ($i \in r'$). Let $T^{\pm} \in \mathrm{M}_{\{o\}\sqcup n\sqcup\{o\}}(E_\Sigma)$ be $N^{\pm}$ with the middle block row replaced by the n rows of the matrix $\langle y; Y\rangle_\sigma$ whose row $\sigma(j)$ is row j of y and row $\sigma(m')$ is row m' of Y, and with the last column z_r replaced by the column with entries 0 at $\sigma(b)$ and $z_{m'}$ at $\sigma(m')$; let $G \in \mathrm{M}_{n'\times(\{o\}\sqcup n\sqcup\{o\})}(\mathcal{O}_\Sigma)$ have row $\sigma'(j)$ equal to $(0, e_{\sigma(j)}^{\mathsf{T}}, 0)$ ($j \in b$) and row $\sigma'(i)$ equal to row i of (K_X, I_X placed in the columns $\sigma(r)$, L_X) ($i \in r'$). Multiplying out, $GT^{\pm}$ has rows $(0, y_j, 0)$ at $\sigma'(j)$ and the rows of $P_X N^{\pm}$ at $\sigma'(r')$, so by Step 3

$$\tilde{M} = G\,T^{\pm}\,R.$$

Up to permutations of rows and columns, G is the block diagonal matrix with blocks 1_b and P_X, so $G \in \mathrm{GL}_{n'}(\mathcal{O}_\Sigma)$, and $RG \in \mathrm{GL}_{n'}(\mathcal{O}_\Sigma)$. Hence $\tilde{M}$ is integral if and only if $T^{\pm}$ is, and $\det\tilde{M} = \det(RG)\det T^{\pm}$ with $\det(RG) \in \mathcal{O}_\Sigma^\times$. $T^{\pm}$ is block upper triangular with diagonal blocks a, $\langle y; Y\rangle_\sigma$, z_* (in some order), so $\det T^{\pm} = a\,z_*\det\langle y; Y\rangle_\sigma$, and $T^{\pm}$ is integral exactly when a, all z_i and $\langle y; Y\rangle_\sigma$ are. As χ^{-1} is multiplicative on $\mathcal{O}_\Sigma$ (a product of integral elements is a unit only if each factor is) and vanishes on $E_\Sigma \setminus \mathcal{O}_\Sigma^\times$, (XD10) gives

$$\Phi^{\mathrm{st}}_{\sigma', y(\mathrm{pr}_i)_\Sigma}(M) = \chi^{-1}(\det(RG))\prod_{i\in q}\mathbf{1}_{\mathcal{O}_\Sigma}(z_i)\cdot\mathbf{1}_{\mathcal{O}_\Sigma}(z_*)\,\chi^{-1}(a)\chi^{-1}(z_*)\cdot\Phi^{\mathrm{st}}_{\sigma,y}(Y).$$

Put $\epsilon = \chi^{-1}(\det(RG))$; RG depends only on A, X, σ, σ' and Σ, and $|\epsilon| = 1$ because χ is unitary on $\mathcal{O}_\Sigma^\times$ ($\chi(t) = \nu(t/\bar{t})$). □

Lemma 6.2.39. *Let R be a finite commutative ring and T a finite set; $\mathrm{M}_{X\times Y}(R)$ denotes the $X \times Y$ matrices over R, $\mathrm{M}_T(R) = \mathrm{M}_{T\times T}(R)$, and $\mathrm{GL}_T(R)$ its group of units. Let $\psi_R\colon R \to \mathbb{C}^\times$ be an additive character which is **primitive**: for every $r \neq 0$ the character $t \mapsto \psi_R(rt)$ is nontrivial. Let $\chi\colon R \to \mathbb{C}$ be a **multiplicative character** ($\chi(1) = 1$, $\chi(rr') = \chi(r)\chi(r')$, and $\chi(r) = 0$ exactly when r is not a unit) which is **primitive**: for every $r \neq 0$ there is $c \in R$ with $\chi(1 + cr) \notin \{0, 1\}$. Put $\chi^{-1}(r) = \chi(r)^{-1}$ for units r and $\chi^{-1}(r) = 0$ otherwise, and*

$$G = \sum_{u\in\mathrm{M}_T(R)}\chi(\det u)\,\psi_R(\mathrm{tr}\,u).$$

Then $|G|^2 = (\#R)^{|T|^2}$, and every function $S\colon \mathrm{M}_T(R) \to \mathbb{C}$ with $S(gu) = \chi(\det g)\,S(u)$ for all $g \in \mathrm{GL}_T(R)$ and $u \in \mathrm{M}_T(R)$ satisfies

$$\sum_{u\in\mathrm{M}_T(R)} S(u)\,\psi_R(\mathrm{tr}(yu)) = G\,\chi^{-1}(\det y)\,S(1) \qquad (y \in \mathrm{M}_T(R)).$$

In particular the left side vanishes when y is not invertible.

Proof. **Step 1: orthogonality and inversion.** For $\rho \in \mathrm{M}_{Y\times X}(R)$, $\mathrm{tr}(\rho v) = \sum_{i,j}\rho_{ji}v_{ij}$, so $\sum_{v\in\mathrm{M}_{X\times Y}(R)}\psi_R(\mathrm{tr}(\rho v)) = \prod_{i,j}\sum_{t\in R}\psi_R(\rho_{ji}t)$, which is $\#\mathrm{M}_{X\times Y}(R)$ if $\rho = 0$ and 0 otherwise, because a nontrivial character of the finite group $(R, +)$ sums to 0 and $t \mapsto \psi_R(\rho_{ji}t)$ is nontrivial for $\rho_{ji} \neq 0$ by primitivity. Consequently, for f on $\mathrm{M}_{X\times Y}(R)$ and $y \in \mathrm{M}_{X\times Y}(R)$, using $\mathrm{tr}(uv) + \mathrm{tr}(yu) = \mathrm{tr}((v + y)u)$,

$$\hat{\hat{f}}(y) = \sum_v f(v)\sum_u\psi_R(\mathrm{tr}((v + y)u)) = \#\mathrm{M}_{X\times Y}(R)\;f(-y).$$

Step 2: equivariance of $\hat{S}$. For $g \in \mathrm{GL}_T(R)$ and $z \in \mathrm{M}_T(R)$, substituting $u \mapsto gu$ (a bijection of $\mathrm{M}_T(R)$),

$$\chi(\det g)\,\hat{S}(zg) = \sum_u S(gu)\,\psi_R(\mathrm{tr}(zgu)) = \hat{S}(z).$$

Step 3: $\hat{S}$ vanishes off $\mathrm{GL}_T(R)$. Let z be a non-invertible matrix. Left multiplication by z on the finite set $\mathrm{M}_T(R)$ is not injective: otherwise it is bijective, some D has $zD = 1$, so $\det z$ is a unit and z is invertible. Hence there is $D \neq 0$ with $zD = 0$; choose an entry $D_{ij} \neq 0$ and, by primitivity of χ, $c \in R$ with $\chi(1 + cD_{ij}) \notin \{0, 1\}$. Let v be the column j of D, so $zv = 0$, and $g = 1 + c\,v\,e_i^{\mathsf{T}}$ (e_i the i-th unit vector). By the matrix determinant lemma $\det g = 1 + c\,e_i^{\mathsf{T}} v = 1 + cD_{ij}$, which is a unit since its χ-value is nonzero; so $g \in \mathrm{GL}_T(R)$, and $zg = z + c\,(zv)\,e_i^{\mathsf{T}} = z$. Step 2 gives $\chi(1 + cD_{ij})\hat{S}(z) = \hat{S}(z)$, hence $\hat{S}(z) = 0$.

Step 4: the formula up to the constant. For $y = g$ invertible, Step 2 with $z = 1$ gives $\hat{S}(g) = \chi(\det g)^{-1}\hat{S}(1)$; with Step 3, $\hat{S}(y) = \chi^{-1}(\det y)\,\hat{S}(1)$ for all y.

Step 5: the constant. The values of χ on the units are roots of unity (the units form a finite group), so $\chi^{-1} = \overline{\chi}$; and $\psi_R(-t) = \overline{\psi_R(t)}$. By Step 1 and Step 4,

$$\#\mathrm{M}_T(R)\,S(1) = \hat{\hat{S}}(-1) = \hat{S}(1)\sum_y \chi^{-1}(\det y)\,\psi_R(-\mathrm{tr}\,y) = \hat{S}(1)\,\overline{G}.$$

The function $S_0(u) = \chi(\det u)$ satisfies the hypothesis, with $S_0(1) = 1$ and $\hat{S}_0(1) = G$; so $|G|^2 = \#\mathrm{M}_T(R) = (\#R)^{|T|^2}$. In particular $G \neq 0$, and for general S, $\hat{S}(1) = \#\mathrm{M}_T(R)\,S(1)/\overline{G} = G\,S(1)$. With Step 4 this is the assertion. □

Lemma 6.2.40. *Let W', W, V', V, A, X be as above and assume that $\#n$ is even. Let ν be an automorphic character of $\mathrm{U}(1)$, $\alpha_W = \nu^{-1}\upsilon^{\#n/2}$ and $\alpha_{W'} = \nu^{-1}\upsilon^{\#n'/2}$. Let $f \in \mathcal{M}_{W',V'}$, $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(t)$, $\kappa \in \mathrm{K}_{\max}(t')$, and for $\alpha \in \mathbb{A}_E$ put*

$$\lambda^{\pm}(\alpha) = \theta^{W,V}_{\alpha_W,\alpha_W}\Big(\mathrm{Ln}^{\pm}_{A,X}(\alpha)(\omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f)\Big)(g,h).$$

If $\lambda^+(0) = \lambda^-(0) = 0$, then for every $a \in \mathbb{A}_E^\times$ the series $\sum_{\xi\in E^\times}\lambda^+(\xi a)$ and $\sum_{\xi\in E^\times}\lambda^-(\xi\bar{a}^{-1})$ converge absolutely and

$$|a|\sum_{\xi\in E^\times}\lambda^+(\xi a) = \sum_{\xi\in E^\times}\lambda^-(\xi\bar{a}^{-1}).$$

Proof. Fix $a \in \mathbb{A}_E^\times$ and put $\varphi = \omega_{\alpha_{W'},\alpha_{W'}}(1,\kappa)f$, $\gamma = \mathrm{lev}_A(1,g)$, $\eta = \mathrm{lev}^{\mathrm{ln}}_X(a,h)$ and $F = \omega_{\alpha_{W'},\alpha_{W'}}(\gamma,\eta)\varphi \in \mathcal{M}_{W',V'}$. As $\#n' = \#n + 2$, $\alpha_{W'} = \alpha_W\upsilon$. By (2) of Lemma 6.2.9 for the character α_W, with $u = 1$ and $c = a$ (so $x = a$), and by the definition of theta kernels ((WC4) of Definition 4.1.15), for every $\xi \in E$

$$\theta_{\mathrm{Hom}(W,V)}(\mathrm{Ln}^{+}_{A,X}(\xi)F) = \frac{|a|^{-\#n/2}}{\alpha_W[\#n](a)}\,\lambda^+(\xi a),$$
$$\theta_{\mathrm{Hom}(W,V)}(\mathrm{Ln}^{-}_{A,X}(\xi)F) = \frac{|a|^{-(\#n+2)/2}}{\alpha_W[\#n](a)}\,\lambda^-(\xi\bar{a}^{-1}).$$

By (2) of Lemma 6.2.24 applied to F, the series $\sum_{\xi\in E}\theta_{\mathrm{Hom}(W,V)}(\mathrm{Ln}^{\pm}_{A,X}(\xi)F)$ converge absolutely and have the same sum. Multiplying by the nonzero number $\alpha_W[\#n](a)\,|a|^{(\#n+2)/2}$, the series $\sum_{\xi\in E}\lambda^+(\xi a)$ and $\sum_{\xi\in E}\lambda^-(\xi\bar{a}^{-1})$ converge absolutely and

$$|a|\sum_{\xi\in E}\lambda^+(\xi a) = \sum_{\xi\in E}\lambda^-(\xi\bar{a}^{-1}).$$

The terms at $\xi = 0$ are $\lambda^+(0) = \lambda^-(0) = 0$, so the same holds for the sums over $\xi \in E^\times$. $\square$

Lemma 6.2.41. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character with semilocal character $\chi = \chi_{\mathfrak{c}}$. For a finite adele a let $a_\Sigma \in E_\Sigma$ be its components at the places of Σ and $\bar{a}$ its conjugate, and put*

$$\Lambda_{\mathfrak{c}}(a) = \chi(a_\Sigma)^{-1}\,\chi\big((\bar{a})_\Sigma\big) \quad \text{if } a \text{ is integral at every finite place and a unit at every place of } \Sigma \cup \bar{\Sigma},$$
$$\Lambda_{\mathfrak{c}}(a) = 0 \quad \text{otherwise.}$$

Let ρ be a Haar measure on $\mathbb{A}_{E,f}^\times$ and $x > 1$. Then the function $a \mapsto \Lambda_{\mathfrak{c}}(a)\,\tilde{\nu}(a)\,|a|^x$ on $\mathbb{A}_{E,f}^\times$ (a read as an idele with infinite components 1) is ρ-integrable, and

$$\int_{\mathbb{A}_{E,f}^\times} \Lambda_{\mathfrak{c}}(a)\,\tilde{\nu}(a)\,|a|^x\,d\rho(a) = \rho(\widehat{\mathcal{O}}_E^\times)\,L^\Sigma(x,\tilde{\nu}).$$

Moreover $x \mapsto L^\Sigma(x,\tilde{\nu})$ is continuous and nowhere zero on $[1,\infty)$.

Proof. Let ψ be the tame part of ν and μ its weight; $\psi_w \neq 1$ exactly for $w \in \Sigma$, and $\Sigma \cap \bar{\Sigma} = \emptyset$, so $\psi_{\bar{w}} = 1$ for $w \in \Sigma$. For a finite adele t write $\bar{t}_{w'} = \overline{t_{\bar{w}'}}$ for the component of $\bar{t}$ at w'.

Step 1: χ and $\tilde{\nu}$ on units. For $z \in \mathrm{U}(1)(\mathbb{A})$ with infinite components 1 and integral finite components, the definition of weight and tame part (Definition 3.1.2) gives $\nu(z) = \prod_w \psi_w(z \bmod w)$. For $t \in \mathcal{O}_\Sigma^\times$, read as a finite adele, $t/\bar{t}$ has components t_w at $w \in \Sigma$, $\overline{t_w}^{-1}$ at $\bar{w}$ and 1 elsewhere, so

$$\chi(t) = \nu(t/\bar{t}) = \prod_{w \in \Sigma} \psi_w(t_w \bmod w).$$

For $u \in \widehat{\mathcal{O}}_E^\times$, likewise $\tilde{\nu}(u) = \nu(u/\bar{u}) = \prod_{w\in\Sigma} \psi_w(u_w \bmod w)\,\psi_w(\bar{u}_w \bmod w)^{-1} = \chi(u_\Sigma)\,\chi((\bar{u})_\Sigma)^{-1}$. Hence

$$\Lambda_{\mathfrak{c}}(u)\,\tilde{\nu}(u) = 1 \qquad (u \in \widehat{\mathcal{O}}_E^\times),$$

and $\tilde{\nu}(u) = 1$ if moreover u has components 1 at $\Sigma \cup \bar{\Sigma}$. The support condition of $\Lambda_{\mathfrak{c}}$ is invariant under multiplication by $\widehat{\mathcal{O}}_E^\times$ and χ is multiplicative, so $\Lambda_{\mathfrak{c}}(au) = \Lambda_{\mathfrak{c}}(a)\Lambda_{\mathfrak{c}}(u)$; with $|u| = 1$ the function $F(a) = \Lambda_{\mathfrak{c}}(a)\tilde{\nu}(a)|a|^x$ satisfies $F(au) = F(a)$ for $u \in \widehat{\mathcal{O}}_E^\times$.

Step 2: the integral. Fix a uniformiser π_w at every finite place w. The support of $\Lambda_{\mathfrak{c}}$ in $\mathbb{A}_{E,f}^\times$ is the disjoint union, over the nonzero integral ideals $\mathfrak{a}$ of $\mathcal{O}_E$ prime to the places of $\Sigma \cup \bar{\Sigma}$, of the cosets $a_{\mathfrak{a}}\widehat{\mathcal{O}}_E^\times$, where $a_{\mathfrak{a}}$ has components $\pi_w^{\mathrm{ord}_w\,\mathfrak{a}}$ at $w \notin \Sigma \cup \bar{\Sigma}$ and 1 at $\Sigma \cup \bar{\Sigma}$. On $a_{\mathfrak{a}}\widehat{\mathcal{O}}_E^\times$ the function F is constant (Step 1) and equal to $F(a_{\mathfrak{a}}) = \tilde{\nu}(a_{\mathfrak{a}})\,\mathrm{N}\mathfrak{a}^{-x}$, as $\Lambda_{\mathfrak{c}}(a_{\mathfrak{a}}) = \chi(1)^{-1}\chi(1) = 1$; by Step 1, $\tilde{\nu}$ is trivial on the units at $w \notin \Sigma \cup \bar{\Sigma}$, so $\tilde{\nu}(a_{\mathfrak{a}}) = \tilde{\nu}(\mathfrak{a})$ is the value of $\tilde{\nu}$ on the ideal $\mathfrak{a}$, independent of the uniformisers. The cosets have measure $\rho(\widehat{\mathcal{O}}_E^\times) \in (0,\infty)$, $\widehat{\mathcal{O}}_E^\times$ being compact and open. By Tonelli, $\int |F|\,d\rho = \rho(\widehat{\mathcal{O}}_E^\times)\sum_{\mathfrak{a}} \mathrm{N}\mathfrak{a}^{-x} < \infty$ (the Dedekind zeta function of E converges for $x > 1$), so F is integrable, and

$$\int F\,d\rho = \rho(\widehat{\mathcal{O}}_E^\times)\sum_{\mathfrak{a}} \tilde{\nu}(\mathfrak{a})\,\mathrm{N}\mathfrak{a}^{-x} = \rho(\widehat{\mathcal{O}}_E^\times)\prod_{w \notin \Sigma\cup\bar{\Sigma}} \big(1 - \tilde{\nu}(\pi_w)\mathrm{N}w^{-x}\big)^{-1} = \rho(\widehat{\mathcal{O}}_E^\times)L^\Sigma(x,\tilde{\nu}),$$

by unique factorisation of ideals and absolute convergence.

Step 3: continuity and nonvanishing. $\tilde{\nu}$ is a unitary Hecke character of E: it is continuous, of modulus 1, and trivial on $E^\times$ because $\xi/\bar{\xi} \in \mathrm{U}(1)(E)$. At the infinite places $\tilde{\nu}(a) = \prod_{v\in\Phi}(a_v/|a_v|)^{2\mu(v)}$ (the weight applied to $a/\bar{a}$), and $\mu(v) > 0$ (XD4), so $\tilde{\nu}$ is not of the form $|\cdot|^{i\tau}$. By Step 1, $\tilde{\nu}$ is ramified at every place of $\Sigma \cup \bar{\Sigma}$ (on units at $w \in \Sigma$ it is $\psi_w(u_w \bmod w) \neq 1$ for suitable u, and at $\bar{w}$ it is $\psi_w(\overline{u_{\bar{w}}} \bmod w)^{-1}$), so the Euler factors of

$L(s,\tilde\nu)$ at these places are 1 and $L^\Sigma(s,\tilde\nu)=L(s,\tilde\nu)$ for $\operatorname{Re}s>1$. By Theorem A.1.12 (3), $L(s,\tilde\nu)$ continues holomorphically to $\operatorname{Re}s\ge 1$ without zeros there; its restriction to $[1,\infty)$ is continuous and nowhere zero, and its value at 1 is the value of (XD7). □

Lemma 6.2.42. *Let W',W,V',V, A, X be as above, let b be an index set with a bijection $\sigma\colon b\sqcup r\to n$, and let $\mu\colon\Phi\to\mathbb{Z}$. Let $\mathfrak{c}=(\nu,\Sigma)$ be a split tame character of weight μ, $\alpha_W=\nu^{-1}\upsilon^{\lfloor\#n/2\rfloor}$, $v\in\mathcal{F}_{W',V'}$ a Schwartz vector, $y\in\mathrm{M}_{b\times n}(E_\Sigma)$, $g\in\mathrm{G}(s)$, and $\pm$ a sign. For real $x>0$ let*

$$k_x(h,\kappa,a)=\prod_{w\in\Phi}\Big(\frac{a_w}{|a_w|}\Big)^{2\mu(w)}|a_w|^{2x}\;\theta^{W,V}_{\alpha_W,\alpha_W}\Big(\mathrm{Ln}^{\pm,\infty}_{A,X}(a)\big(\Omega_{\gamma_\mu,0}(1,\kappa_\infty)v\big)\otimes\big[\Phi^{\mathrm{st}}_{\sigma,y}\big]_\Sigma\Big)(g,h)$$

for $h\in\mathrm{G}(t)$, $\kappa\in\mathrm{K}_{\max}(t')$ with infinite component κ_∞, and $a\in(\mathbb{C}^\times)^\Phi$. Then for every $x>0$: the cone integral $\mathrm{Cone}^{\pm}_{A,X,\mu}(x;v)$ is defined and is a Schwartz vector; k_x is integrable on $[\mathrm{U}(V)]\times\mathrm{K}_{\max}(t')\times(\mathbb{C}^\times)^\Phi$; and

$$\int_{[\mathrm{U}(V)]\times\mathrm{K}_{\max}(t')\times(\mathbb{C}^\times)^\Phi}k_x(h,\kappa,a)\,dh\,d\kappa\,d^\times a=\theta^{\mathrm{lg}}_{\mathfrak{c}}\big(W,V;\mathrm{Cone}^{\pm}_{A,X,\mu}(x;v),\Phi^{\mathrm{st}}_{\sigma,y}\big)(g).$$

Moreover this integral is a continuous function of $x\in[1,\infty)$.

Proof. Fix $x>0$ and the sign $\pm$. For $\kappa\in\mathrm{K}_{\max}(t')$ and $a\in(\mathbb{C}^\times)^\Phi$ put

$$e_x(a)=\prod_w(a_w/|a_w|)^{2\mu(w)}|a_w|^{2x},$$

so $|e_x(a)|=\prod_w|a_w|^{2x}$, and

$$c(\kappa,a)=\mathrm{Ln}^{\pm,\infty}_{A,X}(a)\big(\Omega_{\gamma_\mu,0}(1,\kappa_\infty)v\big)\in\mathcal{F}_{W,V},\qquad F_x(\kappa,a)=e_x(a)\,c(\kappa,a),$$

the integrand of $\mathrm{Cone}^{\pm}_{A,X,\mu}(x;v)$ (LN4). Let $f_0=[\Phi^{\mathrm{st}}_{\sigma,y}]^\sharp\otimes_\Sigma v^0$, so that $w\otimes[\Phi^{\mathrm{st}}_{\sigma,y}]_\Sigma=w\otimes f_0$, and for $h\in\mathrm{G}(t)$ let $\Lambda_h(w)=\theta^{W,V}_{\alpha_W,\alpha_W}(w\otimes f_0)(g,h)$ ($w\in\mathcal{F}_{W,V}$); then $k_x(h,\kappa,a)=\Lambda_h(F_x(\kappa,a))$ by linearity.

Step 1: three facts on Fock models. (a) For $c\in\mathcal{F}_\varepsilon$ and every z, $|c(z)|\le\|c\|_1$, by the reproducing identity of (WR5) and $|\beta^\infty_\varepsilon e^\infty|\le 1$. Hence evaluations are bounded functionals, $\mathcal{F}_\varepsilon$ is closed in $L^1(\nu_\iota)$ (the reproducing identity passes to L^1-limits, which converge pointwise), a Bochner integral $\int F\,d\lambda$ of an integrable $\mathcal{F}_\varepsilon$-valued F lies in $\mathcal{F}_\varepsilon$ with $(\int F\,d\lambda)(z)=\int F(p)(z)\,d\lambda(p)$, and by Tonelli $\mathrm{mom}_N(\int F\,d\lambda)\le\int\mathrm{mom}_N(F(p))\,d\lambda(p)$. (b) Let $u\in\mathrm{U}(\operatorname{diag}\varepsilon)$ with $u^*u=1$. Then $\operatorname{diag}(\varepsilon)u=(u^*)^{-1}\operatorname{diag}(\varepsilon)=u\operatorname{diag}(\varepsilon)$, so u is block diagonal for the signs and u_- is unitary; moreover $\beta^\infty_\varepsilon(uy,z)=\beta^\infty_\varepsilon(y,u^{-1}z)$ and $e^\infty(z-uy)=e^\infty(u^{-1}z-y)$, so the reproducing identity at $u^{-1}z$ gives $(\omega^\infty_\varepsilon(u)c)(z)=\overline{\det u_-}\,c(u^{-1}z)$. As $|\det u_-|=1$ and ν_ι and the norm are invariant under u, $\mathrm{mom}_N(\omega^\infty_\varepsilon(u)c)=\mathrm{mom}_N(c)$. (c) For $\kappa\in\mathrm{K}_{\max}(t')$ the element $u=\bigoplus_w\kappa_w\otimes 1$ of (WR5) acting on $\mathbb{C}^{\Phi\times r'\times n'}$ satisfies $u^*u=1$ and preserves the signs $t'\otimes s'$, and $|\det\kappa_w|=1$; by (b), $\mathrm{mom}_N(\Omega_{\gamma_\mu,0}(1,\kappa_\infty)v)=\mathrm{mom}_N(v)$ for every N.

Step 2: decay of the line terms. By (1) of Lemma 6.2.38, $D^{\pm}_{A,X}$ is adapted at 1 in the basis $m_{A,X}$, so (1) of Lemma 6.2.43, applied with the point b_a (whose norm is $\|a\|$) and with Step 1 (c), gives for all $N,L\in\mathbb{N}$ a constant $C_{N,L}$ with

$$(1+\|a\|)^L\,\mathrm{mom}_N\big(c(\kappa,a)\big)\le C_{N,L}\,\mathrm{mom}_{N+L}(v)\qquad(\kappa,a).\tag{6.10}$$

Step 3: an integral in a. For $x_0\in(0,\infty)$, $x_1\ge x_0$ and an integer $L_0>2x_1$, put $L=dL_0$. Since $1+\|a\|\ge 1+|a_w|$ for each w, $(1+\|a\|)^{-L}\le\prod_w(1+|a_w|)^{-L_0}$, and

$$\int_{(\mathbb{C}^\times)^\Phi}\prod_w(|a_w|^{2x_0}+|a_w|^{2x_1})(1+\|a\|)^{-L}\,d^\times a\le\Big(2\pi\int_0^\infty(\varrho^{2x_0}+\varrho^{2x_1})\varrho^{-1}(1+\varrho)^{-L_0}\,d\varrho\Big)^d<\infty.$$

Step 4: measurability. The map $\kappa \mapsto \Omega_{\gamma_\mu,0}(1,\kappa_\infty)v$ is continuous ((2) of Lemma 4.1.1 for $\omega_\varepsilon^\infty$, and the determinant factors are continuous). By (MX7), $\mathrm{rsl}_{D,m}(b;u') = Z_{D,m}^\infty(\rho^\infty(P_0 b)u')$ for the fixed complex matrix $P_0 = (l_{\mathbb{A}}(m^{-1})^*)_\infty$, where $Z_{D,m}^\infty$ is a bounded operator ((1) of Lemma 6.2.43 with $M = N = 0$ at the point 0) and $(y,u') \mapsto \rho^\infty(y)u'$ is continuous (Proposition 4.1.14: strongly continuous and isometric). Hence c and F_x are continuous on $\mathrm{K}_{\max}(t') \times (\mathbb{C}^\times)^\Phi$, a second countable space, and so strongly measurable. Likewise $(h,w) \mapsto \Lambda_h(w)$ is continuous: $w \mapsto w \otimes f_0$ is linear with $\|w \otimes f_0\| \le \|w\|_1 \|f_0\|$ (ρ is isometric), $(h,f) \mapsto \omega_{\alpha_W,\alpha_W}(g,h)f$ is continuous and $\theta_{\mathrm{Hom}(W,V)}$ is bounded; so k_x is continuous on $\mathrm{G}(t) \times \mathrm{K}_{\max}(t') \times (\mathbb{C}^\times)^\Phi$, and it is left Γ_V-invariant in h by Lemma 6.2.5.

Step 5: the cone integral is defined and is a Schwartz vector. By (6.10) with the L of Step 3 for $x_0 = x_1 = x$,

$$\int \mathrm{mom}_N(F_x(\kappa,a))\, d\kappa\, d^\times a \le C_{N,L}\, \mathrm{mom}_{N+L}(v) \int \prod_w |a_w|^{2x}(1+\|a\|)^{-L}\, d^\times a < \infty$$

for every N, v being a Schwartz vector. For $N = 0$ this is $\int \|F_x\|_1 < \infty$: with Step 4, F_x is Bochner integrable and $\mathrm{Cone}^\pm_{A,X,\mu}(x;v) \in \mathcal{F}_{W,V}$ is defined; by Step 1 (a) all its moments are finite.

Step 6: the kernel is integrable. The function f_0 comes from $\Phi^{\mathrm{st}}_{\sigma,y}$, which is locally constant with compact support (it is supported on integral matrices, where it is χ^{-1} of a polynomial, χ^{-1} being locally constant on $\mathcal{O}_\Sigma^\times$ and 0 elsewhere), so $[\Phi^{\mathrm{st}}_{\sigma,y}]^\sharp \in \mathcal{T}_\Sigma$ by Lemma 6.2.22. If $\Phi^{\mathrm{st}}_{\sigma,y}(Y) \ne 0$, the $n \times n$ matrix formed from the rows of y and of Y (XD10) is invertible over $\mathcal{O}_\Sigma$, so the rows of Y_w are linearly independent and Y_w has a right inverse for every $w \in \Sigma$ ($\Sigma \ne \emptyset$). By (1) of Lemma 6.2.27 with $N = 0$ there are $M, B \in \mathbb{N}$, $C_1 \ge 0$ with $|\Lambda_h(w)| \le C_1\, \mathrm{mom}_M(w)\, \mathrm{ht}_W(g)^B$ for all h and all w with $\mathrm{mom}_M(w) < \infty$. With (6.10),

$$|k_x(h,\kappa,a)| \le C_1 C_{M,L}\, \mathrm{ht}_W(g)^B\, \mathrm{mom}_{M+L}(v) \prod_w |a_w|^{2x}(1+\|a\|)^{-L}. \tag{6.11}$$

The right side does not depend on (h,κ) and is integrable in a (Step 3); $[\mathrm{U}(V)]$ has finite volume (Theorem 4.2.18) and $\mathrm{K}_{\max}(t')$ mass 1. So k_x is integrable on $[\mathrm{U}(V)] \times \mathrm{K}_{\max}(t') \times (\mathbb{C}^\times)^\Phi$.

Step 7: the identity. By (XD9) (with $\alpha = \nu^{-1}\upsilon^{\lfloor \#n/2 \rfloor} = \alpha_W$) and (XD2),

$$\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\mathrm{Cone}^\pm_{A,X,\mu}(x;v),\Phi^{\mathrm{st}}_{\sigma,y})(g) = \int_{[\mathrm{U}(V)]} \Lambda_h(\mathrm{Cone}^\pm_{A,X,\mu}(x;v))\, dh.$$

For fixed h, Λ_h is a bounded linear functional on $\mathcal{F}_{W,V}$ (Step 4), so it commutes with the Bochner integral: $\Lambda_h(\mathrm{Cone}^\pm_{A,X,\mu}(x;v)) = \int k_x(h,\kappa,a)\, d\kappa\, d^\times a$. Fubini's theorem (Step 6) turns the iterated integral into the integral over the product, which is the assertion.

Step 8: continuity on $[1,\infty)$. Let $x_1 > 1$ and L as in Step 3 for $x_0 = 1$, x_1. For $x \in [1,x_1]$, $|a_w|^{2x} \le |a_w|^2 + |a_w|^{2x_1}$, so by (6.11) (with $C_{M,L}$ for this L, independent of x) all k_x, $x \in [1,x_1]$, are dominated by one integrable function (Steps 3 and 6). For fixed (h,κ,a), $x \mapsto k_x(h,\kappa,a) = e_x(a)\Lambda_h(c(\kappa,a))$ is continuous. By dominated convergence the integral is continuous on $[1,x_1]$, and x_1 is arbitrary. □

Lemma 6.2.43. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p an index set, D a splitting of V over V_0 along p, and $m \in \mathrm{GL}_p(\mathbb{A}_E)$ such that D is adapted at 1 in the basis m. Let $\iota_0 = \Phi \times n_0$ and $\varepsilon_0(v,i) = s_0(i)$, so that $\mathcal{F}_{s_0} = \mathcal{F}_{\varepsilon_0}$.*

1. *For all $N, M \in \mathbb{N}$ there is $C \ge 0$ such that*

$$(1+\|x\|)^M\, \mathrm{mom}_N(\mathrm{rsl}_{D,m}(x;v)) \le C\, \mathrm{mom}_{N+M}(v) \qquad (x \in \mathbb{C}^{\Phi \times p},\ v \in \mathcal{F}_s).$$

In particular every real slice of a Schwartz vector is a Schwartz vector.

2. *Let τ be a smooth compactly supported function on $\mathbb{C}^{\Phi\times p}$, g a map from $\mathbb{C}^{\Phi\times p}$ to $\mathrm{U}(\operatorname{diag}\varepsilon_0)$ that is smooth as a matrix-valued function, and $v'\in\mathcal{F}_{s_0}$ a Schwartz vector. There is a Schwartz vector $v\in\mathcal{F}_s$ with*

$$\mathrm{rsl}_{D,m}(x;v)=\tau(x)\,\omega^\infty_{\varepsilon_0}(g(x))v' \qquad \textit{for every } x\in\mathbb{C}^{\Phi\times p}.$$

Proof. Let $\iota=\Phi\times n$ and $\varepsilon(v,i)=s(i)$, so $\mathcal{F}_s=\mathcal{F}_\varepsilon$; write ρ^∞, v_1 for the real Heisenberg operators and the real vacuum of $\mathcal{F}_s$, and ρ^∞_0, v_1^0 for those of $\mathcal{F}_{s_0}$. Put $\beta_1(x,x')=\exp((x'^*x-x^*x')/2)$ for $x,x'\in\mathbb{C}^{\Phi\times p}$.

Step 0: four facts on a Fock model $\mathcal{F}_\varepsilon$. (F1) Since $|\beta^\infty_\varepsilon e^\infty|\le 1$, the reproducing identity gives $|c(z)|\le\|c\|_1$ for $c\in\mathcal{F}_\varepsilon$ and every z. Hence each evaluation $c\mapsto c(z)$ is a bounded linear functional, $\mathcal{F}_\varepsilon$ is closed in $L^1(\nu)$ (the reproducing identity passes to L^1-limits), and the Bochner integral of an integrable $\mathcal{F}_\varepsilon$-valued function F lies in $\mathcal{F}_\varepsilon$ and satisfies $(\int F\,d\mu)(z)=\int F(t)(z)\,d\mu(t)$ for every z. By Tonelli,

$$\mathrm{mom}_N\Big(\int F\,d\mu\Big)\le\int\mathrm{mom}_N(F(t))\,d\mu(t). \tag{M1}$$

(F2) Since $|(\rho^\infty(t)c)(z)|=|c(z-t)|$ and $1+|z+t|\le(1+|t|)(1+|z|)$, $\mathrm{mom}_N(\rho^\infty(t)c)\le(1+|t|)^N\mathrm{mom}_N(c)$. (F3) $\rho^\infty(t)\rho^\infty(t')=\beta^\infty_\varepsilon(t,t')\,\rho^\infty(t+t')$. (F4) For $h\in\mathrm{U}(\operatorname{diag}\varepsilon)$ the block h_- is invertible and

$$\omega^\infty_\varepsilon(h)c=\overline{\det h_-}\int c(t)\,\rho^\infty(ht)v_1\,d\nu(t),$$

a Bochner integral of norm at most $|\det h_-|\,\|c\|_1\|v_1\|_1$: the values at z of both sides agree by (WR5) and (F1), since $(\rho^\infty(ht)v_1)(z)=\beta^\infty_\varepsilon(ht,z)e^\infty(z-ht)$. For the invertibility, the block on the indices of sign -1 of $h^*\mathrm{diag}(\varepsilon)h=\mathrm{diag}(\varepsilon)$ reads $h_-^*h_-=1+h_{+-}^*h_{+-}$, h_{+-} the block of rows of sign $+1$ and columns of sign -1; so $h_-^*h_-$ is positive definite.

Step 1: adapted coordinates. Let G be the complex matrix, block diagonal over Φ, with column blocks $(l_{\mathbb{A}}m^{-*})_\infty$, $(k_{\mathbb{A}}m)_\infty$, $(i_{\mathbb{A}})_\infty$. By adaptedness at 1 in the basis m each block G_v is square with orthonormal columns, hence unitary, and we write $y=G(a,b,w)$ with $a,b\in\mathbb{C}^{\Phi\times p}$, $w\in\mathbb{C}^{\iota_0}$. Then $|y|^2=|a|^2+|b|^2+|w|^2$, ν_ι is the image of $\nu_{\Phi\times p}\otimes\nu_{\Phi\times p}\otimes\nu_{\iota_0}$ (as $\#n=2\#p+\#n_0$), and by (MX1) the functions $a(y)$, $w(y)$ of (MX7) are the coordinates a, w. Since $x_V^*J_Vx_V=\varepsilon_s$, at each $v\in\Phi$ we have $(x_V^{-1}u)_v^*\,\mathrm{diag}(s)\,(x_V^{-1}u')_v=u^*J_Vu'$; with the relations of (MX1) ($l^*J_Vk=1$, $k^*J_Vk=l^*J_Vl=0$, $i^*J_Vi=J_{V_0}$, $i^*J_Vk=i^*J_Vl=0$) and $x_{V_0}^*J_{V_0}x_{V_0}=\varepsilon_{s_0}$ this gives, for $y=G(a,b,w)$ and $y'=G(a',b',w')$,

$$y'^*\mathrm{diag}(\varepsilon)\,y=a'^*b+b'^*a+w'^*\mathrm{diag}(\varepsilon_0)\,w. \tag{C1}$$

Further $\mathrm{rv}(y)=l_{\mathbb{A}}m^{-*}\mathrm{rv}(a)+k_{\mathbb{A}}m\,\mathrm{rv}(b)+i_{\mathbb{A}}\mathrm{rv}(w)$, so $\mathrm{pr}_k\mathrm{rv}(y)=m\,\mathrm{rv}(b)$, $\mathrm{pr}_l\mathrm{rv}(y)=m^{-*}\mathrm{rv}(a)$ and $\chi(y)=\psi_E(\frac{1}{2}\mathrm{rv}(b)^*\mathrm{rv}(a))$. The finite components vanish, and at $v\in\Phi$, with $\jmath_v=i\,\mathrm{Im}\,\jmath_v$ and $2\pi\,\mathrm{Im}(\jmath_v)\kappa_v^2=1$, (WD4) gives the factor $\exp(-2\pi i\cdot 2\,\mathrm{Re}(\frac{\jmath_v}{2}\kappa_v^2b_v^*a_v))=\exp(i\,\mathrm{Im}\,b_v^*a_v)$. Hence

$$\chi(y)=\exp\big((b^*a-a^*b)/2\big). \tag{C2}$$

Finally the matrix $(l_{\mathbb{A}}(m^{-1})^*)_\infty$ of (MX7) maps x to $G(x,0,0)$. For a function c on $\mathbb{C}^\iota$ write $\tilde c(a,b,w)=c(G(a,b,w))$.

Step 2: the slice formula. For $v\in\mathcal{F}_s$ and $x\in\mathbb{C}^{\Phi\times p}$,

$$\mathrm{rsl}_{D,m}(x;v)=\int\tilde v(a,b,w)\,\exp\Big(\frac{b^*a-a^*b}{2}+b^*x-x^*b-|a+x|^2\Big)\,\rho^\infty_0(w)v_1^0\,d\nu(a,b,w). \tag{S}$$

Indeed, by (C1) the exponent of $\beta_\varepsilon^\infty(G(x,0,0),G(a,b,w))$ is $(b^*x-x^*b)/2$, so

$$(\rho^\infty(G(x,0,0))v)^\sim(a,b,w)=e^{(b^*x-x^*b)/2}\,\tilde v(a-x,b,w);$$

insert this in the definition of $Z^\infty_{D,m}$ (MX7) with (C2) and $a(y)=a$, $w(y)=w$, and substitute $a\mapsto a+x$: the exponents add to $(b^*a-a^*b)/2+b^*x-x^*b$ and $e^{-|a|^2}$ becomes $e^{-|a+x|^2}$.

Step 3: (1). The integrand of (S) has modulus $|\tilde v(a,b,w)|\,e^{-|a+x|^2}\,|\rho_0^\infty(w)v_1^0|$. By (M1) and (F2) for $\mathcal{F}_{s_0}$,

$$\mathrm{mom}_N(\mathrm{rsl}_{D,m}(x;v))\le \mathrm{mom}_N(v_1^0)\int|\tilde v(a,b,w)|\,e^{-|a+x|^2}(1+|w|)^N\,d\nu.$$

With $c_M=\sup_{t\ge0}(1+t)^Me^{-t^2}$ and $1+|x|\le(1+|a|)(1+|a+x|)$ we get $(1+|x|)^Me^{-|a+x|^2}\le c_M(1+|a|)^M$, and $|a|,|w|\le|y|$. Therefore (1) holds with $C=c_M\,\mathrm{mom}_N(v_1^0)$, which is finite ($v_1^0$ is a Gaussian). Taking $M=0$: the real slices of a Schwartz vector are Schwartz vectors, and $v\mapsto\mathrm{rsl}_{D,m}(x;v)$ is a bounded operator from $\mathcal{F}_s$ to $\mathcal{F}_{s_0}$.

Step 4: the laws of real slices. For $y_0=G(a',b',w')$, $v\in\mathcal{F}_s$ and $x\in\mathbb{C}^{\Phi\times p}$,

$$\mathrm{rsl}_{D,m}(x;\rho^\infty(y_0)v)=\beta_1(2x+a',b')\,\rho_0^\infty(w')\,\mathrm{rsl}_{D,m}(x+a';v),\tag{L1}$$

$$\mathrm{rsl}_{D,m}(x;v_1)=e^{-|x|^2}\,v_1^0.\tag{L2}$$

For (L1): $(\rho^\infty(y_0)v)(y+y_0)=\beta_\varepsilon^\infty(y_0,y)v(y)$, as $\beta_\varepsilon^\infty(y_0,y_0)=1$. Substitute $(a,b,w)\mapsto(a+a',b+b',w+w')$ in (S). By (C1) the factor $\beta_\varepsilon^\infty(y_0,y)$ is $\exp((a^*b'+b^*a'-a'^*b-b'^*a)/2)$ times $\beta_{\varepsilon_0}^\infty(w',w)$, and by (F3) $\rho_0^\infty(w+w')=\beta_{\varepsilon_0}^\infty(w',w)^{-1}\rho_0^\infty(w')\rho_0^\infty(w)$, so the factors in w cancel. The remaining exponent is

$$\frac{a^*b'+b^*a'-a'^*b-b'^*a}{2}+\frac{(b+b')^*(a+a')-(a+a')^*(b+b')}{2}+(b+b')^*x-x^*(b+b')$$
$$=\frac{b^*a-a^*b}{2}+b^*(x+a')-(x+a')^*b+\frac{b'^*(2x+a')-(2x+a')^*b'}{2},$$

and the Gaussian is $e^{-|a+(x+a')|^2}$. The last term is the exponent of $\beta_1(2x+a',b')$, and $\rho_0^\infty(w')$ is bounded, so it leaves the Bochner integral; what remains is (S) at $x+a'$. For (L2): $\tilde v_1(a,b,w)=e^{-(|a|^2+|b|^2+|w|^2)/2}$. The integrand of (S) is absolutely integrable, and by Fubini the integral over w is $\int v_1^0(w)\rho_0^\infty(w)v_1^0\,d\nu(w)=v_1^0$ (its value at u is the reproducing identity of v_1^0). The exponent in b is $b^*x_1-x_1^*b=2i\,\mathrm{Im}(b^*x_1)$ with $x_1=a/2+x$, and coordinatewise $\pi^{-1}\int_{\mathbb{C}}e^{-|u_1|^2/2}e^{2i\,\mathrm{Im}(\bar u_1c_1)}\,du_1=2e^{-2|c_1|^2}$ for $c_1\in\mathbb{C}$ (Fourier transform of a Gaussian), so with $k_0=\#(\Phi\times p)$ the integral over b is $2^{k_0}e^{-|a+2x|^2/2}$. Since $\frac12|a|^2+\frac12|a+2x|^2+|a+x|^2=2|a+x|^2+|x|^2$ and $\int e^{-2|a|^2}d\nu(a)=2^{-k_0}$, the integral over a gives $2^{k_0}\cdot2^{-k_0}e^{-|x|^2}$.

Two cases of (L1) are used below: with $K=(k_{\mathbb{A}}m)_\infty$ and $I=(i_{\mathbb{A}})_\infty$ (so $Kb'=G(0,b',0)$, $Iw'=G(0,0,w')$), and $e(x,b')=\exp(b'^*x-x^*b')=\beta_1(2x,b')$,

$$\mathrm{rsl}_{D,m}(x;\rho^\infty(Kb')v)=e(x,b')\,\mathrm{rsl}_{D,m}(x;v),\qquad \mathrm{rsl}_{D,m}(x;\rho^\infty(Iw')v)=\rho_0^\infty(w')\,\mathrm{rsl}_{D,m}(x;v).\tag{L3}$$

Step 5: a family of coherent states and its Fourier transform. Let τ, g, v' be as in (2). The function $\sigma(x)=\tau(x)\,e^{|x|^2}\,\overline{\det g(x)_-}$ is smooth with compact support $\mathrm{supp}\,\sigma\subseteq\mathrm{supp}\,\tau$. Put $A(x)=I\,g(x)$, a smooth family of complex $\iota\times\iota_0$ matrices, and $B_0=\max_{x\in\mathrm{supp}\,\sigma}\|A(x)\|$. For $t\in\mathbb{C}^{\iota_0}$ and $x\in\mathbb{C}^{\Phi\times p}$ let

$$H(t,x)=\sigma(x)\,\rho^\infty(A(x)t)v_1\in\mathcal{F}_s,\qquad \widehat H(t,b)=\int H(t,x)\,\overline{e(x,b)}\,d\nu(x)\quad(b\in\mathbb{C}^{\Phi\times p}),$$

a Bochner integral of a continuous compactly supported function (Proposition 4.1.14). We show: for every $N \in \mathbb{N}$ and every integer $j \geq 0$ there is $C_{N,j}$ with

$$\operatorname{mom}_N(\widehat{H}(t,b)) \leq C_{N,j}\,(1+4|b|^2)^{-j}(1+|t|)^{4j+N} \qquad (t,b). \tag{E}$$

By (F1), $\widehat{H}(t,b)(z) = \int \phi_{t,z}(x)\,\overline{e(x,b)}\,d\nu(x)$ with $\phi_{t,z}(x) = \sigma(x)\exp(\Upsilon(A(x)t,z))$, where $\Upsilon(y,z) = (z^*\operatorname{diag}(\varepsilon)y - y^*\operatorname{diag}(\varepsilon)z)/2 - |z-y|^2/2$ is the exponent of the coherent state $\rho^\infty(y)v_1$ at z. In the real coordinates of $x \in \mathbb{C}^{\Phi\times p} = \mathbb{R}^{2k_0}$, $\overline{e(x,b)} = \exp(-2i\operatorname{Im}(b^*x))$ and $\operatorname{Im}(b^*x)$ is a real linear form with gradient of length $|b|$, so the Laplacian Δ_x multiplies $\overline{e(x,b)}$ by $-4|b|^2$. Integrating by parts ($\phi_{t,z}$ is smooth with compact support),

$$(1+4|b|^2)^j\,\widehat{H}(t,b)(z) = \int ((1-\Delta_x)^j\phi_{t,z})(x)\,\overline{e(x,b)}\,d\nu(x).$$

Now $\Upsilon(y,z) = i\operatorname{Im}(z^*\operatorname{diag}(\varepsilon)y) + \operatorname{Re}(z^*y) - |z|^2/2 - |y|^2/2$, so for a multi-index $\delta \neq 0$ and $x \in \operatorname{supp}\sigma$ the derivative $\partial_x^\delta(\Upsilon(A(x)t,z))$ is a sum of $i\operatorname{Im}(z^*\operatorname{diag}(\varepsilon)\,\partial^\delta A(x)t) + \operatorname{Re}(z^*\partial^\delta A(x)t)$ and, by Leibniz, of terms $-\frac{1}{2}\binom{\delta}{\delta'}(\partial^{\delta'}A(x)t)^*(\partial^{\delta-\delta'}A(x)t)$; with $|z| \leq |z-A(x)t| + B_0|t|$ it is bounded by $c_\delta(1+|t|)^2(1+|z-A(x)t|)$, c_δ depending on the derivatives of A of order at most $|\delta|$ on $\operatorname{supp}\sigma$. By the chain and Leibniz rules, $(1-\Delta_x)^j\phi_{t,z}$ is a finite sum of terms $\partial^{\delta_0}\sigma\cdot\prod_{i\geq1}\partial^{\delta_i}\Upsilon\cdot e^\Upsilon$ with $\sum_{i\geq0}|\delta_i| \leq 2j$ and $\delta_i \neq 0$ for $i \geq 1$; since $|e^\Upsilon| = e^{-|z-A(x)t|^2/2}$ and $(1+r)^{2j}e^{-r^2/2} \leq c_j' e^{-r^2/4}$ for $r \geq 0$,

$$\big|(1-\Delta_x)^j\phi_{t,z}(x)\big| \leq c_j''\,(1+|t|)^{4j}\,e^{-|z-A(x)t|^2/4}\,\mathbf{1}_{\operatorname{supp}\sigma}(x).$$

Finally $\int(1+|z|)^N e^{-|z-y|^2/4}d\nu(z) \leq c_N'''(1+|y|)^N$ with $c_N''' = \int(1+|r|)^N e^{-|r|^2/4}d\nu(r)$, and $|A(x)t| \leq B_0|t|$ on $\operatorname{supp}\sigma$. Integrating over z (Tonelli) gives (E) with $C_{N,j} = c_j''\,c_N'''\,(1+B_0)^N\nu(\operatorname{supp}\sigma)$.

Step 6: the vector v and its moments. Fix N and choose j with $2j > N + 2k_0$. Since $1+|b|^2\cdot 4 \geq (1+|b|)^2/2$, (E), (F2) (with $|Kb| = |b|$, K having orthonormal columns) give

$$\iint |v'(t)|\operatorname{mom}_N(\rho^\infty(Kb)\widehat{H}(t,b))\,d\nu(b)\,d\nu(t) \leq 2^j C_{N,j}\int(1+|b|)^{N-2j}d\nu(b)\cdot\operatorname{mom}_{4j+N}(v') < \infty.$$

The map $(t,b) \mapsto v'(t)\rho^\infty(Kb)\widehat{H}(t,b)$ is measurable into $\mathcal{F}_s$ ($\widehat{H}$ is continuous by dominated convergence with the bound of Step 5, and ρ^∞ is isometric and strongly continuous), so for $N = 0$ it is Bochner integrable, and by (F1) and (M1)

$$v = \iint v'(t)\,\rho^\infty(Kb)\,\widehat{H}(t,b)\,d\nu(b)\,d\nu(t) \in \mathcal{F}_s$$

has $\operatorname{mom}_N(v) < \infty$ for every N: v is a Schwartz vector.

Step 7: the slices of v. Fix x. By Step 3, $\operatorname{rsl}_{D,m}(x;\cdot)$ is bounded, so it commutes with the Bochner integrals, and by (L3) and Fubini,

$$\operatorname{rsl}_{D,m}(x;v) = \int v'(t)\operatorname{rsl}_{D,m}\Big(x;\int e(x,b)\,\widehat{H}(t,b)\,d\nu(b)\Big)\,d\nu(t).$$

For fixed t the inner integral is $H(t,x)$: by (F1) its value at z is $\int e(x,b)\widehat{\phi_{t,z}}(b)\,d\nu(b)$ with $\widehat{\phi_{t,z}}(b) = \int\phi_{t,z}\overline{e(\cdot,b)}\,d\nu$, and Fourier inversion on $\mathbb{R}^{2k_0}$, in the normalisation $\phi(x) = \int\widehat{\phi}(b)e(x,b)\,d\nu(b)$ for smooth compactly supported ϕ (the character $e(x,b) = \exp(2i\operatorname{Im}(b^*x))$ and the measure π^{-k_0} times Lebesgue measure in both variables), gives $\phi_{t,z}(x) = H(t,x)(z)$. By (L3) and (L2),

$$\operatorname{rsl}_{D,m}(x;H(t,x)) = \sigma(x)\,\rho_0^\infty(g(x)t)\operatorname{rsl}_{D,m}(x;v_1) = \sigma(x)\,e^{-|x|^2}\,\rho_0^\infty(g(x)t)v_1^0.$$

Integrating against $v'(t)$ and using (F4) for $h = g(x)$,

$$\mathrm{rsl}_{D,m}(x;v) = \sigma(x)\, e^{-|x|^2}\, \overline{\det g(x)_-}^{\,-1}\, \omega_{\varepsilon_0}^{\infty}(g(x))v' = \tau(x)\, \omega_{\varepsilon_0}^{\infty}(g(x))v'.$$

This proves (2). □

Lemma 6.2.44. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set, and (k,l,i) a splitting of V over V_0 along p, with zero term Z and slices $f[x]$; let $Z^\vee$ be the zero term along the opposite splitting (l,k,i). Give $\mathbb{A}_E^p$ the Haar measure dx for which $\mathbb{A}_E^p/E^p$ has volume 1.*

1. *(Mixed law.) Let $\Pi\colon \mathcal{M}_s \to \mathcal{M}_{s_0}$ be a bounded linear map with $\Pi\rho(k_{\mathbb{A}}b) = \Pi$ and $\Pi\rho(i_{\mathbb{A}}w) = \rho(w)\Pi$ for all $b \in \mathbb{A}_E^p$, $w \in \mathbb{A}_E^{n_0}$. Then for $f \in \mathcal{M}_s$, $x,a,b \in \mathbb{A}_E^p$ and $w \in \mathbb{A}_E^{n_0}$,*

$$\Pi\rho(l_{\mathbb{A}}x)\rho(l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b)f = \psi_E\big(\tfrac{1}{2}\jmath\, b^*(2x+a)\big)\, \rho(w)\, \Pi\rho(l_{\mathbb{A}}(x+a))f.$$

   The zero term Z is such a map; for $\Pi = Z$ the identity reads

$$\big(\rho(l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b)f\big)[x] = \psi_E\big(\tfrac{1}{2}\jmath\, b^*(2x+a)\big)\, \rho(w)\, f[x+a].$$

2. *(Summability.) There is $M \geq 0$ with $\sum_{\xi \in E^p} \|f[\xi]\| \leq M\|f\|$ for every $f \in \mathcal{M}_s$.*

3. *(Theta expansion.) For $f \in \mathcal{M}_s$, $\theta_V(f) = \sum_{\xi \in E^p} \theta_{V_0}(f[\xi])$, the series converging absolutely.*

4. *(Integral of the slices.) For $f \in \mathcal{M}_s$ the map $x \mapsto f[x]$ from $\mathbb{A}_E^p$ to $\mathcal{M}_{s_0}$ is continuous and Bochner integrable, and $\int_{\mathbb{A}_E^p} f[x]\, dx = Z^\vee(f)$.*

5. *(Hom slices.) Let T be a space with a point with signs t on n', give $\mathrm{M}_{n'\times p}(\mathbb{A}_E)$ the Haar measure for which $\mathrm{M}_{n'\times p}(E)$ has covolume 1, and let $Z_T^\vee$ be the zero term along the splitting of $\mathrm{Hom}(V,T)$ over $\mathrm{Hom}(V_0,T)$ induced by (l,k,i). For $f \in \mathcal{M}_{V,T}$ and its Hom slices $f[x]$,*

$$\sum_{\xi \in \mathrm{M}_{n'\times p}(E)} \|f[x_T^{-1}\xi]\| < \infty,$$
$$\theta_{\mathrm{Hom}(V,T)}(f) = \sum_{\xi \in \mathrm{M}_{n'\times p}(E)} \theta_{\mathrm{Hom}(V_0,T)}\big(f[x_T^{-1}\xi]\big),$$
$$\int_{\mathrm{M}_{n'\times p}(\mathbb{A}_E)} f[x_T^{-1}x]\, dx = Z_T^\vee(f).$$

Proof. **Step 0: the Heisenberg operators.** The bicharacter $\beta_s(x,y) = \psi_E(\frac{1}{2}\jmath\, y^*\varepsilon_s x)$ is additive in each variable. We have $\psi_E(\bar z) = \psi_E(z)$ (the archimedean factor reads $\mathrm{Re}\, z_v$, and if $z_f - a \in \widehat{\mathcal{O}}_E$ then $\bar z_f - \bar a \in \widehat{\mathcal{O}}_E$ and $\mathrm{Tr}_{E/\mathbb{Q}}\bar a = \mathrm{Tr}_{E/\mathbb{Q}}\, a$), and $\psi_E(z) = 1$ when $\bar z = -z$ (then $\mathrm{Re}\, z_v = 0$, and $a \in E$ with $z_f - a \in \widehat{\mathcal{O}}_E$ has $a + \bar a = (a - z_f) + (\bar a - \bar z_f) \in \widehat{\mathcal{O}}_E \cap F$, so $\mathrm{Tr}_{E/\mathbb{Q}}\, a = \mathrm{Tr}_{F/\mathbb{Q}}(a + \bar a) \in \mathbb{Z}$). As $\bar\jmath = -\jmath$ and $\overline{y^*\varepsilon_s x} = x^*\varepsilon_s y$, we get $\beta_s(y,x) = \beta_s(x,y)^{-1}$ and $\beta_s(z,z) = 1$. From $(\rho(z)f)(x) = \beta_s(z,x)f(x-z)$ one computes

$$\rho(z)\rho(z') = \beta_s(z,z')\, \rho(z+z'), \qquad \rho(z)\rho(z') = \beta_s(z,z')^2\, \rho(z')\rho(z). \tag{6.12}$$

The measure μ_n is a Haar measure of $\mathbb{A}_E^n$, so each $\rho(z)$ is an isometry for the norm of $L^1(\mu_n)$; it preserves $\mathcal{M}_s$ (substitute $y \mapsto y + z$ in the reproducing integral and use (6.12)), and $z \mapsto \rho(z)f$ is continuous (continuity of translation in L^1 and dominated convergence). The frames

satisfy $k_{\mathbb{A}}^*\varepsilon_s k_{\mathbb{A}} = k^*J_V k = 0$, $l_{\mathbb{A}}^*\varepsilon_s l_{\mathbb{A}} = 0$, $l_{\mathbb{A}}^*\varepsilon_s k_{\mathbb{A}} = 1$, $i_{\mathbb{A}}^*\varepsilon_s i_{\mathbb{A}} = \varepsilon_{s_0}$, $i_{\mathbb{A}}^*\varepsilon_s k_{\mathbb{A}} = i_{\mathbb{A}}^*\varepsilon_s l_{\mathbb{A}} = 0$, because $x_V^{-*}\varepsilon_s x_V^{-1} = J_V$ and by the relations of (MX1); hence, for $y = l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b$ and $y' = l_{\mathbb{A}}a' + i_{\mathbb{A}}w' + k_{\mathbb{A}}b'$,

$$\beta_s(y,y') = \psi_E\big(\tfrac{1}{2}\jmath\,(b'^*a + a'^*b)\big)\,\beta_{s_0}(w,w'). \tag{6.13}$$

Finally $\theta_V(\rho(\gamma)f) = \theta_V(f)$ for $\gamma = x_V^{-1}u$, $u \in E^n$ (the canonical theta functional of (WC3) of Definition 4.1.15 is a theta functional of V, Lemma 4.1.17); in particular for $\gamma = k_{\mathbb{A}}\xi$ and $\gamma = l_{\mathbb{A}}\xi$ with $\xi \in E^p$.

Step 1: two laws of the zero term. For $f \in \mathcal{M}_s$, $b \in \mathbb{A}_E^p$, $w \in \mathbb{A}_E^{n_0}$:

$$Z(\rho(k_{\mathbb{A}}b)f) = Z(f), \qquad Z(\rho(i_{\mathbb{A}}w)f) = \rho(w)Z(f). \tag{6.14}$$

By the uniqueness in (7) of Lemma 4.1.1 it suffices to check that the right sides satisfy the defining identity (MX4) for the vectors on the left. For the first: by (6.12) and (6.13), $(\rho(k_{\mathbb{A}}b)f)^\theta(i_{\mathbb{A}}y + k_{\mathbb{A}}b') = \beta_s(i_{\mathbb{A}}y + k_{\mathbb{A}}b', k_{\mathbb{A}}b)\, f^\theta(i_{\mathbb{A}}y + k_{\mathbb{A}}(b'+b)) = f^\theta(i_{\mathbb{A}}y + k_{\mathbb{A}}(b'+b))$, whose mean over $b' \in \mathbb{A}_E^p/E^p$ is that of $f^\theta(i_{\mathbb{A}}y + k_{\mathbb{A}}b')$ by invariance of the Haar measure. For the second: $(\rho(i_{\mathbb{A}}w)f)^\theta(i_{\mathbb{A}}y + k_{\mathbb{A}}b') = \beta_s(i_{\mathbb{A}}y + k_{\mathbb{A}}b', i_{\mathbb{A}}w) f^\theta(i_{\mathbb{A}}(y+w) + k_{\mathbb{A}}b') = \beta_{s_0}(y,w) f^\theta(i_{\mathbb{A}}(y+w) + k_{\mathbb{A}}b')$, whose mean over b' is $\beta_{s_0}(y,w)\theta_{V_0}(\rho(y+w)Z(f)) = \theta_{V_0}(\rho(y)\rho(w)Z(f))$ by (6.12) for V_0.

Step 2: the mixed law (1). Let $y = l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b$ and $u = i_{\mathbb{A}}w + l_{\mathbb{A}}(x+a)$. By (6.12), $\rho(l_{\mathbb{A}}x)\rho(y) = \beta_s(l_{\mathbb{A}}x, y)\rho(k_{\mathbb{A}}b + u) = \beta_s(l_{\mathbb{A}}x, y)\beta_s(k_{\mathbb{A}}b, u)^{-1}\rho(k_{\mathbb{A}}b)\rho(u)$ and $\rho(u) = \beta_s(i_{\mathbb{A}}w, l_{\mathbb{A}}(x+a))^{-1}\rho(i_{\mathbb{A}}w)\rho(l_{\mathbb{A}}(x+a)) = \rho(i_{\mathbb{A}}w)\rho(l_{\mathbb{A}}(x+a))$. By (6.13), $\beta_s(l_{\mathbb{A}}x, y) = \psi_E(\frac{1}{2}\jmath\, b^*x)$ and $\beta_s(k_{\mathbb{A}}b, u)^{-1} = \beta_s(u, k_{\mathbb{A}}b) = \psi_E(\frac{1}{2}\jmath\, b^*(x+a))$. Let Π be as in (1). Applying Π, which absorbs $\rho(k_{\mathbb{A}}b)$ on its right and turns $\rho(i_{\mathbb{A}}w)$ on its right into $\rho(w)$ on its left,

$$\begin{aligned}\Pi\rho(l_{\mathbb{A}}x)\rho(y)f &= \psi_E\big(\tfrac{1}{2}\jmath\, b^*(2x+a)\big)\,\Pi\rho(k_{\mathbb{A}}b)\rho(i_{\mathbb{A}}w)\rho(l_{\mathbb{A}}(x+a))f\\ &= \psi_E\big(\tfrac{1}{2}\jmath\, b^*(2x+a)\big)\,\rho(w)\,\Pi\rho(l_{\mathbb{A}}(x+a))f,\end{aligned}$$

which is (1). By (6.14) the zero term Z satisfies the hypothesis on Π, and $Z\rho(l_{\mathbb{A}}x)F = F[x]$ by (MX4), which gives the second display of (1). In particular $\|f[x]\|$ is continuous in x, $f \mapsto f[x]$ is bounded and linear, and $(\rho(l_{\mathbb{A}}x)f)[\xi] = f[\xi + x]$.

Step 3: superposition. By (1) of Lemma 6.2.12, for $h \in \mathcal{M}_s$ the map $z \mapsto h(z)\rho(z)v^0$ is Bochner integrable in $\mathcal{M}_s$ and

$$h = \int_{\mathbb{A}_E^n} h(z)\,\rho(z)v^0\,d\mu_n(z). \tag{6.15}$$

Step 4: summability (2). Take $g = g_D$ as in (1) of Lemma 6.2.46, so that the splitting is adapted at g in the basis 1, and $u = \omega_s(g)v^0$. By (1), with $\Pi = Z$, the slice maps $f \mapsto f[x]$ satisfy the hypothesis of (2) of Lemma 6.2.46, which gives $C_g \in \mathbb{C}$ with $u[x] = C_g|e_p(x)|^2v^0$ for all x. For $f \in \mathcal{M}_s$ put $h = \omega_s(g^{-1})f$. Applying the bounded operator $\omega_s(g)$ to (6.15) and using $\omega_s(g)\rho(z) = \rho(gz)\omega_s(g)$,

$$f = \int h(z)\,\rho(gz)u\,d\mu_n(z), \qquad \text{so} \qquad f[\xi] = \int h(z)\,(\rho(gz)u)[\xi]\,d\mu_n(z)$$

as $f \mapsto f[\xi]$ is bounded. Write $gz = l_{\mathbb{A}}a_z + i_{\mathbb{A}}w_z + k_{\mathbb{A}}b_z$. By (1), the unitarity of the phase and the isometry $\rho(w_z)$, $\|(\rho(gz)u)[\xi]\| = \|u[\xi + a_z]\| = |C_g|\,|e_p(\xi + a_z)|^2\,\|v^0\|$. Hence, by Tonelli,

$$\sum_{\xi\in E^p}\|f[\xi]\| \le |C_g|\,\|v^0\|\,K\int|h|\,d\mu_n \le |C_g|\,\|v^0\|\,K\,\|\omega_s(g^{-1})\|\,\|f\|, \qquad K = \sup_{a\in\mathbb{A}_E^p}\sum_{\xi\in E^p}|e_p(\xi+a)|^2.$$

K is finite: by (WD5), $|e_p(x)|^2 = \prod_{j\in p}\exp(-2\pi\sum_{v\in\Phi}\mathrm{Im}(\jmath_v)|x_{j,v}|^2)\mathbf{1}[x_{j,f}\in\alpha\widehat{\mathcal{O}}_E]$, so $K\le K_1^{\#p}$ with $K_1=\sup_{a\in\mathbb{A}_E}\sum_{\xi\in E}\exp(-Q(\xi_\infty+a_\infty))\mathbf{1}[\xi+a_f\in\alpha\widehat{\mathcal{O}}_E]$ and $Q(y)=2\pi\sum_v\mathrm{Im}(\jmath_v)|y_v|^2$, a positive definite quadratic form on $E_\infty=\mathbb{C}^\Phi$. For fixed a the ξ counted form an empty set or a coset $\xi_0+\mathfrak{a}$ of the fractional ideal $\mathfrak{a}=E\cap\alpha\widehat{\mathcal{O}}_E$, a lattice in E_∞; so $K_1\le\sup_{y\in E_\infty}\sum_{\lambda\in\mathfrak{a}}e^{-Q(\lambda+y)}$. This sum is $\mathfrak{a}$-periodic in y; for y in a fundamental parallelotope of $Q^{1/2}$-diameter δ we have $Q(\lambda+y)^{1/2}\ge Q(\lambda)^{1/2}-\delta$, and $\sum_\lambda\exp(-\max(0,Q(\lambda)^{1/2}-\delta)^2)<\infty$ since the number of λ with $Q(\lambda)\le r^2$ is $O(r^{2\#\Phi}+1)$.

Step 5: the theta expansion (3). Let $G(b)=f^\theta(k_\mathbb{A}b)=\theta_V(\rho(k_\mathbb{A}b)f)$ for $b\in\mathbb{A}_E^p$. It is continuous (Step 0) and E^p-periodic: for $\gamma\in E^p$, $\rho(k_\mathbb{A}(b+\gamma))=\rho(k_\mathbb{A}\gamma)\rho(k_\mathbb{A}b)$ by (6.12), (6.13), and θ_V absorbs $\rho(k_\mathbb{A}\gamma)$. For $\xi\in E^p$, by (MX4) at $y=0$, (6.12) and Step 0,

$$\begin{aligned}\theta_{V_0}(f[\xi])&=\int_{\mathbb{A}_E^p/E^p}\theta_V\big(\rho(k_\mathbb{A}b)\rho(l_\mathbb{A}\xi)f\big)db=\int_{\mathbb{A}_E^p/E^p}\beta_s(k_\mathbb{A}b,l_\mathbb{A}\xi)^2\,G(b)\,db\\&=\int_{\mathbb{A}_E^p/E^p}\psi_E(\jmath\,\xi^{\mathsf{T}}b)\,G(b)\,db,\end{aligned}$$

the Fourier coefficient of G at the character $b\mapsto\psi_E(\eta^{\mathsf{T}}b)$, $\eta=\jmath\bar\xi$. By (a) of Theorem A.5.2 these characters, for $\eta\in E^p$, are all the characters of $(\mathbb{A}_E/E)^p$, and $\xi\mapsto\jmath\bar\xi$ is a bijection of E^p. As $|\theta_{V_0}(f[\xi])|\le\|\theta_{V_0}\|\,\|f[\xi]\|$, the coefficients are absolutely summable by (2). Fourier inversion at 0 on the compact abelian group $(\mathbb{A}_E/E)^p$ ((b) of Theorem A.5.2, which holds verbatim for this group with dual E^p; Rudin, Fourier analysis on groups, 1.5) gives $\theta_V(f)=G(0)=\sum_\xi\theta_{V_0}(f[\xi])$.

Step 6: the integral of the slices (4). Continuity of $x\mapsto f[x]=Z(\rho(l_\mathbb{A}x)f)$ follows from Step 0. By Step 2 and (2) applied to $\rho(l_\mathbb{A}x)f$, $\sum_\xi\|f[\xi+x]\|\le M\|f\|$ for every x, so $\int_{\mathbb{A}_E^p}\|f[x]\|dx=\int_{\mathbb{A}_E^p/E^p}\sum_\xi\|f[\xi+x]\|dx\le M\|f\|$, and $W=\int f[x]\,dx$ exists. Fix $y\in\mathbb{A}_E^{n_0}$ and the bounded functional $\Theta_y=\theta_{V_0}\circ\rho(y)$. By (1) with $a=x$, $w=y$, $b=0$, $(\rho(l_\mathbb{A}x+i_\mathbb{A}y)f)[\xi]=\rho(y)f[\xi+x]$; so (3) applied to $\rho(i_\mathbb{A}y+l_\mathbb{A}x)f$ gives $\sum_\xi\Theta_y(f[\xi+x])=f^\theta(i_\mathbb{A}y+l_\mathbb{A}x)$. Therefore

$$\theta_{V_0}(\rho(y)W)=\int_{\mathbb{A}_E^p}\Theta_y(f[x])\,dx=\int_{\mathbb{A}_E^p/E^p}\sum_\xi\Theta_y(f[\xi+x])\,dx=\int_{\mathbb{A}_E^p/E^p}f^\theta(i_\mathbb{A}y+l_\mathbb{A}x)\,dx,$$

which is the defining identity (MX4) of the zero term along (l,k,i). By uniqueness, $W=Z^\vee(f)$.

Step 7: Hom slices (5). Let (k',l',i') be the splitting of $\mathrm{Hom}(V,T)$ induced by (k,l,i) (MX4). The point of $\mathrm{Hom}(V,T)$ is $x_T\otimes(x_V^{-1})^{\mathsf{T}}$ ((VP5) of Definition 3.2.2), and $\mathrm{pr}_k=l_\mathbb{A}^*\varepsilon_s=l^*J_Vx_V$; hence $l'_\mathbb{A}=(x_T^{-1}\otimes x_V^{\mathsf{T}})(1\otimes(l^*J_V)^{\mathsf{T}})=x_T^{-1}\otimes\mathrm{pr}_k^{\mathsf{T}}$, which sends an $n'\times p$ matrix X to $x_T^{-1}X\mathrm{pr}_k$. So the slice of f at X for (k',l',i') is the Hom slice $f[x_T^{-1}X]$, and (2), (3), (4) for (k',l',i') give the summability, the expansion, and $\int f[x_T^{-1}X]\,dX=Z'^\vee(f)$, the zero term along (l',k',i'). The splitting of $\mathrm{Hom}(V,T)$ induced by (l,k,i) has the same third frame i' and first frame $J_T^{-1}\otimes(l^*J_V)^{\mathsf{T}}=l'(J_T^{-1}\otimes1_p)$, with $J_T^{-1}\otimes1_p\in\mathrm{GL}_{n'p}(E)$. Since $b\mapsto(J_T^{-1}\otimes1_p)b$ is a continuous automorphism of $\mathbb{A}_E^{n'\times p}/E^{n'\times p}$, it preserves the Haar probability measure, so the two means in (MX4) coincide and $Z'^\vee=Z_T^\vee$. □

Lemma 6.2.45. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set, and (k,l,i) a splitting of V over V_0 along p with zero term Z and slices $f[x]$. Let $g\in\mathrm{G}(s)$, $m\in\mathrm{GL}_p(\mathbb{A}_E)$ and $g_0\in\mathrm{G}(s_0)$ satisfy $g\,k_\mathbb{A}=k_\mathbb{A}m$ and $\mathrm{pr}_i\,g\,i_\mathbb{A}=g_0$, and put $W=\mathrm{pr}_i\,g^{-1}l_\mathbb{A}\in\mathrm{M}_{n_0\times p}(\mathbb{A}_E)$ and $B=\mathrm{pr}_k\,g^{-1}l_\mathbb{A}\in\mathrm{M}_p(\mathbb{A}_E)$.*

1. *For $f\in\mathcal{M}_s$ and $x\in\mathbb{A}_E^p$,*

$$(\omega_s(g)f)[x]=v(\det m)^{-1}|\det m|^{1/2}\,\psi_E(\tfrac12\jmath\,x^*B^*m^*x)\,\omega_{s_0}(g_0)\,\rho(Wx)\,f[m^*x].$$

2. *Let T be a space with a point with signs t on n' and adelic sign matrix ε_t, let α,β be automorphic characters of* $\mathrm{U}(1)$*, $h\in \mathrm{G}(t)$, and put $\chi(y)=\beta(y/\bar y)\,\upsilon(y)^{-\#n'}$ for ideles y of E. For $f\in\mathcal{M}_{V,T}$ and every adelic $n'\times p$ matrix x, the Hom slices satisfy*

$$\begin{aligned}\big(\omega_{\alpha\upsilon^{\#p},\beta}(g,h)f\big)[x]=\chi(\det m)^{-1}|\det m|^{\#n'/2}\,\psi_E\big(\tfrac12 J\,\mathrm{tr}(B\,x^*\varepsilon_t\,x\,m)\big)\\ \omega_{\alpha,\beta}(g_0,h)\,\rho\big(h^{-1}xW^*\varepsilon_{s_0}\big)\,f[h^{-1}xm],\end{aligned}$$

where $\omega_{\alpha,\beta}(g_0,h)$ and ρ act on $\mathcal{M}_{V_0,T}$ and $h^{-1}xW^*\varepsilon_{s_0}$ is read as a vector of $\mathrm{Hom}(V_0,T)$*.*

At $x=0$ these are the laws of the zero terms $Z(\omega_s(g)f)$ and

$$Z_T(\omega_{\alpha\upsilon^{\#p},\beta}(g,h)f)=\chi(\det m)^{-1}|\det m|^{\#n'/2}\omega_{\alpha,\beta}(g_0,h)Z_T(f),$$

Z_T the zero term along the induced splitting of $\mathrm{Hom}(V,T)$*.*

Proof. **Step 1: coordinates of $g^{\pm1}$.** As $g^*\varepsilon_s g=\varepsilon_s$ we have $\varepsilon_s g^{-1}=g^*\varepsilon_s$, so $\mathrm{pr}_l g^{-1}=k_{\mathbb{A}}^*\varepsilon_s g^{-1}=(gk_{\mathbb{A}})^*\varepsilon_s=m^*\mathrm{pr}_l$; likewise $g^{-1}k_{\mathbb{A}}=k_{\mathbb{A}}m^{-1}$ gives $\mathrm{pr}_l g=m^{-*}\mathrm{pr}_l$. Since $\mathrm{pr}_l l_{\mathbb{A}}=1$, $\mathrm{pr}_l i_{\mathbb{A}}=\mathrm{pr}_l k_{\mathbb{A}}=0$, the matrix of g in the adelic basis $(k_{\mathbb{A}},i_{\mathbb{A}},l_{\mathbb{A}})$ is block upper triangular with diagonal blocks m, g_0, m^{-*}; hence

$$\det g=\det m\,\overline{\det m}^{\,-1}\det g_0,\qquad \mathrm{pr}_i\,g^{-1}i_{\mathbb{A}}=g_0^{-1},\tag{6.16}$$

the second because g^{-1} has the inverse block matrix. Moreover $g^{-1}l_{\mathbb{A}}x=l_{\mathbb{A}}m^*x+i_{\mathbb{A}}Wx+k_{\mathbb{A}}Bx$ for $x\in\mathbb{A}_E^p$, as its l-coordinate is $\mathrm{pr}_l g^{-1}l_{\mathbb{A}}x=m^*x$.

Step 2: one space, (1). By (2) of Lemma 4.1.1, $\rho(l_{\mathbb{A}}x)\omega_s(g)=\omega_s(g)\rho(g^{-1}l_{\mathbb{A}}x)$. Hence, by Lemma 6.2.10 and the mixed law,

$$(\omega_s(g)f)[x]=Z(\omega_s(g)\rho(g^{-1}l_{\mathbb{A}}x)f)=\upsilon(\det m)^{-1}|\det m|^{1/2}\,\omega_{s_0}(g_0)(\rho(l_{\mathbb{A}}m^*x+i_{\mathbb{A}}Wx+k_{\mathbb{A}}Bx)f)[0],$$

and $(\rho(l_{\mathbb{A}}m^*x+i_{\mathbb{A}}Wx+k_{\mathbb{A}}Bx)f)[0]=\psi_E(\frac12 J\,(Bx)^*m^*x)\rho(Wx)f[m^*x]$.

Step 3: the pair element on $\mathrm{Hom}(V,T)$**.** Let (k',l',i') be the splitting of $\mathrm{Hom}(V,T)$ induced by (k,l,i). Its adelic frames act on matrices by $l'_{\mathbb{A}}X=x_T^{-1}X\mathrm{pr}_k$, $i'_{\mathbb{A}}Y=Y\mathrm{pr}_i$ and $k'_{\mathbb{A}}X'=\varepsilon_t x_T^*X'\mathrm{pr}_l$ (from (VP5) of Definition 3.2.2, (MX4) and $x_V^{-*}\varepsilon_s=J_V x_V$, $x_T^{-1}J_T^{-1}=\varepsilon_t x_T^*$), and the coordinate $\mathrm{pr}_{i'}$ is $X\mapsto Xi_{\mathbb{A}}$ (as $\mathrm{pr}_k i_{\mathbb{A}}=\mathrm{pr}_l i_{\mathbb{A}}=0$, $\mathrm{pr}_i i_{\mathbb{A}}=1$). The pair map P of (g,h) is $X\mapsto hXg^{-1}$. By Step 1, $Pk'_{\mathbb{A}}X'=h\varepsilon_t x_T^*X'm^*\mathrm{pr}_l=k'_{\mathbb{A}}(MX')$ with $MX'=AX'm^*$, $A=x_T^{-*}\varepsilon_t h\varepsilon_t x_T^*$; so

$$\det M=(\det h)^{\#p}\,\overline{\det m}^{\,\#n'},\qquad |\det M|=|\det m|^{\#n'},$$

as $\det h\,\overline{\det h}=1$. And $\mathrm{pr}_{i'}Pi'_{\mathbb{A}}Y=hY\mathrm{pr}_i g^{-1}i_{\mathbb{A}}=hYg_0^{-1}$ by (6.16), which is the pair map P_0 of (g_0,h) on $\mathrm{Hom}(V_0,T)$.

Step 4: the pair, (2). By (WR4), $\omega_{\alpha\upsilon^{\#p},\beta}(g,h)=\alpha(\det h)\upsilon(\det h)^{\#p}\beta(\det g)^{-1}\omega(P)$ and $\omega_{\alpha,\beta}(g_0,h)=\alpha(\det h)\beta(\det g_0)^{-1}\omega(P_0)$, with ω the Weil representations of $\mathrm{Hom}(V,T)$ and $\mathrm{Hom}(V_0,T)$. Lemma 6.2.10 for (k',l',i'), P, M, P_0 gives $Z_T\omega(P)=\upsilon(\det M)^{-1}|\det M|^{1/2}\omega(P_0)Z_T$. Using (6.16) and $\upsilon(\bar y)=\upsilon(y)^{-1}$ (WD3), the scalar collects to

$$\upsilon(\det h)^{\#p}\,\beta\Big(\frac{\det g}{\det g_0}\Big)^{-1}\upsilon(\det h)^{-\#p}\upsilon(\overline{\det m})^{-\#n'}|\det m|^{\#n'/2}=\chi(\det m)^{-1}|\det m|^{\#n'/2}.$$

So

$$Z_T\,\omega_{\alpha\upsilon^{\#p},\beta}(g,h)=\chi(\det m)^{-1}|\det m|^{\#n'/2}\,\omega_{\alpha,\beta}(g_0,h)\,Z_T.$$

Now the Hom slice at x is $Z_T(\rho(x\mathrm{pr}_k)\cdot)$ (MX4), and $\rho(x\mathrm{pr}_k)\omega(P) = \omega(P)\rho(P^{-1}(x\mathrm{pr}_k))$ with $P^{-1}(x\mathrm{pr}_k) = h^{-1}x\,\mathrm{pr}_k g$. By Step 1, $\mathrm{pr}_k g = l_{\mathbb{A}}^*\varepsilon_s g = (g^{-1}l_{\mathbb{A}})^*\varepsilon_s = m\,\mathrm{pr}_k + W^*\varepsilon_{s_0}\mathrm{pr}_i + B^*\mathrm{pr}_l$ (using $i_{\mathbb{A}}^*\varepsilon_s = \varepsilon_{s_0}\mathrm{pr}_i$ and $k_{\mathbb{A}}^*\varepsilon_s = \mathrm{pr}_l$), so

$$h^{-1}x\,\mathrm{pr}_k g = l'_{\mathbb{A}}X + i'_{\mathbb{A}}Y + k'_{\mathbb{A}}X', \qquad X = x_T h^{-1}xm, \quad Y = h^{-1}xW^*\varepsilon_{s_0}, \quad X' = x_T^{-*}\varepsilon_t h^{-1}xB^*.$$

The mixed law for (k', l', i') (whose slice at X, namely $Z_T(\rho(l'_{\mathbb{A}}X)f) = Z_T(\rho(x_T^{-1}X\mathrm{pr}_k)f)$, is the Hom slice at $x_T^{-1}X = h^{-1}xm$) gives $Z_T(\rho(l'_{\mathbb{A}}X+i'_{\mathbb{A}}Y+k'_{\mathbb{A}}X')f) = \psi_E(\frac{1}{2}\jmath\,\mathrm{tr}(X'^*X))\rho(Y)f[h^{-1}xm]$, and $X'^*X = Bx^*h^{-*}\varepsilon_t h^{-1}xm = Bx^*\varepsilon_t xm$ since $h^{-*}\varepsilon_t h^{-1} = \varepsilon_t$. Combining with the zero-term law proves (2). □

Lemma 6.2.46. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set, and (k, l, i) a splitting of V over V_0 along p.*

1. *Let $x_{\mathbb{H}}$ be the point of the hyperbolic space $\mathbb{H}_p$, $s^{\mathbb{H}}$ its signs ($+1$ on the first and -1 on the second copy of p), and $F_D = (l_{\mathbb{A}}\ k_{\mathbb{A}}\ i_{\mathbb{A}})\,\mathrm{diag}(x_{\mathbb{H}}, 1)$, an adelic matrix with rows n and columns $(p \sqcup p) \sqcup n_0$. There are a bijection $\tau\colon (p\sqcup p)\sqcup n_0 \to n$ with $s(\tau(j)) = (s^{\mathbb{H}} \sqcup s_0)(j)$ for all j, and $g_D \in \mathrm{G}(s)$ whose column $\tau(j)$ is the column j of F_D for every j. For every such g_D the splitting is adapted at g_D in the basis 1.*

2. *Let $m \in \mathrm{GL}_p(\mathbb{A}_E)$ and $g \in \mathrm{G}(s)$ be such that the splitting is adapted at g in the basis m, and let $(\Pi_x)_{x\in\mathbb{A}_E^p}$ be bounded linear maps $\mathcal{M}_s \to \mathcal{M}_{s_0}$ with*

$$\Pi_x\,\rho(l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b) = \psi_E\big(\tfrac{1}{2}\jmath\,b^*(2x+a)\big)\,\rho(w)\,\Pi_{x+a} \qquad (x, a, b \in \mathbb{A}_E^p,\ w \in \mathbb{A}_E^{n_0}).$$

Then there is $C \in \mathbb{C}$ with $\Pi_x\,\omega_s(g)v^0 = C\,|e_p(m^*x)|^2\,v^0$ for all $x \in \mathbb{A}_E^p$, where v^0 on the right is the vacuum vector of $\mathcal{M}_{s_0}$.

Proof. **(1).** By the frame relations and the relation for $x_{\mathbb{H}}$, $F_D^*\varepsilon_s F_D = \varepsilon_{s^{\mathbb{H}}\sqcup s_0}$, and F_D is invertible. At one $v \in \Phi$, $F_{D,v}$ is an invertible congruence between $\mathrm{diag}(s)$ and $\mathrm{diag}(s^{\mathbb{H}} \sqcup s_0)$, so by Sylvester's law of inertia both have the same number of entries $+1$, and a bijection τ with $s \circ \tau = s^{\mathbb{H}} \sqcup s_0$ exists. Let P be the permutation matrix with $P\delta_j = \delta_{\tau(j)}$; the matrices with column $\tau(j)$ equal to column j of F_D are exactly $g_D = F_D P^{-1}$, and $g_D^*\varepsilon_s g_D = P\,\varepsilon_{s^{\mathbb{H}}\sqcup s_0}P^{-1} = \varepsilon_s$ (as $P^{-*} = P$ and $s\circ\tau = s^{\mathbb{H}}\sqcup s_0$), so $g_D \in \mathrm{G}(s)$. Moreover $g_D^{-1}(l_{\mathbb{A}}\ k_{\mathbb{A}}\ i_{\mathbb{A}}) = P\,\mathrm{diag}(x_{\mathbb{H}}^{-1}, 1)$. At $v \in \Phi$, $x_{\mathbb{H},v}^{-1} = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (scalar blocks) is unitary, so the columns are orthonormal. At the finite places $x_{\mathbb{H},f} = \begin{pmatrix} 1 & -c \\ c_0 & c\bar{c}_0 \end{pmatrix}$ has integral entries and determinant $c(c_0 + \bar{c}_0) = c$, a unit, so $x_{\mathbb{H},f}^{-1}$ and $P\,\mathrm{diag}(x_{\mathbb{H},f}^{-1}, 1)$ are integral. Hence the splitting is adapted at g_D in the basis 1.

(2), Step 0: reduction to $m = 1$. Put $l' = l_{\mathbb{A}}m^{-*}$, $k' = k_{\mathbb{A}}m$ and $\Pi'_y = \Pi_{m^{-*}y}$. Then $(l'\ k'\ i_{\mathbb{A}})^*\varepsilon_s(l'\ k'\ i_{\mathbb{A}}) = J_0$, and since $(mb)^*(2m^{-*}y + m^{-*}a) = b^*(2y + a)$ the law reads $\Pi'_y\rho(l'a + i_{\mathbb{A}}w + k'b) = \psi_E(\frac{1}{2}\jmath\,b^*(2y+a))\rho(w)\Pi'_{y+a}$. Steps 1–4 below use only these two facts and the hypothesis that $g^{-1}(l'\ k'\ i_{\mathbb{A}})$ has orthonormal columns at Φ and integral entries; we write them for $m = 1$, and for general m they give $\Pi'_y u = C|e_p(y)|^2 v^0$, that is, $\Pi_x u = C|e_p(m^*x)|^2 v^0$.

(2), Step 1: the adapted frame. Let $G = g^{-1}(l_{\mathbb{A}}\ k_{\mathbb{A}}\ i_{\mathbb{A}})$, with columns indexed by $p \sqcup p \sqcup n_0$, and write $z' = (a; b; w)$ accordingly, so that $gGz' = l_{\mathbb{A}}a + k_{\mathbb{A}}b + i_{\mathbb{A}}w$. As g preserves ε_s, $G^*\varepsilon_s G = J_0$. At $v \in \Phi$, G_v is square with orthonormal columns, hence unitary. At the finite places $\varepsilon_s = 1$ and $J_{0,f} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1 = J_{0,f}^{-1}$, so $G_f^{-1} = J_{0,f}G_f^*$ is integral and $G_f \in \mathrm{GL}_n(\widehat{\mathcal{O}}_E)$.

Therefore $z' \mapsto Gz'$ preserves μ_n (the scalings of (WD6) are scalars), and, in the coordinates $x_{v,j} = \kappa_v u_{v,j}$ at $v \in \Phi$ and $x_f = \alpha\xi$ at the finite places, with $|x|_\infty^2 = \sum_{v,j} |u_{v,j}|^2$,

$$e_n(Gz') = \epsilon(a,b)\, e_{n_0}(w), \qquad \epsilon(a,b) = e^{-(|a|_\infty^2+|b|_\infty^2)/2}\, \psi_0\big(\tfrac{c_0}{2}(a_f'^* b_f' + b_f'^* a_f')\big)\, \mathbf{1}[a_f', b_f' \in \widehat{\mathcal{O}}_E^p],$$

where $a_f = \alpha a_f'$, $b_f = \alpha b_f'$: at $v \in \Phi$ use $|G_v z_v'| = |z_v'|$, and at the finite places $(G_f z')^*(G_f z') = z'^* J_{0,f} z' = a^* b + b^* a + w^* w$ and $G_f z' \in \alpha \widehat{\mathcal{O}}_E^n$ if and only if $z' \in \alpha \widehat{\mathcal{O}}_E^n$.

Step 2: the values lie on the vacuum line. The law at $x = 0$, $b = 0$, $w = 0$ gives $\Pi_y = \Pi_0 \rho(l_{\mathbb{A}} y)$; so $\|\Pi_y\| \le \|\Pi_0\|$ and $y \mapsto \Pi_y F$ is continuous. Let $u = \omega_s(g) v^0$. By (3) of Lemma 6.2.12, $v^0 = \int e_n(y) \rho(y) v^0 \, d\mu_n(y)$; applying $\omega_s(g)$, then Π_x, substituting $y = Gz'$ and using the law,

$$\begin{aligned} \Pi_x u &= \int e_n(y)\, \Pi_x \rho(gy) u \, d\mu_n(y) \\ &= \iint \epsilon(a,b)\, \psi_E\big(\tfrac{1}{2}\jmath\, b^*(2x+a)\big) \Big(\int e_{n_0}(w) \rho(w)\, d\mu_{n_0}(w) \Big) \Pi_{x+a} u \, da\, db, \end{aligned}$$

Fubini being justified by the bound $|e_n(y)|\, \|\Pi_0\|\, \|u\|$ of the integrand, μ_n being the product of μ_p, μ_p, μ_{n_0} in the coordinates (a, b, w). By (3) of Lemma 6.2.12 for s_0, the inner integral sends X to $\langle v^0, X\rangle v^0$. Hence, with the bounded continuous function $\phi(y) = \langle v^0, \Pi_y u\rangle$, $\Pi_x u = \phi(x) v^0$ and

$$\phi(x) = \iint \epsilon(a,b)\, \psi_E\big(\tfrac{1}{2}\jmath\, b^*(2x+a)\big)\, \phi(x+a)\, da\, db. \tag{6.17}$$

Step 3: the integral over b. The integrand of (6.17) is a product of archimedean and finite factors. At each (v, j) the archimedean factor of $\psi_E(\frac{1}{2}\jmath\, b^* y)$ is $\exp((\bar{b}_{v,j} y_{v,j} - \bar{y}_{v,j} b_{v,j})/2)$ in the normalized coordinates (as $\jmath_v \kappa_v^2 = i/(2\pi)$), whose modulus is 1; the Fourier transform of the Gaussian gives $\pi^{-1} \int_{\mathbb{C}} e^{-|t|^2/2} e^{(\bar{t}y - \bar{y}t)/2} dt = 2e^{-|y|^2/2}$. With $y = 2x + a$ and $|a|^2 + |2x+a|^2 = 2|x|^2 + 2|x+a|^2$, the archimedean b-integral times $e^{-|a|_\infty^2/2}$ is $2^N e^{-|x|_\infty^2} e^{-|x+a|_\infty^2}$, $N = \#\Phi\, \#p$. At the finite places, with $x_f = \alpha\xi$, the factor of $\psi_E(\frac{1}{2}\jmath\, b^*(2x+a))$ is $\psi_0(\frac{1}{2} b_f'^*(2\xi + a_f'))$, and $\psi_0(\frac{c_0}{2} a_f'^* b_f') = \psi_0(-\frac{\bar{c}_0}{2} b_f'^* a_f')$ (as $\psi_0(\bar{t}) = \psi_0(t)^{-1}$); since $1 + c_0 - \bar{c}_0 = 2c_0$, the finite factor depending on b_f' is $\psi_0(b_f'^*(\xi + c_0 a_f'))$, whose mean over $b_f' \in \widehat{\mathcal{O}}_E^p$ is $\mathbf{1}[\xi + c_0 a_f' \in \widehat{\mathcal{O}}_E^p]$ by self-duality; for a_f' integral this is $\mathbf{1}[\xi \in \widehat{\mathcal{O}}_E^p]$. Therefore

$$\phi(x) = 2^N e^{-|x|_\infty^2}\, \mathbf{1}\big[\xi \in \widehat{\mathcal{O}}_E^p\big] \int e^{-|x+a|_\infty^2}\, \mathbf{1}\big[a_f' \in \widehat{\mathcal{O}}_E^p\big]\, \phi(x+a)\, da.$$

Step 4: conclusion. For $\xi \in \widehat{\mathcal{O}}_E^p$ the substitution $a \mapsto a - x$ preserves the measure and the condition $a_f' \in \widehat{\mathcal{O}}_E^p$, so the last integral is $K = \int e^{-|a|_\infty^2} \mathbf{1}[a_f' \in \widehat{\mathcal{O}}_E^p] \phi(a)\, da$, independent of x. As $|e_p(x)|^2 = e^{-|x|_\infty^2} \mathbf{1}[\xi \in \widehat{\mathcal{O}}_E^p]$ by (WD5) ($2\pi \operatorname{Im}(\jmath_v)\kappa_v^2 = 1$), we get $\Pi_x u = \phi(x) v^0 = C |e_p(x)|^2 v^0$ with $C = 2^N K$. □

Lemma 6.2.47. *Let G be a locally compact group with a left Haar measure μ, let L and H be unimodular locally compact groups with Haar measures κ and ν, and let $i\colon L \times H \to G$ be a continuous injective homomorphism that is a homeomorphism onto its image, the image being closed. Let $\Lambda \le H$ be a discrete subgroup such that $\Gamma = i(\{1\} \times \Lambda)$ is discrete in G. Let $\rho\colon G \to \mathbb{C}$ be a function with $\operatorname{Re}\rho(g) \ge 0$ for all $g \in G$ and $\operatorname{Re}\rho \ne 0$ on a neighbourhood of some point of G. Then there is $R \in \mathbb{C} \setminus \{0\}$, depending only on $G, L, H, i, \Lambda, \mu, \kappa, \nu, \rho$, with the following property. Let $F\colon G \to \mathbb{C}$ be continuous, left Γ-invariant and μ_Γ-integrable on $\Gamma\backslash G$, and let $c \in \mathbb{C}$. Suppose that every $g \in G$ has some $k \in L \times H$ such that the iterated integral*

$$\int_L \int_{\Lambda\backslash H} F\big(i(l,a)\, i(k)\, g\big)\, d\nu_\Lambda(a)\, d\kappa(l)$$

exists (the inner integral converging absolutely for κ-almost every l and defining a κ-integrable function of l) and equals $\rho(g)\,c$. Then

$$\int_{\Gamma\backslash G} F\,d\mu_\Gamma = R\,c.$$

Proof. Write $M = i(L\times H)$, a closed subgroup of G. As i is an isomorphism of topological groups $L\times H\to M$, the image dm of $\kappa\otimes\nu$ under i is a left Haar measure of M.

A parametrisation of μ. By the existence of Bruhat functions for closed subgroups (Bourbaki, Intégration, ch. VII, §2, no. 4) there is a continuous $\beta_0\ge 0$ on G with $\int_M \beta_0(xm)\,dm = 1$ for every $x\in G$; put $\beta(g) = \beta_0(g^{-1})$, so that $\int_M \beta(m^{-1}g)\,dm = 1$ for every g. For Borel $f\colon G\to[0,\infty]$, Tonelli's theorem and the left invariance of μ (substitute $y = mx$ for fixed m) give

$$\int_G \beta(x)\int_M f(mx)\,dm\,d\mu(x) = \int_M\int_G \beta(m^{-1}y)f(y)\,d\mu(y)\,dm = \int_G f\,d\mu. \tag{6.18}$$

Since $\int_M f(mx)\,dm = \int_L\int_H f(i(l,1)\,i(1,a)\,x)\,d\nu(a)\,d\kappa(l)$ and $i(1,a)i(l,1) = i(l,a)$, (6.18) says that $((x,l),a)\mapsto i(1,a)\,T(x,l)$, with $T(x,l) = i(l,1)x$, carries $\big((\beta\mu)\otimes\kappa\big)\otimes\nu$ to μ; the measure $(\beta\mu)\otimes\kappa$ on $Z = G\times L$ is σ-finite.

Unfolding. Fix F as in the statement. Apply Lemma 6.2.48 to H, the injective continuous homomorphism $a\mapsto i(1,a)$, Λ, $(Z,(\beta\mu)\otimes\kappa)$ and T: applied to $|F|$ (the version in $[0,\infty]$) and Tonelli on $G\times L$ give, with

$$G' = \Big\{x\in G : \int_L\int_{\Lambda\backslash H} \big|F(i(l,a)x)\big|\,d\nu_\Lambda(a)\,d\kappa(l) < \infty\Big\},$$

that $(\beta\mu)(G\setminus G') = 0$; and, applied to F, with $I(x) = \int_{L\times\Lambda\backslash H} F(i(l,a)x)\,d(\kappa\otimes\nu_\Lambda)$ for $x\in G'$ (Fubini),

$$\int_{\Gamma\backslash G} F\,d\mu_\Gamma = \int_{G'} \beta(x)\,I(x)\,d\mu(x), \tag{6.19}$$

the integrand being μ-integrable on G'.

Invariance under M. The quotient measure ν_Λ is invariant under $\Lambda a\mapsto\Lambda a k_2$ for $k_2\in H$: if $\varphi\ge 0$ is Borel on H with $\sum_{\lambda\in\Lambda}\varphi(\lambda a) = 1$ (the indicator of a Borel set meeting each coset Λa exactly once, which exists as Λ is discrete in the second countable group H), so is $a\mapsto\varphi(ak_2^{-1})$, and for Borel $f\ge 0$ on $\Lambda\backslash H$ the right invariance of ν gives $\int_{\Lambda\backslash H} f(\Lambda a k_2)\,d\nu_\Lambda = \int_H \varphi(a) f(\Lambda a k_2)\,d\nu = \int_H \varphi(ak_2^{-1})f(\Lambda a)\,d\nu = \int_{\Lambda\backslash H} f\,d\nu_\Lambda$. Likewise κ is right invariant. For $k = (k_1,k_2)$ we have $i(l,a)\,i(k)\,x = i(lk_1, ak_2)\,x$; hence G' is invariant under left multiplication by M, and $I(i(k)x) = I(x)$ for $x\in G'$ and $k\in L\times H$. Moreover $\mu(G\setminus G') = 0$: by (6.18) with $f = \mathbf{1}_{G\setminus G'}$, which is left M-invariant, $\mu(G\setminus G') = \int_G \beta(x)\mathbf{1}_{G\setminus G'}(x)\,\mathrm{vol}(M)\,d\mu(x) = 0$, as $\beta\mathbf{1}_{G\setminus G'} = 0$ μ-almost everywhere (with $0\cdot\infty = 0$).

The inner integrals. Let $g\in G'$ and let k be as in the hypothesis. Then $i(k)g\in G'$, so the function $(l,\Lambda a)\mapsto F(i(l,a)i(k)g)$ is $\kappa\otimes\nu_\Lambda$-integrable, and by Fubini's theorem its iterated integral, which is $\rho(g)c$ by hypothesis, equals $I(i(k)g) = I(g)$. Hence

$$I(g) = \rho(g)\,c \qquad (g\in G'). \tag{6.20}$$

The constant. Call a pair (F,c) admissible if it satisfies the hypotheses of the statement. If (F,c) is admissible and $c = 0$, then $I = 0$ on G' by (6.20), and (6.19) gives $\int_{\Gamma\backslash G} F\,d\mu_\Gamma = 0 = Rc$

whatever R is. If no admissible pair with $c \neq 0$ exists, put $R = 1$. Otherwise fix one, (F_1, c_1) with $c_1 \neq 0$, with its set G_1' and function I_1, and put

$$R = \int_{G_1'} \beta(x)\, \frac{I_1(x)}{c_1}\, d\mu(x).$$

On G_1' we have $\rho = I_1/c_1$ by (6.20); in particular ρ is left M-invariant on the M-invariant set G_1'. If (F, c) is admissible with $c \neq 0$, then $\beta I/c = \beta\rho = \beta I_1/c_1$ on the μ-conull set $G' \cap G_1'$, so (6.19) gives $\int_{\Gamma\backslash G} F\, d\mu_\Gamma = c \int_{G'} \beta I/c\, d\mu = cR$. So R has the asserted property, and it does not depend on F or c.

$R \neq 0$. Suppose $R = 0$ for the R of the last case. As $\operatorname{Re}\rho \geq 0$, the function $\beta \operatorname{Re}\rho = \operatorname{Re}(\beta I_1/c_1)$ is nonnegative on G_1' with integral $\operatorname{Re} R = 0$, so it vanishes μ-almost everywhere on G_1'. The set $B = \{x \in G_1' : \operatorname{Re}\rho(x) > 0\}$ is left M-invariant, and $\beta \mathbf{1}_B = 0$ μ-almost everywhere; by (6.18) with $f = \mathbf{1}_B$ as before, $\mu(B) = 0$. Let U be a nonempty open set on which $\operatorname{Re}\rho \neq 0$. Then $U \cap G_1' \subset B$ and $\mu(U \cap G_1') = \mu(U) > 0$, a contradiction. Hence $R \neq 0$. □

Lemma 6.2.48. *Let G and H be locally compact groups with left Haar measures μ and ν, let $i\colon H \to G$ be an injective continuous homomorphism, and let $\Lambda \leq H$ be a discrete subgroup such that $\Gamma = i(\Lambda)$ is discrete in G. Let (Z, m) be a σ-finite measure space and $T\colon Z \to G$ a measurable map such that $(z, a) \mapsto i(a)T(z)$ carries $m \otimes \nu$ to μ. Let $F\colon G \to \mathbb{C}$ be Borel, left Γ-invariant and μ_Γ-integrable on $\Gamma\backslash G$. Then for m-almost every z the function $\Lambda a \mapsto F(i(a)T(z))$ is ν_Λ-integrable on $\Lambda\backslash H$, the function $z \mapsto \int_{\Lambda\backslash H} F(i(a)T(z))\, d\nu_\Lambda(a)$ (defined almost everywhere) is m-integrable, and*

$$\int_{\Gamma\backslash G} F\, d\mu_\Gamma = \int_Z \int_{\Lambda\backslash H} F\big(i(a)T(z)\big)\, d\nu_\Lambda(a)\, dm(z).$$

The same identity holds, in $[0, \infty]$, for every Borel left Γ-invariant $F\colon G \to [0, \infty]$.

Proof. **A cutoff for Γ.** The discrete subgroup Γ of the second countable group G is countable. Choose an open neighbourhood U of 1 with $UU^{-1} \cap \Gamma = \{1\}$ and elements $g_1, g_2, \ldots$ with $G = \bigcup_j U g_j$. Each set Ug_j meets each coset Γg in at most one point (if $\gamma u g_j = u' g_j$ with $u, u' \in U$, then $\gamma = u'u^{-1} \in \Gamma \cap UU^{-1}$, so $\gamma = 1$). Hence

$$D = \bigcup_{j \geq 1} \Big(Ug_j \setminus \Gamma\big(Ug_1 \cup \cdots \cup Ug_{j-1}\big) \Big)$$

is a Borel set (a countable union of Borel sets, Γ being countable) meeting every coset Γg in exactly one point. Put $\varphi = \mathbf{1}_D$; then $\sum_{\gamma \in \Gamma} \varphi(\gamma g) = 1$ for every $g \in G$.

The identity for $F \geq 0$. Let $F\colon G \to [0, \infty]$ be Borel and left Γ-invariant. By the defining property of μ_Γ applied to φF,

$$\int_{\Gamma\backslash G} F\, d\mu_\Gamma = \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma} \varphi(\gamma g) F(\gamma g)\, d\mu_\Gamma(\Gamma g) = \int_G \varphi F\, d\mu.$$

As $(z, a) \mapsto i(a)T(z)$ carries $m \otimes \nu$ to μ, and m, ν are σ-finite (H is σ-compact), Tonelli's theorem gives

$$\int_G \varphi F\, d\mu = \int_Z \int_H \varphi\big(i(a)T(z)\big) F\big(i(a)T(z)\big)\, d\nu(a)\, dm(z).$$

Fix z and put $\varphi_z(a) = \varphi(i(a)T(z))$, $F_z(a) = F(i(a)T(z))$. Since i is an injective homomorphism, $\lambda \mapsto i(\lambda)$ is a bijection $\Lambda \to \Gamma$ and $i(\lambda a)T(z) = i(\lambda)\, i(a)T(z)$; therefore $\sum_{\lambda \in \Lambda} \varphi_z(\lambda a) =$

$\sum_{\gamma\in\Gamma}\varphi(\gamma\, i(a)T(z))=1$ and $F_z(\lambda a)=F_z(a)$ for $\lambda\in\Lambda$. By the defining property of ν_Λ,

$$\int_H \varphi_z F_z\,d\nu=\int_{\Lambda\backslash H}\sum_{\lambda\in\Lambda}\varphi_z(\lambda a)F_z(\lambda a)\,d\nu_\Lambda(\Lambda a)=\int_{\Lambda\backslash H}F_z\,d\nu_\Lambda.$$

The left side is a measurable function of z by Tonelli's theorem, hence so is the right side, and combining the three displays proves the identity in $[0,\infty]$.

Integrable F. Apply the case just proved to $|F|$:

$$\int_Z\int_{\Lambda\backslash H}|F(i(a)T(z))|\,d\nu_\Lambda\,dm=\int_{\Gamma\backslash G}|F|\,d\mu_\Gamma<\infty.$$

So the inner function is ν_Λ-integrable for m-almost every z, and the almost everywhere defined function $z\mapsto\int_{\Lambda\backslash H}F(i(a)T(z))\,d\nu_\Lambda$ is dominated by an m-integrable function. Write $F=F_0-F_1+\mathrm{i}F_2-\mathrm{i}F_3$ with F_0,F_1 the positive and negative parts of $\operatorname{Re}F$ and F_2,F_3 those of $\operatorname{Im}F$; each F_j is Borel, left Γ-invariant and bounded by $|F|$, so both sides of the identity are finite for F_j. Applying the case $F\ge0$ to each F_j and taking the same linear combination gives the identity for F. □

Lemma 6.2.49. *Let W and W_0 be spaces with points with signs s on n and s_0 on n_0, and let H be a splitting of W over W_0 along a one-element set, with Levi and radical elements lev_H and rad_H. Let S be a nonempty set of finite places of E, and let $D\colon \mathrm{G}(s)\to\mathbb{C}$ be a continuous function with the following properties.*

1. *$D(\gamma g)=D(g)$ for all $\gamma\in\Gamma_W$ and $g\in\mathrm{G}(s)$.*
2. *$D(g\,(1,a))=D(g)$ for all $g\in\mathrm{G}(s)$ and all $a\in G_f(n)$ whose components at the finite places outside $S\cup\bar S$ are 1.*
3. *If $D(\mathrm{rad}_H(c,b)\,g)=D(g)$ for all adelic $n_0\times1$ matrices c, all $b\in\mathbb{A}_E$ and all $g\in\mathrm{G}(s)$, then $D(\mathrm{lev}_H(t,1))=D(1)$ for all $t\in\mathbb{A}_E^\times$.*

Then D is constant on $\mathrm{G}(s)$.

Proof. Write $H=(k,l,i)$ with adelic frames $k_\mathbb{A},l_\mathbb{A},i_\mathbb{A}$ and coordinates $\mathrm{pr}_k,\mathrm{pr}_l,\mathrm{pr}_i$, and let $Q=(l_\mathbb{A}\mid k_\mathbb{A}\mid i_\mathbb{A})\in\mathrm{GL}_n(\mathbb{A}_E)$; by (MX1) the rows of Q^{-1} are pr_l, pr_k, pr_i. Let $\mathrm{SU}\subseteq\mathrm{G}(s)$ be the kernel of det. Through $g\mapsto x_W g x_W^{-1}$, $\mathrm{G}(s)$ is $\mathrm{U}(W)(\mathbb{A}_F)$, SU is $\mathrm{SU}(W)(\mathbb{A}_F)$ for the special unitary group $\mathrm{SU}(W)$ of the hermitian form J_W on E^n, an algebraic group over F, and $\Gamma_W\cap\mathrm{SU}$ is $\mathrm{SU}(W)(F)$; the topologies agree, since $\mathrm{GL}_n(E\otimes_F\mathbb{A}_F)=\mathrm{GL}_n(\mathbb{A}_E)$ and conjugation by x_W is a homeomorphism.

Step 1: radical elements and the place v. For all c, b the element $\mathrm{rad}_H(c,b)$ lies in SU: by (MX2), $N=\mathrm{rad}_H(c,b)-1$ maps the column space of $l_\mathbb{A}$ into that of $(i_\mathbb{A}\mid k_\mathbb{A})$, that of $i_\mathbb{A}$ into that of $k_\mathbb{A}$, and kills $k_\mathbb{A}$ (as $\mathrm{pr}_l k_\mathbb{A}=0$, $\mathrm{pr}_i k_\mathbb{A}=0$, $\mathrm{pr}_l i_\mathbb{A}=0$); so N is nilpotent and $\det\mathrm{rad}_H(c,b)=1$. Fix $w\in S$ and let v be the place of F below w; the places of E above v are w and $\bar w$, both in $S\cup\bar S$. Let $\mathrm{SU}_v\subseteq\mathrm{SU}$ be the subgroup of elements whose components at all places of E not above v are 1; it corresponds to $\mathrm{SU}(W)(F_v)$, and each of its elements is $(1,a)$ with a as in (2). For $y\in F_v$ let y also denote the adele of E with components y at w, $\bar w$ and 0 elsewhere; then $\bar y=y$ and by (MX2)

$$\mathrm{rad}_H(0,y)=1-\jmath^{-1}y\,k_\mathbb{A}\mathrm{pr}_l\in\mathrm{SU}_v.$$

Since $k_\mathbb{A}\mathrm{pr}_l\,l_\mathbb{A}=k_\mathbb{A}$ has nonzero component at w, some entry of $(k_\mathbb{A}\mathrm{pr}_l)_w$ is nonzero, so the entries of $\mathrm{rad}_H(0,y)_w$ are unbounded as $|y|_v\to\infty$; hence $\mathrm{SU}(W)(F_v)$, a closed subgroup of $\mathrm{GL}_n(E\otimes_F F_v)$ containing an unbounded set, is not compact.

Step 2: strong approximation. The group $\mathrm{SU}(W)$ is connected, simply connected and absolutely almost simple over F: over an algebraic closure of F it becomes SL_n, and $n = n_0 + 2 \geq 2$. By Step 1 and strong approximation (Theorem A.4.8, with $K = F$, $H = \mathrm{SU}(W)$ and the finite set $\{v\}$), $\mathrm{SU}(W)(F)$ is dense in $\mathrm{SU}(W)(\mathbb{A}_F^{\{v\}})$. Hence the products γk with $\gamma \in \Gamma_W \cap \mathrm{SU}$ and $k \in \mathrm{SU}_v$ are dense in SU: given $h \in \mathrm{SU}$ and a neighbourhood of h, which contains a product $N_v \times N^v$ of neighbourhoods of the components of h above v and away from v in $\mathrm{SU}(W)(F_v) \times \mathrm{SU}(W)(\mathbb{A}_F^{\{v\}})$, choose γ with components away from v in N^v and let $k \in \mathrm{SU}_v$ have components $\gamma_u^{-1} h_u$ at the places u above v; then γk has the components of h above v and those of γ elsewhere, so it lies in $N_v \times N^v$.

Step 3: $D(hg) = D(g)$ for all $h \in \mathrm{SU}$ and $g \in \mathrm{G}(s)$. Fix g. For $\gamma \in \Gamma_W \cap \mathrm{SU}$ and $k \in \mathrm{SU}_v$, (1) gives $D(\gamma k g) = D(kg)$, and $kg = g\,(g^{-1}kg)$, where $g^{-1}kg$ has component 1 at every place of E not above v, the infinite places included; so $g^{-1}kg = (1, a)$ with a as in (2), and (2) gives $D(kg) = D(g)$. The function $h \mapsto D(hg)$ on SU is continuous and equals $D(g)$ on the dense set of Step 2, hence everywhere.

Step 4: the torus. By Steps 1 and 3, $D(\mathrm{rad}_H(c, b) g) = D(g)$ for all c, b, g, so (3) gives $D(\mathrm{lev}_H(t, 1)) = D(1)$ for every $t \in \mathbb{A}_E^\times$. By (MX2), $\mathrm{lev}_H(t, 1) = Q\,\mathrm{diag}(\bar{t}^{-1}, t, 1)\,Q^{-1}$, so

$$\det \mathrm{lev}_H(t, 1) = t\,\bar{t}^{-1}.$$

Step 5: every $g \in \mathrm{G}(s)$ is $h\,\mathrm{lev}_H(t, 1)$ with $h \in \mathrm{SU}$ and $t \in \mathbb{A}_E^\times$. From $g^* \varepsilon_s g = \varepsilon_s$, $z = \det g$ satisfies $z\bar{z} = 1$. Every such idele is $t\bar{t}^{-1}$ for some $t \in \mathbb{A}_E^\times$: at each place u of F, $E \otimes_F F_u$ is $\mathbb{C}$, a quadratic field extension of F_u or $F_u \times F_u$ with the exchange involution, and Hilbert's Theorem 90 for this quadratic algebra gives t_u with $z_u = t_u \bar{t}_u^{-1}$; at the almost all finite u that are unramified in E and at which z_u is a unit, t_u can be taken a unit, namely $t_u = (z_{u,1}, 1)$ if $z_u = (z_{u,1}, z_{u,1}^{-1})$ in the split case, and the unit part of t_u in the inert case (a uniformiser of F_u is one of $E \otimes_F F_u$ and is fixed by the involution). So $t = (t_u)$ is an idele. Put $h = g\,\mathrm{lev}_H(t, 1)^{-1}$; by Step 4, $\det h = 1$.

Conclusion. For $g = h\,\mathrm{lev}_H(t, 1)$ as in Step 5, Steps 3 and 4 give $D(g) = D(\mathrm{lev}_H(t, 1)) = D(1)$. □

Lemma 6.2.50. *Let G be a locally compact group with a left Haar measure μ, $H \leq G$ a closed subgroup with a left Haar measure ν, and $\Gamma \leq H$ a subgroup that is discrete in G. Let $F\colon G \to \mathbb{C}$ be continuous, left Γ-invariant and μ_Γ-integrable on $\Gamma\backslash G$. Let (C, κ) be a probability space, $d \geq 0$, $a \in \mathbb{C}$, and $t\colon G \times C \to H$ a map with $b \mapsto t(x, b)$ measurable for every $x \in G$, such that:*

1. *for all $x \in G$ and $b \in C$, the right translation $s \mapsto s\,t(x, b)$ of H carries ν to $d\nu$;*
2. *for all $s \in H$ and $x \in G$, the function $b \mapsto F(s\,t(x, b)\,x)$ is κ-integrable and*

$$\int_C F(s\,t(x, b)\,x)\,d\kappa(b) = a\,F(sx).$$

If $a \neq d$, then $\int_{\Gamma\backslash G} F\,d\mu_\Gamma = 0$.

Proof. Suppose $\int_{\Gamma\backslash G} F\,d\mu_\Gamma \neq 0$; we show $a = d$.

Unfolding along H. By the existence of Bruhat functions for closed subgroups (Bourbaki, Intégration, ch. VII, §2, no. 4) there is a continuous $\beta_0 \geq 0$ on G with $\int_H \beta_0(ys)\,d\nu(s) = 1$ for all $y \in G$; put $\beta(g) = \beta_0(g^{-1})$, so $\int_H \beta(s^{-1}g)\,d\nu(s) = 1$ for all g. For Borel $f \geq 0$ on G, Tonelli's theorem and the left invariance of μ give

$$\int_G \beta(x) \int_H f(sx)\,d\nu(s)\,d\mu(x) = \int_H \int_G \beta(s^{-1}y) f(y)\,d\mu(y)\,d\nu(s) = \int_G f\,d\mu,$$

so $(x,s)\mapsto sx$ carries the σ-finite measure $(\beta\mu)\otimes\nu$ to μ. By Lemma 6.2.48 (with H, the inclusion $H\to G$, $\Lambda=\Gamma$, $Z=G$, $m=\beta\mu$, $T(x)=x$), for $\beta\mu$-almost every x the function $\Gamma s\mapsto F(sx)$ is ν_Γ-integrable on $\Gamma\backslash H$, and

$$\int_{\Gamma\backslash G}F\,d\mu_\Gamma=\int_G\beta(x)\int_{\Gamma\backslash H}F(sx)\,d\nu_\Gamma(s)\,d\mu(x).$$

As the left side is nonzero, there is $x\in G$ such that $s\mapsto F(sx)$ is ν_Γ-integrable and $J=\int_{\Gamma\backslash H}F(sx)\,d\nu_\Gamma(s)\neq0$. Fix this x and write $t_b=t(x,b)$.

The effect of one translation. Let φ be the indicator of a Borel subset of H meeting each coset Γs exactly once (it exists as Γ is discrete in the second countable group H), so that $\sum_{\gamma\in\Gamma}\varphi(\gamma s)=1$ and $\int_{\Gamma\backslash H}f\,d\nu_\Gamma=\int_H\varphi f\,d\nu$ for every Borel left Γ-invariant $f\geq0$ on H. Let $f\geq0$ be such a function and $b\in C$. By (1), $\int_H h(st_b)\,d\nu(s)=d\int_H h\,d\nu$ for Borel $h\geq0$; applied to $h(s)=\varphi(st_b^{-1})f(s)$, and as $s\mapsto\varphi(st_b^{-1})$ is again the indicator of a set meeting each coset once,

$$\int_H\varphi(s)f(st_b)\,d\nu(s)=d\int_H\varphi(st_b^{-1})f(s)\,d\nu(s)=d\int_{\Gamma\backslash H}f\,d\nu_\Gamma.\tag{6.21}$$

Since $s\mapsto F(st_bx)$ is left Γ-invariant, (6.21) with $f(s)=|F(sx)|$, and then with the four positive and negative parts of the real and imaginary parts of $F(sx)$, gives

$$\int_H\varphi(s)\big|F(st_bx)\big|\,d\nu(s)=d\int_{\Gamma\backslash H}|F(sx)|\,d\nu_\Gamma<\infty,\qquad\int_H\varphi(s)F(st_bx)\,d\nu(s)=d\,J.$$

Fubini. The function $(s,b)\mapsto\varphi(s)F(st(x,b)x)$ on $H\times C$ is measurable (F is continuous and $b\mapsto t(x,b)$ measurable), and by the first identity and $\kappa(C)=1$ it is $\nu\otimes\kappa$-integrable. By Fubini's theorem, the second identity and (2),

$$\begin{aligned}d\,J&=\int_C\int_H\varphi(s)F(st_bx)\,d\nu(s)\,d\kappa(b)=\int_H\varphi(s)\int_C F(st(x,b)x)\,d\kappa(b)\,d\nu(s)\\&=a\int_H\varphi(s)F(sx)\,d\nu(s)=a\,J.\end{aligned}$$

As $J\neq0$, $a=d$. □

6.3 The exchange identity

This section proves the exchange identity (Theorem 6.3.2), from the facts of Section 6.2; what the later sections use is that its constant is uniform in the character. The identity belongs to the theory of theta lifts in Witt towers: Rallis's tower property [236], the Siegel–Weil formula and its regularised forms [280, 323, 176, 139], and the Rallis inner product formula [238, 231, 187]. The second term identity and the inner product formula at the boundary of the convergent range [94, 331] are related to the equal-rank constant below, but are not used. The organisation follows [259].

The theta integrals converge because the local data are target-bounded, and their centre coefficients along a splitting are computed by unfolding: for an invertible Gram matrix the coefficient is, by Witt's theorem [330], the integral of one theta term over the quotient by the stabiliser of one frame (Lemma 6.3.25). This is the unitary form of the computation of Fourier coefficients of theta lifts that goes back to [237, 171]; for Fourier–Jacobi and Bessel periods in this language see [96].

The proof has two halves. The first is the exchange for matched functions (Lemma 6.3.3), with a constant whose square is L_* times the equal-rank ratio, in four steps. (1) At the top,

for a large triple, the identity follows from an identity of periods whose value is the equal-rank theta integral (Proposition 6.3.22, Proposition 6.3.6). The coefficients at indices that are not negative at some real place vanish, because an invariant Fock functional of that kind is zero, and the remaining coefficients are controlled by an equivariant lift, which vanishes once the standard datum vanishes at self-dual lattices (Proposition 6.3.33). This rests on a vanishing theorem for singular centre coefficients, which we call Freitag's lemma (Proposition 6.3.36) after Freitag's theory of singular modular forms [90]; see also [188]. In our reading, neither contains it in the form needed. (2) The identity descends from the top by removing hyperbolic planes one at a time (Proposition 6.3.10), with the tower property on the large side and only the zero vector on the small side; the change of point uses Landherr's theorem [180] and the Hasse norm theorem [123]. (3) It passes from standard to all matched data (Lemma 6.3.13). (4) The small side at the adjoint datum is a unimodular multiple of the theta integral of the reversed pair (Lemma 6.3.4).

The second half is the equal-rank constant (Lemma 6.3.15): the ratio is one nonzero constant for every modulus and every character. It depends only on the signs of the space, and it does not vanish identically (Lemma 6.3.16), since otherwise the vanishing would descend along hyperbolic planes to rank 0, where the theta integral is 1; the descent needs only Hecke's theory (Lemma 6.3.62).

Before the law for squares that closes the section comes the explicit matching (Proposition 6.3.67), the identity for class sums over a definite group at polynomial data, which the cost of one lift uses (Lemma 6.6.62). Besides the exchange identity it needs the calculus of theta kernels (Lemma 6.3.71) and an archimedean matching (Lemma 6.3.68), which rests on the density of the Fock vectors of polynomials [14] and on a polarisation identity for products of determinants, from Weyl's dimension formula [151, 92].

Definition 6.3.1. Class sums. Let a be a finite index set, I_a the standard definite space, $\mathbf{K}_a = \mathrm{U}(I_a)(\widehat{\mathcal{O}})$ the stabiliser of the standard lattice in $\mathrm{U}(I_a)(\mathbb{A}_{F,f})$, and $\Gamma \subseteq \mathbf{K}_a$ an open subgroup. For a function F on $\mathrm{U}(I_a)(\mathbb{A}_{F,f})$, left invariant under $\mathrm{U}(I_a)(F)$ and right invariant under Γ, the **class sum** is

$$\sum_{h \bmod \Gamma}^{\mathrm{cl}} F(h) = \sum_{\mathfrak{c}} \frac{F(h_{\mathfrak{c}})}{\#\big(\mathrm{U}(I_a)(F) \cap h_{\mathfrak{c}} \Gamma h_{\mathfrak{c}}^{-1}\big)}, \tag{6.22}$$

the (finite) sum over the double cosets $\mathfrak{c}$ of $\mathrm{U}(I_a)(F)\backslash \mathrm{U}(I_a)(\mathbb{A}_{F,f})/\Gamma$ with representatives $h_{\mathfrak{c}}$. For a theta kernel with second space I_a, a finite point h is read as the adelic point with infinite components 1.

Polynomials and their Fock vectors. For finite index sets r (rows) and c (columns), $\mathbb{C}[x]_{r\times c}$ is the polynomial ring in the variables $x_{\tau,i,k}$, $\tau \in \Phi$, $i \in r$, $k \in c$; for spaces S, T with index sets n_S, n_T a polynomial $P \in \mathbb{C}[x]_{n_T \times n_S}$ has a Fock vector v_P for the pair (S,T), and for a modulus $\mathfrak{m}$ and a function Φ on $n_T \times n_S$ matrices over $R_{\mathfrak{m}}$ we write $\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Phi, P] = \theta^{S,T}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}}(\Phi, v_P))$.

Block signs. Let n, m, l, b, p, a be finite index sets with bijections $j\colon m \sqcup l \to n$, $f\colon p \sqcup l \to b$, $e\colon p \sqcup a \to m$. The signs sgn_j on n are $+1$ on $j(m)$ and -1 on $j(l)$; the signs sgn_f on b are $+1$ on $f(p)$ and -1 on $f(l)$.

Stacked determinants. Let $\iota = (\iota_\tau)_{\tau\in\Phi}$ be finite sets and $\eta = (\eta_{\tau k})_{\tau\in\Phi, k\in\iota_\tau}$ complex $p \times m$ matrices. For τ and k let $\Delta_\tau(\eta_{\tau k})$ be the determinant of the $m \times m$ matrix whose row $e(i)$ is the row i of $\eta_{\tau k}$ for $i \in p$, and whose row $e(i)$ is $(x_{\tau,i,j(k')})_{k'\in m}$ for $i \in a$. The **product of stacked determinants** is

$$\Delta^j(e;\eta) = \prod_{\tau\in\Phi} \prod_{k\in\iota_\tau} \Delta_\tau(\eta_{\tau k}) \ \in \mathbb{C}[x]_{a\times n}. \tag{6.23}$$

Matched polynomial. With the same data, let $y_\tau^{--} = (x_{\tau,j(i),f(i')})_{i,i'\in l}$ and $y_\tau^{++} = (x_{\tau,j(i),f(i')})_{i\in m,i'\in p}$. The **matched polynomial** is

$$Q_{f,j}(\eta) = \prod_{\tau\in\Phi} \det(y_\tau^{--})^{\max(0,\,|\iota_\tau|+|n|-2|l|)} \prod_{\tau\in\Phi}\prod_{k\in\iota_\tau} \det(\eta_{\tau k}\, y_\tau^{++}) \ \in \mathbb{C}[x]_{n\times b}. \tag{6.24}$$

Theorem 6.3.2. *Let W, V, V' be spaces with index sets n, a, b and a bijection $\sigma\colon b\sqcup a\to n$, and let $e\in\mathbb{N}$ with*

$$\operatorname{rk} V = \operatorname{rk} V' + 2e, \qquad \operatorname{neg}(W) = \operatorname{neg}(V) + \operatorname{neg}(V'),$$
$$2\operatorname{neg}(W) + 2 \le \operatorname{rk} W \le 2\mu(v) + 2\operatorname{neg}(W) \quad (v\in\Phi),$$

for a function $\mu\colon\Phi\to\mathbb{Z}$. Let v be a Schwartz vector for (V',W). Then there are a real number $C>0$ and a Schwartz vector x for (W,V) such that for every modulus $\mathfrak{m}$, every χ and every ν tame of weight μ for $(\mathfrak{m},\chi)$ there is $c\in\mathbb{C}$ with

$$|c|^2 = C\cdot L_*(\mathfrak{m},\nu;e) \neq 0$$

and, with $\gamma=\nu^{-1}\upsilon^{\operatorname{rk} W/2}$: for all automorphic characters α,α' of $\mathrm{U}(1)$ with $\alpha[\operatorname{rk} V]\,\alpha'[\operatorname{rk} V'] = \tilde\nu^{-1}$, all Φ, Ψ matched for (σ,χ) modulo $\mathfrak{m}$, and all $g\in\mathrm{U}(W)(\mathbb{A}_F)$,

$$I^{W,V}_{\gamma,\alpha}(\mathrm{mod}_{\mathfrak{m}}(\Phi,x))(g) = c\int_{[\mathrm{U}(V')]} \theta^{V',W}_{\alpha',\gamma}(\mathrm{mod}_{\mathfrak{m}}(\Psi,v))(h,g)\,dh.$$

The constant C and the vector x are chosen before $\mathfrak{m}$ and ν; so $|c|^2$ is proportional to $L_*(\mathfrak{m},\nu;e)$ with a factor independent of the character.

Proof. **Step 1: the parameters.** By σ, $\operatorname{rk} W = \operatorname{rk} V + \operatorname{rk} V' = 2\operatorname{rk} V' + 2e$ is even, so $\operatorname{rk} V - \frac12\operatorname{rk} W = e$. Put $k = \frac12\operatorname{rk} W - \operatorname{neg}(W)$. Then $k\ge 1$ by $2\operatorname{neg}(W)+2\le\operatorname{rk} W$, $\operatorname{rk} W = 2k + 2\operatorname{neg}(W)$, and $k\le\mu(v)$ for all v by $\operatorname{rk} W\le 2\mu(v)+2\operatorname{neg}(W)$; in particular $\mu\ge 1$.

Step 2: the constant. By Lemma 6.3.15 there are a Schwartz vector z for $(\mathbb{H}\oplus W,\mathbb{H}\oplus W)$ and $C_0\neq 0$ with $r_{\mathbb{H}\oplus W}(\nu,z) = C_0$ for every tame ν of weight μ. Apply Lemma 6.3.3 to $(W,U,V) = (W,V',V)$, with this z and $y=v$ (the hypotheses on ranks and signs are those of the theorem). It gives $C_1\in\mathbb{R}$ and a Schwartz vector x for (W,V) such that every tame ν has a number c with

$$C_1|c|^2 = L_*(\mathfrak{m},\nu;e)\,C_0,$$

together with the identity of (2) of that lemma for $\alpha=1$, $\alpha'=\gamma$ and matched functions over $\mathcal{O}_{\mathfrak{m}}$. The right side is not zero ($L_*>0$, $C_0\neq 0$), so $C_1\neq 0$, $c\neq 0$ and $|c|^2 = (C_0/C_1)L_*(\mathfrak{m},\nu;e)$. If $C_0/C_1\le 0$ this contradicts $|c|^2>0$, so no tame ν of weight μ exists and the theorem holds with $C=1$. Otherwise put $C = C_0/C_1 > 0$.

Step 3: data modulo $\mathfrak{m}$. Let Φ, Ψ be matched for (σ,χ) modulo $\mathfrak{m}$, and let $\pi\colon\mathcal{O}_{\mathfrak{m}}\to R_{\mathfrak{m}}$ be the reduction. It is onto, it is compatible with the residue maps (so u is a unit if and only if $\pi(u)$ is), and on units the character of ν at the places of $\mathfrak{m}$ is $\chi\circ\pi$, by (XD4). Let Φ', Ψ' be $\Phi\circ\pi$, $\Psi\circ\pi$ on integral matrices and 0 elsewhere. Then Ψ' is locally constant, (Φ',Ψ') is matched over $\mathcal{O}_{\mathfrak{m}}$ (the conditions of (XD6) pass through the surjection $\mathrm{GL}_n(\mathcal{O}_{\mathfrak{m}})\to\mathrm{GL}_n(R_{\mathfrak{m}})$), and by (XD5) $\mathrm{mod}_{\mathfrak{m}}(\Phi,x) = x\otimes\Phi'\otimes\Phi^\circ$ and $\mathrm{mod}_{\mathfrak{m}}(\Psi,v) = v\otimes\Psi'\otimes\Phi^\circ$. So Step 2 gives the identity for $\alpha=1$, $\alpha'=\gamma$.

Step 4: the splitting characters. Space characters multiply, $\alpha[p]\,\alpha'[q] = (\alpha\alpha')[p+q]$, and $\gamma[\operatorname{rk} W] = \tilde\nu^{-1}$.

Since base change is injective, $\alpha[\operatorname{rk} V]\alpha'[\operatorname{rk} V'] = \tilde{\nu}^{-1}$ holds if and only if $\alpha\alpha' = \gamma$. Twisting the second character of the pair (W, V) by α multiplies its theta kernel by $d = \alpha(\det g)^{-1}$, and twisting the first character of the pair (V', W) from γ to $\alpha' = \gamma\alpha^{-1}$ multiplies its kernel by the same d. Indeed, by (WR4), $\omega_{\alpha_1,\beta_1}(g_1, h_1) = \alpha_1(\det h_1)\beta_1(\det g_1)^{-1}\omega_{1,1}(g_1, h_1)$ for every pair and all characters, and the theta functional is linear; hence, for every datum φ of (W, V), every datum φ' of (V', W), $g \in \mathrm{U}(W)(\mathbb{A}_F)$, $h \in \mathrm{U}(V)(\mathbb{A}_F)$ and $h' \in \mathrm{U}(V')(\mathbb{A}_F)$,

$$\theta^{W,V}_{\gamma,\alpha}(\varphi)(g,h) = \alpha(\det g)^{-1}\theta^{W,V}_{\gamma,1}(\varphi)(g,h), \qquad \theta^{V',W}_{\gamma\alpha^{-1},\gamma}(\varphi')(h',g) = \alpha(\det g)^{-1}\theta^{V',W}_{\gamma,\gamma}(\varphi')(h',g),$$

and integrating over h, resp. h', the theta integral and the right side of Step 3 are multiplied by d. Multiplying the identity of Step 3 by d gives the identity for (α, α'). □

Lemma 6.3.3. *Let W, U, V be spaces with index sets n, b, r and a bijection $\sigma\colon b \sqcup r \to n$, so that $\operatorname{rk} W = \operatorname{rk} U + \operatorname{rk} V$; assume $\operatorname{rk} W$ even and $\operatorname{neg}(W) = \operatorname{neg}(V) + \operatorname{neg}(U)$. Let $\mu\colon \Phi \to \mathbb{Z}_{>0}$, let z be a Schwartz vector for $(\mathbb{H} \oplus W, \mathbb{H} \oplus W)$ and y a Schwartz vector for (U, W). Then there are $C \in \mathbb{R}$ and a Schwartz vector v for (W, V) with the following property. For every modulus $\mathfrak{m}$ whose places are split over F and every ν tame of weight μ for $(\mathfrak{m}, \chi)$, put $\gamma = \nu^{-1}\upsilon^{\operatorname{rk} W/2}$; there is $c \in \mathbb{C}$ with*

1. *$C\,|c|^2 = L_*(\mathfrak{m}, \nu; \operatorname{rk} V - \frac{1}{2}\operatorname{rk} W) \cdot r_{\mathbb{H}\oplus W}(\nu, z)$;*
2. *for all functions Φ on $r \times n$ and Ψ on $n \times b$ matrices over $\prod_{w\in\mathfrak{m}} E_w$, with Ψ locally constant and (Φ, Ψ) matched over $\mathcal{O}_\mathfrak{m}$ for (σ, χ), and all $g \in \mathrm{U}(W)(\mathbb{A}_F)$,*

$$I^{W,V}_{\gamma,1}(v \otimes \Phi \otimes \Phi^\circ)(g) = c \int_{[\mathrm{U}(U)]} \theta^{U,W}_{\gamma,\gamma}(y \otimes \Psi \otimes \Phi^\circ)(h, g)\, dh.$$

Proof. Write $W_0 = \mathbb{H} \oplus W$, with index set $n_0 = (\{o\} \sqcup \{o\}) \sqcup n$ and signs s_0, so $\#n_0 = \#n + 2$ is even, and let A be its tautological splitting over W along $\{o\}$. Tautological splittings are base-adapted: the matrix of adelic frames of that of $\mathbb{H}_p \oplus V_0$ is $\operatorname{diag}(x^{-1}_{\mathbb{H}_p}, 1)$, where $x_{\mathbb{H}_p}$ is unitary at the infinite places and at the finite places integral with determinant $(c\bar{c}_0 + cc_0)^{\#p} = c^{\#p}$, a unit. For a split tame character $\mathfrak{c} = (\nu, \Sigma)$ put

$$\ell(\mathfrak{c}) = \frac{\left|\theta^{\mathrm{lg}}_\mathfrak{c}(W_0, W_0; z, \Psi^{\mathrm{st}}_1)(1)\right|^2}{L_*(\Sigma, \nu; \#n_0/2)},$$

1 the identity $n_0 \times n_0$ matrix; $L_*(\Sigma, \nu; e) > 0$ for $e \geq 0$ by (3) of Theorem A.1.12 (as in (XD7): powers of norms times $|L^\Sigma(j, \tilde{\nu})|^2$, $j \geq 1$, with $\tilde{\nu}$ nontrivial). Let $U^\flat$ and $y^{\mathrm{ad}} \in \mathcal{F}_{W,U^\flat}$ be as in Lemma 6.3.4; $U^\flat$ has signs $-u$, and y^{ad} is a Schwartz vector by (1) of that lemma.

Step 0: moduli and split tame characters. Let $\mathfrak{m}$ be a modulus and ν tame of weight μ for $(\mathfrak{m}, \chi)$. As $\Sigma = \Sigma(\mathfrak{m})$ is nonempty and contains the conjugate of none of its members, $\mu > 0$, and the tame part of ν is nontrivial exactly at Σ, $\mathfrak{c} = (\nu, \Sigma)$ is a split tame character of weight μ ((XD4)). Its semilocal character is χ read through the residue maps: for $t \in \mathcal{O}^\times_\Sigma$, $t/\bar{t}$ has components t_w at $w \in \Sigma$, $\bar{t}_w^{-1}$ at $\bar{w} \notin \Sigma$ (where the tame part is trivial) and 1 elsewhere, so $\nu(t/\bar{t}) = \prod_{w\in\Sigma} \psi_w(t_w \bmod w) = \chi(t \bmod \mathfrak{m})$. Hence matched over $\mathcal{O}_\mathfrak{m}$ for (σ, χ) is matched over $\mathcal{O}_\Sigma$ for $(\sigma, \chi_\mathfrak{c})$, and Φ_χ of (XD8) for W_0 is Ψ^{st}_1 for $\chi_\mathfrak{c}$; with $\gamma_0 = \nu^{-1}\upsilon^{\#n_0/2}$, $I_{W_0}(z) = I^{W_0,W_0}_{\gamma_0,\gamma_0}(z \otimes \Phi_\chi \otimes \Phi^\circ) = \theta^{\mathrm{lg}}_\mathfrak{c}(W_0, W_0; z, \Psi^{\mathrm{st}}_1)$, so $r_{\mathbb{H}\oplus W}(\nu, z) = \ell(\mathfrak{c})$. Finally, for Φ matched with a locally constant Ψ, Φ is locally constant with compact support ((1) of Lemma 6.3.19), so $\Phi^\sharp, \Psi^\sharp \in \mathcal{T}_\Sigma$ ((1) of Lemma 4.1.3) and $v \otimes \Phi \otimes \Phi^\circ = v \otimes [\Phi]_\Sigma$, $y \otimes \Psi \otimes \Phi^\circ = y \otimes [\Psi]_\Sigma$.

Step 1: the exchange identity for $(W, V, U^\flat)$. We show: there is a Schwartz vector $v \in \mathcal{F}_{W,V}$ such that $(\sigma, v, y^{\mathrm{ad}}, \ell)$ satisfy the row-column exchange identity for $(W, V, U^\flat)$ and μ.

Case $b \neq \emptyset$. By (3) of Lemma 6.3.5 (for A and $U^\flat$) there is a Schwartz vector $w \in \mathcal{F}_{W_0,U^\flat}$ with $\mathrm{rsl}_{A,U^\flat}(0; w) = y^{\mathrm{ad}}$. Let $X = \mathbb{H}_b \oplus W_0$, with index set $(b \sqcup b) \sqcup n_0$, D its tautological splitting over W_0 along b, $\bar{D}$ the opposite splitting (base-adapted: the same frame matrix with two blocks exchanged), and $\sigma'\colon b \sqcup (b \sqcup n_0) \to (b \sqcup b) \sqcup n_0$ the associativity bijection. Proposition 6.3.6 for X, $W_1 = W_0$, W, $V' = U^\flat$ (signs $-u$ on the nonempty b), D, $D_1 = A$, σ', μ, w and z gives $\phi_{\mathrm{lg}} \in \mathcal{F}_{X,U^\flat \oplus W_0}$, $\phi_{\mathrm{sm}} \in \mathcal{F}_{X,U^\flat}$ with $\mathrm{rsl}_{\bar{D},U^\flat}(0; \phi_{\mathrm{sm}}) = w$ such that $(\sigma', \phi_{\mathrm{lg}}, \phi_{\mathrm{sm}}, \ell)$ satisfy the row-column exchange identity for $(X, U^\flat \oplus W_0, U^\flat)$ and μ. The composite $A' = \bar{D} \circ A$ is a splitting of X over W along $p = b \sqcup \{o\}$; its matrix of adelic frames is, up to the order of the columns, that of $\bar{D}$ times $\mathrm{diag}(1, 1, M_A)$, M_A that of A, so it is base-adapted. By Lemma 6.3.8, $\mathrm{rsl}_{A',U^\flat}(0; \phi_{\mathrm{sm}}) = \mathrm{rsl}_{A,U^\flat}(0; w) = y^{\mathrm{ad}}$. Apply Proposition 6.3.10 to $W' = X$, W, $V' = U^\flat \oplus W_0$ (index set $b \sqcup n_0$), V, $U^\flat$, p, A', σ, σ': $\#n$ is even, and

$$\mathrm{neg}(U^\flat \oplus W_0) = (\#b - \mathrm{neg}(U)) + (\mathrm{neg}(W) + 1) = \#p + \mathrm{neg}(V)$$

by $\mathrm{neg}(W) = \mathrm{neg}(V) + \mathrm{neg}(U)$. It gives $v \in \mathcal{F}_{W,V}$ with the asserted identity.

Case $b = \emptyset$. Then U and $U^\flat$ are the zero space O, $\mathcal{F}_{W_0,O} = \mathcal{F}_{W,O} = \mathbb{C}$, y and y^{ad} are the same number, and $\mathrm{neg}(W) = \mathrm{neg}(V)$. Let $\sigma_0\colon \emptyset \sqcup n_0 \to n_0$ be the identity. For the empty matrix 0, $\Phi^{\mathrm{st}}_{\sigma_0,0} = \Psi^{\mathrm{st}}_1$ on $n_0 \times n_0$ matrices and $\Psi^{\mathrm{st}}_0 = 1$ on the single $n_0 \times \emptyset$ matrix. By Proposition 6.3.17 ($\#n_0$ even, A base-adapted), $\theta^{\mathrm{lg}}_{\mathfrak{c}}(W_0, W_0; z, \Phi^{\mathrm{st}}_{\sigma_0,0})$ is the constant $c_{\mathfrak{c}} = \theta^{\mathrm{lg}}_{\mathfrak{c}}(W_0, W_0; z, \Psi^{\mathrm{st}}_1)(1)$. In rank 0 the theta model and the Fock model are $\mathbb{C}$, the Weil operators are trivial, $[\mathrm{U}(O)]$ is a point of mass 1 and the characters of the small side, evaluated at determinants of points of $\mathrm{U}(O)$ and v^0, are 1; so $\theta^{\mathrm{sm}}_{\mathfrak{c}}(W_0, O; 1, \Psi^{\mathrm{st}}_0) = 1$. Hence $(\sigma_0, z, 1, \ell)$ satisfy the row-column exchange identity for (W_0, W_0, O) and μ with $C = 1$: $|c_{\mathfrak{c}}|^2 = L_*(\Sigma, \nu; \#n_0 - \#n_0/2)\, \ell(\mathfrak{c})$. By Proposition 6.3.10 for $W' = W_0$, A, $V' = W_0$, V, $U = O$ ($\mathrm{neg}(W_0) = 1 + \mathrm{neg}(V)$) there is $x_0 \in \mathcal{F}_{W,V}$ such that $(\sigma, x_0, \mathrm{rsl}_{A,O}(0; 1), \ell)$ satisfy the identity for (W, V, O); and $\mathrm{rsl}_{A,O}(0; 1) = 1$ by (MX7) (the integral over a point, of mass 1, with the real vacuum 1 of rank 0). Put $v = y\, x_0$. As local data and theta integrals of target-bounded data are linear in the real vector ((WR6) of Definition 3.2.18, (2) of Lemma 6.3.21; the data of (XD9) for the standard data are target-bounded by Lemma 6.2.26 and (2) of Lemma 6.3.31), (σ, v, y, ℓ) satisfy the identity for (W, V, O) with the same constants.

Step 2: all matched functions. Let C be the constant of Step 1, chosen with v before any character. Fix $\mathfrak{m}$, ν, $\mathfrak{c}$ as in Step 0 and let c_0 be the number of (XD12) for $\mathfrak{c}$. Apply Lemma 6.3.18 to $\mathfrak{c}$, W, V, $V' = U^\flat$, σ, v, y^{ad} and c_0: it gives an equivariant lift $(\mathcal{L}, \mathcal{D})$ for $(\mathfrak{c}, W, b)$ with $\Psi \in \mathcal{L}$ and

$$\mathcal{D}\Psi = \theta^{\mathrm{lg}}_{\mathfrak{c}}(W, V; v, \Phi) - c_0\, \theta^{\mathrm{sm}}_{\mathfrak{c}}(W, U^\flat; y^{\mathrm{ad}}, \Psi)$$

whenever Φ and a locally constant Ψ are matched over $\mathcal{O}_\Sigma$ for $(\sigma, \chi_{\mathfrak{c}})$. Let $s_1 \in \mathrm{M}_{b \times n}(\mathcal{O}_\Sigma)$ consist of the rows $\sigma(b)$ of the identity and x_1 of its columns $\sigma(b)$, so $s_1 x_1 = 1$. By Lemma 6.3.31, $(\Phi^{\mathrm{st}}_{\sigma,s_1}, \Psi^{\mathrm{st}}_{s_1})$ is such a pair, so $\Psi^{\mathrm{st}}_{s_1} \in \mathcal{L}$ and $\mathcal{D}\Psi^{\mathrm{st}}_{s_1} = 0$ by (b) of (XD12). The space $\mathcal{P} = \{\Psi \in \mathcal{L} : \mathcal{D}\Psi = 0\}$ consists of locally constant, compactly supported functions ((XD11)(a)), is stable under $y \mapsto \Psi(gy)$, $g \in \mathrm{GL}_n(E_\Sigma)$ ((3) of Lemma 6.3.14), and contains $\Psi^{\mathrm{st}}_{s_1}$; $\chi_{\mathfrak{c}}$ has the residue form of Lemma 6.3.13 (Step 0). By Lemma 6.3.13, every locally constant Ψ matched with some Φ lies in $\mathcal{P}$:

$$\theta^{\mathrm{lg}}_{\mathfrak{c}}(W, V; v, \Phi) = c_0\, \theta^{\mathrm{sm}}_{\mathfrak{c}}(W, U^\flat; y^{\mathrm{ad}}, \Psi), \tag{6.25}$$

and $\Psi \in \mathcal{L}$ is compactly supported and vanishes at the matrices without a left inverse ((XD11)(a)).

Step 3: back to (U,W). Let d be the number of (2) of Lemma 6.3.4 for $\mathfrak{c}$ and y, and $c = c_0 d$, so $|c| = |c_0|$ and (1) is (a) of (XD12): $C|c|^2 = L_*(\Sigma,\nu;\#r - \#n/2)\,\ell(\mathfrak{c}) = L_*(\mathfrak{m},\nu;\operatorname{rk} V - \frac{1}{2}\operatorname{rk} W)\,r_{\mathbb{H}\oplus W}(\nu,z)$. With $\gamma = \nu^{-1}\upsilon^{\#n/2}$, which is the character of the large side of (W,V), (WR4) gives $\omega_{\gamma,1}(g,h) = \gamma(\det g)\,\omega_{\gamma,\gamma}(g,h)$, hence $I^{W,V}_{\gamma,1}(\upsilon\otimes[\Phi]_\Sigma)(g) = \gamma(\det g)\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;\upsilon,\Phi)(g)$. By (6.25) and Lemma 6.3.4,

$$I^{W,V}_{\gamma,1}(\upsilon\otimes\Phi\otimes\Phi^\circ)(g) = c_0\,\gamma(\det g)\,\theta^{\mathrm{sm}}_{\mathfrak{c}}(W,U^\flat;y^{\mathrm{ad}},\Psi)(g) = c\int_{[\mathrm{U}(U)]}\theta^{U,W}_{\gamma,\gamma}(y\otimes\Psi\otimes\Phi^\circ)(h,g)\,dh,$$

which is (2). □

Lemma 6.3.4. *Let W be a space with signs s on n, $\#n$ even, U a space with signs u on b, r an index set and $\sigma\colon b\sqcup r\to n$ a bijection. Let $\epsilon_\flat\in\mathbb{A}_E^\times$ be the idele with components 1 at the infinite places and finite part $\bar{c}\,\bar{\alpha}/\alpha$, and $U^\flat = (-J_U, x_U\epsilon_\flat)$, a space with signs $-u$ on b, with $\mathrm{G}(-u) = \mathrm{G}(u)$ and $[\mathrm{U}(U^\flat)] = [\mathrm{U}(U)]$. For $y\in\mathcal{F}_{U,W}$ (a function of $z\in\mathbb{C}^{\Phi\times(n\times b)}$) let $y^{\mathrm{ad}}(x) = y(x^\natural)$ for $x\in\mathbb{C}^{\Phi\times(b\times n)}$, where $(x^\natural)_{v,i,j} = u_j\,s_i\,\overline{x_{v,j,i}}$ ($v\in\Phi$, $i\in n$, $j\in b$).*

1. *$y^{\mathrm{ad}}\in\mathcal{F}_{W,U^\flat}$, and y^{ad} is a Schwartz vector if y is one.*
2. *Let $\mathfrak{c} = (\nu,\Sigma)$ be a split tame character and $\gamma = \nu^{-1}\upsilon^{\#n/2}$. There is $d\in\mathbb{C}$ with $|d| = 1$ such that for all functions Φ on $\mathrm{M}_{r\times n}(E_\Sigma)$ and Ψ on $\mathrm{M}_{n\times b}(E_\Sigma)$ that are matched over $\mathcal{O}_\Sigma$ for $(\sigma,\chi_{\mathfrak{c}})$, with Ψ locally constant, compactly supported and vanishing at every matrix without a left inverse, and all $g\in\mathrm{G}(s)$,*

$$\gamma(\det g)\,\theta^{\mathrm{sm}}_{\mathfrak{c}}(W,U^\flat;y^{\mathrm{ad}},\Psi)(g) = d\int_{[\mathrm{U}(U)]}\theta^{U,W}_{\gamma,\gamma}(y\otimes[\Psi]_\Sigma)(h,g)\,dh.$$

Proof. **Step 0: the space $U^\flat$.** Write ε_u, ε_s for the adelic sign matrices. Since $\bar{\epsilon}_\flat\epsilon_\flat$ is 1 at the infinite places and $c\bar{c} = -1$ at the finite ones, $(x_U\epsilon_\flat)^*(-J_U)(x_U\epsilon_\flat) = -\bar{\epsilon}_\flat\epsilon_\flat\,\varepsilon_u = \varepsilon_{-u}$: $U^\flat$ is a space with a point with signs $-u$. The conditions $g^*\varepsilon_u g = \varepsilon_u$ and $g^*\varepsilon_{-u}g = \varepsilon_{-u}$ agree, so $\mathrm{G}(-u) = \mathrm{G}(u)$; $\Gamma_{U^\flat} = (x_U\epsilon_\flat)^{-1}\mathrm{U}(-J_U)(F)(x_U\epsilon_\flat) = \Gamma_U$ ($\epsilon_\flat$ is a central scalar and $\mathrm{U}(-J_U) = \mathrm{U}(J_U)$). The Cayley chart and density of $\mathrm{U}(-\operatorname{diag} u)$ at y are those of $\mathrm{U}(\operatorname{diag} u)$ at $-y$ (in Definition 3.2.1, $-J$ and $S(-y) = -S(y)$ give the same D), and $y\mapsto -y$ preserves Lebesgue measure; so the two measures, and the quotient measures on $[\mathrm{U}(U^\flat)] = [\mathrm{U}(U)]$, agree.

Step 1: the adjoint map and three identities. Let ε be the signs $\varepsilon_{ij} = s_i u_j$ of $\mathrm{Hom}(U,W)$ on $n\times b$, and $\varepsilon'_{ji} = -u_j s_i$ those of $\mathrm{Hom}(W,U^\flat)$ on $b\times n$; $N = \#n\cdot\#b$. Let $\lambda = \epsilon_\flat^{-1}$ (components 1 at the infinite places, finite part $\alpha/(\bar{c}\bar{\alpha}) = -c\alpha/\bar{\alpha}$) and

$$\mathrm{ad}(X) = \lambda\,\varepsilon_u X^*\varepsilon_s \qquad (X\in\mathbb{A}_E^{n\times b}),$$

an additive homeomorphism onto $\mathbb{A}_E^{b\times n}$, with entries $\mathrm{ad}(X)_{ji} = \varepsilon_{ij}\overline{X_{ij}}$ at the infinite places and $\lambda\overline{X_{ij}}$ at the finite ones. Its infinite component is the real-linear map A of $\mathbb{C}^{\Phi\times(n\times b)}$ onto $\mathbb{C}^{\Phi\times(b\times n)}$, $(Ax)_{v,j,i} = u_j s_i\overline{x_{v,i,j}}$, whose inverse is $x\mapsto x^\natural$. Note $\lambda\bar{\lambda} = -1$ at the finite places. For a,b,z:

$$\text{(I1) } \beta_{\varepsilon'}(\mathrm{ad}\,a,\mathrm{ad}\,b) = \beta_\varepsilon(a,b), \qquad \text{(I2) } e_N(\mathrm{ad}\,z) = e_N(z), \qquad \text{(I3) } \mathrm{ad}_*\mu_N = \mu_N.$$

(I1): with $w = \bar{b}^{\mathsf{T}}S_\varepsilon a$, the pairing $\overline{\mathrm{ad}\,b}^{\mathsf{T}}S_{\varepsilon'}\mathrm{ad}\,a$ is $\sum -\varepsilon_{ij}b_{ij}\overline{a_{ij}} = -\bar{w}$ at the infinite places and $\bar{\lambda}\lambda\sum b_{ij}\overline{a_{ij}} = -\bar{w}$ at the finite ones; and $\psi_E(\frac{J}{2}(-\bar{w})) = \psi_E(\overline{\frac{J}{2}w}) = \psi_E(\frac{J}{2}w)$. (I2): the archimedean factor of e_N depends on the $|x_{v,k}|$, which A permutes; at the finite places $\mathrm{ad}\,z\in\alpha\widehat{\mathcal{O}}_E^N$ iff $z\in\alpha\widehat{\mathcal{O}}_E^N$ ($\lambda\bar{\alpha}\alpha^{-1} = -c$ is a unit), and then, with $z = \alpha y$, the phases differ by

$\psi_{E,f}(c_0\jmath\,\bar z^{\mathsf T}z) = \psi_{E,f}(\jmath\alpha\bar\alpha\,c_0\bar y^{\mathsf T}y) = 1$, as $c_0\bar y^{\mathsf T}y$ is integral and $\psi_{E,f}(\jmath\alpha\bar\alpha\,\cdot)$ is trivial on $\widehat{\mathcal O}_E$ (self-duality, Lemma 4.1.3). (I3): A is a signed permutation of coordinates composed with conjugation (preserving Lebesgue measure), and in the coordinates z of (WD6) the finite part is $z\mapsto -c\,\bar z^{\mathsf T}$, which preserves the Haar measure. Moreover:

(I4) ad maps the rational vectors of $\mathrm{Hom}(U,W)$ onto those of $\mathrm{Hom}(W,U^\flat)$: using $\varepsilon_u x_U^* = x_U^{-1}J_U^{-1}$ and $x_W^{-*}\varepsilon_s = J_W x_W$, $\mathrm{ad}(x_W^{-1}\xi x_U) = (x_U\epsilon_\flat)^{-1}(J_U^{-1}\xi^*J_W)x_W$ for $\xi\in \mathrm{M}_{n\times b}(E)$, and $\xi\mapsto J_U^{-1}\xi^*J_W$ is a bijection onto $\mathrm{M}_{b\times n}(E)$.

(I5) For $g\in\mathrm{G}(s)$, $h\in\mathrm{G}(u)$: $\mathrm{ad}(gXh^{-1}) = (\varepsilon_u h^{-*}\varepsilon_u)\,\mathrm{ad}(X)\,(\varepsilon_s g^*\varepsilon_s) = h\,\mathrm{ad}(X)\,g^{-1}$. So with k the pair map of (h,g) for (U,W) and k' that of (g,h) for $(W,U^\flat)$, $\mathrm{ad}\circ k = k'\circ\mathrm{ad}$.

Step 2: transport. For f on $\mathbb{A}_E^{n\times b}$ put $Tf = f\circ\mathrm{ad}^{-1}$. If $f\in\mathcal M_\varepsilon$, then by (I3) Tf is μ_N-integrable with $\|Tf\| = \|f\|$, and the substitution $y = \mathrm{ad}^{-1}y'$ in (WD7) with (I1), (I2) shows $Tf\in\mathcal M_{\varepsilon'}$; the same with ad^{-1} gives the inverse $F\mapsto F\circ\mathrm{ad}$. So T is a linear isometry of $\mathcal M_{U,W}$ onto $\mathcal M_{W,U^\flat}$, $Tv^0 = v^0$ by (I2), and by (I1) $(T\rho(z)f)(x) = \beta_\varepsilon(z,\mathrm{ad}^{-1}x)f(\mathrm{ad}^{-1}x - z) = (\rho(\mathrm{ad}\,z)Tf)(x)$:

$$T\rho(z) = \rho(\mathrm{ad}\,z)\,T. \tag{6.26}$$

Step 3: the Weil representations. Let k, k' be as in (I5). By (WR3), the substitution $y = \mathrm{ad}^{-1}y'$ and (I1)–(I3), $T\omega_\varepsilon(k)f = \Delta_\varepsilon(k)\Delta_{\varepsilon'}(k')^{-1}\omega_{\varepsilon'}(k')Tf$ ($\Delta_{\varepsilon'}(k')\neq 0$ by (2) of Lemma 6.2.13). We show $\Delta_{\varepsilon'}(k') = \upsilon(\det k)\Delta_\varepsilon(k)$. Let R be the transposition $X\mapsto X^{\mathsf T}$ of index sets. At $v\in\Phi$, $\mathrm{ad}_v = R\circ E\circ\bar{\cdot}$ with $E = \mathrm{diag}(\varepsilon)$, so $k'_v = R\,E\,\overline{k_v}\,E\,R^{-1} = R\,(k_v^{-1})^{\mathsf T}R^{-1}$, using $k_v^*Ek_v = E$. The block of k'_v on the indices of sign $\varepsilon' = -1$ is thus the transpose of the block of k_v^{-1} on the indices with $\varepsilon = +1$, whose determinant is $\det(k_v)_-/\det k_v$ by Jacobi's identity for complementary minors of an inverse; as $|\det k_v| = 1$, $\overline{\det(k'_v)_-} = \det k_v\cdot\overline{\det(k_v)_-}$. At the finite places $\mathrm{ad}_f = \lambda R\circ\bar{\cdot}$, so $k'_f = R\,\overline{k_f}\,R^{-1}$; with $L = \widehat{\mathcal O}_E^N$, $L\cap\overline{k_f}^{-1}L = \overline{L\cap k_f^{-1}L}$, so $\bar u$ serves for $\overline{k_f}$ when u serves for k_f, and the finite factor of $\Delta_{\varepsilon'}(k')$ is $\upsilon(\det\bar u)|\det u|^{-1/2} = \overline{\Delta(k_f)}$, the multiplier Δ of Lemma 4.1.13 (invariant under the permutation R); since $k_f^*k_f = 1$, Lemma 4.1.13 gives $\overline{\Delta(k_f)} = \upsilon_f(\det k_f)\Delta(k_f)$, υ_f the restriction of υ to the finite ideles (its hypotheses hold by (WD1), (WD3) and Lemma 4.1.3). With $\upsilon(\det k) = \prod_v \det k_v\cdot\upsilon_f(\det k_f)$ (WD3) the claim follows, and

$$T\omega_\varepsilon(k) = \upsilon(\det k)^{-1}\,\omega_{\varepsilon'}(k')\,T,\qquad \det k = \det(g)^{\#b}\det(h)^{-\#n}\ \text{(VP5)}. \tag{6.27}$$

Step 4: the theta functionals. Put $\theta'(F) = \theta_{\mathrm{Hom}(U,W)}(F\circ\mathrm{ad})$ for $F\in\mathcal M_{W,U^\flat}$: a bounded linear functional. For γ a rational vector of $\mathrm{Hom}(U,W)$, (6.26) gives $\theta'(\rho(\mathrm{ad}\,\gamma)F) = \theta_{\mathrm{Hom}(U,W)}(\rho(\gamma)(F\circ\mathrm{ad})) = \theta'(F)$, so by (I4) θ' is invariant under the rational vectors of $\mathrm{Hom}(W,U^\flat)$; and ad induces an isomorphism of the compact quotients by the rational vectors carrying Haar probability measure to Haar probability measure, so

$$\int|\theta'(\rho(x)v^0)|^2dx = \int|\theta_{\mathrm{Hom}(U,W)}(\rho(y)v^0)|^2dy = 1$$

(Lemma 4.1.17). By Lemma 4.1.17 and Lemma 4.1.7, θ' and $\theta_{\mathrm{Hom}(W,U^\flat)}$ differ by a scalar of modulus one: there is z, $|z| = 1$, depending only on U, W and the fixed choices, with

$$\theta_{\mathrm{Hom}(W,U^\flat)}(Tf) = z\,\theta_{\mathrm{Hom}(U,W)}(f)\qquad (f\in\mathcal M_{U,W}). \tag{6.28}$$

Step 5: the theta kernels. Let α_1,β_1 be automorphic characters of U(1), $\alpha_1' = \beta_1^{-1}\upsilon^{\#n}$, $\beta_1' = \alpha_1^{-1}\upsilon^{\#b}$. By (WR4), (6.28), (6.27),

$$\theta^{U,W}_{\alpha_1,\beta_1}(f)(h,g) = \frac{\alpha_1(\det g)}{\beta_1(\det h)}\,z^{-1}\,\frac{\upsilon(\det h)^{\#n}}{\upsilon(\det g)^{\#b}}\,\theta_{\mathrm{Hom}(W,U^\flat)}\big(\omega_{\varepsilon'}(k')Tf\big) = z^{-1}\,\theta^{W,U^\flat}_{\alpha_1',\beta_1'}(Tf)(g,h).$$

Step 6: data, and (1). *Real part.* By (WR5), $\beta^{\infty}_{\varepsilon'}(Ax,Ay)=\beta^{\infty}_{\varepsilon}(x,y)$ (the exponent $\frac{1}{2}(y^*\varepsilon x - x^*\varepsilon y)$ is purely imaginary and is replaced by the conjugate of its negative), $e^{\infty}\circ A = e^{\infty}$ and $A_*\nu=\nu$; so $y\mapsto y\circ A^{-1}=y^{\mathrm{ad}}$ maps $\mathcal{F}_{U,W}$ onto $\mathcal{F}_{W,U^\flat}$, and as $\|Ax\|=\|x\|$ (maximum norm), $\mathrm{mom}_N(y^{\mathrm{ad}})=\mathrm{mom}_N(y)$: this is (1). As T is bounded linear and $v\otimes f$ is a Bochner integral (Lemma 6.2.21), (6.26), $\mathrm{ad}(\mathrm{rv}(t))=\mathrm{rv}(At)$ (κ_v real, $\lambda_\infty=1$) and the substitution $t=A^{-1}t'$ give $T(v\otimes f)=v^{\mathrm{ad}}\otimes Tf$. *Finite part.* Let A_0 be a finite set of finite places containing the conjugate of none of its members, $\Psi_0\in\mathcal{T}_{A_0}$, and $\phi(y)=-c\,y^*$ for finite adelic $n\times b$ matrices y. Then $\mathrm{ad}(\mathrm{fv}(y))=(0,\lambda\bar\alpha y^*)=\mathrm{fv}(\phi(y))$, as $\lambda\bar\alpha=-c\alpha$. For x and y supported on $A_0\cup\bar A_0$, $\overline{\phi(y)}^{\mathsf{T}}(2\phi(x)+\phi(y))$ restricted to $\bar A_0$ is $\bar c c$ times the conjugate of $\bar y^{\mathsf{T}}(2x+y)$ restricted to A_0, and $\phi(x+y)$ is integral at $\bar A_0$ iff $x+y$ is integral at A_0; with (I1)'s computation of $\psi_{E,f}$ this gives $\tau^{(\bar A_0)}_{\phi(y)}\circ\phi=\tau^{(A_0)}_y$. Write $\Psi_0=\sum_y c_y\tau_y$; as $P^{A_0}v^0=v^0$ ((2) of Lemma 6.2.33), (6.26) and $Tv^0=v^0$ give $T(\Psi_0\otimes_{A_0}v^0)=\sum_y c_y\rho(\mathrm{fv}(\phi(y)))P^{\bar A_0}v^0$, and $\sum_y c_y\tau^{(\bar A_0)}_{\phi(y)}=\Psi_0\circ\phi^{-1}$. By (1) of Lemma 4.1.3 the Schrödinger datum does not depend on the representation of a function by the τ_y (a finite representation of a nonzero function is the value of some linear section, and one of 0 is a difference of two representations of one nonzero function); so

$$T\big(v\otimes(\Psi_0\otimes_{A_0}v^0)\big)=v^{\mathrm{ad}}\otimes\big((\Psi_0\circ\phi^{-1})\otimes_{\bar A_0}v^0\big),\qquad \phi^{-1}(\eta)=c\,\eta^*. \tag{6.29}$$

Step 7: matched functions and (2). Let Φ, Ψ be as in (2); by (1) of Lemma 4.1.3, $\Psi^\sharp\in\mathcal{T}_\Sigma$. Let $c_\Sigma\in\mathcal{O}_\Sigma^\times$ be the components of c at Σ. For $X\in\mathrm{M}_{n\times b}(E_\Sigma)$, $\Psi(X)=\chi_{\mathfrak{c}}(c_\Sigma)^{\#b}\Psi(c_\Sigma X)$: if $X=k_b$ for some $k\in\mathrm{GL}_n(\mathcal{O}_\Sigma)$, let k' be k with its columns b multiplied by c_Σ; then $k'_b=c_\Sigma k_b$, $(k'^{-1})_r=(k^{-1})_r$ and $\det k'=c_\Sigma^{\#b}\det k$, and (XD6) for k and k' gives $\chi_{\mathfrak{c}}(\det k)\Psi(k_b)=\Phi((k^{-1})_r)=\chi_{\mathfrak{c}}(c_\Sigma)^{\#b}\chi_{\mathfrak{c}}(\det k)\Psi(c_\Sigma k_b)$; otherwise neither X nor $c_\Sigma X$ is of this form and both values are 0. Now $(\Psi^\sharp\circ\phi^{-1})(\eta)=\Psi(c_\Sigma((\bar\eta)_\Sigma)^{\mathsf{T}})=\chi_{\mathfrak{c}}(c_\Sigma)^{-\#b}(\Psi^{\mathsf{T}})^\flat(\eta)$, so by (6.29) with $A_0=\Sigma$

$$y^{\mathrm{ad}}\otimes[\Psi^{\mathsf{T}}]_{\bar\Sigma}=\chi_{\mathfrak{c}}(c_\Sigma)^{\#b}\,T(y\otimes[\Psi]_\Sigma).$$

Apply Step 5 with $\alpha_1=\beta_1=\gamma$: $\alpha'_1=\nu\upsilon^{\#n/2}$ is the first character of the small side of $(W,U^\flat)$ (XD9), and $\beta'_1=\nu\upsilon^{\#b-\#n/2}$, while the small side has $\upsilon^{\#b}$; by (WR4), $\theta^{W,U^\flat}_{\alpha'_1,\upsilon^{\#b}}=\gamma(\det g)^{-1}\theta^{W,U^\flat}_{\alpha'_1,\beta'_1}$ at (g,h). Hence, with $d=\chi_{\mathfrak{c}}(c_\Sigma)^{\#b}z$, $|d|=1$,

$$\gamma(\det g)\,\theta^{W,U^\flat}_{\alpha'_1,\upsilon^{\#b}}(y^{\mathrm{ad}}\otimes[\Psi^{\mathsf{T}}]_{\bar\Sigma})(g,h)=d\,\theta^{U,W}_{\gamma,\gamma}(y\otimes[\Psi]_\Sigma)(h,g)\qquad(h\in\mathrm{G}(u)=\mathrm{G}(-u)).$$

Integrating over $h\in[\mathrm{U}(U^\flat)]=[\mathrm{U}(U)]$ (Step 0), the integrands being proportional, gives (2); d depends only on $\mathfrak{c}$, U, W and the fixed choices. □

Lemma 6.3.5. *Let X, W_1, V' be spaces with points with index sets m, n, a, let b be an index set, D a base-adapted splitting of X over W_1 along b and $\bar D$ its opposite splitting, and write $\mathbb{C}^{\Phi\times(a\times b)}$ for the space of the arguments of the real slices along V'.*

1. *For every $\phi\in\mathcal{F}_{X,V'}$ and every z, the function $x\mapsto\mathrm{rsl}_{D,V'}(x;\phi)(z)$ is $\nu_{\Phi\times(a\times b)}$-integrable and*
$$\int_{\mathbb{C}^{\Phi\times(a\times b)}}\mathrm{rsl}_{D,V'}(x;\phi)(z)\,d\nu_{\Phi\times(a\times b)}(x)=\mathrm{rsl}_{\bar D,V'}(0;\phi)(z).$$
2. *Let $\tau\ge0$ be a smooth compactly supported real function on $\mathbb{C}^{\Phi\times(a\times b)}$, not identically 0, and $v\in\mathcal{F}_{W_1,V'}$ a Schwartz vector. There are $\kappa\in\mathbb{C}$ and a Schwartz vector $\phi\in\mathcal{F}_{X,V'}$ with*
$$\mathrm{rsl}_{D,V'}(x;\phi)=\tau(x)\,\kappa\,v\quad\text{for every } x,\qquad \mathrm{rsl}_{\bar D,V'}(0;\phi)=v.$$

3. Every Schwartz vector $v \in \mathcal{F}_{W_1,V'}$ is $\mathrm{rsl}_{D,V'}(0;\phi)$ for some Schwartz vector $\phi \in \mathcal{F}_{X,V'}$.

Proof. Throughout, an element c of a Fock model $\mathcal{F}_\varepsilon$ satisfies $|c(z)| \le \|c\|_1$ for every z, by its reproducing identity and $|\beta_\varepsilon^\infty e^\infty| \le 1$; so evaluation at z is a bounded linear functional and commutes with Bochner integrals.

Step 1: a general identity. Let V, V_0 be spaces with points with signs s on n_1 and s_0 on n_0, p an index set, $D' = (k,l,i)$ a splitting of V over V_0 along p and $m \in \mathrm{GL}_p(\mathbb{A}_E)$ such that D' is adapted at 1 in the basis m. Let $\bar D' = (l,k,i)$ and $\bar m = (m^{-1})^*$. We show, for $u \in \mathcal{F}_s$ and every z,

$$\int \mathrm{rsl}_{D',m}(x;u)(z)\, d\nu(x) = Z^\infty_{\bar D',\bar m}(u)(z), \tag{I}$$

the integrand being integrable. Let G be the complex matrix, block diagonal over Φ, with column blocks $(l_{\mathbb{A}}\bar m)_\infty$, $(k_{\mathbb{A}} m)_\infty$, $(i_{\mathbb{A}})_\infty$; by adaptedness it is unitary, and we write $y = G(a,b,w)$ with $a,b \in \mathbb{C}^{\Phi\times p}$, $w \in \mathbb{C}^{\Phi\times n_0}$; $\nu_{\Phi\times n_1}$ is the image of the product of the measures ν in a,b,w. The relations of (MX1) and $x_V^* J_V x_V = \varepsilon_s$ give $\mathrm{pr}_k l_{\mathbb{A}} = 0$, $\mathrm{pr}_k k_{\mathbb{A}} = 1$, $\mathrm{pr}_k i_{\mathbb{A}} = 0$, $\mathrm{pr}_l l_{\mathbb{A}} = 1$, $\mathrm{pr}_l k_{\mathbb{A}} = 0$, $\mathrm{pr}_l i_{\mathbb{A}} = 0$ (for instance $\mathrm{pr}_k k_{\mathbb{A}} = l^* x_V^{-*} \varepsilon_s x_V^{-1} k = l^* J_V k = 1$), and pr_i is 0 on the first two blocks and 1 on the third. Hence, as rv is linear,

$$\mathrm{pr}_k \mathrm{rv}(y) = m\,\mathrm{rv}(b), \quad \mathrm{pr}_l \mathrm{rv}(y) = \bar m\,\mathrm{rv}(a), \quad a(y) = (m^*\mathrm{pr}_l)_\infty y = a, \quad w(y) = w$$

for D' in the basis m, and, since for $\bar D'$ the roles of pr_k and pr_l are exchanged, $\bar a(y) = (\bar m^* \mathrm{pr}_k)_\infty y = b$ and $\bar w(y) = w$ for $\bar D'$ in the basis $\bar m$. The characters are $\chi(y) = \psi_E(\frac{\jmath}{2}\mathrm{rv}(b)^*\mathrm{rv}(a))$ and $\bar\chi(y) = \psi_E(\frac{\jmath}{2}\mathrm{rv}(a)^*\mathrm{rv}(b))$, as $m^*\bar m = 1$. The argument $\frac{\jmath}{2}\mathrm{rv}(b)^*\mathrm{rv}(a)$ has finite part 0 and component $\frac{\jmath_v}{2}\kappa_v^2 b_v^* a_v$ at $v \in \Phi$; with $\jmath_v \kappa_v^2 = i/(2\pi)$ ($\jmath_v$ purely imaginary, $2\pi\,\mathrm{Im}(\jmath_v)\kappa_v^2 = 1$), (WD4) gives

$$\chi(y) = \prod_{v\in\Phi} \exp(-2\pi i\,\mathrm{Re}(\tfrac{i}{2\pi} b_v^* a_v)) = \exp(i\,\mathrm{Im}\, b^*a) = \exp(\tfrac{b^*a - a^*b}{2}), \qquad \bar\chi(y) = \exp(\tfrac{a^*b - b^*a}{2}).$$

By (C1) below, $\beta_\varepsilon^\infty(G(x,0,0), G(a,b,w)) = \exp((b^*x - x^*b)/2)$, where

$$y'^*\mathrm{diag}(\varepsilon)\, y = a'^* b + b'^* a + w'^* \mathrm{diag}(\varepsilon_0)\, w \qquad (y = G(a,b,w),\ y' = G(a',b',w')), \tag{C1}$$

$\varepsilon(v,j) = s(j)$, $\varepsilon_0(v,j) = s_0(j)$; (C1) holds because $(x_V^{-1}u)^*\varepsilon_s(x_V^{-1}u') = u^* J_V u'$ and the relations of (MX1) give the Gram matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & J_{V_0} \end{pmatrix}$ for the columns $l\bar m$, km, i of $x_V\, G$ read rationally, with $x_{V_0}^* J_{V_0} x_{V_0} = \varepsilon_{s_0}$ on the last block. Now $(l_{\mathbb{A}}\bar m)_\infty x = G(x,0,0)$, so by definition $\mathrm{rsl}_{D',m}(x;u) = Z^\infty_{D',m}(\rho^\infty(G(x,0,0))u)$. Evaluating at z, writing $\tilde u(a,b,w) = u(G(a,b,w))$ and substituting $a \mapsto a + x$ (with $\beta_\varepsilon^\infty(t,t) = 1$ and the real linearity of the exponent of β_ε^∞),

$$\mathrm{rsl}_{D',m}(x;u)(z) = \int \tilde u(a,b,w) \exp\Big(\frac{b^*a - a^*b}{2} + b^*x - x^*b - |a+x|^2\Big)\big(\rho_0^\infty(w)v_1\big)(z)\, d\nu(a,b,w),$$

ρ_0^∞, v_1 the real Heisenberg operators and the real vacuum of $\mathcal{F}_{s_0}$. The modulus of the integrand is at most $|\tilde u(a,b,w)| e^{-|a+x|^2}$, as $|(\rho_0^\infty(w)v_1)(z)| = |v_1(z-w)| \le 1$; its integral over (a,b,w,x) is $\|u\|_1 \int e^{-|t|^2} d\nu(t) = \|u\|_1$. So Fubini applies. For fixed a,b, substituting $x = t - a$,

$$\int \exp(b^*x - x^*b - |a+x|^2) d\nu(x) = e^{a^*b - b^*a} \int e^{2i\,\mathrm{Im}(b^*t) - |t|^2} d\nu(t) = e^{a^*b - b^*a} e^{-|b|^2},$$

by the Fourier transform of the Gaussian on each coordinate: $\pi^{-1}\int_{\mathbb{C}} e^{-|t_1|^2} e^{2i\,\mathrm{Im}(\bar{b}_1 t_1)} dt_1 = e^{-|b_1|^2}$ ($2\,\mathrm{Im}(\bar{b}_1 t_1)$ is a real linear form in t_1 with gradient of length $2|b_1|$). Hence

$$\int \mathrm{rsl}_{D',m}(x;u)(z)\,d\nu(x) = \int \tilde{u}(a,b,w)\, e^{(a^*b-b^*a)/2}\, e^{-|b|^2} (\rho_0^\infty(w)v_1)(z)\,d\nu(a,b,w).$$

The right side is $Z^\infty_{\bar{D}',\bar{m}}(u)(z)$: the matrix of (MX5) for $\bar{D}'$ in the basis $\bar{m}$ has the same columns as for D' in m (blocks exchanged), and in its definition $\bar{\chi}(y) = e^{(a^*b-b^*a)/2}$, $e^{-\bar{a}(y)^*\bar{a}(y)} = e^{-|b|^2}$ and $\bar{w}(y) = w$. This proves (I).

Step 2: two invariances of the real zero term. Let $D' = (k,l,i)$ be a splitting and m a basis. (i) If $c \in \mathrm{GL}_p(E)$ and $D'' = (kc, l(c^{-1})^*, i)$ is a splitting, then $Z^\infty_{D'',m} = Z^\infty_{D',cm}$: the coordinates of D'' are $\mathrm{pr}''_l = c^*\mathrm{pr}_l$, $\mathrm{pr}''_k = c^{-1}\mathrm{pr}_k$, $\mathrm{pr}''_i = \mathrm{pr}_i$, so $a''(y) = ((cm)^*\mathrm{pr}_l)_\infty y$, $w'' = w$, and $(\mathrm{pr}''_k\mathrm{rv}(y))^*\mathrm{pr}''_l\mathrm{rv}(y) = (\mathrm{pr}_k\mathrm{rv}(y))^*(c^{-1})^*c^*\mathrm{pr}_l\mathrm{rv}(y) = (\mathrm{pr}_k\mathrm{rv}(y))^*\mathrm{pr}_l\mathrm{rv}(y)$. (ii) If $\delta \in \mathrm{GL}_p(\mathbb{A}_E)$ has unitary infinite components, then $Z^\infty_{D',m\delta} = Z^\infty_{D',m}$: only $a(y)$ changes, to $\delta^*_\infty a(y)$, and $|\delta^*_\infty a(y)| = |a(y)|$.

Step 3: (1). Let $V^X = \mathrm{Hom}(X,V')$, $D' = D_{V'}$ the induced splitting of V^X over $\mathrm{Hom}(W_1,V')$ along $a\times b$ with frames $(K,L,I) = (J_{V'}^{-1}\otimes(k^*J_X)^{\mathsf{T}},\ 1\otimes(l^*J_X)^{\mathsf{T}},\ I)$ by (MX1), $D = (k,l,i)$, and $m = m_{V'} = (x_{V'}^{-1})^*\otimes 1_b$. By Lemma 6.3.53, D' is adapted at 1 in the basis m, so (I) holds for D' with $\bar{D}' = (L,K,I)$ and $\bar{m} = (m^{-1})^* = x_{V'}\otimes 1_b$. The splitting $\bar{D}_{V'}$ induced by $\bar{D} = (l,k,i)$ has frames $J_{V'}^{-1}\otimes(l^*J_X)^{\mathsf{T}} = L\,c$ and $1\otimes(k^*J_X)^{\mathsf{T}} = K\,(c^{-1})^*$ with $c = J_{V'}^{-1}\otimes 1_b \in \mathrm{GL}_{a\times b}(E)$ (as $J_{V'}$ is hermitian) and the same third frame I. By Step 2 (i), $Z^\infty_{\bar{D}_{V'},m} = Z^\infty_{\bar{D}',cm}$. From $x^*_{V'}J_{V'}x_{V'} = \varepsilon_{t'}$ (t' the signs of V') we get $J_{V'}^{-1}(x^*_{V'})^{-1} = x_{V'}\varepsilon_{t'}$, so $cm = \bar{m}\,\delta$ with $\delta = \varepsilon_{t'}\otimes 1_b$, whose infinite components are diagonal with entries ±1; by Step 2 (ii), $Z^\infty_{\bar{D}',cm} = Z^\infty_{\bar{D}',\bar{m}}$. Since $\rho^\infty(0)$ is the identity, $\mathrm{rsl}_{\bar{D},V'}(0;\phi) = Z^\infty_{\bar{D}_{V'},m}(\phi) = Z^\infty_{\bar{D}',\bar{m}}(\phi)$, and (I) for $u = \phi$ is (1).

Step 4: (2). Since τ is continuous, non-negative and not identically 0, $T = \int \tau\,d\nu > 0$; put $\kappa = T^{-1}$. By Lemma 6.3.53, (2) of Lemma 6.2.43 applies to the splitting $D_{V'}$, the basis $m_{V'}$, the smooth compactly supported function $\kappa\tau$, the constant map $x\mapsto 1$ and the Schwartz vector v of $\mathcal{F}_{W_1,V'}$, the Fock model of $\mathrm{Hom}(W_1,V')$ that (MX7) uses. It gives a Schwartz vector ϕ with $\mathrm{rsl}_{D,V'}(x;\phi) = \kappa\tau(x)\,\omega^\infty(1)v = \tau(x)\kappa v$, as $\omega^\infty(1)$ is the identity by the reproducing identity. By (1), $\mathrm{rsl}_{\bar{D},V'}(0;\phi)(z) = \kappa\int\tau\,d\nu\cdot v(z) = v(z)$ for every z.

Step 5: (3). The opposite splitting $\bar{D}$ is base-adapted, its matrix of adelic frames being that of D with two column blocks exchanged, and the opposite splitting of $\bar{D}$ is D. Apply (2) to $\bar{D}$, a smooth compactly supported $\tau\ge 0$ with $\tau(0) > 0$ and v: it gives a Schwartz ϕ with $\mathrm{rsl}_{D,V'}(0;\phi) = v$. □

Proposition 6.3.6. *Let X, W_1, W, V' be spaces with signs r on m, s on n, s_W on n_W, t' on b, with $\#n$ even and b nonempty; let D be a base-adapted splitting of X over W_1 along b and $\bar{D}$ its opposite splitting, D_1 a base-adapted splitting of W_1 over W along a one-element set, $\sigma\colon b\sqcup(b\sqcup n)\to m$ a bijection and $\mu\colon\Phi\to\mathbb{Z}$. Let $v\in\mathcal{F}_{W_1,V'}$ and $v_1\in\mathcal{F}_{W_1,W_1}$ be Schwartz vectors. Then there are $\phi_{\mathrm{lg}}\in\mathcal{F}_{X,V'\oplus W_1}$ and $\phi_{\mathrm{sm}}\in\mathcal{F}_{X,V'}$ with*

$$\mathrm{rsl}_{\bar{D},V'}(0;\phi_{\mathrm{sm}}) = v$$

such that $(\sigma,\phi_{\mathrm{lg}},\phi_{\mathrm{sm}},\ell)$ satisfy the row-column exchange identity for $(X,V'\oplus W_1,V')$ and μ, where

$$\ell(\mathfrak{c}) = \frac{\left|\theta^{\mathrm{lg}}_{\mathfrak{c}}(W_1,W_1;v_1,\Psi_1^{\mathrm{st}})(1)\right|^2}{L_*(\Sigma,\nu;\#n/2)} \qquad (\mathfrak{c} = (\nu,\Sigma)),$$

1 being the identity $n\times n$ matrix in Ψ_1^{st} and the identity of $\mathrm{U}(W_1)(\mathbb{A}_F)$ in the argument.

Proof. Put $j = \#n/2$. If $\mu(w) \le 0$ for some $w \in \Phi$, no split tame character has weight μ ((XD4)); then (3) of Lemma 6.3.5, applied to the base-adapted splitting $\bar{D}$, gives a Schwartz vector ϕ_{sm} with $\mathrm{rsl}_{\bar{D},V'}(0;\phi_{\mathrm{sm}}) = v$, and $\phi_{\mathrm{lg}} = 0$, ϕ_{sm} satisfy the assertion with $C = 0$, the condition on characters being empty. From now on $\mu(w) > 0$ for all w.

Step 1: smooth Witt data. Let $J = \mathrm{diag}(t' \sqcup s)$, a complex $(b \sqcup n) \times (b \sqcup n)$ matrix with $J^* = J = J^{-1}$, $\mathrm{U}(J)$ its unitary group, and $x_0 = (1_b; 0)$, so $x_0^* J x_0 = \mathrm{diag}(t')$ is invertible. By Lemma 6.3.7 there are a smooth map $k\colon \mathrm{M}_{(b\sqcup n)\times b}(\mathbb{C}) \to \mathrm{U}(J)$ and an open neighbourhood $\mathcal{N}_0$ of x_0 such that every $y \in \mathcal{N}_0$ is $k(y)x_0c$ for some $c \in \mathrm{M}_b(\mathbb{C})$. The map $A \mapsto \mathrm{adj}(A)/\det A$ is smooth on the open set of invertible $A \in \mathrm{M}_b(\mathbb{C})$; multiplying it by a smooth function with compact support in that set and equal to 1 on an open neighbourhood $\mathcal{N}_1$ of $\mathrm{diag}(t')$ gives a smooth map $\mathrm{iv}\colon \mathrm{M}_b(\mathbb{C}) \to \mathrm{M}_b(\mathbb{C})$ with $\mathrm{iv}(A) = A^{-1}$ for $A \in \mathcal{N}_1$. Let $\mathcal{N} = \{y \in \mathcal{N}_0 : y^*Jy \in \mathcal{N}_1\}$, an open neighbourhood of x_0. Choose a smooth $\mathrm{bm}\colon \mathrm{M}_{(b\sqcup n)\times b}(\mathbb{C})^\Phi \to [0,1]$ with compact support contained in $\mathcal{N}^\Phi$ and $\mathrm{bm}(x_0,\dots,x_0) = 1$. For $x \in \mathrm{M}_{(b\sqcup n)\times b}(\mathbb{C})$ put

$$\mathrm{pr}(x) = x\,\mathrm{iv}(x^*Jx)\,x^*Jx_0,$$

a smooth map with $\mathrm{pr}(x_0) = x_0$. Three facts are used.

(W1) If $\mathrm{bm}(y) \ne 0$, then for every w: $y_w = k(y_w)x_0c$ for some c, $y_w^*Jy_w$ is invertible and $\mathrm{iv}(y_w^*Jy_w) = (y_w^*Jy_w)^{-1}$. (Indeed $y_w \in \mathcal{N}$.)

(W2) If $C \in \mathrm{GL}_b(\mathbb{C})$ and iv inverts the Gram matrices $G = x^*Jx$ and C^*GC, then $\mathrm{pr}(xC) = xC\,C^{-1}G^{-1}C^{-*}\,C^*x^*Jx_0 = \mathrm{pr}(x)$.

(W3) If $\mathrm{bm}(\mathrm{pr}\circ x) \ne 0$, then for every w there is $c \in \mathrm{M}_b(\mathbb{C})$ with $k(\mathrm{pr}(x_w))\,x_0\,c = x_w$ and $c^*\mathrm{diag}(t')c = x_w^*Jx_w$. Indeed $y = \mathrm{pr}(x_w) = x_wA$ with $A = \mathrm{iv}(x_w^*Jx_w)x_w^*Jx_0$; by (W1), $y = k(y)x_0c_0$ and $\det(A^*x_w^*Jx_wA) = \det(y^*Jy) \ne 0$, so A is invertible; $c = c_0A^{-1}$ gives $k(y)x_0c = x_w$, and $c^*\mathrm{diag}(t')c = (k(y)x_0c)^*J(k(y)x_0c) = x_w^*Jx_w$ as $k(y) \in \mathrm{U}(J)$.

Step 2: the small datum. The arguments of the real slices along V' are families $B = (B_w)_w$ of complex $b\times b$ matrices. Let $\tau_{\mathrm{sm}}(B) = \mathrm{bm}((x_0B_w)_w)\prod_w |\det B_w|^{2(j+\mu(w))}$. It is smooth and non-negative, $\tau_{\mathrm{sm}}(1,\dots,1) = 1$, and it has compact support: $\tau_{\mathrm{sm}}(B) \ne 0$ implies $(x_0B_w)_w \in \mathrm{supp}\,\mathrm{bm}$, and $B_w = x_0^*(x_0B_w)$, so the support lies in the image of a compact set under a continuous map. By (2) of Lemma 6.3.5 there are $\kappa \in \mathbb{C}$ and a Schwartz vector $\phi_{\mathrm{sm}} \in \mathcal{F}_{X,V'}$ with

$$\mathrm{rsl}_{D,V'}(B;\phi_{\mathrm{sm}}) = \tau_{\mathrm{sm}}(B)\,\kappa\,v \quad \text{for all } B, \qquad \mathrm{rsl}_{\bar{D},V'}(0;\phi_{\mathrm{sm}}) = v. \tag{1}$$

Step 3: the large datum. Let e be the bijection of $(b\times n)\sqcup(n\times n)$ onto $(b\sqcup n)\times n$ that is the identity on each part, and $v \boxtimes_e v_1$ the product of Lemma 6.3.51, $z \mapsto v(z_1)v_1(z_2)$; the signs of $\mathrm{Hom}(W_1, V'\oplus W_1)$ restrict to those of $\mathrm{Hom}(W_1,V')$ and $\mathrm{Hom}(W_1,W_1)$ on the two parts, so by (1) of Lemma 6.3.51 $v\boxtimes_e v_1$ is a Schwartz vector of $\mathcal{F}_{W_1,V'\oplus W_1}$. The arguments of the real slices along $V'\oplus W_1$ are families $x = (x_w)_w$ of complex $(b\sqcup n)\times b$ matrices. Put $x'_w = k(\mathrm{pr}(x_w))^{-1}x_w$, let $(x'_w)_1$ be its block of rows b, and

$$\tau_{\mathrm{lg}}(x) = \kappa\,\mathrm{bm}(\mathrm{pr}\circ x)\,\mathrm{bm}(x')\prod_w \overline{\det(x'_w)_1}^{\,2\mu(w)}\prod_w \det k(\mathrm{pr}(x_w))^{j-\mu(w)},$$
$$g(x) = \bigoplus_w k(\mathrm{pr}(x_w))\otimes 1_n.$$

Since $k(y)^{-1} = Jk(y)^*J$ and $|\det k(y)| = 1$ for $k(y) \in \mathrm{U}(J)$ (so negative powers of $\det k(y)$ are powers of its conjugate), τ_{lg} and g are smooth. If $\tau_{\mathrm{lg}}(x) \ne 0$ then $\mathrm{pr}\circ x$ and x' lie in $\mathrm{supp}\,\mathrm{bm}$ and $x_w = k(\mathrm{pr}(x_w))x'_w$, so the support of τ_{lg} lies in the image of $(\mathrm{supp}\,\mathrm{bm})^2$ under the continuous map

$(y,x')\mapsto(k(y_w)x'_w)_w$ and is relatively compact. The matrix $g(x)$ lies in $\mathrm{U}(\mathrm{diag}\,\varepsilon_0)$ for the signs ε_0 of $\mathrm{Hom}(W_1,V'\oplus W_1)$, which are $J\otimes\mathrm{diag}(s)$ at each w: $(k\otimes1)^*(J\otimes\mathrm{diag}\,s)(k\otimes1)=k^*Jk\otimes\mathrm{diag}\,s$. By (WR5), $\Omega_{j-\mu,0}(1,k\circ\mathrm{pr}\circ x)=\prod_w\det k(\mathrm{pr}(x_w))^{j-\mu(w)}\,\omega_{\varepsilon_0}^\infty(g(x))$. By Lemma 6.3.53, (2) of Lemma 6.2.43 applies to the induced splitting of $\mathrm{Hom}(X,V'\oplus W_1)$ and its adapted basis, and gives a Schwartz vector $\phi_{\mathrm{lg}}\in\mathcal{F}_{X,V'\oplus W_1}$ with

$$\mathrm{rsl}_{D,V'\oplus W_1}(x;\phi_{\mathrm{lg}})=\kappa\,\mathrm{bm}(\mathrm{pr}\circ x)\,\mathrm{bm}(x')\prod_w\overline{\det(x'_w)_1}^{2\mu(w)}\,\Omega_{j-\mu,0}(1,k\circ\mathrm{pr}\circ x)(v\boxtimes_e v_1)=:S(x). \tag{2}$$

Step 4: a value of S. Let $x_1=(x_{1,w})_w$ be a family with $x_{1,w}^*Jx_{1,w}=\mathrm{diag}(t')$, $y^1=\mathrm{pr}\circ x_1$, and $C=(C_w)_w$ invertible $b\times b$ matrices. (a) If $\mathrm{bm}(y^1)=0$ then $S(x_1C)=0$. (b) If $u_w\in\mathrm{U}(\mathrm{diag}\,t')$ satisfy $k(y^1_w)x_0u_w=x_{1,w}$ for all w, then with $B_w=u_wC_w$

$$S(x_1C)=\kappa\,\mathrm{bm}(y^1)\,\mathrm{bm}\big((x_0B_w)_w\big)\prod_w\overline{\det B_w}^{2\mu(w)}\,\Omega_{j-\mu,0}(1,k\circ y^1)(v\boxtimes_e v_1).$$

Proof. Let $a=x_1C$. If $\mathrm{bm}((k(y'_w)^{-1}a_w)_w)=0$ for every family y', then $S(a)=0$, its factor $\mathrm{bm}(a')$ being of this form; in (b) the right side vanishes as well, since $k(y^1_w)^{-1}a_w=x_0u_wC_w=x_0B_w$. Otherwise some y' has $\mathrm{bm}((k(y'_w)^{-1}a_w)_w)\neq0$; by (W1) iv inverts the Gram matrix of $k(y'_w)^{-1}a_w$, which is $a_w^*Ja_w=C_w^*\mathrm{diag}(t')C_w$, and it inverts $\mathrm{diag}(t')\in\mathcal{N}_1$; by (W2), $\mathrm{pr}(a_w)=\mathrm{pr}(x_{1,w})=y^1_w$. Then $S(a)$ has the factor $\mathrm{bm}(y^1)$, which gives (a), and in (b) $a'_w=k(y^1_w)^{-1}x_{1,w}C_w=x_0B_w$, $(a'_w)_1=B_w$, which gives the formula.

Step 5: the hypotheses of Proposition 6.3.22. Let $\mathrm{pw}(h)=\mathrm{bm}(\mathrm{pr}\circ(h_w^{-1}x_0)_w)$ for $h\in G_\infty(t'\sqcup s)$, a non-negative continuous function with $\mathrm{pw}(1)=\mathrm{bm}(x_0,\dots,x_0)=1$, hence non-zero near 1; and $I(\mathfrak{c})=\theta^{\mathrm{lg}}_{\mathfrak{c}}(W_1,W_1;v_1,\Psi_1^{\mathrm{st}})(1)$, so that $\ell(\mathfrak{c})=|I(\mathfrak{c})|^2/L_*(\Sigma,\nu;\#n/2)$. The vectors ϕ_{lg}, ϕ_{sm} are Schwartz vectors.

(S), first alternative: if $S(x)\neq0$ then $\mathrm{bm}(\mathrm{pr}\circ x)\neq0$, and (W3) gives c with $x_w^*Jx_w=c^*\mathrm{diag}(t')c$ for every w.

(P): Let $\mathfrak{c}=(\nu,\Sigma)$ have weight μ, $\gamma=\gamma_{\mathfrak{c}}=\nu^{-1}\upsilon^j$, and $h\in G_\infty(t'\sqcup s)$. Let $x_{1,w}=h_w^{-1}x_0$; then $x_{1,w}^*Jx_{1,w}=\mathrm{diag}(t')$, and $y^1=\mathrm{pr}\circ x_1$ has $\mathrm{bm}(y^1)=\mathrm{pw}(h)$.

If $\mathrm{pw}(h)=0$, take $(p_1,p_2)=(1,1)$, so $\hat h=h$. For $B\in\mathrm{GL}_b(\mathbb{A}_E)$ the slice argument is $\hat h^{-1}(B_\infty;0)=(x_{1,w}B_w)_w$, and by Step 4 (a) its slice $S(x_1B_\infty)$ vanishes; both sides of (P) are 0.

If $\mathrm{pw}(h)\neq0$, (W3) applied to x_1 gives u_w with $k(y^1_w)x_0u_w=x_{1,w}$ and $u_w^*\mathrm{diag}(t')u_w=\mathrm{diag}(t')$, i.e. $u_w\in\mathrm{U}(\mathrm{diag}\,t')$. The element $g_w=h_wk(y^1_w)\mathrm{diag}(u_w,1)$ of $\mathrm{U}(J)$ fixes the columns of x_0; writing it in blocks, $g_wx_0=x_0$ gives the first block column $(1;0)$, and the block (b,n) of $g_w^*Jg_w=J$ gives $\mathrm{diag}(t')\,(g_w)_{bn}=0$, so $g_w=\mathrm{diag}(1,q_w)$ with $q_w\in\mathrm{U}(\mathrm{diag}\,s)$. Hence $\mathrm{diag}(u_w,q_w^{-1})\,h_w\,k(y^1_w)=1$. Let $p_1\in\mathrm{G}(t')$, $p_2\in\mathrm{G}(s)$ have infinite components $(u_w)_w$, $(q_w^{-1})_w$ and finite components 1; then $\hat h_wk(y^1_w)=1$ for every w. Fix B,β,f,x,g,y as in (P). The slice argument is $a_w=\hat h_w^{-1}(B_w;0)=h_w^{-1}(u_w^{-1}B_w;0)=x_{1,w}u_w^{-1}B_w$, so Step 4 (b) with $C_w=u_w^{-1}B_w$ and the representation property of $\Omega_{j-\mu,0}$ (Proposition 4.1.14) give

$$w_{\hat h}(B)=\Omega_{j-\mu,0}(1,\hat h)\,S(a)=c_B\,(v\boxtimes_e v_1),\qquad c_B=\kappa\,\mathrm{pw}(h)\,\mathrm{bm}\big((x_0B_w)_w\big)\prod_w\overline{\det B_w}^{2\mu(w)}.$$

By (3) of Lemma 6.3.51, $(v\boxtimes_e v_1)\otimes(f\boxtimes^{\mathrm{tg}}F_{\mathfrak{c}})=(v\otimes f)\boxtimes^{\mathrm{tg}}(v_1\otimes F_{\mathfrak{c}})$; by (b) of Lemma 6.3.52 with the characters γ, β, γ, $\omega_{\gamma,\gamma\beta}(g,\mathrm{diag}(y,u))$ carries it to $\omega_{\gamma,\beta}(g,y)(v\otimes f)\boxtimes^{\mathrm{tg}}\omega_{\gamma,\gamma}(g,u)(v_1\otimes F_{\mathfrak{c}})$; by (2) of Lemma 6.3.51, $\rho((x;0))$ acts by $\rho(x)$ on the first factor and by $\rho(0)=1$ on the second; and by (c) of Lemma 6.3.52 the theta functional of the product is the product of the theta functionals. So the integrand of (P) is

$$c_B\,\theta_{\mathrm{Hom}(W_1,V')}\big(\rho(x)\omega_{\gamma,\beta}(g,y)(v\otimes f)\big)\cdot\theta^{W_1,W_1}_{\gamma,\gamma}\big(v_1\otimes[\Psi_1^{\mathrm{st}}]_\Sigma\big)(g,u).$$

The function Ψ_1^{st} is locally constant with compact support $\mathrm{GL}_n(\mathcal{O}_\Sigma)$ and vanishes off the invertible matrices, so $v_1 \otimes [\Psi_1^{\mathrm{st}}]_\Sigma$ is target-bounded (Lemma 6.2.26) and its theta kernel is integrable over $[\mathrm{U}(W_1)]$ (Lemma 6.3.21). As $\gamma = \nu^{-1}\upsilon^{\lfloor \#n/2 \rfloor}$, its integral is $\theta_{\mathfrak{c}}^{\mathrm{lg}}(W_1, W_1; v_1, \Psi_1^{\mathrm{st}})(g)$, which is $I(\mathfrak{c})$ by Proposition 6.3.17 ($\#n$ even, D_1 base-adapted). Hence the left side of (P) is $c_B\, I(\mathfrak{c})\, \theta_{\mathrm{Hom}(W_1,V')}(\rho(x)\omega_{\gamma,\beta}(g,y)(v\otimes f))$. On the right side, by (1), $\mathrm{rsl}_{D,V'}(B_\infty;\phi_{\mathrm{sm}}) = \mathrm{bm}((x_0B_w)_w)\prod_w |\det B_w|^{2(j+\mu(w))}\kappa v$. With $\zeta_w = \det B_w$, the idele $z_B/\bar{z}_B$ lies in $\mathrm{U}(1)(\mathbb{A})$ with finite components 1 and components $\zeta_w/\bar{\zeta}_w$, so by the weight of ν (Definition 3.1.1) $\tilde{\nu}(z_B) = \prod_w (\zeta_w/\bar{\zeta}_w)^{\mu(w)}$, and

$$\frac{\prod_w |\zeta_w|^{2(j+\mu(w))}}{\tilde{\nu}(z_B)\prod_w |\zeta_w|^{\#n}} = \prod_w (\zeta_w\bar{\zeta}_w)^{\mu(w)}\Big(\frac{\bar{\zeta}_w}{\zeta_w}\Big)^{\mu(w)} = \prod_w \bar{\zeta}_w^{2\mu(w)}.$$

So the right side of (P) is $\kappa\,\mathrm{pw}(h)\, I(\mathfrak{c})\,\mathrm{bm}((x_0B_w)_w)\prod_w \bar{\zeta}_w^{2\mu(w)}\theta_{\mathrm{Hom}(W_1,V')}(\rho(x)\omega_{\gamma,\beta}(g,y)(v\otimes f))$, which is the left side.

Step 6: conclusion. By Proposition 6.3.22, $(\sigma, \phi_{\mathrm{lg}}, \phi_{\mathrm{sm}}, \ell)$ satisfy the row-column exchange identity for $(X, V'\oplus W_1, V')$ and μ, and (1) gives $\mathrm{rsl}_{\bar{D},V'}(0;\phi_{\mathrm{sm}}) = v$. □

Lemma 6.3.7. *Let $m, b \geq 0$, let J be a hermitian $m\times m$ complex matrix, $\mathrm{U}(J) = \{g \in \mathrm{GL}_m(\mathbb{C}) : g^*Jg = J\}$, and let x be an $m \times b$ complex matrix with $\det(x^*Jx) \neq 0$. Then there is a map $k\colon \mathrm{M}_{m\times b}(\mathbb{C}) \to \mathrm{U}(J)$, smooth as a map of real vector spaces into $\mathrm{M}_m(\mathbb{C})$, and a neighbourhood N of x such that for every $y \in N$ there is $c \in \mathrm{M}_b(\mathbb{C})$ with*

$$k(y)\, x\, c = y.$$

Proof. Put $G = x^*Jx$, an invertible hermitian $b\times b$ matrix, $w = G^{-1}x^*$ and $P = xw = xG^{-1}x^*$; then $P^* = P$ and $wJx = 1_b$.

Step 1: a linear map into the Lie algebra of $\mathrm{U}(J)$ that moves x within its span. For $a \in \mathrm{M}_m(\mathbb{C})$ put $L(a) = (aP - Pa^*)J$. The map L is $\mathbb{R}$-linear, $L(1) = 0$, and $L(a)^*J = J(Pa^* - aP)J = -JL(a)$. By induction $(L(a)^*)^jJ = J(-L(a))^j$ for $j \geq 0$, so $\exp(L(a))^*J = J\exp(-L(a))$ and $\exp(L(a))^*J\exp(L(a)) = J$: thus $\exp(L(a)) \in \mathrm{U}(J)$. Moreover $L(a)x = ax - xwa^*Jx$ because $wJx = 1$, so

$$(a - L(a))x = x\,(wa^*Jx). \tag{a}$$

Step 2: a smooth local inverse. Let $T(a) = \exp(L(a))\,(a - L(a))$, a smooth map of the real vector space $\mathrm{M}_m(\mathbb{C})$ to itself with $T(1) = 1$. For $h \in \mathrm{M}_m(\mathbb{C})$, since L is linear, $L(1) = 0$ and the derivative of exp at 0 is the identity,

$$\frac{d}{dt}\Big|_{t=0} T(1+th) = L(h)(1 - L(1)) + \exp(L(1))(h - L(h)) = h,$$

so the derivative of T at 1 is the identity. By the inverse function theorem there are an open neighbourhood U_1 of 1 and a smooth map $A_0\colon U_1 \to \mathrm{M}_m(\mathbb{C})$ with $T(A_0(z)) = z$ for $z \in U_1$. Choose a smooth function $\phi\colon \mathrm{M}_m(\mathbb{C}) \to [0,1]$ with compact support in U_1 and $\phi = 1$ on an open neighbourhood $U_2 \subseteq U_1$ of 1, and let $A(z) = \phi(z)A_0(z)$ for $z \in U_1$ and $A(z) = 0$ otherwise. Then A is smooth on $\mathrm{M}_m(\mathbb{C})$ and $T(A(z)) = z$ for $z \in U_2$.

Step 3: the elements $k(y)$. For $y \in \mathrm{M}_{m\times b}(\mathbb{C})$ put $z(y) = 1 - (x-y)wJ$, an affine function of y with $z(x) = 1$ and, as $wJx = 1$,

$$z(y)\, x = x - (x - y) = y.$$

Let $k(y) = \exp(L(A(z(y))))$. By Step 1, $k(y) \in \mathrm{U}(J)$, and $y \mapsto k(y)$ is smooth as a composite of smooth maps. Let $N = z^{-1}(U_2)$, a neighbourhood of x. For $y \in N$ put $a = A(z(y))$ and $c = wa^*Jx$; by (a) and Step 2,

$$k(y)\,x\,c = \exp(L(a))\,(a - L(a))\,x = T(a)\,x = z(y)\,x = y.$$

□

Lemma 6.3.8. *Let V, V_0, V_1, U be spaces with points, D a base-adapted splitting of V over V_0 along a nonempty index set p and D' a base-adapted splitting of V_0 over V_1 along a nonempty index set q, and let $D \circ D'$ be their composite, a splitting of V over V_1 along $p \sqcup q$. Then for every $\phi \in \mathcal{F}_{V,U}$*

$$\mathrm{rsl}_{D\circ D',U}(0;\phi) = \mathrm{rsl}_{D',U}(0;\mathrm{rsl}_{D,U}(0;\phi)).$$

Proof. Let r be the index set of U. If r is empty, the Fock models of $\mathrm{Hom}(V,U)$, $\mathrm{Hom}(V_0,U)$, $\mathrm{Hom}(V_1,U)$ are $\mathbb{C}$ (functions on a point, ν the unit mass) and each real zero term of (MX7) is the identity (the integral over a point of the value times the real vacuum 1), so there is nothing to prove; let r be nonempty, so that $r \times p$, $r \times q$ are nonempty. By (MX7), $\mathrm{rsl}_{C,U}(0;\phi) = Z^\infty_{C,U}(\phi)$ for every splitting C (the Heisenberg operator at 0 is the identity); so we show $Z^\infty_{D\circ D',U} = Z^\infty_{D',U} \circ Z^\infty_{D,U}$.

Step 1: the induced splittings. Write $D = (k,l,i)$, $D' = (k',l',i')$, so $D \circ D' = ((k \mid ik'),(l \mid il'),ii')$ ((MX5)). By (MX1) the frames induced on $\mathrm{Hom}(V,U)$ are $J_U^{-1} \otimes (k^*J_V)^\mathsf{T}$, $1 \otimes (l^*J_V)^\mathsf{T}$, $1 \otimes (J_{V_0}^{-1}i^*J_V)^\mathsf{T}$, and those induced by D' on $\mathrm{Hom}(V_0,U)$ likewise. Since $(J_{V_0}^{-1}i^*J_V)^\mathsf{T}(k'^*J_{V_0})^\mathsf{T} = (k'^*i^*J_V)^\mathsf{T}$, and similarly for l' and i', the splitting $(D \circ D')_U$ is the composite $D_U \circ D'_U$, once $r \times (p \sqcup q)$ is identified with $(r \times p) \sqcup (r \times q)$; under this identification the adapted basis $(x_U^{-1})^* \otimes 1_{p\sqcup q}$ is the block diagonal matrix of the adapted bases for p and q, and the formulas of (MX7) do not depend on the order of the index set. The composite $D \circ D'$ is base-adapted: its matrix of adelic frames is, up to the order of the columns, that of D times the block diagonal matrix with blocks 1, 1 and the matrix of adelic frames of D' (as $x_V^{-1}ik' = i_{D,\mathbb{A}}x_{V_0}^{-1}k'$ and likewise for l', i'), a product of matrices unitary at the infinite places and integral at the finite places. By Lemma 6.3.53, D_U, D'_U and $(D \circ D')_U$ are adapted at 1 in their adapted bases.

Step 2: the zero terms of the vacuum datum. For each of the three induced splittings C, with zero term Z_C and real zero term Z^∞_C, (2) of Lemma 6.2.20 with $f = v^0$, then $v_1 \otimes v^0 = v^0$ ((4) of Lemma 6.2.21) and $Z_C(v^0) = \mathrm{vf}(C)v^0$ with $\mathrm{vf}(C) \neq 0$ ((4) of Lemma 6.2.20) give, for every Fock vector w of the source of C,

$$Z_C(w \otimes v^0) = Z^\infty_C(w) \otimes Z_C(v_1 \otimes v^0) = \mathrm{vf}(C)\,Z^\infty_C(w) \otimes v^0.$$

By (1) of Lemma 6.2.25, $Z_{D_U \circ D'_U} = Z_{D'_U} \circ Z_{D_U}$. Applied to v^0 this gives $\mathrm{vf}(D_U \circ D'_U) = \mathrm{vf}(D_U)\,\mathrm{vf}(D'_U)$. Applied to $\phi \otimes v^0$, the display (for $C = D_U$, then for $C = D'_U$ and the Fock vector $Z^\infty_{D_U}(\phi)$, and for $C = D_U \circ D'_U$) gives

$$\mathrm{vf}(D_U \circ D'_U)\,Z^\infty_{D_U\circ D'_U}(\phi) \otimes v^0 = \mathrm{vf}(D_U)\,\mathrm{vf}(D'_U)\,Z^\infty_{D'_U}(Z^\infty_{D_U}(\phi)) \otimes v^0.$$

Step 3: conclusion. Dividing by the nonzero vacuum factor,

$$Z^\infty_{D_U\circ D'_U}(\phi) \otimes v^0 = Z^\infty_{D'_U}(Z^\infty_{D_U}(\phi)) \otimes v^0$$

in the theta model of $\mathrm{Hom}(V_1,U)$, and Lemma 6.3.9 gives $Z^\infty_{D\circ D',U}(\phi) = Z^\infty_{D',U}(Z^\infty_{D,U}(\phi))$. □

Lemma 6.3.9. *Let s be signs on an index set n and $\mathcal{F}_\varepsilon$ the Fock model for $\iota = \Phi\times n$, $\varepsilon(v,i) = s_i$. For every $w \in \mathcal{F}_\varepsilon$ and $u \in \mathbb{C}^{\Phi\times n}$,*

$$\big(w \otimes v^0\big)\big(\mathrm{rv}(u)\big) = w(u).$$

In particular, if $w, w' \in \mathcal{F}_\varepsilon$ and $w \otimes v^0 = w' \otimes v^0$, then $w = w'$.

Proof. For $f \in \mathcal{M}_s$ and $x \in \mathbb{A}_E^n$, the identity of (WD7) gives $|f(x)| \le \sup|e_n|\,\|f\|_{L^1(\mu_n)}$, so evaluation at x is a bounded linear functional on $\mathcal{M}_s$ and commutes with the Bochner integral $w \otimes v^0 = \int w(y)\,\rho(\mathrm{rv}(y))v^0\,d\nu_\iota(y)$ ((WR6), (1) of Lemma 6.2.21). With $(\rho(z)f)(x) = \beta_s(z,x)f(x-z)$,

$$\big(w \otimes v^0\big)(\mathrm{rv}(u)) = \int w(y)\,\beta_s\big(\mathrm{rv}(y),\mathrm{rv}(u)\big)\,e_n\big(\mathrm{rv}(u)-\mathrm{rv}(y)\big)\,d\nu_\iota(y).$$

At $v \in \Phi$ the element $\jmath_v$ is purely imaginary with $\operatorname{Im}\jmath_v > 0$ and $\kappa_v^2 = (2\pi\operatorname{Im}\jmath_v)^{-1}$, so $\jmath_v\kappa_v^2 = i/(2\pi)$ and $\operatorname{Im}(\jmath_v)\kappa_v^2 = 1/(2\pi)$. The vectors $\mathrm{rv}(y)$ have components $\kappa_v y_{v,i}$ at the infinite places and 0 at the finite places. Hence, by (WD4) and (WD6),

$$\beta_s\big(\mathrm{rv}(y),\mathrm{rv}(u)\big) = \exp\Big(-2\pi i\sum_{v\in\Phi}\operatorname{Re}\big(\tfrac{i}{2\pi}\,u_v^*\mathrm{diag}(s)\,y_v\big)\Big) = \exp\big(i\operatorname{Im}(u^*\mathrm{diag}(\varepsilon)\,y)\big) = \beta_\varepsilon^\infty(y,u),$$

using $\operatorname{Re}(iz) = -\operatorname{Im} z$ and $\frac{1}{2}(z-\bar z) = i\operatorname{Im} z$ for $z = u^*\mathrm{diag}(\varepsilon)y$; and by (WD5), the finite part 0 lying in $\alpha\widehat{\mathcal{O}}_E^n$,

$$e_n\big(\mathrm{rv}(u-y)\big) = \exp\Big(-\pi\sum_{v,i}\operatorname{Im}(\jmath_v)\kappa_v^2|u_{v,i}-y_{v,i}|^2\Big) = \exp\big(-\tfrac{1}{2}\|u-y\|^2\big) = e^\infty(u-y),$$

$\|\cdot\|$ the euclidean norm. So the integral is $\int w(y)\beta_\varepsilon^\infty(y,u)e^\infty(u-y)\,d\nu_\iota(y)$, which is $w(u)$ by the defining identity of $\mathcal{F}_\varepsilon$ ((WR5)). The last assertion follows by evaluating at the points $\mathrm{rv}(u)$. □

Proposition 6.3.10. *Let W', W, V', V, U be spaces with signs s' on n', s on n, t' on r', t on r, u on b, let p be a nonempty index set and A a base-adapted splitting of W' over W along p. Let $\sigma\colon b\sqcup r \to n$ and $\sigma'\colon b\sqcup r' \to n'$ be bijections, assume $\#n$ even and $\mathrm{neg}(V') = \#p + \mathrm{neg}(V)$, and let $\mu\colon \Phi \to \mathbb{Z}$. If $x \in \mathcal{F}_{W',V'}$, $y \in \mathcal{F}_{W',U}$ and ℓ are such that (σ', x, y, ℓ) satisfy the row-column exchange identity for (W', V', U) and μ, then there is $x_0 \in \mathcal{F}_{W,V}$ such that $(\sigma, x_0, \mathrm{rsl}_{A,U}(0;y), \ell)$ satisfy the row-column exchange identity for (W, V, U) and μ.*

Proof. Since A splits off $\#p$ hyperbolic planes, $\#n' = \#n + 2\#p$, and the bijections give $\#r' = \#n' - \#b = \#r + 2\#p$.

Step 0: reindexing. For a splitting $D = (k, l, i)$ along p and a bijection $e\colon q \to p$ let $D^e = (k\circ e, l\circ e, i)$, the frames with their columns renumbered. It is a splitting along q, base-adapted if D is (its matrix of adelic frames has the same columns), the splitting it induces on $\mathrm{Hom}(V,U)$ and the adapted basis $(x_U^{-1})^*\otimes 1_q$ are those of D renumbered by $1\times e$, and $\mathrm{rsl}_{D^e,U}(0;\cdot) = \mathrm{rsl}_{D,U}(0;\cdot)$: in the formula of (MX7) for the real zero term, $a(y)^*a(y)$ and $(\mathrm{pr}_k\mathrm{rv}(y))^*\mathrm{pr}_l\mathrm{rv}(y)$ are sums over the index set of the splitting, which are unchanged by renumbering, and $w(y)$ does not involve it.

Step 1: one plane. Let $p = \{o\}$, so $\#r' = \#r + 2$ and $\mathrm{neg}(V') = 1 + \mathrm{neg}(V)$. Put $e = \#r - \#n/2$, and $z_e = e+1$, $\pm = +$ if $e \ge 0$, $z_e = -e$, $\pm = -$ if $e < 0$.

(a) *A space with the signs of V' that splits over V.* By (2) of Lemma 6.3.63 (for V, the index set $\{o\}$ and the signs t') there are a space V_1 with signs t' and a base-adapted splitting

X of V_1 over V along $\{o\}$. By (1) of Lemma 6.3.63 there are $\gamma' \in \mathrm{GL}_{r'}(E)$ and $k' \in \mathrm{G}(t')$ with $x_{V'} = \gamma'^{-1} x_{V_1} k'$, and $\mathcal{F}_{W',V'} = \mathcal{F}_{W',V_1}$, $\mathcal{M}_{W',V'} = \mathcal{M}_{W',V_1}$ (same signs).

(b) *The identity for V_1.* Let $(D, (c_\mathfrak{c}))$ be the constants of the hypothesis: $D|c_\mathfrak{c}|^2 = L_*(\Sigma, \nu; e+1)\,\ell(\mathfrak{c})$ (as $\#r' - \#n'/2 = e+1$) and $\theta^{\mathrm{lg}}_\mathfrak{c}(W', V'; x, \Phi^{\mathrm{st}}_{\sigma',z'}) = c_\mathfrak{c}\theta^{\mathrm{sm}}_\mathfrak{c}(W', U; y, \Psi^{\mathrm{st}}_{z'})$ for integral $z' \in \mathrm{M}_{b\times n'}(\mathcal{O}_\Sigma)$. Fix $\mathfrak{c} = (\nu, \Sigma)$ of weight μ and $\alpha = \nu^{-1}\upsilon^{\#n'/2}$. The function $\Phi^{\mathrm{st}}_{\sigma',z'}$ is locally constant, compactly supported and vanishes at the matrices without a right inverse (if it is non-zero at x', the square matrix with the rows of z' and x' is invertible), so $x \otimes [\Phi^{\mathrm{st}}_{\sigma',z'}]_\Sigma$ is target-bounded (Lemma 6.2.26) and its theta kernel is integrable over $[\mathrm{U}(V')]$ (Lemma 6.3.21). By Lemma 6.3.64 (with $W_1 = W'$, $\gamma = 1$, $k = 1$) there is $u_\mathfrak{c} = z\,\alpha(\det k')^{-1}$ with $|u_\mathfrak{c}| = 1$ and $\theta^{\mathrm{lg}}_\mathfrak{c}(W', V'; x, \Phi) = u_\mathfrak{c}\theta^{\mathrm{lg}}_\mathfrak{c}(W', V_1; x, \Phi)$ for these Φ. So with $c_1 = u_\mathfrak{c}^{-1}c_\mathfrak{c}$, $|c_1| = |c_\mathfrak{c}|$ and

$$\theta^{\mathrm{lg}}_\mathfrak{c}(W', V_1; x, \Phi^{\mathrm{st}}_{\sigma',z'}) = c_1\,\theta^{\mathrm{sm}}_\mathfrak{c}(W', U; y, \Psi^{\mathrm{st}}_{z'}) \qquad (z' \in \mathrm{M}_{b\times n'}(\mathcal{O}_\Sigma)). \tag{1}$$

(c) *Constant terms.* Let $C > 0$ be the constant of Proposition 6.2.2 for A, X, σ', σ, μ, and $\gamma_\mathfrak{c}$ its number for $\mathfrak{c}$, $|\gamma_\mathfrak{c}| = C\,L^\sharp_\mathfrak{c}(z_e)$. Put $x_0 = \mathrm{Cone}^\pm_{A,X,\mu}(z_e; x)$, a Schwartz vector of $\mathcal{F}_{W,V}$ by Lemma 6.2.42 ($z_e > 0$); it does not depend on $\mathfrak{c}$. Let $z \in \mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$ and $g \in \mathrm{G}(s)$. The matrix $(\mathrm{pr}_i)_\Sigma$ is integral, the finite part of $\mathrm{pr}_i = \varepsilon_s i^*_{A,\mathbb{A}}\varepsilon_{s'}$ being $i^*_{A,\mathbb{A},f}$ with $i_{A,\mathbb{A},f}$ integral; so $z' = z(\mathrm{pr}_i)_\Sigma$ is integral. By Proposition 6.2.2 at $u = 1$, the definition of $\mathrm{CT}^{\mathrm{pl}}$, (1) at the points $\mathrm{rad}^{\mathrm{ln}}_A(q_1, q_2)\mathrm{lev}_A(1, g)$, and Lemma 6.3.11,

$$\gamma_\mathfrak{c}\,\theta^{\mathrm{lg}}_\mathfrak{c}(W, V; x_0, \Phi^{\mathrm{st}}_{\sigma,z})(g) = \mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(x, z)(1, g) = c_1\,\mathrm{vf}(A_U)\,\theta^{\mathrm{sm}}_\mathfrak{c}(W, U; \mathrm{rsl}_{A,U}(0; y), \Psi^{\mathrm{st}}_z)(g).$$

(d) *The constants are non-zero.* The splitting A_U of $\mathrm{Hom}(W', U)$ over $\mathrm{Hom}(W, U)$ is along $b \times \{o\}$. If $b \neq \varnothing$, $\mathrm{vf}(A_U) \neq 0$ by (4) of Lemma 6.2.20, A_U being adapted at 1 in its adapted basis (Lemma 6.3.53). If $b = \varnothing$, both spaces of homomorphisms have rank 0, so both are the standard space of the empty sign vector; in rank 0 the vacuum measure is the unit point mass, so the theta model is $\mathbb{C}$ with $\rho(0) = 1$, $v^0 = e_0 = 1$ and $\langle f, g\rangle = \bar{f}g$ ((WD5)–(WD7) of Definition 3.2.12, empty sums being 0 and empty products 1), and the canonical theta functional is the identity of $\mathbb{C}$ ((WC1)–(WC3) of Definition 4.1.15: the transporting element of a standard space is 1, and the empty product formula gives $\theta(v^0) = 1$). The quotient $\mathbb{A}^\varnothing_E/E^\varnothing$ in (MX4) is a point of mass 1, so (MX4) reads $\theta(Z(f)) = \theta(f)$, that is $Z(f) = f$, and (MX8) gives $\mathrm{vf}(A_U) = \langle 1, 1\rangle/\langle 1, 1\rangle = 1$. For $w \in \Sigma$ and a unit u_w of $\mathcal{O}_{E,w}$, the idele x_w with component u_w at w and 1 elsewhere has $x_w/\bar{x}_w \in \mathrm{U}(1)(\mathbb{A})$ with components u_w at w, $\bar{u}_w^{-1}$ at $\bar{w} \notin \Sigma$ and 1 elsewhere, so $\tilde{\nu}(x_w) = \psi_w(u_w \bmod w)$, which is not identically 1; hence $\tilde{\nu}$ is ramified at w and not of the form $|\cdot|^{i\tau}$. By (3) of Theorem A.1.12, $L(z, \tilde{\nu}) \neq 0$ for real $z \geq 1$; the Euler factors removed in L^Σ are $(1 - \tilde{\nu}(\varpi_w)\mathrm{N}w^{-z})^{-1}$ at unramified places, finite and non-zero for $z \geq 1$; so $L^\Sigma(z_e, \tilde{\nu}) \neq 0$, $L^\sharp_\mathfrak{c}(z_e) > 0$ and $\gamma_\mathfrak{c} \neq 0$. By (c), $c_0 = c_1\mathrm{vf}(A_U)/\gamma_\mathfrak{c}$ satisfies $\theta^{\mathrm{lg}}_\mathfrak{c}(W, V; x_0, \Phi^{\mathrm{st}}_{\sigma,z}) = c_0\,\theta^{\mathrm{sm}}_\mathfrak{c}(W, U; \mathrm{rsl}_{A,U}(0; y), \Psi^{\mathrm{st}}_z)$ for every integral z.

(e) *The size of c_0.* Put $D_0 = D\,C^2/|\mathrm{vf}(A_U)|^2$, independent of $\mathfrak{c}$. Then

$$D_0|c_0|^2 = D|c_\mathfrak{c}|^2/L^\sharp_\mathfrak{c}(z_e)^2 = L_*(\Sigma, \nu; e+1)\,\ell(\mathfrak{c})/L^\sharp_\mathfrak{c}(z_e)^2,$$

and with $Q = \prod_{w\in\Sigma}\mathrm{N}w$ and $L_j = |L^\Sigma(j, \tilde{\nu})|^2$: if $e \geq 0$, $L_*(e+1) = Q^{(e+1)^2}\prod_{j=1}^{e+1}L_j$ and $L^\sharp(e+1)^2 = L_{e+1}Q^{2e+1}$, so the quotient is $Q^{e^2}\prod_{j=1}^{e}L_j = L_*(e)$; if $e = -k < 0$, $L_*(e+1) = L_*(k-1)^{-1}$, $L^\sharp(k)^2 = L_kQ^{2k-1}$ and $L_*(k) = Q^{k^2}\prod_{j=1}^{k}L_j = L^\sharp(k)^2L_*(k-1)$, so the quotient is $L_*(k)^{-1} = L_*(e)$. Hence $D_0|c_0|^2 = L_*(\Sigma, \nu; \#r - \#n/2)\,\ell(\mathfrak{c})$. Finally $\mathrm{rsl}_{A,U}(0; y)$ is a Schwartz vector by (1) of Lemma 6.2.43 and Lemma 6.3.53. So $(\sigma, x_0, \mathrm{rsl}_{A,U}(0; y), \ell)$ satisfy the row-column exchange identity for (W, V, U) and μ.

Step 2: induction on $\#p$. We prove the statement for all data by strong induction on $\#p \geq 1$. If $\#p = 1$, let $e\colon \{o\} \to p$ be the bijection; A^e is base-adapted, Step 1 applies to

it, and $\mathrm{rsl}_{A^e,U}(0;y) = \mathrm{rsl}_{A,U}(0;y)$ by Step 0. Let $\#p \geq 2$ and write $A = (k,l,i)$. Choose a bijection $e\colon n_1 \sqcup q_1 \to p$ with $q_1 = \{o\}$, so $n_1 \neq \varnothing$ and $\#n_1 < \#p$. Let $Y_0 = \mathbb{H}_{q_1} \oplus W$ (index set $(q_1 \sqcup q_1) \sqcup n$) and R its tautological splitting over W along q_1. Define $B = (k_B, l_B, i_B)$ by $k_B = (k \circ e)|_{n_1}$, $l_B = (l \circ e)|_{n_1}$ and $i_B = ((l \circ e)|_{q_1} \mid (k \circ e)|_{q_1} \mid i)$ (columns indexed by $(q_1 \sqcup q_1) \sqcup n$). The relations of (MX1) for A give those of a splitting of W' over Y_0 along n_1: the Gram matrix of the columns of i_B is $\mathrm{diag}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}, J_W\right) = J_{Y_0}$, these columns are orthogonal to those of k_B, l_B (as $l^* J_{W'} k = 1_p$, $k^* J_{W'} k = l^* J_{W'} l = 0$ and $i^* J_{W'} k = i^* J_{W'} l = 0$), and $\#n' = \#((q_1 \sqcup q_1) \sqcup n) + 2\#n_1$. Its matrix of adelic frames consists of the columns of that of A, the block of $i_{B,\mathbb{A}}$ on $q_1 \sqcup q_1$ multiplied on the right by the point $x_{\mathbb{H}_{q_1}}$; this point is unitary at the infinite places and, at the finite places, integral with determinant $c\bar{c}_0 + cc_0 = c$ a unit ($c_0 + \bar{c}_0 = 1$), so B is base-adapted. The tautological splitting R has the matrix of adelic frames $\mathrm{diag}(x_{\mathbb{H}_{q_1}}^{-1}, 1)$, unitary at the infinite places and integral at the finite places, so R is base-adapted. The composite $B \circ R$ (MX5) has the frames $(k_B \mid i_B k_R) = k \circ e$, $(l_B \mid i_B l_R) = l \circ e$ and the embedding $i_B i_R = i$, so $B \circ R = A^e$. By Step 0 and Lemma 6.3.8,

$$\mathrm{rsl}_{A,U}(0;y) = \mathrm{rsl}_{B\circ R,U}(0;y) = \mathrm{rsl}_{R,U}\big(0; \mathrm{rsl}_{B,U}(0;y)\big).$$

Let $Y_V = \mathbb{H}_{q_1} \oplus V$, with $\mathrm{neg}(Y_V) = 1 + \mathrm{neg}(V)$, so $\mathrm{neg}(V') = \#n_1 + \mathrm{neg}(Y_V)$, and choose a bijection $\tau\colon b \sqcup ((q_1 \sqcup q_1) \sqcup r) \to (q_1 \sqcup q_1) \sqcup n$ (both sides have $\#b + \#r + 2 = \#n + 2$ elements). The rank of Y_0 is even. By the induction hypothesis for B (along n_1, with Y_0, Y_V, τ, σ') there is $x_1 \in \mathcal{F}_{Y_0,Y_V}$ such that $(\tau, x_1, \mathrm{rsl}_{B,U}(0;y), \ell)$ satisfy the identity for (Y_0, Y_V, U) and μ. By Step 1 for R (along q_1, with W, V, σ, τ) there is $x_0 \in \mathcal{F}_{W,V}$ such that $(\sigma, x_0, \mathrm{rsl}_{R,U}(0; \mathrm{rsl}_{B,U}(0;y)), \ell)$ satisfy the identity for (W,V,U) and μ, which is the assertion. □

Lemma 6.3.11. *Let W', W, U be spaces with signs s' on n', s on n, u on b, and A a base-adapted splitting of W' over W along a one-element set. Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, $y \in \mathcal{F}_{W',U}$ a Schwartz vector, $z \in \mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$, and $(\mathrm{pr}_i)_\Sigma \in \mathrm{M}_{n\times n'}(E_\Sigma)$ the matrix of the components at Σ of the finite part of the coordinate pr_i of A. Then for every $g \in \mathrm{G}(s)$*

$$\begin{aligned}&\int_{\mathrm{M}_{n\times 1}(\mathbb{A}_E)/\mathrm{M}_{n\times 1}(E)} \int_{\mathbb{A}_E/E} \theta^{\mathrm{sm}}_{\mathfrak{c}}\big(W', U; y, \Psi^{\mathrm{st}}_{z(\mathrm{pr}_i)_\Sigma}\big)\big(\mathrm{rad}^{\mathrm{ln}}_A(q_1,q_2)\,\mathrm{lev}_A(1,g)\big)\,dq_2\,dq_1 \\ &\qquad = \mathrm{vf}(A_U)\,\theta^{\mathrm{sm}}_{\mathfrak{c}}\big(W, U; \mathrm{rsl}_{A,U}(0;y), \Psi^{\mathrm{st}}_z\big)(g),\end{aligned}$$

the integral being iterated as in the plane constant term of (XD10).

Proof. Write $A = (k,l,i)$ with adelic frames $k_\mathbb{A}, l_\mathbb{A}, i_\mathbb{A}$ and coordinates $\mathrm{pr}_k, \mathrm{pr}_l, \mathrm{pr}_i$; by (MX1) they send $l_\mathbb{A} a + i_\mathbb{A} w + k_\mathbb{A} b$ to b, a, w, so $\mathrm{pr}_k k_\mathbb{A} = \mathrm{pr}_l l_\mathbb{A} = 1$, $\mathrm{pr}_i i_\mathbb{A} = 1$ and the other six products vanish. Put $\Psi' = \Psi^{\mathrm{st}}_{z(\mathrm{pr}_i)_\Sigma}$, $y_0 = \mathrm{rsl}_{A,U}(0;y)$, and let $\lambda_1 = \nu\upsilon^{\lfloor \#n'/2 \rfloor}$, $\lambda_0 = \nu\upsilon^{\lfloor \#n/2 \rfloor}$, $\beta = \upsilon^{\#b}$, so that $\lambda_1 = \lambda_0 \upsilon$ ($\#n' = \#n + 2$). By (XD9) the left side is the iterated integral of $I^{W',U}_{\lambda_1,\beta}(f')(\mathrm{rad}^{\mathrm{ln}}_A(q_1,q_2)\mathrm{lev}_A(1,g))$ with $f' = y \otimes [\Psi'^{\mathsf{T}}]_{\bar{\Sigma}}$, and the right side is $\mathrm{vf}(A_U)\, I^{W,U}_{\lambda_0,\beta}(y_0 \otimes [(\Psi^{\mathrm{st}}_z)^{\mathsf{T}}]_{\bar{\Sigma}})(g)$. The finite part of pr_i is $i^*_{\mathbb{A},f}$, integral since A is base-adapted and $\widehat{\mathcal{O}}_E$ is stable under conjugation; so $z(\mathrm{pr}_i)_\Sigma$ is integral and, by (2) of Lemma 6.3.31, Ψ' and Ψ^{st}_z are locally constant with compact support and vanish at the matrices without a left inverse.

Case $b = \varnothing$. Then $\mathrm{Hom}(W', U)$ and $\mathrm{Hom}(W, U)$ have rank 0: their theta and Fock models are $\mathbb{C}$, the canonical theta functional is the identity of $\mathbb{C}$ (the empty product formula of (WC1)), $\mathrm{G}(\varnothing)$ is trivial with the unit point mass, so $[\mathrm{U}(U)]$ is a point of mass 1, the Weil operators are the identity (in (WR3) the multiplier is an empty product times $\upsilon(1)$ and the integral is over a point), and the characters are evaluated at $\det h = 1$ and, for $\beta = \upsilon^0 = 1$, give 1. In rank 0, $\mathcal{T}_A$ is spanned by $\tau_0 = 1$ and $\Psi \otimes_A f = \Psi(0) f$ (P^A and $\rho(0)$ being the identity), and Ψ'^{T}, $(\Psi^{\mathrm{st}}_z)^{\mathsf{T}}$ take the value $\chi^{-1}(\det(\cdot)) = \chi^{-1}(1) = 1$ at the empty matrix. Hence the integrand of

the left side is the number y at every point, the left side is y, and the right side is $\mathrm{vf}(A_U)\,y_0$; here $\mathrm{vf}(A_U)=1$ ((MX4), (MX8): the zero term along a splitting of the zero space along the empty set is the identity) and $y_0=y$ ((MX7): the real zero term over a point, with real vacuum 1, is the identity). From now on $b\neq\varnothing$, so A_U is along the nonempty set $b\times\{o\}$.

Step 1: the expansion. The function $\phi=\Psi'^{\mathsf T}\circ\bar{\cdot}$ on $\mathrm{M}_{b\times n'}(E_{\bar\Sigma})$, $\phi(w)=\Psi'(\bar w^{\mathsf T})$, is locally constant with compact support, $\phi^\sharp=(\Psi'^{\mathsf T})^\flat$, and it vanishes at every w without a right inverse (if $\phi(w)\neq0$, $\bar w^{\mathsf T}$ has a left inverse). By Lemma 6.2.22 the datum $f'=y\otimes(\phi^\sharp\otimes_{\bar\Sigma}v^0)$ is defined, and it is target-bounded by Lemma 6.2.26 (for the place set $\bar\Sigma$). By Lemma 6.2.4 (for W', W, A, the space U, the characters λ_1,β, f' and the point $\mathrm{lev}_A(1,g)$), the left side equals

$$\int_{[\mathrm U(U)]}\sum_{\xi\in E^b}\mathrm{ct}_A(\xi;F_h)\,dh,\qquad F_h=\omega_{\lambda_1,\beta}\big(\mathrm{lev}_A(1,g),h\big)f',$$

the series converging absolutely. By (MX10), $\mathrm{ct}_A(\xi;F_h)=0$ when $\xi^*J_U\xi\neq0$. We compute the terms of the isotropic ξ.

Step 2: terms of isotropic vectors. Let $\xi\in E^b$ with $\xi^*J_U\xi=0$, $x_\xi=x_U^{-1}\xi$, so $x_\xi^*\varepsilon_u x_\xi=\xi^*J_U\xi=0$ ($x_U^*J_Ux_U=\varepsilon_u$). For an adelic $n\times1$ matrix c let $r=\mathrm{rad}_A(c,0)=1+N$, $N=-i_{\mathbb A}c\,\mathrm{pr}_l+k_{\mathbb A}c^*\varepsilon_s\mathrm{pr}_i-\frac12k_{\mathbb A}c^*\varepsilon_sc\,\mathrm{pr}_l$ (MX2). The relations of the coordinates give $rk_{\mathbb A}=k_{\mathbb A}$, $\mathrm{pr}_ir i_{\mathbb A}=1$ and $\mathrm{rad}_A(c,0)\,\mathrm{rad}_A(-c,0)=1$: with N_- the expression N at $-c$, $N+N_-=-k_{\mathbb A}c^*\varepsilon_sc\,\mathrm{pr}_l$, and in NN_- only the product $k_{\mathbb A}c^*\varepsilon_s\mathrm{pr}_i\cdot i_{\mathbb A}c\,\mathrm{pr}_l=k_{\mathbb A}c^*\varepsilon_sc\,\mathrm{pr}_l$ survives ($\mathrm{pr}_li_{\mathbb A}=\mathrm{pr}_lk_{\mathbb A}=\mathrm{pr}_ik_{\mathbb A}=0$), so $(1+N)(1+N_-)=1$. Hence $r^{-1}=\mathrm{rad}_A(-c,0)$ and $r^{-1}l_{\mathbb A}=l_{\mathbb A}+i_{\mathbb A}c-\frac12k_{\mathbb A}c^*\varepsilon_sc$. By (2) of Lemma 6.2.45 (source W' split by A, target U, the element r with $m=1$, $g_0=1$, $W=\mathrm{pr}_ir^{-1}l_{\mathbb A}=c$, $B=\mathrm{pr}_kr^{-1}l_{\mathbb A}=-\frac12c^*\varepsilon_sc$, the point 1 of $\mathrm G(u)$, characters υ^{-1} and 1), whose phase is 1 because $x_\xi^*\varepsilon_ux_\xi=0$,

$$\big(\omega_{1,1}(r,1)F\big)[x_\xi]=\rho\big(x_\xi c^*\varepsilon_s\big)F[x_\xi]\qquad(F\in\mathcal M_{W',U}).$$

With $c=x_W^{-1}q_1$, (MX6) and (MX10) give, θ_0 the canonical theta functional of $\mathrm{Hom}(W,U)$,

$$\mathrm{ct}_A(\xi;F)=\int_{\mathrm M_{n\times1}(\mathbb A_E)/\mathrm M_{n\times1}(E)}\theta_0\big(\rho(x_\xi(x_W^{-1}q_1)^*\varepsilon_s)\,F[x_\xi]\big)\,dq_1.\tag{1}$$

By (2) of Lemma 6.2.45 for the element $\mathrm{lev}_A(1,g)$ ($m=1$, $g_0=g$; $\mathrm{lev}_A(1,g)^{-1}=\mathrm{lev}_A(1,g^{-1})$ fixes $l_{\mathbb A}$, so $W=B=0$), the point h and the characters λ_0, β (with $\lambda_0\upsilon=\lambda_1$),

$$F_h[x_\xi]=\omega_{\lambda_0,\beta}(g,h)\big(f'[h^{-1}x_\xi]\big).\tag{2}$$

Step 3: Hom slices of f'. Let X be an adelic $b\times1$ matrix. By (1) of Lemma 6.3.12, $f'[X]$ is the slice of f' along A_U at $m_U^{-*}X$, and A_U is adapted at 1 in m_U by Lemma 6.3.53. Its frame $(l_{\mathbb A}m_U^{-*}\mid k_{\mathbb A}m_U\mid i_{\mathbb A})$ for A_U sends (a,b',Y) to $a\,\mathrm{pr}_k+\varepsilon_ub'\mathrm{pr}_l+Y\mathrm{pr}_i$ ($b\times n'$ matrices) ; its finite part at $\bar\Sigma$ sends (a,b',Y) to $w=a\,\mathrm{pr}_k+b'\mathrm{pr}_l+Y\mathrm{pr}_i$ (finite parts at $\bar\Sigma$). Then $\bar w^{\mathsf T}=l_\Sigma\bar a^{\mathsf T}+k_\Sigma\bar b'^{\mathsf T}+i_\Sigma\bar Y^{\mathsf T}$, where $l_\Sigma,k_\Sigma,i_\Sigma$ are the components at Σ of the finite parts of $l_{\mathbb A},k_{\mathbb A},i_{\mathbb A}$ (the finite part of pr_k is $l^*_{\mathbb A,f}$, and so on). The matrix $(l_\Sigma\mid k_\Sigma\mid i_\Sigma)$ is square, integral (base-adapted) and has the integral inverse with block rows $(\mathrm{pr}_l)_\Sigma,(\mathrm{pr}_k)_\Sigma,(\mathrm{pr}_i)_\Sigma$ (finite parts $k^*_{\mathbb A,f}$, $l^*_{\mathbb A,f}$, $i^*_{\mathbb A,f}$); and $(\mathrm{pr}_i)_\Sigma l_\Sigma=(\mathrm{pr}_i)_\Sigma k_\Sigma=0$, $(\mathrm{pr}_i)_\Sigma i_\Sigma=1$. Hence $\bar w^{\mathsf T}$ is integral if and only if a, b', Y are, and then $z(\mathrm{pr}_i)_\Sigma\bar w^{\mathsf T}=z\bar Y^{\mathsf T}$; so

$$\phi(w)=\prod_{j\in b\times\{o\}}\mathbf1_{\mathcal O_{\bar\Sigma}}(a_j)\,\mathbf1_{\mathcal O_{\bar\Sigma}}(b'_j)\cdot\psi_0(Y),\qquad\psi_0(Y)=\Psi^{\mathrm{st}}_z(\bar Y^{\mathsf T}),$$

and $\psi_0^\sharp=((\Psi^{\mathrm{st}}_z)^{\mathsf T})^\flat$. Put $x=\alpha_0^{-1}X$, so that $m_U^{-*}X=m_U^{-*}(\alpha_0x)$, α_0 being the idele of Lemma 6.2.19 (components κ_v at $v\in\Phi$, finite part the Weil idele). By Lemma 6.2.19 (for

$D = A_U$, the place set $\bar\Sigma$, $\Phi = \phi$, $\Psi = \psi_0$, and $f_j \equiv 1$ if the components of x_f at $\bar\Sigma$ are integral, $f_j \equiv 0$ otherwise),

$$f'[X] = c_X \,\mathrm{rsl}_{A,U}(x_\infty; y) \otimes (\psi_0^\sharp \otimes_{\bar\Sigma} v^0) \quad \text{for some } c_X \in \mathbb{C}, \qquad c_0 = \mathrm{vf}(A_U), \tag{3}$$

since at $X = 0$ the indicator is 1 and each $\int_{\mathcal{O}} \psi_{\bar\Sigma}(0)\, dt = 1$.

Step 4: the nonzero isotropic terms vanish. Let $\xi \neq 0$ be isotropic and $h \in \mathrm{G}(u)$. For $H \in \mathcal{M}_{W,U}$ put

$$\Lambda(H) = \int_{\mathrm{M}_{n\times 1}(\mathbb{A}_E)/\mathrm{M}_{n\times 1}(E)} \theta_0\big(\rho(x_\xi(x_W^{-1}q_1)^*\varepsilon_s)\,\omega_{\lambda_0,\beta}(g,h)H\big)\, dq_1.$$

For $z_1 = x_\xi c_1$, $z_2 = x_\xi c_2$ (c_j adelic $1 \times n$) the bicharacter β_{H} of $\mathrm{Hom}(W,U)$ is $\beta_{\mathrm{H}}(z_2, z_1) = \psi_E(\frac{1}{2}\,\mathrm{tr}(z_1^*\varepsilon_u z_2 \varepsilon_s)) = 1$, as $z_1^*\varepsilon_u z_2 = c_1^*(x_\xi^*\varepsilon_u x_\xi)c_2 = 0$; so $\rho(z_1)\rho(z_2) = \rho(z_1 + z_2)$ by (WD7). For $\gamma \in \mathrm{M}_{n\times 1}(E)$, $x_\xi(x_W^{-1}\gamma)^*\varepsilon_s = x_U^{-1}(\xi\gamma^* J_W)x_W$ is a rational vector of $\mathrm{Hom}(W,U)$, under which θ_0 is invariant ((WD9), Lemma 4.1.17). Hence the integrand is periodic; it is continuous ($z \mapsto \rho(z)H'$ is continuous, translation being continuous in L^1 and β_{H} continuous of modulus 1), so Λ is defined, linear, and bounded by $\|\theta_0\|\,\|\omega_{\lambda_0,\beta}(g,h)\|$ times the norm (ρ is isometric; (2) of Lemma 4.1.1). Let $I = \{h^{-1}x_\xi\, c\, g : c \in \mathrm{M}_{1\times n}(\mathbb{A}_E)\}$, an $\mathbb{A}_E$-submodule of $\mathbb{A}_E^{b\times n}$. For $z = h^{-1}x_\xi c g \in I$, (2) of Lemma 4.1.1 and (WR4) give $\omega_{\lambda_0,\beta}(g,h)\rho(z) = \rho(x_\xi c)\,\omega_{\lambda_0,\beta}(g,h)$, and $x_\xi c = x_\xi(x_W^{-1}u_c)^*\varepsilon_s$ with $u_c = x_W\varepsilon_s c^*$; so $\Lambda(\rho(z)H)$ is the integral of the integrand of $\Lambda(H)$ translated by u_c, and $\Lambda(\rho(z)H) = \Lambda(H)$ by invariance of the measure. Next, let y_1 be a finite adelic $b \times n$ matrix with $\psi_0^\sharp(y_1) \neq 0$. Then $\Psi_z^{\mathrm{st}}(\bar y_{1,\Sigma}^{\mathsf{T}}) \neq 0$, so by (2) of Lemma 6.3.31 the matrix $\overline{(y_1)_{\bar w}}^{\mathsf{T}}$ has a left inverse, i.e. $(y_1)_{\bar w}$ has rank $\#b$, for every $w \in \Sigma$. Fix $w \in \Sigma$ and let $a = (h^{-1}x_\xi)_{f,\bar w} \in E_{\bar w}^b$, nonzero as $\xi \neq 0$ and h, x_U are invertible. Then $d = a^*(y_1)_{\bar w}(g_{f,\bar w})^* \neq 0$; take c with component d at $\bar w$ and 0 elsewhere, $z = h^{-1}x_\xi c g$. Then the component at $\bar w$ of $z_f^* y_1 = \mathrm{tr}(z_f^* y_1)$ is $a^*(y_1)_{\bar w} g_{f,\bar w}^* c_{\bar w}^* = dd^* \neq 0$. By Lemma 6.2.35 (signs of $\mathrm{Hom}(W,U)$, the place set $\bar\Sigma$, $\psi_0^\sharp \in \mathcal{T}_{\bar\Sigma}$ by Lemma 6.2.22, I and Λ), $\Lambda(\psi_0^\sharp \otimes_{\bar\Sigma} H) = 0$ for every H. By (1), (2), (3) with $X = h^{-1}x_\xi$, and $y' \otimes (\psi_0^\sharp \otimes_{\bar\Sigma} v^0) = \psi_0^\sharp \otimes_{\bar\Sigma} (y' \otimes v^0)$ ((3) of Lemma 6.2.21: the Schrödinger datum is a finite combination of operators $\rho(\mathrm{fv}(\cdot))P^{\bar\Sigma}$),

$$\mathrm{ct}_A(\xi; F_h) = c_X\, \Lambda(\psi_0^\sharp \otimes_{\bar\Sigma} (\mathrm{rsl}_{A,U}(x_\infty; y) \otimes v^0)) = 0.$$

Step 5: the zero vector. For $\xi = 0$, Step 2 gives $(\omega_{1,1}(r,1)F)[0] = F[0]$ for every q_1, so $\mathrm{ct}_A(0; F_h) = \theta_0(F_h[0])$ (probability measure), and by (2), (3),

$$\mathrm{ct}_A(0; F_h) = \mathrm{vf}(A_U)\,\theta_0\Big(\omega_{\lambda_0,\beta}(g,h)(y_0 \otimes (\psi_0^\sharp \otimes_{\bar\Sigma} v^0))\Big) = \mathrm{vf}(A_U)\,\theta_{\lambda_0,\beta}^{W,U}(y_0 \otimes [(\Psi_z^{\mathrm{st}})^{\mathsf{T}}]_{\bar\Sigma})(g,h).$$

By Steps 1, 4 and 5 the left side of the assertion is $\mathrm{vf}(A_U)\int_{[\mathrm{U}(U)]} \theta_{\lambda_0,\beta}^{W,U}(y_0 \otimes [(\Psi_z^{\mathrm{st}})^{\mathsf{T}}]_{\bar\Sigma})(g,h)\, dh = \mathrm{vf}(A_U)\,\theta_{\mathfrak{c}}^{\mathrm{sm}}(W,U; y_0, \Psi_z^{\mathrm{st}})(g)$, by (XD9). □

Lemma 6.3.12. *Let V, V_0, T be spaces with points with signs s on n, s_0 on n_0 and t on an index set n_T, let p be an index set, D a splitting of V over V_0 along p, and $m_T = (x_T^{-1})^* \otimes 1_p$ its adapted basis along T (MX7).*

1. *For every $f \in \mathcal{M}_{V,T}$ and every adelic $n_T \times p$ matrix X, the Hom slice $f[X]$ is the slice of f along D_T at the vector $m_T^{-*}X$ of $\mathbb{A}_E^{n_T\times p}$.*

2. *If n_T and p are nonempty and D is base-adapted, then $\mathrm{vf}(D_T) \neq 0$, and for every $\phi \in \mathcal{F}_{V,T}$ and $b \in \mathbb{C}^{\Phi\times(n_T\times p)}$*

$$(\phi \otimes v^0)[\mathrm{rv}(b)] = \mathrm{vf}(D_T)\,\mathrm{rsl}_{D,T}(b; \phi) \otimes v^0,$$

the left side being the Hom slice of $\phi \otimes v^0 \in \mathcal{M}_{V,T}$ at the adelic $n_T \times p$ matrix $\mathrm{rv}(b)$*, and the right side a datum of* $\mathcal{M}_{V_0,T}$*.*

Proof. Write $D = (k, l, i)$ and D_T for the induced splitting, along $n_T \times p$.

(1): the Hom slice is a slice along D_T. By (MX1) the frame l of D_T is $1 \otimes (l^* J_V)^{\mathsf{T}}$, so its adelic frame is

$$(x_T^{-1} \otimes x_V^{\mathsf{T}})(1 \otimes (l^* J_V)^{\mathsf{T}}) = x_T^{-1} \otimes (l^* J_V x_V)^{\mathsf{T}} = x_T^{-1} \otimes \mathrm{pr}_k^{\mathsf{T}},$$

since $J_V x_V = (x_V^*)^{-1} \varepsilon_s$ gives $l^* J_V x_V = l_{\mathbb{A}}^* \varepsilon_s = \mathrm{pr}_k$. For an adelic $n_T \times p$ matrix Y this sends Y to $x_T^{-1} Y \mathrm{pr}_k$. As $m_T^{-*} = x_T \otimes 1_p$ sends X to $x_T X$, the slice of $f \in \mathcal{M}_{V,T}$ along D_T at $m_T^{-*} X$ is $Z_{D_T}(\rho(X \mathrm{pr}_k) f)$, which is the Hom slice $f[X]$ of (MX4).

(2). Now $n_T \times p$ is nonempty and D is base-adapted. By Lemma 6.3.53, D_T is adapted at 1 in the basis m_T, and $\mathrm{rsl}_{D,T} = \mathrm{rsl}_{D_T, m_T}$ by (MX7). By (4) of Lemma 6.2.20 for D_T, $Z_{D_T}(v^0) = \mathrm{vf}(D_T)\, v^0$ and $\mathrm{vf}(D_T) \neq 0$. By (1), (3) of Lemma 6.2.20 for D_T, m_T, $f = v^0$ and $y = 0$, then $v_1 \otimes v^0 = v^0$ ((4) of Lemma 6.2.21) and $\rho(0) = 1$,

$$(\phi \otimes v^0)[\mathrm{rv}(b)] = \mathrm{rsl}_{D_T, m_T}(b; \phi) \otimes (v_1 \otimes v^0)[0] = \mathrm{rsl}_{D,T}(b; \phi) \otimes Z_{D_T}(v^0) = \mathrm{vf}(D_T)\, \mathrm{rsl}_{D,T}(b; \phi) \otimes v^0.$$

□

Lemma 6.3.13. *Let Σ be a nonempty finite set of finite places of E, a, b, n finite sets and $\sigma \colon b \sqcup a \to n$ a bijection. Let $\chi \colon \mathcal{O}_\Sigma^\times \to \mathbb{C}^\times$ be a character of the form $\chi(u) = \prod_{w \in \Sigma} \chi_w(u_w \bmod \varpi_w)$ with characters χ_w of $k_w^\times$, extended by 0 to $\mathcal{O}_\Sigma \setminus \mathcal{O}_\Sigma^\times$, and for $s \in \mathrm{M}_{b \times n}(\mathcal{O}_\Sigma)$ let $\Psi_s^{\mathrm{st}}(y) = \chi^{-1}(\det(sy))$ for $y \in \mathrm{M}_{n \times b}(\mathcal{O}_\Sigma)$ (χ^{-1} meaning $u \mapsto \chi(u)^{-1}$ on the units and 0 elsewhere) and $\Psi_s^{\mathrm{st}}(y) = 0$ for the other $y \in \mathrm{M}_{n \times b}(E_\Sigma)$. Let $\mathcal{P}$ be a linear space of locally constant, compactly supported functions on $\mathrm{M}_{n \times b}(E_\Sigma)$ such that $y \mapsto f(gy)$ lies in $\mathcal{P}$ for all $f \in \mathcal{P}$ and $g \in \mathrm{GL}_n(E_\Sigma)$. Let $s \in \mathrm{M}_{b \times n}(\mathcal{O}_\Sigma)$ and $x \in \mathrm{M}_{n \times b}(\mathcal{O}_\Sigma)$ with $sx = 1$, and assume $\Psi_s^{\mathrm{st}} \in \mathcal{P}$. Then $\mathcal{P}$ contains every locally constant function Ψ on $\mathrm{M}_{n \times b}(E_\Sigma)$ that is matched over $\mathcal{O}_\Sigma$ for (σ, χ) with some function Φ.*

Proof. Write $\mathcal{K} = \mathrm{GL}_n(\mathcal{O}_\Sigma)$, a compact group with Haar probability measure dk (left and right invariant), $p = \Psi_s^{\mathrm{st}}$, and $\varpi = (\varpi_w)_{w \in \Sigma}$. Let Ψ be locally constant and matched with Φ.

Step 1: two properties of Ψ. (i) Ψ vanishes off the integral matrices k_b, so it has compact support; being locally constant, it is invariant under an open subgroup of $\mathrm{M}_{n \times b}(E_\Sigma)$, hence there is $c \in \mathbb{N}$ with $\Psi(\varpi^c t + y) = \Psi(y)$ for all y and all integral t. (ii) $\Psi(Yh) = \chi(\det h)^{-1} \Psi(Y)$ for all Y and $h \in \mathrm{GL}_b(\mathcal{O}_\Sigma)$: if $Y = k_b$ with $k \in \mathcal{K}$, let k' be k with the columns $\sigma(b)$ replaced by those of $k_b h$; then $k'_b = Yh$, $(k'^{-1})_a = (k^{-1})_a$ (the rows $\sigma(a)$ of k'^{-1} are those of k^{-1}, as k'^{-1} is k^{-1} with the rows $\sigma(b)$ multiplied by h^{-1}) and $\det k' = \det k \det h$, so $\chi(\det k)\Psi(k_b) = \Phi((k^{-1})_a) = \chi(\det k)\chi(\det h)\Psi(Yh)$; if Y is not of this form, neither is Yh, and both values vanish.

Step 2: an element expanding the kernel of s. The matrix $e = xs$ is idempotent ($sx = 1$). Put $d = \varpi^{-c}(1 - e) + e \in \mathrm{GL}_n(E_\Sigma)$, with inverse $d^{-1} = \varpi^c(1 - e) + e$. Then $dx = x$, so $p(dx) = p(x) = \chi^{-1}(\det(sx)) = 1$.

Step 3: the translates are multiples of Ψ. For $k \in \mathcal{K}$ and y,

$$\Psi(k^{-1}x)\, p(dky) = |p(dky)|\, \Psi(y). \tag{6.30}$$

If $p(dky) = 0$ both sides vanish. Otherwise $z = dky$ is integral and $h = sz \in \mathrm{GL}_b(\mathcal{O}_\Sigma)$, with $|p(z)| = 1$ and $p(z) = \chi(\det h)^{-1}$. Now $y = k^{-1}d^{-1}z = \varpi^c k^{-1}(z - xh) + k^{-1}xh$ with $k^{-1}(z - xh)$ integral, so by Step 1, $\Psi(y) = \Psi(k^{-1}xh) = \chi(\det h)^{-1}\Psi(k^{-1}x) = p(z)\Psi(k^{-1}x)$, which is (6.30).

Step 4: averaging. The map $k \mapsto (y \mapsto p(dky))$ from $\mathcal{K}$ to functions is locally constant: p is invariant under an open subgroup U_0 and supported in a compact set Q, $p(dky) \neq 0$ forces $y \in \bigcup_k k^{-1}d^{-1}Q$, a compact set Q', and for $k'k^{-1} - 1$ in a small enough neighbourhood of 0, $d(k'-k)y = d(k'k^{-1}-1)ky \in U_0$ for all $y \in Q'$. So $k \mapsto \Psi(k^{-1}x)\,p(dk\,\cdot)$ takes finitely many values, each on an open set, and

$$F(y) = \int_{\mathcal{K}} \Psi(k^{-1}x)\,p(dky)\,dk$$

is a finite linear combination of the functions $y \mapsto p(dky)$, which lie in $\mathcal{P}$ by stability; so $F \in \mathcal{P}$. By (6.30), $F(y) = I(y)\Psi(y)$ with $I(y) = \int_{\mathcal{K}} |p(dky)|\,dk$.

Step 5: I is a positive constant on the support of Ψ. If $\Psi(y) \neq 0$, then $y = l_b$ for some $l \in \mathcal{K}$. Also $x = j_b$ for some $j \in \mathcal{K}$: at each $w \in \Sigma$, $sx = 1$ shows that the reduction of x_w modulo ϖ_w has rank $\#b$, so its columns extend to a basis of k_w^n by standard basis vectors; placing the columns of x_w in the positions $\sigma(b)$ and these vectors in the positions $\sigma(a)$ gives j_w with unit determinant. Then $y = lj^{-1}x$, and the substitution $k \mapsto kjl^{-1}$ (right invariance of dk) gives $I(y) = I(x)$. Finally $I(x) > 0$: the integrand $k \mapsto |p(dkx)|$ is locally constant (Step 4), nonnegative, and equal to 1 at $k = 1$ (Step 2), hence positive on an open neighbourhood of 1, which has positive measure.

So $F = I(x)\Psi$ with $I(x) > 0$, and $\Psi = I(x)^{-1}F \in \mathcal{P}$. □

Lemma 6.3.14. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, X a space with signs r on m, b an index set and $(\mathcal{L}, \mathcal{D})$ an equivariant lift for $(\mathfrak{c}, X, b)$; functions are functions on $\mathrm{M}_{m\times b}(E_\Sigma)$.*

1. *Let q be an index set, $U \in \mathrm{M}_{m\times q}(E_\Sigma)$ and $C \in \mathrm{M}_{q\times m}(E_\Sigma)$ with $CU = 0$, and let $\nu_0 \in \mathrm{G}(r)$ have components 1 at the infinite places and at the finite places outside $\Sigma \cup \bar{\Sigma}$, and component $1 + UC$ at Σ. For $\Psi \in \mathcal{L}$ and $x \in \mathrm{G}(r)$ the function $\Psi'(y) = \Psi\big(y + (x^{-1})_\Sigma\, UC\, x_\Sigma\, y\big)$ lies in $\mathcal{L}$, and $\mathcal{D}\Psi'(y'x) = \mathcal{D}\Psi(y'\nu_0^{-1}x)$ for all $y' \in \mathrm{G}(r)$.*

2. *Let D be a splitting of X over a space with a point along p, $T \in \mathrm{M}_p(E)$ and $x \in \mathrm{G}(r)$. For $\Psi \in \mathcal{L}$ the function $M \mapsto \mathcal{D}\Psi(z(M)x)$ is continuous and invariant under $\mathrm{M}_p(E)$, so $\mathrm{coef}(T; \mathcal{D}\Psi)(x)$ is defined. There is a linear functional Λ on the space of all functions with $\Lambda(\Psi) = \mathrm{coef}(T; \mathcal{D}\Psi)(x)$ for $\Psi \in \mathcal{L}$, and for $\Psi \in \mathcal{L}$ and $z \in \mathrm{M}_p(E_\Sigma)$ the function $\Psi_z(y) = \Psi\big(y + (x^{-1}k_{\mathbb{A}})_\Sigma\, z\, (\mathrm{pr}_l x)_\Sigma\, y\big)$ lies in $\mathcal{L}$ and*

$$\Lambda(\Psi_z) = \psi_{E,f}\big(\mathrm{tr}(T\tilde{z})\big)\,\Lambda(\Psi).$$

3. *If $\Psi \in \mathcal{L}$, $\mathcal{D}\Psi = 0$ and $g \in \mathrm{GL}_m(E_\Sigma)$, then $\Phi(y) = \Psi(gy)$ lies in $\mathcal{L}$ and $\mathcal{D}\Phi = 0$.*

Proof. **Step 0: two facts on $\mathrm{G}(r)$.** (a) An element of $\mathrm{G}(r)$ of the form $x_X^{-1}\gamma x_X$ with $\gamma \in \mathrm{GL}_m(E)$ lies in Γ_X: from $x_X^* J_X x_X = \varepsilon_r$ we get $x_X^{-*}\varepsilon_r x_X^{-1} = J_X$, so $(x_X^{-1}\gamma x_X)^*\varepsilon_r(x_X^{-1}\gamma x_X) = \varepsilon_r$ says $\gamma^* J_X \gamma = J_X$, i.e. $\gamma \in \mathrm{U}(X)(F)$. (b) If $g \in \mathrm{G}(r)$ has components 1 at the infinite places, then $g = (1, g_f)$ with $g_f \in G_f(m)$.

Step 1: (1). Let $\Psi \in \mathcal{L}$ and $x \in \mathrm{G}(r)$, and put $g = x^{-1}\nu_0 x \in \mathrm{G}(r)$. Its components at the infinite places and at the finite places outside $\Sigma \cup \bar{\Sigma}$ are 1, so by Step 0 (b) $g = (1, g_f)$ with $g_f \in G_f(m)$ of component 1 at every finite w with $w \notin \Sigma$, $\bar{w} \notin \Sigma$, and

$$(g_f)_\Sigma = (x^{-1})_\Sigma(1+UC)x_\Sigma = 1+(x^{-1})_\Sigma\, UC\, x_\Sigma, \qquad \det(g_f)_\Sigma = \det(1+UC) = \det(1+CU) = 1.$$

So $\Psi'(y) = \Psi((g_f)_\Sigma y)$, and (c) of (XD11) gives $\Psi' \in \mathcal{L}$ and $\mathcal{D}\Psi'(y''g) = \upsilon_{\Sigma,1}(1)^{-\#b}\mathcal{D}\Psi(y'') = \mathcal{D}\Psi(y'')$ for all y''. With $y'' = y'\nu_0^{-1}x$ we have $y''g = y'x$, which is the claim.

Step 2: the coefficient is defined and linear. By (MX2), $z(M) = 1 + k_{\mathbb{A}}(M - M^*)\mathrm{pr}_l$ with $\mathrm{pr}_l k_{\mathbb{A}} = 0$ ((MX1): pr_l reads the coordinate along $l_{\mathbb{A}}$), hence $z(M)z(M') = z(M + M')$, and $M \mapsto z(M)$ is continuous. For $M \in \mathrm{M}_p(E)$, $k_{\mathbb{A}} = x_X^{-1}k$ and $\mathrm{pr}_l = k_{\mathbb{A}}^*\varepsilon_r = k^*J_X x_X$ give $z(M) = x_X^{-1}(1 + k(M - M^*)k^*J_X)x_X$ with a rational matrix in the middle, so $z(M) \in \Gamma_X$ by Step 0 (a). By (b) of (XD11), $\mathcal{D}\Psi$ is continuous and left Γ_X-invariant, so $M \mapsto \mathcal{D}\Psi(z(M)x)$ is continuous and $\mathcal{D}\Psi(z(M + M_0)x) = \mathcal{D}\Psi(z(M_0)z(M)x) = \mathcal{D}\Psi(z(M)x)$ for $M_0 \in \mathrm{M}_p(E)$; it is therefore a continuous function on the compact group $\mathrm{M}_p(\mathbb{A}_E)/\mathrm{M}_p(E)$, and $\mathrm{coef}(T; \mathcal{D}\Psi)(x)$ is an absolutely convergent integral. Since $\mathcal{D}$ is linear, $\Psi \mapsto \mathrm{coef}(T; \mathcal{D}\Psi)(x)$ is a linear functional on the subspace $\mathcal{L}$; extend it (choose a complement of $\mathcal{L}$) to a linear functional Λ on all functions.

Step 3: the shear law of (2). Let $\Psi \in \mathcal{L}$, $z \in \mathrm{M}_p(E_\Sigma)$ and $M' = \tilde{z}$. Since $\tilde{z}$ is supported on Σ and $\tilde{z}^*$ on $\bar{\Sigma}$, and $\Sigma \cap \bar{\Sigma} = \emptyset$, the element $\nu_0 = z(M') \in \mathrm{G}(r)$ has components 1 at the infinite places and outside $\Sigma \cup \bar{\Sigma}$, and component $1 + (k_{\mathbb{A}})_\Sigma z(\mathrm{pr}_l)_\Sigma$ at Σ. Apply (1) with $U = (k_{\mathbb{A}})_\Sigma$, $C = z(\mathrm{pr}_l)_\Sigma$ (so $CU = z(\mathrm{pr}_l k_{\mathbb{A}})_\Sigma = 0$): the function $\Psi'(y) = \Psi(y + (x^{-1}k_{\mathbb{A}})_\Sigma z(\mathrm{pr}_l x)_\Sigma y) = \Psi_z(y)$ lies in $\mathcal{L}$ and $\mathcal{D}\Psi_z(y'x) = \mathcal{D}\Psi(y'z(M')^{-1}x)$. Taking $y' = z(M)$ and substituting $M \mapsto M + M'$ in the invariant measure,

$$\begin{aligned}\Lambda(\Psi_z) &= \int \psi_E(\mathrm{tr}(TM))\,\mathcal{D}\Psi\big(z(M - M')x\big)\,dM \\ &= \psi_E\big(\mathrm{tr}(TM')\big)\int \psi_E(\mathrm{tr}(TM))\,\mathcal{D}\Psi(z(M)x)\,dM = \psi_{E,f}\big(\mathrm{tr}(T\tilde{z})\big)\Lambda(\Psi),\end{aligned}$$

using $z(M)z(M')^{-1} = z(M - M')$, that ψ_E is a character, and that $\mathrm{tr}(T\tilde{z})$ is a finite adele, on which $\psi_E = \psi_{E,f}$.

Step 4: (3). Let $k \in \mathrm{GL}_m(\mathbb{A}_{E,f})$ have component g_w at $w \in \Sigma$, component $(\overline{g_w}^{\mathsf{T}})^{-1}$ at $\bar{w}$ (the conjugation identifying E_w with $E_{\bar{w}}$), and 1 elsewhere; this is well defined as $\Sigma \cap \bar{\Sigma} = \emptyset$. Since $(k^*)_u = \overline{k_{\bar{u}}}^{\mathsf{T}}$, we get $(k^*k)_w = g_w^{-1}g_w = 1$ and $(k^*k)_{\bar{w}} = \overline{g_w^{\mathsf{T}}(g_w^{\mathsf{T}})^{-1}} = 1$, so $k \in G_f(m)$, k has component 1 at the finite w' with $w' \notin \Sigma$, $\bar{w}' \notin \Sigma$, and $k_\Sigma = g$. By (c) of (XD11), $\Phi(y) = \Psi(gy)$ lies in $\mathcal{L}$ and $\mathcal{D}\Phi(y'(1,k)) = \upsilon_{\Sigma,1}(\det g)^{-\#b}\mathcal{D}\Psi(y') = 0$ for all y'; as $y' \mapsto y'(1,k)$ is onto, $\mathcal{D}\Phi = 0$. □

Lemma 6.3.15. *Let W be a space and $k \geq 1$ an integer with $\mathrm{rk}\,W = 2k + 2\,\mathrm{neg}(W)$, and let $\mu\colon \Phi \to \mathbb{Z}$ with $k \leq \mu(v)$ for every $v \in \Phi$. There are a Schwartz vector z for $(\mathbb{H} \oplus W, \mathbb{H} \oplus W)$ and a real number $C \neq 0$ such that*

$$r_{\mathbb{H}\oplus W}(\nu, z) = \frac{|I_{\mathbb{H}\oplus W}(z)(1)|^2}{L_*(\mathfrak{m}, \nu; \frac{1}{2}\,\mathrm{rk}\,W + 1)} = C$$

for every modulus $\mathfrak{m}$, every χ and every ν tame of weight μ for $(\mathfrak{m}, \chi)$.

Proof. **Step 1: for definite J the square of the ratio does not depend on ν.** Let J be a space of rank $2k$ with all signs $+1$, and v a Schwartz vector for $(\mathbb{H} \oplus J, \mathbb{H} \oplus J)$. We show that $r_{\mathbb{H}\oplus J}(\nu, v)^2$ is one number D for all ν.

Apply Lemma 6.3.3 to $(W, U, V) = (J, 0, J)$, with $z = v$ and y the vacuum vector of the one-dimensional Fock space of $\mathrm{Hom}(0, J)$; here $\mathrm{rk}\,V - \frac{1}{2}\,\mathrm{rk}\,W = k$. The pair $(\Phi_\chi, 1)$, with 1 the function on the single $2k \times 0$ matrix, is matched: Φ_χ vanishes off the invertible integral matrices and $\Phi_\chi(\kappa^{-1}) = \chi(\det\kappa)$. On the zero space the theta model is $\mathbb{C}$ with $v^0 = 1$ (the rank-0 case of (WD5)–(WD9) of Definition 3.2.12), its canonical theta functional is the identity of $\mathbb{C}$ ((WC1), and (WC3) with $k_V = 1$, of Definition 4.1.15), and $\omega_{1,1}(g, h)$ is ω of the only element of the trivial group of the empty sign vector, the identity ((WR4) of Definition 3.2.18); so the theta kernel of the characters $(1, 1)$ on the zero space is the value of the datum at 0. This gives $B \in \mathbb{R}$,

a Schwartz vector y' for (J,J) and for each ν a number c_1 with $B|c_1|^2 = L_*(k)\, r_{\mathbb{H}\oplus J}(\nu,v)$ and, writing $\theta = \theta^{J,J}_{\gamma,\gamma}(y'\otimes\Phi_\chi\otimes\Phi^\circ)$,

$$\int_{[\mathrm{U}(J)]} \theta(g,h)\,dh = c_1 \operatorname{vol}_0 \qquad (g\in \mathrm{U}(J)(\mathbb{A}_F)). \tag{a}$$

Here the lemma gives the identity with the characters $(\gamma,1)$ on the pair (J,J) and (γ,γ) on the pair $(0,J)$. By (WR4) of Definition 3.2.18, $\omega_{\alpha,\beta}(g,h) = \alpha(\det h)\beta(\det g)^{-1}\omega_{1,1}(g,h)$ and the theta functional is linear, so $\theta^{J,J}_{\gamma,\gamma}(\varphi)(g,h) = \gamma(\det g)^{-1}\theta^{J,J}_{\gamma,1}(\varphi)(g,h)$, while on the pair $(0,J)$ the kernel of the characters (γ,γ) at (h,g) is $\gamma(\det g)$ times the kernel of $(1,1)$, $\det h = 1$ on $\mathrm{U}(0)$; the latter kernel is the value of the datum at 0. Multiplying the identity of the lemma by $\gamma(\det g)^{-1}$ gives (a).

Apply Lemma 6.3.3 to $(W,U,V) = (J,J,0)$, with $z=v$ and $y=y'$; here $\operatorname{rk} V - \frac{1}{2}\operatorname{rk} W = -k$, and the pair $(1,\Phi_\chi)$ is matched. The Fock space of $\mathrm{Hom}(J,0)$ is one-dimensional, so the theta integral of $(J,0)$ at the vector x that the lemma provides is $\operatorname{vol}_0\xi$ for a number ξ that depends neither on ν nor on g. So there are $C_2\in\mathbb{R}$, $\xi\in\mathbb{C}$ and for each ν a number c_2 with $C_2|c_2|^2 = L_*(-k)\, r_{\mathbb{H}\oplus J}(\nu,v)$ and

$$\operatorname{vol}_0\xi = c_2 \int_{[\mathrm{U}(J)]} \theta(h,g)\,dh \qquad (g\in \mathrm{U}(J)(\mathbb{A}_F)). \tag{b}$$

Integrate (b) over $g\in[\mathrm{U}(J)]$. Since J is definite, $[\mathrm{U}(J)]$ is compact and the kernel is integrable on the product, so by Fubini and (a),

$$\operatorname{vol}_J \operatorname{vol}_0\xi = c_2 \int_{[\mathrm{U}(J)]}\int_{[\mathrm{U}(J)]} \theta(h,g)\,dg\,dh = c_2\, c_1 \operatorname{vol}_0 \operatorname{vol}_J,$$

so $\xi = c_1c_2$. As $L_*(k)L_*(-k)=1$,

$$r_{\mathbb{H}\oplus J}(\nu,v)^2 = \big(L_*(k)\,r\big)\big(L_*(-k)\,r\big) = B\,C_2\,|c_1c_2|^2 = B\,C_2\,|\xi|^2 =: D.$$

Step 2: the first floor. Let J be the standard definite space of rank $2k$. If no tame ν of weight μ exists, any z and $C=1$ serve. Otherwise fix one, ν_0; by Lemma 6.3.16 there is a Schwartz v with $C := r_{\mathbb{H}\oplus J}(\nu_0,v)\neq 0$. By Step 1, $r_{\mathbb{H}\oplus J}(\nu,v)^2 = r_{\mathbb{H}\oplus J}(\nu_0,v)^2$, and both ratios are ≥ 0 by (XD8); so $r_{\mathbb{H}\oplus J}(\nu,v) = C$ for every ν.

Step 3: climbing in the number of negative signs. Induction on $q=\mathrm{neg}(W)$. If $q=0$, then W and J have the same signs, so $\mathbb{H}\oplus W$ has the signs of $\mathbb{H}\oplus J$ (and Fock vectors depend only on the signs). By Lemma 6.3.66(3) there is a space Y with these signs and an adapted decomposition $Y = J\oplus\mathbb{H}$, and Lemma 6.3.66(2), applied to $Y_1 = \mathbb{H}\oplus J$ and to $Y_1 = \mathbb{H}\oplus W$, gives $r_{\mathbb{H}\oplus W}(\nu,v) = r_Y(\nu,v) = r_{\mathbb{H}\oplus J}(\nu,v) = C$ with the v of Step 2. If $q\geq 1$, let $W_0 = \mathbb{H}^{q-1}\oplus J$, so that $\operatorname{rk} W_0 = 2k+2(q-1)$ and $\mathrm{neg}(W_0) = q-1$; the induction hypothesis gives v_0 and $C_0\neq 0$ for $\mathbb{H}\oplus W_0$. By Lemma 6.3.66(3) there is a space Y with the signs of $\mathbb{H}\oplus W$ and an adapted decomposition $Y = (\mathbb{H}\oplus W_0)\oplus\mathbb{H}$ (ranks $2k+2q+2$ on both sides, negative signs $q+1$). By Lemma 6.3.66(1) there are v' and $C'>0$ with $r_Y(\nu,v') = C'\, r_{\mathbb{H}\oplus W_0}(\nu,v_0) = C'C_0$, and by Lemma 6.3.66(2), $r_{\mathbb{H}\oplus W}(\nu,v') = r_Y(\nu,v')$. Take $z=v'$ and $C = C'C_0\neq 0$. □

Lemma 6.3.16. *Let $k\geq 1$, let J be the standard definite space of rank $2k$, and let $\mu\colon\Phi\to\mathbb{Z}$ with $k\leq\mu(v)$ for every $v\in\Phi$. For every ν tame of weight μ for some $(\mathfrak{m},\chi)$ there is a Schwartz vector v for $(\mathbb{H}\oplus J,\mathbb{H}\oplus J)$ with $r_{\mathbb{H}\oplus J}(\nu,v)\neq 0$.*

Proof. Let m be the index set of J ($\#m=2k$), $Y=\mathbb{H}\oplus J$ with index set $n_Y = (\{o\}\sqcup\{o\})\sqcup m$, and let ν be tame of weight μ for $(\mathfrak{m},\chi)$, $\Sigma=\Sigma(\mathfrak{m})$. Since $\mu\geq k\geq 1$, $\mathfrak{c}=(\nu,\Sigma)$ is a split

tame character of weight μ (XD4). Its semilocal character is χ on $\mathcal{O}_\Sigma^\times$: for a unit t of $\mathcal{O}_\Sigma$, read as a finite idele, $t/\bar{t} \in \mathrm{U}(1)(\mathbb{A})$ has components t_w at $w \in \Sigma$, $\bar{t}_w^{-1}$ at $\bar{w} \notin \Sigma$ and 1 elsewhere, so $\nu(t/\bar{t}) = \prod_{w\in\Sigma} \psi_w(t_w \bmod w) = \chi(t)$. Hence the function Φ_χ of (XD8) is Ψ_1^{st} for $\chi_{\mathfrak{c}}$, read through the components at Σ, and $I_Y(z) = \theta_{\mathfrak{c}}^{\mathrm{lg}}(Y, Y; z, \Psi_1^{\mathrm{st}})$ for every Schwartz z (both are $I_{\gamma,\gamma}^{Y,Y}$ of the same datum, $\gamma = \nu^{-1}\upsilon^{k+1}$). Moreover $L_*(\Sigma, \nu; k+1) > 0$: it is a product of powers of norms and of $|L^\Sigma(j, \tilde{\nu})|^2$, $1 \le j \le k+1$, and $L^\Sigma(j, \tilde{\nu}) \neq 0$ for $j \ge 1$ by (3) of Theorem A.1.12, $\tilde{\nu}$ being ramified at the places of Σ (it is ψ_w on the units at $w \in \Sigma$, by the computation above) and the removed Euler factors being finite and non-zero for $\mathrm{Re}\, s \ge 1$.

Suppose, for a contradiction, that $r_Y(\nu, z) = 0$ for every Schwartz z, i.e. $\theta_{\mathfrak{c}}^{\mathrm{lg}}(Y, Y; z, \Psi_1^{\mathrm{st}})(1) = 0$.

Step 1: tautological splittings. For an index set p and a space V_0, the tautological splitting of $\mathbb{H}_p \oplus V_0$ over V_0 is base-adapted: its matrix of adelic frames is $\mathrm{diag}(x_{\mathbb{H}_p}^{-1}, 1)$, where $x_{\mathbb{H}_p}$ is unitary at the infinite places and, at the finite places, integral with determinant $(c\bar{c}_0 + cc_0)^{\#p} = c^{\#p}$, a unit ($c_0 + \bar{c}_0 = 1$). Let R_1 be the tautological splitting of Y over J along $\{o\}$ and R_m that of $X = \mathbb{H}_m \oplus Y$ over Y along m.

Step 2: the equal-rank side vanishes everywhere. By Proposition 6.3.17 ($\#n_Y$ even, R_1 base-adapted), $g \mapsto \theta_{\mathfrak{c}}^{\mathrm{lg}}(Y, Y; z, \Psi_1^{\mathrm{st}})(g)$ is constant, hence 0, for every Schwartz z.

Step 3: vanishing at the top. Let $V_X = J^- \oplus Y$ and $\sigma\colon m \sqcup (m \sqcup n_Y) \to (m \sqcup m) \sqcup n_Y$ the associativity bijection. Let $w \in \mathcal{F}_{X,V_X}$ be a Schwartz vector and v_1 the real vacuum of $\mathcal{F}_{X,J^-}$, a Schwartz vector. We apply Proposition 6.3.22 to X, $W_1 = Y$, $W = J$, $V' = J^-$, $D = R_m$, $D_1 = R_1$, σ, μ, $\phi_{\mathrm{lg}} = w$, $\phi_{\mathrm{sm}} = v_1$ and $\mathrm{pw} \equiv 1$. Condition (S) holds in its second form: the signs of J^- are all -1, Y has a sign $+1$, and $2\,\mathrm{neg}(Y) = 2 \le 2k+2 = \#n_Y$. Condition (P) holds for $(\mathfrak{c}, 0)$ with $(p_1, p_2) = (1, 1)$: the right side is 0, and the left side is 0 by Lemma 6.3.50 (for $W = Y$, $V = J^-$, with Step 2 as its hypothesis), applied to $w_{\hat{h}}(B)$, which is a Schwartz vector. Indeed its real slice factor is Schwartz by (1) of Lemma 6.2.43 and Lemma 6.3.53, and Ω preserves Schwartz vectors: by (WR5), $\omega_\varepsilon^\infty(u)c = \overline{\det u_-} \int c(t)\, \rho^\infty(ut) v_1\, d\nu(t)$, and $\mathrm{mom}_N(\rho^\infty(t)c') \le (1 + \|t\|)^N \mathrm{mom}_N(c')$, so $\mathrm{mom}_N(\omega_\varepsilon^\infty(u)c) \le |\det u_-|\, \mathrm{mom}_N(v_1)(1 + \|u\|)^N \mathrm{mom}_N(c)$, $\|u\|$ the operator norm for the maximum norm. So Proposition 6.3.22 gives c with $|c| = K \cdot 0 = 0$, and therefore $\theta_{\mathfrak{c}}^{\mathrm{lg}}(X, V_X; w, \Phi_{\sigma,z}^{\mathrm{st}}) = 0$ for every Schwartz w and every $z \in \mathrm{M}_{m\times((m\sqcup m)\sqcup n_Y)}(\mathcal{O}_\Sigma)$.

Step 4: descent to $(J, 0)$. Let V_0 be the zero space and $\sigma_0\colon m \sqcup \varnothing \to m$ the identity. Apply Lemma 6.3.54 with $W' = X$, $V' = V_X$, $W = J$, $V = V_0$, $b = m$, $\sigma' = \sigma$, σ_0, $j = 2k+1$ and the given k: the index set of X has $6k+2 = 2k+2j$ elements, $\mathrm{neg}(X) = 2k+1 = j + \mathrm{neg}(J)$, $\mathrm{neg}(V_X) = 2k+1 = j + \mathrm{neg}(V_0)$, $\#m = 2k$ is even, $-k \le 0 - k$ and $k \le \mu$. By Step 3 it gives $\theta_{\mathfrak{c}}^{\mathrm{lg}}(J, V_0; v, \Phi_{\sigma_0,y}^{\mathrm{st}}) = 0$ for every Schwartz $v \in \mathcal{F}_{J,V_0}$ and every integral y.

Step 5: rank 0. $\mathrm{Hom}(J, V_0)$ has the empty index set: its Fock model is $\mathbb{C}$ (functions on a point, ν the unit mass), its theta model is $\mathbb{C}$ with $v^0 = 1$, $\rho(0) = 1$ and Weil operators the identity, and its canonical theta functional is the identity; $\mathrm{U}(V_0)$ is the trivial group and $[\mathrm{U}(V_0)]$ is a point of mass 1. For $y = 1_m$, $\Phi_{\sigma_0,1}^{\mathrm{st}}$ is the function on the single (empty) matrix with value $\chi_{\mathfrak{c}}^{-1}(\det 1_m) = 1$; so $(\Phi_{\sigma_0,1}^{\mathrm{st}})^\sharp = 1 = \tau_0$ in $\mathcal{T}_\Sigma$ ((WR6), $y = 0$ being the only finite adelic vector of rank 0), $(\Phi_{\sigma_0,1}^{\mathrm{st}})^\sharp \otimes_\Sigma v^0 = v^0 = 1$ and $1 \otimes 1 = 1$. Hence $\theta_{\mathfrak{c}}^{\mathrm{lg}}(J, V_0; 1, \Phi_{\sigma_0,1}^{\mathrm{st}})(1) = 1$, contradicting Step 4 for the Schwartz vector $v = 1$. So some Schwartz z has $r_Y(\nu, z) \neq 0$. □

Proposition 6.3.17. *Let W_1 be a space with signs s on n, $\#n$ even, with a base-adapted splitting over a space W along a one-element set. Let $\mathfrak{c}$ be a split tame character and v a Schwartz vector for (W_1, W_1). Then the function $g \mapsto \theta_{\mathfrak{c}}^{\mathrm{lg}}(W_1, W_1; v, \Psi_1^{\mathrm{st}})(g)$ on $\mathrm{G}(s)$, 1 the identity $n \times n$ matrix, is constant.*

Proof. Write $\mathfrak{c} = (\nu, \Sigma)$, $\chi = \chi_{\mathfrak{c}}$, $D_1 = (k, l, i)$ for the splitting and a for the index set of W,

so $\#a = \#n - 2$ is even. Let $\varnothing$ be the empty index set, V_0 the zero space (index set $\varnothing$) and $\sigma\colon \varnothing \sqcup n \to n$ the identity. For the empty matrix $0 \in \mathrm{M}_{\varnothing\times n}(E_\Sigma)$ the standard row datum $\Phi^{\mathrm{st}}_{\sigma,0}$ is the function $x \mapsto \chi^{-1}(\det x)$ on integral $n\times n$ matrices over E_Σ and 0 elsewhere, which is Ψ^{st}_1. Put

$$F = \theta^{\mathrm{lg}}_{\mathfrak{c}}(W_1, W_1; v, \Psi^{\mathrm{st}}_1) = \theta^{\mathrm{lg}}_{\mathfrak{c}}(W_1, W_1; v, \Phi^{\mathrm{st}}_{\sigma,0}),$$

a function on $\mathrm{G}(s)$. We verify the hypotheses of Lemma 6.2.49 for F, the splitting $H = D_1$ and $S = \Sigma$.

Step 1: F is an equivariant lift. Apply Lemma 6.3.18 to $\mathfrak{c}$, the spaces W_1, $V = W_1$ (index set n) and $V' = V_0$ (index set $\varnothing$), the bijection σ, the Schwartz vector v, the real vacuum v_1 of $\mathcal{F}_{W_1,V_0}$ (the Fock vector of the polynomial 1, a Schwartz vector by (XD3)) and $c = 0$. It gives an equivariant lift $(\mathcal{L}, \mathcal{D})$ for $(\mathfrak{c}, W_1, \varnothing)$. Let Ψ_0 be the function with value 1 on the single matrix of $\mathrm{M}_{n\times\varnothing}(E_\Sigma)$. The pair $(\Phi^{\mathrm{st}}_{\sigma,0}, \Psi_0)$ is matched over $\mathcal{O}_\Sigma$ for (σ, χ): $\Phi^{\mathrm{st}}_{\sigma,0}(x) \neq 0$ forces x integral with $\det x \in \mathcal{O}_\Sigma^\times$, i.e. $x = (k^{-1})_n$ for $k = x^{-1} \in \mathrm{GL}_n(\mathcal{O}_\Sigma)$; Ψ_0 vanishes nowhere and every $k_\varnothing$ is the single matrix; and $\Phi^{\mathrm{st}}_{\sigma,0}(k^{-1}) = \chi(\det k^{-1})^{-1} = \chi(\det k) = \chi(\det k)\Psi_0(k_\varnothing)$. So $\Psi_0 \in \mathcal{L}$ and $\mathcal{D}\Psi_0 = F$.

Step 2: hypotheses (1) and (2). By (XD11)(b), F is continuous and left Γ_{W_1}-invariant. Let $g \in G_f(n)$ have component 1 at every finite place w with $w \notin \Sigma$ and $\bar w \notin \Sigma$. By (XD11)(c) the function $\Phi(y) = \Psi_0(g_\Sigma y)$, which is Ψ_0 (there is one matrix), lies in $\mathcal{L}$ and $F(x(1,g)) = \mathcal{D}\Phi(x(1,g)) = \upsilon_{\Sigma,1}(\det g_\Sigma)^{-\#\varnothing}\mathcal{D}\Psi_0(x) = F(x)$. This is (2) of Lemma 6.2.49 for $S = \Sigma$, which is nonempty ((XD4)).

Step 3: hypothesis (3). Apply Proposition 6.2.2 to $W' = V' = W_1$, $W = V = W$, the splittings $A = X = D_1$, the index set $\varnothing$ with the identities $\sigma_0\colon \varnothing \sqcup a \to a$ and $\sigma\colon \varnothing \sqcup n \to n$, and the weight μ of $\mathfrak{c}$: $\#a$ is even and $e = \#a - \#a/2 = \#a/2 \geq 0$, so $z_e = \#a/2 + 1$, the sign is $+$, and $\#r = \#a$ equals the rank of W, so the identity holds at every Levi point. It gives γ with

$$\mathrm{CT}^{\mathrm{pl}}_{D_1,\sigma}(v,0)(u,g) = \gamma\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W, W; \mathrm{Cone}^{+}_{D_1,D_1,\mu}(z_e; v), \Phi^{\mathrm{st}}_{\sigma_0,0})(g) \qquad (u \in \mathbb{A}_E^\times,\ g \in \mathrm{U}(W)(\mathbb{A}_F)),$$

0 the empty matrix in $\mathrm{M}_{\varnothing\times a}(\mathcal{O}_\Sigma)$. The right side does not depend on u. The plane datum at 0 is formed with $\Phi^{\mathrm{st}}_{\sigma,0(\mathrm{pr}_i)_\Sigma} = \Phi^{\mathrm{st}}_{\sigma,0}$ (the matrix $0(\mathrm{pr}_i)_\Sigma$ is empty), so by (XD10)

$$\mathrm{CT}^{\mathrm{pl}}_{D_1,\sigma}(v,0)(u,g) = \int F\big(\mathrm{rad}^{\mathrm{ln}}_{D_1}(q_1,q_2)\,\mathrm{lev}_{D_1}(u,g)\big)\,dq,$$

the integral over the compact quotient with its probability measure. Now assume $F(\mathrm{rad}_{D_1}(c,b)g') = F(g')$ for all adelic c, b and all g'. As $\mathrm{rad}^{\mathrm{ln}}_{D_1}(q_1,q_2) = \mathrm{rad}_{D_1}(x_W^{-1}q_1, 2q_2)$ ((MX10)), the integrand is constant and

$$\mathrm{CT}^{\mathrm{pl}}_{D_1,\sigma}(v,0)(u,g) = F(\mathrm{lev}_{D_1}(u,g)).$$

Hence for $t \in \mathbb{A}_E^\times$,

$$F(\mathrm{lev}_{D_1}(t,1)) = \mathrm{CT}^{\mathrm{pl}}_{D_1,\sigma}(v,0)(t,1) = \mathrm{CT}^{\mathrm{pl}}_{D_1,\sigma}(v,0)(1,1) = F(\mathrm{lev}_{D_1}(1,1)) = F(1),$$

because $\mathrm{lev}_{D_1}(1,1) = k_{\mathbb{A}}\mathrm{pr}_k + i_{\mathbb{A}}\mathrm{pr}_i + l_{\mathbb{A}}\mathrm{pr}_l = 1$ ((MX1), (MX2)). This is (3) of Lemma 6.2.49.

Step 4. By Lemma 6.2.49, F is constant on $\mathrm{G}(s)$. □

Lemma 6.3.18. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character with semilocal character $\chi_{\mathfrak{c}}$, let W, V, V' be spaces with signs s on n and index sets a, b, $\sigma\colon b \sqcup a \to n$ a bijection, $\#n$ even, v, v' Schwartz vectors for (W,V), (W,V'), and $c \in \mathbb{C}$. There is an equivariant lift $(\mathcal{L},\mathcal{D})$ for $(\mathfrak{c}, W, b)$ with the following property: whenever Φ is a function on $\mathrm{M}_{a\times n}(E_\Sigma)$ and Ψ a locally constant*

function on $\mathrm{M}_{n\times b}(E_\Sigma)$ *that are matched over* $\mathcal{O}_\Sigma$ *for* $(\sigma,\chi_{\mathfrak{c}})$ *(*Φ *vanishes off the matrices* $(k^{-1})_a$*,* Ψ *off the* k_b*,* $k\in\mathrm{GL}_n(\mathcal{O}_\Sigma)$*, and* $\Phi((k^{-1})_a)=\chi_{\mathfrak{c}}(\det k)\Psi(k_b)$*), then* $\Psi\in\mathcal{L}$ *and*

$$\mathcal{D}\Psi=\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;v,\Phi)-c\,\theta^{\mathrm{sm}}_{\mathfrak{c}}(W,V';v',\Psi).$$

Proof. Write θ^{lg}, θ^{sm} as in Lemma 6.3.20, and let $\mathcal{B}_a$, $\mathcal{B}_b$ be the spaces of locally constant, compactly supported functions on $\mathrm{M}_{a\times n}(E_\Sigma)$ vanishing off the right invertible matrices, resp. on $\mathrm{M}_{n\times b}(E_\Sigma)$ vanishing off the left invertible ones. By Lemma 6.3.20, θ^{lg} and θ^{sm} are defined on $\mathcal{B}_a$, $\mathcal{B}_b$, linear (they are theta integrals of data depending linearly on the function, (2) of Lemma 6.3.21), and their values are continuous left Γ_W-invariant functions on $\mathrm{G}(s)$.

The space $\mathcal{L}$ *and the map* T*.* Let $\chi=\tilde\nu_{\Sigma,\#n}$, a locally constant homomorphism $E_\Sigma^\times\to\mathbb{C}^\times$ ($\tilde\nu$ is continuous on the totally disconnected group $E_\Sigma^\times$, and so is $|\cdot|$). Apply Lemma 6.3.19 to Σ, χ, a,b,n, σ; it gives a $\mathrm{GL}_n(E_\Sigma)$-stable subspace $\mathcal{Q}\subseteq\mathcal{B}_a\times\mathcal{B}_b$; let $\mathcal{L}$ be the space of second members. By (1) of Lemma 6.3.20, $\Lambda=\theta^{\mathrm{lg}}\colon\mathcal{B}_a\to\{\text{functions on }\mathrm{G}(s)\}$ satisfies $\chi(\det h)\Lambda(\Phi(h\,\cdot))=\Lambda(\Phi)$ for $h\in\mathrm{GL}_a(E_\Sigma)$, so (2) of Lemma 6.3.19 gives a linear $T\colon\mathcal{L}\to\{\text{functions on }\mathrm{G}(s)\}$ with $T\Psi=\theta^{\mathrm{lg}}(\Phi)$ for $(\Phi,\Psi)\in\mathcal{Q}$. Put $\mathcal{D}_0=T-c\,\theta^{\mathrm{sm}}$ on $\mathcal{L}\subseteq\mathcal{B}_b$, and let $\mathcal{D}$ be any linear extension of $\mathcal{D}_0$ to all functions on $\mathrm{M}_{n\times b}(E_\Sigma)$ (extend by 0 on a complement of $\mathcal{L}$).

The four conditions of (XD11). (a) $\mathcal{L}\subseteq\mathcal{B}_b$. (b) For $\Psi\in\mathcal{L}$, $\mathcal{D}\Psi=\theta^{\mathrm{lg}}(\Phi)-c\,\theta^{\mathrm{sm}}(\Psi)$ for some Φ with $(\Phi,\Psi)\in\mathcal{Q}$, continuous and left Γ_W-invariant. (c) Let $g\in G_f(n)$ have components 1 off $\Sigma\cup\bar\Sigma$, $t=\det g_\Sigma$, and $(\Phi,\Psi)\in\mathcal{Q}$. By stability, $g_\Sigma^{-1}\cdot(\Phi,\Psi)=\big(\chi(t)^{-1}\Phi(\cdot\,g_\Sigma^{-1}),\Psi(g_\Sigma\,\cdot)\big)\in\mathcal{Q}$, so $\Psi(g_\Sigma\,\cdot)\in\mathcal{L}$, and by (2) of Lemma 6.3.20

$$\begin{aligned}T\big(\Psi(g_\Sigma\,\cdot)\big)(x(1,g))&=\chi(t)^{-1}\tilde\nu_{\Sigma,\#a}(t)\upsilon(t)^{\#a-\#n}\,T\Psi(x)\\&=\upsilon(t)^{-\#b}|t|^{-\#b/2}\,T\Psi(x)=\upsilon_{\Sigma,1}(t)^{-\#b}\,T\Psi(x),\end{aligned}$$

using $\#n-\#a=\#b$; the small side obeys the same law by (2) of Lemma 6.3.20, hence so does $\mathcal{D}$. (d) is (3) of Lemma 6.3.20.

Matched functions. Let (Φ,Ψ) be matched over $\mathcal{O}_\Sigma$ for $(\sigma,\chi_{\mathfrak{c}})$, Ψ locally constant. For $k\in\mathrm{GL}_n(\mathcal{O}_\Sigma)$, $\det k$ is a unit, so $\chi(\det k)=\tilde\nu(\det k)=\chi_{\mathfrak{c}}(\det k)$; thus (Φ,Ψ) is matched for (σ,χ) in the sense of (1) of Lemma 6.3.19, which gives $(\Phi,\Psi)\in\mathcal{Q}$. So $\Psi\in\mathcal{L}$ and $\mathcal{D}\Psi=T\Psi-c\,\theta^{\mathrm{sm}}(\Psi)=\theta^{\mathrm{lg}}(\Phi)-c\,\theta^{\mathrm{sm}}(\Psi)$. □

Lemma 6.3.19. *Let* Σ *be a finite set of finite places of* E*,* $E_\Sigma=\prod_{w\in\Sigma}E_w$ *and* $\mathcal{O}_\Sigma=\prod_{w\in\Sigma}\mathcal{O}_{E,w}$*, and let* $\chi\colon E_\Sigma^\times\to\mathbb{C}^\times$ *be a locally constant homomorphism. Let* a*,* b*,* n *be finite sets and* $\sigma\colon b\sqcup a\to n$ *a bijection. For* $k\in\mathrm{GL}_n(E_\Sigma)$ *let* k_b *be the* $n\times b$ *matrix formed by the columns* $\sigma(b)$ *of* k*, and* $(k^{-1})_a$ *the* $a\times n$ *matrix formed by the rows* $\sigma(a)$ *of* k^{-1}*. Let* $\mathcal{B}_a$ *be the space of locally constant, compactly supported functions on* $\mathrm{M}_{a\times n}(E_\Sigma)$ *that vanish at every matrix without a right inverse, and* $\mathcal{B}_b$ *the space of locally constant, compactly supported functions on* $\mathrm{M}_{n\times b}(E_\Sigma)$ *that vanish at every matrix without a left inverse. The group* $\mathrm{GL}_n(E_\Sigma)$ *acts on* $\mathcal{B}_a\times\mathcal{B}_b$ *by*

$$g\cdot(\Phi,\Psi)=\big(x\mapsto\chi(\det g)\,\Phi(xg),\ y\mapsto\Psi(g^{-1}y)\big).$$

Then there is a $\mathrm{GL}_n(E_\Sigma)$*-stable linear subspace* $\mathcal{Q}\subseteq\mathcal{B}_a\times\mathcal{B}_b$ *with the following two properties, where* $\mathcal{M}$ *denotes the space of all* Ψ *such that* $(\Phi,\Psi)\in\mathcal{Q}$ *for some* Φ*.*

1. *Let* Φ *be a function on* $\mathrm{M}_{a\times n}(E_\Sigma)$ *and* Ψ *a locally constant function on* $\mathrm{M}_{n\times b}(E_\Sigma)$ *that are matched over* $\mathcal{O}_\Sigma$ *for* (σ,χ)*:* Φ *vanishes off the matrices* $(k^{-1})_a$ *and* Ψ *off the matrices* k_b *with* $k\in\mathrm{GL}_n(\mathcal{O}_\Sigma)$*, and* $\Phi((k^{-1})_a)=\chi(\det k)\,\Psi(k_b)$ *for every* $k\in\mathrm{GL}_n(\mathcal{O}_\Sigma)$*. Then* $(\Phi,\Psi)\in\mathcal{Q}$*.*

2. Let X be a complex vector space and $\Lambda\colon \mathcal{B}_a \to X$ a linear map with

$$\chi(\det h)\,\Lambda(x \mapsto \Phi(hx)) = \Lambda(\Phi) \qquad (h \in \mathrm{GL}_a(E_\Sigma),\ \Phi \in \mathcal{B}_a).$$

Then there is a linear map $T\colon \mathcal{M} \to X$ with $T(\Psi) = \Lambda(\Phi)$ for every $(\Phi, \Psi) \in \mathcal{Q}$.

Proof. For a finite set c, an injection $\iota\colon c \to n$ and $m \in \mathrm{GL}_n(E_\Sigma)$ let m_ι be the $n \times c$ matrix of the columns $\iota(c)$ of m, and let $\mathrm{Prim}(\iota)$ be the set of the m_ι with $m \in \mathrm{GL}_n(\mathcal{O})$. For $u \in \mathrm{GL}_b(E_\Sigma)$ and $v \in \mathrm{GL}_a(E_\Sigma)$ let $d(u,v) \in \mathrm{GL}_n(E_\Sigma)$ act as u on the coordinates $\sigma(b)$ and as v on the coordinates $\sigma(a)$; then $(m\,d(u,v))_b = m_b u$, $((m\,d(u,v))^{-1})_a = v^{-1}(m^{-1})_a$ and $\det(m\,d(u,v)) = \det m \det u \det v$.

Step 1: primitive multiples of a matrix with a left inverse. We show: (1a) $\mathrm{Prim}(\iota)$ is compact and open; (1b) if $y \in \mathrm{M}_{n\times c}(E_\Sigma)$ has a left inverse, there is $g_y \in \mathrm{GL}_c(E_\Sigma)$ with

$$\{g \in \mathrm{GL}_c(E_\Sigma) : yg \in \mathrm{Prim}(\iota)\} = g_y\,\mathrm{GL}_c(\mathcal{O});$$

(1c) the coset $g_y\mathrm{GL}_c(\mathcal{O})$ is a locally constant function of y; (1d) if C is a compact set of matrices with a left inverse and y has a left inverse, then $\{g : yg \in C\}$ lies in a compact subset of $\mathrm{GL}_c(E_\Sigma)$.

(1a) $\mathrm{Prim}(\iota)$ is the image of the compact group $\mathrm{GL}_n(\mathcal{O})$ under a continuous map. If $z = m_\iota$ with $m \in \mathrm{GL}_n(\mathcal{O})$ and $\delta \in \varpi\mathrm{M}_{n\times c}(\mathcal{O})$, the matrix m' obtained from m by adding δ to the columns $\iota(c)$ satisfies $\det m' \in \det m + \varpi\mathcal{O} \subseteq \mathcal{O}^\times$, so $m' \in \mathrm{GL}_n(\mathcal{O})$ and $z + \delta \in \mathrm{Prim}(\iota)$.

(1b) The question is componentwise, so let $\Sigma = \{w\}$ and $\mathcal{O} = \mathcal{O}_{E,w}$. Put $N = yE_w^c \cap \mathcal{O}^n$. The module $\mathcal{O}^n/N$ is finitely generated and torsion-free (if $tv \in N$ with $t \neq 0$ then $v \in yE_w^c$), hence free, so $\mathcal{O}^n = N \oplus N'$ with N, N' free. Every vector of yE_w^c has a nonzero multiple in N, and y is injective, so N has rank $|c|$. Let $m \in \mathrm{GL}_n(\mathcal{O})$ have as columns $\iota(c)$ a basis of N and as its other columns a basis of N'. The columns of m_ι lie in yE_w^c, so $m_\iota = yg_y$ for a matrix g_y, unique as y is injective, and invertible as m_ι has rank $|c|$. If also $yg = m'_\iota$ with $m' \in \mathrm{GL}_n(\mathcal{O})$, put $u = g_y^{-1}g$, so that $m'_\iota = m_\iota u$. The rows $\iota(c)$ of m^{-1} form an integral matrix L with $Lm_\iota = 1$, so $u = Lm'_\iota$ is integral; exchanging m and m', u^{-1} is integral, and $u \in \mathrm{GL}_c(\mathcal{O})$. Conversely, for $u \in \mathrm{GL}_c(\mathcal{O})$ the columns $\iota(c)$ of the product of m with the element of $\mathrm{GL}_n(\mathcal{O})$ that acts as u on the coordinates $\iota(c)$ and as the identity on the others are $m_\iota u = yg_y u$.

(1c) If $yg_y \in \mathrm{Prim}(\iota)$ and y' is close to y, then $y'g_y$ is close to yg_y, hence lies in the open set $\mathrm{Prim}(\iota)$; in particular y' has a left inverse, and $g_{y'}\mathrm{GL}_c(\mathcal{O}) = g_y\mathrm{GL}_c(\mathcal{O})$ by (1b).

(1d) By (1c) and compactness, $z \mapsto g_z\mathrm{GL}_c(\mathcal{O})$ takes finitely many values $g_1\mathrm{GL}_c(\mathcal{O})$, …, $g_r\mathrm{GL}_c(\mathcal{O})$ on C. If $yg \in C$, then $(yg)g_{yg} \in \mathrm{Prim}(\iota)$, so $gg_{yg} \in g_y\mathrm{GL}_c(\mathcal{O})$ by (1b), and g lies in one of the compact sets $g_y\mathrm{GL}_c(\mathcal{O})g_j^{-1}$.

Step 2: the transposed statements for matrices with a right inverse. Let $\mathrm{Prim}_b = \mathrm{Prim}(\sigma|_b)$, the set of the k_b with $k \in \mathrm{GL}_n(\mathcal{O})$, and let Prim_a be the set of the $(k^{-1})_a$ with $k \in \mathrm{GL}_n(\mathcal{O})$, that is, of the rows $\sigma(a)$ of elements of $\mathrm{GL}_n(\mathcal{O})$. Since $m \mapsto m^T$ permutes $\mathrm{GL}_n(\mathcal{O})$, Prim_a consists of the transposes of the elements of $\mathrm{Prim}(\sigma|_a)$, and an $a \times n$ matrix x has a right inverse if and only if x^T has a left inverse. For such x, $hx \in \mathrm{Prim}_a$ if and only if $x^T h^T \in \mathrm{Prim}(\sigma|_a)$, so Step 1 for x^T gives: Prim_a is compact and open; $\{h \in \mathrm{GL}_a(E_\Sigma) : hx \in \mathrm{Prim}_a\} = \mathrm{GL}_a(\mathcal{O})h_x$ with $h_x = g_{x^T}^T$; this coset is a locally constant function of x; and for a compact set C' of matrices with a right inverse, $\{h : hx \in C'\}$ lies in a compact set. Taking $v = 1$ or $u = 1$ in $d(u,v)$ shows that Prim_b is stable under right multiplication by $\mathrm{GL}_b(\mathcal{O})$ and Prim_a under left multiplication by $\mathrm{GL}_a(\mathcal{O})$. The matrix $(k^{-1})_a$ has the right inverse formed by the columns $\sigma(a)$ of k, and k_b has the left inverse formed by the rows $\sigma(b)$ of k^{-1}.

Step 3: the averages $\Psi^\flat$, $\Phi^\flat$. For $\Psi \in \mathcal{B}_b$, y with a left inverse, $\Phi \in \mathcal{B}_a$ and x with a right inverse put

$$\Psi^\flat(y) = \int_{\mathrm{GL}_b(E_\Sigma)} \chi(\det g)\, \Psi(yg)\, dg, \qquad \Phi^\flat(x) = \int_{\mathrm{GL}_a(E_\Sigma)} \chi(\det h)\, \Phi(hx)\, dh.$$

The integrands are continuous, and by (1d) applied to the compact set $C = \{\Psi \neq 0\}$, which consists of matrices with a left inverse, the first vanishes off a compact set; likewise the second by Step 2. So both integrals converge absolutely. Invariance of the Haar measures gives, for $g' \in \mathrm{GL}_b(E_\Sigma)$, $h' \in \mathrm{GL}_a(E_\Sigma)$ and $g \in \mathrm{GL}_n(E_\Sigma)$, with $(\Phi', \Psi') = g \cdot (\Phi, \Psi)$:

$$\begin{aligned} \Psi^\flat(yg') &= \chi(\det g')^{-1}\Psi^\flat(y), & \Phi^\flat(h'x) &= \chi(\det h')^{-1}\Phi^\flat(x), \\ \Psi'^\flat(y) &= \Psi^\flat(g^{-1}y), & \Phi'^\flat(x) &= \chi(\det g)\, \Phi^\flat(xg). \end{aligned} \tag{a}$$

Step 4: the space $\mathcal{Q}$ and its stability. Let $\mathcal{Q}$ be the set of $(\Phi, \Psi) \in \mathcal{B}_a \times \mathcal{B}_b$ with

$$\Phi^\flat((k^{-1})_a) = \chi(\det k)\, \Psi^\flat(k_b) \qquad (k \in \mathrm{GL}_n(E_\Sigma)). \tag{b}$$

The conditions are linear, so $\mathcal{Q}$ is a linear subspace. The spaces $\mathcal{B}_a$, $\mathcal{B}_b$ are stable under the action: $x \mapsto \Phi(xg)$ is locally constant, its support is the image of that of Φ under $x \mapsto xg^{-1}$, and xg^{-1} has a right inverse when x has one; similarly for Ψ. Let $(\Phi, \Psi) \in \mathcal{Q}$, $g \in \mathrm{GL}_n(E_\Sigma)$ and $(\Phi', \Psi') = g \cdot (\Phi, \Psi)$. Since $(k^{-1})_a g = ((g^{-1}k)^{-1})_a$ and $(g^{-1}k)_b = g^{-1}k_b$, (a) and (b) for $g^{-1}k$ give

$$\Phi'^\flat((k^{-1})_a) = \chi(\det g)\, \Phi^\flat(((g^{-1}k)^{-1})_a) = \chi(\det g)\chi(\det g^{-1}k)\, \Psi^\flat(g^{-1}k_b) = \chi(\det k)\, \Psi'^\flat(k_b),$$

so $(\Phi', \Psi') \in \mathcal{Q}$.

Step 5: matched pairs lie in $\mathcal{Q}$ (property 1). Let (Φ, Ψ) be as in property 1. For $m \in \mathrm{GL}_n(\mathcal{O})$, $u \in \mathrm{GL}_b(\mathcal{O})$ and $v \in \mathrm{GL}_a(\mathcal{O})$, the matching identity at $m\, d(u, v)$ and at m gives $\Phi(v^{-1}(m^{-1})_a) = \chi(\det u)\chi(\det v)\, \chi(\det m)\Psi(m_b u)$ and $\Phi((m^{-1})_a) = \chi(\det m)\Psi(m_b)$. With $u = 1$, and with $v = 1$, and since both functions vanish off the stable sets Prim_a, Prim_b,

$$\chi(\det h)\, \Phi(hz) = \Phi(z), \qquad \chi(\det u)\, \Psi(yu) = \Psi(y) \qquad (h \in \mathrm{GL}_a(\mathcal{O}),\ u \in \mathrm{GL}_b(\mathcal{O})) \tag{c}$$

for all z and y.

$\Psi \in \mathcal{B}_b$: it is locally constant, and it vanishes off the compact set Prim_b, whose elements have a left inverse. $\Phi \in \mathcal{B}_a$: it vanishes off the compact open set Prim_a, whose elements have a right inverse, so it suffices to show that Φ is constant near every $z_0 = (k_0^{-1})_a$ with $k_0 \in \mathrm{GL}_n(\mathcal{O})$. Choose $N \geq 1$ with $\chi(1 + \varpi^N\mathcal{O}) = 1$ and Ψ constant on $(k_0)_b + \varpi^N \mathrm{M}_{n\times b}(\mathcal{O})$. For $\varepsilon \in \varpi^N \mathrm{M}_{a\times n}(\mathcal{O})$ let m be k_0^{-1} with ε added to its rows $\sigma(a)$. Then $m \equiv k_0^{-1} \bmod \varpi^N$, so $m \in \mathrm{GL}_n(\mathcal{O})$, $k = m^{-1}$ satisfies $k - k_0 = k(k_0^{-1} - m)k_0 \in \varpi^N \mathrm{M}_n(\mathcal{O})$, $(k^{-1})_a = z_0 + \varepsilon$, and $\det(k_0^{-1}k) \in 1 + \varpi^N\mathcal{O}$. Hence

$$\Phi(z_0 + \varepsilon) = \chi(\det k)\Psi(k_b) = \chi(\det k_0)\, \chi(\det k_0^{-1}k)\, \Psi(k_b) = \chi(\det k_0)\Psi((k_0)_b) = \Phi(z_0).$$

Identity (b): let $k \in \mathrm{GL}_n(E_\Sigma)$, $x = (k^{-1})_a$, $y = k_b$. By Step 2 and (1b) choose h_0, g_0 with $\{h : hx \in \mathrm{Prim}_a\} = \mathrm{GL}_a(\mathcal{O})h_0$ and $\{g : yg \in \mathrm{Prim}_b\} = g_0\mathrm{GL}_b(\mathcal{O})$. As Φ vanishes off Prim_a, right invariance of the Haar measure and (c) give

$$\Phi^\flat(x) = \int_{\mathrm{GL}_a(\mathcal{O})} \chi(\det hh_0)\, \Phi(hh_0x)\, dh = \chi(\det h_0)\, \Phi(h_0x),$$

and likewise $\Psi^\flat(y) = \chi(\det g_0)\Psi(yg_0)$. Put $z = h_0 x \in \mathrm{Prim}_a$ and $y_0 = yg_0 \in \mathrm{Prim}_b$, and choose $m \in \mathrm{GL}_n(\mathcal{O})$ with $(m^{-1})_a = z$. Then $zm_b = 0$ and $zy_0 = h_0(k^{-1})_a k_b g_0 = 0$. The kernel of z has dimension $|b|$ (componentwise), as z has rank $|a|$, and m_b, y_0 have rank $|b|$, so $y_0 = m_b u$ with $u \in \mathrm{GL}_b(E_\Sigma)$. With L the rows $\sigma(b)$ of m^{-1}, $u = Ly_0$ is integral; writing $y_0 = m'_b$ with $m' \in \mathrm{GL}_n(\mathcal{O})$ and L' the rows $\sigma(b)$ of m'^{-1}, $u^{-1} = L'm_b$ is integral; so $u \in \mathrm{GL}_b(\mathcal{O})$. Let $k' = m\,d(u,1)$ and $k'' = k\,d(g_0, h_0^{-1})$. Then $k'_b = m_b u = y_0 = k''_b$ and $(k'^{-1})_a = (m^{-1})_a = z = h_0(k^{-1})_a = (k''^{-1})_a$. So $p = k'^{-1}k''$ has the rows $\sigma(a)$ and the columns $\sigma(b)$ of the identity matrix; in the coordinates ordered by $(\sigma(b), \sigma(a))$ it is block upper unitriangular, and $\det p = 1$. Hence $\det m \det u = \det k' = \det k'' = \det k \det g_0 \det h_0^{-1}$, and by the matching identity at m and by (c),

$$\begin{aligned}\Phi^\flat(x) &= \chi(\det h_0)\chi(\det m)\Psi(m_b) = \chi(\det h_0)\chi(\det m \det u)\Psi(m_b u)\\ &= \chi(\det k)\chi(\det g_0)\Psi(yg_0) = \chi(\det k)\,\Psi^\flat(y).\end{aligned}$$

So $(\Phi, \Psi) \in \mathcal{Q}$.

Step 6: an invariant Λ sees only $\Phi^\flat$ on Prim_a. Let Λ be as in property 2 and $\Phi \in \mathcal{B}_a$. We show that $\mathbf{1}_{\mathrm{Prim}_a}\Phi^\flat \in \mathcal{B}_a$ and

$$\Lambda(\Phi) = \Lambda(\mathbf{1}_{\mathrm{Prim}_a}\Phi^\flat). \tag{d}$$

The set $U = \{\Phi \neq 0\}$ is compact and consists of matrices with a right inverse. By Step 2, $x \mapsto \mathrm{GL}_a(\mathcal{O})h_x$ is locally constant, so it takes finitely many distinct values $\mathrm{GL}_a(\mathcal{O})h_1, \dots, \mathrm{GL}_a(\mathcal{O})h_r$ on U, and for $x \in U$ we have $h_j x \in \mathrm{Prim}_a$ for exactly one j. Let $\Phi_j(x) = \Phi(x)$ if $h_j x \in \mathrm{Prim}_a$ and $\Phi_j(x) = 0$ otherwise; as Prim_a is open and closed, $\Phi_j \in \mathcal{B}_a$, and $\Phi = \sum_j \Phi_j$. The function $\Phi'_j(x) = \Phi_j(h_j^{-1}x)$ lies in $\mathcal{B}_a$ and vanishes off Prim_a, and the invariance of Λ at $h = h_j$ gives $\chi(\det h_j)\Lambda(\Phi_j) = \Lambda(\Phi'_j)$. So $\Lambda(\Phi) = \Lambda(\Phi^\circ)$ with $\Phi^\circ = \sum_j \chi(\det h_j)^{-1}\Phi'_j$, which vanishes off Prim_a.

Choose $N \geq 1$ such that Φ° is invariant under translation by $\varpi^N \mathrm{M}_{a\times n}(\mathcal{O})$, and let H be the group of $h \in \mathrm{GL}_a(\mathcal{O})$ with $h \equiv 1 \bmod \varpi^N$ and $\chi(\det h) = 1$, an open normal subgroup of $\mathrm{GL}_a(\mathcal{O})$ (the intersection of a principal congruence subgroup and the kernel of the locally constant homomorphism $\chi \circ \det$). For $h \in H$ we have $\Phi^\circ(hx) = \Phi^\circ(x)$ for all x: for integral x, $hx - x = (h-1)x \in \varpi^N \mathrm{M}_{a\times n}(\mathcal{O})$; for non-integral x, hx is not integral and both values vanish, Prim_a consisting of integral matrices. Let $h'_1, \dots, h'_s$ represent $\mathrm{GL}_a(\mathcal{O})/H$ and $\Phi^{\circ\circ}(x) = s^{-1}\sum_i \chi(\det h'_i)\Phi^\circ(h'_i x)$. By the invariance of Λ, $\Lambda(\Phi^{\circ\circ}) = \Lambda(\Phi^\circ) = \Lambda(\Phi)$. Since H is normal, $h \mapsto \chi(\det h)\Phi^\circ(hx)$ is constant on the cosets $h'_i H$, each of volume $1/s$, so

$$\Phi^{\circ\circ}(x) = \int_{\mathrm{GL}_a(\mathcal{O})} \chi(\det h)\,\Phi^\circ(hx)\,dh.$$

Let $x \in \mathrm{Prim}_a$; then $\{h' : h'x \in \mathrm{Prim}_a\} = \mathrm{GL}_a(\mathcal{O})$. By left invariance of the Haar measure,

$$\Phi^{\circ\circ}(x) = \sum_j \chi(\det h_j)^{-1} \int_{\mathrm{GL}_a(\mathcal{O})} \chi(\det h)\Phi_j(h_j^{-1}hx)\,dh = \sum_j \int_{h_j^{-1}\mathrm{GL}_a(\mathcal{O})} \chi(\det h')\Phi_j(h'x)\,dh'.$$

If $\Phi_j(h'x) \neq 0$ then $h_j h'x \in \mathrm{Prim}_a$, so $h_j h' \in \mathrm{GL}_a(\mathcal{O})$ and $h' \in h_j^{-1}\mathrm{GL}_a(\mathcal{O})$. Hence the j-th integral is the integral over all of $\mathrm{GL}_a(E_\Sigma)$, and $\Phi^{\circ\circ}(x) = \sum_j \Phi_j^\flat(x) = \Phi^\flat(x)$. For $x \notin \mathrm{Prim}_a$, $hx \notin \mathrm{Prim}_a$ for $h \in \mathrm{GL}_a(\mathcal{O})$, so $\Phi^{\circ\circ}(x) = 0$. Thus $\Phi^{\circ\circ} = \mathbf{1}_{\mathrm{Prim}_a}\Phi^\flat$, which lies in $\mathcal{B}_a$, and (d) holds.

Step 7: the factorisation (property 2). Let (Φ, Ψ) and (Φ', Ψ) lie in $\mathcal{Q}$. Every $x \in \mathrm{Prim}_a$ is $(k^{-1})_a$ for some $k \in \mathrm{GL}_n(\mathcal{O})$, and (b) for both pairs gives $\Phi^\flat(x) = \chi(\det k)\Psi^\flat(k_b) = \Phi'^\flat(x)$.

So $\mathbf{1}_{\mathrm{Prim}_a}\Phi^\flat = \mathbf{1}_{\mathrm{Prim}_a}\Phi'^\flat$, and $\Lambda(\Phi) = \Lambda(\Phi')$ by (d). Hence $T(\Psi) = \Lambda(\Phi)$, for any Φ with $(\Phi,\Psi) \in \mathcal{Q}$, is a well-defined map on $\mathcal{M}$; it is linear because $\mathcal{Q}$ is a linear subspace and Λ is linear. Together with Steps 4 and 5 this proves the lemma. □

Lemma 6.3.20. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character and W, V, V' spaces with index sets n, a, b, $\#n$ even. Let v, v' be Schwartz vectors for (W,V), (W,V'); let Φ be a locally constant, compactly supported function on $\mathrm{M}_{a\times n}(E_\Sigma)$ vanishing off the right invertible matrices, and Ψ one on $\mathrm{M}_{n\times b}(E_\Sigma)$ vanishing off the left invertible matrices. Write $\theta^{\mathrm{lg}}(\Phi) = \theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;v,\Phi)$ and $\theta^{\mathrm{sm}}(\Psi) = \theta^{\mathrm{sm}}_{\mathfrak{c}}(W,V';v',\Psi)$; these theta integrals converge absolutely and are continuous and left Γ_W-invariant. Then for $x \in \mathrm{G}(s)$:*

1. *for $h \in \mathrm{GL}_a(E_\Sigma)$, $\tilde\nu_{\Sigma,\#n}(\det h)\,\theta^{\mathrm{lg}}\big(y \mapsto \Phi(hy)\big) = \theta^{\mathrm{lg}}(\Phi)$;*
2. *for $g \in G_f(n)$ with components 1 at the finite places outside $\Sigma\cup\bar\Sigma$, with $t = \det g_\Sigma$,*

$$\theta^{\mathrm{lg}}\big(y\mapsto\Phi(yg_\Sigma^{-1})\big)\big(x(1,g)\big) = \tilde\nu_{\Sigma,\#a}(t)\,\upsilon(t)^{\#a-\#n}\,\theta^{\mathrm{lg}}(\Phi)(x),$$
$$\theta^{\mathrm{sm}}\big(y\mapsto\Psi(g_\Sigma y)\big)\big(x(1,g)\big) = \upsilon_{\Sigma,1}(t)^{-\#b}\,\theta^{\mathrm{sm}}(\Psi)(x);$$

3. *for $k \in \mathbf{K}_0(n)$ with components 1 at the places of Σ, $\theta^{\mathrm{lg}}(\Phi)(x(1,k)) = \theta^{\mathrm{lg}}(\Phi)(x)$ and $\theta^{\mathrm{sm}}(\Psi)(x(1,k)) = \theta^{\mathrm{sm}}(\Psi)(x)$.*

Proof. Put $\alpha = \nu^{-1}\upsilon^{\#n/2}$ (the characters of the large side), $\alpha' = \nu\upsilon^{\#n/2}$, $\beta' = \upsilon^{\#b}$ (those of the small side). By (WD3), $\upsilon(\bar x) = \upsilon(x)^{-1}$, so $\tilde\upsilon = \upsilon^2$; hence the space characters are $\alpha[\#n] = \tilde\nu^{-1}$, $\alpha[\#a] = \tilde\nu^{-1}\upsilon^{\#n-\#a}$ and $\beta'[\#b] = \upsilon^{\#b}$.

Data and convergence. The large side is the theta integral of $\varphi(\Phi) = v\otimes(\Phi^\sharp\otimes_\Sigma v^0)$ and the small side that of $\varphi'(\Psi) = v'\otimes(F_\Psi\otimes_{\bar\Sigma} v^0)$, $F_\Psi = (\Psi^{\mathsf T})^\flat$, $F_\Psi(y) = \Psi\big(((\bar y)_\Sigma)^{\mathsf T}\big)$, a function of $y_{\bar\Sigma}$; these functions lie in $\mathcal{T}_\Sigma$, $\mathcal{T}_{\bar\Sigma}$ by Lemma 6.2.22. $\Psi^{\mathsf T}$ vanishes off the right invertible matrices, and so does its conjugate, so by Lemma 6.2.26 (for $A=\Sigma$, resp. $A=\bar\Sigma$ and the function $z\mapsto\Psi^{\mathsf T}(\bar z)$) both data are target-bounded, and (2) of Lemma 6.3.21 gives convergence, continuity, left Γ_W-invariance and linearity in the datum; (3) there gives $I(\omega(g',h')\varphi)(x) = I(\varphi)(xg')$ for target-bounded φ. The real part of a datum is not touched by finite points: by Lemma 4.1.5, $\omega((1,k),(1,k'))(v\otimes f) = v\otimes\omega((1,k),(1,k'))f$.

(1). Let $k \in G_f(a)$ have component h_w at $w\in\Sigma$, the component $\overline{(h_w^{-1})^{\mathsf T}}$ at $\bar w$ and 1 elsewhere; it is unitary (w is split), and $k_\Sigma = h$. By (4) of Lemma 4.1.1 for the pair (W,V), the points $(1,k^{-1})$ and $\Psi = \Phi^\sharp$, $\omega_{\alpha,\alpha}((1,1),(1,k^{-1}))(\Phi^\sharp\otimes_\Sigma v^0) = \tilde\nu(\det h)|\det h|^{\#n/2}\,(\Phi(h\,\cdot))^\sharp\otimes_\Sigma v^0$ (the factor is $\alpha[\#n](\det_\Sigma k^{-1})|\det_\Sigma k^{-1}|^{-\#n/2}$, and $\Phi^\sharp(ky) = \Phi(hy_\Sigma)$). So $\omega_{\alpha,\alpha}(1,k^{-1})\varphi(\Phi) = \tilde\nu_{\Sigma,\#n}(\det h)\varphi(\Phi(h\,\cdot))$, and $\theta^{\mathrm{lg}}(\Phi)(x) = I(\omega_{\alpha,\alpha}(1,k^{-1})\varphi(\Phi))(x) = \tilde\nu_{\Sigma,\#n}(\det h)\theta^{\mathrm{lg}}(\Phi(h\,\cdot))(x)$.

(2), large side. By (4) of Lemma 4.1.1 with $(1,g)$ on W, $(1,1)$ on V and the function $\Phi_1 = \Phi(\cdot\,g_\Sigma^{-1})$, $\omega_{\alpha,\alpha}((1,g),1)\varphi(\Phi_1) = \alpha[\#a](t)^{-1}|t|^{\#a/2}\varphi(\Phi_1(\cdot\,g_\Sigma)) = \tilde\nu(t)\upsilon(t)^{\#a-\#n}|t|^{\#a/2}\varphi(\Phi)$, and $\theta^{\mathrm{lg}}(\Phi_1)(x(1,g)) = I(\omega_{\alpha,\alpha}((1,g),1)\varphi(\Phi_1))(x)$ gives the first identity.

(2), small side. At a split place, unitarity gives $g_{\bar w} = \overline{(g_w^{-1})^{\mathsf T}}$, so $\det_{\bar\Sigma} g = \bar t^{-1}$ and the components at Σ of $\bar g$ form the matrix $(g_\Sigma^{-1})^{\mathsf T}$. Hence $F_{\Psi_1}(yg) = \Psi_1\big(((\bar y)_\Sigma(g_\Sigma^{-1})^{\mathsf T})^{\mathsf T}\big) = F_{\Psi_1(g_\Sigma^{-1}\cdot)}(y)$. By (4) of Lemma 4.1.1 for (W,V'), $A=\bar\Sigma$, the factor is $\beta'[\#b](\bar t^{-1})^{-1}|\bar t^{-1}|^{\#b/2} = \upsilon(t)^{-\#b}|t|^{-\#b/2} = \upsilon_{\Sigma,1}(t)^{-\#b}$ (as $\upsilon(\bar t^{-1}) = \upsilon(t)$), so $\omega_{\alpha',\beta'}((1,g),1)\varphi'(\Psi_1) = \upsilon_{\Sigma,1}(t)^{-\#b}\varphi'(\Psi_1(g_\Sigma^{-1}\cdot))$. With $\Psi_1 = \Psi(g_\Sigma\cdot)$ and $\theta^{\mathrm{sm}}(\Psi_1)(x(1,g)) = I(\omega_{\alpha',\beta'}((1,g),1)\varphi'(\Psi_1))(x)$ this is the second identity.

(3). Let $A\in\{\Sigma,\bar\Sigma\}$, $F\in\mathcal{T}_A$, and write $F\otimes_A v^0 = \sum_y c_y(F)\rho(\mathrm{fv}(y))P^A v^0$ with y supported at $A\cup\bar A$ ((WR6)). The vectors $\mathrm{fv}(y)$ and $\mathrm{rv}(z)$ have disjoint supports, so their bicharacter is 1 and $\rho(\mathrm{fv}(y))$, P^A commute with every $\rho(\mathrm{rv}(z))$ (Heisenberg relation, (1) of Lemma 4.1.10);

hence $v \otimes (F \otimes_A v^0) = \sum_y c_y(F)\rho(\mathrm{fv}(y))P^A(v \otimes v^0)$. The point $(1,k)$ acts on vectors of Hom by $X \mapsto Xk^{-1}$, which fixes the vectors supported at $A \cup \bar{A}$ (k is 1 at Σ, hence by unitarity at $\bar{\Sigma}$); by (2) of Lemma 4.1.1 $\omega((1,k),1)$ commutes with $\rho(\mathrm{fv}(y))$ and with P^A, and by (6) there $\omega_{\alpha_1,\beta_1}((1,k),1)(v \otimes v^0) = \beta_1(\det k)^{-1}v \otimes v^0$. For $\beta_1 = \alpha$ or β', $\beta_1(\det k) = 1$: $\det k \in \mathrm{U}(1)(\mathbb{A})$ has infinite components 1, unit finite components and component 1 at $\Sigma \cup \bar{\Sigma}$, so $\nu(\det k) = \prod_w \psi_w(\det k \bmod w) = 1$ (tame part trivial off Σ) and $v(\det k) = 1$ by (WD3). So $\omega((1,k),1)$ fixes both data, and (3) of Lemma 6.3.21 gives the invariance. □

Lemma 6.3.21. *Let S, T be spaces with signs s on n and s' on n', n' nonempty, and α, β automorphic characters of* $\mathrm{U}(1)$.

1. *The target-bounded data form a linear subspace of $\mathcal{M}_{S,T}$, stable under $\omega_{\alpha,\beta}(g',h')$ for all $(g',h') \in \mathrm{G}(s) \times \mathrm{G}(s')$.*
2. *For target-bounded φ and $g \in \mathrm{G}(s)$, the function $h \mapsto \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)$ is continuous, bounded and left Γ_T-invariant, so $I^{S,T}_{\alpha,\beta}(\varphi)(g)$ converges absolutely; the function $I^{S,T}_{\alpha,\beta}(\varphi)$ is continuous on $\mathrm{G}(s)$ and left Γ_S-invariant; and $\varphi \mapsto I^{S,T}_{\alpha,\beta}(\varphi)$ is linear on the target-bounded data.*
3. *For target-bounded φ, $(g',h') \in \mathrm{G}(s) \times \mathrm{G}(s')$ and $g \in \mathrm{G}(s)$,*
$$I^{S,T}_{\alpha,\beta}(\omega_{\alpha,\beta}(g',h')\varphi)(g) = I^{S,T}_{\alpha,\beta}(\varphi)(gg').$$

Proof. Write $\theta(\varphi) = \theta^{S,T}_{\alpha,\beta}(\varphi)$, $I = I^{S,T}_{\alpha,\beta}$, $\omega = \omega_{\alpha,\beta}$, and x_S for the point of S.

Step 1: heights under translation and on compact sets. For $g \in \mathrm{G}(s)$ the lattice of S_g is $L(g) = \{u \in E^n : ((x_S g)^{-1}u)_f \in \widehat{\mathcal{O}}_E^n\}$ and $\lambda_S(g)$ is the infimum of 1 and of $\|((x_S g)^{-1}u)_\infty\|$ over $u \in L(g) \setminus \{0\}$. Let $g' \in \mathrm{G}(s)$ and choose a positive integer N with $g'_f\widehat{\mathcal{O}}_E^n \subset N^{-1}\widehat{\mathcal{O}}_E^n$. If $u \in L(gg')$ then $(x_S g)^{-1}u \in g'_f\widehat{\mathcal{O}}_E^n$ at the finite places, so $Nu \in L(g)$, and $\|((x_S gg')^{-1}u)_\infty\| \geq \|g'_\infty\|^{-1}\|((x_S g)^{-1}u)_\infty\| \geq \|g'_\infty\|^{-1}N^{-1}\lambda_S(g)$ (operator norm). Hence, as $\lambda_S \leq 1$,

$$\mathrm{ht}_S(gg') \leq c(g')\,\mathrm{ht}_S(g), \qquad c(g') = \max(1, N\|g'_\infty\|). \tag{6.31}$$

For a compact $K \subset \mathrm{G}(s)$, c is bounded on K (g'_f ranges over a compact set, so one N serves, and $\|g'_\infty\|$ is bounded), so by (6.31) with $g = 1$ the height is bounded on K.

Step 2: (1). If $|\theta(\varphi)(g,h)| \leq C\,\mathrm{ht}_S(g)^B$ and $|\theta(\varphi')(g,h)| \leq C'\mathrm{ht}_S(g)^{B'}$ for all characters, then, as $\mathrm{ht}_S \geq 1$ and θ is linear in the datum, $|\theta(a\varphi+a'\varphi')(g,h)| \leq (|a|C+|a'|C')\mathrm{ht}_S(g)^{\max(B,B')}$. For (g',h'), $\theta(\omega(g',h')\varphi)(g,h) = \theta(\varphi)(gg',hh')$ (as ω is a representation; the characters in (XD2) are those of ω, and for other characters α',β' the operator $\omega_{\alpha',\beta'}(g',h')$ differs from $\omega(g',h')$ by the scalar $(\alpha'\alpha^{-1})(\det h')(\beta'\beta^{-1})(\det g')^{-1}$ of modulus 1), so by (6.31) $|\theta_{\alpha',\beta'}(\omega(g',h')\varphi)(g,h)| \leq Cc(g')^B\mathrm{ht}_S(g)^B$.

Step 3: (2). The map $(g,h) \mapsto \omega(g,h)\varphi$ is continuous and $\theta_{\mathrm{Hom}(S,T)}$ is bounded, so $\theta(\varphi)$ is continuous on $\mathrm{G}(s) \times \mathrm{G}(s')$; by Lemma 6.2.5 it is invariant under $\Gamma_S \times \Gamma_T$ on the left. For fixed g it is bounded in h by $C\,\mathrm{ht}_S(g)^B$, and $[\mathrm{U}(T)]$ has finite volume (Theorem 4.2.18), so $I(\varphi)(g)$ converges absolutely and $I(\varphi)(\gamma g) = I(\varphi)(g)$ for $\gamma \in \Gamma_S$. For g in a compact neighbourhood K of a point the integrands are bounded by $C\sup_K \mathrm{ht}_S^B < \infty$ (Step 1), a constant integrable on $[\mathrm{U}(T)]$; dominated convergence gives the continuity of $I(\varphi)$. Linearity on the subspace of (1) is linearity of the integral.

Step 4: (3). By Step 2, $\omega(g',h')\varphi$ is target-bounded and

$$I(\omega(g',h')\varphi)(g)=\int_{[\mathrm{U}(T)]}\theta(\varphi)(gg',hh')\,dh.$$

The group $\mathrm{U}(T)(\mathbb{A}_F)$ is unimodular (it is reductive), so the quotient measure on $[\mathrm{U}(T)]$ is invariant under right translation by h', and the integral is $I(\varphi)(gg')$. □

Proposition 6.3.22. *Let X, W_1, W, V' be spaces with signs r on m, s on n, s_W on n_W, t' on b, with $\#n$ even and b nonempty; let D be a base-adapted splitting of X over W_1 along b, D_1 a base-adapted splitting of W_1 over W along a one-element set, $\sigma\colon b\sqcup(b\sqcup n)\to m$ a bijection and $\mu\colon\Phi\to\mathbb{Z}$. Let $\phi_{\mathrm{lg}}\in\mathcal{F}_{X,V'\oplus W_1}$ and $\phi_{\mathrm{sm}}\in\mathcal{F}_{X,V'}$ be Schwartz vectors, $\mathrm{pw}\colon G_\infty(t'\sqcup s)\to[0,\infty)$ a function that does not vanish on a neighbourhood of 1. For a split tame character $\mathfrak{c}=(\nu,\Sigma)$ put $\gamma_{\mathfrak{c}}=\nu^{-1}\upsilon^{\#n/2}$ and $F_{\mathfrak{c}}=(\Psi_1^{\mathrm{st}})^\sharp\otimes_\Sigma v^0\in\mathcal{M}_{W_1,W_1}$, 1 the identity $n\times n$ matrix and Ψ_1^{st} formed with $\chi_{\mathfrak{c}}$; for $B\in\mathrm{GL}_b(\mathbb{A}_E)$ let z_B be the idele with components $\det B_v$ at $v\in\Phi$ and 1 at the finite places. For a split tame character $\mathfrak{c}=(\nu,\Sigma)$ of weight μ and $I_0\in\mathbb{C}$ consider the condition*

(P) (period identity for $(\mathfrak{c},I_0)$) for every $h\in G_\infty(t'\sqcup s)$ there is $(p_1,p_2)\in\mathrm{G}(t')\times\mathrm{G}(s)$ such that, with $\hat h=\mathrm{diag}(p_{1,\infty},p_{2,\infty})\,h\in G_\infty(t'\sqcup s)$, for all $B\in\mathrm{GL}_b(\mathbb{A}_E)$, all automorphic characters β of $\mathrm{U}(1)$, all $f\in\mathcal{M}_{W_1,V'}$, $x\in\mathrm{M}_{b\times n}(\mathbb{A}_E)$, $g\in\mathrm{G}(s)$ and $y\in\mathrm{G}(t')$,

$$\begin{aligned}&\int_{[\mathrm{U}(W_1)]}\theta_{\mathrm{Hom}(W_1,V'\oplus W_1)}\Big(\rho\big((x;0)\big)\,\omega_{\gamma_{\mathfrak{c}},\gamma_{\mathfrak{c}}\beta}\big(g,\mathrm{diag}(y,u)\big)\big(w_{\hat h}(B)\otimes(f\boxtimes^{\mathrm{tg}}F_{\mathfrak{c}})\big)\Big)\,du\\&\quad=\frac{\mathrm{pw}(h)\,I_0}{\tilde\nu(z_B)\prod_{v\in\Phi}|\det B_v|^{\#n}}\,\theta_{\mathrm{Hom}(W_1,V')}\Big(\rho(x)\,\omega_{\gamma_{\mathfrak{c}},\beta}(g,y)\big(\mathrm{rsl}_{D,V'}(B_\infty;\phi_{\mathrm{sm}})\otimes f\big)\Big),\end{aligned}$$

where $w_{\hat h}(B)=\Omega_{\#n/2-\mu,0}(1,\hat h)\,\mathrm{rsl}_{D,V'\oplus W_1}\big(\hat h^{-1}(B_\infty;0);\phi_{\mathrm{lg}}\big)\in\mathcal{F}_{W_1,V'\oplus W_1}$, the argument of the slice having components $\hat h_v^{-1}(B_v;0)$, $v\in\Phi$;

and assume

(S) (support or signs) either every $x=(x_v)_{v\in\Phi}$, x_v complex $(b\sqcup n)\times b$ matrices, with $\mathrm{rsl}_{D,V'\oplus W_1}(x;\phi_{\mathrm{lg}})\neq0$ has, at every $v\in\Phi$, $x_v^*\mathrm{diag}(t'\sqcup s)\,x_v=c^*\mathrm{diag}(t')\,c$ for some $c\in\mathrm{M}_b(\mathbb{C})$; or $t'(i)=-1$ for all $i\in b$, $s(j)=1$ for some $j\in n$, and $2\,\mathrm{neg}(W_1)\le\#n$.

Then there is $K>0$ such that for every split tame character $\mathfrak{c}=(\nu,\Sigma)$ of weight μ and every $I_0\in\mathbb{C}$ for which (P) holds there is $c\in\mathbb{C}$ with $|c|=K|I_0|$ and

$$\theta^{\mathrm{lg}}_{\mathfrak{c}}(X,V'\oplus W_1;\phi_{\mathrm{lg}},\Phi^{\mathrm{st}}_{\sigma,z})=c\,\theta^{\mathrm{sm}}_{\mathfrak{c}}(X,V';\phi_{\mathrm{sm}},\Psi^{\mathrm{st}}_z)\qquad\text{for every } z\in\mathrm{M}_{b\times m}(\mathcal{O}_\Sigma).$$

In particular, if I is a complex function on the split tame characters of weight μ such that (P) holds for $(\mathfrak{c},I(\mathfrak{c}))$ for every $\mathfrak{c}$, then $(\sigma,\phi_{\mathrm{lg}},\phi_{\mathrm{sm}},\ell)$ satisfy the row-column exchange identity for $(X,V'\oplus W_1,V')$ and μ, where $\ell(\mathfrak{c})=|I(\mathfrak{c})|^2/L_*(\Sigma,\nu;\#n/2)$.

Proof. Throughout, $i\colon\mathrm{G}(t')\times\mathrm{G}(s)\to\mathrm{G}(t'\sqcup s)$ is the block diagonal map $(y,y')\mapsto\mathrm{diag}(y,y')$, $\kappa_\alpha\in\mathbb{A}_E^\times$ is the idele with components κ_v at $v\in\Phi$ and finite part α, and, for a split tame character $\mathfrak{c}=(\nu,\Sigma)$, $z_0=(\mathrm{pr}_l)_\Sigma\in\mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$ for the coordinate pr_l of D (integral, D being base-adapted), $\alpha_{\mathfrak{c}}=\nu^{-1}\upsilon^{\lfloor\#m/2\rfloor}$, $\alpha'_{\mathfrak{c}}=\nu\upsilon^{\lfloor\#m/2\rfloor}$, $\beta'=\upsilon^{\#b}$, and for $z\in\mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$

$$\Theta^{\mathrm{lg}}_z=\theta^{\mathrm{lg}}_{\mathfrak{c}}(X,V'\oplus W_1;\phi_{\mathrm{lg}},\Phi^{\mathrm{st}}_{\sigma,z})=I_{\alpha_{\mathfrak{c}},\alpha_{\mathfrak{c}}}(f_z),\qquad\Theta^{\mathrm{sm}}_z=\theta^{\mathrm{sm}}_{\mathfrak{c}}(X,V';\phi_{\mathrm{sm}},\Psi^{\mathrm{st}}_z)=I_{\alpha'_{\mathfrak{c}},\beta'}(f'_z),$$

with $f_z=\phi_{\mathrm{lg}}\otimes[\Phi^{\mathrm{st}}_{\sigma,z}]_\Sigma$ and $f'_z=\phi_{\mathrm{sm}}\otimes[(\Psi^{\mathrm{st}}_z)^{\mathsf T}]_{\bar\Sigma}$. Centre coefficients, radical and Levi elements are those of D.

Step 0: the constants. Put $\varrho(h) = \mathrm{pw}(h_\infty)$ for $h = (h_\infty, u) \in \mathrm{G}(t' \sqcup s)$ if $\mathrm{diag}(u_1, u_2)\, u \in \mathbf{K}_0(b \sqcup n)$ for some $(u_1, u_2) \in G_f(b) \times G_f(n)$, and $\varrho(h) = 0$ otherwise. Then $\varrho \geq 0$, and $\varrho \neq 0$ on the neighbourhood $N \times \mathbf{K}_0(b \sqcup n)$ of 1, N a neighbourhood of 1 in $G_\infty(t' \sqcup s)$ on which $\mathrm{pw} \neq 0$ (take $u_1 = u_2 = 1$). The groups $\mathrm{G}(t')$, $\mathrm{G}(s)$ are unimodular (products of unitary groups of local fields, almost all compact), i is a continuous injective homomorphism and a homeomorphism onto its closed image, Γ_{W_1} is discrete, and $i(\{1\} \times \Gamma_{W_1})$ lies in the discrete group $\Gamma_{V' \oplus W_1}$, the point of $V' \oplus W_1$ being $\mathrm{diag}(x_{V'}, x_{W_1})$. Let $R \neq 0$ be the constant of Lemma 6.2.47 for $G = \mathrm{G}(t' \sqcup s)$, $L = \mathrm{G}(t')$, $H = \mathrm{G}(s)$ with their measures, this i, $\Lambda = \Gamma_{W_1}$ and $\rho = \varrho$. Let $K_1 > 0$ be the constant of Lemma 6.3.23 (whose hypotheses are those of the present statement, (IP) being (P)), and $K = |R| K_1 > 0$. Neither depends on a character.

Fix a split tame character $\mathfrak{c} = (\nu, \Sigma)$ of weight μ and I_0 such that (P) holds; let C_1 be as in Lemma 6.3.23, $|C_1| = K_1 |I_0|$, and $c = RC_1$, so that $|c| = K|I_0|$.

Step 1: the coefficients at self-dual lattices. Let $T \in \mathrm{M}_b(E)$, $p = \mathrm{rad}(c', z)\, \mathrm{lev}(q, g)$ with $c' \in \mathrm{M}_{n \times b}(\mathbb{A}_E)$, $z \in \mathrm{M}_b(\mathbb{A}_E)$, $q \in \mathrm{GL}_b(\mathbb{A}_E)$, $g \in \mathrm{G}(s)$, and assume (a) or (b) of (VL) of Proposition 6.3.33. We show

$$\mathrm{coef}\big(\jmath T; \Theta^{\mathrm{lg}}_{z_0}\big)(p) = c\, \mathrm{coef}\big(\jmath T; \Theta^{\mathrm{sm}}_{z_0}\big)(p). \tag{6.32}$$

By (1) of Lemma 6.3.24, f_{z_0} has integrable terms at p for D, $V' \oplus W_1$, $\alpha_{\mathfrak{c}}, \alpha_{\mathfrak{c}}$, and f'_{z_0} for D, V', $\alpha'_{\mathfrak{c}}, \beta'$; so both coefficients are defined by (1) of Lemma 6.3.25 (weight $y = 1$; the integral over a fundamental domain is the integral over the quotient).

Case (b): $\det T \neq 0$ *and* T *is not represented by* t'. The right side of (6.32) vanishes by (2) of Lemma 6.3.24. Under the first alternative of (S) so does the left side, by the same item. Under the second alternative apply Lemma 6.3.27 to X, $Y = V' \oplus W_1$ (signs $t' \sqcup s$), $Z = W_1$, the nonempty b, D, the character $\nu_0 = \nu^{-1} \upsilon^{\#n/2}$ of weight $\mu_0 = \#n/2 - \mu$ (υ has weight 1, (WD3) of Definition 3.2.12), T and f_{z_0}: $\delta = \nu_0 \upsilon^{\#b} = \alpha_{\mathfrak{c}}$ since $\lfloor \#m/2 \rfloor = \#b + \#n/2$; $s(j) = 1$ for some j; $\#b + \mathrm{neg}(W_1)$ is the number of signs -1 of $t' \sqcup s$ as $t' \equiv -1$; $\mu_0(\tau) = \#n/2 - \mu(\tau) < \#n/2 \leq \#n - \mathrm{neg}(W_1) = \#\{j : s_j = 1\}$ as $\mu(\tau) > 0$ ((XD4)) and $2\,\mathrm{neg}(W_1) \leq \#n$; T is invertible and not represented by $t' = -1$; and the terms are integrable. So the left side vanishes too.

Case (a): $T = J_{V'}$ *and* $q = \kappa_\alpha x_{V'} Y$ *with* $(Y_f k)^* (Y_f k) = 1$ *for some* $k \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$. Let $x_0 = (1; 0) \in \mathrm{M}_{(b \sqcup n) \times b}(E)$, with Gram matrix $x_0^* \mathrm{diag}(J_{V'}, J_{W_1}) x_0 = J_{V'}$, invertible. A rational point γ of $\mathrm{U}(V' \oplus W_1)$ with $\gamma x_0 = x_0$ has first block column $(1; 0)$, and the block (b, n) of $\gamma^* \mathrm{diag}(J_{V'}, J_{W_1}) \gamma = \mathrm{diag}(J_{V'}, J_{W_1})$ reads $J_{V'} \gamma_{bn} = 0$; so $\gamma = \mathrm{diag}(1, \gamma_2)$ with γ_2 a rational point of $\mathrm{U}(W_1)$, and conversely. Hence the group Γ_{x_0} of (2) of Lemma 6.3.25 is $i(\{1\} \times \Gamma_{W_1})$, and that lemma gives

$$\mathrm{coef}(\jmath J_{V'}; \Theta^{\mathrm{lg}}_{z_0})(p) = \int_{i(\{1\} \times \Gamma_{W_1}) \backslash \mathrm{G}(t' \sqcup s)} F(h)\, dh, \qquad F(h) = \mathrm{term}^{\mathrm{lg}}_{x_0}(p, h),$$

F being left invariant and integrable on the quotient. For the small side take $x_0 = 1 \in \mathrm{M}_b(E)$: a rational point of $\mathrm{U}(V')$ fixing 1 is 1, so

$$\mathrm{coef}(\jmath J_{V'}; \Theta^{\mathrm{sm}}_{z_0})(p) = \int_{\mathrm{G}(t')} \mathrm{term}^{\mathrm{sm}}_1(p, y)\, dy =: \Sigma_{\mathrm{sm}}.$$

The function F is continuous: $h \mapsto \omega_{\alpha_{\mathfrak{c}}, \alpha_{\mathfrak{c}}}(p, h) f_{z_0}$ is continuous ((2) of Lemma 4.1.1), and the theta term is this vector followed by a Heisenberg operator (an isometry of the theta model), the zero term ((7) of Lemma 4.1.1, bounded) and a theta functional (bounded, (WD9) of Definition 3.2.12). By Lemma 6.3.23, for every $h \in \mathrm{G}(t' \sqcup s)$ there is (k_1, k_2) with

$$\int_{\mathrm{G}(t')} \int_{\Gamma_{W_1} \backslash \mathrm{G}(s)} F\big(i(y, y')\, i(k_1, k_2)\, h\big)\, dy'\, dy = \varrho(h)\, C_1 \Sigma_{\mathrm{sm}},$$

the iterated integral existing. By Lemma 6.2.47 with the constant $C_1\Sigma_{\mathrm{sm}}$, the integral of F over the quotient is $RC_1\Sigma_{\mathrm{sm}} = c\,\Sigma_{\mathrm{sm}}$, which is (6.32).

Step 2: the equivariant lift. The rank $\#m = 2\#b + \#n$ is even. Apply Lemma 6.3.18 to $\mathfrak{c}$, $W = X$ (signs r on m), $V = V' \oplus W_1$ (index set $b \sqcup n$), V' (index set b), σ, $v = \phi_{\mathrm{lg}}$, $v' = \phi_{\mathrm{sm}}$ and the number c: it gives an equivariant lift $(\mathcal{L}, \mathcal{D})$ for $(\mathfrak{c}, X, b)$ with $\Psi \in \mathcal{L}$ and $\mathcal{D}\Psi = \theta^{\mathrm{lg}}_{\mathfrak{c}}(X, V' \oplus W_1; \phi_{\mathrm{lg}}, \Phi) - c\,\theta^{\mathrm{sm}}_{\mathfrak{c}}(X, V'; \phi_{\mathrm{sm}}, \Psi)$ whenever Φ and a locally constant Ψ are matched over $\mathcal{O}_\Sigma$ for $(\sigma, \chi_{\mathfrak{c}})$. By Lemma 6.3.31, for $z \in \mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$ the pair $(\Phi^{\mathrm{st}}_{\sigma,z}, \Psi^{\mathrm{st}}_z)$ is such a pair; hence

$$\Psi^{\mathrm{st}}_z \in \mathcal{L}, \qquad \mathcal{D}\Psi^{\mathrm{st}}_z = \Theta^{\mathrm{lg}}_z - c\,\Theta^{\mathrm{sm}}_z. \tag{6.33}$$

Step 3: vanishing at self-dual lattices. By (2) of Lemma 6.3.31, $\Phi^{\mathrm{st}}_{\sigma,z}$ and the function $y \mapsto (\Psi^{\mathrm{st}}_z)^{\mathsf{T}}(\bar{y})$ on $\mathrm{M}_{b\times m}(E_{\bar\Sigma})$ (whose transfer to finite adelic matrices is the function of the conjugate local datum, (XD9)) are locally constant with compact support and vanish at the matrices without a right inverse. So f_z and f'_z are target-bounded (Lemma 6.2.26, with $A = \Sigma$, resp. $A = \bar\Sigma$), and Θ^{lg}_z, Θ^{sm}_z are continuous and left Γ_X-invariant ((2) of Lemma 6.3.21; the targets $V' \oplus W_1$, V' have nonempty index sets). By (1) of Lemma 6.3.32, (6.33) and (6.32),

$$\mathrm{coef}(\jmath T; \mathcal{D}\Psi^{\mathrm{st}}_{z_0})(p) = \mathrm{coef}(\jmath T; \Theta^{\mathrm{lg}}_{z_0})(p) - c\,\mathrm{coef}(\jmath T; \Theta^{\mathrm{sm}}_{z_0})(p) = 0$$

for every (T, p) as in (VL) of Proposition 6.3.33. With $\Psi_0 = \Psi^{\mathrm{st}}_{z_0} \in \mathcal{L}$, Proposition 6.3.33 (for D, D_1, V') gives $\mathcal{D}\Psi = 0$ for every locally constant Ψ matched with some function; for $\Psi = \Psi^{\mathrm{st}}_z$ (Lemma 6.3.31) and (6.33), $\Theta^{\mathrm{lg}}_z = c\,\Theta^{\mathrm{sm}}_z$ for every $z \in \mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$. This is the main assertion.

Step 4: the row-column exchange identity. Let I be as in the last assertion and $c(\mathfrak{c})$ the number of Step 3 for $(\mathfrak{c}, I(\mathfrak{c}))$, so $|c(\mathfrak{c})| = K|I(\mathfrak{c})|$. The value $L_*(\Sigma, \nu; \#n/2)$ is positive: it is a power of $\prod_{w\in\Sigma} \mathrm{N}w$ times the numbers $|L^\Sigma(j, \tilde\nu)|^2$, $1 \le j \le \#n/2$, and $L^\Sigma(j, \tilde\nu) \ne 0$ for $j \ge 1$ by (3) of Theorem A.1.12 ($\tilde\nu$ is nontrivial, being $\psi_w \ne 1$ on the units at $w \in \Sigma$; the removed Euler factors are finite and nonzero for $\mathrm{Re}\, s \ge 1$); so ℓ is defined. With $a = b \sqcup n$ the index set of $V' \oplus W_1$, $\#a - \lfloor \#m/2 \rfloor = \#b + \#n - (\#b + \#n/2) = \#n/2$, and with $C = K^{-2}$

$$C\,|c(\mathfrak{c})|^2 = |I(\mathfrak{c})|^2 = L_*(\Sigma, \nu; \#a - \lfloor \#m/2 \rfloor)\,\ell(\mathfrak{c}),$$

which is (a) of (XD12); (b) of (XD12) is the main assertion, and ϕ_{lg}, ϕ_{sm} are Schwartz vectors. □

Lemma 6.3.23. *Let X, W_1, V' be spaces with signs r on m, s on n, t' on b, with $\#n$ even and b nonempty; let D be a base-adapted splitting of X over W_1 along b, $\sigma\colon b \sqcup (b \sqcup n) \to m$ a bijection and $\mu\colon \Phi \to \mathbb{Z}$. Let $\phi_{\mathrm{lg}} \in \mathcal{F}_{X,V'\oplus W_1}$ and $\phi_{\mathrm{sm}} \in \mathcal{F}_{X,V'}$ be Schwartz vectors and $\mathrm{pw}\colon G_\infty(t' \sqcup s) \to [0, \infty)$ a function. For a split tame character $\mathfrak{c} = (\nu, \Sigma)$ put $\gamma_{\mathfrak{c}} = \nu^{-1}\upsilon^{\#n/2}$, $F_{\mathfrak{c}} = (\Psi^{\mathrm{st}}_1)^\sharp \otimes_\Sigma v^0 \in \mathcal{M}_{W_1,W_1}$ (1 the identity $n \times n$ matrix, Ψ^{st}_1 formed with $\chi_{\mathfrak{c}}$), $z_0 = (\mathrm{pr}_l)_\Sigma \in \mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$ for the coordinate pr_l of D, $\alpha_{\mathfrak{c}} = \nu^{-1}\upsilon^{\lfloor \#m/2 \rfloor}$, $\alpha'_{\mathfrak{c}} = \nu\,\upsilon^{\lfloor \#m/2 \rfloor}$, $\beta' = \upsilon^{\#b}$; let $\mathrm{term}^{\mathrm{lg}}_x$ be the theta terms along D of $\phi_{\mathrm{lg}} \otimes [\Phi^{\mathrm{st}}_{\sigma,z_0}]_\Sigma \in \mathcal{M}_{X,V'\oplus W_1}$ for $V' \oplus W_1$ and the characters $(\alpha_{\mathfrak{c}}, \alpha_{\mathfrak{c}})$, and $\mathrm{term}^{\mathrm{sm}}_x$ those of $\phi_{\mathrm{sm}} \otimes [(\Psi^{\mathrm{st}}_{z_0})^{\mathsf{T}}]_{\bar\Sigma} \in \mathcal{M}_{X,V'}$ for V' and $(\alpha'_{\mathfrak{c}}, \beta')$. For $B \in \mathrm{GL}_b(\mathbb{A}_E)$ let z_B be the idele with components $\det B_v$ at $v \in \Phi$ and 1 at the finite places. For a split tame character $\mathfrak{c}$ of weight μ and $I_0 \in \mathbb{C}$ consider the condition*

(IP) for every $h \in G_\infty(t' \sqcup s)$ there is $(p_1, p_2) \in \mathrm{G}(t') \times \mathrm{G}(s)$ such that, with $\hat h = \mathrm{diag}(p_{1,\infty}, p_{2,\infty})\,h$, for all $B \in \mathrm{GL}_b(\mathbb{A}_E)$, all automorphic characters β of $\mathrm{U}(1)$, all $f \in \mathcal{M}_{W_1,V'}$, $x \in \mathrm{M}_{b\times n}(\mathbb{A}_E)$, $g \in \mathrm{G}(s)$ and $y \in \mathrm{G}(t')$,

$$\int_{[\mathrm{U}(W_1)]} \theta_{\mathrm{Hom}(W_1, V'\oplus W_1)}\Big(\rho((x;0))\,\omega_{\gamma_{\mathfrak{c}},\gamma_{\mathfrak{c}}\beta}(g, \mathrm{diag}(y, u))\big(w_{\hat h}(B) \otimes (f \boxtimes^{\mathrm{tg}} F_{\mathfrak{c}})\big)\Big)\,du$$
$$= \frac{\mathrm{pw}(h)\,I_0}{\tilde\nu(z_B)\prod_{v\in\Phi} |\det B_v|^{\#n}}\,\theta_{\mathrm{Hom}(W_1,V')}\Big(\rho(x)\,\omega_{\gamma_{\mathfrak{c}},\beta}(g, y)\big(\mathrm{rsl}_{D,V'}(B_\infty; \phi_{\mathrm{sm}}) \otimes f\big)\Big),$$

where $w_{\hat h}(B) = \Omega_{\#n/2-\mu,0}(1,\hat h)\,\mathrm{rsl}_{D,V'\oplus W_1}(\hat h^{-1}(B_\infty;0);\phi_{\mathrm{lg}})$.

Then there is $K_1>0$ *with the following property. Let* $\mathfrak{c}$ *be a split tame character of weight* μ *and* $I_0\in\mathbb{C}$ *such that (IP) holds. There is* $C_1\in\mathbb{C}$ *with* $|C_1|=K_1|I_0|$ *such that for all* $c\in \mathrm{M}_{n\times b}(\mathbb{A}_E)$, $z\in \mathrm{M}_b(\mathbb{A}_E)$, $g\in \mathrm{G}(s)$, $Y\in \mathrm{GL}_b(\mathbb{A}_E)$ *and* $k\in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ *with* $(Y_fk)^*(Y_fk)=1$, *with* $p=\mathrm{rad}(c,z)\,\mathrm{lev}(\kappa_\alpha x_{V'}Y,g)$, κ_α *the idele with components* κ_v *at* $v\in\Phi$ *and finite part* α, $x_0=(1;0)\in \mathrm{M}_{(b\sqcup n)\times b}(E)$ *and* $1\in \mathrm{M}_b(E)$, *and for every* $h=(h_\infty,u)\in \mathrm{G}(t'\sqcup s)$, *there is* $(k_1,k_2)\in \mathrm{G}(t')\times\mathrm{G}(s)$ *such that: for every* $y\in \mathrm{G}(t')$ *the function* $y'\mapsto \mathrm{term}^{\mathrm{lg}}_{x_0}(p,\mathrm{diag}(y,y')\,\mathrm{diag}(k_1,k_2)\,h)$ *is integrable on* $[\mathrm{U}(W_1)]$, *its integral is an integrable function of* $y\in\mathrm{G}(t')$, *and*

$$\int_{\mathrm{G}(t')}\int_{[\mathrm{U}(W_1)]}\mathrm{term}^{\mathrm{lg}}_{x_0}(p,\mathrm{diag}(y,y')\,\mathrm{diag}(k_1,k_2)\,h)\,dy'\,dy=\varrho(h)\,C_1\int_{\mathrm{G}(t')}\mathrm{term}^{\mathrm{sm}}_1(p,y)\,dy,$$

where $\varrho(h)=\mathrm{pw}(h_\infty)$ *if* $\mathrm{diag}(u_1,u_2)\,u\in\mathbf{K}_0(b\sqcup n)$ *for some* $(u_1,u_2)\in G_f(b)\times G_f(n)$, *and* $\varrho(h)=0$ *otherwise.*

Proof. **Step 0: notation and the constant.** Write $D=(k,l,i)$ with adelic frames $k_{\mathbb{A}},l_{\mathbb{A}},i_{\mathbb{A}}$, $U=V'\oplus W_1$ (signs $t'\sqcup s$ on $b\sqcup n$, point $x_U=\mathrm{diag}(x_{V'},x_{W_1})$), D_U, $D_{V'}$ the splittings of $\mathrm{Hom}(X,U)$, $\mathrm{Hom}(X,V')$ induced by D, with adapted bases m_U, $m_{V'}$. The adelic matrix $P_0=(k_{\mathbb{A}}\mid i_{\mathbb{A}}\mid l_{\mathbb{A}})$ has integral entries at the finite places (D is base-adapted), and so has its inverse, whose block rows $\mathrm{pr}_k=l^*_{\mathbb{A}}\varepsilon_r$, $\mathrm{pr}_i=\varepsilon_s i^*_{\mathbb{A}}\varepsilon_r$, $\mathrm{pr}_l=k^*_{\mathbb{A}}\varepsilon_r$ are conjugate transposes of integral matrices there ($\varepsilon=1$ at the finite places); so $P_{0,\Sigma}\in\mathrm{GL}_m(\mathcal{O}_\Sigma)$ and $z_0=(\mathrm{pr}_l)_\Sigma$ is integral. Let $\epsilon_\sigma\in\{\pm1\}$ be the sign such that, for every $b\times m$ matrix z_1 and $(b\sqcup n)\times m$ matrix w, the $m\times m$ matrix with row $\sigma(j)$ equal to row j of z_1 ($j\in b$) and row $\sigma(i')$ equal to row i' of w ($i'\in b\sqcup n$) has determinant $\epsilon_\sigma\det\binom{w}{z_1}$, the rows of w placed first. By (MX2), $\mathrm{rad}(c,z)$ and $\mathrm{lev}(q,g)$ fix and move the frame as follows: $p\,k_{\mathbb{A}}=k_{\mathbb{A}}q$ and $\mathrm{pr}_i\,p\,i_{\mathbb{A}}=g$ for $p=\mathrm{rad}(c,z)\mathrm{lev}(q,g)$, and $\mathrm{pr}_ip^{-1}l_{\mathbb{A}}=g^{-1}c$.

Fix the data c,z,g,Y,k of the statement, $q=\kappa_\alpha x_{V'}Y$, $p=\mathrm{rad}(c,z)\mathrm{lev}(q,g)$, $x_c=x_{V'}^{-1}c^*\varepsilon_s\in \mathrm{M}_{b\times n}(\mathbb{A}_E)$, $d_0=\det(\kappa_\alpha x_{V'})$. Let $\mathfrak{c}=(\nu,\Sigma)$ be split tame of weight μ, $\chi=\chi_{\mathfrak{c}}$, $\mathfrak{m}=\prod_{w\in\Sigma}w$ (a modulus with $\Sigma(\mathfrak{m})=\Sigma$), $f=\phi_{\mathrm{lg}}\otimes[\Phi]_\Sigma$ with $\Phi=\Phi^{\mathrm{st}}_{\sigma,z_0}$, $f'=\phi_{\mathrm{sm}}\otimes[\Phi']_{\bar\Sigma}$ with $\Phi'=(\Psi^{\mathrm{st}}_{z_0})^{\mathsf{T}}$. The vacuum factors $\mathrm{vf}_U=\mathrm{vf}(D_U)$, $\mathrm{vf}_{V'}=\mathrm{vf}(D_{V'})$ are nonzero by (2) of Lemma 6.3.12 (b, $b\sqcup n$ nonempty). Put

$$K_1=\Big|\frac{\mathrm{vf}_U}{\mathrm{vf}_{V'}}\Big|\,|d_0|^{\#n/2}>0,\qquad \varepsilon'=\chi(\epsilon_\sigma\det P_{0,\Sigma}^{-1})^{-1},\qquad C_1=\frac{\mathrm{vf}_U}{\mathrm{vf}_{V'}}\,\varepsilon'\,\tilde\nu(d_0)\,|d_0|^{\#n/2}\,I_0.$$

K_1 depends only on D, V', W_1; as $|\varepsilon'|=|\tilde\nu(d_0)|=1$, $|C_1|=K_1|I_0|$; and C_1 does not depend on c,z,g,Y,k, ϵ_σ, P_0, d_0 being fixed.

Step 1: the semilocal character. By (XD4) and Definition 3.1.1,

$$\nu(u)=\prod_v u_v^{\mu(v)}\prod_{w\in\Sigma}\psi_w(u_w \bmod w)$$

for $u\in\mathrm{U}(1)(\mathbb{A})$ with integral finite components. For $t\in\mathcal{O}_\Sigma^\times$, $t/\bar t$ has components t_w at $w\in\Sigma$, $\overline{t_w}^{-1}$ at $\bar w\notin\Sigma$ and 1 elsewhere, so $\chi(t)=\prod_{w\in\Sigma}\psi_w(t_w \bmod w)$: χ depends on t mod $\mathfrak{m}$. Consequently: (a) for $\kappa\in\mathbf{K}_0(b\sqcup n)$ (infinite components 1), $\nu(\det\kappa)=\chi((\det\kappa)_\Sigma)$ and $\upsilon(\det\kappa)=1$ (WD3); (b) for a finite idele d with unit components, $\tilde\nu(d)=\nu(d/\bar d)=\prod_{w\in\Sigma}\psi_w(d_w)\psi_w(\overline{d_{\bar w}})^{-1}=\chi(d_\Sigma)\,\chi((\bar d)_\Sigma)^{-1}$; (c) for $u\in\mathrm{U}(1)(\mathbb{A})$, $\tilde\nu(u)=\nu(u^2)$ as $\bar u=u^{-1}$.

Step 2: invariance of f under the standard level. The function Φ on integral matrices is $\bar\Phi(x \bmod \mathfrak{m})$, $\bar\Phi(\bar x)=\bar\chi^{-1}(\det\bar M)$ with M formed as in (XD10) and $\bar\chi^{-1}$ the reduction of χ^{-1} (zero at nonunits), by Step 1; so $\Phi^\sharp=\bar\Phi^\natural$. Let $\kappa\in\mathbf{K}_0(b\sqcup n)$. Replacing the rows x by

$\bar\kappa^{-1}x$ multiplies $\det\bar M$ by $\det\bar\kappa^{-1}$, so $\bar\Phi(\bar\kappa^{-1}\bar x)=\chi((\det\kappa)_\Sigma)\bar\Phi(\bar x)$. By (6) of Lemma 4.1.1 (pair (X,U), $k=1$, $k'=\kappa$) and Step 1 (a), as $\alpha_{\mathfrak c}=\nu^{-1}\upsilon^{\lfloor\#m/2\rfloor}$,

$$\omega_{\alpha_{\mathfrak c},\alpha_{\mathfrak c}}(1,(1,\kappa))f=\chi((\det\kappa)_\Sigma)^{-1}\chi((\det\kappa)_\Sigma)\,f=f. \tag{6.34}$$

Step 3: the terms at the parabolic point. By (1) of Lemma 6.3.12, the Hom slice at X is the slice along the induced splitting at $m^{-*}X=x_UX$ (resp. $x_{V'}X$). Apply (2) of Lemma 6.2.45 to $V=X$, $V_0=W_1$, D, the element p ($m=q$, $g_0=g$, $W=g^{-1}c$) and the Hom slice at $x_U^{-1}x_0=(x_{V'}^{-1};0)$, resp. $x_{V'}^{-1}\cdot1$, both with Gram matrix $J_{V'}$ for ε_U, resp. $\varepsilon_{t'}$. Then $\rho(h^{-1}XW^*\varepsilon_s)$ follows $\omega(g,h)$ as $\rho(XW^*\varepsilon_sg^{-1})=\rho((x_c;0))$, resp. $\rho(x_c)$ ((2) of Lemma 4.1.1; $W^*\varepsilon_sg^{-1}=c^*\varepsilon_s$), and $h^{-1}Xq=\kappa_\alpha h^{-1}(Y;0)$, resp. $\kappa_\alpha y^{-1}Y$. With $\gamma_{\mathfrak c}=\alpha_{\mathfrak c}\upsilon^{-\#b}$ (as $\lfloor\#m/2\rfloor=\#b+\#n/2$) and $\alpha'_{\mathfrak c}\upsilon^{-\#b}=\nu\upsilon^{\#n/2}$, this gives, for $h\in\mathrm G(t'\sqcup s)$ and $y\in\mathrm G(t')$,

$$\begin{aligned}\mathrm{term}^{\mathrm{lg}}_{x_0}(p,h)&=\kappa_{\mathrm{lg}}\,\theta_{\mathrm{Hom}(W_1,U)}\Big(\rho((x_c;0))\,\omega_{\gamma_{\mathfrak c},\gamma_{\mathfrak c}\upsilon^{\#b}}(g,h)\\&\qquad f[\kappa_\alpha h^{-1}(Y;0)]\Big),\\ \mathrm{term}^{\mathrm{sm}}_1(p,y)&=\kappa_{\mathrm{sm}}\,\nu(\det y)^2\,\theta_{\mathrm{Hom}(W_1,V')}\Big(\rho(x_c)\,\omega_{\gamma_{\mathfrak c},\upsilon^{\#b}}(g,y)\\&\qquad f'[\kappa_\alpha y^{-1}Y]\Big),\end{aligned}$$

using $\omega_{\nu\upsilon^{\#n/2},\upsilon^{\#b}}(g,y)=\nu(\det y)^2\omega_{\gamma_{\mathfrak c},\upsilon^{\#b}}(g,y)$ (WR4), where, with the same phase

$$\psi=\psi_E(\tfrac12\jmath\,\mathrm{tr}(\mathrm{pr}_kp^{-1}l_{\mathbb A}\,J_{V'}\,q))$$

on both sides and the characters

$$\begin{aligned}\chi_{\mathrm{lg}}(y)&=\alpha_{\mathfrak c}(y/\bar y)\upsilon(y)^{-\#b-\#n}=\tilde\nu(y)^{-1}\upsilon(y)^{\#b},\\ \chi_{\mathrm{sm}}(y)&=\upsilon(y/\bar y)^{\#b}\upsilon(y)^{-\#b}=\upsilon(y)^{\#b}\end{aligned}$$

of that law,

$$\begin{aligned}&\kappa_{\mathrm{lg}}=\chi_{\mathrm{lg}}(\det q)^{-1}|\det q|^{(\#b+\#n)/2}\psi,\qquad \kappa_{\mathrm{sm}}=\chi_{\mathrm{sm}}(\det q)^{-1}|\det q|^{\#b/2}\psi,\\ &\frac{\kappa_{\mathrm{lg}}}{\kappa_{\mathrm{sm}}}=\tilde\nu(\det q)\,|\det q|^{\#n/2}.\end{aligned}$$

Step 4: the slices. The frames of D_U act on coordinates by $l'_{\mathbb A}m_U^{-*}a=a\,\mathrm{pr}_k$, $k'_{\mathbb A}m_Uz'=\varepsilon_Uz'\mathrm{pr}_l$, $i'_{\mathbb A}Y'=Y'\mathrm{pr}_i$ (from (MX1), (VP5) of Definition 3.2.2, $m_U^{-*}a=x_Ua$, $m_Uz'=x_U^{-*}z'$); likewise for $D_{V'}$. At the finite places $\varepsilon=1$, so the frame of Lemma 6.2.19 sends (a,z',Y') to $M'=a\,\mathrm{pr}_k+z'\mathrm{pr}_l+Y'\mathrm{pr}_i$, and $M'P_0=(a\mid Y'\mid z')$.

Large side. For M' over E_Σ the matrix M of (XD10) for $\Phi(M')$ satisfies, as $z_0P_{0,\Sigma}=(0\mid0\mid1_b)$, $\det(MP_{0,\Sigma})=\epsilon_\sigma\det\begin{pmatrix}a&Y'&z'\\0&0&1_b\end{pmatrix}=\epsilon_\sigma\det(a\mid Y')$, so $\det M=\epsilon_\sigma\det P_{0,\Sigma}^{-1}\det(a\mid Y')$; and M is integral exactly when $(a\mid Y'\mid z')$ is ($P_{0,\Sigma}$ unimodular). Hence

$$\Phi(M')=\prod_{i\in(b\sqcup n)\times b}\mathbf1_{\mathcal O_\Sigma}(z'_i)\cdot\Psi_a(Y'),\qquad \Psi_a(Y')=\varepsilon'\,\mathbf1[a,Y'\text{ integral}]\,\chi^{-1}(\det(a\mid Y')),$$

χ^{-1} being multiplicative on $\mathcal O_\Sigma$ (zero at nonunits). By Lemma 6.3.53, D_U is adapted at 1 in the basis m_U, along the nonempty $(b\sqcup n)\times b$. So Lemma 6.2.19 (with $f_i=1$, $\alpha_0=\kappa_\alpha$, and

the character integrals $\int_{\mathcal{O}_\Sigma}\psi_\Sigma(at)\,dt=\mathbf{1}[a\in\mathcal{O}_\Sigma]$) gives, for an adelic $(b\sqcup n)\times b$ matrix a with finite part a_f,

$$f[\kappa_\alpha a]=\mathrm{vf}_U\,\mathbf{1}[a_f\text{ integral at the finite places outside }\Sigma]\,\mathrm{rsl}_{D,U}(a_\infty;\phi_{\mathrm{lg}})\otimes[\Psi_{a_{f,\Sigma}}]_\Sigma.$$

If $a_f=(B_f;0)$ with $B_f\in\mathrm{GL}_b(\widehat{\mathcal{O}}_E)$, then $\det((B_f;0)_\Sigma\mid Y')=\det B_\Sigma\det Y'_2$, Y'_2 the rows n of Y', so $\Psi_{a_{f,\Sigma}}=\varepsilon'\chi^{-1}(\det B_\Sigma)\,(\mathbf{1}\otimes\Psi_1^{\mathrm{st}})$ (the product of the constant 1 on the rows b and Ψ_1^{st} on the rows n, on integral matrices); both factors are functions modulo $\mathfrak{m}$ (Step 1), and by (a) of Lemma 6.3.80 (with $P_1=P_2=1$) and $v_1\otimes F=F$, $[\mathbf{1}\otimes\Psi_1^{\mathrm{st}}]_\Sigma=v^0\boxtimes^{\mathrm{tg}}F_{\mathfrak{c}}$, v^0 the vacuum of $\mathcal{M}_{W_1,V'}$. So

$$f[\kappa_\alpha a]=\mathrm{vf}_U\,\varepsilon'\chi^{-1}(\det B_\Sigma)\,\mathrm{rsl}_{D,U}(a_\infty;\phi_{\mathrm{lg}})\otimes(v^0\boxtimes^{\mathrm{tg}}F_{\mathfrak{c}})\qquad(a_f=(B_f;0),\ B_f\in\mathrm{GL}_b(\widehat{\mathcal{O}}_E)),\tag{6.35}$$

and $f[\kappa_\alpha a]=0$ if a_f is not integral.

Small side. Read $[\Phi']_{\bar\Sigma}$ as the local datum at $\bar\Sigma$ of $G(y_{\bar\Sigma})=\Phi'((\bar y)_\Sigma)$. For M' over $E_{\bar\Sigma}$ and $w\in\Sigma$, $(z_0)_w$ is the conjugate of $((k_{\mathbb{A}})_{\bar w})^{\mathsf{T}}$, so $z_0\,\overline{M'}^{\mathsf{T}}$ has at w the conjugate of $(M'_{\bar w}(k_{\mathbb{A}})_{\bar w})^{\mathsf{T}}=a_{\bar w}^{\mathsf{T}}$ ($M'k_{\mathbb{A}}=a$). Hence $G(M')=\prod_i\mathbf{1}_{\mathcal{O}_{\bar\Sigma}}(z'_i)\cdot\mathbf{1}[a,Y'\text{ integral}]\,\chi^{-1}((\overline{\det a})_\Sigma)$, and Lemma 6.2.19 for $\bar\Sigma$ and $D_{V'}$ (adapted at 1 in $m_{V'}$ by Lemma 6.3.53), with $\mathbf{1}_{\bar\Sigma}\otimes_{\bar\Sigma}v^0=v^0$, gives for an invertible adelic $b\times b$ matrix B

$$f'[\kappa_\alpha B]=\mathrm{vf}_{V'}\,\mathbf{1}[B_f\text{ integral}]\,\chi^{-1}((\overline{\det B_f})_\Sigma)\,\mathrm{rsl}_{D,V'}(B_\infty;\phi_{\mathrm{sm}})\otimes v^0.\tag{6.36}$$

Step 5: the finite part of the point. Let $h_0=(h_\infty,u)\in\mathrm{G}(t'\sqcup s)$. Apply (IP) to h_∞: it gives (p_1,p_2) and $\hat h=\mathrm{diag}(p_{1,\infty},p_{2,\infty})h_\infty$. *Case A: no $(u_1,u_2)\in G_f(b)\times G_f(n)$ has* $\mathrm{diag}(u_1,u_2)u\in\mathbf{K}_0(b\sqcup n)$; then $\varrho(h_0)=0$. Take $(k_1,k_2)=(1,1)$. For $y\in\mathrm{G}(t')$, $y'\in\mathrm{G}(s)$ let $h=\mathrm{diag}(y,y')h_0$, $a=h^{-1}(Y;0)$, so $a_f=h_f^{-1}(Y_f;0)$. Suppose a_f integral. Then $Z=a_fk$ is integral with $Z^*Z=(Y_fk)^*(Y_fk)=1$, and so is $Z_0=(1;0)$; both have the integral left inverses Z^*, Z_0^*. By (2) of Lemma 3.2.6 there is $g_1\in\mathrm{GL}_{b\sqcup n}(\widehat{\mathcal{O}}_E)$ with $g_1^*g_1=1$ and $g_1Z=Z_0$. The unitary matrix $U_0=g_1h_f^{-1}\mathrm{diag}(Y_fk,1)$ fixes $(1;0)$, so $U_0^*=U_0^{-1}$ fixes it too and $U_0=\mathrm{diag}(1,w')$ with $w'\in G_f(n)$. Then $h_f=\mathrm{diag}(Y_fk,w'^{-1})g_1$ and $u=\mathrm{diag}(y_f,y'_f)^{-1}\mathrm{diag}(Y_fk,w'^{-1})g_1$, so $\mathrm{diag}(u_1,u_2)u=g_1\in\mathbf{K}_0(b\sqcup n)$ with $u_1=(y_f^{-1}Y_fk)^{-1}\in G_f(b)$, $u_2=w'y'_f\in G_f(n)$, against the assumption. So a_f is not integral, $f[\kappa_\alpha a]=0$ by Step 4, and $\mathrm{term}^{\mathrm{lg}}_{x_0}(p,h)=0$ by Step 3 for all y,y'. The identity of the statement holds with both sides 0. *Case B:* $\mathrm{diag}(u_1,u_2)u=\kappa\in\mathbf{K}_0(b\sqcup n)$ *for some* (u_1,u_2). Then $\varrho(h_0)=\mathrm{pw}(h_\infty)$. Take $k_1=(p_{1,\infty},u_1)\in\mathrm{G}(t')$, $k_2=(p_{2,\infty},u_2)\in\mathrm{G}(s)$, so that $\mathrm{diag}(k_1,k_2)h_0=(\hat h,1)(1,\kappa)$. By (6.34), for y,y',

$$\mathrm{term}^{\mathrm{lg}}_{x_0}(p,\mathrm{diag}(y,y')\mathrm{diag}(k_1,k_2)h_0)=\mathrm{term}^{\mathrm{lg}}_{x_0}(p,\mathrm{diag}(y,y')(\hat h,1)).$$

Step 6: the inner integral in Case B. Fix $y\in\mathrm{G}(t')$ and put $B=y^{-1}Y\in\mathrm{GL}_b(\mathbb{A}_E)$. For $h=\mathrm{diag}(y,y')(\hat h,1)$, $h^{-1}(Y;0)$ has infinite part $\hat h^{-1}(B_\infty;0)$ and finite part $(B_f;0)$, independent of y'. If B_f is not integral, both $\mathrm{term}^{\mathrm{lg}}_{x_0}(p,h)$ and $\mathrm{term}^{\mathrm{sm}}_1(p,y)$ vanish (Steps 3, 4). Let B_f be integral; as $B_fk=y_f^{-1}Y_fk$ is unitary, $(B_fk)^{-1}=(B_fk)^*$ is integral, $B_f\in\mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ and $\det B_f$ is a unit idele. By (5) of Lemma 4.1.1 ($\gamma_{\mathfrak{c}}$ has weight $\#n/2-\mu$; the finite part of $(\hat h,1)$ is 1), $\omega_{\gamma_{\mathfrak{c}},\gamma_{\mathfrak{c}}\upsilon^{\#b}}(1,(\hat h,1))$ sends $w\otimes F$ to $(\Omega_{\#n/2-\mu,0}(1,\hat h)w)\otimes F$. With Step 3, (6.35) and $\omega(g,h)=\omega(g,\mathrm{diag}(y,y'))\,\omega(1,(\hat h,1))$,

$$\mathrm{term}^{\mathrm{lg}}_{x_0}(p,h)=\kappa_{\mathrm{lg}}\,\mathrm{vf}_U\,\varepsilon'\chi^{-1}(\det B_\Sigma)\,\theta_{\mathrm{Hom}(W_1,U)}\Big(\rho((x_c;0))\,\omega_{\gamma_{\mathfrak{c}},\gamma_{\mathfrak{c}}\upsilon^{\#b}}(g,\mathrm{diag}(y,y'))\\(w_{\hat h}(B)\otimes(v^0\boxtimes^{\mathrm{tg}}F_{\mathfrak{c}}))\Big).$$

This is a continuous function of y', left invariant under Γ_{W_1} (by (2) of Lemma 6.3.25, $\mathrm{diag}(1,\gamma)$ fixing x_0 for $\gamma \in \Gamma_{W_1}$), and bounded by (1) of Lemma 6.3.24; $[\mathrm{U}(W_1)]$ has finite volume (Theorem 4.2.18; a point if n is empty), so it is integrable there. By (IP) with B, $\beta = \upsilon^{\#b}$, the vacuum $v^0 \in \mathcal{M}_{W_1,V'}$, $x = x_c$, g and y, and by (6.36) and Step 3,

$$\int_{[\mathrm{U}(W_1)]} \mathrm{term}^{\mathrm{lg}}_{x_0}(p,h)\,dy' = \frac{\kappa_{\mathrm{lg}}\,\mathrm{vf}_U\,\varepsilon'\chi^{-1}(\det B_\Sigma)\,\mathrm{pw}(h_\infty)\,I_0}{\tilde\nu(z_B)\prod_{v\in\Phi}|\det B_v|^{\#n}} \cdot \frac{\mathrm{term}^{\mathrm{sm}}_1(p,y)}{\kappa_{\mathrm{sm}}\,\nu(\det y)^2\,\mathrm{vf}_{V'}\,\chi^{-1}((\overline{\det B_f})_\Sigma)}.$$

Now $\det q = d_0 \det y \det B$, $\det B = z_B \det B_f$ as ideles, $|\det y| = |\det B_f| = 1$ and $|z_B|^{\#n/2} = \prod_v |\det B_v|^{\#n}$; by Step 3 and Step 1 (b), (c),

$$\frac{\kappa_{\mathrm{lg}}}{\kappa_{\mathrm{sm}}} = \tilde\nu(d_0)\,\nu(\det y)^2\,\tilde\nu(z_B)\,\tilde\nu(\det B_f)\,|d_0|^{\#n/2}\prod_v |\det B_v|^{\#n},$$
$$\chi^{-1}(\det B_\Sigma)\,\tilde\nu(\det B_f) = \chi^{-1}((\overline{\det B_f})_\Sigma).$$

So the scalar is $\frac{\mathrm{vf}_U}{\mathrm{vf}_{V'}}\varepsilon'\tilde\nu(d_0)|d_0|^{\#n/2}\mathrm{pw}(h_\infty)I_0 = \varrho(h_0)\,C_1$, and for every $y \in \mathrm{G}(t')$

$$\int_{[\mathrm{U}(W_1)]} \mathrm{term}^{\mathrm{lg}}_{x_0}(p, \mathrm{diag}(y,y')\mathrm{diag}(k_1,k_2)h_0)\,dy' = \varrho(h_0)\,C_1\,\mathrm{term}^{\mathrm{sm}}_1(p,y), \tag{6.37}$$

also when B_f is not integral (both sides 0).

Step 7: conclusion. By (1) of Lemma 6.3.24, f' has integrable terms at p for D, V', $\alpha'_{\mathfrak{c}}, \beta'$; by (2) of Lemma 6.3.25 for $x_0 = 1 \in \mathrm{M}_b(E)$, whose stabiliser in the rational points of $\mathrm{U}(V')$ is trivial, $y \mapsto \mathrm{term}^{\mathrm{sm}}_1(p,y)$ is integrable on $\mathrm{G}(t')$. In Case B, (6.37) shows that the inner integral exists for every y and is an integrable function of y, and integrating (6.37) over $\mathrm{G}(t')$ gives the identity of the statement; in Case A everything is 0. □

Lemma 6.3.24. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, X a space with signs r on m, W_1 a space with signs s on n, V' a space with signs t' on a nonempty index set b, D a base-adapted splitting of X over W_1 along b with coordinate pr_l, $z_0 = (\mathrm{pr}_l)_\Sigma \in \mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$, and $\sigma\colon b \sqcup (b \sqcup n) \to m$ a bijection. Let $\phi \in \mathcal{F}_{X,V'\oplus W_1}$ and $\phi' \in \mathcal{F}_{X,V'}$ be Schwartz vectors, put $\alpha = \nu^{-1}\upsilon^{\lfloor \#m/2\rfloor}$, $\alpha' = \nu\,\upsilon^{\lfloor \#m/2\rfloor}$, $\beta' = \upsilon^{\#b}$,*

$$f = \phi \otimes \big[\Phi^{\mathrm{st}}_{\sigma,z_0}\big]_\Sigma \in \mathcal{M}_{X,V'\oplus W_1}, \qquad f' = \phi' \otimes \big[(\Psi^{\mathrm{st}}_{z_0})^{\mathsf{T}}\big]_{\bar\Sigma} \in \mathcal{M}_{X,V'},$$

and let $p = \mathrm{rad}(c,z)\,\mathrm{lev}(q,g)$ with $c \in \mathrm{M}_{n\times b}(\mathbb{A}_E)$, $z \in \mathrm{M}_b(\mathbb{A}_E)$, $q \in \mathrm{GL}_b(\mathbb{A}_E)$, $g \in \mathrm{G}(s)$ (elements of D).

1. *f has integrable terms at p for D, $V' \oplus W_1$, α, α, and f' has integrable terms at p for D, V', α', β'; moreover the sums $\sum_x |\mathrm{term}_x(f)(p,h)|$ over $x \in \mathrm{M}_{(b\sqcup n)\times b}(E)$ and $\sum_x |\mathrm{term}_x(f')(p,h')|$ over $x \in \mathrm{M}_b(E)$ (theta terms for the same data) are bounded functions of $h \in \mathrm{G}(t' \sqcup s)$, resp. $h' \in \mathrm{G}(t')$.*

2. *Let $T \in \mathrm{M}_b(E)$ be such that T, read as an adelic matrix, is not represented by the signs t'. Then $\mathrm{coef}\big(\jmath T; \theta^{\mathrm{sm}}_{\mathfrak{c}}(X, V'; \phi', \Psi^{\mathrm{st}}_{z_0})\big)(p) = 0$. If moreover every $x = (x_v)_{v\in\Phi}$, x_v complex $(b \sqcup n) \times b$ matrices, with $\mathrm{rsl}_{D,V'\oplus W_1}(x;\phi) \neq 0$ has, at every $v \in \Phi$, $x_v^*\mathrm{diag}(t' \sqcup s)\,x_v = e^*\mathrm{diag}(t')\,e$ for some $e \in \mathrm{M}_b(\mathbb{C})$, then $\mathrm{coef}\big(\jmath T; \theta^{\mathrm{lg}}_{\mathfrak{c}}(X, V' \oplus W_1; \phi, \Phi^{\mathrm{st}}_{\sigma,z_0})\big)(p) = 0$.*

Proof. Write $D = (k,l,i)$ with adelic frames $k_{\mathbb{A}}, l_{\mathbb{A}}, i_{\mathbb{A}}$, $\Phi = \Phi^{\mathrm{st}}_{\sigma,z_0}$, $\Phi' = (\Psi^{\mathrm{st}}_{z_0})^{\mathsf{T}}$, $U = V' \oplus W_1$ (signs $t' \sqcup s$ on $b \sqcup n$, point $x_U = \mathrm{diag}(x_{V'}, x_{W_1})$), and D_U, $D_{V'}$ for the splittings of $\mathrm{Hom}(X,U)$, $\mathrm{Hom}(X,V')$ induced by D. By (2) of Lemma 6.3.31, Φ and $\Psi^{\mathrm{st}}_{z_0}$ are locally constant with compact

support; so $\Phi^\sharp \in \mathcal{T}_\Sigma$ and $\Phi'^\flat \in \mathcal{T}_{\bar\Sigma}$ ($y \mapsto (\bar y)_\Sigma$ is a homeomorphism from the components at $\bar\Sigma$ to those at Σ), and $f = \phi \otimes (\Phi^\sharp \otimes_\Sigma v^0)$, $f' = \phi' \otimes (\Phi'^\flat \otimes_{\bar\Sigma} v^0)$; $\bar\Sigma$ contains the conjugate of none of its members, as Σ does not.

Step 1: the point on the frame. By (MX2), $\mathrm{lev}(q,g)k_\mathbb{A} = k_\mathbb{A} q$, $\mathrm{lev}(q,g)i_\mathbb{A} = i_\mathbb{A} g$, $\mathrm{rad}(c,z)k_\mathbb{A} = k_\mathbb{A}$ and $\mathrm{rad}(c,z)i_\mathbb{A} = i_\mathbb{A} + k_\mathbb{A} c^* \varepsilon_s$, since $\mathrm{pr}_k k_\mathbb{A} = 1$, $\mathrm{pr}_i k_\mathbb{A} = \mathrm{pr}_l k_\mathbb{A} = 0$, $\mathrm{pr}_i i_\mathbb{A} = 1$, $\mathrm{pr}_k i_\mathbb{A} = \mathrm{pr}_l i_\mathbb{A} = 0$. Hence

$$p\,k_\mathbb{A} = k_\mathbb{A} q, \qquad p\,i_\mathbb{A} = i_\mathbb{A} g + k_\mathbb{A} c^* \varepsilon_s g, \qquad p^{-1}(k_\mathbb{A} \mid i_\mathbb{A}) = (k_\mathbb{A} \mid i_\mathbb{A})\,Q^{-1}, \quad Q = \begin{pmatrix} q & c^*\varepsilon_s g \\ 0 & g \end{pmatrix}, \tag{6.38}$$

Q being invertible. Let c_p be the finite part of $p^{-1}(k_\mathbb{A} \mid i_\mathbb{A})$. The adelic $m \times m$ matrix $P_0 = (k_\mathbb{A} \mid i_\mathbb{A} \mid l_\mathbb{A})$ is invertible, the coordinates pr_k, pr_i, pr_l forming the block rows of its inverse; and $\mathrm{pr}_l P_0 = (0 \mid 0 \mid 1_b)$.

Step 2: the support of $\Phi^\sharp$. Let η be a finite adelic $(b \sqcup n) \times m$ matrix with $\Phi^\sharp(\eta) = \Phi(\eta_\Sigma) \neq 0$, and $w \in \Sigma$. By (XD10) the $m \times m$ matrix M whose row $\sigma(j)$ is row j of z_0 ($j \in b$) and whose row $\sigma(i')$ is row i' of η_Σ ($i' \in b \sqcup n$) has $\det M \in \mathcal{O}_\Sigma^\times$, so $\det M_w \neq 0$. As $(z_0)_w = ((\mathrm{pr}_l)_f)_w$, the rows $\sigma(j)$, $j \in b$, of $M_w(P_0)_w$ are those of $(0 \mid 0 \mid 1_b)$ and its other rows form $(\eta_w(k_\mathbb{A} \mid i_\mathbb{A})_w \mid \eta_w(l_\mathbb{A})_w)$; expanding along the first group of rows, $\det(M_w(P_0)_w) = \pm \det(\eta_w(k_\mathbb{A} \mid i_\mathbb{A})_w)$. Hence $\eta_w(k_\mathbb{A} \mid i_\mathbb{A})_w$ is invertible, and by (6.38) so is the square matrix $(\eta\, c_p)_w = \eta_w(k_\mathbb{A} \mid i_\mathbb{A})_w (Q^{-1})_w$.

Step 3: the support of $\Phi'^\flat$. Let η be a finite adelic $b \times m$ matrix with $\Phi'^\flat(\eta) = \Psi^{\mathrm{st}}_{z_0}(y) \neq 0$, $y = ((\bar\eta)_\Sigma)^\mathsf{T}$, and $w \in \Sigma$. By (XD10), y is integral and $\det(z_0 y) \in \mathcal{O}_\Sigma^\times$. Now $(z_0)_w = ((k_\mathbb{A})_f^*)_w$ is the conjugate of $(((k_\mathbb{A})_f)_{\bar w})^\mathsf{T}$ and y_w that of $(\eta_{\bar w})^\mathsf{T}$, so $(z_0 y)_w$ is the conjugate of $(\eta_{\bar w}((k_\mathbb{A})_f)_{\bar w})^\mathsf{T}$, and $\eta_{\bar w}(k_\mathbb{A})_{\bar w} \in \mathrm{GL}_b(E_{\bar w})$. With R_0 its inverse, $\eta_{\bar w}(k_\mathbb{A} \mid i_\mathbb{A})_{\bar w}$ has the right inverse $\binom{R_0}{0}$, and by (6.38) $(\eta\, c_p)_{\bar w}$ has the right inverse $Q_{\bar w}\binom{R_0}{0}$; here $\bar w \in \bar\Sigma$.

Step 4: (1). By Step 2 the hypothesis of Lemma 6.3.26 holds for $S = X$, $S_0 = W_1$, D (along the nonempty b), $T = U$ (nonempty index set), the characters α, α, $A = \Sigma$, $\Psi = \Phi^\sharp$, the Schwartz vector ϕ and the point p; by Step 3 it holds for $T = V'$, α', β', $A = \bar\Sigma$, $\Psi = \Phi'^\flat$, ϕ' and p. Items (1) and (2) of Lemma 6.3.26 give (1).

Step 5: (2), the small side. Let $\mathfrak{F}$ be a Borel set meeting each coset of $\Gamma_{V'}$ in $\mathrm{G}(t')$ once; the integral over $[\mathrm{U}(V')]$ of a left $\Gamma_{V'}$-invariant integrable function is its integral over $\mathfrak{F}$, so $\theta^{\mathrm{sm}}_{\mathfrak{c}}(X, V'; \phi', \Psi^{\mathrm{st}}_{z_0})$ is the function Θ_1 of Lemma 6.3.25 for $W = X$, $W_0 = W_1$, D, $V = V'$, α', β', f' and the weight $y = 1$ (the kernel is left $\Gamma_{V'}$-invariant in h by Lemma 6.2.5). By (1) and (1) of Lemma 6.3.25 at $g = p$, the coefficient is the sum over the $x \in \mathrm{M}_b(E)$ with $x^* J_{V'} x = T$ of integrals of theta terms. For such x, $J_{V'} = x_{V'}^{-*} \varepsilon_{t'} x_{V'}^{-1}$ (as $x_{V'}^* J_{V'} x_{V'} = \varepsilon_{t'}$) gives $T = e^* \varepsilon_{t'} e$ with $e = x_{V'}^{-1} x$, so $T_v = e_v^* \mathrm{diag}(t') e_v$ at every $v \in \Phi$: T would be represented by t'. The sum is empty, and the coefficient is 0.

Step 6: (2), the large side. As in Step 5, by (1) and (1) of Lemma 6.3.25 for $V = U$, α, α, f and $y = 1$, the coefficient of the large side at p is $\sum_x \int_{\mathfrak{F}} \mathrm{term}_x(f)(p,h)\,dh$ over the $x \in \mathrm{M}_{(b\sqcup n)\times b}(E)$ with $x^* J_U x = T$, $\mathfrak{F}$ now for Γ_U; we show that every such term vanishes. Let $h \in \mathrm{G}(t' \sqcup s)$. By (6.38), $p\,k_\mathbb{A} = k_\mathbb{A} q$ and $\mathrm{pr}_i\, p\, i_\mathbb{A} = g$; so (2) of Lemma 6.2.45 (for $V = X$, $V_0 = W_1$, the element p with $m = q$, $g_0 = g$, the target U, the characters $\alpha \upsilon^{-\#b}, \alpha$ and h) expresses $(\omega_{\alpha,\alpha}(p,h)f)[x_U^{-1}x]$ as a scalar times $\omega_{\alpha\upsilon^{-\#b},\alpha}(g,h)\rho(\cdot)\,f[Y]$, $Y = h^{-1} x_U^{-1} x q$. It suffices to show $f[Y] = 0$.

By (1) of Lemma 6.3.12, $f[Y]$ is the slice of f along D_U at $x_U Y = (x_U \otimes 1_b) Y = m_U^{-*} Y$. Let $\tilde Y \in \mathbb{C}^{\Phi \times ((b\sqcup n)\times b)}$ have components $\tilde Y_v = \kappa_v^{-1} Y_v$ ($v \in \Phi$), and Y_f the finite part of Y; then $m_U^{-*} Y = m_U^{-*} \mathrm{rv}(\tilde Y) + (x_U Y)_f$, the second summand with components 0 at the infinite places. The splitting D_U is adapted at 1 in the basis m_U by Lemma 6.3.53 (D being base-adapted),

along the nonempty $(b \sqcup n) \times b$; so (3) of Lemma 6.2.20 gives

$$f[Y] = \mathrm{rsl}_{D,U}(\tilde{Y};\phi) \otimes (v_1 \otimes (\Phi^\sharp \otimes_\Sigma v^0))[(x_U Y)_f],$$

the outer slice along D_U. Now $h^* \varepsilon_U h = \varepsilon_U$ and $x_U^{-*} \varepsilon_U x_U^{-1} = J_U$ give $Y^* \varepsilon_U Y = q^* x^* J_U x q = q^* T q$, so at every $v \in \Phi$

$$\tilde{Y}_v^* \,\mathrm{diag}(t' \sqcup s)\, \tilde{Y}_v = \kappa_v^{-2}\, q_v^* T_v q_v.$$

If $\mathrm{rsl}_{D,U}(\tilde{Y};\phi) \neq 0$, the hypothesis gives $e_v \in \mathrm{M}_b(\mathbb{C})$ with $\kappa_v^{-2} q_v^* T_v q_v = e_v^* \mathrm{diag}(t') e_v$, i.e. $T_v = (\kappa_v e_v q_v^{-1})^* \mathrm{diag}(t')(\kappa_v e_v q_v^{-1})$, for every $v \in \Phi$, and T would be represented by t'. So the real slice is 0, hence $f[Y] = 0$ (the datum $w \otimes F$ of (WR6) is linear in w), every term vanishes, and the coefficient of the large side at p is 0. □

Lemma 6.3.25. *Let W, W_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set, (k,l,i) a splitting of W over W_0 along p, V a space with a point with signs t on m and point x_V, α, β automorphic characters of $\mathrm{U}(1)$ and $f \in \mathcal{M}_{W,V}$, with theta terms $\mathrm{term}_x(g,h) = \mathrm{term}_x(f)(g,h)$ of (MX6) for the splitting (k,l,i) of W, the space V and α, β. Fix a Borel set $\mathfrak{F} \subset \mathrm{G}(t)$ meeting each coset $\Gamma_V h$ exactly once. Let $y\colon \mathrm{G}(t) \to \mathbb{C}$ be continuous and left Γ_V-invariant, let $g \in \mathrm{G}(s)$, and assume that f has integrable terms at g with weight y. Put $\Theta_y(g') = \int_{\mathfrak{F}} \theta_{\alpha,\beta}^{W,V}(f)(g',h)\, y(h)\, dh$ for $g' \in \mathrm{G}(s)$.*

1. *For every $\tau \in \mathrm{M}_p(E)$ the centre coefficient $\mathrm{coef}(\jmath\tau;\Theta_y)(g)$ is defined, and*

$$\mathrm{coef}(\jmath\tau;\Theta_y)(g) = \sum_{x \in \mathrm{M}_{m\times p}(E),\ x^* J_V x = \tau} \int_{\mathfrak{F}} \mathrm{term}_x(g,h)\, y(h)\, dh,$$

   the series converging absolutely.

2. *Let $x_0 \in \mathrm{M}_{m\times p}(E)$ have invertible Gram matrix $\tau_0 = x_0^* J_V x_0$. Let $\Gamma_{x_0} = \{x_V^{-1}\gamma x_V : \gamma \in \mathrm{U}(V)(F),\ \gamma x_0 = x_0\}$. Then $h \mapsto \mathrm{term}_{x_0}(g,h) y(h)$ is left Γ_{x_0}-invariant and integrable on $\Gamma_{x_0}\backslash \mathrm{G}(t)$ for the quotient measure of dh, and*

$$\mathrm{coef}(\jmath\tau_0;\Theta_y)(g) = \int_{\Gamma_{x_0}\backslash \mathrm{G}(t)} \mathrm{term}_{x_0}(g,h)\, y(h)\, dh.$$

Proof. By (MX6) with $R = \mathfrak{F}$, $\int_{\mathfrak{F}} \sum_x |\mathrm{term}_x(g,h) y(h)|\, dh < \infty$; this is the hypothesis used below. Write $\beta(x) = x^* J_V x$ and note $(x_V^{-1}x)^* \varepsilon_t (x_V^{-1}x) = \beta(x)$, as $x_V^{-*}\varepsilon_t x_V^{-1} = J_V$.

Step 1: expansion and the centre. By (5) of Lemma 6.2.44 applied to $\omega_{\alpha,\beta}(g',h)f$, $\theta_{\alpha,\beta}^{W,V}(f)(g',h) = \sum_x \mathrm{term}_x(g',h)$, absolutely. For an adelic $p \times p$ matrix M, the element $z(M)$ of (MX2) satisfies $z(M)k_{\mathbb{A}} = k_{\mathbb{A}}$ and $\mathrm{pr}_i z(M) i_{\mathbb{A}} = 1$ (as $\mathrm{pr}_l k_{\mathbb{A}} = \mathrm{pr}_l i_{\mathbb{A}} = 0$), and $z(M)z(M') = z(M+M')$, so $z(M)^{-1} l_{\mathbb{A}} = l_{\mathbb{A}} - k_{\mathbb{A}}(M - M^*)$: in (2) of Lemma 6.2.45 (with $\alpha\upsilon^{-\#p}$ in place of α, $h = 1$) its data are $m = 1$, $g_0 = 1$, $W = 0$, $B = M^* - M$. As $\omega_{\alpha,\beta}(z(M)g,h) = \omega_{\alpha,\beta}(z(M),1)\,\omega_{\alpha,\beta}(g,h)$, that law at $x_V^{-1}x$ gives

$$\mathrm{term}_x(z(M)g,h) = \psi_E\big(\tfrac{1}{2}\jmath\,\mathrm{tr}((M^* - M)\beta(x))\big)\mathrm{term}_x(g,h) = \psi_E\big(-\mathrm{tr}(\jmath\beta(x)M)\big)\mathrm{term}_x(g,h),$$

because $\mathrm{tr}(M^*\beta(x)) = \overline{\mathrm{tr}(\beta(x)M)}$ for hermitian $\beta(x)$ and $\psi_E(\bar{z}) = \psi_E(z)$, $\bar{\jmath} = -\jmath$.

Step 2: (1). Put $I_x = \int_{\mathfrak{F}} \mathrm{term}_x(g,h) y(h)\, dh$; by the hypothesis $\sum_x |I_x| < \infty$. By Step 1, $|\theta_{\alpha,\beta}^{W,V}(f)(z(M)g,h)y(h)| \le \sum_x |\mathrm{term}_x(g,h)y(h)|$, so $\Theta_y(z(M)g)$ is defined, and by dominated convergence

$$\Theta_y(z(M)g) = \sum_x \psi_E\big(-\mathrm{tr}(\jmath\beta(x)M)\big) I_x,$$

a continuous function of M, bounded by $\sum_x |I_x|$ and invariant under $\mathrm{M}_p(E)$ (ψ_E is trivial on E). Integrating against $\psi_E(\mathrm{tr}(\jmath\tau M))$ over the compact group $\mathrm{M}_p(\mathbb{A}_E)/\mathrm{M}_p(E)$ with its probability measure, and exchanging sum and integral (the sum of the absolute values is $\sum_x |I_x| < \infty$),

$$\mathrm{coef}(\jmath\tau;\Theta_y)(g) = \sum_x I_x \int_{\mathrm{M}_p(\mathbb{A}_E)/\mathrm{M}_p(E)} \psi_E\big(\mathrm{tr}(\jmath(\tau-\beta(x))M)\big)\,dM.$$

For $U \in \mathrm{M}_p(E)$, $M \mapsto \psi_E(\mathrm{tr}(UM))$ is a character of this compact group; if $U_{ab} \neq 0$, its restriction to the matrices with one nonzero entry in place (b,a) is $z \mapsto \psi_E(U_{ab}z)$, nontrivial by (a) of Theorem A.5.2. So the inner integral is 1 if $\jmath(\tau-\beta(x)) = 0$, that is $\beta(x) = \tau$, and 0 otherwise. This is (1).

Step 3: equivariance of the terms. Let $\gamma \in \mathrm{U}(V)(F)$ and $\gamma_{\mathbb{A}} = x_V^{-1}\gamma x_V \in \Gamma_V$. By (2) of Lemma 6.2.45 with $g = 1$ (so $m = 1$, $g_0 = 1$, $W = B = 0$) and $h = \gamma_{\mathbb{A}}$, $(\omega_{\alpha,\beta}(1,\gamma_{\mathbb{A}})F)[X] = \omega_{\alpha\upsilon^{-\#p},\beta}(1,\gamma_{\mathbb{A}})F[\gamma_{\mathbb{A}}^{-1}X]$, and $\gamma_{\mathbb{A}}^{-1}x_V^{-1}\gamma x = x_V^{-1}x$. By Lemma 6.2.5 for $\mathrm{Hom}(W_0,V)$ and the automorphic characters $\alpha\upsilon^{-\#p}$, β,

$$\mathrm{term}_{\gamma x}(g,\gamma_{\mathbb{A}}h) = \mathrm{term}_x(g,h). \tag{6.39}$$

Step 4: (2). Let $X_0 = \{x : \beta(x) = \tau_0\}$. Each $x \in X_0$ is injective (τ_0 is invertible), so $x_0c \mapsto xc$ is an isometry from x_0E^p into (E^m, u'^*J_Vu); by Witt's theorem A.6.7 it extends to some $\gamma \in \mathrm{U}(V)(F)$ with $\gamma x_0 = x$. Conversely $\beta(\gamma x_0) = \tau_0$. So $\gamma_{\mathbb{A}} \mapsto \gamma^{-1}x_0$ is a bijection $\Gamma_{x_0}\backslash\Gamma_V \to X_0$. Put $\Phi_x(h) = \mathrm{term}_x(g,h)y(h)$; as y is Γ_V-invariant, (6.39) gives $\Phi_{\gamma x}(\gamma_{\mathbb{A}}h) = \Phi_x(h)$; in particular Φ_{x_0} is left Γ_{x_0}-invariant and $\Phi_{x_0}(\gamma_{\mathbb{A}}h) = \Phi_{\gamma^{-1}x_0}(h)$, so

$$\sum_{\delta\in\Gamma_{x_0}\backslash\Gamma_V} \Phi_{x_0}(\delta h) = \sum_{x\in X_0} \Phi_x(h),$$

and likewise with absolute values. For Borel $\Psi \geq 0$ on $\mathrm{G}(t)$ left invariant under Γ_{x_0}, the defining properties of the two quotient measures (written with $\Psi = \sum_{\gamma'\in\Gamma_{x_0}}(\varphi\Psi)(\gamma'\,\cdot)$, φ the indicator of a Borel set meeting each coset $\Gamma_{x_0}h$ once) give $\int_{\Gamma_{x_0}\backslash\mathrm{G}(t)}\Psi = \int_{\Gamma_V\backslash\mathrm{G}(t)}\sum_{\delta\in\Gamma_{x_0}\backslash\Gamma_V}\Psi(\delta h)$, and the latter is $\int_{\mathfrak{F}}\sum_\delta \Psi(\delta h)\,dh$. Applied to $|\Phi_{x_0}|$ this shows $\int_{\Gamma_{x_0}\backslash\mathrm{G}(t)}|\Phi_{x_0}| \leq \int_{\mathfrak{F}}\sum_x|\Phi_x| < \infty$; applied to the positive and negative parts of the real and imaginary parts of Φ_{x_0} (each Γ_{x_0}-invariant) it gives $\int_{\Gamma_{x_0}\backslash\mathrm{G}(t)}\Phi_{x_0} = \int_{\mathfrak{F}}\sum_{x\in X_0}\Phi_x\,dh = \sum_{x\in X_0} I_x$, which is $\mathrm{coef}(\jmath\tau_0;\Theta_y)(g)$ by (1). □

Lemma 6.3.26. *Let S, S_0, T be spaces with points with signs s on n, s_0 on n_0 and t on n', with n' nonempty; let p be a nonempty index set and D a splitting of S over S_0 along p with adelic frames $k_{\mathbb{A}}$, $i_{\mathbb{A}}$. Let α,β be automorphic characters of $\mathrm{U}(1)$, A a finite set of finite places of E containing the conjugate of none of its members, $\Psi \in \mathcal{T}_A$, $v \in \mathcal{F}_{S,T}$ a Schwartz vector, $f = v \otimes (\Psi \otimes_A v^0) \in \mathcal{M}_{S,T}$ and $g \in \mathrm{G}(s)$. Let c_g be the finite part of the adelic $n \times (p \sqcup n_0)$ matrix $g^{-1}(k_{\mathbb{A}} \mid i_{\mathbb{A}})$, and assume that for every finite adelic $n' \times n$ matrix η with $\Psi(\eta) \neq 0$ there is $w \in A$ such that the $n' \times (p \sqcup n_0)$ matrix $(\eta\, c_g)_w$ over E_w has a right inverse. Then, with the theta terms of f for D, T, α, β:*

1. *there is $C \geq 0$ with $\sum_{x\in\mathrm{M}_{n'\times p}(E)}|\mathrm{term}_x(f)(g,h)| \leq C$ for every $h \in \mathrm{G}(t)$;*
2. *f has integrable terms at g for D, T, α, β.*

Proof. Write $D = (k,l,i)$, $\mathrm{term}_x(\varphi)(h) = \mathrm{term}_x(\varphi)(g,h)$ for $\varphi \in \mathcal{M}_{S,T}$ (g is fixed), $\omega = \omega_{\alpha,\beta}$, and D_T for the splitting of $\mathrm{Hom}(S,T)$ over $\mathrm{Hom}(S_0,T)$ along $n' \times p$ induced by D (MX1), with adelic frames $k'_{\mathbb{A}}$, $i'_{\mathbb{A}}$. By (1) of Lemma 6.3.12, for $x \in \mathrm{M}_{n'\times p}(E)$ the Hom slice $F[x_T^{-1}x]$ is the slice of F along D_T at $(x_T \otimes 1_p)(x_T^{-1}x) = x$.

Step 1: rational points permute the terms. Let $\gamma \in \Gamma_T$ and $\gamma_0 = x_T \gamma x_T^{-1} \in \mathrm{U}(T)(F)$. We show $\mathrm{term}_{\gamma_0 x}(\varphi)(\gamma h) = \mathrm{term}_x(\varphi)(h)$. By (2) of Lemma 6.2.45 for the element $1 \in \mathrm{G}(s)$ (so $m = 1$, $g_0 = 1$, $W = B = 0$), the characters $\alpha\upsilon^{-\#_p}, \beta$ and the element γ of $\mathrm{G}(t)$, $(\omega(1,\gamma)F)[X] = \omega_{\alpha\upsilon^{-\#_p},\beta}(1,\gamma)\, F[\gamma^{-1}X]$, and $\gamma^{-1}x_T^{-1}\gamma_0 x = x_T^{-1}x$. As $\omega(g,\gamma h) = \omega(1,\gamma)\omega(g,h)$ ((2) of Lemma 4.1.1) and $\theta_0 \circ \omega_{\alpha\upsilon^{-\#_p},\beta}(1,\gamma) = \theta_0$ by Lemma 6.2.5 for (S_0, T),

$$\mathrm{term}_{\gamma_0 x}(\varphi)(\gamma h) = \theta_0\Big(\omega_{\alpha\upsilon^{-\#_p},\beta}(1,\gamma)\big((\omega(g,h)\varphi)[x_T^{-1}x]\big)\Big) = \mathrm{term}_x(\varphi)(h).$$

Step 2: growth in the height. We show that there are $C_1 \geq 0$ and $B \in \mathbb{N}$ with

$$\sum_{x \in \mathrm{M}_{n'\times p}(E)} |\mathrm{term}_x(\varphi)(h)| \leq C_1\, \|\varphi\|\, \mathrm{ht}_T(h)^B \qquad (\varphi \in \mathcal{M}_{S,T},\ h \in \mathrm{G}(t)). \tag{6.40}$$

By (2) of Lemma 6.2.44 for D_T (along the nonempty $n' \times p$) there is $M \geq 0$ with $\sum_x \|F[x_T^{-1}x]\| \leq M\|F\|$ for all $F \in \mathcal{M}_{S,T}$. Let $\Omega_T \subseteq \mathrm{G}(t)$ be the compact set of Lemma 6.2.29 for T, and $B_T = \{(y,1) : y \in G_\infty(t),\ \|y_v\| \leq 2\ (v \in \Phi)\}$, compact since $\mathrm{U}(\mathrm{diag}\, t)$ is closed and $\|y^{-1}\| = \|\mathrm{diag}(t) y^* \mathrm{diag}(t)\| = \|y\|$ on it. The set $K = \{(g,1)\} \cup (\{1\} \times (\Omega_T \cup B_T))$ is compact; for each φ the map $\kappa \mapsto \omega_{1,1}(\kappa)\varphi$ is continuous ((2) of Lemma 4.1.1), hence bounded on K, and the uniform boundedness principle gives $C_0 = \max(1, \sup_{\kappa\in K} \|\omega_{1,1}(\kappa)\|) < \infty$. Choose $B \in \mathbb{N}$ with $2^B \geq C_0$. Given h, Lemma 6.2.29 writes $h = \gamma z (y_1 \cdots y_k, 1)$ with $\gamma \in \Gamma_T$, $z \in \Omega_T$, $(y_j, 1) \in B_T$ and $2^k \leq \mathrm{ht}_T(h)$. Put $h' = \gamma^{-1}h$. By Step 1 applied to $\gamma_0^{-1}x$, $\mathrm{term}_x(\varphi)(h) = \mathrm{term}_{\gamma_0^{-1}x}(\varphi)(h')$, and $x \mapsto \gamma_0^{-1}x$ is a bijection of $\mathrm{M}_{n'\times p}(E)$; so

$$\sum_x |\mathrm{term}_x(\varphi)(h)| = \sum_x |\mathrm{term}_x(\varphi)(h')| \leq \|\theta_0\|\, M\, \|\omega(g,h')\|\, \|\varphi\|.$$

By (WR4), $\|\omega(g,h')\| = \|\omega_{1,1}(g,h')\|$, and $(g,h') = (g,1)(1,z)\prod_j (1,(y_j,1))$ is a product of $k+2$ elements of K; so $\|\omega(g,h')\| \leq C_0^{k+2} \leq 4^B\, \mathrm{ht}_T(h)^B$. This is (6.40) with $C_1 = 4^B\|\theta_0\|M$.

Step 3: a short frame. The finite part of the adelic matrix $(x_S g)^{-1}(k \mid i)$ has finitely many entries, so there is an integer $N_0 \geq 1$ with $N_0\,((x_S g)^{-1}(k \mid i))_f$ integral. Put $c = N_0(k \mid i) \in \mathrm{M}_{n\times(p\sqcup n_0)}(E)$: its columns lie in the lattice of S_g; let $Z \geq 0$ be the largest of their lengths. Since $k_{\mathbb{A}} = x_S^{-1}k$ and $i_{\mathbb{A}} = x_S^{-1} i x_{S_0}$,

$$c' = (x_S g)^{-1} c = N_0\, g^{-1}(k_{\mathbb{A}} \mid i_{\mathbb{A}})\, \mathrm{diag}(1_p, x_{S_0}^{-1}), \qquad c'_f = N_0\, c_g\, \mathrm{diag}(1_p, x_{S_0,f}^{-1}). \tag{6.41}$$

Moreover $c^* J_S k = N_0 \begin{pmatrix} k^* J_S k \\ i^* J_S k \end{pmatrix} = 0$ by (MX1).

Step 4: invariance along the frame. Let $d \in \mathrm{M}_{n'\times 1}(E)$, $\mathcal{X} \subset \mathrm{M}_{n'\times p}(E)$ finite and $u_x \in \mathbb{C}$ $(x \in \mathcal{X})$; let $\ell = \sum_{x\in\mathcal{X}} u_x\, \theta_0(\,\cdot\,[x_T^{-1}x])$, a bounded linear functional on $\mathcal{M}_{S,T}$. We show $\ell(\rho(z)F) = \ell(F)$ for $F \in \mathcal{M}_{S,T}$ and $z = x_T^{-1} y x_S$, $y = dY c^* J_S$, $Y \in \mathrm{M}_{1\times(p\sqcup n_0)}(E)$, i.e. $z \in Z_{d,c}(E)$ in the notation of Lemma 6.2.31. By Step 3, $yk = 0$. By (MX1) every adelic vector is $l_{\mathbb{A}}a + i_{\mathbb{A}}w + k_{\mathbb{A}}b$ with (a,w,b) its coordinates, so $1 = l_{\mathbb{A}}\mathrm{pr}_l + i_{\mathbb{A}}\mathrm{pr}_i + k_{\mathbb{A}}\mathrm{pr}_k$, and $z k_{\mathbb{A}} = x_T^{-1} y k = 0$ gives

$$z = Y_1 \mathrm{pr}_i + x_T^{-1}(yl)\mathrm{pr}_l = i'_{\mathbb{A}} Y_1 + k'_{\mathbb{A}} X', \qquad Y_1 = x_T^{-1}(yi)x_{S_0}, \quad X' = J_T\, yl \in \mathrm{M}_{n'\times p}(E).$$

By the mixed law ((1) of Lemma 6.2.44) for D_T, at the slice point x, with $a = 0$, $w = Y_1$, $b = X'$,

$$(\rho(z)F)[x_T^{-1}x] = \psi_E\big(\jmath\,\mathrm{tr}(X'^* x)\big)\, \rho(Y_1)\, F[x_T^{-1}x] = \rho(Y_1)\, F[x_T^{-1}x],$$

as $\jmath\,\mathrm{tr}(X'^{*}x) \in E$ and ψ_E is trivial on E. Since Y_1 is the rational vector of $\mathrm{Hom}(S_0, T)$ attached to $yi \in \mathrm{M}_{n'\times n_0}(E)$, $\theta_0(\rho(Y_1)G) = \theta_0(G)$, and the claim follows term by term.

Step 5: (1). Let $f_0 = \Psi \otimes_A v^0$, so $f = v \otimes f_0$. Let C' be the constant of (2) of Lemma 6.2.31 for S, T, α, β, A, Ψ, the index sets $\{1\}$ (columns of d) and $p \sqcup n_0$ (columns of c), $N = 0$, $Y_0 = C_1$, $Z_0 = Z$ and $A_0 = B$. Fix h, $H = \mathrm{ht}_T(h) \geq 1$ and a finite $\mathcal{X} \subset \mathrm{M}_{n'\times p}(E)$; choose u_x with $|u_x| = 1$ and $u_x\,\mathrm{term}_x(f)(h) = |\mathrm{term}_x(f)(h)|$, and let ℓ be as in Step 4. Then $\ell(\omega(g,h)\varphi) = \sum_{x\in\mathcal{X}} u_x\,\mathrm{term}_x(\varphi)(h)$, so by (6.40) $|\ell(\omega(g,h)\varphi)| \leq C_1\|\varphi\|H^B$ for all φ: the growth hypothesis of (2) of Lemma 6.2.31 holds with $r = 1$. Let $H > 2$. By Lemma 6.2.30 there is a nonzero d in the lattice of T_h with $d^*J_Td = 0$ and length at most $2/H$. We check that (d, c) is a short pair for $(g, h, \ell, 2/H, Z)$: (i) holds by the choices of d and c; (ii) as $d^*J_Td = 0$; (iii) is Step 4. For (iv), first alternative, let $d' = (x_Th)^{-1}d$ and η with $\Psi(\eta) \neq 0$; take $w \in A$ and a right inverse R of $(\eta c_g)_w$ as in the hypothesis. The entries of the row $(d'^{*}_f)_w$ are the conjugates of those of $(d'_f)_{\bar w} = ((x_Th)^{-1})_{\bar w}\,d \neq 0$, so $(d'^{*}_f)_w \neq 0$, and $(d'^{*}_f)_w(\eta c_g)_wR = (d'^{*}_f)_w$ shows $(d'^{*}_f)_w(\eta c_g)_w \neq 0$. By (6.41), $(d'^{*}_f\eta c'_f)_w = N_0\,(d'^{*}_f)_w(\eta c_g)_w\,\mathrm{diag}(1_p, (x^{-1}_{S_0,f})_w) \neq 0$. The product of the lengths is $2Z/H = 2Z_0r/H$. Hence (2) of Lemma 6.2.31 (with $\mathrm{mom}_B(v) < \infty$, v being a Schwartz vector) gives, for every $H \geq 1$,

$$\sum_{x\in\mathcal{X}} |\mathrm{term}_x(f)(h)| = \ell(\omega(g,h)f) \leq C'\,\mathrm{mom}_B(v).$$

The bound does not depend on $\mathcal{X}$ or h; taking the supremum over $\mathcal{X}$ proves (1) with $C = C'\mathrm{mom}_B(v)$.

Step 6: (2). Each $h \mapsto \mathrm{term}_x(f)(h)$ is continuous ((2) of Lemma 4.1.1, the Hom slice and θ_0 being bounded linear), so $h \mapsto \sum_x |\mathrm{term}_x(f)(h)|$ is Borel. Let $R \subset \mathrm{G}(t)$ be a measurable set of representatives of $\Gamma_T\backslash\mathrm{G}(t)$. By the defining property of the quotient measure ($\sum_{\gamma\in\Gamma_T} \mathbf{1}_R(\gamma h) = 1$), the measure of R is the volume of $[\mathrm{U}(T)]$, which is finite by Theorem 4.2.18 (n' is nonempty and the measure of $\mathrm{G}(t)$ corresponds to a Haar measure of $\mathrm{U}(T)(\mathbb{A}_F)$). By (1), $\int_R \sum_x |\mathrm{term}_x(f)(h)|\,dh \leq C\,\mathrm{vol}([\mathrm{U}(T)]) < \infty$, which is (2) (weight 1). □

Lemma 6.3.27. *Let X, Y, Z be spaces with points with signs r on m, s on n, t on a, let b be a nonempty finite index set and D a splitting of X over Z along b. Let $\mu\colon \Phi \to \mathbb{Z}$ with $\mu(\tau) < \#\{j \in a : t_j = 1\}$ for every $\tau \in \Phi$, and assume that $s_i = 1$ for some $i \in n$ and $\#b + \#\{j \in a : t_j = -1\} = \#\{k \in n : s_k = -1\}$. Let ν be an automorphic character of $\mathrm{U}(1)$ of weight μ and $\delta = \nu\upsilon^{\#b}$. Let $T \in \mathrm{M}_b(E)$ be invertible and, read as an adelic matrix, not represented by the signs -1 on b. Let $f \in \mathcal{M}_{X,Y}$ and $g \in \mathrm{G}(r)$ be such that f has integrable terms at g (for D, Y, δ, δ), and put $\Theta(g') = \int_{[\mathrm{U}(Y)]} \theta^{X,Y}_{\delta,\delta}(f)(g', h)\,dh$. Then*

$$\mathrm{coef}(\jmath T; \Theta)(g) = 0.$$

Proof. Write $\omega = \omega_{\delta,\delta}$ on $\mathcal{M}_{X,Y}$, $\omega' = \omega_{\nu,\delta}$ on $\mathcal{M}_{Z,Y}$, and $\mathrm{term}_x = \mathrm{term}_x(f)$ for the theta terms of (MX6) (for D, Y, δ, δ).

Step 1: unfolding. By (1) of Lemma 6.3.25 (with $(W, W_0, V) = (X, Z, Y)$, $p = b$, $\alpha = \beta = \delta$, weight $y = 1$; the hypothesis is the integrability of the terms), $\mathrm{coef}(\jmath T; \Theta)(g)$ is the sum over the $x \in \mathrm{M}_{n\times b}(E)$ with $x^*J_Yx = T$ of $\int \mathrm{term}_x(g, h)\,dh$ over a set of representatives of $[\mathrm{U}(Y)]$. If there is no such x the coefficient is 0. Otherwise fix one, x_0; as T is invertible, (2) of Lemma 6.3.25 gives

$$\mathrm{coef}(\jmath T; \Theta)(g) = \int_{\Gamma_{x_0}\backslash\mathrm{G}(s)} \mathrm{term}_{x_0}(g, h)\,dh, \tag{6.42}$$

$h \mapsto \mathrm{term}_{x_0}(g,h)$ being left Γ_{x_0}-invariant and integrable on the quotient, $\Gamma_{x_0} = \{x_Y^{-1}\gamma x_Y : \gamma \in \mathrm{U}(Y)(F),\ \gamma x_0 = x_0\}$. Put $x' = x_Y^{-1}x_0$ and $S = \{h \in \mathrm{G}(s) : hx' = x'\}$, a closed subgroup containing Γ_{x_0}.

Step 2: the stabiliser acts through the smaller pair. Let $k \in S$, $h \in \mathrm{G}(s)$, and $\varphi_h = (\omega(g,h)f)[x'] \in \mathcal{M}_{Z,Y}$. Then

$$\mathrm{term}_{x_0}(g,kh) = \theta_{\mathrm{Hom}(Z,Y)}\big(\omega'(1,k)\varphi_h\big). \tag{6.43}$$

Indeed, (2) of Lemma 6.2.45 for $(V,V_0,T) = (X,Z,Y)$, the element 1 of $\mathrm{G}(r)$ (so $m=1$, $g_0=1$, $W=0$, $B=0$), the characters (ν,δ) (so $\nu\upsilon^{\#b} = \delta$) and $h=k$ gives $(\omega(1,k)F)[x] = \omega'(1,k)F[k^{-1}x]$ for every $F \in \mathcal{M}_{X,Y}$ and adelic x; take $F = \omega(g,h)f$, $x = x'$, and use $\omega(1,k)\omega(g,h) = \omega(g,kh)$ ((2) of Lemma 4.1.1) and $k^{-1}x' = x'$.

Step 3: a compact subgroup of S. We have $x'^{*}\varepsilon_s x' = x_0^{*}J_Y x_0 = T$. As T is not represented by the signs -1, there is $\tau \in \Phi$ such that the complex matrix x'_τ has Gram matrix $x'^{*}_\tau \mathrm{diag}(s) x'_\tau = T_\tau$ not of the form $-c^{*}c$, i.e. not negative semidefinite. Lemma 6.3.28 (for the pair (Z,Y), μ, τ, b and $x = x'_\tau$; its hypotheses are those of the statement) gives $u \in \mathrm{U}(\mathrm{diag}\, s)$ such that every continuous linear form on $\mathcal{F}_{Z,Y}$ invariant under $\Omega_{\mu,0}(1,c_{[\tau]})$ for all $c \in K_s$ with $cux'_\tau = ux'_\tau$ is zero. Let $C_0 = \{c \in K_s : cux'_\tau = ux'_\tau\}$, compact, and $C = \{(u^{-1}cu)_{[\tau]} : c \in C_0\}$; its elements fix x', so C is a compact subgroup of S. Let dc be its Haar probability measure (Theorem A.5.1) and $w = (u^{-1})_{[\tau]} \in \mathrm{G}(s)$.

Step 4: the mean over C vanishes. Let $k \in S$ and $A(\varphi) = \int_C \theta_{\mathrm{Hom}(Z,Y)}(\omega'(1,kc)\varphi)\,dc$ for $\varphi \in \mathcal{M}_{Z,Y}$. The integrand is continuous in c ((2) of Lemma 4.1.1), and $\sup_{c\in C}\|\omega'(1,kc)\| < \infty$ by the uniform boundedness principle on the Banach space $\mathcal{M}_{Z,Y}$, so A is a bounded linear functional; by invariance of dc, $A(\omega'(1,p)\varphi) = A(\varphi)$ for $p \in C$. Fix $f' \in \mathcal{M}_{Z,Y}$ and put $\ell(v) = A(\omega'(1,w)(v \otimes f'))$ for $v \in \mathcal{F}_{Z,Y}$. It is continuous: $\|v \otimes f'\| \le \|v\|_{L^1}\|f'\|$, since ρ preserves the L^1 norm. By Lemma 4.1.5 (the real parts act on the Fock factor; ν has weight μ, and the weight of δ does not enter at $(1,y)$), $\omega'(1,y)(v \otimes f') = (\Omega_{\mu,0}(1,y)v) \otimes f'$ for $y \in G_\infty(s)$. For $c \in C_0$, $p = w\,c_{[\tau]}w^{-1} = (u^{-1}cu)_{[\tau]} \in C$, so

$$\ell(\Omega_{\mu,0}(1,c_{[\tau]})v) = A\big(\omega'(1,w)\omega'(1,c_{[\tau]})(v\otimes f')\big) = A\big(\omega'(1,p)\omega'(1,w)(v\otimes f')\big) = \ell(v).$$

Hence $\ell = 0$, i.e. $A\big((\Omega_{\mu,0}(1,w)v)\otimes f'\big) = 0$ for all v, f'; as $\Omega_{\mu,0}(1,w)$ is invertible, $A(v\otimes f') = 0$ for all $v \in \mathcal{F}_{Z,Y}$, $f' \in \mathcal{M}_{Z,Y}$, in particular for $v = v_P$, $f' = \rho_f(y)v^0$. By Lemma 6.3.30, $A = 0$. With (6.43),

$$\int_C \mathrm{term}_{x_0}(g,kch)\,dc = A(\varphi_h) = 0 \qquad (k \in S,\ h \in \mathrm{G}(s)). \tag{6.44}$$

Step 5: the averaging criterion. Apply Lemma 6.2.50 with $G = \mathrm{G}(s)$ and its measure, $H = S$ with a left Haar measure, $\Gamma = \Gamma_{x_0}$ (discrete, contained in S), $F(h) = \mathrm{term}_{x_0}(g,h)$ (continuous, by (2) of Lemma 4.1.1 and the boundedness of the slice map and of the theta functional; left Γ-invariant and integrable on $\Gamma\backslash G$ by Step 1), (C,dc), $t(x,c) = c$, $d = 1$ and $a = 0$: right translation by an element of the compact subgroup C preserves the left Haar measure of S, the modular function of S being a continuous homomorphism to $\mathbb{R}_{>0}$, trivial on compact subgroups; and (6.44) is hypothesis (2) with $a = 0$. As $0 \neq 1$, the integral (6.42) vanishes. □

Lemma 6.3.28. *Let S, T be spaces with points with signs t on a and s on n, let $\mu\colon \Phi \to \mathbb{Z}$, $\tau \in \Phi$ and b a finite index set. Assume $\mu(\tau) < \#\{j \in a : t_j = 1\}$, $s_i = 1$ for some $i \in n$, and $\#b + \#\{j \in a : t_j = -1\} = \#\{k \in n : s_k = -1\}$. Let x be a complex $n \times b$ matrix such that $x^{*}\mathrm{diag}(s)\,x$ is not negative semidefinite. Let $K_s = \{c \in \mathrm{U}(\mathrm{diag}\, s) : c^{*}c = 1\}$, and for $c \in \mathrm{U}(\mathrm{diag}\, s)$ let $c_{[\tau]} \in G_\infty(s)$ have component c at τ and 1 at the other places. Then there is $u \in \mathrm{U}(\mathrm{diag}\, s)$ such that every continuous linear form ℓ on $\mathcal{F}_{S,T}$ with*

$$\ell(\Omega_{\mu,0}(1,c_{[\tau]})w) = \ell(w) \qquad (w \in \mathcal{F}_{S,T},\ c \in K_s,\ cux = ux)$$

is zero.

Proof. Write $D = \operatorname{diag}(s)$, $h(y,y') = y^*Dy'$ on $\mathbb{C}^n$, and $N = \#\{j : t_j = 1\} - \mu(\tau) \geq 1$. The variables of $\mathcal{F}_{S,T}$ are $X_{v,k,j}$ ($v \in \Phi$, $k \in n$, $j \in a$).

Step 1: the isometry u. As x^*Dx is not negative semidefinite, some $v \in \mathbb{C}^b$ has $h(xv,xv) > 0$; rescale so that $h(xv,xv) = 1 = h(e_i,e_i)$. By Witt's theorem A.6.7 (for $\mathbb{C}$ with complex conjugation, the nondegenerate form h, the line $\mathbb{C}xv$ and the isometric map $xv \mapsto e_i$) there is an isometry u of h, i.e. $u \in \mathrm{U}(\operatorname{diag} s)$, with $uxv = e_i$. Put $y = ux$ and $C = \{c \in K_s : cy = y\}$.

Step 2: a negative unit vector for each point. Let $z \in \mathbb{C}^{\Phi\times n\times a}$. There is $r \in \mathbb{C}^n$ with $r_k = 0$ whenever $s_k = 1$, $\sum_k r_k\bar{r}_k = 1$, $\sum_k r_k y_{kl} = 0$ for $l \in b$, and $\sum_k \bar{r}_k z_{\tau,k,j} = 0$ for j with $t_j = -1$. Indeed, the linear map $f\colon r \mapsto ((\sum_k r_k y_{kl})_{l\in b}, (\sum_k r_k \bar{z}_{\tau,k,j})_{t_j=-1}, (r_k)_{s_k=1})$ goes from $\mathbb{C}^n$ to a space of dimension $\#b + \#\{t_j = -1\} + \#\{s_k = 1\} = \#n$. If f were injective it would be onto, and some r would have $f(r) = (0,0,(1)_k)$; then $r_i = 1$, but $\sum_k r_k(yv)_k = \sum_l v_l \sum_k r_k y_{kl} = 0$ while $yv = e_i$ gives $\sum_k r_k(yv)_k = r_i$. So f has a nonzero kernel vector; normalise it, and conjugate the second condition.

Step 3: the rotations. Let r be as in Step 2, $\Pi = \bar{r}r^{\mathsf{T}}$ (entries $\bar{r}_k r_l$) and, for $|\eta| = 1$, $c_\eta = 1 + (\bar{\eta}-1)\Pi$. Then $\Pi^* = \Pi = \Pi^2$ and $D\Pi = \Pi D = -\Pi$ (as r lives on the indices of sign -1), so with $E \in \{1, D\}$ and $E\Pi = \Pi E = \epsilon\Pi$, $c_\eta^* E c_\eta = E + \epsilon\Pi((\bar{\eta}-1) + (\eta-1) + (\eta-1)(\bar{\eta}-1)) = E$. Hence $c_\eta \in K_s$, $c_\eta^{-1} = c_\eta^* = 1 + (\eta-1)\Pi$, and $c_\eta y = y$ since $r^{\mathsf{T}}y = 0$; so $c_\eta \in C$. The block of c_η of sign $+1$ is the identity, and the block of sign -1 is a rank-one perturbation of the identity of determinant $1 + (\bar{\eta}-1)r^{\mathsf{T}}\bar{r} = \bar{\eta} = \eta^{-1}$.

Step 4: the law of the rotations. Every $c \in K_s$ commutes with D ($c^*c = 1$ and $c^*Dc = D$), so $c_{[\tau]}$ is block diagonal for the signs and (5) of Lemma 4.1.1 applies to $\Omega_{\mu,0}(1, c_{[\tau]})$: it maps v_P to a scalar times $v_{P'}$, P' obtained from P by a linear substitution of the variables; in particular it maps Fock vectors of homogeneous polynomials of degree d to such vectors. For $c = c_\eta$:

$$\Omega_{\mu,0}(1, (c_\eta)_{[\tau]})v_P = \eta^N v_{P_\eta},$$

where P_η is P after $X_{\tau,k,j} \mapsto X_{\tau,k,j} + (\bar{\eta}-1)r_k \sum_i \bar{r}_i X_{\tau,i,j}$ for $t_j = -1$ and $X_{\tau,k,j} \mapsto X_{\tau,k,j} + (\eta-1)\bar{r}_k \sum_i r_i X_{\tau,i,j}$ for $t_j = 1$, the variables at $v \neq \tau$ being unchanged.

Step 5: no nonzero fixed polynomial. Let P satisfy $\Omega_{\mu,0}(1, c_{[\tau]})v_P = v_P$ for all $c \in C$. Fix z and choose r by Step 2. For $|\eta| = 1$, $c_\eta \in C$ gives $v_P = \eta^N v_{P_\eta}$, hence $P = \eta^N P_\eta$ ($P \mapsto v_P$ is injective: $v_P(x) = P(\zeta(x))e^{-|x|^2/2}$ with ζ bijective). At z the substitution for $t_j = -1$ adds $(\bar{\eta}-1)r_k \sum_i \bar{r}_i z_{\tau,i,j} = 0$; so $P(z) = \eta^N P(z + (\eta-1)z')$, z' having entries $\bar{r}_k \sum_i r_i z_{\tau,i,j}$ at (τ,k,j) with $t_j = 1$ and 0 elsewhere. The polynomial $g(\eta) = \eta^N P(z + (\eta-1)z')$ equals the constant $P(z)$ on the unit circle, hence everywhere, and $g(0) = 0$ as $N \geq 1$. So $P(z) = 0$ for all z, and $P = 0$.

Step 6: averaging over C. C is a closed subgroup of the compact group K_s, hence compact; let dc be its Haar probability measure (Theorem A.5.1). For $d \geq 0$ let W_d be the finite-dimensional space of Fock vectors of homogeneous polynomials of degree d; by Step 4 it is stable under $\rho(c) = \Omega_{\mu,0}(1, c_{[\tau]})$, $c \in C$, and $c \mapsto \rho(c)|_{W_d}$ is continuous by (2) of Lemma 4.1.1. Let $M = \int_C \rho(c)|_{W_d}\, dc \in \operatorname{End}(W_d)$. For $w \in W_d$, $Mw \in W_d$ is fixed by every $\rho(c')$, $c' \in C$ (invariance of dc), so $Mw = 0$ by Step 5. For ℓ as in the statement and $w \in W_d$, $\ell(w) = \int_C \ell(\rho(c)w)\, dc = \ell(Mw) = 0$.

Step 7. Every polynomial is a finite sum of homogeneous ones, so $\ell(v_P) = 0$ for all P; by Lemma 6.3.29 and the continuity of ℓ, $\ell = 0$. □

Lemma 6.3.29. *Let ε be signs on a finite index set ι. For every polynomial $P \in \mathbb{C}[X_i : i \in \iota]$ the Fock vector v_P lies in $\mathcal{F}_\varepsilon$, and the Fock vectors of polynomials are dense in $\mathcal{F}_\varepsilon$ both for the*

norm of $L^1(\nu_\iota)$ and for the norm of its inner product.

Proof. Write $N = \#\iota$, $\zeta(x) = (\zeta_i(x))_i$ (complex conjugation in the coordinates of sign $+1$), so that ζ is a real-linear bijection of $\mathbb{C}^\iota$ preserving $|x|$ and the measure ν_ι, and let $\mathcal{F}_N$ be the Fock space of entire functions of Theorem A.5.11, with norm $\|f\|^2 = \pi^{-N}\int |f(z)|^2 e^{-|z|^2} d\lambda(z)$.

The kernel in the coordinates ζ. For each index i the exponent of $\beta^\infty_\varepsilon(y,x)e^\infty(x-y)$ contributes $\frac{1}{2}\varepsilon_i(\bar{x}_i y_i - \bar{y}_i x_i) - \frac{1}{2}|x_i - y_i|^2$, which is $\bar{x}_i y_i - \frac{1}{2}|x_i|^2 - \frac{1}{2}|y_i|^2$ if $\varepsilon_i = 1$ and $x_i\bar{y}_i - \frac{1}{2}|x_i|^2 - \frac{1}{2}|y_i|^2$ if $\varepsilon_i = -1$; in both cases it is $\zeta_i(x)\overline{\zeta_i(y)} - \frac{1}{2}|x_i|^2 - \frac{1}{2}|y_i|^2$. Hence, for $c\colon \mathbb{C}^\iota \to \mathbb{C}$ and $f(z) = c(\zeta^{-1}z)e^{|z|^2/2}$, after the substitution $w = \zeta(y)$ the reproducing identity of (WR5) reads

$$f(z) = \int f(w)\, e^{-|w|^2}\, e^{\sum_i z_i \bar{w}_i}\, d\nu_\iota(w) \qquad (z \in \mathbb{C}^\iota), \tag{6.45}$$

and $\int |c|^2 d\nu_\iota = \|f\|^2$, $\int |c|\, d\nu_\iota = \pi^{-N}\int |f| e^{-|z|^2/2} d\lambda$.

Polynomials. For $c = v_P$ we have $f = P$. The integral $\pi^{-N}\int |P| e^{-|z|^2/2} d\lambda$ is finite, and (6.45) holds for $f = P$: expanding $e^{\sum z_i\bar{w}_i} = \sum_\alpha z^\alpha \bar{w}^\alpha/\alpha!$ (absolutely convergent, dominated by $e^{|z||w|}$, which is integrable against $|P(w)|e^{-|w|^2}$), the orthogonality of the monomials in (FK1) of Theorem A.5.11, $\pi^{-N}\int w^\beta \bar{w}^\alpha e^{-|w|^2} d\lambda = \alpha!\,\delta_{\alpha\beta}$, gives $\int w^\beta e^{-|w|^2} e^{\sum z_i\bar{w}_i} d\nu_\iota(w) = z^\beta$. So $v_P \in \mathcal{F}_\varepsilon$.

Density in $L^1(\nu_\iota)$. Let K be the integral operator $(Kg)(x) = \int g(y)\beta^\infty_\varepsilon(y,x)e^\infty(x-y)\, d\nu_\iota(y)$; its kernel has modulus $e^{-|x-y|^2/2}$ and $\int e^{-|x|^2/2} d\nu_\iota = 2^N$, so $\|Kg\|_{L^1} \le 2^N\|g\|_{L^1}$. Let $c \in \mathcal{F}_\varepsilon$, so $c = Kc$, and for $R > 0$ let c_R be c times the indicator of $\{|y| \le R\}$; then $\|c - Kc_R\|_{L^1} = \|K(c - c_R)\|_{L^1} \le 2^N\|c - c_R\|_{L^1} \to 0$. By the computation of the kernel, $(Kc_R)(x) = e^{-|x|^2/2}F_R(\zeta(x))$ with $F_R(z) = \int_{|y|\le R} c(y)e^{-|y|^2/2}e^{\sum_i z_i\overline{\zeta_i(y)}} d\nu_\iota(y) = \sum_\alpha a_\alpha z^\alpha$, where $|a_\alpha| \le \|c\|_{L^1} R^{|\alpha|}/\alpha!$ (expand the exponential; the series converges absolutely, uniformly on $|y| \le R$). Let $P_M = \sum_{|\alpha|\le M} a_\alpha z^\alpha$. Since $\pi^{-1}\int_{\mathbb{C}} |z|^k e^{-|z|^2/2} d\lambda = 2^{1+k/2}\Gamma(1 + k/2)$,

$$\|Kc_R - v_{P_M}\|_{L^1} \le \|c\|_{L^1} \sum_{|\alpha|>M} \prod_{i\in\iota} \frac{R^{\alpha_i}\, 2^{1+\alpha_i/2}\,\Gamma(1+\alpha_i/2)}{\alpha_i!},$$

the tail of a convergent series (the one-variable series $\sum_k R^k 2^{k/2}\Gamma(1 + k/2)/k!$ converges, as $\Gamma(1 + k/2)/k! \le 1/\Gamma(1 + k/2)$ up to a factor 2^k). So $v_{P_M} \to Kc_R$, and the Fock vectors of polynomials are dense in L^1 norm.

Density for the inner product. Let $c \in \mathcal{F}_\varepsilon$ and f as above. By (6.45),

$$f(z) = \int c(y) e^{-|y|^2/2} e^{\sum_i z_i\overline{\zeta_i(y)}} d\nu_\iota(y);$$

the integrand is entire in z and bounded by $|c(y)|e^{|z|^2/2}$ (as $|z||y| - |y|^2/2 \le |z|^2/2$), with c integrable, so f is entire, and $\|f\|^2 = \int |c|^2 d\nu_\iota < \infty$ (c is bounded by $\int |c|\, d\nu_\iota$, the kernel having modulus $e^{-|x-y|^2/2} \le 1$, so c is square integrable). Thus $f \in \mathcal{F}_N$, and by (FK1) of Theorem A.5.11 the partial sums P_M of its expansion in the orthonormal basis $z^\alpha/\sqrt{\alpha!}$ converge to f in $\mathcal{F}_N$. The P_M are polynomials, and $\int |c - v_{P_M}|^2 d\nu_\iota = \|f - P_M\|^2 \to 0$. □

Lemma 6.3.30. *Let s be signs on a finite index set n, Y a Banach space and $L\colon \mathcal{M}_s \to Y$ a bounded linear map with $L(v_P \otimes \rho_f(y)v^0) = 0$ for every polynomial P in the variables indexed by $\Phi \times n$ and every $y \in \mathbb{A}^n_{E,f}$. Then $L = 0$.*

Proof. **Step 1: $v_1 \otimes \rho_f(y)v^0 = \rho_f(y)v^0$.** By (3) and (4) of Lemma 6.2.21, $v_1 \otimes \rho_f(y)v^0 = \rho_f(y)(v_1 \otimes v^0) = \rho_f(y)v^0$.

Step 2: translates of the real vacuum are limits of Fock vectors of polynomials. For each coordinate $c = (w, i)$ of sign σ one computes

$$\beta_\varepsilon^\infty(t_0, t)\, v_1(t - t_0) = e^{-|t_{0,c}|^2/2} e^{-|t_c|^2/2} \exp(\zeta_c(t)\eta_c)$$

with $\eta_c = t_{0,c}$ if $\sigma = 1$ and $\eta_c = \bar{t}_{0,c}$ if $\sigma = -1$ (ζ as in (WR5)). Hence $\rho^\infty(t_0)v_1 = v_1(t_0)\exp(\langle\zeta, \eta\rangle)\, e^\infty$ with $\langle\zeta, \eta\rangle = \sum_c \zeta_c\eta_c$, and with $P_K(X) = \sum_{k\le K}(\sum_c \eta_c X_c)^k/k!$ the Fock vectors $v_1(t_0)v_{P_K}$ converge to $\rho^\infty(t_0)v_1$ in $L^1(\nu)$ by dominated convergence (dominant $e^{|\eta|\,|t|}e^{-|t|^2/2}$).

Step 3: conclusion. Let $z \in \mathbb{A}_E^n$; write $z = \mathrm{rv}(t_0)+\mathrm{fv}(y)$ with t_0 the normalized coordinates of z_∞ and $y = \alpha^{-1}z_f$. By (3) of Lemma 6.2.21 and (WD8), $\rho(z)v^0 = c\,\rho(\mathrm{rv}(t_0))\rho_f(y)v^0$ for a scalar c, and by Step 1, (2) of Lemma 6.2.21 and Step 2, with $F = \rho_f(y)v^0$,

$$L(\rho(z)v^0) = c\,L(\rho(\mathrm{rv}(t_0))(v_1 \otimes F)) = c\,L((\rho^\infty(t_0)v_1) \otimes F) = c\,v_1(t_0)\lim_{K\to\infty} L(v_{P_K} \otimes F) = 0,$$

as $w \mapsto L(w \otimes F)$ is bounded on $L^1(\nu)$. By superposition and boundedness,

$$Lh = \int h(z)L(\rho(z)v^0)\, d\mu_n(z) = 0 \qquad \text{for every } h \in \mathcal{M}_s.$$

□

Lemma 6.3.31. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, a, b, n finite index sets, $\sigma\colon b \sqcup a \to n$ a bijection and $s \in \mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$.*

1. *$\Phi^{\mathrm{st}}_{\sigma,s}$ and Ψ^{st}_s are matched over $\mathcal{O}_\Sigma$ for $(\sigma, \chi_{\mathfrak{c}})$.*
2. *Ψ^{st}_s is locally constant with compact support on $\mathrm{M}_{n\times b}(E_\Sigma)$ and vanishes at every matrix without a left inverse; $\Phi^{\mathrm{st}}_{\sigma,s}$ is locally constant with compact support on $\mathrm{M}_{a\times n}(E_\Sigma)$ and vanishes at every matrix without a right inverse.*

Proof. Write $\chi = \chi_{\mathfrak{c}}$. For $x \in \mathrm{M}_{a\times n}(E_\Sigma)$ let $M(x)$ be the $n \times n$ matrix whose row $\sigma(j)$ is row j of s ($j \in b$) and whose row $\sigma(i)$ is row i of x ($i \in a$), so that $\Phi^{\mathrm{st}}_{\sigma,s}(x) = \chi^{-1}(\det M(x))$ if $M(x)$ is integral and 0 otherwise. As s is integral, $M(x)$ is integral exactly when x is. Recall that χ^{-1} vanishes off $\mathcal{O}_\Sigma^\times$.

The row datum is supported on the rows of inverses. If $\Phi^{\mathrm{st}}_{\sigma,s}(x) \neq 0$, then $M(x)$ is integral with $\det M(x) \in \mathcal{O}_\Sigma^\times$, so $M(x) \in \mathrm{GL}_n(\mathcal{O}_\Sigma)$; for $k = M(x)^{-1} \in \mathrm{GL}_n(\mathcal{O}_\Sigma)$ the rows $\sigma(a)$ of $k^{-1} = M(x)$ form x, that is $x = (k^{-1})_a$.

The column datum is supported on the columns of $\mathrm{GL}_n(\mathcal{O}_\Sigma)$. If $\Psi^{\mathrm{st}}_s(y) \neq 0$, then y is integral and $sy \in \mathrm{GL}_b(\mathcal{O}_\Sigma)$. Fix $w \in \Sigma$ and let $\bar{y}_w$ be the reduction of y_w modulo ϖ_w. Since $(sy)^{-1}s\,y = 1$ with $(sy)^{-1}s$ integral, $\bar{y}_w$ has rank $\#b$, so its columns extend to a basis of k_w^n by $\#a$ standard basis vectors e_j. Let $g_w \in \mathrm{M}_n(\mathcal{O}_{E,w})$ have the columns of y_w in the positions $\sigma(b)$ and these e_j in the positions $\sigma(a)$; its determinant is a unit, being nonzero modulo ϖ_w. Then $k = (g_w)_{w\in\Sigma} \in \mathrm{GL}_n(\mathcal{O}_\Sigma)$ and $y = k_b$ (k_w is the residue field).

The values. Let $k \in \mathrm{GL}_n(\mathcal{O}_\Sigma)$ and $M = M((k^{-1})_a)$, an integral matrix. The rows $\sigma(a)$ of Mk are the rows $\sigma(a)$ of $k^{-1}k = 1$, that is the standard rows $e^{\mathsf{T}}_{\sigma(i)}$, and its rows $\sigma(b)$ are those of sk. Expanding along the standard rows, $\det(Mk)$ is the determinant of the submatrix of Mk on the rows and columns $\sigma(b)$, which is sk_b. Hence $\det M \cdot \det k = \det(sk_b)$ with $\det k \in \mathcal{O}_\Sigma^\times$, and k_b is integral. If $\det(sk_b)$ is a unit, so is $\det M$, and $\chi^{-1}(\det M) = \chi(\det k)\chi^{-1}(\det(sk_b))$ by multiplicativity of χ on $\mathcal{O}_\Sigma^\times$; otherwise both sides vanish. Thus $\Phi^{\mathrm{st}}_{\sigma,s}((k^{-1})_a) = \chi(\det k)\,\Psi^{\mathrm{st}}_s(k_b)$, which completes the proof of (1).

Item (2). The integral matrices form a compact open subset of $\mathrm{M}_{n\times b}(E_\Sigma)$ off which Ψ_s^{st} vanishes. On it, $y \mapsto \det(sy)$ is continuous into $\mathcal{O}_\Sigma$; the units form an open and closed subset of $\mathcal{O}_\Sigma$ (the nonunits are the elements with some component in $\varpi_w \mathcal{O}_{E,w}$), and χ is locally constant on the units by the sidecalc above; so Ψ_s^{st} is locally constant. If $\Psi_s^{\mathrm{st}}(y) \neq 0$, then $(sy)^{-1}s$ is a left inverse of y. The same arguments apply to $x \mapsto \chi^{-1}(\det M(x))$ on the compact open set of integral x; if $\Phi_{\sigma,s}^{\mathrm{st}}(x) \neq 0$, then $M(x)$ is invertible, and the columns $\sigma(a)$ of $M(x)^{-1}$ form a right inverse of x: the rows $\sigma(a)$ of $M(x)M(x)^{-1} = 1$, taken in the columns $\sigma(a)$, read $x\,(M(x)^{-1})_{\cdot,\sigma(a)} = 1$. □

Lemma 6.3.32. *Let W be a space with a point with signs s on n, H a splitting of W over W_0 along p with adelic frame $k_\mathbb{A}$, radical centre elements $z(M)$ and centre coefficients* $\mathrm{coef}(T;\cdot)$*, and let $D, D' \colon \mathrm{G}(s) \to \mathbb{C}$ be continuous and left invariant under Γ_W. Let $T \in \mathrm{M}_p(E)$.*

1. *$z(M)z(M') = z(M+M')$ for adelic M, M', and $z(M) \in \Gamma_W$ for $M \in \mathrm{M}_p(E)$. For every $g \in \mathrm{G}(s)$ the function $M \mapsto D(z(M)g)$ is continuous and $\mathrm{M}_p(E)$-periodic, so* $\mathrm{coef}(T;D)(g)$ *is defined;* $\mathrm{coef}(T;D)$ *is continuous on* $\mathrm{G}(s)$*;* $\mathrm{coef}(T;D+cD') = \mathrm{coef}(T;D) + c\,\mathrm{coef}(T;D')$ *for $c \in \mathbb{C}$; and for $Y \in \mathrm{G}(s)$ the function $D_Y = D(\cdot\, Y)$ satisfies* $\mathrm{coef}(T;D_Y)(g) = \mathrm{coef}(T;D)(gY)$*.*

2. *For every adelic $p \times p$ matrix M and $g \in \mathrm{G}(s)$,*

$$\psi_E(\mathrm{tr}(TM))\,\mathrm{coef}(T;D)(z(M)g) = \mathrm{coef}(T;D)(g).$$

3. *If $g \in \mathrm{G}(s)$ and $c \in \mathrm{GL}_p(\mathbb{A}_E)$ satisfy $gk_\mathbb{A} = k_\mathbb{A} c$, then $g\,z(M) = z(cMc^*)\,g$ for every adelic M.*

4. *If $\gamma \in \Gamma_W$ and $c \in \mathrm{M}_p(E)$ satisfy $\gamma k_\mathbb{A} = k_\mathbb{A} c$, then* $\mathrm{coef}(T;D)(\gamma g) = \mathrm{coef}(c^*Tc;D)(g)$ *for every g; in particular* $\mathrm{coef}(T;D)(\gamma g) = \mathrm{coef}(T;D)(g)$ *when $c = 1$.*

Proof. Write $x = x_W$, $\varepsilon = \varepsilon_s$, and let k be the rational frame of H, so $k_\mathbb{A} = x^{-1}k$ and $\mathrm{pr}_l = k_\mathbb{A}^* \varepsilon = k^* J_W x$ (as $(x^{-1})^*\varepsilon = J_W x$). By (MX1), $\mathrm{pr}_l k_\mathbb{A} = 0$. For $g \in \mathrm{G}(s)$, $\varepsilon g = (g^*)^{-1}\varepsilon$, hence

$$\mathrm{pr}_l\, g = (g^{-1}k_\mathbb{A})^*\varepsilon. \tag{6.46}$$

(1). As $\mathrm{pr}_l k_\mathbb{A} = 0$, $(1 + k_\mathbb{A} X \mathrm{pr}_l)(1 + k_\mathbb{A} X' \mathrm{pr}_l) = 1 + k_\mathbb{A}(X + X')\mathrm{pr}_l$; with $X = M - M^*$ this is $z(M)z(M') = z(M+M')$. For $M \in \mathrm{M}_p(E)$, $x\,z(M)\,x^{-1} = 1 + k(M - M^*)k^* J_W$ has entries in E, and $z(M) \in \mathrm{G}(s)$ (MX2), so $z(M) \in \Gamma_W$. Hence $D(z(M+\gamma)g) = D(z(\gamma)z(M)g) = D(z(M)g)$ for $\gamma \in \mathrm{M}_p(E)$; continuity holds as $(M, g) \mapsto z(M)g$ is continuous. The integrand $\psi_E(\mathrm{tr}(TM))D(z(M)g)$ of (MX3) is then continuous and periodic (ψ_E is trivial on E and $\mathrm{tr}(T\gamma) \in E$), so the coefficient is the integral of a bounded continuous function over a relatively compact Borel fundamental domain Q of $\mathrm{M}_p(E)$ in $\mathrm{M}_p(\mathbb{A}_E)$ (e.g. the product of the domains $P \times \widehat{\mathcal{O}}_E$, P a half-open fundamental parallelepiped of $\mathcal{O}_E$). For g in a compact neighbourhood K_g the integrand is bounded on $\bar{Q} \times K_g$, and dominated convergence gives continuity of $\mathrm{coef}(T;D)$. Linearity is linearity of the integral, and $\mathrm{coef}(T;D_Y)(g) = \int \psi_E(\mathrm{tr}(TM))D(z(M)gY)\,dM = \mathrm{coef}(T;D)(gY)$.

(2). By (1), $\mathrm{coef}(T;D)(z(M_0)g) = \int \psi_E(\mathrm{tr}(TM))D(z(M+M_0)g)\,dM$, and the invariance of the Haar measure of the quotient under translation by M_0 turns this into $\int \psi_E(\mathrm{tr}(T(M - M_0)))D(z(M)g)\,dM = \psi_E(\mathrm{tr}(TM_0))^{-1}\mathrm{coef}(T;D)(g)$.

(3). By (6.46) applied to g^{-1}, $\mathrm{pr}_l g^{-1} = (gk_\mathbb{A})^*\varepsilon = c^*\mathrm{pr}_l$. Hence $g\,z(M)\,g^{-1} = 1 + gk_\mathbb{A}(M - M^*)\mathrm{pr}_l g^{-1} = 1 + k_\mathbb{A} c(M - M^*)c^*\mathrm{pr}_l = z(cMc^*)$.

(4). Since γ is invertible, so is c. The map $\phi(N) = cNc^*$ is a continuous additive automorphism of $\mathrm{M}_p(\mathbb{A}_E)$ with $\phi(\mathrm{M}_p(E)) = \mathrm{M}_p(E)$; it induces an automorphism of the compact group $\mathrm{M}_p(\mathbb{A}_E)/\mathrm{M}_p(E)$, which preserves its Haar probability measure by uniqueness. By (3) and the invariance of D, $D(z(cNc^*)\gamma g) = D(\gamma z(N) g) = D(z(N)g)$, and $\mathrm{tr}(TcNc^*) = \mathrm{tr}(c^*TcN)$. Substituting $M = \phi(N)$ in (MX3),

$$\mathrm{coef}(T; D)(\gamma g) = \int \psi_E\big(\mathrm{tr}(c^*TcN)\big) D(z(N)g)\, dN = \mathrm{coef}(c^*Tc; D)(g).$$

□

Proposition 6.3.33. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, X a space with signs r on m, W_1 a space with signs s on n, W a space, b an index set and $(\mathcal{L}, \mathcal{D})$ an equivariant lift for $(\mathfrak{c}, X, b)$. Let D be a base-adapted splitting of X over W_1 along b and D_1 a base-adapted splitting of W_1 over W along a one-element set; radical elements, Levi elements and centre coefficients are those of D. Let $\Psi_0 = \Psi^{\mathrm{st}}_{(\mathrm{pr}_l)_\Sigma}$ be the standard column datum, for $\chi_{\mathfrak{c}}$, of the matrix $(\mathrm{pr}_l)_\Sigma \in \mathrm{M}_{b\times m}(\mathcal{O}_\Sigma)$ of the coordinate pr_l of D, and assume $\Psi_0 \in \mathcal{L}$. Let V' be a space with signs t' on b, and let $\kappa_\alpha \in \mathbb{A}_E^\times$ be the idele with components κ_v at $v \in \Phi$ and finite part α. Assume*

(VL) (vanishing at self-dual lattices) $\mathrm{coef}(\jmath T; \mathcal{D}\Psi_0)\big(\mathrm{rad}(c,z)\,\mathrm{lev}(q,g)\big) = 0$ *for all $T \in \mathrm{M}_b(E)$, $c \in \mathrm{M}_{n\times b}(\mathbb{A}_E)$, $z \in \mathrm{M}_b(\mathbb{A}_E)$, $q \in \mathrm{GL}_b(\mathbb{A}_E)$ and $g \in \mathrm{G}(s)$ such that either*

- *(a) $T = J_{V'}$ and $q = \kappa_\alpha\, x_{V'} Y$ for some $Y \in \mathrm{GL}_b(\mathbb{A}_E)$ for which there is $k \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ with $(Y_f k)^*(Y_f k) = 1$; or*
- *(b) $\det T \neq 0$ and T, read as an adelic matrix, is not represented by the signs t'.*

Then $\mathcal{D}\Psi = 0$ for every locally constant function Ψ on $\mathrm{M}_{m\times b}(E_\Sigma)$ for which there are an index set a, a bijection $\sigma\colon b \sqcup a \to m$ and a function Φ on $\mathrm{M}_{a\times m}(E_\Sigma)$ such that Φ and Ψ are matched over $\mathcal{O}_\Sigma$ for $(\sigma, \chi_{\mathfrak{c}})$.

Proof. Write $f = \mathcal{D}\Psi_0$, a continuous, left Γ_X-invariant function on $\mathrm{G}(r)$ ((XD11)(b)), and $(1, g)$ for the point with infinite component 1 and finite component g.

Step 1: reduction. By Lemma 6.3.34 it suffices to show $\mathrm{coef}(T; f)((1, g_0)) = 0$ for every $T \in \mathrm{M}_b(E)$ with $T^* = -T$ and $\tau = ((\jmath\alpha\bar{\alpha})^{-1}T)_\Sigma \in \mathrm{GL}_b(\mathcal{O}_\Sigma)$ and every $g_0 \in G_f(m)$ with components 1 at the places of Σ. Fix such T and g_0; $\det T \neq 0$ since $\det \tau$ is a unit.

Step 2: a line and the remaining planes. Write $D = (k, l, i)$ and $D_1 = (k_1, l_1, i_1)$, with W on the index set n_W. Let $W_0 = \mathbb{H}_b \oplus W$, with index set $(b \sqcup b) \sqcup n_W$ and point $x_{W_0} = \mathrm{diag}(x_{\mathbb{H}_b}, x_W)$, and let B be its tautological splitting over W along b: l_B, k_B are the identity blocks of the first and second copy of b, so that k_B is the block of signs -1. Put

$$A = (ik_1,\ il_1,\ (l \mid k \mid ii_1)),$$

the embedding sending the first copy of b to l, the second to k and n_W to ii_1. The relations (MX1) of D and D_1 give $(ik_1)^* J_X (ik_1) = k_1^* J_{W_1} k_1 = 0$, likewise for il_1, $(il_1)^* J_X (ik_1) = 1$, $(l \mid k \mid ii_1)^* J_X (l \mid k \mid ii_1) = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_W) = J_{W_0}$, $(l \mid k \mid ii_1)^* J_X\, ik_1 = (0; 0; i_1^* J_{W_1} k_1) = 0$ and likewise for il_1; and $\#m = 2\#b + \#n = (2\#b + \#n_W) + 2$. So A is a splitting of X over W_0 along a one-element set, and $i_A k_B = k = k_D$. Its matrix of adelic frames is, up to the order of the columns, $(l_{D,\mathbb{A}} \mid k_{D,\mathbb{A}} \mid i_{D,\mathbb{A}}) \cdot \mathrm{diag}\big(x_{\mathbb{H}_b}, (l_{1,\mathbb{A}} \mid k_{1,\mathbb{A}} \mid i_{1,\mathbb{A}})\big)$ (as $x_X^{-1} i x_{W_1} = i_{D,\mathbb{A}}$ and $x_X^{-1}(l \mid k) = (l_{D,\mathbb{A}} \mid k_{D,\mathbb{A}})$); both factors are unitary at the infinite places (D, D_1 are base-adapted, and $x_{\mathbb{H}_b}$ is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ there) and integral at the finite places (the finite component

of $x_{\mathbb{H}_b}$ is $\begin{pmatrix} 1 & -c \\ c_0 & c\bar{c}_0 \end{pmatrix}$). So A is base-adapted. The matrix of adelic frames of B is $\mathrm{diag}(x_{\mathbb{H}_b}^{-1}, 1)$, unitary at the infinite places and integral at the finite places, since $\det x_{\mathbb{H}_b}$ has finite part $(c\bar{c}_0 + cc_0)^{\#b} = c^{\#b}$, a unit ($c_0 + \bar{c}_0 = 1$). So B is base-adapted.

Step 3: Freitag's lemma. Let S be the set of finite places w with $w \notin \Sigma \cup \bar{\Sigma}$ (so $S = \bar{S}$), $\mathrm{e} = \mathrm{e}_{S \cup \bar{S}}$, $U = \bar{\Sigma}$ (nonempty) and H the frame stabiliser of D off S. We apply (2) of Proposition 6.3.36 to X, A, B, D, the index set b, T, S, U and f. The function f is continuous and left Γ_X-invariant, and right invariant under every $k \in \mathbf{K}_0(m)$ with components 1 at the places outside S, i.e. at $\Sigma \cup \bar{\Sigma}$, by (XD11)(d). The partial means of f are invariant under the A-shears, resp. the $\bar{A}$-shears, for $\bar{\Sigma}$ at every point of H by Lemma 6.3.47 (with $k_D = i_A k_B$ and $\Psi_0 \in \mathcal{L}$).

It remains to check the last hypothesis. Let $x \in \mathrm{G}(r)$ and $\gamma \in \mathrm{M}_b(\mathbb{A}_E)$ with $x k_{D,\mathbb{A}} = k_{D,\mathbb{A}} q_0$, $q_0 = \mathrm{e}\gamma + (1 - \mathrm{e})$, and $T\{\gamma\}$ unimodular on $S \cup \bar{S}$. The matrix q_0 is invertible: by unitarity of x and $l_{D,\mathbb{A}}^* \varepsilon_r k_{D,\mathbb{A}} = 1$, $(x l_{D,\mathbb{A}})^* \varepsilon_r k_{D,\mathbb{A}} q_0 = 1$, so q_0 has a left inverse over the commutative ring $\mathbb{A}_E$. By (3) of Lemma 6.3.48, $x = \mathrm{rad}(c, z)\,\mathrm{lev}(q_0, g)$ for some adelic c, z and some $g \in \mathrm{G}(s)$. Put $\beta = \jmath^{-1} T \in \mathrm{M}_b(E)$, $\det \beta \neq 0$, and $q = \kappa_\alpha^{-1} q_0$. Since $\delta_\jmath = \jmath \kappa_\alpha \bar{\kappa}_\alpha$ ((MX9): components $\jmath_v \kappa_v^2$ at $v \in \Phi$, finite part $\jmath\alpha\bar{\alpha}$),

$$q^* \beta q = \delta_\jmath^{-1} q_0^* T q_0 = T\{q_0\}.$$

Its finite part is unimodular at every finite place: at $w \in S$ it is the component of $T\{\gamma\}$ ($q_0 = \gamma$ at $S = \bar{S}$); at $w \in \Sigma$ it is τ_w ($q_0 = 1$ at $\Sigma \cup \bar{\Sigma}$); at $w \in \bar{\Sigma}$ it is the component of $\delta_\jmath^{-1} T$, which is hermitian ($T^* = -T$, $\bar{\jmath} = -\jmath$), hence the conjugate transpose of its component $\tau_{\bar{w}}$ at $\bar{w} \in \Sigma$. And $T\{q_0\}$ is hermitian for the same reason. By (1) of Lemma 3.2.6 (E contains a primitive 5040th root of unity) there is $k \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ with $k^*(q^* \beta q)_f k = 1$. The hypothesis (VL) is the hypothesis of Lemma 6.3.49 for f, D, V'; that lemma gives

$$\mathrm{coef}(T; f)(x) = \mathrm{coef}(\jmath\beta; f)\big(\mathrm{rad}(c, z)\,\mathrm{lev}(\kappa_\alpha q, g)\big) = 0.$$

So Proposition 6.3.36 gives $\mathrm{coef}(T; f) = 0$ on H.

Step 4: conclusion. The point $(1, g_0)$ lies in H: its infinite component is 1, its components at Σ are 1, and so are those at $\bar{\Sigma}$, since $g_0^* g_0 = 1$ and the component of g_0^* at $\bar{w}$ is the conjugate transpose of the component of g_0 at w. So $\mathrm{coef}(T; f)((1, g_0)) = 0$, and Step 1 concludes. □

Lemma 6.3.34. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, X a space with a point with signs r on m, b an index set, $(\mathcal{L}, \mathcal{D})$ an equivariant lift for $(\mathfrak{c}, X, b)$, and D a base-adapted splitting of X over a space with a point along b, with coordinate pr_l and centre coefficients* coef*. Assume $\Psi_0 = \Psi^{\mathrm{st}}_{(\mathrm{pr}_l)_\Sigma} \in \mathcal{L}$ and*

$$\mathrm{coef}(T; \mathcal{D}\Psi_0)((1, g)) = 0$$

for every $T \in \mathrm{M}_b(E)$ with $T^* = -T$ and $\big((\jmath\alpha\bar{\alpha})^{-1} T\big)_\Sigma \in \mathrm{GL}_b(\mathcal{O}_\Sigma)$ and every $g \in G_f(m)$ whose components at the places of Σ are 1. Then $\mathcal{D}\Psi = 0$ for every locally constant function Ψ on $\mathrm{M}_{m \times b}(E_\Sigma)$ for which there are an index set a, a bijection $\sigma \colon b \sqcup a \to m$ and a function Φ such that Φ and Ψ are matched over $\mathcal{O}_\Sigma$ for $(\sigma, \chi_\mathfrak{c})$.

Proof. Let $k_\mathbb{A}, l_\mathbb{A}, i_\mathbb{A}$ be the adelic frames of D and $\mathrm{pr}_k, \mathrm{pr}_l, \mathrm{pr}_i$ its coordinates, $\varepsilon = \varepsilon_r$, and $\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$ for finite adeles t; a matrix z over E_Σ is read as the finite adelic matrix $\tilde{z}$ with components z at Σ and 0 elsewhere.

Step 1: the frame at Σ. As D is base-adapted, $(l_\mathbb{A} \mid k_\mathbb{A} \mid i_\mathbb{A})$ has integral finite part; its inverse is the matrix of the rows $\mathrm{pr}_l = k_\mathbb{A}^* \varepsilon$, $\mathrm{pr}_k = l_\mathbb{A}^* \varepsilon$, $\mathrm{pr}_i = \varepsilon_{s_0} i_\mathbb{A}^* \varepsilon$ ((MX1)), whose finite parts are conjugate transposes of integral matrices, hence integral. So $Q_\Sigma = (l_\mathbb{A} \mid k_\mathbb{A} \mid i_\mathbb{A})_\Sigma \in$

$\mathrm{GL}_m(\mathcal{O}_\Sigma)$; take $K = (l_\mathbb{A})_\Sigma$ and $L = (k_\mathbb{A})_\Sigma$, two disjoint blocks of $\#b$ columns of Q_Σ, so that the corresponding rows of Q_Σ^{-1} are $K' = (\mathrm{pr}_l)_\Sigma$ and $L' = (\mathrm{pr}_k)_\Sigma$, and $\Psi_0 = \Psi^{\mathrm{st}}_{K'}$. For $z \in \mathrm{M}_b(E_\Sigma)$ put $S_z(y) = \Psi_0(y + LzK'y)$.

Step 2: the additive character. By Lemma 4.1.3, $\widehat{\mathcal{O}}_E$ is its own annihilator under $(u,v) \mapsto \psi_0(\bar{u}v)$. Hence $e(t) = \psi_0(\tilde{t})$ is a continuous additive character of E_Σ with conductor $\mathcal{O}_\Sigma$: if $t \in \mathcal{O}_\Sigma$ then $\tilde{t}\mathcal{O}_\Sigma \subseteq \widehat{\mathcal{O}}_E$ and $e(t\mathcal{O}_\Sigma) = 1$; if $e(t\mathcal{O}_\Sigma) = 1$ then $\psi_0(\tilde{t}v) = \psi_0(\tilde{t}v_\Sigma) = 1$ for every $v \in \widehat{\mathcal{O}}_E$ ($\tilde{t}$ vanishes off Σ), so $\bar{\tilde{t}}$ annihilates $\widehat{\mathcal{O}}_E$, hence is integral, and $t \in \mathcal{O}_\Sigma$.

Step 3: the coefficient functionals are admissible. Let $T \in \mathrm{M}_b(E)$ and $g \in G_f(m)$ with components 1 at Σ, and $x = (1,g)$; then $x_\Sigma = 1$, $(x^{-1}k_\mathbb{A})_\Sigma = L$ and $(\mathrm{pr}_l x)_\Sigma = K'$. By (2) of Lemma 6.3.14 there is a linear functional $\Lambda_{T,g}$ with $\Lambda_{T,g}(\Psi) = \mathrm{coef}(T; \mathcal{D}\Psi)(x)$ for $\Psi \in \mathcal{L}$; moreover $S_z \in \mathcal{L}$ for every z (the functions Ψ_z there, for $\Psi = \Psi_0$), and

$$\Lambda_{T,g}(S_z) = \psi_{E,f}\big(\mathrm{tr}(T\tilde{z})\big)\Lambda_{T,g}(S_0) = e\big(\mathrm{tr}(\tau z)\big)\Lambda_{T,g}(S_0), \qquad \tau = \big((\jmath\alpha\bar{\alpha})^{-1}T\big)_\Sigma,$$

since $\psi_{E,f}(\mathrm{tr}(T\tilde{z})) = \psi_0(\mathrm{tr}((\jmath\alpha\bar{\alpha})^{-1}T\tilde{z}))$ and $\tilde{z}$ vanishes off Σ. If $T^* = -T$ and $\tau \in \mathrm{GL}_b(\mathcal{O}_\Sigma)$, then $\Lambda_{T,g}(S_0) = \mathrm{coef}(T; \mathcal{D}\Psi_0)(x) = 0$ by hypothesis. So for skew-hermitian T the restriction of $\Lambda_{T,g}$ to $\mathcal{L}$ is admissible in the sense of Lemma 6.3.35 (for e, Q_Σ, K, L).

Step 4: functions killed by the admissible functionals. Let $f \in \mathcal{L}$ be killed by every admissible functional on $\mathcal{L}$; we show $\mathcal{D}f = 0$. By Step 3, $\mathrm{coef}(T; \mathcal{D}f)((1,g)) = 0$ for every skew-hermitian $T \in \mathrm{M}_b(E)$ and every g as there. Fix such a g and let $\varphi(M) = \mathcal{D}f(z(M)(1,g))$ for adelic $b \times b$ matrices M; φ is continuous and $\mathrm{M}_b(E)$-periodic ((1) of Lemma 6.3.32; $\mathcal{D}f$ is continuous and left Γ_X-invariant by (XD11)(b)). Its Fourier coefficient at the character $M \mapsto \psi_E(\mathrm{tr}(TM))$ is $\mathrm{coef}(T; \mathcal{D}f)((1,g))$. These vanish also for T not skew-hermitian: as $z(M) = 1 + k_\mathbb{A}(M - M^*)\mathrm{pr}_l$ depends only on $M - M^*$, $\varphi(M+H) = \varphi(M)$ for adelic $H = H^*$, so the coefficient at T equals $\psi_E(\mathrm{tr}(TH))$ times itself (translation invariance of the measure); if it is nonzero, then $\psi_E(\mathrm{tr}(T(Z+Z^*))) = 1$ for all adelic Z, and since $\psi_E(\bar{y}) = \psi_E(y)$ ((WD4) of Definition 3.2.12: $\mathrm{Re}\,\bar{y}_v = \mathrm{Re}\,y_v$, $\mathrm{Tr}_{E/\mathbb{Q}}\,\bar{a} = \mathrm{Tr}_{E/\mathbb{Q}}\,a$), $\psi_E(\mathrm{tr}(TZ^*)) = \psi_E(\mathrm{tr}(T^*Z))$ and $\psi_E(\mathrm{tr}((T+T^*)Z)) = 1$ for all Z, which forces $T + T^* = 0$ by (a) of Theorem A.5.2 applied to each entry. By (a) of Theorem A.5.2 (on the b^2 entries), the characters $M \mapsto \psi_E(\mathrm{tr}(TM))$, $T \in \mathrm{M}_b(E)$, are all the characters of the compact group $\mathrm{M}_b(\mathbb{A}_E)/\mathrm{M}_b(E)$; a continuous function on a compact abelian group whose Fourier coefficients all vanish is 0 (the characters span a dense subalgebra of the continuous functions, by the Stone–Weierstrass theorem). So $\varphi = 0$, in particular $\mathcal{D}f((1,g)) = 0$. By Lemma 4.2.64 (for $R = \Sigma$) the products $\gamma(1,g)$, $\gamma \in \Gamma_X$, g with components 1 at $\Sigma \cup \bar{\Sigma}$, are dense in $\mathrm{G}(r)$; $\mathcal{D}f$ vanishes on them by left invariance and is continuous, so $\mathcal{D}f = 0$.

Step 5: conclusion. Let $\mathcal{P} = \{\Psi \in \mathcal{L} : \mathcal{D}\Psi = 0\}$. It is a linear space of locally constant, compactly supported functions ((XD11)(a)), and $y \mapsto \Psi(hy)$ lies in $\mathcal{P}$ for $\Psi \in \mathcal{P}$, $h \in \mathrm{GL}_m(E_\Sigma)$ ((3) of Lemma 6.3.14). The space $\mathcal{M} = \mathcal{L}$ contains every S_z (Step 3), and $\mathcal{P}$ contains every $f \in \mathcal{L}$ killed by the admissible functionals (Step 4). The character $\chi_\mathfrak{c}$ has the form required in Lemma 6.3.35: on $\mathcal{O}_\Sigma^\times$, $\chi_\mathfrak{c}(u) = \nu(u/\bar{u}) = \prod_{w\in\Sigma}\psi_w(u_w \bmod w)$ with $\psi_w \neq 1$, ψ the tame part of ν (the idele $u/\bar{u}$ has components u_w at $w \in \Sigma$, $\bar{u}_w^{-1}$ at $\bar{w} \notin \Sigma$, where $\psi_{\bar{w}} = 1$, and 1 elsewhere). By Lemma 6.3.35, $\mathcal{P}$ contains every locally constant Ψ matched over $\mathcal{O}_\Sigma$ for $(\sigma, \chi_\mathfrak{c})$ with some Φ; for these, $\mathcal{D}\Psi = 0$. □

Lemma 6.3.35. *Let Σ be a nonempty finite set of finite places of E and b, n finite sets. Let $\chi\colon \mathcal{O}_\Sigma^\times \to \mathbb{C}^\times$ be a character of the form $\chi(u) = \prod_{w\in\Sigma}\chi_w(u_w \bmod \varpi_w)$ with nontrivial characters χ_w of $k_w^\times$. For $s \in \mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$ let Ψ^{st}_s be the standard column datum of (XD10) formed with χ in place of the semilocal character: $\Psi^{\mathrm{st}}_s(y) = \chi(\det(sy))^{-1}$ if $y \in \mathrm{M}_{n\times b}(\mathcal{O}_\Sigma)$ and $\det(sy) \in \mathcal{O}_\Sigma^\times$, and $\Psi^{\mathrm{st}}_s(y) = 0$ otherwise. Let $e\colon E_\Sigma \to \mathbb{C}^\times$ be a continuous additive character with conductor*

$\mathcal{O}_\Sigma$: for $t \in E_\Sigma$, $e(t\mathcal{O}_\Sigma) = 1$ if and only if $t \in \mathcal{O}_\Sigma$. Let $k \in \mathrm{GL}_n(\mathcal{O}_\Sigma)$, let K and L be the $n \times b$ matrices formed by two disjoint sets of $|b|$ columns of k, and K', L' the $b \times n$ matrices formed by the corresponding rows of k^{-1}. For $z \in \mathrm{M}_b(E_\Sigma)$ let $S_z(y) = \Psi^{\mathrm{st}}_{K'}(y + LzK'y)$.

Let $\mathcal{P}$ be a linear space of locally constant, compactly supported functions on $\mathrm{M}_{n\times b}(E_\Sigma)$ such that $y \mapsto f(gy)$ lies in $\mathcal{P}$ for all $f \in \mathcal{P}$ and $g \in \mathrm{GL}_n(E_\Sigma)$. Let $\mathcal{M}$ be a linear space of functions on $\mathrm{M}_{n\times b}(E_\Sigma)$ that contains every S_z, and assume that $\mathcal{P}$ contains every $f \in \mathcal{M}$ with $\Lambda(f) = 0$ for every linear functional Λ on $\mathcal{M}$ that is admissible*, that is, for which there is $T \in \mathrm{M}_b(E_\Sigma)$ with*

$$\Lambda(S_z) = e(\mathrm{tr}(Tz))\,\Lambda(S_0) \quad (z \in \mathrm{M}_b(E_\Sigma)) \qquad \text{and} \qquad \Lambda(S_0) = 0 \ \text{ if } T \in \mathrm{GL}_b(\mathcal{O}_\Sigma).$$

Then $\mathcal{P}$ contains every locally constant function Ψ on $\mathrm{M}_{n\times b}(E_\Sigma)$ that, for some finite set a, bijection $\sigma\colon b \sqcup a \to n$ and function Φ, is matched with Φ over $\mathcal{O}_\Sigma$ for (σ, χ) in the sense of (XD6) (the words of (XD6) over the ring $\mathcal{O}_\Sigma$, with this χ).

Proof. Let $\bar\chi$ be the function on $\bar R$ equal to $\prod_w \chi_w(t_w)^{-1}$ at units $t = (t_w)_w$ and to 0 at non-units. Then $\bar\chi$ is multiplicative, $\bar\chi(\bar u) = \chi(u)^{-1}$ for $u \in \mathcal{O}^\times$, and $\bar\chi(\bar t) = 0$ for $t \in \mathcal{O} \smallsetminus \mathcal{O}^\times$. Consequently, for $s \in \mathrm{M}_{b\times n}(\mathcal{O})$ and integral y, $\Psi^{\mathrm{st}}_s(y) = \bar\chi(\overline{\det(sy)})$. The block relations are $K'K = L'L = 1$ and $K'L = L'K = 0$, and K, L, K', L' are integral.

Step 1: two properties of a matched Ψ. Let Ψ, Φ, a, σ be as in the statement. For $m \in \mathrm{GL}_n(\mathcal{O})$ and $u \in \mathrm{GL}_b(\mathcal{O})$ let $m' \in \mathrm{GL}_n(\mathcal{O})$ be m times the matrix acting as u on the coordinates $\sigma(b)$ and as the identity on $\sigma(a)$; then $m'_b = m_b u$, the rows $\sigma(a)$ of m'^{-1} and of m^{-1} coincide, and $\det m' = \det m \det u$. The matching identity at m and at m' gives $\chi(\det u)\Psi(m_b u) = \Psi(m_b)$. As Ψ vanishes off the set Prim_b of the m_b, which is stable under right multiplication by $\mathrm{GL}_b(\mathcal{O})$,

$$\Psi(yu) = \chi(\det u)^{-1}\Psi(y) \qquad (u \in \mathrm{GL}_b(\mathcal{O}),\ y \in \mathrm{M}_{n\times b}(E_\Sigma)). \tag{a}$$

Since Prim_b is compact, Ψ is locally constant with compact support, so there is $c \geq 1$ with $\Psi(y + \varpi^c t) = \Psi(y)$ for all y and all integral t.

Step 2: a multiple of a standard datum in $\mathcal{P}$ forces $\Psi \in \mathcal{P}$. Let $p \in \mathcal{P}$, $s \in \mathrm{M}_{b\times n}(\mathcal{O})$, $x \in \mathrm{M}_{n\times b}(\mathcal{O})$ and $r\colon \mathrm{M}_{n\times b}(E_\Sigma) \to \mathbb{Z}_{\geq 0}$ with $sx = 1$, $p = r \cdot \Psi^{\mathrm{st}}_s$ and $r(x) \neq 0$. We show $\Psi \in \mathcal{P}$. Put $Q = xs$, so $Q^2 = Q$, and $d = \varpi^{-c}(1-Q) + Q \in \mathrm{GL}_n(E_\Sigma)$, with inverse $\varpi^c(1-Q) + Q$; then $dx = x$ and $sd = s$.

First, for $h \in \mathrm{GL}_n(\mathcal{O})$ and all y,

$$\Psi(h^{-1}x)\,p(dhy) = |p(dhy)|\,\Psi(y). \tag{b}$$

If $p(dhy) = 0$ both sides vanish. Otherwise $z = dhy$ is integral and $g' = sz$ has a unit determinant, so $g' \in \mathrm{GL}_b(\mathcal{O})$, and $g' = sdhy = shy$. Then $z - xg' = \varpi^{-c}(1-Q)hy$, so $y = \varpi^c h^{-1}(z - xg') + h^{-1}xg'$ with $h^{-1}(z - xg')$ integral. By Step 1 and (a), $\Psi(y) = \Psi(h^{-1}xg') = \chi(\det g')^{-1}\Psi(h^{-1}x)$. Moreover $p(z) = r(z)\chi(\det g')^{-1}$, and $|\chi(\det g')| = 1$ as χ has finite image. Hence both sides of (b) equal $r(z)\Psi(y)$.

Let $F_h(y) = \Psi(h^{-1}x)\,p(dhy)$; each F_h lies in $\mathcal{P}$ by the stability of $\mathcal{P}$. The function $h \mapsto F_h$ is constant on the cosets of an open normal subgroup H of $\mathrm{GL}_n(\mathcal{O})$: $h \mapsto \Psi(h^{-1}x)$ is locally constant; p is invariant under translation by $\varpi^N \mathrm{M}_{n\times b}(\mathcal{O})$ for some N and vanishes off a compact set C_p, the y with $dhy \in C_p$ for some $h \in \mathrm{GL}_n(\mathcal{O})$ form a bounded set B, and for $h' = hh_1$ with $h_1 \equiv 1 \bmod \varpi^{N'}$ and N' large, $dh'y - dhy = dh(h_1 - 1)y \in \varpi^N \mathrm{M}_{n\times b}(\mathcal{O})$ for $y \in B$, while for $y \notin B$ both values vanish. Hence

$$G'(y) = \int_{\mathrm{GL}_n(\mathcal{O})} F_h(y)\,dh$$

is a finite linear combination of the F_h and lies in $\mathcal{P}$. By (b), $G'(y) = I(y)\Psi(y)$ with $I(y) = \int_{\mathrm{GL}_n(\mathcal{O})} |p(dhy)|\, dh$. The matrix x is j_b for some $j \in \mathrm{GL}_n(\mathcal{O})$: $\mathcal{O}^n = x\mathcal{O}^b \oplus (\ker s \cap \mathcal{O}^n)$ (write $v = xsv + (v - xsv)$), the second summand is a direct summand of a free module over the product of discrete valuation rings $\mathcal{O}$, hence free of rank $|n| - |b|$ componentwise, and j has the columns of x at $\sigma(b)$ and a basis of the second summand elsewhere. If $\Psi(y) \neq 0$ then $y = l_b$ with $l \in \mathrm{GL}_n(\mathcal{O})$, and the substitution $h \mapsto hjl^{-1}$, which preserves the Haar measure, gives $I(y) = \int |p(dhjl^{-1}l_b)|\, dh = I(j_b) = I(x)$, as $jl^{-1}l_b = j_b$. So $G' = I(x)\Psi$. Finally $I(x) > 0$: the integrand is continuous and non-negative, and at $h = 1$ it is $|p(x)| = r(x)\,|\chi(\det sx)|^{-1} = r(x) \neq 0$. Therefore $\Psi = I(x)^{-1}G' \in \mathcal{P}$.

Step 3: a Gauss sum over $\bar{R}$. Let $\bar{e}(\bar{t}) = e(\varpi^{-1}t)$ for $t \in \mathcal{O}$; it is well defined, e being trivial on $\mathcal{O}$, and is an additive character of $\bar{R}$. For $\bar{T} \in \mathrm{M}_b(\bar{R})$ put

$$\Gamma(\bar{T}) = \sum_{\bar{r} \in \mathrm{M}_b(\bar{R})} \bar{\chi}(\det \bar{r})\, \bar{e}(\mathrm{tr}(\bar{T}\bar{r})).$$

We show $\Gamma(\bar{T}) = 0$ if $\det \bar{T}$ is not a unit. Write $\bar{e}(t) = \prod_w \bar{e}_w(t_w)$ with $\bar{e}_w(t_w) = \bar{e}$ of the element with component t_w at w and 0 elsewhere, and $\bar{\chi} = \prod_w \bar{\chi}_w$ with $\bar{\chi}_w$ equal to χ_w^{-1} on $k_w^\times$ and 0 at 0. Then $\Gamma(\bar{T}) = \prod_w \Gamma_w(\bar{T}_w)$ with $\Gamma_w(\bar{T}_w) = \sum_{\bar{r} \in \mathrm{M}_b(k_w)} \bar{\chi}_w(\det \bar{r})\bar{e}_w(\mathrm{tr}(\bar{T}_w\bar{r}))$. If $\det \bar{T}$ is not a unit, some $\bar{T}_w$ is singular; choose $0 \neq v \in k_w^b$ with $\bar{T}_w v = 0$, $\gamma \in k_w^\times$ with $\chi_w(\gamma) \neq 1$, and $f \in k_w^b$ with $f^T v = \gamma^{-1} - 1$. Then $u = 1 + vf^T$ has $\det u = \gamma^{-1}$ and $\bar{T}_w u = \bar{T}_w$, and the bijection $\bar{r} \mapsto u\bar{r}$ of $\mathrm{M}_b(k_w)$ gives $\Gamma_w(\bar{T}_w) = \bar{\chi}_w(\gamma^{-1})\Gamma_w(\bar{T}_w) = \chi_w(\gamma)\Gamma_w(\bar{T}_w)$, so $\Gamma_w(\bar{T}_w) = 0$.

Step 4: the Gauss sum of shears lies in $\mathcal{P}$. For $\bar{r} \in \mathrm{M}_b(\bar{R})$ choose an integral lift $l(\bar{r})$, with $l(1) = 1$, and put

$$\Gamma_0 = \sum_{\bar{r} \in \mathrm{M}_b(\bar{R})} \bar{\chi}(\det \bar{r})\, S_{\varpi^{-1}l(\bar{r})} \in \mathcal{M}.$$

Let Λ be admissible, with matrix T; we show $\Lambda(\Gamma_0) = 0$. Suppose not. Then $\Lambda(S_0) \neq 0$. For integral z we have $S_z = S_0$: $K'(y + LzK'y) = K'y$ since $K'L = 0$, and $y + LzK'y$ is integral exactly when y is. So $e(\mathrm{tr}(Tz)) = 1$ for every integral z; taking for z an element $\alpha \in \mathcal{O}$ at the place (j, i) and 0 elsewhere gives $e(\alpha T_{ij}) = 1$ for all $\alpha \in \mathcal{O}$, so $T_{ij} \in \mathcal{O}$ by the conductor property, and T is integral. Then $e(\mathrm{tr}(T\varpi^{-1}l(\bar{r}))) = \bar{e}(\mathrm{tr}(\bar{T}\bar{r}))$, and $\Lambda(\Gamma_0) = \Gamma(\bar{T})\Lambda(S_0)$. By Step 3, $\det \bar{T}$ is a unit, so $T \in \mathrm{GL}_b(\mathcal{O})$, and admissibility gives $\Lambda(S_0) = 0$, a contradiction. Hence every admissible Λ kills Γ_0, and $\Gamma_0 \in \mathcal{P}$ by hypothesis.

Step 5: a translate of Γ_0 is a multiple of a standard datum. Let $g = -\varpi^{-1} \cdot 1_n$ and $G(y) = \Gamma_0(gy)$, which lies in $\mathcal{P}$. First, for integral x' and integral $\rho \in \mathrm{M}_b(\mathcal{O})$,

$$\Psi^{\mathrm{st}}_{L'}(L\rho K'x' - \varpi x') = \bar{\chi}(\det \bar{\rho})\, \Psi^{\mathrm{st}}_{K'}(x'). \tag{c}$$

Indeed the argument $y' = L\rho K'x' - \varpi x'$ is integral and $L'y' = \rho K'x' - \varpi L'x'$, so $\overline{L'y'} = \bar{\rho}\,\overline{K'x'}$; take $\bar{\chi} \circ \det$. Now fix y and $\bar{r}$, and put $x' = gy + L\varpi^{-1}l(\bar{r})K'gy$; the term of $\bar{r}$ in $G(y)$ is $\bar{\chi}(\det \bar{r})\Psi^{\mathrm{st}}_{K'}(x')$. If it is non-zero, x' is integral. Since $K'L = 0$, $K'x' = K'gy = -\varpi^{-1}K'y$, and

$$L\,l(\bar{r})K'x' - \varpi x' = -\varpi^{-1}L\,l(\bar{r})K'y + y + \varpi^{-1}L\,l(\bar{r})K'y = y,$$

so by (c) the term equals $\Psi^{\mathrm{st}}_{L'}(y)$. Hence $G = r \cdot \Psi^{\mathrm{st}}_{L'}$, where $r(y)$ is the number of $\bar{r}$ whose term at y is non-zero. Let $x = L - \varpi K$: it is integral and $L'x = L'L - \varpi L'K = 1$. At $y = x$ the term of $\bar{r} = 1$ has $gx = -\varpi^{-1}L + K$, $K'gx = 1$ and $x' = -\varpi^{-1}L + K + \varpi^{-1}L = K$, so it equals $\Psi^{\mathrm{st}}_{K'}(K) = \chi(\det K'K)^{-1} = 1$, and $r(x) \neq 0$.

Step 6: conclusion. Step 2 applies to $p = G$, $s = L'$, $x = L - \varpi K$ and the function r of Step 5, and gives $\Psi \in \mathcal{P}$. □

This group proves Freitag's lemma (Proposition 6.3.36), the vanishing theorem on which the stable-space step rests (Proposition 6.3.33). If the partial means of a centre coefficient along a line are invariant under the shears at a nonempty set of places, and the coefficient vanishes at the points whose Siegel bracket is unimodular, then it vanishes on the stabiliser of the frame away from S: shear invariance makes the coefficient singular along the line, and a singular coefficient is determined by its values at the unimodular points. The proof spreads shear invariance to the whole radical by strong approximation, reduces every point to a straight point by an Iwasawa decomposition for the flag, and shows that at a straight point of least dual index the periods of the shifted mean force the Siegel bracket and its inverse to be integral. The key is that denominator indices decrease strictly under a non-unimodular integral change of basis (Lemma 6.3.45). In the proof of Proposition 6.3.33, the shear hypothesis is Lemma 6.3.47, and the vanishing hypothesis comes from the self-dual lattices through Lemma 6.3.49.

The rest of the group collects lemmas used earlier in the section, in the order of their first use, among them a smooth local form of Witt's theorem for isotropic vectors (Lemma 6.3.57; [330]). It closes with the explicit matching and its archimedean inputs.

Proposition 6.3.36. *Let W be a space with a point with signs s on n, A a base-adapted splitting of W over a space with a point W_0 along a one-element set, B a base-adapted splitting of W_0 over W_2 along an index set p, and D a splitting of W over W_1 along p whose rational frame is $k_D = i_A k_B$; all centre coefficients below are taken along D. Let $T \in \mathrm{M}_p(E)$ with $\det T \neq 0$, S a set of finite places of E, U a nonempty set of finite places of E, $\mathrm{e} = \mathrm{e}_{S \cup \bar{S}}$, and H the frame stabiliser of D off S. Let $f\colon \mathrm{G}(s) \to \mathbb{C}$ be continuous, left invariant under Γ_W, and right invariant under every $k \in \mathbf{K}_0(n)$ (read in $\mathrm{G}(s)$ with infinite components 1) whose component is 1 at every finite place w with $w \notin S$ and $\bar{w} \notin S$.*

1. *For every $x \in \mathrm{G}(s)$ the coefficient $\mathrm{coef}(T; f)(x)$ is defined, and the functions*

$$b \mapsto \mathrm{coef}(T; f)(z^{\mp}_{A,B}(b)\, x)$$

on $\mathrm{M}_{1\times p}(\mathbb{A}_E)$ are continuous and invariant under $\mathrm{M}_{1\times p}(E)$; so the partial means

$$\mathrm{coef}^{\pm}_{A,B,D}(T; f)(x)$$

are defined.

2. *Assume that $\mathrm{coef}^{+}_{A,B,D}(T; f)$ is invariant under the A-shears for U, and $\mathrm{coef}^{-}_{A,B,D}(T; f)$ under the $\bar{A}$-shears for U, at every point of H. Assume further that $\mathrm{coef}(T; f)(X) = 0$ for every $X \in \mathrm{G}(s)$ and $\gamma \in \mathrm{M}_p(\mathbb{A}_E)$ such that $X k_{D,\mathbb{A}} = k_{D,\mathbb{A}}(\mathrm{e}\gamma + (1 - \mathrm{e}))$ and $T\{\gamma\}$ is unimodular on $S \cup \bar{S}$. Then $\mathrm{coef}(T; f)(g) = 0$ for every $g \in H$.*

Proof. Write $f^{\pm} = \mathrm{coef}^{\pm}_{A,B,D}(T; f)$, $z^{\pm} = z^{\pm}_{A,B}$, $k_D = k_{D,\mathbb{A}}$ (so $k_D = i_{A,\mathbb{A}} k_{B,\mathbb{A}}$), $F_1 = (k_{A,\mathbb{A}} \mid k_D)$, and for a straight point $P = (X, u, t, t', \gamma)$ of (A, B, D) off S put $\varphi^{+}_P(a) = f^{+}(z^{+}(a)X)$, $\varphi^{-}_P(b) = f^{-}(z^{-}(b)X)$. The opposite splitting $\bar{A}$ is base-adapted (the same columns in another order), has the same embedding $i_{\mathbb{A}}$, and its shifts, partial means and shifted means are those of A with the signs exchanged ((MX5), (MX11)); so every result below applied to $\bar{A}$ in place of A concerns f^{-} and the $\bar{A}$-shears.

(1) is (1) of Lemma 6.3.32 (for D) and (2) of Lemma 6.3.37.

(2). Suppose that $\mathrm{coef}(T; f)(g) \neq 0$ for some $g \in H$.

There is a straight point with nonzero shifted mean. By (4) of Lemma 6.3.37 ($\det T \neq 0$), $a \mapsto f^{+}(z^{+}(a)g)$ is not identically zero; choose a with $f^{+}(z^{+}(a)g) \neq 0$. By (1) of Lemma 6.3.38

there are $\kappa \in \mathrm{G}(s)$ with finite part in $\mathbf{K}_0(n)$ and blocks with $z^+(a)g\kappa F_1 = F_1 \begin{pmatrix} \alpha & \beta \\ 0 & \gamma_0 \end{pmatrix}$. The point $z^+(a)g$ lies in H, as $z^+(a)$ fixes k_D ((1) of Lemma 6.3.37). By (2) of Lemma 6.3.39 (with $X = g = z^+(a)g$) there is a straight point P_0 of (A, B, D) off S with $\varphi^+_{P_0}(0) \neq 0$. So the set Σ of straight points P of (A, B, D) off S with $\varphi^+_P \not\equiv 0$ is nonempty. For $P \in \Sigma$ the matrix γ is invertible ((1) of Lemma 6.3.39), so $T\{\gamma\}$ is invertible and the dual index of γ on $S \cup \bar{S}$ for T is a positive integer (MX12). Choose $P = (X, u, t, t', \gamma) \in \Sigma$ with the least dual index.

Supports. By Lemma 6.3.43 (its minimality hypothesis is the choice of P), $\gamma^{-1}a^*t$ is integral on $S \cup \bar{S}$ whenever $\varphi^+_P(a) \neq 0$. The opposite $\bar{P} = (X, u, t', t, \gamma)$ is a straight point of $(\bar{A}, B, D)$ off S, and its first shifted mean for $\bar{A}$ is φ^-_P. Let Q be a straight point of $(\bar{A}, B, D)$ off S whose first shifted mean for $\bar{A}$ is not identically zero. Its opposite $\bar{Q}$ is a straight point of (A, B, D) off S with the same matrix γ_Q, and that first shifted mean is $\varphi^-_{\bar{Q}}$; by (4) of Lemma 6.3.37, $\varphi^+_Q \equiv 0$ would force $\varphi^-_{\bar{Q}} \equiv 0$, so $\bar{Q} \in \Sigma$ and the dual index of γ_Q is at least that of γ. Hence Lemma 6.3.43 applies to $\bar{A}$ and $\bar{P}$ (with the hypothesis on f^- and the $\bar{A}$-shears): $\gamma^{-1}b^*t'$ is integral on $S \cup \bar{S}$ whenever $\varphi^-_P(b) \neq 0$.

Unimodularity. By (1) of Lemma 6.3.39, $T\{\gamma\}$ is integral on $S \cup \bar{S}$; by (2) of Lemma 6.3.46 and the two support statements, $T\{\gamma\}$ is invertible with $T\{\gamma\}^{-1}$ integral on $S \cup \bar{S}$. So $T\{\gamma\}$ is unimodular on $S \cup \bar{S}$ (MX9).

Contradiction. By (1) of Lemma 6.3.39 there are $Y \in H$ and $v \in \mathrm{G}(s)$ with $\mathrm{e}Y = \mathrm{e}v$, $vk_D = k_D\gamma$ and $\mathrm{coef}(T; f)(Y) \neq 0$. As e is a scalar,

$$Yk_D = \mathrm{e}\,Yk_D + Y(1 - \mathrm{e})k_D = \mathrm{e}\,vk_D + (1 - \mathrm{e})k_D = k_D(\mathrm{e}\gamma + (1 - \mathrm{e})),$$

and the hypothesis gives $\mathrm{coef}(T; f)(Y) = 0$, a contradiction. Hence $\mathrm{coef}(T; f)(g) = 0$ for every $g \in H$. □

Lemma 6.3.37. *Let W be a space with a point with signs s on n, A a splitting of W over W_0 along a one-element set, B a splitting of W_0 over W_2 along p, D a splitting of W over W_1 along p with rational frame $k_D = i_A k_B$, and write $z^\pm = z^\pm_{A,B}$, z_D for the radical centre elements of D and $\mathrm{coef}(T; \cdot)$ for the centre coefficients along D. Let $f\colon \mathrm{G}(s) \to \mathbb{C}$ be continuous and left invariant under Γ_W, and $T \in \mathrm{M}_p(E)$.*

1. *For adelic $1 \times p$ matrices a, a', b and adelic $p \times p$ matrices M: $z^\pm(a + a') = z^\pm(a)z^\pm(a')$; $z^\pm(a) \in \Gamma_W$ if a is rational; $z^\pm(a)k_{D,\mathbb{A}} = k_{D,\mathbb{A}}$; $z^\pm(a)z_D(M) = z_D(M)z^\pm(a)$; and $z^+(a)z^-(b) = z_D(-a^*b)\, z^-(b)\, z^+(a)$.*

2. *For $x \in \mathrm{G}(s)$ the functions $b \mapsto \mathrm{coef}(T; f)(z^\mp(b)x)$ are continuous and $\mathrm{M}_{1\times p}(E)$-periodic, and the partial means*

$$f^+(x) = \int \mathrm{coef}(T; f)(z^-(b)x)\, db, \qquad f^-(x) = \int \mathrm{coef}(T; f)(z^+(b)x)\, db$$

   (over $\mathrm{M}_{1\times p}(\mathbb{A}_E)/\mathrm{M}_{1\times p}(E)$) are continuous functions of x, with $f^+(z^-(b)x) = f^+(x)$, $f^-(z^+(b)x) = f^-(x)$ and $\psi_E(\mathrm{tr}(TM))\, f^\pm(z_D(M)x) = f^\pm(x)$.

3. *The shifted means $\varphi^+_x(a) = f^+(z^+(a)x)$ and $\varphi^-_x(b) = f^-(z^-(b)x)$ satisfy $\varphi^+_{z^+(a)x}(y) = \varphi^+_x(y + a)$ and $\varphi^-_{z^+(a)x}(b) = \psi_E(-\mathrm{tr}(Ta^*b))\, \varphi^-_x(b)$.*

4. *Let $\det T \neq 0$, $x, x' \in \mathrm{G}(s)$, $c \in \mathbb{C}$ with $\varphi^+_{x'} = c\,\varphi^+_x$. Then $\mathrm{coef}(T; f)(z^-(b)z^+(a)x') = c\,\mathrm{coef}(T; f)(z^-(b)z^+(a)x)$ for all a, b, and $\varphi^-_{x'} = c\,\varphi^-_x$. In particular, if $\varphi^+_x = 0$, then $\mathrm{coef}(T; f)(x) = 0$ and $\varphi^-_x = 0$.*

5. *All of the above holds with A replaced by its opposite splitting, which exchanges z^+ and z^-, f^+ and f^-, φ^+ and φ^-.*

Proof. Write $k_{\mathbb{A}}, l_{\mathbb{A}}, i_{\mathbb{A}}, \mathrm{pr}_k, \mathrm{pr}_l$ for the adelic frames and coordinates of A, $\varepsilon = \varepsilon_s$, $K = i_{\mathbb{A}} k_{B,\mathbb{A}} = k_{D,\mathbb{A}}$ and $\Lambda = K^* \varepsilon$, which is the coordinate pr_l of D. By (MX1) for A and B: $\Lambda K = k^*_{B,\mathbb{A}} \varepsilon_{W_0} k_{B,\mathbb{A}} = 0$, $\Lambda k_{\mathbb{A}} = \Lambda l_{\mathbb{A}} = 0$, $\mathrm{pr}_k K = \mathrm{pr}_l K = 0$, $\mathrm{pr}_k k_{\mathbb{A}} = \mathrm{pr}_l l_{\mathbb{A}} = 1$, $\mathrm{pr}_k l_{\mathbb{A}} = \mathrm{pr}_l k_{\mathbb{A}} = 0$. The composite $A \circ B$ has adelic frame $(k_{\mathbb{A}} \mid K)$ and coordinate pr_l with rows pr_l, Λ; for $\bar{A} \circ B$ these are $(l_{\mathbb{A}} \mid K)$ and pr_k, Λ. As $M_b - M_b^*$ has the blocks b at (o,p) and $-b^*$ at (p,o),

$$\begin{aligned} z^-(b) &= 1 + N_b, & N_b &= k_{\mathbb{A}} b \Lambda - K b^* \mathrm{pr}_l; \\ z^+(a) &= 1 + P_a, & P_a &= l_{\mathbb{A}} a \Lambda - K a^* \mathrm{pr}_k; \\ z_D(M) &= 1 + K(M - M^*)\Lambda. \end{aligned}$$

(1). Additivity and rationality are (1) of Lemma 6.3.32 for the splittings $A \circ B$ and $\bar{A} \circ B$ ($M_{a+a'} = M_a + M_{a'}$). $P_a K = N_b K = 0$ by the relations, so $z^{\pm}$ fix K; then (3) of Lemma 6.3.32 (with $c = 1$) gives the commutation with $z_D(M)$. From the relations, $P_a N_b = -K a^* b \Lambda$ and $N_b P_a = -K b^* a \Lambda$, so $z^+(a) z^-(b) = 1 + P_a + N_b - K a^* b \Lambda$ and $z^-(b) z^+(a) = 1 + Q$ with $Q = P_a + N_b - K b^* a \Lambda$. Since $\Lambda Q = 0$, $z_D(-a^*b)(1+Q) = 1 + Q + K(-a^*b + b^*a)\Lambda = z^+(a) z^-(b)$.

(2). $\mathrm{coef}(T; f)$ is continuous by (1) of Lemma 6.3.32, hence so are the two functions of b. For rational γ, $z^{\mp}(b+\gamma)x = z^{\mp}(\gamma) z^{\mp}(b) x$ with $z^{\mp}(\gamma) \in \Gamma_W$ fixing $k_{D,\mathbb{A}}$, so (4) of Lemma 6.3.32 with $c = 1$ gives periodicity. The integrals are over a relatively compact fundamental domain of a jointly continuous function, so $f^{\pm}$ are continuous (dominated convergence on compact neighbourhoods). Invariance under $z^{\mp}(b)$ is the translation invariance of the measure with additivity. By (1) and (2) of Lemma 6.3.32, $\mathrm{coef}(T; f)(z^-(b) z_D(M) x) = \mathrm{coef}(T; f)(z_D(M) z^-(b) x) = \psi_E(\mathrm{tr}(TM))^{-1} \mathrm{coef}(T; f)(z^-(b) x)$; integrate over b. Likewise for f^-.

(3). The first law is additivity of z^+. By (1), $z^-(b) z^+(a) = z_D(a^*b) z^+(a) z^-(b)$, so by (2)

$$\begin{aligned} \varphi^-_{z^+(a)x}(b) &= f^-(z_D(a^*b) z^+(a) z^-(b) x) = \psi_E(\mathrm{tr}(Ta^*b))^{-1} f^-(z^+(a) z^-(b) x) \\ &= \psi_E(-\mathrm{tr}(Ta^*b)) \varphi^-_x(b). \end{aligned}$$

(4). Let $a_0 \in \mathrm{M}_{1\times p}(E)$. By (1), $z^-(b) z^+(a_0) = z_D(a_0^* b) z^+(a_0) z^-(b)$; by (2) and (4) of Lemma 6.3.32 ($z^+(a_0)$ is rational and fixes $k_{D,\mathbb{A}}$),

$$\varphi^+_x(a_0 + a) = \int \psi_E(-\mathrm{tr}(Ta_0^* b)) \, \mathrm{coef}(T; f)(z^-(b) z^+(a) x) \, db. \tag{6.47}$$

Fix a and let

$$F(b) = \mathrm{coef}(T; f)(z^-(b) z^+(a) x') - c \, \mathrm{coef}(T; f)(z^-(b) z^+(a) x),$$

continuous and periodic by (2). By (6.47) and the hypothesis, $\int \psi_E(b\,R) F(b)\, db = 0$ for $R = -Ta_0^*$ ($\mathrm{tr}(Ta_0^* b) = bTa_0^*$). As T is invertible, R runs over all of E^p when a_0 runs over $\mathrm{M}_{1\times p}(E)$, and by (a) of Theorem A.5.2 on each coordinate the $b \mapsto \psi_E(bR)$ are all the continuous characters of $(\mathbb{A}_E/E)^p$. Hence $F = 0$, which is the first claim. By (1) and (2) of Lemma 6.3.32, $\mathrm{coef}(T; f)(z^+(a) z^-(b) y) = \psi_E(\mathrm{tr}(Ta^*b)) \, \mathrm{coef}(T; f)(z^-(b) z^+(a) y)$ for every y; so the first claim gives $\mathrm{coef}(T; f)(z^+(a) z^-(b) x') = c \, \mathrm{coef}(T; f)(z^+(a) z^-(b) x)$, and integration over a gives $\varphi^-_{x'}(b) = c \, \varphi^-_x(b)$. With $x' = x$ and $c = 0$: $\mathrm{coef}(T; f)(x)$ (the case $a = b = 0$) and φ^-_x vanish.

(5). By (MX5), $z^{\pm}_{\bar{A},B} = z^{\mp}_{A,B}$, the opposite of $\bar{A}$ is A, and $\bar{A}$ has the same embedding $i_{\mathbb{A}}$, so $k_D = i_{\bar{A}} k_B$; the proof applies verbatim. □

Lemma 6.3.38. *Let W be a space with a point with signs s on n, A a base-adapted splitting of W over W_0 (signs on n_0) along a one-element set, B a base-adapted splitting of W_0 over W_2 along p, $k_D = i_{A,\mathbb{A}} k_{B,\mathbb{A}}$, and let $F_1 = (k_{A,\mathbb{A}} \mid k_D)$ and $F_2 = (l_{A,\mathbb{A}} \mid k_D)$ be the adelic frames k of $A \circ B$ and of $\bar{A} \circ B$. Block matrices below have the blocks 1×1, $1 \times p$, $p \times 1$, $p \times p$.*

1. *For every $g \in \mathrm{G}(s)$ there are $\kappa \in \mathrm{G}(s)$ with finite part in $\mathbf{K}_0(n)$, $a \in \mathbb{A}_E$, $\beta \in \mathrm{M}_{1\times p}(\mathbb{A}_E)$ and $\gamma \in \mathrm{M}_p(\mathbb{A}_E)$ such that*
$$g\kappa\, F_1 = F_1 \begin{pmatrix} a & \beta \\ 0 & \gamma \end{pmatrix}.$$

2. *If $g \in \mathrm{G}(s)$ satisfies $gF_1 = F_1 \begin{pmatrix} a & \beta \\ 0 & \gamma \end{pmatrix}$, then there are $c \in \mathrm{M}_{n_0\times 1}(\mathbb{A}_E)$, $\zeta \in \mathbb{A}_E$, $u \in \mathrm{G}(s)$ and $t, t' \in \mathbb{A}_E$ with $g = \mathrm{rad}_A(c,\zeta)\, u$, $uF_1 = F_1 \,\mathrm{diag}(t,\gamma)$, $uF_2 = F_2 \,\mathrm{diag}(t',\gamma)$ and $\bar{t}'t = 1$.*

Proof. Write $k = k_{A,\mathbb{A}}$, $l = l_{A,\mathbb{A}}$, $i = i_{A,\mathbb{A}}$, $\mathrm{pr}_k, \mathrm{pr}_i$ for the coordinates of A, and $\varepsilon = \varepsilon_s$.

(2). The first column of the hypothesis is $gk = ka$. Since g preserves the form, $(gl)^*\varepsilon ka = (gl)^*\varepsilon gk = l^*\varepsilon k = 1$, so a is a unit and g lies in the group P_X of Lemma 6.2.16 for $X = A$ (base-adapted). By (2) of Lemma 6.2.16, $g = n\,\mathrm{lev}_A(m,h)$ with $n = \mathrm{rad}_A(c,\zeta)$ for some c, ζ, $m = \mathrm{pr}_k gk = a$ and $h = \mathrm{pr}_i gi \in \mathrm{G}(s_0)$. As $\mathrm{pr}_i k = 0$ and $\mathrm{pr}_i i = 1$ (MX1), the second block of the hypothesis, $gk_D = k\beta + k_D\gamma$, gives $hk_{B,\mathbb{A}} = \mathrm{pr}_i gk_D = k_{B,\mathbb{A}}\gamma$. Let $u = \mathrm{lev}_A(m,h)$. By (MX2), $uk = km$, $ul = l\,\bar{m}^{-1}$ and $uk_D = i\,h\,k_{B,\mathbb{A}} = k_D\gamma$; so $uF_1 = F_1\,\mathrm{diag}(m,\gamma)$ and $uF_2 = F_2\,\mathrm{diag}(\bar{m}^{-1},\gamma)$. With $t = m$, $t' = \bar{m}^{-1}$ we have $\bar{t}'t = m^{-1}m = 1$.

(1). Let $C = A \circ B$ (MX5); it has frames $k_C = F_1$, $l_C = (l \mid i\,l_{B,\mathbb{A}})$, $i_C = i\,i_{B,\mathbb{A}}$, and up to the order of the columns $(l_C \mid k_C \mid i_C) = (l \mid k \mid i)\,\mathrm{diag}(1, 1, (l_{B,\mathbb{A}} \mid k_{B,\mathbb{A}} \mid i_{B,\mathbb{A}}))$, a product of two matrices that are unitary at every infinite place and integral at the finite places (A, B base-adapted). The relations of (MX1) for C give, for the form $y^*\varepsilon z$, $k_C^*\varepsilon k_C = l_C^*\varepsilon l_C = 0$, $l_C^*\varepsilon k_C = 1$, $i_C^*\varepsilon k_C = i_C^*\varepsilon l_C = 0$ and $i_C^*\varepsilon i_C = \varepsilon_{s_2}$ (s_2 the signs of W_2). Put $J = 1 + \#p$ and let j be the composition $(1, \#p)$ of J, or (1) if p is empty. Let v be a place of F, G_v the component of $\mathrm{G}(s)$ at v, $K_v \subset G_v$ the component at v of the unitary matrices (if $v \in \Phi$) or of $\mathbf{K}_0(n)$ (if v is finite), and $P_v \subset G_v$ the stabiliser of the flag $W_{1,v} \subset W_{2,v}$ spanned by the first column, resp. all columns, of $(F_1)_v$.

For $v \in \Phi$, $G_v = \mathrm{U}(\mathrm{diag}\, s)$, and the columns $e_1 = k_v$, $(e_2, \dots, e_J) = (k_D)_v$, $(f_1, \dots, f_J) = (l_C)_v$, $(u_c) = (i_C)_v$ form a basis with the relations of (i) of Lemma 6.2.17 for the form $y^*\mathrm{diag}(s)z$, orthonormal for the standard form; so K_v is the stabiliser of the majorant, the signature (p_v, q_v) of $\mathrm{diag}(s)$ has $\min(p_v, q_v) \geq J$ (the span of the e and f has signature (J, J)), and (i) of Lemma 6.2.17 gives $G_v = P_vK_v$. For finite v, $E \otimes_F F_v$ is an unramified quadratic field extension of F_v or $F_v \times F_v$ (Lemma 3.2.9); G_v is the unitary group of y^*z on $(E \otimes_F F_v)^n$ ($\varepsilon = 1$ there), K_v is the stabiliser of the self-dual lattice of integral vectors, and $W_{1,v} \subset W_{2,v}$ is an isotropic flag of free submodules of ranks 1 and J (the columns of $(F_1)_v$ are part of the invertible matrix $(l_C \mid k_C \mid i_C)_v$, and $k_C^*\varepsilon k_C = 0$); (ii) of Lemma 6.2.17 gives $G_v = P_vK_v$.

Let $g \in \mathrm{G}(s)$, and let R be the finite set of finite places v of F at which g_v is not integral. For $v \in \Phi \cup R$ write $g_v = \pi_v\kappa_v$ with $\pi_v \in P_v$, $\kappa_v \in K_v$; for the other v put $\pi_v = 1$, $\kappa_v = g_v \in K_v$. Then $\pi = (\pi_v)$ and $\kappa' = (\kappa_v)$ lie in $\mathrm{G}(s)$, $g = \pi\kappa'$, and $\kappa = \kappa'^{-1}$ has finite part in $\mathbf{K}_0(n)$. At every place π_v preserves $W_{1,v}$ and $W_{2,v}$, so $\pi_v(F_1)_v = (F_1)_v \begin{pmatrix} a_v & \beta_v \\ 0 & \gamma_v \end{pmatrix}$, with $a_v = 1$, $\beta_v = 0$, $\gamma_v = 1$ for $v \notin \Phi \cup R$; hence a, β, γ are adelic and $g\kappa F_1 = \pi F_1 = F_1 \begin{pmatrix} a & \beta \\ 0 & \gamma \end{pmatrix}$. □

Lemma 6.3.39. *Let W be a space with a point with signs s on n, A a base-adapted splitting of W over W_0 along a one-element set, B a base-adapted splitting of W_0 over W_2 along p, and D a splitting of W over W_1 along p with $k_{D,\mathbb{A}} = i_{A,\mathbb{A}} k_{B,\mathbb{A}}$; write $k_D = k_{D,\mathbb{A}}$, $F_1 = k_{A\circ B,\mathbb{A}} = (k_{A,\mathbb{A}} \mid k_D)$, $z^{\pm} = z^{\pm}_{A,B}$. Let S be a set of finite places of E, $\mathrm{e} = \mathrm{e}_{S\cup\bar{S}}$, H the frame stabiliser of D off S, $T \in \mathrm{M}_p(E)$, and $f\colon \mathrm{G}(s) \to \mathbb{C}$ continuous, left invariant under Γ_W, and right invariant under every $k \in \mathbf{K}_0(n)$ (read in $\mathrm{G}(s)$ with infinite components 1) whose component is 1 at every finite place outside $S \cup \bar{S}$. Write $f^+ = \mathrm{coef}^+_{A,B,D}(T; f)$.*

1. *Let (X, u, t, t', γ) be a straight point of (A, B, D) off S. Then γ is invertible and $uk_D = k_D\gamma$. If $f^+(z^+(a)X) \neq 0$ for some $a \in \mathrm{M}_{1\times p}(\mathbb{A}_E)$, then there are $Y \in H$ and $v \in \mathrm{G}(s)$ with $\mathrm{e}Y = \mathrm{e}v$, $vk_D = k_D\gamma$ and $\mathrm{coef}(T; f)(Y) \neq 0$; and $T\{\gamma\}$ is integral on $S \cup \bar{S}$.*

2. *Let U be a nonempty set of finite places of E, and assume that f^+ is invariant under the A-shears for U at every point of H. Let $X \in H$ and $g \in \mathrm{G}(s)$ with $\mathrm{e}X = \mathrm{e}g$, let $\kappa \in \mathrm{G}(s)$ have finite part in $\mathbf{K}_0(n)$, and let*

$$g\kappa F_1 = F_1 \begin{pmatrix} a & \beta \\ 0 & \gamma \end{pmatrix} \qquad (a \in \mathbb{A}_E,\ \beta \in \mathrm{M}_{1\times p}(\mathbb{A}_E),\ \gamma \in \mathrm{M}_p(\mathbb{A}_E)).$$

   If $f^+(X) \neq 0$, then there is a straight point (X', u', t, t', γ) of (A, B, D) off S, with this γ, such that $f^+(X') \neq 0$.

Proof. For $g \in \mathrm{G}(s)$ let $g^{(S)}$ be as in Lemma 6.3.40; then $\mathrm{e}g^{(S)} = \mathrm{e}g$ and $(1-\mathrm{e})g^{(S)} = g^{(S)}(1-\mathrm{e}) = 1 - \mathrm{e}$, since e is a scalar with $\mathrm{e}^2 = \mathrm{e}$. The finite components of $k_D = i_{A,\mathbb{A}} k_{B,\mathbb{A}}$ are integral, A and B being base-adapted. By (1) of Lemma 6.3.37, $z^{\pm}(a)k_D = k_D$ for all a.

(1). Let l_C be the adelic frame l of $C = A \circ B$; by (MX1), $l_C^* \varepsilon_s F_1 = 1$. The frame condition $uF_1 = F_1 \,\mathrm{diag}(t, \gamma)$ gives $uk_D = k_D\gamma$ and $\mathrm{diag}(t, \gamma) = l_C^* \varepsilon_s u F_1$, which is invertible with inverse $l_C^* \varepsilon_s u^{-1} F_1$ (as $u^{-1}F_1 = F_1 \,\mathrm{diag}(t, \gamma)^{-1}$ placewise); so γ is invertible. Let $f^+(z^+(a)X) \neq 0$. By (MX11) $f^+(z^+(a)X)$ is the integral of $b \mapsto \mathrm{coef}(T; f)(z^-(b)z^+(a)X)$, so this function does not vanish at some b. Put $Y = z^-(b)z^+(a)X$ and $v = z^-(b)z^+(a)u$. As $z^{\pm}$ fix k_D and $X \in H$, $Y(1 - \mathrm{e})k_D = (1 - \mathrm{e})k_D$, so $Y \in H$; $\mathrm{e}Y = z^-(b)z^+(a)\,\mathrm{e}X = z^-(b)z^+(a)\,\mathrm{e}u = \mathrm{e}v$; $vk_D = z^-(b)z^+(a)k_D\gamma = k_D\gamma$; and $\mathrm{coef}(T; f)(Y) \neq 0$. By (4) of Lemma 6.3.40 (with $X = Y$, $u = v$), $T\{\gamma\}$ is integral on $S \cup \bar{S}$.

(2). By (2) of Lemma 6.3.38, $g\kappa = \mathrm{rad}_A(c, \zeta)\, u'$ with $u'F_1 = F_1 \,\mathrm{diag}(t, \gamma)$, $u'k_{\bar{A}\circ B,\mathbb{A}} = k_{\bar{A}\circ B,\mathbb{A}} \,\mathrm{diag}(t', \gamma)$ and $\bar{t}'t = 1$. By (1) of Lemma 6.2.16 (for $X = A$, base-adapted) the radical elements of A form a group, so $h = \mathrm{rad}_A(c, \zeta)^{-1} = \mathrm{rad}_A(c', \zeta')$ for some c', ζ'. By (MX2), $h - 1 = -i_{\mathbb{A}} c' \mathrm{pr}_l + k_{\mathbb{A}} c'^* \varepsilon_{s_0} \mathrm{pr}_i - k_{\mathbb{A}} Y' \mathrm{pr}_l$ with $Y' = \frac{1}{2}(\jmath^{-1}(\zeta' + \bar{\zeta}') + c'^* \varepsilon_{s_0} c')$ (frames and coordinates of A), and as $\bar{\mathrm{e}} = \mathrm{e} = \mathrm{e}^2$ the same formula for $(\mathrm{e}c', \mathrm{e}\zeta')$ gives $\mathrm{e}(h - 1)$; so $h^{(S)} = 1 + \mathrm{e}(h - 1) = \mathrm{rad}_A(\mathrm{e}c', \mathrm{e}\zeta')$. Put

$$X_1 = X\kappa^{(S)}, \qquad X' = h^{(S)} X_1.$$

Then $X_1(1 - \mathrm{e})k_D = X(1 - \mathrm{e})k_D = (1 - \mathrm{e})k_D$ and $X'(1 - \mathrm{e})k_D = h^{(S)}(1 - \mathrm{e})k_D = (1 - \mathrm{e})k_D$, so $X_1, X' \in H$; and $\mathrm{e}X' = h^{(S)}\,\mathrm{e}X\,\kappa^{(S)} = h^{(S)}\,\mathrm{e}g\kappa = \mathrm{e}\,hg\kappa = \mathrm{e}u'$, using $\mathrm{e}X = \mathrm{e}g$, $\mathrm{e}\kappa^{(S)} = \mathrm{e}\kappa$ and $\mathrm{e}h^{(S)} = \mathrm{e}h$. So (X', u', t, t', γ) is a straight point of (A, B, D) off S. By (2) of Lemma 6.3.41 (with $g = X_1 \in H$ and $\mathrm{rad}_A(\mathrm{e}c', \mathrm{e}\zeta')X_1 = X' \in H$), $f^+(X') = f^+(X_1)$. By (3) of Lemma 6.3.40 applied to each point $z^-(b)X$, $\mathrm{coef}(T; f)(z^-(b)X\kappa^{(S)}) = \mathrm{coef}(T; f)(z^-(b)X)$, and integrating over b, $f^+(X_1) = f^+(X) \neq 0$. □

Lemma 6.3.40. *Let W be a space with a point with signs s on n, D a splitting of W over W_1 along p with adelic frame $k_D = k_{D,\mathbb{A}}$, $f\colon \mathrm{G}(s) \to \mathbb{C}$ continuous and left invariant under Γ_W, $T \in \mathrm{M}_p(E)$, S a set of finite places of E and $\mathrm{e} = \mathrm{e}_{S\cup\bar{S}}$; centre coefficients are taken along D. For $g \in \mathrm{G}(s)$ let $g^{(S)}$ be the matrix $\mathrm{e}g + (1-\mathrm{e})$ (the components of g at the places of $S \cup \bar{S}$, and 1 at all other places).*

1. *$g^{(S)} \in \mathrm{G}(s)$, $(gh)^{(S)} = g^{(S)}h^{(S)}$, and $z_D(M)^{(S)} = z_D(\mathrm{e}M)$ for adelic $p \times p$ matrices M.*
2. *Let $X, g \in \mathrm{G}(s)$ with $\mathrm{e}X = \mathrm{e}g$ and $c \in \mathrm{M}_p(\mathbb{A}_E)$ with $gk_D = k_D c$. Then for every adelic $p \times p$ matrix M*
$$\psi_E\big(\mathrm{e}\,\mathrm{tr}(c^*TcM)\big)\,\mathrm{coef}(T;f)\big(X\,z_D(\mathrm{e}M)\big) = \mathrm{coef}(T;f)(X).$$
3. *Assume that f is right invariant under every $k \in \mathbf{K}_0(n)$ (read in $\mathrm{G}(s)$ with infinite components 1) whose component is 1 at every finite place outside $S\cup\bar{S}$. Then $\mathrm{coef}(T;f)(X\kappa^{(S)}) = \mathrm{coef}(T;f)(X)$ for all $X \in \mathrm{G}(s)$ and all $\kappa \in \mathrm{G}(s)$ whose finite part lies in $\mathbf{K}_0(n)$.*
4. *Assume the hypothesis of (3) and that the finite components of k_D have entries in $\widehat{\mathcal{O}}_E$. Let $X, u \in \mathrm{G}(s)$ with $\mathrm{e}X = \mathrm{e}u$, $\gamma \in \mathrm{M}_p(\mathbb{A}_E)$ with $uk_D = k_D\gamma$, and $\mathrm{coef}(T;f)(X) \neq 0$. Then $T\{\gamma\}$ is integral on $S \cup \bar{S}$.*

Proof. Write $\varepsilon = \varepsilon_s$ and $\Lambda = k_D^*\varepsilon$, the coordinate pr_l of D. The adele e satisfies $\mathrm{e}^2 = \mathrm{e}$, $\bar{\mathrm{e}} = \mathrm{e}$ (the set $S\cup\bar{S}$ is stable under conjugation) and commutes with all adelic matrices; for $g \in \mathrm{G}(s)$, $g^{-1} = \varepsilon g^*\varepsilon$.

(1). As $\mathrm{e}(1-\mathrm{e}) = 0$, $(\mathrm{e}g^* + 1 - \mathrm{e})\varepsilon(\mathrm{e}g + 1 - \mathrm{e}) = \mathrm{e}g^*\varepsilon g + (1-\mathrm{e})\varepsilon = \varepsilon$, and $(\mathrm{e}g + 1 - \mathrm{e})$ is invertible with inverse $\mathrm{e}g^{-1} + 1 - \mathrm{e}$; so $g^{(S)} \in \mathrm{G}(s)$, and multiplicativity is the same computation. By (MX2), $z_D(M)^{(S)} = 1 + \mathrm{e}k_D(M - M^*)\Lambda = 1 + k_D(\mathrm{e}M - (\mathrm{e}M)^*)\Lambda = z_D(\mathrm{e}M)$.

(2). From $\mathrm{e}X = \mathrm{e}g$ we get $\mathrm{e}X^* = \mathrm{e}g^*$ and $\mathrm{e}X^{-1} = \varepsilon(\mathrm{e}X^*)\varepsilon = \mathrm{e}g^{-1}$. Hence $\mathrm{e}Xk_D = \mathrm{e}k_Dc$ and $\mathrm{e}X^{-1}k_D = \mathrm{e}k_Dc^{-1}$ (c is invertible: with l_D the adelic frame $l_{D,\mathbb{A}}$, $l_D^*\varepsilon k_D = 1$ by (MX1), and $c = l_D^*\varepsilon g k_D$; at each place g is invertible and k_D has independent columns, so c is invertible there, $g^{-1}k_D = k_Dc^{-1}$, and $c^{-1} = l_D^*\varepsilon g^{-1}k_D$ is adelic), and by (6.46) of Lemma 6.3.32, $\mathrm{e}\Lambda X = \mathrm{e}(X^{-1}k_D)^*\varepsilon = \mathrm{e}(c^{-1})^*\Lambda$. Therefore, with $N = M - M^*$,

$$X\,z_D(\mathrm{e}M) = X + \mathrm{e}Xk_DN\Lambda = X + \mathrm{e}k_DcN\Lambda, \qquad z_D(\mathrm{e}Mc^*)\,X = X + \mathrm{e}k_DcNc^*\Lambda X = X + \mathrm{e}k_DcN\Lambda,$$

so $X\,z_D(\mathrm{e}M) = z_D(\mathrm{e}cMc^*)X$. By (2) of Lemma 6.3.32 for D,

$$\psi_E(\mathrm{tr}(T\mathrm{e}cMc^*))\,\mathrm{coef}(T;f)(z_D(\mathrm{e}cMc^*)X) = \mathrm{coef}(T;f)(X),$$

and $\mathrm{tr}(TcMc^*) = \mathrm{tr}(c^*TcM)$.

(3). $\kappa^{(S)}$ has infinite components 1, finite part in $\mathbf{K}_0(n)$ and component 1 at every finite place outside $S\cup\bar{S}$; so $f(z_D(M)X\kappa^{(S)}) = f(z_D(M)X)$ for every M, and (MX3) gives the claim.

(4). Fix $i, j \in p$ and $a \in \widehat{\mathcal{O}}_E$, read as the adele with infinite components 0, and let $M = aE_{ji}$ (E_{ji} the matrix unit). At the finite places $\varepsilon = 1$, so $\Lambda = k_D^*$ there, and the finite part of $z_D(M) = 1 + k_D(M - M^*)\Lambda$ has entries in $\widehat{\mathcal{O}}_E$; it lies in $G_f(n)$, so its inverse, its conjugate transpose, is integral too, and the finite part of $z_D(M)$ lies in $\mathbf{K}_0(n)$. By (1), (3) and (2) (with $g = u$, $c = \gamma$),

$$\begin{aligned}\mathrm{coef}(T;f)(X) &= \mathrm{coef}(T;f)\big(Xz_D(M)^{(S)}\big) = \mathrm{coef}(T;f)\big(Xz_D(\mathrm{e}M)\big)\\ &= \psi_E\big(\mathrm{e}\,\mathrm{tr}(\gamma^*T\gamma M)\big)^{-1}\mathrm{coef}(T;f)(X),\end{aligned}$$

and as $\mathrm{coef}(T;f)(X) \neq 0$, $\psi_E(\mathrm{e}\,\mathrm{tr}(\gamma^* T\gamma M)) = 1$. Here $\mathrm{tr}(\gamma^* T\gamma M) = a\,(\gamma^* T\gamma)_{ij} = a\,\delta_\jmath T\{\gamma\}_{ij}$. The adele $\mathrm{e}\,\delta_\jmath T\{\gamma\}_{ij} a$ has infinite components 0, and the finite part of $\delta_\jmath$ is $\jmath\alpha\bar{\alpha}$; so by (WD4), with d the finite part of $\mathrm{e}\,T\{\gamma\}_{ij}$, $\psi_0(da) = 1$. This holds for all $a \in \widehat{\mathcal{O}}_E$, so $\bar{d}$ lies in the annihilator of $\widehat{\mathcal{O}}_E$ under $(u,v) \mapsto \psi_0(\bar{u}v)$, which is $\widehat{\mathcal{O}}_E$ by Lemma 4.1.3; hence $d \in \widehat{\mathcal{O}}_E$, that is, $T\{\gamma\}_{ij}$ is integral on $S \cup \bar{S}$. □

Lemma 6.3.41. *Let W be a space with a point with signs s on n, A a splitting of W over W_0 (signs s_0 on n_0) along a one-element set, B a splitting of W_0 over W_2 along p, and D a splitting of W over W_1 along p with $k_{D,\mathbb{A}} = i_{A,\mathbb{A}} k_{B,\mathbb{A}}$. Let $f\colon \mathrm{G}(s) \to \mathbb{C}$ be continuous and left invariant under Γ_W, $T \in \mathrm{M}_p(E)$, S a set of finite places of E, U a nonempty set of finite places of E, and H the frame stabiliser of D off S. Assume that $\mathrm{coef}^+_{A,B,D}(T;f)$ is invariant under the A-shears for U at every point of H. Then:*

1. *for $x \in H$ and $\zeta \in \mathbb{A}_E$,*

$$\mathrm{coef}^+_{A,B,D}(T;f)(z_A(\zeta)x) = \mathrm{coef}^+_{A,B,D}(T;f)(x),$$

   and $\mathrm{coef}^+_{A,B,D}(T;f)(x)$ is the centre coefficient of f along $A \circ B$ at $0 \oplus T$ at x;

2. *for $g \in H$, $c \in \mathrm{M}_{n_0\times 1}(\mathbb{A}_E)$ and $\beta \in \mathbb{A}_E$ with $\mathrm{rad}_A(c,\beta)\,g \in H$,*

$$\mathrm{coef}^+_{A,B,D}(T;f)(\mathrm{rad}_A(c,\beta)\,g) = \mathrm{coef}^+_{A,B,D}(T;f)(g).$$

Proof. Write $f^+ = \mathrm{coef}^+_{A,B,D}(T;f)$, $C(x)$ for the centre coefficient of f along $A \circ B$ at $0 \oplus T$, $z^- = z^-_{A,B}$, $\mathrm{rad} = \mathrm{rad}_A$, $k = k_{A,\mathbb{A}}$, $i = i_{A,\mathbb{A}}$, $k_D = k_{D,\mathbb{A}}$, $k_B = k_{B,\mathbb{A}}$, $\mathrm{pr}_l, \mathrm{pr}_i$ for the coordinates of A, $\varepsilon_0 = \varepsilon_{s_0}$, $\Lambda = k_D^* \varepsilon_s$ and $\mathrm{e} = \mathrm{e}_U$. By (MX1), read adelically: $\mathrm{pr}_l k = \mathrm{pr}_l i = 0$, $\mathrm{pr}_i k = 0$, $\mathrm{pr}_i i = 1$, $\Lambda k = 0$, and $\mathrm{pr}_l k_D = 0$. Every $g \in \mathrm{G}(s)$ with $gk = k$ commutes with every $z_A(\zeta)$ ((3) of Lemma 6.3.32 for A with $c = 1$), and $z_A(\zeta)k_D = k_D$ as $\mathrm{pr}_l k_D = 0$; so $z_A(\zeta) \in H$, H being a group.

Step 1: radical elements. By (MX2), $\mathrm{rad}(c,\beta) = 1 + N(c,\beta)$ with $N(c,\beta) = -ic\,\mathrm{pr}_l + kc^*\varepsilon_0 \mathrm{pr}_i - kY\,\mathrm{pr}_l$, $Y = \frac{1}{2}(\jmath^{-1}(\beta + \bar{\beta}) + c^*\varepsilon_0 c)$. The relations give $\mathrm{rad}(c,\beta)k = k$, $\mathrm{rad}(c,\beta)i = i + kc^*\varepsilon_0$, and $N(c,\beta)N(c',\beta') = -k\,c^*\varepsilon_0 c'\,\mathrm{pr}_l$; comparing with $N(c+c', \beta+\beta'+\jmath t)$ for $t = c^*\varepsilon_0 c'$ (so $c'^*\varepsilon_0 c = \bar{t}$, ε_0 being real diagonal, and $\bar{\jmath} = -\jmath$),

$$\mathrm{rad}(c,\beta)\,\mathrm{rad}(c',\beta') = \mathrm{rad}\big(c + c',\ \beta + \beta' + \jmath\, c^*\varepsilon_0 c'\big). \tag{6.48}$$

Also $\mathrm{rad}(0,\beta) = 1 - k\frac{1}{2}\jmath^{-1}(\beta + \bar{\beta})\mathrm{pr}_l = z_A(-(2\jmath)^{-1}\beta)$, since $M = -(2\jmath)^{-1}\beta$ has $M - \bar{M} = -(2\jmath)^{-1}(\beta + \bar{\beta})$.

Step 2: rational centre elements. Let $\zeta_0 \in E$. Then $z_A(\zeta_0) \in \Gamma_W$ ((1) of Lemma 6.3.32 for A). For adelic b, $z^-(b) = 1 + kb\Lambda - k_D b^* \mathrm{pr}_l$ fixes k, so it commutes with $z_A(\zeta_0)$; as $z_A(\zeta_0)$ fixes k_D, (4) of Lemma 6.3.32 for D with $c = 1$ gives $\mathrm{coef}(T;f)(z^-(b)z_A(\zeta_0)x) = \mathrm{coef}(T;f)(z_A(\zeta_0)z^-(b)x) = \mathrm{coef}(T;f)(z^-(b)x)$ (coefficients along D), and integration over b gives $f^+(z_A(\zeta_0)x) = f^+(x)$ for every $x \in \mathrm{G}(s)$.

Step 3: local centre elements are shears. For $y \in \mathbb{A}_E$, $\overline{\mathrm{e}y} = \mathrm{e}_{\bar{U}}\bar{y}$, so

$$z_A(\mathrm{e}y) = 1 + k(\mathrm{e}y - \overline{\mathrm{e}y})\mathrm{pr}_l = 1 + \mathrm{e}_{\bar{U}}\,k\,(-\bar{y}\,\mathrm{pr}_l) - \mathrm{e}\,(-ky\,\mathrm{pr}_l),$$

and $(-\bar{y}\,\mathrm{pr}_l)k = 0$: $z_A(\mathrm{e}y)$ is an A-shear for U.

Step 4: strong approximation. Fix $w_0 \in U$. By Theorem A.4.7 for the number field E, the additive group $\mathbb{G}_a^m$ and the set $\{w_0\}$, the set $E^m + E^m_{w_0}$ is dense in $\mathbb{A}_E^m$, where $E^m_{w_0}$ is

embedded as the adelic vectors with components 0 outside w_0; these lie in $\mathrm{e}\,\mathbb{A}_E^m$. Hence for any $Z \in \mathrm{GL}_m(\mathbb{A}_E)$ the set $\{Z\gamma + \mathrm{e}y : \gamma \in E^m,\ y \in \mathbb{A}_E^m\}$ is dense in $\mathbb{A}_E^m$, being the image of a dense set under the homeomorphism $v \mapsto Zv$, which commutes with multiplication by e.

Step 5: proof of (1). Let $x \in H$ and $\varphi(\zeta) = f^+(z_A(\zeta)x)$, continuous by (2) of Lemma 6.3.37. For $\zeta_0 \in E$ and $y \in \mathbb{A}_E$, $z_A(\zeta_0 + \mathrm{e}y) = z_A(\zeta_0)z_A(\mathrm{e}y)$, so by Step 2, Step 3 and the hypothesis at x, $\varphi(\zeta_0 + \mathrm{e}y) = f^+(z_A(\mathrm{e}y)x) = f^+(x)$. By Step 4 ($m = 1$, $Z = 1$) and continuity, $\varphi \equiv f^+(x)$. Then (2) of Lemma 6.3.42 gives $C(x) = \int_{\mathbb{A}_E/E} \varphi(\zeta)\, d\zeta = f^+(x)$.

Step 6: $C(\mathrm{rad}(c,\beta)g) = C(g)$ for $g \in H$ and all c, β. (a) C is continuous by (1) of Lemma 6.3.32 for $A \circ B$, and $C(z_A(\zeta)x) = C(x)$ for all x: by (1) of Lemma 6.3.42 with $b = c = N = 0$, $z_A(\zeta) = z_{A\circ B}(\zeta \oplus 0)$, and (2) of Lemma 6.3.32 for $A \circ B$ applies with $\mathrm{tr}((0 \oplus T)(\zeta \oplus 0)) = 0$. (b) Let $Y \in E^{n_0}$ and $Y_0 = x_{W_0}^{-1} Y$. Conjugated by x_W, using $x_W k = k_{\mathrm{rat}}$, $x_W i = i_{\mathrm{rat}} x_{W_0}$, $\mathrm{pr}_l x_W^{-1} = k_{\mathrm{rat}}^* J_W$ and $\mathrm{pr}_i x_W^{-1} = \varepsilon_0 x_{W_0}^* i_{\mathrm{rat}}^* J_W$ (k_{rat}, i_{rat} the rational frames of A), the three terms of $N(Y_0, 0)$ become $-i_{\mathrm{rat}} Y k_{\mathrm{rat}}^* J_W$, $k_{\mathrm{rat}} Y^* i_{\mathrm{rat}}^* J_W$ and $-\frac{1}{2} k_{\mathrm{rat}} (Y^* J_{W_0} Y) k_{\mathrm{rat}}^* J_W$, all over E; so $\mathrm{rad}(Y_0, 0) \in \Gamma_W$. By Step 1 it fixes k and sends $k_D = i k_B$ to $k_D + k\, Y_0^* \varepsilon_0 k_B = k_D + k\, v$ with $v = Y^* J_{W_0} k_{B,\mathrm{rat}} \in \mathrm{M}_{1\times p}(E)$: it acts on the frame $(k \mid k_D)$ of $A \circ B$ by $u = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$, and $u^*(0 \oplus T)u = 0 \oplus T$. By (4) of Lemma 6.3.32 for $A \circ B$, $C(\mathrm{rad}(Y_0, 0)x) = C(x)$ for all x. (c) Let $y \in \mathrm{M}_{n_0 \times 1}(\mathbb{A}_E)$ and $\nu = \mathrm{rad}(\mathrm{e}y, 0)$. As $(\mathrm{e}y)^* = \mathrm{e}_{\bar{U}} y^*$,

$$\nu = 1 + \mathrm{e}_{\bar{U}}\, k\, c_1 - \mathrm{e}\,(i y\, \mathrm{pr}_l), \qquad c_1 = y^* \varepsilon_0 (\mathrm{pr}_i - \tfrac{1}{2} \mathrm{e}\, y\, \mathrm{pr}_l), \qquad c_1 k = 0,$$

an A-shear for U; ν fixes k, so it commutes with every $z_A(\zeta)$. For $g \in H$, $z_A(\zeta)g \in H$, and by (2) of Lemma 6.3.42 and the hypothesis at $z_A(\zeta)g$,

$$C(\nu g) = \int f^+(z_A(\zeta)\nu g) d\zeta = \int f^+(\nu\, z_A(\zeta) g) d\zeta = \int f^+(z_A(\zeta) g) d\zeta = C(g).$$

(d) For $c = Y_0 + \mathrm{e}y$ and $\beta \in \mathbb{A}_E$ put $\beta' = \beta - \jmath\, Y_0^* \varepsilon_0\, \mathrm{e}y$. By (6.48), $\mathrm{rad}(c, \beta) = \mathrm{rad}(Y_0, 0)\, \mathrm{rad}(0, \beta')\, \nu$, and $\mathrm{rad}(0, \beta') = z_A(-(2\jmath)^{-1}\beta')$; by (b), (a) and (c), $C(\mathrm{rad}(c,\beta)g) = C(\nu g) = C(g)$ for $g \in H$. (e) By Step 4 ($m = \#n_0$, $Z = x_{W_0}^{-1}$) such c are dense in $\mathrm{M}_{n_0\times 1}(\mathbb{A}_E)$, and $c \mapsto C(\mathrm{rad}(c,\beta)g)$ is continuous; so the identity holds for all c.

Step 7: proof of (2). Let $g \in H$ and $h = \mathrm{rad}(c, \beta)$ with $hg \in H$. By (1) at hg and at g and by Step 6, $f^+(hg) = C(hg) = C(g) = f^+(g)$. □

Lemma 6.3.42. *Let W be a space with a point with signs s on n, A a splitting of W over W_0 along a one-element set o, B a splitting of W_0 over W_2 along p, and D a splitting of W over W_1 along p with $k_{D,\mathbb{A}} = i_{A,\mathbb{A}} k_{B,\mathbb{A}}$; write $z^- = z^-_{A,B}$. Let $f\colon \mathrm{G}(s) \to \mathbb{C}$ be continuous and left invariant under Γ_W, $T \in \mathrm{M}_p(E)$, and $0 \oplus T$ the block diagonal $(o \sqcup p) \times (o \sqcup p)$ matrix with blocks 0 and T.*

1. *For adelic matrices ζ (1×1), b ($1 \times p$), c ($p \times 1$) and N ($p \times p$),*

$$z_{A\circ B} \begin{pmatrix} \zeta & b \\ c & N \end{pmatrix} = z_D(N)\, z^-(b - c^*)\, z_A(\zeta).$$

2. *For every $g \in \mathrm{G}(s)$ the function $\zeta \mapsto \mathrm{coef}^+_{A,B,D}(T; f)(z_A(\zeta)g)$ on $\mathbb{A}_E$ is continuous and E-periodic, and, the coefficient on the left being taken along $A \circ B$,*

$$\mathrm{coef}(0 \oplus T; f)(g) = \int_{\mathbb{A}_E/E} \mathrm{coef}^+_{A,B,D}(T; f)(z_A(\zeta)g)\, d\zeta.$$

Proof. Write $k = k_{A,\mathbb{A}}$, pr_l for the coordinate pr_l of A, $\varepsilon = \varepsilon_s$, $k_D = k_{D,\mathbb{A}}$ and $\Lambda = k_D^*\varepsilon$, the coordinate pr_l of D. By the relations of (MX1) for A and B (read adelically through x_W, x_{W_0}, as in the proof of Lemma 6.3.37): $\mathrm{pr}_l k = 0$, $\mathrm{pr}_l k_D = 0$, $\Lambda k = 0$, $\Lambda k_D = k_{B,\mathbb{A}}^* \varepsilon_{s_0} k_{B,\mathbb{A}} = 0$. The composite $A \circ B$ has adelic frame $(k \mid k_D)$ and coordinate pr_l with the rows pr_l, Λ (MX5).

(1). Put $\beta = b - c^*$, so that $c - b^* = -\beta^*$. By (MX2) for $A \circ B$,

$$z_{A\circ B}\begin{pmatrix} \zeta & b \\ c & N \end{pmatrix} = 1 + k(\zeta - \bar{\zeta})\mathrm{pr}_l + k\beta\Lambda - k_D\beta^*\mathrm{pr}_l + k_D(N - N^*)\Lambda.$$

On the other side $z_A(\zeta) = 1 + k(\zeta - \bar{\zeta})\mathrm{pr}_l$, $z_D(N) = 1 + k_D(N - N^*)\Lambda$, and $z^-(\beta) = 1 + N_\beta$ with $N_\beta = k\beta\Lambda - k_D\beta^*\mathrm{pr}_l$ (the matrix $M_\beta - M_\beta^*$ of (MX5) has the blocks β at (o,p) and $-\beta^*$ at (p,o)). The relations give $N_\beta k = 0$ and $\Lambda N_\beta = 0$, $\Lambda k = 0$; hence $N_\beta\, k(\zeta - \bar{\zeta})\mathrm{pr}_l = 0$ and $k_D(N - N^*)\Lambda\big(N_\beta + k(\zeta - \bar{\zeta})\mathrm{pr}_l\big) = 0$, so the product $z_D(N)z^-(\beta)z_A(\zeta)$ is $1 + k_D(N - N^*)\Lambda + N_\beta + k(\zeta - \bar{\zeta})\mathrm{pr}_l$, which is the display.

(2). Write Q_X for the quotient of the adelic matrices of a shape X by the rational ones, with its Haar probability measure. By (1) of Lemma 6.3.32 for the splitting $A \circ B$, the function $\Phi(M) = \psi_E(\mathrm{tr}((0 \oplus T)M))\, f(z_{A\circ B}(M)g)$ is continuous and periodic under the rational $(o \sqcup p) \times (o \sqcup p)$ matrices, and $\mathrm{coef}(0 \oplus T; f)(g)$ is its integral over $Q_{o\sqcup p}$. The decomposition into the four blocks identifies $Q_{o\sqcup p}$ with $Q_{1\times 1} \times Q_{1\times p} \times Q_{p\times 1} \times Q_{p\times p}$ as compact groups, so its Haar probability measure is the product of the four (uniqueness of Haar measure), and by Fubini for the bounded continuous Φ the integral may be taken in the order N, then (b,c), then ζ. As $\mathrm{tr}((0 \oplus T)M) = \mathrm{tr}(TN)$, (1) gives

$$\Phi\begin{pmatrix} \zeta & b \\ c & N \end{pmatrix} = \psi_E(\mathrm{tr}(TN))\, f(z_D(N)\, y), \qquad y = z^-(b - c^*)\, z_A(\zeta)\, g,$$

whose integral over $N \in Q_{p\times p}$ is $\mathrm{coef}(T; f)(y)$, the coefficient along D (MX3). By (2) of Lemma 6.3.37 the function $F_\zeta(\beta) = \mathrm{coef}(T; f)(z^-(\beta)z_A(\zeta)g)$ is continuous and periodic under $\mathrm{M}_{1\times p}(E)$. The map $(b,c) \mapsto b - c^*$ is a continuous surjective homomorphism $\mathrm{M}_{1\times p}(\mathbb{A}_E) \times \mathrm{M}_{p\times 1}(\mathbb{A}_E) \to \mathrm{M}_{1\times p}(\mathbb{A}_E)$ carrying rational points onto rational points, so it induces a continuous surjective homomorphism $\pi\colon Q_{1\times p} \times Q_{p\times 1} \to Q_{1\times p}$ of compact groups; the image of the Haar probability measure under π is a probability measure invariant under all translations (each is π of a translation), hence the Haar probability measure. Therefore

$$\int_{Q_{1\times p}\times Q_{p\times 1}} F_\zeta(b - c^*)\, db\, dc = \int_{Q_{1\times p}} F_\zeta(\beta)\, d\beta = \mathrm{coef}^+_{A,B,D}(T; f)(z_A(\zeta)g)$$

by (MX11). This is, for every adelic ζ, the integral of $\Phi(\zeta, \cdot)$ over the last three blocks, so it is continuous in ζ (dominated convergence for the bounded continuous Φ on the compact quotients) and E-periodic (as Φ is periodic in the block ζ); integrating over $\zeta \in Q_{1\times 1} = \mathbb{A}_E/E$ gives the identity of (2). □

Lemma 6.3.43. *Let W, A, B, D, S, H, T and f be as in Lemma 6.3.39, with $\det T \neq 0$, write $f^+ = \mathrm{coef}^+_{A,B,D}(T; f)$ and $z^+ = z^+_{A,B}$, let U be a nonempty set of finite places of E, and assume that f^+ is invariant under the A-shears for U at every point of H. Let (X, u, t, t', γ) be a straight point of (A, B, D) off S such that every straight point $(X_1, u_1, t_1, t_1', \gamma_1)$ of (A, B, D) off S for which $a \mapsto f^+(z^+(a)X_1)$ is not identically zero has a dual index of γ_1 on $S \cup \bar{S}$ for T at least that of γ. Then $\gamma^{-1}a^*t$ is integral on $S \cup \bar{S}$ for every $a \in \mathrm{M}_{1\times p}(\mathbb{A}_E)$ with $f^+(z^+(a)X) \neq 0$.*

Proof. Write $k = k_{A,\mathbb{A}}$, $l = l_{A,\mathbb{A}}$, $k_D = k_{D,\mathbb{A}}$, pr_k for the coordinate of A, $\Lambda = k_D^* \varepsilon_s$, $F_1 = (k \mid k_D)$, $\mathrm{e} = \mathrm{e}_{S \cup \bar{S}}$. By (1) of Lemma 6.3.39, γ is invertible, and so is $T\{\gamma\}$ ($\det T \neq 0$), so the dual indices are defined; the same holds for every straight point. By (MX2) and (MX5) for $\bar{A} \circ B$, whose adelic frame k is $(l \mid k_D)$ with coordinate pr_l given by the rows $l^* \varepsilon_s = \mathrm{pr}_k$ and Λ, $z^+(a) = 1 + la\Lambda - k_D a^* \mathrm{pr}_k$, and by (MX1) $\Lambda k = 0$, $\Lambda k_D = 0$, $\mathrm{pr}_k k = 1$, $\mathrm{pr}_k k_D = 0$; hence $z^+(a)k = k - k_D a^*$ and $z^+(a)k_D = k_D$, that is

$$z^+(a) F_1 = F_1 \begin{pmatrix} 1 & 0 \\ -a^* & 1 \end{pmatrix}.$$

Suppose $f^+(z^+(a)X) \neq 0$ and $\gamma^{-1} a^* t$ is not integral on $S \cup \bar{S}$. Let $h = z^+(a)u$; then

$$hF_1 = z^+(a) F_1 \operatorname{diag}(t, \gamma) = F_1 \begin{pmatrix} t & 0 \\ -a^* t & \gamma \end{pmatrix}.$$

By (1) of Lemma 6.3.38 there are $\kappa \in \mathrm{G}(s)$ with finite part in $\mathbf{K}_0(n)$ and a', β', γ' with $h\kappa F_1 = F_1 \begin{pmatrix} a' & \beta' \\ 0 & \gamma' \end{pmatrix}$. The point $X_a = z^+(a)X$ lies in H ($z^+(a)$ fixes k_D), satisfies $\mathrm{e} X_a = z^+(a)\,\mathrm{e}u = \mathrm{e}h$, and $f^+(X_a) \neq 0$. By (2) of Lemma 6.3.39 (with $X = X_a$, $g = h$) there is a straight point $(X', u', t'', t''', \gamma')$ of (A, B, D) off S with $f^+(X') \neq 0$, that is, $a'' \mapsto f^+(z^+(a'')X')$ does not vanish at $a'' = 0$; by (1) of Lemma 6.3.39, $T\{\gamma'\}$ is integral on $S \cup \bar{S}$. Since $\gamma^{-1}(-a^* t) = -\gamma^{-1} a^* t$ is not integral on $S \cup \bar{S}$, Lemma 6.3.44 (with $\tau = t$, $x = -a^* t$) shows that the dual index of γ' is smaller than that of γ, contrary to the minimality of (X, u, t, t', γ). □

Lemma 6.3.44. *Let W be a space with a point with signs s on n, A a base-adapted splitting of W over W_0 along a one-element set, B a base-adapted splitting of W_0 over W_2 along p, $F_1 = (k_{A,\mathbb{A}} \mid i_{A,\mathbb{A}} k_{B,\mathbb{A}})$, $T \in \mathrm{M}_p(E)$ with $\det T \neq 0$, and S a set of finite places of E. Let $h, \kappa \in \mathrm{G}(s)$, the finite part of κ in $\mathbf{K}_0(n)$, with*

$$hF_1 = F_1 \begin{pmatrix} \tau & 0 \\ x & \gamma \end{pmatrix}, \qquad h\kappa F_1 = F_1 \begin{pmatrix} a & \beta \\ 0 & \gamma' \end{pmatrix}$$

(blocks 1×1, $1 \times p$, $p \times 1$, $p \times p$ over $\mathbb{A}_E$), where γ is invertible, $T\{\gamma'\}$ is integral on $S \cup \bar{S}$, and $\gamma^{-1} x$ is not integral on $S \cup \bar{S}$. Then γ', $T\{\gamma\}$ and $T\{\gamma'\}$ are invertible, and the dual index of γ' on $S \cup \bar{S}$ for T is smaller than that of γ.

Proof. Let $C = A \circ B$, with frames $k_C = F_1$ and l_C (MX5), and $\varepsilon = \varepsilon_s$; by (MX1), $l_C^* \varepsilon F_1 = 1$. Write B_0, B_1 for the two block matrices of the hypothesis. As $hF_1 = F_1 B_0$ and $h\kappa F_1 = F_1 B_1$ with h, $h\kappa$ invertible, $B_1 = l_C^* \varepsilon h \kappa F_1$ is invertible at every place, with inverse $l_C^* \varepsilon (h\kappa)^{-1} F_1$, and $(h\kappa)^{-1} F_1 = F_1 B_1^{-1}$. Hence

$$\kappa^{-1} F_1 = (h\kappa)^{-1} h F_1 = (h\kappa)^{-1} F_1 B_0 = F_1\, a_0, \qquad a_0 = B_1^{-1} B_0 = l_C^* \varepsilon\, \kappa^{-1} F_1.$$

At the finite places $\varepsilon = 1$, and $l_C = (l_{A,\mathbb{A}} \mid i_{A,\mathbb{A}} l_{B,\mathbb{A}})$, F_1 and κ^{-1} are integral there (A, B are base-adapted, and κ^{-1} has finite part in $\mathbf{K}_0(n)$); so a_0 is integral at all finite places, in particular on $S \cup \bar{S}$. Write $a_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The lower block row of $B_0 = B_1 a_0$ reads $x = \gamma' a_{21}$, $\gamma = \gamma' a_{22}$. As γ is invertible, so are γ' and a_{22}, and $T\{\gamma\}$, $T\{\gamma'\}$ are invertible since $\det T \neq 0$ and δ_J is an idele. Moreover $\gamma^{-1} x = a_{22}^{-1} a_{21}$. If $(\det a_{22})^{-1}$ were integral on $S \cup \bar{S}$, then so would be $a_{22}^{-1} = (\det a_{22})^{-1} \operatorname{adj}(a_{22})$ and $\gamma^{-1} x$, contrary to the hypothesis. Finally $T\{\gamma\} = T\{\gamma' a_{22}\} = a_{22}^*\, T\{\gamma'\}\, a_{22}$, δ_J being a scalar. With d_S as in Lemma 6.3.45, the two dual indices are $d_S(T\{\gamma'\})$ and $d_S(T\{\gamma\})$. By (2) of Lemma 6.3.45 with $C = T\{\gamma'\}$ and $r = a_{22}$, both integral on $S \cup \bar{S}$, $d_S(T\{\gamma'\}) < d_S(a_{22}^* T\{\gamma'\} a_{22}) = d_S(T\{\gamma\})$. □

Lemma 6.3.45. *For a finite adele a of E let $\mathrm{den}(a)$ be its denominator index, the index in $\widehat{\mathcal{O}}_E$ of $H(a) = \{y \in \widehat{\mathcal{O}}_E : ay \in \widehat{\mathcal{O}}_E\}$. For a set S of finite places of E, an adele x and an invertible adelic square matrix C, let x_S be the finite part of $\mathrm{e}_{S\cup\bar{S}}\, x$ and $d_S(C) = \mathrm{den}\big((\det C)_S^{-1}\big)$; so for $T \in \mathrm{M}_p(E)$ and $\gamma \in \mathrm{M}_p(\mathbb{A}_E)$ with $T\{\gamma\}$ invertible, $d_S(T\{\gamma\})$ is the dual index of γ on $S \cup \bar{S}$ for T.*

1. *$\mathrm{den}(a)$ is finite for every finite adele a. If $a = ub$ with $u \in \widehat{\mathcal{O}}_E$, and some $y \in \widehat{\mathcal{O}}_E$ has $ay \in \widehat{\mathcal{O}}_E$ and by $\notin \widehat{\mathcal{O}}_E$, then $\mathrm{den}(a) < \mathrm{den}(b)$.*
2. *Let C, r be adelic $p\times p$ matrices, integral on $S\cup\bar{S}$, such that r^*Cr is invertible and $(\det r)^{-1}$ is not integral on $S \cup \bar{S}$. Then $d_S(C) < d_S(r^*Cr)$.*

Proof. **(1).** $\widehat{\mathcal{O}}_E$ is a compact open subring of the finite adeles, and $H(a)$ is the intersection of $\widehat{\mathcal{O}}_E$ with the preimage of the open set $\widehat{\mathcal{O}}_E$ under the continuous map $y \mapsto ay$; so $H(a)$ is an open subgroup of the compact group $\widehat{\mathcal{O}}_E$, of finite index. If $by \in \widehat{\mathcal{O}}_E$ then $ay = u(by) \in \widehat{\mathcal{O}}_E$, so $H(b) \subset H(a)$; the given y lies in $H(a) \setminus H(b)$. Hence $[H(a) : H(b)] \geq 2$ and $\mathrm{den}(b) = \mathrm{den}(a)\,[H(a) : H(b)] > \mathrm{den}(a)$.

(2). Write $\mathrm{e} = \mathrm{e}_{S\cup\bar{S}}$; as $\mathrm{e}^2 = \mathrm{e}$, $(xy)_S = x_S y_S$, and an adele x is integral on $S \cup \bar{S}$ exactly when $x_S \in \widehat{\mathcal{O}}_E$. Since $S \cup \bar{S}$ is stable under conjugation, $\bar{x}$ is integral on $S \cup \bar{S}$ when x is; so $\det r$, $\det r^* = \overline{\det r}$ and $\det C$ are integral on $S \cup \bar{S}$. As $\det(r^*Cr)$ is a unit, so are $\det r$, $\det r^*$ and $\det C$. Put

$$a = \big((\det C)^{-1}\big)_S, \qquad b = \big(\det(r^*Cr)^{-1}\big)_S, \qquad u = (\det r^* \det r)_S, \qquad y = (\det r^* \det C)_S.$$

Then $u, y \in \widehat{\mathcal{O}}_E$; $(\det C)^{-1} = \det r^* \det r\, \det(r^*Cr)^{-1}$ gives $a = ub$; $ay = (\det r^*)_S \in \widehat{\mathcal{O}}_E$; and $by = ((\det r)^{-1})_S \notin \widehat{\mathcal{O}}_E$ by hypothesis. By (1), $d_S(C) = \mathrm{den}(a) < \mathrm{den}(b) = d_S(r^*Cr)$. □

Lemma 6.3.46. *Let W, A, B, D, S, T and f be as in Lemma 6.3.39, with $\det T \neq 0$; write $\mathrm{e} = \mathrm{e}_{S\cup\bar{S}}$, $z^{\pm} = z^{\pm}_{A,B}$ and $f^{\pm} = \mathrm{coef}^{\pm}_{A,B,D}(T; f)$. Let (X, u, t, t', γ) be a straight point of (A, B, D) off S, and put $\varphi^+(a) = f^+(z^+(a)X)$ and $\varphi^-(b) = f^-(z^-(b)X)$ for $a, b \in \mathrm{M}_{1\times p}(\mathbb{A}_E)$.*

1. *Assume that $\gamma^{-1}b^*t'$ is integral on $S \cup \bar{S}$ whenever $\varphi^-(b) \neq 0$. Let $x \in \mathrm{M}_{p\times 1}(\mathbb{A}_E)$ with $T\{\gamma\}x$ integral on $S \cup \bar{S}$, and $a_x = \mathrm{e}\, t'x^*\gamma^*$. Then $\varphi^+(y + a_x) = \varphi^+(y)$ for every $y \in \mathrm{M}_{1\times p}(\mathbb{A}_E)$.*
2. *Assume that φ^+ is not identically zero, that $\gamma^{-1}a^*t$ is integral on $S\cup\bar{S}$ whenever $\varphi^+(a) \neq 0$, and that $\gamma^{-1}b^*t'$ is integral on $S \cup \bar{S}$ whenever $\varphi^-(b) \neq 0$. Then $T\{\gamma\}$ is invertible and $T\{\gamma\}^{-1}$ is integral on $S \cup \bar{S}$.*

Proof. By (1) of Lemma 6.3.39, γ is invertible, hence so is $T\{\gamma\} = \delta_\jmath^{-1}\gamma^*T\gamma$. The adele e is real, idempotent, and supported on the finite places; integrality on $S \cup \bar{S}$ is preserved by sums, products and conjugate transposes ($S \cup \bar{S}$ is stable under conjugation).

(1). By (3) of Lemma 6.3.37, $f^-(z^-(b)z^+(a)X) = \psi_E(-\mathrm{tr}(Ta^*b))\,\varphi^-(b)$ for all a, b. Take $a = a_x$, so $a_x^* = \mathrm{e}\,\gamma x\bar{t}'$, and put $c_b = (\gamma^{-1}b^*t')^*\,(T\{\gamma\}x) = \delta_\jmath^{-1}\bar{t}'\,bT\gamma x$ (a 1×1 matrix, read as an adele). Then

$$\mathrm{tr}(Ta_x^*b) = \mathrm{e}\,\bar{t}'\,bT\gamma x = \mathrm{e}\,\delta_\jmath\, c_b.$$

If $\varphi^-(b) \neq 0$, then $\gamma^{-1}b^*t'$ and $T\{\gamma\}x$ are integral on $S \cup \bar{S}$, so c_b is, and the adele $-\mathrm{e}\delta_\jmath c_b$ has infinite components 0 and finite part $\jmath\alpha\bar{\alpha}r$ with r the finite part of $-\mathrm{e}c_b$, which lies in $\widehat{\mathcal{O}}_E$; by (WD4), $\psi_E(-\mathrm{e}\delta_\jmath c_b) = \psi_0(r) = 1$. Hence $f^-(z^-(b)z^+(a_x)X) = \varphi^-(b)$ for every b (both

sides vanish where $\varphi^-(b) = 0$). By (5) of Lemma 6.3.37, (4) holds for the opposite splitting $\bar{A}$, for which the roles of f^+, f^- and of the two shifted means are exchanged; applied with $x' = z^+(a_x)X$, $x = X$ and $c = 1$ it gives $f^+(z^+(y)z^+(a_x)X) = f^+(z^+(y)X)$ for every y. As $z^+(y + a_x) = z^+(y)z^+(a_x)$ ((1) of Lemma 6.3.37), this is $\varphi^+(y + a_x) = \varphi^+(y)$.

(2). Fix $j \in p$ and let x be the j-th column of $T\{\gamma\}^{-1}$; then $T\{\gamma\}x$ is the j-th unit column, integral. Choose y with $\varphi^+(y) \neq 0$; by (1), $\varphi^+(y+a_x) = \varphi^+(y) \neq 0$. By hypothesis $\gamma^{-1}(y+a_x)^*t$ and $\gamma^{-1}y^*t$ are integral on $S \cup \bar{S}$, hence so is their difference

$$\gamma^{-1}a_x^*t = \gamma^{-1}\,\mathrm{e}\,\gamma x\,\bar{t}'t = \mathrm{e}\,x,$$

using $\bar{t}'t = 1$. So x is integral on $S \cup \bar{S}$; as j was arbitrary, so is $T\{\gamma\}^{-1}$. □

Lemma 6.3.47. *Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character, X a space with a point with signs r on m, b an index set and $(\mathcal{L}, \mathcal{D})$ an equivariant lift for $(\mathfrak{c}, X, b)$. Let D be a base-adapted splitting of X over a space with a point W_1 along b with coordinate $\mathrm{pr}_{l,D}$, and assume $\Psi_0 = \Psi^{\mathrm{st}}_{(\mathrm{pr}_{l,D})_\Sigma} \in \mathcal{L}$. Let A be a splitting of X over a space with a point W_0 along a one-element set and B a splitting of W_0 over a space with a point along b, with rational frames satisfying $k_D = i_A k_B$. Let S be the set of finite places w of E with $w \notin \Sigma \cup \bar{\Sigma}$, H the frame stabiliser of D off S, and $T \in \mathrm{M}_b(E)$. Then at every point of H, $\mathrm{coef}^+_{A,B,D}(T; \mathcal{D}\Psi_0)$ is invariant under the A-shears for $\bar{\Sigma}$, and $\mathrm{coef}^-_{A,B,D}(T; \mathcal{D}\Psi_0)$ is invariant under the $\bar{A}$-shears for $\bar{\Sigma}$.*

Proof. For $\Psi \in \mathcal{L}$ write $f^\pm_\Psi = \mathrm{coef}^\pm_{A,B,D}(T; \mathcal{D}\Psi)$. As $\mathcal{D}\Psi$ is continuous and left Γ_X-invariant ((XD11)(b)), (2) of Lemma 6.3.37 shows that the partial means of $\mathcal{D}\Psi$ are defined everywhere, with $f^+_\Psi(z^-_{A,B}(a)x) = f^+_\Psi(x)$; and $\Psi \mapsto f^\pm_\Psi(x)$ is linear on $\mathcal{L}$ ($\mathcal{D}$ is linear, and so are the centre coefficients by (1) of Lemma 6.3.32 and the integral). In this proof $k_A = k_{A,\mathbb{A}}$ and $k_D = k_{D,\mathbb{A}}$ denote the adelic frames and k^E_A, k^E_D, i^E_A, k^E_B the rational ones (so $k^E_D = i^E_A k^E_B$); write $\mathrm{pr}_l = \mathrm{pr}_{l,D}$, $\varepsilon = \varepsilon_r$, and $s_0 = (\mathrm{pr}_l)_\Sigma$, so $\Psi_0 = \Psi^{\mathrm{st}}_{s_0}$; s_0 is integral because D is base-adapted and $\mathrm{pr}_l = k^*_D\varepsilon$ has finite part $k^*_{D,f}$.

Step 1: two orthogonality relations. $\mathrm{pr}_l k_A = 0$: indeed $\mathrm{pr}_l k_A = (x_X^{-1}k^E_D)^*\varepsilon\, x_X^{-1}k^E_A = (k^E_D)^* J_X k^E_A$ (as $(x_X^{-1})^*\varepsilon x_X^{-1} = J_X$), and $(k^E_D)^*J_X k^E_A = (k^E_B)^*(i^E_A)^*J_X k^E_A = 0$ by (MX1) for A. Moreover, for $g \in H$ and every place $w \in \Sigma$, $(\mathrm{pr}_l g)_w = (\mathrm{pr}_l)_w$: from $g^*\varepsilon g = \varepsilon$, $\mathrm{pr}_l g = (g^{-1}k_D)^*\varepsilon$, and at w the conjugate transpose reads the component at $\bar{w} \in \bar{\Sigma}$, where $g^{-1}k_D = k_D$ (g fixes the components of k_D at the places of $\Sigma \cup \bar{\Sigma}$, the complement of $S \cup \bar{S}$ among the finite places).

Step 2: a linear form and its invariance. Fix $x \in \mathrm{G}(r)$ with $(\mathrm{pr}_l x)_\Sigma = s_0$, and put $u = (x^{-1}k_A)_\Sigma \in \mathrm{M}_{m\times 1}(E_\Sigma)$. Let $\Lambda(\Psi) = f^+_\Psi(x)$ on $\mathcal{L}$. We claim: for $\Phi \in \mathcal{L}$ and $w \in \mathrm{M}_{1\times b}(E_\Sigma)$, the function $\Phi_w(y) = \Phi(y + u\,w\,s_0\,y)$ lies in $\mathcal{L}$ and $\Lambda(\Phi_w) = \Lambda(\Phi)$. Let $N \in \mathrm{M}_{1\times b}(\mathbb{A}_E)$ have components w at Σ and 0 elsewhere, and $\nu_1 = z^-_{A,B}(N) = z_{A\circ B}(M_N)$. The frames of $A \circ B$ are $k_{A\circ B} = (k_A \mid k_D)$ and its coordinate $\mathrm{pr}_{l,A\circ B} = k^*_{A\circ B}\varepsilon$ has the rows $\mathrm{pr}_{l,A}$ and pr_l ((MX1), (MX5)); with (MX2),

$$\nu_1 = 1 + k_A N\,\mathrm{pr}_l - k_D N^* \mathrm{pr}_{l,A}.$$

At a place $w' \in \Sigma$ the component of N^* is the conjugate transpose of that of N at $\bar{w}' \notin \Sigma$, which is 0; so ν_1 has component $1 + (k_A)_{w'}w_{w'}(\mathrm{pr}_l)_{w'}$ at $w' \in \Sigma$ and 1 at the infinite places and at the finite places outside $\Sigma \cup \bar{\Sigma}$. By Step 1, $C = w(\mathrm{pr}_l)_\Sigma$ and $U = (k_A)_\Sigma$ satisfy $CU = 0$. By (1) of Lemma 6.3.14, $\Phi'(y) = \Phi(y + (x^{-1})_\Sigma UC\,x_\Sigma y)$ lies in $\mathcal{L}$ and $\mathcal{D}\Phi'(y'x) = \mathcal{D}\Phi(y'\nu_1^{-1}x)$ for all y'; and $(x^{-1})_\Sigma UCx_\Sigma = u\,w\,(\mathrm{pr}_l x)_\Sigma = u\,w\,s_0$, so $\Phi' = \Phi_w$. Taking $y' = z_D(M)z^-_{A,B}(a)$, and using $z^-_{A,B}(a)z^-_{A,B}(-N) = z^-_{A,B}(a - N)$ ((1) of Lemma 6.3.37),

$$\mathrm{coef}(T; \mathcal{D}\Phi_w)(z^-_{A,B}(a)x) = \mathrm{coef}(T; \mathcal{D}\Phi)(z^-_{A,B}(a - N)x),$$

and integrating over a modulo $\mathrm{M}_{1\times b}(E)$ (translation invariance of the probability measure) gives $\Lambda(\Phi_w) = \Lambda(\Phi)$.

Step 3: averaging over integral shears. Let u, x be as in Step 2 and $r \in \mathrm{M}_{1\times m}(E_\Sigma)$ with $ru = 0$, and assume $s_0 u = 0$. For $t \in \mathcal{K} = \mathrm{M}_{m\times b}(\mathcal{O}_\Sigma)$ (a compact group with Haar probability dt) let $\tau_t(y) = y + u\,(rt)\,s_0\,y$. For $\Phi \in \mathcal{L}$ put $\bar\Phi(y) = \int_{\mathcal{K}} \Phi(\tau_t y)\,dt$. We claim $\bar\Phi \in \mathcal{L}$ and $\Lambda(\bar\Phi) = \Lambda(\Phi)$. As $s_0 u = 0$, $s_0\tau_t(y) = s_0 y$ and $\tau_{-t}\tau_t = \mathrm{id}$. Φ is locally constant with compact support Q ((XD11)(a)), so there is an open subgroup U_0 of $\mathrm{M}_{m\times b}(E_\Sigma)$ with $\Phi(y+\delta) = \Phi(y)$ for all y and $\delta \in U_0$. If $\Phi(\tau_t y) \neq 0$ then $y = \tau_{-t}(\tau_t y)$ lies in the compact set $Q' = \{\tau_{-t}(q) : t \in \mathcal{K}, q \in Q\}$. Choose an open subgroup $U_1 \subseteq \mathcal{K}$ with $u\,r\,\delta\,s_0\,y \in U_0$ for $\delta \in U_1$, $y \in Q'$. For $t - t' \in U_1$ and $y \in Q'$, $\Phi(\tau_{t'} y) = \Phi(\tau_t y + ur(t'-t)s_0 y) = \Phi(\tau_t y)$, and both vanish for $y \notin Q'$. So $t \mapsto \Phi\circ\tau_t$ is constant on the finitely many cosets $t_1U_1, \dots, t_NU_1$ of U_1 in $\mathcal{K}$, and $\bar\Phi = N^{-1}\sum_j \Phi\circ\tau_{t_j}$. Each $\Phi\circ\tau_{t_j}$ is Φ_w for $w = rt_j$, so it lies in $\mathcal{L}$ with the same value of Λ (Step 2); by linearity the claim follows.

Step 4: the shear of the standard datum. With u, r as in Step 3 let $\Psi'(y) = \Psi_0(y + u\,r\,y)$, and assume $\Psi' \in \mathcal{L}$. We show $\bar\Psi' = \bar\Psi_0$, hence $\Lambda(\Psi') = \Lambda(\Psi_0)$. Fix y. Since $ru = 0$, $\Psi'(\tau_t y) = \Psi_0(y + u(rts_0 y + ry))$ and $\Psi_0(\tau_t y) = \Psi_0(y + u\,rts_0 y)$. If $\Psi_0(y+u\rho) \neq 0$ for some $\rho \in \mathrm{M}_{1\times b}(E_\Sigma)$, then $y+u\rho$ is integral and $s_0(y+u\rho) = s_0 y \in \mathrm{GL}_b(\mathcal{O}_\Sigma)$; then $t_0 = (y+u\rho)(s_0 y)^{-1} \in \mathcal{K}$ satisfies $u\,rt_0 s_0 y = ur(y+u\rho) = ury$, so $\Psi'(\tau_t y) = \Psi_0(\tau_{t+t_0} y)$ and the two means agree by translation invariance of dt. Otherwise all values $\Psi_0(y+u\rho)$ vanish, and so do both integrands.

Step 5: the shears. Let $g \in H$ and ν an A-shear for $\bar\Sigma$: $\nu = 1 + \mathrm{e}_\Sigma k_A c - \mathrm{e}_{\bar\Sigma} a$ with $ck_A = 0$. Its components are $1 + (k_A)_w c_w$ at $w \in \Sigma$ and 1 at the infinite places and at the finite places outside $\Sigma \cup \bar\Sigma$. Put $x = \nu g$. By Step 1, $(\mathrm{pr}_l \nu)_\Sigma = (\mathrm{pr}_l)_\Sigma + (\mathrm{pr}_l k_A)_\Sigma c_\Sigma = s_0$ and $(\mathrm{pr}_l x)_\Sigma = (\mathrm{pr}_l g)_\Sigma = s_0$ (the components at Σ of $\mathrm{pr}_l \nu g$ are products of those of $\mathrm{pr}_l\nu$ and g). Hence $u = (x^{-1}k_A)_\Sigma$ satisfies $s_0 u = (\mathrm{pr}_l x\, x^{-1} k_A)_\Sigma = (\mathrm{pr}_l k_A)_\Sigma = 0$, and $r = (cx)_\Sigma$ satisfies $ru = (ck_A)_\Sigma = 0$. By (1) of Lemma 6.3.14 with $U = (k_A)_\Sigma$, $C = c_\Sigma$ ($CU = 0$), $\nu_0 = \nu$ and the point x: $\Psi'(y) = \Psi_0(y + (x^{-1})_\Sigma (k_A)_\Sigma c_\Sigma x_\Sigma y) = \Psi_0(y + ury)$ lies in $\mathcal{L}$ and $\mathcal{D}\Psi'(y'x) = \mathcal{D}\Psi_0(y'\nu^{-1}x) = \mathcal{D}\Psi_0(y'g)$; taking $y' = z_D(M) z^-_{A,B}(a)$ and integrating, $f^+_{\Psi'}(x) = f^+_{\Psi_0}(g)$. By Step 4, $f^+_{\Psi'}(x) = \Lambda(\Psi') = \Lambda(\Psi_0) = f^+_{\Psi_0}(\nu g)$. So $f^+_{\Psi_0}(\nu g) = f^+_{\Psi_0}(g)$: this is the first assertion.

Step 6: the opposite splitting. The opposite splitting $\bar A$ has the same embedding $i_{\bar A} = i_A$, so $k_D = i_{\bar A} k_B$, and its frame is $k_{\bar A} = l_A$. Steps 1–5 for $\bar A$ in place of A show that $\mathrm{coef}^+_{\bar A,B,D}(T; \mathcal{D}\Psi_0)$ is invariant under the $\bar A$-shears for $\bar\Sigma$ at the points of H; and $\mathrm{coef}^+_{\bar A,B,D} = \mathrm{coef}^-_{A,B,D}$ ((MX11)). □

Lemma 6.3.48. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p an index set and $D = (k,l,i)$ a splitting of V over V_0 along p, with its Levi and radical elements.*

1. *For $m, m' \in \mathrm{GL}_p(\mathbb{A}_E)$ and $g_0, g_0' \in \mathrm{G}(s_0)$, $\mathrm{lev}(m,g_0)\,\mathrm{lev}(m',g_0') = \mathrm{lev}(mm', g_0 g_0')$, and $\mathrm{lev}(m,g_0)\,k_{\mathbb{A}} = k_{\mathbb{A}} m$.*

2. *For $m \in \mathrm{GL}_p(\mathbb{A}_E)$, $g_0 \in \mathrm{G}(s_0)$, an adelic $n_0 \times p$ matrix c and an adelic $p \times p$ matrix b,*

$$\mathrm{lev}(m,g_0)\,\mathrm{rad}(c,b)\,\mathrm{lev}(m,g_0)^{-1} = \mathrm{rad}(g_0 c m^*,\ mbm^*).$$

3. *If $x \in \mathrm{G}(s)$ and $q \in \mathrm{GL}_p(\mathbb{A}_E)$ satisfy $x\,k_{\mathbb{A}} = k_{\mathbb{A}} q$, then $x = \mathrm{rad}(c,b)\,\mathrm{lev}(q,g_0)$ for some adelic c, b and some $g_0 \in \mathrm{G}(s_0)$.*

4. *For $m \in \mathrm{GL}_p(E)$, $\mathrm{lev}(m,1) \in \Gamma_V$.*

Proof. Write $\varepsilon = \varepsilon_s$, $\varepsilon_0 = \varepsilon_{s_0}$. From (MX1): $\mathrm{pr}_k = l_{\mathbb{A}}^*\varepsilon$, $\mathrm{pr}_l = k_{\mathbb{A}}^*\varepsilon$, $\mathrm{pr}_i = \varepsilon_0 i_{\mathbb{A}}^*\varepsilon$, and the frame $(l_{\mathbb{A}} \mid k_{\mathbb{A}} \mid i_{\mathbb{A}})$ is invertible with inverse the matrix of the rows pr_l, pr_k, pr_i; so $\mathrm{pr}_k k_{\mathbb{A}} = 1$, $\mathrm{pr}_l l_{\mathbb{A}} = 1$, $\mathrm{pr}_i i_{\mathbb{A}} = 1$, the other products of a coordinate with a frame vanish, and $k_{\mathbb{A}}\mathrm{pr}_k + l_{\mathbb{A}}\mathrm{pr}_l + i_{\mathbb{A}}\mathrm{pr}_i = 1$. The relations of (MX1) read $k_{\mathbb{A}}^*\varepsilon k_{\mathbb{A}} = l_{\mathbb{A}}^*\varepsilon l_{\mathbb{A}} = 0$, $l_{\mathbb{A}}^*\varepsilon k_{\mathbb{A}} = 1$, $i_{\mathbb{A}}^*\varepsilon i_{\mathbb{A}} = \varepsilon_0$, $i_{\mathbb{A}}^*\varepsilon k_{\mathbb{A}} = i_{\mathbb{A}}^*\varepsilon l_{\mathbb{A}} = 0$, as $x_V^* J_V x_V = \varepsilon$.

(1). By definition $\mathrm{lev}(m, g_0)$ sends $k_{\mathbb{A}}a \mapsto k_{\mathbb{A}}ma$, $l_{\mathbb{A}}a \mapsto l_{\mathbb{A}}(m^{-1})^*a$ and $i_{\mathbb{A}}w \mapsto i_{\mathbb{A}}g_0 w$. Composing on the frame gives $\mathrm{lev}(mm', g_0 g_0')$, since $(m^{-1})^*(m'^{-1})^* = ((mm')^{-1})^*$; and the frame determines the matrix.

(2). Put $\lambda = \mathrm{lev}(m, g_0)$, so $\lambda^{-1} = \mathrm{lev}(m^{-1}, g_0^{-1})$ by (1). From the action on the frame, $\lambda i_{\mathbb{A}} = i_{\mathbb{A}} g_0$, $\lambda k_{\mathbb{A}} = k_{\mathbb{A}} m$, $\mathrm{pr}_l\lambda^{-1} = m^*\mathrm{pr}_l$ and $\mathrm{pr}_i\lambda^{-1} = g_0^{-1}\mathrm{pr}_i$. With $X(c,b) = \frac{1}{2}(\jmath^{-1}(b+b^*) + c^*\varepsilon_0 c)$, conjugating the four terms of $\mathrm{rad}(c,b)$ gives

$$\lambda\,\mathrm{rad}(c,b)\,\lambda^{-1} = 1 - i_{\mathbb{A}}(g_0 c m^*)\mathrm{pr}_l + k_{\mathbb{A}}\, mc^*\varepsilon_0 g_0^{-1}\,\mathrm{pr}_i - k_{\mathbb{A}}\, mX(c,b)m^*\,\mathrm{pr}_l.$$

With $c' = g_0 c m^*$: $g_0^*\varepsilon_0 g_0 = \varepsilon_0$ gives $\varepsilon_0 g_0^{-1} = g_0^*\varepsilon_0$, so $mc^*\varepsilon_0 g_0^{-1} = c'^*\varepsilon_0$, and $c'^*\varepsilon_0 c' = mc^*\varepsilon_0 c m^*$; as $\jmath$ is a scalar, $mX(c,b)m^* = X(c', mbm^*)$. This is $\mathrm{rad}(c', mbm^*)$.

(3). Let $x_1 = \mathrm{lev}(q,1)^{-1}x$; by (1), $x_1 k_{\mathbb{A}} = k_{\mathbb{A}}$. Unitarity $x_1^*\varepsilon x_1 = \varepsilon$ gives $\varepsilon x_1 = (x_1^{-1})^*\varepsilon$, hence $\mathrm{pr}_l x_1 = k_{\mathbb{A}}^*(x_1^{-1})^*\varepsilon = (x_1^{-1}k_{\mathbb{A}})^*\varepsilon = \mathrm{pr}_l$. Therefore $x_1 i_{\mathbb{A}} = i_{\mathbb{A}} g_0 + k_{\mathbb{A}}\varphi$ with $g_0 = \mathrm{pr}_i x_1 i_{\mathbb{A}}$ and $\varphi = \mathrm{pr}_k x_1 i_{\mathbb{A}}$, as $\mathrm{pr}_l x_1 i_{\mathbb{A}} = \mathrm{pr}_l i_{\mathbb{A}} = 0$. Applying $x_1^*\varepsilon x_1 = \varepsilon$ to $i_{\mathbb{A}}$ and using the relations above, $g_0^*\varepsilon_0 g_0 = \varepsilon_0$, so $g_0 \in \mathrm{G}(s_0)$. Let $x_2 = \mathrm{lev}(1, g_0)^{-1}x_1$; then $x_2 k_{\mathbb{A}} = k_{\mathbb{A}}$, $\mathrm{pr}_l x_2 = \mathrm{pr}_l$ (as before), $x_2 i_{\mathbb{A}} = i_{\mathbb{A}} + k_{\mathbb{A}}\varphi$, and $x_2 l_{\mathbb{A}} = l_{\mathbb{A}} + i_{\mathbb{A}}\psi + k_{\mathbb{A}}\chi$ with $\psi = \mathrm{pr}_i x_2 l_{\mathbb{A}}$, $\chi = \mathrm{pr}_k x_2 l_{\mathbb{A}}$. Unitarity of x_2 on the pairs $(i_{\mathbb{A}}, l_{\mathbb{A}})$ and $(l_{\mathbb{A}}, l_{\mathbb{A}})$ gives

$$\varepsilon_0\psi + \varphi^* = 0, \qquad \chi + \chi^* + \psi^*\varepsilon_0\psi = 0.$$

Put $c = -\psi$, so $\varphi = c^*\varepsilon_0$; $S = -\chi - \frac{1}{2}c^*\varepsilon_0 c$ satisfies $S + S^* = 0$, and $b = \jmath S$ satisfies $b + b^* = 2\jmath S$ ($\bar{\jmath} = -\jmath$), i.e. $X(c,b) = S + \frac{1}{2}c^*\varepsilon_0 c = -\chi$. Then $\mathrm{rad}(c,b)$ sends $k_{\mathbb{A}} \mapsto k_{\mathbb{A}}$, $i_{\mathbb{A}} \mapsto i_{\mathbb{A}} + k_{\mathbb{A}}c^*\varepsilon_0$ and $l_{\mathbb{A}} \mapsto l_{\mathbb{A}} - i_{\mathbb{A}}c - k_{\mathbb{A}}X(c,b)$, as does x_2; so $x_2 = \mathrm{rad}(c,b)$ and, by (1) and (2), $x = \mathrm{lev}(q, g_0)\,\mathrm{rad}(c,b) = \mathrm{rad}(g_0 c q^*, qbq^*)\,\mathrm{lev}(q, g_0)$.

(4). By (VP2) of Definition 3.2.2 it suffices that $x_V\,\mathrm{lev}(m,1)\,x_V^{-1}$ is a rational point of $\mathrm{U}(V)$. From $x_V^* J_V x_V = \varepsilon$, $(x_V^{-1})^*\varepsilon x_V^{-1} = J_V$ and $x_{V_0}\varepsilon_0 x_{V_0}^* = J_{V_0}^{-1}$, so

$$x_V\,\mathrm{lev}(m,1)\,x_V^{-1} = km\,l^* J_V + i\,J_{V_0}^{-1} i^* J_V + l\,(m^{-1})^* k^* J_V,$$

a matrix over E; it lies in $\mathrm{U}(V)$ because $\mathrm{lev}(m,1) \in \mathrm{G}(s)$ ((MX2)). □

Lemma 6.3.49. *Let X be a space with a point with signs r on m, D a splitting of X over a space with a point W_1 (signs s on n) along b, V' a space with a point with signs t' on b, $\kappa_\alpha \in \mathbb{A}_E^\times$ the idele with components κ_v at $v \in \Phi$ and finite part α, and $f\colon \mathrm{G}(r) \to \mathbb{C}$ continuous and left invariant under Γ_X. Assume that $\mathrm{coef}(\jmath T; f)(\mathrm{rad}(c,z)\,\mathrm{lev}(q,g)) = 0$ for all $T \in \mathrm{M}_b(E)$, $c \in \mathrm{M}_{n\times b}(\mathbb{A}_E)$, $z \in \mathrm{M}_b(\mathbb{A}_E)$, $q \in \mathrm{GL}_b(\mathbb{A}_E)$, $g \in \mathrm{G}(s)$ such that either (a) $T = J_{V'}$ and $q = \kappa_\alpha x_{V'} Y$ for some $Y \in \mathrm{GL}_b(\mathbb{A}_E)$ for which some $k \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ has $(Y_f k)^*(Y_f k) = 1$, or (b) $\det T \neq 0$ and T is not represented by the signs t' (centre coefficients, radical and Levi elements of D). Let $\beta \in \mathrm{M}_b(E)$ with $\det\beta \neq 0$, $q \in \mathrm{GL}_b(\mathbb{A}_E)$ and $k \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ with $k^*(q^*\beta q)_f k = 1$. Then*

$$\mathrm{coef}(\jmath\beta; f)(\mathrm{rad}(c,z)\,\mathrm{lev}(\kappa_\alpha q, g)) = 0 \qquad (c \in \mathrm{M}_{n\times b}(\mathbb{A}_E),\ z \in \mathrm{M}_b(\mathbb{A}_E),\ g \in \mathrm{G}(s)).$$

Proof. Write $p = \mathrm{rad}(c,z)\,\mathrm{lev}(\kappa_\alpha q, g)$. If β, read as an adelic matrix, is not represented by t', then $T = \beta$ and the point p (with $\kappa_\alpha q$ in the place of q) are as in (b), and the coefficient vanishes. Assume now that β is represented by t': for every $v \in \Phi$ there is $a_v \in \mathrm{M}_b(\mathbb{C})$ with $a_v^* \mathrm{diag}(t') a_v = \beta_v$, and a_v is invertible as $\det \beta \neq 0$. In particular β is hermitian (its images under the embeddings at $v \in \Phi$ are).

A point for β. Let $x_\beta \in \mathrm{GL}_b(\mathbb{A}_E)$ have components a_v^{-1} at $v \in \Phi$ and finite part $q_f k$. Then $x_\beta^* \beta x_\beta$ is $(a_v^{-1})^* a_v^* \mathrm{diag}(t') a_v a_v^{-1} = \mathrm{diag}(t')$ at $v \in \Phi$ and $k^*(q^*\beta q)_f k = 1$ at the finite places, so $V_\beta = (\beta, x_\beta)$ is a space with a point with signs t'. By Lemma 4.1.8 (for V_β and V') there are $\gamma \in \mathrm{GL}_b(E)$ with $\gamma^* J_{V'} \gamma = \beta$ and $w \in \mathrm{G}(t')$ with $x_\beta = \gamma^{-1} x_{V'} w$.

Moving the index to the point. By (1) and (4) of Lemma 6.3.48, $\lambda = \mathrm{lev}(\gamma, 1) \in \Gamma_X$ and $\lambda k_{\mathbb{A}} = k_{\mathbb{A}} \gamma$, $k_{\mathbb{A}}$ the adelic frame of D. By (4) of Lemma 6.3.32 (with the rational matrix γ; $\jmath$ is a scalar, so $\gamma^*(\jmath J_{V'})\gamma = \jmath\beta$),

$$\mathrm{coef}(\jmath\beta; f)(p) = \mathrm{coef}(\jmath J_{V'}; f)(\lambda p).$$

By (2) and (1) of Lemma 6.3.48, $\lambda p = \mathrm{rad}(c\gamma^*, \gamma z \gamma^*)\,\lambda\,\mathrm{lev}(\kappa_\alpha q, g) = \mathrm{rad}(c\gamma^*, \gamma z\gamma^*)\,\mathrm{lev}(\kappa_\alpha \gamma q, g)$, the idele κ_α being a scalar.

The self-dual test. Put $Y = x_{V'}^{-1}\gamma q$, so $\kappa_\alpha \gamma q = \kappa_\alpha x_{V'} Y$. From $x_\beta = \gamma^{-1} x_{V'} w$ we get $x_{V'}^{-1}\gamma = w x_\beta^{-1}$, so $Y = w x_\beta^{-1} q$ and $Y_f k = w_f (q_f k)^{-1} q_f k = w_f$, and $w_f \in G_f(b)$ satisfies $w_f^* w_f = 1$. So $(J_{V'}, \lambda p)$ is as in (a), and $\mathrm{coef}(\jmath J_{V'}; f)(\lambda p) = 0$. □

Lemma 6.3.50. *Let W, V be spaces with signs s on n and t on r, $\mathfrak{c} = (\nu, \Sigma)$ a split tame character, $\gamma = \nu^{-1} \upsilon^{\lfloor \#n/2 \rfloor}$ and $F_\mathfrak{c} = (\Psi_1^{\mathrm{st}})^\sharp \otimes_\Sigma v^0 \in \mathcal{M}_{W,W}$ (1 the identity $n \times n$ matrix). Assume that $\theta_\mathfrak{c}^{\mathrm{lg}}(W, W; v, \Psi_1^{\mathrm{st}})(g) = 0$ for every Schwartz vector $v \in \mathcal{F}_{W,W}$ and every $g \in \mathrm{G}(s)$. Then for every Schwartz vector $w \in \mathcal{F}_{W,V\oplus W}$, every automorphic character β of $\mathrm{U}(1)$, $f \in \mathcal{M}_{W,V}$, $x \in \mathrm{M}_{r\times n}(\mathbb{A}_E)$, $g \in \mathrm{G}(s)$ and $y \in \mathrm{G}(t)$, the function of l below is integrable on $[\mathrm{U}(W)]$ and*

$$\int_{[\mathrm{U}(W)]} \theta_{\mathrm{Hom}(W,V\oplus W)}\Big(\rho((x;0))\,\omega_{\gamma,\gamma\beta}(g, \mathrm{diag}(y,l))\,(w \otimes (f \boxtimes^{\mathrm{tg}} F_\mathfrak{c}))\Big)\,dl = 0.$$

Proof. Let e be the bijection of $(r \times n) \sqcup (n \times n)$ onto $(r \sqcup n) \times n$ that is the identity on each part, and for $u \in \mathbb{C}^{\Phi \times ((r\sqcup n)\times n)}$ let (u_1, u_2) be its two parts. Fix β, f, x, g, y and, for $l \in \mathrm{G}(s)$, let

$$S_l(c) = \theta_{\mathrm{Hom}(W,V\oplus W)}\Big(\rho((x;0))\omega_{\gamma,\gamma\beta}(g, \mathrm{diag}(y,l))(c \otimes (f \boxtimes^{\mathrm{tg}} F_\mathfrak{c}))\Big),$$
$$T(c_1) = \theta_{\mathrm{Hom}(W,V)}(\rho(x)\omega_{\gamma,\beta}(g,y)(c_1 \otimes f)),$$
$$L_l(c_2) = \theta_{\mathrm{Hom}(W,W)}(\omega_{\gamma,\gamma}(g,l)(c_2 \otimes F_\mathfrak{c})) = \theta_{\gamma,\gamma}^{W,W}(c_2 \otimes [\Psi_1^{\mathrm{st}}]_\Sigma)(g,l),$$

bounded linear functionals on $\mathcal{F}_{W,V\oplus W}$, $\mathcal{F}_{W,V}$, $\mathcal{F}_{W,W}$ (composites of $c \mapsto c \otimes F$, bounded by (1) of Lemma 6.2.21, the bounded operators ω and ρ of Proposition 4.1.14, and the bounded theta functionals).

Step 1: coherent states. For u as above, $(\rho^\infty(u)v_1)(z) = \beta^\infty(u,z)e^\infty(z-u)$ is a product over the coordinates, so $\rho^\infty(u)v_1 = \rho^\infty(u_1)v_1 \boxtimes_e \rho^\infty(u_2)v_1$ in the notation of Lemma 6.3.51. By (3) and (2) of Lemma 6.3.51, (b) and (c) of Lemma 6.3.52 (characters γ, β, γ) and $\rho(0) = 1$,

$$S_l(\rho^\infty(u)v_1) = T(\rho^\infty(u_1)v_1)\,L_l(\rho^\infty(u_2)v_1).$$

Step 2: a Schwartz vector for (W, W). By the reproducing identity,

$$w = \int w(u)\,\rho^\infty(u)v_1\,d\nu(u),$$

a Bochner integral (ρ^∞ is isometric and strongly continuous, w integrable). Put

$$v_w = \int w(u)\, T(\rho^\infty(u_1)v_1)\, \rho^\infty(u_2)v_1\, d\nu(u) \in \mathcal{F}_{W,W},$$

a Bochner integral of norm at most $\|T\|\,\|v_1\|_1^2\|w\|_1$. It is a Schwartz vector: by Tonelli and the evaluation of the integrand,

$$\mathrm{mom}_N(v_w) \le \|T\|\|v_1\|_1 \int |w(u)|\,\mathrm{mom}_N(\rho^\infty(u_2)v_1)\, d\nu(u),$$

and $\mathrm{mom}_N(\rho^\infty(t)c) \le (1+\|t\|)^N \mathrm{mom}_N(c)$ (as $|(\rho^\infty(t)c)(z)| = |c(z-t)|$ and $1+\|z+t\| \le (1+\|t\|)(1+\|z\|)$), so $\mathrm{mom}_N(v_w) \le \|T\|\|v_1\|_1 \mathrm{mom}_N(v_1)\,\mathrm{mom}_N(w) < \infty$, using $\|u_2\| \le \|u\|$.

Step 3: the integral. Bounded linear functionals commute with Bochner integrals, so by Steps 1 and 2, for every l,

$$S_l(w) = \int w(u)\, T(\rho^\infty(u_1)v_1) L_l(\rho^\infty(u_2)v_1)\, d\nu(u) = L_l(v_w) = \theta^{W,W}_{\gamma,\gamma}(v_w \otimes [\Psi_1^{\mathrm{st}}]_\Sigma)(g,l).$$

The function Ψ_1^{st} is locally constant, compactly supported and zero off the invertible matrices, so $v_w \otimes [\Psi_1^{\mathrm{st}}]_\Sigma$ is target-bounded (Lemma 6.2.26) and $l \mapsto S_l(w)$ is integrable on $[\mathrm{U}(W)]$ (Lemma 6.3.21); as $\gamma = \nu^{-1}\upsilon^{\lfloor \#n/2 \rfloor}$, its integral is $\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,W; v_w, \Psi_1^{\mathrm{st}})(g) = 0$ by hypothesis. □

Lemma 6.3.51. *Let s_1, s_2, s be signs on index sets n_1, n_2, n and $e\colon n_1 \sqcup n_2 \to n$ a bijection with $s \circ e = s_1 \sqcup s_2$. Let $\iota_j = \Phi \times n_j$, $\iota = \Phi \times n$ with the signs $\varepsilon_j(v,i) = s_j(i)$, $\varepsilon(v,i) = s(i)$, and for z in $\mathbb{C}^\iota$ or $\mathbb{A}_E^n$ let (z_1, z_2) be the restrictions of $z \circ e$ (of $(v,i) \mapsto z_{v,e(i)}$ in the first case) to n_1 and n_2. For $c_1 \in \mathcal{F}_{\varepsilon_1}$ and $c_2 \in \mathcal{F}_{\varepsilon_2}$ let $c_1 \boxtimes_e c_2$ be the function $z \mapsto c_1(z_1)c_2(z_2)$ on $\mathbb{C}^\iota$.*

1. *$c_1 \boxtimes_e c_2 \in \mathcal{F}_\varepsilon$, $\|c_1 \boxtimes_e c_2\|_1 = \|c_1\|_1\|c_2\|_1$ and $\mathrm{mom}_N(c_1 \boxtimes_e c_2) \le \mathrm{mom}_N(c_1)\,\mathrm{mom}_N(c_2)$ for every N; in particular $c_1 \boxtimes_e c_2$ is a Schwartz vector if c_1 and c_2 are.*
2. *For $z \in \mathbb{A}_E^n$, $f_1 \in \mathcal{M}_{s_1}$ and $f_2 \in \mathcal{M}_{s_2}$, $\rho(z)(f_1 \boxtimes_e f_2) = \rho(z_1)f_1 \boxtimes_e \rho(z_2)f_2$.*
3. *For $f_1 \in \mathcal{M}_{s_1}$ and $f_2 \in \mathcal{M}_{s_2}$, $(c_1 \boxtimes_e c_2) \otimes (f_1 \boxtimes_e f_2) = (c_1 \otimes f_1) \boxtimes_e (c_2 \otimes f_2)$.*

Proof. **(1).** Write $x = (x_1, x_2)$, $y = (y_1, y_2)$ for points of $\mathbb{C}^\iota$ in the coordinates given by e. The exponent of $\beta^\infty_\varepsilon(y,x)$ is $(x^*\mathrm{diag}(\varepsilon)y - y^*\mathrm{diag}(\varepsilon)x)/2$, a sum over the coordinates, and $\varepsilon \circ e = \varepsilon_1 \sqcup \varepsilon_2$; so $\beta^\infty_\varepsilon(y,x) = \beta^\infty_{\varepsilon_1}(y_1,x_1)\beta^\infty_{\varepsilon_2}(y_2,x_2)$, likewise $e^\infty(x-y) = e^\infty(x_1-y_1)e^\infty(x_2-y_2)$, and ν_ι is the image of $\nu_{\iota_1} \otimes \nu_{\iota_2}$ ($\#\iota = \#\iota_1 + \#\iota_2$). By Tonelli $c = c_1 \boxtimes_e c_2$ is ν_ι-integrable with $\|c\|_1 = \|c_1\|_1\|c_2\|_1$. Since $|\beta^\infty e^\infty| \le 1$, the function $y \mapsto c(y)\beta^\infty_\varepsilon(y,x)e^\infty(x-y)$ is integrable, and by Fubini and the reproducing identities of c_1 and c_2

$$\begin{aligned}\int c(y)\beta^\infty_\varepsilon(y,x)e^\infty(x-y)\, d\nu_\iota(y) &= \prod_{j=1,2} \int c_j(y_j)\beta^\infty_{\varepsilon_j}(y_j,x_j)e^\infty(x_j-y_j)\, d\nu_{\iota_j}(y_j)\\ &= c_1(x_1)c_2(x_2) = c(x).\end{aligned}$$

So $c \in \mathcal{F}_\varepsilon$. For the moments, $\|x\| = \max(\|x_1\|, \|x_2\|)$ for the maximum norm, so $1+\|x\| \le (1+\|x_1\|)(1+\|x_2\|)$, and Tonelli gives $\mathrm{mom}_N(c) \le \mathrm{mom}_N(c_1)\mathrm{mom}_N(c_2)$.

(2). By (WD6), $\beta_s(z,x) = \psi_E(\frac{J}{2}\sum_{i\in n} \bar{x}_i S_i z_i)$ with S_i the sign s_i at the infinite places and 1 at the finite places; as ψ_E is additive and $s \circ e = s_1 \sqcup s_2$, $\beta_s(z,x) = \beta_{s_1}(z_1,x_1)\beta_{s_2}(z_2,x_2)$. Hence

$$(\rho(z)(f_1 \boxtimes_e f_2))(x) = \beta_{s_1}(z_1,x_1)f_1(x_1-z_1)\,\beta_{s_2}(z_2,x_2)f_2(x_2-z_2) = (\rho(z_1)f_1 \boxtimes_e \rho(z_2)f_2)(x).$$

(3). The map $(F_1, F_2) \mapsto F_1 \boxtimes_e F_2$ from $\mathcal{M}_{s_1} \times \mathcal{M}_{s_2}$ to $\mathcal{M}_s$ is bilinear with $\|F_1 \boxtimes_e F_2\|_1 = \|F_1\|_1 \|F_2\|_1$, since μ_n is the image of $\mu_{n_1} \otimes \mu_{n_2}$ under e ((WD6); Tonelli). For $y \in \mathbb{C}^\iota$, $\mathrm{rv}(y)$ splits as $(\mathrm{rv}(y_1), \mathrm{rv}(y_2))$, so by (2)

$$(c_1 \boxtimes_e c_2)(y)\, \rho(\mathrm{rv}(y))(f_1 \boxtimes_e f_2) = \big(c_1(y_1)\rho(\mathrm{rv}(y_1))f_1\big) \boxtimes_e \big(c_2(y_2)\rho(\mathrm{rv}(y_2))f_2\big).$$

The two factors are Bochner integrable in y_1, y_2 (the Heisenberg operators are isometric and strongly continuous, Proposition 4.1.14, and c_j is integrable). A bounded bilinear map commutes with Bochner integrals in each variable, and the integral over y is the iterated integral over (y_1, y_2) (Fubini for the Bochner integral of an absolutely integrable function). Hence the integral over y against ν_ι, which is $(c_1 \boxtimes_e c_2) \otimes (f_1 \boxtimes_e f_2)$ by (WR6), equals $(c_1 \otimes f_1) \boxtimes_e (c_2 \otimes f_2)$. □

Lemma 6.3.52. *Let α, β, β' be automorphic characters of $\mathrm{U}(1)$. For $g_1 \in \mathrm{G}(s_1)$, $g_2 \in \mathrm{G}(s_2)$ write $\mathrm{diag}(g_1, g_2) \in \mathrm{G}(s_1 \sqcup s_2)$ for the block diagonal element.*

(a) (Source sums.) Let S_1, S_2, T be spaces with points with signs s_1, s_2, s'. For $f \in \mathcal{M}_{S_1,T}$, $f' \in \mathcal{M}_{S_2,T}$, $g_i \in \mathrm{G}(s_i)$ and $h \in \mathrm{G}(s')$,

$$\omega_{\beta\beta',\alpha}(\mathrm{diag}(g_1, g_2), h)(f \boxtimes^{\mathrm{sc}} f') = \omega_{\beta,\alpha}(g_1, h) f \boxtimes^{\mathrm{sc}} \omega_{\beta',\alpha}(g_2, h) f'.$$

(b) (Target sums.) Let S, T_1, T_2 be spaces with points with signs s, s_1, s_2. For $f \in \mathcal{M}_{S,T_1}$, $f' \in \mathcal{M}_{S,T_2}$, $g \in \mathrm{G}(s)$ and $h_i \in \mathrm{G}(s_i)$,

$$\omega_{\alpha,\beta\beta'}(g, \mathrm{diag}(h_1, h_2))(f \boxtimes^{\mathrm{tg}} f') = \omega_{\alpha,\beta}(g, h_1) f \boxtimes^{\mathrm{tg}} \omega_{\alpha,\beta'}(g, h_2) f'.$$

(c) (Theta values: exact factorisation.) In the situation of (a), respectively (b), for all data

$$\theta_{\mathrm{Hom}(S_1 \oplus S_2, T)}(f \boxtimes^{\mathrm{sc}} f') = \theta_{\mathrm{Hom}(S_1,T)}(f)\, \theta_{\mathrm{Hom}(S_2,T)}(f'),$$
$$\theta_{\mathrm{Hom}(S, T_1 \oplus T_2)}(f \boxtimes^{\mathrm{tg}} f') = \theta_{\mathrm{Hom}(S,T_1)}(f)\, \theta_{\mathrm{Hom}(S,T_2)}(f').$$

(d) (Theta kernels: exact factorisation.) Consequently

$$\theta^{S_1 \oplus S_2, T}_{\beta\beta',\alpha}(f \boxtimes^{\mathrm{sc}} f')(\mathrm{diag}(g_1, g_2), h) = \theta^{S_1,T}_{\beta,\alpha}(f)(g_1, h)\, \theta^{S_2,T}_{\beta',\alpha}(f')(g_2, h),$$

$$\theta^{S, T_1 \oplus T_2}_{\alpha,\beta\beta'}(f \boxtimes^{\mathrm{tg}} f')(g, \mathrm{diag}(h_1, h_2)) = \theta^{S,T_1}_{\alpha,\beta}(f)(g, h_1)\, \theta^{S,T_2}_{\alpha,\beta'}(f')(g, h_2).$$

Proof. **Products for one space.** Let $\sigma_1, \sigma_2, \sigma$ be signs on index sets m_1, m_2, m and $e\colon m_1 \sqcup m_2 \to m$ a bijection with $\sigma \circ e = \sigma_1 \sqcup \sigma_2$; for $p_i \in \mathrm{G}(\sigma_i)$ let $p \in \mathrm{G}(\sigma)$ be the element that is block diagonal with blocks p_1, p_2 through e. We show

$$\omega_\sigma(p)(F_1 \boxtimes_e F_2) = \omega_{\sigma_1}(p_1)F_1 \boxtimes_e \omega_{\sigma_2}(p_2)F_2 \qquad (F_i \in \mathcal{M}_{\sigma_i}). \tag{6.49}$$

The multiplier is multiplicative: at $v \in \Phi$ the block of p_v on the indices of sign -1 is, through e, block diagonal with blocks those of $p_{1,v}$, $p_{2,v}$, so its determinant is the product; if $u_i \in \mathrm{GL}_{m_i}(\mathbb{A}_{E,f})$ satisfy $u_i \widehat{\mathcal{O}}_E^{m_i} = \widehat{\mathcal{O}}_E^{m_i} \cap p_{i,f}^{-1} \widehat{\mathcal{O}}_E^{m_i}$, the block diagonal u (through e) satisfies the same for p, and $\det u = \det u_1 \det u_2$; as υ and $|\cdot|$ are multiplicative, $\Delta_\sigma(p) = \Delta_{\sigma_1}(p_1)\Delta_{\sigma_2}(p_2)$. Writing $x = (x_1, x_2)$, $y = (y_1, y_2)$ through e, we have $py = (p_1 y_1, p_2 y_2)$, and by (WR6) e_m, β_σ, μ_m are $e_{m_1} e_{m_2}$, $\beta_{\sigma_1}\beta_{\sigma_2}$, $\mu_{m_1} \otimes \mu_{m_2}$. The integrand of (WR3) for $\omega_\sigma(p)(F_1 \boxtimes_e F_2)$ at x is therefore the product of the integrands for $\omega_{\sigma_i}(p_i)F_i$ at x_i; it is absolutely integrable, as $|\beta| = 1$, $|e_m| \le 1$ and $F_i \in L^1(\mu_{m_i})$, so Fubini's theorem gives (6.49).

(a). Let $\sigma = s' \otimes (s_1 \sqcup s_2)$ on $n' \times (n_1 \sqcup n_2)$, $\sigma_i = s' \otimes s_i$, and e the bijection of the source product (WR6); $\sigma \circ e = \sigma_1 \sqcup \sigma_2$ by (VP5). The pair map of $(\mathrm{diag}(g_1, g_2), h)$ is $X \mapsto hX\,\mathrm{diag}(g_1, g_2)^{-1}$, which on $X = (X_1 \mid X_2)$ is $(hX_1 g_1^{-1} \mid hX_2 g_2^{-1})$: through e it is block diagonal with the pair maps p_i of (g_i, h). By (WR4) of Definition 3.2.18,

$$\omega_{\beta\beta',\alpha}(\mathrm{diag}(g_1, g_2), h) = \beta(\det h)\beta'(\det h)\,\alpha(\det g_1)^{-1}\alpha(\det g_2)^{-1}\,\omega_\sigma(p),$$

and (6.49) with

$$\omega_{\beta,\alpha}(g_1, h) = \beta(\det h)\alpha(\det g_1)^{-1}\omega_{\sigma_1}(p_1),$$
$$\omega_{\beta',\alpha}(g_2, h) = \beta'(\det h)\alpha(\det g_2)^{-1}\omega_{\sigma_2}(p_2)$$

gives (a).

(b). The same with $\sigma = (s_1 \sqcup s_2) \otimes s$ on $(n_1' \sqcup n_2') \times n$, the bijection of the target product, and the pair map $Y \mapsto \mathrm{diag}(h_1, h_2) Y g^{-1}$, which on Y with row blocks Y_1, Y_2 is $(h_1 Y_1 g^{-1}; h_2 Y_2 g^{-1})$; the characters are $\alpha(\det h_1)\alpha(\det h_2)\,\beta(\det g)^{-1}\beta'(\det g)^{-1}$.

(c). Through the bijection e of (a), $\mathrm{Hom}(S_1 \oplus S_2, T)$ is the block orthogonal sum of $\mathrm{Hom}(S_1, T)$ and $\mathrm{Hom}(S_2, T)$: by (VP5) its matrix $J_T \otimes (J_{S_1 \oplus S_2}^{-1})^{\mathsf{T}}$ and its point $x_T \otimes (x_{S_1 \oplus S_2}^{-1})^{\mathsf{T}}$, with $J_{S_1 \oplus S_2} = \mathrm{diag}(J_{S_1}, J_{S_2})$ and $x_{S_1 \oplus S_2} = \mathrm{diag}(x_{S_1}, x_{S_2})$, are block diagonal with blocks $J_T \otimes (J_{S_i}^{-1})^{\mathsf{T}}$, $x_T \otimes (x_{S_i}^{-1})^{\mathsf{T}}$, and the signs agree; and $f \boxtimes^{\mathrm{sc}} f'$ is the coefficient product through e. The factorisation clause of Lemma 4.1.17, read through the sign-preserving bijection e, gives the first identity. For target sums the bijection of (b) exhibits $\mathrm{Hom}(S, T_1 \oplus T_2)$, with matrix $\mathrm{diag}(J_{T_1}, J_{T_2}) \otimes (J_S^{-1})^{\mathsf{T}}$ and point $\mathrm{diag}(x_{T_1}, x_{T_2}) \otimes (x_S^{-1})^{\mathsf{T}}$, as the block orthogonal sum of $\mathrm{Hom}(S, T_i)$, and the same clause gives the second identity.

(d). By (WC4) of Definition 4.1.15, $\theta^{S,T}_{\alpha,\beta}(\varphi)(g, h) = \theta_{\mathrm{Hom}(S,T)}(\omega_{\alpha,\beta}(g, h)\varphi)$; apply (a), respectively (b), and then (c) to the two factors. □

Lemma 6.3.53. *Let V, V_0, U be spaces with points with signs s on n, s_0 on n_0 and t on r, p an index set, and D a base-adapted splitting of V over V_0 along p. Then the splitting D_U of* $\mathrm{Hom}(V, U)$ *over* $\mathrm{Hom}(V_0, U)$ *along $r \times p$ induced by D is adapted at* 1 *in the adapted basis $m_U = (x_U^{-1})^* \otimes 1_p$.*

Proof. Let $D = (k, l, i)$, with adelic frames $k_{\mathbb{A}} = x_V^{-1} k$, $l_{\mathbb{A}} = x_V^{-1} l$, $i_{\mathbb{A}} = x_V^{-1} i x_{V_0}$, and let $B_D = (l_{\mathbb{A}} \mid k_{\mathbb{A}} \mid i_{\mathbb{A}})$; by hypothesis B_D has orthonormal columns at every infinite place and entries in $\widehat{\mathcal{O}}_E$ at the finite places. Write $V^U = \mathrm{Hom}(V, U)$ and $V_0^U = \mathrm{Hom}(V_0, U)$; their points are $x_{V^U} = x_U \otimes (x_V^{-1})^{\mathsf{T}}$ and $x_{V_0^U} = x_U \otimes (x_{V_0}^{-1})^{\mathsf{T}}$, so $x_{V^U}^{-1} = x_U^{-1} \otimes x_V^{\mathsf{T}}$. By (MX1) the frames of D_U are $K = J_U^{-1} \otimes (k^* J_V)^{\mathsf{T}}$, $L = 1_r \otimes (l^* J_V)^{\mathsf{T}}$ and $I = 1_r \otimes (J_{V_0}^{-1} i^* J_V)^{\mathsf{T}}$, with adelic frames $x_{V^U}^{-1} K$, $x_{V^U}^{-1} L$ and $x_{V^U}^{-1} I x_{V_0^U}$. We have $m_U^{-1} = x_U^* \otimes 1_p$, hence $(m_U^{-1})^* = x_U \otimes 1_p$.

Step 1: three identities. From $x_V^* J_V x_V = \varepsilon_s$, $x_U^* J_U x_U = \varepsilon_t$ and $x_{V_0}^* J_{V_0} x_{V_0} = \varepsilon_{s_0}$ we get $J_V x_V = (x_V^*)^{-1}\varepsilon_s$, $x_U^{-1} J_U^{-1} (x_U^*)^{-1} = \varepsilon_t$ and $x_{V_0}^{-1} J_{V_0}^{-1} = \varepsilon_{s_0} x_{V_0}^*$. Hence

$$k^* J_V x_V = k_{\mathbb{A}}^* \varepsilon_s, \qquad l^* J_V x_V = l_{\mathbb{A}}^* \varepsilon_s, \qquad x_{V_0}^{-1} J_{V_0}^{-1} i^* J_V x_V = \varepsilon_{s_0} (x_V^{-1} i x_{V_0})^* \varepsilon_s = \varepsilon_{s_0} i_{\mathbb{A}}^* \varepsilon_s.$$

Step 2: the three column blocks. By the rules for Kronecker products and Step 1,

$$x_{V^U}^{-1} L\,(m_U^{-1})^* = x_U^{-1} x_U \otimes x_V^{\mathsf{T}} (l^* J_V)^{\mathsf{T}} = 1_r \otimes (l^* J_V x_V)^{\mathsf{T}} = 1_r \otimes \varepsilon_s \bar{l}_{\mathbb{A}},$$

$$x_{V^U}^{-1} K\, m_U = x_U^{-1} J_U^{-1} (x_U^{-1})^* \otimes (k^* J_V x_V)^{\mathsf{T}} = \varepsilon_t \otimes \varepsilon_s \bar{k}_{\mathbb{A}},$$

$$x_{V^U}^{-1} I\, x_{V_0^U} = x_U^{-1} x_U \otimes \left(x_{V_0}^{-1} J_{V_0}^{-1} i^* J_V x_V\right)^{\mathsf{T}} = 1_r \otimes \varepsilon_s \bar{i}_{\mathbb{A}} \varepsilon_{s_0}.$$

So the matrix of (MX5) for D_U at $g=1$ in the basis m_U, with column blocks indexed by $r\times p$, $r\times p$, $r\times n_0$, is

$$Q=\left(1_r\otimes\varepsilon_s\bar{l}_{\mathbb{A}}\;\middle|\;\varepsilon_t\otimes\varepsilon_s\bar{k}_{\mathbb{A}}\;\middle|\;1_r\otimes\varepsilon_s\bar{i}_{\mathbb{A}}\varepsilon_{s_0}\right)=\left(1_r\otimes\varepsilon_s\right)\left(1_r\otimes\bar{B}_D\right)R_{\pm},$$

where the columns of $1_r\otimes\bar{B}_D$ are indexed by $r\times(p\sqcup p\sqcup n_0)=(r\times p)\sqcup(r\times p)\sqcup(r\times n_0)$ and $R_{\pm}$ is the block diagonal matrix with blocks 1, $\varepsilon_t\otimes 1_p$, $1_r\otimes\varepsilon_{s_0}$: a diagonal matrix with entries ±1 at the infinite places and 1 at the finite places.

Step 3: orthonormality and integrality. At an infinite place v the matrices $(1_r\otimes\varepsilon_s)_v$ and $(R_{\pm})_v$ are diagonal with entries ±1, so

$$Q_v^*Q_v=(R_{\pm})_v(1_r\otimes\bar{B}_{D,v}^*\bar{B}_{D,v})(R_{\pm})_v=(R_{\pm})_v(1_r\otimes\overline{B_{D,v}^*B_{D,v}})(R_{\pm})_v=(R_{\pm})_v^2=1,$$

since $B_{D,v}^*B_{D,v}=1$. At the finite places ε_s, ε_t, ε_{s_0} are the identity, so every entry of Q_f is 0 or the conjugate of an entry of $B_{D,f}$; complex conjugation preserves $\widehat{\mathcal{O}}_E$, so Q_f has entries in $\widehat{\mathcal{O}}_E$. Hence D_U is adapted at 1 in the basis m_U. □

Lemma 6.3.54. *Let $k\in\mathbb{N}$ and $\mu\colon\Phi\to\mathbb{Z}$ with $k\le\mu(w)$ for every $w\in\Phi$, and let $\mathfrak{c}=(\nu,\Sigma)$ be a split tame character of weight μ. Let W', V', W, V be spaces with signs s' on n', t' on r', s on n, t on r, let b be an index set with bijections $\sigma\colon b\sqcup r\to n$ and $\sigma'\colon b\sqcup r'\to n'$, and let $j\ge1$ with*

$$\#n'=\#n+2j,\qquad \mathrm{neg}(W')=j+\mathrm{neg}(W),\qquad \mathrm{neg}(V')=j+\mathrm{neg}(V),$$

$\#n$ even and $-k\le\#r-\#n/2$. Suppose that $\theta_{\mathfrak{c}}^{\mathrm{lg}}(W',V';v',\Phi^{\mathrm{st}}_{\sigma',y'})=0$ for every Schwartz vector $v'\in\mathcal{F}_{W',V'}$ and every $y'\in\mathrm{M}_{b\times n'}(\mathcal{O}_\Sigma)$. Then $\theta_{\mathfrak{c}}^{\mathrm{lg}}(W,V;v,\Phi^{\mathrm{st}}_{\sigma,y})=0$ for every Schwartz vector $v\in\mathcal{F}_{W,V}$ and every $y\in\mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$.

Proof. **Step 1: one plane, with splittings.** Let A be a base-adapted splitting of W' over W and X one of V' over V, both along $\{o\}$, and let (W',V',σ') be vanishing; we show that (W,V,σ) is vanishing. Fix a Schwartz vector $v\in\mathcal{F}_{W,V}$, $y\in\mathrm{M}_{b\times n}(\mathcal{O}_\Sigma)$, $g\in\mathrm{G}(s)$, and put $I=\theta_{\mathfrak{c}}^{\mathrm{lg}}(W,V;v,\Phi^{\mathrm{st}}_{\sigma,y})(g)$ and $e=\#r-\#n/2\ge-k$. By Lemma 6.3.55 there are a real-analytic $B\colon\mathbb{R}\to(0,\infty)$ and a Schwartz vector $v'\in\mathcal{F}_{W',V'}$ with $\theta_{\mathfrak{c}}^{\mathrm{lg}}(W,V;\mathrm{Cone}^+_{A,X,\mu}(x;v'),\Phi^{\mathrm{st}}_{\sigma,y})(g)=B(x)I$ for all real $x>0$. By Lemma 6.3.62 there is a real-analytic Λ on $(-k,\infty)$ with $\Lambda(x)=L^\Sigma(x,\tilde{\nu})$ for $x\ge1$ and $\Lambda(i)\ne0$ for every integer $i>-k$. Let $f=\varphi^{\mathrm{pl}}_{A,\sigma'}(v',y)$ and $F(x)=\int_Y Z_x(f;1,g)$, with Y, its measure and Z_x as in Lemma 6.2.36. By Lemma 6.2.36 (with $u=1$), F is real-analytic on $\mathbb{R}$, and with its constants ϵ_+, $|\epsilon_+|=C_+>0$, α_0 and $Q=\prod_{w\in\Sigma}\mathrm{N}w$,

$$F(x)=\epsilon_+\,|\alpha_0|^xQ^{x-1/2}L^\Sigma(x,\tilde{\nu})\,B(x)\,I\qquad(x\ge1).$$

The function $R(x)=\epsilon_+|\alpha_0|^xQ^{x-1/2}\Lambda(x)B(x)I$ is real-analytic on $(-k,\infty)$ ($|\alpha_0|^x=e^{x\log|\alpha_0|}$, $Q^{x-1/2}=e^{(x-1/2)\log Q}$) and agrees with F on $[1,2]$; by the identity theorem for real-analytic functions on the connected interval $(-k,\infty)$, $F=R$ there. On the other hand, the matrix $(\mathrm{pr}_i)_\Sigma$ of A is integral (A is base-adapted), so $y(\mathrm{pr}_i)_\Sigma\in\mathrm{M}_{b\times n'}(\mathcal{O}_\Sigma)$ and, (W',V',σ') being vanishing, the integrand of $\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v',y)(1,g)$ vanishes identically; so $\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v',y)(1,g)=0$. By Lemma 6.2.3 (its integrability hypothesis holds by Lemma 6.2.36), $\mathrm{CT}^{\mathrm{pl}}_{A,\sigma'}(v',y)(1,g)=c_0F(e+1)$ with $c_0\ne0$, so $F(e+1)=0$. As $e+1>-k$, $R(e+1)=0$. Here $\epsilon_+\ne0$, $|\alpha_0|^{e+1}>0$, $Q^{e+1/2}>0$, $\Lambda(e+1)\ne0$ ($e+1$ is an integer $>-k$) and $B(e+1)>0$; hence $I=0$.

Step 2: spaces with the same signs. Let (W',V',σ') be vanishing and W_1', V_1' spaces with the signs of W', V'. By (1) of Lemma 6.3.63 there are $\gamma\in\mathrm{GL}_{n'}(E)$, $k_1\in\mathrm{G}(s')$ with $x_{W_1'}=\gamma^{-1}x_{W'}k_1$, and γ', k_1' likewise for V_1', V'. For a Schwartz v' and integral y', the datum

$v' \otimes [\Phi^{\mathrm{st}}_{\sigma',y'}]_\Sigma$ (the same element of $\mathcal{M}_{W',V'} = \mathcal{M}_{W_1',V_1'}$) is target-bounded by Lemma 6.2.26, $\Phi^{\mathrm{st}}_{\sigma',y'}$ being locally constant, compactly supported and zero off the right-invertible matrices; so its theta kernel is integrable over $[\mathrm{U}(V')]$ (Lemma 6.3.21), and by Lemma 6.3.64 there is u with $|u| = 1$ and $\theta^{\mathrm{lg}}_{\mathfrak{c}}(W_1', V_1'; v', \Phi^{\mathrm{st}}_{\sigma',y'})(g) = u\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W', V'; v', \Phi^{\mathrm{st}}_{\sigma',y'})(k_1 g) = 0$. So (W_1', V_1', σ') is vanishing.

Step 3: induction on j. For $j = 1$: $\#n' = \#n + 2$, $\mathrm{neg}(W') = 1 + \mathrm{neg}(W)$, so (2) of Lemma 6.3.63 gives a space W_1 with the signs s' and a base-adapted splitting A of W_1 over W along $\{o\}$; as $\#r' = \#n' - \#b = \#r + 2$ and $\mathrm{neg}(V') = 1 + \mathrm{neg}(V)$, it gives likewise V_1 with the signs t' and X for V_1 over V. By Step 2, (W_1, V_1, σ') is vanishing, and by Step 1 so is (W, V, σ). Let the assertion hold for j and let the data satisfy the hypotheses for $j+1$. Put $Y_W = \mathbb{H}_{\{o\}} \oplus W$ and $Y_V = \mathbb{H}_{\{o\}} \oplus V$, with index sets $(\{o\} \sqcup \{o\}) \sqcup n$ and $(\{o\} \sqcup \{o\}) \sqcup r$, and choose a bijection $\tau\colon b \sqcup ((\{o\} \sqcup \{o\}) \sqcup r) \to (\{o\} \sqcup \{o\}) \sqcup n$ (both sides have $\#n + 2$ elements). Then $\#n' = (\#n + 2) + 2j$, $\mathrm{neg}(W') = j + \mathrm{neg}(Y_W)$, $\mathrm{neg}(V') = j + \mathrm{neg}(Y_V)$, $\#n + 2$ is even and $(\#r + 2) - (\#n + 2)/2 = e + 1 \geq -k$; by the assertion for j, (Y_W, Y_V, τ) is vanishing. The tautological splittings R_W of Y_W over W and R_V of Y_V over V along $\{o\}$ are base-adapted: their matrices of adelic frames are $\mathrm{diag}(x_{\mathbb{H}}^{-1}, 1)$, where the point $x_{\mathbb{H}}$ of $\mathbb{H}_{\{o\}}$ is unitary at the infinite places and integral at the finite places with determinant $c\bar{c}_0 + cc_0 = c$, a unit ($c_0 + \bar{c}_0 = 1$). By Step 1 for R_W, R_V, (W, V, σ) is vanishing. □

Lemma 6.3.55. *Let W', W, V', V, A, X be as above with $\#n$ even, let b be an index set with a bijection $\sigma\colon b \sqcup r \to n$, and let $\mu\colon \Phi \to \mathbb{Z}$. There is a real analytic function $B\colon \mathbb{R} \to (0, \infty)$ such that every Schwartz vector $v \in \mathcal{F}_{W,V}$ has a Schwartz vector $v' \in \mathcal{F}_{W',V'}$ with*

$$\theta^{\mathrm{lg}}_{\mathfrak{c}}(W, V; \mathrm{Cone}^+_{A,X,\mu}(z; v'), \Phi^{\mathrm{st}}_{\sigma,y})(g) = B(z)\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W, V; v, \Phi^{\mathrm{st}}_{\sigma,y})(g)$$

for all real $z > 0$, all split tame characters $\mathfrak{c} = (\nu, \Sigma)$ of weight μ, all $y \in \mathrm{M}_{b\times n}(E_\Sigma)$ and all $g \in \mathrm{G}(s)$. In particular, for every $z > 0$ every Schwartz vector v has a Schwartz vector v' for which this identity holds with $B(z)$ replaced by 1.

Proof. Put $\gamma(w) = \#n/2 - \mu(w)$ and let ρ be the function of Lemma 6.3.56. For $a \in (\mathbb{C}^\times)^\Phi$ write $|a|^{2z} = \prod_{w\in\Phi} |a_w|^{2z}$ and $\ell(a) = \sum_{w\in\Phi} \log|a_w|$.

Step 1: the factor B. Put $B(z) = \int \rho(u, a)\,|a|^{2z}\,du\,d^\times a$, the integral over $\mathrm{K}_{\max}(t') \times (\mathbb{C}^\times)^\Phi$. On the compact support K_ρ of ρ, $|\ell| \leq L$ for some L. The integrand is continuous, ≥ 0, with compact support, and positive on the nonempty open set where $\rho > 0$, which has positive measure (du is a Haar measure and $d^\times a$ has a positive continuous density); so $0 < B(z) < \infty$. Since $|a|^{2z} = e^{2z\ell(a)} = \sum_{k\geq 0} (2z)^k \ell(a)^k/k!$ and $\sum_k |2z|^k |\ell|^k \rho/k! \leq e^{2|z|L}\rho$ is integrable, $B(z) = \sum_{k\geq 0} c_k z^k$ with $c_k = \frac{2^k}{k!} \int \rho\,\ell^k$, $|c_k| \leq \frac{(2L)^k}{k!} \int \rho$; this power series converges on $\mathbb{R}$, so B is real analytic.

Step 2: real points do not change the large side. Let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character of weight μ and $\alpha = \nu^{-1} \upsilon^{\#n/2}$; α has weight $-\mu + \#n/2 = \gamma$. The function $\Phi = \Phi^{\mathrm{st}}_{\sigma,y}$ on $\mathrm{M}_{r\times n}(E_\Sigma)$ is locally constant ($\chi_{\mathfrak{c}}$ is continuous on the totally disconnected group $\mathcal{O}_\Sigma^\times$), compactly supported (on integral matrices), and vanishes at every matrix x without a right inverse: if $\Phi(x) \neq 0$, the $n \times n$ matrix with the rows of y and of x is invertible, so the rows of x are linearly independent. Let $f = \Phi^\sharp \otimes_\Sigma v^0$. By Lemma 6.2.26, $w \otimes f$ is target-bounded for every Schwartz vector $w \in \mathcal{F}_{W,V}$. For $h \in G_\infty(t)$, (5) of Lemma 4.1.1 gives $\omega_{\alpha,\alpha}((1,1),(h,1))(w \otimes f) = (\Omega_{\gamma,\gamma}(1,h)w) \otimes f = (\Omega_{\gamma,0}(1,h)w) \otimes f$ (the factor of the first argument 1 is 1), and (3) of Lemma 6.3.21 with the points 1 and $(h, 1)$ gives (if $r = \varnothing$, $G_\infty(t)$ is trivial and there is nothing to prove)

$$\theta^{\mathrm{lg}}_{\mathfrak{c}}(W, V; \Omega_{\gamma,0}(1,h)w, \Phi)(g) = I^{W,V}_{\alpha,\alpha}(\omega_{\alpha,\alpha}((1,1),(h,1))(w \otimes f))(g) = \theta^{\mathrm{lg}}_{\mathfrak{c}}(W, V; w, \Phi)(g),$$

the integral converging absolutely ((2) of Lemma 6.3.21).

Step 3: the identity. Let v be a Schwartz vector and v' the Schwartz vector of Lemma 6.3.56 for v. Fix $z>0$, $\mathfrak{c}$, y, g and let k_z be the integrand of Lemma 6.2.42 for v', the sign $+$ and $x=z$. By Lemma 6.3.56, for each (u,a) there is $h=h(u,a)\in G_\infty(t)$ with

$$\prod_w (a_w/|a_w|)^{2\mu(w)}\mathrm{Ln}^{+,\infty}_{A,X}(a)(\Omega_{\gamma_\mu,0}(1,u_\infty)v')=\rho(u,a)\Omega_{\gamma,0}(1,h)v;$$

as $w\mapsto w\otimes f$ and the theta kernel are linear,

$$k_z(h',u,a)=\rho(u,a)\,|a|^{2z}\,\theta^{W,V}_{\alpha,\alpha}\big((\Omega_{\gamma,0}(1,h)v)\otimes f\big)(g,h')\qquad (h'\in \mathrm{G}(t)).$$

By Lemma 6.2.42, k_z is integrable on $[\mathrm{U}(V)]\times \mathrm{K}_{\max}(t')\times(\mathbb{C}^\times)^\Phi$ and its integral is the left side of the statement. By Fubini's theorem and Step 2 (for every (u,a) the integral over h' converges absolutely and equals $\rho(u,a)|a|^{2z}\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;v,\Phi)(g)$), the integral is $B(z)\,\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;v,\Phi)(g)$.

Step 4: the normalised form. For $z>0$ apply Step 3 to the Schwartz vector $B(z)^{-1}v$; by the linearity of $w\mapsto\theta^{\mathrm{lg}}_{\mathfrak{c}}(W,V;w,\Phi)(g)$ on Schwartz vectors ((2) of Lemma 6.3.21 and Step 2), its vector v' satisfies the identity with the factor 1. □

Lemma 6.3.56. *Let W',W,V',V, A, X be as above with $\#n$ even, let $\mu\colon\Phi\to\mathbb{Z}$, and put $\gamma(w)=\#n/2-\mu(w)$ for $w\in\Phi$. There is a continuous function $\rho\geq 0$ with compact support on $\mathrm{K}_{\max}(t')\times(\mathbb{C}^\times)^\Phi$, not identically 0, such that every Schwartz vector $v\in\mathcal{F}_{W,V}$ has a Schwartz vector $v'\in\mathcal{F}_{W',V'}$ with the following property: for every $(u,a)\in\mathrm{K}_{\max}(t')\times(\mathbb{C}^\times)^\Phi$ there is $h\in G_\infty(t)$ with*

$$\prod_{w\in\Phi}\Big(\frac{a_w}{|a_w|}\Big)^{2\mu(w)}\ \mathrm{Ln}^{+,\infty}_{A,X}(a)(\Omega_{\gamma_\mu,0}(1,u_\infty)v')=\rho(u,a)\ \Omega_{\gamma,0}(1,h)\,v.$$

Proof. Put $\gamma(w)=\#n/2-\mu(w)$. As A splits off one hyperbolic plane, $\#n'=\#n+2$, so $\gamma_\mu(w)=\lfloor\#n'/2\rfloor-\mu(w)=\gamma(w)+1$. Write Z^∞ for $Z^\infty_{X_W,m^{\mathrm{tg}}_W}$, m^{tg}_W the basis of Lemma 6.3.58, and for $w\in\Phi$ let $x_{B,w}\in\mathbb{C}^{r'}$ be the component at w of the adelic frame $k_{X,\mathbb{A}}$ of X; for $a\in\mathbb{C}^\Phi$ and $u\in\mathrm{K}_{\max}(t')$ let $u^{-1}a\,x_B$ be the family $(u_w^{-1}a_wx_{B,w})_w$.

Step 1: the line vector. The relations $k_X^*J_{V'}k_X=0$, $l_X^*J_{V'}k_X=1$ of (MX1) and $x_{V'}^*J_{V'}x_{V'}=\varepsilon_{t'}$ give $k^*_{X,\mathbb{A}}\varepsilon_{t'}k_{X,\mathbb{A}}=0$ and $l^*_{X,\mathbb{A}}\varepsilon_{t'}k_{X,\mathbb{A}}=1$. At $w\in\Phi$: $x^*_{B,w}\mathrm{diag}(t')x_{B,w}=0$, and $x_{B,w}\neq 0$. Put $c_w=|x_{B,w}|^2>0$.

Step 2: a Witt family and a bump. For each w, Lemma 6.3.57 gives a smooth $k_w\colon\mathbb{C}^{r'}\to\mathrm{U}(\mathrm{diag}\,t')$ and an open neighbourhood N_w of $x_{B,w}$ with $k_w(y)x_{B,w}=y$ for isotropic $y\in N_w$. The set $\mathcal{G}$ of families $x=(x_w)_w$ with $x_w\in N_w$ and $\frac12c_w<|x_w|^2<2c_w$ for all w is an open neighbourhood of $x_B=(x_{B,w})_w$; choose a smooth β on $\mathbb{C}^{\Phi\times(r'\times\{o\})}$ with values in $[0,1]$, compact support in $\mathcal{G}$ and $\beta(x_B)=1$, and put $k(x)=(k_w(x_w))_w\in G_\infty(t')$. Then: (B1) if $\beta(x)\neq0$ and every x_w is isotropic, $k_w(x_w)x_{B,w}=x_w$; (B2) if $\beta(x)\neq0$, $\frac12c_w\leq|x_w|^2\leq2c_w$ for all w.

Step 3: the function ρ. Let $\rho(u,a)=\beta(u^{-1}a\,x_B)\prod_w|a_w|^{\#n}$ on $\mathrm{K}_{\max}(t')\times(\mathbb{C}^\times)^\Phi$. It is continuous, non-negative, and $\rho(1,(1)_w)=\beta(x_B)=1$. Since $u_w^*u_w=1$, $|u_w^{-1}a_wx_{B,w}|^2=|a_w|^2c_w$; so by (B2) $\rho(u,a)\neq0$ implies $\frac12\leq|a_w|^2\leq2$ for all w, and the support of ρ lies in the compact set $\mathrm{K}_{\max}(t')\times\prod_w\{a_w:\frac12\leq|a_w|^2\leq2\}$.

Step 4: the vectors v^B and v'. Let $v\in\mathcal{F}_{W,V}$ be a Schwartz vector. By Lemma 6.3.58, (2) of Lemma 6.2.43 applies to X_W and m^{tg}_W (the Fock models of (MX7) for $\mathrm{Hom}(W,V')$ and $\mathrm{Hom}(W,V)$ being $\mathcal{F}_{W,V'}$ and $\mathcal{F}_{W,V}$); with a smooth compactly supported τ with $\tau(0)=1$ and the constant map 1 it gives a Schwartz $v^B\in\mathcal{F}_{W,V'}$ whose real slice at 0 is $\tau(0)\,\omega^\infty(1)v=v$; as

$\rho^\infty(0)=1$, this slice is $Z^\infty(v^B)$, so $Z^\infty(v^B)=v$. Next, let $\tau'(x)=\beta(x)\prod_w \det k_w(x_w)^{\gamma(w)}$ and $g'(x)=\bigoplus_w k_w(x_w)\otimes 1_n$. Since $|\det k_w(y)|=1$ for $k_w(y)\in \mathrm{U}(\mathrm{diag}\, t')$, negative powers of $\det k_w$ are powers of its conjugate, so τ' is smooth with compact support; g' is smooth with values in the unitary group of the signs of $\mathrm{Hom}(W,V')$, as $(k\otimes 1)^*(\mathrm{diag}\, t'\otimes \mathrm{diag}\, s)(k\otimes 1)=\mathrm{diag}\, t'\otimes \mathrm{diag}\, s$. By Lemma 6.3.53 and (2) of Lemma 6.2.43 for the splitting induced by A on $\mathrm{Hom}(W',V')$ there is a Schwartz $v'\in\mathcal{F}_{W',V'}$ with

$$\mathrm{rsl}_{A,V'}(x;v')=\tau'(x)\,\omega^\infty(g'(x))\,v^B=\beta(x)\,\Omega_{\gamma,0}(1,k(x))v^B \qquad \text{for all } x, \tag{1}$$

the last equality by (WR5).

Step 5: the identity. Fix (u,a) and put $y=u^{-1}a\,x_B$, so that $u_\infty^{-1}(a\,x_B)=y$. By Lemma 6.3.59, Lemma 6.3.60 (with the weight γ; $\gamma_\mu=\gamma+1$), (1) and the representation property of $\Omega_{\gamma,0}$ (Proposition 4.1.14),

$$\mathrm{Ln}^{+,\infty}_{A,X}(a)(\Omega_{\gamma_\mu,0}(1,u_\infty)v')=Z^\infty\Big(\Omega_{\gamma,0}(1,u_\infty)\,\mathrm{rsl}_{A,V'}(y;v')\Big)=\beta(y)\,Z^\infty\Big(\Omega_{\gamma,0}(1,u_\infty k(y))v^B\Big).$$

If $\beta(y)=0$ then $\rho(u,a)=0$ and the assertion holds with $h=1$. Otherwise each $y_w=u_w^{-1}a_w x_{B,w}$ is isotropic, u_w preserving the form $\mathrm{diag}(t')$, so by (B1) $k_w(y_w)x_{B,w}=y_w$, and $g=u_\infty k(y)$ satisfies $g_w x_{B,w}=u_w y_w=a_w x_{B,w}$. By Lemma 6.3.61 ($\#n$ even) there is $h\in G_\infty(t)$ with $Z^\infty(\Omega_{\gamma,0}(1,g)v^B)=\prod_w(a_w/|a_w|)^{-2\mu(w)}|a_w|^{\#n}\,\Omega_{\gamma,0}(1,h)Z^\infty(v^B)$, and $Z^\infty(v^B)=v$. Hence

$$\prod_w\Big(\frac{a_w}{|a_w|}\Big)^{2\mu(w)}\mathrm{Ln}^{+,\infty}_{A,X}(a)(\Omega_{\gamma_\mu,0}(1,u_\infty)v')=\beta(y)\prod_w|a_w|^{\#n}\,\Omega_{\gamma,0}(1,h)v=\rho(u,a)\,\Omega_{\gamma,0}(1,h)v.$$

□

Lemma 6.3.57. *Let ε be signs on a finite index set a, $J=\mathrm{diag}(\varepsilon)$ the complex $a\times a$ matrix and $\mathrm{U}(J)=\{g\in \mathrm{GL}_a(\mathbb{C}): g^*Jg=J\}$. Let $x_0\in\mathbb{C}^a$ be nonzero with $x_0^*Jx_0=0$. Then there are a map $k\colon \mathbb{C}^a\to \mathrm{U}(J)$, smooth as a map of real vector spaces into $\mathrm{M}_a(\mathbb{C})$, and a neighbourhood N of x_0 such that*

$$k(x)\,x_0=x \qquad \textit{for every } x\in N \textit{ with } x^*Jx=0.$$

Proof. **Step 1: quasi-reflections.** For $z\in\mathbb{C}^a$ and $c\in\mathbb{C}^\times$ with $c+\bar c=z^*Jz$ put $R_{z,c}=1-c^{-1}zz^*J$. For $v,w\in\mathbb{C}^a$, using $(z^*Jv)^-=v^*Jz$,

$$(R_{z,c}v)^*J(R_{z,c}w)=v^*Jw+(v^*Jz)(z^*Jw)\big(-\bar c^{-1}-c^{-1}+|c|^{-2}z^*Jz\big)=v^*Jw,$$

since $|c|^{-2}(z^*Jz-c-\bar c)=0$. So $R_{z,c}^*JR_{z,c}=J$, and $R_{z,c}\in\mathrm{U}(J)$ (J being invertible, so is $R_{z,c}$).

Step 2: a smooth family carrying $y_0=Jx_0$. Let $y_0=Jx_0$. Then $y_0^*Jy_0=x_0^*J^3x_0=x_0^*Jx_0=0$ and $y_0^*J=x_0^*$. For $x\in\mathbb{C}^a$ put $\sigma(x)=\frac12 x^*Jx-x^*x_0$ and $z(x)=y_0-x$. Then

$$\sigma(x)+\overline{\sigma(x)}=x^*Jx-x^*x_0-x_0^*x=z(x)^*Jz(x),$$

because $z^*Jz=y_0^*Jy_0-y_0^*Jx-x^*Jy_0+x^*Jx$ with $y_0^*Jx=x_0^*x$ and $x^*Jy_0=x^*x_0$. The function σ is smooth (a real polynomial) and $\sigma(x_0)=-|x_0|^2\neq 0$, so there is $\delta>0$ with $\sigma\neq 0$ on the ball $B(x_0,2\delta)$. Let b be a smooth function on $\mathbb{C}^a$ with values in $[0,1]$, equal to 1 on $B(x_0,\delta)$ and 0 off $B(x_0,2\delta)$, and $\pi(x)=x_0+b(x)(x-x_0)$; then π is smooth, $\pi(x)\in B(x_0,2\delta)$ for all x, and $\pi(x)=x$ on $B(x_0,\delta)$. Put $k_1(x)=R_{z(\pi(x)),\sigma(\pi(x))}\in\mathrm{U}(J)$; its entries are polynomials in the real coordinates of $\pi(x)$ divided by the non-vanishing smooth function $\sigma(\pi(x))$, so k_1 is smooth.

If $x \in B(x_0,\delta)$ and $x^*Jx = 0$, then $\pi(x) = x$, $\sigma(x) = -x^*x_0$ and $z(x)^*Jy_0 = y_0^*Jy_0 - x^*Jy_0 = -x^*x_0 = \sigma(x)$, so

$$k_1(x)\,y_0 = y_0 - \sigma(x)^{-1}(z(x)^*Jy_0)z(x) = y_0 - z(x) = x.$$

Step 3: carrying x_0. By Step 2 at $x = x_0$, $k_1(x_0)y_0 = x_0$, so $k_1(x_0)^{-1}x_0 = y_0$. Let $k(x) = k_1(x)k_1(x_0)^{-1}$, smooth with values in $\mathrm{U}(J)$, and $N = B(x_0,\delta)$. For isotropic $x \in N$, $k(x)x_0 = k_1(x)y_0 = x$. □

Lemma 6.3.58. *Let Y, Y_0, T be spaces with points with signs σ on n_Y, σ_0 on n_0 and τ on n_T, p an index set and D a base-adapted splitting of Y over Y_0 along p. Then the splitting of* $\mathrm{Hom}(T,Y)$ *over* $\mathrm{Hom}(T,Y_0)$ *along $p \times n_T$ induced on the target by D is adapted at* 1 *in the basis*

$$m_T^{\mathrm{tg}} = 1_p \otimes \bar{x}_T\,\varepsilon_\tau \in \mathrm{GL}_{p\times n_T}(\mathbb{A}_E).$$

Proof. Let $D = (k,l,i)$ with adelic frames $k_{\mathbb{A}} = x_Y^{-1}k$, $l_{\mathbb{A}} = x_Y^{-1}l$, $i_{\mathbb{A}} = x_Y^{-1}ix_{Y_0}$, and $B_D = (l_{\mathbb{A}} \mid k_{\mathbb{A}} \mid i_{\mathbb{A}})$, with orthonormal columns at the infinite places and entries in $\widehat{\mathcal{O}}_E$ at the finite places. The points of $\mathrm{Hom}(T,Y)$ and $\mathrm{Hom}(T,Y_0)$ are $x_Y \otimes (x_T^{-1})^{\mathsf{T}}$ and $x_{Y_0} \otimes (x_T^{-1})^{\mathsf{T}}$, and by (LN1) the frames of the induced splitting are $k \otimes J_T^{\mathsf{T}}$, $l \otimes 1$, $i \otimes 1$. Its adelic frames are therefore

$$(x_Y^{-1} \otimes x_T^{\mathsf{T}})(k \otimes J_T^{\mathsf{T}}) = k_{\mathbb{A}} \otimes (J_T x_T)^{\mathsf{T}}, \qquad (x_Y^{-1} \otimes x_T^{\mathsf{T}})(l \otimes 1) = l_{\mathbb{A}} \otimes x_T^{\mathsf{T}},$$

$$(x_Y^{-1} \otimes x_T^{\mathsf{T}})(i \otimes 1)(x_{Y_0} \otimes (x_T^{-1})^{\mathsf{T}}) = i_{\mathbb{A}} \otimes 1.$$

Put $m = m_T^{\mathrm{tg}}$. Since $(J_T x_T)^{\mathsf{T}}\bar{x}_T = (x_T^* J_T x_T)^{\mathsf{T}} = \varepsilon_\tau$, we get $(k_{\mathbb{A}} \otimes (J_T x_T)^{\mathsf{T}})\,m = k_{\mathbb{A}} \otimes \varepsilon_\tau^2 = k_{\mathbb{A}} \otimes 1$. Since $m^{-1} = 1 \otimes \varepsilon_\tau \bar{x}_T^{-1}$ and $(\bar{x}_T^{-1})^* = (x_T^{-1})^{\mathsf{T}}$, $(m^{-1})^* = 1 \otimes (x_T^{-1})^{\mathsf{T}}\varepsilon_\tau$, so $(l_{\mathbb{A}} \otimes x_T^{\mathsf{T}})(m^{-1})^* = l_{\mathbb{A}} \otimes \varepsilon_\tau$. Hence the matrix of (MX5) at 1 in the basis m is, with the column blocks indexed by $p \times n_T$, $p \times n_T$ and $n_0 \times n_T$,

$$Q = \left(l_{\mathbb{A}} \otimes \varepsilon_\tau \;\middle|\; k_{\mathbb{A}} \otimes 1 \;\middle|\; i_{\mathbb{A}} \otimes 1\right) = (B_D \otimes 1_{n_T})\,R_\pm,$$

where the columns of $B_D \otimes 1_{n_T}$ are indexed by $(p \sqcup p \sqcup n_0) \times n_T = (p \times n_T) \sqcup (p \times n_T) \sqcup (n_0 \times n_T)$ and $R_\pm$ is block diagonal with the blocks $1_p \otimes \varepsilon_\tau$, 1, 1, diagonal with entries ± 1 at the infinite places and 1 at the finite places. At an infinite place v, $Q_v^*Q_v = (R_\pm)_v(B_{D,v}^* B_{D,v} \otimes 1)(R_\pm)_v = (R_\pm)_v^2 = 1$. At the finite places every entry of Q is an entry of B_D or 0, hence in $\widehat{\mathcal{O}}_E$. So the induced splitting is adapted at 1 in the basis m. □

Lemma 6.3.59. *Let W', W, V', V, A, X be as above, and for $w \in \Phi$ let $x_{B,w} \in \mathbb{C}^{r'}$ be the component at w of the adelic frame $k_{X,\mathbb{A}}$ of X. For $a \in \mathbb{C}^\Phi$ let $a\,x_B \in \mathbb{C}^{\Phi \times (r' \times \{o\})}$ be the family $(a_w x_{B,w})_{w\in\Phi}$. Then for every $v \in \mathcal{F}_{W',V'}$*

$$\mathrm{Ln}_{A,X}^{+,\infty}(a)\,v = Z^\infty_{X_W, m_W^{\mathrm{tg}}}\big(\mathrm{rsl}_{A,V'}(a\,x_B; v)\big).$$

Proof. Fix $a \in \mathbb{C}^\Phi$ and let $\alpha = \mathrm{rv}(a) \in \mathbb{A}_E$, the adele with components $\kappa_w a_w$ at $w \in \Phi$ and 0 at the finite places. Fix $v \in \mathcal{F}_{W',V'}$ and put $F = v \otimes v^0 \in \mathcal{M}_{W',V'}$. Since X splits a hyperbolic plane off V', $\#r' = \#r + 2 \geq 2$. By (LN3), $\mathrm{Ln}_{A,X}^{+,\infty}(a)v = \mathrm{rsl}_{D^+,m}(b_a; v)$, and D^+ is a splitting of $\mathrm{Hom}(W',V')$ over $\mathrm{Hom}(W,V)$ along the nonempty set $q \sqcup \{*\}$.

Step 1: the slice along D^+ in one step. By (1) of Lemma 6.2.38, D^+ is adapted at 1 in the basis m. By (3) of Lemma 6.2.20 (for D^+, m, $f = v^0$, $b = b_a$, $y = 0$), $v_1 \otimes v^0 = v^0$ ((4) of Lemma 6.2.21) and (4) of Lemma 6.2.20,

$$F\big[m^{-*}\mathrm{rv}(b_a)\big]_{D^+} = \mathrm{Ln}^{+,\infty}_{A,X}(a)v \otimes Z_{D^+}(v^0) = \mathrm{vf}(D^+)\,\mathrm{Ln}^{+,\infty}_{A,X}(a)v \otimes v^0, \qquad \mathrm{vf}(D^+) \neq 0, \quad (1)$$

the slice being taken along D^+. The basis m is block diagonal with the block 1 on $\{*\}$, and $\mathrm{rv}(b_a)$ has the entry α at $*$ and 0 elsewhere; so $m^{-*}\mathrm{rv}(b_a) = \alpha e_*$, e_* the unit vector at $*$.

Step 2: slices along D^+ at multiples of e_* are line terms. Let $f \in \mathcal{M}_{W',V'}$ and $c \in \mathbb{A}_E$. Write x for the point of $\mathrm{Hom}(W',V')$, $x_{\mathbb{H}}$ for that of $\mathbb{H}_{\{*\}} \oplus \mathrm{Hom}(W,V)$, and i^{pl}, l^R for the embedding of D^{pl} and the frame l of R^+. By (MX5) the frame l of D^+ has the column $i^{\mathrm{pl}}l^R$ at $*$, so its adelic frame sends ce_* to $x^{-1}i^{\mathrm{pl}}l^Rc = i^{\mathrm{pl}}_{\mathbb{A}}l^R_{\mathbb{A}}c$, with $i^{\mathrm{pl}}_{\mathbb{A}} = x^{-1}i^{\mathrm{pl}}x_{\mathbb{H}}$ and $l^R_{\mathbb{A}} = x^{-1}_{\mathbb{H}}l^R$. By (1) of Lemma 6.2.25, $Z_{D^+} = Z_{R^+} \circ Z^{\mathrm{pl}}$, and by (1) of Lemma 6.2.44 for $\Pi = Z^{\mathrm{pl}}$ (at $x = a = b = 0$; D^{pl} is along the nonempty set q), $Z^{\mathrm{pl}}\rho(i^{\mathrm{pl}}_{\mathbb{A}}w) = \rho(w)Z^{\mathrm{pl}}$. Hence

$$f[ce_*]_{D^+} = Z_{R^+}Z^{\mathrm{pl}}(\rho(i^{\mathrm{pl}}_{\mathbb{A}}l^R_{\mathbb{A}}c)f) = Z_{R^+}(\rho(l^R_{\mathbb{A}}c)Z^{\mathrm{pl}}f) = (Z^{\mathrm{pl}}f)[c]_+ = \mathrm{Ln}^+_{A,X}(c)f,$$

the last step by (1) of Lemma 6.2.24. By (LN2), $\mathrm{Ln}^+_{A,X}(c)f = Z^{\mathrm{tg}}_{X,W}(f[c\,k_{X,\mathbb{A}}])$, the inner slice the Hom slice along the splitting $A_{V'}$ of $\mathrm{Hom}(W',V')$ over $\mathrm{Hom}(W,V')$ induced by A.

Step 3: the zero term on the target of a vacuum datum. We show: for every $\psi \in \mathcal{F}_{W,V'}$,

$$Z^{\mathrm{tg}}_{X,W}(\psi \otimes v^0) = \mathrm{vf}(X_W)\,Z^{\infty}_{X_W,m^{\mathrm{tg}}_W}(\psi) \otimes v^0, \qquad Z^{\mathrm{tg}}_{X,W}(v^0) = \mathrm{vf}(X_W)\,v^0, \qquad \mathrm{vf}(X_W) \neq 0. \quad (2)$$

The splitting X_W is along $\{o\} \times n$. If $n \neq \varnothing$: X_W is adapted at 1 in the basis m^{tg}_W by Lemma 6.3.58, and (2) of Lemma 6.2.20 with $f = v^0$, $v_1 \otimes v^0 = v^0$ and (4) of Lemma 6.2.20 give (2). If $n = \varnothing$: $\mathrm{Hom}(W,V')$ and $\mathrm{Hom}(W,V)$ have rank 0, so both are the standard space of the empty sign vector, with theta model $\mathbb{C}$, $\rho(0) = 1$, $v^0 = 1$, $\langle f,g\rangle = \bar{f}g$ ((WD5)–(WD7), empty sums 0 and empty products 1) and the identity of $\mathbb{C}$ as canonical theta functional ((WC1)–(WC3)); the quotient in (MX4) is a point of mass 1, so (MX4) reads $Z^{\mathrm{tg}}_{X,W}(f) = f$ and $\mathrm{vf}(X_W) = 1$ by (MX8). The Fock models are $\mathbb{C}$ (functions on a point, ν the unit mass, real vacuum 1), the datum $\psi \otimes v^0$ is the number ψ, and the real zero term of (MX7) is the integral over a point of $\psi \cdot 1 \cdot 1 \cdot 1$, so $Z^{\infty}_{X_W,m^{\mathrm{tg}}_W}(\psi) = \psi$; (2) holds.

Step 4: the line term in two steps. The adelic $r' \times 1$ matrix $\alpha\,k_{X,\mathbb{A}}$ has components $\kappa_w a_w x_{B,w}$ at $w \in \Phi$ and 0 at the finite places, so $\alpha\,k_{X,\mathbb{A}} = \mathrm{rv}(a\,x_B)$. By (2) of Lemma 6.3.12 (for W', W, $T = V'$ with the nonempty index set r', and A along $\{o\}$),

$$F[\alpha\,k_{X,\mathbb{A}}] = (v \otimes v^0)[\mathrm{rv}(a\,x_B)] = \mathrm{vf}(A_{V'})\,\mathrm{rsl}_{A,V'}(a\,x_B; v) \otimes v^0, \qquad \mathrm{vf}(A_{V'}) \neq 0.$$

Applying $Z^{\mathrm{tg}}_{X,W}$, Step 2 (with $c = \alpha$) and (2) (with $\psi = \mathrm{rsl}_{A,V'}(a\,x_B; v)$) give

$$F[\alpha e_*]_{D^+} = \mathrm{vf}(A_{V'})\,\mathrm{vf}(X_W)\,Z^{\infty}_{X_W,m^{\mathrm{tg}}_W}\big(\mathrm{rsl}_{A,V'}(a\,x_B; v)\big) \otimes v^0. \quad (3)$$

Step 5: the vacuum factors. Apply Step 2 with $c = 0$ to $f = v^0$: $Z_{D^+}(v^0) = Z^{\mathrm{tg}}_{X,W}(v^0[0])$, and the Hom slice at 0 is $v^0[0] = Z_{A_{V'}}(v^0) = \mathrm{vf}(A_{V'})v^0$ ((4) of Lemma 6.2.20 for $A_{V'}$, which is along the nonempty set $r' \times \{o\}$ and adapted at 1 in its adapted basis by Lemma 6.3.53). By (2) and (MX8), $\mathrm{vf}(D^+) = \mathrm{vf}(A_{V'})\,\mathrm{vf}(X_W)$.

Step 6: conclusion. By (1), Step 1 and (3), with Step 5 and the linearity of $\phi \mapsto \phi \otimes v^0$ ((WR6)),

$$\mathrm{vf}(D^+)\Big(\mathrm{Ln}^{+,\infty}_{A,X}(a)v - Z^{\infty}_{X_W,m^{\mathrm{tg}}_W}\big(\mathrm{rsl}_{A,V'}(a\,x_B; v)\big)\Big) \otimes v^0 = 0.$$

As $\mathrm{vf}(D^+) \neq 0$, Lemma 6.3.9 for the signs of $\mathrm{Hom}(W,V)$ gives the assertion. □

Lemma 6.3.60. *Let W', W, V' and A be as above, $\gamma\colon \Phi \to \mathbb{Z}$, $h \in G_\infty(t')$, $x = (x_w)_{w\in\Phi} \in \mathbb{C}^{\Phi\times(r'\times\{o\})}$ (each x_w a vector of $\mathbb{C}^{r'}$) and $h^{-1}x = (h_w^{-1}x_w)_w$. Then for every $v \in \mathcal{F}_{W',V'}$*

$$\mathrm{rsl}_{A,V'}(x;\Omega_{\gamma+1,0}(1,h)v) = \Omega_{\gamma,0}(1,h)\,\mathrm{rsl}_{A,V'}(h^{-1}x;v),$$

with Ω the real Weil representation of the pair (W',V') on the left and of (W,V') on the right, 1 the identity of $G_\infty(s')$, resp. $G_\infty(s)$.

Proof. If r' is empty, $G_\infty(t')$ is trivial and both sides are $\mathrm{rsl}_{A,V'}(x;v)$; let r' be nonempty, so that $A_{V'}$ is a splitting along the nonempty set $r'\times\{o\}$.

Step 1: reduction to the weights $(1,0)$**.** By (WR5),

$$\Omega_{\gamma+1,0}(1,h) = \prod_w \det(h_w)^{\gamma(w)}\,\Omega_{1,0}(1,h) \quad\text{and}\quad \Omega_{\gamma,0}(1,h) = \prod_w \det(h_w)^{\gamma(w)}\,\Omega_{0,0}(1,h),$$

and $\mathrm{rsl}_{A,V'}(x;\cdot)$ is linear ((MX7), a Bochner integral linear in the vector). So it suffices to prove

$$\mathrm{rsl}_{A,V'}(x;\Omega_{1,0}(1,h)v) = \Omega_{0,0}(1,h)\,\mathrm{rsl}_{A,V'}(h^{-1}x;v).$$

Step 2: real slices as Hom slices. Apply (2) of Lemma 6.3.12 to the base-adapted splitting A of W' over W along $\{o\}$ and $T = V'$; the index sets r' and $\{o\}$ are nonempty. So $\mathrm{vf} = \mathrm{vf}(A_{V'}) \neq 0$, and for every $\phi \in \mathcal{F}_{W',V'}$ and $b \in \mathbb{C}^{\Phi\times(r'\times\{o\})}$

$$(\phi\otimes v^0)[\mathrm{rv}(b)] = \mathrm{vf}\,\mathrm{rsl}_{A,V'}(b;\phi)\otimes v^0, \tag{6.50}$$

the left side a Hom slice.

Step 3: the law of Hom slices. Let $\hat{h} = (h,1) \in \mathrm{G}(t')$. By (5) of Lemma 4.1.1 (the characters υ and 1 have weights 1 and 0), $\omega_{\upsilon,1}(1,\hat{h})(\upsilon\otimes v^0) = (\Omega_{1,0}(1,h)v)\otimes v^0$. Apply (2) of Lemma 6.2.45 to the splitting A along $\{o\}$, $T = V'$, the characters $\alpha = 1$ (so that $\alpha\upsilon^{\#\{o\}} = \upsilon$) and $\beta = 1$, the element $g = 1$ (with $m = 1$, $g_0 = 1$, $W = \mathrm{pr}_i l_{A,\mathbb{A}} = 0$, $B = \mathrm{pr}_k l_{A,\mathbb{A}} = 0$, $\chi(1) = 1$) and $\hat{h}$: for every adelic X,

$$\big(\omega_{\upsilon,1}(1,\hat{h})(\upsilon\otimes v^0)\big)[X] = \omega_{1,1}(1,\hat{h})\big((\upsilon\otimes v^0)[\hat{h}^{-1}X]\big),$$

$\omega_{1,1}$ on the right being that of the pair (W,V'). At $X = \mathrm{rv}(b)$, $\hat{h}^{-1}X = \mathrm{rv}(h^{-1}b)$ ($\hat{h}$ has finite components 1 and κ_υ is a scalar). By (6.50) on both sides and (5) of Lemma 4.1.1 for the pair (W,V') (weights $0,0$),

$$\mathrm{vf}\,\mathrm{rsl}_{A,V'}\big(b;\Omega_{1,0}(1,h)v\big)\otimes v^0 = \mathrm{vf}\,\big(\Omega_{0,0}(1,h)\,\mathrm{rsl}_{A,V'}(h^{-1}b;v)\big)\otimes v^0.$$

As $\mathrm{vf}\neq 0$, Lemma 6.3.9 gives the identity of Step 1 at $x = b$. □

Lemma 6.3.61. *Let W, V', V and X be as above with $\#n$ even, $\mu\colon \Phi\to\mathbb{Z}$ and $\gamma(w) = \#n/2 - \mu(w)$. For $w\in\Phi$ let $x_{B,w}\in\mathbb{C}^{r'}$ be the component at w of $k_{X,\mathbb{A}}$. Let $g\in G_\infty(t')$ and $a\in(\mathbb{C}^\times)^\Phi$ with $g_w x_{B,w} = a_w x_{B,w}$ for every $w\in\Phi$. Then there is $h\in G_\infty(t)$ such that for every $v\in\mathcal{F}_{W,V'}$*

$$Z^\infty_{X_W,m_W^{\mathrm{tg}}}\big(\Omega_{\gamma,0}(1,g)v\big) = \prod_{w\in\Phi}\Big(\frac{a_w}{|a_w|}\Big)^{-2\mu(w)}|a_w|^{\#n}\;\Omega_{\gamma,0}(1,h)\,Z^\infty_{X_W,m_W^{\mathrm{tg}}}(v),$$

with Ω the real Weil representation of the pair (W,V') on the left and of (W,V) on the right.

Proof. Write $Z^\infty = Z^\infty_{X_W, m_W^{\mathrm{tg}}}$ and $\hat g = (g,1) \in \mathrm{G}(t')$, and let $\hat a \in \mathbb{A}_E^\times$ have components a_w at $w \in \Phi$ and 1 at the finite places. Then $\hat g k_{X,\mathbb{A}} = k_{X,\mathbb{A}} \hat a$.

Step 1: the point h. By (3) of Lemma 6.3.48, $\hat g = \mathrm{rad}(c,z)\,\mathrm{lev}(\hat a, h_0)$ with $h_0 \in \mathrm{G}(t)$, and $h_0 = \mathrm{pr}_i \hat g i_{X,\mathbb{A}}$ since $\mathrm{pr}_i \,\mathrm{rad}(c,z)\, i_{X,\mathbb{A}} = 1$ ((MX2)); its finite components are $\mathrm{pr}_i i_{X,\mathbb{A}} = 1$, so $h_0 = (h,1)$ with $h \in G_\infty(t)$. In the frame $(l_{X,\mathbb{A}} \mid k_{X,\mathbb{A}} \mid i_{X,\mathbb{A}})$ the radical element is unipotent (it maps $k_{X,\mathbb{A}}$ to itself, $i_{X,\mathbb{A}}$ into $i_{X,\mathbb{A}} + k_{X,\mathbb{A}}(\cdot)$ and $l_{X,\mathbb{A}}$ into $l_{X,\mathbb{A}}$ + (the span of $i_{X,\mathbb{A}}, k_{X,\mathbb{A}}$)) and the Levi element is block diagonal with blocks $\bar{\hat a}^{-1}$, $\hat a$, h_0; so $\det g_w = (a_w/\bar a_w)\det h_w$ for $w \in \Phi$.

Step 2: reduction to the weights $(0,0)$. By (WR5), $\Omega_{\gamma,0}(1,g) = \prod_w \det(g_w)^{\gamma(w)} \Omega_{0,0}(1,g)$ and $\Omega_{\gamma,0}(1,h) = \prod_w \det(h_w)^{\gamma(w)}\Omega_{0,0}(1,h)$, and Z^∞ is linear. With $a_w/\bar a_w = (a_w/|a_w|)^2$ and $2\gamma(w) - \#n = -2\mu(w)$, the assertion follows from

$$Z^\infty\big(\Omega_{0,0}(1,g)v\big) = \prod_{w\in\Phi} \Big(\frac{a_w}{|a_w|}\Big)^{-\#n} |a_w|^{\#n}\; \Omega_{0,0}(1,h)\, Z^\infty(v). \tag{6.51}$$

If n is empty, all Fock models involved are $\mathbb{C}$, $\Omega_{0,0}$ and Z^∞ are the identity, and (6.51) holds; let n be nonempty, so that X_W is a splitting along the nonempty set $\{o\} \times n$.

Step 3: the adelic law. Apply (1) of Lemma 6.2.9 to $D = X$ (with $Y = V'$, $Y_0 = V$, $p = \{o\}$), $T = W$, the element $\hat g$ with $m = \hat a$ and h_0, the point $1 \in \mathrm{G}(s)$ and the characters $\alpha = \beta = 1$, and to $F = v \otimes v^0 \in \mathcal{M}_{W,V'}$:

$$Z^{\mathrm{tg}}_{X,W}\big(\omega_{1,v}(1,\hat g)(v\otimes v^0)\big) = 1[\#n](\hat a)\,|\hat a|^{\#n/2}\,\omega_{1,1}(1,h_0)\, Z^{\mathrm{tg}}_{X,W}(v\otimes v^0),$$

with $1[\#n](\hat a) = \upsilon(\hat a)^{-\#n} = \prod_w (a_w/|a_w|)^{-\#n}$ and $|\hat a|^{\#n/2} = \prod_w |a_w|^{\#n}$ (the module at a complex place is the square of the absolute value). By (5) of Lemma 4.1.1 (the characters have weight 0 on the target side; the source point is 1), $\omega_{1,\upsilon}(1,\hat g)(v\otimes v^0) = (\Omega_{0,0}(1,g)v)\otimes v^0$ and $\omega_{1,1}(1,h_0)(w'\otimes v^0) = (\Omega_{0,0}(1,h)w')\otimes v^0$. By Lemma 6.3.58, X_W is adapted at 1 in m_W^{tg}; so (2) and (4) of Lemma 6.2.20 and $v_1 \otimes v^0 = v^0$ ((4) of Lemma 6.2.21) give $Z^{\mathrm{tg}}_{X,W}(\phi\otimes v^0) = \mathrm{vf}\, Z^\infty(\phi)\otimes v^0$ for every Fock vector ϕ, with $\mathrm{vf} = \mathrm{vf}(X_W) \neq 0$. Hence

$$\mathrm{vf}\, Z^\infty\big(\Omega_{0,0}(1,g)v\big)\otimes v^0 = \prod_w \Big(\frac{a_w}{|a_w|}\Big)^{-\#n} |a_w|^{\#n}\, \mathrm{vf}\,\big(\Omega_{0,0}(1,h) Z^\infty(v)\big)\otimes v^0,$$

and Lemma 6.3.9 gives (6.51). □

Lemma 6.3.62. *Let $\mu\colon \Phi \to \mathbb{Z}$ and $k \in \mathbb{N}$ with $k \le \mu(v)$ for every $v \in \Phi$, and let $\mathfrak{c} = (\nu, \Sigma)$ be a split tame character of weight μ. Then there is a real-analytic function $\Lambda\colon (-k,\infty) \to \mathbb{C}$ such that $\Lambda(x) = L^\Sigma(x, \tilde\nu)$ for every real $x \ge 1$ and $\Lambda(j) \neq 0$ for every integer $j > -k$.*

Proof. Let $\psi = (\psi_w)_w$ be the tame part of ν, so $\psi_w \neq 1$ exactly for $w \in \Sigma$. Complex conjugation of E extends to $\mathbb{A}_E$; for a place w of E let $\bar w$ be the conjugate place. For a finite place w it induces an isomorphism $\sigma_w\colon E_w \to E_{\bar w}$ with $\sigma_w(\mathcal{O}^\times_{E,w}) = \mathcal{O}^\times_{E,\bar w}$, and hence an isomorphism of residue fields. As F is totally real and E totally imaginary, every infinite place $v \in \Phi$ is complex and fixed by conjugation, which acts on $E_v = \mathbb{C}$ (through the embedding of the CM type) as complex conjugation. For a place w let $\iota_w\colon E_w^\times \to \mathbb{A}_E^\times$ be the inclusion with components 1 at the other places and $\tilde\nu_w = \tilde\nu \circ \iota_w$.

Step 1: $\tilde\nu$ is a unitary Hecke character of E. The map $x \mapsto x/\bar x$ is a continuous homomorphism $\mathbb{A}_E^\times \to \mathrm{U}(1)(\mathbb{A})$ (as $(x/\bar x)\overline{(x/\bar x)} = 1$) that maps $E^\times$ into $\mathrm{U}(1)(E)$. Since ν is continuous, of modulus 1 and trivial on $\mathrm{U}(1)(E)$, $\tilde\nu$ is a continuous character of $\mathbb{A}_E^\times/E^\times$ of modulus 1.

Step 2: the local components. (a) Let $v \in \Phi$ and $a \in \mathbb{C}^\times$. The idele $\iota_v(a)/\overline{\iota_v(a)}$ has the component $a/\bar a$ at v and 1 elsewhere; it lies in $\mathrm{U}(1)(\mathbb{A})$ and its finite components are integral,

so the definition of weight and tame part (Definition 3.1.2) gives

$$\tilde{\nu}_v(a) = \Big(\frac{a}{\bar{a}}\Big)^{\mu(v)} = \Big(\frac{a}{|a|}\Big)^{2\mu(v)}.$$

(b) Let w be a finite place and $u \in \mathcal{O}_{E,w}^\times$. If $\bar{w} = w$, the idele $\iota_w(u)/\overline{\iota_w(u)}$ has the single component $u/\sigma_w(u) \in \mathcal{O}_{E,w}^\times$ at w and 1 elsewhere, so $\tilde{\nu}_w(u) = \psi_w((u/\sigma_w(u)) \bmod w)$. If $\bar{w} \neq w$, it has the components u at w and $\sigma_w(u)^{-1}$ at $\bar{w}$, so

$$\tilde{\nu}_w(u) = \psi_w(u \bmod w)\, \psi_{\bar{w}}(\sigma_w(u)^{-1} \bmod \bar{w}).$$

Since Σ contains the conjugate of none of its members, $w \in \Sigma$ implies $\bar{w} \neq w$ and $\bar{w} \notin \Sigma$. Hence: if $w \notin \Sigma \cup \bar{\Sigma}$, then $\psi_w = \psi_{\bar{w}} = 1$ and $\tilde{\nu}_w$ is trivial on $\mathcal{O}_{E,w}^\times$; if $w \in \Sigma$, then $\tilde{\nu}_w(u) = \psi_w(u \bmod w)$, which is not identically 1 because $\psi_w \neq 1$ and reduction $\mathcal{O}_{E,w}^\times \to k_w^\times$ is onto; if $w = \bar{w}_0$ with $w_0 \in \Sigma$, then $\psi_w = 1$ and $\tilde{\nu}_w(u) = \psi_{w_0}(\sigma_w(u) \bmod w_0)^{-1}$, again not identically 1 since σ_w maps $\mathcal{O}_{E,w}^\times$ onto $\mathcal{O}_{E,w_0}^\times$. So $\tilde{\nu}$ is ramified exactly at the places of $\Sigma \cup \bar{\Sigma}$, the Euler product of $L(s, \tilde{\nu})$ runs over the finite places $w \notin \Sigma \cup \bar{\Sigma}$, and

$$L(s, \tilde{\nu}) = L^\Sigma(s, \tilde{\nu}) \qquad (\operatorname{Re} s > 1).$$

(c) By (a), $\tilde{\nu}_v$ is the character $a \mapsto (a/|a|)^{2\mu(v)}$ of $\mathbb{C}^\times$, whose local L-factor is $\Gamma_{\mathbb{C}}(s + \mu(v))$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ ([296], Section 2.5: the factor of $a \mapsto (a/|a|)^\ell$ is $\Gamma_{\mathbb{C}}(s + |\ell|/2)$). Hence

$$L_\infty(s, \tilde{\nu}) = L_\infty(s, \tilde{\nu}^{-1}) = \prod_{v \in \Phi} \Gamma_{\mathbb{C}}(s + \mu(v)).$$

Step 3: continuation. $\tilde{\nu}$ is not of the form $|\cdot|_{\mathbb{A}}^{i\tau}$: on the unit circle of $\mathbb{C}^\times$ at $v \in \Phi$ the character $|\iota_v(\cdot)|_{\mathbb{A}}^{i\tau} = |\cdot|^{2i\tau}$ is trivial, whereas $\tilde{\nu}_v(e^{i\vartheta}) = e^{2i\mu(v)\vartheta}$ is not, as $\mu(v) \geq k \geq 0$ and $\mu(v) > 0$ by (XD4). The same holds for $\tilde{\nu}^{-1} = \overline{\tilde{\nu}}$. By Theorem A.1.12 (2), whose poles occur only in the case excluded here (item (1)), $\xi(s, \tilde{\nu})$ continues to an entire function. As Γ has no zeros, $1/L_\infty(s, \tilde{\nu})$ is entire, so

$$L(s, \tilde{\nu}) = \big(|d_E| \operatorname{N}\mathfrak{f}(\tilde{\nu})\big)^{-s/2}\, \xi(s, \tilde{\nu})\, L_\infty(s, \tilde{\nu})^{-1}$$

continues to an entire function, again written $L(s, \tilde{\nu})$; the same argument applies to $\overline{\tilde{\nu}}$. By Step 2 (b) this continuation agrees with $L^\Sigma(s, \tilde{\nu})$ on $\operatorname{Re} s > 1$, so its value at $s = 1$ is the value $L^\Sigma(1, \tilde{\nu})$ of (XD7). Put $\Lambda(x) = L(x, \tilde{\nu})$ for real $x > -k$. The restriction of an entire function to an open interval of $\mathbb{R}$ is real-analytic (at each real point its Taylor series converges on a disc), and $\Lambda(x) = L^\Sigma(x, \tilde{\nu})$ for $x \geq 1$.

Step 4: nonvanishing at the integers $j > -k$. If $j \geq 1$, then $\Lambda(j) = L(j, \tilde{\nu}) \neq 0$ by Theorem A.1.12 (3). Let $j \leq 0$ and $j > -k$. By Theorem A.1.12 (2), $\xi(s, \tilde{\nu}) = W(\tilde{\nu})\, \xi(1 - s, \overline{\tilde{\nu}})$ and $W(\tilde{\nu})W(\overline{\tilde{\nu}}) = 1$, so $W(\tilde{\nu}) \neq 0$. At $s = 1 - j \geq 1$ the three factors of $\xi(s, \overline{\tilde{\nu}})$ are holomorphic: $L(1 - j, \overline{\tilde{\nu}}) \neq 0$ by Theorem A.1.12 (3), $L_\infty(1 - j, \overline{\tilde{\nu}}) = \prod_v \Gamma_{\mathbb{C}}(1 - j + \mu(v))$ is finite and nonzero because $1 - j + \mu(v) \geq 1$, and the power of $|d_E| \operatorname{N}\mathfrak{f}(\overline{\tilde{\nu}})$ is positive. Hence $\xi(j, \tilde{\nu}) = W(\tilde{\nu})\, \xi(1 - j, \overline{\tilde{\nu}}) \neq 0$. On the other side, $j + \mu(v) \geq j + k > 0$ for every $v \in \Phi$, so $L_\infty(s, \tilde{\nu})$ is holomorphic at $s = j$ and $\xi(j, \tilde{\nu}) = (|d_E| \operatorname{N}\mathfrak{f}(\tilde{\nu}))^{j/2} L_\infty(j, \tilde{\nu})\, L(j, \tilde{\nu})$. Therefore $\Lambda(j) = L(j, \tilde{\nu}) \neq 0$. □

Lemma 6.3.63. *1. Let V, V_1 be spaces with points with the same signs s on n. There are $\gamma \in \mathrm{GL}_n(E)$ with $\gamma^* J_{V_1} \gamma = J_V$ and $k \in \mathrm{G}(s)$ with $x_V = \gamma^{-1} x_{V_1} k$; and for every such pair (γ, k), $\Gamma_V = k^{-1} \Gamma_{V_1} k$.*

2. *Let W be a space with a point with signs s on n, let p, r be finite index sets and t signs on r with $\#r = \#n + 2\#p$ and $\#\{\rho : t_\rho = -1\} = \#p + \#\{i : s_i = -1\}$. There are a space with a point W' with signs t and a base-adapted splitting of W' over W along p.*

Proof. (1). This is Lemma 4.1.8.

(2). Let s'' be the signs of $W'' = \mathbb{H}_p \oplus W$ on $(p \sqcup p) \sqcup n$: $+1$ on the first copy of p, -1 on the second, s on n. It has $\#r$ indices and $\#p + \#\{s_i = -1\}$ signs -1, so there is a bijection $\epsilon\colon r \to (p \sqcup p) \sqcup n$ with $s'' \circ \epsilon = t$. For a matrix M indexed by $(p \sqcup p) \sqcup n$ let M^ϵ be the matrix $(M_{\epsilon(\rho),\epsilon(\rho')})_{\rho,\rho' \in r}$, and similarly for one-sided reindexing of rows. Put $W' = (J^\epsilon_{W''}, x^\epsilon_{W''})$; as reindexing is conjugation by a permutation matrix, $(x^\epsilon_{W''})^* J^\epsilon_{W''} x^\epsilon_{W''} = (\varepsilon_{s''})^\epsilon = \varepsilon_t$, so W' is a space with a point with signs t. Let l, k, i be the $r \times p$, $r \times p$, $r \times n$ matrices obtained by reindexing the rows of the column blocks of the identity matrix for the first copy of p, the second copy and n. The conditions of (MX1) are invariant under the simultaneous reindexing, and in the original order they read off the blocks of $J_{W''} = \operatorname{diag}(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), J_W)$: $l^* J l = 0$, $k^* J k = 0$, $l^* J k = 1$, $i^* J i = J_W$, $i^* J k = i^* J l = 0$, and $\#r = \#n + 2\#p$. So (k, l, i) is a splitting of W' over W along p. In the original order the matrix with column blocks $l_{\mathbb{A}} = x^{-1} l$, $k_{\mathbb{A}} = x^{-1} k$, $i_{\mathbb{A}} = x^{-1} i x_W$ ($x = x_{W''}$) is $\operatorname{diag}(x^{-1}_{\mathbb{H}_p}, 1_n)$. At an infinite place $x_{\mathbb{H}_p} = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ (blocks scalar) is real orthogonal, so its inverse has orthonormal columns. At the finite places $x_{\mathbb{H}_p} = \left(\begin{smallmatrix} 1 & -c \\ c_0 & c\bar{c}_0 \end{smallmatrix}\right)$ has integral entries and determinant $c\bar{c}_0 + cc_0 = c$, a unit ($c_0 + \bar{c}_0 = 1$ by (WD1)), so its inverse $c^{-1} \left(\begin{smallmatrix} c\bar{c}_0 & c \\ -c_0 & 1 \end{smallmatrix}\right)$ is integral. Reindexing rows permutes the entries, so the splitting is base-adapted. □

Lemma 6.3.64. *Let W, W_1 be spaces with points with signs s on n and V, V_1 spaces with points with signs t on a. Let $\gamma \in \mathrm{GL}_n(E)$, $k \in \mathrm{G}(s)$, $\gamma' \in \mathrm{GL}_a(E)$, $k' \in \mathrm{G}(t)$ with $x_W = \gamma^{-1} x_{W_1} k$ and $x_V = \gamma'^{-1} x_{V_1} k'$. There is $z \in \mathbb{C}$ with $|z| = 1$ such that for all automorphic characters α, β of $\mathrm{U}(1)$, all $\varphi \in \mathcal{M}_{W,V} = \mathcal{M}_{W_1,V_1}$, $g \in \mathrm{G}(s)$ and $h \in \mathrm{G}(t)$, with $u = z\,\beta(\det k)\,\alpha(\det k')^{-1}$,*

$$\theta^{W,V}_{\alpha,\beta}(\varphi)(g,h) = u\,\theta^{W_1,V_1}_{\alpha,\beta}(\varphi)(kg, k'h);$$

moreover $h \mapsto \theta^{W,V}_{\alpha,\beta}(\varphi)(g,h)$ is integrable on $[\mathrm{U}(V)]$ if and only if $h \mapsto \theta^{W_1,V_1}_{\alpha,\beta}(\varphi)(kg,h)$ is integrable on $[\mathrm{U}(V_1)]$, and then $I^{W,V}_{\alpha,\beta}(\varphi)(g) = u\, I^{W_1,V_1}_{\alpha,\beta}(\varphi)(kg)$.

Proof. Write $H = \mathrm{Hom}(W,V)$, $H_1 = \mathrm{Hom}(W_1,V_1)$ (signs $t \otimes s$ on $a \times n$, the same model $\mathcal{M}$), $N = a \times n$, and $P = k' \otimes (k^{-1})^{\mathsf{T}} \in \mathrm{G}(t \otimes s)$, the image of (k,k') under the pair map.

Rational vectors. The points are $x_H = x_V \otimes (x_W^{-1})^{\mathsf{T}}$ and $x_{H_1} = x_{V_1} \otimes (x_{W_1}^{-1})^{\mathsf{T}}$, and $(x_W^{-1})^{\mathsf{T}} = (k^{-1} x_{W_1}^{-1} \gamma)^{\mathsf{T}}$, so

$$x_H = (\gamma'^{-1} \otimes \gamma^{\mathsf{T}})\, x_{H_1}\, P.$$

As $\gamma'^{-1} \otimes \gamma^{\mathsf{T}} \in \mathrm{GL}_N(E)$ permutes E^N, the rational vectors satisfy $H(E) = \{x_H^{-1} u : u \in E^N\} = P^{-1} H_1(E)$.

Comparison of the theta functionals. Let $\theta' = \theta_{H_1} \circ \omega_{t\otimes s}(P)$. By (2) of Lemma 4.1.1, $\omega_{t\otimes s}(P)$ is a bounded operator of $\mathcal{M}$ preserving the inner product, so $\langle \omega_{t\otimes s}(P) v^0, \omega_{t\otimes s}(P) v^0 \rangle = \langle v^0, v^0 \rangle = 1$ ((1) of Lemma 4.1.10), and $\omega_{t\otimes s}(P) \rho(y) = \rho(Py)\, \omega_{t\otimes s}(P)$ for $y \in \mathbb{A}_E^N$; and $x \mapsto Px$ is an additive homeomorphism of $\mathbb{A}_E^N$ carrying $H(E)$ onto $H_1(E)$. By (3) of Lemma 6.3.65 for $V = H$, $V' = H_1$, $\tau = P$ and $\Pi = \omega_{t\otimes s}(P)$, there is z_1 with $|z_1| = 1$ and $\theta' = z_1 \theta_H$; so $\theta_H = z\,\theta'$ with $z = z_1^{-1}$, $|z| = 1$.

Kernels. By (WR4), $\omega_{\alpha,\beta}(g,h) = \alpha(\det h)\beta(\det g)^{-1}\omega_{t\otimes s}(h\otimes(g^{-1})^{\mathsf{T}})$, and $P\,(h\otimes(g^{-1})^{\mathsf{T}}) = k'h\otimes((kg)^{-1})^{\mathsf{T}}$. Hence

$$\theta^{W,V}_{\alpha,\beta}(\varphi)(g,h) = z\,\alpha(\det h)\beta(\det g)^{-1}\theta_{H_1}\big(\omega_{t\otimes s}(k'h\otimes((kg)^{-1})^{\mathsf{T}})\varphi\big) = z\,\frac{\beta(\det k)}{\alpha(\det k')}\,\theta^{W_1,V_1}_{\alpha,\beta}(\varphi)(kg,k'h).$$

Integrals. By (1) of Lemma 6.3.63, for the pair (γ',k') (it qualifies: $J_V = (x_V^{-1})^*\varepsilon_t x_V^{-1} = \gamma'^*(x_{V_1}^{-1})^*(k'^{-1})^*\varepsilon_t k'^{-1}x_{V_1}^{-1}\gamma' = \gamma'^* J_{V_1}\gamma'$ as $k'\in \mathrm{G}(t)$), $\Gamma_V = k'^{-1}\Gamma_{V_1}k'$. Left translation by k' preserves the measure of $\mathrm{G}(t)$ and carries Γ_V-orbits to Γ_{V_1}-orbits, so $\Gamma_V h\mapsto \Gamma_{V_1}k'h$ carries the quotient measure of $[\mathrm{U}(V)]$ to that of $[\mathrm{U}(V_1)]$. With the identity of the kernels, $|u|=1$, this gives the equivalence of integrability and the identity of the theta integrals. □

Lemma 6.3.65. *Let ι, ι' be index sets with $\#\iota = \#\iota' = N$, ε signs on ι and ε' signs on ι', $W = \mathbb{A}_E^{\iota}$ and $W' = \mathbb{A}_E^{\iota'}$.*

1. *Let $\sigma\colon W\to W'$ be an additive homeomorphism with*

$$\beta_{\varepsilon'}(\sigma x,\sigma y) = \beta_\varepsilon(x,y),\qquad e_N(\sigma x) = e_N(x)\qquad (x,y\in W),\qquad \sigma_*\mu_N = \mu_N.$$

   Then $\Pi f = f\circ\sigma^{-1}$ defines a linear bijection $\Pi\colon \mathcal{M}_\varepsilon\to\mathcal{M}_{\varepsilon'}$ that preserves the norm and the inner product, with $\Pi v^0 = v^0$ and $\Pi\rho(y) = \rho(\sigma y)\Pi$ for $y\in W$.

2. *In (1), let moreover $k\in\mathrm{G}(\varepsilon)$ and $k'\in\mathrm{G}(\varepsilon')$ satisfy $\sigma(kx) = k'\sigma(x)$ for all $x\in W$. Then*

$$\Pi\,\omega_\varepsilon(k) = \frac{\Delta_\varepsilon(k)}{\Delta_{\varepsilon'}(k')}\,\omega_{\varepsilon'}(k')\,\Pi.$$

3. *Let V, V' be spaces with points with signs ε and ε', $\tau\colon W\to W'$ an additive homeomorphism with $\tau(V(E)) = V'(E)$, and $\Pi\colon\mathcal{M}_\varepsilon\to\mathcal{M}_{\varepsilon'}$ a bounded linear map with $\langle\Pi v^0,\Pi v^0\rangle = 1$ and $\Pi\rho(y) = \rho(\tau y)\Pi$ for $y\in W$. Then $\theta'\circ\Pi$ is a theta functional of V for every theta functional θ' of V', and there is $z\in\mathbb{C}$ with $|z|=1$ such that $\theta_{V'}(\Pi f) = z\,\theta_V(f)$ for all $f\in\mathcal{M}_\varepsilon$.*

4. *Let δ be signs on ι and $k\in\mathrm{G}(\delta)$. Then $(k^{-1})^{\mathsf{T}}\in\mathrm{G}(-\delta)$ and*

$$\Delta_{-\delta}\big((k^{-1})^{\mathsf{T}}\big) = \upsilon(\det k)\,\Delta_\delta(k).$$

Proof. **(1).** Let $f\in\mathcal{M}_\varepsilon$. As σ is a homeomorphism, Πf is measurable, and $\sigma_*\mu_N = \mu_N$ gives $\int|\Pi f|\,d\mu_N = \int|f|\,d\mu_N$ and $\int\overline{\Pi f}\,\Pi g\,d\mu_N = \int\bar f g\,d\mu_N$ for $f,g\in\mathcal{M}_\varepsilon$. For $x'\in W'$, the reproducing identity of (WD7) at $\sigma^{-1}x'$, the substitution $y=\sigma^{-1}y'$ (which preserves μ_N) and the two identities of the hypothesis, applied to $\sigma^{-1}y'$, $\sigma^{-1}x'$ and to $\sigma^{-1}x'-\sigma^{-1}y' = \sigma^{-1}(x'-y')$ (additivity), give

$$(\Pi f)(x') = \int f(y)\,\beta_\varepsilon(y,\sigma^{-1}x')\,e_N(\sigma^{-1}x'-y)\,d\mu_N(y) = \int(\Pi f)(y')\,\beta_{\varepsilon'}(y',x')\,e_N(x'-y')\,d\mu_N(y'),$$

so $\Pi f\in\mathcal{M}_{\varepsilon'}$. The map $\sigma^{-1}\colon W'\to W$ is an additive homeomorphism satisfying the same three hypotheses with the roles of ε and ε' exchanged (apply them to $\sigma^{-1}x'$, $\sigma^{-1}y'$; and $(\sigma^{-1})_*\mu_N = \mu_N$), so $f'\mapsto f'\circ\sigma$ maps $\mathcal{M}_{\varepsilon'}$ into $\mathcal{M}_\varepsilon$; it is inverse to Π, and Π is a linear bijection preserving norm and inner product. Next, $\Pi v^0 = e_N\circ\sigma^{-1} = e_N = v^0$. Finally, for $y\in W$,

$$(\Pi\rho(y)f)(x') = \beta_\varepsilon(y,\sigma^{-1}x')\,f(\sigma^{-1}x'-y) = \beta_{\varepsilon'}(\sigma y,x')\,(\Pi f)(x'-\sigma y) = (\rho(\sigma y)\Pi f)(x').$$

(2). By (2) of Lemma 6.2.13, $\Delta_{\varepsilon'}(k') \neq 0$. For $f \in \mathcal{M}_\varepsilon$ and $x' \in W'$, (WR3), the substitution $y = \sigma^{-1}y'$, the identity $k\sigma^{-1}y' = \sigma^{-1}(k'y')$ (from $\sigma(kx) = k'\sigma(x)$ with $x = \sigma^{-1}y'$) and the hypotheses of (1) give

$$\begin{aligned}(\Pi\omega_\varepsilon(k)f)(x') &= \Delta_\varepsilon(k)\int(\Pi f)(y')\,\beta_\varepsilon(\sigma^{-1}(k'y'),\sigma^{-1}x')\,e_N(\sigma^{-1}(x'-k'y'))\,d\mu_N(y')\\ &= \Delta_\varepsilon(k)\int(\Pi f)(y')\,\beta_{\varepsilon'}(k'y',x')\,e_N(x'-k'y')\,d\mu_N(y'),\end{aligned}$$

which is $\Delta_\varepsilon(k)\Delta_{\varepsilon'}(k')^{-1}(\omega_{\varepsilon'}(k')\Pi f)(x')$ by (WR3).

(3). Let θ' be a theta functional of V' and $\theta'' = \theta' \circ \Pi$, a bounded linear functional on $\mathcal{M}_\varepsilon$. For $\gamma \in V(E)$, $\tau\gamma \in V'(E)$, so $\theta''(\rho(\gamma)f) = \theta'(\rho(\tau\gamma)\Pi f) = \theta''(f)$. The quotients $W/V(E)$ and $W'/V'(E)$ are compact (Lemma 4.1.10, (2)). As τ is additive, bijective and $\tau(V(E)) = V'(E)$ (so $\tau^{-1}(V'(E)) = V(E)$), it induces a continuous bijective homomorphism $\bar\tau\colon W/V(E) \to W'/V'(E)$; a continuous bijection of a compact space onto a Hausdorff space is a homeomorphism, and the image under $\bar\tau$ of the invariant probability measure of $W/V(E)$ is an invariant probability measure of $W'/V'(E)$, hence its invariant probability measure. With $\Pi\rho(x)v^0 = \rho(\tau x)\Pi v^0$ and (2) of Lemma 4.1.10 for θ' and $u = v = \Pi v^0$,

$$\begin{aligned}\int_{W/V(E)}\left|\theta''(\rho(x)v^0)\right|^2 dx &= \int_{W/V(E)}\left|\theta'(\rho(\tau x)\Pi v^0)\right|^2 dx\\ &= \int_{W'/V'(E)}\left|\theta'(\rho(y)\Pi v^0)\right|^2 dy = \langle \Pi v^0, \Pi v^0\rangle = 1.\end{aligned}$$

So θ'' is a theta functional of V (WD9). By Lemma 4.1.17, $\theta_{V'}$ is a theta functional of V' and θ_V one of V; so $\theta_{V'} \circ \Pi$ and θ_V are theta functionals of V, and by Lemma 4.1.7 they differ by a scalar z of absolute value one.

(4). Write $k^{-\mathsf{T}} = (k^{-1})^{\mathsf{T}}$, $I_- = \{i \in \iota : \delta_i = -1\}$ and $I_+ = \iota \smallsetminus I_-$. *Membership.* As $\varepsilon_{-\delta} = -\varepsilon_\delta$ at the infinite places and 1 at the finite places, $\mathrm{G}(-\delta) = \mathrm{G}(\delta)$. At $v \in \Phi$, with $D = \mathrm{diag}(\delta)$ ($D = D^{-1}$ real), $k_v^* D k_v = D$ gives $k_v^{-1}D(k_v^{-1})^* = D$, and conjugating entrywise, $\overline{k_v^{-1}}\,D\,(k_v^{-1})^{\mathsf{T}} = D$, that is $(k_v^{-\mathsf{T}})^* D k_v^{-\mathsf{T}} = D$. At the finite places $k_f^* k_f = 1$ gives $k_f^{-\mathsf{T}} = \overline{k_f}$ and $\overline{k_f}^{\,*}\,\overline{k_f} = \overline{k_f^* k_f} = 1$. So $k^{-\mathsf{T}} \in \mathrm{G}(\delta) = \mathrm{G}(-\delta)$.

Infinite places. The indices of sign -1 for $-\delta$ are those of I_+. By Jacobi's theorem on the complementary minors of an inverse matrix, $\det((k_v^{-1})_{I_+}) = \det((k_v)_{I_-})/\det k_v$ (a determinant over the empty set being 1), and $(k_v^{-\mathsf{T}})_{I_+} = \big((k_v^{-1})_{I_+}\big)^{\mathsf{T}}$ has the same determinant. As $|\det k_v| = 1$ ($k_v^* D k_v = D$), $\overline{\det(k_v)}^{-1} = \det k_v$, so

$$\prod_{v\in\Phi}\overline{\det\big((k_v^{-\mathsf{T}})_{I_+}\big)} = \prod_{v\in\Phi}\det k_v \cdot \prod_{v\in\Phi}\overline{\det\big((k_v)_{I_-}\big)}:$$

the archimedean factor of $\Delta_{-\delta}(k^{-\mathsf{T}})$ is $\prod_v \det k_v$ times that of $\Delta_\delta(k)$ (WR3).

Finite places. Let $L = \widehat{\mathcal{O}}_E^{\,\iota}$ and $u \in \mathrm{GL}_N(\mathbb{A}_{E,f})$ with $uL = L \cap k_f^{-1}L$. Conjugation is a ring automorphism of $\mathbb{A}_{E,f}$ preserving $\widehat{\mathcal{O}}_E$, so $\bar u L = L \cap \overline{k_f}^{-1}L$, and $\bar u$ is an admissible choice in (WR3) for $k_f^{-\mathsf{T}} = \overline{k_f}$. Conjugation permutes the finite places, carrying the place w to $\bar w$ of the same norm, so $|\det\bar u| = |\det u|$; and $\upsilon(\bar x) = \upsilon(x)^{-1} = \overline{\upsilon(x)}$ (WD3). Hence the finite factor of $\Delta_{-\delta}(k^{-\mathsf{T}})$ is $\overline{\Delta(k_f)}$, where $\Delta(k_f) = \upsilon_f(\det u)|\det u|^{-1/2}$ is the finite factor of $\Delta_\delta(k)$ and $\upsilon_f(y) = \upsilon((1,y))$ ($(1,y)$ the idele with infinite components 1). This Δ is the one of Lemma 4.1.13 for $\jmath = \jmath$, the Weil idele α and υ_f, whose hypotheses hold: $\bar\jmath = -\jmath$ and $c_0 \in \mathcal{O}_E$ by (WD1), $\widehat{\mathcal{O}}_E$ is self-dual for $(z,w) \mapsto \psi_{E,f}(\jmath\alpha\bar\alpha\bar z w)$ by Lemma 4.1.3, and υ_f is continuous, of absolute value

1, trivial on $\widehat{\mathcal{O}}_E^\times$ and satisfies $\upsilon_f(\bar t) = \upsilon_f(t)^{-1}$ by (WD3). As $k_f^* k_f = 1$, Lemma 4.1.13 gives $\overline{\Delta(k_f)} = \upsilon_f(\det k_f)\,\Delta(k_f)$.

Conclusion. The idele $\det k$ is the product of $((\det k_v)_v, 1)$, whose finite components are units, and of $(1, \det k_f)$; by (WD3), $\upsilon(\det k) = \prod_v \det k_v \cdot \upsilon_f(\det k_f)$, as $|\det k_v| = 1$. Multiplying the two factors, $\Delta_{-\delta}(k^{-\mathsf T}) = \upsilon(\det k)\,\Delta_\delta(k)$. □

Lemma 6.3.66. *Call a decomposition $Y = W \oplus \mathbb{H}$ of a space Y, with $\mathbb{H}$ spanned by isotropic vectors l, l' with $\langle l, l'\rangle = 1$,* adapted *if the frames are orthonormal at the infinite places and integral at the finite places. Let $\mu\colon \Phi \to \mathbb{Z}_{>0}$; ν runs over the characters tame of weight μ for some $(\mathfrak m, \chi)$.*

1. *Let $Y = W \oplus \mathbb{H}$ be adapted with $\operatorname{rk} W$ even. For every Schwartz vector v for (W, W) there are a Schwartz vector v' for (Y, Y) and $C > 0$ with $r_Y(\nu, v') = C\, r_W(\nu, v)$ for all ν.*
2. *Let $Y = W \oplus \mathbb{H}$ be adapted with $\operatorname{rk} Y$ even, and let Y_1 be a space with the signs of Y. Then $r_{Y_1}(\nu, v) = r_Y(\nu, v)$ for every Schwartz vector v for (Y, Y) and all ν.*
3. *For every space W_1 and every sign vector s' of length $\operatorname{rk} W_1 + 2$ with $\operatorname{neg}(s') = \operatorname{neg}(W_1) + 1$ there are a space Y with signs s' and an adapted decomposition $Y = W_1 \oplus \mathbb{H}$.*

Proof. **Step 0: the equal-rank integrals as large sides.** An adapted decomposition $Y = W \oplus \mathbb{H}$ is a base-adapted splitting (k, l, i) of Y over W along a one-element set: $\mathbb{H}$ is spanned by the isotropic vectors l, k with $l^* J_Y k = 1$, and the frames are orthonormal at the infinite and integral at the finite places. Let ν be tame of weight μ for $(\mathfrak m, \chi)$; then $\mathfrak c = (\nu, \Sigma(\mathfrak m))$ is a split tame character of weight μ ($\Sigma(\mathfrak m)$ contains the conjugate of none of its places, $\mu > 0$). For $u \in \mathcal{O}_\Sigma^\times$ the component of $u/\bar u$ at $\bar w$ ($w \in \Sigma$) is the inverse of the conjugate of u_w, and $\psi_{\bar w} = 1$, so by the definition of the tame part (Definition 3.1.2) $\chi_{\mathfrak c}(u) = \nu(u/\bar u) = \prod_{w\in\Sigma} \psi_w(u_w \bmod w) = \chi(u \bmod \mathfrak m)$; hence Φ_χ of (XD8) is the standard column datum Ψ_1^{st} of (XD10), and, for every space Z of even rank, $I_Z(v) = \theta_{\mathfrak c}^{\mathrm{lg}}(Z, Z; v, \Psi_1^{\mathrm{st}})$ (both with the character $\nu^{-1}\upsilon^{\operatorname{rk} Z/2}$). For the empty index set $\varnothing$ and the identity $\sigma_Z\colon \varnothing \sqcup n_Z \to n_Z$ of the index set of Z, the standard row datum at the empty matrix is $\Phi^{\mathrm{st}}_{\sigma_Z, 0} = \Psi_1^{\mathrm{st}}$.

(1) Let n, n' be the index sets of W, Y, $\#n = 2d$, $\#n' = 2d+2$, and A the splitting. By Proposition 6.2.2 for $W' = V' = Y$, $W = V = W$, the splittings $A = X = A$, the index set $\varnothing$ with σ_W, σ_Y and the weight μ ($\#n$ even; $e = 2d - d = d \geq 0$, $z_e = d+1$, the sign $+$; the ranks of V and W agree), there is $C_1 > 0$ such that every split tame character $\mathfrak c$ of weight μ has $\gamma_{\mathfrak c}$ with $|\gamma_{\mathfrak c}| = C_1 L_{\mathfrak c}^\sharp(d+1)$ and $\mathrm{CT}^{\mathrm{pl}}_{A,\sigma_Y}(v', 0)(1,1) = \gamma_{\mathfrak c}\theta_{\mathfrak c}^{\mathrm{lg}}(W, W; \mathrm{Cone}^+_{A,A,\mu}(d+1; v'), \Psi_1^{\mathrm{st}})(1)$ for every Schwartz vector v' for (Y, Y). By Lemma 6.3.55 (normalised form, $z = d+1 > 0$, the index set $\varnothing$, σ_W) there is a Schwartz vector v' for (Y, Y) with $\theta_{\mathfrak c}^{\mathrm{lg}}(W, W; \mathrm{Cone}^+_{A,A,\mu}(d+1; v'), \Psi_1^{\mathrm{st}}) = \theta_{\mathfrak c}^{\mathrm{lg}}(W, W; v, \Psi_1^{\mathrm{st}})$ for all $\mathfrak c$. By Proposition 6.3.17 (Y has even rank and the base-adapted splitting A) the function $F_{\mathfrak c} = \theta_{\mathfrak c}^{\mathrm{lg}}(Y, Y; v', \Psi_1^{\mathrm{st}})$ is constant; the plane datum at the empty matrix is formed with Ψ_1^{st}, so $\mathrm{CT}^{\mathrm{pl}}_{A,\sigma_Y}(v', 0)(1,1)$, the mean of $F_{\mathfrak c}$ over a compact quotient with its probability measure, is $F_{\mathfrak c}(1)$. Hence $I_Y(v')(1) = \gamma_{\mathfrak c} I_W(v)(1)$ by Step 0, and $|I_Y(v')(1)|^2 = C_1^2 L_{\mathfrak c}^\sharp(d+1)^2 |I_W(v)(1)|^2$. With $S = \Sigma(\mathfrak m)$,

$$\frac{L_*(S, \nu; d+1)}{L_*(S, \nu; d)} = \Big(\prod_{w\in S} \mathrm{N}w\Big)^{(d+1)^2 - d^2} \big|L^S(d+1, \tilde\nu)\big|^2 = \big|L^S(d+1, \tilde\nu)\big|^2 \prod_{w\in S} (\mathrm{N}w)^{2(d+1)-1} = L_{\mathfrak c}^\sharp(d+1)^2,$$

so $r_Y(\nu, v') = C_1^2 L_{\mathfrak c}^\sharp(d+1)^2 |I_W(v)(1)|^2 / L_*(S, \nu; d+1) = C_1^2\, r_W(\nu, v)$, and $C = C_1^2 > 0$ does not depend on ν.

(2) Let ν and $\mathfrak{c}$ be as in Step 0. By (1) of Lemma 6.3.63 there are γ_0 and k with $x_{Y_1} = \gamma_0^{-1} x_Y k$; by Lemma 6.3.64, applied with $(W,V) = (Y_1, Y_1)$, $(W_1, V_1) = (Y, Y)$ and the pair (γ_0, k) on both sides, the integral defining $I_{Y_1}(v)(1)$ converges exactly when that defining $I_Y(v)(k)$ does, and then $I_{Y_1}(v)(1) = u\, I_Y(v)(k)$ with $|u| = 1$. By Proposition 6.3.17 (Y has even rank and the base-adapted splitting $Y = W \oplus \mathbb{H}$, Step 0), $I_Y(v)(k) = I_Y(v)(1)$. So $|I_{Y_1}(v)(1)| = |I_Y(v)(1)|$ and $r_{Y_1}(\nu, v) = r_Y(\nu, v)$.

(3) This is (2) of Lemma 6.3.63 for a one-element index set p: a splitting (k, l, i) of Y over W_1 along p is a decomposition $Y = iW_1 \oplus \mathbb{H}$ with $\mathbb{H}$ spanned by the isotropic vectors l, k, $l^* J_Y k = 1$, and base-adapted means adapted. □

Proposition 6.3.67. *Let n, m, l, b, p, a be finite index sets with bijections $j\colon m \sqcup l \to n$, $f\colon p \sqcup l \to b$, $e\colon p \sqcup a \to m$, $\sigma\colon b \sqcup a \to n$. Let W be a space with index set n and signs sgn_j, V' a space with index set b and signs sgn_f, and I_a the standard definite space. Let $d \geq 0$ with $|a| = |b| + 2d$, assume $2|l| + 2 \leq |n|$, and let $\iota = (\iota_\tau)_{\tau \in \Phi}$ be finite sets. Then there is $C \in \mathbb{R}$ (possibly 0) with the following property. For every modulus $\mathfrak{m}$, every character χ of $R_{\mathfrak{m}}$ and every automorphic character α of $\mathrm{U}(1)$ tame of weight $\tau \mapsto |l| - |\iota_\tau|$ for $(\mathfrak{m}, \chi)$ there is $c \in \mathbb{C}$ with*

$$L_*(\mathfrak{m}, \alpha^{-1} \upsilon^{|b|+d}; d) \neq 0, \qquad L_*(\mathfrak{m}, \alpha^{-1} \upsilon^{|b|+d}; d)\, |c|^2 = C,$$

such that for every family $\eta = (\eta_{\tau k})_{\tau \in \Phi,\, k \in \iota_\tau}$ of complex $p \times m$ matrices, all Φ, Ψ matched for (σ, χ^{-1}) modulo $\mathfrak{m}$, every automorphic character β of $\mathrm{U}(1)$ and every $g \in \mathrm{U}(W)(\mathbb{A}_F)$,

$$c \cdot \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} \theta^{W, I_a}_{\alpha, \beta; \mathfrak{m}}[\Phi, \Delta^j(e; \eta)](g, h) = \int_{[\mathrm{U}(V')]} \theta^{V', W}_{\alpha/\beta,\, \alpha; \mathfrak{m}}[\Psi, Q_{f,j}(\eta)](x, g)\, dx.$$

Proof. The set a is nonempty: otherwise $|b| = |a| - 2d \leq 0$, so $b = \emptyset$, $d = 0$ and $|n| = |a| + |b| = 0 < 2|l| + 2$. Let $c_a > 0$ be the total mass of the measure of $G_\infty(s_a^+)$.

Step 1: the archimedean matching. Write $\gamma_0(\tau) = |l| - |\iota_\tau|$, $\mathcal{H}_1 = \mathcal{F}_{W, I_a}$ for the Fock model of the pair (W, I_a), $\omega^\infty = \Omega_{\gamma_0, 0}$ for its real Weil representation of $G_\infty(\mathrm{sgn}_j) \times G_\infty(s_a^+)$, and v_P for the Fock vector of a polynomial P in either pair. By Lemma 6.3.68, for the index sets, the bijections j, f, e, the spaces W, V' and the sets ι of the statement, there is $Q_0 \in \mathbb{C}[x]_{n \times b}$ such that every $v \in \mathcal{H}_1$ has a number $\varkappa(v)$ with: for every family $(\lambda_{g'})_{g' \in G_\infty(\mathrm{sgn}_j)}$ of continuous linear forms on $\mathcal{H}_1$ and every linear form λ' on $\mathbb{C}[x]_{n \times b}$ such that (H1) $\lambda_{g'}(\omega^\infty(x_1, x_2) w) = \lambda_{g' x_1}(w)$ for all $g', x_1 \in G_\infty(\mathrm{sgn}_j)$, $x_2 \in G_\infty(s_a^+)$ and $w \in \mathcal{H}_1$, and (H2) $\lambda_x(v) = \lambda'(Q)$ whenever $\Omega_{\gamma_0, 0}(1, x) v_{Q_0} = v_Q$ in the Fock model $\mathcal{F}_{V', W}$ of the pair (V', W), one has $\varkappa(v)\, \lambda_1(v_{\Delta^j(e;\eta)}) = \lambda'(Q_{f,j}(\eta))$ for every η.

Step 2: the exchange identity and the constant. Apply Theorem 6.3.2 to W, $V = I_a$, V', σ, $e = d$, the weight $\mu(\tau) = |\iota_\tau| - |l| + |b| + d$ and the Schwartz vector v_{Q_0} ((XD3): Fock vectors of polynomials are Schwartz vectors). Its hypotheses hold:

It gives $C_0 > 0$ and a Schwartz vector x_0 for (W, I_a). Put $C = c_a^2 |\varkappa(x_0)|^2 / C_0$.

Let $\mathfrak{m}, \chi, \alpha$ be as in the statement and $\nu = \alpha^{-1} \upsilon^{|b|+d}$; it is tame of weight μ for $(\mathfrak{m}, \chi^{-1})$, and $\nu^{-1} \upsilon^{|n|/2} = \alpha$ because $|n| = |a| + |b| = 2(|b| + d)$. Indeed, by (WD3) of Definition 3.2.12, υ restricted to $\mathrm{U}(1)(\mathbb{A})$ has weight 1 and trivial tame part; if α has the tame part ψ ($\psi_w \neq 1$ exactly for $w \in \Sigma(\mathfrak{m})$, and $\chi(u) = \prod_w \psi_w(u \bmod w)$ on units), then ν has weight $-\gamma_0 + |b| + d = \mu$ and tame part ψ^{-1}, which gives χ^{-1} on units. Moreover the pair of characters $(1, \alpha)$ on (I_a, V') satisfies the compatibility $1[|a|]\, \alpha[|b|] = \tilde{\nu}^{-1}$ of Theorem 6.3.2: the base change of υ on $\mathrm{U}(1)(\mathbb{A})$ is $x \mapsto \upsilon(x/\bar{x}) = \upsilon(x)^2$, as $\upsilon(\bar{x}) = \upsilon(x)^{-1}$ (WD3), so $\tilde{\nu}^{-1} = \tilde{\alpha}\, \upsilon^{-2(|b|+d)}$, while

$1[|a|]\,\alpha[|b|] = \upsilon^{-|a|}\tilde{\alpha}\,\upsilon^{-|b|} = \tilde{\alpha}\,\upsilon^{-2(|b|+d)}$ since $|a|+|b| = 2(|b|+d)$. Theorem 6.3.2 gives c_0 with $|c_0|^2 = C_0 L_*(\mathfrak{m},\nu;d) \neq 0$ and, for all Φ,Ψ matched for (σ,χ^{-1}) and all $g' \in \mathrm{U}(W)(\mathbb{A}_F)$,

$$I_{\alpha,1}^{W,I_a}\big(\mathrm{mod}_{\mathfrak{m}}(\Phi,x_0)\big)(g') = c_0 \int_{[\mathrm{U}(V')]} \theta_{\alpha,\alpha;\mathfrak{m}}^{V',W}[\Psi,Q_0](x,g')\,dx. \tag{6.52}$$

Put $c = c_a\varkappa(x_0)/c_0$. As $C_0 > 0$ and $|c_0|^2 \neq 0$, $L_*(\mathfrak{m},\nu;d) \neq 0$, and $L_*(\mathfrak{m},\nu;d)\,|c|^2 = c_a^2|\varkappa(x_0)|^2 L_*(\mathfrak{m},\nu;d)/(C_0 L_*(\mathfrak{m},\nu;d)) = C$, with $\nu = \alpha^{-1}\upsilon^{|b|+d}$.

Step 3: the identity for $\beta = 1$. Fix η, matched Φ,Ψ and g. For $g' \in G_\infty(\mathrm{sgn}_j)$ write $(g',1)$ for the adelic point with infinite component g' and finite component 1, and for $w \in \mathcal{H}_1$ and $Q \in \mathbb{C}[x]_{n\times b}$ let

$$\lambda_{g'}(w) = I_{\alpha,1}^{W,I_a}\big(\mathrm{mod}_{\mathfrak{m}}(\Phi,w)\big)\big(g(g',1)\big), \qquad \lambda'(Q) = c_0\int_{[\mathrm{U}(V')]} \theta_{\alpha,\alpha;\mathfrak{m}}^{V',W}[\Psi,Q](x,g)\,dx.$$

(a) *The forms.* By (3) of Lemma 6.3.71 for the pair (W,I_a) ($a \neq \emptyset$) and the characters α, 1 of weights γ_0, 0, each $\lambda_{g'}$ is a continuous linear form on $\mathcal{H}_1$, and $\lambda_{g'}(\omega^\infty(x_1,x_2)w) = \lambda_{g'x_1}(w)$, because $g(g',1)(x_1,1) = g(g'x_1,1)$; this is (H1). The function Ψ vanishes off the matrices k_b formed by the columns b of some $k \in \mathrm{GL}_n(R_{\mathfrak{m}})$, and k_b has the left inverse formed by the rows b of k^{-1}; so by (2) of Lemma 6.3.71 for the pair (V',W) the integral defining $\lambda'(Q)$ converges absolutely and λ' is a linear form on $\mathbb{C}[x]_{n\times b}$.

(b) *(H2) for $v = x_0$.* Let $g' \in G_\infty(\mathrm{sgn}_j)$ and Q with $\Omega_{\gamma_0,0}(1,g')v_{Q_0} = v_Q$. The characters α,α of the pair (V',W) have weights γ_0,γ_0, and $\Omega_{\gamma_0,\gamma_0}(1,g') = \Omega_{\gamma_0,0}(1,g')$, the factor $\prod_v \det(x_v)^{-\gamma'(v)}$ of (WR5) being 1 at $x = 1$. So (1) of Lemma 6.3.71 for (V',W) gives $\theta_{\alpha,\alpha;\mathfrak{m}}^{V',W}[\Psi,Q_0](x,g(g',1)) = \theta_{\alpha,\alpha;\mathfrak{m}}^{V',W}[\Psi,Q](x,g)$ for every x, and (6.52) at the point $g(g',1)$ gives $\lambda_{g'}(x_0) = \lambda'(Q)$.

(c) *The class sum.* By (5) of Lemma 6.3.71 (W has the signs sgn_j and $\gamma_0(\tau) = |l| - |\iota_\tau|$), $\Omega_{\gamma_0,0}(1,y)v_{\Delta^j(e;\eta)} = v_{\Delta^j(e;\eta)}$ for all $y \in G_\infty(s_a^+)$. For $u \in \mathrm{GL}_a(R_{\mathfrak{m}})$ let $D_u \in \mathrm{GL}_n(R_{\mathfrak{m}})$ act as the identity on the basis vectors $\sigma(b)$ and as u on $\sigma(a)$. For $k \in \mathrm{GL}_n(R_{\mathfrak{m}})$, the columns b of kD_u are those of k, the rows a of $(kD_u)^{-1} = D_u^{-1}k^{-1}$ form $u^{-1}(k^{-1})_a$, and $\det(kD_u) = \det k\det u$; so, Φ and Ψ being matched for (σ,χ^{-1}), $\Phi(u^{-1}(k^{-1})_a) = \chi(\det k\det u)^{-1}\Psi(k_b) = \chi(\det u)^{-1}\Phi((k^{-1})_a)$, while for X not of the form $(k^{-1})_a$ neither is $u^{-1}X$ (as $X = u\,(k'^{-1})_a = ((k'D_u^{-1})^{-1})_a$ otherwise), and both values are 0. Hence $\Phi(u^{-1}X) = \chi(\det u)^{-1}\Phi(X)$ for all X. For $k \in \mathbf{K}_a$ the determinant $\det k \in \mathrm{U}(1)(\mathbb{A})$ has integral finite components and infinite components 1, so $\alpha(\det k) = \prod_w \psi_w(\det k \bmod w) = \chi(\det \bar{k})$ (Definition 3.1.1 and (XD4)), $\bar{k}$ the reduction modulo $\mathfrak{m}$; taking $u = \bar{k}$, $\alpha(\det k)\Phi(\bar{k}^{-1}X) = \Phi(X)$. So (4) of Lemma 6.3.71 applies to $P = \Delta^j(e;\eta)$ and gives

$$\lambda_1(v_{\Delta^j(e;\eta)}) = I_{\alpha,1}^{W,I_a}\big(\mathrm{mod}_{\mathfrak{m}}(\Phi,v_{\Delta^j(e;\eta)})\big)(g) = c_a \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} \theta_{\alpha,1;\mathfrak{m}}^{W,I_a}\big[\Phi,\Delta^j(e;\eta)\big](g,h).$$

(d) *Conclusion.* By (a) and (b), Step 1 applies to $v = x_0$ and gives $\varkappa(x_0)\lambda_1(v_{\Delta^j(e;\eta)}) = \lambda'(Q_{f,j}(\eta))$. Inserting (c) and dividing by $c_0 \neq 0$,

$$c \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} \theta_{\alpha,1;\mathfrak{m}}^{W,I_a}\big[\Phi,\Delta^j(e;\eta)\big](g,h) = \int_{[\mathrm{U}(V')]} \theta_{\alpha,\alpha;\mathfrak{m}}^{V',W}\big[\Psi,Q_{f,j}(\eta)\big](x,g)\,dx.$$

Step 4: the character β. By (XD2) of Definition 3.2.22, (g,h) acts on the datum by $\alpha(\det h)\beta(\det g)^{-1}$ times the Weil operator, and the theta functional is linear; hence

$$\theta_{\alpha\varrho,\beta\varrho'}^{S,T}(\varphi)(g,h) = \varrho(\det h)\varrho'(\det g)^{-1}\theta_{\alpha,\beta}^{S,T}(\varphi)(g,h)$$

for automorphic characters ϱ, ϱ' of U(1) ((WR4) of Definition 3.2.18). So replacing the second character 1 on the left (it twists U(W)) by β, and the first character α on the right (it twists U(W) there) by α/β, multiplies both sides by $\beta(\det g)^{-1}$. This gives the statement. □

Lemma 6.3.68. *Let n, m, l, b, p, a be finite index sets with bijections $j\colon m \sqcup l \to n$, $f\colon p \sqcup l \to b$, $e\colon p \sqcup a \to m$; let W be a space with signs sgn_j on n and V' a space with signs sgn_f on b; let $\iota = (\iota_\tau)_{\tau\in\Phi}$ be finite sets and $\gamma_0(\tau) = \#l - \#\iota_\tau$. Then there is $Q_0 \in \mathbb{C}[x]_{n\times b}$ such that every $v \in \mathcal{F}_{W,I_a}$ has a number $\varkappa(v) \in \mathbb{C}$ with the following property. Let $(\lambda_g)_{g\in G_\infty(\mathrm{sgn}_j)}$ be continuous linear forms on $\mathcal{F}_{W,I_a}$ and λ' a linear form on $\mathbb{C}[x]_{n\times b}$ such that*

$$\lambda_g(\Omega_{\gamma_0,0}(x_1,x_2)w) = \lambda_{gx_1}(w) \qquad (g, x_1 \in G_\infty(\mathrm{sgn}_j),\ x_2 \in G_\infty(s_a^+),\ w \in \mathcal{F}_{W,I_a}),$$

and $\lambda_x(v) = \lambda'(Q)$ whenever $x \in G_\infty(\mathrm{sgn}_j)$ and $Q \in \mathbb{C}[x]_{n\times b}$ satisfy $\Omega_{\gamma_0,0}(1,x)v_{Q_0} = v_Q$ in $\mathcal{F}_{V',W}$. Then for every family $\eta = (\eta_{\tau k})_{\tau\in\Phi,\, k\in\iota_\tau}$ of complex $p\times m$ matrices

$$\varkappa(v)\,\lambda_1(v_{\Delta^j(e;\eta)}) = \lambda'(Q_{f,j}(\eta)).$$

Proof. Notation. For $k \in G_\infty(s_m^+)$ let $k^\sharp \in G_\infty(\mathrm{sgn}_j)$ act as k on $j(m)$ and as the identity on $j(l)$, so $(kk')^\sharp = k^\sharp k'^\sharp$; put $\chi_m(k) = \prod_\tau \det(k_\tau)^{\#\iota_\tau}$. For a family η and k write ηk^{T} for the family $(\eta_{\tau i}k_\tau^{\mathsf{T}})$, similarly $\eta(k^{-1})^{\mathsf{T}}$. Let η^0, $t(z)$, $\Omega^\circ(z)$ be as in Lemma 6.3.69 for $n, m, l, p, a, j, e, W, \iota$. Put $Q_0 = Q_{f,j}(\eta^0)$, fix v and let $\varkappa(v) = \kappa(v)$ be the number of Lemma 6.3.69. Let (λ_g), λ' satisfy the two hypotheses; call them the equivariance and the matching hypothesis.

Step 1: two laws of the real Weil representation. All elements below are block diagonal for the signs, so (5) of Lemma 4.1.1 applies. For $k \in G_\infty(s_m^+)$, $u \in G_\infty(s_a^+)$ and every family η,

$$\Omega_{\gamma_0,0}(k^\sharp, u)\, v_{\Delta^j(e;\eta(k^{-1})^{\mathsf{T}})} = \chi_m(k)^{-1}\, v_{\Delta^j(e;\eta)} \quad \text{in } \mathcal{F}_{W,I_a}, \tag{6.53}$$

$$\Omega_{\gamma_0,0}(1, k^\sharp)\, v_{Q_{f,j}(\eta)} = \chi_m(k)^{-1}\, v_{Q_{f,j}(\eta k^{\mathsf{T}})} \quad \text{in } \mathcal{F}_{V',W}. \tag{6.54}$$

Step 2: the identity for unitary translates of η^0. Let $k \in G_\infty(s_m^+)$ and $\xi = \eta^0 k^{\mathsf{T}}$. By (6.53) for η^0, k^{-1}, $u = 1$, and since $\Omega_{\gamma_0,0}$ is a representation ((2) of Lemma 4.1.1), $v_{\Delta^j(e;\xi)} = \chi_m(k)\,\Omega_{\gamma_0,0}(k^\sharp, 1)v_{\Delta^j(e;\eta^0)}$, so by equivariance

$$\lambda_1(v_{\Delta^j(e;\xi)}) = \ell(v_{\Delta^j(e;\eta^0)}), \qquad \ell = \chi_m(k)\,\lambda_{k^\sharp}. \tag{6.55}$$

By equivariance with $x_1 = 1$, ℓ is a continuous linear form invariant under $\Omega_{\gamma_0,0}(1,u)$, $u \in G_\infty(s_a^+)$, so Lemma 6.3.69 gives $\varkappa(v)\,\ell(v_{\Delta^j(e;\eta^0)}) = \frac{1}{2\pi}\int_0^{2\pi} \ell(\Omega^\circ(e^{i\theta})v)\,d\theta$. Let $|z| = 1$. By equivariance, $\ell(\Omega^\circ(z)v) = z^{-\#a\sum_\tau \#\iota_\tau}\chi_m(k)\,\lambda_{(kt(z))^\sharp}(v)$. Since $t(z)$ is diagonal with entry z^{-2} at the places $e(p)$ where the rows of η^0 have their 1, $\eta^0(kt(z))^{\mathsf{T}} = \eta^0 t(z)k^{\mathsf{T}} = z^{-2}\xi$, and each factor $\det(\xi_{\tau i}y_\tau^{++})$ of $Q_{f,j}$ is a $p\times p$ determinant, so $Q_{f,j}(z^{-2}\xi) = z^{-2\#p\sum_\tau \#\iota_\tau}Q_{f,j}(\xi)$. By (6.54) for $kt(z)$, $\Omega_{\gamma_0,0}(1,(kt(z))^\sharp)v_{Q_0} = v_{Q'}$ with $Q' = \chi_m(kt(z))^{-1}z^{-2\#p\sum \#\iota_\tau}Q_{f,j}(\xi)$, and the matching hypothesis gives $\lambda_{(kt(z))^\sharp}(v) = \lambda'(Q')$. As $\chi_m(t(z)) = z^{-(2\#p+\#a)\sum_\tau \#\iota_\tau}$,

$$\ell(\Omega^\circ(z)v) = z^{-\#a\sum \#\iota_\tau}\, z^{(2\#p+\#a)\sum \#\iota_\tau}\, z^{-2\#p\sum \#\iota_\tau}\,\lambda'(Q_{f,j}(\xi)) = \lambda'(Q_{f,j}(\xi))$$

for all z, so the circle mean is $\lambda'(Q_{f,j}(\xi))$, and with (6.55)

$$\varkappa(v)\,\lambda_1(v_{\Delta^j(e;\eta^0k^{\mathsf{T}})}) = \lambda'(Q_{f,j}(\eta^0k^{\mathsf{T}})) \qquad (k \in G_\infty(s_m^+)). \tag{6.56}$$

Step 3: polarisation. Let η be any family. Lemma 6.3.70, for $r = e|_p$ (row i of $\eta^0_{\tau k}$ is the basis vector at $e(i)$) and $\xi = \eta$, gives a finitely supported c on $G_\infty(s^+_m)$ with $\prod_{\tau,i} \det(\eta_{\tau i} Y_\tau) = \sum_k c(k) \prod_\tau \det(\eta^0 k_\tau^{\mathsf T} Y_\tau)^{\#\iota_\tau}$ over every commutative $\mathbb{C}$-algebra. Taking $Y_\tau = y_\tau^{++}$ in $\mathbb{C}[x]_{n\times b}$ and multiplying by the factor $\prod_\tau \det(y_\tau^{--})^{\max(0,\#\iota_\tau+\#n-2\#l)}$, which does not depend on the family, gives $Q_{f,j}(\eta) = \sum_k c(k) Q_{f,j}(\eta^0 k^{\mathsf T})$. For the stacked determinants fix complex values of the variables of $\mathbb{C}[x]_{a\times n}$ and τ, and let Z_τ be the complex $a \times m$ matrix $(x_{\tau,i,j(k')})$. There are an $m\times p$ matrix Y_τ and $d_\tau \in \mathbb{C}$ with $\Delta_\tau(\zeta) = d_\tau \det(\zeta Y_\tau)$ at this point for all $p\times m$ matrices ζ: if the rows of Z_τ are dependent, both sides vanish for $d_\tau = 0$; otherwise choose $g \in \mathrm{GL}_m(\mathbb{C})$ whose row $e(i)$ is row i of Z_τ for $i \in a$; multiplying the matrix of $\Delta_\tau(\zeta)$ on the right by g^{-1} turns the rows $e(a)$ into the basis vectors at $e(a)$, and expanding along them gives $\Delta_\tau(\zeta) = \det g \cdot \det(\zeta g^{-1} J)$, J the $m \times p$ matrix whose column i is the basis vector at $e(i)$; take $Y_\tau = g^{-1}J$, $d_\tau = \det g$. Hence at this point $\Delta^j(e;\eta) = \prod_\tau d_\tau^{\#\iota_\tau} \prod_{\tau,i} \det(\eta_{\tau i} Y_\tau) = \sum_k c(k) \Delta^j(e;\eta^0 k^{\mathsf T})$; as the point is arbitrary, $\Delta^j(e;\eta) = \sum_k c(k) \Delta^j(e;\eta^0 k^{\mathsf T})$ as polynomials. Since $P \mapsto \lambda_1(v_P)$ and λ' are linear, (6.56) gives

$$\varkappa(v)\lambda_1\big(v_{\Delta^j(e;\eta)}\big) = \sum_k c(k)\, \varkappa(v)\lambda_1\big(v_{\Delta^j(e;\eta^0 k^{\mathsf T})}\big) = \sum_k c(k)\, \lambda'\big(Q_{f,j}(\eta^0 k^{\mathsf T})\big) = \lambda'\big(Q_{f,j}(\eta)\big).$$

□

Lemma 6.3.69. *Let n, m, l, p, a be finite index sets with bijections $j\colon m \sqcup l \to n$ and $e\colon p \sqcup a \to m$, W a space with signs sgn_j on n, $\iota = (\iota_\tau)_{\tau\in\Phi}$ finite sets and $\gamma_0(\tau) = \#l - \#\iota_\tau$. For $y \in G_\infty(s^+_m)$ let $y^\sharp \in G_\infty(\mathrm{sgn}_j)$ act as y on $j(m)$ (through j) and as the identity on $j(l)$. For $z \in \mathbb{C}$ with $|z| = 1$ let $t(z) \in G_\infty(s^+_m)$ be diagonal at every place, with entry z^{-2} at $e(k)$, $k \in p$, and z^{-1} at $e(i)$, $i \in a$, and put*

$$\Omega^\circ(z) = z^{-\#a \sum_\tau \#\iota_\tau}\, \Omega_{\gamma_0,0}\big(t(z)^\sharp,\, z^{-1} 1_a\big) \quad \text{on } \mathcal{F}_{W,I_a}.$$

Let η^0 be the family with $\eta^0_{\tau k}$ equal, for all τ and $k \in \iota_\tau$, to the $p \times m$ matrix whose row i is the standard basis vector at $e(i)$. Then for every $w \in \mathcal{F}_{W,I_a}$ there is $\kappa(w) \in \mathbb{C}$ such that every continuous linear form ℓ on $\mathcal{F}_{W,I_a}$ with $\ell(\Omega_{\gamma_0,0}(1,u)w') = \ell(w')$ for all $u \in G_\infty(s^+_a)$ and $w' \in \mathcal{F}_{W,I_a}$ satisfies

$$\kappa(w)\, \ell\big(v_{\Delta^j(e;\eta^0)}\big) = \frac{1}{2\pi} \int_0^{2\pi} \ell\big(\Omega^\circ(e^{i\theta}) w\big)\, d\theta.$$

Proof. Write $\mathcal{F} = \mathcal{F}_{W,I_a}$; its variables are $x_{\tau,i,k}$ ($\tau \in \Phi$, $i \in a$, $k \in n$). Call a continuous linear form ℓ as in the statement *invariant*, and put $\mathrm{Mn}(w,\ell) = \frac{1}{2\pi}\int_0^{2\pi} \ell(\Omega^\circ(e^{i\theta})w)\, d\theta$. By (2) of Lemma 4.1.1, $\Omega_{\gamma_0,0}$ is a continuous representation by bounded operators, so the integrand is continuous; the operators $\Omega^\circ(z)$, $|z| = 1$, are bounded uniformly, by M say (for the inner-product norm they are isometries; for the norm of $L^1(\nu)$, in which $\mathcal{F}$ is closed because the reproducing kernel is bounded, by the uniform boundedness principle, each orbit being bounded on the compact circle); hence $|\mathrm{Mn}(w,\ell) - \mathrm{Mn}(w',\ell)| \le M\|\ell\|\, \|w - w'\|$. Let $\mathrm{cols} = j(e(a)) \subseteq n$.

Step 1: the torus on Fock vectors of polynomials. Let $P \in \mathbb{C}[x]_{a\times n}$. The elements $t(z)^\sharp$ and $z^{-1}1_a$ are block diagonal for the signs, so (5) of Lemma 4.1.1 applies with $S = W$, $T = I_a$, $\gamma = \gamma_0$, $\gamma' = 0$:

Hence $\Omega^\circ(z) v_P = v_{P_z}$, where P_z is P with every variable $x_{\tau,i,k}$, $k \notin \mathrm{cols}$, multiplied by z. Writing $P = \sum_\alpha c_\alpha x^\alpha$ (finite sum) and $d(\alpha)$ for the total degree of x^α in the variables with $k \notin \mathrm{cols}$, $\ell(\Omega^\circ(z)v_P) = \sum_\alpha c_\alpha z^{d(\alpha)} \ell(v_{x^\alpha})$ is a polynomial in z, and its mean over the circle is its constant term:

$$\mathrm{Mn}(v_P, \ell) = \ell(v_{P_0}), \tag{6.57}$$

P_0 being P with the variables $x_{\tau,i,k}$, $k \notin \mathrm{cols}$, set to 0.

Step 2: averaging over the definite group. Let $z_{\tau,i,i'}$ $(i,i' \in a)$ be variables of square matrices Z_τ and ϱ the algebra map $z_{\tau,i,i'} \mapsto x_{\tau,i,j(e(i'))}$; then $P_0 = \varrho(q)$ for a unique polynomial q in the $z_{\tau,i,i'}$. For $u \in G_\infty(s_a^+)$ write q^u for q with Z_τ replaced by $u_\tau^{\mathsf T} Z_\tau$ and $\chi(u) = \prod_\tau \det(u_\tau)^{\#\iota_\tau}$. By (5) of Lemma 4.1.1 with $(x,y) = (1,u)$ the scalar is $\prod_\tau \det(u_\tau)^{\gamma_0(\tau)-\#l} = \chi(u)^{-1}$ and the substitution replaces the $a \times n$ matrix of variables at τ by $u_\tau^{\mathsf T}$ times it on the columns $j(m)$, so $\Omega_{\gamma_0,0}(1,u)v_{\varrho(q)} = \chi(u)^{-1} v_{\varrho(q^u)}$. For invariant ℓ the linear form $M(Q) = \ell(v_{\varrho(Q)})$ therefore satisfies

$$M(q^u) = \chi(u)\, M(q) \qquad (u \in G_\infty(s_a^+)). \tag{6.58}$$

The coefficients of q^u are polynomials in the entries of u; integrating them against $\chi(u)^{-1}$ for the Haar probability measure du of the compact group $G_\infty(s_a^+)$ (Theorem A.5.1) gives a polynomial $\bar q$ with $M'(\bar q) = \int \chi(u)^{-1} M'(q^u)\, du$ for every linear form M' (a finite sum of coefficients). By (6.58), $M(\bar q) = M(q)$. Evaluation at $Z_\tau = x_\tau^{\mathsf T}$, for $x \in G_\infty(s_a^+)$, is a linear form, and q^u evaluated there is q evaluated at $(x_\tau u_\tau)^{\mathsf T}$; by invariance of du under $u \mapsto x^{-1}u$ and multiplicativity of χ,

$$\bar q((x_\tau^{\mathsf T})_\tau) = \int \chi(u)^{-1} q(((x_\tau u_\tau)^{\mathsf T})_\tau) du = \chi(x) \int \chi(u)^{-1} q((u_\tau^{\mathsf T})_\tau) du = \kappa_q\, \chi(x),$$

with κ_q the last integral. So the polynomial $\bar q - \kappa_q \prod_\tau \det(Z_\tau)^{\#\iota_\tau}$ vanishes at the transposes of all families of unitary matrices; as the unitary group is Zariski dense in $\mathrm{GL}_{\#a}(\mathbb{C})$ (Weyl's unitarian trick: a polynomial vanishing on U vanishes on $\exp(\mathrm{M}_{\#a}(\mathbb{C}))$, the exponentials of complex combinations of hermitian matrices, which contain a neighbourhood of 1), it is 0. Hence $M(q) = \kappa_q\, M\big(\prod_\tau \det(Z_\tau)^{\#\iota_\tau}\big)$.

With (6.57): $\mathrm{Mn}(v_P, \ell) = \kappa_q\, \ell(v_{\Delta^j(e;\eta^0)})$ for every invariant ℓ, κ_q depending only on P and ι.

Step 3: all vectors. The Fock vectors of polynomials are dense in $\mathcal F$ by Lemma 6.3.29. Put $D = v_{\Delta^j(e;\eta^0)}$. If $\ell(D) = 0$ for every invariant ℓ, Step 2 gives $\mathrm{Mn}(v_P, \ell) = 0$ for all P and invariant ℓ, hence $\mathrm{Mn}(w, \ell) = 0$ for all w by density and continuity, and $\kappa(w) = 0$ works. Otherwise fix an invariant ℓ_0 with $\ell_0(D) = 1$ and put $\kappa(w) = \mathrm{Mn}(w, \ell_0)$, a continuous linear form in w. For invariant ℓ the continuous linear form $w \mapsto \mathrm{Mn}(w,\ell) - \kappa(w)\ell(D)$ vanishes at every v_P (Step 2 for ℓ and for ℓ_0 gives $\mathrm{Mn}(v_P,\ell) = \kappa_q \ell(D)$ and $\kappa(v_P) = \kappa_q$), hence on $\mathcal F$. □

Lemma 6.3.70. *Let p, m be finite index sets, $r\colon p \to m$ an injective map, and η^0 the complex $p \times m$ matrix whose row k is the standard basis vector at $r(k)$. Let $\iota = (\iota_\tau)_{\tau\in\Phi}$ be finite sets and $\xi = (\xi_{\tau k})_{\tau\in\Phi,\, k\in\iota_\tau}$ complex $p \times m$ matrices. Then there is a finitely supported function c on $G_\infty(s_m^+)$ such that for every commutative $\mathbb{C}$-algebra S and all $m \times p$ matrices Y_τ $(\tau \in \Phi)$ over S*

$$\prod_{\tau\in\Phi} \prod_{k\in\iota_\tau} \det(\xi_{\tau k} Y_\tau) = \sum_{u \in G_\infty(s_m^+)} c(u) \prod_{\tau\in\Phi} \det(\eta^0 u_\tau^{\mathsf T} Y_\tau)^{\#\iota_\tau}.$$

Proof. If $m = \emptyset$ then $p = \emptyset$, both sides are 1 for c the indicator of the identity. Let $m \neq \emptyset$ and write $d_\tau = \#\iota_\tau$.

Reduction to generic matrices. Let R be the polynomial ring over $\mathbb{C}$ in variables $y_{\tau,i,k}$ $(\tau \in \Phi,\ i \in m,\ k \in p)$ and $Y_\tau^{\mathrm g} = (y_{\tau,i,k})_{i,k}$. If the identity holds in R for the matrices $Y_\tau^{\mathrm g}$, the $\mathbb{C}$-algebra map $R \to S$ sending $y_{\tau,i,k}$ to the (i,k) entry of Y_τ gives it for (Y_τ), since both sides are polynomials in the entries with complex coefficients and c does not depend on S or Y. So it suffices to find c with the identity in R.

A module containing the products. Number m as $1,\dots,M$ $(M = \#m \geq 1)$ so that p goes to $1,\dots,P$ $(P = \#p \leq M)$ through a bijection, fixed once. For each τ let X_τ be an $M \times M$ matrix

of indeterminates and read Y^{g}_τ as its first P columns. Let $\nu_\tau \in \mathbb{Z}^M$ have entries d_τ in the first P places and 0 in the others; it is nonincreasing with last entry ≥ 0. By Theorem A.5.19 (with $K = \mathbb{C}$, $k = M$) the space $\nabla(\nu_\tau)$ of polynomials in X_τ homogeneous of degree d_τ in each of the first P columns and of degree 0 in the others (hence independent of them), and unchanged when a multiple of an earlier column is added to a later one, is an irreducible representation of $\mathrm{GL}_M(\mathbb{C})$ under $(g \cdot Q)(X_\tau) = Q(g^{\mathsf{T}} X_\tau)$. For a complex $p \times m$ matrix ζ put $f_\zeta(X_\tau) = \det(\zeta Y^{\mathrm{g}}_\tau)$. Column b of $\zeta Y^{\mathrm{g}}_\tau$ is ζ times column b of Y^{g}_τ, so f_ζ is linear in each of the first P columns of X_τ and independent of the others, and adding a multiple of column a to column $b > a$ of X_τ adds a multiple of column a to column b of $\zeta Y^{\mathrm{g}}_\tau$ (or changes nothing, if $b > P$), which leaves the determinant unchanged. Hence $\prod_{k \in \iota_\tau} f_{\zeta_k} \in \nabla(\nu_\tau)$ for any ζ_k. Moreover the first P columns of $g^{\mathsf{T}} X_\tau$ are $g^{\mathsf{T}} Y^{\mathrm{g}}_\tau$, so

$$g \cdot f_\zeta = f_{\zeta g^{\mathsf{T}}} \qquad (g \in \mathrm{GL}_M(\mathbb{C})). \tag{6.59}$$

Irreducibility of the product. Let $V \subseteq \mathbb{C}[X_\tau : \tau \in \Phi]$ be the span of the products $\prod_\tau Q_\tau$ with $Q_\tau \in \nabla(\nu_\tau)$; as the variables of different τ are disjoint, multiplication identifies $\bigotimes_\tau \nabla(\nu_\tau)$ with V, compatibly with the action of $\mathrm{GL}_M(\mathbb{C})^\Phi$, $g = (g_\tau)$ acting on the factor τ through g_τ. Each $\nabla(\nu_\tau)$ is irreducible and finite-dimensional over the algebraically closed field $\mathbb{C}$, so by Burnside's theorem the span of the operators of $\mathrm{GL}_M(\mathbb{C})$ on it is its full endomorphism algebra; hence the span of the operators of $\mathrm{GL}_M(\mathbb{C})^\Phi$ on V is $\bigotimes_\tau \operatorname{End} \nabla(\nu_\tau) = \operatorname{End} V$, and V is irreducible.

Unitary translates span V. Let $F_0 = \prod_\tau f_{\eta^0}^{d_\tau} \in V$. It is nonzero: at $Y^{\mathrm{g}}_\tau = (\eta^0)^{\mathsf{T}}$ for all τ its value is 1, because $\eta^0 (\eta^0)^{\mathsf{T}} = 1_p$ (r is injective). Let λ be a linear form on V vanishing on every $u \cdot F_0$, $u \in G_\infty(s^+_m)$ (read in $\mathrm{GL}_M(\mathbb{C})^\Phi$ through the numbering). Extend λ to a linear form $\tilde\lambda$ on the finite-dimensional space of polynomials in the X_τ of degree at most $\sum_\tau P d_\tau$, which contains $F_0(g^{\mathsf{T}} X)$ for every family g of complex $M \times M$ matrices. Then $h(g) = \tilde\lambda(F_0((g_\tau^{\mathsf{T}} X_\tau)_\tau))$ is a polynomial in the entries of $g \in \mathrm{M}_M(\mathbb{C})^\Phi$ that vanishes on $G_\infty(s^+_m)$. It vanishes identically: for a basis $H_1, \dots, H_{M^2}$ of the real space of hermitian $M \times M$ matrices, $z \mapsto h\big((\exp(i \sum_k z_{\tau k} H_k))_\tau\big)$ is an entire function on $\mathbb{C}^{\Phi \times M^2}$ vanishing on the real points, hence zero; since $\{i \sum_k z_k H_k : z \in \mathbb{C}^{M^2}\} = \mathrm{M}_M(\mathbb{C})$, h vanishes on $\exp(\mathrm{M}_M(\mathbb{C}))^\Phi$, which contains a neighbourhood of 1, so $h = 0$. Hence λ vanishes on $g \cdot F_0$ for all $g \in \mathrm{GL}_M(\mathbb{C})^\Phi$, whose span is a nonzero invariant subspace of V, equal to V by irreducibility. So $\lambda = 0$, and the $u \cdot F_0$ ($u \in G_\infty(s^+_m)$) span V.

Conclusion. $F_\xi = \prod_\tau \prod_{k \in \iota_\tau} f_{\xi_{\tau k}}$ lies in V, so $F_\xi = \sum_u c(u)\, u \cdot F_0$ for a finitely supported c. By (6.59), $u \cdot F_0 = \prod_\tau f_{\eta^0 u_\tau^{\mathsf{T}}}^{d_\tau}$. Both sides of this identity involve only the first P columns of the X_τ, so it is the asserted identity in R. □

Lemma 6.3.71. *Let S, T be spaces with signs s on n_S and s' on n_T, let α, β be automorphic characters of $\mathrm{U}(1)$ of weights γ, γ', and let $\mathfrak{m}$ be a modulus. For $x \in G_\infty(s)$ write $(x, 1)$ for the point of $\mathrm{G}(s)$ with infinite component x and finite component 1, and likewise for T.*

(1) For every function Ψ on $\mathrm{M}_{n_T \times n_S}(R_\mathfrak{m})$, $v \in \mathcal{F}_{S,T}$, $x \in G_\infty(s)$, $y \in G_\infty(s')$, $g \in \mathrm{G}(s)$ and $h \in \mathrm{G}(s')$,

$$\theta^{S,T}_{\alpha,\beta}\big(\mathrm{mod}_\mathfrak{m}(\Psi, v)\big)\big(g(x,1), h(y,1)\big) = \theta^{S,T}_{\alpha,\beta}\big(\mathrm{mod}_\mathfrak{m}(\Psi, \Omega_{\gamma,\gamma'}(x,y)v)\big)(g,h).$$

(2) Let Ψ vanish at every matrix without a left inverse over $R_\mathfrak{m}$, $P \in \mathbb{C}[x]_{n_T \times n_S}$ and $h \in \mathrm{G}(s')$. Then $\kappa(g) = \theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi, P](g,h)$ is continuous and left Γ_S-invariant on $\mathrm{G}(s)$, and $\sup_g |\kappa(g)| \operatorname{ht}_S(g)^B < \infty$ for every $B \in \mathbb{N}$. In particular κ is bounded and integrable on $[\mathrm{U}(S)]$, and $P \mapsto \int_{[\mathrm{U}(S)]} \theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi, P](x,h)\, dx$ is a linear form on $\mathbb{C}[x]_{n_T \times n_S}$.

(3) Let $T = \mathrm{I}_a$ with a nonempty and let Φ be a function on $\mathrm{M}_{a \times n_S}(R_\mathfrak{m})$. For $g \in \mathrm{G}(s)$ the integral $\lambda_g(w) = I^{S,\mathrm{I}_a}_{\alpha,\beta}\big(\mathrm{mod}_\mathfrak{m}(\Phi, w)\big)(g)$ converges absolutely for every $w \in \mathcal{F}_{S,\mathrm{I}_a}$, λ_g is a

continuous linear form on $\mathcal{F}_{S,\mathrm{I}_a}$, and

$$\lambda_g(\Omega_{\gamma,\gamma'}(x_1,x_2)w) = \lambda_{g(x_1,1)}(w) \qquad (x_1 \in G_\infty(s),\ x_2 \in G_\infty(s_a^+),\ w \in \mathcal{F}_{S,\mathrm{I}_a}).$$

(4) In (3), let $P \in \mathbb{C}[x]_{a\times n_S}$ satisfy $\Omega_{\gamma,\gamma'}(1,y)v_P = v_P$ for all $y \in G_\infty(s_a^+)$, and let

$$\alpha(\det k)\,\Phi(\bar{k}^{-1}X) = \Phi(X)$$

for all $k \in \mathbf{K}_a$ and X, $\bar{k}$ the reduction of k modulo $\mathfrak{m}$. Then for every $g \in \mathrm{G}(s)$ the function

$$h \mapsto \theta^{S,\mathrm{I}_a}_{\alpha,\beta;\mathfrak{m}}[\Phi,P](g,h)$$

of the finite points h is left $\mathrm{U}(\mathrm{I}_a)(F)$-invariant and right $\mathbf{K}_a$-invariant, and

$$I^{S,\mathrm{I}_a}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}}(\Phi,v_P))(g) = c_a \sum^{\mathrm{cl}}_{h \bmod \mathbf{K}_a} \theta^{S,\mathrm{I}_a}_{\alpha,\beta;\mathfrak{m}}[\Phi,P](g,h),$$

where $c_a > 0$ is the total mass of the measure of $G_\infty(s_a^+)$.

(5) Let n,m,l,p,a be finite index sets with bijections $j\colon m\sqcup l \to n$, $e\colon p\sqcup a \to m$, let $\iota = (\iota_\tau)_{\tau\in\Phi}$ be finite sets and $\eta = (\eta_{\tau k})$ complex $p\times m$ matrices, and let S have the signs sgn_j on $n_S = n$ and $\gamma(\tau) = \#l - \#\iota_\tau$. Then for the pair (S,I_a) and every γ', $\Omega_{\gamma,\gamma'}(1,y)v_{\Delta^j(e;\eta)} = v_{\Delta^j(e;\eta)}$ for all $y \in G_\infty(s_a^+)$.

Proof. Write $\theta = \theta^{S,T}_{\alpha,\beta}$, $\omega = \omega_{\alpha,\beta}$ and $A = \Sigma(\mathfrak{m})$. For a function Ψ on $\mathrm{M}_{n_T\times n_S}(R_{\mathfrak{m}})$ put $f_\Psi = \Psi^\natural \otimes_A v^0 \in \mathcal{M}_{S,T}$ ($\Psi^\natural \in \mathcal{T}_A$ by (6) of Lemma 4.1.1), so that $\mathrm{mod}_{\mathfrak{m}}(\Psi,v) = v \otimes f_\Psi$ with f_Ψ independent of v.

Step 1: (1). By (5) of Lemma 4.1.1 with $k = k' = 1$, $\omega((x,1),(y,1))(v\otimes f_\Psi) = (\Omega_{\gamma,\gamma'}(x,y)v)\otimes f_\Psi$, since $\omega((1,1),(1,1))$ is the identity. As ω is a representation,

$$\theta(v\otimes f_\Psi)(g(x,1),h(y,1)) = \theta_{\mathrm{Hom}(S,T)}(\omega(g,h)\,\omega((x,1),(y,1))(v\otimes f_\Psi)) = \theta((\Omega_{\gamma,\gamma'}(x,y)v)\otimes f_\Psi)(g,h).$$

Step 2: (2). The map $(g,h) \mapsto \omega(g,h)\varphi$ is continuous and $\theta_{\mathrm{Hom}(S,T)}$ is a bounded linear functional, so $\theta(\varphi)$ is continuous on $\mathrm{G}(s)\times\mathrm{G}(s')$ for every datum φ; it is left invariant under $\Gamma_S\times\Gamma_T$ by Lemma 6.2.5. The Fock vector v_P is a Schwartz vector ((XD3)), so $\mathrm{mom}_N(v_P) < \infty$ for every N. Given B, (1) of Lemma 6.3.74 with $M = B$ gives B', N, C with $|\kappa(g)|\,\mathrm{ht}_S(g)^B \le C\,\mathrm{mom}_N(v_P)\,\mathrm{ht}_T(h)^{B'}$ for all g. For $B = 0$ this bounds κ; as κ is continuous, left Γ_S-invariant and bounded and $[\mathrm{U}(S)]$ has finite volume (Theorem 4.2.18; for $n_S = \emptyset$ the group is trivial), κ is integrable on $[\mathrm{U}(S)]$. The datum $v_P \otimes f_\Psi = \int v_P(y)\rho(\mathrm{rv}(y))f_\Psi\,d\nu(y)$ is linear in v_P, $P \mapsto v_P$ is linear, and $\theta_{\mathrm{Hom}(S,T)}\circ\omega(x,h)$ is linear; so the integral is linear in P.

Step 3: (3). Put $f = f_\Phi$ and $\varphi_w = w\otimes f$. The operators $\rho(z)$ are isometries of $\mathcal{M}_{S,\mathrm{I}_a}$ (Lemma 4.1.10, (1)), so the Bochner integral $\varphi_w = \int w(y)\rho(\mathrm{rv}(y))f\,d\nu(y)$ satisfies $\|\varphi_w\| \le \|w\|\,\|f\|$, $\|w\|$ the L^1 norm of the Fock model; and $w \mapsto \varphi_w$ is linear. The space I_a is definite at every infinite place, so $\Gamma_{\mathrm{I}_a}\backslash\mathrm{G}(s_a^+)$ is compact (Theorem A.4.5); as the quotient map is open, finitely many images of relatively compact open sets cover it, and there is a compact $C \subseteq \mathrm{G}(s_a^+)$ with $\Gamma_{\mathrm{I}_a}C = \mathrm{G}(s_a^+)$. Fix g. For each $\varphi \in \mathcal{M}_{S,\mathrm{I}_a}$ the continuous map $h \mapsto \omega(g,h)\varphi$ is bounded on C; $\mathcal{M}_{S,\mathrm{I}_a}$ is a Banach space (a closed subspace of L^1, Lemma 4.1.10 (1)), so by the uniform boundedness principle $M_g = \sup_{h\in C}\|\omega(g,h)\| < \infty$. Hence $|\theta(\varphi_w)(g,h)| \le \|\theta_{\mathrm{Hom}(S,\mathrm{I}_a)}\|\,M_g\,\|f\|\,\|w\|$ for $h \in C$, and by the left Γ_{I_a}-invariance of Step 2 for all h. The function $h \mapsto \theta(\varphi_w)(g,h)$ is continuous, left Γ_{I_a}-invariant and bounded, and $[\mathrm{U}(\mathrm{I}_a)]$ has finite volume vol;

so $\lambda_g(w)$ converges absolutely, λ_g is linear, and $|\lambda_g(w)| \le \mathrm{vol}\cdot\|\theta_{\mathrm{Hom}(S,\mathrm{I}_a)}\|\,M_g\,\|f\|\,\|w\|$, so λ_g is continuous. By (1), $\theta(\varphi_{\Omega_{\gamma,\gamma'}(x_1,x_2)w})(g,h) = \theta(\varphi_w)(g(x_1,1),h(x_2,1))$. The group $\mathrm{G}(s_a^+)$ is unimodular (the adelic points of the reductive group $\mathrm{U}(\mathrm{I}_a)$), so the quotient measure of $[\mathrm{U}(\mathrm{I}_a)]$ is invariant under right translation by $(x_2,1)$, and integrating over h gives $\lambda_g(\Omega_{\gamma,\gamma'}(x_1,x_2)w) = \lambda_{g(x_1,1)}(w)$.

Step 4: (4). Fix g and put $F(h) = \theta(\varphi_{v_P})(g,h)$ on $\mathrm{G}(s_a^+) = G_\infty(s_a^+)\times G_f(a)$. By Step 2, F is continuous and left Γ_{I_a}-invariant. For $y \in G_\infty(s_a^+)$ and a finite point h_f, $(y,h_f) = (1,h_f)(y,1)$, and (1) with $x=1$ and the invariance of v_P give $F(y,h_f) = F(1,h_f)$. The point of I_a is 1, so $\mathbf{K}_a$ is the standard level $\mathbf{K}_0(a)$ of I_a, and for $k \in \mathbf{K}_a$, (6) of Lemma 4.1.1 (with the element 1 of $\mathbf{K}_0(n_S)$) gives $\omega((1,1),(1,k))\varphi_{v_P} = \alpha(\det k)\,v_P \otimes f_{\Phi'}$ with $\Phi'(X) = \Phi(\bar{k}^{-1}X)$; by hypothesis $\alpha(\det k)\Phi' = \Phi$, so $\omega((1,1),(1,k))\varphi_{v_P} = \varphi_{v_P}$ and $F(h(1,k)) = F(h)$. In particular $h_f \mapsto F(1,h_f)$ is left invariant under the finite components of Γ_{I_a}, which are the elements of $\mathrm{U}(\mathrm{I}_a)(F)$, and right $\mathbf{K}_a$-invariant. By (2) of Lemma 6.4.74 with $n=a$, $V=\mathbb{C}$, $f=F$ and $\mathrm{K} = \mathrm{K}' = \mathbf{K}_0(a)$, the function $H(h_f) = \int_{G_\infty(s_a^+)} F(y,h_f)\,dy = F(1,h_f)$ (dy the Haar probability measure) satisfies

$$I^{S,\mathrm{I}_a}_{\alpha,\beta}(\varphi_{v_P})(g) = \int_{[\mathrm{U}(\mathrm{I}_a)]} F = c_a\,\mathrm{CI}_{\mathbf{K}_0(a)}(H),$$

and by (1) of Lemma 6.4.74 with $\mathrm{K} = \mathrm{K}' = \mathbf{K}_0(a)$, $\mathrm{CI}_{\mathbf{K}_0(a)}(H) = \sum_{[h]} H(h)/\#(\Gamma_f \cap h\mathbf{K}_a h^{-1})$, the sum over $\Gamma_f\backslash G_f(a)/\mathbf{K}_a$; this is the class sum (6.22) of $h \mapsto F(1,h) = \theta^{S,\mathrm{I}_a}_{\alpha,\beta;\mathfrak{m}}[\Phi,P](g,h)$.

Step 5: (5). Apply the second formula of (5) of Lemma 4.1.1 to the pair (S,I_a), $x=1$ and $y \in G_\infty(s_a^+)$; both are block diagonal for the partitions by sign, the block of sign -1 of y_τ is empty and that of sign $+1$ is y_τ. The scalar factor is

$$\prod_{\tau\in\Phi} \det(y_\tau)^{\gamma(\tau)-\#\{i\in n:\mathrm{sgn}_j(i)=-1\}} = \prod_{\tau\in\Phi} \det(y_\tau)^{(\#l-\#\iota_\tau)-\#l} = \prod_{\tau\in\Phi} \det(y_\tau)^{-\#\iota_\tau},$$

the factors of $x=1$ being 1. For $i' \in a$ and $k \in n$ with $\mathrm{sgn}_j(k) = +1$ the substitution sends $X_{\tau,i',k}$ to $\sum_{i\in a}(y_\tau)_{ii'}X_{\tau,i,k}$. The polynomial $\Delta_\tau(\eta_{\tau k})$ contains only the variables $X_{\tau,i,j(k')}$, $i \in a$, $k' \in m$, and $\mathrm{sgn}_j(j(k')) = +1$; so the $a\times m$ matrix $X_\tau = (X_{\tau,i,j(k')})$ becomes $y_\tau^{\mathsf{T}}X_\tau$, and the $m\times m$ matrix whose determinant is $\Delta_\tau(\eta_{\tau k})$ is multiplied on the left by the matrix D_τ with $(D_\tau)_{e(i'),e(i)} = (y_\tau)_{ii'}$ for $i,i' \in a$, $(D_\tau)_{e(i'),e(i')} = 1$ for $i' \in p$ and all other entries 0, of determinant $\det y_\tau$. Hence $\Delta^j(e;\eta)$ becomes $\prod_\tau \det(y_\tau)^{\#\iota_\tau}\Delta^j(e;\eta)$, and $\Omega_{\gamma,\gamma'}(1,y)v_{\Delta^j(e;\eta)} = \prod_\tau \det(y_\tau)^{-\#\iota_\tau}\prod_\tau \det(y_\tau)^{\#\iota_\tau}\,v_{\Delta^j(e;\eta)} = v_{\Delta^j(e;\eta)}$. □

The last group turns the exchange identity into a law for squares of theta lifts (Proposition 6.3.72). For $W = U_1 \oplus U_2$ and spaces A, A' with $\mathrm{rk}\,A = \mathrm{rk}\,A' + 2t$, a form on U_1 can be lifted up to A and back to U_2, or down to A' and up to U_2. For matched data the second composite is κ/c_i times the first, with $|c_i|^2 = C\,L_*(\mathfrak{m}_i,\nu_i;t) \neq 0$ and $C>0$ and κ independent of i. The number κ may be 0, which is harmless where the law is used, since the scalar of the auxiliary forms is chosen afterwards (Proposition 6.4.6). In our reading this is a seesaw identity in the sense of Kudla [171], with the exchange identity supplying the constant and its size. It is used for the squares of the grids of Section 6.4, where the moduli of the constants are the turn ratios, and for the squares of the triangle of lifts, where they give the chain ratios. The proof applies the exchange identity at the Fock vector of the product of the polynomials of the four lifts (Lemma 6.3.73); the decay of the kernels, the factorisation of data and kernels over orthogonal sums, and the approximation of Schwartz vectors by Fock vectors of polynomials (Lemma 6.3.79) make this legitimate.

Proposition 6.3.72. *Let A', A, U_1, U_2 be spaces with index sets n, n', n_1, n_2, let $W = U_1 \oplus U_2$ and let $\sigma\colon n \sqcup n' \to n_1 \sqcup n_2$ be a bijection. Let $t \in \mathbb{N}$ and $\mu\colon \Phi \to \mathbb{Z}$. Let I be a set and, for*

$i \in I$, ν_i tame of weight μ for $(\mathfrak{m}_i, \chi_i)$. Let the pairs (U_1, A), (U_2, A), (A', U_1), (A', U_2) carry characters $(a_{1,i}, b_{1,i})$, $(a_{2,i}, b_{2,i})$, $(b'_{1,i}, a'_{1,i})$, $(b'_{2,i}, a'_{2,i})$ of $\mathrm{U}(1)$ and polynomials P_1, P_2, P'_1, P'_2 (real data: their Fock vectors). Assume:

(OS1) $\operatorname{rk} A = \operatorname{rk} A' + 2t$, $\operatorname{neg}(W) = \operatorname{neg}(A) + \operatorname{neg}(A')$, *and* $2\operatorname{neg}(W) + 2 \le \operatorname{rk} W \le 2\mu(v) + 2\operatorname{neg}(W)$ *for* $v \in \Phi$;

(OS2) *the characters of each pair have weights independent of i, and for types e, e', e_1, e_2 on A, A', U_1, U_2 the Fock vectors of P_1, P_2, P'_1, P'_2 are edges for the weights of the characters of their pairs (the weight of the first character first) from e_1 to e, from $\bar{e}_2$ to $\bar{e}$, from $\bar{e}'$ to $\bar{e}_1$ and from e' to e_2;*

(OS3) *each of $a_{1,i}[\operatorname{rk} U_1]a_{2,i}[\operatorname{rk} U_2]$, $a'_{1,i}[\operatorname{rk} U_1]a'_{2,i}[\operatorname{rk} U_2]$, $b_{1,i}[\operatorname{rk} A]b'_{1,i}[\operatorname{rk} A']$, $b_{2,i}[\operatorname{rk} A]b'_{2,i}[\operatorname{rk} A']$ equals $\tilde{\nu}_i^{-1}$.*

For $i \in I$ and a matrix x (indices $n' \times (n_1 \sqcup n_2)$) over $R_i = R_{\mathfrak{m}_i}$ let $\Theta_i(x)\colon \mathcal{A}_{e_1}(U_1) \to \mathcal{A}_{e_2}(U_2)$ be the theta lift from U_1 to A followed by the transposed lift from A to U_2, with the indicator functions of the two column blocks of x as data modulo $\mathfrak{m}_i$; for a matrix y (indices $(n_1 \sqcup n_2) \times n$) let $\Theta'_i(y)$ be the transposed lift from U_1 to A' followed by the lift from A' to U_2, with the indicator functions of the two row blocks of y. Then:

1. *$\Theta_i(kx) = \chi_i(\det k)\,\Theta_i(x)$ and $\Theta'_i(yk) = \chi_i(\det k)\,\Theta'_i(y)$ for invertible k over R_i;*
2. *there are $C > 0$ and $\kappa \in \mathbb{C}$, possibly 0, both independent of i, and for each i a number c_i with*
$$|c_i|^2 = C \cdot L_*(\mathfrak{m}_i, \nu_i; t) \neq 0, \qquad \sum_y \Psi'(y)\,\Theta'_i(y) = \frac{\kappa}{c_i} \sum_x \Psi(x)\,\Theta_i(x)$$
for all Ψ, Ψ' matched for (σ, χ_i) modulo $\mathfrak{m}_i$.

In particular, for every $f \in \mathcal{A}_{e_1}(U_1)$ and every linear functional l on $\mathcal{A}_{e_2}(U_2)$, the functions $L(x) = l(\Theta_i(x)f)$ and $L'(y) = l(\Theta'_i(y)f)$ satisfy the square law with constant κ/c_i: both transform by $\chi_i \circ \det$, and $\sum \Psi' L' = (\kappa/c_i) \sum \Psi L$ for matched Ψ, Ψ'.

Proof. **Step 1: the characters.** Base change is injective on automorphic characters of $\mathrm{U}(1)$: every $u \in \mathrm{U}(1)(\mathbb{A})$ is $x/\bar{x}$ for an idele x of E (Hilbert's Theorem 90 at each place of F, with x_v a unit wherever u_v is a unit and E/F is unramified at v, which holds at almost all v), so $\tilde{\alpha} = \tilde{\beta}$ forces $\alpha(u) = \tilde{\alpha}(x) = \tilde{\beta}(x) = \beta(u)$. By (OS1) and σ, $\operatorname{rk} W = \operatorname{rk} A + \operatorname{rk} A' = 2\operatorname{rk} A' + 2t$ is even; put $\gamma_i = \nu_i^{-1}\upsilon^{\operatorname{rk} W/2}$. As $\upsilon(\bar{x}) = \upsilon(x)^{-1}$, the base change of υ restricted to $\mathrm{U}(1)(\mathbb{A})$ is υ^2, so $\tilde{\gamma}_i = \tilde{\nu}_i^{-1}\upsilon^{\operatorname{rk} W}$. Since $\alpha[p] = \tilde{\alpha}\upsilon^{-p}$, (OS3) gives $(a_{1,i}a_{2,i})^{\sim} = \tilde{\nu}_i^{-1}\upsilon^{\operatorname{rk} W} = \tilde{\gamma}_i$ and likewise for $a'_{1,i}a'_{2,i}$, and $(b_{1,i}b'_{1,i})^{\sim} = \tilde{\nu}_i^{-1}\upsilon^{\operatorname{rk} A + \operatorname{rk} A'} = (b_{2,i}b'_{2,i})^{\sim}$. Hence $a_{1,i}a_{2,i} = a'_{1,i}a'_{2,i} = \gamma_i$ and $b_{1,i}b'_{1,i} = b_{2,i}b'_{2,i}$. Put $\nu_{0,i} = b_{1,i}/b_{2,i}$; then $b_{2,i}\nu_{0,i} = b_{1,i}$ and $b'_{1,i}\nu_{0,i} = b'_{2,i}$.

Step 2: the exchange identity for $W = U_1 \oplus U_2$. Apply Theorem 6.3.2 to $(W, V, V') = (U_1 \oplus U_2, A, A')$ with $e = t$ and v the Fock vector of the product $P'_1P'_2$ (in the variables of the row blocks), a Schwartz vector for (A', W); (OS1) gives its hypotheses. It yields $C > 0$, a Schwartz vector w for (W, A) and for each i a number c_i with $|c_i|^2 = C\,L_*(\mathfrak{m}_i, \nu_i; t) \neq 0$ such that, for the splitting characters $(b_{1,i}, b'_{1,i})$ (allowed by (OS3)) and matched Ψ, Ψ',

$$I^{W,A}_{\gamma_i, b_{1,i}}(\mathrm{mod}_{\mathfrak{m}_i}(\Psi, w))(g) = c_i \int_{[\mathrm{U}(A')]} \theta^{A',W}_{b'_{1,i},\gamma_i}(\mathrm{mod}_{\mathfrak{m}_i}(\Psi', v))(h, g)\,dh \qquad (g \in \mathrm{U}(W)(\mathbb{A}_F)).$$

Let κ_w be the number of (2) of Lemma 6.3.73 for w (it does not depend on i), and put $\kappa = \kappa_w$.

Step 3: integration against $f_1 \otimes f_2$. Fix i, matched Ψ, Ψ', a form f_1 of type e_1 on U_1 and a form f_2 of type $\bar{e}_2$ on U_2. If $\Psi(x) \neq 0$, x is the block of rows n' of some k^{-1}, $k \in \mathrm{GL}_{n_1 \sqcup n_2}(R_i)$, and has the right inverse formed by the columns n' of k; if $\Psi'(y) \neq 0$, y is the block of columns n of some k and has a left inverse. The data are linear in the functions: $\mathrm{mod}_{\mathfrak{m}_i}(\Psi, w) = \sum_x \Psi(x)\mathrm{mod}_{\mathfrak{m}_i}(\delta_x, w)$, and likewise for Ψ'. Take $g = \mathrm{diag}(g_1, g_2)$ in Step 2, multiply by $f_1(g_1)f_2(g_2)\nu_{0,i}(\det g_2)$ and integrate over $[\mathrm{U}(U_1)] \times [\mathrm{U}(U_2)]$. On the left, the integrand is $\sum_x \Psi(x)\mathrm{Up}_{i,x}(w)$ in the notation of Lemma 6.3.73 (as $\gamma_i = a_{1,i}a_{2,i}$); each term with $\Psi(x) \neq 0$ is integrable on the triple quotient by (1) there, so by Fubini's theorem and (2) there the integral is $\kappa_w \sum_x \Psi(x) \int_{[\mathrm{U}(U_2)]} f_2 \cdot \Theta_i(x) f_1$. On the right, for y with $\Psi'(y) \neq 0$ and row blocks y_1, y_2, $\delta_y = \delta_{y_1} \otimes \delta_{y_2}$, so (a) of Lemma 6.3.80 gives $\mathrm{mod}_{\mathfrak{m}_i}(\delta_y, v) = \mathrm{mod}_{\mathfrak{m}_i}(\delta_{y_1}, v_{P_1'}) \boxtimes^{\mathrm{tg}} \mathrm{mod}_{\mathfrak{m}_i}(\delta_{y_2}, v_{P_2'})$, and the exact factorisation (d) of Lemma 6.3.52 (target sums, the character $b'_{1,i}$ on $\mathrm{U}(W)$ and $a'_{1,i}a'_{2,i} = \gamma_i$ on $\mathrm{U}(A')$), with the twist law for $b'_{2,i} = b'_{1,i}\nu_{0,i}$, gives

$$\nu_{0,i}(\det g_2)\, \theta^{A',W}_{b'_{1,i},\gamma_i}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_y, v))(h, \mathrm{diag}(g_1, g_2)) = K'_1(h, g_1)K'_2(h, g_2),$$

$K'_j = \theta^{A',U_j}_{b'_{j,i},a'_{j,i}}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_{y_j}, v_{P_j'}))$. By (1) of Lemma 6.3.73 the product with $f_1(g_1)f_2(g_2)$ is integrable on $[\mathrm{U}(U_1)] \times [\mathrm{U}(U_2)] \times [\mathrm{U}(A')]$; by Fubini's theorem its integral is $\int_{[\mathrm{U}(A')]} L'^{\mathsf{t}}_{1,i,y} f_1 \cdot L'^{\mathsf{t}}_{2,i,y} f_2$, with $L'^{\mathsf{t}}_{2,i,y}$ the transposed lift from U_2 to A' of the kernel K'_2; by (2) of Lemma 6.3.76 for the pair (A', U_2), the form $L'^{\mathsf{t}}_{1,i,y} f_1$ of type e' and the form f_2 of type $\bar{e}_2$, this is $\int_{[\mathrm{U}(U_2)]} f_2 \cdot \Theta'_i(y) f_1$. Hence

$$\kappa_w \sum_x \Psi(x) \int_{[\mathrm{U}(U_2)]} f_2 \cdot \Theta_i(x) f_1 = c_i \sum_y \Psi'(y) \int_{[\mathrm{U}(U_2)]} f_2 \cdot \Theta'_i(y) f_1.$$

By (3) of Lemma 6.3.73, $F = \sum_y \Psi'(y)\Theta'_i(y)f_1 - \frac{\kappa}{c_i}\sum_x \Psi(x)\Theta_i(x)f_1$ is a form of type e_2; as $c_i \neq 0$ the identity gives $\int f_2 F = 0$ for all forms f_2 of type $\bar{e}_2$, in particular for $f_2 = \bar{F}$. So $\int_{[\mathrm{U}(U_2)]} |F|^2 = 0$, and $F = 0$ since F is continuous and the quotient measure gives positive mass to the image of every nonempty open set. This is (2), with C and κ independent of i.

Step 4: equivariance. By Step 1, $(a_{1,i}a_{2,i})^{-1} = (a'_{1,i}a'_{2,i})^{-1} = \gamma_i^{-1} = \nu_i \upsilon^{-\mathrm{rk}\, W/2}$, which has the tame part of ν_i (υ has trivial tame part), so it is tame for $(\mathfrak{m}_i, \chi_i)$. (3) of Lemma 6.3.73 gives (1).

The square law. Apply (1) and (2) to f and follow by l. □

Lemma 6.3.73. *Let A, A', U_1, U_2 be spaces with signs s' on n', s on n, s_1 on n_1, s_2 on n_2, let $W = U_1 \oplus U_2$ (index set $n_1 \sqcup n_2$) and write $\mathrm{diag}(g_1, g_2)$ for the block diagonal point of $\mathrm{U}(W)(\mathbb{A}_F)$. Let e, e', e_1, e_2 be types on A, A', U_1, U_2 and I a set; for $i \in I$ let $\mathfrak{m}_i$ be a modulus, $R_i = R_{\mathfrak{m}_i}$, and let $a_{1,i}, b_{1,i}, a_{2,i}, b_{2,i}, a'_{1,i}, b'_{1,i}, a'_{2,i}, b'_{2,i}$ be automorphic characters of $\mathrm{U}(1)$ whose weights $\gamma_1, \delta_1, \gamma_2, \delta_2, \gamma'_1, \delta'_1, \gamma'_2, \delta'_2$ do not depend on i; put $\nu_{0,i} = b_{1,i}/b_{2,i}$. Let $P_1 \in \mathbb{C}[x]_{n' \times n_1}$, $P_2 \in \mathbb{C}[x]_{n' \times n_2}$, $P'_1 \in \mathbb{C}[x]_{n_1 \times n}$, $P'_2 \in \mathbb{C}[x]_{n_2 \times n}$ be such that v_{P_1} is an edge for (γ_1, δ_1) from e_1 to e (pair (U_1, A)), v_{P_2} one for (γ_2, δ_2) from $\bar{e}_2$ to $\bar{e}$ (pair (U_2, A)), $v_{P'_1}$ one for (δ'_1, γ'_1) from $\bar{e}'$ to $\bar{e}_1$ (pair (A', U_1)) and $v_{P'_2}$ one for (δ'_2, γ'_2) from e' to e_2 (pair (A', U_2)). For $x \in \mathrm{M}_{n' \times (n_1 \sqcup n_2)}(R_i)$ with column blocks x_1, x_2 let $L_{1,i,x}$ be the lift from U_1 to A with the kernel $\theta^{U_1,A}_{a_{1,i},b_{1,i}}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_{x_1}, v_{P_1}))$, let $L_{2,i,x}$ and $L^{\mathsf{t}}_{2,i,x}$ be the lift from U_2 to A and the transposed lift from A to U_2 with the kernel $\theta^{U_2,A}_{a_{2,i},b_{2,i}}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_{x_2}, v_{P_2}))$ (as in Lemma 6.3.76), and $\Theta_i(x) = L^{\mathsf{t}}_{2,i,x} \circ L_{1,i,x}$. For $y \in \mathrm{M}_{(n_1 \sqcup n_2) \times n}(R_i)$ with row blocks y_1, y_2 let $\Theta'_i(y) = L'_{2,i,y} \circ L'^{\mathsf{t}}_{1,i,y}$, where $L'^{\mathsf{t}}_{1,i,y}$ is the transposed lift from U_1 to A' with the kernel $\theta^{A',U_1}_{b'_{1,i},a'_{1,i}}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_{y_1}, v_{P'_1}))$ and $L'_{2,i,y}$ the lift from A' to U_2 with the kernel $\theta^{A',U_2}_{b'_{2,i},a'_{2,i}}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_{y_2}, v_{P'_2}))$. For $w \in \mathcal{F}_{W,A}$ and forms f_1 of type e_1 on U_1 and f_2 of type $\bar{e}_2$ on U_2 put*

$$\mathrm{Up}_{i,x}(w)(g_1, g_2, h) = f_1(g_1)f_2(g_2)\, \nu_{0,i}(\det g_2)\, \theta^{W,A}_{a_{1,i}a_{2,i},\, b_{1,i}}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_x, w))(\mathrm{diag}(g_1, g_2), h).$$

(1) (Integrability.) Let f_1, f_2 be forms of types e_1, $\bar{e}_2$, and α, β, ν' automorphic characters of $\mathrm{U}(1)$*. If x has a right inverse over R_i, there are $N \in \mathbb{N}$ and $D \geq 0$ such that for every $w \in \mathcal{F}_{W,A}$ with $\mathrm{mom}_N(w) < \infty$ the function*

$$f_1(g_1) f_2(g_2) \nu'(\det g_2) \theta^{W,A}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_x, w))(\mathrm{diag}(g_1, g_2), h)$$

is integrable on $[\mathrm{U}(U_1)] \times [\mathrm{U}(U_2)] \times [\mathrm{U}(A)]$ and bounded by $D\,\mathrm{mom}_N(w)$. If y has a left inverse over R_i, the same holds for

$$f_1(g_1) f_2(g_2) \nu'(\det g_2) \theta^{A',W}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_y, w'))(h, \mathrm{diag}(g_1, g_2))$$

on $[\mathrm{U}(U_1)] \times [\mathrm{U}(U_2)] \times [\mathrm{U}(A')]$, $w' \in \mathcal{F}_{A',W}$.

(2) (The factor κ.) For every Schwartz vector $w \in \mathcal{F}_{W,A}$ there is $\kappa \in \mathbb{C}$, possibly 0, such that for all $i \in I$, all x with a right inverse over R_i and all forms f_1 of type e_1 on U_1 and f_2 of type $\bar{e}_2$ on U_2

$$\int_{[\mathrm{U}(U_1)]\times[\mathrm{U}(U_2)]\times[\mathrm{U}(A)]} \mathrm{Up}_{i,x}(w) = \kappa \int_{[\mathrm{U}(A)]} L_{1,i,x} f_1 \cdot L_{2,i,x} f_2 = \kappa \int_{[\mathrm{U}(U_2)]} f_2 \cdot \Theta_i(x) f_1.$$

(3) (Equivariance.) $\Theta_i(x)$ maps forms of type e_1 on U_1 to forms of type e_2 on U_2, and so does $\Theta'_i(y)$. If $(a_{1,i} a_{2,i})^{-1}$ is tame of some weight for $(\mathfrak{m}_i, \chi)$, then $\Theta_i(kx) = \chi(\det k)\,\Theta_i(x)$ for every $k \in \mathrm{GL}_{n'}(R_i)$; if $(a'_{1,i} a'_{2,i})^{-1}$ is tame of some weight for $(\mathfrak{m}_i, \chi)$, then $\Theta'_i(yk) = \chi(\det k)\,\Theta'_i(y)$ for every $k \in \mathrm{GL}_n(R_i)$.

Proof. **Step 1: heights on W.** For $g_j \in \mathrm{U}(U_j)(\mathbb{A}_F)$, $\mathrm{ht}_W(\mathrm{diag}(g_1, g_2)) \leq \mathrm{ht}_{U_1}(g_1)\,\mathrm{ht}_{U_2}(g_2)$. Indeed the matrix and the point of W are block diagonal, so a vector $u = (u_1, u_2)$ lies in the lattice of the moved space $W_{\mathrm{diag}(g_1,g_2)}$ exactly when each u_j lies in that of $(U_j)_{g_j}$, and the euclidean norm of the infinite components of $(x_W \mathrm{diag}(g_1, g_2))^{-1} u$ is at least that for any nonzero u_j; so it is at least $\min(\lambda_{U_1}(g_1), \lambda_{U_2}(g_2)) \geq \lambda_{U_1}(g_1)\lambda_{U_2}(g_2)$, as $\lambda_{U_j} \leq 1$. Hence $\lambda_W(\mathrm{diag}(g_1, g_2)) \geq \lambda_{U_1}(g_1)\lambda_{U_2}(g_2)$.

Step 2: (1). By (ED4)(a) for the half edges v_{P_1} from e_1 and v_{P_2} from $\bar{e}_2$, the types e_1 and $\bar{e}_2$ are cusp kernels of order 0, so by Lemma 6.3.75 $\sup |f_j|\,\mathrm{ht}^B_{U_j} < \infty$ for every B. If x has a right inverse, δ_x vanishes at every matrix without a right inverse, and (2) of Lemma 6.3.74 for $(S, T) = (W, A)$ and $M = 0$ gives B, N, C with $|\theta^{W,A}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}_i}(\delta_x, w))(g, h)| \leq C\,\mathrm{mom}_N(w)\,\mathrm{ht}_W(g)^B$ for all w with $\mathrm{mom}_N(w) < \infty$. By Step 1 and $|\nu'| = 1$, the function of (1) is bounded by $D\,\mathrm{mom}_N(w)$ with $D = C \sup |f_1| \mathrm{ht}^B_{U_1} \sup |f_2| \mathrm{ht}^B_{U_2}$. It is continuous, and left invariant under $\Gamma_{U_1} \times \Gamma_{U_2} \times \Gamma_A$: $\mathrm{diag}(\gamma_1, \gamma_2) \in \Gamma_W$ for $\gamma_j \in \Gamma_{U_j}$, the kernel is invariant by Lemma 6.2.5, the forms are invariant, and $\nu'(\det \gamma_2) = 1$ as $\det \gamma_2 \in \mathrm{U}(1)(E)$. The three quotients have finite volume (Theorem 4.2.18), so the function is integrable on their product. For y with a left inverse the same argument applies with (1) of Lemma 6.3.74 for $(S, T) = (A', W)$ and $M = 0$.

Step 3: kernels of products of polynomials. Fix i and x with a right inverse, and drop i from the notation. Let $Q_1 \in \mathbb{C}[x]_{n' \times n_1}$, $Q_2 \in \mathbb{C}[x]_{n' \times n_2}$, and $Q_1 Q_2$ their product in the variables of the column blocks. As $\delta_x = \delta_{x_1} \otimes \delta_{x_2}$, (b) of Lemma 6.3.80 gives $\mathrm{mod}_{\mathfrak{m}}(\delta_x, v_{Q_1Q_2}) = \mathrm{mod}_{\mathfrak{m}}(\delta_{x_1}, v_{Q_1}) \boxtimes^{\mathrm{sc}} \mathrm{mod}_{\mathfrak{m}}(\delta_{x_2}, v_{Q_2})$, and the exact factorisation (d) of Lemma 6.3.52 (source sums, with the characters a_1, a_2 on $\mathrm{U}(A)$ and b_1 on $\mathrm{U}(W)$) gives

$$\theta^{W,A}_{a_1 a_2, b_1}(\mathrm{mod}_{\mathfrak{m}}(\delta_x, v_{Q_1Q_2}))(\mathrm{diag}(g_1, g_2), h) = K_1(g_1, h)\,\theta^{U_2,A}_{a_2,b_1}(\mathrm{mod}_{\mathfrak{m}}(\delta_{x_2}, v_{Q_2}))(g_2, h),$$

$K_1 = \theta^{U_1,A}_{a_1,b_1}(\mathrm{mod}_{\mathfrak{m}}(\delta_{x_1}, v_{Q_1}))$. By (WR4), $\theta^{U_2,A}_{a_2,b_1} = \nu_0(\det g_2)^{-1} \theta^{U_2,A}_{a_2,b_2}$; with

$$K_2 = \theta^{U_2,A}_{a_2,b_2}(\mathrm{mod}_{\mathfrak{m}}(\delta_{x_2}, v_{Q_2})),$$
$$\mathrm{Up}_x(v_{Q_1Q_2})(g_1, g_2, h) = \big(f_1(g_1) K_1(g_1, h)\big)\big(f_2(g_2) K_2(g_2, h)\big).$$

This is integrable by (1); by Fubini's theorem its integral is $\int_{[\mathrm{U}(A)]} L_1^{Q_1} f_1 \cdot L_2^{Q_2} f_2$, $L_j^{Q_j}$ the lift with the Fock vector v_{Q_j} in place of v_{P_j}. By (4) of Lemma 6.3.76 there are continuous compactly supported a_{Q_1}, a'_{Q_2} on $G_\infty(s')$, depending only on the weights, the types, P_j and Q_j (not on i, x, f_1, f_2), with $L_1^{Q_1} f_1 = R(a_{Q_1}) L_1 f_1$ and $L_2^{Q_2} f_2 = R(a'_{Q_2}) L_2 f_2$, where $L_1 f_1$ is a form of type e and $L_2 f_2$ one of type $\bar e$ on A ((1) of Lemma 6.3.76).

Step 4: pairing of real convolutions. By (ED4)(a) for the half edge v_{P_1} for $(y,x) \mapsto \Omega(x,y)$ from $\bar e$ to $\bar e_1$, $\bar e$ is a cusp kernel of order 0, in particular an integrable idempotent kernel with $\bar e(1) \neq 0$. For forms F_1 of type e and F_2 of type $\bar e$ on A, $\overline{F_1}$ is a form of type $\bar e$ ($R(\bar e)\overline{F_1} = \overline{R(e)F_1}$), so Lemma 6.3.78 for $\bar e$ gives $\int_{[\mathrm{U}(A)]} F_1(g) F_2(g(z,1))\, dg = \frac{e(z)}{\overline{e(1)}} \int F_1 F_2$ for $z \in G_\infty(s')$. For continuous compactly supported a, a', Fubini's theorem (the forms are bounded, the quotient has finite volume) and the right invariance of the quotient measure under $(x',1)^{-1}$ give

$$\int R(a)F_1 \cdot R(a')F_2 = \iint a(x')a'(y') \int F_1(g) F_2(g(x'^{-1}y',1))\, dg\, dx'\, dy' = r(a,a') \int F_1 F_2,$$
$$r(a,a') = \iint a(x')a'(y') \frac{e(x'^{-1}y')}{\overline{e(1)}}\, dx'\, dy'.$$

With Step 3, $\int \mathrm{Up}_{i,x}(v_{Q_1Q_2}) = \rho(Q_1,Q_2)\, \pi_{i,x}(f_1,f_2)$, where $\rho(Q_1,Q_2) = r(a_{Q_1}, a'_{Q_2})$ does not depend on (i,x,f_1,f_2) and $\pi_{i,x}(f_1,f_2) = \int_{[\mathrm{U}(A)]} L_{1,i,x} f_1 \cdot L_{2,i,x} f_2$.

Step 5: (2). Call a quadruple $j = (i,x,f_1,f_2)$ as in (2) a case, and write $J_j(w) = \int \mathrm{Up}_{i,x}(w)$ for w with finite moments; by (1) it is defined, linear in w, and $|J_j(w)| \leq \mathrm{vol}_j\, D_j \,\mathrm{mom}_{N_j}(w)$, vol_j the volume of the triple quotient. Every polynomial in $\mathbb{C}[x]_{n' \times (n_1 \sqcup n_2)}$ is a finite sum of products $Q_1 Q_2$ (monomials factor along the column blocks), so by Step 4 and linearity every polynomial Q has a number $\rho(Q)$ with $J_j(v_Q) = \rho(Q)\pi_j$ for all cases j. Let w be a Schwartz vector and (Q_k) the polynomials of Lemma 6.3.79; then $|J_j(w) - \rho(Q_k)\pi_j| = |J_j(w - v_{Q_k})| \leq \mathrm{vol}_j D_j \,\mathrm{mom}_{N_j}(w - v_{Q_k}) \to 0$ for every case j. If $\pi_j = 0$ for all cases, then $J_j(w) = 0$ for all j and $\kappa = 0$ serves. Otherwise fix j_0 with $\pi_{j_0} \neq 0$; then $\rho(Q_k) \to \kappa := J_{j_0}(w)/\pi_{j_0}$, and for every case $J_j(w) = \lim_k \rho(Q_k)\pi_j = \kappa \pi_j$. This is the first identity of (2). The second is (2) of Lemma 6.3.76 for the pair (U_2, A), the form f_2 of type $\bar e_2$ and the form $L_{1,i,x} f_1$ of type $e = \overline{\bar e}$: $\int_{[\mathrm{U}(A)]} L_{1,i,x} f_1 \cdot L_{2,i,x} f_2 = \int_{[\mathrm{U}(U_2)]} f_2 \cdot L^{\mathsf{t}}_{2,i,x}(L_{1,i,x} f_1)$.

Step 6: (3). By (1) of Lemma 6.3.76, $L_{1,i,x}$ maps forms of type e_1 to forms of type e, and $L^{\mathsf{t}}_{2,i,x}$ maps forms of type $\overline{\bar e} = e$ to forms of type $\overline{\bar e_2} = e_2$; likewise $L'^{\mathsf{t}}_{1,i,y}$ maps forms of type $\overline{\bar e_1} = e_1$ to forms of type $\overline{\bar e'} = e'$ and $L'_{2,i,y}$ forms of type e' to forms of type e_2. Let $k \in \mathrm{GL}_{n'}(R_i)$. Every $w \in \Sigma(\mathfrak{m}_i)$ lies over a place of F that splits as $w\bar w$ ($\bar w \notin \Sigma(\mathfrak{m}_i)$, so $\bar w \neq w$). Choose $\kappa_w \in \mathrm{GL}_{n'}(\mathcal{O}_{E,w})$ reducing to the w-component of k (lift the entries; the determinant is a unit because its reduction is), and let $\hat k \in \underline{G_f(n')}$ have the component κ_w at $w \in \Sigma(\mathfrak{m}_i)$, the component at $\bar w$ determined by $\hat k^* \hat k = 1$ (it is $\overline{(\kappa_w^{-1})^{\mathsf{T}}}$, read at $\bar w$), and the component 1 elsewhere. Then $\hat k$ is integral, $\hat k \in \mathbf{K}_0(n')$, and its reduction modulo $\mathfrak{m}_i$ is k. If μ has weight μ_∞ and tame part ψ with $\psi_{w'} \neq 1$ exactly for $w' \in \Sigma(\mathfrak{m}_i)$, and χ is the character of R_i it defines ((XD4)), then $u = \det \hat k$ has infinite components 1 and integral finite components, so by Definition 3.1.2 $\mu(u) = \prod_{w \in \Sigma(\mathfrak{m}_i)} \psi_w(\det \kappa_w \bmod w) = \chi(\det k)$ (the components at $\bar w$ contribute $\psi_{\bar w} = 1$). By (6) of Lemma 4.1.1 for the pair (U_j, A),

$$\omega_{a_j,b_j}((1,1),(1,\hat k^{-1}))\mathrm{mod}_{\mathfrak{m}_i}(\delta_{kx_j}, v_{P_j}) = a_j(\det \hat k)^{-1} \mathrm{mod}_{\mathfrak{m}_i}(\delta_{x_j}, v_{P_j}),$$

since $\delta_{kx_j}(kX) = \delta_{x_j}(X)$. By (3) of Lemma 6.3.76 (with the points 1 of $\mathrm{U}(U_j)$ and $\hat k^{-1}$ of $\mathrm{U}(A)$), $L_{1,kx} f_1(h(1,\hat k^{-1})) = a_1(\det \hat k)^{-1} L_{1,x} f_1(h)$, i.e. $L_{1,kx} f_1 = a_1(\det \hat k)^{-1}(L_{1,x} f_1)(\cdot(1,\hat k))$, and

$$L^{\mathsf{t}}_{2,kx} F = a_2(\det \hat k)^{-1} L^{\mathsf{t}}_{2,x}(F(\cdot(1,\hat k^{-1})))$$

for forms F of type e. Hence

$$\Theta_i(kx)f_1 = (a_1a_2)^{-1}(\det\hat{k})\,\Theta_i(x)f_1 = \chi(\det k)\Theta_i(x)f_1$$

by the preceding paragraph for $\mu = (a_1a_2)^{-1}$. For $k \in \mathrm{GL}_n(R_i)$ let $\hat{k} \in \mathbf{K}_0(n)$ be constructed in the same way. By (6) of Lemma 4.1.1 for the pair (A', U_j), whose second character a'_j twists $\mathrm{U}(A')$,

$$\omega_{b'_j,a'_j}((1,\hat{k}),(1,1))\mathrm{mod}_{\mathfrak{m}_i}(\delta_{y_jk}, v_{P'_j}) = a'_j(\det\hat{k})^{-1}\mathrm{mod}_{\mathfrak{m}_i}(\delta_{y_j}, v_{P'_j}),$$

since $\delta_{y_jk}(Xk) = \delta_{y_j}(X)$. By (3) of Lemma 6.3.76, $L'^{\mathfrak{t}}_{1,yk}f_1 = a'_1(\det\hat{k})^{-1}(L'^{\mathfrak{t}}_{1,y}f_1)(\cdot(1,\hat{k})^{-1})$ and

$$L'_{2,yk}F = a'_2(\det\hat{k})^{-1}L'_{2,y}(F(\cdot(1,\hat{k})))$$

for forms F of type e'; hence

$$\Theta'_i(yk)f_1 = (a'_1a'_2)^{-1}(\det\hat{k})\,\Theta'_i(y)f_1 = \chi(\det k)\Theta'_i(y)f_1.$$

□

Lemma 6.3.74. *Let S, T be spaces with signs s on n_S and s' on n_T, let α, β be automorphic characters of $\mathrm{U}(1)$, $\mathfrak{m}$ a modulus, and Ψ a function on $\mathrm{M}_{n_T\times n_S}(R_\mathfrak{m})$. Write $K_v(g,h) = \theta^{S,T}_{\alpha,\beta}(\mathrm{mod}_\mathfrak{m}(\Psi, v))(g,h)$ for $v \in \mathcal{F}_{S,T}$, $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$.*

(1) If Ψ vanishes at every matrix without a left inverse in $\mathrm{M}_{n_S\times n_T}(R_\mathfrak{m})$, then for every $M \in \mathbb{N}$ there are $B, N \in \mathbb{N}$ and $C \geq 0$ such that for every v with $\mathrm{mom}_N(v) < \infty$ and all g, h

$$|K_v(g,h)|\,\mathrm{ht}_S(g)^M \leq C\,\mathrm{mom}_N(v)\,\mathrm{ht}_T(h)^B.$$

(2) If Ψ vanishes at every matrix without a right inverse in $\mathrm{M}_{n_S\times n_T}(R_\mathfrak{m})$, then for every $M \in \mathbb{N}$ there are $B, N \in \mathbb{N}$ and $C \geq 0$ such that for every v with $\mathrm{mom}_N(v) < \infty$ and all g, h

$$|K_v(g,h)|\,\mathrm{ht}_T(h)^M \leq C\,\mathrm{mom}_N(v)\,\mathrm{ht}_S(g)^B.$$

(3) For every Ψ there are $B, N \in \mathbb{N}$ and $C \geq 0$ such that for every v with $\mathrm{mom}_N(v) < \infty$ and all g, h, $|K_v(g,h)| \leq C\,\mathrm{mom}_N(v)\,\big(\mathrm{ht}_S(g)\,\mathrm{ht}_T(h)\big)^B$.

Proof. Put $A = \Sigma(\mathfrak{m})$, a nonempty finite set of finite places containing the conjugate of none of its members (XD4), and $f_0 = \Psi^\natural \otimes_A v^0$, so that $\mathrm{mod}_\mathfrak{m}(\Psi, v) = v \otimes f_0$ and $\Psi^\natural \in \mathcal{T}_A$ by (4) of Lemma 4.1.3. Fix $w \in A$, and write $\#n_S$, $\#n_T$ for the numbers of columns and rows.

Step 1: the support of $\Psi^\natural$ at w. Let η be a finite adelic $n_T \times n_S$ matrix with $\Psi^\natural(\eta) \neq 0$. Then the components of η at A are integral and $x = \eta_A \bmod \mathfrak{m}$ has $\Psi(x) \neq 0$. The reduction $r\colon R_\mathfrak{m} = \mathcal{O}_E/\mathfrak{m} \to \mathcal{O}_E/w = k_w$ is a ring homomorphism, and $r(x)$ is the reduction modulo the maximal ideal of $\eta_w \in \mathrm{M}_{n_T\times n_S}(\mathcal{O}_{E,w})$ (the isomorphism $\mathcal{O}_E/\mathfrak{m} \cong \prod_{w'\in A}\mathcal{O}_{E,w'}/w'^{e_{w'}}$ is compatible with the reductions).

- If x has a left inverse $L \in \mathrm{M}_{n_S\times n_T}(R_\mathfrak{m})$, then $r(L)\,r(x) = 1$, so $r(x)$ has rank $\#n_S$ over k_w and some $\#n_S \times \#n_S$ minor of η_w is a unit of $\mathcal{O}_{E,w}$; hence η_w has rank $\#n_S$ over the field E_w and has a left inverse in $\mathrm{M}_{n_S\times n_T}(E_w)$.
- If x has a right inverse, the same argument with $r(x)\,r(R) = 1$ shows that η_w has rank $\#n_T$ and a right inverse in $\mathrm{M}_{n_S\times n_T}(E_w)$.

Step 2: (1) and (2). Under the hypothesis of (1), Step 1 shows that η_w has a left inverse over E_w whenever $\Psi^\natural(\eta) \neq 0$. Clause (2) of Lemma 6.2.27, for S, T, A, $\Psi^\natural$, w and the exponent M, gives $B, N \in \mathbb{N}$ and $C \geq 0$ with $|K_v(g,h)|\,\mathrm{ht}_S(g)^M \leq C\,\mathrm{mom}_N(v)\,\mathrm{ht}_T(h)^B$ for all v with $\mathrm{mom}_N(v) < \infty$ and all g, h (and all α, β); this is (1). Under the hypothesis of (2), Step 1 gives right inverses, and clause (1) of Lemma 6.2.27 gives (2) in the same way.

Step 3: (3). By Lemma 6.2.28 there are $Y \geq 0$ and $B \in \mathbb{N}$ with $|\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)| \leq Y\|\varphi\|(\mathrm{ht}_S(g)\mathrm{ht}_T(h))^B$ for all $\varphi \in \mathcal{M}_{S,T}$. By (1) of Lemma 6.2.21, $\|v \otimes f_0\| \leq \|v\|_{L^1(\nu)}\|f_0\| = \mathrm{mom}_0(v)\|f_0\|$. Hence (3) holds with $N = 0$ and $C = Y\|f_0\|$. □

Lemma 6.3.75. *Let V be a space with a point with signs s on n, $B \in \mathbb{N}$, e a cusp kernel of order B on $G_\infty(s)$, and F a form of type e on V. Then*

$$\sup_{g \in \mathrm{G}(s)} |F(g)|\,\mathrm{ht}_V(g)^A < \infty \qquad \textit{for every } A \in \mathbb{N}.$$

Proof. Let $A \in \mathbb{N}$, let $K \subseteq G_f(n)$ be an open subgroup under which F is right invariant, and put $c = \sup|F| < \infty$ (F is bounded).

If $n = \emptyset$, the lattice of every moved space V_g is $\{0\}$, so $\lambda_V(g) = 1$ and $\mathrm{ht}_V(g) = 1$, and the supremum is c. Let $N = \#n \geq 1$ and number n as $\{1, \dots, N\}$.

The framed space of V. By (VP3) of Definition 3.2.2 V is a framed hermitian space U (E^n with the form h_V, the signs s, the columns of $x_{V,v}$ as frame at $v \in \Phi$ and the finite isometry $z \mapsto x_{V,f}z$; a framed space of rank N in the sense of (FT1) of Definition 3.2.4), with group $\mathrm{G}(s) = G_\infty(s) \times G_f(n)$, coordinate map $c_V(u) = x_V^{-1}u$ and rational points $\Gamma_U = \Gamma_V$, and $\lambda_V(g)$ is the first minimum of (VP4) for both readings:

$$\lambda_U(g) = \lambda_V(g) = \mathrm{ht}_V(g)^{-1} \qquad (g \in \mathrm{G}(s)).$$

The kernel and the form. Take for the Haar measures of (EK1) of Definition 5.6.2 the Cayley measures of the factors $\mathrm{U}(\mathrm{diag}\,s)$, the measure of $G_\infty(s)$ in (ED1). Since $(zx)^{-1} = x^{-1}z^{-1}$ and $\|x\| = \|x\|_\infty$, the conditions of (ED1) on e are those of (EK3) and (EK4)(a)–(c), so e is a cusp kernel of order B in the sense of (EK4); and $R(e)$ of (ED2) is the real convolution $R(e)$ of (EK2). Thus F is continuous, bounded by c, left Γ_U-invariant, right K-invariant and satisfies $R(e)F = F$ (e is integrable).

Conclusion. By Lemma 5.8.7 for E, N, s, U, B, e, K and A there is $R \geq 0$ with $|F(g)|\lambda_U(g)^{-A} \leq c\,R$ for all g, that is $|F(g)|\,\mathrm{ht}_V(g)^A \leq c\,R$ on $\mathrm{G}(s)$. □

Lemma 6.3.76. *Let S, T be spaces with signs s on n_S and s' on n_T, let α, β be automorphic characters of $\mathrm{U}(1)$ of weights γ, γ', $\mathfrak{m}$ a modulus, Ψ a function on $\mathrm{M}_{n_T \times n_S}(R_\mathfrak{m})$, and P a polynomial whose Fock vector $v_P \in \mathcal{F}_{S,T}$ is an edge for the weights (γ, γ') from e to e'. Put $\varphi = \mathrm{mod}_\mathfrak{m}(\Psi, v_P)$, $\kappa(g,h) = \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)$, and for functions F on $\mathrm{G}(s)$ and F' on $\mathrm{G}(s')$*

$$LF(h) = \int_{[\mathrm{U}(S)]} \kappa(g,h)F(g)\,dg, \qquad L^{\mathsf{t}}F'(g) = \int_{[\mathrm{U}(T)]} \kappa(g,h)F'(h)\,dh.$$

(1) For every form F of type e on S the integral $LF(h)$ converges absolutely for every h, and LF is a form of type e' on T. For every form F' of type $\bar{e}'$ on T the integral $L^{\mathsf{t}}F'(g)$ converges absolutely for every g, and $L^{\mathsf{t}}F'$ is a form of type $\bar{e}$ on S.

(2) For forms F of type e on S and F' of type $\bar{e}'$ on T, the function $(g,h) \mapsto \kappa(g,h)F(g)F'(h)$ is integrable on $[\mathrm{U}(S)] \times [\mathrm{U}(T)]$, and $\int_{[\mathrm{U}(T)]} F' \cdot LF = \int_{[\mathrm{U}(S)]} F \cdot L^{\mathsf{t}}F'$.

(3) *Let $k \in G_f(n_S)$, $k' \in G_f(n_T)$, $c \in \mathbb{C}$ and Ψ' a function on $\mathrm{M}_{n_T \times n_S}(R_\mathfrak{m})$ with*

$$\omega_{\alpha,\beta}((1,k),(1,k'))\varphi = c \bmod_\mathfrak{m}(\Psi', v_P),$$

and let L', $L'^{\mathfrak{t}}$ be formed with Ψ' in place of Ψ. Then for forms F of type e on S and F' of type $\bar{e}'$ on T the right translates $F(\cdot(1,k))$, $F'(\cdot(1,k'))$ are forms of the same types, and

$$LF(h(1,k')) = c\, L'(F(\cdot(1,k)))(h), \qquad L^{\mathfrak{t}}F'(g(1,k)) = c\, L'^{\mathfrak{t}}(F'(\cdot(1,k')))(g).$$

(4) *For every polynomial Q there is a continuous compactly supported function a_Q on $G_\infty(s')$, depending only on (γ,γ'), e, e', P and Q, such that for every form F of type e on S the integral $L_QF(h) = \int_{[\mathrm{U}(S)]} \theta^{S,T}_{\alpha,\beta}(\bmod_\mathfrak{m}(\Psi, v_Q))(g,h)F(g)\,dg$ converges absolutely and $L_QF = R(a_Q)\,LF$.*

Proof. Write $\omega = \omega_{\alpha,\beta}$, $\theta = \theta_{\mathrm{Hom}(S,T)}$ and $f = \Psi^\natural \otimes_{\Sigma(\mathfrak{m})} v^0$, so $\varphi = v_P \otimes f$.

Step 1: bounds. By (ED4)(a) for the half edge v_P from e to e', e is a cusp kernel of order 0, and by (ED4)(a) for the half edge from $\bar{e}'$ to $\bar{e}$, so is $\bar{e}'$. By Lemma 6.3.75, $\sup|F|\,\mathrm{ht}_S^A < \infty$ and $\sup|F'|\,\mathrm{ht}_T^A < \infty$ for forms F of type e on S, F' of type $\bar{e}'$ on T and every A. The Fock vector v_P is a Schwartz vector ((XD3)), so (3) of Lemma 6.3.74 gives B and C' with $|\kappa(g,h)| \le C'(\mathrm{ht}_S(g)\mathrm{ht}_T(h))^B$, and likewise for v_Q with some B_Q, C'_Q. Hence $|\kappa(g,h)F(g)| \le C''\mathrm{ht}_T(h)^B$, $|\kappa(g,h)F'(h)| \le C''\mathrm{ht}_S(g)^B$ and $|\kappa(g,h)F(g)F'(h)| \le C''$ for a constant C''. The function κ is continuous (as $(g,h) \mapsto \omega(g,h)\varphi$ is continuous and θ is bounded) and left invariant under $\Gamma_S \times \Gamma_T$ (Lemma 6.2.5); F, F' are continuous and left invariant. As $[\mathrm{U}(S)]$ and $[\mathrm{U}(T)]$ have finite volume (Theorem 4.2.18), the integrals $LF(h)$, $L^{\mathfrak{t}}F'(g)$, $L_QF(h)$ converge absolutely and the function of (2) is integrable on the product.

Step 2: a level of the datum. The characters α, β are continuous and the determinant is continuous, so there are open subgroups $U \subseteq \mathbf{K}_0(n_S)$ and $U' \subseteq \mathbf{K}_0(n_T)$ on which $\beta \circ \det = 1$, resp. $\alpha \circ \det = 1$, and whose elements reduce to 1 modulo $\mathfrak{m}$. For $k \in U$, $k' \in U'$, (6) of Lemma 4.1.1 gives $\omega((1,k),(1,k'))\varphi = \alpha(\det k')\beta(\det k)^{-1} v_P \otimes (\Psi^\natural \otimes v^0) = \varphi$, since $\Psi_{k,k'} = \Psi$ when $\bar{k} = 1$ and $\bar{k}' = 1$.

Step 3: (1). Let F be a form of type e. For $k' \in U'$, as ω is a representation, $LF(h(1,k')) = \int \theta(\omega(g,h)\,\omega((1,1),(1,k'))\varphi)F(g)\,dg = LF(h)$; so LF has the level U'. By Lemma 6.3.77, (ω,θ) is a theta kernel for (S,T) on $\mathrm{Hom}(S,T)$ with real part $\Omega_{\gamma,\gamma'}$, and $LF = L_{v_P \otimes f}F$ is its lift with the datum $v_P \otimes f$ ((ED3)). By (ED4)(c), LF is a form of type e' on T. The pair $(\omega^{\mathrm{sw}},\theta)$ with $\omega^{\mathrm{sw}}(h,g) = \omega(g,h)$ is a theta kernel for (T,S) on $\mathrm{Hom}(S,T)$ with real part $(y,x) \mapsto \Omega_{\gamma,\gamma'}(x,y)$: the homomorphism property, the continuity, the rational invariance and the identity $\omega^{\mathrm{sw}}((y,1),(x,1))(v \otimes f) = (\Omega_{\gamma,\gamma'}(x,y)v) \otimes f$ are those of (ω,θ). Its lift of F' with the datum $v_P \otimes f$ is $L^{\mathfrak{t}}F'$, which has the level U as above. By (ED4)(c) for the half edge v_P for $(y,x) \mapsto \Omega_{\gamma,\gamma'}(x,y)$ from $\bar{e}'$ to $\bar{e}$, $L^{\mathfrak{t}}F'$ is a form of type $\bar{e}$ on S.

Step 4: (2). By Step 1 and Fubini's theorem, both iterated integrals of $\kappa(g,h)F(g)F'(h)$ over $[\mathrm{U}(S)] \times [\mathrm{U}(T)]$ equal its integral, and they are $\int F' \cdot LF$ and $\int F \cdot L^{\mathfrak{t}}F'$.

Step 5: (3). Right translation by $(1,k)$ commutes with left translation by Γ_S and with right translation by the points $(x,1)$, $x \in G_\infty(s)$, and it carries a level U_0 of F to the open subgroup kU_0k^{-1}; so $F(\cdot(1,k))$ is continuous, bounded, left Γ_S-invariant, has a level and satisfies $R(e)F(\cdot(1,k)) = F(\cdot(1,k))$: it is a form of type e, and likewise for F'. The group $\mathrm{G}(s)$ is unimodular, so the quotient measure is invariant under the right translation $g \mapsto g(1,k)$, and

$$\begin{aligned} LF(h(1,k')) &= \int \theta(\omega(g(1,k),h(1,k'))\varphi)F(g(1,k))\,dg \\ &= c\int \theta(\omega(g,h)\bmod_\mathfrak{m}(\Psi',v_P))F(g(1,k))\,dg = c\,L'(F(\cdot(1,k)))(h). \end{aligned}$$

The second identity follows in the same way from the substitution $h \mapsto h(1, k')$ in $[\mathrm{U}(T)]$.

Step 6: (4). By (ED4)(d) for the half edge v_P for $\Omega_{\gamma,\gamma'}$ from e to e', there is a continuous compactly supported a_Q on $G_\infty(s')$ with $L_{v_Q \otimes f''} F'' = R(a_Q) L_{v_P \otimes f''} F''$ for all spaces with points of signs s, s', η, all theta kernels with real part $\Omega_{\gamma,\gamma'}$, all f'' and all forms F'' of type e; it depends only on $\Omega_{\gamma,\gamma'}$, e, e', v_P and Q. For the theta kernel (ω, θ) and $f'' = f$, $L_{v_Q \otimes f} F = L_Q F$ (convergent by Step 1) and $L_{v_P \otimes f} F = LF$. □

Lemma 6.3.77. *Let S, T be spaces with points, and α, β automorphic characters of* U(1) *of weights γ, γ'. Then $(\omega_{\alpha,\beta}, \theta_{\mathrm{Hom}(S,T)})$ is a theta kernel for (S, T) on* $\mathrm{Hom}(S, T)$*, and it has real part $\Omega_{\gamma,\gamma'}$.*

Proof. The space $\mathrm{Hom}(S, T)$ has signs $\eta(i, j) = s'(i)s(j)$ on $\iota = n' \times n$ ((VP5) of Definition 3.2.2), and $\mathcal{M}_{S,T} = \mathcal{M}_\eta$ (WR4). **Homomorphism and continuity.** By (2) of Lemma 4.1.1, $\omega_{\alpha,\beta}$ is a representation of $\mathrm{G}(s) \times \mathrm{G}(s')$ by bounded operators of $\mathcal{M}_\eta$: it is ω_η composed with the pair map, a continuous homomorphism, times the continuous character $(g, h) \mapsto \alpha(\det h)\beta(\det g)^{-1}$, and $(g, f) \mapsto \omega_\eta(g)f$ is jointly continuous; so $(p, \varphi) \mapsto \omega_{\alpha,\beta}(p)\varphi$ is continuous. **The theta functional.** $\theta_{\mathrm{Hom}(S,T)}$ is a bounded linear functional on $\mathcal{M}_\eta$ ((WC3) of Definition 4.1.15). For $\gamma \in \Gamma_S$, $\gamma' \in \Gamma_T$, $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$ and $\varphi \in \mathcal{M}_\eta$, apply Lemma 6.2.5 to the vector $\omega_{\alpha,\beta}(g, h)\varphi$: since $\omega_{\alpha,\beta}(\gamma g, \gamma' h) = \omega_{\alpha,\beta}(\gamma, \gamma')\omega_{\alpha,\beta}(g, h)$,

$$\theta_{\mathrm{Hom}(S,T)}\big(\omega_{\alpha,\beta}(\gamma g, \gamma' h)\varphi\big) = \theta_{\mathrm{Hom}(S,T)}\big(\omega_{\alpha,\beta}(g, h)\varphi\big).$$

So $(\omega_{\alpha,\beta}, \theta_{\mathrm{Hom}(S,T)})$ is a theta kernel for (S, T) on $\mathrm{Hom}(S, T)$ in the sense of (ED3). **Real part.** The Fock model of (ED3) for $V = \mathrm{Hom}(S, T)$ is $\mathcal{F}_\varepsilon$ with $\varepsilon(v, i, j) = s'(i)s(j)$, which is $\mathcal{F}_{S,T}$ of (WR5). By the first identity of (5) of Lemma 4.1.1 with $k = 1$, $k' = 1$, for $x \in G_\infty(s)$, $y \in G_\infty(s')$, $v \in \mathcal{F}_{S,T}$ and $f \in \mathcal{M}_\eta$, $\omega_{\alpha,\beta}((x, 1), (y, 1))(v \otimes f) = (\Omega_{\gamma,\gamma'}(x, y)v) \otimes f$, as $\omega_{\alpha,\beta}(1, 1)$ is the identity. Hence the theta kernel has real part $\Omega_{\gamma,\gamma'}$. □

Lemma 6.3.78. *Let V be a space with a point with signs s, e an idempotent kernel on $G_\infty(s)$, F and F' forms of type e on V, and $x \in G_\infty(s)$. Then*

$$\big\langle F,\ F'(\,\cdot\,(x, 1))\big\rangle_V = \frac{\overline{e(x)}}{e(1)}\,\langle F, F'\rangle_V.$$

Proof. **Preliminaries.** $G_\infty(s)$ is a finite product of real unitary groups $\mathrm{U}(p, q)$, which are reductive and hence unimodular; so dx is invariant under left and right translations and under $x \mapsto x^{-1}$. $\mathrm{G}(s)$ is unimodular and Γ_V is a discrete subgroup, so the quotient measure on $\Gamma_V \backslash \mathrm{G}(s)$ is invariant under right translation by $\mathrm{G}(s)$. It is finite: $\mathrm{U}(V)$ is reductive over F and has no nontrivial F-rational character, since its characters factor through $\det\colon \mathrm{U}(V) \to \mathrm{U}(1)$ and the torus $\mathrm{U}(1) = \ker \mathrm{N}_{E/F}$ is anisotropic over F; hence $[\mathrm{U}(V)]$ has finite volume by [37]. Consequently $\langle T, T'\rangle_V$ is defined and finite for bounded measurable left Γ_V-invariant T, T', with $|\langle T, T'\rangle_V| \le \mathrm{vol}([\mathrm{U}(V)]) \sup|T| \sup|T'|$, and $\langle T_y, T'\rangle_V = \langle T, T'_{y^{-1}}\rangle_V$ for $y \in G_\infty(s)$, by the substitution $g \mapsto g(y^{-1}, 1)$. The kernel e is integrable (ED1), so every form T of type e satisfies $R(e)T = T$ (ED2); forms are bounded and continuous, and a translate T_y of a form is again bounded, continuous and left invariant under Γ_V.

Step 1: $R(e)$ is self-adjoint. Let T, T' be bounded continuous functions on $\mathrm{G}(s)$, left invariant under Γ_V. The function $(g, y) \mapsto \overline{T(g)}\, e(y)\, T'(g(y, 1))$ on $[\mathrm{U}(V)] \times G_\infty(s)$ is bounded

by $\sup|T|\sup|T'|\,|e(y)|$, which is integrable for the product of the finite quotient measure and dx. By Fubini,

$$\langle T, R(e)T'\rangle_V = \int_{G_\infty(s)} e(y)\,\langle T, T'_y\rangle_V\,dy = \int_{G_\infty(s)} e(y)\,\langle T_{y^{-1}}, T'\rangle_V\,dy = \int_{G_\infty(s)} e(y^{-1})\,\langle T_y, T'\rangle_V\,dy,$$

by the translation rule of the preliminaries and the substitution $y \mapsto y^{-1}$. As $e(y^{-1}) = \overline{e(y)}$ (ED1), the last integral is $\int \overline{e(y)}\,\langle T_y, T'\rangle_V\,dy$, which by Fubini in the same way equals $\int_{[\mathrm{U}(V)]} \overline{\int e(y)T(g(y,1))\,dy}\;T'(g)\,dg = \langle R(e)T, T'\rangle_V$.

Step 2: the rank one formula. Let T be a form of type e and $x \in G_\infty(s)$. For $g \in \mathrm{G}(s)$, since $R(e)T = T$,

$$(R(e)T_x)(g) = \int e(z)\,T(g(zx,1))\,dz = \int e(z)\int e(y)\,T(g(zxy,1))\,dy\,dz.$$

The integrand is bounded by $\sup|T|\,|e(z)|\,|e(y)|$, integrable on $G_\infty(s)^2$. Substituting $y' = zxy$ in the inner integral (left invariance of dy) and applying Fubini,

$$\begin{aligned}(R(e)T_x)(g) &= \int\Big(\int e(z)\,e\big((zx)^{-1}y'\big)\,dz\Big)T(g(y',1))\,dy' \\ &= \frac{e(x^{-1})}{e(1)}\int e(y')\,T(g(y',1))\,dy' = \frac{\overline{e(x)}}{e(1)}\,T(g),\end{aligned}$$

by the idempotent law of (ED1), then $R(e)T = T$ and $e(x^{-1}) = \overline{e(x)}$.

Step 3: conclusion. $F = R(e)F$, Step 1 applied to $T = F$, $T' = F'_x$, and Step 2 applied to $T = F'$ give

$$\langle F, F'_x\rangle_V = \langle R(e)F, F'_x\rangle_V = \langle F, R(e)F'_x\rangle_V = \Big\langle F, \frac{\overline{e(x)}}{e(1)}F'\Big\rangle_V = \frac{\overline{e(x)}}{e(1)}\langle F, F'\rangle_V,$$

the product being linear in its second argument. □

Lemma 6.3.79. *Let S, T be spaces and $v \in \mathcal{F}_{S,T}$ a Schwartz vector. There is a sequence of polynomials P_k such that $\mathrm{mom}_N(v - v_{P_k}) \to 0$ as $k \to \infty$, for every $N \in \mathbb{N}$.*

Proof. Let $d = \#\iota$; if $d = 0$ then $\mathcal{F}_\varepsilon = \mathbb{C}$, every vector is v_P for a constant P, and there is nothing to prove. Let $d \geq 1$ and write $|\cdot|$ for the euclidean norm of $\mathbb{C}^\iota$; two norms on $\mathbb{C}^\iota$ are equivalent, so $(1+\|z\|)^N \leq C_N(1+|z|)^N$ and conversely, and it suffices to prove the claim for the moments formed with $|\cdot|$.

Step 1: the reproducing kernel. For $x, y \in \mathbb{C}^\iota$ put $\sigma(y,x) = \zeta(y)^*\zeta(x) = \sum_{i\in\iota}\overline{\zeta(y)_i}\,\zeta(x)_i$. Expanding $|x-y|^2 = |x|^2+|y|^2-x^*y-y^*x$ in the exponent of $\beta^\infty_\varepsilon(y,x)e^\infty(x-y)$, the contribution of a coordinate i is $\bar{x}_i y_i$ if $\varepsilon_i = 1$ and $\bar{y}_i x_i$ if $\varepsilon_i = -1$, so

$$\beta^\infty_\varepsilon(y,x)\,e^\infty(x-y) = e^\infty(x)\,e^\infty(y)\,\exp\big(\sigma(y,x)\big), \qquad |\sigma(y,x)| \leq |x|\,|y|.$$

Hence, by the reproducing identity of $v \in \mathcal{F}_\varepsilon$,

$$v(x) = e^\infty(x)\int v(y)\,e^\infty(y)\,\exp\big(\sigma(y,x)\big)\,d\nu(y) \qquad (x \in \mathbb{C}^\iota).$$

Step 2: the polynomials. For $k \geq 1$ let

$$P_k(X) = \int v(y)\,e^\infty(y)\sum_{j<k}\frac{1}{j!}\Big(\sum_{i\in\iota}\overline{\zeta(y)_i}\,X_i\Big)^j d\nu(y),$$

a polynomial of degree less than k: expanding the powers, its coefficients are integrals of $v(y)e^{\infty}(y)$ times monomials in the $\overline{\zeta(y)}_i$, which converge because $e^{\infty}(y)|y|^j$ is bounded and v is integrable. By (WR5), $v_{P_k}(x) = P_k(\zeta(x))e^{\infty}(x)$, so with $R_k(\sigma) = \sum_{j\geq k}\sigma^j/j!$, Step 1 gives

$$v(x) - v_{P_k}(x) = e^{\infty}(x)\int v(y)\,e^{\infty}(y)\,R_k\big(\sigma(y,x)\big)\,d\nu(y).$$

Step 3: domination. Let $N \in \mathbb{N}$. By Tonelli,

$$\begin{aligned}\mathrm{mom}_N(v - v_{P_k}) &\leq \iint g_k(x,y)\,d\nu(y)\,d\nu(x),\\ g_k(x,y) &= (1+|x|)^N e^{\infty}(x)e^{\infty}(y)\,|v(y)|\,\big|R_k(\sigma(y,x))\big|.\end{aligned}$$

For fixed (x,y), $R_k(\sigma(y,x)) \to 0$ as $k\to\infty$ (tail of a convergent series), and $|R_k(\sigma)| \leq e^{|\sigma|} \leq e^{|x||y|}$. Since $-\frac{1}{2}|x|^2 - \frac{1}{2}|y|^2 + |x||y| = -\frac{1}{2}(|x|-|y|)^2$,

$$g_k(x,y) \leq G(x,y) = (1+|x|)^N e^{-(|x|-|y|)^2/2}\,|v(y)|.$$

In polar coordinates on $\mathbb{C}^{\iota} = \mathbb{R}^{2d}$, with $r = |y|$ and $1+\varrho \leq (1+r)(1+|\varrho - r|)$,

$$\begin{aligned}\int (1+|x|)^N e^{-(|x|-r)^2/2}d\nu(x) &= c_d\int_0^{\infty}(1+\varrho)^N \varrho^{2d-1}e^{-(\varrho-r)^2/2}d\varrho\\ &\leq c_d(1+r)^{N+2d-1}\int_{\mathbb{R}}(1+|u|)^{N+2d-1}e^{-u^2/2}du,\end{aligned}$$

c_d the ν-measure factor of the unit sphere. Hence $\iint G \leq C\,\mathrm{mom}_{N+2d-1}(v) < \infty$, v being a Schwartz vector. By dominated convergence $\iint g_k \to 0$, so $\mathrm{mom}_N(v - v_{P_k}) \to 0$. The polynomials P_k do not depend on N, which proves the claim for every N. □

Lemma 6.3.80. *Let $\mathfrak{m}$ be a modulus.*

(a) Let S, T_1, T_2 be spaces with points with signs on n, n_1, n_2. For $i = 1,2$ let $P_i \in \mathbb{C}[x]_{n_i\times n}$ and let Φ_i be a function on $\mathrm{M}_{n_i\times n}(R_{\mathfrak{m}})$. Let $P_1P_2 \in \mathbb{C}[x]_{(n_1\sqcup n_2)\times n}$ be the product of P_1 and P_2 read in the variables of the rows n_1 and n_2, and $\Phi_1\otimes\Phi_2$ the function $y \mapsto \Phi_1(y_1)\Phi_2(y_2)$ on $\mathrm{M}_{(n_1\sqcup n_2)\times n}(R_{\mathfrak{m}})$, y_i the matrix of the rows n_i of y. Then, in $\mathcal{M}_{S,T_1\oplus T_2}$,

$$\mathrm{mod}_{\mathfrak{m}}(\Phi_1\otimes\Phi_2,\, v_{P_1P_2}) = \mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_{P_1}) \boxtimes^{\mathrm{tg}} \mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_{P_2}).$$

(b) Let S_1, S_2, T be spaces with points with signs on n_1, n_2, n'. For $i = 1,2$ let $P_i \in \mathbb{C}[x]_{n'\times n_i}$ and let Φ_i be a function on $\mathrm{M}_{n'\times n_i}(R_{\mathfrak{m}})$. Let $P_1P_2 \in \mathbb{C}[x]_{n'\times(n_1\sqcup n_2)}$ be the product of P_1 and P_2 read in the variables of the columns n_1 and n_2, and $\Phi_1\otimes\Phi_2$ the function $x \mapsto \Phi_1(x_1)\Phi_2(x_2)$ on $\mathrm{M}_{n'\times(n_1\sqcup n_2)}(R_{\mathfrak{m}})$, x_i the matrix of the columns n_i of x. Then, in $\mathcal{M}_{S_1\oplus S_2,T}$,

$$\mathrm{mod}_{\mathfrak{m}}(\Phi_1\otimes\Phi_2,\, v_{P_1P_2}) = \mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_{P_1}) \boxtimes^{\mathrm{sc}} \mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_{P_2}).$$

Proof. The two cases are proved together. Let $A = \Sigma(\mathfrak{m})$ and $N = n_1 \sqcup n_2$. In case (a) the space is $\mathrm{Hom}(S, T_1\oplus T_2)$, its coordinates are those of $N\times n$ matrices and the two blocks are the rows n_1, n_2; in case (b) the space is $\mathrm{Hom}(S_1\oplus S_2, T)$, its coordinates are those of $n'\times N$ matrices and the two blocks are the columns n_1, n_2. Read every function of such matrices (archimedean, finite or adelic, and on Φ times the coordinates for the Fock model) as a function of the pair of its two blocks, and write $f_1\boxtimes f_2$ for $(x_1,x_2)\mapsto f_1(x_1)f_2(x_2)$; thus $\boxtimes^{\mathrm{tg}}$ in case (a) and $\boxtimes^{\mathrm{sc}}$ in

case (b) are $\boxtimes$ for the identity bijection on each part. By (VP5) of Definition 3.2.2 the signs of the space, $(i,j)\mapsto t(i)s(j)$, restrict on the two blocks to those of $\mathrm{Hom}(S,T_1)$ and $\mathrm{Hom}(S,T_2)$ in case (a), and of $\mathrm{Hom}(S_1,T)$ and $\mathrm{Hom}(S_2,T)$ in case (b); and the theta models and Fock models depend only on the signs.

Step 1: factorisation of the ingredients. The bicharacters β of (WD6) and β^∞ of (WR5), the vacua e of (WD5) and e^∞, the phase of (WD8), the functions τ_y of (WR6), the holomorphic coordinates ζ of (WR5), the maps fv, rv and the measures μ, ν are sums, products or product measures over the coordinates, and the coordinates of the space are the disjoint union of those of the two blocks. Hence: (i) $\rho(z)(f_1\boxtimes f_2)=\rho(z_1)f_1\boxtimes\rho(z_2)f_2$ for $z=(z_1,z_2)$; (ii) $v^0=v^0\boxtimes v^0$; (iii) $v_{P_1P_2}=v_{P_1}\boxtimes v_{P_2}$; (iv) $\tau_{(y_1,y_2)}=\tau_{y_1}\boxtimes\tau_{y_2}$; (v) $(\Phi_1\otimes\Phi_2)^\natural=\Phi_1^\natural\boxtimes\Phi_2^\natural$, since the components at A of a matrix are integral exactly when those of both blocks are, and then the reduction modulo $\mathfrak{m}$ is taken blockwise.

Step 2: Schrödinger data. If $\Psi=\sum_y a_y\tau_y$ is any finite expansion, then $\Psi\otimes_A f=\sum_y a_y\rho(\mathrm{fv}(y))P^Af$: a linear section taking Ψ to (a_y) exists (extend from a basis containing Ψ), and the datum does not depend on the section. Expand $\Phi_i^\natural=\sum_y a_y^i\tau_y$ ($\mathcal{T}_A$ is spanned by the τ_y). By (iv), (v), $(\Phi_1\otimes\Phi_2)^\natural=\sum a^1_{y_1}a^2_{y_2}\tau_{(y_1,y_2)}$, so by $P^Av^0=v^0$, (i) and (ii),

$$(\Phi_1\otimes\Phi_2)^\natural\otimes_A v^0=\sum a^1_{y_1}a^2_{y_2}\,\rho(\mathrm{fv}(y_1))v^0\boxtimes\rho(\mathrm{fv}(y_2))v^0=(\Phi_1^\natural\otimes_A v^0)\boxtimes(\Phi_2^\natural\otimes_A v^0).$$

Step 3: data of products. Let v_i be Fock vectors and f_i vectors of the theta models of the two blocks. Evaluation at a point is a bounded functional ($|f(x)|=|\langle\rho(x)e,f\rangle|\le\|f\|$ as $|e|\le1$), so it commutes with the Bochner integral defining $v\otimes f$, and

$$\begin{aligned}((v_1\boxtimes v_2)\otimes(f_1\boxtimes f_2))(x)&=\int v_1(t_1)v_2(t_2)\,\beta(\mathrm{rv}(t),x)\,(f_1\boxtimes f_2)(x-\mathrm{rv}(t))\,d\nu(t)\\&=(v_1\otimes f_1)(x_1)\,(v_2\otimes f_2)(x_2)\end{aligned}$$

by Step 1 (factorisation of β, rv and ν) and Fubini (the integrand is bounded by

$$|v_1(t_1)||v_2(t_2)|\,\|f_1\|\|f_2\|,$$

which is integrable). So

$$(v_1\boxtimes v_2)\otimes(f_1\boxtimes f_2)=(v_1\otimes f_1)\boxtimes(v_2\otimes f_2).$$

Conclusion. By (iii), Step 2 and Step 3,

$$\mathrm{mod}_{\mathfrak{m}}(\Phi_1\otimes\Phi_2,v_{P_1P_2})=(v_{P_1}\boxtimes v_{P_2})\otimes((\Phi_1^\natural\otimes_A v^0)\boxtimes(\Phi_2^\natural\otimes_A v^0))=\mathrm{mod}_{\mathfrak{m}}(\Phi_1,v_{P_1})\boxtimes\mathrm{mod}_{\mathfrak{m}}(\Phi_2,v_{P_2}),$$

which is $\boxtimes^{\mathrm{tg}}$ in case (a) and $\boxtimes^{\mathrm{sc}}$ in case (b). □

6.4 Auxiliary forms span the target forms

This section proves that auxiliary forms span the target forms (Lemma 6.4.5), for a scaling γ of the twists with $\gamma(1)=1$ and $|\gamma(\vartheta)|=\mathrm{N}Q(\vartheta)^{e/2}\prod_{0\le i,j<m}\tau_{ij}(\vartheta)$. It follows from a statement about all forms of principal level on Y_0 (Proposition 6.4.6), which the good projection carries to the target forms.

The difficulty is that the chain Θ_ϑ runs through indefinite groups and integrality must hold for every twist at once, while integrality and size can be read off only on definite groups, where forms are algebraic modular forms [110]. The proof therefore moves the chain to definite spaces

through two grids of hermitian spaces (Definition 3.2.28), whose row m consists of definite spaces. A path through the grids pairs a form on Y_m with a form on the second grid, by moving both along its two halves and taking a period where they meet. On the first path the pairing is a period of a flat restriction, which is onto by deep-level vanishing (Lemma 6.4.9). Each turn of a path across a unit square is an instance of the one-square identity and costs one turn ratio, so the m^2 turns to the last path cost $\prod_{i,j}\tau_{ij}$ (Proposition 6.4.87). On the last path all spaces are definite, and the periods are finite sums with one denominator for all twists (Proposition 6.4.80). In our reading this is the device of the seesaw identity of Section 4, carried out in two directions on a grid rather than along one chain.

The section opens with the discrete series of $\mathrm{U}(p,q)$ and its parameters (Definition 6.4.1; see [119], and [163] for lowest K-types), followed by the objects of the first and the last path. The last path is defined for arbitrary weight parameters (Definition 6.4.4); right after it the weights are fixed, once and for all, as the numbers of Lemma 6.4.3. For Weyl modules realised by products of minors and their integral structure see [91, 3, 67, 151]. Deep level is measured with the sizes of Definition 5.6.3 (Section 5.6), and the type kernels are built from Flensted-Jensen's functions [81].

Definition 6.4.1. Let $n \geq 1$ and $j = (j_1, \dots, j_n) \in \{\pm 1\}^n$ a **signature vector**, with $p(j) = \#\{a : j_a = 1\}$, $q(j) = \#\{a : j_a = -1\}$ and $q_0(j) = \min(p(j), q(j))$. Put $J_j = \mathrm{diag}(j_1, \dots, j_n)$ and

$$\mathrm{U}(J_j) = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^* J_j g = J_j\} \cong \mathrm{U}(p(j), q(j)),$$

with maximal compact subgroup $K_j = \mathrm{U}(J_j) \cap \{g : gJ_j = J_j g\} \cong \mathrm{U}(p(j)) \times \mathrm{U}(q(j))$ and diagonal maximal torus $T = \{\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})\} \subset K_j$. A weight $\mu \in \mathbb{R}^n$ is the character $\mathrm{diag}(e^{i\theta_a}) \mapsto e^{i\sum_a \mu_a \theta_a}$ of T. Put

$$\rho_c(j)_a = \tfrac{1}{2}\big(\#\{b > a : j_b = j_a\} - \#\{b < a : j_b = j_a\}\big),$$
$$\rho_n(j)_a = \tfrac{1}{2}\big(\#\{b > a : j_b \neq j_a\} - \#\{b < a : j_b \neq j_a\}\big).$$

(Hc1) A **Harish-Chandra parameter** is $\Xi \in \mathbb{R}^n$ with $\Xi_1 > \Xi_2 > \cdots > \Xi_n$ and $\Xi_a - \frac{n-1}{2} \in \mathbb{Z}$ for all a.

(Hc2) Ξ has **gap at least** h for j if $|\Xi_a - \Xi_b| \geq h$ whenever $j_a \neq j_b$.

(Hc3) Its **Blattner parameter** (lowest K-type) is $\beta_j(\Xi) = \Xi + \rho_n(j) - \rho_c(j)$, and the compact dominance order is: $\mu \preceq \mu'$ if $\mu' - \mu$ is a nonnegative integer combination of $e_a - e_b$ with $a < b$, $j_a = j_b$.

(Hc4) The **discrete-series gap threshold** is $h_{\mathrm{DS}}(j) = \max\big(n\,p(j)\,q(j) + 1,\ 16\,n\,q_0(j) + 8\sqrt{2q_0(j)}\big)$.

A unitary representation $(\pi, \mathcal{H})$ of $\mathrm{U}(J_j)$ on a nonzero Hilbert space is a **discrete series with parameter** Ξ if it is irreducible, its matrix coefficients are square-integrable, it contains an irreducible K_j-subrepresentation of highest weight $\beta_j(\Xi)$ (highest with respect to $\preceq$), and every K_j-type μ occurring in π satisfies $\|\mu + 2\rho_c(j)\| \geq \|\beta_j(\Xi) + 2\rho_c(j)\|$.

Definition 6.4.2. Let D be admissible data, ϑ an m-fold twist and $z \in \{\mathrm{I}, \mathrm{II}\}$; the first column of the grid z consists of the nodes $(z, 0, j)$, $0 \leq j \leq m$, with the spaces Y_j ($z = \mathrm{I}$) and $\tilde{Y}_0^j$ ($z = \mathrm{II}$), which are the spaces $V^0_{z,j}$ of (TV1). For $j \geq 0$ let $e^{\mathrm{col}}_{z,j}$ be the edge $((z,0,j),(z,0,j+1))$.

(FY1) Transposed point lifts and the transposed chain. For $j < m$, with $S = V^0_{z,j}$, $T = V^0_{z,j+1}$ and ν_1, ν_2, φ those of the triangle lift $\mathrm{Tl}^0_{\vartheta,z,j}$ ((TV3)), the **transposed point lift**

sends $F \in \mathcal{F}(z,0,j+1)$ to the function

$$\mathrm{Pt}_{\vartheta,z,j}F\,(g) = \#\mathrm{GL}_{n_{z,0,j}}(\mathcal{O}_E/\mathfrak{t}_\vartheta(z)) \int_{[\mathrm{U}(T)]} \theta^{S,T}_{\nu_1,\nu_2}(\varphi)(g,h)\,F(h)\,dh$$

on the adelic group of S. The **transposed chain** is $\Theta^\flat_{\vartheta,z,m} = \mathrm{id}$ and $\Theta^\flat_{\vartheta,z,j} = \mathrm{Pt}_{\vartheta,z,j} \circ \Theta^\flat_{\vartheta,z,j+1}$ $(j < m)$; write $\Theta^\flat = \Theta^\flat_{1,\mathrm{II},0}$.

(FY2) Forms of deep level. For $0 \le i \le m$, $\mathcal{F}_\mathfrak{b}(i)$ is the space of the grid forms $\varphi \in \mathcal{F}(\mathrm{II},0,i)$ on $\tilde{Y}^i_0$ with $\varphi(gk) = \varphi(g)$ for all g and all $k \in \mathbf{K}_0(n_{\mathrm{II},0,i})$ whose reduction modulo $\mathfrak{b}$ is 1.

(FY3) Flat restrictions. For an m-fold twist ϑ let $\mathrm{K}_\vartheta(Y_0) \subseteq \mathbf{K}_0(n_{\mathrm{I},0,0})$ be the group of the elements whose reduction modulo every chosen place and every place of Q is 1, and $\mathcal{F}^+_\vartheta(Y_0)$ the forms on Y_0 of type e_λ, $\lambda = \frac{1}{2}p_{\mathrm{I},0,0}$, fixed by $\mathrm{K}_\vartheta(Y_0)$. For a real point $g \in G_\infty(s)$ of $\tilde{Y}^0_0 = Y_0 \oplus \langle 1\rangle$, a linear map $r\colon \mathcal{F}_\mathfrak{b}(m) \to \mathcal{A}_M(Y_0)$ is a **flat restriction** at g if for all $\varphi \in \mathcal{F}_\mathfrak{b}(m)$, all m-fold twists ϑ and all $y \in \mathcal{F}^+_\vartheta(Y_0)$

$$\langle y, r\varphi\rangle_{Y_0} = \int_{[\mathrm{U}(Y_0)]} \overline{y(x)}\;(\Theta^\flat\varphi)\big(\mathrm{diag}(x,1)\,(g,1)\big)\,dx.$$

(FY4) The point data of the first path. For an edge tuple S for ϑ, the **point datum** of the first path is

$$\Psi^1(S) = \prod_z \prod_{0\le j<m} \Phi^{\mathrm{pt}}_j(S(e^{\mathrm{col}}_{z,j}) \bmod \mathfrak{t}_\vartheta(z)),$$

Φ^{pt}_j the point datum of (TV3) for the triangle z and the matrix reduced from $R_{\mathfrak{m}_\vartheta}$ to R_ϑ. A **level coset family** x is a choice, for each z and $0 \le j < m$, of a coset $x_{z,j}$ in $\mathrm{GL}_{n_{z,0,j}}(R_{\mathfrak{m}_\vartheta})$ of the trivial subgroup if $z = \mathrm{I}$ and of the kernel of the reduction modulo $\mathfrak{b}$ if $z = \mathrm{II}$. Fix a representative of each coset; the **family** g_x of x is the family of invertible matrices modulo $\mathfrak{m}_\vartheta$ at the nodes with the representative of $x_{z,j}$ at $(z,0,j)$, except $g_x(\mathrm{II},0,0) = g_x(\mathrm{I},0,0)\oplus 1$, and 1 at all other nodes. The **first-path datum** is

$$\Psi^0(S) = \sum_x \epsilon_w(g_x)\,\Psi^1(g_x \cdot S),$$

the sum over the level coset families, w the set of edges of the first path.

Lemma 6.4.3. *Let D be admissible data and $z \in \{\mathrm{I},\mathrm{II}\}$. For $0 \le i \le m$ let N_i be the rank of the space at the node (z,i,m), n_i the index set of its basis, and $\gamma_i = \gamma_{z,i}$, so that $\gamma_i(N)(v) = \lfloor (N+[\neg z])/2\rfloor - \kappa_{i+1}(v)$ is the weight of the character of the lift move along $((z,i,m),(z,i+1,m))$ at an end of rank N. There are natural numbers $\kappa(v,h)$ ($v \in \Phi$, $h \ge 1$; the number of columns of height h) and integers $e_i(v)$ ($0 \le i \le m$, $v \in \Phi$) such that:*

(1) for $0 \le i < m$: $N_i < N_{i+1}$, $\kappa(v,h) = 0$ for $N_i < h < N_{i+1}$, and $e_i(v) = \gamma_i(N_{i+1})(v) + \kappa(v,N_i)$, for every v;

(2) for $0 \le i < m$: $\gamma_i(N_i) = e_{i+1}$;

(3) for $0 \le i \le m$, the family $(a \mapsto -p_{z,i,m}(v)(a))_{v\in\Phi}$ of maps $n_i \to \mathbb{Z}$ is a doubled Harish-Chandra parameter of (κ, e_i) on n_i, κ read in the rank N_i;

(4) for $0 \le i \le m$, $\sum_{v\in\Phi}\sum_{1\le h<N_i} h\,\kappa(v,h) \le b\,r^4$;

(5) for $0 \le i \le m$, $|e_i(v)| \le 20b$ for every $v \in \Phi$.

Proof. Fix $v \in \Phi$ and z; write $c_j = \kappa_j(v)$, $R = r^8$, $[z] \in \{0,1\}$, and $d_i = 1$ if $z = \mathrm{I}$ and i is odd, $d_i = 0$ otherwise (the δ of (GP5) at the node (z,i,m), as m is even). Put $q_i = -p_{z,i,m}(v)\colon n_i \to \mathbb{Z}$. By (GP5) its entries are:

- at the down index (l,c), $1 \le l \le m$, $0 \le c < t - [z]$: $2c_l - t + [z] + 1 + 2c + d_i$;
- at the base index j' with $k-1 \le j' \le r - 1 + [z]$, $j = j' + 2 - k \in \{1, \dots, m + [z]\}$: $2c_j - 2R - [z] + d_i$;
- at the base index $j' < k - 1$: $2j' - k + 2 - [z] + d_i$;
- at the up index (l,c), $1 \le l \le i$, $0 \le c < t + [z]$: $-2c_l - t - [z] + 1 + 2c + d_i$.

The ranks are $N_i = r + (i+m)t$ for $z = \mathrm{I}$ and $N_i = r + 1 + i(t+1) + m(t-1)$ for $z = \mathrm{II}$, so $N_{i+1} = N_i + t + [z]$.

Step 1: the entries of q_i are distinct and $\equiv N_i - 1 \pmod 2$. All entries are $\equiv [z] + d_i \pmod 2$ (t odd); and $N_i - 1 \equiv i \equiv d_i$ for $z = \mathrm{I}$, $N_i - 1 \equiv r \equiv 1$ for $z = \mathrm{II}$ (r, t odd, m even). Within one group (a down block, the small base indices $j' < k-1$, an up block) consecutive entries differ by 2. Each group lies within distance $t+k+2$ of its centre: $2c_l = 2\kappa(v) + (4l+2)R$ for the down block l, $2\kappa(v) + 4jR$ for the base entry j, 0 for the small base entries, $-2\kappa(v) - (4l+2)R$ for the up block l; distinct centres differ by at least $2R > 2(t+k+2)$. So the entries are distinct, and in decreasing order the groups are: down block m (for $z = \mathrm{II}$ preceded by the base entry $m+1$), base entry m, down block $m-1$, …, down block 1, base entry 1, the small base entries, up block 1, …, up block i.

Step 2: definition of κ and e_i. Let $u_0^{(i)} > u_1^{(i)} > \cdots > u_{N_i-1}^{(i)}$ be the entries of q_i. The entries of q_i are those of q_{i+1} outside the up block $i+1$, shifted by $d_i - d_{i+1}$, and by Step 1 the up block $i+1$ consists of the $t + [z]$ smallest entries of q_{i+1}. Hence $u_p^{(i)} = u_p^{(m)} + d_i - d_m$ for $p < N_i$. Define, for $1 \le h < N_m$ and $h \ge N_m$ respectively,

$$\kappa(v,h) = \tfrac{1}{2}(u_{h-1}^{(m)} - u_h^{(m)}) - 1, \qquad \kappa(v,h) = 0, \qquad \text{and} \qquad e_i(v) = \tfrac{1}{2}(u_{N_i-1}^{(i)} + N_i - 1).$$

By Step 1 these are natural numbers and integers, and $\kappa(v,h) = \frac{1}{2}(u_{h-1}^{(i)} - u_h^{(i)}) - 1$ for $1 \le h < N_i$.

(3). With $\lambda(p) = \sum_{p<h<N_i} \kappa(v,h)$, the numbers $\mathrm{hc}(p) = 2(e_i(v) + \lambda(p)) + N_i - 1 - 2p$ satisfy $\mathrm{hc}(N_i - 1) = 2e_i(v) - N_i + 1 = u_{N_i-1}^{(i)}$ and $\mathrm{hc}(p) - \mathrm{hc}(p+1) = 2\kappa(v, p+1) + 2 = u_p^{(i)} - u_{p+1}^{(i)}$; so $\mathrm{hc}(p) = u_p^{(i)}$ for all p. Thus q_i is injective and each of its values is some $\mathrm{hc}(p)$, which is (3) at v.

(1), (2). $N_i < N_{i+1}$. For $N_i < h < N_{i+1}$, $u_{h-1}^{(i+1)}$ and $u_h^{(i+1)}$ are consecutive entries of the up block $i+1$, so $\kappa(v,h) = 0$. Let $s_i = u_{N_i-1}^{(i)}$ be the smallest entry of q_i; for $i \ge 1$ it is the entry $(i,0)$ of the up block i, $s_i = -2c_i - t - [z] + 1 + d_i$. The largest entry of the up block $i+1$ of q_{i+1} is $-2c_{i+1} + t + [z] - 1 + d_{i+1}$, and the entry of q_{i+1} just above it is $s_i + d_{i+1} - d_i$. Hence

$$\kappa(v,N_i) = \tfrac{1}{2}(s_i - d_i + 2c_{i+1} - t - [z] - 1), \qquad e_i(v) - \kappa(v,N_i) = \tfrac{1}{2}(N_i + t + [z] + d_i) - c_{i+1},$$

$$e_{i+1}(v) = \tfrac{1}{2}(s_{i+1} + N_{i+1} - 1) = \tfrac{1}{2}(N_i + d_{i+1}) - c_{i+1}.$$

It remains to check $\frac{1}{2}(N_i + t + [z] + d_i) = \lfloor \frac{1}{2}(N_{i+1} + [\neg z]) \rfloor$ and $\frac{1}{2}(N_i + d_{i+1}) = \lfloor \frac{1}{2}(N_i + [\neg z]) \rfloor$. For $z = \mathrm{II}$, $d = 0$ and N_i is even, $t+1$ even, so both sides agree. For $z = \mathrm{I}$, $N_i \equiv i + 1$: if i is even, $N_i + t$ is even, $d_i = 0$, $d_{i+1} = 1$, N_i odd; if i is odd, $N_i + t$ is odd, $d_i = 1$, $d_{i+1} = 0$, N_i even; in each case both identities hold. This is (1) and (2) at v.

(4), (5). By Lemma 4.2.67, $f \le x^{12}$, $r \le 3x^5$, $t \le 15x^5$, $m \le 2\sqrt{x}$, $R \le 3^8 x^{40}$, $x^{10^{12}} \le b$, and $r \ge x^5$ by (AD2); also $\kappa(v) \le b$ since $x \le x_1$. Every entry of every q_i ($i \le m$) has absolute value

at most $2c_{m+1}+t+k+2 \le 2b+2(2m+3)R+t+k+2 \le 3b$. Since $\kappa(v,h)=\frac{1}{2}(u_{h-1}-u_h)-1$, summation by parts gives

$$\sum_{1\le h<N_i} h\,\kappa(v,h) \le \tfrac{1}{2}\sum_{h=1}^{N_i-1} h\,(u^{(i)}_{h-1}-u^{(i)}_h) = \tfrac{1}{2}\sum_{p=0}^{N_i-1}(u^{(i)}_p - u^{(i)}_{N_i-1}) \le 3bN_i.$$

With $N_i \le r+1+2mt \le 64x^{11/2}$, the sum over $v\in\Phi$ is at most $3b\,x^{12}\cdot 64x^{11/2} = 192\,x^{35/2}b \le x^{20}b \le b\,r^4$, as $x^{5/2}\ge 192$. Finally $|e_i(v)| \le \frac{1}{2}(3b+N_i) \le 2b \le 20b$, as $N_i \le b$. □

Definition 6.4.4. Let D be admissible data. For $z\in\{\mathrm{I},\mathrm{II}\}$ and $i\ge 0$ let $n_{z,i}$ be the index set of the basis of the space at the node (z,i,m) and $N_{z,i}=\#n_{z,i}$; for $i\le m$ these spaces are positive definite ((GP4)), so their adelic unitary group is $G_\infty(1)\times G_f(n_{z,i})$. The edges of the half z of the last path are $e_{z,i}=((z,i,m),(z,i+1,m))$, $0\le i<m$.

(LQ1) Weights. For $0\le i\le m$ let $l_{z,i}=-p_{z,i,m}$ be the family of the maps $a\mapsto -p_{z,i,m}(v)(a)$ on $n_{z,i}$ ($v\in\Phi$). For each z let natural numbers $\kappa_z(v,h)$ ($v\in\Phi$, $h\ge 1$) and integers $e^{\det}_{z,i}(v)$ ($v\in\Phi$, $0\le i\le m$) be given, the **weight parameters** of the half z (the $e^{\det}_{z,i}$ are the determinant exponents; the letter $e_{z,i}$ alone denotes an edge), with the following two properties:

(i) $\kappa_z(v,h)=0$ for $N_{z,i}<h<N_{z,i+1}$, for every $v\in\Phi$ and $0\le i<m$;

(ii) for $0\le i\le m$, $l_{z,i}$ is a doubled Harish-Chandra parameter of $(\kappa_z, e^{\det}_{z,i})$ on $n_{z,i}$, κ_z read in the rank $N_{z,i}$.

This definition uses no other property of them. For $0\le i\le m$ the weight at (z,i,m) is $(\kappa_z,e^{\det}_{z,i})$, κ_z read in the rank $N_{z,i}$; its Weyl module is $\mathbb{V}_{z,i}=\mathbb{V}_{\kappa_z}$ in the rank $N_{z,i}$, with index set $I_{z,i}$, and its algebraic forms are $\mathcal{A}_{z,i}=\mathcal{A}_{n_{z,i}}(\kappa_z,e^{\det}_{z,i})$.

(LQ2) Levels. For an m-fold twist ϑ, the **level** $\mathrm{K}_\vartheta(z,i)\subseteq\mathbf{K}_0(n_{z,i})$ is the group of the elements whose reduction modulo $\mathfrak{m}_\vartheta$ is 1 if $(z,i)=(\mathrm{I},0)$, and of those whose reduction modulo $\mathfrak{b}$ is 1 otherwise; the reduction modulo an ideal $\mathfrak{a}$ is the map $\mathbf{K}_0(n)\subseteq\mathrm{GL}_n(\widehat{\mathcal{O}}_E)\to\mathrm{GL}_n(\mathcal{O}_E/\mathfrak{a})$.

(LQ3) Class integrals. Let n be an index set, $\mathrm{K}\subseteq\mathbf{K}_0(n)$ an open subgroup and F a function on $G_f(n)$ with values in a complex vector space, left invariant under the finite components of the rational points of I_n and right invariant under an open subgroup of $G_f(n)$. With $\mathrm{K}_F=\{k\in\mathrm{K}: F(gk)=F(g)\text{ for all }g\}$, the **class integral** of F at the level K is

$$\mathrm{CI}_\mathrm{K}(F)=[\mathrm{K}:\mathrm{K}_F]^{-1}\sum_{[h]}\frac{1}{\#\Gamma_h}F(h),$$

the class sum of Definition 3.2.23 for the level K_F.

(LQ4) Algebraic theta lifts. Let $0\le i<m$, $n=n_{z,i}$, $n'=n_{z,i+1}$, $N=N_{z,i}$, $N'=N_{z,i+1}$, and suppose $N\le N'$, so that by (i) of (LQ1) the Schur matrix of κ_z from the rank N to the rank N' is defined ((WM2)); let $\mathcal{M}_{n,n'}$ be the theta model of $\mathrm{Hom}(\mathrm{I}_n,\mathrm{I}_{n'})$, $\theta_{n,n'}$ its canonical theta functional. Let $X=(X_\tau)_{\tau\in\Phi}$ be the $n'\times n$ matrices of the variables $X_{\tau,b,a}$ ($b\in n'$, $a\in n$) of the Fock model of $\mathrm{Hom}(\mathrm{I}_n,\mathrm{I}_{n'})$, and $P_{i'i}$ the entry (i',i) of the Schur matrix $S_{\kappa_z}(X)$ from the rank N to the rank N' ($i'\in I_{z,i+1}$, $i\in I_{z,i}$). For $f\in\mathcal{M}_{n,n'}$ the **jet matrix** $\mathrm{jm}(f)\in\mathrm{M}_{I_{z,i+1}\times I_{z,i}}(\mathbb{C})$ has the entries

$$\mathrm{jm}(f)_{i'i}=\theta_{n,n'}(v_{P_{i'i}(\eta_P X)}\otimes f),$$

$P(\eta_P X)$ the polynomial obtained by substituting $\eta_P X_{\tau,b,a}$ for $X_{\tau,b,a}$. For an m-fold twist ϑ let α and β be the edge characters of the lift move along $e_{z,i}$ at the ends of ranks N and N' ((GP6) of Definition 3.2.28), and write $(1,g)$ for the adelic point with archimedean component 1. The **algebraic theta lift** along $e_{z,i}$ with datum f is the partial map sending a function $\eta\colon G_f(n) \to \mathbb{V}_{z,i}$ to the function on $G_f(n')$

$$\mathrm{Al}_{\vartheta,z,i}(f)\eta\,(h) = \mathrm{CI}_{\mathrm{K}_\vartheta(z,i)}\Big(g \mapsto \mathrm{jm}\big(\omega_{\alpha,\beta}((1,g),(1,h))f\big)\,\eta(g)\Big).$$

It is defined at η when, for every $h \in G_f(n')$, the integrand $g \mapsto \mathrm{jm}(\omega_{\alpha,\beta}((1,g),(1,h))f)\,\eta(g)$ satisfies the hypotheses of (LQ3) for the index set n. For an edge tuple S for ϑ, the **edge datum** is $\varphi^S_{z,i} = c_s^{NN'}\,\Phi^\natural \otimes_{\Sigma(\mathfrak{m}_\vartheta)} v^0 \in \mathcal{M}_{n,n'}$, where Φ is the indicator function of the matrix $S(e_{z,i})$ on the $n' \times n$ matrices over $R_{\mathfrak{m}_\vartheta}$.

(LQ5) Algebraic forms of grid forms and chains of lifts. The **algebraic form** of a grid form $y \in \mathcal{F}(z,0,m)$ is $y^\flat = y^\flat_{l_{z,0}}$, the algebraic form of the function y on $G_\infty(1) \times G_f(n_{z,0})$ for $l_{z,0}$ and the weight $(\kappa_z, e^{\det}_{z,0})$ ((WM7)). For an m-fold twist ϑ, an edge tuple S for ϑ and $y \in \mathcal{F}(z,0,m)$, the **chain of algebraic lifts** is

$$\mathrm{Lc}^S_{z,0}(y) = y^\flat, \qquad \mathrm{Lc}^S_{z,i+1}(y) = \mathrm{Al}_{\vartheta,z,i}(\varphi^S_{z,i})\,\mathrm{Lc}^S_{z,i}(y) \quad (0 \le i < m),$$

where $\mathrm{Lc}^S_{z,i+1}(y)$ is defined when $\mathrm{Lc}^S_{z,i}(y)$ is defined and $\mathrm{Al}_{\vartheta,z,i}(\varphi^S_{z,i})$ is defined at it ((LQ4)).

(LQ6) Branching pairings at (m,m). Let $n = n_{\mathrm{I},m}$ and $n' = n_{\mathrm{II},m}$, identified with $n \sqcup \{*\}$, $*$ the added line of (GP2); for g an $n \times n$ matrix and u a scalar, $\mathrm{diag}(g,u)$ is the $n' \times n'$ block matrix. Put $l = l_{\mathrm{I},m}$, $l' = l_{\mathrm{II},m}$ and let B_0 be the number of boxes $\sum_\tau \sum_{1 \le h < N'} h\,\kappa_{\mathrm{II}}(\tau,h)$, $N' = \#n'$. An integer matrix B with rows $I_{\mathrm{I},m}$ and columns $I_{\mathrm{II},m}$ is a **branching pairing at** (m,m) if:

(a) there are integers N_τ ($\tau \in \Phi$) with $|N_\tau| \le B_0 + |e^{\det}_{\mathrm{II},m}(\tau)|$ such that for every commutative ring S with an involution $x \mapsto \bar{x}$, every field R of characteristic 0, all ring homomorphisms $f_\tau\colon S \to R$ ($\tau \in \Phi$), every $g \in \mathrm{M}_n(S)$ with $\bar{g}^{\mathsf{T}} g = 1$, every $u \in S$ with $\bar{u}u = 1$ and all $v' \in R^{I_{\mathrm{I},m}}$, $v \in R^{I_{\mathrm{II},m}}$,

$$\big(P_{\kappa_{\mathrm{I}}, e^{\det}_{\mathrm{I},m}}(f(g))\,v'\big)^{\mathsf{T}} B\, P_{\kappa_{\mathrm{II}}, e^{\det}_{\mathrm{II},m}}(f(\mathrm{diag}(g,u)))\,v = \prod_\tau f_\tau(u)^{N_\tau}\; v'^{\mathsf{T}} B v,$$

where $f(x)$ is the family $(f_\tau(x))_\tau$ of entrywise images;

(b) for every $M \in \mathbb{N}$ with $B_0 \le M$ and all $v' \in \mathbb{V}_{\mathrm{I},m}$, $v \in \mathbb{V}_{\mathrm{II},m}$, $|v'^{\mathsf{T}} B v| \le (M^M)^3\,\|v'\|_{\kappa_{\mathrm{I}}} \|v\|_{\kappa_{\mathrm{II}}}$;

(c) for every $g \in G_\infty(1)$ for n' there is $c \in \mathbb{C}$ such that for all $\eta \in \mathcal{A}_{\mathrm{I},m}$, $\varphi \in \mathcal{A}_{\mathrm{II},m}$, $u \in \mathrm{U}(1)(\mathbb{A}_f)$ and every open subgroup $\mathrm{K} \subseteq \mathbf{K}_0(n)$,

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} F^l_\eta(x)\,F^{l'}_\varphi\big(\mathrm{diag}(x,(1,u))\,(g,1)\big)\,dx = \frac{c}{[\mathbf{K}_0(n) : \mathrm{K}]}\,\mathrm{CI}_{\mathrm{K}}\Big(h \mapsto \eta(h)^{\mathsf{T}} B\,\varphi(\mathrm{diag}(h,u))\Big).$$

(LQ7) The last-path sum. For an m-fold twist ϑ and $h \in G_f(n_{\mathrm{I},m})$ the **block point** is $\mathrm{bp}_\vartheta(h) = \mathrm{diag}(h, \mathrm{flp}^m_\vartheta) \in G_f(n_{\mathrm{II},m})$. For a branching pairing B at (m,m), a function Ψ on the edge tuples for ϑ and a pair $y = (y_{\mathrm{I}}, y_{\mathrm{II}})$ with $y_z \in \mathcal{F}(z,0,m)$, the **last-path sum** is

$$\mathrm{Ls}_B(\Psi, y) = \sum_S \Psi(S)\;\mathrm{CI}_{\mathrm{K}_\vartheta(\mathrm{I},m)}\Big(h \mapsto \mathrm{Lc}^S_{\mathrm{I},m}(y_{\mathrm{I}})(h)^{\mathsf{T}} B\,\mathrm{Lc}^S_{\mathrm{II},m}(y_{\mathrm{II}})(\mathrm{bp}_\vartheta(h))\Big),$$

the sum over the edge tuples S for ϑ supported on the edges of the last path; it is defined when, for every such S, both chains $\mathrm{Lc}^S_{z,m}(y_z)$ are defined and the integrand of its class integral satisfies the hypotheses of (LQ3). The pair y is **fixed by the outer levels** if $y^\flat_z(gk) = y^\flat_z(g)$ for all g, all $k \in \mathrm{K}_\vartheta(z,0)$ and both z.

From now on, for all admissible data D and both $z \in \{\mathrm{I},\mathrm{II}\}$, the weight parameters of the half z in (LQ1) of Definition 6.4.4 are fixed as a choice of the numbers provided by Lemma 6.4.3 for D and z: $\kappa_z(v,h) = \kappa(v,h)$ ($v \in \Phi$, $h \geq 1$) and $e^{\det}_{z,i}(v) = e_i(v)$ ($v \in \Phi$, $0 \leq i \leq m$). In that lemma $N_i = N_{z,i}$ and $n_i = n_{z,i}$, and the family of its item (3) is $l_{z,i}$; so property (i) of (LQ1) is part of its item (1) and property (ii) is its item (3). Its item (1) also gives $N_{z,i} < N_{z,i+1}$ for $0 \leq i < m$, so the algebraic theta lifts of (LQ4) are defined along every edge of the last path; its items (2), (4) and (5) hold for the fixed weight parameters.

Lemma 6.4.5. *Let D be admissible data. There are $e \in \mathbb{Z}$ and a function γ from the m-fold twists to $\mathbb{C} \setminus \{0\}$ with $\gamma(1) = 1$ and*

$$|\gamma(\vartheta)| = \mathrm{N}Q(\vartheta)^{e/2} \prod_{0 \leq i,j < m} \tau_{ij}(\vartheta) \qquad \text{for every } m\text{-fold twist } \vartheta,$$

such that $\mathcal{T}$ is spanned by the target forms that are auxiliary for γ.

Proof. Let c be the function of Proposition 6.4.6 and $F = \prod_{0 \leq i,j < m} \tau_{ij}$.

The scalars. Put $\gamma(\vartheta) = c(\vartheta)/c(1)$; then $\gamma(\vartheta) \neq 0$ and $\gamma(1) = 1$. Each τ_{ij} is a product of ratios, so $F \geq 0$, and $F(1) = 1$, since $r_p(1) = 1$ for every critical pair p by Lemma 6.1.7(b) (so $\tau_{ij}(1) = 1$, Lemma 6.1.7(c)). By the choice of c there are $C \in \mathbb{R}$ and $e \in \mathbb{Z}$ with $|c(\vartheta)|^2 = C\,\mathrm{N}Q(\vartheta)^e F(\vartheta)^2$ for all ϑ; at $\vartheta = 1$, where $Q = \emptyset$, this gives $C = |c(1)|^2 > 0$. Hence $|\gamma(\vartheta)|^2 = \mathrm{N}Q(\vartheta)^e F(\vartheta)^2$, and taking nonnegative square roots, $|\gamma(\vartheta)| = \mathrm{N}Q(\vartheta)^{e/2} F(\vartheta)$.

Projection to the target. Let $P = |\bar{G}|\, e_0$, acting on $\mathcal{A}_M(Y_0)$. Since e_0 is an idempotent and $\mathcal{T} = e_0 \mathcal{A}_M(Y_0)$, the map P is onto $\mathcal{T}$. By Proposition 6.4.6 the forms y with (i) and (ii) span $\mathcal{A}_M(Y_0)$, so the forms Py for such y span $\mathcal{T}$. It remains to show that each such Py is auxiliary for γ.

(AF1). Let ϑ be an m-fold twist and $\iota \mid \ell$ with $\ell \nmid \mathrm{N}Q(\vartheta)$. By (i), $\gamma(\vartheta)\Theta_\vartheta y$ is integral at ι. By Lemma 4.5.1, applied with $c = \gamma(\vartheta)$, $z = y$ and the identity element $s = 1$ of $\bar{G}$, the form $\gamma(\vartheta)\Theta_\vartheta(|\bar{G}|e_0 y) = \gamma(\vartheta)\Theta_\vartheta(Py)$ is integral at ι.

(AF2). Let σ be an automorphism of $\mathbb{C}$. By (ii), every conjugated value $\sigma((\Theta_1 y)(g))$, $g \in \mathrm{U}(Y_m)(\mathbb{A}_f)$, has squared Weyl norm at most e^B with $B = x^{202}b^2$; so Lemma 4.5.2, applied with $z = y$ and $s = 1$, gives $\|Py\|^2_\sigma = \||\bar{G}|e_0 y\|^2_\sigma \leq \exp(x^{99} + x^{202}b^2)$. Finally $x^{99} + x^{202}b^2 \leq 2x_1^2 b^2$, because $x^{10^6} \leq x_1$ (Proposition 2.3.5) and $x, b \geq 1$ give $x^{99} \leq x_1^2 b^2$ and $x^{202} \leq x_1^2$. So $\|Py\|^2_\sigma \leq \exp(2x_1^2 b^2)$. □

The first group proves Proposition 6.4.6. A short form of deep level is restricted flatly and rescaled by a hook number times the denominator of the last path; the integrality of the rescaled chain and the bound for its conjugates then follow from those of the last-path sums, through an integral invariant pairing of dual Weyl modules that detects integrality up to the hook number (Lemma 6.4.73). The key statement is Lemma 6.4.7: the pairing of a vector with the value at a point of the rescaled chain of a restriction is one constant times a last-path sum at a delta form. It composes the first-path period (Lemma 6.4.8), the transfer of periods along the turns (Proposition 6.4.87) and the last-path periods (Proposition 6.4.80), with Schur's lemma for the Weyl module [91].

The surjective flat restriction rests on an abstract principle for triangles of arrows whose squares commute up to scalars: weighted chains are injective, and the Gram form of a full chain is the product of the Gram operators of the first arrows (Lemma 6.4.59). The lifts, descents and ascents of the triangle satisfy its conditions (Proposition 6.4.11), the squares commuting by the one-square identity, and the moduli of the square constants of the twisted triangle are the chain ratios. For the injectivity of theta lifts through the inner product formula see [237, 187]. All moves are theta lifts whose real data are joint harmonics, in our reading the harmonic polynomials of the archimedean theta correspondence [157, 135, 136], and on the last path they are multiples of chains of algebraic theta lifts (Lemma 6.4.68). Finally, forms of deep level on the outer definite space are spanned by short forms (Lemma 6.4.77), by the finiteness of class numbers [33, 37].

Proposition 6.4.6. *Let D be admissible data, and let Θ_ϑ also denote the same composite of the m lifts on all of $\mathcal{A}_M(Y_0)$. There is a function c from the m-fold twists to $\mathbb{C}^\times$, of absolute value $\prod_{0\le i,j<m}\tau_{ij}$ up to a norm power, such that $\mathcal{A}_M(Y_0)$ is spanned by the forms y with:*

(i) $(c(\vartheta)/c(1))\,\Theta_\vartheta y$ is integral at ι for every m-fold twist ϑ and every prime identification $\iota \mid \ell$ with $\ell \nmid \mathrm{N}Q(\vartheta)$;

(ii) for every automorphism σ of $\mathbb{C}$ and every $g \in \mathrm{U}(Y_m)(\mathbb{A}_f)$, the conjugate $\sigma((\Theta_1 y)(g))$ of the value at g, a vector of the Weyl module, has squared norm at most $\exp(x^{202}b^2)$ for the Gram matrix of the Weyl module.

Proof. Let $n = n_{\mathrm{I},0}$ be the index set of Y_m, $(\kappa', e') = (\kappa_{\mathrm{I}}, e^{\det}_{\mathrm{I},0})$, and $P' = P_{\kappa',e'}$, $P_Y = P_{\kappa_Y,e_Y}$.

Step 1: the choices. By Lemma 6.4.78 fix a branching pairing B at (m,m). By (3) of Lemma 6.4.3 for $z = \mathrm{I}$, $i = 0$, $-p_{\mathrm{I},0,m}$ is a doubled Harish-Chandra parameter of (κ', e'); so Lemma 6.4.73 for n, (κ_Y, e_Y) with $l = p_{\mathrm{I},0,m}$ and (κ', e') gives an integer matrix A with rows $I_{\kappa'}$ and columns I_{κ_Y} and an integer matrix C with (1)–(4) there; let $h = h_{\kappa'} \ge 1$ be the hook number. Let c, (Ψ_ϑ), β, r be as in (3) of Lemma 6.4.7 for B and A, and N as in (b) of Proposition 6.4.80 for B. Put $s = 1$, $\varepsilon = 0$ if $\beta = 0$, and $s = hN/\beta$, $\varepsilon = 1$ otherwise; then $s \ne 0$ and $s\beta = h\varepsilon N$. By linearity of Θ_ϑ, for every m-fold twist ϑ, $\varphi \in \mathcal{F}_{\mathfrak{b}}(m)$, $x \in G_f(n)$ and $v \in \mathbb{V}_{\kappa'}$,

$$v^{\mathsf{T}} A \,\frac{c(\vartheta)}{c(1)}\,\big(\Theta_\vartheta(s\, r\varphi)\big)(x) = h\,\varepsilon\, N\,\mathrm{Ls}_B\big(\Psi_\vartheta, (\Delta^\vartheta_{x,v}, \varphi)\big). \tag{6.60}$$

Here every last-path sum is defined, by (ii) of Lemma 6.4.79. Call $\varphi \in \mathcal{F}_{\mathfrak{b}}(m)$ *short* if $\varphi^\flat$ is integral at every prime identification and $\|\sigma(\varphi^\flat(g))\|^2_{\kappa_{\mathrm{II}}} \le \exp(x^{99}b^2)$ for all σ, g. We show that $y = s\,r\varphi$ has (i) and (ii) for every short φ. This proves the proposition: the short forms span $\mathcal{F}_{\mathfrak{b}}(m)$ (Lemma 6.4.77), r is linear and onto $\mathcal{A}_M(Y_0)$, and $s \ne 0$, so the forms $s\,r\varphi$ with φ short span $\mathcal{A}_M(Y_0)$.

Step 2: (i). Let ϑ be an m-fold twist, $\iota \mid \ell$ with $\ell \nmid \mathrm{N}Q$, φ short and $g \in \mathrm{U}(Y_m)(\mathbb{A}_f) = G_f(n)$. Put $\Theta' = \frac{c(\vartheta)}{c(1)}(\Theta_\vartheta(s\,r\varphi))(g)$ and $\zeta = P_Y(f_\iota(g^{-1}))\,\iota(\Theta')$. For an integer vector $w \in \mathbb{Z}^{I_{\kappa'}}$ let $v_w = \iota^{-1}(P'(f_\iota(g))w)$, so that $P'(f_\iota(g^{-1}))\iota(v_w) = w$ has integral coordinates. By (2) of Lemma 6.4.7, $\delta^\vartheta_{g,v_w} = (\Delta^\vartheta_{g,v_w})^\flat$ is integral at ι; $\varphi^\flat$ is integral at ι; the pair $(\Delta^\vartheta_{g,v_w}, \varphi)$ is fixed by the outer levels by (1) there; and Ψ_ϑ is a path datum on the last path. So by (b) of Proposition 6.4.80, $N\,\mathrm{Ls}_B(\Psi_\vartheta, (\Delta^\vartheta_{g,v_w}, \varphi))$ is integral at ι, and by (6.60), $\iota(v_w^{\mathsf{T}} A\,\Theta') \in h\mathcal{O}$. By (1) of Lemma 6.4.73 over $R = \overline{\mathbb{Q}}_\ell$ for the family $f_\iota(g^{-1})$ of invertible matrices, and as A has integer entries,

$$w^{\mathsf{T}} A\,\zeta = \big(P'(f_\iota(g^{-1}))\iota(v_w)\big)^{\mathsf{T}} A\, P_Y(f_\iota(g^{-1}))\,\iota(\Theta') = \iota(v_w^{\mathsf{T}} A\,\Theta') \in h\mathcal{O}$$

for every w. In particular every coordinate of $A\zeta$ lies in $h\mathcal{O}$, so by (3) there, $h\zeta = CA\zeta$ has coordinates in $h\mathcal{O}$ (C is an integer matrix), and ζ has coordinates in $\mathcal{O}$. As g was arbitrary, $\frac{c(\vartheta)}{c(1)}\Theta_\vartheta y$ is integral at ι.

Step 3: (ii). Let $\vartheta = 1$, σ a field automorphism of $\mathbb{C}$, φ short, $g \in G_f(n)$ and $\Theta'' = (\Theta_1(s\, r\varphi))(g)$. Let $v \in \mathbb{V}_{\kappa'}$. By (2) of Lemma 6.4.7, $\|\sigma(\delta^1_{g,v}(g'))\|^2 \le \|\sigma(v)\|^2$ for all g'; φ is short; the pair is fixed by the outer levels; and Ψ_1 is a path datum of bound $V_1^{m^2}T_1$. By (c) of Proposition 6.4.80 and (6.60) (with $c(1)/c(1) = 1$, $|\sigma(\varepsilon)| \le 1$, and σ fixing the integers h, N),

$$|\sigma(v^{\mathsf{T}} A\, \Theta'')| \le hN \exp(x^{200}b^2) \exp(x^{99}b^2/2)\, \|\sigma(v)\|_{\kappa'} = K\, \|\sigma(v)\|_{\kappa'}.$$

By (4) of Lemma 6.4.73 for $\sigma(\Theta'')$ there is $w \in \mathbb{V}_{\kappa'}$ with $\|w\|_{\kappa'} = \|\sigma(\Theta'')\|_{\kappa_Y}$ and $u^{\mathsf{T}} A\, \sigma(\Theta'') = \langle w, u\rangle_{\kappa'}$ for all u. For $u = w$ and $v = \sigma^{-1}(w)$ (coordinatewise), as A has integer entries,

$$\|w\|^2_{\kappa'} = w^{\mathsf{T}} A\, \sigma(\Theta'') = \sigma\big(v^{\mathsf{T}} A\, \Theta''\big) \le K\, \|w\|_{\kappa'},$$

so $\|\sigma(\Theta'')\|^2_{\kappa_Y} = \|w\|^2_{\kappa'} \le K^2$.

Step 4: the constant. By (4) of Lemma 6.4.3 for $z = \mathrm{I}$, $i = 0$, the boxes of κ' in the rank of Y_m number at most $L = br^4$, so $h \le (L^L)^3$ by (5) of Lemma 6.4.73; and $N \le \exp(x^{99}b^2)$. With $\log L \le \log b + 4\log r \le b + 4r \le 5br$ and $r \le 3x^5$,

$$\log K^2 \le 6L \log L + 3x^{99}b^2 + 2x^{200}b^2 \le 30 \cdot 3^5 x^{25} b^2 + 3x^{99}b^2 + 2x^{200}b^2 \le x^{202}b^2,$$

the last step as $x \ge 3$ (Lemma 2.3.6). Hence $\|\sigma((\Theta_1 y)(g))\|^2 \le \exp(x^{202}b^2)$, which is (ii). □

Lemma 6.4.7. *Let D be admissible data, B a branching pairing at (m, m), $n = n_{\mathrm{I},0}$, and A an integer matrix with rows $I_{\mathrm{I},0}$ and columns I_{κ_Y} satisfying (1) and (2) of Lemma 6.4.73 for (κ_Y, e_Y) with $l = p_{\mathrm{I},0,m}$ and $(\kappa', e') = (\kappa_{\mathrm{I}}, e^{\det}_{\mathrm{I},0})$. For an m-fold twist ϑ put $V_\vartheta = \#(\mathcal{O}_E/\mathfrak{m}_\vartheta)^{2n_m}$ and $T_\vartheta = \#(\mathcal{O}_E/\mathfrak{m}_\vartheta)^{8mn^2_m}$. For $x \in G_f(n)$ and $v \in \mathbb{V}_{\mathrm{I},0}$ let $\delta^\vartheta_{x,v}$ be the function on $G_f(n)$ with $\delta^\vartheta_{x,v}(\gamma_f x k) = \rho_{\kappa_{\mathrm{I}}, e^{\det}_{\mathrm{I},0}}(\gamma_\infty)v$ for rational points γ of I_n and $k \in \mathrm{K}_\vartheta(\mathrm{I}, 0)$, and 0 off $\Gamma_f x \mathrm{K}_\vartheta(\mathrm{I}, 0)$. Then:*

(1) $\delta^\vartheta_{x,v}$ is a well-defined element of $\mathcal{A}_{\mathrm{I},0}$, there is a unique $\Delta^\vartheta_{x,v} \in \mathcal{F}(\mathrm{I}, 0, m)$ with $(\Delta^\vartheta_{x,v})^\flat = \delta^\vartheta_{x,v}$, and for every $\varphi \in \mathcal{F}_{\mathfrak{b}}(m)$ the pair $(\Delta^\vartheta_{x,v}, \varphi)$ is fixed by the outer levels;

(2) if $\iota \mid \ell$ is a prime identification with $\ell \nmid \mathrm{N}Q$ and $P_{\kappa_{\mathrm{I}}, e^{\det}_{\mathrm{I},0}}(f_\iota(x^{-1}))\, \iota(v)$ has integral coordinates, then $\delta^\vartheta_{x,v}$ is integral at ι; and $\|\sigma(\delta^\vartheta_{x,v}(g))\|_{\kappa_{\mathrm{I}}} \le \|\sigma(v)\|_{\kappa_{\mathrm{I}}}$ for every field automorphism σ of $\mathbb{C}$ and every g;

(3) there are a function c from the m-fold twists to $\mathbb{C}^\times$ of absolute value $\prod_{0 \le i,j < m} \tau_{ij}$ up to a norm power, path data Ψ_ϑ of bound $V_\vartheta^{m^2}T_\vartheta$ on the edges of the last path, a number $\beta \in \mathbb{C}$ and a surjective linear map $r\colon \mathcal{F}_{\mathfrak{b}}(m) \to \mathcal{A}_M(Y_0)$ such that for every m-fold twist ϑ, $\varphi \in \mathcal{F}_{\mathfrak{b}}(m)$, $x \in G_f(n)$ and $v \in \mathbb{V}_{\mathrm{I},0}$,

$$v^{\mathsf{T}} A\, \frac{c(\vartheta)}{c(1)}\, (\Theta_\vartheta r\varphi)(x) = \beta\, \mathrm{Ls}_B\big(\Psi_\vartheta, (\Delta^\vartheta_{x,v}, \varphi)\big).$$

Proof. Write $n = n_{\mathrm{I},0}$, $\mathrm{K} = \mathrm{K}_\vartheta(\mathrm{I}, 0)$, $\mathrm{ix}_\vartheta = [\mathbf{K}_0(n) : \mathrm{K}]$, $p = p_{\mathrm{I},0,m}$, $l = l_{\mathrm{I},0} = -p$, $(\kappa', e') = (\kappa_{\mathrm{I}}, e^{\det}_{\mathrm{I},0})$, $\rho' = \rho_{\kappa',e'}$, $\rho_Y = \rho_{\kappa_Y, e_Y}$, and Γ for the rational points of I_n. By (3) of Lemma 6.4.3, l is a doubled Harish-Chandra parameter of (κ', e').

Step 1: trivial stabilisers. Every element of K reduces to 1 modulo $\mathfrak{m}_\vartheta$, hence modulo every chosen place w (which divides $\mathfrak{m}_\vartheta$), so it acts trivially on L_w/wL_w for the lattice $L=\mathcal{O}_E^n$ of I_n, whose stabiliser is $\mathbf{K}_0(n)$. By (c) of Lemma 4.6.2, $\Gamma_f\cap h\mathrm{K}h^{-1}=\{1\}$ for every $h\in G_f(n)$.

Step 2: (1). If $\gamma_f x k=\gamma'_f x k'$ with $\gamma,\gamma'\in\Gamma$, $k,k'\in\mathrm{K}$, then $(\gamma'^{-1}\gamma)_f=xk'k^{-1}x^{-1}$, so $\gamma=\gamma'$ by Step 1; hence $\delta^\vartheta_{x,v}$ is well defined. It is right invariant under the open subgroup K, and $\delta^\vartheta_{x,v}(\gamma'_f h)=\rho'(\gamma'_\infty)\delta^\vartheta_{x,v}(h)$ for every rational point γ' (both sides vanish off $\Gamma_f x\mathrm{K}$, and on it $\rho'((\gamma'\gamma)_\infty)=\rho'(\gamma'_\infty)\rho'(\gamma_\infty)$), so $\delta^\vartheta_{x,v}\in\mathcal{A}_{\mathrm{I},0}$. By (4) of Lemma 6.4.68 there is a unique $\Delta^\vartheta_{x,v}\in\mathcal{F}(\mathrm{I},0,m)$ with $(\Delta^\vartheta_{x,v})^\flat=\delta^\vartheta_{x,v}$, and $\Delta^\vartheta_{x,v}=F^l_{\delta^\vartheta_{x,v}}$. Its algebraic form is fixed by $\mathrm{K}=\mathrm{K}_\vartheta(\mathrm{I},0)$. For $\varphi\in\mathcal{F}_\mathfrak{b}(m)$ and $k\in\mathrm{K}_\vartheta(\mathrm{II},0)$, the kernel of the reduction modulo $\mathfrak{b}$, $\varphi(y,gk)=\varphi(y,g)$ by (FY2), so $\varphi^\flat(gk)=\varphi^\flat(g)$ by the formula of (WM7). So the pair is fixed by the outer levels.

Step 3: (2). Let $h=\gamma_f x k$. For $\gamma\in\mathrm{GL}_n(E)$, $f_{\iota,\tau}(\gamma_f)=\iota(\overline{\tau(\gamma)})$ entrywise, and $P_{\kappa',e'}$ has integer coefficients and integer powers of the determinant, so $\iota(\rho'(\gamma_\infty)v)=\iota(P_{\kappa',e'}(\bar\gamma_\infty)v)=P_{\kappa',e'}(f_\iota(\gamma_f))\iota(v)$ and

$$P_{\kappa',e'}(f_\iota(h^{-1}))\,\iota\big(\delta^\vartheta_{x,v}(h)\big)=P_{\kappa',e'}(f_\iota(k^{-1}))\,P_{\kappa',e'}(f_\iota(x^{-1}))\,\iota(v).$$

As $k\in\mathbf{K}_0(n)\subseteq\mathrm{GL}_n(\widehat{\mathcal{O}}_E)$, $f_\iota(k^{-1})$ is integral with unit determinant, so $P_{\kappa',e'}(f_\iota(k^{-1}))$ has integral entries, and the right side is integral under the hypothesis of (2). Off $\Gamma_f x\mathrm{K}$ the value is 0. So $\delta^\vartheta_{x,v}$ is integral at ι. For the norms: $\sigma(\rho'(\gamma_\infty)v)=P_{\kappa',e'}(u)\,\sigma(v)$ with $u_\tau=\sigma(\overline{\tau(\gamma)})$, the image of the unitary matrix γ under an embedding of the CM field E, which commutes with complex conjugation, so $u\in\prod_\tau\mathrm{U}(N)$. By (WM3), $\|\xi\|^2_{\kappa'}=\sum_T|\xi(\Delta_T)|^2$ over the increasing fillings, ξ read as the linear form $b_i\mapsto\xi_i$ on $\nabla_{\kappa'}$; $S_{\kappa'}(u)\xi$ is the form $f\mapsto\xi(f(u\,\cdot))$, and by the Cauchy–Binet formula $(\Delta_T(u\,\cdot))_T$ is the image of $(\Delta_T)_T$ under the tensor product of the exterior powers of the u_τ, a unitary matrix in the basis of increasing fillings. With $|\det u_\tau|=1$ this gives $\|P_{\kappa',e'}(u)\xi\|_{\kappa'}=\|\xi\|_{\kappa'}$, so $\|\sigma(\delta^\vartheta_{x,v}(g))\|_{\kappa'}\in\{0,\|\sigma(v)\|_{\kappa'}\}$.

Step 4: transfer to the last path. Let Ψ^0, g, α, r be as in Lemma 6.4.8, and c as in Proposition 6.4.87. The transfer of periods from the first to the last path with bound V^{m^2}, applied to $N(\vartheta)=T_\vartheta$ and Ψ^0, gives path data Ψ_ϑ of bound $V_\vartheta^{m^2}T_\vartheta$ on the edges of the last path such that $\Psi\propto_{\mathrm{last}}\Xi$ implies $\Psi^0\propto_{\mathrm{first}}\Xi/c$. The last-path sums used here are defined: for $y'\in\mathcal{F}(\mathrm{I},0,m)$ and $\varphi\in\mathcal{F}(\mathrm{II},0,m)$, $\mathrm{Ls}_B(\Psi_\vartheta,(y',\varphi))$ is defined by (ii) of Lemma 6.4.79. Let $\Xi(\vartheta,y',\varphi)=\mathrm{Ls}_B(\Psi_\vartheta,(y',\varphi))/\mathrm{ix}_\vartheta$. By (a) of Proposition 6.4.80, $\Psi\propto_{\mathrm{last}}\Xi$; so there is $\beta_1\in\mathbb{C}$ with $\mathrm{per}_{\mathrm{first}}(\Psi^0;g)(\vartheta,y',\varphi)=\beta_1\Xi(\vartheta,y',\varphi)/c(\vartheta)$ for every period triple, and by Lemma 6.4.8

$$\int_{[\mathrm{U}(Y_m)]}y'(x')\,F^p_{\Theta_\vartheta r\varphi}(x')\,dx'=\frac{\alpha\beta_1}{\mathrm{ix}_\vartheta\,c(\vartheta)}\,\mathrm{Ls}_B(\Psi_\vartheta,(y',\varphi)).\tag{6.61}$$

Step 5: the pairing with a delta form. Fix ϑ, φ, x, v and put $\Theta=\Theta_\vartheta r\varphi$, an algebraic form of weight (κ_Y,e_Y) on $\mathrm{U}(Y_m)=\mathrm{U}(\mathrm{I}_n)$ with $\Theta(hk)=\chi_\vartheta(k)\Theta(h)$ for $k\in\mathrm{K}_\vartheta$, χ_ϑ the character of K_ϑ, by (c) of Lemma 6.4.75; it is right invariant under K: every $k\in\mathrm{K}$ reduces to 1 at the chosen places and at the places of Q, so $k\in\mathrm{K}_\vartheta$, and $\chi_\vartheta(k)=1$ since every diagonal block of the reduction of k at a level place is 1. Let $f(y,h)=F^l_{\delta^\vartheta_{x,v}}(y,h)\,F^p_\Theta(y,h)$, continuous, left invariant under Γ (by the transformation laws of algebraic forms and the multiplicativity of ρ) and right K-invariant in h. By (2) of Lemma 6.4.73, $F^l_\delta(y,h)=\delta(h)^{\mathsf T}A\,\rho_Y(y)v_p^+$, and by (WM7) $F^p_\Theta(y,h)=\langle v_p^+,\rho_Y(y)^{-1}\Theta(h)\rangle_{\kappa_Y}$; so $\int_{G_\infty(1)}f(y,h)\,dy=\delta(h)^{\mathsf T}A\,T_0\,\Theta(h)$, dy the Haar probability measure, with

$$T_0\xi=\int_{G_\infty(1)}\rho_Y(y)v_p^+\,\langle v_p^+,\rho_Y(y)^{-1}\xi\rangle_{\kappa_Y}\,dy.$$

Substituting $y=y_0y'$ shows $T_0\rho_Y(y_0)=\rho_Y(y_0)T_0$. The module $\mathbb{V}_{\kappa_Y}$ is irreducible under $\rho_Y(G_\infty(1))$: it is irreducible under $\prod_\tau\mathrm{GL}_N(\mathbb{C})$ (the Weyl module of highest weight λ°, [91],

Ch. 8), and a subspace stable under $\prod_\tau \mathrm{U}(N)$ is stable under its Zariski closure $\prod_\tau \mathrm{GL}_N(\mathbb{C})$, the action being polynomial. By Schur's lemma $T_0 = t\cdot 1$, and taking traces, $t\,\dim \mathbb{V}_{\kappa_Y} = \langle v_p^+, v_p^+\rangle_{\kappa_Y}$. This is positive: $\xi \mapsto F_{\kappa_Y}\xi = (\xi(\Delta_T))_T$ is injective, as the Δ_T of increasing fillings span ∇_{κ_Y}, so $\langle\cdot,\cdot\rangle_{\kappa_Y}$ is positive definite; and $v_p^+ \neq 0$, as for the filling T whose sets $T(\tau,h,j)$ are the h indices with the h largest values of p_τ, the minors of y^p in $\Delta_T(y^p)$ are determinants of permutation matrices, so $\Delta_T(y^p) = \pm 1$, and Δ_T is an integer combination of the b_i. Hence $t \neq 0$ does not depend on ϑ, φ, x, v. By (2) of Lemma 6.4.74 at the level K,

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} f = \frac{c_n t}{\mathrm{ix}_\vartheta}\,\mathrm{CI}_\mathrm{K}(F), \qquad F(h) = \delta^\vartheta_{x,v}(h)^\mathsf{T} A\,\Theta(h).$$

F is right K-invariant, so by (LQ3) and Step 1, $\mathrm{CI}_\mathrm{K}(F) = \sum_{[h]} F(h)$ over $\Gamma_f\backslash G_f(n)/\mathrm{K}$, and as $\delta^\vartheta_{x,v}$ vanishes off $\Gamma_f x\mathrm{K}$ and $\delta^\vartheta_{x,v}(x) = v$, $\mathrm{CI}_\mathrm{K}(F) = v^\mathsf{T} A\,\Theta(x)$.

Step 6: (3). By Step 2, $y' = \Delta^\vartheta_{x,v} = F^l_{\delta^\vartheta_{x,v}}$ is a conjugate form on Y_m; comparing Step 5 with (6.61),

$$v^\mathsf{T} A\,\frac{c(\vartheta)}{c(1)}\,(\Theta_\vartheta r\varphi)(x) = \frac{\alpha\beta_1}{c_n\,t\,c(1)}\,\mathrm{Ls}_B\big(\Psi_\vartheta, (\Delta^\vartheta_{x,v}, \varphi)\big),$$

which is (3) with $\beta = \alpha\beta_1/(c_n t c(1))$; c has absolute value $\prod \tau_{ij}$ up to a norm power by Proposition 6.4.87, and r is linear and onto by Lemma 6.4.8. □

Lemma 6.4.8. *Let D be admissible data and, for an m-fold twist ϑ, $T_\vartheta = \#(\mathcal{O}_E/\mathfrak{m}_\vartheta)^{8mn^2_m}$. There are path data Ψ^0_ϑ of bound T_ϑ on the edges of the first path, a real point g of $\mathrm{U}(\tilde{Y}^0_0)$, a number $\alpha \in \mathbb{C}$ and a surjective linear map $r\colon \mathcal{F}_\mathfrak{b}(m) \to \mathcal{A}_M(Y_0)$ such that for every m-fold twist ϑ, every conjugate form y' on Y_m and every $\varphi \in \mathcal{F}_\mathfrak{b}(m)$,*

$$\int_{[\mathrm{U}(Y_m)]} y'(x)\,F^{p_{\mathrm{I},0,m}}_{\Theta_\vartheta r\varphi}(x)\,dx = \alpha\,\mathrm{per}_{\mathrm{first}}(\Psi^0; g)(\vartheta, y', \varphi).$$

Proof. **Step 1: the choices.** Let g and r be a real point of $\tilde{Y}^0_0$ and a surjective flat restriction $r\colon \mathcal{F}_\mathfrak{b}(m) \to \mathcal{A}_M(Y_0)$ at g, as given by Lemma 6.4.9; let C be the constant of Lemma 6.4.65, put $\alpha = C$, and let Ψ^0_ϑ be the first-path datum of (FY4), a path datum of bound T_ϑ on the edges of the first path by (1) of Lemma 6.4.65. It remains to prove the identity. Fix an m-fold twist ϑ, a conjugate form $y' \in \mathcal{F}(\mathrm{I},0,m)$ on Y_m and $\varphi \in \mathcal{F}_\mathfrak{b}(m)$, and put $y = r\varphi \in \mathcal{A}_M(Y_0)$.

Step 2: transposition down the column of Y. By (a) of Lemma 6.4.67, $y \in \mathcal{F}^0_\triangle(\mathrm{I},0)$, and by (b) of Lemma 6.4.67, $F^{p_{\mathrm{I},0,m}}_{\Theta_\vartheta y} = \mathrm{Wc}^0_{\vartheta,\mathrm{I},m} y$. By (3) of Lemma 6.4.63 for $z = \mathrm{I}$ and $b = m$, where $\Theta^\flat_{\vartheta,\mathrm{I},m} = \mathrm{id}$, $V^0_{\mathrm{I},m} = Y_m$ and $V^0_{\mathrm{I},0} = Y_0$,

$$\int_{[\mathrm{U}(Y_m)]} y'(x)\,F^{p_{\mathrm{I},0,m}}_{\Theta_\vartheta y}(x)\,dx = \int_{[\mathrm{U}(Y_m)]} y' \cdot \mathrm{Wc}^0_{\vartheta,\mathrm{I},m} y = \int_{[\mathrm{U}(Y_0)]} u\,y, \qquad u = \Theta^\flat_{\vartheta,\mathrm{I},0}\,y', \tag{6.62}$$

the integrals being absolutely convergent.

Step 3: $\bar{u}$ is a test form of the flat restriction. By (1) of Lemma 6.4.63 (with $b = 0 < m$), $u \in \mathcal{F}(\mathrm{I},0,0)$ and $u(h(1,k)) = u(h)$ for every $k \in \mathbf{K}_0(n_{\mathrm{I},0,0})$ whose reduction modulo $\mathfrak{t}_\vartheta(\mathrm{I}) = \pi_M \prod_{w\in Q} w$ is 1. The generator π_M of the product of the chosen places and the places of Q give pairwise distinct places (the chosen places lie above primes dividing M, and $\mathrm{N}Q$ is prime to M), each to the first power, so by the Chinese remainder theorem an element of $\mathbf{K}_0(n_{\mathrm{I},0,0})$ reduces to 1 modulo $\mathfrak{t}_\vartheta(\mathrm{I})$ if and only if it reduces to 1 modulo every chosen place and every place of Q; that is, the group of these k is $\mathrm{K}_\vartheta(Y_0)$. By (b) of Lemma 6.4.15 (offset 0, $p = (\mathrm{I},0,0)$, $\lambda = \frac{1}{2}p_{\mathrm{I},0,0}$), the grid forms at $(\mathrm{I},0,0)$ have the type $\bar{e}_\lambda$; as the conditions of (ED2) are invariant

under complex conjugation, $\bar{u}$ is a form of type e_λ on Y_0, and it is right invariant under $\mathrm{K}_\vartheta(Y_0)$; so $\bar{u} \in \mathcal{F}^+_\vartheta(Y_0)$.

Step 4: the flat restriction. By (FY3) for φ, ϑ and $\bar{u}$,

$$\int_{[\mathrm{U}(Y_0)]} u\,(r\varphi) = \langle \bar{u}, r\varphi\rangle_{Y_0} = \int_{[\mathrm{U}(Y_0)]} u(x)\,(\Theta^\flat\varphi)\big(\mathrm{diag}(x,1)\,(g,1)\big)\,dx = \big\langle u,\ \Theta^\flat_{1,\mathrm{II},0}\varphi\big\rangle_{0,g},$$

the last equality being the definition of the period $\langle\cdot,\cdot\rangle_{0,g}$ in (3.3) for $l=0$, where $Y_0^0 = Y_0$, the restriction is along $x \mapsto \mathrm{diag}(x,1)$ into $\mathrm{U}(\tilde{Y}_0^0)$ and the power $\mathrm{flp}_\vartheta^0 = 1$ of the flip idele acts trivially.

Step 5: the period of the first path. As $\varphi \in \mathcal{F}_\mathfrak{b}(m) \subseteq \mathcal{F}(\mathrm{II},0,m)$, (2) of Lemma 6.4.65 gives $\langle u, \Theta^\flat_{1,\mathrm{II},0}\varphi\rangle_{0,g} = C\,\mathrm{per}_{\mathrm{first}}(\Psi^0;g)(\vartheta,y',\varphi)$. With (6.62) and Step 4,

$$\int_{[\mathrm{U}(Y_m)]} y'(x)\,F^{p_{\mathrm{I},0,m}}_{\Theta_\vartheta r\varphi}(x)\,dx = \alpha\,\mathrm{per}_{\mathrm{first}}(\Psi^0;g)(\vartheta,y',\varphi),$$

which is the assertion; r is linear and surjective by Step 1. □

Lemma 6.4.9. *Let D be admissible data. There are a real point $g \in G_\infty(s)$ of $\tilde{Y}_0^0$ and a flat restriction $r\colon \mathcal{F}_\mathfrak{b}(m) \to \mathcal{A}_M(Y_0)$ at g that is surjective.*

Proof. Write $\mathcal{A} = \mathcal{A}_M(Y_0)$, $n_0 = n_{\mathrm{I},0,0}$, $e = e_\varsigma$, and for a real point g of $\tilde{Y}_0^0$, a bounded continuous left Γ_{Y_0}-invariant function y on the adelic group of Y_0 and $\psi \in \mathcal{F}_\mathfrak{b}(0)$

$$P_g(y,\psi) = \int_{[\mathrm{U}(Y_0)]} \overline{y(x)}\,\psi\big(\mathrm{diag}(x,1)\,(g,1)\big)\,dx.$$

Step 0: P_g is defined. As $\tilde{Y}_0^0 = Y_0 \oplus \langle 1\rangle$ is an orthogonal sum ((GP2); block diagonal matrix and point, (VP5)), $\mathrm{diag}(\gamma,1)$ is a rational point of $\tilde{Y}_0^0$ for every rational point γ of Y_0, so $x \mapsto \psi(\mathrm{diag}(x,1)(g,1))$ is left Γ_{Y_0}-invariant; the integrand is continuous and bounded, and $[\mathrm{U}(Y_0)]$ has finite volume (Theorem 4.2.18), so the integral converges absolutely. P_g is conjugate-linear in y and linear in ψ.

Step 1: the representing map. By (b) of Lemma 6.4.15 for the triangle I at $(0,0)$, the Fock vector of the joint harmonic is an edge from $e_{\frac{1}{2}p_{\mathrm{I},0,0}} = e$, so e is a cusp kernel of order 0 ((a) of (ED4)); with the open subgroup K_M, Lemma 6.4.64 shows that $\mathcal{A}$ is finite-dimensional. The Petersson product is positive definite on $\mathcal{A}$: if $\langle y,y\rangle = 0$ then $y = 0$ almost everywhere, hence everywhere by continuity. So for every $\psi \in \mathcal{F}_\mathfrak{b}(0)$ there is a unique $R_g\psi \in \mathcal{A}$ with $\langle y, R_g\psi\rangle = P_g(y,\psi)$ for all $y \in \mathcal{A}$, and $\psi \mapsto R_g\psi$ is linear by uniqueness.

Step 2: the representing identity for all test forms. Let ϑ be an m-fold twist, $y \in \mathcal{F}^+_\vartheta(Y_0)$ and $\psi \in \mathcal{F}_\mathfrak{b}(0)$; we show $\langle y, R_g\psi\rangle = P_g(y,\psi)$. For $k \in \mathbf{K}_0(n_0)$ put $y_k(x) = y(x(1,k))$.

(i) Let $k \in \mathrm{K}_M$. At each chosen place w (with $\bar{w} \neq w$, as $\mathfrak{b}$ is a modulus) an element of $\mathbf{K}_0(n_0)$ is determined at the pair $\{w,\bar{w}\}$ by its component at w, which may be any element of $\mathrm{GL}_{n_0}(\mathcal{O}_{E,w})$. As $\mathcal{O}_{E,w}$ is local, every lift of an invertible matrix modulo $\mathfrak{b}\mathcal{O}_{E,w}$ is invertible; so there is $u \in \mathbf{K}_0(n_0)$ with $u_w \equiv k_w^{-1} \pmod{\mathfrak{b}\mathcal{O}_{E,w}}$ at every chosen place w and component 1 at all places other than the chosen places and their conjugates. Then ku reduces to 1 modulo $\mathfrak{b}$; u reduces to 1 at every chosen place (as k does), and to 1 at every place of Q, since the places of Q lie above primes prime to M and so are none of the chosen places or their conjugates. Hence $u \in \mathrm{K}_\vartheta(Y_0)$, $y(x(1,u)) = y(x)$, and $y_{ku} = y_k$. Substituting $x \mapsto x(1,ku)^{-1}$ in $P_g(y_{ku},\psi)$ (the quotient measure is right invariant) and using that $(1,ku)$ commutes with $(g,1)$,

$$P_g(y_k,\psi) = P_g(y_{ku},\psi) = \int_{[\mathrm{U}(Y_0)]} \overline{y(x)}\,\psi\big(\mathrm{diag}(x,1)(g,1)\,(1,\mathrm{diag}((ku)^{-1},1))\big)\,dx = P_g(y,\psi),$$

because $\mathrm{diag}((ku)^{-1},1) \in \mathbf{K}_0(n_{\mathrm{II},0,0})$ reduces to 1 modulo $\mathfrak{b}$ and $\psi \in \mathcal{F}_{\mathfrak{b}}(0)$ ((FY2)). Likewise $\langle y_k, R_g\psi\rangle = \langle y, R_g\psi\rangle$, by the substitution $x \mapsto x(1,k)^{-1}$ and the right K_M-invariance of $R_g\psi$.

(ii) Let $\bar{y}(x) = \int_{\mathrm{K}_M} y_k(x)\,dk$, dk the Haar probability measure of the compact open group K_M. It is continuous, bounded, left Γ_{Y_0}-invariant and right K_M-invariant, and $R(e)\bar{y} = \bar{y}$ by Fubini's theorem, because $R(e)$ acts in the infinite component and $R(e)y_k = y_k$ for every k; so $\bar{y} \in \mathcal{A}$. By Fubini's theorem on $[\mathrm{U}(Y_0)] \times \mathrm{K}_M$ (bounded integrands, finite measures) and (i),

$$\langle y, R_g\psi\rangle = \int_{\mathrm{K}_M} \langle y_k, R_g\psi\rangle\,dk = \langle \bar{y}, R_g\psi\rangle = P_g(\bar{y},\psi) = \int_{\mathrm{K}_M} P_g(y_k,\psi)\,dk = P_g(y,\psi),$$

the middle equality by Step 1.

Step 3: a flat restriction at every point. By (1) of Lemma 6.4.63 for $z = \mathrm{II}$, the trivial twist and $b = 0 < m$, where $\mathfrak{t}_1(\mathrm{II}) = \mathfrak{b}$, $\Theta^\flat = \Theta^\flat_{1,\mathrm{II},0}$ maps $\mathcal{F}(\mathrm{II},0,m) \supseteq \mathcal{F}_{\mathfrak{b}}(m)$ linearly into the grid forms at $(\mathrm{II},0,0)$ fixed by the elements reducing to 1 modulo $\mathfrak{b}$, that is, into $\mathcal{F}_{\mathfrak{b}}(0)$. Put $r_g = R_g \circ \Theta^\flat\colon \mathcal{F}_{\mathfrak{b}}(m) \to \mathcal{A}$, a linear map. By Step 2 with $\psi = \Theta^\flat\varphi$, $\langle y, r_g\varphi\rangle_{Y_0} = \int_{[\mathrm{U}(Y_0)]} \overline{y(x)}\,(\Theta^\flat\varphi)(\mathrm{diag}(x,1)(g,1))\,dx$ for all φ, ϑ and $y \in \mathcal{F}^+_\vartheta(Y_0)$: r_g is a flat restriction at g ((FY3)).

Step 4: surjectivity at the point of deep-level vanishing. Let $g = \omega$ be the real point of $\tilde{Y}^0_0$ given by Lemma 6.4.98, and $r = r_g$. As $\mathcal{A}$ is finite-dimensional with a positive definite inner product, it suffices that a $y \in \mathcal{A}$ orthogonal to $r(\mathcal{F}_{\mathfrak{b}}(m))$ vanishes. For such y and every $\psi \in \mathcal{F}_{\mathfrak{b}}(0)$ choose, by (b) of Lemma 6.4.10, $\varphi \in \mathcal{F}_{\mathfrak{b}}(m)$ with $\Theta^\flat\varphi = \psi$; then $0 = \langle y, r\varphi\rangle = P_\omega(y,\psi)$ by Step 1. Taking complex conjugates,

$$\int_{[\mathrm{U}(Y_0)]} y(x)\,\overline{\psi\big((x\oplus 1)\,\omega\big)}\,dx = 0 \qquad \text{for every } \psi \in \mathcal{F}_{\mathfrak{b}}(0),$$

and $\mathcal{F}_{\mathfrak{b}}(0)$ is the space of the grid forms at $(\mathrm{II},0,0)$ of the deep level, the forms of Lemma 6.4.98. By Lemma 6.4.98, $y = 0$. So r is surjective, and (g,r) is as asserted. □

Lemma 6.4.10. *Let D be admissible data.*

(a) The weighted chain $\mathrm{Wc}^0_{1,\mathrm{II},m}$ of row 0 of the triangle II*, for the trivial twist, is injective on $\mathcal{F}^{\mathrm{bs}}_{\mathrm{II}}$.*

(b) $\Theta^\flat(\mathcal{F}_{\mathfrak{b}}(m)) = \mathcal{F}_{\mathfrak{b}}(0)$.

Proof. Write $z = \mathrm{II}$ throughout.

(a). By (c) of Lemma 6.4.15 choose a left pair $\Phi_{i,n}$ at every position (i,n) with $i+n<m$. By (a) of Proposition 6.4.11 for these left pairs, the data I (the m-fold twists), $\mathcal{H}_{i,n} = \mathcal{F}^i_\triangle(z,n)$, $a^\vartheta_{i,n} = \mathrm{Tl}^i_{\vartheta,z,n}$ for $i+n<m$, $w_\vartheta(n) = \#\mathrm{GL}_{n_n}(R_\vartheta)$, $V_0 = \mathcal{F}^{\mathrm{bs}}_z$ and u_i the inclusion satisfy the hypotheses and (Qu1)–(Qu3) of Lemma 6.4.59. Its weighted chain of row 0 for the trivial twist satisfies $X^1_{0,0} = \mathrm{id}$ and, for $n<m$, $X^1_{0,n+1} = \#\mathrm{GL}_{n_n}(\mathcal{O}_E/\mathfrak{t}_1(z))\,\mathrm{Tl}^0_{1,z,n} \circ X^1_{0,n}$, the recursion of $\mathrm{Wc}^0_{1,z,n}$ in (TV3); so $X^1_{0,n} = \mathrm{Wc}^0_{1,z,n}$ for $n \le m$. By (1) of Lemma 6.4.59 with $i=0$, $n=m$, the map $\mathrm{Wc}^0_{1,z,m} \circ u_0$ is injective, which is (a).

(b), the inclusion $\subseteq$. By (1) of Lemma 6.4.63 for the trivial twist and $b = 0 < m$, $\Theta^\flat$ maps $\mathcal{F}(z,0,m)$, hence $\mathcal{F}_{\mathfrak{b}}(m)$, linearly into the grid forms at $(z,0,0)$ fixed by the elements of $\mathbf{K}_0(n_{z,0,0})$ that reduce to 1 modulo $\mathfrak{b}$, that is into $\mathcal{F}_{\mathfrak{b}}(0)$ ((FY2)).

(b), finite dimension. Let $\lambda = \frac{1}{2}p_{z,0,0}$. By (b) of Lemma 6.4.15 at $(0,0)$, e_λ is a cusp kernel of order 0 ((a) of (ED4)); the conditions of (ED1) are invariant under complex conjugation (a replaced by $\bar{a}$), so $\bar{e}_\lambda$ is one too. By the same clause (offset 0) the grid forms at $(z,0,0)$ are the

forms of type $\bar{e}_\lambda$, so $\mathcal{F}_{\mathfrak{b}}(0)$ is finite-dimensional by Lemma 6.4.64, and the Petersson product is positive definite on it (a continuous F with $\int |F|^2 = 0$ vanishes).

(b), the inclusion $\supseteq$. Let $\mathcal{V} = \Theta^\flat(\mathcal{F}_{\mathfrak{b}}(m)) \subseteq \mathcal{F}_{\mathfrak{b}}(0)$; by finite dimension $\mathcal{F}_{\mathfrak{b}}(0) = \mathcal{V} \oplus (\mathcal{V}^\perp \cap \mathcal{F}_{\mathfrak{b}}(0))$, so it suffices that $\psi \in \mathcal{F}_{\mathfrak{b}}(0)$ with $\langle \psi, v\rangle = 0$ for all $v \in \mathcal{V}$ vanishes. Put $y = \bar{\psi}$. As ψ is a form of type $\bar{e}_\lambda$ and the conditions of (ED2) are invariant under complex conjugation, y is a form of type e_λ on $V^0_{z,0}$, that is $y \in \mathcal{F}^0_\triangle(z,0)$ ((TV2)), and it is right invariant under the elements reducing to 1 modulo $\mathfrak{b}$, so $y \in \mathcal{F}^{\mathrm{bs}}_z$ ((TV4)). Let $U = \mathrm{Wc}^0_{1,z,m} y$, a triangle form at $(0,m)$ right invariant under the elements of $\mathbf{K}_0(n_{z,0,m})$ reducing to 1 modulo $\mathfrak{b}$ ((2) of Lemma 6.4.63, $m \geq 1$), and $\varphi = \bar{U}$; U is a form of type e_{λ_m}, $\lambda_m = \frac{1}{2} p_{z,0,m}$ ((TV2)), and by (b) of Lemma 6.4.15 (offset 0) the grid forms at $(z,0,m)$ are the forms of type $\bar{e}_{\lambda_m}$, so φ is a grid form there, so $\varphi \in \mathcal{F}_{\mathfrak{b}}(m)$ ((FY2)). By (3) of Lemma 6.4.63 with $b = m$ ($\Theta^\flat_{1,z,m} = \mathrm{id}$),

$$\langle U, U\rangle = \int_{[\mathrm{U}(\tilde{Y}^m_0)]} \varphi\, U = \int_{[\mathrm{U}(\tilde{Y}^0_0)]} (\Theta^\flat \varphi)\, y = \langle \psi, \Theta^\flat \varphi\rangle = 0,$$

as $\Theta^\flat \varphi \in \mathcal{V}$. So $U = 0$ by continuity, $y = 0$ by (a), and $\psi = 0$. □

Proposition 6.4.11. *Let D be admissible data, $z \in \{\mathrm{I}, \mathrm{II}\}$, and for $i + n < m$ let $\Phi_{i,n}$ be a left pair at (i,n). For every m-fold twist ϑ there are functions $e_\vartheta(i,n)$ on $\mathrm{M}_{n_{n+1}\times n_n}(R_\vartheta)$ ($i+n<m$) with $e_\vartheta(i,n+1) = e_\vartheta(i,n)^\natural$ whenever $i+n+1<m$, such that the following holds for the data: I the set of m-fold twists; $\mathcal{H}_{i,n} = \mathcal{F}^i_\triangle(z,n)$ with the Petersson product of $V^i_{z,n}$; $a^\vartheta_{i,n} = \mathrm{Tl}^i_{\vartheta,z,n}$ and $b^\vartheta_{i,n} = \mathrm{Ds}^i_{\vartheta,z,n}(\Phi_{i,n}, e_\vartheta(i,n))$ if $i+n<m$, and $a^\vartheta_{i,n} = 0$, $b^\vartheta_{i,n} = 0$ otherwise; $w_\vartheta(n) = \#\mathrm{GL}_{n_n}(R_\vartheta)$; $V_0 = \mathcal{F}^{\mathrm{bs}}_z$ with the Petersson product of $V^0_{z,0}$; and $u_i \colon V_0 \to \mathcal{H}_{i,0}$ the inclusion, $\mathcal{F}^i_\triangle(z,0)$ being $\mathcal{F}^0_\triangle(z,0)$ (Lemma 6.4.15(a)).*

(a) The maps $a^\vartheta_{i,n}$, $b^\vartheta_{i,n}$ are linear maps $\mathcal{H}_{i,n} \to \mathcal{H}_{i,n+1}$, $\mathcal{H}_{i,n+1} \to \mathcal{H}_{i+1,n}$, the u_i are linear isometries, and (Qu1)–(Qu3) hold.

(b) Let $z = \mathrm{I}$ and $\mathcal{F}^{\mathrm{bs}}_z \neq 0$, let $\mathcal{M}(c,g)$ mean that c has absolute value g up to a norm power (a unital and multiplicative relation), and $f_{r,k} = q_{rk}$ for $1 \leq r < k \leq m$. Then (Qx1) and (Qx2) hold.

Proof. **Step 0: the data.** The left pairs exist by Lemma 6.4.15(c). For $i < m$ and each ϑ let $e_\vartheta(i,0) = \Psi_0$ be the function of Lemma 6.4.54 for $\Phi_{i,0}$, and define $e_\vartheta(i,n+1) = e_\vartheta(i,n)^\natural$ recursively for $i+n+1<m$. By Lemma 6.4.51(a), (b) the maps of the statement take $\mathcal{H}_{i,n}$ to $\mathcal{H}_{i,n+1}$ and $\mathcal{H}_{i,n+1}$ to $\mathcal{H}_{i+1,n}$ (the zero maps trivially), and they are linear; u_i is an isometry by Lemma 6.4.15(a). Write $w = w_\vartheta$, $a_{i,n} = a^\vartheta_{i,n}$, $b_{i,n} = b^\vartheta_{i,n}$ and $X_{i,n}$ for the weighted chains of Lemma 6.4.59.

Step 1: (Qu1). Let $i+n+2 \leq m$. By Lemma 6.4.12 for $\Phi_0 = \Phi_{i,n}$, $\Phi_1 = \Phi_{i,n+1}$ there are $\kappa_{i,n} \in \mathbb{C}$ and $C_{i,n} > 0$, independent of ϑ, and $c_{i,n}(\vartheta)$ with $|c_{i,n}(\vartheta)|^2 = C_{i,n} L_*(\Sigma(\mathfrak{t}_\vartheta(z)), \nu_\vartheta; t') \neq 0$ such that, with $p = e_\vartheta(i,n)$ and $p^\natural = e_\vartheta(i,n+1)$,

$$w(n)\, a_{i+1,n} \circ b_{i,n} = \frac{\kappa_{i,n}}{c_{i,n}(\vartheta)}\, w(n+2)\, b_{i,n+1} \circ a_{i,n+1} \qquad \text{on } \mathcal{H}_{i,n+1}, \tag{6.63}$$

which is (Qu1) with $\gamma = \kappa_{i,n} w(n+2)/c_{i,n}(\vartheta)$.

Step 2: (Qu2). For $i < m$, $y \in V_0$ and $x \in \mathcal{H}_{i,1}$, Lemma 6.4.54 gives $\langle x, a_{i,0} u_i y\rangle = \langle b_{i,0} x, u_{i+1} y\rangle$; it holds for every $y \in \mathcal{F}^0_\triangle(z,0)$.

Step 3: (Qu3). Let $i < m$, $y \in V_0$ and $v = b_{i,0} a_{i,0} y \in \mathcal{H}_{i+1,0} = \mathcal{F}^0_\triangle(z,0)$. For every $x \in \mathcal{F}^0_\triangle(z,0)$, Step 2 with $a_{i,0} y$ in place of x and x in place of y gives $\langle a_{i,0} y, a_{i,0} x\rangle = \langle v, x\rangle$, the

Petersson products of $V^{i+1}_{z,0}$ and $V^0_{z,0}$ agreeing (Lemma 6.4.15(a)). The first lift of row i is $a_{i,0}$ on the forms at $(0,0)$ (TV5); so Lemma 6.4.58(c) gives $v \in \mathcal{F}^{\mathrm{bs}}_z$, i.e. $b_{i,0}a_{i,0}u_i y = u_{i+1}v \in u_{i+1}(V_0)$. If $v = 0$, then $\langle a_{i,0}y, a_{i,0}y\rangle = \langle v, y\rangle = 0$, so $a_{i,0}y = 0$ (a continuous function of Petersson norm 0 vanishes, the quotient measure charging nonempty open sets), and $y = 0$ by Lemma 6.4.58(b).

Step 4: if $V_0 \neq 0$, every $\kappa_{i,k}$ is nonzero. Let $i+k+2 \le m$ with $\kappa_{i,k} = 0$; fix ϑ, and write $\gamma_n = \kappa_{i,n}w(n+2)/c_{i,n}(\vartheta)$ for $n \le k$. Put $c_0 = 1/w(0)$ and $c_{n+1} = c_n\gamma_n/w(n+1)$. For $n \le k$, $X_{i+1,n}\, b_{i,0}a_{i,0} = c_n\, b_{i,n}X_{i,n+1}$ on $\mathcal{H}_{i,0}$: for $n = 0$ both sides are $b_{i,0}a_{i,0}$, and if it holds for $n < k$, then by (6.63) and $X_{i,n+2} = w(n+1)a_{i,n+1}X_{i,n+1}$,

$$\begin{aligned} X_{i+1,n+1}b_{i,0}a_{i,0} &= w(n)\, a_{i+1,n}X_{i+1,n}b_{i,0}a_{i,0} = c_n\, w(n)\, a_{i+1,n}b_{i,n}X_{i,n+1} \\ &= c_n\gamma_n\, b_{i,n+1}a_{i,n+1}X_{i,n+1} = c_{n+1}\, b_{i,n+1}X_{i,n+2}. \end{aligned}$$

With $n = k$ and (6.63) at (i,k), where $\gamma_k = 0$: $X_{i+1,k+1}b_{i,0}a_{i,0} = c_k w(k)\, a_{i+1,k}b_{i,k}X_{i,k+1} = c_k\gamma_k\, b_{i,k+1}a_{i,k+1}X_{i,k+1} = 0$. Let $y \in V_0$; by Step 3 there is $x \in V_0$ with $u_{i+1}x = b_{i,0}a_{i,0}u_i y$, so $X_{i+1,k+1}u_{i+1}x = 0$, and as $(i+1)+(k+1) \le m$, Lemma 6.4.59(1) (whose hypotheses are Steps 1–3) gives $x = 0$; then $y = 0$ by Step 3. So $V_0 = 0$.

Step 5: (Qx1). Let $z = \mathrm{I}$ (so $t' = t$, $\Sigma(\mathfrak{t}_\vartheta(z)) = S_M \cup Q$ with S_M the chosen places), $V_0 \neq 0$ and $i+n+2 \le m$. By Step 4, $\kappa_{i,n} \neq 0$; put $\beta(\vartheta) = \overline{c_{i,n}(\vartheta)}/\overline{\kappa_{i,n}} \neq 0$. Multiplying (6.63) by $c_{i,n}(\vartheta)/\kappa_{i,n} = \overline{\beta(\vartheta)}$ gives $w(n)\,\overline{\beta(\vartheta)}\, a_{i+1,n}b_{i,n} = w(n+2)\, b_{i,n+1}a_{i,n+1}$, the identity of (Qx1). Its modulus: with $\nu_q = \mu^0_{i+n+2}(\mu^0_{i+1})^{-1}$, so that $\nu_\vartheta = (\nu_q\vartheta^{\,n+1})^{-1}$, and $\vartheta^{\,n+1}$ having the modulus Q of ϑ, the definitions of the ratios give

$$q_{i+1,i+n+2}(\vartheta)\prod_{z'=1}^{t}|L^{S_M}(z',\widetilde{\nu_q^{-1}})| = \prod_{z'=1}^{t}|L^{S_M\cup Q}(z',\tilde\nu_\vartheta)|,$$

provided the factors $L^{S_M}(z',\widetilde{\nu_q^{-1}})$ are nonzero. They are: $\widetilde{\nu_q^{-1}}$, $x \mapsto \nu_q^{-1}(x/\bar x)$, is a unitary Hecke character of E; at a chosen place w and an idele x with a unit a at w and 1 elsewhere, $x/\bar x$ is integral with component a at w and 1 at the infinite places, so $\widetilde{\nu_q^{-1}}(x) = \omega_w^{-(n+1)}(a \bmod w)$ (the tame part of ν_q^{-1} is $\omega^{-(n+1)}$, trivial at $\bar w$), which is not identically 1 by Lemma 6.7.3, as $1 \le n+1 < m$; so the character is ramified, not of the form $|\cdot|^{i\tau}_{\mathbb{A}}$, and by Theorem A.1.12(3) its L-function has no zeros on $\mathrm{Re}\, s \ge 1$, while the finitely many removed Euler factors are finite and nonzero there. Hence, by (XD7),

$$|\beta(\vartheta)|^2 = \frac{C_{i,n}}{|\kappa_{i,n}|^2}\,(\mathrm{N}S_M\cdot\mathrm{N}Q)^{t^2}\prod_{z'=1}^{t}\left|L^{S_M\cup Q}(z',\tilde\nu_\vartheta)\right|^2 = C'\,(\mathrm{N}Q)^{t^2}\,q_{i+1,i+n+2}(\vartheta)^2,$$

$\mathrm{N}S_M = \prod_{w\in S_M}\mathrm{N}w$, with $C' = C_{i,n}|\kappa_{i,n}|^{-2}(\mathrm{N}S_M)^{t^2}\prod_{z'}|L^{S_M}(z',\widetilde{\nu_q^{-1}})|^2$ independent of ϑ. So $\mathcal{M}(\beta, f_{i+1,i+n+2})$. The relation $\mathcal{M}$ is unital ($C = 1$, exponent 0) and multiplicative (multiply the constants, add the exponents).

Step 6: the central translations. Let $z = \mathrm{I}$, $i' + n' < m$, and $u \in G_f(n_{n'})$, $u' \in G_f(n_{n'+1})$ the scalar matrices of the norm-one idele $\zeta = \overline{\alpha_{\mathrm{W}}t}/(\alpha_{\mathrm{W}}t)$ of Lemma 6.4.60. By (XD2), $((1,u),(1,u'))$ acts on the datum φ of $\mathrm{Tl}^{i'}_{\vartheta,z,n'}$ by $\alpha(\zeta^{N_{n'+1}})\beta(\zeta^{N_{n'}})^{-1}$, (α,β) its characters, times the Weil operator of $X \mapsto u'Xu^{-1} = X$, which is the identity ((2) of Lemma 4.1.1); so $\omega((1,u),(1,u'))\varphi = \eta\varphi$ with $|\eta| = 1$, and $\omega((1,u),1)\varphi = \eta\,\omega(1,(1,u'^{-1}))\varphi$. For a triangle form y at (i',n'), the substitution $g \mapsto g(1,u)$ in $[\mathrm{U}(V^{i'}_{z,n'})]$ gives

$$\mathrm{Tl}^{i'}_{\vartheta,z,n'}(\rho(u^{-1})y)(h) = \int\theta(\omega(g,h)\,\omega((1,u),1)\varphi)y(g)\,dg = \eta\,\mathrm{Tl}^{i'}_{\vartheta,z,n'}y(h(1,u'^{-1})),$$

i.e. $\mathrm{Tl}^{i'}_{\vartheta,z,n'}\circ\rho(u^{-1}) = \eta\,\rho(u'^{-1})\circ\mathrm{Tl}^{i'}_{\vartheta,z,n'}$.

Step 7: (Qx2). Let $z = \mathrm{I}$, $i+n+2 \le m$, and let $\Phi'_{i,n}$, $\Phi'_{i,n+1}$ be reversed pairs (Lemma 6.4.15(c)). By Lemma 6.4.60 at (i,n) and at $(i,n+1)$ and Lemma 6.4.62 there are c_1, c_2, γ of absolute value 1 up to a norm power with, for every ϑ and $\Psi = e_\vartheta(i,n)$ (so $\Psi^\natural = e_\vartheta(i,n+1)$): $\langle b_{i,n}x, y'\rangle = c_1(\vartheta)\langle x, A_0\rho(u^{-1})y'\rangle$ for $x \in \mathcal{H}_{i,n+1}$, $y' \in \mathcal{H}_{i+1,n}$; $\langle x', A_1\rho(u'^{-1})y\rangle = c_2(\vartheta)\langle b_{i,n+1}x', y\rangle$ for $x' \in \mathcal{H}_{i,n+2}$, $y \in \mathcal{H}_{i+1,n+1}$; and $a_{i,n+1}A_0 = \gamma(\vartheta)A_1a_{i+1,n}$ on $\mathcal{H}_{i+1,n}$; here $A_0 = \mathrm{As}^i_{\vartheta,z,n}(\Phi'_{i,n}, \widehat{\Psi})$, $A_1 = \mathrm{As}^i_{\vartheta,z,n+1}(\Phi'_{i,n+1}, \widehat{\Psi^\natural})$, and u, u' are as in Step 6 for $n' = n$. Let $\kappa = c_1\gamma\eta c_2$, which satisfies $\mathcal{M}(\kappa, 1)$ ($\mathcal{M}(\eta,1)$ as $|\eta| = 1$). Let $z' \in \mathcal{H}_{i,n+2}$, and let $x \in \mathcal{H}_{i,n+1}$ be the vector of Lemma 6.4.51(d) for $a_{i,n+1}$, so that $\langle a_{i,n+1}y, z'\rangle = \langle y, x\rangle$ for all y, hence $\langle x, y\rangle = \langle z', a_{i,n+1}y\rangle$. For $y' \in \mathcal{H}_{i+1,n}$, using Step 6 for $(i',n') = (i+1,n)$,

$$\begin{aligned}\langle b_{i,n}x, y'\rangle &= c_1\langle z', a_{i,n+1}A_0\rho(u^{-1})y'\rangle = c_1\gamma\langle z', A_1a_{i+1,n}\rho(u^{-1})y'\rangle\\ &= c_1\gamma\eta\langle z', A_1\rho(u'^{-1})a_{i+1,n}y'\rangle = \kappa(\vartheta)\langle b_{i,n+1}z', a_{i+1,n}y'\rangle,\end{aligned}$$

which is (Qx2). □

Lemma 6.4.12. *Let D be admissible data, $z \in \{\mathrm{I},\mathrm{II}\}$, $i,n \ge 0$ with $i+n+2 \le m$, Φ_0 a left pair at (i,n) and Φ_1 a left pair at $(i,n+1)$. Write $n_{n+1} = n_n \sqcup B_{n+1}$ and $n_{n+2} = n_{n+1} \sqcup B_{n+2}$, B_l the down block l, and identify B_{n+2} with B_{n+1} by $(n+2,c) \mapsto (n+1,c)$. For a function p on $\mathrm{M}_{n_{n+1}\times n_n}(R_\vartheta)$ let $p^\natural$ be its next datum formed for $(I,J,K) = (n_{n+1}, n_n, B_{n+1})$, read through this identification as a function on $\mathrm{M}_{n_{n+2}\times n_{n+1}}(R_\vartheta)$. Then there are $\kappa \in \mathbb{C}$ and $C > 0$, independent of ϑ, and for every m-fold twist ϑ a number c_ϑ with*

$$|c_\vartheta|^2 = C\,L_*\big(\Sigma(\mathfrak{t}_\vartheta(z)), \nu_\vartheta; t'\big) \ne 0,$$

such that for every m-fold twist ϑ and every function p on $\mathrm{M}_{n_{n+1}\times n_n}(R_\vartheta)$, on $\mathcal{F}^i_\triangle(z, n+1)$,

$$\#\mathrm{GL}_{n_n}(R_\vartheta)\ \mathrm{Tl}^{i+1}_{\vartheta,z,n} \circ \mathrm{Ds}^i_{\vartheta,z,n}(\Phi_0, p) = \frac{\kappa}{c_\vartheta}\ \#\mathrm{GL}_{n_{n+2}}(R_\vartheta)\ \mathrm{Ds}^i_{\vartheta,z,n+1}(\Phi_1, p^\natural) \circ \mathrm{Tl}^i_{\vartheta,z,n+1}.$$

Proof. **Step 1: the one-square identity for the square at (i,n).** Let I_{tw} be the set of m-fold twists. We apply Proposition 6.3.72 to the spaces $A' = V^{i+1}_{z,n}$, $A = V^i_{z,n+2}$, $U_1 = V^i_{z,n+1}$, $U_2 = V^{i+1}_{z,n+1}$ with index sets n_n, n_{n+2}, n_{n+1}, n_{n+1}, the integer t', the weight $\mu = (n+1)\Delta$, the family $\vartheta \mapsto \nu_\vartheta$, tame of weight μ for $(\mathfrak{t}_\vartheta(z), \chi^{\mathrm{sq}}_\vartheta)$ by Lemma 6.4.14(c), the characters of Lemma 6.4.14(a) on the four pairs, the polynomials $P_1 = H_{i,n+1}$ and $P'_2 = H_{i+1,n}$, the joint harmonics of the triangle lifts at $(i,n+1)$ and $(i+1,n)$, $P_2 = \Phi_1$, $P'_1 = \Phi_0$, and the bijection $\sigma\colon n_n \sqcup n_{n+2} \to n_{n+1} \sqcup n_{n+1}$ that is the identity from $n_{n+2} \supseteq n_{n+1}$ onto the first copy, maps n_n identically into the second copy and maps B_{n+2} onto B_{n+1} in the second copy by the identification of the statement. Its hypotheses: (OS1) and (OS3) are Lemma 6.4.14(a). (OS2): the weights of the characters do not depend on ϑ (Lemma 6.4.14(c)); with the types $e_1 = e_{\lambda_1}$, $e = e_\lambda$, $e' = e_{\lambda'}$, $e_2 = e_{\lambda_2}$ of the triangle forms on U_1, A, A', U_2 ($\lambda_1 = \frac12 p^i_{z,0,n+1}$, $\lambda = \frac12 p^i_{z,0,n+2}$, $\lambda' = \frac12 p^{i+1}_{z,0,n}$, $\lambda_2 = \frac12 p^{i+1}_{z,0,n+1}$), v_{P_1} is an edge from e_1 to e and $v_{P'_2}$ one from e' to e_2 for the weights of the characters of their pairs by Lemma 6.4.15(b) (as $i+n+1 < m$); v_{Φ_1} is an edge from $e_{-\lambda_2} = \overline{e_2}$ to $e_{-\lambda} = \overline{e}$, and v_{Φ_0} one from $e_{-\lambda'} = \overline{e'}$ to $e_{-\lambda_1} = \overline{e_1}$, for the weights $(\delta_i(N_{n+1}), \delta_i(N_{n+2}))$ and $(\delta_i(N_n), \delta_i(N_{n+1}))$ of the left-pair characters, by (TV6). For a matrix x over R_ϑ with rows n_{n+2} and columns $n_{n+1} \sqcup n_{n+1}$, with column blocks x_1, x_2, the composite $\Theta_\vartheta(x)$ of Proposition 6.3.72 is $\mathrm{Ds}^i_{\vartheta,z,n+1}(\Phi_1, \mathbf{1}_{x_2}) \circ \mathrm{Tl}^i_{\vartheta,z,n+1}[\mathbf{1}_{x_1}]$, the transposed lift from A to U_2 being the descent (TV7), the integral over $[\mathrm{U}(A)]$ against the theta kernel of the pair (U_2, A); for y with rows $n_{n+1} \sqcup n_{n+1}$, columns n_n and row blocks y_1, y_2, $\Theta'_\vartheta(y) = \mathrm{Tl}^{i+1}_{\vartheta,z,n}[\mathbf{1}_{y_2}] \circ \mathrm{Ds}^i_{\vartheta,z,n}(\Phi_0, \mathbf{1}_{y_1})$. These maps take triangle forms to triangle forms by Lemma 6.4.51. So Proposition 6.3.72 gives $C > 0$ and

$\kappa \in \mathbb{C}$ independent of ϑ and numbers c_ϑ with $|c_\vartheta|^2 = C\, L_*(\Sigma(\mathfrak{t}_\vartheta(z)), \nu_\vartheta; t') \neq 0$, such that for every form $f \in \mathcal{F}^i_\triangle(z, n+1)$ and every linear functional l on $\mathcal{F}^{i+1}_\triangle(z, n+1)$ the functions $L(x) = l(\Theta_\vartheta(x)f)$ and $L'(y) = l(\Theta'_\vartheta(y)f)$ satisfy: $L(kx) = \chi^{\mathrm{sq}}_\vartheta(\det k)L(x)$ for $k \in \mathrm{GL}_{n_{n+2}}(R_\vartheta)$, $L'(yk) = \chi^{\mathrm{sq}}_\vartheta(\det k)L'(y)$ for $k \in \mathrm{GL}_{n_n}(R_\vartheta)$, and $\sum_y \Psi'(y)L'(y) = \frac{\kappa}{c_\vartheta}\sum_x \Psi(x)L(x)$ for all Ψ, Ψ' matched for $(\sigma, \chi^{\mathrm{sq}}_\vartheta)$.

Step 2: the square law at the inclusion. Put $R = R_\vartheta$, $I = n_{n+1}$, $J = n_n$, $K = B_{n+1}$, so that $J \sqcup K = n_{n+1}$, $I \sqcup K$ is n_{n+2} through the identification of the statement, and $W = I \sqcup J \sqcup K$ is $n_{n+1} \sqcup n_{n+1}$ (I the first copy, $J \sqcup K$ the second); under these identifications the canonical bijection $J \sqcup (I \sqcup K) \to W$ of Lemma 6.4.13 is σ. By Step 1, L and L' satisfy (i) and (ii) of Lemma 6.4.13 for the character $\chi^{\mathrm{sq}}_\vartheta$ and $c = \kappa/c_\vartheta$. Part (1) there gives, for every function p on $\mathrm{M}_{I\times J}(R)$,

$$\#\mathrm{GL}_{n_n}(R)\sum_y p(y)\, L'\begin{pmatrix} y \\ \iota_J \end{pmatrix} = \frac{\kappa}{c_\vartheta}\,\#\mathrm{GL}_{n_{n+2}}(R)\sum_x p^\natural(x)\, L(\iota_I \;\; x).$$

Here $\iota_J \in \mathrm{M}_{n_{n+1}\times n_n}(R)$ and $\iota_I \in \mathrm{M}_{n_{n+2}\times n_{n+1}}(R)$ are the matrices with entry 1 at (c, c) and 0 elsewhere, whose indicator functions are the point data Φ^{pt}_n and Φ^{pt}_{n+1} of (TV3). So $L'\binom{y}{\iota_J} = l(\mathrm{Tl}^{i+1}_{\vartheta,z,n}\,\mathrm{Ds}^i_{\vartheta,z,n}(\Phi_0, \mathbf{1}_y)f)$ and $L(\iota_I\; x) = l\big(\mathrm{Ds}^i_{\vartheta,z,n+1}(\Phi_1, \mathbf{1}_x)\,\mathrm{Tl}^i_{\vartheta,z,n+1}f\big)$.

Step 3: summing the data. By Lemma 6.4.51(c), $\sum_y p(y)\mathrm{Ds}^i_{\vartheta,z,n}(\Phi_0, \mathbf{1}_y) = \mathrm{Ds}^i_{\vartheta,z,n}(\Phi_0, p)$ and $\sum_x p^\natural(x)\mathrm{Ds}^i_{\vartheta,z,n+1}(\Phi_1, \mathbf{1}_x) = \mathrm{Ds}^i_{\vartheta,z,n+1}(\Phi_1, p^\natural)$; with the linearity of Tl and of l, Step 2 reads

$$l\Big(\#\mathrm{GL}_{n_n}(R)\,\mathrm{Tl}^{i+1}_{\vartheta,z,n}\mathrm{Ds}^i_{\vartheta,z,n}(\Phi_0, p)f - \frac{\kappa}{c_\vartheta}\#\mathrm{GL}_{n_{n+2}}(R)\,\mathrm{Ds}^i_{\vartheta,z,n+1}(\Phi_1, p^\natural)\mathrm{Tl}^i_{\vartheta,z,n+1}f\Big) = 0$$

for every linear functional l on $\mathcal{F}^{i+1}_\triangle(z, n+1)$. Linear functionals separate the points of a vector space (extend a basis containing a given nonzero vector), so the vector in brackets is 0 for every f, which is the claim. □

Lemma 6.4.13. *Let R be a finite commutative ring and χ a multiplicative character of R. For finite sets I', J', K' and a function p on $\mathrm{M}_{I'\times J'}(R)$, the **next datum** $p^\natural$ is the function on $\mathrm{M}_{(I'\sqcup K')\times(J'\sqcup K')}(R)$ given on block matrices by*

$$p^\natural\begin{pmatrix} \eta & s \\ \zeta & d \end{pmatrix} = \begin{cases} p(-\eta)/\#\mathrm{M}_{I'\times K'}(R) & \text{if } \zeta = 0 \text{ and } d = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $\iota_{J'} \in \mathrm{M}_{(J'\sqcup K')\times J'}(R)$ is the matrix $\binom{1}{0}$. For an additive character ψ_R and a function f on $\mathrm{M}_{X\times Y}(R)$ put $\hat{f}(\rho) = \sum_{v\in\mathrm{M}_{X\times Y}(R)} f(v)\,\psi_R(\mathrm{tr}(\rho v))$ for $\rho \in \mathrm{M}_{Y\times X}(R)$.

Let I, J, K be finite sets, $W = I \sqcup J \sqcup K$, and $\sigma\colon J \sqcup (I \sqcup K) \to W$ the canonical bijection. Let $c \in \mathbb{C}$ and let L on $\mathrm{M}_{(I\sqcup K)\times W}(R)$ and L' on $\mathrm{M}_{W\times J}(R)$ satisfy

(i) $L(gx) = \chi(\det g)\,L(x)$ for $g \in \mathrm{GL}_{I\sqcup K}(R)$, and $L'(yh) = \chi(\det h)\,L'(y)$ for $h \in \mathrm{GL}_J(R)$;

(ii) $\sum_y \Psi(y)\,L'(y) = c\sum_x \Phi(x)\,L(x)$ for all Φ on $\mathrm{M}_{(I\sqcup K)\times W}(R)$ and Ψ on $\mathrm{M}_{W\times J}(R)$ matched for (σ, χ).

Then:

1. *for every function p on $\mathrm{M}_{I\times J}(R)$, with $p^\natural$ formed for (I, J, K),*

$$\#\mathrm{GL}_J(R)\sum_{y\in\mathrm{M}_{I\times J}(R)} p(y)\,L'\begin{pmatrix} y \\ \iota_J \end{pmatrix} = c\,\#\mathrm{GL}_{I\sqcup K}(R)\sum_{x\in\mathrm{M}_{(I\sqcup K)\times(J\sqcup K)}(R)} p^\natural(x)\,L(\iota_I \;\; x),$$

where $\binom{y}{\iota_J}$ has the rows y (index I) above ι_J (index $J \sqcup K$), and $(\iota_I \;\; x)$ has the columns ι_I (index I) before x (index $J \sqcup K$);

2. *suppose moreover that χ is primitive, ψ_R is a primitive additive character and $J = I \sqcup K$. There is $\gamma \in \mathbb{C}$, depending only on R, χ, ψ_R, $|I|$ and $|K|$, with*

$$|\gamma|^2 \, (\#R)^{|K|(|K|+2|I|)} = 1,$$

such that for all c, L, L' as above and every function p on $\mathrm{M}_{(I\sqcup K)\times I}(R)$, with $p^\natural$ formed for $(I \sqcup K, I, K)$,

$$\sum_{y \in \mathrm{M}_{I\times(I\sqcup K)}(R)} \widehat{\overline{p}}(y) \, L'\begin{pmatrix} y \\ \iota_J \end{pmatrix} = \gamma \, c \sum_{x \in \mathrm{M}_{(I\sqcup K)\times((I\sqcup K)\sqcup K)}(R)} \widehat{p^\natural}(x) \, L(\iota_I \;\; x).$$

Proof. **Step 1: two block identities.** Let $k \in \mathrm{GL}_W(R)$, let $x = (x_1 \; x_2) \in \mathrm{M}_{a\times W}(R)$ be the rows a of k^{-1} (x_1 on the columns b, x_2 on a) and $y = \binom{y_1}{y_2} \in \mathrm{M}_{W\times b}(R)$ the columns b of k (y_1 on the rows b). Since $k^{-1}k = 1$, the rows a of $k^{-1}k$ are $(0 \;\; 1)$; hence $x_1y_1 + x_2y_2 = 0$, and

$$\begin{pmatrix} 1 & 0 \\ x_1 & x_2 \end{pmatrix} k = \begin{pmatrix} y_1 & * \\ 0 & 1 \end{pmatrix}, \qquad \text{so} \qquad \det x_2 \cdot \det k = \det y_1.$$

Step 2: block functions are matched. Let δ be a function on $\mathrm{M}_{a\times b}(R)$, and put $\Phi_\delta(x) = \chi^{-1}(\det x_2)\,\delta(x_2^{-1}x_1)$ if x_2 is invertible and 0 otherwise, and $\Psi_\delta(y) = \chi^{-1}(\det y_1)\,\delta(-y_2y_1^{-1})$ if y_1 is invertible and 0 otherwise. They are matched for (σ, χ). Support: if $\Phi_\delta(x) \neq 0$, then x_2 is invertible and x is the rows a of k^{-1} for $k = \left(\begin{smallmatrix} 1 & 0 \\ x_1 & x_2 \end{smallmatrix}\right)^{-1}$; if $\Psi_\delta(y) \neq 0$, then y_1 is invertible and y is the columns b of $k = \left(\begin{smallmatrix} y_1 & 0 \\ y_2 & 1 \end{smallmatrix}\right)$. Values: let k, x, y be as in Step 1. If x_2 is invertible, so is y_1 (Step 1, $\det k$ a unit), and $y_2 = -x_2^{-1}x_1y_1$, so $-y_2y_1^{-1} = x_2^{-1}x_1$; then $\chi(\det k)\Psi_\delta(y) = \chi(\det k)\chi^{-1}(\det x_2)\chi^{-1}(\det k)\,\delta(x_2^{-1}x_1) = \Phi_\delta(x)$. If x_2 is not invertible, then $\det y_1 = \det x_2 \det k$ is not a unit and both values are 0.

Step 3: the law at a point. Let $w \in \mathrm{M}_{a\times b}(R)$ and δ the indicator function of $\{w\}$. By (ii) for $(\Phi_\delta, \Psi_\delta)$,

$$\sum_{y_1} \chi^{-1}(\det y_1) \sum_{y_2} \delta(-y_2y_1^{-1}) L'\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c \sum_{x_2} \chi^{-1}(\det x_2) \sum_{x_1} \delta(x_2^{-1}x_1) \, L(x_1 \; x_2),$$

the outer sums running over the invertible y_1, resp. x_2, since χ^{-1} vanishes at non-units. For $y_1 = h \in \mathrm{GL}_b(R)$ the inner sum on the left has the single term $y_2 = -wh$, and $L'\left(\begin{smallmatrix} h \\ -wh \end{smallmatrix}\right) = L'\left(\left(\begin{smallmatrix} 1 \\ -w \end{smallmatrix}\right)h\right) = \chi(\det h)L'\left(\begin{smallmatrix} 1 \\ -w \end{smallmatrix}\right)$ by (i); for $x_2 = g$ the inner sum on the right has the single term $x_1 = gw$, and $L(gw \; g) = \chi(\det g)L(w \; 1)$. Hence

$$\#\mathrm{GL}_b(R) \; L'\begin{pmatrix} 1 \\ -w \end{pmatrix} = c \, \#\mathrm{GL}_a(R) \; L(w \;\; 1). \tag{6.64}$$

Step 4: part (1). Return to the order $I \sqcup J \sqcup K$ of W. Let $\eta \in \mathrm{M}_{I\times J}(R)$, $s \in \mathrm{M}_{I\times K}(R)$, and take $w = \binom{\eta}{0} \in \mathrm{M}_{(I\sqcup K)\times J}(R)$ in (6.64). In the order $I \sqcup J \sqcup K$ of the rows, $\left(\begin{smallmatrix} 1 \\ -w \end{smallmatrix}\right)$ is $\binom{-\eta}{\iota_J}$; in that order of the columns, $(w \;\; 1)$ is $\left(\iota_I \;\; \left(\begin{smallmatrix} \eta & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$. The matrix $u = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_{I\sqcup K}(R)$ has determinant 1, fixes ι_I and maps $\left(\begin{smallmatrix} \eta & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \eta & s \\ 0 & 1 \end{smallmatrix}\right)$; by (i),

$$\#\mathrm{GL}_J(R) \, L'\begin{pmatrix} -\eta \\ \iota_J \end{pmatrix} = c \, \#\mathrm{GL}_{I\sqcup K}(R) \, L\Big(\iota_I \;\; \begin{pmatrix} \eta & s \\ 0 & 1 \end{pmatrix}\Big) \qquad \text{for all } \eta, s.$$

By the definition of $p^\natural$,

$$\sum_x p^\natural(x)L(\iota_I\, x) = \sum_{\eta,s} \frac{p(-\eta)}{\#\mathrm{M}_{I\times K}(R)} L(\iota_I \left(\begin{smallmatrix}\eta & s\\ 0 & 1\end{smallmatrix}\right)).$$

Multiplying by $c\,\#\mathrm{GL}_{I\sqcup K}(R)$ and inserting the last display, the sum over s cancels the denominator, and the result is $\#\mathrm{GL}_J(R)\sum_\eta p(-\eta)L'\binom{-\eta}{\iota_J}$, which is the left side of (1) after $y=-\eta$.

Step 5: the transform of a next datum. Let ψ_R be primitive, I', J', K' finite sets and q a function on $\mathrm{M}_{I'\times J'}(R)$. For $\rho = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{M}_{(J'\sqcup K')\times(I'\sqcup K')}(R)$ and $v = \left(\begin{smallmatrix}\eta & s\\ 0 & 1\end{smallmatrix}\right)$, $\mathrm{tr}(\rho v) = \mathrm{tr}(A\eta)+\mathrm{tr}(Cs)+\mathrm{tr}\,D$. Summing first over s, by orthogonality of characters ($\sum_s \psi_R(\mathrm{tr}(Cs))$ is a product over the entries of sums $\sum_{t\in R}\psi_R(C_{ji}t)$, so it is $\#\mathrm{M}_{I'\times K'}(R)$ for $C=0$ and 0 otherwise, ψ_R being primitive), and then over η, substituting $\eta\mapsto -\eta$,

$$\widehat{q^\natural}(\rho) = \begin{cases} \hat q(-A)\,\psi_R(\mathrm{tr}\,D) & C=0,\\ 0 & C\neq 0.\end{cases}$$

Step 6: part (2). Here $J = I\sqcup K$, and $\overline{p^\natural} = (\overline{p})^\natural$. Let G be the Gauss sum of Lemma 6.2.39 for (R,χ,ψ_R) and the set K, so $|G|^2 = (\#R)^{|K|^2}$, and put $q = \widehat{\overline{p}}$, a function on $\mathrm{M}_{I\times(I\sqcup K)}(R)$. For $\eta\in\mathrm{M}_{I\times(I\sqcup K)}(R)$, $s\in\mathrm{M}_{I\times K}(R)$ and $d\in\mathrm{M}_K(R)$ put $F_{\eta,s}(d) = L(\iota_I \left(\begin{smallmatrix}\eta & s\\ 0 & d\end{smallmatrix}\right))$. For $g\in \mathrm{GL}_K(R)$ the block matrix $\mathrm{diag}(1,g)\in\mathrm{GL}_{I\sqcup K}(R)$ fixes ι_I and maps $\left(\begin{smallmatrix}\eta & s\\ 0 & d\end{smallmatrix}\right)$ to $\left(\begin{smallmatrix}\eta & s\\ 0 & gd\end{smallmatrix}\right)$, so $F_{\eta,s}(gd) = \chi(\det g)F_{\eta,s}(d)$ by (i), and Lemma 6.2.39 at $y=1$ gives $\sum_d F_{\eta,s}(d)\psi_R(\mathrm{tr}\,d) = G\,F_{\eta,s}(1)$. By Step 5 (for $(I',J',K') = (I\sqcup K, I, K)$; the block A is indexed by $I\times(I\sqcup K)$) and the definition of $q^\natural$ (formed for $(I, I\sqcup K, K)$),

$$\sum_x \widehat{\overline{p^\natural}}(x)L(\iota_I\, x) = \sum_{\eta,s} q(-\eta)\sum_d F_{\eta,s}(d)\psi_R(\mathrm{tr}\,d) = G\,\#\mathrm{M}_{I\times K}(R)\sum_x q^\natural(x)\,L(\iota_I\, x).$$

By part (1) applied to q, and $\#\mathrm{GL}_J(R) = \#\mathrm{GL}_{I\sqcup K}(R)\neq 0$, $\sum_y q(y)L'\binom{y}{\iota_J} = c\sum_x q^\natural(x)L(\iota_I\, x)$. So the assertion holds with $\gamma = \left(G\,\#\mathrm{M}_{I\times K}(R)\right)^{-1}$, and $|\gamma|^{-2} = (\#R)^{|K|^2}(\#R)^{2|I||K|}$. □

Lemma 6.4.14. *Let D be admissible data, $z\in\{\mathrm{I},\mathrm{II}\}$, $[z]\in\{0,1\}$ with $[\mathrm{II}]=1$, $t' = t-[z]$, and $i,n\geq 0$ with $i+n+2\leq m$. For an m-fold twist ϑ let $\vartheta' = \vartheta$ if $z=\mathrm{I}$ and $\vartheta' = 1$ if $z=\mathrm{II}$, let $\epsilon_\vartheta(N)$ be the edge character with $c=\neg z$ and $l = n+i+1$ at the rank N, $\chi_\vartheta(N) = \chi^{\mathrm{lp}}_{\vartheta,z,i}(N)$, and*

$$\nu_\vartheta = \left(\mu^0_{n+i+2}\left(\mu^0_{i+1}\right)^{-1}\vartheta'^{\,n+1}\right)^{-1}, \qquad \mu = (n+1)\Delta \text{ (constant on } \Phi).$$

(a) (Left squares.) Put $U_1 = V^i_{z,n+1}$, $A = V^i_{z,n+2}$, $A' = V^{i+1}_{z,n}$, $U_2 = V^{i+1}_{z,n+1}$, $W = U_1\oplus U_2$, and give the pairs (U_1,A), (U_2,A), (A',U_1), (A',U_2) the characters $(\epsilon_\vartheta(N_{n+1}),\epsilon_\vartheta(N_{n+2}))$, $(\chi_\vartheta(N_{n+1}),\chi_\vartheta(N_{n+2}))$, $(\chi_\vartheta(N_n),\chi_\vartheta(N_{n+1}))$, $(\epsilon_\vartheta(N_n),\epsilon_\vartheta(N_{n+1}))$. Then $\mathrm{rk}\,A = \mathrm{rk}\,A' + 2t'$, $\mathrm{neg}(W) = \mathrm{neg}(A)+\mathrm{neg}(A')$, $2\,\mathrm{neg}(W)+2\leq \mathrm{rk}\,W\leq 2\mu+2\,\mathrm{neg}(W)$, and, writing (a_k,b_k) for the characters of (U_k,A) and (b'_k,a'_k) for those of (A',U_k),

$$a_1[\mathrm{rk}\,U_1]\,a_2[\mathrm{rk}\,U_2] = a'_1[\mathrm{rk}\,U_1]\,a'_2[\mathrm{rk}\,U_2] = b_1[\mathrm{rk}\,A]\,b'_1[\mathrm{rk}\,A'] = b_2[\mathrm{rk}\,A]\,b'_2[\mathrm{rk}\,A'] = \tilde\nu_\vartheta^{-1}.$$

(b) (Reversed squares.) Let $z=\mathrm{I}$. Put $U_1 = V^{i+1}_{z,n}$, $A = V^{i+1}_{z,n+1}$, $A' = V^i_{z,n+1}$, $U_2 = V^i_{z,n+2}$, $W = U_1\oplus U_2$, and give (U_1,A), (U_2,A), (A',U_1), (A',U_2) the characters $(\epsilon_\vartheta(N_n),\epsilon_\vartheta(N_{n+1}))$, $(\chi_\vartheta(N_{n+2}),\chi_\vartheta(N_{n+1}))$, $(\chi_\vartheta(N_{n+1}),\chi_\vartheta(N_n))$, $(\epsilon_\vartheta(N_{n+1}),\epsilon_\vartheta(N_{n+2}))$. Then $\mathrm{rk}\,A = \mathrm{rk}\,A'$, $\mathrm{neg}(W) = \mathrm{neg}(A)+\mathrm{neg}(A')$, $2\,\mathrm{neg}(W)+2\leq\mathrm{rk}\,W\leq 2\mu+2\,\mathrm{neg}(W)$, and the four products of (a), formed with these characters, equal $\tilde\nu_\vartheta^{-1}$.

(c) ν_ϑ is tame of weight μ for $(\mathfrak{t}_\vartheta(z), \chi_\vartheta^{\mathrm{sq}})$, where $\chi_\vartheta^{\mathrm{sq}}(u) = \prod_{w\in\Sigma(\mathfrak{t}_\vartheta(z))} \psi_w^\nu(u \bmod w)$ for the tame part ψ^ν of ν_ϑ; its weight and the weights of all the characters in (a) and (b) do not depend on ϑ.

Proof. **Ranks.** $N_{n+2} = N_n + 2t'$, and $N_n + N_{n+2} = 2N_{n+1}$. In (a), $\operatorname{rk} W = 2N_{n+1}$ and $\operatorname{rk} A - \operatorname{rk} A' = N_{n+2} - N_n = 2t'$; in (b), $\operatorname{rk} W = N_n + N_{n+2} = 2N_{n+1}$ and $\operatorname{rk} A = \operatorname{rk} A' = N_{n+1}$.

Signs. By (TV1) the space $V^o_{z,c}$ has the signs with offset o of the node $(z,0,c)$: by (GP5) every block index has the sign $+1$, and the base index β has the sign $(-1)^{o+e+\min(e\dot{-}o,\,c)}$, $e = \max(\beta+1-k, 0)$. Fix β and write $\pi(o,c) \in \{0,1\}$ for the parity of $o+e+\min(e\dot{-}o, c)$, so that $\pi(o,c) = 1$ exactly when β has the sign -1 in $V^o_{z,c}$. We claim $\pi(i,n+1) + \pi(i+1,n+1) = \pi(i,n+2) + \pi(i+1,n)$. Put $a = i+e$ and $d = e\dot{-}i$, so that $e\dot{-}(i+1) = d\dot{-}1$. If $d \le n+1$, then $\min(d, n+1) = \min(d, n+2) = d$ and $d\dot{-}1 \le n$, so $\min(d\dot{-}1, n+1) = \min(d\dot{-}1, n)$, and the two sides agree term by term. If $d \ge n+2$, then $d\dot{-}1 = d-1 \ge n+1$, the left side is the sum of the parities of $a+n+1$ and $a+n+2$ and the right side that of $a+n+2$ and $a+n+1$. Summing over β gives $\operatorname{neg}(V^i_{z,n+1}) + \operatorname{neg}(V^{i+1}_{z,n+1}) = \operatorname{neg}(V^i_{z,n+2}) + \operatorname{neg}(V^{i+1}_{z,n})$, which is $\operatorname{neg}(W) = \operatorname{neg}(A) + \operatorname{neg}(A')$ in (a), and, read as $\operatorname{neg}(V^{i+1}_{z,n}) + \operatorname{neg}(V^i_{z,n+2}) = \operatorname{neg}(V^{i+1}_{z,n+1}) + \operatorname{neg}(V^i_{z,n+1})$, the same in (b).

The inequalities. Only base indices have negative signs, so $\operatorname{neg}(W) \le 2(r+[z]) \le 2r+2$, while $\operatorname{rk} W = 2N_{n+1} \ge 2(r+t') \ge 2(6r-1)$; as $r \ge 3$ ((AD2)), $2\operatorname{neg}(W) + 2 \le 4r+6 \le \operatorname{rk} W$. Also $N_{n+1} \le r+1+(n+1)\,5r \le (n+1)(6r+1) \le (n+1)\,2r^8 = \mu$, so $\operatorname{rk} W \le 2\mu \le 2\mu + 2\operatorname{neg}(W)$.

The characters. By (GP6) and (TV6), $\epsilon_\vartheta(N) = \lambda\, \upsilon^{f(N)}$ and $\chi_\vartheta(N) = \lambda_0^{-1} \upsilon^{N-f(N)}$ with $\lambda = \mu^0_{n+i+2} \vartheta'^{\,n+i+2}$ and $\lambda_0 = \mu^0_{i+1} \vartheta'^{\,i+1}$, so $\lambda\lambda_0^{-1} = \nu_\vartheta^{-1}$. By (WD3), $\upsilon(\bar{x}) = \upsilon(x)^{-1}$, so the base change of υ restricted to $\mathrm{U}(1)(\mathbb{A})$ is $x \mapsto \upsilon(x/\bar{x}) = \upsilon(x)^2$; with $\alpha[p] = \tilde{\alpha}\,\upsilon^{-p}$ and $(\alpha\beta)^\sim = \tilde{\alpha}\tilde{\beta}$,

$$\epsilon_\vartheta(N)[p]\ \chi_\vartheta(N')[p'] = \tilde{\nu}_\vartheta^{-1}\, \upsilon^{\,2f(N)+2N'-2f(N')-p-p'}.$$

As $f(N+2t') = f(N) + t'$, the exponent vanishes in each of the eight cases: in (a), (N, N', p, p') is $(N_{n+1}, N_{n+1}, N_{n+1}, N_{n+1})$ for the first two products, $(N_{n+2}, N_n, N_{n+2}, N_n)$ for the third and $(N_n, N_{n+2}, N_n, N_{n+2})$ for the fourth; in (b) it is $(N_n, N_{n+2}, N_n, N_{n+2})$, $(N_{n+2}, N_n, N_{n+2}, N_n)$, $(N_{n+1}, N_{n+1}, N_{n+1}, N_{n+1})$ and $(N_{n+1}, N_{n+1}, N_{n+1}, N_{n+1})$.

Tameness and weights. By Definition 3.2.26, μ_j^0 has weight $-\kappa_j$ and tame part ω^j; ϑ' has weight 0; weights add and tame parts multiply under products. So ν_ϑ has weight $\kappa_{n+i+2} - \kappa_{i+1} = (n+1)\Delta = \mu$ and tame part $\psi^\nu = \omega^{-(n+1)}\psi'^{-(n+1)}$, ψ' the tame part of ϑ' (trivial if $z = \mathrm{II}$). At a chosen place w, $\psi_w = 1$ (the places of Q are prime to M, Definition 3.1.2) and $\omega_w = \eta_w^{q_2}$ has order q_2; as $1 \le n+1 < m < q_2$ ($q_2 = (q_0 d)^{s+6} \ge q_0 \ge 18480$ and $m \le 2\sqrt{x}$ is a power of 2 with $2^s = m^{22}$, so $m < 2^{s+6} \le q_2$), $\psi_w^\nu \ne 1$. At $w \in Q$ (only if $z = \mathrm{I}$), $\omega_w = 1$ and $\psi_w^{-(n+1)} \ne 1$, ϑ being an m-fold twist and $1 \le n+1 \le m$. At all other places $\psi_w^\nu = 1$. So the tame part of ν_ϑ is nontrivial exactly at $\Sigma(\mathfrak{t}_\vartheta(z))$ ((TV2): the chosen places, and the places of Q if $z = \mathrm{I}$), which with (XD4) is the tameness claimed. The weights of $\epsilon_\vartheta(N)$ and $\chi_\vartheta(N)$ are $\gamma_{\neg z, n+i+1}(N)$ and $\delta_i(N)$ ((GP6), (TV6)), independent of ϑ. □

Lemma 6.4.15. *Let D be admissible data and $z \in \{\mathrm{I}, \mathrm{II}\}$. For a node p write N_p for its rank and, if p has offset 0, $e_p = e_{-\frac{1}{2}p_{z,i,j}}$ for the type of the grid forms $\mathcal{F}(p)$, $p = (z,i,j)$.*

(a) For every $i \ge 0$, the triangle forms at $(i,0)$ are the triangle forms at $(0,0)$: a function on the common adelic group of $V^0_{z,0}$ and $V^i_{z,0}$ lies in $\mathcal{F}^i_\triangle(z,0)$ if and only if it lies in $\mathcal{F}^0_\triangle(z,0)$, and the Petersson products of $V^i_{z,0}$ and of $V^0_{z,0}$ agree on these forms.

(b) Let $o, a, b \ge 0$ with $a \le m$, $b+1+o \le m$, and $o = 0$ or $a = 0$; let $p = (z,a,b)$, $q = (z,a,b+1)$, $\gamma = \gamma_{\neg z, b+o}(N_p)$, $\gamma' = \gamma_{\neg z, b+o}(N_q)$, $\lambda = \frac{1}{2}p^o_{z,a,b}$, $\lambda' = \frac{1}{2}p^o_{z,a,b+1}$, and let

H be the joint harmonic (GP7) for the signs with offset o of p and q, the weights (γ,γ') and the doubled parameters $(2\lambda,2\lambda')$, a polynomial in the entries of $q\times p$ matrices. Then v_H is an edge for the weights (γ,γ') from e_λ to $e_{\lambda'}$. For $o=0$, H is the joint harmonic of the transposed move between p and q, and $e_\lambda=\overline{e_p}$, $e_{\lambda'}=\overline{e_q}$. For $a=0$, $o=i$, $b=n$ with $i+n<m$, H is the joint harmonic of the triangle lift at (i,n): its Fock vector is an edge for the weights $(\gamma_{\neg z,n+i}(N_n),\gamma_{\neg z,n+i}(N_{n+1}))$ from e_λ to $e_{\lambda'}$, where $\lambda=\frac{1}{2}p^i_{z,0,n}$ and $\lambda'=\frac{1}{2}p^i_{z,0,n+1}$.

(c) For all $i,n\geq 0$ with $i+n<m$ there are a left pair and a reversed pair at (i,n) ((TV6) of Definition 3.2.30).

(d) Let $0\leq a<m$, $0\leq b\leq m$, $p=(z,a,b)$ and $q=(z,a+1,b)$. The Fock vector of the joint harmonic of the lift move from p to q is an edge for the weights $(\gamma_{z,a}(N_p),\gamma_{z,a}(N_q))$ from e_p to e_q.

(e) Let O be a descent pair with ranks $1\leq r<n$ that fits a parameter $l=(l_\tau)_{\tau\in\Phi}$, let $\xi_O=v^+_{l_O}$, and put $\gamma(\tau)=r_-(O)$ and $\gamma'(\tau)=\lfloor n/2\rfloor+\mathrm{wt}_O(\tau)$ for $\tau\in\Phi$. Then the Fock vector $v_{\mathrm{Har}_O(\xi_O)}$ of the pair $(W(O),\mathrm{I}_n)$ is an edge for the weights (γ,γ') from $e_{l/2}$ to $e_{l_O/2}$ (types on $G_\infty(\varepsilon_O)$ and $G_\infty(\mathbf{1}_n)$).

Proof. Every node has a nonempty index set: its base $\{0,\dots,r-1+[z]\}$ is nonempty (GP5).

(a): the opposite space carries the same forms. Let $V=(J_V,x_V)$ be a space with a point with signs s on n and $V^-=(-J_V,c\,x_V)$ its opposite space, with signs $-s$; here c is a scalar idele with infinite components 1. We show that $\mathrm{G}(-s)=\mathrm{G}(s)$, that the two measures agree, that $\Gamma_{V^-}=\Gamma_V$, and that the type kernel e_λ of any family of parameters λ is the same function for the signs $-s$ as for s.

- Groups. A matrix g satisfies $g^*\varepsilon_{-s}g=\varepsilon_{-s}$ if and only if $g^*\varepsilon_s g=\varepsilon_s$, as $\varepsilon_{-s}=-\varepsilon_s$ at the infinite places and $\varepsilon_{\pm s}=1$ at the finite places; so $\mathrm{U}(\mathrm{diag}(-s))=\mathrm{U}(\mathrm{diag}\,s)$, $G_\infty(-s)=G_\infty(s)$, and $G_f(n)$ does not depend on the signs.
- Measures. For the hermitian involution $J=\mathrm{diag}\,s$ write $D_J(y)=1-\frac{1}{2}JS(y)$ and $\kappa_J(y)=2D_J(y)^{-1}-1$. Since S is linear, $D_{-J}(y)=1+\frac{1}{2}JS(y)=D_J(-y)$ and $\kappa_{-J}(y)=\kappa_J(-y)$. Hence the Cayley measure of $\mathrm{U}(-J)$, the image under κ_{-J} of $|\det D_{-J}(y)|^{-2N}\mathrm{vol}\,\mathrm{U}(N)^{-1}dy$, is the image under κ_J of the image of $|\det D_J(y)|^{-2N}\mathrm{vol}\,\mathrm{U}(N)^{-1}dy$ under $y\mapsto -y$, which preserves Lebesgue measure; so the two Cayley measures agree, and so do the measures of G_∞, $G_f(n)$ and G.
- Rational points. $\mathrm{U}(V^-)(F)=\{g\in\mathrm{GL}_n(E):g^*(-J_V)g=-J_V\}=\mathrm{U}(V)(F)$, and as c is central in $\mathrm{GL}_n(\mathbb{A}_E)$, $\Gamma_{V^-}=(c\,x_V)^{-1}\mathrm{U}(V)(F)(c\,x_V)=x_V^{-1}\mathrm{U}(V)(F)x_V=\Gamma_V$. So the quotients $[\mathrm{U}(V^-)]$ and $[\mathrm{U}(V)]$ and their quotient measures agree.
- Type kernels. In Definition 3.2.3 the signs ε enter only through the group $\mathrm{U}(\mathrm{diag}\,\varepsilon)$, its subgroup K_ε of unitary matrices with its Haar probability measure, the block-diagonal part bd_ε and the counts $2\rho^c_l$, and these depend only on the relations $\varepsilon_i=\varepsilon_j$, which are the same for ε and $-\varepsilon$. So Flensted-Jensen's function c_λ for $-s$ is that for s, on the same group; by the previous items its formal degree for the Cayley measure is the same; hence e_λ of (FT3) is the same function on $G_\infty(s)=G_\infty(-s)$.

Every clause of (ED2) (continuity, boundedness, left invariance under the rational points, a level, $R(e)F=F$ with the measure of G_∞) and the Petersson product are thus the same for

V^- and V. Now the node $(z,0,0)$ has no block indices, its parameter has $\delta = 0$, and the offset changes, at a base index i, only the sign, from $(-1)^e$ to $(-1)^{o+e+\min(e-o,0)} = (-1)^o(-1)^e$; so $p^i_{z,0,0} = p^0_{z,0,0}$, and the triangle forms at $(i,0)$ are the forms of type e_λ, $\lambda = \frac{1}{2}p^0_{z,0,0}$, on $V^i_{z,0}$. As $V^{i+1}_{z,0} = (V^i_{z,0})^-$ ((TV1)), (a) follows by induction on i.

(b): transposed moves with offset. By Lemma 6.4.16(ii) the families $2\lambda = p^o_{z,a,b}$ and $2\lambda' = p^o_{z,a,b+1}$ correspond for the weights (γ,γ') and the signs with offset o of p and q. By Lemma 6.4.18(iii) for these data (index sets nonempty), v_H is an edge for the weights (γ,γ') from e_λ to $e_{\lambda'}$. For $o = 0$ the data of H are those that (GP7) assigns to the transposed move between p and q (the signs of p and q, $\gamma_{\neg z,b}$ at the two ranks, the doubled parameters $p_{z,a,b}$, $p_{z,a,b+1}$, $q \times p$ matrices); and Lemma 6.4.18(i) gives $e_{-\lambda} = \overline{e_\lambda}$, hence $e_\lambda = \overline{e_{-\lambda}} = \overline{e_p}$, and likewise $e_{\lambda'} = \overline{e_q}$. For $a = 0$, $o = i$, $b = n$, the hypothesis $b + 1 + o \le m$ is $i + n < m$, and the data of H are those of the triangle lift at (i,n) in (TV3): the spaces $V^i_{z,n}$, $V^i_{z,n+1}$ carry the signs with offset i of $(z,0,n)$, $(z,0,n+1)$ (TV1), the weights are $\gamma_{\neg z,n+i}(N_n)$, $\gamma_{\neg z,n+i}(N_{n+1})$, and the doubled parameters $p^i_{z,0,n}$, $p^i_{z,0,n+1}$.

(c): left and reversed pairs. Let $i + n < m$, $\mu = \frac{1}{2}p^{i+1}_{z,0,n}$, $\mu' = \frac{1}{2}p^i_{z,0,n+1}$, s_1, s_2 the signs with offset $i+1$ of $(z,0,n)$ and with offset i of $(z,0,n+1)$, and $\delta = \delta_i(N_n)$, $\delta' = \delta_i(N_{n+1})$. By Lemma 6.4.16(iii), $l = -2\mu$ and $l' = -2\mu'$ correspond for the weights (δ,δ') and the signs (s_1,s_2). By Lemma 6.4.18(i), $e_{-\mu} = e_{l/2}$ and $e_\mu = e_{-l/2}$ are cusp kernels of order 0 with $e_\mu = \overline{e_{-\mu}}$, so $e_{-\mu} = \overline{e_\mu}$; likewise for μ'. Let H_1 be the joint harmonic (GP7) for $(s_1,s_2,\delta,\delta',l,l')$, a polynomial in the entries of $n_{n+1} \times n_n$ matrices; by Lemma 6.4.18(iii), v_{H_1} is an edge for the weights (δ,δ') from $e_{-\mu}$ to $e_{-\mu'}$. So H_1 is a left pair at (i,n). The block rule of (WM8) is symmetric: exchanging (n,s,l,γ) with (n',t,l',γ') leaves the gap condition (a) unchanged, maps the shifted entries $\tilde{l}$, $\tilde{l}'$ to $\tilde{l}'$, $\tilde{l}$, and exchanges the two cases of (b) and the two sides of the equalities in (c). Hence l' and l correspond for the weights (δ',δ) and the signs (s_2,s_1), and by Lemma 6.4.18(iii) the joint harmonic H_2 for $(s_2,s_1,\delta',\delta,l',l)$, a polynomial in the entries of $n_n \times n_{n+1}$ matrices, has a Fock vector that is an edge for the weights (δ',δ) from $e_{-\mu'}$ to $e_{-\mu}$; with the kernel conditions shown above, H_2 is a reversed pair at (i,n).

(d): lift moves. By Lemma 6.4.16(i), $-p_{z,a,b}$ and $-p_{z,a+1,b}$ correspond for the weights $(\gamma_{z,a}(N_p),\gamma_{z,a}(N_q))$ and the signs of p and q; these, with the doubled parameters $-p_{z,a,b}$, $-p_{z,a+1,b}$, are the data that (GP7) assigns to the lift move from p to q. By Lemma 6.4.18(iii) its Fock vector is an edge for these weights from $e_{-\frac{1}{2}p_{z,a,b}} = e_p$ to $e_{-\frac{1}{2}p_{z,a+1,b}} = e_q$.

(e): fitted descent pairs. By (i) of the fitting condition (DP11), l and l_O correspond for the weights (γ,γ') and the signs $(\varepsilon_O, \mathbf{1}_n)$; the index sets $\{1,\dots,r\}$ and $\{1,\dots,n\}$ are nonempty. By Lemma 6.4.47, $\mathrm{Har}_O(\xi_O) = \prod_{\tau\in\Phi} P_\tau$ with P_τ in the variables of τ a joint harmonic for the signs $(\varepsilon_O,\mathbf{1}_n)$, the weights $(\gamma(\tau),\gamma'(\tau))$ and the parameters $(l_\tau, l_{O,\tau})$. By Lemma 6.4.18(iii), $v_{\mathrm{Har}_O(\xi_O)}$ is an edge for the weights (γ,γ') from $e_{l/2}$ to $e_{l_O/2}$. □

Lemma 6.4.16. *Let D be admissible data and $z \in \{\mathrm{I},\mathrm{II}\}$. For a node p and an offset $o \ge 0$ write s^o_p for its signs with offset o and N_p for its rank.*

(i) Let $0 \le a < m$, $0 \le b \le m$, $p = (z,a,b)$, $q = (z,a+1,b)$. Then $-p_{z,a,b}$ and $-p_{z,a+1,b}$ correspond for the weights $(\gamma_{z,a}(N_p),\gamma_{z,a}(N_q))$ and the signs (s^0_p, s^0_q).

(ii) Let $o,a,b \ge 0$ with $a \le m$, $b+1+o \le m$, and $o = 0$ or $a = 0$; let $p = (z,a,b)$, $q = (z,a,b+1)$. Then $p^o_{z,a,b}$ and $p^o_{z,a,b+1}$ correspond for the weights $(\gamma_{\neg z,b+o}(N_p),\gamma_{\neg z,b+o}(N_q))$ and the signs (s^o_p, s^o_q).

(iii) Let $i,n \ge 0$ with $i+n < m$, $p = (z,0,n)$, $q = (z,0,n+1)$. Then $-p^{i+1}_{z,0,n}$ and $-p^i_{z,0,n+1}$ correspond for the weights $(\delta_i(N_p),\delta_i(N_q))$ and the signs (s^{i+1}_p, s^i_q).

Proof. Fix $\tau \in \Phi$ and z; write $\kappa_l = \kappa_l(\tau)$, $\varsigma_i = \varsigma_i(\tau)$ and $[z] \in \{0,1\}$. A pair of families corresponds if the block rule holds at every τ, so it suffices to prove the three block rules at τ.

Sizes. By (AD1) and $C_P \geq 6$, $\log x \geq 100 \log\log b \geq 6 \cdot 10^{14}$, so $r \geq x^5 \geq 10^4$ and $m \leq 2\sqrt{x} \leq r$. Every node (z,i,j) with $i,j \leq m$ has rank $N \leq r+1+2m(t+1) \leq 14r^2$, so for two such nodes $500(N_p+N_q)^3 \leq 500 \cdot 28^3 r^6 < r^8 \leq \Delta - t - 2$. Further:

(F1) $\kappa_{l'} - \kappa_l = (l'-l)\Delta$, and $\kappa_l \geq \kappa_1 \geq \Delta$ for $l \geq 1$;

(F2) ς is strictly decreasing, $2\varsigma_i \leq k-2$, and for $0 \leq i \leq r$, $l \geq 1$: $\varsigma_i + \kappa_l \geq \frac{\Delta}{2}$ if $i < k+l-1$, and $\varsigma_i + \kappa_l \leq -\frac{\Delta}{2}$ if $i \geq k+l-1$.

The centre system of the grid z at τ. Let $\Sigma = \{2\varsigma_i + [z] : 0 \leq i < r + [z]\}$ (so $\#\Sigma = r + [z]$ by (F2)), and let C consist of the up centres U_l with $z_{U_l} = \kappa_l$, $d_{U_l} = t + [z]$ and the down centres D_l with $z_{D_l} = -\kappa_l$, $d_{D_l} = t - [z]$, for $1 \leq l \leq m$. It satisfies (a)–(d) of Lemma 6.4.17 with $h = \Delta - t - 2$: (a) $t \pm [z] \equiv 1 + [z] \equiv r + [z]$, as r and t are odd; (b) $2\varsigma_i + [z] + r + [z] \equiv r \equiv 1$; (c) for $y = 2\varsigma_i + [z] \in \Sigma$: $2\kappa_l - y \geq 2\Delta - k + 1 \geq h + t + 1$ by (F1), (F2), and $|y + 2\kappa_l| \geq 2|\varsigma_i + \kappa_l| - 1 \geq \Delta - 1 \geq h + t$ by (F2); (d) an entry y of a block $b_{c'}$ has $|y - 2z_{c'}| \leq t$, and $|2z_c - 2z_{c'}| \geq 2\Delta$ for $c \neq c'$ by (F1) (centres of one kind differ by multiples of Δ, of different kinds by $\kappa_l + \kappa_{l'} \geq 2\Delta$), so $|y - 2z_c| \geq 2\Delta - t \geq h + t + 1 \geq h + d_c$.

Claim. Let $o, a, b \geq 0$ with $a \leq m$, $o + b \leq m$, and $o = 0$ or $a = 0$, and put $B(o,a,b) = \{U_l : 1 \leq l \leq a\} \cup \{D_l : o+1 \leq l \leq o+b\} \subseteq C$. Then $p^o_{z,a,b}(\tau)$ on the index set of the node (z,a,b), with the signs $s^o_{(z,a,b)}$, is a parameter of $B(o,a,b)$ with common sign $(-1)^o$.

Proof of the claim. Write $B = B(o,a,b)$. The rank $r + [z] + a(t + [z]) + b(t - [z])$ of the node is n_B. As $\#\Sigma = r + [z]$ is odd exactly for $z = \mathrm{I}$ and $\#B = a + b$, λ_B is the number δ of (GP5). So by (GP5) the numbers $p^o_{z,a,b}(\tau)(x) + \lambda_B$ are: $2\varsigma_i + [z]$ at the base index i; $2\kappa_l + t + [z] - 1 - 2c$ $(0 \leq c < t + [z])$ at the up index (l,c), that is the block b_{U_l}; and $-2\kappa_{l+o} + t - [z] - 1 - 2c$ $(0 \leq c < t - [z])$ at the down index (l,c), that is the block $b_{D_{l+o}}$. Their set is E_B. For the signs, let y be the number at an index x and count the centres outside B that count for y (up and $y > 2z_c$, or down and $y < 2z_c$).

- Up index (l,c), $l \leq a$ (so $a \geq 1$ and $o = 0$): $0 < y \leq 2\kappa_l + t$; an up centre $U_{l'} \notin B$ has $l' > l$ and $2\kappa_{l'} \geq 2\kappa_l + 2\Delta > y$, and no down centre counts as $y > 0$. So $\mathrm{sg}_B(y) = 1 = s^o(x)/(-1)^o$.
- Down index (l,c), $1 \leq l \leq b$: $|y + 2\kappa_{o+l}| \leq t$ and $y < 0$, so no up centre counts; a down centre $D_{l'} \notin B$ has $l' \leq o$ or $l' > o + b$, and by (F1) $-2\kappa_{l'} \geq -2\kappa_{o+l} + 2\Delta > y$ in the first case and $-2\kappa_{l'} < y$ in the second. So exactly the o centres $D_1, \ldots, D_o$ count, and $(-1)^o \mathrm{sg}_B(y) = 1 = s^o(x)$.
- Base index i, $y = 2\varsigma_i + [z] \leq k-1$: no up centre counts, by (F1). By (F2), $D_{l'}$ counts, that is $2(\varsigma_i + \kappa_{l'}) + [z] < 0$, exactly when $i \geq k + l' - 1$, that is $l' \leq e$ with $e = \max(i + 1 - k, 0)$; here $e \leq r + 1 - k = m$. The $D_{l'}$ with $1 \leq l' \leq e$ and $o + 1 \leq l' \leq o + b$ are $\min(e \dot{-} o, b)$ in number, so $e - \min(e \dot{-} o, b)$ centres count and $(-1)^o \mathrm{sg}_B(y) = (-1)^{o+e+\min(e\dot{-}o,b)} = s^o(x)$.

This proves the claim. Note also that replacing the signs s by $-s$ and the common sign ϵ by $-\epsilon$ keeps the property of being a parameter of B.

Weights. For a node of rank N, $\gamma_{c,l}(N) = \lfloor (N + [\neg c])/2 \rfloor - \kappa_{l+1}$ (GP6). In the grid $\tilde{Y}$ ($z = \mathrm{II}$) every rank $r + 1 + a(t+1) + b(t-1)$ is even. Hence $\gamma_{z,a}(N) = -\kappa_{a+1} + \lceil N/2 \rceil$ (for $z = \mathrm{I}$ by definition, for $z = \mathrm{II}$ as N is even) and $\gamma_{\neg z,l}(N) = -\kappa_{l+1} + \lfloor N/2 \rfloor$ (likewise).

(i). Here $o = 0$, $a + 1 \leq m$, $b \leq m$; $B(0, a+1, b) = B(0,a,b) \cup \{U_{a+1}\}$ with $U_{a+1} \notin B(0,a,b)$, and by the claim $p_{z,a,b}(\tau)$ and $p_{z,a+1,b}(\tau)$ are parameters of these sets with common sign 1.

As $h \geq 500(N_p + N_q)^3$, (i) of Lemma 6.4.17 for the up centre U_{a+1} gives the block rule for $(-p_{z,a,b}(\tau), -p_{z,a+1,b}(\tau))$, the weights

$$(-\kappa_{a+1} + \lceil N_p/2 \rceil, -\kappa_{a+1} + \lceil N_q/2 \rceil) = (\gamma_{z,a}(N_p), \gamma_{z,a}(N_q))$$

and the signs (s_p^0, s_q^0).

(ii). Here $a \leq m$, $o + b + 1 \leq m$, and $o = 0$ or $a = 0$, so the claim applies to (o, a, b) and $(o, a, b+1)$, with common sign $(-1)^o$; $B(o, a, b+1) = B(o, a, b) \cup \{D_{o+b+1}\}$, and $D_{o+b+1} \notin B(o, a, b)$. (i) of Lemma 6.4.17 for the down centre D_{o+b+1} gives the block rule for $(p^o_{z,a,b}(\tau), p^o_{z,a,b+1}(\tau))$, the weights

$$(-\kappa_{b+o+1} + \lfloor N_p/2 \rfloor, -\kappa_{b+o+1} + \lfloor N_q/2 \rfloor) = (\gamma_{\neg z,b+o}(N_p), \gamma_{\neg z,b+o}(N_q))$$

and the signs (s_p^o, s_q^o).

(iii). Here $i + n < m$. By the claim, $p^{i+1}_{z,0,n}(\tau)$ with the signs s_p^{i+1} is a parameter of $B(i + 1, 0, n) = \{D_{i+2}, \dots, D_{i+n+1}\}$ with common sign $(-1)^{i+1}$, hence with the signs $-s_p^{i+1}$ a parameter of the same set with common sign $(-1)^i$; and $p^i_{z,0,n+1}(\tau)$ with the signs s_q^i is a parameter of $B(i, 0, n+1) = B(i+1, 0, n) \cup \{D_{i+1}\}$ with common sign $(-1)^i$. (i) of Lemma 6.4.17 for the down centre D_{i+1} gives the block rule for $(p^{i+1}_{z,0,n}(\tau), p^i_{z,0,n+1}(\tau))$, the weights $(\gamma_{\neg z,i}(N_p), \gamma_{\neg z,i}(N_q))$ and the signs $(-s_p^{i+1}, s_q^i)$. By (ii) of Lemma 6.4.17, $(-p^{i+1}_{z,0,n}(\tau), -p^i_{z,0,n+1}(\tau))$ satisfies the block rule for the weights $(N_p - \gamma_{\neg z,i}(N_p), N_q - \gamma_{\neg z,i}(N_q)) = (\delta_i(N_p), \delta_i(N_q))$ and the signs (s_p^{i+1}, s_q^i). □

Lemma 6.4.17. *Let $\Sigma \subset \mathbb{Z}$ be a finite set and C a finite set; for $c \in C$ let $z_c \in \mathbb{Z}$, $d_c \in \mathbb{N}$ and a kind,* up *or* down*, be given, and put $b_c = \{2z_c + d_c - 1 - 2\beta : 0 \leq \beta < d_c\} = 2z_c + [d_c]$. Let $h \geq 1$ be an integer with*

(a) $d_c \equiv \#\Sigma \pmod 2$ for every $c \in C$;

(b) $y + \#\Sigma$ is odd for every $y \in \Sigma$;

(c) $|y - 2z_c| \geq h + d_c$ for every $c \in C$ and $y \in \Sigma$;

(d) $|y - 2z_c| \geq h + d_c$ for all $c \neq c'$ in C and $y \in b_{c'}$.

For $B \subseteq C$ put $E_B = \Sigma \cup \bigcup_{c \in B} b_c$, $n_B = \#\Sigma + \sum_{c \in B} d_c$, $\lambda_B = 1$ if $\#\Sigma$ and $\#B$ are both odd and $\lambda_B = 0$ otherwise, and for $y \in \mathbb{Z}$

$$\mathrm{sg}_B(y) = (-1)^{\#\{c \in C \setminus B:\ c \text{ is up and } y > 2z_c, \text{ or } c \text{ is down and } y < 2z_c\}}.$$

A map $l\colon I \to \mathbb{Z}$ on a finite set I with signs $s\colon I \to \{\pm 1\}$ is a parameter of B with common sign $\epsilon \in \{\pm 1\}$ *if $\#I = n_B$, $\{l(x) + \lambda_B : x \in I\} = E_B$, and $s(x) = \epsilon\, \mathrm{sg}_B(l(x) + \lambda_B)$ for every $x \in I$.*

(i) Let $B \subseteq C$, $c \in C \setminus B$ and $B' = B \cup \{c\}$; let l on I with signs s be a parameter of B and l' on I' with signs s' a parameter of B', with the same common sign; let $N = \#I$, $N' = \#I'$ and $h \geq 500(N + N')^3$. If c is down, the pair (l, l') satisfies the block rule for the weights $(z_c + \lfloor N/2 \rfloor,\ z_c + \lfloor N'/2 \rfloor)$ and the signs (s, s'). If c is up, the pair $(-l, -l')$ satisfies the block rule for the weights $(-z_c + \lceil N/2 \rceil,\ -z_c + \lceil N'/2 \rceil)$ and the signs (s, s').

(ii) Let $l\colon I \to \mathbb{Z}$ with signs s and $l'\colon I' \to \mathbb{Z}$ with signs t, on finite sets with $N = \#I$, $N' = \#I'$, satisfy the block rule for the weights (γ, γ') and the signs (s, t). Then $(-l, -l')$ satisfies the block rule for the weights $(N - \gamma, N' - \gamma')$ and the signs $(-s, t)$.

Proof. Throughout, a centre $c \notin B$ *counts for* y if c is up and $y > 2z_c$, or c is down and $y < 2z_c$; so $\mathrm{sg}_B(y) = (-1)^{\#\{c \in C \setminus B:\ c \text{ counts for } y\}}$.

Step 1: the entries. Let $B \subseteq C$ and $c \in C \setminus B$. Every $y \in E_B$ satisfies $|y - 2z_c| \geq h + d_c$, by (c) if $y \in \Sigma$ and by (d) if $y \in b_{c'}$ with $c' \in B$, $c' \neq c$. Every element y' of b_c satisfies $|y' - 2z_c| \leq d_c - 1$. Hence $b_c \cap E_B = \emptyset$, and by induction on $\#B$ the sets Σ and b_c ($c \in B$) are pairwise disjoint; as $\#b_c = d_c$, $\#E_B = n_B$. If l on I is a parameter of B, then $x \mapsto l(x) + \lambda_B$ maps I, of size n_B, onto E_B, of size n_B; so it is bijective and l is injective.

Step 2: parities. Every $y \in E_B$ has $y + \#\Sigma$ odd: for $y \in \Sigma$ by (b), and for $y = 2z_c + d_c - 1 - 2\beta \in b_c$ because $y + \#\Sigma \equiv d_c - 1 + \#\Sigma \equiv 1$ by (a). By (a), $n_B \equiv \#\Sigma\,(1 + \#B) \pmod 2$, so $n_B \equiv \#\Sigma - \lambda_B \pmod 2$ (both sides are 0 for $\#\Sigma$ even; for $\#\Sigma$ odd, $\lambda_B \equiv \#B$). Hence for a parameter l of B, $l(x) = (l(x) + \lambda_B) - \lambda_B \equiv \#\Sigma - 1 - \lambda_B \equiv n_B - 1 \pmod 2$.

Step 3: the gap. Let l on I with signs s be a parameter of B with common sign ϵ, and $x, x' \in I$ with $s(x) \neq s(x')$. With $y = l(x) + \lambda_B$, $y' = l(x') + \lambda_B$ we get $\mathrm{sg}_B(y) \neq \mathrm{sg}_B(y')$, so some $c \in C \setminus B$ counts for exactly one of y, y'; then $2z_c$ lies weakly between y and y', and both are at distance at least $h + d_c \geq h > 0$ from $2z_c$ by Step 1. So y and y' lie strictly on opposite sides of $2z_c$ and $|l(x) - l(x')| = |y - y'| \geq 2h$. With Steps 1 and 2, l has gap h for s in the sense of (WM8), hence gap h' for every $h' \leq h$.

Step 4: the lowerings. In (i), $N = n_B$ and $N' = n_{B'} = N + d_c$. We claim $N' \bmod 2 = \lambda_B$ and $N \bmod 2 = \lambda_{B'}$. If $\#\Sigma$ is even, $\lambda_B = \lambda_{B'} = 0$ and $N \equiv N' \equiv 0$ by Step 2. If $\#\Sigma$ is odd, $\lambda_{B'} = 1 - \lambda_B$ since $\#B' = \#B + 1$; by Step 2, $N \equiv 1 - \lambda_B = \lambda_{B'}$, and $N' \equiv N + d_c \equiv N + 1 \equiv \lambda_B$ by (a).

Step 5: signs under insertion. As $C \setminus B$ is the disjoint union of $C \setminus B'$ and $\{c\}$, $\mathrm{sg}_B(y) = -\mathrm{sg}_{B'}(y)$ if c counts for y, and $\mathrm{sg}_B(y) = \mathrm{sg}_{B'}(y)$ otherwise.

Step 6: (i) for a down centre. Let $\gamma = z_c + \lfloor N/2 \rfloor$, $\gamma' = z_c + \lfloor N'/2 \rfloor$. Since $N' - 2\lfloor N'/2 \rfloor = N' \bmod 2 = \lambda_B$ and $N - 2\lfloor N/2 \rfloor = \lambda_{B'}$ (Step 4), the shifted entries of (WM8) are

$$\tilde{l}(x) = l(x) - 2\gamma' + N' = l(x) + \lambda_B - 2z_c, \qquad \tilde{l}'(x') = l'(x') - 2\gamma + N = l'(x') + \lambda_{B'} - 2z_c.$$

(a) of (WM8): by Step 3 (for B and for B'), l and l' have gap $h \geq 500(N + N')^3$ for s and s'. (b) of (WM8): $N \leq N'$, and $\tilde{l}'(I') = E_{B'} - 2z_c = (E_B - 2z_c) \cup (b_c - 2z_c) = \tilde{l}(I) \cup [d_c] = \tilde{l}(I) \cup [N' - N]$. (c) of (WM8): if $\tilde{l}(x) = \tilde{l}'(x')$, then $y = l(x) + \lambda_B = l'(x') + \lambda_{B'}$, $s(x) = \epsilon\,\mathrm{sg}_B(y)$ and $s'(x') = \epsilon\,\mathrm{sg}_{B'}(y)$; as c is down, it counts for y exactly when $y < 2z_c$, that is when $\tilde{l}(x) < 0$. By Step 5, $s(x) = s'(x')$ if $\tilde{l}(x) > 0$ and $s(x) = -s'(x')$ if $\tilde{l}(x) < 0$.

Step 7: (i) for an up centre. Let $\Gamma = -z_c + \lceil N/2 \rceil$, $\Gamma' = -z_c + \lceil N'/2 \rceil$. As $2\lceil N'/2 \rceil - N' = \lambda_B$ and $2\lceil N/2 \rceil - N = \lambda_{B'}$, the shifted entries of $(-l, -l')$ are

$$-l(x) - 2\Gamma' + N' = -(l(x) + \lambda_B - 2z_c), \qquad -l'(x') - 2\Gamma + N = -(l'(x') + \lambda_{B'} - 2z_c).$$

(a): $-l$ and $-l'$ are injective, have the parities of l and l' and the same absolute differences, so Step 3 gives the gaps. (b): the shifted entries of $-l'$ form $-(E_{B'} - 2z_c) = -(E_B - 2z_c) \cup (-[d_c])$, which is the set of shifted entries of $-l$ together with $[d_c] = [N' - N]$, as $[d] = -[d]$. (c): for a common shifted entry $u = -(y - 2z_c)$, with y as in Step 6, the up centre c counts for y exactly when $y > 2z_c$, that is when $u < 0$; by Step 5 the signs agree when $u > 0$ and are opposite when $u < 0$.

(ii). The shifted entries of $-l$ and $-l'$ for the weights $(N - \gamma, N' - \gamma')$ are $-l(a) - 2(N' - \gamma') + N' = -\tilde{l}(a)$ and $-l'(b) - 2(N - \gamma) + N = -\tilde{l}'(b)$, with $\tilde{l}, \tilde{l}'$ the shifted entries of (l, l') for (γ, γ'). (a): a gap condition depends only on $|l(a) - l(a')|$, the parity of $l(a)$ and whether $s(a) \neq s(a')$; all are unchanged by $l \mapsto -l$, $s \mapsto -s$. (b): negate both sides of (b) for (l, l') and use $[d] = -[d]$. (c): if $-\tilde{l}(a) = -\tilde{l}'(b) > 0$, then $\tilde{l}(a) < 0$, so $s(a) = -t(b)$, that is $(-s)(a) = t(b)$; if $-\tilde{l}(a) < 0$, then $s(a) = t(b)$, that is $(-s)(a) = -t(b)$. □

Lemma 6.4.18. *Let n, n' be nonempty finite index sets with signs s on n and t on n', let $\gamma, \gamma' \colon \Phi \to \mathbb{Z}$, and let $l = (l_v)_{v\in\Phi}$, $l' = (l'_v)_{v\in\Phi}$ be families of maps $l_v \colon n \to \mathbb{Z}$, $l'_v \colon n' \to \mathbb{Z}$ that correspond for the weights (γ, γ') and the signs (s, t).*

(i) $e_{l/2}$ and $e_{-l/2}$ are cusp kernels of order 0 on $G_\infty(s)$ with $e_{-l/2} = \overline{e_{l/2}}$, and $e_{l'/2}$, $e_{-l'/2}$ are cusp kernels of order 0 on $G_\infty(t)$ with $e_{-l'/2} = \overline{e_{l'/2}}$.

(ii) For every $v \in \Phi$ there is a joint harmonic for the signs (s, t), the weights $(\gamma(v), \gamma'(v))$ and the parameters (l_v, l'_v).

(iii) For every $v \in \Phi$ let H_v be a joint harmonic for the signs (s, t), the weights $(\gamma(v), \gamma'(v))$ and the parameters (l_v, l'_v), written in the variables $X_{v,a,c}$ ($a \in n'$, $c \in n$), and $H = \prod_{v\in\Phi} H_v$ (for instance the joint harmonic of (GP7) for $(s, t, \gamma, \gamma', l, l')$, by (ii)). Then the Fock vector v_H is an edge for the weights (γ, γ') from $e_{l/2}$ to $e_{l'/2}$.

Proof. Write $N = \#n$, $N' = \#n'$.

(i). By (a) of the block rule (WM8), every l_v has gap $h = 500(N + N')^3$ for s, and so has $-l_v$ (the conditions are invariant under $l \mapsto -l$). As $h \geq 500N^3$, Lemma 5.6.21 with $B = 0$ shows that $e_{l/2}$ and $e_{-l/2}$ are cusp kernels of order 0 on $G_\infty(s)$. By (ii) of Lemma 5.6.29, $c_{-l_v/2} = \overline{c_{l_v/2}}$ on $\mathrm{U}(\operatorname{diag} s)$; hence $|c_{-l_v/2}| = |c_{l_v/2}|$, so $d_{-l_v/2} = d_{l_v/2}$, a real number, and $e_{-l/2}(x) = \prod_v d_{l_v/2}\, c_{l_v/2}(x_v) = \overline{e_{l/2}(x)}$. The same argument applies to l' on $G_\infty(t)$ (gap $h \geq 500N'^3$ for t). By (ED1), moreover, $e(x^{-1}) = \overline{e(x)}$ for each of these kernels, so

$$e_{l/2}(x^{-1}) = e_{-l/2}(x), \qquad e_{-l'/2}(y^{-1}) = e_{l'/2}(y) \qquad (x \in G_\infty(s),\ y \in G_\infty(t)). \tag{6.65}$$

(ii). For every v the pair (l_v, l'_v) satisfies the block rule for the weights $(\gamma(v), \gamma'(v))$ and the signs (s, t), and Lemma 6.4.19 gives the joint harmonic.

(iii), Step 1: the local facts. Let $\iota = n' \times n$, $\varepsilon(v, a, c) = t(a)s(c)$ on $\Phi \times \iota$, $\Omega = \Omega_{\gamma,\gamma'}$ on $\mathcal{F}_\varepsilon$, and P, P' the operators of (o) of Proposition 6.4.21 for H, with the kernels $e_{-l/2}$ on $G_\infty(s)$ and $e_{l'/2}$ on $G_\infty(t)$; its hypotheses hold. Put $\Omega^{\mathrm{sw}}(y, x) = \Omega(x, y)$ on $G_\infty(t) \times G_\infty(s)$. By (2) of Lemma 4.1.1, Ω is a representation of $G_\infty(s) \times G_\infty(t)$ and $y \mapsto \Omega(1, y)u$, $x \mapsto \Omega(x, 1)u$ are continuous. Every $c \in \mathcal{F}_\varepsilon$ is bounded with $\sup|c| \leq \|c\|$, since $|\beta^\infty_\varepsilon| = 1$ and $0 < e^\infty \leq 1$ in the reproducing identity of (WR5); so $|\langle c, c'\rangle| \leq \|c\|\,\|c'\|$ and $u \mapsto \langle c, u\rangle$ is a bounded linear functional.

Step 2: lifts by kernels with a real part. Let (σ, σ') be (s, t) or (t, s), Ω_0 the map Ω or Ω^{sw} on $G_\infty(\sigma) \times G_\infty(\sigma')$ accordingly, and k_0 a cusp kernel of order 0 on $G_\infty(\sigma)$ such that $x \mapsto k_0(x^{-1})\Omega_0(x, 1)u$ is Bochner integrable for every $u \in \mathcal{F}_\varepsilon$; put $P_0 u = \int k_0(x^{-1})\Omega_0(x, 1)u\, dx$. Let S, T, V be spaces with points with signs σ, σ', and $\eta(a, c) = t(a)s(c)$ on ι (in both cases, so that the Fock model of (ED3) is $\mathcal{F}_\varepsilon$); let (ω, θ) be a theta kernel for (S, T) on V with real part Ω_0, $f \in \mathcal{M}_\eta$, and F a form of type k_0 on S.

(L1) By Lemma 6.4.45 there are C, A with $|\theta(\omega(g, h)\varphi)| \leq C\|\varphi\|(\mathrm{ht}_S(g)\mathrm{ht}_T(h))^A$, and by Lemma 6.3.75 $D = \sup_g |F(g)|\mathrm{ht}_S(g)^A < \infty$. Hence $|\theta(\omega(g, h)\varphi)F(g)| \leq CD\,\mathrm{ht}_T(h)^A\|\varphi\|$ for all g, h, φ. The integrand is continuous and $[\mathrm{U}(S)]$ has finite volume (Theorem 4.2.18); so for each h, $\Lambda_h(\varphi) = L_\varphi F(h)$ is a bounded linear functional on $\mathcal{M}_\eta$, of norm at most $CD\,\mathrm{vol}([\mathrm{U}(S)])\,\mathrm{ht}_T(h)^A$.

(L2) As ω is a homomorphism, $L_\varphi F(hh') = L_{\omega(1,h')\varphi}F(h)$; so $L_\varphi F(h) = \Lambda_1(\omega(1, h)\varphi)$ is continuous in h, and it is left invariant under Γ_T by the invariance of (ED3).

(L3) By the real part, $\omega(1,(y,1))(u\otimes f)=(\Omega_0(1,y)u)\otimes f$, hence

$$L_{(\Omega_0(1,y)u)\otimes f}F(h)=L_{u\otimes f}F(h(y,1));$$

likewise $\omega((x,1),1)(u\otimes f)=(\Omega_0(x,1)u)\otimes f$.

(L4) The map $u\mapsto u\otimes f$ is bounded linear from $\mathcal{F}_\varepsilon$ to $\mathcal{M}_\eta$, of norm at most $\|f\|$ ((1) of Lemma 6.2.21). So if k is a function on $G_\infty(\sigma')$ and $y\mapsto k(y)\Omega_0(1,y)u$ is Bochner integrable, then $y\mapsto k(y)L_{u\otimes f}F(h(y,1))$ is integrable and its integral is $\Lambda_h\big((\int k(y)\Omega_0(1,y)u\,dy)\otimes f\big)$, by (L3); the same holds in the first variable.

(L5) *Projection.* For every $u\in\mathcal{F}_\varepsilon$, $L_{u\otimes f}F=L_{(P_0u)\otimes f}F$. Fix h and put $\varphi=u\otimes f$. By (L4) in the first variable, (L3) and the right invariance of the quotient measure (substituting $g\mapsto g(x,1)^{-1}$),

$$\begin{aligned}\Lambda_h((P_0u)\otimes f)&=\int k_0(x^{-1})\,\Lambda_h(\omega((x,1),1)\varphi)\,dx\\&=\int k_0(x^{-1})\int_{[\mathrm{U}(S)]}\theta(\omega(g,h)\varphi)\,F(g(x^{-1},1))\,dg\,dx.\end{aligned}$$

Substitute $x\mapsto x^{-1}$; the double integrand becomes $k_0(x)\,\theta(\omega(g,h)\varphi)\,F(g(x,1))$. It is absolutely integrable on $G_\infty(\sigma)\times[\mathrm{U}(S)]$: writing $\theta(\omega(g,h)\varphi)=\theta(\omega(g(x,1),h)\,\omega((x^{-1},1),1)\varphi)$, (L1) bounds its modulus by $CD\,\mathrm{ht}_T(h)^A\|f\|\,|k_0(x)|\,\|\Omega_0(x^{-1},1)u\|$, whose integral over x is $\int|k_0(x^{-1})|\,\|\Omega_0(x,1)u\|\,dx<\infty$, and $[\mathrm{U}(S)]$ has finite volume. By Fubini and $R(k_0)F=F$ (k_0 is integrable and F is a form of type k_0), the integral is $\int_{[\mathrm{U}(S)]}\theta(\omega(g,h)\varphi)F(g)\,dg=L_\varphi F(h)$.

Step 3: the first half edge, for Ω from $e_{l/2}$ to $e_{l'/2}$. Take $(\sigma,\sigma')=(s,t)$, $\Omega_0=\Omega$, $k_0=e_{l/2}$: by (6.65), $k_0(x^{-1})=e_{-l/2}(x)$, so $P_0=P$, and (o) of Proposition 6.4.21 gives the integrability. (a): $e_{l/2}$ is a cusp kernel of order 0 by (i). (b): by (i), $\overline{e_{l/2}}=e_{-l/2}$, and the bounded functional $\langle v_H,\cdot\rangle$ commutes with the Bochner integral, so the zeta integral is $e_{l/2}(1)^{-1}\langle v_H,Pv_H\rangle$, nonzero by (a) of Proposition 6.4.21 ($e_{l/2}(1)\neq0$ by (ED1)). (c): let $S,T,V,(\omega,\theta),f,F$ be as in (ED4)(c) and $F_H=L_{v_H\otimes f}F$. By (L5), $F_H=L_{(Pv_H)\otimes f}F$. By (o) for $u=Pv_H$, (L4) with $k=e_{l'/2}$ and (b) of Proposition 6.4.21, for every h, $y\mapsto e_{l'/2}(y)F_H(h(y,1))$ is integrable and

$$R(e_{l'/2})F_H(h)=\Lambda_h((P'Pv_H)\otimes f)=\Lambda_h((Pv_H)\otimes f)=F_H(h).$$

If F_H is right invariant under an open subgroup K', it is moreover continuous and left invariant under Γ_T (L2), and Lemma 6.4.46 (with the cusp kernel $e_{l'/2}$ of order 0, by (i)) shows that $F_H=R(e_{l'/2})F_H$ is a form of type $e_{l'/2}$ on T. (d): for a polynomial Q let a be the function on $G_\infty(t)$ of (c) of Proposition 6.4.21; it depends on Q only. As a is continuous with compact support and $y\mapsto\Omega(1,y)Pv_H$ is continuous, $y\mapsto a(y)\Omega(1,y)Pv_H$ is Bochner integrable. By (L5), Proposition 6.4.21 (c), (L4) and (L5) again,

$$L_{v_Q\otimes f}F(h)=\Lambda_h((Pv_Q)\otimes f)=\int a(y)\,L_{(Pv_H)\otimes f}F(h(y,1))\,dy=R(a)F_H(h).$$

Step 4: the second half edge, for Ω^{sw} from $\overline{e_{l'/2}}=e_{-l'/2}$ to $\overline{e_{l/2}}=e_{-l/2}$. Take $(\sigma,\sigma')=(t,s)$, $\Omega_0=\Omega^{\mathrm{sw}}$, $k_0=e_{-l'/2}$: by (6.65), $k_0(y^{-1})=e_{l'/2}(y)$, so $P_0=P'$, integrable by (o). (a): $e_{-l'/2}$ is a cusp kernel of order 0 by (i). (b): the zeta integral $\int\langle v_H,\Omega(1,y)v_H\rangle\overline{e_{-l'/2}(y)}\,dy/e_{-l'/2}(1)$ equals $e_{-l'/2}(1)^{-1}\langle v_H,P'v_H\rangle\neq0$ by (i) and (a) of Proposition 6.4.21. (c): with $F_H=L_{v_H\otimes f}F$ for a form F of type $e_{-l'/2}$, (L5) gives $F_H=L_{(P'v_H)\otimes f}F$, and (o) for $u=P'v_H$, (L4) with

$k = e_{-l/2}$ and $PP'v_H = P'v_H$ ((b) of Proposition 6.4.21) give $R(e_{-l/2})F_H = F_H$ with integrable integrands; with a level, Lemma 6.4.46 for the cusp kernel $e_{-l/2}$ of order 0 shows that F_H is a form of type $e_{-l/2}$. (d): with the function b on $G_\infty(s)$ of (c) of Proposition 6.4.21, as in Step 3, $L_{v_Q\otimes f}F = R(b)F_H$.

So v_H is a half edge for Ω from $e_{l/2}$ to $e_{l'/2}$ and a half edge for Ω^{sw} from $\overline{e_{l'/2}}$ to $\overline{e_{l/2}}$: an edge for the weights (γ, γ'). □

Lemma 6.4.19. *Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma, \gamma' \in \mathbb{Z}$, and let $\lambda\colon I \to \mathbb{Z}$, $\lambda'\colon I' \to \mathbb{Z}$ be such that (λ, λ') satisfies the block rule for the weights (γ, γ') and the signs (s, t). Then there is a joint harmonic for the signs (s, t), the weights (γ, γ') and the parameters (λ, λ').*

Proof. We construct H as a product of determinants and check the conditions of (HV7). Condition (a) of (HV7) is the hypothesis, so it remains to find H satisfying (b): jointly highest of weights $(\mu_{\lambda'}, \mu_{-\lambda})$ for $(\lambda', -\lambda)$, for the row and column units of (s, t) with the shifts $(\gamma, -\gamma')$.

Step 1: the conditions in coordinates. For a row $a \in I'$, a column $c \in I$ and $\sigma \in \{\pm 1\}$ let $D^\sigma_a = \sum_{c'\colon t(a)s(c')=\sigma} X_{ac'}\partial_{ac'}$ and $D^\sigma_c = \sum_{a'\colon t(a')s(c)=\sigma} X_{a'c}\partial_{a'c}$ be the Euler operators of row a and of column c in the variables of sign σ. From (HV5) with the shifts $(\gamma, -\gamma')$,

$$\begin{aligned}
E^{\mathrm{row}}_{aa} &= D^+_a - D^-_a + \gamma - \#\{c : t(a)s(c) = -1\},\\
E^{\mathrm{row}}_{aa'} &= \sum_{c\colon t(a)s(c)=1} X_{ac}\partial_{a'c} - \sum_{c\colon t(a)s(c)=-1} X_{a'c}\partial_{ac} \qquad (a \neq a',\ t(a) = t(a')),\\
E^{\mathrm{col}}_{cc} &= D^-_c - D^+_c - \gamma' + \#\{a : t(a)s(c) = -1\},\\
E^{\mathrm{col}}_{cc'} &= -\sum_{a\colon t(a)s(c)=1} X_{ac'}\partial_{ac} + \sum_{a\colon t(a)s(c)=-1} X_{ac}\partial_{ac'} \qquad (c \neq c',\ s(c) = s(c')).
\end{aligned}$$

By the definitions of ν and ν' in Lemma 6.4.20, a polynomial H is K-highest of weight $\mu_{\lambda'}$ for λ' for the row units if and only if

$$(D^+_a - D^-_a)H = \nu(a)H \ (a \in I'), \qquad E^{\mathrm{row}}_{aa'}H = 0 \ \text{ when } t(a) = t(a'),\ \lambda'(a') < \lambda'(a), \tag{6.66}$$

and K-highest of weight $\mu_{-\lambda}$ for $-\lambda$ for the column units if and only if

$$(D^-_c - D^+_c)H = \nu'(c)H \ (c \in I), \qquad E^{\mathrm{col}}_{cc'}H = 0 \ \text{ when } s(c) = s(c'),\ \lambda(c) < \lambda(c'). \tag{6.67}$$

Step 2: layers. By (d) of Lemma 6.4.20 and the uniqueness of partners in (a) there, every row a with $\nu(a) \neq 0$ is matched with exactly one column c_a, and $t(a)s(c_a) = \operatorname{sgn}\nu(a)$. For $m \in \mathbb{N}$ and $\tau, \tau' \in \{\pm 1\}$ let the *layer* $L_{m,\tau,\tau'}$ be the set of matched pairs (a, c) with $|\nu(a)| > m$, $t(a) = \tau$ and $s(c) = \tau'$; let $R(L)$ and $C(L)$ be its sets of rows and of columns, so that $a \mapsto c_a$ is a bijection $R(L) \to C(L)$. Put $D_L = \det(X_{a\,c_b})_{a,b\in R(L)}$ (one ordering of $R(L)$ for rows and columns; a simultaneous reordering does not change it), $D_\emptyset = 1$, and

$$H = \prod_{m\geq 0} \prod_{\tau,\tau'\in\{\pm1\}} D_{L_{m,\tau,\tau'}},$$

a finite product, since $L_{m,\tau,\tau'} = \emptyset$ for $m \geq \max_a |\nu(a)|$. The following hold.

(L1) D_L is the generic determinant in the distinct variables X_{ac}, $a \in R(L)$, $c \in C(L)$, hence nonzero; it involves no other variable, is homogeneous of degree one in each row of $R(L)$ and in each column of $C(L)$, and all its variables have the sign $\tau\tau'$.

(L2) A row a lies in $R(L_{m,\tau,\tau'})$ if and only if $\nu(a) \neq 0$, $(\tau,\tau') = (t(a), s(c_a))$ and $m < |\nu(a)|$: it lies in exactly $|\nu(a)|$ layers, all with columns of sign $s(c_a)$ and variables of sign $\operatorname{sgn}\nu(a)$. A column lies in some $C(L)$ if and only if it is c_a for a row a with $\nu(a) \neq 0$, and then it lies exactly in the layers of a.

(L3) Let (a,c) be matched with $\nu(a) \neq 0$ and let $a' \in I'$ satisfy $t(a') = t(a)$ and $t(a)s(c)(\lambda'(a') - \lambda'(a)) > 0$. Then a' lies in every layer that contains a. Indeed, by (e) of Lemma 6.4.20, a' is matched with a column c' with $s(c') = s(c)$ and $|\nu(a')| \geq |\nu(a)| > 0$; so $c' = c_{a'}$, and by (L2) the layers of a' are the $L_{m,t(a),s(c)}$ with $m < |\nu(a')|$, which include those of a.

(L4) Let (a,c) be matched with $\nu(a) \neq 0$, $p = t(a)s(c)$, and let $c' \in I$ satisfy $s(c') = s(c)$ and $p(\lambda(c') - \lambda(c)) > 0$. Then c' lies in every layer that contains c. Indeed, by (a) of Lemma 6.4.20, $p = \operatorname{sgn}\tilde{\lambda}(c)$ and $p\tilde{\lambda}(c) > |N' - N|$; as $\tilde{\lambda} - \lambda$ is constant, $p\tilde{\lambda}(c') > p\tilde{\lambda}(c) > |N' - N|$. A column matched with no row has $|\tilde{\lambda}| < N - N' \leq |N' - N|$ by (a) there, so c' is matched with a row a', and $t(a')s(c') = \operatorname{sgn}\tilde{\lambda}(c') = p$; hence $t(a') = ps(c) = t(a)$. As $\tilde{\lambda}'(a') = \tilde{\lambda}(c')$, $\tilde{\lambda}'(a) = \tilde{\lambda}(c)$ and $\tilde{\lambda}' - \lambda'$ is constant, $p(\lambda'(a') - \lambda'(a)) = p(\tilde{\lambda}(c') - \tilde{\lambda}(c)) > 0$. By (L3), a' lies in every layer containing a; these layers contain c', the partner of a', and they are the layers containing c by (L2).

Step 3: $H \neq 0$, termwise harmonicity and the weights. H is a product of nonzero polynomials by (L1), so $H \neq 0$. By (L1) and (L2), the variables of row a occurring in H come from the layers of a, whose columns have the one sign $s(c_a)$, and the variables of a column $c = c_a$ come from the same layers, whose rows have the one sign $t(a)$. So no monomial of H contains two variables of one row from columns of opposite signs, or two variables of one column from rows of opposite signs: H is termwise harmonic. By (L1) and (L2), H is homogeneous of degree $|\nu(a)|$ in row a, in variables of the single sign $\operatorname{sgn}\nu(a)$ when $\nu(a) \neq 0$; hence $(D_a^+ - D_a^-)H = \nu(a)H$, also when $\nu(a) = 0$ (row a does not occur). For a column $c = c_a$ with $\nu(a) \neq 0$, H is homogeneous of degree $|\nu(a)|$ in column c, in variables of sign $\operatorname{sgn}\nu(a)$, so $(D_c^- - D_c^+)H = -\nu(a)H = \nu'(c)H$ by (b) of Lemma 6.4.20. Any other column c does not occur in H, and $\nu'(c) = 0$: if c is matched with no row, by (c) of Lemma 6.4.20; if c is matched with a row a, then $\nu(a) = 0$ and $\nu'(c) = -\nu(a) = 0$ by (b) there.

Step 4: a replacement rule. Let L be a layer and $b \neq b'$ rows with $b' \in R(L)$, and let $\delta = \sum_{c \in C(L)} X_{bc}\partial_{b'c}$. By the Laplace expansion along row b', δD_L is the determinant of the matrix of D_L with row b' replaced by $(X_{bc})_{c \in C(L)}$; it is 0 if $b \in R(L)$, the matrix then having two equal rows. In the same way, for columns $e \neq e'$ with $e' \in C(L)$, $\sum_{a \in R(L)} X_{ae}\partial_{ae'} D_L$ is D_L with column e' replaced by column e, and is 0 if $e \in C(L)$. Moreover $\partial_{ac} D_L = 0$ unless $a \in R(L)$ and $c \in C(L)$. Every operator T below is a sum of derivations $X_{bc}\partial_{b'c}$, so $TH = \sum_L (\prod_{L' \neq L} D_{L'}) T D_L$, the sum over the layers.

Step 5: the raising conditions. Let $a \neq a'$ with $t(a) = t(a')$ and $\lambda'(a') < \lambda'(a)$. Let $T_1 = \sum_{c:\, t(a)s(c) = 1} X_{ac}\partial_{a'c}$ and $T_2 = \sum_{c:\, t(a)s(c) = -1} X_{a'c}\partial_{ac}$, so $E^{\mathrm{row}}_{aa'} = T_1 - T_2$. If $T_1 D_L \neq 0$ for a layer L, then $a' \in R(L)$ and some column of L has $t(a')s(c) = 1$; all columns of L have the sign $s(c_{a'})$, so $t(a')s(c_{a'}) = 1$, T_1 contains $X_{ac}\partial_{a'c}$ for every $c \in C(L)$, and $T_1 D_L$ is D_L with row a' replaced by row a. But (L3) for the pair $(a', c_{a'})$ and the row a applies, since $t(a')s(c_{a'})(\lambda'(a) - \lambda'(a')) = \lambda'(a) - \lambda'(a') > 0$, so $a \in R(L)$ and $T_1 D_L = 0$ by Step 4, a contradiction. If $T_2 D_L \neq 0$, then $a \in R(L)$, $t(a)s(c_a) = -1$, and $T_2 D_L$ is D_L with row a replaced by row a'; (L3) for (a, c_a) and a' applies, since $t(a)s(c_a)(\lambda'(a') - \lambda'(a)) = \lambda'(a) - \lambda'(a') > 0$, so $a' \in R(L)$ and $T_2 D_L = 0$, a contradiction. Hence $T_1 H = T_2 H = 0$ and $E^{\mathrm{row}}_{aa'} H = 0$.

Let $c \neq c'$ with $s(c) = s(c')$ and $\lambda(c) < \lambda(c')$. Let $T_3 = \sum_{a:\, t(a)s(c) = 1} X_{ac'}\partial_{ac}$ and $T_4 = \sum_{a:\, t(a)s(c) = -1} X_{ac}\partial_{ac'}$, so $E^{\mathrm{col}}_{cc'} = -T_3 + T_4$. If $T_3 D_L \neq 0$, then $c \in C(L)$, say $c = c_b$ with $b \in R(L)$; all rows of L have the sign $t(b)$, so $t(b)s(c) = 1$ and $T_3 D_L$ is D_L with column c replaced

by column c'; (L4) for (b,c), $p=1$, and c' gives $c'\in C(L)$, so $T_3D_L=0$, a contradiction. If $T_4D_L\neq 0$, then $c'=c_b\in C(L)$ with $t(b)s(c')=-1$, and T_4D_L is D_L with column c' replaced by column c; (L4) for (b,c'), $p=-1$, and c applies, since $-(\lambda(c)-\lambda(c'))>0$, so $c\in C(L)$ and $T_4D_L=0$, a contradiction. Hence $E^{\mathrm{col}}_{cc'}H=0$.

Step 6: conclusion. By Steps 3 and 5, H satisfies (6.66) and (6.67), is nonzero and termwise harmonic; so it is jointly highest of weights $(\mu_{\lambda'},\mu_{-\lambda})$ for $(\lambda',-\lambda)$ for the row and column units of (s,t) with the shifts $(\gamma,-\gamma')$. With the block rule, it is a joint harmonic. □

Lemma 6.4.20. *Let I, I' be nonempty finite sets with signs s on I and t on I', $N=\#I$, $N'=\#I'$, $\gamma,\gamma'\in\mathbb{Z}$, and let $\lambda\colon I\to\mathbb{Z}$, $\lambda'\colon I'\to\mathbb{Z}$ be such that (λ,λ') satisfies the block rule for the weights (γ,γ') and the signs (s,t). Let $\tilde\lambda(c)=\lambda(c)-2\gamma'+N'$ and $\tilde\lambda'(a)=\lambda'(a)-2\gamma+N$ be the shifted entries. Call $(a,c)\in I'\times I$ **matched** if $\tilde\lambda'(a)=\tilde\lambda(c)$ and $t(a)s(c)\,\tilde\lambda'(a)>|N'-N|$, and put*

$$\nu(a)=\mu_{\lambda'}(a)-\gamma+\#\{c\in I: t(a)s(c)=-1\}\quad(a\in I'),$$
$$\nu'(c)=\mu_{-\lambda}(c)+\gamma'-\#\{a\in I': t(a)s(c)=-1\}\quad(c\in I),$$

with $\mu_{\lambda'}$ the lowest K-type of λ' for the signs t and $\mu_{-\lambda}$ that of $-\lambda$ for the signs s. Then:

(a) every row $a\in I'$ is matched with at most one column and every column with at most one row; a row a matched with no column has $|\tilde\lambda'(a)|<N'-N$, and a column c matched with no row has $|\tilde\lambda(c)|<N-N'$; for a matched pair (a,c), $\tilde\lambda'(a)\neq 0$ and $t(a)s(c)$ is the sign of $\tilde\lambda'(a)$;

(b) $\nu(a)=-\nu'(c)$ for every matched pair (a,c);

(c) for $u\in\{\pm1\}$: if $a\in I'$ and $u\tilde\lambda'(a)>N-N'$, then $u\,\nu(a)\geq 0$; if $c\in I$ and $-u\tilde\lambda(c)>N'-N$, then $u\,\nu'(c)\geq 0$. In particular $\nu(a)=0$ for every row matched with no column and $\nu'(c)=0$ for every column matched with no row;

(d) if $\nu(a)\neq 0$, then a is matched with a column c, and $t(a)s(c)$ is the sign of $\nu(a)$;

(e) if (a,c) is matched, $\nu(a)\neq 0$, and $a'\in I'$ satisfies $t(a')=t(a)$ and $t(a)s(c)(\lambda'(a')-\lambda'(a))>0$, then a' is matched with a column c' with $s(c')=s(c)$, and $|\nu(a)|\leq|\nu(a')|$.

Proof. **Step 0: parities and the lowest K-types.** Let $h=500(N+N')^3\geq N+N'$. By (a) of the block rule, λ is injective, $\lambda(c)\equiv N-1 \pmod 2$, and $|\lambda(c)-\lambda(d)|\geq 2h$ whenever $s(c)\neq s(d)$; likewise for λ' with N' and t. Hence $\tilde\lambda$ and $\tilde\lambda'$ are injective, $\tilde\lambda(c)\equiv N-N'-1$ and $\tilde\lambda'(a)\equiv N'-N-1 \pmod 2$. For an injective integer vector l on a set n with signs ε and $l_i\equiv\#n-1 \pmod 2$, Definition 3.2.3 gives $2\rho_l(i)=\sum_j\operatorname{sgn}(l_i-l_j)\equiv\#n-1\equiv l_i$, so the floor in l_i^ρ is exact and

$$2\mu_l(i)=l_i+\sum_j\operatorname{sgn}(l_i-l_j)-2\sum_{j:\varepsilon_j=\varepsilon_i}\operatorname{sgn}(l_i-l_j)=l_i+\sum_{j\in n}\varepsilon_i\varepsilon_j\operatorname{sgn}(l_j-l_i).\tag{6.68}$$

Applied to λ' and to $-\lambda$, with $2\#\{c: t(a)s(c)=-1\}=N-\sum_c t(a)s(c)$ and $2\#\{a: t(a)s(c)=-1\}=N'-\sum_a t(a)s(c)$,

$$2\nu(a)=\tilde\lambda'(a)+\sum_{j\in I'}t(a)t(j)\operatorname{sgn}\bigl(\lambda'(j)-\lambda'(a)\bigr)-\sum_{c\in I}t(a)s(c),\tag{6.69}$$

$$2\nu'(c)=-\tilde\lambda(c)+\sum_{e\in I}s(c)s(e)\operatorname{sgn}\bigl(\lambda(c)-\lambda(e)\bigr)+\sum_{a\in I'}t(a)s(c).\tag{6.70}$$

Step 1: (a). The partners are unique because $\tilde\lambda$ and $\tilde\lambda'$ are injective, and for a matched pair $t(a)s(c)\tilde\lambda'(a) > |N'-N| \ge 0$ gives the last assertion. Suppose $N \le N'$ and put $d = N'-N$. By (b) of the block rule, $\tilde\lambda'(I') = \tilde\lambda(I) \cup [d]$; as $\#\tilde\lambda'(I') = N' = \#\tilde\lambda(I) + \#[d]$, the union is disjoint. The set $[d]$ consists of the integers $x \equiv d-1 \pmod 2$ with $|x| < d$. Let $c \in I$; then $\tilde\lambda(c) = \tilde\lambda'(a)$ for some a, and $\tilde\lambda'(a) \notin [d]$ with $\tilde\lambda'(a) \equiv d-1$, so $|\tilde\lambda'(a)| > d = |N'-N|$ (the value d has the wrong parity). By (c) of the block rule $t(a) = s(c)$ if $\tilde\lambda'(a) > 0$ and $t(a) = -s(c)$ if $\tilde\lambda'(a) < 0$, so $t(a)s(c)\tilde\lambda'(a) = |\tilde\lambda'(a)| > |N'-N|$ and (a,c) is matched. A row a has $\tilde\lambda'(a) \in \tilde\lambda(I)$, and then it is matched as just shown, or $\tilde\lambda'(a) \in [d]$, i.e. $|\tilde\lambda'(a)| < N'-N$. If $N' \le N$, the same argument with $\tilde\lambda(I) = \tilde\lambda'(I') \cup [N-N']$ shows that every row is matched and that a column matched with no row has $|\tilde\lambda(c)| < N-N'$ ((c) of the block rule is symmetric in the two sides).

Step 2: a counting principle. Let (A,B) be (I',I) or (I,I'), so that by (a) the matched pairs form a bijection between subsets of A and B. If $f\colon A \to \mathbb{Z}$ and $g\colon B \to \mathbb{Z}$ satisfy $g(b) \le f(a)$ for matched a, b, $f(a) \ge 0$ for unmatched a and $g(b) \le 0$ for unmatched b, then

$$\sum_{b\in B} g(b) \le \sum_{b \text{ matched}} g(b) \le \sum_{b \text{ matched}} f(\text{partner of } b) = \sum_{a \text{ matched}} f(a) \le \sum_{a\in A} f(a). \tag{6.71}$$

Step 3: (b). Let (a,c) be matched, $x = \tilde\lambda'(a) = \tilde\lambda(c)$ and $p = t(a)s(c) = \operatorname{sgn} x$, so $|x| > |N'-N|$. As $\tilde\lambda' - \lambda'$ and $\tilde\lambda - \lambda$ are constant, $\operatorname{sgn}(\lambda'(j) - \lambda'(a)) = -\operatorname{sgn}(x - \tilde\lambda'(j))$ and $\operatorname{sgn}(\lambda(c) - \lambda(e)) = \operatorname{sgn}(x - \tilde\lambda(e))$. Put $f(j) = -s(c)t(j) + t(a)t(j)\operatorname{sgn}(x - \tilde\lambda'(j))$ on I' and $g(e) = s(c)s(e)\operatorname{sgn}(x - \tilde\lambda(e)) - t(a)s(e)$ on I. Adding (6.69) and (6.70),

$$2\nu(a) + 2\nu'(c) = \sum_{e\in I} g(e) - \sum_{j\in I'} f(j).$$

For a matched pair (j,e) let $y = \tilde\lambda'(j) = \tilde\lambda(e)$, $q = t(j)s(e) = \operatorname{sgn} y$ and $\sigma = \operatorname{sgn}(x-y)$; with $s(c) = pt(a)$ and $s(e) = qt(j)$, $g(e) = t(a)t(j)\,q(p\sigma - 1)$ and $f(j) = t(a)t(j)(\sigma - p)$. If $q = p$ these agree since $p^2 = 1$; if $q \ne p$, then x and y have opposite signs, so $\sigma = p$ and both vanish. For an unmatched row j, $|\tilde\lambda'(j)| < N'-N < |x|$, so $\operatorname{sgn}(x - \tilde\lambda'(j)) = p$ and $f(j) = 0$; for an unmatched column e, $|\tilde\lambda(e)| < N-N' < |x|$, so $g(e) = p^2 t(a)s(e) - t(a)s(e) = 0$. By (6.71) for (f,g) and for $(-f,-g)$, $\sum g = \sum f$, hence $\nu(a) = -\nu'(c)$.

Step 4: (c) for rows. Let $u\tilde\lambda'(a) > N-N'$. For $j \in I'$ the number $u\tilde\lambda'(j) + N' - N$ is odd (Step 0); write it $2v(j)+1$. Then v is injective, $v(a) \ge 0$, and with $S = \{0,\dots,v(a)\}$ at most $v(a)+1$ rows have $v(j) \in S$. Put $f(j) = ut(a)t(j)\operatorname{sgn}(\lambda'(j) - \lambda'(a)) + 2[v(j) \in S] - 1 \ge -2$ and $g(c) = ut(a)s(c) - 1 \le 0$. Let (j,c) be matched. If $ut(a)s(c) = -1$, then $g(c) = -2 \le f(j)$. Otherwise $s(c) = ut(a)$ and $g(c) = 0$. If $t(j) = t(a)$, then $u\tilde\lambda'(j) = t(j)s(c)\tilde\lambda'(j) > |N'-N|$, so $v(j) \ge 0$; either $v(j) \le v(a)$ and $f(j) \ge 0$, or $u(\lambda'(j) - \lambda'(a)) > 0$ and the first term of $f(j)$ is 1, so $f(j) \ge 0$. If $t(j) = -t(a)$, then $-u\tilde\lambda'(j) > |N'-N|$, so $u\tilde\lambda'(j) < N-N' < u\tilde\lambda'(a)$, and the first term is $u\cdot(-1)\cdot(-u) = 1$. Let j be unmatched: $|\tilde\lambda'(j)| < N'-N$ gives $v(j) \ge 0$; if $v(j) > v(a)$, then $N-N' < u\tilde\lambda'(a) < u\tilde\lambda'(j) < N'-N$, so $0 < |\lambda'(j) - \lambda'(a)| < 2N' < 2h$, hence $t(j) = t(a)$ and the first term is 1; in all cases $f(j) \ge 0$. By (6.71) with $(A,B) = (I',I)$,

$$u\sum_c t(a)s(c) - N \le u\sum_j t(a)t(j)\operatorname{sgn}(\lambda'(j) - \lambda'(a)) + 2(v(a)+1) - N',$$

and (6.69) gives $2u\nu(a) \ge u\tilde\lambda'(a) - 2v(a) - 2 + N' - N = -1$. As $\nu(a)$ is an integer, $u\nu(a) \ge 0$.

Step 5: (c) for columns. Let $-u\tilde\lambda(c) > N'-N$. For $e \in I$ the number $-u\tilde\lambda(e) + N - N'$ is odd; write it $2w(e)+1$, so w is injective, $w(c) \ge 0$, and at most $w(c)+1$ columns have $w(e) \in S' = \{0,\dots,w(c)\}$. Put $f'(e) = us(c)s(e)\operatorname{sgn}(\lambda(c) - \lambda(e)) + 2[w(e) \in S'] - 1 \ge -2$ on I and $g'(a) = -ut(a)s(c) - 1 \le 0$ on I'. Let (a,e) be matched. If $-ut(a)s(c) = -1$, $g'(a) = -2 \le$

$f'(e)$. Otherwise $t(a) = -us(c)$, $g'(a) = 0$, and matching reads $-us(c)s(e)\tilde{\lambda}(e) > |N' - N|$. If $s(e) = s(c)$, then $-u\tilde{\lambda}(e) > |N' - N|$, so $w(e) \geq 0$, and either $w(e) \leq w(c)$ or $u(\lambda(c) - \lambda(e)) > 0$ and the first term is 1; so $f'(e) \geq 0$. If $s(e) = -s(c)$, then $-u\tilde{\lambda}(e) < N' - N < -u\tilde{\lambda}(c)$, so $\operatorname{sgn}(\lambda(c) - \lambda(e)) = -u$ and the first term is 1. Let e be unmatched: $|\tilde{\lambda}(e)| < N - N'$ gives $w(e) \geq 0$; if $w(e) > w(c)$, then $N' - N < -u\tilde{\lambda}(c) < -u\tilde{\lambda}(e) < N - N'$, so $0 < |\lambda(c) - \lambda(e)| < 2N < 2h$, hence $s(e) = s(c)$ and the first term is 1; so $f'(e) \geq 0$. By (6.71) with $(A, B) = (I, I')$,

$$-u \sum_a t(a)s(c) - N' \leq u \sum_e s(c)s(e) \operatorname{sgn}(\lambda(c) - \lambda(e)) + 2(w(c) + 1) - N,$$

and (6.70) gives $2u\nu'(c) \geq -u\tilde{\lambda}(c) - 2w(c) - 2 + N - N' = -1$, so $u\nu'(c) \geq 0$. Finally, for an unmatched row $|\tilde{\lambda}'(a)| < N' - N$ gives $u\tilde{\lambda}'(a) > N - N'$ for both u, so $\nu(a) = 0$; for an unmatched column $|\tilde{\lambda}(c)| < N - N'$ gives $-u\tilde{\lambda}(c) > N' - N$ for both u, so $\nu'(c) = 0$.

Step 6: (d). If $\nu(a) \neq 0$, then a is matched with some c by (c), and $p = t(a)s(c) = \operatorname{sgn} \tilde{\lambda}'(a)$ satisfies $p\tilde{\lambda}'(a) = |\tilde{\lambda}'(a)| > |N' - N| \geq N - N'$; by (c), $p\nu(a) \geq 0$, so p is the sign of $\nu(a)$.

Step 7: (e). First, $\mu = \mu_{\lambda'}$ is dominant: if $t(b) = t(b')$ and $\lambda'(b') < \lambda'(b)$, then $\mu(b') \leq \mu(b)$. Indeed, by (6.68), $2\mu(b) - 2\mu(b') = \lambda'(b) - \lambda'(b') + \sum_j t(b)t(j)(\operatorname{sgn}(\lambda'(j) - \lambda'(b)) - \operatorname{sgn}(\lambda'(j) - \lambda'(b')))$; the bracket is -1 for $j \in \{b, b'\}$, -2 for $\lambda'(b') < \lambda'(j) < \lambda'(b)$ and 0 otherwise, so the sum is $-2 - 2m_= + 2m_\neq$, $m_=$, $m_\neq$ the numbers of rows strictly between of the sign of b and of the other sign; and $\lambda'(b) - \lambda'(b') \geq 2(m_= + m_\neq + 1)$, the entries being distinct of one parity. Hence $2\mu(b) - 2\mu(b') \geq 4m_\neq \geq 0$. Now let (a, c), a' be as in (e) and $p = t(a)s(c)$, the sign of $\nu(a)$ by (d) (the partner of a being unique). As $t(a') = t(a)$, $\nu(a') - \nu(a) = \mu(a') - \mu(a)$, and $p(\lambda'(a') - \lambda'(a)) > 0$ with dominance gives $p(\mu(a') - \mu(a)) \geq 0$. So $p\nu(a') \geq p\nu(a) = |\nu(a)| > 0$: $\nu(a') \neq 0$ has the sign p and $|\nu(a')| \geq |\nu(a)|$. By (d), a' is matched with a column c' with $t(a')s(c') = p = t(a)s(c)$, so $s(c') = s(c)$. □

This group supplies the local theta correspondence at the infinite places for a product of joint harmonics (Proposition 6.4.21), which Lemma 6.4.18 needs. For the operators P and P' obtained by integrating the real Weil representation against the type kernels of the two groups, the Fock vector v_H of a product of joint harmonics has positive projections, each projection is reproduced by the partner kernel, and the projections of v_H generate those of all Fock vectors of polynomials. In our reading this is an explicit form of the archimedean theta correspondence between discrete series of two unitary groups [136]: the joint harmonic is a common lowest vector, in the spirit of [157], and reproduction says that the discrete series of parameters l and l' correspond (see [186, 229]). It is proved in full, with Flensted-Jensen's functions [81] for the discrete series of Harish-Chandra [119].

The statement reduces to one place, since the Weil representation acts place by place on pure tensors (Lemma 6.4.23) and its derivative on the Fock vectors of polynomials is a quadratic oscillator operator [14, 135]. At one place (Proposition 6.4.26) each property has its own proof. Positivity comes from the positivity of oscillator coefficients on harmonic polynomial vectors (Proposition 6.4.29) and an average over the compact group. Reproduction is a rigidity statement (Theorem 6.4.36), made in the context printed before Proposition 6.4.26: a vector of the lowest K-type that is killed by the compact raising operators, with Casimir eigenvalue at least κ_l, has Flensted-Jensen's function as its matrix coefficient. The Casimir value is computed in two ways, the second by matching the Casimir operators of the two groups of the pair; in our reading this is the classical coincidence of the Casimir operators of a dual pair on the oscillator representation (see [136]). Generation is Howe's argument (Lemma 6.4.40), with the Fischer decomposition [135, 137]. The group ends with the remaining lemmas of the spanning proof.

Proposition 6.4.21. *Let n, n' be nonempty finite sets with signs s on n and t on n', let $\gamma, \gamma' \colon \Phi \to \mathbb{Z}$, and let $l = (l_v)_{v \in \Phi}$, $l' = (l'_v)_{v \in \Phi}$ be families of maps $l_v \colon n \to \mathbb{Z}$, $l'_v \colon n' \to \mathbb{Z}$.*

For every $v \in \Phi$ let H_v be a joint harmonic for the signs (s,t), the weights $(\gamma(v),\gamma'(v))$ and the parameters (l_v,l'_v), written in the variables $X_{v,a,c}$ ($a \in n'$, $c \in n$), and $H = \prod_{v\in\Phi} H_v$. Let $\varepsilon(v,a,c) = t(a)s(c)$ on $\iota = \Phi \times n' \times n$, $\Omega = \Omega_{\gamma,\gamma'}$, $e = e_{-l/2}$ on $G_\infty(s)$ and $e' = e_{l'/2}$ on $G_\infty(t)$. Then:

(o) for every $u \in \mathcal{F}_\varepsilon$ the functions $x \mapsto e(x)\,\Omega(x,1)u$ on $G_\infty(s)$ and $y \mapsto e'(y)\,\Omega(1,y)u$ on $G_\infty(t)$ are Bochner integrable in $\mathcal{F}_\varepsilon$; write

$$Pu = \int e(x)\Omega(x,1)u\,dx \quad \text{and} \quad P'u = \int e'(y)\Omega(1,y)u\,dy;$$

(a) $\mathrm{Re}\langle v_H, Pv_H\rangle > 0$ and $\mathrm{Re}\langle v_H, P'v_H\rangle > 0$;

(b) $P'Pv_H = Pv_H$ and $PP'v_H = P'v_H$;

(c) for every polynomial Q in the variables $X_{v,a,c}$ there are continuous compactly supported functions a on $G_\infty(t)$ and b on $G_\infty(s)$ with

$$Pv_Q = \int_{G_\infty(t)} a(y)\,\Omega(1,y)Pv_H\,dy, \qquad P'v_Q = \int_{G_\infty(s)} b(x)\,\Omega(x,1)P'v_H\,dx.$$

Proof. **Step 0: the places.** By (FT3) of Definition 3.2.4, $e(x) = \prod_v e_v(x_v)$ and $e'(y) = \prod_v e'_v(y_v)$, where $e_v = d_{-l_v/2}\overline{c_{-l_v/2}}$ on $\mathrm{U}(\mathrm{diag}\,s)$ and $e'_v = d_{l'_v/2}\overline{c_{l'_v/2}}$ on $\mathrm{U}(\mathrm{diag}\,t)$, the formal degrees being taken for the Cayley measures. For each v, the operators π_v, π'_v are those of Proposition 6.4.26 for the signs (s,t) and the weights $(\gamma(v),\gamma'(v))$, and H_v is a joint harmonic for the parameters (l_v,l'_v); write p_v, p'_v for the operators of Proposition 6.4.26 (o) at v. By (a) of Lemma 6.4.23, $v_H = \otimes(v_{H_v})_v$.

(o). Let $u \in \mathcal{F}_\varepsilon$. The functions e and $x \mapsto \Omega(x,1)u$ are continuous ((i) of Lemma 6.4.22), so $x \mapsto e(x)\Omega(x,1)u$ is strongly measurable on the σ-compact group $G_\infty(s)$, and by (ii) of Lemma 6.4.22 with $y = 1$,

$$\|e(x)\Omega(x,1)u\| \le 2^{\#\iota}\,\|u\| \prod_{v\in\Phi} |e_v(x_v)|\,\|x_v\|^A,$$

whose integral is $2^{\#\iota}\|u\|\prod_v \int |e_v(g)|\|g\|^A dg < \infty$ by Tonelli's theorem and (o) of Proposition 6.4.26. So the function is Bochner integrable; the same argument applies to $y \mapsto e'(y)\Omega(1,y)u$.

Step 1: projections of pure tensors. Let $w = (w_v)$ with $w_v \in \mathcal{F}_v$. By (o) of Proposition 6.4.26, $g \mapsto e_v(g)\pi_v(g)w_v$ and $h \mapsto e'_v(h)\pi'_v(h)w_v$ are Bochner integrable, so (d) of Lemma 6.4.23 with $k_v = e_v$, respectively $k_v = e'_v$, gives

$$P(\otimes w) = \otimes(p_v w_v)_v, \qquad P'(\otimes w) = \otimes(p'_v w_v)_v.$$

(a). By Step 1, $Pv_H = \otimes(p_v v_{H_v})_v$, and by (a) of Lemma 6.4.23, $\langle v_H, Pv_H\rangle = \prod_v \langle v_{H_v}, p_v v_{H_v}\rangle$, a product of positive real numbers by (a) of Proposition 6.4.26. So $\langle v_H, Pv_H\rangle > 0$, in particular its real part is positive; likewise for P'.

(b). By Step 1 twice and (b) of Proposition 6.4.26 at every place, $P'Pv_H = \otimes(p'_v p_v v_{H_v})_v = \otimes(p_v v_{H_v})_v = Pv_H$, and $PP'v_H = \otimes(p_v p'_v v_{H_v})_v = \otimes(p'_v v_{H_v})_v = P'v_H$.

(c). First let $Q = \prod_v Q_v$, where Q_v is a polynomial in the variables $X_{v,a,c}$ ($a \in n'$, $c \in n$) of the place v. By (a) of Lemma 6.4.23 and Step 1, $Pv_Q = \otimes(p_v v_{Q_v})_v$, and by (c) of Proposition 6.4.26 there are smooth compactly supported α_v on $\mathrm{U}(\mathrm{diag}\,t)$ with $p_v v_{Q_v} =$

$\int \alpha_v(h)\pi'_v(h)\,p_v v_{H_v}\,dh$. The functions $h \mapsto \alpha_v(h)\pi'_v(h)p_v v_{H_v}$ are continuous with compact support, hence Bochner integrable, so (d) of Lemma 6.4.23 for the second group with $k_v = \alpha_v$ and $w_v = p_v v_{H_v}$, together with Step 1 for Pv_H, gives

$$\int_{G_\infty(t)} a_Q(y)\,\Omega(1,y)\,Pv_H\,dy = \otimes\Big(\int \alpha_v(h)\pi'_v(h)p_v v_{H_v}\,dh\Big)_v = \otimes(p_v v_{Q_v})_v = Pv_Q,$$
$$a_Q(y) = \prod_v \alpha_v(y_v),$$

and a_Q is continuous with compact support. A general polynomial Q is a finite sum $\sum_i Q_i$ of monomials, each a product over the places of monomials in the variables of one place; as $Q \mapsto Pv_Q$ is linear, $a = \sum_i a_{Q_i}$ satisfies the first identity. The second identity follows in the same way from the second identity of (c) of Proposition 6.4.26, with smooth compactly supported β_v on $\mathrm{U}(\operatorname{diag} s)$, $b_Q(x) = \prod_v \beta_v(x_v)$, (d) of Lemma 6.4.23 for the first group and Step 1 for P'. □

Lemma 6.4.22. *Let n, n' be finite index sets with signs s on n and t on n', $\gamma, \gamma' \colon \Phi \to \mathbb{Z}$, $\iota = \Phi \times n' \times n$ with $\varepsilon(v,a,c) = t(a)s(c)$, and $\Omega = \Omega_{\gamma,\gamma'}$. Then:*

(i) $x \mapsto \Omega(x,1)$ and $y \mapsto \Omega(1,y)$ are representations of $G_\infty(s)$ and $G_\infty(t)$ by bounded operators of $\mathcal{F}_\varepsilon$ that commute with each other, preserve the inner product of $\mathcal{F}_\varepsilon$, and are strongly continuous: $x \mapsto \Omega(x,1)w$ and $y \mapsto \Omega(1,y)w$ are continuous for every $w \in \mathcal{F}_\varepsilon$;

(ii) for $x \in G_\infty(s)$, $y \in G_\infty(t)$, the operator norm on $\mathcal{F}_\varepsilon$ satisfies

$$\|\Omega(x,y)\| \le 2^{\#\iota} \prod_{v\in\Phi} \left(\|x_v\|\,\|y_v\|\right)^{\#n\cdot\#n'} \le 2^{\#\iota} \max_{v\in\Phi} \left(\|x_v\|\,\|y_v\|\right)^{\#\iota}.$$

Proof. By (WR5), $\Omega(x,y) = \delta(x,y)\,\omega_\varepsilon^\infty(u(x,y))$ with $\delta(x,y) = \prod_v \det(y_v)^{\gamma(v)} \det(x_v)^{-\gamma'(v)}$ and $u(x,y) = \bigoplus_v y_v \otimes (x_v^{-1})^{\mathsf{T}} \in \mathrm{U}(J_\varepsilon)$ ((VP5) of Definition 3.2.2). For $g \in \mathrm{U}(J_\sigma)$, $g^* J_\sigma g = J_\sigma$ gives $|\det g| = 1$ and $g^{-1} = J_\sigma g^* J_\sigma$, so $\|g^{-1}\| = \|g\|$ and $1 = \|gg^{-1}\| \le \|g\|^2$, i.e. $\|g\| \ge 1$; in particular $|\delta(x,y)| = 1$.

(i). By (2) of Lemma 4.1.1, Ω is a representation of $G_\infty(s) \times G_\infty(t)$; as $(x,1)$ and $(1,y)$ commute in this product, so do $\Omega(x,1)$ and $\Omega(1,y)$, and $x \mapsto \Omega(x,1)$, $y \mapsto \Omega(1,y)$ are representations. By the same clause each $\omega_\varepsilon^\infty(u)$ is bounded and preserves the inner product, and $(u,w) \mapsto \omega_\varepsilon^\infty(u)w$ is continuous; since $|\delta| = 1$, $\Omega(x,y)$ preserves the inner product, and since δ and u are continuous, $x \mapsto \Omega(x,1)w$ and $y \mapsto \Omega(1,y)w$ are continuous.

(ii). Let $u \in \mathrm{U}(J_\varepsilon)$, $c \in \mathcal{F}_\varepsilon$ and ι_- the indices of sign -1. The exponent of $\beta_\varepsilon^\infty(z,x)$ is $(x^* J_\varepsilon z - z^* J_\varepsilon x)/2$, which is purely imaginary, so $|\beta_\varepsilon^\infty| = 1$, and $e^\infty > 0$. By the integral formula of (WR5) and Tonelli's theorem,

$$\|\omega_\varepsilon^\infty(u)c\| \le |\det u_-| \iint |c(z)|\,e^\infty(x - uz)\,d\nu_\iota(z)\,d\nu_\iota(x) = |\det u_-| \int e^\infty\,d\nu_\iota\;\|c\|,$$

using the translation invariance of ν_ι. Here $\int e^\infty d\nu_\iota = \prod_{i\in\iota} \pi^{-1} \int_{\mathbb{C}} e^{-|w|^2/2}\,dw = 2^{\#\iota}$, and $|\det w_-| \le \|w_-\|^m \le \|w\|^m$ for a matrix w whose block w_- on the indices of sign -1 has size m, w_- being a compression of w. For $u = u(x,y)$ the matrix is block diagonal for $\iota = \bigsqcup_v \iota_v$, $\iota_v = \{v\} \times n' \times n$, with blocks $u_v = y_v \otimes (x_v^{-1})^{\mathsf{T}}$, so u_- is the direct sum of the $(u_v)_-$ and $|\det u_-| = \prod_v |\det(u_v)_-| \le \prod_v \|u_v\|^{\#n\cdot\#n'}$, using $\|u_v\| \ge 1$ (u_v is unitary for the signs ε on ι_v). The operator norm of a Kronecker product is the product of the operator norms, so $\|u_v\| = \|y_v\|\,\|x_v^{-1}\| = \|y_v\|\,\|x_v\|$. With $|\delta| = 1$ this gives the first inequality of (ii); the second follows since every factor $\|x_v\|\,\|y_v\|$ is at least 1 and $\#\iota = \#\Phi \cdot \#n \cdot \#n'$. □

Lemma 6.4.23. *Let n, n' be finite index sets with signs s on n and t on n', $\gamma,\gamma'\colon \Phi\to\mathbb{Z}$, $\iota=\Phi\times n'\times n$ with $\varepsilon(v,a,c)=t(a)s(c)$, $\Omega=\Omega_{\gamma,\gamma'}$ on $\mathcal{F}_\varepsilon$, and for $v\in\Phi$ let $\iota_v=n'\times n$ with $\varepsilon_v(a,c)=t(a)s(c)$, $\mathcal{F}_v=\mathcal{F}_{\varepsilon_v}$,*

$$\pi_v(g)=\det(g)^{-\gamma'(v)}\,\omega^\infty_{\varepsilon_v}(1_{n'}\otimes(g^{-1})^{\mathsf T})\quad(g\in\mathrm U(\operatorname{diag}s)),$$
$$\pi'_v(h)=\det(h)^{\gamma(v)}\,\omega^\infty_{\varepsilon_v}(h\otimes 1_n)\quad(h\in\mathrm U(\operatorname{diag}t)).$$

For a family $w=(w_v)_{v\in\Phi}$ with $w_v\in\mathcal{F}_v$ let $\otimes w$ be the function $z\mapsto\prod_v w_v(z_v)$ on $\mathbb{C}^\iota$, where $z_v=(z_{v,a,c})_{(a,c)\in\iota_v}$. Then:

(a) $\otimes w\in\mathcal{F}_\varepsilon$, $\|\otimes w\|=\prod_v\|w_v\|$ and $\langle\otimes w,\otimes w'\rangle=\prod_v\langle w_v,w'_v\rangle$; for polynomials P_v in the variables $X_{a,c}$, $\otimes(v_{P_v})_v=v_P$ with $P=\prod_v P_v(X_{v,\cdot,\cdot})$;

(b) $\Omega(x,y)(\otimes w)=\otimes\big(\pi_v(x_v)\pi'_v(y_v)w_v\big)_v$ for $x\in G_\infty(s)$, $y\in G_\infty(t)$;

(c) for every v, π_v and π'_v are commuting representations of $\mathrm U(\operatorname{diag}s)$ and $\mathrm U(\operatorname{diag}t)$ by bounded operators of $\mathcal{F}_v$ that preserve its inner product, are strongly continuous and satisfy $\|\pi_v(g)\|\le 2^{\#\iota}\|g\|^{\#n\cdot\#n'}$, $\|\pi'_v(h)\|\le 2^{\#\iota}\|h\|^{\#n\cdot\#n'}$; and for X, Y with $X^*\operatorname{diag}(s)+\operatorname{diag}(s)X=0$, $Y^*\operatorname{diag}(t)+\operatorname{diag}(t)Y=0$ and every polynomial P in the variables $X_{a,c}$, the maps $\tau\mapsto\pi_v(\exp\tau X)v_P$ and $\tau\mapsto\pi'_v(\exp\tau Y)v_P$ are differentiable at 0 in $\mathcal{F}_v$ with derivatives v_Q and $v_{Q'}$, $Q=\sum_{c,c'}X_{cc'}E^{\mathrm{col}}_{cc'}P$, $Q'=\sum_{a,a'}Y_{aa'}E^{\mathrm{row}}_{aa'}P$, the units being those of the signs (s,t) with the shifts $(\gamma(v),-\gamma'(v))$;

(d) if for every v a measurable k_v on $\mathrm U(\operatorname{diag}s)$ makes $g\mapsto k_v(g)\pi_v(g)w_v$ Bochner integrable, then $x\mapsto\prod_v k_v(x_v)\,\Omega(x,1)(\otimes w)$ is Bochner integrable on $G_\infty(s)$ and

$$\int_{G_\infty(s)}\prod_v k_v(x_v)\,\Omega(x,1)(\otimes w)\,dx=\otimes\Big(\int k_v(g)\pi_v(g)w_v\,dg\Big)_v;$$

the same holds for $G_\infty(t)$, π'_v and $\Omega(1,y)$.

Proof. **Factorisation of the ingredients.** Under $\mathbb{C}^\iota=\prod_v\mathbb{C}^{\iota_v}$, $\nu_\iota=\bigotimes_v\nu_{\iota_v}$, and, the exponents being sums over the coordinates, $\beta^\infty_\varepsilon(y,z)=\prod_v\beta^\infty_{\varepsilon_v}(y_v,z_v)$ and $e^\infty(z)=\prod_v e^\infty(z_v)$; also $|\beta^\infty|=1$.

(a). By Tonelli, $\int|\otimes w|\,d\nu_\iota=\prod_v\|w_v\|<\infty$. The reproducing identity of (WR5) for $\otimes w$ at z is the product over v of those of the w_v at z_v, by Fubini (the integrand is integrable, being bounded in modulus by $\prod_v|w_v(y_v)|$). So $\otimes w\in\mathcal{F}_\varepsilon$ with the stated norm; the inner product factors by Fubini in the same way. For polynomials, $v_P(z)=P(\zeta(z))e^\infty(z)=\prod_v P_v(\zeta(z_v))e^\infty(z_v)$, ζ being taken coordinatewise.

(b). By (WR5), $\Omega(x,y)=\delta\,\omega^\infty_\varepsilon(u)$ with $\delta=\prod_v\det(y_v)^{\gamma(v)}\det(x_v)^{-\gamma'(v)}$ and $u=\bigoplus_v u_v$, $u_v=y_v\otimes(x_v^{-1})^{\mathsf T}$, block diagonal for $\iota=\bigsqcup_v\iota_v$; so its block on the indices of sign -1 is the direct sum of those of the u_v, and $\overline{\det u_-}=\prod_v\overline{\det(u_v)_-}$. In the integral formula of (WR5) for $\omega^\infty_\varepsilon(u)(\otimes w)(z)$ the integrand is $\prod_v w_v(y_v)\beta^\infty_{\varepsilon_v}(u_vy_v,z_v)e^\infty(z_v-u_vy_v)$, integrable by Tonelli, so by Fubini $\omega^\infty_\varepsilon(u)(\otimes w)=\otimes(\omega^\infty_{\varepsilon_v}(u_v)w_v)_v$. By (2) of Lemma 4.1.1, $\omega^\infty_{\varepsilon_v}$ is a representation, so $\omega^\infty_{\varepsilon_v}(u_v)=\omega^\infty_{\varepsilon_v}(y_v\otimes 1)\,\omega^\infty_{\varepsilon_v}(1\otimes(x_v^{-1})^{\mathsf T})$; distributing δ over the places gives (b).

(c). Fix v. For $u\ne v$ let $e_u=e^\infty\in\mathcal{F}_u$ (the real vacuum v_1; $\|e_u\|=2^{\#\iota_u}>0$, $\langle e_u,e_u\rangle>0$), and let $j\colon\mathcal{F}_v\to\mathcal{F}_\varepsilon$ send w_0 to $\otimes w$ with $w_v=w_0$, $w_u=e_u$. By (a), j is linear, $\|j(w_0)\|=C\|w_0\|$ and $\langle j(w_0),j(w'_0)\rangle=C'\langle w_0,w'_0\rangle$ with constants $C,C'>0$, and $j(v_P)=v_{P(X_{v,\cdot,\cdot})}$. For $g\in\mathrm U(\operatorname{diag}s)$ let $\hat g\in G_\infty(s)$ have component g at v and 1 elsewhere, and similarly $\hat h$; as $\pi_u(1)=\pi'_u(1)=1$, (b) gives $j(\pi_v(g)w_0)=\Omega(\hat g,1)j(w_0)$ and $j(\pi'_v(h)w_0)=\Omega(1,\hat h)j(w_0)$. Since

j is injective and the properties of Lemma 6.4.22 (i) hold for Ω, π_v and π'_v are commuting representations preserving the inner product, and they are strongly continuous since $\|\pi_v(g)w_0 - \pi_v(g_0)w_0\| = C^{-1}\|\Omega(\hat{g},1)j(w_0) - \Omega(\hat{g}_0,1)j(w_0)\|$. By the first inequality of (ii) of Lemma 6.4.22, in which the factors at the places $u \neq v$ are $\|1\| = 1$, $\|\pi_v(g)w_0\| = C^{-1}\|\Omega(\hat{g},1)j(w_0)\| \leq 2^{\#\iota}\|g\|^{\#n\cdot\#n'}\|w_0\|$, and likewise for π'_v. For the derivatives let $\hat{X}$ have component X at v and 0 elsewhere; then $\exp\tau\hat{X} = \widehat{\exp\tau X}$, and by Lemma 6.4.24 the curve $\Omega(\exp\tau\hat{X},1)j(v_P) = j(\pi_v(\exp\tau X)v_P)$ is differentiable at 0 with derivative $v_{\rho_{\mathrm{col}}(\hat{X})P(X_{v,\cdot,\cdot})} = j(v_Q)$, the units at the place v acting on the variables at v only. As $\|j(w_0)\| = C\|w_0\|$, the difference quotients of $\tau \mapsto \pi_v(\exp\tau X)v_P$ converge to v_Q. The same argument with $\hat{Y}$ and the row units gives $v_{Q'}$.

(d). Put $f_v(g) = k_v(g)\pi_v(g)w_v$, Bochner integrable on $\mathrm{U}(\mathrm{diag}\, s)$. By (b),

$$\prod_v k_v(x_v)\Omega(x,1)(\otimes w) = \otimes(f_v(x_v))_v.$$

The map $(w_v)_v \mapsto \otimes w$ is multilinear with $\|\otimes w\| = \prod_v \|w_v\|$ by (a), hence a bounded multilinear map; so $x \mapsto \otimes(f_v(x_v))$ is strongly measurable and $\|\otimes(f_v(x_v))\| = \prod_v \|f_v(x_v)\|$ is integrable for the product measure by Tonelli. A bounded linear map commutes with Bochner integrals; applying this to $\otimes$ in one variable at a time, the others fixed, and using Fubini's theorem for Bochner integrals on the product of the groups (induction on $\#\Phi$), $\int \otimes(f_v(x_v))\,dx = \otimes(\int f_v)_v$. The second group is treated in the same way with π'_v. □

Lemma 6.4.24. *Let n, n' be finite index sets with signs s on n and t on n', $\gamma,\gamma' \colon \Phi \to \mathbb{Z}$, $\iota = \Phi \times n' \times n$ with $\varepsilon(v,a,c) = t(a)s(c)$, and $\Omega = \Omega_{\gamma,\gamma'}$ on $\mathcal{F}_\varepsilon$. For $v \in \Phi$ let $E^{\mathrm{row}}_{v,aa'}$ and $E^{\mathrm{col}}_{v,cc'}$ be the row and column units of the signs (s,t) with the shifts $(\gamma(v), -\gamma'(v))$, acting on the variables $X_{v,a,c}$ of $\mathbb{C}[X_\iota]$. Let $X = (X_v)_{v\in\Phi}$ and $Y = (Y_v)_{v\in\Phi}$ be families of matrices with $X_v^*\mathrm{diag}(s) + \mathrm{diag}(s)X_v = 0$ and $Y_v^*\mathrm{diag}(t) + \mathrm{diag}(t)Y_v = 0$. Then for every polynomial $P \in \mathbb{C}[X_\iota]$ the maps $\tau \mapsto \Omega(\exp\tau X,1)v_P$ and $\tau \mapsto \Omega(1,\exp\tau Y)v_P$ from $\mathbb{R}$ to $\mathcal{F}_\varepsilon$ are differentiable at $\tau = 0$, with derivatives*

$$v_{\rho_{\mathrm{col}}(X)P}, \quad \rho_{\mathrm{col}}(X) = \sum_{v\in\Phi}\sum_{c,c'\in n}(X_v)_{cc'}E^{\mathrm{col}}_{v,cc'}, \qquad v_{\rho_{\mathrm{row}}(Y)P}, \quad \rho_{\mathrm{row}}(Y) = \sum_{v\in\Phi}\sum_{a,a'\in n'}(Y_v)_{aa'}E^{\mathrm{row}}_{v,aa'}.$$

Proof. By (WR5), $\Omega(x,y) = \prod_v \det(y_v)^{\gamma(v)}\det(x_v)^{-\gamma'(v)}\,\omega_\varepsilon^\infty(\bigoplus_v y_v \otimes (x_v^{-1})^{\mathsf{T}})$, the matrix $y_v \otimes (x_v^{-1})^{\mathsf{T}}$ acting on $n' \times n$ matrices by $Z \mapsto y_v Z x_v^{-1}$, so $(y_v \otimes (x_v^{-1})^{\mathsf{T}})_{(a,c),(a',c')} = (y_v)_{aa'}(x_v^{-1})_{c'c}$.

The first group. Put $Z_X = \bigoplus_v 1_{n'} \otimes (-X_v^{\mathsf{T}})$, with entries $(Z_X)_{(v,a,c),(v',a',c')} = -\delta_{vv'}\delta_{aa'}(X_v)_{c'c}$. Then

$$\exp(\tau Z_X) = \bigoplus_v 1 \otimes (\exp(\tau X_v)^{-1})^{\mathsf{T}},$$

which lies in $\mathrm{U}(\mathrm{diag}\,\varepsilon)$ for every τ by (VP5) of Definition 3.2.2, as $\exp(\tau X_v) \in \mathrm{U}(\mathrm{diag}\, s)$; differentiating $\exp(\tau Z_X)^*\mathrm{diag}(\varepsilon)\exp(\tau Z_X) = \mathrm{diag}(\varepsilon)$ at $\tau = 0$ gives $Z_X^*\mathrm{diag}(\varepsilon) + \mathrm{diag}(\varepsilon)Z_X = 0$. As $\det\exp(\tau X_v) = e^{\tau\,\mathrm{tr}\,X_v}$,

$$\Omega(\exp\tau X,1) = e^{-\tau\sum_v \gamma'(v)\,\mathrm{tr}\,X_v}\,\omega_\varepsilon^\infty(\exp\tau Z_X).$$

By Lemma 6.4.25 and the product rule (a smooth scalar factor equal to 1 at 0 times a curve differentiable at 0), the derivative at 0 is v_{P_1} with

$$\begin{aligned} P_1 &= \sum_{i,j\in\iota}(Z_X)_{ij}\mathrm{ra}_i\mathrm{lo}_j P - \sum_v \gamma'(v)\,\mathrm{tr}(X_v)P \\ &= -\sum_v\sum_{c,c'}(X_v)_{cc'}\sum_{a\in n'}\mathrm{ra}_{v,a,c'}\mathrm{lo}_{v,a,c}P - \sum_v\gamma'(v)\,\mathrm{tr}(X_v)P, \end{aligned}$$

after exchanging the names c, c'. For one variable, $\mathrm{lo}\,\mathrm{ra} - \mathrm{ra}\,\mathrm{lo} = 1$ ($[\partial, X] = 1$ for sign 1, $[X, -\partial] = 1$ for sign -1), and operators in different variables commute, so $\mathrm{lo}_{v,a,c}\mathrm{ra}_{v,a,c'} = \mathrm{ra}_{v,a,c'}\mathrm{lo}_{v,a,c} + \delta_{cc'}$. By (HV5) with the shifts $(\gamma(v), -\gamma'(v))$,

$$E^{\mathrm{col}}_{v,cc'} = -\sum_{a\in n'} \mathrm{lo}_{v,a,c}\mathrm{ra}_{v,a,c'} + (-\gamma'(v) + \#n')\delta_{cc'} = -\sum_{a\in n'} \mathrm{ra}_{v,a,c'}\mathrm{lo}_{v,a,c} - \gamma'(v)\delta_{cc'}.$$

Multiplying by $(X_v)_{cc'}$ and summing gives $P_1 = \rho_{\mathrm{col}}(X)P$.

The second group. Put $Z_Y = \bigoplus_v Y_v \otimes 1_n$, with entries $(Z_Y)_{(v,a,c),(v',a',c')} = \delta_{vv'}(Y_v)_{aa'}\delta_{cc'}$. As above $\exp(\tau Z_Y) = \bigoplus_v \exp(\tau Y_v) \otimes 1 \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$, Z_Y satisfies $Z_Y^*\mathrm{diag}(\varepsilon) + \mathrm{diag}(\varepsilon)Z_Y = 0$, and $\Omega(1, \exp\tau Y) = e^{\tau\sum_v \gamma(v)\,\mathrm{tr}\,Y_v}\omega_\varepsilon^\infty(\exp\tau Z_Y)$. By Lemma 6.4.25 and the product rule the derivative at 0 is v_{P_2} with

$$P_2 = \sum_v \sum_{a,a'} (Y_v)_{aa'} \sum_{c\in n} \mathrm{ra}_{v,a,c}\mathrm{lo}_{v,a',c}P + \sum_v \gamma(v)\,\mathrm{tr}(Y_v)P = \sum_v \sum_{a,a'} (Y_v)_{aa'} E^{\mathrm{row}}_{v,aa'}P = \rho_{\mathrm{row}}(Y)P,$$

since by (HV5) $E^{\mathrm{row}}_{v,aa'} = \sum_c \mathrm{ra}_{v,a,c}\mathrm{lo}_{v,a',c} + \gamma(v)\delta_{aa'}$. □

Lemma 6.4.25. *Let ι be a finite index set with signs ε, and for $i \in \iota$ let A_i be the multiplication by X_i if $\varepsilon_i = 1$ and $-\partial/\partial X_i$ if $\varepsilon_i = -1$, and B_i be $\partial/\partial X_i$ if $\varepsilon_i = 1$ and the multiplication by X_i if $\varepsilon_i = -1$, operators on $\mathbb{C}[X_i : i \in \iota]$. Let $Y \in \mathrm{M}_\iota(\mathbb{C})$ satisfy $Y^*\mathrm{diag}(\varepsilon) + \mathrm{diag}(\varepsilon)Y = 0$. Then for every polynomial $P \in \mathbb{C}[X_i : i \in \iota]$ the map $t \mapsto \omega_\varepsilon^\infty(\exp(tY))\,v_P$ from $\mathbb{R}$ to $\mathcal{F}_\varepsilon$ is differentiable at $t = 0$ for the norm of $\mathcal{F}_\varepsilon$, with derivative $v_{P'}$, where*

$$P' = \sum_{i,j\in\iota} Y_{ij}\, A_iB_jP.$$

Proof. Put $J = \mathrm{diag}(\varepsilon)$ and $N = \#\iota$. If $N = 0$, the group and its Lie algebra are trivial, and both derivatives in the assertion are zero. Suppose $N > 0$. Write $u(t) = \exp(tY)$ and $\delta(t) = \overline{\det u(t)_-}$. Differentiation of $u(t)^*Ju(t)$ shows that $u(t) \in \mathrm{U}(J)$. In the following integrals the measure in each complex vector variable is $\nu_\iota = \pi^{-N}\,d\mathrm{Leb}$.

We first differentiate in the stated L^1 norm. By (WR5), the representative of $\omega_\varepsilon^\infty(u(t))v_P$ is the integral, in y, of

$$K_t(x,y) = \delta(t)P(\zeta(y))\exp\left(-\frac{|y|^2 + |x - u(t)y|^2}{2} + \frac{x^*Ju(t)y - (u(t)y)^*Jx}{2}\right).$$

This expression is continuously differentiable in the real variable t. Choose a closed interval about zero on which $\|u(t)\| \le 2$. On this interval u', δ and δ' are bounded. The last term in the exponent is purely imaginary. Moreover

$$|x|^2 + |y|^2 \le 9(|y|^2 + |x - u(t)y|^2),$$

since $|x|^2 \le 2|x-u(t)y|^2+8|y|^2$. If D is the degree of P (take $D = 0$ when $P = 0$), differentiation of the exponent and the polynomial growth bound for P therefore give a constant C such that

$$|\partial_t K_t(x,y)| \le C(1 + |x| + |y|)^{D+2}\exp(-(|x|^2 + |y|^2)/18). \tag{1}$$

Indeed the differentiated exponent has absolute value at most a constant times $|x||y| + |y|^2$. The right side of (1) is integrable on the product of the two complex vector spaces. The fundamental

theorem of calculus bounds every sufficiently small nonzero difference quotient $(K_t - K_0)/t$ by (1). Pointwise convergence and dominated convergence on the product space consequently imply

$$\left\| \frac{\int K_t(\,\cdot\,,y)\,d\nu_\iota(y) - \int K_0(\,\cdot\,,y)\,d\nu_\iota(y)}{t} - \int \partial_t K_0(\,\cdot\,,y)\,d\nu_\iota(y) \right\|_{L^1} \longrightarrow 0. \tag{2}$$

Here the norm is bounded by the product integral of the absolute difference of the integrands, which tends to zero by the same domination. Thus (2) proves norm differentiability, rather than merely differentiability of matrix coefficients.

We evaluate its derivative. Put $z = \zeta(x)$ and $\eta_i(y) = \overline{\zeta_i(y)}$. For a polynomial Q define

$$\mathcal{I}_z(Q) = \int Q(\zeta(y)) \exp\left(-|y|^2 + \sum_i z_i \eta_i(y) \right) d\nu_\iota(y).$$

The elementary complex Gaussian moments give

$$\mathcal{I}_z(Q) = Q(z), \qquad \int \eta_i(y) Q(\zeta(y)) e^{-|y|^2 + z\cdot\eta(y)} d\nu_\iota(y) = \partial_i Q(z), \tag{3}$$

and insertion of $\eta_i \eta_j$ gives $\partial_i \partial_j Q(z)$. To check normalization, in one variable

$$\pi^{-1} \int_{\mathbb{C}} \bar{w}^a w^b e^{-|w|^2}\, d\mathrm{Leb}(w) = \mathbf{1}_{a=b} a!$$

by polar coordinates. Expanding the exponential proves the first identity in (3), coordinate by coordinate; conjugating a coordinate preserves Lebesgue measure and covers both signs. Differentiating in z proves the other identities. These operations are justified uniformly on compact sets of z by a polynomial times $e^{-|y|^2 + M|y|}$, which is integrable.

At $t = 0$ the kernel is

$$K_0(x,y) = e^{-|x|^2/2} P(\zeta(y)) e^{-|y|^2 + z\cdot\eta(y)}.$$

In particular its integral is $v_P(x)$. The derivative of the scalar is

$$\delta'(0) = \overline{\mathrm{tr}(Y_{--})} = -\mathrm{tr}(Y_{--}), \tag{4}$$

because $Y^* J + JY = 0$ makes every diagonal entry of Y purely imaginary. The derivative of the remaining exponent at zero is

$$\begin{aligned} T(x,y) &= x^* \frac{1+J}{2} Y y + y^* J Y \frac{1-J}{2} x - \tfrac{1}{2} y^* (Y + Y^*) y \\ &= \sum_{i,j} Y_{ij} \left(\mathbf{1}_{\varepsilon_i = 1} \bar{x}_i y_j + \varepsilon_i \mathbf{1}_{\varepsilon_j = -1} \bar{y}_i x_j - \mathbf{1}_{\varepsilon_i \neq \varepsilon_j} \bar{y}_i y_j \right). \end{aligned} \tag{5}$$

For the first equality differentiate the Gaussian and phase in K_t and use $Y^* = -JYJ$; that identity also gives the second equality.

Apply (3) to each coefficient of Y_{ij} in (5), with the factor $P(\zeta(y))$. For $(\varepsilon_i, \varepsilon_j) = (+,+)$ its integral is $z_i \partial_j P(z)$. For $(+,-)$ its three terms give respectively $z_i z_j P(z)$, $z_i z_j P(z)$ and $-z_i z_j P(z)$, whose sum is $z_i z_j P(z)$. For $(-,+)$ it is $-\partial_i \partial_j P(z)$. For $(-,-)$ it is $-z_j \partial_i P(z)$. Adding (4), the polynomial multiplying $e^{-|x|^2/2}$ in the derivative is therefore

$$\sum_{++} Y_{ij} X_i \partial_j P + \sum_{+-} Y_{ij} X_i X_j P - \sum_{-+} Y_{ij} \partial_i \partial_j P - \sum_{--} Y_{ij} (X_j \partial_i P + \mathbf{1}_{i=j} P).$$

Here each sum ranges over the indicated signs of (i,j). This is exactly $\sum_{i,j} Y_{ij} A_i B_j P$, since in the last case $A_i B_j P = -\partial_i (X_j P)$. In particular the negative-block vacuum scalar has not been discarded. The derivative in (2) is thus $v_{P'}$. This is a Fock vector, and the Fock norm is the inherited L^1 norm, so (2) is precisely the differentiability asserted. □

Scope. This convention is in force in, and only in, the rigidity statements of the local theta correspondence at one real place that are placed under it; the text opening their part of the argument names them. Within those results the letters below have the meaning fixed here and no other; in particular G, K, X, M, A, B, π, ρ and ι are local to this context and do not denote the global objects written with the same letters elsewhere. Every further hypothesis (the parameter l and its gap, highest vectors and weights, Casimir eigenvalues, Haar measures) is stated in each result.

The rigidity context consists of the following data.

- **The group.** A nonempty finite index set n (an integer $n \geq 1$ is read as $\{1, \dots, n\}$), $N = \#n$, signs $\varepsilon \in \{\pm 1\}^n$, $J = \mathrm{diag}(\varepsilon)$, the group $G = \mathrm{U}(J) = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^* J g = J\}$, its **block compact subgroup** $K = G \cap \mathrm{U}(N)$ (the elements of G preserving the blocks of indices of sign $+1$ and of sign -1), and its real Lie algebra $\mathfrak{u}(J) = \{Z \in \mathrm{M}_n(\mathbb{C}) : Z^* J + J Z = 0\}$. For $g \in G$, $\|g\|$ is the operator norm of g on $\mathbb{C}^n$ with the euclidean norm.
- **The representation.** A complex Banach space X, a real number $A \geq 0$, a constant $C_\pi \in (0, \infty)$, and a homomorphism $\pi \colon G \to \mathcal{L}(X)$ into the bounded complex-linear operators which is strongly continuous ($g \mapsto \pi(g)x$ is continuous for every $x \in X$) and satisfies
$$\|\pi(g)\| \leq C_\pi \|g\|^A \qquad (g \in G).$$
- **The invariant form.** A continuous Hermitian form B on X, conjugate-linear in its first and linear in its second argument, positive definite ($B(x,x) > 0$ for $x \neq 0$), with $B(\pi(g)x, \pi(g)x') = B(x,x')$ for all $g \in G$ and $x, x' \in X$.
- **The compatible module.** A complex vector space M with a Lie algebra representation $\rho \colon \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(M)$, and a complex-linear map $\iota \colon M \to X$ such that for every $Z \in \mathfrak{u}(J)$ and $w \in M$
$$\left.\frac{d}{dt}\right|_{t=0} \pi\big(\exp(tZ)\big)\iota(w) = \iota\big(\rho(Z)w\big),$$
the derivative taken in the norm of X.

Proposition 6.4.26. *Let I, I' be nonempty finite sets with signs s on I and t on I', $N = \#I$, $N' = \#I'$, let $\gamma, \gamma' \in \mathbb{Z}$, $\lambda \colon I \to \mathbb{Z}$, $\lambda' \colon I' \to \mathbb{Z}$, and let H be a joint harmonic for the signs (s,t), the weights (γ, γ') and the parameters (λ, λ'). Let $\varepsilon(a,c) = t(a)s(c)$ on $I' \times I$, $\mathcal{F} = \mathcal{F}_\varepsilon$, and*

$$\pi(x) = \det(x)^{-\gamma'} \omega_\varepsilon^\infty\big(1 \otimes (x^{-1})^{\mathsf{T}}\big) \ \ (x \in \mathrm{U}(\mathrm{diag}\, s)), \qquad \pi'(y) = \det(y)^{\gamma} \omega_\varepsilon^\infty(y \otimes 1) \ \ (y \in \mathrm{U}(\mathrm{diag}\, t)).$$

With the Cayley measures dx, dy of the two groups, let $e(x) = d\,\overline{c_{-\lambda/2}(x)}$ with $d^{-1} = \int |c_{-\lambda/2}|^2 dx$, and $e'(y) = d'\,\overline{c_{\lambda'/2}(y)}$ with $d'^{-1} = \int |c_{\lambda'/2}|^2 dy$. Then:

(o) $\int |e(x)|\,\|x\|^{NN'} dx < \infty$ and $\int |e'(y)|\,\|y\|^{NN'} dy < \infty$; for every $u \in \mathcal{F}$ the functions $x \mapsto e(x)\pi(x)u$ and $y \mapsto e'(y)\pi'(y)u$ are Bochner integrable in $\mathcal{F}$, and $pu = \int e(x)\pi(x)u\,dx$, $p'u = \int e'(y)\pi'(y)u\,dy$ define bounded operators;

(a) $\langle v_H, p v_H \rangle$ and $\langle v_H, p' v_H \rangle$ are positive real numbers;

(b) $p' p v_H = p v_H$ and $p p' v_H = p' v_H$;

(c) for every polynomial Q in the variables X_{ac} ($a \in I'$, $c \in I$) there are smooth compactly supported functions α on $\mathrm{U}(\mathrm{diag}\, t)$ and β on $\mathrm{U}(\mathrm{diag}\, s)$ with

$$p v_Q = \int \alpha(y)\,\pi'(y)\,p v_H\,dy, \qquad p' v_Q = \int \beta(x)\,\pi(x)\,p' v_H\,dx.$$

Proof. **Step 0: the Fock space.** For $c \in \mathcal{F}$, the reproducing identity of (WR5) with $|\beta^\infty_\varepsilon| = 1$ and $0 < e^\infty \le 1$ gives $|c(x)| \le \|c\|$ for every x. Hence $|\langle c, c'\rangle| \le \int |c||c'|\,d\nu \le \|c\|\,\|c'\|$, so $\langle\,,\rangle$ is a continuous hermitian form, positive definite because $\langle c, c\rangle = 0$ forces $c = 0$ almost everywhere and then everywhere by the reproducing identity. $\mathcal{F}$ is complete: if $c_k \to c$ in $L^1(\nu)$ with $c_k \in \mathcal{F}$, then $\tilde c(x) = \int c(y)\beta^\infty_\varepsilon(y,x)e^\infty(x-y)\,d\nu(y)$ is the pointwise limit of the $c_k(x)$ (the kernel is bounded by 1), so $\tilde c = c$ almost everywhere, $\tilde c$ satisfies the reproducing identity, and $c_k \to \tilde c \in \mathcal{F}$.

Step 1: the two oscillator representations. Apply Lemma 6.4.23 with the constant weights γ, γ' on Φ, $n = I$, $n' = I'$, and any $v \in \Phi$: its π_v, π'_v, $\mathcal{F}_v$ are π, π', $\mathcal{F}$. By its (c), π and π' are commuting strongly continuous representations by bounded operators preserving $\langle\,,\rangle$, $\|\pi(x)\| \le C\|x\|^A$, $\|\pi'(y)\| \le C\|y\|^A$ with $C = 2^{\#\Phi\cdot A}$, and for $X \in \mathfrak{u}(\operatorname{diag} s)$, $Y \in \mathfrak{u}(\operatorname{diag} t)$ and every polynomial P, $\frac{d}{d\tau}\big|_0 \pi(e^{\tau X})v_P = v_{\rho_{\mathrm{col}}(X)P}$ and $\frac{d}{d\tau}\big|_0 \pi'(e^{\tau Y})v_P = v_{\rho_{\mathrm{row}}(Y)P}$. The maps ρ_{row}, ρ_{col} are representations of the Lie algebras $\mathfrak{gl}(I')$, $\mathfrak{gl}(I)$: with the identity $[AB, CD] = A[B,C]D + C[A,D]B$ (valid when $[A,C] = [B,D] = 0$) and the canonical commutation relations $[\mathrm{lo}_{ac}, \mathrm{ra}_{a'c'}] = \delta_{aa'}\delta_{cc'}$, raising operators commuting among themselves and lowering operators too,

$$[E^{\mathrm{row}}_{ab}, E^{\mathrm{row}}_{cd}] = \sum_{x,y}[\mathrm{ra}_{ax}\mathrm{lo}_{bx}, \mathrm{ra}_{cy}\mathrm{lo}_{dy}] = \delta_{bc}\sum_x \mathrm{ra}_{ax}\mathrm{lo}_{dx} - \delta_{ad}\sum_x \mathrm{ra}_{cx}\mathrm{lo}_{bx} = \delta_{bc}E^{\mathrm{row}}_{ad} - \delta_{ad}E^{\mathrm{row}}_{cb}$$

(the shift terms cancel), so $[\rho_{\mathrm{row}}(Z), \rho_{\mathrm{row}}(W)] = \rho_{\mathrm{row}}(ZW - WZ)$; for the columns, (a) of Lemma 6.4.27 writes $E^{\mathrm{col}}_{cd} = -\check E^{\mathrm{row}}_{dc}$ with row units $\check E^{\mathrm{row}}$ of the transposed setting, which satisfy the same relations, and $[E^{\mathrm{col}}_{ab}, E^{\mathrm{col}}_{cd}] = [\check E^{\mathrm{row}}_{ba}, \check E^{\mathrm{row}}_{dc}] = \delta_{bc}E^{\mathrm{col}}_{ad} - \delta_{ad}E^{\mathrm{col}}_{cb}$.

Step 2: the data of the one-group lemmas. For the column group $\mathrm{U}(\operatorname{diag} s)$ take $X = \mathcal{F}$, $B = \langle\,,\rangle$ (Step 0), π, the Haar measure dx (Lemma 5.2.41), $M = \mathbb{C}[X]$, $\rho = \rho_{\mathrm{col}}$, the map ι, the growth exponent A (Step 1) and the parameter $l = -\lambda$. By (a) of (WM8) for the block rule ((a) of (HV7)), λ has gap $500(N+N')^3 \ge 500N^3 + NN'$, hence $-\lambda$ has large gap A for s; the kernel $d_{l/2}\overline{c_{l/2}}$ of the lemmas is e; by (b) of (HV7), H is K-highest of weight $\mu_{-\lambda}$ for the order $-\lambda$ for ρ_{col}. For the row group $\mathrm{U}(\operatorname{diag} t)$ take π', dy, $\rho = \rho_{\mathrm{row}}$, ι and $l = \lambda'$, which has gap $500(N+N')^3 \ge 500N'^3 + NN'$, hence large gap A for t; the kernel is e', and H is K-highest of weight $\mu_{\lambda'}$ for λ' for ρ_{row}. We use Lemma 6.4.28, Lemma 6.4.30, Lemma 6.4.33, Lemma 6.4.34 and Theorem 6.4.36 for these data, possibly with ι replaced by $p\circ\iota$ or $p'\circ\iota$. By (i) of Lemma 6.4.28, p commutes with every $\pi'(y)$ and p' with every $\pi(x)$ (Step 1); so for $Y \in \mathfrak{u}(\operatorname{diag} t)$, $\pi'(e^{\tau Y})pv_P = p\,\pi'(e^{\tau Y})v_P$ has derivative $pv_{\rho_{\mathrm{row}}(Y)P}$ at 0, p being bounded: $p\circ\iota$ is a derived map for $(\pi', \rho_{\mathrm{row}})$, and likewise $p'\circ\iota$ for $(\pi, \rho_{\mathrm{col}})$.

(o). This is (i) of Lemma 6.4.28 for both groups.

(a). $H \ne 0$ is termwise harmonic, so by Proposition 6.4.29 $\langle v_H, \pi(z)v_H\rangle > 0$ for every positive definite $z \in \mathrm{U}(\operatorname{diag} s)$, and Lemma 6.4.30 gives $\mathrm{Re}\langle v_H, pv_H\rangle > 0$. By (i) of Lemma 6.4.28, $\langle v_H, pv_H\rangle = \langle pv_H, pv_H\rangle$ is real and ≥ 0; so it is a positive real number. The same argument applies to p'.

Step 3: Casimir values. Let $\Omega_{\mathrm{row}} = \sum_{a,a'} E^{\mathrm{row}}_{aa'}E^{\mathrm{row}}_{a'a}$, $\Omega_{\mathrm{col}} = \sum_{c,c'} E^{\mathrm{col}}_{cc'}E^{\mathrm{col}}_{c'c}$, and let c_{row}, c_{col}, b_{pair}, $\mathrm{Cas}_{\mathrm{row}}(w)$, $\mathrm{Cas}_{\mathrm{col}}(w)$, κ_l and $\kappa_l(w)$ be as in Lemma 6.4.35. Expanding, $\mathrm{Cas}_{\mathrm{row}}(w) = \Omega_{\mathrm{row}} - 2w\sum_a E^{\mathrm{row}}_{aa} + N'w^2$ and $\mathrm{Cas}_{\mathrm{col}}(w) = \Omega_{\mathrm{col}} - 2w\sum_c E^{\mathrm{col}}_{cc} + Nw^2$, and on H the diagonal units act by $\mu_{\lambda'}$, $\mu_{-\lambda}$. By Lemma 6.4.34 for the column group, $pv_{\Omega_{\mathrm{col}}P} = \kappa_{-\lambda}pv_P$ for every P, so $pv_{\mathrm{Cas}_{\mathrm{col}}(c_{\mathrm{col}})H} = \kappa_{-\lambda}(c_{\mathrm{col}})pv_H$. By (1) and (3) of Lemma 6.4.35, $pv_{\mathrm{Cas}_{\mathrm{row}}(c_{\mathrm{row}})H} = (\kappa_{-\lambda}(c_{\mathrm{col}}) - b_{\mathrm{pair}})pv_H = \kappa_{\lambda'}(c_{\mathrm{row}})pv_H$, and therefore

$$pv_{\Omega_{\mathrm{row}}H} = \Big(\kappa_{\lambda'}(c_{\mathrm{row}}) + 2c_{\mathrm{row}}\sum_a \mu_{\lambda'}(a) - N'c^2_{\mathrm{row}}\Big)pv_H = \kappa_{\lambda'}\,pv_H.$$

Symmetrically, Lemma 6.4.34 for the row group gives $p'v_{\Omega_{\mathrm{row}}P} = \kappa_{\lambda'}p'v_P$, hence $p'v_{\mathrm{Cas}_{\mathrm{row}}(c_{\mathrm{row}})H} = \kappa_{\lambda'}(c_{\mathrm{row}})p'v_H$, $p'v_{\mathrm{Cas}_{\mathrm{col}}(c_{\mathrm{col}})H} = (\kappa_{\lambda'}(c_{\mathrm{row}}) + b_{\mathrm{pair}})p'v_H = \kappa_{-\lambda}(c_{\mathrm{col}})p'v_H$, and $p'v_{\Omega_{\mathrm{col}}H} = \kappa_{-\lambda}\,p'v_H$.

Step 4: rigidity. By Steps 2 and 3, Theorem 6.4.36 applies to the row group with ι replaced by $p\circ\iota$, the vector H and $c = \kappa_{\lambda'}$: for every measurable f on $\mathrm{U}(\mathrm{diag}\,t)$ with $\int |f(y)|\|y\|^A dy < \infty$ and $\int f(y)e'(y^{-1}g)\,dy = e'(g)$ for all g,

$$\int f(y)\pi'(y)\,pv_H\,dy = pv_H;$$

and it applies to the column group with $p'\circ\iota$, H and $c = \kappa_{-\lambda}$: $\int f(x)\pi(x)\,p'v_H\,dx = p'v_H$ for every such f on $\mathrm{U}(\mathrm{diag}\,s)$ reproducing e.

(b). $f = e'$ is admissible: its moment is finite by (o), and $\int e'(y)e'(y^{-1}g)\,dy = e'(g)$ by (i) of Lemma 5.6.22 (with $x = 1$; λ' has gap $500N'^3$). So $p'pv_H = pv_H$; likewise $pp'v_H = p'v_H$ with $f = e$.

(c), first identity. Let $S(\varphi) = \int \varphi(y)\pi'(y)\,pv_H\,dy$ for smooth compactly supported φ on $\mathrm{U}(\mathrm{diag}\,t)$, and let G_1 be the set of polynomials Q with $pv_Q = S(\varphi)$ for some such φ, a linear subspace. We verify the hypotheses of (JC) of Lemma 6.4.40.

- G_1 is stable under the row units. Let $pv_Q = S(\varphi)$ and $Y \in \mathfrak{u}(\mathrm{diag}\,t)$. The curve $\pi'(e^{\tau Y})pv_Q$ has derivative $pv_{\rho_{\mathrm{row}}(Y)Q}$ at 0 (Step 2) and, being $\pi'(e^{\tau Y})S(\varphi)$, derivative $S(\varphi_Y)$ by (iii) of Lemma 6.4.28; so $pv_{\rho_{\mathrm{row}}(Y)Q} = S(\varphi_Y)$. Every $Z \in \mathrm{M}_{I'}(\mathbb{C})$ is $Y_1 + iY_2$ with $Y_1 = \frac{1}{2}(Z - JZ^*J)$ and $Y_2 = \frac{1}{2i}(Z + JZ^*J)$ in $\mathfrak{u}(J)$, $J = \mathrm{diag}\,t$; by linearity $pv_{\rho_{\mathrm{row}}(Z)Q} = S(\varphi_{Y_1} + i\varphi_{Y_2})$. Taking for Z the matrix units, $E^{\mathrm{row}}_{aa'}Q \in G_1$.

- $H \in G_1$: by (ii) of Lemma 6.4.28 there is a smooth compactly supported β' on $\mathrm{U}(\mathrm{diag}\,t)$ reproducing e', and its moment is finite; by Step 4, $S(\beta') = pv_H$.

- (JC1): let $s(c) \neq s(d)$, $-\lambda(c) < -\lambda(d)$, and P be K-highest of weight $\nu = \mu_{-\lambda} + e_c - e_d$ for $-\lambda$ for ρ_{col}. Apply Lemma 6.4.33 to the column group with $l = -\lambda$ and $o = \lambda$: $o_x < o_y$ whenever $l_x > l_y$, and $\sum_x o_x(\nu_x - \mu_l(x)) = \lambda(c) - \lambda(d) > 0$. It gives $pv_{E^{\mathrm{col}}_{dc}P} = p\,\iota(\rho_{\mathrm{col}}(E_{dc})P) = 0 = S(0)$.

- (JC2): let $s(c) = s(d)$ and $-\lambda(d) < -\lambda(c)$. By Lemma 6.4.34 for the column group ($l = -\lambda$) there is a linear map P from $\mathcal{H}_{c_{\lambda/2}}$ to bounded operators with $\mathsf{P}(c_{\lambda/2}) = p$ and $\mathsf{P}(f)v_{\rho_{\mathrm{col}}(Z)P} = -\mathsf{P}(Zf)v_P$. By (iii) of Lemma 5.6.24 for the parameter λ (which has gap 0), $E_{dc}c_{\lambda/2} = 0$, as $d \neq c$, $s(d) = s(c)$ and $\lambda(d) > \lambda(c)$. Hence $pv_{E^{\mathrm{col}}_{dc}P} = 0 = S(0)$ for every P.

- (JC3): by the same two results, $E_{cc}c_{\lambda/2} = \mu_\lambda(c)c_{\lambda/2}$ and $pv_{E^{\mathrm{col}}_{cc}P - \mu_{-\lambda}(c)P} = -(\mu_\lambda(c) + \mu_{-\lambda}(c))pv_P = 0$ by (e) of Lemma 6.4.27.

By (JC) of Lemma 6.4.40, G_1 consists of all polynomials; take $\alpha = \varphi$.

(c), second identity. Let $S'(\varphi) = \int \varphi(x)\pi(x)\,p'v_H\,dx$ for smooth compactly supported φ on $\mathrm{U}(\mathrm{diag}\,s)$ and G_2 the set of Q with $p'v_Q = S'(\varphi)$ for some φ. As above, with the groups exchanged: G_2 is stable under the column units (derivative of $\pi(e^{\tau X})p'v_Q$, Step 2, and (iii) of Lemma 6.4.28); $H \in G_2$ (a smooth bump reproducing e, (ii) of Lemma 6.4.28, and Step 4). (JR1): for $t(a) \neq t(b)$, $\lambda'(a) < \lambda'(b)$ and P K-highest of weight $\mu_{\lambda'} + e_a - e_b$ for λ' for ρ_{row}, Lemma 6.4.33 for the row group with $l = \lambda'$ and $o = -\lambda'$ ($\sum_x o_x(\nu_x - \mu_l(x)) = \lambda'(b) - \lambda'(a) > 0$) gives $p'v_{E^{\mathrm{row}}_{ba}P} = 0$. (JR2): for $t(a) = t(b)$ and $\lambda'(b) < \lambda'(a)$, Lemma 6.4.34 for the row group ($l = \lambda'$, $\mathsf{P}(c_{-\lambda'/2}) = p'$) and (iii) of Lemma 5.6.24 for $-\lambda'$ ($b \neq a$, $t(b) = t(a)$, $-\lambda'(b) > -\lambda'(a)$) give $E_{ba}c_{-\lambda'/2} = 0$ and $p'v_{E^{\mathrm{row}}_{ba}P} = 0$. (JR3): $E_{aa}c_{-\lambda'/2} = \mu_{-\lambda'}(a)c_{-\lambda'/2}$, so $p'v_{E^{\mathrm{row}}_{aa}P - \mu_{\lambda'}(a)P} =$

$-(\mu_{-\lambda'}(a)+\mu_{\lambda'}(a))p'v_P=0$ by (e) of Lemma 6.4.27. By (JR) of Lemma 6.4.40, G_2 consists of all polynomials; take $\beta=\varphi$. □

Lemma 6.4.27. *Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma,\gamma'\in\mathbb{Z}$, and identify the polynomial rings $\mathbb{C}[X_{ac}: a\in I',\, c\in I]$ and $\mathbb{C}[Y_{ca}: c\in I,\, a\in I']$ by $Y_{ca}=X_{ac}$. Let $E^{\mathrm{row}}_{aa'}$, $E^{\mathrm{col}}_{cc'}$ be the row and column units of the signs (s,t) with the shifts $(\gamma,-\gamma')$ and $\Delta^{\mathrm{row}}_{aa'}$, $\Delta^{\mathrm{col}}_{cc'}$ the row and column Laplacians of (s,t) (rows I', columns I, variables X_{ac}); let $\check{E}^{\mathrm{row}}_{cc'}$, $\check{E}^{\mathrm{col}}_{aa'}$ be the row and column units of the signs (t,s) with the shifts $(\gamma',-\gamma)$ and $\check{\Delta}^{\mathrm{row}}_{cc'}$, $\check{\Delta}^{\mathrm{col}}_{aa'}$ the row and column Laplacians of (t,s) (rows I, columns I', variables Y_{ca}). Then:*

(a) $\check{E}^{\mathrm{row}}_{cc'}=-E^{\mathrm{col}}_{c'c}$ for all $c,c'\in I$, and $\check{E}^{\mathrm{col}}_{aa'}=-E^{\mathrm{row}}_{a'a}$ for all $a,a'\in I'$;

(b) $\check{\Delta}^{\mathrm{row}}_{cc'}=\Delta^{\mathrm{col}}_{c'c}$ whenever $s(c)\neq s(c')$, and $\check{\Delta}^{\mathrm{col}}_{aa'}=\Delta^{\mathrm{row}}_{a'a}$ whenever $t(a)\neq t(a')$; a polynomial is termwise harmonic, respectively Laplace harmonic, for (t,s) if and only if it is so for (s,t);

(c) for maps $o,\nu\colon I\to\mathbb{Z}$, a polynomial is K-highest of weight ν for o for the column units $(E^{\mathrm{col}}_{cc'})$ if and only if it is K-highest of weight $-\nu$ for $-o$ for the row units $(\check{E}^{\mathrm{row}}_{cc'})$; for maps $o',\nu'\colon I'\to\mathbb{Z}$, a polynomial is K-highest of weight ν' for o' for the row units $(E^{\mathrm{row}}_{aa'})$ if and only if it is K-highest of weight $-\nu'$ for $-o'$ for the column units $(\check{E}^{\mathrm{col}}_{aa'})$;

(d) for $\lambda\colon I\to\mathbb{Z}$ and $\lambda'\colon I'\to\mathbb{Z}$, a polynomial H is a joint harmonic for the signs (s,t), the weights (γ,γ') and the parameters (λ,λ') if and only if it is a joint harmonic for the signs (t,s), the weights (γ',γ) and the parameters (λ',λ);

(e) if J is a nonempty finite set with signs and $l\colon J\to\mathbb{Z}$ is injective with $l(j)\equiv\#J-1 \pmod 2$ for all j (for instance λ or λ' when (λ,λ') satisfies the block rule), then $\mu_{-l}=-\mu_l$.

Proof. **Step 1: the elementary operators.** In the setting (t,s) the variable Y_{ca} lies in the row c of sign $s(c)$ and the column a of sign $t(a)$, so its sign is $s(c)t(a)$, the sign of X_{ac} in the setting (s,t). Hence the raising and lowering operators of Y_{ca} are ra_{ac} and lo_{ac}. Operators attached to different variables commute, and for each variable

$$\mathrm{lo}_{ac}\,\mathrm{ra}_{ac}=\mathrm{ra}_{ac}\,\mathrm{lo}_{ac}+1.$$

So $\mathrm{lo}_{ac'}\mathrm{ra}_{ac}=\mathrm{ra}_{ac}\mathrm{lo}_{ac'}+\delta_{cc'}$ and $\mathrm{lo}_{ac}\mathrm{ra}_{a'c}=\mathrm{ra}_{a'c}\mathrm{lo}_{ac}+\delta_{aa'}$.

Step 2: (a). In the setting (t,s) there are N rows, so (HV5) with the shifts $(\gamma',-\gamma)$ gives

$$\check{E}^{\mathrm{row}}_{cc'}=\sum_{a\in I'}\mathrm{ra}_{ac}\,\mathrm{lo}_{ac'}+\gamma'\delta_{cc'},\qquad \check{E}^{\mathrm{col}}_{aa'}=-\sum_{c\in I}\mathrm{lo}_{ac}\,\mathrm{ra}_{a'c}+(-\gamma+N)\delta_{aa'},$$

while in the setting (s,t) with the shifts $(\gamma,-\gamma')$

$$E^{\mathrm{col}}_{c'c}=-\sum_{a\in I'}\mathrm{lo}_{ac'}\,\mathrm{ra}_{ac}+(-\gamma'+N')\delta_{cc'},\qquad E^{\mathrm{row}}_{a'a}=\sum_{c\in I}\mathrm{ra}_{a'c}\,\mathrm{lo}_{ac}+\gamma\delta_{aa'}.$$

By Step 1, $E^{\mathrm{col}}_{c'c}=-\sum_a\mathrm{ra}_{ac}\mathrm{lo}_{ac'}-N'\delta_{cc'}+(-\gamma'+N')\delta_{cc'}=-\check{E}^{\mathrm{row}}_{cc'}$, and $\check{E}^{\mathrm{col}}_{aa'}=-\sum_c\mathrm{ra}_{a'c}\mathrm{lo}_{ac}-N\delta_{aa'}+(-\gamma+N)\delta_{aa'}=-E^{\mathrm{row}}_{a'a}$.

Step 3: (b). By (HV6) for (t,s), for $s(c)\neq s(c')$, $\check{\Delta}^{\mathrm{row}}_{cc'}=\sum_{a:\,s(c)t(a)=-1}\partial_{ac}\partial_{ac'}$, while $\Delta^{\mathrm{col}}_{c'c}=\sum_{a:\,t(a)s(c')=1}\partial_{ac'}\partial_{ac}$; the two index sets agree because $s(c')=-s(c)$. For $t(a)\neq t(a')$, $\check{\Delta}^{\mathrm{col}}_{aa'}=\sum_{c:\,s(c)t(a)=1}\partial_{ac}\partial_{a'c}$ and $\Delta^{\mathrm{row}}_{a'a}=\sum_{c:\,t(a')s(c)=-1}\partial_{a'c}\partial_{ac}$, with the same index sets as

$t(a') = -t(a)$. The two families of conditions defining termwise harmonic polynomials for (t,s) (two variables of one column of Y from rows of opposite signs, two variables of one row of Y from columns of opposite signs) are those for (s,t) in the other order, and the Laplacians of (t,s) are those of (s,t) by the identities just shown; so both notions agree.

Step 4: (c). Let x be a polynomial. By (a), $E^{\mathrm{col}}_{cc}x = \nu(c)x$ is $\check{E}^{\mathrm{row}}_{cc}x = -\nu(c)x$, and $E^{\mathrm{col}}_{cc'}x = 0$ is $\check{E}^{\mathrm{row}}_{c'c}x = 0$. The condition "$E^{\mathrm{col}}_{cc'}x = 0$ whenever $s(c) = s(c')$ and $o(c') < o(c)$" therefore reads "$\check{E}^{\mathrm{row}}_{jj'}x = 0$ whenever $s(j) = s(j')$ and $(-o)(j') < (-o)(j)$" (with $j = c'$, $j' = c$). This is the first equivalence; the second is the same computation with $\check{E}^{\mathrm{col}}_{aa'} = -E^{\mathrm{row}}_{a'a}$.

Step 5: (d). The block rule (WM8) for (λ,λ') with the weights (γ,γ') and the signs (s,t) has the shifted entries $\tilde{\lambda}(c) = \lambda(c) - 2\gamma' + N'$ and $\tilde{\lambda}'(a) = \lambda'(a) - 2\gamma + N$. For (λ',λ) with the weights (γ',γ) and the signs (t,s) (first index set I', second I) the shifted entries are $\lambda'(a) - 2\gamma + N = \tilde{\lambda}'(a)$ and $\lambda(c) - 2\gamma' + N' = \tilde{\lambda}(c)$. Conditions (a), (b), (c) of (WM8) for the second datum are those for the first with the two sides exchanged, and each of them is unchanged by this exchange (the gap $500(N+N')^3$ is symmetric, the two cases of (b) are exchanged, and $s(c) = \pm t(a)$ is symmetric). So (λ,λ') satisfies the block rule for (γ,γ'), (s,t) if and only if (λ',λ) does for (γ',γ), (t,s).

(e). Under the block rule, λ is injective with $\lambda(c) \equiv N-1 \pmod 2$ by (a) of (WM8), and likewise λ' with N'. For a vector l as in (e), Definition 3.2.3 gives $2\rho_{-l} = -2\rho_l$ and $2\rho^{\mathrm{c}}_{-l} = -2\rho^{\mathrm{c}}_l$; and $l(j) + 2\rho_l(j)$ is even, because $2\rho_l(j)$ is a sum of $\#J - 1$ terms ± 1 (l injective), hence $\equiv \#J - 1 \equiv l(j)$. So $(-l)^\rho = -l^\rho$ and

$$\mu_{-l} = -\mu_l.$$

End of (d). Now let the block rule hold. By (HV7), H is a joint harmonic for (s,t), (γ,γ'), (λ,λ') if and only if $H \neq 0$ is termwise harmonic, K-highest of weight $\mu_{\lambda'}$ for λ' for $(E^{\mathrm{row}}_{aa'})$ and K-highest of weight $\mu_{-\lambda}$ for $-\lambda$ for $(E^{\mathrm{col}}_{cc'})$; and H is a joint harmonic for (t,s), (γ',γ), (λ',λ) if and only if $H \neq 0$ is termwise harmonic for (t,s), K-highest of weight μ_λ for λ for $(\check{E}^{\mathrm{row}}_{cc'})$ and K-highest of weight $\mu_{-\lambda'}$ for $-\lambda'$ for $(\check{E}^{\mathrm{col}}_{aa'})$, both pairs of units having the shifts of the statement. By (b) and (c), with $-\mu_{-\lambda} = \mu_\lambda$ and $-\mu_{\lambda'} = \mu_{-\lambda'}$, the two lists of conditions coincide. □

Lemma 6.4.28. *Let $n \geq 1$, $\varepsilon \in \{\pm 1\}^n$, $J = \mathrm{diag}(\varepsilon)$, $G = \mathrm{U}(J)$ with a Haar measure dg, $\mathfrak{u}(J) = \{Y : Y^*J + JY = 0\}$, and $\|g\|$ the operator norm. Let X be a complex Banach space, B a continuous positive definite hermitian form on X, conjugate linear in its first variable, and $\pi\colon G \to \mathrm{GL}(X)$ a representation by bounded operators, strongly continuous, with $B(\pi(g)x, \pi(g)y) = B(x,y)$ and $\|\pi(g)\| \leq C\|g\|^A$ for some $C > 0$, $A \in \mathbb{N}$ and all g.*

(i) Let $l \in \mathbb{Z}^n$ have large gap A for ε, $\lambda = \frac{1}{2}l$, $d_\lambda = (\int_G |c_\lambda|^2 dg)^{-1}$ and $e_\lambda = d_\lambda \overline{c_\lambda}$. Then $\int_G |e_\lambda(g)|\,\|g\|^A dg < \infty$; for every $x \in X$ the function $g \mapsto e_\lambda(g)\pi(g)x$ is Bochner integrable, and $\pi(e_\lambda)x = \int_G e_\lambda(g)\pi(g)x\,dg$ defines a bounded operator with

$$B(y, \pi(e_\lambda)x) = B(\pi(e_\lambda)y, x), \qquad \pi(e_\lambda)\pi(e_\lambda) = \pi(e_\lambda),$$
$$B(x, \pi(e_\lambda)x) = B(\pi(e_\lambda)x, \pi(e_\lambda)x) \geq 0$$

for all $x, y \in X$; and $\pi(e_\lambda)$ commutes with every bounded operator on X that commutes with all $\pi(g)$.

(ii) In the situation of (i) there is a smooth compactly supported $\beta\colon G \to \mathbb{C}$ with

$$\int_G \beta(y)\, e_\lambda(y^{-1}g)\, dy = e_\lambda(g)$$

for every $g \in G$.

(iii) For $w \in X$ and a smooth compactly supported $\varphi\colon G \to \mathbb{C}$ put $S_w(\varphi) = \int_G \varphi(y)\pi(y)w\,dy$ and, for $Y \in \mathfrak{u}(J)$, $\varphi_Y(y) = \frac{d}{dt}\big|_{t=0}\varphi(e^{-tY}y)$. Then $\tau \mapsto \pi(\exp \tau Y)S_w(\varphi)$ is differentiable at $\tau = 0$ with derivative $S_w(\varphi_Y)$, and φ_Y is smooth and compactly supported.

Proof. **(i).** The vector l has gap $h = 500n^3 + A$; let $L = \max_i |l_i|$ and $c = A + 8n^2 + 1 \le h$. By (1) of Lemma 5.7.38, $|c_\lambda(g)| \le (2/\|g\|)^c$, so

$$|e_\lambda(g)|\,\|g\|^A \le d_\lambda 2^c \|g\|^{A-c} = 2^A\, d_\lambda \Big(\frac{2}{\|g\|}\Big)^{8n^2+1},$$

which is integrable with $d_\lambda > 0$ by (3) of Lemma 5.7.38. By (i) of Lemma 5.6.22 (l has gap $500n^3$), e_λ is continuous, $e_\lambda(g^{-1}) = \overline{e_\lambda(g)}$ and $\int_G e_\lambda(z)e_\lambda(z^{-1}y)\,dz = e_\lambda(y)$ (take $x = 1$ there). The function $g \mapsto e_\lambda(g)\pi(g)x$ is continuous on the σ-compact group G, hence strongly measurable, with norm at most $C|e_\lambda(g)|\|g\|^A\|x\|$; so it is Bochner integrable and $\|\pi(e_\lambda)\| \le C\int |e_\lambda|\|g\|^A dg$.

Self-adjointness. By invariance of B and of dg under inversion,

$$\begin{aligned} B(y, \pi(e_\lambda)x) &= \int e_\lambda(g)B(\pi(g^{-1})y, x)\,dg = \int e_\lambda(g^{-1})B(\pi(g)y, x)\,dg \\ &= \int \overline{e_\lambda(g)}B(\pi(g)y, x)\,dg = B(\pi(e_\lambda)y, x). \end{aligned}$$

Idempotence. $\pi(e_\lambda)\pi(e_\lambda)x = \iint e_\lambda(g)e_\lambda(h)\pi(gh)x\,dh\,dg$, where the integrand is bounded in norm by $C|e_\lambda(g)||e_\lambda(h)|\|g\|^A\|h\|^A\|x\|$ ($\|gh\| \le \|g\|\|h\|$), integrable for the product measure. The substitution $h = g^{-1}k$ preserves the product measure, and Fubini's theorem for Bochner integrals gives $\int(\int e_\lambda(g)e_\lambda(g^{-1}k)\,dg)\pi(k)x\,dk = \int e_\lambda(k)\pi(k)x\,dk = \pi(e_\lambda)x$. Hence $B(x, \pi(e_\lambda)x) = B(x, \pi(e_\lambda)\pi(e_\lambda)x) = B(\pi(e_\lambda)x, \pi(e_\lambda)x) \ge 0$. If T is bounded and commutes with every $\pi(g)$, then $T\pi(e_\lambda)x = \int e_\lambda(g)T\pi(g)x\,dg = \pi(e_\lambda)Tx$.

(ii). The gap h exceeds $8n^2$, so (ii) of Lemma 5.7.36 gives a smooth compactly supported β with $c_\lambda(g) = \int \beta(y)c_\lambda(gy)\,dy$ for all g. Multiplying by d_λ and conjugating, $e_\lambda(g) = \int \overline{\beta(y)}e_\lambda(gy)\,dy$. At g^{-1}, after conjugation and with $\overline{e_\lambda(z)} = e_\lambda(z^{-1})$,

$$e_\lambda(g) = \overline{e_\lambda(g^{-1})} = \int \beta(y)\,\overline{e_\lambda(g^{-1}y)}\,dy = \int \beta(y)\,e_\lambda(y^{-1}g)\,dy.$$

(iii). By (i) of Lemma 5.7.36 (whose definition of φ_Y is the present one), φ_Y is smooth with support in that of φ. The integrand of $S_w(\varphi)$ is continuous with compact support. By left invariance of dy,

$$\pi(e^{\tau Y})S_w(\varphi) = \int \varphi(y)\pi(e^{\tau Y}y)w\,dy = \int \varphi(e^{-\tau Y}y)\pi(y)w\,dy.$$

Since $\frac{d}{d\sigma}\varphi(e^{-\sigma Y}y) = \varphi_Y(e^{-\sigma Y}y)$, for $0 < |\tau| \le 1$

$$r_\tau(y) = \frac{\varphi(e^{-\tau Y}y) - \varphi(y)}{\tau} - \varphi_Y(y) = \frac{1}{\tau}\int_0^\tau \big(\varphi_Y(e^{-\sigma Y}y) - \varphi_Y(y)\big)\,d\sigma,$$

which vanishes outside the compact set $K = \{e^{\sigma Y} : |\sigma| \le 1\} \cdot \operatorname{supp}\varphi$. The function $(\sigma, y) \mapsto \varphi_Y(e^{-\sigma Y}y)$ is uniformly continuous on $[-1, 1] \times K$ and φ_Y vanishes off K, so $\sup_y |r_\tau(y)| \to 0$ as $\tau \to 0$. Therefore

$$\Big\|\frac{\pi(e^{\tau Y})S_w(\varphi) - S_w(\varphi)}{\tau} - S_w(\varphi_Y)\Big\| = \Big\|\int_K r_\tau(y)\pi(y)w\,dy\Big\| \le \sup_y |r_\tau(y)| \int_K \|\pi(y)w\|\,dy \to 0.$$

□

Proposition 6.4.29. *Let I, I' be nonempty finite index sets with signs s on I and t on I', and let $\gamma, \gamma' \in \mathbb{Z}$. Put $\iota = I' \times I$ and $\varepsilon(a,c) = t(a)s(c)$. On the SAME Fock model $\mathcal{F}_\varepsilon$ put*

$$\pi(x) = \det(x)^{-\gamma'}\,\omega_\varepsilon^\infty(1 \otimes (x^{-1})^{\mathsf{T}}), \qquad \pi'(y) = \det(y)^{\gamma}\,\omega_\varepsilon^\infty(y \otimes 1).$$

If $P \in \mathbb{C}[X_{ac} : a \in I', c \in I]$ is nonzero and termwise harmonic, then

$$\langle v_P, \pi(z) v_P\rangle \in \mathbb{R}_{>0} \quad \textit{for every positive-definite hermitian } z \in \mathrm{U}(\mathrm{diag}\, s),$$

and

$$\langle v_P, \pi'(w) v_P\rangle \in \mathbb{R}_{>0} \quad \textit{for every positive-definite hermitian } w \in \mathrm{U}(\mathrm{diag}\, t).$$

Proof. **Polynomial inner products in the given model.** All inner products below are the integrals in (WR5), conjugate-linear in their first argument. They are taken in the prescribed L^1 Fock model; no replacement by a Hilbert representation is made. For multi-indices $\mathbf{a}, \mathbf{b}$ the elementary one-variable Gaussian integrals give

$$\langle v_{X^{\mathbf{a}}}, v_{X^{\mathbf{b}}}\rangle = \begin{cases} \prod_i a_i! & \mathbf{a} = \mathbf{b}, \\ 0 & \mathbf{a} \neq \mathbf{b}. \end{cases}$$

Indeed angular integration gives orthogonality in each coordinate, and radial integration of $|z|^{2m} e^{-|z|^2}$ with measure $\pi^{-1} dz$ gives $m!$. Conjugating a coordinate when its sign is positive leaves this integral unchanged. Thus, writing $(Q,P)_{\mathrm{F}} = \langle v_Q, v_P\rangle$ locally in this proof,

$$(X_i Q, P)_{\mathrm{F}} = (Q, \partial_i P)_{\mathrm{F}}, \qquad (P,P)_{\mathrm{F}} > 0 \quad (P \neq 0).$$

For a matrix M let $L_M P$ mean the substitution $X_j \mapsto \sum_i M_{ij} X_i$. The chain rule in the displayed Gaussian pairing, or its expansion on symmetric tensors degree by degree, gives

$$L_M L_N = L_{MN}, \qquad (L_M Q, P)_{\mathrm{F}} = (Q, L_{M^*} P)_{\mathrm{F}}.$$

For completeness the degree-d monomials identify this pairing with $d!$ times the induced pairing on symmetric tensors: the tensor $M^{\otimes d}$ has adjoint $(M^*)^{\otimes d}$, and restriction to symmetric tensors proves the last equality. Distinct degrees are orthogonal. These are finite-dimensional polynomial identities.

The harmonic coefficient formula. We prove the needed formula on a finite coordinate set J with signs ϵ_i, allowing an arbitrary finite labeling d_i. Suppose that $U \in \mathrm{U}(\mathrm{diag}\,\epsilon)$ is block diagonal for these labels: $U_{ij} = 0$ if $d_i \neq d_j$. Put

$$C_{ij} = \begin{cases} U_{ij} & \epsilon_i = \epsilon_j = +1, \\ \bar{U}_{ij} & \epsilon_i = \epsilon_j = -1, \\ 0 & \epsilon_i \neq \epsilon_j, \end{cases} \qquad A = C^{-1}, \qquad \delta = \det(U_{--})^{-1}.$$

Both sign principal blocks are invertible: from $U^* \mathrm{diag}(\epsilon) U = \mathrm{diag}(\epsilon)$, $U_{--}^* U_{--} = 1 + U_{+-}^* U_{+-}$ and $U_{++}^* U_{++} = 1 + U_{-+}^* U_{-+}$. Thus C and A are defined and are block diagonal jointly for labels and signs.

Call a polynomial label-harmonic here if $\partial_i \partial_j P = 0$ whenever $d_i = d_j$ and $\epsilon_i \neq \epsilon_j$. We claim that for two such polynomials,

$$\langle v_Q, \omega_\epsilon^\infty(U) v_P\rangle = \delta\,(L_A Q, P)_{\mathrm{F}}. \tag{$*$}$$

We give the coefficient recurrences and induction proving this claim.

Use the raising and lowering operators R_i, B_i of (HV5), with coordinate signs ϵ_i. The derivative at zero of the Heisenberg operator in direction $z \in \mathbb{C}^J$ on a polynomial vector is the vector of

$$D_z P = \sum_i (z_i R_i P - \epsilon_i \bar{z}_i B_i P).$$

Indeed, for real τ, set $\zeta_i(x) = \bar{x}_i$ for $\epsilon_i = +1$ and $\zeta_i(x) = x_i$ for $\epsilon_i = -1$. The definitions give the exact identity

$$(\rho(\tau z) v_P)(x) = e^{-|x|^2/2} \exp\left(-\frac{\tau^2 |z|^2}{2} + \tau \sum_{\epsilon_i = +1} \bar{x}_i z_i + \tau \sum_{\epsilon_i = -1} x_i \bar{z}_i \right) P(\zeta(x) - \tau \zeta(z)).$$

Its derivative at $\tau = 0$ is $\sum_{\epsilon_i=+1}(z_i X_i P - \bar{z}_i \partial_i P) + \sum_{\epsilon_i=-1}(\bar{z}_i X_i P - z_i \partial_i P)$, which is precisely the displayed D_z. Thus neither a factor 2 nor a sign is suppressed. The polynomial operators obey $[\partial_i, X_j] = \delta_{ij}$; their Fischer adjoints are $R_i^* = \epsilon_i B_i$ and $B_i^* = \epsilon_i R_i$. These follow separately from $(X_i)^* = \partial_i$: for a negative coordinate $R_i = -\partial_i$, $B_i = X_i$, so both adjoints acquire the minus sign. The differentiation is valid in the L^1 norm: on a bounded interval of the translation parameter every differentiated expression is bounded by a fixed polynomial times $\exp(-|x|^2/4 + C)$, an integrable function. Unitarity and Heisenberg covariance of the actual real Weil action are Lemma 4.1.1(2). Differentiating covariance and moving the skew-adjoint Heisenberg derivative across the inner product, first for $z = e_j$ and then for $z = ie_j$, gives, with $B_U(Q, P) = \langle v_Q, \omega_\epsilon^\infty(U) v_P \rangle$,

$$B_U(Q, R_j P) = \sum_i U_{ij} \epsilon_i B_U(B_i Q, P),$$
$$\epsilon_j B_U(Q, B_j P) = \sum_i \bar{U}_{ij} B_U(R_i Q, P).$$

These two equalities also follow directly by comparing the coefficients of z_j and $\bar{z}_j$ in $B_U(Q, D_z P) = -B_U(D_{Uz} Q, P)$; conjugate-linearity is in the first entry.

We now solve these recurrences explicitly. Fix j and suppose

$$\partial_k Q = 0 \quad \text{if } d_k = d_j \text{ and } \epsilon_k \neq \epsilon_j. \tag{a}$$

The polynomial P in this calculation is arbitrary. Put $S_j = \{i : d_i = d_j,\ \epsilon_i = \epsilon_j\}$ and, for every index k,

$$w_k = B_U(X_k Q, P), \qquad b_i = B_U(Q, \partial_i P).$$

For every $i \in S_j$, condition (a) also holds with i in place of j.

If $\epsilon_i = +1$, the second recurrence, with $B_i = \partial_i$, gives

$$\begin{aligned} b_i &= \sum_{\epsilon_k = +1} \bar{U}_{ki} w_k - \sum_{\epsilon_k = -1} \bar{U}_{ki} B_U(\partial_k Q, P) \\ &= \sum_{k \in S_j} \bar{U}_{ki} w_k = \sum_{k \in S_j} \bar{C}_{ki} w_k. \end{aligned} \tag{b+}$$

In the first sum, a term outside S_j has $d_k \neq d_i$ and therefore $U_{ki} = 0$. In the second sum, if $d_k = d_i$ its derivative is zero by (a), and if $d_k \neq d_i$ its matrix entry is zero. This accounts for every opposite-sign summand, including its minus sign.

If $\epsilon_i = -1$, use instead $R_i = -\partial_i$ in the first recurrence:

$$\begin{aligned} b_i &= -B_U(Q, R_i P) \\ &= -\sum_{\epsilon_k=+1} U_{ki} B_U(\partial_k Q, P) + \sum_{\epsilon_k=-1} U_{ki} w_k \\ &= \sum_{k\in S_j} U_{ki} w_k = \sum_{k\in S_j} \bar{C}_{ki} w_k. \end{aligned} \tag{b-}$$

Again every first-sum term vanishes either by (a) or by the label condition, and the second sum has no other surviving label. Here $C_{ki} = \bar{U}_{ki}$ on the negative block, which explains the final conjugation. We have proved, in both cases and for every $i \in S_j$,

$$b_i = \sum_{k\in S_j} \bar{C}_{ki} w_k. \tag{b}$$

The matrix C is a direct sum over the joint label-and-sign classes. Each summand is invertible: C is invertible, and its determinant is the product of the determinants of these finite diagonal blocks. Its inverse A is the direct sum of their inverses, as is verified by multiplying those blocks with C. In particular $A_{ij} = 0$ for $i \notin S_j$. Multiply (b) by $\bar{A}_{ij}$ and sum over $i \in S_j$. The calculation is

$$\begin{aligned} \sum_{i\in S_j} \bar{A}_{ij} b_i &= \sum_{k\in S_j} \overline{\sum_{i\in S_j} C_{ki} A_{ij}}\, w_k \\ &= \sum_{k\in S_j} \overline{(CA)_{kj}}\, w_k = \sum_{k\in S_j} \delta_{kj} w_k = w_j. \end{aligned}$$

All sums are finite. The extension from S_j to all indices adds only zero entries of A. Thus the required solved recurrence is

$$B_U(X_j Q, P) = \sum_i \bar{A}_{ij} B_U(Q, \partial_i P). \tag{**}$$

This computation proves the inverse orientation directly: the system is $b_{S_j} = (C_{S_j})^* w_{S_j}$ and its inverse has entry $((C_{S_j})^*)^{-1}_{ji} = \bar{A}_{ij}$. No inverse mixes a discarded label or sign, and no harmonicity assumption on P has been used in (**).

The constant coefficient is $B_U(1,1) = \delta$. To calculate it, set $H = (1 + U^*U)/2$. Evaluation at zero is pairing with the vacuum by the reproducing formula. Hence the literal integral kernel gives

$$B_U(1,1) = \overline{\det U_{--}} \int e^{-y^* H y}\, \pi^{-\#J} dy = \frac{\overline{\det U_{--}}}{\det H} = \frac{1}{\det U_{--}}.$$

Here $\det H = |\det U_{--}|^2$. Explicitly, with $E = \mathrm{diag}(\epsilon)$, put $D_0 = \mathrm{diag}(U_{++}, U_{--}) = (U + EUE)/2$. Then

$$U^* D_0 = (U^*U + U^*EUE)/2 = (U^*U + 1)/2 = H.$$

Since $U^*EU = E$, one has $|\det U| = 1$, whence

$$\det H = \det U_{++} \det U_{--} / \det U.$$

The complementary-minor formula gives $\det U_{++}/\det U = \det(U^{-1})_{--}$, and $U^{-1} = EU^*E$ makes the latter $\overline{\det U_{--}}$. The Gaussian integral follows by a unitary diagonalization of the positive matrix H and one-dimensional integration.

A label-harmonic polynomial is a sum of monomials which contain no pair of variables with the same label and opposite signs. This is because each prohibited second derivative sends different monomials containing that pair to different monomials, with nonzero integer coefficients. For a nonconstant allowed monomial X_jQ, the remaining Q has no prohibited opposite-sign variable relative to j. The inverse retains the label condition, since $(U^{-1})_{ik} = \epsilon_i\epsilon_k\bar{U}_{ki}$. Apply (**) for U^{-1} and second polynomial 1; every $\partial_i 1$ is exactly zero. By unitarity $B_U(1, X_jQ) = \overline{B_{U^{-1}}(X_jQ, 1)} = 0$. Together with the constant calculation this proves $B_U(1, P) = \delta(1, P)_{\mathrm{F}}$ for every label-harmonic P.

Now induct on the degree of an allowed monomial Q, simultaneously for every label-harmonic P. Its derivatives remain label-harmonic. For X_jQ, (**) and the induction hypothesis give

$$\begin{aligned} B_U(X_jQ, P) &= \delta \sum_i \bar{A}_{ij}(L_AQ, \partial_i P)_{\mathrm{F}} \\ &= \delta \sum_i \bar{A}_{ij}(X_iL_AQ, P)_{\mathrm{F}} = \delta(L_A(X_jQ), P)_{\mathrm{F}}. \end{aligned}$$

Linearity in P and conjugate-linearity in Q extend this identity to their monomial sums. This proves (*), with every determinant and conjugation retained.

Positivity for a positive element. If U is additionally positive-definite hermitian, its two sign principal blocks are positive definite, as is the entrywise conjugate of the negative block. Consequently $C > 0$, $A = C^{-1} > 0$ and $\det U_{--} > 0$. Choose the positive square root $S = A^{1/2}$. It is invertible, $S^* = S$, and the polynomial adjoint identity gives

$$(L_AP, P)_{\mathrm{F}} = (L_SL_SP, P)_{\mathrm{F}} = (L_SP, L_SP)_{\mathrm{F}} > 0 \qquad (P \neq 0).$$

The last inequality holds because L_S is invertible, with inverse $L_{S^{-1}}$. Thus (*) gives

$$\langle v_P, \omega_\epsilon^\infty(U) v_P \rangle = \frac{(L_SP, L_SP)_{\mathrm{F}}}{\det U_{--}} > 0$$

for every nonzero label-harmonic polynomial. The empty sign block has determinant 1, so this also covers definite signatures.

The two actual oscillator factors. For the column action take $U = 1 \otimes (z^{-1})^{\mathsf{T}}$ and labels $d(a, c) = a$. The matrix is positive-definite hermitian: inverse, transpose and tensoring with the identity preserve this property. It is unitary for the signs $t(a)s(c)$ and is block diagonal in the row labels. Within one row, opposite coordinate signs mean $s(c) \neq s(c')$, so the second termwise-harmonic condition of (HV6) is exactly the required label-harmonicity. The preceding formula applies to this U, not to $z \otimes 1$.

For the row action take $U = w \otimes 1$ and labels $d(a, c) = c$. It is likewise positive-definite hermitian and unitary for the same coordinate signs. Within a column, opposite signs mean $t(a) \neq t(a')$, so the first termwise-harmonic condition gives the needed hypothesis. Thus the same formula applies to the row factor.

Finally $\det z$ and $\det w$ are positive real numbers, while their unitary defining equations give $|\det z| = |\det w| = 1$. Therefore both determinants equal 1. The twists $\det(z)^{-\gamma'}$ and $\det(w)^{\gamma}$ are exactly 1, irrespective of the integer exponents. There is no omitted phase. The two asserted strict inequalities follow in the original Fock inner product. □

Lemma 6.4.30. *Let $n \geq 1$ be read as $\{1, \ldots, n\}$ in that context, assume $A \in \mathbb{N}$, and let $l \in \mathbb{Z}^n$ have large gap A for s, with $\lambda = l/2$. Suppose that $v \in M$ is K-highest of weight μ_l for the order l, that is,*

$$\rho(E_{aa})v = \mu_l(a)v, \qquad \rho(E_{ab})v = 0 \quad \text{if } s_a = s_b \text{ and } l_b < l_a.$$

If $B(\iota v, \pi(z)\iota v)$ is a strictly positive real number for every positive-definite matrix $z \in G$, then $g \mapsto e_\lambda(g)\pi(g)x$ is Bochner integrable for every $x \in X$, its integral defines a bounded operator $\pi(e_\lambda)$, and

$$\operatorname{Re} B\big(\iota v, \pi(e_\lambda)\iota v\big) > 0.$$

Proof. Put $K = K_s$, $\mu = \mu_l$, $x = \iota v$, $m = m_\mu$ and

$$D = \int_K |m(k)|^2 \, dk > 0, \qquad a(u, z) = m_{-l^\rho}(u^* z u).$$

The measure dk has mass one; the measure dg of G is the Cayley measure used to define d_λ and e_λ. By Lemma 5.2.41 it is invariant on both sides.

The kernel has the moment required for the integrated representation. Let $c = 8n^2 + 1$ and choose an integer $L \geq \max_a |l_a|$. Since $c \leq 500n^3$, the gap $500n^3 + A$ and Lemma 5.7.38(1),(3) give $d_\lambda > 0$ and

$$|e_\lambda(g)| \, \|g\|^A \leq 2^A d_\lambda (2/\|g\|)^c, \qquad Q_A := \int_G |e_\lambda(g)| \, \|g\|^A \, dg < \infty.$$

Indeed use the coefficient bound with exponent $A + c$; its product with $\|g\|^A$ is the displayed bound. More explicitly the cited integral estimate gives

$$Q_A \leq 8 \, 2^{A+c} \big(2000(n + L + 1)^6\big)^{2n^2}.$$

Apply Lemma 6.4.28(i) to the same Banach representation, invariant form, integer exponent A, Cayley Haar measure and gap $500n^3 + A$. Its vector integrals define the bounded operator

$$Ty := \int_G e_\lambda(g)\pi(g)y \, dg = \pi(e_\lambda)y, \qquad \|T\| \leq C_\pi Q_A.$$

The norm bound follows by integrating the growth bound. Bounded linear functionals commute with this Bochner integral, so $f(Ty) = \int_G e_\lambda(g) f(\pi(g)y) \, dg$.

A compact highest-weight average reproduces x. Apply Lemma 6.4.31 with $I = \{1, \dots, n\}$, $\nu = \mu_l$ and the representation, form, module and map of the statement. Large gap implies that the entries of l are distinct. The norm-derived action and the two highest-vector equations are precisely its hypotheses; no compact-finiteness assumption is needed. Since its a_ν is the number D above, it gives

$$\int_K \overline{m(k)} \, \pi(k)x \, dk = Dx. \tag{6.72}$$

In particular, for $w \in X$ and $g \in G$,

$$\int_K \overline{m(k)} \, B(w, \pi(gk)x) \, dk = DB(w, \pi(g)x), \tag{6.73}$$

$$\int_K m(k) \, B(x, \pi(k^{-1})w) \, dk = DB(x, w). \tag{6.74}$$

The first identity follows by applying the bounded functional $B(w, \pi(g)\,\cdot\,)$ to (6.72); the second follows from invariance of B, conjugate linearity in its first argument, and the same identity.

The double compact average is strictly positive. Define the integrable scalar function

$$F(g) = e_\lambda(g) B(x, \pi(g)x), \qquad \mathcal{A}(g) = \int_{K \times K} F(k_1^{-1} g k_2) \, dk_1 \, dk_2.$$

The moment bound and continuity of B prove integrability of F. Bi-invariance of dg and Tonelli give $\int_{K\times K}\int_G |F(k_1^{-1}gk_2)|\,dg\,dk_1dk_2 = \int_G |F|$. Consequently $\mathcal{A}$ is integrable, Fubini applies, and

$$B(x,Tx) = \int_G F(g)\,dg = \int_G \mathcal{A}(g)\,dg. \tag{6.75}$$

Fix $g \in G$. By Lemma 5.6.30(ii), write $g = k_0 z$ with $k_0 \in K$ and $z \in G$ positive definite. For $u \in K$ the matrix u^*zu is positive definite, so all its principal minors and its determinant are positive. Thus $a(u,z) > 0$, even when some powers in m_{-l^ρ} are negative. The function $u \mapsto a(u,z)^{-1}$ is continuous on K.

Apply Lemma 5.6.26(i),(ii) with the block-diagonal factors $k_1^{-1}k_0$ and k_2^{-1} and the accretive matrix z; positive definiteness implies accretivity. It gives the exact polar identity

$$\overline{c_\lambda(k_1^{-1}k_0zk_2)} = \frac{1}{D}\int_K \frac{m(u^*k_0^*k_1)\,m(k_2^*u)}{a(u,z)}\,du.$$

Here conjugation of the numerator uses $m(y^*) = \overline{m(y)}$ from Lemma 5.6.27(i), and the denominator is real. For this fixed z every function in the resulting triple integral over K^3 is continuous, with nonvanishing denominator. Hence it is absolutely integrable and the order of integration may be exchanged.

For each fixed u, replace k_1 by k_0ua and k_2 by ub in the inner integral; Haar probability is unchanged. The two minor factors become $m(a)$ and $\overline{m(b)}$, and the matrix coefficient becomes $B(x,\pi(a^{-1}u^{-1}zub)x)$. Integrating first in a by (6.74), and then in b by (6.73), gives

$$\mathcal{A}(g) = d_\lambda D\int_K \frac{B(x,\pi(u^{-1}zu)x)}{m_{-l^\rho}(u^*zu)}\,du.$$

The numerator is a strictly positive real number by hypothesis, since $u^{-1} = u^*$ and $u^*zu \in G$ is positive definite. The denominator, d_λ and D are strictly positive. The integrand is continuous and strictly positive on the nonempty compact group K, whose measure is one. Therefore $\mathcal{A}(g)$ is a strictly positive real number for every $g \in G$.

Finally $\operatorname{Re}\mathcal{A}$ is integrable and strictly positive everywhere, and dg is a nonzero Haar measure. Its integral is strictly positive. Taking real parts in (6.75) yields $\operatorname{Re} B(x,\pi(e_\lambda)x) > 0$, as required. □

Lemma 6.4.31. *Let I be a nonempty finite set, $s \in \{\pm1\}^I$, $J = \operatorname{diag}(s)$, $G = \mathrm{U}(J)$ and $K = K_s$. Let $l \in \mathbb{Z}^I$ have distinct entries and let $\nu \in \mathbb{Z}^I$. Let X be a complex Banach space, B a continuous positive-definite hermitian form on X, conjugate-linear in its first argument, and $\pi\colon G \to \mathrm{GL}(X)$ a strongly continuous representation by bounded complex-linear operators preserving B. Let M be a complex vector space, $\rho\colon \mathfrak{gl}(I,\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(M)$ a Lie algebra representation and $\iota\colon M \to X$ linear. Suppose that*

$$\left.\frac{d}{dt}\right|_{t=0} \pi(\exp(tY))\iota(w) = \iota(\rho(Y)w) \qquad (Y^*J + JY = 0,\ w \in M)$$

in the norm of X. Let $v \in M$ satisfy

$$\rho(E_{aa})v = \nu_a v, \qquad \rho(E_{ab})v = 0 \quad (s_a = s_b,\ l_a > l_b).$$

Then:

1. *If $\iota v \neq 0$, then $\nu_a \geq \nu_b$ whenever $s_a = s_b$ and $l_a > l_b$.*

2. *The number $a_\nu = \int_K |m_\nu(k)|^2\,dk$ is strictly positive, the following Bochner integral exists, and*

$$\int_K \overline{m_\nu(k)}\,\pi(k)\iota v\,dk = a_\nu\,\iota v.$$

Proof. Put $x = \iota v$, $m = m_\nu$, $D = \int_K |m(k)|^2\,dk$, and $I_\pm = \{a \in I : s_a = \pm 1\}$. The function m is continuous on K: its defining principal-minor factors are polynomials and its only negative powers are powers of the nonzero determinant. Since $m(1) = 1$, positivity of Haar measure on nonempty open sets gives $D > 0$. The map $k \mapsto \overline{m(k)}\pi(k)x$ is continuous on the compact group K, hence Bochner integrable. If $x = 0$, the averaging assertion is immediate. Assume from now on that $x \neq 0$.

Positivity of the form forces compact dominance. Differentiate

$$B(\pi(\exp(tY))\iota(w), \pi(\exp(tY))\iota(w')) = B(\iota(w), \iota(w'))$$

at zero for $Y^*J + JY = 0$. This yields

$$B(\iota(\rho(Y)w), \iota(w')) + B(\iota(w), \iota(\rho(Y)w')) = 0.$$

For $s_a = s_b$, the matrices $E_{ab} - E_{ba}$ and $i(E_{ab} + E_{ba})$ are in this real Lie algebra. Taking their two identities and using conjugate linearity in the first variable gives

$$B(\iota(\rho(E_{ab})w), \iota(w')) = B(\iota(w), \iota(\rho(E_{ba})w')).$$

Fix $l_a > l_b$ in a sign block. The highest-vector equation and $[E_{ab}, E_{ba}] = E_{aa} - E_{bb}$ therefore give

$$0 \le B(\iota(\rho(E_{ba})v), \iota(\rho(E_{ba})v)) = B(x, \iota(\rho(E_{ab})\rho(E_{ba})v)) = (\nu_a - \nu_b)B(x, x).$$

As $B(x, x) > 0$, this proves (i).

The highest vector lies in one compact irreducible module. This argument does not require the norm of X to come from B. Complete X for the norm $B(y, y)^{1/2}$, and denote this Hilbert space by $\mathcal{H}$. The natural map $X \to \mathcal{H}$ is continuous and injective, since B is continuous and positive definite. The operators $\pi(k)$ extend to unitary operators $U(k)$ on $\mathcal{H}$. Strong continuity first holds on the image of X and then on $\mathcal{H}$, by density and $\|U(k)\| = 1$. The derivative hypothesis passes to $\mathcal{H}$.

For completeness, the compact-type decomposition needed here follows from the quoted Peter–Weyl statement for class functions, without assuming completeness for the norm defined by B. Choose one representative of each irreducible unitary isomorphism class of K. For such a finite-dimensional representation σ, of dimension d_σ, set

$$P^\sigma_{ij} = d_\sigma \int_K \overline{\sigma_{ij}(k)}\,U(k)\,dk, \qquad P_\sigma = \sum_i P^\sigma_{ii}.$$

These are strong operator integrals. Schur orthogonality gives

$$P^\sigma_{ij}P^\tau_{ab} = \begin{cases} \delta_{ja}P^\sigma_{ib}, & \sigma \simeq \tau, \\ 0, & \sigma \not\simeq \tau, \end{cases} \qquad U(h)P^\sigma_{ij} = \sum_a \sigma_{ai}(h)P^\sigma_{aj}.$$

These identities can be obtained directly by substituting $k = h^{-1}g$ in the double integrals: the average of a linear map between irreducible unitary spaces is an intertwiner, hence is zero between inequivalent representations and is its trace divided by d_σ times the identity on one

irreducible space. Also $(P^\sigma_{ij})^* = P^\sigma_{ji}$. Thus P_σ is an orthogonal projection, commutes with $U(K)$, and the orbit of $P_\sigma y$ lies in the finite span of the vectors $P^\sigma_{ij} y$.

The highest-weight classification for $K \cong \mathrm{U}(\#I_+) \times \mathrm{U}(\#I_-)$ ([163], compact-group highest-weight theory) now applies to that finite-dimensional orbit. Since P_σ commutes with $U(K)$, it commutes with the infinitesimal action on every vector where the latter is defined, and hence with its complex-linear extension. Therefore $P_\sigma x$ has weight ν and is killed by the compact raising operators. In an irreducible compact module a nonzero vector with these properties has its highest weight; consequently $P_\sigma x = 0$ unless σ has highest weight ν. Such an irreducible representation exists by the tensor construction below and is unique by highest-weight classification; denote it by σ_ν.

Here is the completeness step. Put $y = x - P_{\sigma_\nu} x$ in $\mathcal{H}$, so $P_\sigma y = 0$ for every σ. The continuous class function

$$H(k) = \int_K \langle y, U(hkh^{-1}) y \rangle_{\mathcal{H}} \, dh$$

has zero inner product with every irreducible character: invariance of the character and Fubini reduce that inner product to $d_\sigma^{-1} \langle y, P_\sigma y \rangle_{\mathcal{H}} = 0$. By Peter–Weyl for class functions ([163], Chapter I, §5), $H = 0$ almost everywhere. A continuous function zero almost everywhere for Haar measure on a compact group is identically zero, since every nonempty open set has positive measure. Hence $0 = H(1) = \|y\|^2_{\mathcal{H}}$ and $x = P_{\sigma_\nu} x$.

The integrals defining $P^\sigma_{ij} x$ also exist in X, because their integrands are continuous on K. Injectivity of $X \to \mathcal{H}$ transfers the preceding equality to X. Thus the cyclic K-module generated by x is finite-dimensional, is of highest weight ν, and is irreducible: in an isotypic sum its highest vectors are $h_\nu \otimes b$, and the orbit of any such vector spans $\sigma_\nu \otimes \mathbb{C} b$.

We identify its normalized highest coefficient explicitly. In a sign block with indices $i_1, \dots, i_p$ in decreasing l order and dominant entries $\nu_1 \geq \cdots \geq \nu_p$, take the tensor product of $\det^{\nu_p}$ with

$$\bigotimes_{j=1}^{p-1} \left(\bigwedge^j \mathbb{C}^p \right)^{\otimes(\nu_j - \nu_{j+1})} .$$

The tensor of the vectors $e_{i_1} \wedge \cdots \wedge e_{i_j}$ and the determinant vector is a unit highest vector. Its weight is ν, and its matrix coefficient is $\det(k)^{\nu_p} \prod_{j<p} \Delta_{\{i_1,\dots,i_j\}}(k)^{\nu_j - \nu_{j+1}}$. Take the tensor product over the two sign blocks, omitting an empty block. The displayed coefficient is precisely $m_\nu(k)$: principal minors of a block-diagonal matrix split over the two blocks, so the defining level-set product for m_ν gives these two products. The highest-weight space of this tensor product at weight ν is one-dimensional; its cyclic submodule is therefore the irreducible highest-weight module of weight ν, by finite-dimensional complete reducibility and highest-weight classification.

It follows that in a unitary realization of σ_ν, with unit highest vector h_ν, one has $m(k) = \langle h_\nu, \sigma_\nu(k) h_\nu \rangle$. Schur orthogonality, or the matrix-unit calculation above with h_ν as the first basis vector, gives $D = 1/d_{\sigma_\nu}$ and $\int_K \overline{m(k)} \sigma_\nu(k) h_\nu \, dk = D h_\nu$. Apply this on the cyclic module of x to obtain the asserted identity. □

Theorem 6.4.32. *Let $j \in \{\pm 1\}^n$, $K = K_j \cong \mathrm{U}(p(j)) \times \mathrm{U}(q(j))$, $\mathfrak{k}_{\mathbb{C}} = \{Y \in M_n(\mathbb{C}) : Y J_j = J_j Y\}$ (the complexified Lie algebra of K, spanned by the matrix units E_{ab} with $j_a = j_b$) and $K_{\mathbb{C}} = \{y \in \mathrm{GL}_n(\mathbb{C}) : y J_j = J_j y\} \cong \mathrm{GL}_{p(j)}(\mathbb{C}) \times \mathrm{GL}_{q(j)}(\mathbb{C})$. For a finite-dimensional continuous representation (τ, W) of K let $d\tau$ be its differential, extended complex-linearly to $\mathfrak{k}_{\mathbb{C}}$, and for $\eta \in \mathbb{Z}^n$ put $W(\eta) = \{x \in W : d\tau(E_{aa}) x = \eta_a x \text{ for all } a\}$. Let Q^+ be the set of nonnegative integer combinations of the vectors $e_a - e_b$ with $a < b$, $j_a = j_b$.*

1. *(Weights.) $W = \bigoplus_\eta W(\eta)$, and $d\tau(E_{ab}) W(\eta) \subseteq W(\eta + e_a - e_b)$ for $j_a = j_b$. W is a direct sum of irreducible subrepresentations. If W is irreducible, there is a unique $\mu \in \mathbb{Z}^n$, its*

highest weight, such that $W(\mu) \neq 0$ and $d\tau(E_{ab})W(\mu) = 0$ for all $a < b$ with $j_a = j_b$; then $\dim W(\mu) = 1$, W is spanned by $\tau(K)W(\mu)$, and $\mu - \eta \in Q^+$ for every η with $W(\eta) \neq 0$. Two irreducible representations are isomorphic if and only if their highest weights agree.

2. *(Brauer–Klimyk.) If W is irreducible with highest weight μ and U is a finite-dimensional continuous representation of K, every irreducible subrepresentation of $W \otimes U$ has highest weight $\mu + \omega$ for some ω with $U(\omega) \neq 0$.*

3. *(Holomorphic extension.) τ extends uniquely to a holomorphic representation $\tilde{\tau}$ of $K_{\mathbb{C}}$, with $\tilde{\tau}(e^Y) = e^{d\tau(Y)}$ for $Y \in \mathfrak{k}_{\mathbb{C}}$. If τ is unitary for an inner product on W, then $\tilde{\tau}(y^*) = \tilde{\tau}(y)^*$ for all $y \in K_{\mathbb{C}}$.*

Proof. The highest-weight theorem for a compact connected Lie group is [164], Theorem 5.110 and the properties in Theorem 5.5; complete reducibility is Corollary 4.7. Apply them to $K = \mathrm{U}(p) \times \mathrm{U}(q)$. Its torus characters are exactly $\mathrm{diag}(e^{i\theta_a}) \mapsto e^{i\sum_a \eta_a \theta_a}$ with $\eta \in \mathbb{Z}^n$. Their differentials give $d\tau(E_{aa}) = \eta_a$ on the corresponding spaces. The bracket $[E_{cc}, E_{ab}] = (\delta_{ca} - \delta_{cb})E_{ab}$ gives the weight shift. The positive roots in the two factors are exactly the stated $e_a - e_b$; the highest-weight theorem therefore gives all of item 1, including the cone Q^+ and the one-dimensional highest-weight space. Its orbit spans by irreducibility. An empty block contributes the trivial group and no roots.

For two irreducible factors the tensor-constituent assertion is precisely [164], Proposition 9.72, printed p.598. Decompose U into irreducibles by Corollary 4.7 and apply that proposition to each direct summand of $W \otimes U$. Every constituent lies in such a summand: its projections to the summands are intertwining maps, and a nonzero projection is injective on an irreducible constituent. This proves item 2.

We supply the extension and adjoint deductions explicitly. Complexify $d\tau$ and integrate it on the simply connected covering group of $K_{\mathbb{C}}$, using Lie-algebra integration. The resulting homomorphism is holomorphic because its differential is complex-linear. Polar decomposition gives a deformation retraction $K_{\mathbb{C}} \to K$; hence the inclusion induces an isomorphism on fundamental groups. The inverse image of K in the simply connected covering group is its connected simply connected cover. On it the integrated homomorphism equals the pullback of τ, since both have the same differential on its real Lie algebra and the group is connected. In particular it is trivial on the covering kernel, which is contained in this inverse image. It descends to a holomorphic representation $\widetilde{\tau}$ on $K_{\mathbb{C}}$. Uniqueness follows since its differential is forced to be the complexification of $d\tau$, and a homomorphism on a connected Lie group is determined by its differential. The exponential identity follows from integration on one-parameter subgroups. These are the usual complexification construction; see [164], Chapter I, Sections 11–12, and Chapter VII, Section 1.

If τ is unitary and $H = H^*$ lies in $\mathfrak{k}_{\mathbb{C}}$, then iH lies in the compact Lie algebra, so $d\tau(iH)$ is skew-adjoint. Complex linearity gives $d\tau(H) = d\tau(iH)/i$, which is self-adjoint. Write $y = k \exp H$ by blockwise polar decomposition. Then

$$\widetilde{\tau}(y^*) = e^{d\tau(H)}\tau(k)^{-1} = \big(\tau(k)e^{d\tau(H)}\big)^* = \widetilde{\tau}(y)^*.$$

This proves item 3 without an unstated analytic continuation principle. □

Lemma 6.4.33. *Give K Haar probability measure dk, and fix a Haar measure dg on G. Assume $A \in \mathbb{N}$, and let $l \in \mathbb{Z}^n$ have gap $500N^3 + A$ for ε. Put $\lambda = l/2$ and*

$$e_\lambda(g) = d_\lambda \overline{c_\lambda(g)}, \qquad d_\lambda = \left(\int_G |c_\lambda(g)|^2 \, dg \right)^{-1}.$$

Then the Bochner integral

$$T_\lambda x = \int_G e_\lambda(g)\pi(g)x\,dg$$

exists for every $x \in X$ and defines a bounded linear operator. Let $\nu \in \mathbb{Z}^n$ and let $v \in M$ be K-highest of weight ν for l, meaning

$$\rho(E_{aa})v = \nu_a v, \qquad \rho(E_{ab})v = 0 \quad (\varepsilon_a = \varepsilon_b,\ l_a > l_b).$$

If there is $o \in \mathbb{Z}^n$ such that $o_a < o_b$ whenever $l_a > l_b$ and

$$\sum_{a\in n} o_a(\nu_a - \mu_l(a)) > 0,$$

then

$$T_\lambda \pi(y)\iota(v) = 0 \quad (y \in G), \qquad T_\lambda \iota(\rho(Y)v) = 0 \quad (Y \in \mathfrak{gl}_n(\mathbb{C})).$$

Proof. **The weighted moment defines a bounded operator.** Apply Lemma 6.4.28(i): the same Banach representation, invariant form, integer growth exponent A, Haar measure and gap $500N^3 + A$ supply all its hypotheses. Enumerating the finite index set preserves these data and the operator norm. It gives

$$Q_A := \int_G |e_\lambda(g)|\|g\|^A\,dg < \infty, \qquad T_\lambda x = \int_G e_\lambda(g)\pi(g)x\,dg, \qquad \|T_\lambda\| \le C_\pi Q_A.$$

The last inequality follows by integrating the given growth bound.

Compact averaging reproduces the highest vector. Put $u = \iota(v)$. If $u = 0$, the orbit assertion is immediate, and the derivative argument below also applies. Suppose $u \neq 0$. Apply Lemma 6.4.31 with the signs ε, the order given by l, and the representation, Hermitian form, module and map (π, B, M, ρ, ι) of the statement. The entries of l are distinct by its gap hypothesis. The strong continuity, continuity and positivity of B, invariance of B and norm-derived action required there are exactly the hypotheses stated here. No compact-finiteness hypothesis is imposed. The lemma proves

$$\nu_a \ge \nu_b \quad (\varepsilon_a = \varepsilon_b,\ l_a > l_b)$$

and the Banach-valued averaging identity

$$\int_K \overline{m_\nu(k)}\,\pi(k)u\,dk = a_\nu u, \qquad a_\nu = \int_K |m_\nu(k)|^2\,dk > 0. \tag{1}$$

For later use, $m_\nu(k^{-1}) = \overline{m_\nu(k)}$: since $k^{-1} = k^*$, each principal minor and the determinant in the defining product for m_ν is conjugated.

The same average annihilates the shifted kernel. The dominance just proved and the assumed vector o are exactly the hypotheses of Lemma 5.6.31(ii). Thus, for every $h, y \in G$,

$$\int_K \overline{m_\nu(k)}\,c_\lambda(hky^{-1})\,dk = 0.$$

Invert k in the Haar probability measure and use the preceding identity for $m_\nu(k^{-1})$ and the definition of e_λ. Since d_λ is real, this gives

$$\int_K \overline{m_\nu(k)}\,e_\lambda(hk^{-1}y^{-1})\,dk = d_\lambda \overline{\int_K \overline{m_\nu(k)}c_\lambda(hky^{-1})\,dk} = 0. \tag{2}$$

Fix $y \in G$. Equation (1), boundedness of T_λ and the representation law imply

$$a_\nu T_\lambda \pi(y)u = \int_K \int_G \overline{m_\nu(k)} e_\lambda(g)\pi(gyk)u \, dg \, dk.$$

This is an absolutely integrable Banach-valued double integral. Indeed, $\|k\| = 1$ and submultiplicativity give the upper bound

$$C_\pi \|u\| \|y\|^A \left(\int_K |m_\nu(k)| \, dk \right) \left(\int_G |e_\lambda(g)| \|g\|^A \, dg \right) < \infty$$

for its integral of norms. The group $G = \mathrm{U}(J)$ is unimodular, so for fixed k the substitution $h = gyk$ preserves dg. The resulting double integral is still absolutely integrable. Fubini and (2) give

$$a_\nu T_\lambda \pi(y)u = \int_G \left(\int_K \overline{m_\nu(k)} e_\lambda(hk^{-1}y^{-1}) \, dk \right) \pi(h)u \, dh = 0.$$

As $a_\nu > 0$, this proves the orbit assertion. The sign of the separating inequality has not changed: it is $o \cdot (\nu - \mu_l) > 0$, with o reversing the order of l.

Differentiate the zero orbit and complexify. For $Z^*J + JZ = 0$, the function $t \mapsto T_\lambda \pi(\exp(tZ))u$ is identically zero by the orbit assertion. The assumed derivative formula and boundedness of T_λ therefore give $T_\lambda \iota(\rho(Z)v) = 0$. For an arbitrary complex matrix Y put

$$Z_1 = (Y - JY^*J)/2, \qquad Z_2 = (Y + JY^*J)/(2\mathrm{i}).$$

Both satisfy $Z_r^* J + JZ_r = 0$, and $Y = Z_1 + \mathrm{i}Z_2$. Complex linearity of ρ, ι and T_λ proves the asserted vanishing for every Y. □

Lemma 6.4.34. *Let dg be a Haar measure on G. Assume $A \in \mathbb{N}$, and let $l \in \mathbb{Z}^n$ have large gap A; put $\lambda = l/2$, $d_\lambda = (\int_G |c_\lambda(g)|^2 \, dg)^{-1}$, $e_\lambda = d_\lambda \overline{c_\lambda}$, and*

$$\kappa_l = \frac{1}{4} \sum_{a=1}^{n} (l_a^2 - (2\rho_l(a))^2).$$

There is a complex linear map

$$P : \mathcal{H}_{c_{-\lambda}} \longrightarrow \mathcal{L}(X), \qquad P(f)w = d_\lambda \int_G f(g)\pi(g)w \, dg,$$

where each displayed integral is a Bochner integral and each $P(f)$ is bounded. It satisfies

$$P(c_{-\lambda}) = \pi(e_\lambda), \qquad P(f)\iota(\rho(Z)v) = -P(Zf)\iota(v) \quad (Z \in \mathfrak{gl}_n(\mathbb{C}),\ f \in \mathcal{H}_{c_{-\lambda}},\ v \in M),$$

where

$$\pi(e_\lambda)w = \int_G e_\lambda(g)\pi(g)w \, dg.$$

Consequently, with E_{ab} the matrix units and $\Omega_\rho = \sum_{a,b} \rho(E_{ab})\rho(E_{ba})$,

$$\pi(e_\lambda)\iota(\Omega_\rho v) = \kappa_l \, \pi(e_\lambda)\iota(v) \qquad (v \in M).$$

Proof. Write $c_0 = c_{-\lambda}$, $\mathcal{H} = \mathcal{H}_{c_0}$, and $h = 500n^3 + A$. The parameter $-l$ has the same gap h, and $h > 8n^2$. The group G is unimodular, so dg is invariant under right as well as left translations. Also $g^{-1} = Jg^*J$, whence $\|g^{-1}\| = \|g\| \geq 1$.

Every function in the cyclic module has the required moments. Set $r = A + 8n^2 + 1$. Since $n \geq 1$, $r \leq h$. By Lemma 5.7.38(1), applied to $-l$,

$$|c_0(g)| \leq (2/\|g\|)^r.$$

Part (3) of that lemma, with exponent $r - A = 8n^2 + 1$ and $L = \max_a |l_a|$, shows that $(2/\|g\|)^{r-A}$ is integrable; the positive formal degree can be divided out. By Lemma 5.7.36(ii), for every $f \in \mathcal{H}$ there is a smooth compactly supported φ_f with

$$f(g) = \int_G \varphi_f(y) c_0(gy)\, dy.$$

The inequality $\|g\| \leq \|gy\|\|y^{-1}\|$ gives

$$|f(g)| \leq K_f (2/\|g\|)^r, \qquad K_f = \int_G |\varphi_f(y)|\, \|y^{-1}\|^r\, dy < \infty. \tag{1}$$

Consequently

$$I_f := \int_G |f(g)| \|g\|^A\, dg \leq K_f 2^A \int_G (2/\|g\|)^{r-A}\, dg < \infty. \tag{2}$$

The same conclusions hold for Zf, for every complex matrix Z, since $\mathcal{H}$ is stable under complexified right derivatives by Lemma 5.7.36(ii). These functions are smooth by Lemma 5.6.24(i),(ii).

Vector integration defines the bounded operators. For fixed $f \in \mathcal{H}$ and $w \in X$, the vector-valued function $g \mapsto f(g)\pi(g)w$ is continuous and has norm at most $C_\pi |f(g)| \|g\|^A \|w\|$. It is strongly measurable: G is second countable, and a continuous image of G has separable range. Thus (2) makes its Bochner integral well defined. Define $P(f)$ by the formula in the statement. Linearity of the integral gives linearity in w, and

$$\|P(f)w\| \leq d_\lambda C_\pi I_f \|w\|.$$

Here $d_\lambda > 0$ by Lemma 5.7.38(3). Thus $P(f)$ is a bounded operator. Linearity of the integral also gives complex linearity of $f \mapsto P(f)$. This construction integrates vectors; it does not require the map $g \mapsto \pi(g)$ to be measurable for the operator norm. By Lemma 5.6.29(ii), $c_0 = \overline{c_\lambda}$, so $d_\lambda c_0 = e_\lambda$ and $P(c_0) = \pi(e_\lambda)$.

Right translation proves integration by parts without a boundary term. First let $Z \in \mathfrak{u}(J)$, $f \in \mathcal{H}$, and $v \in M$. Define continuous X-valued functions

$$F(g) = f(g)\pi(g)\iota(v), \qquad Q(g) = (Zf)(g)\pi(g)\iota(v) + f(g)\pi(g)\iota(\rho(Z)v).$$

Both are Bochner integrable by (2). The derivative hypothesis at 0, the representation law, and $e^{(t+s)Z} = e^{tZ}e^{sZ}$ give, for every $g \in G$ and $t \in \mathbb{R}$,

$$\frac{d}{dt}\pi(ge^{tZ})\iota(v) = \pi(ge^{tZ})\iota(\rho(Z)v).$$

By the right-derivative identity in Lemma 5.6.24(i) and the product rule,

$$\frac{d}{dt}F(ge^{tZ}) = Q(ge^{tZ}). \tag{3}$$

The derivative is continuous in t.

For completeness, (3) has a common integrable bound for $0 \le t \le 1$. Put

$$D_Z = \max_{0\le t\le 1} \max(\|e^{tZ}\|, \|e^{-tZ}\|),$$

which is finite and at least 1. For $u = f$ or $u = Zf$, (1) and $\|g\| \le \|ge^{tZ}\| D_Z$ imply

$$|u(ge^{tZ})| \le K_u D_Z^r (2/\|g\|)^r, \qquad \|\pi(ge^{tZ})\| \le C_\pi D_Z^A \|g\|^A.$$

Hence $\|Q(ge^{tZ})\|$ is bounded by

$$C_\pi D_Z^{r+A} 2^A \big(K_{Zf}\|\iota(v)\| + K_f \|\iota(\rho(Z)v)\|\big)(2/\|g\|)^{r-A},$$

an integrable function of g. Thus the Banach-valued fundamental theorem of calculus applies pointwise to (3), and Bochner Fubini applies to its integral over $G \times [0,1]$. Right invariance gives

$$\begin{aligned}\int_G Q(g)\,dg &= \int_0^1 \int_G Q(ge^{tZ})\,dg\,dt \\ &= \int_G \int_0^1 Q(ge^{tZ})\,dt\,dg \\ &= \int_G (F(ge^Z) - F(g))\,dg = 0.\end{aligned}$$

The last two integrals exist by the integrability of F and right invariance. Multiplication by d_λ now gives

$$P(f)\iota(\rho(Z)v) = -P(Zf)\iota(v). \tag{4}$$

No differentiation of an improper integral or unproved vanishing at infinity has been used.

Every $Z \in \mathfrak{gl}_n(\mathbb{C})$ has the decomposition

$$Z = Z_1 + iZ_2, \quad Z_1 = \tfrac{1}{2}(Z - JZ^*J), \quad Z_2 = \tfrac{1}{2i}(Z + JZ^*J), \qquad Z_1, Z_2 \in \mathfrak{u}(J).$$

The action ρ, ι, P, and the complexified right derivative are complex linear. Applying (4) to Z_1, Z_2 therefore proves it for every complex matrix Z.

Two integrations by parts give the Casimir eigenvalue. For every a, b, apply (4) first with $f = c_0$, $Z = E_{ab}$, and v replaced by $\rho(E_{ba})v$, and then with $f = E_{ab}c_0$, $Z = E_{ba}$:

$$P(c_0)\iota(\rho(E_{ab})\rho(E_{ba})v) = P\big(E_{ba}(E_{ab}c_0)\big)\iota(v).$$

Summing and interchanging the finite indices a, b on the right yields

$$\pi(e_\lambda)\iota(\Omega_\rho v) = P\left(\sum_{a,b} E_{ab}(E_{ba}c_0)\right)\iota(v).$$

By Lemma 5.7.36(v), applied to $-l$, the function in parentheses is $\kappa_{-l}c_0$. Since $2\rho_{-l} = -2\rho_l$, the displayed formula for the Casimir value gives $\kappa_{-l} = \kappa_l$. Complex linearity of P and $P(c_0) = \pi(e_\lambda)$ prove the final identity. □

Lemma 6.4.35. *Let I, I' be nonempty finite sets with signs s on I and t on I', put $N = \#I$, $N' = \#I'$, and let $\gamma, \gamma' \in \mathbb{Z}$. On $\mathbb{C}[X_{ac} : a \in I', c \in I]$ use the row and column units with shifts $(\gamma, -\gamma')$. Put*

$$c_{\mathrm{row}} = \gamma - \frac{N}{2}, \qquad c_{\mathrm{col}} = -\gamma' + \frac{N'}{2}, \qquad b_{\mathrm{pair}} = \frac{NN'(N'-N)}{4}.$$

For $w \in \mathbb{C}$ define

$$\mathrm{Cas}_{\mathrm{row}}(w) = \sum_{a,a' \in I'} (E^{\mathrm{row}}_{aa'} - w\delta_{aa'}\mathrm{id})(E^{\mathrm{row}}_{a'a} - w\delta_{a'a}\mathrm{id}),$$
$$\mathrm{Cas}_{\mathrm{col}}(w) = \sum_{c,c' \in I} (E^{\mathrm{col}}_{cc'} - w\delta_{cc'}\mathrm{id})(E^{\mathrm{col}}_{c'c} - w\delta_{c'c}\mathrm{id}),$$

where multiplication means composition of operators. For either $J = I$ or $J = I'$, with its given signs, and an integer parameter l on J, put

$$\kappa_l = \frac{1}{4}\sum_{j \in J}(l(j)^2 - (2\rho_l(j))^2), \qquad \kappa_l(w) = \kappa_l - 2w\sum_{j \in J}\mu_l(j) + \#J\, w^2.$$

Then:

1. *The following identity holds on every polynomial:*

$$\mathrm{Cas}_{\mathrm{col}}(c_{\mathrm{col}}) = \mathrm{Cas}_{\mathrm{row}}(c_{\mathrm{row}}) + b_{\mathrm{pair}}\mathrm{id}.$$

2. *If l has gap 0, then for every $w \in \mathbb{C}$,*

$$\kappa_l(w) = \sum_{j \in J}\left(\frac{l(j)}{2} - w\right)^2 - \frac{\#J((\#J)^2 - 1)}{12}.$$

3. *If $\lambda : I \to \mathbb{Z}$ and $\lambda' : I' \to \mathbb{Z}$ satisfy the block rule for (γ, γ') and (s, t), then*

$$\kappa_{-\lambda}(c_{\mathrm{col}}) = \kappa_{\lambda'}(c_{\mathrm{row}}) + b_{\mathrm{pair}}.$$

Proof. **The two operator sums have the same nonconstant terms.** Composition of polynomial operators satisfies

$$\mathrm{lo}_{ac}\mathrm{ra}_{bd} = \mathrm{ra}_{bd}\mathrm{lo}_{ac} + \delta_{ab}\delta_{cd}\mathrm{id}.$$

For a positive variable this is the product rule $\partial(XP) = X\partial P + P$; for a negative variable it is $X(-\partial P) = -\partial(XP) + P$. Operators belonging to different variables commute. In particular all the raising operators commute with one another, as do all the lowering operators.

Put, only within this proof,

$$D_2 = \sum_{a,c}\mathrm{ra}_{ac}\mathrm{lo}_{ac}, \qquad D_4 = \sum_{a,b,c,d}\mathrm{ra}_{ac}\mathrm{ra}_{bd}\mathrm{lo}_{bc}\mathrm{lo}_{ad}.$$

The defining row units give

$$E^{\mathrm{row}}_{ab} - c_{\mathrm{row}}\delta_{ab}\mathrm{id} = \sum_{c}\mathrm{ra}_{ac}\mathrm{lo}_{bc} + \frac{N}{2}\delta_{ab}\mathrm{id}.$$

For the column units, commuting the lowering operator past the raising operator gives

$$E^{\mathrm{col}}_{cd} = -\sum_{a}\mathrm{ra}_{ad}\mathrm{lo}_{ac} - \gamma'\delta_{cd}\mathrm{id}, \qquad E^{\mathrm{col}}_{cd} - c_{\mathrm{col}}\delta_{cd}\mathrm{id} = -\left(\sum_{a}\mathrm{ra}_{ad}\mathrm{lo}_{ac} + \frac{N'}{2}\delta_{cd}\mathrm{id}\right).$$

Indeed each diagonal term in the commutation contributes 1, so the N' such terms cancel the $+N'$ in the defining column shift.

For the row sum, the only interchange required to put each product in normal order is

$$\mathrm{ra}_{ac}\mathrm{lo}_{bc}\mathrm{ra}_{bd}\mathrm{lo}_{ad} = \mathrm{ra}_{ac}\mathrm{ra}_{bd}\mathrm{lo}_{bc}\mathrm{lo}_{ad} + \delta_{cd}\mathrm{ra}_{ac}\mathrm{lo}_{ad}.$$

Summing the second term over a, b, c, d gives $N'D_2$: setting $d = c$ leaves the free row index b, with N' choices. The two cross terms from the added scalar $N/2$ give ND_2, and the product of the two scalars gives $N'N^2/4$ copies of the identity. Consequently

$$\mathrm{Cas}_{\mathrm{row}}(c_{\mathrm{row}}) = D_4 + (N + N')D_2 + \frac{N'N^2}{4}\mathrm{id}. \tag{1}$$

For the column sum the two minus signs cancel. Its product without the diagonal scalars expands into

$$\sum_{c,d,a,b} \mathrm{ra}_{ad}\mathrm{lo}_{ac}\mathrm{ra}_{bc}\mathrm{lo}_{bd} = \sum_{c,d,a,b} \mathrm{ra}_{ad}\mathrm{ra}_{bc}\mathrm{lo}_{ac}\mathrm{lo}_{bd} + \sum_{c,d,a} \mathrm{ra}_{ad}\mathrm{lo}_{ad}.$$

The last sum is ND_2, because c is free. In the first sum exchange c, d and then commute the two lowering operators. It becomes exactly D_4. The scalar cross terms now give $N'D_2$, and the scalar product gives $N(N')^2/4$ copies of the identity. Thus

$$\mathrm{Cas}_{\mathrm{col}}(c_{\mathrm{col}}) = D_4 + (N + N')D_2 + \frac{N(N')^2}{4}\mathrm{id}. \tag{2}$$

Subtracting (1) from (2) proves (i). All sums here are finite and all equalities are identities of operators on the full polynomial ring; no harmonicity assumption is used.

The centering of a parameter gives the stated quadratic value. First note the elementary operator identity, valid for any square family of operators indexed by a finite set J:

$$\sum_{i,j}(E_{ij} - w\delta_{ij}\mathrm{id})(E_{ji} - w\delta_{ji}\mathrm{id}) = \sum_{i,j} E_{ij}E_{ji} - 2w\sum_i E_{ii} + \#J\, w^2\mathrm{id}.$$

Both mixed sums are $-w\sum_i E_{ii}$, since their Kronecker symbols force $i = j$. On a vector of diagonal weight μ_l and full Casimir value κ_l, this identity gives precisely $\kappa_l(w)$.

Write $r = \#J$ and order the distinct entries of l increasingly as $l(j_0) < \cdots < l(j_{r-1})$. By the definition of the two counts,

$$2\rho_l(j_b) = 2b - r + 1 \quad (0 \le b < r).$$

The identities $\sum_{b=0}^{r-1} b = r(r-1)/2$ and $\sum_{b=0}^{r-1} b^2 = r(r-1)(2r-1)/6$ therefore give

$$\sum_{j\in J}(2\rho_l(j))^2 = 4\frac{r(r-1)(2r-1)}{6} - 4(r-1)\frac{r(r-1)}{2} + r(r-1)^2 = \frac{r(r^2-1)}{3}. \tag{3}$$

The gap-zero parity condition makes $l(j) + 2\rho_l(j)$ even, because both summands are congruent to $r - 1$ modulo 2. Thus the integer part in l^ρ is exact, and $\mu_l(j) = l(j)/2 + \rho_l(j) - 2\rho_l^{\mathrm{c}}(j)$. The sums of the entries of $2\rho_l$ and of $2\rho_l^{\mathrm{c}}$ are both zero: in their double sums the contribution of the ordered pair (i, j) is the negative of that of (j, i), and equality of signs is a symmetric condition. Hence

$$2\sum_j \mu_l(j) = \sum_j l(j). \tag{4}$$

Substituting (3)–(4) into the definition of $\kappa_l(w)$, and expanding each square, gives

$$\kappa_l(w) = \frac{1}{4}\sum_j l(j)^2 - \frac{r(r^2-1)}{12} - w\sum_j l(j) + rw^2 = \sum_j (l(j)/2 - w)^2 - \frac{r(r^2-1)}{12}.$$

This proves (ii) for every complex w.

The unmatched entries account for the entire scalar difference. Assume the block rule. Use its shifted entries

$$\widetilde{\lambda}'(a) = \lambda'(a) - 2\gamma + N, \qquad \widetilde{\lambda}(c) = \lambda(c) - 2\gamma' + N'.$$

They are injective on their respective index sets. The matching of Lemma 6.4.20(a) is a partial bijection, with equal shifted entries at each matched pair.

Suppose first that $N' \geq N$, and put $d = N' - N$. Every column is matched: an unmatched column would have $|\widetilde{\lambda}(c)| < N - N' = -d$, which is impossible. There are therefore exactly d unmatched rows. Each has $|\widetilde{\lambda}'(a)| < d$ by the same lemma. Its shifted entry is congruent to $d-1$ modulo 2, since

$$\lambda'(a) - 2\gamma + N \equiv N' - 1 + N \equiv N' - N - 1 \pmod 2.$$

If $d = 0$ there are no unmatched rows. If $d > 0$, the integers of that parity and modulus below d are exactly $d-1, d-3, \ldots, 1-d$, a set of cardinality d. The d unmatched entries are distinct, so they fill this set. Cancelling the equal squares at matched pairs gives

$$\sum_a \widetilde{\lambda}'(a)^2 - \sum_c \widetilde{\lambda}(c)^2 = \sum_{b=0}^{d-1} (d - 1 - 2b)^2 = \frac{d(d^2-1)}{3}. \tag{5}$$

The last equality follows by the two finite-sum formulas used in (3); it also holds for $d = 0$.

If $N' < N$, the same argument with the index sets exchanged gives $\sum_c \widetilde{\lambda}(c)^2 - \sum_a \widetilde{\lambda}'(a)^2 = (N - N')((N-N')^2 - 1)/3$. Negating it gives (5) with the signed integer $d = N' - N$. Thus (5) holds in both rank orders.

Finally,

$$\lambda'(a)/2 - c_{\mathrm{row}} = \widetilde{\lambda}'(a)/2, \qquad -\lambda(c)/2 - c_{\mathrm{col}} = -\widetilde{\lambda}(c)/2.$$

The block rule supplies gap zero for λ' and $-\lambda$, so (ii) applies to both. Using (5) gives

$$\begin{aligned} \kappa_{\lambda'}(c_{\mathrm{row}}) - \kappa_{-\lambda}(c_{\mathrm{col}}) &= \frac{d(d^2-1) - N'((N')^2-1) + N(N^2-1)}{12} \\ &= \frac{d^3 - (N')^3 + N^3}{12} = -\frac{NN'd}{4}. \end{aligned}$$

Here the last equality is the expansion of $d = N' - N$. Rearranging proves (iii), with exactly the scalar in (i). □

Theorem 6.4.36. *Assume $A \in \mathbb{N}$, and let $l \in \mathbb{Z}^n$ have large gap A for $\varepsilon \in \{\pm 1\}^n$. Put*

$$\kappa_l = \frac{1}{4}\sum_a (l_a^2 - (2\rho_l(a))^2), \quad d_l = \left(\int_G |c_{l/2}(g)|^2\, dg\right)^{-1}, \quad e_l(g) = d_l\, \overline{c_{l/2}(g)}.$$

Write $\Omega = \sum_{a,b} \rho(E_{ab})\rho(E_{ba})$. Suppose $v \in M$ satisfies

$$\rho(E_{aa})v = \mu_l(a)v, \qquad \rho(E_{ab})v = 0 \quad (\varepsilon_a = \varepsilon_b,\ l_a > l_b), \qquad \iota(\Omega v) = c\,\iota(v)$$

for a real number $c \geq \kappa_l$. Then

$$B(\iota(v), \pi(g)\iota(v)) = B(\iota(v), \iota(v))c_{l/2}(g) \qquad (g \in G).$$

Moreover, for every measurable $f : G \to \mathbb{C}$ with $\int_G |f(g)|\|g\|^A\,dg < \infty$ and

$$\int_G f(y)e_l(y^{-1}g)\,dy = e_l(g) \qquad (g \in G),$$

the Bochner integral satisfies

$$\int_G f(g)\pi(g)\iota(v)\,dg = \iota(v).$$

Proof. **The derived action descends to a positive inner product space.** Write $x = \iota(v)$ and $q = B(x, x) \geq 0$. If $x = 0$, both conclusions are immediate, so assume $x \neq 0$. Equip X, for the moment, with the possibly incomplete inner product B; this changes neither its vectors nor its linear maps. Differentiate invariance along $\exp(tY)$, using boundedness of B and the assumed derivative identity. For $Y^*J + JY = 0$ this gives

$$B(\iota(\rho(Y)u), \iota(w)) = -B(\iota(u), \iota(\rho(Y)w)).$$

Complexifying, since every matrix Z decomposes as $Z = (Z - JZ^*J)/2 + i(Z + JZ^*J)/(2i)$ with both summands in the real Lie algebra, yields

$$B(\iota(\rho(Z)u), \iota(w)) = B(\iota(u), \iota(\rho(JZ^*J)w)). \tag{1}$$

If $\iota(u) = 0$, apply (1) with $w = \rho(Z)u$. Positive definiteness implies $\iota(\rho(Z)u) = 0$. Thus the action descends to $\iota(M)$, and restricts to its submodule H_x generated by x. On this pre-Hilbert space its elementary operators have adjoints $E_{ab}^* = \varepsilon_a\varepsilon_b E_{ba}$. The full Casimir is central: expanding $[E_{cd}, \sum_{a,b} E_{ab}E_{ba}]$ gives two sums that cancel term by term. Hence it acts as c on H_x, since it acts as c on its generator. In particular its quadratic form is bounded below by κ_l.

The comparison module has a normalized generator. Let $\mathcal{H}_c$ be the derivative module of $c_{l/2}$ in Lemma 5.7.36, with inner product

$$\langle u, w\rangle_D = d_l \int_G \overline{u(g)}w(g)\,dg.$$

The large gap is greater than $8n^2$, so Lemma 5.6.23 gives $d_l > 0$. Write $b = c_{l/2}$ as an element of this module. By Lemma 5.7.36, $\|b\|_D = 1$, $|u(g)| \leq \|u\|_D$, the adjoints have the same signs, b is highest of weight μ_l, and the full Casimir is κ_l. For every $u \in \mathcal{H}_c$,

$$\langle b, u\rangle_D = u(1). \tag{2}$$

Indeed $u = f_\varphi$ for a compactly supported smooth φ, by that lemma; substitute its integral formula, use Fubini with the bound $|\varphi(y)||b(z)|$, and apply Schur orthogonality in Lemma 5.6.23 with $x = 1$. This gives (2) exactly.

On $H = H_x \oplus \mathcal{H}_c$ take the orthogonal direct sum inner product and the componentwise Lie algebra action. It has the signed adjoints and full Casimir lower bound κ_l. The two vectors

$$z = (x, b), \qquad a = (x, -qb)$$

are highest of weight μ_l, and $\langle a, z\rangle = q - q = 0$.

Compact orthogonality propagates through every filtration level. Let $F_0 = U(\mathfrak{k}_{\mathbb{C}})z$ and $F_{s+1} = F_s + \sum_{i,j} E_{ij}F_s$. Apply Lemma 6.4.37 to the direct-sum module and its highest

vectors z, a. Their common weight is μ_l, the signed adjoints and full Casimir lower bound were checked above, and $\langle a, z\rangle = 0$. Thus

$$a \perp F_s \qquad (s \geq 0). \tag{3}$$

The scalar coefficient difference vanishes. For $g \in G$ define the linear functional on H

$$L_g(u, w) = B(x, \pi(g)u) - q\, w(g).$$

It satisfies the coefficient derivative identity at every real time along every $Y \in \mathfrak{u}(J)$. On the first summand this follows from the representation law and the derivative hypothesis, by shifting the time origin; on the second it is the right derivative of the locally holomorphic functions of Lemma 5.7.36. Moreover, invariance of B, Cauchy–Schwarz and the evaluation bound give

$$|L_g(u, w)| \leq (\sqrt{q} + q)\|(u, w)\|_H$$

uniformly in g. Equations (2) and (3) give $L_1(u, w) = \langle a, (u, w)\rangle = 0$ on every F_s. The coefficient uniqueness assertion of Lemma 6.4.39, applied to z, therefore gives $L_g(z) = 0$ for every g. This is the claimed coefficient identity.

The convolution identity fixes the vector. Strong measurability follows from the continuous orbit map and the measurable scalar f. The growth bound and the moment assumption make $g \mapsto f(g)\pi(g)x$ Bochner integrable. By inversion in Lemma 5.7.40, the assumed convolution identity at g^{-1}, divided by $d_l > 0$, is

$$\int_G f(y) c_{l/2}(gy)\, dy = c_{l/2}(g). \tag{4}$$

All these scalar integrals are absolutely convergent, since $\|y\| \geq 1$ and $|c_{l/2}| \leq 1$. Put $r = \int f(y)\pi(y)x\, dy - x$. For each g, invariance, the coefficient identity and (4) give

$$B(\pi(g)x, r) = q\left(\int_G f(y) c_{l/2}(g^{-1}y)\, dy - c_{l/2}(g^{-1})\right) = 0.$$

Hermitian symmetry gives $B(r, \pi(g)x) = 0$. Boundedness of B permits it to pass through the Bochner integral in its second variable, so

$$B(r, r) = \int_G f(y) B(r, \pi(y)x)\, dy - B(r, x) = 0.$$

Positive definiteness forces $r = 0$, proving the second conclusion. □

Lemma 6.4.37. *Let $n \geq 1$, let $\varepsilon \in \{\pm 1\}^n$, and let $l \in \mathbb{Z}^n$ have large gap A for an integer $A \geq 0$. Put*

$$\kappa_l = \frac{1}{4}\sum_{i=1}^{n} (l_i^2 - (2\rho_l(i))^2).$$

Let H be a complex pre-Hilbert space, with inner product conjugate-linear in the first variable, and let $\rho : \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(H)$ be a Lie algebra representation. Write $D_{ij} = \rho(E_{ij})$ and $\Omega = \sum_{i,j} D_{ij} D_{ji}$. Suppose

$$\langle D_{ij} u, w\rangle = \varepsilon_i \varepsilon_j \langle u, D_{ji} w\rangle, \qquad \mathrm{Re}\langle u, \Omega u\rangle \geq \kappa_l \|u\|^2$$

for all $u, w \in H$ and all i, j. Let $z, a \in H$ satisfy, for $v = z, a$,

$$D_{ii} v = \mu_l(i) v, \qquad D_{ij} v = 0 \quad (\varepsilon_i = \varepsilon_j,\ l_i > l_j).$$

Let $F_0(z)$ be the span of all compact words $D_{i_1j_1}\cdots D_{i_rj_r}z$, where $\varepsilon_{i_\nu}=\varepsilon_{j_\nu}$, including the empty word, and recursively set

$$F_{r+1}(z)=F_r(z)+\sum_{i,j}D_{ij}F_r(z).$$

If $\langle a,z\rangle=0$, then $a\perp F_r(z)$ for every integer $r\ge 0$.

Proof. Write $\mu=\mu_l$, $r_i=2\rho_l^{\mathrm{c}}(i)$, $R_i=2\rho_l(i)$ and

$$C_K=\sum_{\varepsilon_i=\varepsilon_j}D_{ij}D_{ji},\qquad q(\nu)=\sum_i\nu_i(\nu_i+r_i),\qquad q_0=q(\mu).$$

All compact sums below run over ordered pairs (i,j) with $\varepsilon_i=\varepsilon_j$. The matrix-unit commutator is

$$[D_{ij},D_{pq}]=\delta_{jp}D_{iq}-\delta_{qi}D_{pj}. \tag{1}$$

Here and below the adjoint of a matrix unit means the algebraic adjoint $D_{ij}^*=\varepsilon_i\varepsilon_jD_{ji}$, defined on all of H; no completeness or boundedness of these units is required. For a fixed compact unit $X=D_{ij}$, expansion of its commutator with C_K gives four sums:

$$\begin{aligned}[X,C_K]=&\sum_{\varepsilon_q=\varepsilon_j}D_{iq}D_{qj}-\sum_{\varepsilon_p=\varepsilon_i}D_{pj}D_{ip}\\&+\sum_{\varepsilon_p=\varepsilon_j}D_{pj}D_{ip}-\sum_{\varepsilon_q=\varepsilon_i}D_{iq}D_{qj}=0.\end{aligned}$$

The cancellation uses $\varepsilon_i=\varepsilon_j$, so the first and fourth indexing sets agree, as do the second and third. Thus C_K commutes with every compact unit.

Suppose u is compact highest of real weight ν. For each compact pair with $l_i>l_j$, $D_{ij}u=0$ and

$$(D_{ij}D_{ji}+D_{ji}D_{ij})u=[D_{ij},D_{ji}]u=(\nu_i-\nu_j)u.$$

The diagonal terms are $D_{ii}^2u=\nu_i^2u$. Every unordered off-diagonal compact pair occurs once in this calculation, and $\sum_{\varepsilon_i=\varepsilon_j,\ l_i>l_j}(\nu_i-\nu_j)=\sum_i r_i\nu_i$. Consequently $C_Ku=q(\nu)u$. For arbitrary u, compact adjointness gives the separate norm identity

$$\langle u,C_Ku\rangle=\sum_{\varepsilon_i=\varepsilon_j}\langle u,D_{ij}D_{ji}u\rangle=\sum_{\varepsilon_i=\varepsilon_j}\|D_{ji}u\|^2. \tag{2}$$

In particular $C_K=q_0$ on $F_0(z)$. By Lemma 6.4.38, every $F_j(z)$ is finite-dimensional; its hypotheses hold since large gap includes distinctness and the required parity of l, and the integer vector μ has a finite coordinate bound. The commutators (1) show that every $F_j(z)$ is compact-invariant. Thus its orthogonal complement is also compact-invariant. Let P be the orthogonal projection onto $F_0(z)$. It exists in the pre-Hilbert space since this subspace is finite-dimensional, and commutes with every compact unit.

The first new compact types have a uniform Casimir gap. We first prove that any nonzero compact highest vector of weight $\mu+\eta$ satisfies

$$0\le n^3+\sum_i\eta_i(\eta_i+l_i). \tag{3}$$

Let $u\neq 0$ be compact highest of weight $\nu=\mu+\eta$; we use (3) only for integer η. The diagonal weights are real. For each noncompact pair with $l_i>l_j$, (1) gives

$$D_{ij}D_{ji}+D_{ji}D_{ij}=2D_{ij}D_{ji}-D_{ii}+D_{jj},\qquad \langle u,D_{ij}D_{ji}u\rangle=-\|D_{ji}u\|^2.$$

Let $\mathcal{P} = \{(i,j) : \varepsilon_i \neq \varepsilon_j,\ l_i > l_j\}$; it selects exactly one ordering of every noncompact unordered pair. Since

$$\sum_{(i,j)\in\mathcal{P}} (\nu_i - \nu_j) = \sum_i (R_i - r_i)\nu_i,$$

adding the compact and noncompact contributions to the full Casimir gives the exact equality

$$\begin{aligned}\mathrm{Re}\langle u, \Omega u\rangle &= \left(q(\nu) - \sum_i (R_i - r_i)\nu_i\right)\|u\|^2 - 2\sum_{(i,j)\in\mathcal{P}} \|D_{ji}u\|^2 \\ &= \left(\sum_i \nu_i(\nu_i + 2r_i - R_i)\right)\|u\|^2 - 2\sum_{(i,j)\in\mathcal{P}} \|D_{ji}u\|^2.\end{aligned}$$

The lower bound on Ω, nonnegativity of all squared norms, and $\|u\|^2 > 0$ therefore imply

$$\kappa_l \leq \sum_i \nu_i(\nu_i + 2r_i - R_i).$$

Parity in the parameter gives $2\mu_i = l_i + R_i - 2r_i$ without a rounding term. Thus

$$\nu_i(\nu_i + 2r_i - R_i) = (\mu_i + \eta_i)(l_i - \mu_i + \eta_i) = \eta_i(\eta_i + l_i) + \mu_i(l_i - \mu_i).$$

Writing $S = \sum_i \eta_i(\eta_i + l_i)$, subtracting $\kappa_l = \sum_i (l_i^2 - R_i^2)/4$ gives

$$0 \leq S - \frac{1}{4}\sum_i (l_i - 2\mu_i)^2 + \frac{1}{4}\sum_i R_i^2 \leq S + \frac{1}{4}n^3.$$

Here $|R_i| \leq n$ was used. It follows in particular that $0 \leq S + n^3$, which is (3).

Put $U = F_1(z) \cap F_0(z)^\perp$. A nonzero compact highest vector $u \in U$ pairs nontrivially with some $D_{ij}z$. To see this, interchange compact units using (1), putting compact lowering factors on the left. This expresses $F_1(z)$ as the span of compact lowering words applied to z or to a vector $D_{ij}z$. If all those short words were orthogonal to u, move the leftmost lowering factor across the inner product: its compact raising adjoint kills u. Thus u would be orthogonal to $F_1(z)$, a contradiction. The diagonal units are self-adjoint, so a nonzero pairing forces the weight of u to be $\mu + e_i - e_j$. The pair must be noncompact, since $D_{ij}z \in F_0(z)$ for a compact pair. In particular $i \neq j$. By (3),

$$l_i - l_j \geq -n^3 - 2.$$

Large gap gives $|l_i - l_j| \geq 1000n^3$; hence $l_i - l_j \geq 1000n^3$. Direct subtraction gives

$$q(\mu + e_i - e_j) - q_0 = l_i - l_j + R_i - R_j - r_i + r_j + 2 \geq 1000n^3 - 4(n-1) + 2 \geq 4n + 2. \quad (4)$$

Here a noncompact pair entails $n \geq 2$.

Consequently $C_K \geq q_0 + 4n + 2$ on U. For completeness, diagonalize the self-adjoint C_K on the finite-dimensional compact-invariant U. Every nonzero eigenspace is compact-invariant and contains a highest vector: simultaneously diagonalize the commuting self-adjoint diagonal units, and choose a weight maximizing its pairing with l. A nonzero raising image would increase that pairing. Formula (4) bounds the eigenvalue of this eigenspace. This proves the stated operator inequality.

A projection inequality controls the first commutator. For $v \perp F_0(z)$ define

$$\begin{aligned}B(v) &= \sum_{p,q} \|PD_{qp}v\|^2, \\ T(v) &= \sum_{p,q}\sum_{\varepsilon_i=\varepsilon_j} \mathrm{Re}\langle D_{ji}PD_{qp}v,\ P[D_{qp}, D_{ji}]v\rangle, \\ W(v) &= \sum_{p,q}\sum_{\varepsilon_i=\varepsilon_j} \|P[D_{ji}, D_{qp}]v\|^2.\end{aligned}$$

We claim $B(v) \le T(v)$. Only the orthogonal projection of v onto $F_1(z)$ contributes to these three quantities: if $v \perp F_1(z)$ then $D_{pq}v \perp F_0(z)$ by the adjoint identity, and the same holds for any commutator in (1). We may therefore assume $v \in U$.

The positive form $\sum_{p,q}\langle PD_{qp}u, PD_{qp}v\rangle$ has operator

$$D^{(1)} = \sum_{p,q} \varepsilon_p\varepsilon_q D_{pq}PD_{qp}.$$

We verify the invariant contraction that will also be needed at later levels. If D is any operator commuting with every compact unit, put

$$\mathcal{A}(D) = \sum_{p,q} \varepsilon_p\varepsilon_q D_{pq}DD_{qp}.$$

For a compact $X = D_{ij}$ the term $D_{pq}[X, D]D_{qp}$ vanishes. Expanding the other two terms by (1) gives

$$\begin{aligned}[X, \mathcal{A}(D)] = &\sum_q \varepsilon_j\varepsilon_q D_{iq}DD_{qj} - \sum_p \varepsilon_p\varepsilon_i D_{pj}DD_{ip} \\ &+ \sum_p \varepsilon_p\varepsilon_j D_{pj}DD_{ip} - \sum_q \varepsilon_i\varepsilon_q D_{iq}DD_{qj} = 0.\end{aligned} \tag{5a}$$

The first and fourth sums cancel term by term because $\varepsilon_i = \varepsilon_j$, and so do the second and third. Hence $\mathcal{A}(D)$ commutes with all compact units and therefore with C_K. Taking $D = P$ proves this assertion for $D^{(1)}$.

More generally, suppose an indexed family $\mathcal{J}$ has form operator D: $\sum_{J\in\mathcal{J}}\langle Ju, Jv\rangle = \langle u, Dv\rangle$, and D commutes with compact units. Then

$$\begin{aligned}\sum_J \sum_{\varepsilon_i=\varepsilon_j} \langle JD_{ji}u, JD_{ji}v\rangle &= \sum_{\varepsilon_i=\varepsilon_j} \langle D_{ji}u, DD_{ji}v\rangle \\ &= \sum_{\varepsilon_i=\varepsilon_j} \langle u, D_{ij}DD_{ji}v\rangle \\ &= \langle u, DC_Kv\rangle = \sum_J \langle Ju, JC_Kv\rangle.\end{aligned} \tag{5b}$$

The second equality uses the positive compact sign; the third uses $[D, D_{ij}] = 0$. This does not require the individual J to commute with the compact units.

Here $\mathcal{J}$ consists of all PD_{qp} and its form operator is $D^{(1)}$: indeed $\langle PD_{qp}u, PD_{qp}v\rangle = \varepsilon_p\varepsilon_q\langle u, D_{pq}PD_{qp}v\rangle$. Let Q_U be the orthogonal projection onto the finite-dimensional space U. The compact stability of U and its orthogonal complement shows that Q_U commutes with each compact unit and with C_K. The operators

$$D_U = Q_U D^{(1)}|_U, \qquad C_U = C_K|_U$$

are self-adjoint on U, and D_U is positive. They commute: for $u \in U$,

$$D_UC_Uu = Q_UD^{(1)}C_Ku = Q_UC_KD^{(1)}u = C_KQ_UD^{(1)}u = C_UD_Uu.$$

Thus each eigenspace of C_U is preserved by D_U; diagonalizing the positive D_U on each such space gives an orthonormal basis e_α with $C_Ue_\alpha = \theta_\alpha e_\alpha$, $D_Ue_\alpha = d_\alpha e_\alpha$, $\theta_\alpha \ge q_0 + 4n + 2$ and $d_\alpha \ge 0$. For $v = \sum_\alpha v_\alpha e_\alpha$ one consequently has

$$\langle v, D_UC_Uv\rangle = \sum_\alpha d_\alpha\theta_\alpha|v_\alpha|^2 \ge (q_0 + 4n + 2)\sum_\alpha d_\alpha|v_\alpha|^2 = (q_0 + 4n + 2)B(v).$$

Apply (5b) to the family PD_{qp} and put $u = v$. Since $v \in U$, its left side is the right side below, while its right side is $\langle v, D_U C_U v\rangle$. We obtain

$$(q_0 + 4n + 2)B(v) \le \sum_{p,q} \sum_{\varepsilon_i = \varepsilon_j} \|PD_{qp}D_{ji}v\|^2. \tag{5}$$

Expand $PD_{qp}D_{ji}v = D_{ji}PD_{qp}v + P[D_{qp}, D_{ji}]v$. By (2) on the range of P, the right side of (5) is $q_0 B(v) + 2T(v) + W(v)$. On the other hand (1), the inequality $\|u - w\|^2 \le 2\|u\|^2 + 2\|w\|^2$, and extension of the compact sum to all (i, j) give

$$W(v) \le 2 \sum_{p,q,i,j} \left(\delta_{iq}\|PD_{jp}v\|^2 + \delta_{pj}\|PD_{qi}v\|^2\right) = 4nB(v).$$

The two sums each equal $nB(v)$. Subtracting these bounds from (5) gives $2B(v) \le 2T(v)$, proving the claim.

Finite detecting families propagate this inequality. Use the family consisting of P at level zero, and at level $k+1$ use all operators JD_{qp}, where J belongs to the level-k family. Write $\mathcal{J}_k$ for this finite indexed family and

$$b_k(v) = \sum_{J \in \mathcal{J}_k} \|Jv\|^2.$$

We prove the following four properties by induction:

(a) Every $J \in \mathcal{J}_k$ kills $F_k(z)^\perp$; if $v \perp F_j(z)$ for every $j < k$ and every $Jv = 0$, then $v \perp F_k(z)$.

(b) The form $\sum_J \langle Ju, Jv\rangle = \langle u, D^{(k)}v\rangle$ has a positive operator $D^{(k)}$ commuting with the compact units.

(c) For $v \perp F_k(z)$ and each $J \in \mathcal{J}_k$,

$$2\sum_{p,q} \|JD_{qp}v\|^2 \le 2\sum_{p,q} \sum_{\varepsilon_i = \varepsilon_j} \mathrm{Re}\langle D_{ji}JD_{qp}v,\ J[D_{qp}, D_{ji}]v\rangle.$$

(d) If $v \perp F_j(z)$ for every $j < k$, then

$$(q_0 + 2k)b_k(v) \le \sum_{J \in \mathcal{J}_k} \sum_{\varepsilon_i = \varepsilon_j} \left(\|JD_{ji}v\|^2 - \|[D_{ji}, J]v\|^2\right). \tag{6}$$

At level zero, (a) is the projection identity and (b) follows from $P^* = P$. Property (c) is the projection inequality just proved. For (d), P commutes with compact units and $C_K = q_0$ on its range, so (6) is equality.

Here are the induction details. The adjoint relation gives

$$v \perp F_{k+1}(z) \quad \Longrightarrow \quad D_{pq}v \perp F_k(z). \tag{7}$$

This proves the killing assertion of (a) at the next level. For detection, suppose v is orthogonal to the earlier levels and $JD_{qp}v = 0$ for all J, p, q. Apply the preceding detection assertion to $D_{qp}v$, using (7) for its earlier levels. Thus every $D_{qp}v$ is orthogonal to $F_k(z)$. Taking adjoints shows $v \perp D_{pq}F_k(z)$; together with $v \perp F_k(z)$ this proves detection at level $k + 1$. The next form operator is

$$D^{(k+1)} = \sum_{p,q} \varepsilon_p \varepsilon_q D_{pq} D^{(k)} D_{qp}.$$

To check that this is its form operator, expand

$$\sum_{J\in\mathcal{J}_k}\sum_{p,q}\langle JD_{qp}u, JD_{qp}v\rangle = \sum_{p,q}\langle D_{qp}u, D^{(k)}D_{qp}v\rangle$$
$$= \sum_{p,q}\varepsilon_p\varepsilon_q\langle u, D_{pq}D^{(k)}D_{qp}v\rangle.$$

It is positive because at $u = v$ the left side is a sum of squared norms, and is self-adjoint by polarization of this Hermitian form. Formula (5a), applied to $D = D^{(k)}$, proves its commutation with each compact unit. This proves (b) at the next level with no additional invariance hypothesis.

For (c) at the next level take $v \perp F_{k+1}(z)$ and fix $J \in \mathcal{J}_k$, D_{qp}. For every matrix unit, and hence every matrix Y, $J\rho(Y)v = 0$ by (7) and (a). Consequently

$$J\rho(Y)D_{qp}v = JD_{qp}\rho(Y)v, \tag{8}$$

because their difference is $J\rho([Y, E_{qp}])v$. For clarity fix the prepended unit $E_0 = D_{qp}$, and use $E = D_{ba}$ for the units summed in (c) and $C = D_{ji}$ for its compact units. Applying the preceding (c) to $E_0v \perp F_k(z)$ gives

$$2\sum_{a,b}\|JEE_0v\|^2 \le 2\sum_{a,b}\sum_{\varepsilon_i=\varepsilon_j}\mathrm{Re}\langle CJEE_0v, J[E,C]E_0v\rangle.$$

Equation (8) holds for every Lie algebra element, so it applies both to E and to $[E, C]$ (a linear combination of matrix units by (1)). It changes this inequality into

$$2\sum_{a,b}\|JE_0Ev\|^2 \le 2\sum_{a,b}\sum_{\varepsilon_i=\varepsilon_j}\mathrm{Re}\langle CJE_0Ev, JE_0[E,C]v\rangle.$$

This is exactly (c) for the member JE_0 of $\mathcal{J}_{k+1}$.

It remains to prove (d) at level $k+1$. Let $v \perp F_j(z)$ for every $j < k+1$. For $j < k$ one has $j+1 < k+1$, so (7) applied at index j gives $D_{qp}v \perp F_j(z)$. Thus (d) at level k applies to every $D_{qp}v$. Sum its inequalities over all p, q. In the following displays $\sum_{J,E,C}$ means $J \in \mathcal{J}_k$, all $E = D_{qp}$, and all compact $C = D_{ji}$, with each family indexed including repetitions. The result is

$$(q_0 + 2k)b_{k+1}(v) \le \sum_{J,E,C}(\|JCEv\|^2 - \|[C,J]Ev\|^2). \tag{9a}$$

Moreover $v \perp F_k(z)$, so (c) at level k applies to v. Summing it over J gives

$$2b_{k+1}(v) \le 2\sum_{J,E,C}\mathrm{Re}\langle CJEv, J[E,C]v\rangle. \tag{9b}$$

For each fixed J, E, C put $u_1 = JCEv$, $u_2 = CJEv$, $u_3 = J[E,C]v$. Then $JECv = u_1 + u_3$, $[C,J]Ev = u_2 - u_1$, and $[C,JE]v = u_2 - u_1 - u_3$. Expanding squared norms therefore gives

$$\begin{aligned}\|JECv\|^2 - \|[C,JE]v\|^2 &= \|u_1+u_3\|^2 - \|u_2-u_1-u_3\|^2\\ &= \|u_1\|^2 - \|u_2-u_1\|^2 + 2\,\mathrm{Re}\langle u_2, u_3\rangle\\ &= \|JCEv\|^2 - \|[C,J]Ev\|^2 + 2\,\mathrm{Re}\langle CJEv, J[E,C]v\rangle.\end{aligned} \tag{9}$$

In the second equality, the terms $\|u_3\|^2$ cancel and the two mixed terms combine to $2\,\mathrm{Re}\langle u_1 + (u_2 - u_1), u_3\rangle$. Add (9a) and (9b), and apply (9) in every summand. This yields

$$\begin{aligned}(q_0 + 2(k+1))b_{k+1}(v) &\le \sum_{J,E,C}(\|(JE)Cv\|^2 - \|[C,JE]v\|^2)\\ &= \sum_{J'\in\mathcal{J}_{k+1}}\sum_{\varepsilon_i=\varepsilon_j}(\|J'D_{ji}v\|^2 - \|[D_{ji},J']v\|^2).\end{aligned}$$

The last equality is exactly the indexed definition $J' = JD_{qp}$, not an assertion that these operators are distinct. This proves (d), completing the induction.

The compact eigenvalue forces orthogonality at every level. Compact word interchange, with lowering factors first and raising factors last, shows $F_0(z)$ is spanned by compact lowering words applied to z. In the pairing with a, a nonempty lowering word has a raising adjoint killing a; the empty word pairs to zero by hypothesis. Hence $a \perp F_0(z)$. The same highest-vector calculation used above gives $C_K a = q_0 a$.

Proceed by induction on k. Suppose $a \perp F_j(z)$ for all $j < k$. Property (b) and compact adjointness give the exact identity

$$\sum_{J\in\mathcal{J}_k}\sum_{\varepsilon_i=\varepsilon_j}\|JD_{ji}a\|^2 = \operatorname{Re}\sum_J\langle Ja, JC_K a\rangle = q_0 b_k(a). \tag{10}$$

The first equality is (5b) for $\mathcal{J} = \mathcal{J}_k$, $D = D^{(k)}$ and $u = v = a$; property (b) supplies its exact commutation hypothesis. The last equality uses $C_K a = q_0 a$ and the reality of q_0. Combining (6) with (10), and discarding the subtracted nonnegative squares in (6), gives

$$(q_0 + 2k)b_k(a) \le q_0 b_k(a).$$

For $k > 0$ this implies $b_k(a) = 0$, so every $Ja = 0$. Property (a) now gives $a \perp F_k(z)$. The case $k = 0$ was already established. This proves the assertion for all k. □

Lemma 6.4.38. *Let $n \ge 1$, let $\varepsilon_a \in \{1, -1\}$, and let $l \in \mathbb{Z}^n$ have pairwise distinct entries with $l_a \equiv n - 1 \pmod 2$. Let H be a complex pre-Hilbert space and let $\rho : \mathfrak{gl}_n(\mathbb{C}) \to \operatorname{End}_{\mathbb{C}}(H)$ be a Lie algebra representation. Write $T_{ab} = \rho(E_{ab})$, and suppose*

$$\langle T_{ab}u, v\rangle = \varepsilon_a\varepsilon_b\langle u, T_{ba}v\rangle, \qquad \operatorname{Re}\langle u, \Omega u\rangle \ge \kappa_l\|u\|^2, \quad \Omega = \sum_{a,b} T_{ab}T_{ba}, \quad \kappa_l = \frac{1}{4}\sum_a\bigl(l_a^2 - (2\rho_l(a))^2\bigr)$$

for all $u, v \in H$. Let $x \in H$ satisfy

$$T_{aa}x = \nu_a x, \qquad T_{ab}x = 0 \quad (\varepsilon_a = \varepsilon_b,\ l_a > l_b), \qquad \nu \in \mathbb{Z}^n.$$

Let F_0 be the smallest subspace containing x and invariant under all T_{ab} with $\varepsilon_a = \varepsilon_b$, and define

$$F_{s+1} = F_s + \sum_{a,b} T_{ab}F_s \qquad (s \ge 0).$$

For every integer $R \ge 0$ with $|\nu_a| \le R$, every $s \ge 0$, every $u \in F_s$, and every complex $n \times n$ matrix Z, the space F_s is finite-dimensional, $\rho(Z)u \in F_{s+1}$, and

$$\|\rho(Z)u\| \le (n+1)^4(R + 1 + s)\|Z\|_{\mathrm{op}}\|u\|.$$

Proof. **Compact words give finite-dimensional filtration spaces.** Put $\mathfrak{k}_{\mathbb{C}} = \operatorname{span}\{E_{ab} : \varepsilon_a = \varepsilon_b\}$ and

$$\Omega_K = \sum_{\varepsilon_a=\varepsilon_b} T_{ab}T_{ba}, \qquad c_K(\mu) = \sum_a \mu_a^2 + \sum_{\substack{\varepsilon_a=\varepsilon_b\\ l_a>l_b}}(\mu_a - \mu_b).$$

The commutator relations $[T_{ab}, T_{cd}] = \delta_{bc}T_{ad} - \delta_{da}T_{cb}$ show that Ω_K commutes with $\mathfrak{k}_{\mathbb{C}}$. Pairing the two orders of each off-diagonal compact pair, and using the highest-vector equations, gives $\Omega_K x = c_K(\nu)x$. Thus $\Omega_K = c_K(\nu)$ on F_0. Its quadratic form is

$$\langle v, \Omega_K v\rangle = \sum_{\varepsilon_a=\varepsilon_b}\|T_{ba}v\|^2. \tag{1}$$

If $x = 0$, all assertions are immediate; assume otherwise. In particular $c_K(\nu) \geq 0$.

Order the compact matrix units with the lowering units $l_a < l_b$ first, diagonal units next, and raising units last. Repeatedly use the displayed commutator relation to interchange adjacent units in a word. The interchange term has the same length and the commutator term has smaller length. Induction on length and then on the number of inversions therefore expresses F_0 as the span of compact lowering words applied to x: the raising factors kill x and diagonal factors act by scalars. Every word vector is a simultaneous diagonal eigenvector, of weight $\mu = \nu + \sum(e_a - e_b)$, by commuting each T_{cc} through the word. A compact lowering word of length q has

$$l \cdot \mu \leq l \cdot \nu - q.$$

For a nonzero such word vector v, (1) and its diagonal terms give $\sum_a \mu_a^2 \leq c_K(\nu)$. Consequently

$$q \leq l \cdot \nu + \|l\|_2 \sqrt{c_K(\nu)}.$$

There are only finitely many words of bounded length in the finitely many compact lowering units. This proves that F_0 is finite-dimensional. The recursion then proves the same for every F_s. The commutator relation also proves inductively that every F_s is $\mathfrak{k}_{\mathbb{C}}$-invariant.

The compact Casimir is bounded above on F_s. We first record how a highest vector in F_s detects a short word. The same word interchanges as above, now commuting compact units past arbitrary units, show that F_s is spanned by vectors

$$T_{a_1 b_1} \cdots T_{a_q b_q}\, \rho(w)x, \tag{2}$$

where all the displayed units are compact lowering units and w is a word of length at most s in arbitrary matrix units. Indeed a commutator of a compact unit with a matrix unit is a sum of matrix units, so it never increases the number of arbitrary factors; compact raising and diagonal factors can be moved to the right until they act on x. If a nonzero compact highest vector $z \in F_s$ were orthogonal to every $\rho(w)x$ with $\text{length}(w) \leq s$, it would be orthogonal to every vector (2): moving its leftmost lowering factor across the inner product produces a raising factor that kills z. This would give $z \perp F_s$, a contradiction. Hence some such word has $\langle \rho(w)x, z\rangle \neq 0$.

The diagonal operators are self-adjoint, so vectors of different diagonal weights are orthogonal. The weight μ of this highest vector therefore equals the weight of a detecting word. Each factor changes each weight coordinate by at most one, whence

$$|\mu_a| \leq R + s. \tag{3}$$

This conclusion applies to every compact highest vector in F_s.

The self-adjoint operator Ω_K preserves the finite-dimensional space F_s, and each of its eigenspaces is compact-invariant. Each nonzero eigenspace contains a compact highest vector: the commuting self-adjoint diagonal units admit a simultaneous orthogonal eigenspace decomposition; among its finitely many weights choose one maximizing $l \cdot \mu$. A nonzero raising image would have strictly larger pairing, which is impossible. On that highest vector the calculation preceding (1) gives the eigenvalue $c_K(\mu)$. Put $T = R + s$. By (3), and because each coordinate occurs with coefficient of absolute value at most n in the linear part of c_K,

$$c_K(\mu) \leq nT(T + n).$$

The spectral theorem on F_s now yields

$$\langle u, \Omega_K u\rangle \leq nT(T + n)\|u\|^2 \qquad (u \in F_s). \tag{4}$$

The full Casimir controls all matrix units. The adjoint identities give

$$\sum_{a,b} \|T_{ab}u\|^2 = 2\langle u, \Omega_K u\rangle - \mathrm{Re}\langle u, \Omega u\rangle.$$

Since $|2\rho_l(a)| \le n$, one has $\kappa_l \ge -n^3/4$. Therefore (4) gives

$$\sum_{a,b} \|T_{ab}u\|^2 \le \left(2nT(T+n) + \frac{n^3}{4}\right)\|u\|^2. \tag{5}$$

Each entry satisfies $|Z_{ab}| \le \|Z\|_{\mathrm{op}}$. The triangle inequality and Cauchy–Schwarz over the n^2 pairs imply

$$\|\rho(Z)u\|^2 \le n^2\|Z\|_{\mathrm{op}}^2 \sum_{a,b} \|T_{ab}u\|^2.$$

For $T \ge 0$ and $n \ge 1$,

$$n^2\left(2nT(T+n) + \frac{n^3}{4}\right) \le \left(2n^4 + \frac{n^5}{4}\right)(T+1)^2 \le 3n^5(T+1)^2 \le (n+1)^8(T+1)^2.$$

Combining this with (5) and taking square roots proves the claimed norm bound. Finally $\rho(Z) = \sum_{a,b} Z_{ab}T_{ab}$, so its image of F_s lies in F_{s+1} by the defining recursion. □

Lemma 6.4.39. *Let $n \ge 1$, let $\varepsilon \in \{\pm 1\}^n$, and let $l \in \mathbb{Z}^n$ have large gap A for some integer $A \ge 0$. Put $J = \mathrm{diag}(\varepsilon)$, $G = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^*Jg = J\}$ and*

$$\kappa_l = \frac{1}{4}\sum_a (l_a^2 - (2\rho_l(a))^2).$$

Let H be a complex pre-Hilbert space, with inner product conjugate-linear in the first variable, and let $\rho : \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(H)$ be a Lie algebra representation. Write $T_{ab} = \rho(E_{ab})$ and $\Omega = \sum_{a,b} T_{ab}T_{ba}$. Assume that for all $u, w \in H$,

$$\langle T_{ab}u, w\rangle = \varepsilon_a\varepsilon_b\langle u, T_{ba}w\rangle, \qquad \mathrm{Re}\langle u, \Omega u\rangle \ge \kappa_l\|u\|^2.$$

Let $x \in H$ and $\nu \in \mathbb{Z}^n$ satisfy

$$T_{aa}x = \nu_a x, \qquad T_{ab}x = 0 \quad (\varepsilon_a = \varepsilon_b,\ l_a > l_b).$$

Let $\mathfrak{k}_{\mathbb{C}}$ be spanned by the E_{ab} with $\varepsilon_a = \varepsilon_b$, set $F_0 = U(\mathfrak{k}_{\mathbb{C}})x$, and recursively set

$$F_{s+1} = F_s + \sum_{a,b} T_{ab}F_s \qquad (s \ge 0).$$

Suppose that $C_g : H \to \mathbb{C}$ is complex linear for every $g \in G$, and that there is one $D \ge 0$ such that

$$|C_g(u)| \le D\|u\| \qquad (g \in G,\ u \in H).$$

Assume the derivative identity

$$\frac{d}{dt}C_{g\exp(tY)}(u) = C_{g\exp(tY)}(\rho(Y)u)$$

for every $g \in G$, $u \in H$, $Y \in \mathfrak{u}(J) = \{Y : Y^*J + JY = 0\}$ and every real t. If $C_1(u) = 0$ for every $s \ge 0$ and $u \in F_s$, then $C_g(x) = 0$ for every $g \in G$.

Proof. **Repeated derivatives have a common radius of control.** Choose an integer $R \geq 0$ with $|\nu_a| \leq R$ for every a. The large-gap hypotheses in particular imply that the entries of l are distinct and have parity $n-1$. Thus Lemma 6.4.38 applies to the given module and filtration. For a fixed $Y \in \mathfrak{u}(J)$ put

$$a = (n+1)^4 \|Y\|_{\mathrm{op}}, \qquad b = \max\{a, 1\}.$$

It follows by induction that, for $u \in F_s$ and $k \geq 0$,

$$\rho(Y)^k u \in F_{s+k}, \qquad \|\rho(Y)^k u\| \leq a^k \prod_{j=0}^{k-1} (R+1+s+j)\, \|u\|. \tag{1}$$

For $k = 0$ the product and a^0 mean 1; the induction step applies the growth estimate on F_{s+k}.

Fix $g \in G$ and $u \in H$, and write $f_u(t) = C_{g\exp(tY)}(u)$. The derivative assumption gives

$$f_u^{(k)}(t) = C_{g\exp(tY)}(\rho(Y)^k u) \qquad (k \geq 0). \tag{2}$$

Indeed each right-hand side is differentiable, by applying the same assumption to the vector $\rho(Y)^k u$. It is therefore continuous, and induction proves both that f_u is smooth and that (2) holds. This argument uses only derivatives of scalar functions; it requires no completion of H.

Suppose that C_g vanishes on every F_s. For $u \in F_s$, all the derivatives (2) at zero vanish, by (1). Taylor's integral remainder on the real interval from zero to t, applied to the real and imaginary parts, gives for every integer $k \geq 1$

$$\begin{aligned} |C_{g\exp(tY)}(u)| &\leq \frac{|t|^k}{k!} \sup_{r \text{ between } 0 \text{ and } t} |C_{g\exp(rY)}(\rho(Y)^k u)| \\ &\leq D\|u\|\, (a|t|)^k \frac{\prod_{j=0}^{k-1}(R+1+s+j)}{k!}. \end{aligned} \tag{3}$$

The first inequality also follows directly by taking the absolute value in the complex-valued integral remainder; thus no extra factor is needed. If $a = 0$, (1) makes the right-hand side zero. The assertion at $t = 0$ is immediate. If $a > 0$ and $0 < |t| \leq 1/(2b)$, the ratio of successive nonnegative factors on the last line of (3) is

$$a|t|\, \frac{R+1+s+k}{k+1} \;\longrightarrow\; a|t| \leq \frac{1}{2}.$$

Consequently that line tends to zero as $k \to \infty$. We have proved

$$C_g|_{F_s} = 0 \text{ for all } s \quad \Longrightarrow \quad C_{g\exp(tY)}|_{F_s} = 0 \text{ for all } s \quad \left(|t| \leq \frac{1}{2b}\right). \tag{4}$$

Although the bound in (3) depends on s, the permitted radius in (4) does not.

Vanishing propagates along a full one-parameter subgroup. Choose an integer $N \geq 2b$. Apply (4) successively with base points $g\exp(jY/N)$, for $j = 0, \ldots, N-1$, and $t = 1/N$. The uniform bound by D is valid at all these base points, and the derivative identity is assumed there as well. The one-parameter subgroup law gives

$$C_g|_{F_s} = 0 \text{ for all } s \quad \Longrightarrow \quad C_{g\exp Y}|_{F_s} = 0 \text{ for all } s. \tag{5}$$

Every group element is a product of two such exponentials. Let $g \in G$, put $Q = g^* g$, and let $h = Q^{1/2}$ be its positive Hermitian square root. The relation $g^* J g = J$ gives $JQJ = Q^{-1}$. By diagonalizing the positive Hermitian matrix Q, functional calculus gives

$$JhJ = h^{-1}, \qquad J(\log h)J = -\log h.$$

Hence $Y_2 = \log h$ satisfies $Y_2^* J + J Y_2 = 0$. Put $k = g h^{-1}$. Direct calculation shows $k^* k = 1$ and $k^* J k = J$, so k commutes with J. Thus k is block unitary for the positive and negative eigenspaces of J. Diagonalize its two unitary blocks, choose real arguments of their eigenvalues, and obtain a block skew-Hermitian matrix Y_1 with $\exp Y_1 = k$. It commutes with J, so $Y_1 \in \mathfrak{u}(J)$ as well. Therefore

$$g = \exp Y_1 \exp Y_2.$$

Starting with the assumed vanishing of C_1 on every F_s, apply (5) first with Y_1 and then with Y_2. It follows that C_g vanishes on every F_s. Finally $x \in F_0$, and hence $C_g(x) = 0$, as required. □

Lemma 6.4.40. *Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma, \gamma' \in \mathbb{Z}$, $\lambda\colon I \to \mathbb{Z}$, $\lambda'\colon I' \to \mathbb{Z}$, and let H be a joint harmonic for the signs (s,t), the weights (γ, γ') and the parameters (λ, λ'). Let $E^{\mathrm{row}}_{aa'}$ and $E^{\mathrm{col}}_{cc'}$ be the row and column units of (s,t) with the shifts $(\gamma, -\gamma')$, acting on $\mathbb{C}[X] = \mathbb{C}[X_{ac} : a \in I', c \in I]$, and let e_j denote the indicator function of an index j. Let $G \subseteq \mathbb{C}[X]$ be a linear subspace containing H.*

(JR) Suppose that $E^{\mathrm{col}}_{cc'} G \subseteq G$ for all $c, c' \in I$, and that

- *(JR1) $E^{\mathrm{row}}_{ba} P \in G$ whenever $t(a) \neq t(b)$, $\lambda'(a) < \lambda'(b)$ and P is K-highest of weight $\mu_{\lambda'} + e_a - e_b$ for λ' for the row units;*
- *(JR2) $E^{\mathrm{row}}_{ba} P \in G$ for every polynomial P whenever $t(a) = t(b)$ and $\lambda'(b) < \lambda'(a)$;*
- *(JR3) $E^{\mathrm{row}}_{aa} P - \mu_{\lambda'}(a) P \in G$ for every polynomial P and every $a \in I'$.*

Then $G = \mathbb{C}[X]$.

(JC) Suppose that $E^{\mathrm{row}}_{aa'} G \subseteq G$ for all $a, a' \in I'$, and that

- *(JC1) $E^{\mathrm{col}}_{dc} P \in G$ whenever $s(c) \neq s(d)$, $-\lambda(c) < -\lambda(d)$ and P is K-highest of weight $\mu_{-\lambda} + e_c - e_d$ for $-\lambda$ for the column units;*
- *(JC2) $E^{\mathrm{col}}_{dc} P \in G$ for every polynomial P whenever $s(c) = s(d)$ and $-\lambda(d) < -\lambda(c)$;*
- *(JC3) $E^{\mathrm{col}}_{cc} P - \mu_{-\lambda}(c) P \in G$ for every polynomial P and every $c \in I$.*

Then $G = \mathbb{C}[X]$.

Here $\mu_{\lambda'}$ is the lowest K-type of λ' for the signs t and $\mu_{-\lambda}$ that of $-\lambda$ for the signs s.

Proof. **Proof of (JR).** Every polynomial has a total degree, so it suffices to show $\mathrm{Pol}_{\le j} \subseteq G$ for every $j \in \mathbb{Z}$, by induction on j; for $j < 0$, $\mathrm{Pol}_{\le j} = 0$. Let $j \ge 0$, assume $\mathrm{Pol}_{\le j-1} \subseteq G$, and suppose $\mathrm{Pol}_{\le j} \not\subseteq G$. Put $W = G^{\perp_j}$; by (i) of Lemma 6.4.41, $W \neq 0$, W is finite-dimensional, and $G \cap W = 0$.

Step 1: W contains a nonzero Q with $E^{\mathrm{col}}_{cc'} Q = 0$ whenever $s(c) = s(c')$ and $-\lambda(c') < -\lambda(c)$. Let $s(c) = s(d)$. By (a) of Lemma 6.4.27, $\check{E}^{\mathrm{row}}_{cd} G = E^{\mathrm{col}}_{dc} G \subseteq G$, so by (ii) of Lemma 6.4.41 for the transposed setting (with $N = G$) $\check{E}^{\mathrm{row}}_{dc} W \subseteq W$, i.e. $E^{\mathrm{col}}_{cd} W \subseteq W$. By (iv) of Lemma 6.4.41 for the transposed setting and the order λ on its rows I, there are $Q \in W \setminus \{0\}$ and ν with Q K-highest of weight ν for λ for $(\check{E}^{\mathrm{row}}_{cc'})$; by (c) of Lemma 6.4.27, Q is K-highest of weight $-\nu$ for $-\lambda$ for $(E^{\mathrm{col}}_{cc'})$, which contains the claim.

Step 2: Q is K-highest of weight $\mu_{\lambda'}$ for λ' for the row units. Let $a \in I'$. By (JR3), $E^{\mathrm{row}}_{aa} P = (E^{\mathrm{row}}_{aa} P - \mu_{\lambda'}(a) P) + \mu_{\lambda'}(a) P \in G$ for $P \in G$, so $E^{\mathrm{row}}_{aa} G \subseteq G$ and, by (ii) of Lemma 6.4.41 with $N = G$, $E^{\mathrm{row}}_{aa} W \subseteq W$. Hence $E^{\mathrm{row}}_{aa} Q - \mu_{\lambda'}(a) Q$ lies in W, and in G by (JR3), so it is 0. Let $t(a) = t(b)$ and $\lambda'(b) < \lambda'(a)$. By (JR2), E^{row}_{ba} maps $\mathbb{C}[X]$ into G, so by (ii) of Lemma 6.4.41 with

$N = \mathbb{C}[X]$, $E^{\mathrm{row}}_{ab}Q$ lies in $\mathbb{C}[X]^{\perp j}$, which is 0 (an element of it lies in $\mathrm{Pol}_{\leq j}$ and is orthogonal to itself). So $E^{\mathrm{row}}_{ab}Q = 0$.

Step 3: the column Laplacians kill Q. Let $s(c) \neq s(d)$. By (b) of Lemma 6.4.27, $\Delta^{\mathrm{col}}_{cd}Q = \check{\Delta}^{\mathrm{row}}_{dc}Q$. By (iii) of Lemma 6.4.41 for the transposed setting (rows d, c of opposite signs, $P = Q \in G^{\perp j}$), $R = \check{\Delta}^{\mathrm{row}}_{dc}Q$ lies in G, and $\check{E}^{\mathrm{row}}_{cd}R = -E^{\mathrm{col}}_{dc}R \in G$ by stability; hence $R = 0$.

Step 4: the row Laplacians kill Q. By (a) of (HV7) the pair (λ, λ') satisfies the block rule, so the shifted entries $\tilde{\lambda}'$, the matched pairs and the integers $\nu(a)$ of Lemma 6.4.20 are defined, and λ' is injective. By Step 2 and Lemma 6.4.42 (order λ', weight $\mu_{\lambda'}$) it suffices to show: if $t(a) \neq t(b)$ and $R = \Delta^{\mathrm{row}}_{ab}Q$ is K-highest of weight $\mu_{\lambda'} + e_a - e_b$ for λ', then $R = 0$. As $a \neq b$, either $\lambda'(a) < \lambda'(b)$ or $\lambda'(b) < \lambda'(a)$.

If $\lambda'(a) < \lambda'(b)$: $R \in G$ by (iii) of Lemma 6.4.41, and $E^{\mathrm{row}}_{ba}R \in G$ by (JR1); so $R = 0$ by (iii) of Lemma 6.4.41.

If $\lambda'(b) < \lambda'(a)$: first, $\tilde{\lambda}'(a) > N - N'$ or $-\tilde{\lambda}'(b) > N - N'$. Indeed, by (a) of Lemma 6.4.20, either a is matched with no column, and then $|\tilde{\lambda}'(a)| < N' - N$ gives $\tilde{\lambda}'(a) > N - N'$; or a is matched with a column c and $\sigma(a,c)\tilde{\lambda}'(a) > |N' - N|$: if $\sigma(a,c) = 1$, $\tilde{\lambda}'(a) > N - N'$; if $\sigma(a,c) = -1$, then $-\tilde{\lambda}'(b) > -\tilde{\lambda}'(a) > |N' - N| \geq N - N'$, because $\tilde{\lambda}' - \lambda'$ is constant. Next apply Lemma 6.4.43 with $(R, C, \tau, \kappa, l) = (I', I, t, s, \lambda')$ and $Y = X$, so $\theta = \sigma$ and $\mathrm{Pl}^q_{x \leftarrow x'} = \sum_{c:\, \sigma(x',c)=q} X_{xc}\partial_{x'c}$. Its hypothesis (i) is $\Delta^{\mathrm{col}}_{cc'}Q = 0$ ($s(c) \neq s(c')$), which is Step 3. For $x \neq x'$ with $t(x) = t(x')$ one has $E^{\mathrm{row}}_{xx'} = \mathrm{Pl}^+_{x \leftarrow x'} - \mathrm{Pl}^-_{x' \leftarrow x}$ (formula of (ii) of Lemma 6.4.41, the variables X_{xc}, $X_{x'c}$ being distinct), so its hypothesis (ii) follows from Step 2. Moreover $E^{\mathrm{row}}_{xx} = \mathrm{Pl}^+_{x \leftarrow x} - \mathrm{Pl}^-_{x \leftarrow x} + \gamma - \#\{c : \sigma(x,c) = -1\}$, so $E^{\mathrm{row}}_{xx}Q = \mu_{\lambda'}(x)Q$ gives $(\mathrm{Pl}^+_{x \leftarrow x} - \mathrm{Pl}^-_{x \leftarrow x})Q = \nu(x)Q$ with $\nu(x) = \mu_{\lambda'}(x) - \gamma + \#\{c : \sigma(x,c) = -1\}$ as in Lemma 6.4.20. For $u \in \{\pm 1\}$ and $x \in I'$ with $u\tilde{\lambda}'(x) > N - N'$, (c) of Lemma 6.4.20 gives $u\nu(x) \geq 0$, and (b) of Lemma 6.4.43 gives $\partial_{xc}Q = 0$ for every c with $\sigma(x,c) = -u$. If $\tilde{\lambda}'(a) > N - N'$, take $(x,u) = (a,1)$: $\partial_{ac}Q = 0$ whenever $\sigma(a,c) = -1$, so every term $\partial_{bc}\partial_{ac}Q$ of $R = \sum_{c:\, \sigma(a,c)=-1} \partial_{ac}\partial_{bc}Q$ vanishes. If $-\tilde{\lambda}'(b) > N - N'$, take $(x,u) = (b,-1)$: $\partial_{bc}Q = 0$ whenever $\sigma(b,c) = 1$, which holds for every c in the sum, as $\sigma(b,c) = -\sigma(a,c)$. In both cases $R = 0$.

Step 5: contradiction. By Steps 3 and 4, Q is Laplace harmonic for (s,t); by Step 2 it is K-highest of weight $\mu_{\lambda'}$ for λ' for the row units; by Step 1, $E^{\mathrm{col}}_{cc'}Q = 0$ whenever $s(c) = s(c')$ and $-\lambda(c') < -\lambda(c)$. By Lemma 6.4.44, $Q = aH$ for some $a \in \mathbb{C}$, so $Q \in G$. As $Q \in W$ and $G \cap W = 0$, $Q = 0$, contradicting Step 1. Hence $\mathrm{Pol}_{\leq j} \subseteq G$, and (JR) follows.

Proof of (JC). By (d) of Lemma 6.4.27, H is a joint harmonic for the signs (t,s), the weights (γ', γ) and the parameters (λ', λ), written in the variables Y_{ca}; its row units are the $\check{E}^{\mathrm{row}}_{cc'}$, its column units the $\check{E}^{\mathrm{col}}_{aa'}$, its row parameter is λ on I with the signs s, and its lowest K-type there is μ_λ, the lowest K-type of λ for the signs s. Moreover $\mu_\lambda = -\mu_{-\lambda}$: the polynomial $H \neq 0$ is K-highest of weight $\mu_{-\lambda}$ for $(E^{\mathrm{col}}_{cc'})$, hence of weight $-\mu_{-\lambda}$ for $(\check{E}^{\mathrm{row}}_{cc'})$ by (c) of Lemma 6.4.27, and of weight μ_λ for $(\check{E}^{\mathrm{row}}_{cc'})$ as a joint harmonic of the transposed data; compare the eigenvalues of $\check{E}^{\mathrm{row}}_{cc}$ on H. We check the hypotheses of (JR) for these data and G, using (a) and (c) of Lemma 6.4.27. Stability: $\check{E}^{\mathrm{col}}_{aa'}G = E^{\mathrm{row}}_{a'a}G \subseteq G$. (JR1): let $s(c) \neq s(d)$, $\lambda(c) < \lambda(d)$, and P be K-highest of weight $\mu_\lambda + e_c - e_d$ for λ for $(\check{E}^{\mathrm{row}})$; then P is K-highest of weight $\mu_{-\lambda} + e_d - e_c$ for $-\lambda$ for (E^{col}), and $-\lambda(d) < -\lambda(c)$, so (JC1) with (c,d) exchanged gives $E^{\mathrm{col}}_{cd}P \in G$, i.e. $\check{E}^{\mathrm{row}}_{dc}P \in G$. (JR2): let $s(c) = s(d)$ and $\lambda(d) < \lambda(c)$; then $-\lambda(c) < -\lambda(d)$, and (JC2) with (c,d) exchanged gives $E^{\mathrm{col}}_{cd}P \in G$, i.e. $\check{E}^{\mathrm{row}}_{dc}P \in G$, for every P. (JR3): $\check{E}^{\mathrm{row}}_{cc}P - \mu_\lambda(c)P = -(E^{\mathrm{col}}_{cc}P - \mu_{-\lambda}(c)P) \in G$ by (JC3). So (JR) for the transposed data gives $G = \mathbb{C}[Y] = \mathbb{C}[X]$. □

Lemma 6.4.41. *Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma, \gamma'' \in \mathbb{Z}$, and let $E^{\mathrm{row}}_{aa'}$ $(a, a' \in I')$ be the row units of (s,t) with the shifts (γ, γ'') and $\Delta^{\mathrm{row}}_{ab}$ $(t(a) \neq t(b))$ the row Laplacians, operators on $\mathbb{C}[X] = \mathbb{C}[X_{ac} : a \in I', c \in I]$. For $p = \sum_m p_m X^m$ and $q = \sum_m q_m X^m$*

(multi-indices $m \in \mathbb{N}^{I' \times I}$, $m! = \prod_{a,c} m_{ac}!$) let $\langle p, q\rangle_{\mathrm{Fi}} = \sum_m m! \overline{p_m}\, q_m$ be the Fischer inner product. For $j \in \mathbb{Z}$ let $\mathrm{Pol}_{\le j}$ be the space of polynomials of total degree at most j ($\mathrm{Pol}_{\le j} = 0$ for $j < 0$), and for a subspace $G \subseteq \mathbb{C}[X]$ let $G^{\perp j} = \{p \in \mathrm{Pol}_{\le j} : \langle q, p\rangle_{\mathrm{Fi}} = 0$ for all $q \in G \cap \mathrm{Pol}_{\le j}\}$. Then:

(i) $\mathrm{Pol}_{\le j} = (G \cap \mathrm{Pol}_{\le j}) \oplus G^{\perp j}$ *for every subspace G and every j; in particular $G \cap G^{\perp j} = 0$, and $G^{\perp j} \neq 0$ if $\mathrm{Pol}_{\le j} \not\subseteq G$;*

(ii) if $t(a) = t(a')$, then $E^{\mathrm{row}}_{aa'}$ maps $\mathrm{Pol}_{\le j}$ into itself and $\langle q, E^{\mathrm{row}}_{aa'} p\rangle_{\mathrm{Fi}} = \langle E^{\mathrm{row}}_{a'a} q, p\rangle_{\mathrm{Fi}}$ for all p, q; consequently, for subspaces G, N with $E^{\mathrm{row}}_{a'a} N \subseteq G$, the operator $E^{\mathrm{row}}_{aa'}$ maps $G^{\perp j}$ into $N^{\perp j}$;

(iii) let $j \in \mathbb{Z}$, let G be a subspace containing $\mathrm{Pol}_{\le j-1}$, $P \in G^{\perp j}$, $a, b \in I'$ with $t(a) \neq t(b)$, and $Q = \Delta^{\mathrm{row}}_{ab} P$; then $Q \in G$, and if $E^{\mathrm{row}}_{ba} Q \in G$, then $Q = 0$;

(iv) if $W \neq 0$ is a finite-dimensional subspace of $\mathbb{C}[X]$ with $E^{\mathrm{row}}_{aa'} W \subseteq W$ whenever $t(a) = t(a')$, then for every map $o\colon I' \to \mathbb{Z}$ there are $x \in W \setminus \{0\}$ and $\nu\colon I' \to \mathbb{Z}$ such that x is K-highest of weight ν for o for the row units $(E^{\mathrm{row}}_{aa'})$.

Proof. **Step 0: the Fischer product.** $\langle\,,\rangle_{\mathrm{Fi}}$ is conjugate linear in the first and linear in the second variable, hermitian and positive definite, and distinct monomials are orthogonal. For every variable X_i,

$$\langle X_i p, q\rangle_{\mathrm{Fi}} = \langle p, \partial_i q\rangle_{\mathrm{Fi}}, \qquad \langle p, X_i q\rangle_{\mathrm{Fi}} = \langle \partial_i p, q\rangle_{\mathrm{Fi}}.$$

(i). $\mathrm{Pol}_{\le j}$ is finite-dimensional and the restriction of $\langle\,,\rangle_{\mathrm{Fi}}$ to it is positive definite, so $\mathrm{Pol}_{\le j}$ is the orthogonal direct sum of its subspace $G \cap \mathrm{Pol}_{\le j}$ and the orthogonal complement of that subspace in $\mathrm{Pol}_{\le j}$, which is $G^{\perp j}$. An element of $G \cap G^{\perp j}$ is orthogonal to itself, hence 0; and if $G^{\perp j} = 0$, then $\mathrm{Pol}_{\le j} = G \cap \mathrm{Pol}_{\le j} \subseteq G$.

(ii). Let $t(a) = t(a')$, so $\sigma(a, c) = \sigma(a', c)$ for every c. By (HV5),

$$E^{\mathrm{row}}_{aa'} = \sum_{c\colon \sigma(a,c)=1} X_{ac}\partial_{a'c} - \sum_{c\colon \sigma(a,c)=-1} \partial_{ac} X_{a'c} + \gamma\delta_{aa'}.$$

Each term maps a homogeneous polynomial to a homogeneous polynomial of the same degree, so $\mathrm{Pol}_{\le j}$ is preserved. By Step 0, $\langle q, X_{ac}\partial_{a'c} p\rangle_{\mathrm{Fi}} = \langle \partial_{ac} q, \partial_{a'c} p\rangle_{\mathrm{Fi}} = \langle X_{a'c}\partial_{ac} q, p\rangle_{\mathrm{Fi}}$ and $\langle q, \partial_{ac} X_{a'c} p\rangle_{\mathrm{Fi}} = \langle X_{ac} q, X_{a'c} p\rangle_{\mathrm{Fi}} = \langle \partial_{a'c} X_{ac} q, p\rangle_{\mathrm{Fi}}$; as γ is real and the formula for $E^{\mathrm{row}}_{a'a}$ is the displayed one with a, a' exchanged, $\langle q, E^{\mathrm{row}}_{aa'} p\rangle_{\mathrm{Fi}} = \langle E^{\mathrm{row}}_{a'a} q, p\rangle_{\mathrm{Fi}}$. Now let $E^{\mathrm{row}}_{a'a} N \subseteq G$ and $y \in G^{\perp j}$. Then $E^{\mathrm{row}}_{aa'} y \in \mathrm{Pol}_{\le j}$, and for $z \in N \cap \mathrm{Pol}_{\le j}$ the vector $E^{\mathrm{row}}_{a'a} z$ lies in $G \cap \mathrm{Pol}_{\le j}$, so $\langle z, E^{\mathrm{row}}_{aa'} y\rangle_{\mathrm{Fi}} = \langle E^{\mathrm{row}}_{a'a} z, y\rangle_{\mathrm{Fi}} = 0$. Hence $E^{\mathrm{row}}_{aa'} y \in N^{\perp j}$.

(iii). Let $t(a) \neq t(b)$, so $\sigma(b, c) = -\sigma(a, c)$. By (HV5) and (HV6),

$$E^{\mathrm{row}}_{ba} = \sum_{c\colon \sigma(a,c)=-1} X_{ac}X_{bc} - \sum_{c\colon \sigma(a,c)=1} \partial_{ac}\partial_{bc}, \qquad Q = \Delta^{\mathrm{row}}_{ab} P = \sum_{c\colon \sigma(a,c)=-1} \partial_{ac}\partial_{bc} P.$$

As $P \in \mathrm{Pol}_{\le j}$, $Q \in \mathrm{Pol}_{\le j-2} \subseteq G$, and $\sum_{c\colon \sigma(a,c)=1} \partial_{ac}\partial_{bc} Q \in \mathrm{Pol}_{\le j-4} \subseteq G$. Suppose $E^{\mathrm{row}}_{ba} Q \in G$. Then

$$A = \sum_{c\colon \sigma(a,c)=-1} X_{ac}X_{bc} Q = E^{\mathrm{row}}_{ba} Q + \sum_{c\colon \sigma(a,c)=1} \partial_{ac}\partial_{bc} Q$$

lies in $G \cap \mathrm{Pol}_{\le j}$, so $\langle A, P\rangle_{\mathrm{Fi}} = 0$ because $P \in G^{\perp j}$. By Step 0,

$$\langle A, P\rangle_{\mathrm{Fi}} = \sum_{c\colon \sigma(a,c)=-1} \langle Q, \partial_{bc}\partial_{ac} P\rangle_{\mathrm{Fi}} = \langle Q, Q\rangle_{\mathrm{Fi}}.$$

Hence $Q = 0$.

(iv). For a multi-index m define $w_m\colon I' \to \mathbb{Z}$ by

$$w_m(a) = \sum_{c\colon \sigma(a,c)=1} m_{ac} - \sum_{c\colon \sigma(a,c)=-1} (m_{ac}+1) + \gamma,$$

and let M_w be the span of the monomials X^m with $w_m = w$; then $\mathbb{C}[X] = \bigoplus_w M_w$. By the formula of (ii) with $a' = a$, using $\partial_{ac}X_{ac} = X_{ac}\partial_{ac} + 1$, E^{row}_{aa} acts on M_w as the scalar $w(a)$. Let $t(a) = t(a')$ and $a \neq a'$. Each term $X_{ac}\partial_{a'c}$ ($\sigma(a,c) = 1$) and $-\partial_{ac}X_{a'c} = -X_{a'c}\partial_{ac}$ ($\sigma(a,c) = -1$) of $E^{\mathrm{row}}_{aa'}$ sends X^m to a multiple of $X^{m+e_{(a,c)}-e_{(a',c)}}$, respectively $X^{m+e_{(a',c)}-e_{(a,c)}}$, whose weight is $w_m + e_a - e_{a'}$ in both cases (a variable of sign 1 moves from row a' to row a, or one of sign -1 from row a to row a'). So $E^{\mathrm{row}}_{aa'}M_w \subseteq M_{w+e_a-e_{a'}}$.

Let W be as in (iv). Then $W = \bigoplus_w (W \cap M_w)$: if $x \in W$ has the nonzero components x_w for w in a finite set S, and $w_0 \in S$, choose for every $w \in S \smallsetminus \{w_0\}$ some a_w with $w(a_w) \neq w_0(a_w)$; the operator $\prod_{w\neq w_0}(E^{\mathrm{row}}_{a_wa_w} - w(a_w))/(w_0(a_w) - w(a_w))$ maps W into W, kills x_w for $w \neq w_0$ and fixes x_{w_0}, so $x_{w_0} \in W$. Since $W \neq 0$ is finite-dimensional, the set S_W of weights w with $W \cap M_w \neq 0$ is finite and nonempty. Choose $\nu \in S_W$ with $\phi(\nu) = \sum_a o(a)\nu(a)$ maximal and $x \in W \cap M_\nu$, $x \neq 0$. Then $E^{\mathrm{row}}_{aa}x = \nu(a)x$ for every a. If $t(a) = t(a')$ and $o(a') < o(a)$, then $E^{\mathrm{row}}_{aa'}x \in W \cap M_{\nu+e_a-e_{a'}}$ and $\phi(\nu + e_a - e_{a'}) = \phi(\nu) + o(a) - o(a') > \phi(\nu)$, so $\nu + e_a - e_{a'} \notin S_W$ and $E^{\mathrm{row}}_{aa'}x = 0$. So x is K-highest of weight ν for o. □

Lemma 6.4.42. *Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma, \gamma'' \in \mathbb{Z}$, and let $E^{\mathrm{row}}_{aa'}$ be the row units of (s,t) with the shifts (γ,γ'') and $\Delta^{\mathrm{row}}_{ab}$ ($t(a) \neq t(b)$) the row Laplacians. Let $o, \nu\colon I' \to \mathbb{Z}$, for $a \in I'$ let $e_a\colon I' \to \mathbb{Z}$ be its indicator function, and let $P \in \mathbb{C}[X_{ac} : a \in I',\, c \in I]$ be K-highest of weight ν for o for the row units. Suppose that for all $a, b \in I'$ with $t(a) \neq t(b)$, if $\Delta^{\mathrm{row}}_{ab}P$ is K-highest of weight $\nu + e_a - e_b$ for o for the row units, then $\Delta^{\mathrm{row}}_{ab}P = 0$. Then $\Delta^{\mathrm{row}}_{ab}P = 0$ for all $a, b \in I'$ with $t(a) \neq t(b)$.*

Proof. **Step 1: commutators.** The raising operators commute with each other, the lowering operators commute with each other, and $[\mathrm{lo}_{ac}, \mathrm{ra}_{a'c'}] = \delta_{aa'}\delta_{cc'}$. For $t(a) \neq t(b)$ and c with $\sigma(a,c) = -1$ one has $\sigma(b,c) = 1$, $\partial_{ac} = -\mathrm{ra}_{ac}$ and $\partial_{bc} = \mathrm{lo}_{bc}$, so

$$\Delta^{\mathrm{row}}_{ab} = -\sum_{c\colon \sigma(a,c)=-1} \mathrm{ra}_{ac}\,\mathrm{lo}_{bc}.$$

For operators with $[A,C] = [B,D] = 0$ one has $[AB,CD] = A[B,C]D + C[A,D]B$. Applied to $A = \mathrm{ra}_{xc}$, $B = \mathrm{lo}_{zc}$, $C = \mathrm{ra}_{ac'}$, $D = \mathrm{lo}_{bc'}$ it gives $[\mathrm{ra}_{xc}\mathrm{lo}_{zc}, \mathrm{ra}_{ac'}\mathrm{lo}_{bc'}] = \delta_{za}\delta_{cc'}\,\mathrm{ra}_{xc}\mathrm{lo}_{bc'} - \delta_{xb}\delta_{cc'}\,\mathrm{ra}_{ac'}\mathrm{lo}_{zc}$. Let $t(x) = t(z)$ and $t(a) \neq t(b)$. Summing over c and over c' with $\sigma(a,c') = -1$ (the scalar term of E^{row}_{xz} is central),

$$[E^{\mathrm{row}}_{xz}, \Delta^{\mathrm{row}}_{ab}] = -\delta_{za}\sum_{c\colon \sigma(a,c)=-1} \mathrm{ra}_{xc}\mathrm{lo}_{bc} + \delta_{xb}\sum_{c\colon \sigma(a,c)=-1} \mathrm{ra}_{ac}\mathrm{lo}_{zc}.$$

If $z = a$, then $t(x) = t(a)$, so for these c, $\mathrm{ra}_{xc} = -\partial_{xc}$ and $\mathrm{lo}_{bc} = \partial_{bc}$, and the first sum is $-\Delta^{\mathrm{row}}_{xb}$. If $x = b$, then $t(z) = t(b)$, so $\mathrm{lo}_{zc} = \partial_{zc}$, $\mathrm{ra}_{ac} = -\partial_{ac}$, and the second sum is $-\Delta^{\mathrm{row}}_{az}$. Hence

$$E^{\mathrm{row}}_{xz}\Delta^{\mathrm{row}}_{ab} = \Delta^{\mathrm{row}}_{ab}E^{\mathrm{row}}_{xz} + \delta_{za}\Delta^{\mathrm{row}}_{xb} - \delta_{xb}\Delta^{\mathrm{row}}_{az} \qquad (t(x) = t(z),\ t(a) \neq t(b)), \tag{6.76}$$

where $\Delta^{\mathrm{row}}_{xb}$ and $\Delta^{\mathrm{row}}_{az}$ occur only when $t(x) = t(a) \neq t(b)$, respectively $t(a) \neq t(b) = t(z)$, so they are defined.

Step 2: a minimal nonvanishing Laplacian is K-highest. Suppose some $\Delta^{\mathrm{row}}_{ab}P$ with $t(a) \neq t(b)$ is nonzero. Among the finitely many such pairs choose one with $o(b) - o(a)$ minimal;

then $\Delta^{\mathrm{row}}_{a'b'}P = 0$ whenever $t(a') \neq t(b')$ and $o(b') - o(a') < o(b) - o(a)$. Put $Q = \Delta^{\mathrm{row}}_{ab}P$. For $x \in I'$, (6.76) with $z = x$ and $E^{\mathrm{row}}_{xx}P = \nu(x)P$ gives $E^{\mathrm{row}}_{xx}Q = \nu(x)Q + \delta_{xa}\Delta^{\mathrm{row}}_{ab}P - \delta_{xb}\Delta^{\mathrm{row}}_{ab}P = (\nu + e_a - e_b)(x)\,Q$. Let $t(x) = t(z)$ and $o(z) < o(x)$. Then $E^{\mathrm{row}}_{xz}P = 0$, and (6.76) gives $E^{\mathrm{row}}_{xz}Q = \delta_{za}\Delta^{\mathrm{row}}_{xb}P - \delta_{xb}\Delta^{\mathrm{row}}_{az}P$. If $z = a$, then $o(b) - o(x) < o(b) - o(a)$, so $\Delta^{\mathrm{row}}_{xb}P = 0$; if $x = b$, then $o(z) - o(a) < o(b) - o(a)$, so $\Delta^{\mathrm{row}}_{az}P = 0$. Hence $E^{\mathrm{row}}_{xz}Q = 0$, and Q is K-highest of weight $\nu + e_a - e_b$ for o. By hypothesis $Q = 0$, a contradiction. So all $\Delta^{\mathrm{row}}_{ab}P$ vanish. □

Lemma 6.4.43. *Let R, C be finite sets with signs $\tau\colon R \to \{\pm1\}$, $\kappa\colon C \to \{\pm1\}$, let Y_{xy} ($x \in R$, $y \in C$) be variables, $\theta(x,y) = \tau(x)\kappa(y)$, $\partial_{xy} = \partial/\partial Y_{xy}$, and $l\colon R \to \mathbb{Z}$ injective. For $x, x' \in R$ with $\tau(x) = \tau(x')$ and $q \in \{\pm1\}$ put $\mathrm{Pl}^q_{x\leftarrow x'} = \sum_{y\in C:\ \theta(x',y)=q} Y_{xy}\partial_{x'y}$, an operator on $\mathbb{C}[Y]$. Let $F \in \mathbb{C}[Y]$ satisfy*

(i) $\sum_{x''\in R:\ \theta(x'',y)=1} \partial_{x''y}\partial_{x''y'}F = 0$ *whenever* $\kappa(y) \neq \kappa(y')$;

(ii) $\mathrm{Pl}^+_{x\leftarrow x'}F = \mathrm{Pl}^-_{x'\leftarrow x}F$ *whenever* $\tau(x) = \tau(x')$ *and* $l(x') < l(x)$.

Then:

(a) $\mathrm{Pl}^q_{x\leftarrow x'}F = 0$ *whenever* $\tau(x) = \tau(x')$ *and* $q\,(l(x) - l(x')) > 0$;

(b) if $x \in R$, $k \in \mathbb{Z}$ and $u \in \{\pm1\}$ satisfy $(\mathrm{Pl}^+_{x\leftarrow x} - \mathrm{Pl}^-_{x\leftarrow x})F = kF$ and $uk \geq 0$, then $\partial_{xy}F = 0$ for every y with $\theta(x,y) = -u$, and $\mathrm{Pl}^u_{x\leftarrow x}F = uk\,F$.

Proof. **The Fischer product.** For $p = \sum_m p_m Y^m$, $p' = \sum_m p'_m Y^m$ (multi-indices m, $m! = \prod m_i!$) put $\langle p, p'\rangle = \sum_m m!\,\overline{p_m}\,p'_m$, a positive definite hermitian form with $\langle Y_i p, p'\rangle = \langle p, \partial_i p'\rangle$ and $\langle p, Y_i p'\rangle = \langle \partial_i p, p'\rangle$.

Orthogonality. Let $\tau(x) = \tau(x')$, so $\theta(x,\cdot) = \theta(x',\cdot)$. Put $u_x(y,y') = \partial_{xy}\partial_{xy'}F$ and $M(q,x,x') = \sum_{y'\colon \theta(x,y')=-q}\sum_{y\colon \theta(x,y)=q}\langle u_x(y,y'), u_{x'}(y,y')\rangle$. In each term $y \neq y'$, so $\partial_{x'y'}(Y_{xy}\,\cdot) = Y_{xy}\partial_{x'y'}$, and the adjointness gives

$$\langle \mathrm{Pl}^{-q}_{x'\leftarrow x}F, \mathrm{Pl}^q_{x\leftarrow x'}F\rangle = \sum_{y',y}\langle \partial_{xy'}F,\ Y_{xy}\partial_{x'y'}\partial_{x'y}F\rangle = M(q,x,x').$$

As $u_x(y,y') = u_x(y',y)$, $M(-q,x,x') = M(q,x,x')$. Each $M(1,x,x')$ is real and ≥ 0: if $l(x') < l(x)$ it is $\|\mathrm{Pl}^+_{x\leftarrow x'}F\|^2$ by (ii); if $x' = x$ it is a sum of squared norms; if $l(x) < l(x')$ it is $M(-1,x,x') = \langle \mathrm{Pl}^+_{x'\leftarrow x}F, \mathrm{Pl}^-_{x\leftarrow x'}F\rangle = \|\mathrm{Pl}^-_{x\leftarrow x'}F\|^2$ by (ii) for (x',x). The x' with $\tau(x') = \tau(x)$ are those with $\theta(x',y) = 1$ for the y with $\theta(x,y) = 1$, so

$$\sum_{x'\colon \tau(x')=\tau(x)} M(1,x,x') = \sum_{y'\colon \theta(x,y')=-1}\ \sum_{y\colon \theta(x,y)=1}\Big\langle u_x(y,y'),\ \sum_{x''\colon \theta(x'',y)=1}\partial_{x''y}\partial_{x''y'}F\Big\rangle = 0$$

by (i), as $\kappa(y) \neq \kappa(y')$. Hence all $M(1,x,x')$ vanish, and

$$\langle \mathrm{Pl}^{-q}_{x'\leftarrow x}F, \mathrm{Pl}^q_{x\leftarrow x'}F\rangle = 0 \qquad (\tau(x) = \tau(x'),\ q = \pm1). \tag{6.77}$$

(a). By (ii), for (x,x') if $q = 1$ and for (x',x) if $q = -1$, $\mathrm{Pl}^q_{x\leftarrow x'}F = \mathrm{Pl}^{-q}_{x'\leftarrow x}F$; so $\|\mathrm{Pl}^q_{x\leftarrow x'}F\|^2 = 0$ by (6.77).

(b). Put $D^q = \mathrm{Pl}^q_{x\leftarrow x} = \sum_{y\colon \theta(x,y)=q} Y_{xy}\partial_{xy}$. The hypothesis gives $D^uF = D^{-u}F + ukF$, and (6.77) with $x' = x$, $q = u$ gives

$$0 = \langle D^{-u}F, D^uF\rangle = \|D^{-u}F\|^2 + uk\,\langle D^{-u}F, F\rangle, \qquad \langle D^{-u}F, F\rangle = \sum_{y\colon \theta(x,y)=-u}\|\partial_{xy}F\|^2 \geq 0.$$

Both terms are ≥ 0, so $D^{-u}F = 0$; then $\sum_{y\colon \theta(x,y)=-u}\|\partial_{xy}F\|^2 = \langle D^{-u}F, F\rangle = 0$, and $D^uF = ukF$. □

Lemma 6.4.44. *Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma, \gamma' \in \mathbb{Z}$, $\lambda\colon I \to \mathbb{Z}$, $\lambda'\colon I' \to \mathbb{Z}$, and let Q be a joint harmonic for the signs (s,t), the weights (γ,γ') and the parameters (λ,λ'). Let P be a polynomial in the variables X_{ac} ($a \in I'$, $c \in I$) such that P is Laplace harmonic for (s,t) ((HV6)), P is K-highest of weight $\mu_{\lambda'}$ for λ' for the row units of (s,t) with the shifts $(\gamma,-\gamma')$, and $C_{cc'}P = 0$ whenever $s(c) = s(c')$ and $-\lambda(c') < -\lambda(c)$, ($C_{cc'}$) the column units with these shifts. Then $P = aQ$ for some $a \in \mathbb{C}$.*

Proof. **Step 1: a linear class containing P and Q.** For a row a and $q \in \{\pm 1\}$ let $D_a^q = \sum_{c\colon \sigma(a,c)=q} X_{ac}\partial_{ac}$. By (HV5), $E_{aa}^{\mathrm{row}} = D_a^+ - D_a^- + \gamma - \#\{c : \sigma(a,c) = -1\}$, and for $a \neq b$ with $t(a) = t(b)$ and $c \neq d$ with $s(c) = s(d)$,

$$E_{ab}^{\mathrm{row}} = \sum_{c\colon \sigma(a,c)=1} X_{ac}\partial_{bc} - \sum_{c\colon \sigma(a,c)=-1} X_{bc}\partial_{ac}, \qquad E_{cd}^{\mathrm{col}} = -\sum_{a\colon \sigma(a,c)=1} X_{ad}\partial_{ac} + \sum_{a\colon \sigma(a,c)=-1} X_{ac}\partial_{ad}.$$

With ν as in Lemma 6.4.20, let $\mathcal{K}$ be the set of polynomials F with

(K1) $\Delta_{ab}^{\mathrm{row}}F = 0$ ($t(a) \neq t(b)$) and $\Delta_{cd}^{\mathrm{col}}F = 0$ ($s(c) \neq s(d)$);

(K2) $(D_a^+ - D_a^-)F = \nu(a)F$ for every row a;

(K3) $E_{ab}^{\mathrm{row}}F = 0$ whenever $t(a) = t(b)$ and $\lambda'(b) < \lambda'(a)$;

(K4) $E_{cd}^{\mathrm{col}}F = 0$ whenever $s(c) = s(d)$ and $\lambda(c) < \lambda(d)$.

$\mathcal{K}$ is a vector space; (K2) is $E_{aa}^{\mathrm{row}}F = \mu_{\lambda'}(a)F$ by the formula above, so $P \in \mathcal{K}$. Also $Q \in \mathcal{K}$: it is jointly highest of weights $(\mu_{\lambda'}, \mu_{-\lambda})$ for $(\lambda', -\lambda)$ ((b) of (HV7)), and since it is termwise harmonic every term $\partial_{ac}\partial_{bc}Q$ ($t(a) \neq t(b)$) and $\partial_{ac}\partial_{ad}Q$ ($s(c) \neq s(d)$) of the Laplacians vanishes. By (a) of the block rule, λ and λ' are injective.

Step 2: the rows. Let $F \in \mathcal{K}$. Apply Lemma 6.4.43 with $(R, C, \tau, \kappa, l) = (I', I, t, s, \lambda')$ and $Y = X$, so $\theta = \sigma$ and $\mathrm{Pl}_{a\leftarrow b}^q = \sum_{c\colon \sigma(b,c)=q} X_{ac}\partial_{bc}$: its (i) is $\Delta_{cc'}^{\mathrm{col}}F = 0$ (K1), and its (ii) is (K3), as $E_{ab}^{\mathrm{row}} = \mathrm{Pl}_{a\leftarrow b}^+ - \mathrm{Pl}_{b\leftarrow a}^-$ by Step 1. With (K2) ($D_a^q = \mathrm{Pl}_{a\leftarrow a}^q$, $k = \nu(a)$) it gives:

(R1) if $t(a) = t(b)$ and $q(\lambda'(a) - \lambda'(b)) > 0$, then $\mathrm{Pl}_{a\leftarrow b}^q F = 0$;

(R2) if $u \in \{\pm 1\}$ and $u\nu(a) \geq 0$, then $\partial_{ac}F = 0$ whenever $\sigma(a,c) = -u$, and $D_a^u F = u\nu(a)F$.

Step 3: the columns. Apply Lemma 6.4.43 with $(R, C, \tau, \kappa, l) = (I, I', s, t, \lambda)$ and $Y_{cb} = X_{bc}$, so $\theta(c,b) = \sigma(b,c)$ and $\mathrm{Pl}_{c\leftarrow c'}^q = \sum_{b\colon \sigma(b,c')=q} X_{bc}\partial_{bc'}$: its (i) reads $\sum_{c\colon \sigma(b,c)=1} \partial_{bc}\partial_{b'c}F = 0$ for $t(b) \neq t(b')$, which is $\Delta_{b'b}^{\mathrm{row}}F = 0$ (K1), since $\sigma(b', c) = -1$ for these c; its (ii) is (K4), as $E_{c'c}^{\mathrm{col}} = -\mathrm{Pl}_{c\leftarrow c'}^+ + \mathrm{Pl}_{c'\leftarrow c}^-$ by Step 1. By (a) there:

(C1) if $s(c) = s(c')$ and $q(\lambda(c) - \lambda(c')) > 0$, then $\sum_{b\colon \sigma(b,c')=q} X_{bc}\partial_{bc'}F = 0$.

Step 4: a nonzero element of $\mathcal{K}$ does not vanish at the matching matrix. Let x be the matrix with $x_{ac} = 1$ if (a,c) is matched and 0 otherwise; each row and each column of x has at most one entry 1 ((a) of Lemma 6.4.20). Let $F \in \mathcal{K}$, $F \neq 0$, and suppose $F(x) = 0$. Let $G(Y) = F(x + Y)$, G_d its nonzero homogeneous component of least degree ($d \geq 1$), and Y_{rc} a variable occurring in a monomial of G_d with nonzero coefficient, so $\partial G_d/\partial Y_{rc} \neq 0$; let $\varsigma = \sigma(r,c)$. A contradiction follows if (A) $\partial_{rc}F = 0$, or (B) there are an operator $T = \sum_j X_{\alpha_j}\partial_{\beta_j}$ and a constant k with $(T-k)F = 0$ and $\sum_j x_{\alpha_j}\partial_{\beta_j} = \partial_{rc}$: in the variables Y, $T - k = \partial/\partial Y_{rc} + \sum_j Y_{\alpha_j}\partial/\partial Y_{\beta_j} - k$, and the component of degree $d-1$ of $(T-k)G = 0$ is $\partial G_d/\partial Y_{rc} = 0$. By (a) of Lemma 6.4.20, either r is matched with no column and $|\tilde{\lambda}'(r)| < N' - N$, or r is matched with a column d_r, $p = \sigma(r, d_r) = \operatorname{sgn}\tilde{\lambda}'(r)$ and $|\tilde{\lambda}'(r)| > |N' - N|$.

Case 1: r unmatched, or $\varsigma = -p$. Then $-\varsigma\tilde\lambda'(r) > N - N'$, so $-\varsigma\nu(r) \geq 0$ by (c) of Lemma 6.4.20, and (R2) with $u = -\varsigma$ gives $\partial_{rc}F = 0$: (A).

Case 2: $\varsigma = p$ and $d_r = c$. Then $\varsigma\tilde\lambda'(r) > N - N'$, so $\varsigma\nu(r) \geq 0$, and (R2) with $u = \varsigma$ gives $(D_r^\varsigma - \varsigma\nu(r))F = 0$; row r of x has its only 1 in column c, which occurs in D_r^ς: (B).

Case 3: $\varsigma = p$, $d_r \neq c$, $\varsigma(\lambda(d_r) - \lambda(c)) > 0$. Here $s(d_r) = s(c)$. By (C1) with $(c, c') = (d_r, c)$ and $q = \varsigma$, $T = \sum_{b:\,\sigma(b,c)=\varsigma} X_{bd_r}\partial_{bc}$ kills F; column d_r of x has its only 1 in row r, which occurs in T: (B).

Case 4: $\varsigma = p$, $d_r \neq c$, $\varsigma(\lambda(c) - \lambda(d_r)) > 0$. Then $\varsigma\tilde\lambda(c) > \varsigma\tilde\lambda(d_r) = \varsigma\tilde\lambda'(r) > |N' - N|$; a column matched with no row has $|\tilde\lambda| < N - N' \leq |N' - N|$, so c is matched with a row $a \neq r$, with $\sigma(a, c) = \operatorname{sgn}\tilde\lambda(c) = \varsigma$, hence $t(a) = t(r)$, and $\varsigma(\lambda'(a) - \lambda'(r)) = \varsigma(\tilde\lambda(c) - \tilde\lambda(d_r)) > 0$. By (R1) with $(a, b) = (a, r)$ and $q = \varsigma$, $T = \mathrm{Pl}^\varsigma_{a\leftarrow r} = \sum_{c':\,\sigma(r,c')=\varsigma} X_{ac'}\partial_{rc'}$ kills F; row a of x has its only 1 in column c, which occurs in T: (B).

So $F(x) \neq 0$.

Step 5: conclusion. $Q \neq 0$ lies in $\mathcal{K}$, so $Q(x) \neq 0$; with $a = P(x)/Q(x)$, $P - aQ \in \mathcal{K}$ vanishes at x, hence $P = aQ$ by Step 4. □

Lemma 6.4.45. *Let S, T, V be spaces with points with signs s on n, s' on n', η on ι, and let (ω, θ) be a theta kernel for (S, T) on V. There are $C \geq 0$ and $A \in \mathbb{N}$ such that*

$$|\theta(\omega(g, h)\varphi)| \leq C\,\|\varphi\|\,(\mathrm{ht}_S(g)\,\mathrm{ht}_T(h))^A \qquad (g \in \mathrm{G}(s),\ h \in \mathrm{G}(s'),\ \varphi \in \mathcal{M}_\eta).$$

Proof. **Operators on a compact set.** Let $\Omega_S \subseteq \mathrm{G}(s)$ and $\Omega_T \subseteq \mathrm{G}(s')$ be the compact sets of Lemma 6.2.29 for S and T. The set $B_S = \{(x, 1) : x \in G_\infty(s),\ \|x_v\| \leq 2\ (v \in \Phi)\}$ is compact: $\mathrm{U}(\operatorname{diag} s)$ is closed in $\mathrm{M}_n(\mathbb{C})$, and on it $\|x^{-1}\| = \|\operatorname{diag}(s)x^*\operatorname{diag}(s)\| = \|x\|$, so $\{x \in \mathrm{U}(\operatorname{diag} s) : \|x\| \leq 2\}$ is a compact subset of $\mathrm{GL}_n(\mathbb{C})$; likewise $B_T \subseteq \mathrm{G}(s')$. Let $\mathcal{K} = (\Omega_S \cup B_S \cup \{1\}) \times (\Omega_T \cup B_T \cup \{1\})$, a compact subset of $\mathrm{G}(s) \times \mathrm{G}(s')$. For each $\varphi \in \mathcal{M}_\eta$ the map $k \mapsto \omega(k)\varphi$ is continuous ((ED3)), hence bounded on $\mathcal{K}$. The space $\mathcal{M}_\eta$ is a Banach space, a closed subspace of $L^1(\mu)$ by (1) of Lemma 4.1.10, and every $\omega(k)$ is a bounded operator; so the uniform boundedness principle gives $C_0 = \max(1, \sup_{k\in\mathcal{K}}\|\omega(k)\|) < \infty$. Choose $A \in \mathbb{N}$ with $2^A \geq C_0$.

The bound. Let $g \in \mathrm{G}(s)$ and $h \in \mathrm{G}(s')$. By Lemma 6.2.29, $g = \gamma z(x_1 \cdots x_k, 1)$ and $h = \delta z'(y_1 \cdots y_l, 1)$ with $\gamma \in \Gamma_S$, $\delta \in \Gamma_T$, $z \in \Omega_S$, $z' \in \Omega_T$, all $(x_i, 1) \in B_S$ and $(y_j, 1) \in B_T$, $2^k \leq \mathrm{ht}_S(g)$ and $2^l \leq \mathrm{ht}_T(h)$. The invariance of (ED3), applied to $\gamma^{-1}g$ and $\delta^{-1}h$, gives $\theta(\omega(g, h)\varphi) = \theta(\omega(\gamma^{-1}g, \delta^{-1}h)\varphi)$. As ω is a homomorphism on the product group,

$$\omega(\gamma^{-1}g, \delta^{-1}h) = \omega(z, 1)\prod_{i=1}^{k}\omega((x_i, 1), 1)\cdot\omega(1, z')\prod_{j=1}^{l}\omega(1, (y_j, 1)),$$

a product of $k + l + 2$ operators $\omega(\kappa)$ with $\kappa \in \mathcal{K}$. Hence

$$|\theta(\omega(g, h)\varphi)| \leq \|\theta\|\,2^{A(k+l+2)}\|\varphi\| \leq 4^A\|\theta\|\,(\mathrm{ht}_S(g)\,\mathrm{ht}_T(h))^A\|\varphi\|,$$

and $C = 4^A\|\theta\|$ works. □

Lemma 6.4.46. *Let V be a space with a point with signs s on n, e a cusp kernel of order 0 on $G_\infty(s)$, $K \subseteq G_f(n)$ an open subgroup, and F a continuous function on $\mathrm{G}(s)$, left invariant under Γ_V and right invariant under K, such that $x \mapsto e(x)F(g(x,1))$ is integrable on $G_\infty(s)$ for every $g \in \mathrm{G}(s)$. Then $R(e)F$ is a form of type e on V, right invariant under K.*

Proof. By (VP3) of Definition 3.2.2 V is a framed hermitian space U (with E^n as underlying space, the columns of $(x_V)_v$ as frames at $v \in \Phi$ and the finite isometry $z \mapsto x_{V,f}z$; a framed space of rank $\#n$ in the sense of (FT1) of Definition 3.2.4), with the signs s at every infinite place, $G^U = \mathrm{G}(s) = G_\infty(s) \times G_f(n)$ and $\Gamma_U = \Gamma_V$; so Lemma 5.8.29 applies to V. For $t > 0$ let $F_t = F \cdot \mathbf{1}_{\{|F| \le t\}}$. It is measurable ($F$ is continuous), bounded by t, left invariant under Γ_V and right invariant under K, as $|F|$ is. By Lemma 5.8.29, $R(e)F_t$ is defined everywhere, continuous, bounded, left invariant under Γ_V, right invariant under K, and $R(e)R(e)F_t = R(e)F_t$; since e is integrable, $R(e)F_t$ is a form of type e on V, right invariant under K ((ED2)). Let $\mathcal{W}$ be the space of these forms; it is finite-dimensional by Lemma 6.4.64.

Fix $g \in \mathrm{G}(s)$. For every x, $|e(x)F_t(g(x,1))| \le |e(x)F(g(x,1))|$, an integrable function of x by hypothesis, and $F_t(g(x,1)) \to F(g(x,1))$ as $t \to \infty$. By dominated convergence $R(e)F_t(g) \to R(e)F(g)$. So $R(e)F$ is the pointwise limit of elements of $\mathcal{W}$. In the space of all complex functions on $\mathrm{G}(s)$ with the topology of pointwise convergence, a Hausdorff topological vector space, the finite-dimensional subspace $\mathcal{W}$ is closed. Hence $R(e)F \in \mathcal{W}$. □

Lemma 6.4.47. *Let O be a descent pair with ranks $1 \le r < n$ that fits a parameter $l = (l_\tau)_{\tau\in\Phi}$, and let $\xi_O = v^+_{l_O}$. Then $\mathrm{Har}_O(\xi_O) = \prod_{\tau\in\Phi} P_\tau$, where each P_τ is a polynomial in the variables $X_{\tau,i,c}$ ($1 \le i \le n$, $1 \le c \le r$) that is a joint harmonic for the signs $(\varepsilon_O, \mathbf{1}_n)$, the weights $(r_-(O), \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau))$ and the parameters $(l_\tau, l_{O,\tau})$.*

Proof. Fix τ where it is not displayed; write $\kappa^\pm = \kappa^\pm_O$, $r_\pm = r_\pm(O)$, $\gamma = r_-$, $\gamma' = \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)$.

Step 1: the highest vector. $l_{O,\tau}$ is strictly decreasing (DP11), so $\#\{a' : l_{O,\tau}(a') > l_{O,\tau}(a)\} = a - 1$ and, by (WM6), $\xi_O = \mathrm{ev}_{\kappa_O}(y_{\mathrm{id}})$, y_{id} the array with entry 1 at row a, column $a-1$. As $\kappa_O(\tau,i) = \kappa^+(\tau,i) + \kappa^-(\tau,n-i)$ (DP3), (c) of Lemma 3.2.20 gives $C_O\xi_O = \mathrm{ev}_{\kappa^+}(y_{\mathrm{id}}) \otimes \mathrm{ev}_{\kappa^-}(y_{w_0})$, y_{w_0} with entry 1 at row a, column $n-a$. By (DP8),

$$\mathrm{Har}_O(\xi_O) = \mathrm{Let}_1\big(\mathrm{Pol}_{\kappa^+}(\mathrm{ev}_{\kappa^+}(y_{\mathrm{id}}))\big) \cdot \mathrm{Let}_{-1}\big(\mathrm{Pol}_{\kappa^-}(\mathrm{ev}_{\kappa^-}(y_{w_0}))\big).$$

Step 2: the polynomial of an evaluation vector at a permutation array. Let κ be a weight of rank n and, for every τ, π_τ a bijection of $\{1,\dots,n\}$ onto $\{0,\dots,n-1\}$; let y^π have entry 1 at $(\tau, a, \pi_\tau(a))$ and 0 elsewhere. By Lemma 6.4.50 for the rows $R = \{1,\dots,n\}$ in their natural order,

$$\mathrm{Pol}_\kappa(\mathrm{ev}_\kappa(y^\pi)) = \prod_\tau \prod_{1\le h<n} M_{\tau,h}^{\kappa(\tau,h)}, \qquad M_{\tau,h} = \det\big(y_{\tau,\pi_\tau^{-1}(i),j}\big)_{0\le i,j<h}.$$

Step 3: the staircase. By (ii) of (DP11) and the injectivity of l_τ ((a) of the block rule, (i) of (DP11)), ord_O restricted to the columns of sign σ is a bijection onto $\{0,\dots,r_\sigma - 1\}$; let $c_\sigma(j)$ be the column of sign σ with $\mathrm{ord}_O = j$. So $\mathrm{Let}_\sigma(y_{\tau,i,j}) = X_{\tau,i,c_\sigma(j)}$ for $j < r_\sigma$ and $= 0$ for $j \ge r_\sigma$. For y_{id}, $\pi_\tau(a) = a - 1$; for y_{w_0}, $\pi_\tau(a) = n - a$. By Steps 1 and 2 and since Let_σ is a homomorphism of algebras, $\mathrm{Har}_O(\xi_O) = \prod_\tau P_\tau$ with

$$P_\tau = \prod_{\eta=\pm1} \prod_{1\le p<n} D_{\tau,\eta,p}^{\kappa^\eta(\tau,p)}, \qquad D_{\tau,1,p} = \det(X_{\tau,i+1,c_1(j)})_{i,j<p}, \quad D_{\tau,-1,p} = \det(X_{\tau,n-i,c_{-1}(j)})_{i,j<p},$$

where for $p > r_\eta$ the factor is 1: then a column $j \ge r_\eta$ of the minor is mapped to 0, but $\kappa^\eta(\tau,p) = 0$ by (iii) of (DP11). This is the polynomial P of Lemma 6.4.48 for $R = \{1,\dots,n\}$,

$C = \{1, \dots, r\}$, $\sigma = \varepsilon_O$, $l' = l_{O,\tau}$, $l = l_\tau$, $k(\eta, p) = \kappa^\eta(\tau, p)$ and the shifts $(\gamma_1, \gamma_2) = (\gamma, -\gamma')$: as $l_{O,\tau}$ is strictly decreasing, $x_1(a) = a - 1$ and $x_{-1}(a) = n - a$, so the rows of $D_{\tau,\eta,p}$ are the a_i of that lemma; its o is ord_O by (ii) of (DP11); $k(\eta, p) = 0$ for $p > r_\eta$ by (iii); and if $i + j > n$ then, as $r_+ + r_- = r < n$, $i > r_+$ or $j > r_-$, so $k(1, i) = 0$ or $k(-1, j) = 0$. By Lemma 6.4.48, P_τ is jointly highest of weights (ν, w) for $(l_{O,\tau}, -l_\tau)$ for the row and column units of $(\varepsilon_O, \mathbf{1}_n)$ with the shifts $(\gamma, -\gamma')$, with ν, w as there.

Step 4: the row weight is the lowest K-type of $l_{O,\tau}$. With $\gamma = r_-$,

$$\nu(a) = \sum_{a-1<p<n} \kappa^+(\tau, p) - \sum_{n-a<p<n} \kappa^-(\tau, p).$$

For the signs $\mathbf{1}_n$, $2\rho^{\mathrm{c}} = 2\rho$ in Definition 3.2.3, and as $l_{O,\tau}$ is strictly decreasing, $2\rho_{l_{O,\tau}}(a) = (n - a) - (a - 1)$. With $l_{O,\tau}(a) = 2(e_O(\tau) + \lambda_{O,\tau}(a - 1)) + n + 1 - 2a$ this gives $l^\rho_a = e_O(\tau) + \lambda_{O,\tau}(a - 1) + n + 1 - 2a$ and $\mu_{l_{O,\tau}}(a) = e_O(\tau) + \lambda_{O,\tau}(a - 1)$. By (DP3) and the head of Definition 3.2.19,

$$\begin{aligned} e_O(\tau) + \lambda_{O,\tau}(a - 1) &= -\sum_{1\le i<n} \kappa^-(\tau, i) + \sum_{a-1<h<n} \big(\kappa^+(\tau, h) + \kappa^-(\tau, n - h)\big) \\ &= \sum_{a-1<h<n} \kappa^+(\tau, h) - \sum_{n-a<i<n} \kappa^-(\tau, i), \end{aligned}$$

as $\sum_{a-1<h<n} \kappa^-(\tau, n - h) = \sum_{1\le i\le n-a} \kappa^-(\tau, i)$. So $\nu = \mu_{l_{O,\tau}}$.

Step 5: the column weight, and conclusion. By (i) of (DP11), $(l_\tau, l_{O,\tau})$ satisfies the block rule for the weights (γ, γ') and the signs $(\varepsilon_O, \mathbf{1}_n)$, so by Lemma 6.4.19 there is a joint harmonic Q for these signs, weights and parameters. By Steps 3 and 4, P_τ is termwise harmonic, K-highest of weight $\mu_{l_{O,\tau}}$ for $l_{O,\tau}$ for the row units, and killed by the column raising operators for $-l_\tau$; by Lemma 6.4.49, $P_\tau = aQ$ with $a \in \mathbb{C}$, and $a \ne 0$ as $P_\tau \ne 0$. For each column c the column unit C_{cc} multiplies P_τ by $w(c)$ and Q by $\mu_{-l_\tau}(c)$ ((b) of (HV7) for Q), so $w = \mu_{-l_\tau}$. Hence P_τ is jointly highest of weights $(\mu_{l_{O,\tau}}, \mu_{-l_\tau})$ for $(l_{O,\tau}, -l_\tau)$ with the shifts $(\gamma, -\gamma')$, and with the block rule it is a joint harmonic for $(\varepsilon_O, \mathbf{1}_n)$, (γ, γ'), $(l_\tau, l_{O,\tau})$. □

Lemma 6.4.48. *Let R and C be nonempty finite sets, $N = \#R$, σ signs on C, $\mathbf{1}_R$ the constant sign $+1$ on R, and $\gamma_1, \gamma_2 \in \mathbb{Z}$; consider the polynomials in the variables X_{ac} ($a \in R$, $c \in C$) with the row and column units of the signs $(\sigma, \mathbf{1}_R)$ with the shifts (γ_1, γ_2). Let $l' \colon R \to \mathbb{Z}$ and $l \colon C \to \mathbb{Z}$ be injective; for $\eta = \pm 1$ and $a \in R$ put $x_\eta(a) = \#\{a' \in R : \eta l'(a') > \eta l'(a)\}$, and for $c \in C$ put $o(c) = \#\{d \in C : \sigma(d) = \sigma(c),\ \sigma(c)l(c) < \sigma(c)l(d)\}$. Let $k(\eta, p) \in \mathbb{N}$ ($\eta = \pm 1$, $1 \le p < N$) satisfy $k(\eta, p) = 0$ whenever $p > \#\{c : \sigma(c) = \eta\}$, and $k(1, i) = 0$ or $k(-1, j) = 0$ whenever $i + j > N$. For $p \le \#\{c : \sigma(c) = \eta\}$ let $D_{\eta,p} = \det(X_{a_i,c_j})_{0\le i,j<p}$, where $a_i \in R$ is the row with $x_\eta(a_i) = i$ and $c_j \in C$ the column with $\sigma(c_j) = \eta$ and $o(c_j) = j$, and let*

$$P = \prod_{\eta=\pm1} \prod_{1\le p<N} D_{\eta,p}^{k(\eta,p)}.$$

Then P is jointly highest of weights (ν, w) for $(l', -l)$, where

$$\begin{aligned} \nu(a) &= \gamma_1 - \#\{c : \sigma(c) = -1\} + \sum_{\eta=\pm1} \eta \sum_{x_\eta(a)<p<N} k(\eta, p), \\ w(c) &= \gamma_2 + N\,[\sigma(c) = -1] - \sigma(c) \sum_{o(c)<p<N} k(\sigma(c), p), \end{aligned}$$

$[\sigma(c) = -1]$ being 1 if $\sigma(c) = -1$ and 0 otherwise.

Proof. Write $C_\pm = \{c \in C : \sigma(c) = \pm 1\}$ and $\partial_{ac} = \partial/\partial X_{ac}$. A factor $D_{\eta,p}$ with $k(\eta,p) > 0$ is defined, as then $p \le \#C_\eta$; factors with $k(\eta,p) = 0$ are 1. For $D = D_{\eta,p}$ let $\mathrm{rows}(D) = \{a \in R : x_\eta(a) < p\}$ and $\mathrm{cols}(D) = \{c \in C_\eta : o(c) < p\}$, the rows and columns of the $p \times p$ matrix whose determinant is D. The maps x_η and $o|_{C_\eta}$ are bijections onto $\{0,\dots,N-1\}$ and $\{0,\dots,\#C_\eta - 1\}$, l' and l being injective.

Step 1: the units as differential operators. From (HV5), using $\partial_{ac} X_{a'c'} = X_{a'c'}\partial_{ac} + \delta_{aa'}\delta_{cc'}$: for $a, a' \in R$,

$$E^{\mathrm{row}}_{aa'} = \sum_{c\in C_+} X_{ac}\partial_{a'c} - \sum_{c\in C_-} X_{a'c}\partial_{ac} + (\gamma_1 - \#C_-)\,\delta_{aa'}, \tag{6.78}$$

and for $c, c' \in C$ of the same sign,

$$E^{\mathrm{col}}_{cc'} = -\sum_{a\in R} X_{ac'}\partial_{ac} + \gamma_2\,\delta_{cc'} \quad (c,c' \in C_+), \qquad E^{\mathrm{col}}_{cc'} = \sum_{a\in R} X_{ac}\partial_{ac'} + (\gamma_2 + N)\,\delta_{cc'} \quad (c,c' \in C_-). \tag{6.79}$$

In each case the operator minus its scalar part is a derivation δ of $\mathbb{C}[X]$, so

$$\delta(P) = P\sum_{\eta,p} k(\eta,p)\,\delta(D_{\eta,p})/D_{\eta,p}$$

in the field of fractions; it suffices to compute δ on the factors. Two facts on a determinant D of a matrix of distinct variables are used: $\sum_{c\in\mathrm{cols}(D)} X_{bc}\partial_{b'c} D$ for $b' \in \mathrm{rows}(D)$ is the determinant with row b' replaced by row b (in the columns of D), which is 0 if $b \in \mathrm{rows}(D)$, $b \ne b'$, and D if $b = b'$; it is 0 if $b' \notin \mathrm{rows}(D)$; and the same holds for columns. Variables outside the rows or columns of D do not occur in D.

Step 2: the weights. Let $a \in R$. By (6.78) with $a' = a$ and Step 1, $\sum_{c\in C_+} X_{ac}\partial_{ac} D_{+,p} = [x_+(a) < p]\,D_{+,p}$ and $\sum_{c\in C_-} X_{ac}\partial_{ac} D_{-,p} = [x_-(a) < p]\,D_{-,p}$, while the sum over the columns of the other sign kills the factor. Hence

$$E^{\mathrm{row}}_{aa} P = \Big(\gamma_1 - \#C_- + \sum_{x_+(a)<p<N} k(1,p) - \sum_{x_-(a)<p<N} k(-1,p)\Big)P = \nu(a)\,P.$$

Let $c \in C_+$. Only the factors $D_{+,p}$ involve column c, each of degree $[o(c) < p]$ in it, so $E^{\mathrm{col}}_{cc} P = (\gamma_2 - \sum_{o(c)<p<N} k(1,p))P = w(c)P$. For $c \in C_-$ likewise $E^{\mathrm{col}}_{cc} P = (\gamma_2 + N + \sum_{o(c)<p<N} k(-1,p))P = w(c)P$.

Step 3: the raising conditions. Let $a \ne a'$ in R with $l'(a') < l'(a)$; then $x_+(a) < x_+(a')$ and $x_-(a') < x_-(a)$. On $D_{+,p}$ the operator (6.78) acts as $\sum_{c\in C_+} X_{ac}\partial_{a'c}$, which is 0 unless $a' \in \mathrm{rows}(D_{+,p})$; but then $x_+(a) < x_+(a') < p$, so $a \in \mathrm{rows}(D_{+,p})$ and the result is 0 by Step 1. On $D_{-,p}$ it acts as $-\sum_{c\in C_-} X_{a'c}\partial_{ac}$, which is 0 unless $a \in \mathrm{rows}(D_{-,p})$, and then $x_-(a') < x_-(a) < p$ gives $a' \in \mathrm{rows}(D_{-,p})$ and 0 again. So $E^{\mathrm{row}}_{aa'} P = 0$, and P is K-highest of weight ν for l' for the row units (all rows have the sign $+1$). Let $c \ne c'$ in C with $\sigma(c) = \sigma(c')$ and $-l(c') < -l(c)$, i.e. $l(c) < l(c')$. If both lie in C_+, then $o(c') < o(c)$, and $-\sum_a X_{ac'}\partial_{ac}$ is 0 on every factor unless $c \in \mathrm{cols}(D_{+,p})$, in which case $o(c') < o(c) < p$ puts c' among the columns and the result is 0. If both lie in C_-, then $o(c) < o(c')$, and $\sum_a X_{ac}\partial_{ac'}$ is 0 unless $c' \in \mathrm{cols}(D_{-,p})$, in which case c is a column too and the result is 0. So P is K-highest of weight w for $-l$ for the column units.

Step 4: $P \ne 0$ and termwise harmonicity. Each factor is a nonzero polynomial, so $P \ne 0$. All rows have the sign $+1$, so the first condition of termwise harmonicity is empty. Let $a \in R$. If a is a row of a factor $D_{1,p}$ and of a factor $D_{-1,q}$ with $k(1,p) > 0$ and $k(-1,q) > 0$, then $p \ge x_+(a) + 1$ and $q \ge x_-(a) + 1$, and $x_+(a) + x_-(a) = N - 1$, so $p + q \ge N + 1 > N$, which

the hypothesis on k excludes. So the variables of row a occurring in P all have columns of one sign, every monomial of P has this property, and $\partial^2 P/\partial X_{ac}\partial X_{ac'} = 0$ whenever $\sigma(c) \neq \sigma(c')$.

By Steps 2 to 4, P is jointly highest of weights (ν, w) for $(l', -l)$ (HV7). □

Lemma 6.4.49. *Let I, I' be nonempty finite sets with signs s on I and t on I', let $\gamma, \gamma' \in \mathbb{Z}$, $\lambda\colon I \to \mathbb{Z}$, $\lambda'\colon I' \to \mathbb{Z}$, and let Q be a joint harmonic for the signs (s,t), the weights (γ, γ') and the parameters (λ, λ'). Let P be a polynomial in the variables X_{ac} ($a \in I'$, $c \in I$) that is termwise harmonic, K-highest of weight $\mu_{\lambda'}$ for λ' for the row units of (s,t) with the shifts $(\gamma, -\gamma')$, and that satisfies $C_{cc'}P = 0$ whenever $s(c) = s(c')$ and $-\lambda(c') < -\lambda(c)$, $(C_{cc'})$ the column units with these shifts. Then $P = aQ$ for some $a \in \mathbb{C}$.*

Proof. As P is termwise harmonic, $\partial^2 P/\partial X_{ac}\partial X_{bc} = 0$ whenever $t(a) \neq t(b)$ and $\partial^2 P/\partial X_{ac}\partial X_{ad} = 0$ whenever $s(c) \neq s(d)$; every term of $\Delta^{\mathrm{row}}_{ab}P$ ($t(a) \neq t(b)$) and of $\Delta^{\mathrm{col}}_{cd}P$ ($s(c) \neq s(d)$) is of this form, so these Laplacians kill P. The remaining hypotheses of Lemma 6.4.44 (row weight $\mu_{\lambda'}$, compact row and column raising operators) are those of the statement, so $P = aQ$ for some $a \in \mathbb{C}$ by that lemma. □

Lemma 6.4.50. *Let Φ be a finite set, R a totally ordered finite set of $N \geq 1$ rows (the order used for the minors of (WM1)), and κ a weight in rank N at every $\tau \in \Phi$ with rows R. For every $\tau \in \Phi$ let $\pi_\tau\colon R \to \{0, \dots, N-1\}$ be a bijection, and let y^π be the array with $y^\pi_{\tau,a,c} = 1$ if $c = \pi_\tau(a)$ and $y^\pi_{\tau,a,c} = 0$ otherwise. Then*

$$\mathrm{Pol}_\kappa(\mathrm{ev}_\kappa(y^\pi)) = \prod_{\tau\in\Phi}\prod_{1\leq h<N} M_{\tau,h}^{\kappa(\tau,h)}, \qquad M_{\tau,h} = \det\bigl(y_{\tau,\pi_\tau^{-1}(i),j}\bigr)_{0\leq i,j<h}.$$

Proof. By (WM4) and (WM3), and since $\Delta_T = \sum_i (F_\kappa)_{Ti} b_i$ as polynomials and hence also at y,

$$\mathrm{Pol}_\kappa(\mathrm{ev}_\kappa(y)) = \sum_{i,j} (F_\kappa^{\mathsf{T}} F_\kappa)_{ij}\, b_j(y)\, b_i = \sum_{T\in\mathcal{C}_\kappa} \Delta_T(y)\, \Delta_T$$

for every array y. Let $T \in \mathcal{C}_\kappa$ and let $T(\tau,h,j) = \{a_0 < \dots < a_{h-1}\}$ be one of its sets. The matrix $(y^\pi_{\tau,a_k,c})_{0\leq k,c<h}$ has in row k a single nonzero entry 1, in the column $\pi_\tau(a_k)$, if $\pi_\tau(a_k) < h$, and no nonzero entry otherwise. So it is a permutation matrix if $\pi_\tau(T(\tau,h,j)) = \{0,\dots,h-1\}$, and it has a zero row otherwise. Hence $\Delta_T(y^\pi) \neq 0$ only for the one filling with $T(\tau,h,j) = A_{\tau,h} = \pi_\tau^{-1}\{0,\dots,h-1\}$ for all (τ,h,j), and then $\Delta_{\tau,A_{\tau,h}}(y^\pi) = \epsilon_{\tau,h}$ is the sign of the permutation $k \mapsto \pi_\tau(a_k)$ of $\{0,\dots,h-1\}$. Therefore

$$\mathrm{Pol}_\kappa(\mathrm{ev}_\kappa(y^\pi)) = \prod_{\tau,h} \bigl(\epsilon_{\tau,h}\, \Delta_{\tau,A_{\tau,h}}\bigr)^{\kappa(\tau,h)},$$

and $\epsilon_{\tau,h}\Delta_{\tau,A_{\tau,h}} = M_{\tau,h}$: permuting the rows of the minor $\Delta_{\tau,A_{\tau,h}}$ from the increasing order $a_0 < \dots < a_{h-1}$ to the order $\pi_\tau^{-1}(0), \dots, \pi_\tau^{-1}(h-1)$ multiplies it by $\epsilon_{\tau,h}$. □

Lemma 6.4.51. *Let D be admissible data, $z \in \{\mathrm{I}, \mathrm{II}\}$, $i, n \geq 0$ with $i + n < m$, and ϑ an m-fold twist.*

*(a) For every function Ψ on $\mathrm{M}_{n_{n+1}\times n_n}(R_\vartheta)$, the **datum lift** $\mathrm{Tl}^i_{\vartheta,z,n}[\Psi] = L_\varphi$, for the theta kernel and the characters of the triangle lift at (i,n) and $\varphi = \mathrm{mod}_{\mathfrak{t}_\vartheta(z)}(\Psi, v_H)$, maps $\mathcal{F}^i_\triangle(z,n)$ to $\mathcal{F}^i_\triangle(z,n+1)$; $\mathrm{Tl}^i_{\vartheta,z,n}$ is the case of the point datum.*

(b) *For a left pair* Φ *at* (i,n) *and every function* Ψ *on* $\mathrm{M}_{n_{n+1}\times n_n}(R_\vartheta)$, $\mathrm{Ds}^i_{\vartheta,z,n}(\Phi,\Psi)$ *maps* $\mathcal{F}^i_\triangle(z,n+1)$ *to* $\mathcal{F}^{i+1}_\triangle(z,n)$*; for a reversed pair* Φ' *at* (i,n) *and every function* Ψ' *on* $\mathrm{M}_{n_n\times n_{n+1}}(R_\vartheta)$, $\mathrm{As}^i_{\vartheta,z,n}(\Phi',\Psi')$ *maps* $\mathcal{F}^{i+1}_\triangle(z,n)$ *to* $\mathcal{F}^i_\triangle(z,n+1)$.

(c) *All these maps are linear in the form and in the finite datum: for instance* $\mathrm{Ds}^i_{\vartheta,z,n}(\Phi,\Psi)=\sum_\eta \Psi(\eta)\,\mathrm{Ds}^i_{\vartheta,z,n}(\Phi,\mathbf{1}_\eta)$, $\mathbf{1}_\eta$ *the indicator function of* η.

(d) *For every function* Ψ *as in (a) and every* $x\in\mathcal{F}^i_\triangle(z,n+1)$ *there is* $x^\dagger\in\mathcal{F}^i_\triangle(z,n)$ *with* $\langle \mathrm{Tl}^i_{\vartheta,z,n}[\Psi]\,y,\ x\rangle=\langle y,\ x^\dagger\rangle$ *for every* $y\in\mathcal{F}^i_\triangle(z,n)$*, the Petersson products of* $V^i_{z,n+1}$ *and* $V^i_{z,n}$.

Proof. Write $\mathfrak{t}=\mathfrak{t}_\vartheta(z)$ and $P=\Sigma(\mathfrak{t})$, a nonempty set of places containing the conjugate of none of its members (the chosen places and, if $z=\mathrm{I}$, the places of Q: (DP1) of Definition 3.2.23 through (LF4) of Definition 3.2.26).

Step 1: the characters are trivial at integral points that are 1 modulo $\mathfrak{t}$. The characters involved are edge characters (GP6) with $c=\neg z$ and left-pair characters $\chi^{-1}\upsilon^N$ with χ an edge character (TV6). By Definition 3.2.26, μ^0_{l+1} has a tame part trivial off the chosen places, υ has trivial tame part, and ϑ^{l+1} (present only if $\neg z=\mathrm{II}$, i.e. $z=\mathrm{I}$) has a tame part trivial off Q; tame parts multiply under products and inverses. So each character α involved has a tame part ψ with $\psi_w=1$ for $w\notin P$. For an index set n' and $k\in\mathbf{K}_0(n')$ with $k\equiv\mathbf{1}\pmod{\mathfrak{t}}$, $u=\det k$ has infinite components 1, unit finite components and $u\bmod w=1$ for $w\in P$, so $\alpha(u)=1$ by the definition of weight and tame part. The group $K_\mathfrak{t}(n')$ of such k is open in $G_f(n')$ (it contains the product of the principal congruence subgroups at the places below P with the integral points elsewhere).

Step 2: the data are fixed by these groups. Let (S,T) be the pair of spaces of one of the maps, with index sets a, b and characters (α,β), and $\varphi=\mathrm{mod}_\mathfrak{t}(\Psi,\upsilon)$ its datum, υ the Fock vector of the polynomial. For $k\in K_\mathfrak{t}(a)$ and $k'\in K_\mathfrak{t}(b)$, (6) of Lemma 4.1.1 gives $\omega_{\alpha,\beta}((1,k),(1,k'))\varphi=\alpha(\det k')\beta(\det k)^{-1}\,\mathrm{mod}_\mathfrak{t}(\Psi_{k,k'},\upsilon)$ with $\Psi_{k,k'}(X)=\Psi(\bar{k}'^{-1}X\bar{k})=\Psi(X)$, the reductions modulo $\mathfrak{t}$ being $\mathbf{1}$; by Step 1, $\omega_{\alpha,\beta}((1,k),(1,k'))\varphi=\varphi$.

Step 3: the images have a level. For a lift L_φ and a form F on S, $h\in\mathrm{G}$ (the adelic group of T) and $k'\in K_\mathfrak{t}(b)$,

$$L_\varphi F(h(1,k'))=\int_{[\mathrm{U}(S)]}\theta\big(\omega_{\alpha,\beta}(g,h)\,\omega_{\alpha,\beta}(1,(1,k'))\varphi\big)F(g)\,dg=L_\varphi F(h)$$

by Step 2 with $k=\mathbf{1}$. For a descent or an ascent, F is a function on the adelic group of T and the map is $g\mapsto\int_{[\mathrm{U}(T)]}\theta(\omega_{\alpha,\beta}(g,h)\varphi)F(h)\,dh$; the same computation with $k\in K_\mathfrak{t}(a)$, $k'=\mathbf{1}$ shows that its value is right invariant under $K_\mathfrak{t}(a)$.

Step 4: the images are triangle forms. By Lemma 6.3.77, $(\omega_{\alpha,\beta},\theta_{\mathrm{Hom}(S,T)})$ is a theta kernel for (S,T) on $\mathrm{Hom}(S,T)$ with real part $\Omega_{\gamma,\gamma'}$, (γ,γ') the weights of (α,β); by Lemma 6.4.53 (exchanging the two factors), $((h,g)\mapsto\omega_{\alpha,\beta}(g,h),\theta_{\mathrm{Hom}(S,T)})$ is a theta kernel for (T,S) on the same space with real part $(y,x)\mapsto\Omega_{\gamma,\gamma'}(x,y)$, whose lift is the map

$$F\mapsto\big(g\mapsto\int_{[\mathrm{U}(T)]}\theta(\omega_{\alpha,\beta}(g,h)\varphi)F(h)\,dh\big).$$

(a): by Lemma 6.4.15(b), v_H is an edge for the weights of the characters of the triangle lift from e_λ to $e_{\lambda'}$, $\lambda=\frac12 p^i_{z,0,n}$, $\lambda'=\frac12 p^i_{z,0,n+1}$; by (ED4)(c) for its first half edge and Step 3, the image of a form of type e_λ is a form of type $e_{\lambda'}$. (b): for a left pair Φ, v_Φ is an edge for the weights

of $(\chi^{\mathrm{lp}}_{\vartheta,z,i}(N_n), \chi^{\mathrm{lp}}_{\vartheta,z,i}(N_{n+1}))$ from $e_{-\mu}$ to $e_{-\mu'}$, $\mu = \frac{1}{2}p^{i+1}_{z,0,n}$, $\mu' = \frac{1}{2}p^{i}_{z,0,n+1}$, with $e_{-\mu} = \overline{e_\mu}$ and $e_{-\mu'} = \overline{e_{\mu'}}$ (TV6); its second half edge is for $(y,x) \mapsto \Omega(x,y)$ from $\overline{e_{-\mu'}} = e_{\mu'}$ to $\overline{e_{-\mu}} = e_\mu$, so by (ED4)(c) for the exchanged kernel and Step 3 the descent maps forms of type $e_{\mu'}$ on $V^i_{z,n+1}$ to forms of type e_μ on $V^{i+1}_{z,n}$. For a reversed pair the same argument, with $v_{\Phi'}$ an edge from $e_{-\mu'}$ to $e_{-\mu}$ for the weights $(\delta_i(N_{n+1}), \delta_i(N_n))$, shows that the ascent maps forms of type e_μ on $V^{i+1}_{z,n}$ to forms of type $e_{\mu'}$ on $V^i_{z,n+1}$.

Step 5: (c). The map $\Psi \mapsto \Psi^\natural$ is linear, so $\Psi \mapsto \mathrm{mod}_{\mathfrak{t}}(\Psi, v) = v \otimes (\Psi^\natural \otimes_P v^0)$ is linear ((WR6) of Definition 3.2.18); θ and $\omega_{\alpha,\beta}(g,h)$ are linear, so the kernel of Ψ is $\sum_\eta \Psi(\eta)$ times the kernel of $\mathbf{1}_\eta$, pointwise; each summand, multiplied by a triangle form, is integrable by Lemma 6.4.52 (the types e_λ, $e_{\mu'}$, e_μ are cusp kernels of order 0 by (ED4)(a) of the edges of Step 4), so the integral of the sum is the sum of the integrals. Writing $\Psi = \sum_\eta \Psi(\eta)\mathbf{1}_\eta$ gives (c); linearity in the form is linearity of the integral.

Step 6: (d). Let $(S,T) = (V^i_{z,n}, V^i_{z,n+1})$, $\theta(g,h) = \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)$ the kernel of $\mathrm{Tl}^i_{\vartheta,z,n}[\Psi]$, and put $x^\dagger(g) = \int_{[\mathrm{U}(T)]} \overline{\theta(g,h)}\, x(h)\, dh$. By Lemma 6.4.52 the function $(g,h) \mapsto \theta(g,h)y(g)\overline{x(h)}$ is integrable, and Fubini gives $\langle \mathrm{Tl}^i_{\vartheta,z,n}[\Psi]y, x\rangle = \int_{[\mathrm{U}(S)]} \overline{y(g)}\, x^\dagger(g)\, dg = \langle y, x^\dagger\rangle$. Now $x^\dagger = \overline{G}$ with $G(g) = \int_{[\mathrm{U}(T)]} \theta(g,h)\overline{x(h)}\, dh$, the lift of $\overline{x}$ for the exchanged kernel of Step 4. The function $\overline{x}$ is a form of type $\overline{e_{\lambda'}}$ (complex conjugation preserves continuity, boundedness, left invariance and levels, and $R(\overline{e})\overline{F} = \overline{R(e)F}$); G has a level by Step 3; and the second half edge of v_H is a half edge for the exchanged real part from $\overline{e_{\lambda'}}$ to $\overline{e_\lambda}$, so G is a form of type $\overline{e_\lambda}$ by (ED4)(c), and $x^\dagger = \overline{G}$ is a form of type e_λ, i.e. $x^\dagger \in \mathcal{F}^i_\triangle(z,n)$. □

Lemma 6.4.52. *Let S, T be spaces with signs s on n and s' on n', α, β automorphic characters of $\mathrm{U}(1)$, $\mathfrak{m}$ a modulus, Ψ a function on $\mathrm{M}_{n'\times n}(R_\mathfrak{m})$, P a polynomial and $\varphi = \mathrm{mod}_\mathfrak{m}(\Psi, v_P)$. Let e, e' be cusp kernels of order 0 on $G_\infty(s)$, $G_\infty(s')$, and F, F' forms of type e on S and of type e' on T. Then for every $h \in \mathrm{G}(s')$ the function $g \mapsto \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)\, F(g)$ is integrable on $[\mathrm{U}(S)]$, and for every $g \in \mathrm{G}(s)$ the function $h \mapsto \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)\, F'(h)$ is integrable on $[\mathrm{U}(T)]$; moreover $(g,h) \mapsto \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)\, F(g)\, F'(h)$ is integrable on $[\mathrm{U}(S)] \times [\mathrm{U}(T)]$.*

Proof. Write $\kappa(g,h) = \theta^{S,T}_{\alpha,\beta}(\varphi)(g,h) = \theta_{\mathrm{Hom}(S,T)}(\omega_{\alpha,\beta}(g,h)\varphi)$. The datum $\varphi = \mathrm{mod}_\mathfrak{m}(\Psi, v_P)$ is an element of $\mathcal{M}_{S,T}$ ((XD5), (WR6)), so $\|\varphi\| < \infty$.

Continuity and invariance. By (2) of Lemma 4.1.1, $(x,f) \mapsto \omega_{s'\otimes s}(x)f$ is continuous; the pair map $(g,h) \mapsto h \otimes (g^{-1})^{\mathsf{T}}$ and the characters $\alpha(\det h)$, $\beta(\det g)^{-1}$ are continuous, and $\theta_{\mathrm{Hom}(S,T)}$ is bounded. Hence κ is continuous on $\mathrm{G}(s) \times \mathrm{G}(s')$. By Lemma 6.2.5, $\kappa(\gamma g, \gamma' h) = \kappa(g,h)$ for $\gamma \in \Gamma_S$, $\gamma' \in \Gamma_T$; forms are left invariant under the rational points (ED2). So the three functions of the statement are continuous and invariant under Γ_S, Γ_T, $\Gamma_S \times \Gamma_T$ respectively, and define continuous, hence measurable, functions on $[\mathrm{U}(S)]$, $[\mathrm{U}(T)]$ and $[\mathrm{U}(S)] \times [\mathrm{U}(T)]$.

Bounds. By Lemma 6.2.28 there are $Y \geq 0$ and $A \in \mathbb{N}$ with $|\kappa(g,h)| \leq Y\|\varphi\|\, (\mathrm{ht}_S(g)\, \mathrm{ht}_T(h))^A$ for all g, h. By Lemma 6.3.75, applied with $B = 0$ to the cusp kernels e, e' of order 0 and the forms F, F', the numbers $C_F = \sup_g |F(g)|\, \mathrm{ht}_S(g)^A$ and $C_{F'} = \sup_h |F'(h)|\, \mathrm{ht}_T(h)^A$ are finite. Hence, as forms are bounded (ED2),

$$\begin{aligned} |\kappa(g,h)F(g)| &\leq Y\|\varphi\|\, C_F\, \mathrm{ht}_T(h)^A, \\ |\kappa(g,h)F'(h)| &\leq Y\|\varphi\|\, C_{F'}\, \mathrm{ht}_S(g)^A, \\ |\kappa(g,h)F(g)F'(h)| &\leq Y\|\varphi\|\, C_F C_{F'} \end{aligned}$$

for all $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$.

Integrability. By Theorem 4.2.18 (for $\mathrm{U}(S)$ and $\mathrm{U}(T)$ with the Haar measures of (VP1) of Definition 3.2.2; a space of rank 0 has the trivial group and nothing to prove), $[\mathrm{U}(S)]$ and $[\mathrm{U}(T)]$

have finite volume, and so has their product. A measurable function bounded by a constant is integrable on a space of finite measure. For fixed h the first bound is a constant in g, for fixed g the second is a constant in h, and the third is a constant; so the three functions are integrable on $[\mathrm{U}(S)]$, on $[\mathrm{U}(T)]$ and on $[\mathrm{U}(S)] \times [\mathrm{U}(T)]$. □

Lemma 6.4.53. *Let S, T, V be spaces with points with signs s on n, s' on n', η on ι, let $\varepsilon(v,i) = \eta_i$, and let (ω, θ) be a theta kernel for (S,T) on V with real part Ω. Put $\omega^{\mathrm{ex}}(h,g) = \omega(g,h)$ for $h \in \mathrm{G}(s')$, $g \in \mathrm{G}(s)$, and $\Omega^{\mathrm{ex}}(y,x) = \Omega(x,y)$ for $y \in G_\infty(s')$, $x \in G_\infty(s)$. Then $(\omega^{\mathrm{ex}}, \theta)$ is a theta kernel for (T,S) on V with real part Ω^{ex}, and the lift with datum $\varphi \in \mathcal{M}_\eta$ of a function F on $\mathrm{G}(s')$ is*

$$g \longmapsto \int_{[\mathrm{U}(T)]} \theta(\omega(g,h)\varphi) F(h)\, dh \qquad (g \in \mathrm{G}(s)).$$

Proof. Let $\sigma\colon \mathrm{G}(s') \times \mathrm{G}(s) \to \mathrm{G}(s) \times \mathrm{G}(s')$, $\sigma(h,g) = (g,h)$; it is an isomorphism of topological groups, and $\omega^{\mathrm{ex}} = \omega \circ \sigma$.

Homomorphism and continuity. As a composite of homomorphisms, ω^{ex} is a homomorphism from $\mathrm{G}(s') \times \mathrm{G}(s)$ to the bounded operators of $\mathcal{M}_\eta$. The map $(p,\varphi) \mapsto \omega^{\mathrm{ex}}(p)\varphi$ is the composite of the homeomorphism $\sigma \times \mathrm{id}$ with the continuous map $(q,\varphi) \mapsto \omega(q)\varphi$ of (ED3), hence continuous.

Invariance of the functional. The functional θ is bounded and linear on $\mathcal{M}_\eta$. For $\gamma' \in \Gamma_T$, $\gamma \in \Gamma_S$, $h \in \mathrm{G}(s')$, $g \in \mathrm{G}(s)$ and $\varphi \in \mathcal{M}_\eta$, the invariance of (ω, θ) gives

$$\theta(\omega^{\mathrm{ex}}(\gamma' h, \gamma g)\varphi) = \theta(\omega(\gamma g, \gamma' h)\varphi) = \theta(\omega(g,h)\varphi) = \theta(\omega^{\mathrm{ex}}(h,g)\varphi).$$

So $(\omega^{\mathrm{ex}}, \theta)$ is a theta kernel for (T,S) on V; the signs of T, S, V are s', s, η, and ε is unchanged.

Real part. Ω^{ex} maps $G_\infty(s') \times G_\infty(s)$ to bounded operators of $\mathcal{F}_\varepsilon$, and for $y \in G_\infty(s')$, $x \in G_\infty(s)$, $v \in \mathcal{F}_\varepsilon$, $f \in \mathcal{M}_\eta$,

$$\omega^{\mathrm{ex}}\big((y,1),(x,1)\big)(v \otimes f) = \omega\big((x,1),(y,1)\big)(v \otimes f) = \big(\Omega(x,y)v\big) \otimes f = \big(\Omega^{\mathrm{ex}}(y,x)v\big) \otimes f.$$

The lift. By (ED3) for the pair (T,S), the lift of F with datum φ is

$$g \mapsto \int_{[\mathrm{U}(T)]} \theta(\omega^{\mathrm{ex}}(h,g)\varphi) F(h)\, dh,$$

and $\omega^{\mathrm{ex}}(h,g) = \omega(g,h)$. □

Lemma 6.4.54. *Let D be admissible data, $z \in \{\mathrm{I}, \mathrm{II}\}$, $0 \le i < m$, Φ a left pair at $(i,0)$ and ϑ an m-fold twist. There is a function Ψ_0 on $\mathrm{M}_{n_1 \times n_0}(R_\vartheta)$ such that for all $y \in \mathcal{F}^i_\triangle(z,0)$ and $x \in \mathcal{F}^i_\triangle(z,1)$*

$$\big\langle x,\ \mathrm{Tl}^i_{\vartheta,z,0}\, y \big\rangle_{V^i_{z,1}} = \big\langle \mathrm{Ds}^i_{\vartheta,z,0}(\Phi, \Psi_0)\, x,\ y \big\rangle_{V^{i+1}_{z,0}},$$

where on the right y is read as a triangle form at $(i+1,0)$ (Lemma 6.4.15(a)).

Proof. Write $S = V^i_{z,0}$, $S^- = V^{i+1}_{z,0}$ (its opposite space, with the same adelic group, (TV1)), $T = V^i_{z,1}$, $\mathfrak{t} = \mathfrak{t}_\vartheta(z)$, ν_1, ν_2 the edge characters with $c = \neg z$, $l = i$ at the ranks N_0, N_1, H the joint harmonic and $\varphi = \mathrm{mod}_{\mathfrak{t}}(\Phi_0^{\mathrm{pt}}, v_H)$ the datum of $\mathrm{Tl}^i_{\vartheta,z,0}$, $\theta = \theta^{S,T}_{\nu_1,\nu_2}(\varphi)$, and $\lambda = \frac{1}{2}p^i_{z,0,0}$, $\lambda' = \frac{1}{2}p^i_{z,0,1}$, so that the triangle forms at $(i,0)$, $(i+1,0)$ and $(i,1)$ are the forms of type e_λ, e_λ and $e_{\lambda'}$ (Lemma 6.4.15(a)).

Step 1: Fubini. By Lemma 6.4.15(b), v_H is an edge from e_λ to $e_{\lambda'}$, so e_λ and $\overline{e_{\lambda'}}$ are cusp kernels of order 0 ((ED4)(a) of its two half edges), and so is $e_{\lambda'}$ (every clause of (ED1) is preserved by complex conjugation). By Lemma 6.4.52, $(g,h)\mapsto \theta(g,h)\,y(g)\,\overline{x(h)}$ is integrable on $[\mathrm{U}(S)]\times[\mathrm{U}(T)]$, and Fubini gives

$$\langle x, \mathrm{Tl}^i_{\vartheta,z,0} y\rangle_T = \int_{[\mathrm{U}(T)]} \overline{x(h)} \int_{[\mathrm{U}(S)]} \theta(g,h)\,y(g)\,dg\,dh = \int_{[\mathrm{U}(S)]} \overline{D(g)}\,y(g)\,dg,$$
$$D(g) = \int_{[\mathrm{U}(T)]} \overline{\theta(g,h)}\,x(h)\,dh.$$

Step 2: the conjugate kernel. By Lemma 6.4.55 there are c with $|c|=1$ and $\Psi'(X) = \overline{\Phi_0^{\mathrm{pt}}(u^{-1}X)}$ with $\overline{\theta(g,h)} = c\,\theta^{S^-,T}_{\nu_1^{-1}\upsilon^{N_0},\,\nu_2^{-1}\upsilon^{N_1}}(\mathrm{mod}_{\mathfrak{t}}(\Psi', v_{\bar H}))(g,h)$. By (TV6), $\nu_1^{-1}\upsilon^{N_0} = \chi^{\mathrm{lp}}_{\vartheta,z,i}(N_0)$ and $\nu_2^{-1}\upsilon^{N_1} = \chi^{\mathrm{lp}}_{\vartheta,z,i}(N_1)$, the characters of the descents at $(i,0)$. For a polynomial Q write D_Q for the map $x\mapsto (g\mapsto \int_{[\mathrm{U}(T)]}\theta^{S^-,T}_{\chi^{\mathrm{lp}}(N_0),\chi^{\mathrm{lp}}(N_1)}(\mathrm{mod}_{\mathfrak{t}}(\Psi', v_Q))(g,h)\,x(h)\,dh)$; so $D = c\,D_{\bar H}x$, and $D_\Phi = \mathrm{Ds}^i_{\vartheta,z,0}(\Phi,\Psi')$ (TV7).

Step 3: trading $\bar H$ for the left pair. By Lemma 6.3.77 and Lemma 6.4.53 (exchanging the two factors), the theta kernel of the pair (S^-,T) with these characters gives a theta kernel for (T,S^-) whose real part is $(y',x')\mapsto \Omega_{\delta,\delta'}(x',y')$, $(\delta,\delta') = (\delta_i(N_0),\delta_i(N_1))$, and D_Q is its lift with datum $\mathrm{mod}_{\mathfrak{t}}(\Psi', v_Q)$. By (TV6), v_Φ is an edge for (δ,δ') from $e_{-\mu}$ to $e_{-\mu'}$, $\mu = \frac12 p^{i+1}_{z,0,0} = \lambda$ ((GP5): no block indices at column 0), $\mu' = \frac12 p^i_{z,0,1} = \lambda'$, with $e_{-\mu} = \overline{e_\mu}$; its second half edge is a half edge for the exchanged real part from $\overline{e_{-\mu'}} = e_{\lambda'}$ to $\overline{e_{-\mu}} = e_\lambda$, and e_λ is an idempotent kernel (Step 1). The lift $D_\Phi x$ has a level (Lemma 6.4.51(b)). So Lemma 6.4.57, with $P=\bar H$, the form x of type $e_{\lambda'}$ on T and the form y of type e_λ on S^-, gives $\kappa\in\mathbb{C}$, depending only on $\bar H$, Φ and e_λ, with $\langle y, D_{\bar H}x\rangle_{S^-} = \kappa\,\langle y, D_\Phi x\rangle_{S^-}$.

Step 4: conclusion. The Petersson products of S and S^- agree (Lemma 6.4.15(a)). By Steps 1–3,

$$\langle x, \mathrm{Tl}^i_{\vartheta,z,0} y\rangle_T = \langle D, y\rangle_{S^-} = \bar c\,\overline{\langle y, D_{\bar H}x\rangle} = \bar c\,\bar\kappa\,\overline{\langle y, D_\Phi x\rangle}$$
$$= \big\langle c\kappa\,\mathrm{Ds}^i_{\vartheta,z,0}(\Phi,\Psi')x,\ y\big\rangle_{S^-},$$

and $c\kappa\,\mathrm{Ds}^i_{\vartheta,z,0}(\Phi,\Psi') = \mathrm{Ds}^i_{\vartheta,z,0}(\Phi, c\kappa\Psi')$ by Lemma 6.4.51(c). So $\Psi_0 = c\kappa\Psi'$. □

Lemma 6.4.55. *Let S, T be spaces with signs s on n and s' on n', and S^- the opposite space of S, whose adelic group $\mathrm{G}(-s)$ is $\mathrm{G}(s)$. There is $c_{S,T}\in\mathbb{C}$ with $|c_{S,T}|=1$ such that for all automorphic characters α, β of $\mathrm{U}(1)$, every modulus $\mathfrak{m}$, every function Ψ on $\mathrm{M}_{n'\times n}(R_{\mathfrak{m}})$, every polynomial P and all $g\in\mathrm{G}(s)$, $h\in\mathrm{G}(s')$,*

$$\overline{\theta^{S,T}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}}(\Psi, v_P))(g,h)} = c_{S,T}\,\theta^{S^-,T}_{\alpha^{-1}\upsilon^{\#n},\,\beta^{-1}\upsilon^{\#n'}}(\mathrm{mod}_{\mathfrak{m}}(\Psi', v_{\bar P}))(g,h),$$

where $\Psi'(X) = \overline{\Psi(u^{-1}X)}$, $u\in R_{\mathfrak{m}}^\times$ is the reduction of c modulo $\mathfrak{m}$, and $\bar P$ is P with conjugated coefficients (its Fock vector taken for the signs of $\mathrm{Hom}(S^-,T)$).

Proof. Write $\iota = n'\times n$ and $\varepsilon = s'\otimes s$ (the signs of $\mathrm{Hom}(S,T)$), so that $\mathrm{Hom}(S^-,T)$ has the signs $-\varepsilon$. Let σ_c and cj be those of (WR7) of Definition 3.2.18, $\sigma_c(x) = (x_\infty, c^{-1}x_f)$ on $\mathbb{A}_E^\iota$ and $(\mathrm{cj}\,f)(x) = \overline{f(\sigma_c x)}$, and $c_{S,T}$ that of Lemma 6.4.56; by (1) of Lemma 6.4.56, cj is an antilinear isometry of $\mathcal{M}_\varepsilon$ onto $\mathcal{M}_{-\varepsilon}$ with $\mathrm{cj}\,v^0 = v^0$ and $\mathrm{cj}\,\rho(z) = \rho(\sigma_c^{-1}z)\,\mathrm{cj}$, and $|c_{S,T}|=1$.

Step 1: the datum. Let $A = \Sigma(\mathfrak{m})$ and $\varphi = v_P\otimes(\Psi^\natural\otimes_A v^0)$ (XD5). *Real part.* $v\otimes f = \int v(t)\rho(\mathrm{rv}(t))f\,d\nu(t)$ is a Bochner integral (Lemma 6.2.21); cj is real-linear and bounded and $\sigma_c^{-1}\mathrm{rv}(t) = \mathrm{rv}(t)$, so $\mathrm{cj}(v\otimes f) = \bar v\otimes \mathrm{cj}\,f$, $\bar v$ the pointwise conjugate, read in the Fock model of

$-\varepsilon$. For $v = v_P$, $\overline{v_P(x)} = \bar{P}(\overline{\zeta_\varepsilon(x)})e^\infty(x)$ and $\overline{\zeta_\varepsilon(x)} = \zeta_{-\varepsilon}(x)$ (WR5), so $\overline{v_P} = v_{\bar{P}}$ for the signs of $\mathrm{Hom}(S^-, T)$. *Finite part.* Write $\Psi^\natural = \sum_y c_y\tau_y$ (WR6); as $P^A v^0 = v^0$ ((2) of Lemma 6.2.33), $\Psi^\natural \otimes_A v^0 = \sum_y c_y\rho(\mathrm{fv}(y))v^0$, and as $\sigma_c^{-1}\mathrm{fv}(y) = \mathrm{fv}(cy)$,

$$\mathrm{cj}(\Psi^\natural \otimes_A v^0) = \sum_y \bar{c}_y\,\rho(\mathrm{fv}(cy))v^0.$$

For $x \in \mathbb{A}_{E,f}^\iota$, $c^{-1}x+y$ is integral at A exactly when $x+cy$ is, and $-\bar{y}^\mathsf{T}(2c^{-1}x+y) = \overline{cy}^\mathsf{T}(2x+cy)$ (as $-c^{-1} = \bar{c}$, $\bar{c}c = -1$); hence $\overline{\tau_y(c^{-1}x)} = \tau_{cy}(x)$, and $\sum_y \bar{c}_y\tau_{cy}(x) = \overline{\Psi^\natural(c^{-1}x)} = \Psi'^\natural(x)$ with $\Psi'(X) = \overline{\Psi(u^{-1}X)}$, u the reduction of c. By (1) of Lemma 4.1.3 the Schrödinger datum does not depend on the representation of $\Psi'^\natural$ by the τ_y (every finite representation of a nonzero function is the value of some linear section; a representation of 0 is the difference of two representations of one nonzero function), so $\mathrm{cj}(\Psi^\natural \otimes_A v^0) = \Psi'^\natural \otimes_A v^0$ and

$$\mathrm{cj}\,\mathrm{mod}_\mathfrak{m}(\Psi, v_P) = \mathrm{mod}_\mathfrak{m}(\Psi', v_{\bar{P}}),$$

the right side formed for $\mathrm{Hom}(S^-, T)$.

Step 2: conclusion. By (4) of Lemma 6.4.56 for $f = \mathrm{mod}_\mathfrak{m}(\Psi, v_P)$ and Step 1,

$$\overline{\theta^{S,T}_{\alpha,\beta}(\mathrm{mod}_\mathfrak{m}(\Psi, v_P))(g,h)} = c_{S,T}\,\theta^{S^-,T}_{\alpha^{-1}\upsilon^{\#n},\,\beta^{-1}\upsilon^{\#n'}}(\mathrm{mod}_\mathfrak{m}(\Psi', v_{\bar{P}}))(g,h).$$

□

Lemma 6.4.56. *Let S, T be spaces with points with signs s on n and s' on n', $\iota = n' \times n$, and $\varepsilon = s' \otimes s$ the signs of $\mathrm{Hom}(S,T)$, so that $\mathrm{Hom}(S^-,T)$ has the signs $-\varepsilon$. Let σ_c and the conjugate $\mathrm{cj}\,f$ of a function f on $\mathbb{A}_E^\iota$, $(\mathrm{cj}\,f)(x) = \overline{f(\sigma_c x)}$, be those of (WR7). Then:*

1. *cj is an antilinear bijection of $\mathcal{M}_{S,T} = \mathcal{M}_\varepsilon$ onto $\mathcal{M}_{S^-,T} = \mathcal{M}_{-\varepsilon}$ with $\|\mathrm{cj}\,f\| = \|f\|$, $\langle \mathrm{cj}\,f, \mathrm{cj}\,f'\rangle = \overline{\langle f, f'\rangle}$ and $\mathrm{cj}\,v^0 = v^0$, and $\mathrm{cj}\,\rho(z) = \rho(\sigma_c^{-1}z)\,\mathrm{cj}$ for $z \in \mathbb{A}_E^\iota$;*
2. *for automorphic characters α, β of $\mathrm{U}(1)$, $g \in \mathrm{G}(s)$ and $h \in \mathrm{G}(s')$,*

$$\mathrm{cj}\,\omega_{\alpha,\beta}(g,h) = \omega_{\alpha^{-1}\upsilon^{\#n},\,\beta^{-1}\upsilon^{\#n'}}(g,h)\,\mathrm{cj},$$

   the right side the Weil representation of the pair (S^-, T);
3. *there is $c_{S,T} \in \mathbb{C}$ with $|c_{S,T}| = 1$, depending only on S and T, such that*

$$\overline{\theta_{\mathrm{Hom}(S,T)}(f)} = c_{S,T}\,\theta_{\mathrm{Hom}(S^-,T)}(\mathrm{cj}\,f) \qquad (f \in \mathcal{M}_{S,T});$$

4. *consequently, for all α, β, g, h as in (2) and $f \in \mathcal{M}_{S,T}$,*

$$\overline{\theta^{S,T}_{\alpha,\beta}(f)(g,h)} = c_{S,T}\,\theta^{S^-,T}_{\alpha^{-1}\upsilon^{\#n},\,\beta^{-1}\upsilon^{\#n'}}(\mathrm{cj}\,f)(g,h).$$

Proof. Write $N = \#\iota$ and $\mathcal{X} = \mathbb{A}_E^\iota$; $\bar{c}c = -1$ and $c^{-1} = -\bar{c}$.

Step 1: three identities. For $a, b, z \in \mathcal{X}$:

$$\text{(I1) } \overline{\beta_\varepsilon(\sigma_c a, \sigma_c b)} = \beta_{-\varepsilon}(a,b), \qquad \text{(I2) } \overline{e_N(\sigma_c z)} = e_N(z), \qquad \text{(I3) } (\sigma_c)_*\mu_N = \mu_N.$$

(I1): $\overline{\beta_\varepsilon(\sigma_c a, \sigma_c b)} = \psi_E(-\frac{1}{2}\,\overline{\sigma_c b}^\mathsf{T} S\,\sigma_c a)$ by (WD6), S the diagonal matrix of (WD6) for the signs ε. At the infinite places σ_c is the identity and $S = \varepsilon$, which gives the infinite part $-\frac{1}{2}\bar{b}^\mathsf{T}\varepsilon a =$

$\frac{\jmath}{2}\bar{b}^{\mathsf{T}}(-\varepsilon)a$ of the argument of $\beta_{-\varepsilon}$; at the finite places $S=1$ and $\overline{c^{-1}}c^{-1}=(\bar{c}c)^{-1}=-1$, which gives $\frac{\jmath}{2}\bar{b}^{\mathsf{T}}a$, the finite part of that argument. (I2): the archimedean factor of e_N is real and does not see σ_c; at the finite places $c^{-1}z\in\alpha\widehat{\mathcal{O}}_E^{\iota}$ if and only if $z\in\alpha\widehat{\mathcal{O}}_E^{\iota}$ (c is a unit), and then $\overline{\psi_{E,f}(\frac{c_0\jmath}{2}\overline{c^{-1}z}^{\mathsf{T}}c^{-1}z)}=\overline{\psi_{E,f}(-\frac{c_0\jmath}{2}\bar{z}^{\mathsf{T}}z)}=\psi_{E,f}(\frac{c_0\jmath}{2}\bar{z}^{\mathsf{T}}z)$ (WD5). (I3): σ_c is the identity on the archimedean factor and multiplication by a unit idele on the finite one, which preserves its Haar measure (WD6).

Step 2: the antilinear map; proof of (1). If $f\in\mathcal{M}_\varepsilon$, then $\mathrm{cj}\,f$ is μ_N-integrable with $\|\mathrm{cj}\,f\|=\|f\|$ by (I3), and by (WD7), the substitution $y=\sigma_c y'$ (I3), (I1) and (I2),

$$\begin{aligned}(\mathrm{cj}\,f)(x)&=\int\overline{f(\sigma_c y')}\,\overline{\beta_\varepsilon(\sigma_c y',\sigma_c x)}\,\overline{e_N(\sigma_c(x-y'))}\,d\mu_N(y')\\&=\int(\mathrm{cj}\,f)(y')\,\beta_{-\varepsilon}(y',x)\,e_N(x-y')\,d\mu_N(y'),\end{aligned}$$

so $\mathrm{cj}\,f\in\mathcal{M}_{-\varepsilon}$. The same argument with σ_c^{-1} (again $\bar{c}c=-1$) shows that $g\mapsto\overline{g(\sigma_c^{-1}\cdot)}$ maps $\mathcal{M}_{-\varepsilon}$ to $\mathcal{M}_\varepsilon$; it is inverse to cj, so cj is an antilinear bijection. By (I3), $\langle\mathrm{cj}\,f,\mathrm{cj}\,f'\rangle=\int f(\sigma_c x)\overline{f'(\sigma_c x)}\,d\mu_N(x)=\int f\,\overline{f'}\,d\mu_N=\overline{\langle f,f'\rangle}$. By (I2), $\mathrm{cj}\,v^0=v^0$. By (I1), $(\mathrm{cj}\,\rho(z)f)(x)=\overline{\beta_\varepsilon(\sigma_c\sigma_c^{-1}z,\sigma_c x)}\,\overline{f(\sigma_c(x-\sigma_c^{-1}z))}=\beta_{-\varepsilon}(\sigma_c^{-1}z,x)\,(\mathrm{cj}\,f)(x-\sigma_c^{-1}z)=(\rho(\sigma_c^{-1}z)\mathrm{cj}\,f)(x)$.

Step 3: the Weil operators; proof of (2). Let $k\in\mathrm{G}(\varepsilon)$; it lies in $\mathrm{G}(-\varepsilon)$ too (the same matrices: $\varepsilon_{-\varepsilon}=-\varepsilon_\varepsilon$ at the infinite places and 1 at the finite places), and k commutes with σ_c (k acts by an adelic matrix, σ_c by a scalar idele). By (WR3), the substitution $y=\sigma_c y'$, (I1)–(I3),

$$\mathrm{cj}\,\omega_\varepsilon(k)f=\frac{\overline{\Delta_\varepsilon(k)}}{\Delta_{-\varepsilon}(k)}\,\omega_{-\varepsilon}(k)\,\mathrm{cj}\,f\qquad(f\in\mathcal{M}_\varepsilon),$$

where $\Delta_{-\varepsilon}(k)\neq0$ by (2) of Lemma 6.2.13. We show $\overline{\Delta_\varepsilon(k)}=\upsilon(\det k)\,\Delta_{-\varepsilon}(k)$. *Infinite places.* For $k_v\in\mathrm{U}(\mathrm{diag}\,\varepsilon)$, $k_v^{-1}=\mathrm{diag}(\varepsilon)k_v^*\mathrm{diag}(\varepsilon)$, whose block on the indices of sign -1 is $((k_v)_-)^*$; by Jacobi's identity for complementary minors of an inverse, $\overline{\det(k_v)_-}=\det((k_v^{-1})_-)=\det(k_v)_+/\det k_v$, $(k_v)_+$ the block of sign $+1$. As $|\det k_v|=1$, $\det(k_v)_-=\det k_v\cdot\overline{\det(k_v)_+}$. The archimedean factor of $\overline{\Delta_\varepsilon(k)}$ is $\prod_v\det(k_v)_-$, that of $\Delta_{-\varepsilon}(k)$ is $\prod_v\overline{\det(k_v)_+}$ (the block of sign -1 for $-\varepsilon$), and their ratio is $\prod_v\det k_v$. *Finite places.* The finite factor $\upsilon(\det u)|\det u|^{-1/2}$ of $\Delta_{\pm\varepsilon}(k)$ does not depend on the signs; it is the multiplier $\Delta(k_f)$ of Lemma 4.1.13 for υ_f the restriction of υ to the finite ideles (which has modulus 1, is trivial on the unit ideles by (WD3), and satisfies $\upsilon_f(\bar{t})=\upsilon_f(t)^{-1}$), whose hypotheses on $\jmath$, α, c_0 hold by (WD1) and Lemma 4.1.3; as $k_f^*k_f=1$, Lemma 4.1.13 gives $\overline{\Delta(k_f)}=\upsilon_f(\det k_f)\Delta(k_f)$. Since $\upsilon(\det k)=\prod_v\det k_v/|\det k_v|\cdot\upsilon_f(\det k_f)$ (WD3), the claim follows, and

$$\mathrm{cj}\,\omega_\varepsilon(k)=\upsilon(\det k)\,\omega_{-\varepsilon}(k)\,\mathrm{cj}\qquad(k\in\mathrm{G}(\varepsilon)).\tag{6.80}$$

Now let $k=h\otimes(g^{-1})^{\mathsf{T}}$, the image of (g,h) under the pair map of (S,T), which is also that of (S^-,T); $\det k=\det(h)^{\#n}\det(g)^{-\#n'}$ ((VP5) of Definition 3.2.2). By (WR4), the antilinearity of cj and (6.80),

$$\begin{aligned}\mathrm{cj}\,\omega_{\alpha,\beta}(g,h)&=\overline{\alpha(\det h)}\,\beta(\det g)\,\upsilon(\det h)^{\#n}\upsilon(\det g)^{-\#n'}\,\omega_{-\varepsilon}(k)\,\mathrm{cj}\\&=(\alpha^{-1}\upsilon^{\#n})(\det h)\,\big((\beta^{-1}\upsilon^{\#n'})(\det g)\big)^{-1}\omega_{-\varepsilon}(k)\,\mathrm{cj},\end{aligned}$$

which is $\omega_{\alpha^{-1}\upsilon^{\#n},\beta^{-1}\upsilon^{\#n'}}(g,h)\,\mathrm{cj}$ for the pair (S^-,T) by (WR4).

Step 4: the theta functionals; proof of (3). Let X be the point of $\mathrm{Hom}(S,T)$; that of $\mathrm{Hom}(S^-,T)$ is $x_T\otimes((cx_S)^{-1})^{\mathsf{T}}=c^{-1}X$ ((VP5) of Definition 3.2.2), so its rational vectors are

$cX^{-1}u = \sigma_c^{-1}(X^{-1}u)$ ($u \in E^\iota$): σ_c^{-1} maps the rational vectors $V(E)$ of $\mathrm{Hom}(S,T)$ onto those $V^-(E)$ of $\mathrm{Hom}(S^-,T)$. Put $\theta'(g) = \overline{\theta_{\mathrm{Hom}(S,T)}(\mathrm{cj}^{-1}g)}$ for $g \in \mathcal{M}_{-\varepsilon}$. It is linear (a composite of two antilinear maps and a linear one) and bounded (cj^{-1} is an isometry by (1)). By (1), $\mathrm{cj}^{-1}\rho(\sigma_c^{-1}\gamma) = \rho(\gamma)\mathrm{cj}^{-1}$, so $\theta'(\rho(\sigma_c^{-1}\gamma)g) = \theta'(g)$ for $\gamma \in V(E)$: θ' is invariant under $V^-(E)$. As $\mathrm{cj}^{-1}v^0 = v^0$, and σ_c induces an isomorphism of the compact groups $\mathcal{X}/V^-(E) \to \mathcal{X}/V(E)$ carrying invariant probability measure to invariant probability measure,

$$\begin{aligned}\int_{\mathcal{X}/V^-(E)} |\theta'(\rho(x)v^0)|^2\,dx &= \int_{\mathcal{X}/V^-(E)} |\theta_{\mathrm{Hom}(S,T)}(\rho(\sigma_c x)v^0)|^2\,dx\\ &= \int_{\mathcal{X}/V(E)} |\theta_{\mathrm{Hom}(S,T)}(\rho(y)v^0)|^2\,dy = 1,\end{aligned}$$

the first equality by $\mathrm{cj}^{-1}\rho(x) = \rho(\sigma_c x)\mathrm{cj}^{-1}$ (from (1)), the last by Lemma 4.1.17. So θ' and $\theta_{\mathrm{Hom}(S^-,T)}$ are normalised theta functionals of $\mathrm{Hom}(S^-,T)$ (Lemma 4.1.17), and by Lemma 4.1.7 there is $c_{S,T} \in \mathbb{C}$, $|c_{S,T}| = 1$, with $\theta' = c_{S,T}\,\theta_{\mathrm{Hom}(S^-,T)}$. Applied to $g = \mathrm{cj}\,f$ this is (3); $c_{S,T}$ depends only on S, T and the fixed theta functionals.

Step 5: proof of (4). By the definition of theta kernels (WC4), (3) and (2),

$$\begin{aligned}\overline{\theta^{S,T}_{\alpha,\beta}(f)(g,h)} &= \overline{\theta_{\mathrm{Hom}(S,T)}(\omega_{\alpha,\beta}(g,h)f)}\\ &= c_{S,T}\,\theta_{\mathrm{Hom}(S^-,T)}(\mathrm{cj}\,\omega_{\alpha,\beta}(g,h)f)\\ &= c_{S,T}\,\theta_{\mathrm{Hom}(S^-,T)}(\omega_{\alpha^{-1}\upsilon^{\#n},\beta^{-1}\upsilon^{\#n'}}(g,h)\mathrm{cj}\,f),\end{aligned}$$

which is the right side of (4). □

Lemma 6.4.57. *Let s, s', η, ε and Ω be as in (ED3), e a type for the signs s, e' an idempotent kernel on $G_\infty(s')$, and $\Phi \in \mathcal{F}_\varepsilon$ a half edge for Ω from e to e'. For every polynomial P there is $\kappa_P \in \mathbb{C}$ such that, for all spaces with points S, T, V with signs s, s', η, every theta kernel for (S,T) on V with real part Ω, every $f \in \mathcal{M}_\eta$, every form F of type e on S for which $L_{\Phi\otimes f}F$ has a level, and every form G of type e' on T,*

$$\langle G,\ L_{v_P\otimes f}F\rangle_T = \kappa_P\,\langle G,\ L_{\Phi\otimes f}F\rangle_T.$$

Proof. By (ED4)(d) there is a continuous compactly supported a on $G_\infty(s')$, depending only on P (and on Φ), with $L_{v_P\otimes f}F = R(a)L_{\Phi\otimes f}F$ for all data as in the statement. Put

$$\kappa_P = \mu = \frac{1}{e'(1)}\int_{G_\infty(s')} a(v)\,e'(v^{-1})\,dv,$$

a convergent integral (a is bounded with compact support and e' is integrable, hence so is $v \mapsto e'(v^{-1})$). Fix the data and write $L = L_{\Phi\otimes f}F$; by (ED4)(c) it is a form of type e', so $R(e')L = L$, and $R(e')G = G$; both are bounded and continuous.

Step 1: adjoints of the operators $R(k)$. For an integrable k on $G_\infty(s')$ and bounded measurable functions X, Y on $\Gamma_T\backslash\mathrm{G}(s')$, $\langle R(k)X,Y\rangle_T = \langle X,R(k^*)Y\rangle_T$ with $k^*(x) = \overline{k(x^{-1})}$. Indeed $(x,g) \mapsto \overline{k(x)X(g(x,1))}\,Y(g)$ is integrable on $G_\infty(s')\times[\mathrm{U}(T)]$ ($[\mathrm{U}(T)]$ has finite volume), so by Fubini, the invariance of the quotient measure under right translation by $(x,1)$ and then the inversion $x \mapsto x^{-1}$,

$$\begin{aligned}\langle R(k)X,Y\rangle_T &= \int \overline{k(x)}\int_{[\mathrm{U}(T)]} \overline{X(g)}\,Y(g(x^{-1},1))\,dg\,dx\\ &= \int_{[\mathrm{U}(T)]} \overline{X(g)}\int \overline{k(x^{-1})}\,Y(g(x,1))\,dx\,dg = \langle X,R(k^*)Y\rangle_T.\end{aligned}$$

As e' is an idempotent kernel, $e'(x^{-1}) = \overline{e'(x)}$, so $e'^{*} = e'$.

Step 2: $R(e')R(a)R(e')X = \mu\, R(e')X$ for bounded measurable X. For $g \in \mathrm{G}(s')$,

$$R(e')R(a)R(e')X(g) = \iiint e'(u)\, a(v)\, e'(w)\, X\big(g(uvw,1)\big)\, dw\, dv\, du,$$

the integrand being absolutely integrable on $G_\infty(s')^3$ (X bounded, e' and a integrable). Substituting $y = uvw$ in the innermost integral (left invariance of dw) and applying Fubini, the value is $\int\!\int a(v)\big(\int e'(u)\, e'((uv)^{-1}y)\, du\big)X(g(y,1))\, dy\, dv$, and by the defining identity of an idempotent kernel ((ED1), with $x = v$) the inner integral is $e'(v^{-1})e'(y)/e'(1)$. So the value is $\mu \int e'(y)X(g(y,1))\, dy = \mu\, R(e')X(g)$.

Step 3: the constant. By Steps 1 and 2,

$$\langle G, R(a)L\rangle_T = \langle R(e')G, R(a)R(e')L\rangle_T = \langle G, R(e')R(a)R(e')L\rangle_T = \mu\,\langle G, R(e')L\rangle_T = \mu\,\langle G, L\rangle_T,$$

and $\mu = \kappa_P$ depends only on a and e', that is on P, Φ and e'. □

Lemma 6.4.58. *Let D be admissible data, $z \in \{\mathrm{I}, \mathrm{II}\}$ and $0 \le i < m$. For an m-fold twist ϑ let $a_\vartheta \colon \mathcal{F}^0_\triangle(z,0) \to \mathcal{F}^i_\triangle(z,1)$ be the first lift of row i, Σ_ϑ the set of finite places of F below no place of $\Sigma(\mathfrak{t}_\vartheta(z))$, b_ϑ the edge character with $c = \neg z$ and $l = i$ at the rank N_1, and R_{Σ_ϑ} the Rallis series of $(V^i_{z,0}, V^i_{z,1}, b_\vartheta, \Sigma_\vartheta)$, acting on functions on the common adelic group of $V^0_{z,0}$ and $V^i_{z,0}$. Then there is $A_{z,i} \in \mathbb{C} \setminus \{0\}$, independent of ϑ, such that for every m-fold twist ϑ:*

(a) for every base form $y \in \mathcal{F}^{\mathrm{bs}}_z$ and every $x \in \mathcal{F}^0_\triangle(z,0)$,

$$\begin{aligned}\#\mathrm{GL}_{n_0}(R_\vartheta)\,\langle a_\vartheta x, a_\vartheta y\rangle_{V^i_{z,1}} &= \frac{A_{z,i}}{\#\mathrm{M}_{n_1\times n_0}(R_\vartheta)}\,\langle x, R_{\Sigma_\vartheta} y\rangle_{V^0_{z,0}} \\ &= \sum_q \frac{A_{z,i}}{\#\mathrm{M}_{n_1\times n_0}(R_\vartheta)}\,\mathrm{mc}_{b_\vartheta,\Sigma_\vartheta}(u_q)\,\langle x, y(\,\cdot\, u_q)\rangle_{V^0_{z,0}},\end{aligned}$$

the series absolutely convergent, q running over the cosets of K_{Σ_ϑ};

(b) a_ϑ is injective on $\mathcal{F}^{\mathrm{bs}}_z$;

(c) if $y \in \mathcal{F}^{\mathrm{bs}}_z$ and $v \in \mathcal{F}^0_\triangle(z,0)$ satisfy $\langle v, x\rangle_{V^0_{z,0}} = \langle a_\vartheta y, a_\vartheta x\rangle_{V^i_{z,1}}$ for every $x \in \mathcal{F}^0_\triangle(z,0)$, then $v \in \mathcal{F}^{\mathrm{bs}}_z$.

Proof. Fix D, z, $i < m$ and an m-fold twist ϑ. Write $\mathfrak{t} = \mathfrak{t}_\vartheta(z)$, $P = \Sigma(\mathfrak{t})$, $R = R_\vartheta$, $N = \#\mathrm{M}_{n_1\times n_0}(R)$, $S = V^i_{z,0}$, $T = V^i_{z,1}$, a and $b = b_\vartheta$ the edge characters with $c = \neg z$, $l = i$ at the ranks N_0 and N_1, H the joint harmonic of the triangle lift at $(i,0)$, $x_0 \in \mathrm{M}_{n_1\times n_0}(R)$ the matrix with entry 1 at (c,c) ($c \in n_0$) and 0 elsewhere, $\Phi = \Phi_0^{\mathrm{pt}}$ its indicator function, $f = \Phi^\natural \otimes_P v^0$ and $\varphi = \mathrm{mod}_{\mathfrak{t}}(\Phi, v_H) = v_H \otimes f$. By (TV3) and (TV5), $a_\vartheta x = L_\varphi x$ for the theta kernel $\theta^{S,T}_{a,b}$, the copy of x at $(i,0)$ being x itself on the common group (TV5). Let $\mathbf{1}$ also denote identity matrices, and put $J = \{k \in \mathbf{K}_0(N_0) : k \equiv \mathbf{1} \pmod{\mathfrak{t}}\}$.

Step 0: the places of $\mathfrak{t}$, reduction, and the level J. By (TV2), P consists of the chosen places and, if $z = \mathrm{I}$, the places of Q; these are the chosen and twist places of the level system of the descent pair $O_{1,\vartheta}$ ((LF4) of Definition 3.2.26), so by (DP1) of Definition 3.2.23 they are distinct and none is the conjugate of itself or of another; $P \neq \emptyset$, as $\pi_M\bar\pi_M = p_0p_1$ ((AD5)) makes π_M a nonunit. Hence each $w \in P$ lies over a place of F split in E, distinct w lie over distinct places, and for every index set n the component of $G_f(n)$ at the place below w is identified by projection to w with $\mathrm{GL}_n(E_w)$ (the condition $\bar g^{\mathsf{T}} g = \mathbf{1}$ determines the component

at $\bar{w}$), that of $\mathbf{K}_0(n)$ with $\mathrm{GL}_n(\mathcal{O}_{E,w})$. By the Chinese remainder theorem $R = \prod_{w\in P} \mathcal{O}_{E,w}/w^{e_w}$ (e_w the exponent of w in $\mathfrak{t}$), and a matrix over $\mathcal{O}_{E,w}$ whose determinant is a unit modulo w^{e_w} is invertible over $\mathcal{O}_{E,w}$. So reduction modulo $\mathfrak{t}$ maps $\mathbf{K}_0(n)$ onto $\mathrm{GL}_n(R)$, with kernel the elements $\equiv \mathbf{1} \pmod{\mathfrak{t}}$; in particular $[\mathbf{K}_0(N_0) : J] = \#\mathrm{GL}_{n_0}(R)$, and J is open. Every chosen place divides $\mathfrak{t}$ ((TV2): $\pi_M \mid \mathfrak{t}$ if $z = \mathrm{I}$, $\mathfrak{t} = \mathfrak{b} = \pi_M^b$ if $z = \mathrm{II}$), so an element of J reduces to $\mathbf{1}$ modulo every chosen place and, if $z = \mathrm{II}$, modulo $\mathfrak{b}$: $J \subseteq \mathrm{K}_z^{\mathrm{bs}}$ (TV4).

Step 1: inner products of data modulo $\mathfrak{t}$. For functions Ψ_1, Ψ_2 on $\mathrm{M}_{n_1\times n_0}(R)$,

$$\langle \Psi_1^\natural \otimes_P v^0, \Psi_2^\natural \otimes_P v^0\rangle = \frac{1}{N}\sum_X \overline{\Psi_1(X)}\,\Psi_2(X).$$

Indeed, by (1) and (2) of Lemma 6.2.33 and $\langle v^0, v^0\rangle = 1$ (Lemma 6.2.13) the left side is $\int_{X_P} \overline{\Psi_1^\natural}\Psi_2^\natural$; the integrand vanishes off the matrices integral at every finite place, a set of mass 1, and is there a function of the reduction modulo $\mathfrak{t}$; reduction is a continuous surjective homomorphism of compact groups onto $\mathrm{M}_{n_1\times n_0}(R)$, so it carries the Haar probability measure to the uniform measure. In particular $\langle f, f\rangle = 1/N$.

Step 2: the characters at integral points. Tame parts multiply under products of characters (their defining formula in Definition 3.1.2 is multiplicative). By (GP6) and Definition 3.2.26, a and b are $\mu_{i+1}^0 \vartheta^{i+1}$ (if $z = \mathrm{I}$) or μ_{i+1}^0 (if $z = \mathrm{II}$) times a power of υ; μ_{i+1}^0 has tame part ω^{i+1}, trivial off the chosen places, υ has trivial tame part (LF2), and ϑ^{i+1} has a tame part trivial off Q. So a and b have tame parts ψ with $\psi_w = 1$ for $w \notin P$. Let $k \in \mathbf{K}_0(n)$ ($n = n_0$ or n_1) with $k \equiv \mathbf{1} \pmod{\mathfrak{t}}$. Then $u = \det k \in \mathrm{U}(1)(\mathbb{A})$ has infinite components 1, unit finite components and $u \bmod w = 1$ for $w \in P$, so $a(u) = b(u) = 1$ by the definition of weight and tame part.

Step 3: f is sharp for J off the places below P, with mass $c = 1/(N\,\#\mathrm{GL}_{n_0}(R))$. Let S_0 be the set of places of F below P; Σ_ϑ is the set of the other finite places. We check (SH1)–(SH4) of Lemma 6.5.12 for f, J and the characters a, b. (SH1): $\langle f, f\rangle = 1/N \neq 0$ (Step 1) and $c = \langle f, f\rangle/[\mathbf{K}_0(N_0) : J]$ (Step 0); $v_1 \otimes f = f$ by (4) of Lemma 6.2.33; and f is a vacuum above Σ_ϑ: a vector $y \in \mathcal{O}_E^{n_1\times n_0}$ has components 0 at the places not above Σ_ϑ exactly when its components at $P\cup\bar{P}$ are 0, and then $\rho_f(y)f = f$ by (5) of Lemma 6.2.33. (SH2): let $k \in G_f(n_0)$ with $k_v \in K_v$ for $v \in \Sigma_\vartheta$. If $k \in \mathbf{K}_0(N_0)$, then by $f = v_1 \otimes f$ and (6) of Lemma 4.1.1 (with $\alpha = 1$, $\beta = b$, $k' = \mathbf{1}$),

$$\omega_{1,b}((1,k),1)f = b(\det k)^{-1}\,\Phi_k^\natural \otimes_P v^0, \qquad \Phi_k(X) = \Phi(X\bar{k}),$$

$\bar{k}$ the reduction of k modulo $\mathfrak{t}$; by Step 1 the coefficient $\langle f, \omega_{1,b}((1,k),1)f\rangle$ is

$$b(\det k)^{-1}N^{-1}\sum_X \overline{\Phi(X)}\Phi(X\bar{k}).$$

A term is nonzero only for $X = x_0 = x_0\bar{k}$, and then $\bar{k} = x_0^{\mathsf{T}}x_0\bar{k} = x_0^{\mathsf{T}}x_0 = \mathbf{1}$. So the coefficient is 0 if $k \notin J$, and $b(\det k)^{-1}N^{-1} = \langle f, f\rangle$ if $k \in J$ (Step 2). If $k \notin \mathbf{K}_0(N_0)$, write $k = k_1k_2$ with k_1 having the components of k at the places of S_0 and $\mathbf{1}$ elsewhere, k_2 the components of k elsewhere and $\mathbf{1}$ at S_0; then $k_2 \in J$, so $\omega_{1,b}((1,k_2),1)f = f$ by the case just treated (where $\Phi_{k_2} = \Phi$ and $b(\det k_2) = 1$), and $\omega_{1,b}((1,k),1)f = \omega_{1,b}((1,k_1),1)f$ by (2) of Lemma 4.1.1. By (4) of Lemma 4.1.1 (with $A = P$, $h = \mathbf{1}$) this is a scalar times $\Phi'\otimes_P v^0$ with $\Phi'(x) = \Phi^\natural(xk_1)$, and by (1), (2) of Lemma 6.2.33 the coefficient is that scalar times $\int_{X_P} \overline{\Phi^\natural(x)}\Phi^\natural(xk_1)\,dx$. The integrand vanishes everywhere: otherwise, for each $w \in P$, both x_w and x_wk_w are integral and reduce to x_0 modulo w^{e_w}, so $B = x_0^{\mathsf{T}}x_w$ and $B' = x_0^{\mathsf{T}}x_wk_w$ lie in $\mathrm{GL}_{n_0}(\mathcal{O}_{E,w})$ (they reduce to $\mathbf{1}$), $L = B^{-1}x_0^{\mathsf{T}}$ and $L' = B'^{-1}x_0^{\mathsf{T}}$ are integral left inverses of x_w and x_wk_w, and $k_w = L\,x_wk_w$, $k_w^{-1} = L'x_w$ are

integral; with Step 0 the components of k at S_0 are integral, and with the hypothesis at Σ_ϑ, $k \in \mathbf{K}_0(N_0)$, a contradiction. So the coefficient is 0. (SH3): $f = \Phi^\natural \otimes_P v^0$ with $\Phi^\natural \in \mathcal{T}_P$ ((6) of Lemma 4.1.1), P finite, nonempty and without conjugates (Step 0), and for $w \in P$, $\Phi^\natural(x) \neq 0$ forces x_w integral with $x_w \equiv x_0$, so x_w has the left inverse $(x_0^\intercal x_w)^{-1} x_0^\intercal$ as above. (SH4): for $k \in J$ and $k' \in \mathbf{K}_0(N_1)$ with $k' \equiv \mathbf{1} \pmod{\mathfrak{t}}$ (open subgroups, Step 0), (6) of Lemma 4.1.1 gives $\omega_{a,b}((1,k),(1,k'))f = a(\det k')\, b(\det k)^{-1}\, \Phi^\natural_{k,k'} \otimes_P v^0$ with $\Phi_{k,k'}(X) = \Phi(\bar{k}'^{-1} X \bar{k}) = \Phi(X)$, and $a(\det k') = b(\det k) = 1$ by Step 2; so f is fixed.

Step 4: (a) and (b). We apply Lemma 6.5.12 to S, T (signs on n_0 and n_1, with $2N_0 = 2(r + [z]) < r + t = N_1$, $[z] \in \{0,1\}$, $t = 5r$), the weights $\gamma = \gamma_{\neg z,i}(N_0)$, $\gamma' = \gamma_{\neg z,i}(N_1)$, the types $e = e_\lambda$, $e' = e_{\lambda'}$ with $\lambda = \frac{1}{2} p^i_{z,0,0}$, $\lambda' = \frac{1}{2} p^i_{z,0,1}$, the vector $v = v_H$ (a Schwartz vector, (XD3) of Definition 3.2.22, and an edge for (γ, γ') from e to e' by Lemma 6.4.15(b) with $n = 0$, as $i < m$), the characters a, b (of weights γ, γ' by (GP6)), S_0, J, the mass c and $F = f$, which is sharp by Step 3. It gives a constant $A \neq 0$ built from S, T, v_H, γ, γ' and e alone; none of these depends on ϑ ((TV1)–(TV3); the weights of the edge characters do not involve the twist), so $A_{z,i} = A$ is independent of ϑ. By Lemma 6.4.15(a) a form $x \in \mathcal{F}^0_\triangle(z,0)$ is a form of type e on S, and a base form y is moreover right invariant under $J \subseteq \mathrm{K}^{\mathrm{bs}}_z$ (Step 0). Hence $\langle a_\vartheta x, a_\vartheta y\rangle_T = A\, c\, \langle x, R_{\Sigma_\vartheta} y\rangle_S = \sum_q A\, c\, \mathrm{mc}_{b,\Sigma_\vartheta}(u_q) \langle x, y(\,\cdot\, u_q)\rangle_S$, absolutely convergent; multiplying by $\#\mathrm{GL}_{n_0}(R) = 1/(Nc)$ and replacing $\langle\cdot,\cdot\rangle_S$ by $\langle\cdot,\cdot\rangle_{V^0_{z,0}}$ (Lemma 6.4.15(a)) gives (a). The lift L_φ is injective on the forms of type e right invariant under J, which gives (b).

Step 5: (c). Let $u \in \mathrm{K}^{\mathrm{bs}}_z \subseteq \mathbf{K}_0(N_0)$ and write ρ for right translation by finite adelic points. By Step 0 choose $g \in \mathbf{K}_0(N_1)$ whose reduction modulo $\mathfrak{t}$ is $\bar{u} \oplus \mathbf{1}$, block diagonal for $n_1 = n_0 \sqcup (n_1 \setminus n_0)$. Then $\bar{g}^{-1} X \bar{u} = x_0$ holds exactly for $X = \bar{g} x_0 \bar{u}^{-1} = x_0$, so $\Phi_{u,g} = \Phi$ in (6) of Lemma 4.1.1, which gives $\omega_{a,b}((1,u),(1,g))\varphi = \lambda\varphi$ with $\lambda = a(\det g)\, b(\det u)^{-1}$, $|\lambda| = 1$. As $\omega_{a,b}$ is a representation of the product group, $\omega_{a,b}((1,u^{-1}),1)\varphi = \lambda^{-1}\omega_{a,b}(1,(1,g))\varphi$. For a form x of type e on S and $h \in \mathrm{G}(s')$, the substitution $g' \mapsto g'(1,u)^{-1}$, which preserves the quotient measure of $[\mathrm{U}(S)]$ ($\mathrm{G}(s)$ is unimodular), gives

$$L_\varphi(\rho(u)x)(h) = \int_{[\mathrm{U}(S)]} \theta\big(\omega_{a,b}(g',h)\, \omega_{a,b}((1,u^{-1}),1)\varphi\big)\, x(g')\, dg' = \lambda^{-1}\, L_\varphi x\big(h(1,g)\big),$$

that is $\rho(g)\, a_\vartheta x = \lambda\, a_\vartheta(\rho(u)x)$. Right translation by $(1,u)$ maps $\mathcal{F}^0_\triangle(z,0)$ bijectively onto itself (it preserves continuity, boundedness, left invariance under the rational points and the existence of a level, and commutes with $R(e)$, which acts by translation at the infinite places), and right translations preserve Petersson products. Now let y, v be as in (c); $\rho(u)y = y$. For every x,

$$\begin{aligned}\langle \rho(u)v, \rho(u)x\rangle &= \langle v, x\rangle = \langle a_\vartheta y, a_\vartheta x\rangle = \langle \rho(g) a_\vartheta y, \rho(g) a_\vartheta x\rangle \\ &= |\lambda|^2 \langle a_\vartheta(\rho(u)y), a_\vartheta(\rho(u)x)\rangle = \langle a_\vartheta y, a_\vartheta(\rho(u)x)\rangle = \langle v, \rho(u)x\rangle.\end{aligned}$$

As $\rho(u)$ is onto, $\rho(u)v - v$ is orthogonal to every triangle form at $(0,0)$, in particular to itself; it is continuous and the quotient measure charges every nonempty open set, so $\rho(u)v = v$. Hence $v \in \mathcal{F}^{\mathrm{bs}}_z$. □

Lemma 6.4.59. *Let $m \geq 0$ be an integer and I a set. Let V_0 and $\mathcal{H}_{i,n}$ ($i, n \in \mathbb{Z}_{\geq 0}$) be complex inner product spaces, and for $\vartheta \in I$ and $i, n \geq 0$ let*

$$a^\vartheta_{i,n} \colon \mathcal{H}_{i,n} \to \mathcal{H}_{i,n+1}, \qquad b^\vartheta_{i,n} \colon \mathcal{H}_{i,n+1} \to \mathcal{H}_{i+1,n}$$

*be linear maps, $w_\vartheta(n)$ positive integers, and $u_i \colon V_0 \to \mathcal{H}_{i,0}$ linear isometries. The **weighted chain** of row i is $X^\vartheta_{i,0} = \mathrm{id}$ and $X^\vartheta_{i,n+1} = w_\vartheta(n)\, a^\vartheta_{i,n} \circ X^\vartheta_{i,n} \colon \mathcal{H}_{i,0} \to \mathcal{H}_{i,n+1}$. Assume, for every $\vartheta \in I$:*

(Qu1) *for $i+n+2 \le m$ there is $\gamma \in \mathbb{C}$ with $w_\vartheta(n)\, a^\vartheta_{i+1,n} \circ b^\vartheta_{i,n} = \gamma\, b^\vartheta_{i,n+1} \circ a^\vartheta_{i,n+1}$ on $\mathcal{H}_{i,n+1}$;*

(Qu2) *for $i < m$, $y \in V_0$ and $z \in \mathcal{H}_{i,1}$: $\langle z, a^\vartheta_{i,0} u_i y\rangle = \langle b^\vartheta_{i,0} z, u_{i+1} y\rangle$;*

(Qu3) *for $i < m$, $b^\vartheta_{i,0} a^\vartheta_{i,0} u_i(V_0) \subseteq u_{i+1}(V_0)$, and $b^\vartheta_{i,0} a^\vartheta_{i,0} u_i y = 0$ only for $y = 0$.*

Then:

1. *for $i+n \le m$ and $\vartheta \in I$ the map $X^\vartheta_{i,n} \circ u_i$ is injective.*
2. *Let $\mathcal{M}$ be a relation between functions $c\colon I \to \mathbb{C}$ and $f\colon I \to \mathbb{R}$ (read "c has modulus f") such that $\mathcal{M}(1,1)$ holds and $\mathcal{M}(c,f)$, $\mathcal{M}(c',f')$ imply $\mathcal{M}(cc', ff')$. Let $f_{r,k}\colon I \to \mathbb{R}$ be given for $1 \le r < k \le m$, and assume:*

   (Qx1) *for $i+n+2 \le m$ there is $\beta\colon I \to \mathbb{C}\setminus\{0\}$ with $\mathcal{M}(\beta, f_{i+1,i+n+2})$ and, for every ϑ, $w_\vartheta(n)\,\overline{\beta(\vartheta)}\, a^\vartheta_{i+1,n} \circ b^\vartheta_{i,n} = w_\vartheta(n+2)\, b^\vartheta_{i,n+1} \circ a^\vartheta_{i,n+1}$ on $\mathcal{H}_{i,n+1}$;*

   (Qx2) *for $i+n+2 \le m$ there is $\kappa\colon I \to \mathbb{C}$ with $\mathcal{M}(\kappa, 1)$ such that for every ϑ and every $z \in \mathcal{H}_{i,n+2}$ some $x \in \mathcal{H}_{i,n+1}$ satisfies $\langle a^\vartheta_{i,n+1} y, z\rangle = \langle y, x\rangle$ for all $y \in \mathcal{H}_{i,n+1}$ and $\langle b^\vartheta_{i,n} x, y'\rangle = \kappa(\vartheta)\,\langle b^\vartheta_{i,n+1} z, a^\vartheta_{i+1,n} y'\rangle$ for all $y' \in \mathcal{H}_{i+1,n}$;*

   (Qx3) *U_0 is a complex vector space, $s\colon U_0 \to V_0$ a map, $S_\vartheta(i)$ ($i < m$) endomorphisms of U_0, and $C_i\colon I \to \mathbb{C}$ functions with $\mathcal{M}(C_i, 1)$, such that for $i < m$, $x \in V_0$, $y \in U_0$:*

$$w_\vartheta(0)\,\langle a^\vartheta_{i,0} u_i x,\ a^\vartheta_{i,0} u_i\, s(y)\rangle = C_i(\vartheta)\,\langle x,\ s(S_\vartheta(i) y)\rangle.$$

   Then for every $0 \le i \le m$ there is $\kappa_i\colon I \to \mathbb{C}$ with $\mathcal{M}\big(\kappa_i, \prod_{i<r<k\le m} f_{r,k}\big)$ such that, with $n = m - i$, for all ϑ, $x \in V_0$, $y \in U_0$,

$$\langle X^\vartheta_{i,n} u_i x,\ X^\vartheta_{i,n} u_i\, s(y)\rangle = \kappa_i(\vartheta)\,\frac{w_\vartheta(0)\, w_\vartheta(1)^n}{w_\vartheta(n)}\,\Big\langle x,\ s(S_\vartheta(i) S_\vartheta(i+1)\cdots S_\vartheta(m-1)\, y)\Big\rangle.$$

Products of functions on I are pointwise; an empty product is 1, an empty composite of endomorphisms is the identity.

Proof. **Step 1: moving $b \circ a$ past a chain.** Let $i \ge 0$ and $n \ge 0$ with $i+n < m$. For $k < n$ let γ_k be a scalar given by (Qu1) for the square (i,k) (admissible, since $i+k+2 \le i+n+1 \le m$). Put $c_0 = 1/w(0)$ and $c_{k+1} = c_k \gamma_k / w(k+1)$. We claim

$$X_{i+1,n} \circ b_{i,0} \circ a_{i,0} = c_n\, b_{i,n} \circ X_{i,n+1} \qquad \text{on } \mathcal{H}_{i,0}. \tag{6.81}$$

For $n = 0$ the right side is $\frac{1}{w(0)} b_{i,0}(w(0) a_{i,0}) = b_{i,0} a_{i,0}$. If (6.81) holds for n and $i+n+1 < m$, then for $y \in \mathcal{H}_{i,0}$, with $Y = X_{i,n+1} y$,

$$\begin{aligned} X_{i+1,n+1} b_{i,0} a_{i,0} y &= w(n)\, a_{i+1,n} X_{i+1,n} b_{i,0} a_{i,0} y = c_n\, w(n)\, a_{i+1,n} b_{i,n} Y \\ &= c_n \gamma_n\, b_{i,n+1} a_{i,n+1} Y = \frac{c_n \gamma_n}{w(n+1)}\, b_{i,n+1} X_{i,n+2} y, \end{aligned}$$

by (Qu1) and $X_{i,n+2} = w(n+1) a_{i,n+1} X_{i,n+1}$; this is (6.81) for $n+1$.

Step 2: part (1). Induction on n, for all i at once. For $n = 0$, $X_{i,0} u_i = u_i$ is an isometry. Let $i+n+1 \le m$ and $X_{i,n+1} u_i y = 0$. By (Qu3) (as $i < m$) there is $x \in V_0$ with $u_{i+1} x = b_{i,0} a_{i,0} u_i y$. By (6.81) (as $i+n < m$), $X_{i+1,n} u_{i+1} x = c_n\, b_{i,n} X_{i,n+1} u_i y = 0$; since $(i+1)+n \le m$, the induction hypothesis gives $x = 0$. Hence $b_{i,0} a_{i,0} u_i y = 0$, and $y = 0$ by (Qu3).

Step 3: the constants of part (2). Assume (Qx1)–(Qx3); (Qx1) implies (Qu1) with $\gamma = w(n+2)/\overline{\beta(\vartheta)}$. For a square (i,k) with $i+k+2 \le m$ fix $\beta_{i,k}$ as in (Qx1) and $\kappa'_{i,k}$ as in (Qx2). For $i+n<m$ put $\beta^{(i,n)} = \prod_{k<n} \beta_{i,k}$ and $\kappa^{(i,n)} = \prod_{k<n} \kappa'_{i,k}$. By the multiplicativity of $\mathcal{M}$ and since the square (i,k) has modulus $f_{i+1,i+k+2}$,

$$\mathcal{M}\Big(\beta^{(i,n)}, \prod_{i+1<k\le i+1+n} f_{i+1,k}\Big), \qquad \mathcal{M}\big(\kappa^{(i,n)}, 1\big).$$

With $\gamma_k = w(k+2)/\overline{\beta_{i,k}}$ in Step 1 the recursion telescopes: $c_n = \frac{1}{w(0)} \prod_{k<n} \frac{w(k+2)}{w(k+1)\,\overline{\beta_{i,k}}} = \frac{w(n+1)}{w(0)\,w(1)\,\overline{\beta^{(i,n)}}}$, a nonzero number.

Step 4: adjointness along a chain. For $i+n<m$, $y \in V_0$ and $z \in \mathcal{H}_{i,n+1}$,

$$\big\langle z,\, a_{i,n} X_{i,n} u_i y \big\rangle = \kappa^{(i,n)} \big\langle b_{i,n} z,\, X_{i+1,n} u_{i+1} y \big\rangle. \tag{6.82}$$

For $n=0$ this is (Qu2). Let it hold for n, with $i+n+1<m$, and let $z \in \mathcal{H}_{i,n+2}$. Take $x \in \mathcal{H}_{i,n+1}$ as in (Qx2) for the square (i,n). Then $\langle z, a_{i,n+1} v\rangle = \overline{\langle a_{i,n+1} v, z\rangle} = \overline{\langle v, x\rangle} = \langle x, v\rangle$ for $v \in \mathcal{H}_{i,n+1}$, so, writing $Y = X_{i+1,n} u_{i+1} y$,

$$\begin{aligned} \big\langle z, a_{i,n+1} X_{i,n+1} u_i y \big\rangle &= w(n) \big\langle x, a_{i,n} X_{i,n} u_i y \big\rangle = w(n)\, \kappa^{(i,n)} \langle b_{i,n} x, Y\rangle \\ &= w(n)\, \kappa^{(i,n)} \kappa'_{i,n} \big\langle b_{i,n+1} z, a_{i+1,n} Y \big\rangle, \end{aligned}$$

which is $\kappa^{(i,n+1)} \langle b_{i,n+1} z, X_{i+1,n+1} u_{i+1} y\rangle$.

Step 5: one row down. Let $i+n<m$, $x, z \in V_0$, and $x' \in V_0$ with $u_{i+1} x' = b_{i,0} a_{i,0} u_i x$ ((Qu3)). Put $X = X_{i,n+1} u_i x$, $Y = X_{i+1,n} u_{i+1} z$. By (6.82) with z replaced by X,

$$\langle X, X_{i,n+1} u_i z\rangle = w(n) \langle X, a_{i,n} X_{i,n} u_i z\rangle = w(n)\, \kappa^{(i,n)} \langle b_{i,n} X, Y\rangle.$$

By (6.81) and Step 3, $X_{i+1,n} u_{i+1} x' = c_n\, b_{i,n} X$, so $\langle X_{i+1,n} u_{i+1} x', Y\rangle = \overline{c_n} \langle b_{i,n} X, Y\rangle$ with $\overline{c_n} = w(n+1)/(w(0)w(1)\beta^{(i,n)}) \neq 0$. Hence

$$\big\langle X_{i,n+1} u_i x,\, X_{i,n+1} u_i z \big\rangle = \kappa^{(i,n)} \beta^{(i,n)}\, \frac{w(0)\, w(1)\, w(n)}{w(n+1)}\, \big\langle X_{i+1,n} u_{i+1} x',\, X_{i+1,n} u_{i+1} z \big\rangle. \tag{6.83}$$

Step 6: part (2). Descending induction on i from $i=m$, with $n = m-i$. For $i=m$ take $\kappa_m = 1$; both sides are $\langle x, s(y)\rangle$ (the factor is $w(0)/w(0) = 1$). Let $i<m$, $n+1 = m-i$, and let κ_{i+1} be given. For $x \in V_0$ take x' as in Step 5 and put $T = S(i+1)\cdots S(m-1)$. Applying (6.83) (with $z = s(y)$; here $i+n<m$) and then the statement for $i+1$,

$$\big\langle X_{i,n+1} u_i x, X_{i,n+1} u_i s(y)\big\rangle = \kappa^{(i,n)} \beta^{(i,n)} \kappa_{i+1}\, \frac{w(0)\, w(1)^{n+1}}{w(n+1)} \cdot w(0)\, \langle x', s(Ty)\rangle.$$

Since u_{i+1} is an isometry and by (Qu2) with $z = a_{i,0} u_i x$,

$$\langle x', s(Ty)\rangle = \big\langle b_{i,0} a_{i,0} u_i x,\, u_{i+1} s(Ty)\big\rangle = \big\langle a_{i,0} u_i x,\, a_{i,0} u_i s(Ty)\big\rangle,$$

and by (Qx3) $w(0)\langle x', s(Ty)\rangle = C_i\, \langle x, s(S(i)Ty)\rangle$. So the statement holds for i with $\kappa_i = \kappa^{(i,n)} \beta^{(i,n)} \kappa_{i+1} C_i$. Its modulus: $\mathcal{M}(\kappa^{(i,n)}, 1)$, $\mathcal{M}(C_i, 1)$, $\mathcal{M}(\beta^{(i,n)}, \prod_{i+1<k\le m} f_{i+1,k})$ (Step 3, $i+1+n = m$) and $\mathcal{M}(\kappa_{i+1}, \prod_{i+1<r<k\le m} f_{r,k})$ give, by multiplicativity, $\mathcal{M}(\kappa_i, \prod_{i<r<k\le m} f_{r,k})$, the factors with $r = i+1$ being those of $\beta^{(i,n)}$. $\square$

Lemma 6.4.60. *Let D be admissible data, $z = \mathrm{I}$, $i, n \geq 0$ with $i + n < m$, Φ a left pair and Φ' a reversed pair at (i, n). For an m-fold twist ϑ let $u_\vartheta \in G_f(n_n)$ be the scalar matrix of the finite idele $\overline{\alpha_{\mathrm{W}} t}/(\alpha_{\mathrm{W}} t)$, t the idele of (TV7) for ϑ, and ρ right translation. There are functions c_1, c_2 from the m-fold twists to $\mathbb{C}$, each of absolute value 1 up to a norm power, such that for every m-fold twist ϑ, every function Ψ on $\mathrm{M}_{n_{n+1} \times n_n}(R_\vartheta)$, every $x \in \mathcal{F}^i_\triangle(z, n+1)$ and every $y \in \mathcal{F}^{i+1}_\triangle(z, n)$,*

$$\langle \mathrm{Ds}^i_{\vartheta,z,n}(\Phi, \Psi)x,\ y\rangle = c_1(\vartheta)\,\langle x,\ \mathrm{As}^i_{\vartheta,z,n}(\Phi', \widehat{\Psi})\,\rho(u_\vartheta^{-1})y\rangle,$$
$$\langle x,\ \mathrm{As}^i_{\vartheta,z,n}(\Phi', \widehat{\Psi})\,\rho(u_\vartheta^{-1})y\rangle = c_2(\vartheta)\,\langle \mathrm{Ds}^i_{\vartheta,z,n}(\Phi, \Psi)x,\ y\rangle,$$

the Petersson products being those of $V^{i+1}_{z,n}$ and of $V^i_{z,n+1}$.

Proof. Write $V = V^{i+1}_{z,n}$, $W = V^i_{z,n+1}$, $\mathfrak{t} = \mathfrak{t}_\vartheta(z)$, $u = u_\vartheta$, $k = N_n N_{n+1}$, $\mu = \frac{1}{2}p^{i+1}_{z,0,n}$, $\mu' = \frac{1}{2}p^i_{z,0,n+1}$; the triangle forms on V and W are the forms of type e_μ and $e_{\mu'}$, which are cusp kernels of order 0 together with $e_{-\mu} = \overline{e_\mu}$, $e_{-\mu'} = \overline{e_{\mu'}}$ ((ED4)(a) of the edges of (TV6)). For a polynomial P write

$$K_P(g, h) = \theta^{V,W}_{\chi(N_n),\chi(N_{n+1})}(\mathrm{mod}_{\mathfrak{t}}(\Psi, v_P))(g, h),$$
$$K'_P(h, g) = \theta^{W,V}_{\chi(N_{n+1}),\chi(N_n)}(\mathrm{mod}_{\mathfrak{t}}(\widehat{\Psi}, v_P))(h, g)$$

(g on V, h on W), and D_P, A_P for the maps

$$x \mapsto \int_{[\mathrm{U}(W)]} K_P(\cdot, h)x(h)\,dh \quad \text{and} \quad y \mapsto \int_{[\mathrm{U}(V)]} K'_P(\cdot, g)y(g)\,dg;$$

by (TV7), $D_\Phi = \mathrm{Ds}^i_{\vartheta,z,n}(\Phi, \Psi)$ and $A_{\Phi'} = \mathrm{As}^i_{\vartheta,z,n}(\Phi', \widehat{\Psi})$, the idele t and the character ψ_ϑ of (TV7) being those of Lemma 6.4.61 for $\mathfrak{m} = \mathfrak{t}$.

Step 0: reversal and the norm of $\#R_\vartheta$. By Lemma 6.4.61 (with $(\alpha, \beta) = (\chi(N_n), \chi(N_{n+1}))$, $D = \Psi$) there is $\tau = \tau(\vartheta)$ with $|\tau|^2(\#R_\vartheta)^k = 1$ (so $\tau \neq 0$) and $\overline{K_P(g, h)} = \tau\, K'_{P^\flat}(h, g(1, u))$ for every polynomial P. Moreover $\#R_\vartheta = \#(\mathcal{O}_E/\pi_M\mathcal{O}_E)\cdot \mathrm{N}Q$ (Chinese remainder theorem, (TV2), the places of Q prime to π_M), so $|\tau|^2 = C_0\,(\mathrm{N}Q)^{-k}$ with $C_0 = \#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^{-k}$, independent of ϑ. Right translation by the central element $(1, u^{\pm 1})$ maps triangle forms on V to triangle forms (it preserves continuity, boundedness, left invariance under the rational points and the existence of a level, and commutes with $R(e_\mu)$, which acts by right translation at the infinite places) and preserves Petersson products (the quotient measure of $[\mathrm{U}(V)]$ is right invariant, the adelic group being unimodular). The map $P \mapsto P^\flat$ for the pair (V, W) followed by the same map for (W, V) is the identity (the coefficients are conjugated twice and $X_{v,a,b} \mapsto (-s_b s'_a)(-s'_a s_b)X_{v,a,b}$), so $\Phi' = Q^\flat$ for $Q = \Phi'^\flat$ (formed for (W, V)).

Step 1: the first identity. By Lemma 6.4.52 the kernels are jointly integrable against products of triangle forms, so Fubini applies below. With $y' = \rho(u^{-1})y$ and the substitution $g \mapsto g(1, u)^{-1}$ (invariance of the quotient measure),

$$\begin{aligned}\langle D_\Phi x, y\rangle_V &= \int_{[\mathrm{U}(V)]} y(g) \int_{[\mathrm{U}(W)]} \overline{K_\Phi(g, h)}\,\overline{x(h)}\,dh\,dg \\ &= \tau \int_{[\mathrm{U}(W)]} \overline{x(h)} \int_{[\mathrm{U}(V)]} K'_{\Phi^\flat}(h, g)\,y'(g)\,dg\,dh = \tau\,\langle x, A_{\Phi^\flat}y'\rangle_W.\end{aligned}$$

By Lemma 6.3.77 and Lemma 6.4.53 (exchanging the two factors), A_P is the lift for a theta kernel for (V, W) whose real part is the exchanged $\Omega_{\delta,\delta'}$, $(\delta, \delta') = (\delta_i(N_{n+1}), \delta_i(N_n))$. The reversed pair $v_{\Phi'}$ is an edge for (δ, δ') from $e_{-\mu'}$ to $e_{-\mu}$ (TV6); its second half edge is a half edge for the exchanged real part from $\overline{e_{-\mu}} = e_\mu$ to $\overline{e_{-\mu'}} = e_{\mu'}$, an idempotent kernel, and $A_{\Phi'}y'$ has a level

(Lemma 6.4.51(b)). So Lemma 6.4.57 (with $P = \Phi^\flat$, the form y' of type e_μ on V and $G = x$ of type $e_{\mu'}$ on W) gives κ_1, independent of ϑ and Ψ, with $\langle x, A_{\Phi^\flat} y'\rangle_W = \kappa_1 \langle x, A_{\Phi'} y'\rangle_W$. Hence the first identity holds with $c_1(\vartheta) = \tau\kappa_1$, and $|c_1|^2 = |\kappa_1|^2 C_0(\mathrm{N}Q)^{-k}$: absolute value 1 up to a norm power.

Step 2: the second identity. By Step 0 applied to Q, $K'_{\Phi'}(h, g(1,u)) = \tau^{-1}\overline{K_Q(g,h)}$. With the substitution $g \mapsto g(1,u)$,

$$A_{\Phi'} y'(h) = \int_{[\mathrm{U}(V)]} K'_{\Phi'}(h, g(1,u))\, y(g)\, dg = \tau^{-1} \int_{[\mathrm{U}(V)]} \overline{K_Q(g,h)}\, y(g)\, dg,$$

so by Fubini $\langle x, A_{\Phi'} y'\rangle_W = \tau^{-1} \int_{[\mathrm{U}(V)]} y(g)\overline{D_Q x(g)}\, dg = \tau^{-1}\langle D_Q x, y\rangle_V$. The left pair v_Φ is an edge for $(\delta_i(N_n), \delta_i(N_{n+1}))$ from $e_{-\mu}$ to $e_{-\mu'}$; its second half edge is for the exchanged real part from $e_{\mu'}$ to e_μ, an idempotent kernel, and $D_\Phi x$ has a level (Lemma 6.4.51(b)); D_P is the lift for the exchanged kernel of the pair (V, W) (Lemma 6.3.77, Lemma 6.4.53). So Lemma 6.4.57 (with $P = Q$, the form x and $G = y$) gives κ_2, independent of ϑ and Ψ, with $\langle y, D_Q x\rangle_V = \kappa_2 \langle y, D_\Phi x\rangle_V$, i.e. $\langle D_Q x, y\rangle_V = \overline{\kappa_2}\langle D_\Phi x, y\rangle_V$. Hence the second identity holds with $c_2(\vartheta) = \tau^{-1}\overline{\kappa_2}$, and $|c_2|^2 = |\kappa_2|^2 C_0^{-1}(\mathrm{N}Q)^k$. □

Lemma 6.4.61. *Let S, T be spaces with signs s on n and s' on n', α, β automorphic characters of $\mathrm{U}(1)$, $\mathfrak{m}$ a modulus and t a finite idele, integral, with components 1 at the finite places outside $\Sigma(\mathfrak{m})$, generating $\mathfrak{m}\widehat{\mathcal{O}}_E$. Let $\psi_\mathfrak{m}$ be the additive character of $R_\mathfrak{m}$ with $\psi_\mathfrak{m}(x \bmod \mathfrak{m}) = \psi_{E,f}(\jmath\, \alpha_\mathrm{W}\bar\alpha_\mathrm{W}\, t^{-1} x)$ for $x \in \widehat{\mathcal{O}}_E$; for a function D on $\mathrm{M}_{n'\times n}(R_\mathfrak{m})$ let $\widehat{D}(\rho) = \sum_x \overline{D(x)}\, \psi_\mathfrak{m}(\mathrm{tr}(\rho x))$ on $\mathrm{M}_{n\times n'}(R_\mathfrak{m})$; for a polynomial P in the variables $X_{v,a,b}$ ($v \in \Phi$, $a \in n'$, $b \in n$) let $P^\flat$ be the polynomial obtained from P by conjugating its coefficients and substituting $X_{v,a,b} \mapsto -s'_a\, s_b\, X_{v,b,a}$; and let $u_\mathfrak{m} \in G_f(n)$ be the scalar matrix of the finite idele $\overline{\alpha_\mathrm{W} t}/(\alpha_\mathrm{W} t)$, of norm one. There is $c \in \mathbb{C}$ with $|c|^2\, (\#R_\mathfrak{m})^{\#n\cdot\#n'} = 1$ such that for all D, P, $g \in \mathrm{G}(s)$ and $h \in \mathrm{G}(s')$*

$$\overline{\theta^{S,T}_{\alpha,\beta}(\mathrm{mod}_\mathfrak{m}(D, v_P))(g,h)} = c\, \theta^{T,S}_{\beta,\alpha}(\mathrm{mod}_\mathfrak{m}(\widehat{D}, v_{P^\flat}))(h,\, g\,(1, u_\mathfrak{m})).$$

Proof. Write $\varepsilon = s' \otimes s$, $\varepsilon(a,b) = s'_a s_b$ on $n' \times n$. Then $\mathrm{Hom}(S^-, T)$ has the signs $-\varepsilon$ on $n' \times n$ and $\mathrm{Hom}(T, S)$ the signs $\varepsilon'(b,a) = s_b s'_a$ on $n \times n'$ (VP5); their vectors are read as $n' \times n$ and $n \times n'$ adelic matrices. Put $N = \#n \cdot \#n'$ and $A = \Sigma(\mathfrak{m})$; as A contains the conjugate of none of its members, $A \cap \bar A = \emptyset$. The letter c also denotes the idele with infinite components 1 and finite component c; $c^{-1} = -\bar c$. Let $u_\mathrm{W}, u_t \in \mathrm{G}(s)$ be the scalar matrices with infinite components 1 and finite components $\bar\alpha_\mathrm{W}/\alpha_\mathrm{W}$ and $\bar t/t$ (both of norm one), so that $u_t u_\mathrm{W} = (1, u_\mathfrak{m})$, and let $S' = S_{u_\mathrm{W}} = (J_S, x_S u_\mathrm{W})$ be the moved space (VP4), with signs s.

Step 1: the flip. For an adelic $n' \times n$ matrix X let $X^\dagger$ be the $n \times n'$ matrix with components

$$(X^\dagger)_v = -\,\mathrm{diag}(s)\, X_v^*\, \mathrm{diag}(s') \quad (v \in \Phi), \qquad (X^\dagger)_f = c\,\alpha_\mathrm{W}\bar\alpha_\mathrm{W}^{-1}\, X_f^*,$$

that is $X^\dagger = \mu\, \varepsilon_s X^* \varepsilon_{s'}$ for the scalar idele μ with infinite components -1 and finite component $c\alpha_\mathrm{W}\bar\alpha_\mathrm{W}^{-1}$; entrywise $(X^\dagger)_{v,b,a} = -s_b s'_a \overline{X_{v,a,b}}$. It is additive and a homeomorphism, with inverse $Y \mapsto Y^\ddagger$, $(Y^\ddagger)_v = -\,\mathrm{diag}(s') Y_v^*\, \mathrm{diag}(s)$, $(Y^\ddagger)_f = -c\,\alpha_\mathrm{W}\bar\alpha_\mathrm{W}^{-1} Y_f^*$. It has five properties.

(F1) $\beta_{\varepsilon'}(x^\dagger, y^\dagger) = \beta_{-\varepsilon}(x, y)$ for $x, y \in \mathbb{A}_E^{n'\times n}$. Put $Q_v = \sum_{a,b} \varepsilon(a,b)\overline{y_{v,a,b}} x_{v,a,b}$ for $v \in \Phi$ and $Q_f = \sum_{a,b} \overline{y_{a,b}} x_{a,b}$ (finite components). By (WD4), (WD6),

$$\beta_{-\varepsilon}(x,y) = \exp(-2\pi i \sum_v 2\,\mathrm{Re}(-\jmath_v Q_v/2))\psi_{E,f}(\jmath Q_f/2).$$

For the flipped vectors, at $v \in \Phi$ the term of index (b,a) is

$$\varepsilon'(b,a)\overline{(y^\dagger)_{v,b,a}}(x^\dagger)_{v,b,a} = \varepsilon(a,b)y_{v,a,b}\overline{x_{v,a,b}},$$

with sum $\overline{Q_v}$, and at the finite places the factor is $\overline{c\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}^{-1}}\, c\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}^{-1} = \bar c c = -1$, with sum $-\overline{Q_f}$. Since $\bar\jmath = -\jmath$, $\mathrm{Re}(\jmath_v\overline{Q_v}) = \mathrm{Re}(\overline{\jmath_v}Q_v) = \mathrm{Re}(-\jmath_v Q_v)$ and $-\jmath\overline{Q_f}/2 = \overline{\jmath Q_f/2}$; and $\psi_{E,f}(\bar y_0) = \psi_{E,f}(y_0)$, because $y_0 - a \in \widehat{\mathcal{O}}_E$ with $a \in E$ gives $\bar y_0 - \bar a \in \widehat{\mathcal{O}}_E$ and $\mathrm{Tr}_{E/\mathbb{Q}}\,\bar a = \mathrm{Tr}_{E/\mathbb{Q}}\,a$. This gives (F1).

(F2) $e_N(x^\dagger) = e_N(x)$. The archimedean factor of e_N sees only $|x_{v,a,b}| = |(x^\dagger)_{v,b,a}|$. At the finite places write $x_f = \alpha_{\mathrm{W}} z$; then $(x^\dagger)_f = c\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}^{-1}\bar\alpha_{\mathrm{W}} z^* = \alpha_{\mathrm{W}}\, cz^*$, so $(x^\dagger)_f \in \alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E^N$ if and only if $x_f \in \alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E^N$. In that case $\sum\overline{(x^\dagger)_f}(x^\dagger)_f = \bar c c\sum x_f\overline{x_f} = -M$ with $M = \sum\overline{x_f}x_f = \alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}\sum\bar z z \in \alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E$. By Lemma 4.1.3, $c_0\jmath M \in \jmath\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1}\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1} + \widehat{\mathcal{O}}_E$, on which $\psi_{E,f}$ is trivial ($\mathrm{Tr}_{E/\mathbb{Q}}\,\mathfrak{D}_{E/\mathbb{Q}}^{-1} \subseteq \mathbb{Z}$). Hence $\psi_{E,f}(-c_0\jmath M/2) = \psi_{E,f}(c_0\jmath M/2)\,\psi_{E,f}(-c_0\jmath M) = \psi_{E,f}(c_0\jmath M/2)$, and the finite factors of (WD5) agree.

(F3) The image of μ_N under $x \mapsto x^\dagger$ is μ_N. In the coordinates of (WD6) (archimedean coordinates divided by κ_v, which is real, and finite coordinates divided by α_{W}) the map is $u \mapsto (-s_b s'_a\overline{u_{v,a,b}})_{v,b,a}$ on $\mathbb{C}^{\Phi\times n'\times n}$, a real-linear isometry (a permutation of coordinates, conjugations and signs) that preserves Lebesgue measure, and $z \mapsto cz^*$ on the finite coordinates, a permutation of coordinates composed with the adelic conjugation (a ring automorphism of $\mathbb{A}_{E,f}$ mapping $\widehat{\mathcal{O}}_E$ onto itself) and multiplication by the unit c, each of which preserves the Haar measure giving $\widehat{\mathcal{O}}_E^N$ mass 1.

(F4) $X \mapsto X^\dagger$ maps the rational vectors of $\mathrm{Hom}(S^-,T)$ onto those of $\mathrm{Hom}(T,S')$. The point of $\mathrm{Hom}(S^-,T)$ is the map $X \mapsto x_T X(cx_S)^{-1}$ (VP5), so its rational vectors are the $X = c\,x_T^{-1}\xi_0 x_S$ with $\xi_0 \in \mathrm{M}_{n'\times n}(E)$; those of $\mathrm{Hom}(T,S')$, whose point is $Y \mapsto x_S u_{\mathrm{W}} Y x_T^{-1}$, are the $u_{\mathrm{W}}^{-1}x_S^{-1}\eta_0 x_T$ with $\eta_0 \in \mathrm{M}_{n\times n'}(E)$. From $x_S^* J_S x_S = \varepsilon_s$ and $x_T^* J_T x_T = \varepsilon_{s'}$ (VP2), $x_S^* = \varepsilon_s x_S^{-1} J_S^{-1}$ and $(x_T^*)^{-1} = J_T x_T \varepsilon_{s'}$, so for $X = c\,x_T^{-1}\xi_0 x_S$

$$\varepsilon_s X^* \varepsilon_{s'} = \bar c\,\varepsilon_s\,\varepsilon_s x_S^{-1} J_S^{-1}\xi_0^* J_T x_T \varepsilon_{s'}\,\varepsilon_{s'} = \bar c\, x_S^{-1}(J_S^{-1}\xi_0^* J_T)x_T.$$

The scalar idele $\mu\bar c$ has infinite components -1 and finite component $\bar c c\,\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}^{-1} = -\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}^{-1}$, i.e. $\mu\bar c = -u_{\mathrm{W}}^{-1}$. Hence $X^\dagger = u_{\mathrm{W}}^{-1}x_S^{-1}(-J_S^{-1}\xi_0^* J_T)x_T$, and $\xi_0 \mapsto -J_S^{-1}\xi_0^* J_T$ is a bijection of $\mathrm{M}_{n'\times n}(E)$ onto $\mathrm{M}_{n\times n'}(E)$.

(F5) For $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$: $(hXg^{-1})^\dagger = g\,X^\dagger h^{-1}$. Indeed $g^*\varepsilon_s g = \varepsilon_s$ gives $\varepsilon_s(g^{-1})^* = g\varepsilon_s$, $h^*\varepsilon_{s'}h = \varepsilon_{s'}$ gives $h^*\varepsilon_{s'} = \varepsilon_{s'}h^{-1}$, and μ is a scalar, so $(hXg^{-1})^\dagger = \mu\varepsilon_s(g^{-1})^* X^* h^*\varepsilon_{s'} = g\,\mu\varepsilon_s X^*\varepsilon_{s'}\,h^{-1}$.

Step 2: the transport. For a function f on $\mathbb{A}_E^{n'\times n}$ put $(\Pi f)(Y) = f(Y^\ddagger)$, the composite of f with the inverse of the flip $X \mapsto X^\dagger$. By (F1), (F2) and (F3) the flip, an additive homeomorphism, satisfies the hypotheses of (1) of Lemma 6.3.65 for the signs $-\varepsilon$ and ε'; so Π is a linear bijection of $\mathcal{M}_{-\varepsilon}$ onto $\mathcal{M}_{\varepsilon'}$ preserving norm and inner product, $\Pi v^0 = v^0$, and

$$\Pi\rho(y_0) = \rho(y_0^\dagger)\,\Pi \qquad (y_0 \in \mathbb{A}_E^{n'\times n}). \tag{6.84}$$

Let $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$, k the pair map of (g,h) for (S^-,T), $X \mapsto hXg^{-1}$, and k' the pair map of (h,g) for (T,S), $Y \mapsto gYh^{-1}$. By (F5), $(kX)^\dagger = k'X^\dagger$ for every X, so (2) of Lemma 6.3.65 gives

$$\Pi\,\omega_{-\varepsilon}(k) = \frac{\Delta_{-\varepsilon}(k)}{\Delta_{\varepsilon'}(k')}\,\omega_{\varepsilon'}(k')\,\Pi.$$

We show $\Delta_{\varepsilon'}(k') = \upsilon(\det k)\,\Delta_{-\varepsilon}(k)$. As matrices, $k_{(a,b),(a_1,b_1)} = h_{aa_1}(g^{-1})_{b_1 b}$ and $k'_{(b,a),(b_1,a_1)} = g_{bb_1}(h^{-1})_{a_1 a}$; since k^{-1} is $X \mapsto h^{-1}Xg$, $(k^{-1})_{(a_1,b_1),(a,b)} = k'_{(b,a),(b_1,a_1)}$: after the identification

$(b,a) \leftrightarrow (a,b)$, k' is the transpose of k^{-1}. The identification is a relabelling of the index set by a bijection; it carries the signs ε' to ε, the block of sign -1 at an infinite place to the corresponding block, and $\widehat{\mathcal{O}}_E^{n\times n'}$ to $\widehat{\mathcal{O}}_E^{n'\times n}$ (so an element u of (WR3) to its relabelling, of the same determinant); hence it does not change the multiplier, and $\Delta_{\varepsilon'}(k') = \Delta_\varepsilon((k^{-1})^{\mathsf{T}})$ with $k \in \mathrm{G}(-\varepsilon)$. By (4) of Lemma 6.3.65 for $\delta = -\varepsilon$, this is $\upsilon(\det k)\,\Delta_{-\varepsilon}(k)$, which is the claim, and $\Pi\omega_{-\varepsilon}(k) = \upsilon(\det k)^{-1}\omega_{\varepsilon'}(k')\Pi$. With $\det k = \det(h)^{\#n}\det(g)^{-\#n'}$ (VP5) and (WR4), for all automorphic characters α_1, β_1 of $\mathrm{U}(1)$,

$$\begin{aligned}\Pi\,\omega^{S^-,T}_{\alpha_1,\beta_1}(g,h) &= \alpha_1(\det h)\beta_1(\det g)^{-1}\upsilon(\det h)^{-\#n}\upsilon(\det g)^{\#n'}\omega_{\varepsilon'}(k')\,\Pi \\ &= \omega^{T,S}_{\alpha_2,\beta_2}(h,g)\,\Pi, \qquad \alpha_2 = \beta_1^{-1}\upsilon^{\#n'},\ \beta_2 = \alpha_1^{-1}\upsilon^{\#n}.\end{aligned} \tag{6.85}$$

Step 3: theta functionals and kernels. By (F4) the flip maps the rational vectors of $\mathrm{Hom}(S^-,T)$ onto those of $\mathrm{Hom}(T,S')$, a space with a point with the signs ε'; Π is bounded, $\langle \Pi v^0, \Pi v^0\rangle = \langle v^0, v^0\rangle = 1$ ((1) of Lemma 6.2.13), and (6.84) holds. So (3) of Lemma 6.3.65, for $V = \mathrm{Hom}(S^-,T)$, $V' = \mathrm{Hom}(T,S')$ and τ the flip, gives $z_0 \in \mathbb{C}$, $|z_0| = 1$, with $\theta_{\mathrm{Hom}(T,S')}(\Pi F) = z_0\,\theta_{\mathrm{Hom}(S^-,T)}(F)$ for all F. Next, the point of $\mathrm{Hom}(T,S')$ is that of $\mathrm{Hom}(T,S)$ composed with the pair map $k_{\mathrm{W}}\colon Y \mapsto u_{\mathrm{W}}Y$ of $(1,u_{\mathrm{W}})$ for (T,S), so $\mathrm{Hom}(T,S')$ is the moved space $\mathrm{Hom}(T,S)_{k_{\mathrm{W}}}$ (VP4). If k_0 is a transporting element of $\mathrm{Hom}(T,S)$ with point X_0, then $k_0k_{\mathrm{W}}(X_0k_{\mathrm{W}})^{-1}u = k_0X_0^{-1}u$, so k_0k_{W} is one of the moved space, and by Lemma 4.1.17 (independence of the transporting element) and (2) of Lemma 4.1.1, $\theta_{\mathrm{Hom}(T,S')}(G) = \theta_{\mathrm{Hom}(T,S)}(\omega_{\varepsilon'}(k_{\mathrm{W}})G)$. As $\omega^{T,S}_{\alpha_2,\beta_2}(1,u_{\mathrm{W}}) = \alpha_2(\det u_{\mathrm{W}})\,\omega_{\varepsilon'}(k_{\mathrm{W}})$ (WR4) and u_{W} is central, (WC4), (6.85) and (2) of Lemma 4.1.1 give, for $F \in \mathcal{M}_{-\varepsilon}$, $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$,

$$\theta^{S^-,T}_{\alpha_1,\beta_1}(F)(g,h) = z_0^{-1}\,\theta_{\mathrm{Hom}(T,S')}(\omega^{T,S}_{\alpha_2,\beta_2}(h,g)\Pi F) = z_0^{-1}\alpha_2(\det u_{\mathrm{W}})^{-1}\,\theta^{T,S}_{\alpha_2,\beta_2}(\Pi F)(h, g\,u_{\mathrm{W}}). \tag{6.86}$$

Step 4: the data. *Real part.* For $\tau \in \mathbb{C}^{\Phi\times n'\times n}$ let $\tau^\sharp \in \mathbb{C}^{\Phi\times n\times n'}$, $\tau^\sharp_{v,b,a} = -s_bs'_a\overline{\tau_{v,a,b}}$; then $\mathrm{rv}(\tau)^\dagger = \mathrm{rv}(\tau^\sharp)$ (κ_v is real and the finite components are 0), $\tau \mapsto \tau^\sharp$ preserves ν, and its inverse is φ_∞, $(\varphi_\infty\tau')_{v,a,b} = -s'_as_b\overline{\tau'_{v,b,a}}$. For an integrable w_0 on $\mathbb{C}^{\Phi\times n'\times n}$ and $f \in \mathcal{M}_{-\varepsilon}$, the Bochner integral of (1) of Lemma 6.2.21, the boundedness of Π and (6.84) give

$$\Pi(w_0\otimes f) = \int w_0(\tau)\,\rho(\mathrm{rv}(\tau^\sharp))\Pi f\,d\nu(\tau) = \int w_0(\varphi_\infty\tau')\,\rho(\mathrm{rv}(\tau'))\Pi f\,d\nu(\tau') = (w_0\circ\varphi_\infty)\otimes\Pi f.$$

For the Fock vector $v_{\bar P}$ of the signs $-\varepsilon$: $\zeta_{v,a,b}(x)$ is $\overline{x_{v,a,b}}$ if $s'_as_b = -1$ and $x_{v,a,b}$ if $s'_as_b = 1$, while for ε', $\zeta'_{v,b,a}(y)$ is $\overline{y_{v,b,a}}$ if $s_bs'_a = 1$ and $y_{v,b,a}$ otherwise (WR5). In both cases $\zeta_{v,a,b}(\varphi_\infty y) = -s'_as_b\,\zeta'_{v,b,a}(y)$, and $e^\infty(\varphi_\infty y) = e^\infty(y)$; hence $v_{\bar P}\circ\varphi_\infty = v_{P^\flat}$.

Class data as translates. For $r \in \mathrm{M}_{n'\times n}(\widehat{\mathcal{O}}_E)$ let δ_r be the indicator function of the class of r on $\mathrm{M}_{n'\times n}(R_{\mathfrak{m}})$, and y_r the finite adelic $n'\times n$ matrix with components $-t^{-1}r$ at the places of A and 0 elsewhere. As $\bar y_r$ vanishes at A, (2) of Lemma 4.1.3 gives $R_A(y_r)\mathbf{1}_A = B_r$, $B_r(x) = \mathbf{1}_A(x+y_r)$, the indicator of the x whose components at A lie in $t^{-1}r$ plus integral matrices; by (1) of Lemma 4.1.3 and (2) of Lemma 6.2.33, $\mathbf{1}_A\otimes_A v^0 = P^Av^0 = v^0$, so $\rho(\mathrm{fv}(y_r))v^0 = B_r\otimes_A v^0$. The finite component of u_t is t^{-1} at the places of A and 1 outside $A\cup\bar A$, so $B_r(xu_t) = 1$ exactly when the components of x at A lie in $r + t\cdot$(integral matrices), i.e. when x is integral at A and congruent to r modulo $\mathfrak{m}$: $B_r(xu_t) = \delta_r^\natural(x)$. Let $\alpha' = \alpha^{-1}\upsilon^{\#n}$, $\beta' = \beta^{-1}\upsilon^{\#n'}$. By (4) and (5) of Lemma 4.1.1 (the latter with real components 1), for every Fock vector v of $\mathrm{Hom}(S^-,T)$,

$$\omega^{S^-,T}_{\alpha',\beta'}(u_t,1)(v\otimes\rho(\mathrm{fv}(y_r))v^0) = C_1\,\mathrm{mod}_{\mathfrak{m}}(\delta_r,v), \qquad C_1 = \beta'[\#n'](\det_A u_t)^{-1}\,|\det_A u_t|^{\#n'/2},$$

where $\det_A u_t$ has components $t^{-\#n}$ at A and 1 elsewhere; as t generates $\mathfrak{m}$ at A, $|\det_A u_t| = (\#R_{\mathfrak{m}})^{\#n}$, so $|C_1|^2 = (\#R_{\mathfrak{m}})^N$, and C_1 depends neither on r nor on v. By (WC4) and (2) of

Lemma 4.1.1,

$$\theta^{S^-,T}_{\alpha',\beta'}(\mathrm{mod}_{\mathfrak{m}}(\delta_r,v))(g,h)=C_1^{-1}\,\theta^{S^-,T}_{\alpha',\beta'}(v\otimes\rho(\mathrm{fv}(y_r))v^0)(g\,u_t,h).$$

The flipped translate. $\mathrm{fv}(y)^\dagger=(0,c\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}^{-1}(\alpha_{\mathrm{W}}y)^*)=\mathrm{fv}(cy^*)$. So by (6.84) and $\Pi v^0=v^0$, $\Pi\rho(\mathrm{fv}(y_r))v^0=\rho(\mathrm{fv}(y_r'))v^0$ with $y_r'=cy_r^*$, which vanishes at A. By (2) of Lemma 4.1.3 and $\mathbf{1}_A\otimes_A v^0=v^0$, $\rho(\mathrm{fv}(y_r'))v^0=(R_A(y_r')\mathbf{1}_A)\otimes_A v^0$ with

$$\begin{aligned}(R_A(y_r')\mathbf{1}_A)(x)&=\psi_0\big(\textstyle\sum_{b,a}\overline{(y_r')_{b,a}}\,(x_{b,a})_A\big)\mathbf{1}_A(x)=\psi_0\big(\bar c\,\mathrm{tr}(y_rx_A)\big)\mathbf{1}_A(x)\\&=\psi_0\big(-t^{-1}\bar c\,\mathrm{tr}(r\,x_A)\big)\mathbf{1}_A(x),\end{aligned}$$

$\psi_0(\cdot)=\psi_{E,f}(\jmath\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}\cdot)$. For x integral at A, $-\bar c\,\mathrm{tr}(rx_A)$ is an integral finite adele, so by the definition of $\psi_{\mathfrak{m}}$ this is $\psi_{\mathfrak{m}}(-\bar c\,\mathrm{tr}(rx_A)\bmod\mathfrak{m})$: $R_A(y_r')\mathbf{1}_A=\Psi_r^\natural$ for the function $\Psi_r(Z)=\psi_{\mathfrak{m}}(-\bar c\,\mathrm{tr}(Zr))$ on $\mathrm{M}_{n\times n'}(R_{\mathfrak{m}})$. With the real part, $\Pi(v_{\bar P}\otimes\rho(\mathrm{fv}(y_r))v^0)=\mathrm{mod}_{\mathfrak{m}}(\Psi_r,v_{P^\flat})$, formed for $\mathrm{Hom}(T,S)$.

Step 5: conclusion. (6.86) for $\alpha_1=\alpha'$, $\beta_1=\beta'$ has $\alpha_2=\beta$, $\beta_2=\alpha$. With Step 4 and $u_tu_{\mathrm{W}}=(1,u_{\mathfrak{m}})$, for every r, P, g, h,

$$\theta^{S^-,T}_{\alpha',\beta'}(\mathrm{mod}_{\mathfrak{m}}(\delta_r,v_{\bar P}))(g,h)=C_1^{-1}z_0^{-1}\beta(\det u_{\mathrm{W}})^{-1}\,\theta^{T,S}_{\beta,\alpha}(\mathrm{mod}_{\mathfrak{m}}(\Psi_r,v_{P^\flat}))(h,g\,(1,u_{\mathfrak{m}})).$$

Let D be a function on $\mathrm{M}_{n'\times n}(R_{\mathfrak{m}})$ and $D'(X)=\overline{D(c^{-1}X)}$. Then $D'=\sum_{[r]}D'([r])\,\delta_r$ over the classes $[r]$, and $\Phi_0\mapsto\mathrm{mod}_{\mathfrak{m}}(\Phi_0,v)$ is linear ($\Phi_0\mapsto\Phi_0^\natural$ is linear, $\Psi\otimes_A f$ does not depend on the section by (1) of Lemma 4.1.3 and is therefore linear in Ψ, and $v\otimes\cdot$ is linear), as are the theta kernels. Hence the last identity holds with δ_r replaced by D' and Ψ_r by $\sum_{[r]}D'([r])\Psi_r$, and, substituting $r=cr_1$ (a bijection of the classes) and using $-\bar cc=1$,

$$\sum_{[r]}\overline{D([c^{-1}r])}\,\psi_{\mathfrak{m}}\big(-\bar c\,\mathrm{tr}(Zr)\big)=\sum_{[r_1]}\overline{D([r_1])}\,\psi_{\mathfrak{m}}\big(\mathrm{tr}(Zr_1)\big)=\widehat{D}(Z).$$

Finally, by Lemma 6.4.55 (with $\Psi'=D'$, the reduction of c being the unit there),

$$\overline{\theta^{S,T}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}}(D,v_P))(g,h)}=c_{S,T}\,\theta^{S^-,T}_{\alpha',\beta'}(\mathrm{mod}_{\mathfrak{m}}(D',v_{\bar P}))(g,h)$$

with $|c_{S,T}|=1$. So the identity of the statement holds for all D, P, g, h with the constant $c_{S,T}\,C_1^{-1}z_0^{-1}\beta(\det u_{\mathrm{W}})^{-1}$, whose modulus squared is $(\#R_{\mathfrak{m}})^{-\#n\cdot\#n'}$. □

Lemma 6.4.62. *Let D be admissible data, $z=\mathrm{I}$, $i,n\geq0$ with $i+n+2\leq m$, Φ_0' a reversed pair at (i,n) and Φ_1' a reversed pair at $(i,n+1)$. There is a function γ from the m-fold twists to $\mathbb{C}$, of absolute value 1 up to a norm power, such that for every m-fold twist ϑ and every function Ψ on $\mathrm{M}_{n_{n+1}\times n_n}(R_\vartheta)$, on $\mathcal{F}^{i+1}_{\triangle}(z,n)$,*

$$\mathrm{Tl}^i_{\vartheta,z,n+1}\circ\mathrm{As}^i_{\vartheta,z,n}(\Phi_0',\widehat{\Psi})=\gamma(\vartheta)\,\mathrm{As}^i_{\vartheta,z,n+1}(\Phi_1',\widehat{\Psi^\natural})\circ\mathrm{Tl}^{i+1}_{\vartheta,z,n},$$

$\Psi^\natural$ the next datum of Ψ as in Lemma 6.4.12, a function on $\mathrm{M}_{n_{n+2}\times n_{n+1}}(R_\vartheta)$.

Proof. Fix ϑ where it is not quantified, and write $R=R_\vartheta$, $P=\Sigma(\mathfrak{t}_\vartheta(\mathrm{I}))$.

Step 0: R is a product of residue fields and $\chi_\vartheta^{\mathrm{sq}}$ is primitive. The primes p_0,p_1 are $\equiv1$ modulo q_2^2 ((AD5)), hence prime to d and to every prime factor of $q_1=q_0T(s)$ (these are the prime factors of q_0, as $2,3\mid q_0$; Definition 2.3.1), all of which divide $q_2=(q_0d)^{s+6}$; so p_0,p_1 are unramified in $E=K(\zeta_{q_1})$. As $\pi_M\bar\pi_M=p_0p_1$ in $\mathcal{O}_K$ with $p_0\neq p_1$, neither p_ϵ

is inert in K (p_ϵ^2 would divide p_0p_1), so $\pi_M\mathcal{O}_K$ is a product of two distinct primes of degree one, and $\pi_M\mathcal{O}_E$ is the product of the chosen places, each with exponent 1. With (TV2) and $\mathrm{N}Q$ prime to M (Definition 3.1.2), $\mathfrak{t}_\vartheta(\mathrm{I}) = \prod_{w\in P} w$ and $R = \prod_{w\in P} k_w$. By Lemma 6.4.14(c), $\chi^{\mathrm{sq}}_\vartheta(u) = \prod_{w\in P}\psi^\nu_w(u_w)$ with every ψ^ν_w nontrivial. Let $a \in R$, $a \neq 0$, with $a_w \neq 0$; take $v \in k_w^\times$ with $\psi^\nu_w(v) \neq 1$ and $c \in R$ with $c_w = (v-1)/a_w$ and $c_{w'} = 0$ for $w' \neq w$; then $\chi^{\mathrm{sq}}_\vartheta(1+ca) = \psi^\nu_w(v) \notin \{0,1\}$. So $\chi^{\mathrm{sq}}_\vartheta$ is primitive; ψ_ϑ is primitive by the context of (TV7) (self-duality of $\widehat{\mathcal{O}}_E$ for $z \mapsto \psi_{E,f}(\jmath\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}z)$). Moreover $\#R = \#(\mathcal{O}_E/\pi_M\mathcal{O}_E)\cdot \mathrm{N}Q$.

Step 1: the one-square identity for the reversed square. Apply Proposition 6.3.72 to $U_1 = V^{i+1}_{z,n}$, $A = V^{i+1}_{z,n+1}$, $A' = V^i_{z,n+1}$, $U_2 = V^i_{z,n+2}$ (index sets n_n, n_{n+1}, n_{n+1}, n_{n+2}), the integer 0, the weight $\mu = (n+1)\Delta$, the family ν_ϑ (tame for $(\mathfrak{t}_\vartheta(\mathrm{I}), \chi^{\mathrm{sq}}_\vartheta)$ by Lemma 6.4.14(c)), the characters of Lemma 6.4.14(b), the polynomials $P_1 = H_{i+1,n}$, $P_2' = H_{i,n+1}$ (joint harmonics of the triangle lifts at $(i+1,n)$ and $(i,n+1)$), $P_2 = \Phi_1'$, $P_1' = \Phi_0'$, and the bijection $\sigma\colon n_{n+1}\sqcup n_{n+1} \to n_n \sqcup n_{n+2}$ that maps the first copy identically into $n_{n+2} \supseteq n_{n+1}$, and the second copy $n_{n+1} = n_n \sqcup B_{n+1}$ by the identity on n_n (onto the first summand) and $B_{n+1} \to B_{n+2}$, $(n+1,c) \mapsto (n+2,c)$. (OS1), (OS3): Lemma 6.4.14(b). (OS2): the weights do not depend on ϑ; with the types e_1, e, e', e_2 of the triangle forms on U_1, A, A', U_2, $v_{H_{i+1,n}}$ is an edge from e_1 to e and $v_{H_{i,n+1}}$ one from e' to e_2 (Lemma 6.4.15(b), as $i+n+1 < m$); the reversed pair Φ_1' at $(i,n+1)$ is an edge for $(\delta_i(N_{n+2}), \delta_i(N_{n+1}))$, the weights of the characters of (U_2, A), from $e_{-\mu'} = \overline{e_2}$ to $e_{-\mu} = \overline{e}$ ($\mu = \frac12 p^{i+1}_{z,0,n+1}$, $\mu' = \frac12 p^i_{z,0,n+2}$), and Φ_0' at (i,n) is an edge from $\overline{e'}$ to $\overline{e_1}$ for the weights of (A', U_1) (TV6). The composites are $\Theta_\vartheta(x) = \mathrm{As}^i_{\vartheta,z,n+1}(\Phi_1', \mathbf{1}_{x_2}) \circ \mathrm{Tl}^{i+1}_{\vartheta,z,n}[\mathbf{1}_{x_1}]$ for x with rows n_{n+1} and column blocks x_1 (columns n_n), x_2 (columns n_{n+2}), and $\Theta'_\vartheta(y) = \mathrm{Tl}^i_{\vartheta,z,n+1}[\mathbf{1}_{y_2}] \circ \mathrm{As}^i_{\vartheta,z,n}(\Phi_0', \mathbf{1}_{y_1})$ for y with columns n_{n+1} and row blocks y_1 (rows n_n), y_2 (rows n_{n+2}), the transposed lifts being the ascents of (TV7); they map triangle forms to triangle forms (Lemma 6.4.51). As $L_*(\cdot,\cdot;0) = 1$ ((XD7)), Proposition 6.3.72 gives $\kappa \in \mathbb{C}$ and $C > 0$, independent of ϑ, and c_ϑ with $|c_\vartheta|^2 = C$, such that for every $f \in \mathcal{F}^{i+1}_\triangle(z,n)$ and every linear functional l on $\mathcal{F}^i_\triangle(z, n+2)$, $L(x) = l(\Theta_\vartheta(x)f)$ and $L'(y) = l(\Theta'_\vartheta(y)f)$ satisfy (i) and (ii) of Lemma 6.4.13 with $c = \kappa/c_\vartheta$, for the sets $I = n_n$, $K = B_{n+1}$, $J = I \sqcup K = n_{n+1}$, $W = I \sqcup J \sqcup K$ identified with $n_n \sqcup n_{n+2}$ through $J \sqcup K \cong n_{n+2}$ as above (so that the canonical bijection of Lemma 6.4.13 is σ).

Step 2: the Fourier form of the square law. By Step 0, part (2) of Lemma 6.4.13 applies with $\psi_R = \psi_\vartheta$: there is $\gamma_0 = \gamma_0(\vartheta)$ with $|\gamma_0|^2(\#R)^{|K|(|K|+2|I|)} = 1$ such that for every $p = \Psi$ on $\mathrm{M}_{(I\sqcup K)\times I}(R) = \mathrm{M}_{n_{n+1}\times n_n}(R)$, with $p^\natural$ formed for $(I\sqcup K, I, K) = (n_{n+1}, n_n, B_{n+1})$, i.e. $\Psi^\natural$ of the statement,

$$\sum_y \widehat{\Psi}(y)\, L'\begin{pmatrix} y \\ \iota_J \end{pmatrix} = \gamma_0 \frac{\kappa}{c_\vartheta} \sum_x \widehat{\Psi^\natural}(x)\, L(\iota_I \;\; x),$$

since $\widehat{\overline{p}}$ in the notation of Lemma 6.4.13 is the Fourier datum of (TV7). Here $\iota_J \in \mathrm{M}_{n_{n+2}\times n_{n+1}}(R)$ and $\iota_I \in \mathrm{M}_{n_{n+1}\times n_n}(R)$ are the inclusion matrices, whose indicators are the point data of the triangle lifts at $(i,n+1)$ and $(i+1,n)$; so $L'\binom{y}{\iota_J} = l(\mathrm{Tl}^i_{\vartheta,z,n+1}\mathrm{As}^i_{\vartheta,z,n}(\Phi_0', \mathbf{1}_y)f)$ and $L(\iota_I\; x) = l(\mathrm{As}^i_{\vartheta,z,n+1}(\Phi_1', \mathbf{1}_x)\mathrm{Tl}^{i+1}_{\vartheta,z,n}f)$.

Step 3: conclusion. By the linearity of Lemma 6.4.51(c) and of l, l annihilates

$$\mathrm{Tl}^i_{\vartheta,z,n+1}\mathrm{As}^i_{\vartheta,z,n}(\Phi_0', \widehat{\Psi})f - \gamma(\vartheta)\mathrm{As}^i_{\vartheta,z,n+1}(\Phi_1', \widehat{\Psi^\natural})\mathrm{Tl}^{i+1}_{\vartheta,z,n}f$$

with $\gamma = \gamma_0\kappa/c_\vartheta$; as linear functionals separate points, the identity holds. Finally $|\gamma(\vartheta)|^2 = |\kappa|^2C^{-1}(\#R)^{-e}$ with $e = |K|(|K|+2|I|)$ independent of ϑ and $\#R = \#(\mathcal{O}_E/\pi_M\mathcal{O}_E)\,\mathrm{N}Q$ (Step 0), so γ has absolute value 1 up to a norm power (the constant $|\kappa|^2C^{-1}\#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^{-e}$ and the exponent $-e$). □

Lemma 6.4.63. *Let D be admissible data, $z \in \{\mathrm{I},\mathrm{II}\}$ and ϑ an m-fold twist. For $0 \le b \le m$ let K_b be the group of the $k \in \mathbf{K}_0(n_b)$ whose reduction modulo $\mathfrak{t}_\vartheta(z)$ is 1.*

(1) For $0 \le b \le m$, $\Theta^\flat_{\vartheta,z,b}$ is a linear map from $\mathcal{F}(z,0,m)$ to $\mathcal{F}(z,0,b)$, and if $b < m$ every $\Theta^\flat_{\vartheta,z,b}F$ is right invariant under K_b.

(2) For $0 \le b \le m$, $\mathrm{Wc}^0_{\vartheta,z,b}$ is a linear map from $\mathcal{F}^0_\triangle(z,0)$ to $\mathcal{F}^0_\triangle(z,b)$, and if $b \ge 1$ every $\mathrm{Wc}^0_{\vartheta,z,b}y$ is right invariant under K_b.

(3) For $0 \le b \le m$, $F \in \mathcal{F}(z,0,m)$ and $y \in \mathcal{F}^0_\triangle(z,0)$ the function $(\Theta^\flat_{\vartheta,z,b}F)\,(\mathrm{Wc}^0_{\vartheta,z,b}y)$ is integrable on $[\mathrm{U}(V^0_{z,b})]$, and

$$\int_{[\mathrm{U}(V^0_{z,b})]} (\Theta^\flat_{\vartheta,z,b}F)\,(\mathrm{Wc}^0_{\vartheta,z,b}y) = \int_{[\mathrm{U}(V^0_{z,0})]} (\Theta^\flat_{\vartheta,z,0}F)\,y.$$

Proof. Write $\mathfrak{t} = \mathfrak{t}_\vartheta(z)$ and $c_b = \#\mathrm{GL}_{n_b}(\mathcal{O}_E/\mathfrak{t})$.

Step 1: one edge. Fix $b < m$ and put $S = V^0_{z,b}$, $T = V^0_{z,b+1}$; let ν_1,ν_2, H and $\varphi = \mathrm{mod}_\mathfrak{t}(\Phi^{\mathrm{pt}}_b, v_H)$ be the characters, the joint harmonic and the datum of $\mathrm{Tl}^0_{\vartheta,z,b}$ ((TV3)), $\lambda = \frac12 p_{z,0,b}$, $\lambda' = \frac12 p_{z,0,b+1}$, $\kappa(g,h) = \theta^{S,T}_{\nu_1,\nu_2}(\varphi)(g,h)$, and L, $L^{\mathfrak{t}}$ the two integral operators of Lemma 6.3.76 for these data. By (GP6), ν_1 and ν_2 have the weights $\gamma = \gamma_{\neg z,b}(N_b)$ and $\gamma' = \gamma_{\neg z,b}(N_{b+1})$, and by (b) of Lemma 6.4.15 (with $i = 0$, $n = b$) the Fock vector v_H is an edge for the weights (γ,γ') from e_λ to $e_{\lambda'}$; so Lemma 6.3.76 applies with $e = e_\lambda$, $e' = e_{\lambda'}$, $P = H$, $\mathfrak{m} = \mathfrak{t}$, $\Psi = \Phi^{\mathrm{pt}}_b$. The triangle forms at $(0,b)$ and $(0,b+1)$ are the forms of type e_λ on S and of type $e_{\lambda'}$ on T ((TV2), offset 0). By the same clause read with offset 0, the type of the grid forms at $(z,0,b)$ is $\bar e_\lambda$ and that of the grid forms at $(z,0,b+1)$ is $\bar e_{\lambda'}$: the grid forms there are the forms of type $\bar e_\lambda$ on S and of type $\bar e_{\lambda'}$ on T. Since $\theta^{S,T}_{\nu_1,\nu_2}(\varphi)(g,h) = \theta_{\mathrm{Hom}(S,T)}(\omega_{\nu_1,\nu_2}(g,h)\varphi)$ ((XD2)), the lift L_φ of (ED3) is L, so $\mathrm{Tl}^0_{\vartheta,z,b} = L$, and $\mathrm{Pt}_{\vartheta,z,b} = c_b\,L^{\mathfrak{t}}$ by (FY1). By (1) and (2) of Lemma 6.3.76: $\mathrm{Tl}^0_{\vartheta,z,b}$ maps $\mathcal{F}^0_\triangle(z,b)$ to $\mathcal{F}^0_\triangle(z,b+1)$ and $\mathrm{Pt}_{\vartheta,z,b}$ maps $\mathcal{F}(z,0,b+1)$ to $\mathcal{F}(z,0,b)$, both linearly (the integrals are linear in the form); and for $w \in \mathcal{F}^0_\triangle(z,b)$, $G \in \mathcal{F}(z,0,b+1)$ the function $(g,h) \mapsto \kappa(g,h)w(g)G(h)$ is integrable on $[\mathrm{U}(S)] \times [\mathrm{U}(T)]$ and

$$\int_{[\mathrm{U}(T)]} G \cdot \mathrm{Tl}^0_{\vartheta,z,b}w = \int_{[\mathrm{U}(S)]} w \cdot L^{\mathfrak{t}}G. \tag{6.87}$$

By Fubini's theorem the integrability on the product gives the integrability of

$$h \mapsto G(h)\int \kappa(g,h)w(g)\,dg = G(h)\,(\mathrm{Tl}^0_{\vartheta,z,b}w)(h)$$

on $[\mathrm{U}(T)]$.

Step 2: levels. Keep $b < m$. Let $k \in \mathrm{K}_b$ and $k' \in \mathrm{K}_{b+1}$. The finite components of $\det k$ and $\det k'$ are integral and their infinite components are 1, so by the definition of weight and tame part, $\nu_i(\det k) = \prod_w \psi^{(i)}_w(\det k \bmod w)$ with $\psi^{(i)}$ the tame part of ν_i, and likewise for k'. By (GP6), ν_i is $\mu^0_{b+1}\vartheta^{\,b+1}\upsilon^e$ if $\neg z = \mathrm{II}$ and $\mu^0_{b+1}\upsilon^e$ if $\neg z = \mathrm{I}$, for an integer e; as the tame part of a product is the product of the tame parts, μ^0_{b+1} has tame part ω^{b+1}, trivial off the chosen places, υ has trivial tame part, and the tame part of ϑ is trivial off Q, the components $\psi^{(i)}_w \ne 1$ occur only at chosen places, and at places of Q only for $z = \mathrm{I}$. All these places divide $\mathfrak{t}$ ($\mathfrak{t} = \pi_M \prod_{w\in Q} w$ for $z = \mathrm{I}$, $\mathfrak{t} = \mathfrak{b}$, the deep level, for $z = \mathrm{II}$), and $k \equiv 1$, $k' \equiv 1$ modulo $\mathfrak{t}$, so $\det k \equiv \det k' \equiv 1$

modulo every such place and $\nu_i(\det k) = \nu_i(\det k') = 1$. By (6) of Lemma 4.1.1 for $A = \Sigma(\mathfrak{t})$, and since the reductions of k, k' modulo $\mathfrak{t}$ are 1,

$$\omega_{\nu_1,\nu_2}\big((1,k),(1,1)\big)\varphi = \nu_2(\det k)^{-1}\varphi = \varphi, \qquad \omega_{\nu_1,\nu_2}\big((1,1),(1,k')\big)\varphi = \nu_1(\det k')\varphi = \varphi.$$

By (3) of Lemma 6.3.76 with $c = 1$ and $\Psi' = \Psi$: $(\mathrm{Tl}^0_{\vartheta,z,b}w)(h(1,k')) = (\mathrm{Tl}^0_{\vartheta,z,b}w)(h)$ and $(L^{\mathsf{t}}G)(g(1,k)) = (L^{\mathsf{t}}G)(g)$. So every $\mathrm{Tl}^0_{\vartheta,z,b}w$ is right invariant under K_{b+1} and every $\mathrm{Pt}_{\vartheta,z,b}G$ under K_b.

Step 3: (1) and (2). By (FY1), $\Theta^\flat_{\vartheta,z,m} = \mathrm{id}$ and $\Theta^\flat_{\vartheta,z,b} = \mathrm{Pt}_{\vartheta,z,b} \circ \Theta^\flat_{\vartheta,z,b+1}$; by decreasing induction on b, Steps 1 and 2 give (1). By (TV3), $\mathrm{Wc}^0_{\vartheta,z,0} = \mathrm{id}$ and $\mathrm{Wc}^0_{\vartheta,z,b+1} = c_b\,\mathrm{Tl}^0_{\vartheta,z,b} \circ \mathrm{Wc}^0_{\vartheta,z,b}$; by induction on b, Steps 1 and 2 give (2).

Step 4: (3). Induction on b. For $b = 0$ the two sides agree, and the integrand is integrable: as $m \geq 1$, $\Theta^\flat_{\vartheta,z,0}F = c_0\,L^{\mathsf{t}}G$ with $G = \Theta^\flat_{\vartheta,z,1}F$ for the operator L^{t} of the edge $b = 0$, and $y \cdot L^{\mathsf{t}}G$ is integrable on $[\mathrm{U}(V^0_{z,0})]$ by Fubini's theorem, as in Step 1. Let $b < m$ and put $G = \Theta^\flat_{\vartheta,z,b+1}F \in \mathcal{F}(z,0,b+1)$ and $w = \mathrm{Wc}^0_{\vartheta,z,b}y \in \mathcal{F}^0_\triangle(z,b)$. By Step 1 the function $G \cdot \mathrm{Wc}^0_{\vartheta,z,b+1}y = c_b\,G \cdot \mathrm{Tl}^0_{\vartheta,z,b}w$ is integrable on $[\mathrm{U}(T)]$, and by (6.87)

$$\int_{[\mathrm{U}(T)]} G \cdot \mathrm{Wc}^0_{\vartheta,z,b+1}y = c_b \int_{[\mathrm{U}(S)]} w \cdot L^{\mathsf{t}}G = \int_{[\mathrm{U}(S)]} (\mathrm{Pt}_{\vartheta,z,b}G)\,w = \int_{[\mathrm{U}(S)]} (\Theta^\flat_{\vartheta,z,b}F)\,(\mathrm{Wc}^0_{\vartheta,z,b}y),$$

the function $w \cdot L^{\mathsf{t}}G$ being integrable by Fubini's theorem as in Step 1. The induction hypothesis turns the right side into $\int_{[\mathrm{U}(V^0_{z,0})]} (\Theta^\flat_{\vartheta,z,0}F)\,y$. □

Lemma 6.4.64. *Let V be a space with a point with signs s on n, e a cusp kernel of order 0 on $G_\infty(s)$ and $K \subseteq G_f(n)$ an open subgroup. Then the space of the forms of type e on V that are right invariant under K is finite-dimensional.*

Proof. Let $N = \#n$, $G_\infty = G_\infty(s)$ with its measure (the product of the Cayley measures, a Haar measure), $\Gamma = \Gamma_V$, and $\mathcal{A}$ the space of the forms of type e on V right invariant under K. By (VP3) of Definition 3.2.2 V is a framed hermitian space U (with E^n as underlying space, the columns of $(x_V)_v$ as frames at $v \in \Phi$ and the finite isometry $z \mapsto x_{V,f}z$; a framed space of rank N in the sense of (FT1)), with group $\mathrm{G}(s)$ and $\Gamma_U = \Gamma$, and $\lambda_V(g)$ is the first minimum of (VP4) for both readings: $\lambda_U(g) = \lambda_V(g)$. The conditions (ED1) on e coincide with (EK3), (EK4) of Definition 5.6.2 for the signs s, the Haar measures being the Cayley measures and $\|x\| = \prod_v \|x_v\|$ the archimedean size; and the convolution $R(e)$ of (ED2) is that of (EK2). So every $F \in \mathcal{A}$ is a continuous bounded function on $\mathrm{G}(s) = G^U$, left Γ_U-invariant, right K-invariant, with $R(e)F = F$.

By Lemma 5.8.7, applied to U (whose signs are s at every infinite place), e (order $B = 0$), K and $A = 1$, there is $R_1 \geq 0$ with $|F(g)|\lambda_U(g)^{-1} \leq R_1 \sup|F|$ for all $F \in \mathcal{A}$ and g; put $R_0 = \max(R_1, 1)$. By Lemma 5.8.20 with $T = 2R_0$ there is a compact $C \subseteq \mathrm{G}(s)$ such that every g with $\lambda_U(g)^{-1} \leq 2R_0$ has $\gamma g \in C$ for some $\gamma \in \Gamma$. As e is integrable, left translation is continuous in $L^1(G_\infty)$, so there is a neighbourhood W_∞ of 1 in G_∞ with $\int_{G_\infty} |e(u^{-1}h) - e(h)|\,dh \leq \frac{1}{2}$ for $u \in W_\infty$; put $W = W_\infty \times K$, a neighbourhood of 1 in $\mathrm{G}(s)$. For $F \in \mathcal{A}$ with $M = \sup|F|$, $g \in \mathrm{G}(s)$ and $w = (u,k) \in W$, by right K-invariance, $F = R(e)F$ and the substitution $h \mapsto u^{-1}h$ (left invariance of the Haar measure),

$$|F(gw) - F(g)| = \Big|\int_{G_\infty} (e(u^{-1}h) - e(h))F(g(h,1))\,dh\Big| \leq \tfrac{1}{2}M.$$

Cover C by finitely many $a_1W, \ldots, a_rW$ with $a_j \in C$. We show that $F \mapsto (F(a_1), \ldots, F(a_r))$ is injective on $\mathcal{A}$; then $\dim \mathcal{A} \leq r$. Let $F \in \mathcal{A}$ vanish at every a_j and suppose $M = \sup|F| > 0$.

Choose g with $|F(g)| > M/2$. Then $\lambda_U(g)^{-1} \le R_1 M/|F(g)| < 2R_0$, so $\gamma g \in C$ for some $\gamma \in \Gamma$, say $\gamma g = a_j w$ with $w \in W$. By Γ-invariance and the display, $|F(g)| = |F(a_j w) - F(a_j)| \le M/2$, a contradiction. Hence $F = 0$. □

Lemma 6.4.65. *Let D be admissible data and, for an m-fold twist ϑ, $T_\vartheta = \#(\mathcal{O}_E/\mathfrak{m}_\vartheta)^{8mn_m^2}$. There is $C \in \mathbb{C}$ such that for every m-fold twist ϑ:*

(1) the first-path datum Ψ^0_ϑ is a path datum of bound T_ϑ on the edges of the first path;

(2) for every real point g of $\tilde{Y}^0_0$, every $y' \in \mathcal{F}(\mathrm{I}, 0, m)$ and every $\varphi \in \mathcal{F}(\mathrm{II}, 0, m)$,

$$\langle \Theta^\flat_{\vartheta,\mathrm{I},0}\, y',\ \Theta^\flat_{1,\mathrm{II},0}\, \varphi \rangle_{0,g} = C \operatorname{per}_{\mathrm{first}}(\Psi^0; g)(\vartheta, y', \varphi).$$

Proof. Fix an m-fold twist ϑ; write $\mathfrak{m} = \mathfrak{m}_\vartheta = \mathfrak{b} \prod_{w \in Q} w$ ($\mathfrak{b}$ the deep level, a positive power of π_M), $\mathfrak{t}_z = \mathfrak{t}_\vartheta(z)$ ($\mathfrak{t}_\mathrm{I} = \pi_M \prod_{w \in Q} w$, $\mathfrak{t}_\mathrm{II} = \mathfrak{b}$, independent of ϑ), $n_{z,b} = n_{z,0,b}$, $e_{z,b}$ for the edge $((z,0,b),(z,0,b+1))$, w for the edges of the first path, $c_b(z) = \#\mathrm{GL}_{n_{z,b}}(\mathcal{O}_E/\mathfrak{t}_z)$, and $\mathcal{X}$ for the set of level coset families. All tuples below are supported on w. The ranks of the nodes $(z,0,b)$, $b \le m$, are at most $r + 1 + mt = n_m$.

Step 1: the moves. Let $b < m$, $S = V^0_{z,b}$, $T = V^0_{z,b+1}$, ν_1, ν_2, H those of $\mathrm{Tl}^0_{\vartheta,z,b}$, and for a function Ψ on $\mathrm{M}_{n_{z,b+1} \times n_{z,b}}(R_\mathfrak{m})$ let $L^\mathfrak{t}_\Psi G(g) = \int_{[\mathrm{U}(T)]} \theta^{S,T}_{\nu_1,\nu_2}(\mathrm{mod}_\mathfrak{m}(\Psi, v_H))(g,h) G(h)\, dh$. By (GP6), (GP7) the transposed move along $e_{z,b}$ with matrix y is $L^\mathfrak{t}_{\mathbf{1}_y}$ (same characters, harmonic and spaces as the triangle lift, (TV3)). As in (GP6) ν_1, ν_2 have the weights of the edge, and by (b) of Lemma 6.4.15 (v_H is an edge from e_λ to $e_{\lambda'}$, $\lambda = \frac{1}{2} p_{z,0,b}$, $\lambda' = \frac{1}{2} p_{z,0,b+1}$, and the grid forms at $(z,0,b)$, $(z,0,b+1)$ have the types $\bar{e}_\lambda$, $\bar{e}_{\lambda'}$), (1) of Lemma 6.3.76 shows that every $L^\mathfrak{t}_\Psi$ maps grid forms at $(z,0,b+1)$ to grid forms at $(z,0,b)$, the integrals converging absolutely. Since $\mathfrak{m} \subseteq \mathfrak{t}_z$, Lemma 6.4.66 gives $\mathrm{mod}_{\mathfrak{t}_z}(\Phi^{\mathrm{pt}}_b, v_H) = \mathrm{mod}_\mathfrak{m}(\Phi^{\mathrm{pt}}_b \circ \mathrm{red}, v_H)$, red the reduction from $R_\mathfrak{m}$ to $R_{\mathfrak{t}_z}$; the datum $\mathrm{mod}_\mathfrak{m}(\Psi, v_H)$ is linear in Ψ ((XD5), (WR6)) and the theta kernel is linear in the datum, so by (FY1)

$$\mathrm{Pt}_{\vartheta,z,b} = c_b(z) \sum_y \Phi^{\mathrm{pt}}_b(y \bmod \mathfrak{t}_z)\, L^\mathfrak{t}_{\mathbf{1}_y}, \tag{6.88}$$

y over $\mathrm{M}_{n_{z,b+1} \times n_{z,b}}(R_\mathfrak{m})$; the same holds for $\mathrm{Pt}_{1,\mathrm{II},b}$ with these moves for ϑ, because $\mathfrak{t}_1(\mathrm{II}) = \mathfrak{t}_\vartheta(\mathrm{II}) = \mathfrak{b}$ and the characters of the column of $\tilde{Y}$ ($c = \mathrm{I}$ in (GP6)) do not involve the twist.

Step 2: expansion of the left side. Composing (6.88) down the column, by induction on $m - b$, $\Theta^\flat_{\vartheta,\mathrm{I},0} = \prod_{b<m} c_b(\mathrm{I}) \sum_S \prod_{b<m} \Phi^{\mathrm{pt}}_b(S(e_{\mathrm{I},b}) \bmod \mathfrak{t}_\mathrm{I})\, \mathrm{fm}_\mathrm{I}(S)$, the sum over the tuples supported on the column, and likewise for $\Theta^\flat_{1,\mathrm{II},0}$. A tuple supported on w is the same as a pair of tuples supported on the two columns, and the period is bilinear, so with $\Pi = \prod_{b<m} c_b(\mathrm{I}) c_b(\mathrm{II})$ and $\mathrm{pd}(S) = \langle \mathrm{fm}_\mathrm{I}(S) y', \mathrm{fm}_\mathrm{II}(S) \varphi \rangle_{0,g}$,

$$\langle \Theta^\flat_{\vartheta,\mathrm{I},0} y', \Theta^\flat_{1,\mathrm{II},0} \varphi \rangle_{0,g} = \Pi \sum_S \Psi^1(S)\, \mathrm{pd}(S). \tag{6.89}$$

Step 3: translation of a tuple. Each $\Sigma(\mathfrak{m})$ is the set of places of a modulus, so its places w' satisfy $\bar{w}' \ne w'$, and the component of an element of $\mathbf{K}_0(n)$ at the pair $\{w', \bar{w}'\}$ is determined by any element of $\mathrm{GL}_n(\mathcal{O}_{E,w'})$ at w'; as $\mathcal{O}_{E,w'}$ is local, every element of $\mathrm{GL}_n(R_\mathfrak{m})$ is the reduction of some $k \in \mathbf{K}_0(n)$ (a **lift**). For k integral with infinite components 1, the definition of weight and tame part gives $\nu_i(\det k) = \prod_{w'} \psi_{w'}(\det k \bmod w')$, ψ the common tame part of ν_1, ν_2 (they differ by a power of υ, of trivial tame part), which is nontrivial only at places dividing $\mathfrak{m}$; so $\nu_i(\det k) = \chi_e(\det \bar{k})$ for the edge e, $\bar{k}$ the reduction modulo $\mathfrak{m}$. Let h be a family of invertible matrices modulo $\mathfrak{m}$ at the nodes, equal to 1 at $(z,0,m)$ for both z and with

$h(\mathrm{II},0,0) = h(\mathrm{I},0,0) \oplus 1$. Choose lifts $k_{z,b}$ of $h(z,0,b)$ with $k_{z,m} = 1$ and $k_{\mathrm{II},0} = \mathrm{diag}(k_{\mathrm{I},0}, 1)$. For a tuple S let $u_b(S) =$ fm of y' (resp. φ) down to $(z,0,b)$. We claim

$$u_b(h\cdot S)(g'(1,k_{z,b})) = \epsilon_b\, u_b(S)(g') \qquad \text{with} \qquad \epsilon_b = \prod_{b\le i<m} \chi_{e_{z,i}}(\det h(z,0,i+1)/\det h(z,0,i)).$$

For $b = m$ both sides are the form itself. If it holds at $b+1$: by (6) of Lemma 4.1.1,

$$\omega_{\nu_1,\nu_2}((1,k_{z,b}),(1,k_{z,b+1}))\,\mathrm{mod}_{\mathfrak{m}}(\mathbf{1}_y, v_H) = \nu_1(\det k_{z,b+1})\nu_2(\det k_{z,b})^{-1}\mathrm{mod}_{\mathfrak{m}}(\mathbf{1}_{\bar{k}_{z,b+1}y\bar{k}_{z,b}^{-1}}, v_H),$$

and for $y = (h\cdot S)(e_{z,b}) = h(z,0,b+1)^{-1}S(e_{z,b})h(z,0,b)$ the new matrix is $S(e_{z,b})$ and the factor is $\chi_{e_{z,b}}(\det h(z,0,b+1)/\det h(z,0,b))$; so (3) of Lemma 6.3.76 and the induction hypothesis give the claim at b. At $b = 0$, substituting $x \mapsto x(1,k_{\mathrm{I},0})^{-1}$ in the period (right invariance of the quotient measure; $(1,\mathrm{diag}(k,1))$ commutes with $(g,1)$ and $\mathrm{diag}(x(1,k)^{-1},1) = \mathrm{diag}(x,1)(1,\mathrm{diag}(k,1))^{-1}$; $\mathrm{flp}^0_\vartheta = 1$),

$$\mathrm{pd}(h\cdot S) = \epsilon_w(h)\,\mathrm{pd}(S). \tag{6.90}$$

Step 4: the sum over the translates. Every family g_x ($x \in \mathcal{X}$) is as in Step 3, and so is g_x^{-1}. As $S \mapsto g_x\cdot S$ is a bijection of the tuples supported on w with inverse $S \mapsto g_x^{-1}\cdot S$, and ϵ_w is multiplicative, (6.90) gives $\sum_S \epsilon_w(g_x)\Psi^1(g_x\cdot S)\,\mathrm{pd}(S) = \sum_S \Psi^1(S)\,\mathrm{pd}(S)$ for every x, hence

$$\mathrm{per}_{\mathrm{first}}(\Psi^0; g)(\vartheta, y', \varphi) = \sum_S \Psi^0(S)\,\mathrm{pd}(S) = \#\mathcal{X}\sum_S \Psi^1(S)\,\mathrm{pd}(S).$$

Step 5: the constant. $\#\mathcal{X} = \prod_{b<m}\#\mathrm{GL}_{n_{\mathrm{I},b}}(R_{\mathfrak{m}})\cdot\#(\mathrm{GL}_{n_{\mathrm{II},b}}(R_{\mathfrak{m}})/\ker)$, ker the kernel of the surjective reduction modulo $\mathfrak{b}$, so the second factor is $c_b(\mathrm{II})$. For a modulus $\mathfrak{a}$ and $n \ge 1$, $\#\mathrm{GL}_n(\mathcal{O}_E/\mathfrak{a}) = \#(\mathcal{O}_E/\mathfrak{a})^{n^2}\prod_{w'\in\Sigma(\mathfrak{a})}\prod_{i=1}^n(1-\mathrm{N}w'^{-i})$ (the reduction $\mathrm{GL}_n(\mathcal{O}/w'^e) \to \mathrm{GL}_n(k_{w'})$ is onto with kernel of order $\mathrm{N}w'^{(e-1)n^2}$). As $\mathfrak{t}_{\mathrm{I}}$ and $\mathfrak{m}$ have the same places, $c_b(\mathrm{I}) = \kappa_b\,\#\mathrm{GL}_{n_{\mathrm{I},b}}(R_{\mathfrak{m}})$ with $\kappa_b = (\#(\mathcal{O}_E/\pi_M)/\#(\mathcal{O}_E/\mathfrak{b}))^{n_{\mathrm{I},b}^2}$, the factors of the places of Q cancelling ; κ_b does not depend on ϑ. So $\Pi = C\,\#\mathcal{X}$ with $C = \prod_{b<m}\kappa_b$, and (6.89) with Step 4 gives (2).

Step 6: (1). (PD1): $\Psi^1(g_x\cdot S)$ depends only on the matrices of S on w. (PD2): let g be admissible. Define σ on $\mathcal{X}$ by $(\sigma x)_{z,b} = g(z,0,b)\,x_{z,b}$ (left multiplication permutes the cosets). Then $h = g_{\sigma x}^{-1} g\, g_x$ is, at the nodes of the first path, in the level of the node: 1 at $(\mathrm{I},0,b)$, $b < m$, by the choice of representatives; at $(\mathrm{I},0,m)$ and $(\mathrm{II},0,m)$ it is g there, in the level of these nodes by admissibility (trivial, resp. reducing to 1 modulo $\mathfrak{b}$); at $(\mathrm{II},0,b)$, $1 \le b < m$, it reduces to 1 modulo $\mathfrak{b}$; at $(\mathrm{II},0,0)$ it is $h(\mathrm{I},0,0)\oplus 1 = 1$, by (FY4) and admissibility. For such h, $\Psi^1(h\cdot S) = \Psi^1(S)$ (the reductions modulo $\mathfrak{t}_z$ of the matrices do not change) and $\epsilon_w(h) = 1$ (on the column of Y, $h = 1$ at both ends of each edge; on the column of $\tilde{Y}$, χ_e is built from the tame part of μ^0_{b+1}, nontrivial only at chosen places, where $\det h \equiv 1$). With $h'\cdot(h\cdot S) = (hh')\cdot S$,

$$\epsilon_w(g)\Psi^0(g\cdot S) = \sum_x \epsilon_w(gg_x)\Psi^1((gg_x)\cdot S) = \sum_x \epsilon_w(g_{\sigma x})\epsilon_w(h)\Psi^1(h\cdot(g_{\sigma x}\cdot S)) = \Psi^0(S).$$

(PD3): if $\Psi^0(S) \ne 0$, some $\Psi^1(g_x\cdot S) \ne 0$; the vertical edges of w are those of the column of Y, and there $M = (g_x\cdot S)(e)$ reduces modulo $\mathfrak{t}_{\mathrm{I}}$ to the inclusion J, so $\det(J^{\mathsf{T}}M)$ is a unit modulo every place of $\mathfrak{t}_{\mathrm{I}}$, which are the places of $\mathfrak{m}$, hence a unit of $R_{\mathfrak{m}}$; so M has a left inverse L, and $S(e) = g_x(q)\,M\,g_x(p)^{-1}$ has the left inverse $g_x(p)Lg_x(q)^{-1}$. (PD4): Ψ^1 takes the values $0,1$ and $\epsilon_w(g_x)$ is a root of unity (a value of characters of the finite group $R_{\mathfrak{m}}^\times$), so $\Psi^0(S)$ is a sum of at most $\#\mathcal{X}$ roots of unity, and $\#\mathcal{X} \le \prod_{z,b<m}\#\mathrm{M}_{n_{z,b}}(R_{\mathfrak{m}}) \le (\#R_{\mathfrak{m}}^{(2n_m)^2})^{2m} = T_\vartheta$. □

Lemma 6.4.66. *Let S, T be spaces with points with index sets n, n', let $\mathfrak{m}$, $\mathfrak{m}'$ be moduli with $\mathfrak{m}' \subseteq \mathfrak{m}$, and let Φ be a function on $\mathrm{M}_{n'\times n}(R_{\mathfrak{m}})$. Let Φ' be the function $X \mapsto \Phi(X \bmod \mathfrak{m})$ on $\mathrm{M}_{n'\times n}(R_{\mathfrak{m}'})$. Then $\mathrm{mod}_{\mathfrak{m}'}(\Phi', v) = \mathrm{mod}_{\mathfrak{m}}(\Phi, v)$ for every $v \in \mathcal{F}_{S,T}$.*

Proof. Let $A = \Sigma(\mathfrak{m})$ and $A' = \Sigma(\mathfrak{m}')$. Since $\mathfrak{m}' \subseteq \mathfrak{m}$, $A \subseteq A'$; let $R = A' \setminus A$ and $\mathbf{1}_R$ the indicator of the finite adelic matrices integral at R. Since A' contains the conjugate of none of its members, $\bar{R}$ meets neither A nor $\bar{A}$.

Step 1: $\Phi'^{\natural} = \Phi^{\natural}\mathbf{1}_R$. A finite adelic matrix x is integral at A' exactly when it is integral at A and at R, and then $\Phi'(x_{A'} \bmod \mathfrak{m}') = \Phi(x_A \bmod \mathfrak{m})$, as the reduction $R_{\mathfrak{m}'} \to R_{\mathfrak{m}}$ only reads the components at A.

Step 2: $\tau_y^{A'} = \tau_y^A \mathbf{1}_R$ for y supported on $A \cup \bar{A}$, where τ_y^A, $\tau_y^{A'}$ are the functions τ_y of (WR6) for A and for A'. Indeed $y_R = 0$ and $\bar{y}_R = \overline{y_{\bar{R}}} = 0$, so $\bar{y}^{\mathsf{T}}(2x+y)_{A'} = \bar{y}^{\mathsf{T}}(2x+y)_A$, and $\mathbf{1}_{A'}(x+y) = \mathbf{1}_A(x+y)\mathbf{1}_R(x)$.

Step 3. Expand $\Phi^{\natural} = \sum_y a_y \tau_y^A$ with y supported on $A \cup \bar{A}$. By Steps 1 and 2, $\Phi'^{\natural} = \sum_y a_y \tau_y^{A'}$. For any finite expansion $\Psi = \sum a_y \tau_y$ one has $\Psi \otimes_A f = \sum a_y \rho(\mathrm{fv}(y)) P^A f$ (a linear section taking Ψ to (a_y) exists, and the datum does not depend on the section). Hence, using $P^{A'}v^0 = v^0 = P^A v^0$,

$$\Phi'^{\natural} \otimes_{A'} v^0 = \sum_y a_y \, \rho(\mathrm{fv}(y)) v^0 = \Phi^{\natural} \otimes_A v^0,$$

and applying $v \otimes (\cdot)$ gives $\mathrm{mod}_{\mathfrak{m}'}(\Phi', v) = \mathrm{mod}_{\mathfrak{m}}(\Phi, v)$. □

Lemma 6.4.67. *Let D be admissible data.*

(a) $\mathcal{A}_M(Y_0) \subseteq \mathcal{F}^0_{\triangle}(\mathrm{I}, 0)$.

(b) For every m-fold twist ϑ and every $y \in \mathcal{A}_M(Y_0)$, lift 1, …, lift m applied in turn to y are defined (each defining integral converges absolutely at every point), so $\Theta_\vartheta y$ is defined, and $F^{p_{\mathrm{I},0,m}}_{\Theta_\vartheta y} = \mathrm{Wc}^0_{\vartheta,\mathrm{I},m}\, y$ as functions on $G_\infty(1) \times G_f(n)$.

Proof. **(a).** By (TV2) of Definition 3.2.30, $\mathcal{F}^0_{\triangle}(\mathrm{I}, 0)$ is the space of forms of type e_λ ((ED2) of Definition 3.2.29) on $V^0_{\mathrm{I},0} = Y_0$ with $\lambda = \frac{1}{2} p_{\mathrm{I},0,0}$. At the node $(\mathrm{I}, 0, 0)$ there are no blocks and $\delta = 0$, so $p_{\mathrm{I},0,0}(v)(i) = 2\varsigma_i(v)$ and $\lambda = \varsigma$ (GP5); the signs of the node, $(-1)^e$ at the base index i with $e = \max(i+1-k, 0)$, are the signs of the fixed frame of Y_0 in the numbering of Definition 3.2.31 ($(-1)^c$ at the vector of number $k-1+c$, $0 < c \leq m-1$, and $+1$ elsewhere). By Definition 3.2.16, $\mathcal{A}_M(Y_0) = \mathcal{A}_{\mathrm{K}_M}(Y_0, \varsigma)$, the forms of type e_ς in the sense of (FT2) on the framed space Y_0 that are right invariant under K_M. Through the frame, which is the point of Y_0, the group $G_\infty \times G_f$ and the rational points Γ_{Y_0} of (VP3) of Definition 3.2.2 (those of (FT1)) are $\mathrm{G}(s) = G_\infty(s) \times G_f(r)$ and Γ_{Y_0} of (VP1), (VP2) of Definition 3.2.2, the measure dx of (FT2) is the measure of $G_\infty(s)$ of (ED2) (products of Cayley measures), and the conditions of (FT2) (continuous, bounded, left invariant under the rational points, right invariant under an open subgroup of G_f, and $\int e(x) F(g(x,1)) dx = F(g)$ when e is integrable) are those of (ED2) for the same kernel e_ς. Hence $\mathcal{A}_M(Y_0) \subseteq \mathcal{F}^0_{\triangle}(\mathrm{I}, 0)$.

(b). Fix ϑ and $y \in \mathcal{A}_M(Y_0)$, write $\mathfrak{t} = \mathfrak{t}_\vartheta(\mathrm{I})$, $N_j = \mathrm{rk}\, Y_j$, $p = p_{\mathrm{I},0,m}$, and let $F_0 = y$ and F_j be lift j applied to F_{j-1}. We show by induction that $F_j = \mathrm{Wc}^0_{\vartheta,\mathrm{I},j}\, y \in \mathcal{F}^0_{\triangle}(\mathrm{I}, j)$ for $0 \leq j \leq m$; for $j = 0$ this is (a). Let $1 \leq j \leq m$ and assume it for $j-1$.

- *The data of lift j are those of the triangle lift at $(0, j-1)$.* The triangle lift $\mathrm{Tl}^0_{\vartheta,\mathrm{I},j-1}$ has $S = V^0_{\mathrm{I},j-1} = Y_{j-1}$ and $T = V^0_{\mathrm{I},j} = Y_j$ (TV1); its characters are the edge characters (GP6) with $c = \mathrm{II}$ and $l = j-1$ at the ranks N_{j-1} and N_j, namely $\mu_j^0 \vartheta^j \upsilon^{\lfloor N_{j-1}/2 \rfloor} = \nu_j \upsilon^{\lfloor N_{j-1}/2 \rfloor}$ and $\nu_j \upsilon^{\lfloor N_j/2 \rfloor}$, in the order of lift j; its joint harmonic is the fixed (GP7) harmonic for the signs of Y_{j-1} and Y_j, the weights $\gamma_{\mathrm{II},j-1}(N_{j-1})$, $\gamma_{\mathrm{II},j-1}(N_j)$ and the doubled parameters $p^0_{\mathrm{I},0,j-1} = p_{\mathrm{I},0,j-1}$, $p_{\mathrm{I},0,j}$, which is H_j; and its point datum Φ^{pt}_{j-1}, the indicator of the matrix

with entry 1 at (a,a) for the indices a of Y_{j-1}, which are the first indices of Y_j (GP5), is $\mathbf{1}_j^{\mathrm{inc}}$; the modulus is $\mathfrak{t}$ in both. So the theta kernels and data agree, and lift j of a function F for which the integral converges absolutely is $c_j\, L_\varphi F$, L_φ the lift of (ED3) used by $\mathrm{Tl}^0_{\vartheta,\mathrm{I},j-1}$ and c_j the index in $\mathbf{K}_0(n_{j-1})$ of its elements reducing to 1 at the level places.

- *Convergence.* By Lemma 6.4.15(b) (offset 0, $a=0$, $b=j-1$, $j-1<m$) the Fock vector of H_j is an edge from e_λ, $\lambda=\frac{1}{2}p_{\mathrm{I},0,j-1}$, so e_λ is a cusp kernel of order 0 ((ED4)(a)). As F_{j-1} is a form of type e_λ on Y_{j-1}, Lemma 6.4.52 shows that the integral defining lift j of F_{j-1} converges absolutely for every h.

- *The index.* The level places are the chosen places and the places of Q, whose prime ideals are distinct with product $\mathfrak{t}$ and which contain the conjugate of none of their members ($\mathfrak{t}$ is a modulus, (TV2), (XD4)). By Lemma 6.4.76, $c_j=\#\mathrm{GL}_{n_{j-1}}(\mathcal{O}_E/\mathfrak{t})$.

Hence $F_j=\#\mathrm{GL}_{n_{j-1}}(\mathcal{O}_E/\mathfrak{t})\,\mathrm{Tl}^0_{\vartheta,\mathrm{I},j-1}F_{j-1}=\mathrm{Wc}^0_{\vartheta,\mathrm{I},j}\,y$ by (TV3), and $F_j\in\mathcal{F}^0_\triangle(\mathrm{I},j)$ by (a) of Lemma 6.4.51 (with $i=0$, $n=j-1<m$) and linearity. This completes the induction.

So $F=F_m=\mathrm{Wc}^0_{\vartheta,\mathrm{I},m}\,y$ is a form of type $e_{p/2}$ on $V^0_{\mathrm{I},m}=Y_m=\mathrm{I}_n$ (TV1), (TV2). The doubled parameter $l=2\lambda(Y_m)$ of the chain is p , and p is a doubled Harish-Chandra parameter of (κ_Y,e_Y) . By Lemma 6.4.70, $F=F^p_\eta$ for an algebraic form η of weight (κ_Y,e_Y), and by (b) of Lemma 6.4.72, $\Theta_\vartheta y=F^\flat_p=(F^p_\eta)^\flat_p=\eta$. Hence $F^p_{\Theta_\vartheta y}=F^p_\eta=F=\mathrm{Wc}^0_{\vartheta,\mathrm{I},m}\,y$. □

Lemma 6.4.68. *Let D be admissible data and $z\in\{\mathrm{I},\mathrm{II}\}$.*

(1) For $0\le j\le m$ the space at the node (z,j,m) is the standard definite space $\mathrm{I}_{n_{z,j}}$, so that its grid forms are functions on $G_\infty(1)\times G_f(n_{z,j})$; and for every m-fold twist ϑ, $u\in\mathcal{F}(\mathrm{I},m,m)$, $v\in\mathcal{F}(\mathrm{II},m,m)$ and every real point g of $\mathrm{U}(\tilde{Y}^m_m)$, with $n=n_{\mathrm{I},m}$,

$$\langle u,v\rangle_{m,g}=\int_{[\mathrm{U}(\mathrm{I}_n)]}u(x)\,v\big(\mathrm{diag}(x,(1,\mathrm{flp}^m_\vartheta))\,(g,1)\big)\,dx.$$

(2) For every m-fold twist ϑ, edge tuple S for ϑ, $y\in\mathcal{F}(z,0,m)$ and $0\le j\le m$, the chain $\mathrm{Lc}^S_{z,j}(y)$ is defined (for $j<m$ and every h, the integrand of the class integral of (LQ4) that gives $\mathrm{Lc}^S_{z,j+1}(y)(h)$ satisfies the hypothesis of (LQ3)) and $\mathrm{Lc}^S_{z,j}(y)\in\mathcal{A}_{z,j}$.

(3) For $1\le j\le m$ there is $c_{z,j}\in\mathbb{C}$ such that for every m-fold twist ϑ, edge tuple S for ϑ and $y\in\mathcal{F}(z,0,m)$, the image of y under the composite of the lift moves along $e_{z,0},\dots,e_{z,j-1}$ with the matrices $S(e_{z,0}),\dots,S(e_{z,j-1})$ is the function $F^{l_{z,j}}_\eta$ with

$$\eta=\frac{c_{z,j}}{[\mathbf{K}_0(n_{z,0}):\mathrm{K}_\vartheta(z,0)]}\,\mathrm{Lc}^S_{z,j}(y).$$

(4) The map $y\mapsto y^\flat$ is a linear bijection from $\mathcal{F}(z,0,m)$ onto $\mathcal{A}_{z,0}$, and $y=F^{l_{z,0}}_{y^\flat}$ for every $y\in\mathcal{F}(z,0,m)$.

Proof. **(1).** By (GP4) the node (z,j,m) has $\lfloor[z]/2\rfloor=0$ negative signs, so all its signs are $+1$, and by (GP8) its space is the standard definite space $\mathrm{I}_{n_{z,j}}$; its adelic unitary group is $G_\infty(1)\times G_f(n_{z,j})$. Let $n=n_{\mathrm{I},m}$. By (GP2), $\tilde{Y}^m_m=Y^m_m\oplus\langle(-1)^m\rangle$, and $m=2^{e+1}$ is even by (AD2), so the added line has sign $+1$; by (GP8) and the numbering of (GP5), $\tilde{Y}^m_m$ is the standard definite space on $n\sqcup\{*\}$, $*$ the added line, and $\mathrm{U}(Y^m_m)\to\mathrm{U}(\tilde{Y}^m_m)$ is $x\mapsto\mathrm{diag}(x,1)$. By (3.3) with $l=m$, $\langle u,v\rangle_{m,g}$ is the integral over $[\mathrm{U}(\mathrm{I}_n)]$ of u against the restriction of the right translate

of v by $t=(g,1)\cdot\mathrm{diag}(1,(1,\mathrm{flp}_\vartheta^m))$, the real point g together with the finite element flp_ϑ^m on the added line. As $(g,1)$ has finite component 1 and $\mathrm{diag}(1,(1,\mathrm{flp}_\vartheta^m))$ has archimedean component 1, $\mathrm{diag}(x,1)\,t$ has archimedean component $\mathrm{diag}(x_\infty,1)g$ and finite component $\mathrm{diag}(x_f,\mathrm{flp}_\vartheta^m)$, that is $\mathrm{diag}(x,1)t=\mathrm{diag}(x,(1,\mathrm{flp}_\vartheta^m))(g,1)$. This is the formula of (1).

(4). By (1) the space at $(z,0,m)$ is $\mathrm{I}_{n_{z,0}}$, and by definition $\mathcal{F}(z,0,m)$ is the space of forms on it of the type of the negated grid parameter, $e_{l_{z,0}/2}$, with $l_{z,0}=-p_{z,0,m}$ (LQ1). By (3) of Lemma 6.4.3, $l_{z,0}$ is a doubled Harish-Chandra parameter of $(\kappa_z,e_{z,0}^{\det})$. Hence, by Lemma 6.4.70, $\eta\mapsto F_\eta^{l_{z,0}}$ maps $\mathcal{A}_{z,0}$ into $\mathcal{F}(z,0,m)$ and onto it; it is linear, since $F_\eta^l(x,g)=\langle v_l^+,\rho_{\kappa,e}(x^{-1})\eta(g)\rangle_\kappa$ is linear in η. By (b) of Lemma 6.4.72, $(F_\eta^{l_{z,0}})^\flat=\eta$ for every $\eta\in\mathcal{A}_{z,0}$, so the map is injective, hence a linear bijection whose inverse is $y\mapsto y^\flat$ (LQ5). In particular $y^\flat\in\mathcal{A}_{z,0}$ and $y=F_{y^\flat}^{l_{z,0}}$ for every $y\in\mathcal{F}(z,0,m)$.

The lift moves of the last path. Fix ϑ, an edge tuple S and $0\le i<m$; put $N_i=\#n_{z,i}$. By the definition of the lift move and (XD5), the lift move along $e_{z,i}$ with the matrix $S(e_{z,i})$ is L_φ for the theta kernel $(\omega_{\alpha,\beta},\theta)$ with α,β the edge characters at its ends and the datum $\varphi=\mathrm{mod}_{\mathfrak{m}_\vartheta}(\mathbf{1}_{S(e_{z,i})},v_H)=v_H\otimes(\Phi^\natural\otimes_{\Sigma(\mathfrak{m}_\vartheta)}v^0)$, H the joint harmonic of the move and Φ the indicator of $S(e_{z,i})$. By (LQ4), $\varphi=v_H\otimes f$ with $f=c_s^{-N_iN_{i+1}}\varphi_{z,i}^S$ ($c_s\ne0$). As jm, $\omega_{\alpha,\beta}$ and the class integral are linear, $\mathrm{Al}_{\vartheta,z,i}(f)=c_s^{-N_iN_{i+1}}\mathrm{Al}_{\vartheta,z,i}(\varphi_{z,i}^S)$, and both are linear in η. So Lemma 6.4.69 gives, for every $\eta\in\mathcal{A}_{z,i}$, that the integrands defining $\mathrm{Al}_{\vartheta,z,i}(\varphi_{z,i}^S)\eta$ satisfy the hypotheses of (LQ3), that $\mathrm{Al}_{\vartheta,z,i}(\varphi_{z,i}^S)\eta\in\mathcal{A}_{z,i+1}$, and

$$(\text{lift move along } e_{z,i} \text{ with } S(e_{z,i}))(F_\eta^{l_{z,i}})=\frac{c_s^{-N_iN_{i+1}}\beta_{z,i}^{\mathrm{h}}}{[\mathbf{K}_0(n_{z,i}):\mathrm{K}_\vartheta(z,i)]}\,F^{l_{z,i+1}}_{\mathrm{Al}_{\vartheta,z,i}(\varphi_{z,i}^S)\eta}.\qquad(*)$$

(2). For $0\le j<m$, $N_{z,j}<N_{z,j+1}$ by (1) of Lemma 6.4.3, so the algebraic theta lift along $e_{z,j}$ is the partial map of (LQ4). By induction on j: $\mathrm{Lc}_{z,0}^S(y)=y^\flat$ is defined and lies in $\mathcal{A}_{z,0}$ by (4); if $\mathrm{Lc}_{z,j}^S(y)$ is defined and lies in $\mathcal{A}_{z,j}$ and $j<m$, the preceding paragraph with $i=j$ and $\eta=\mathrm{Lc}_{z,j}^S(y)$ shows that the integrand of the class integral giving $\mathrm{Lc}_{z,j+1}^S(y)(h)$ satisfies the hypotheses of (LQ3) for every h, so that $\mathrm{Al}_{\vartheta,z,j}(\varphi_{z,j}^S)$ is defined at $\mathrm{Lc}_{z,j}^S(y)$ and $\mathrm{Lc}_{z,j+1}^S(y)$ is defined (LQ5), and that $\mathrm{Lc}_{z,j+1}^S(y)\in\mathcal{A}_{z,j+1}$.

(3). For $(z,i)\ne(\mathrm{I},0)$ the level $\mathrm{K}_\vartheta(z,i)$ is the kernel of the reduction modulo $\mathfrak{b}$ on $\mathbf{K}_0(n_{z,i})$ (LQ2), which does not depend on ϑ; it is open in the compact group $\mathbf{K}_0(n_{z,i})$, so $k_{z,i}=[\mathbf{K}_0(n_{z,i}):\mathrm{K}_\vartheta(z,i)]$ is a positive integer independent of ϑ for $i\ge1$. Put

$$c_{z,j}=\prod_{0\le i<j}c_s^{-N_iN_{i+1}}\beta_{z,i}^{\mathrm{h}}\cdot\prod_{1\le i<j}k_{z,i}^{-1}\qquad(1\le j\le m),$$

which depends on neither ϑ, S nor y. Let $M_j(y)$ be the image of y under the composite of the lift moves along $e_{z,0},\dots,e_{z,j-1}$ with the matrices $S(e_{z,0}),\dots,S(e_{z,j-1})$. We show by induction on $j\ge1$ that $M_j(y)$ is defined and equals $F_{\eta_j}^{l_{z,j}}$ with $\eta_j=c_{z,j}[\mathbf{K}_0(n_{z,0}):\mathrm{K}_\vartheta(z,0)]^{-1}\mathrm{Lc}_{z,j}^S(y)$. For $j=1$: $y=F^{l_{z,0}}_{\mathrm{Lc}_{z,0}^S(y)}$ by (4), and $(*)$ with $i=0$ gives $M_1(y)=F_{\eta_1}^{l_{z,1}}$. If the claim holds for $j<m$, then $\eta_j\in\mathcal{A}_{z,j}$ by (2), and $(*)$ with $i=j$, $\eta=\eta_j$ and the linearity of $\mathrm{Al}_{\vartheta,z,j}(\varphi_{z,j}^S)$ give

$$M_{j+1}(y)=\frac{c_s^{-N_jN_{j+1}}\beta_{z,j}^{\mathrm{h}}}{k_{z,j}}\,F^{l_{z,j+1}}_{\mathrm{Al}_{\vartheta,z,j}(\varphi_{z,j}^S)\eta_j}=F_{\eta_{j+1}}^{l_{z,j+1}},$$

since $\mathrm{Al}_{\vartheta,z,j}(\varphi_{z,j}^S)\mathrm{Lc}_{z,j}^S(y)=\mathrm{Lc}_{z,j+1}^S(y)$ by (LQ5). This is (3). □

Lemma 6.4.69. *Let D be admissible data, $z \in \{\mathrm{I},\mathrm{II}\}$, $0 \le i < m$, $n = n_{z,i}$, $n' = n_{z,i+1}$, and let H be the joint harmonic of the lift move along $e_{z,i}$. There is $\beta^{\mathrm{h}}_{z,i} \in \mathbb{C}$ such that for every m-fold twist ϑ, with α and β the edge characters of the lift move along $e_{z,i}$ at its ends of ranks $\#n$ and $\#n'$, every modulus $\mathfrak{m}$, every function Ψ on the $n' \times n$ matrices over $R_{\mathfrak{m}}$ with $f = \Psi^\natural \otimes_{\Sigma(\mathfrak{m})} v^0 \in \mathcal{M}_{n,n'}$, and every $\eta \in \mathcal{A}_{z,i}$:*

(a) for every $h \in G_f(n')$ the function $g \mapsto \mathrm{jm}(\omega_{\alpha,\beta}((1,g),(1,h))f)\,\eta(g)$ on $G_f(n)$ satisfies the hypotheses of (LQ3), so that the partial map $\mathrm{Al}_{\vartheta,z,i}(f)$ of (LQ4) is defined at η, and $\mathrm{Al}_{\vartheta,z,i}(f)\eta \in \mathcal{A}_{z,i+1}$;

(b) for the theta kernel $(\omega_{\alpha,\beta},\theta_{n,n'})$ of $\mathrm{Hom}(\mathrm{I}_n,\mathrm{I}_{n'})$, the integral defining $L_{v_H\otimes f}F_\eta^{l_{z,i}}$ converges absolutely at every point, and

$$L_{v_H\otimes f}\,F_\eta^{l_{z,i}} = \frac{\beta^{\mathrm{h}}_{z,i}}{[\mathbf{K}_0(n) : \mathrm{K}_\vartheta(z,i)]}\,F^{l_{z,i+1}}_{\mathrm{Al}_{\vartheta,z,i}(f)\eta}.$$

Proof. Fix z, i and put $N = \#n$, $N' = \#n'$, $\kappa = \kappa_z$, $e = e^{\det}_{z,i}$, $e' = e^{\det}_{z,i+1}$, $l = l_{z,i}$, $l' = l_{z,i+1}$, $\gamma' = \gamma_{z,i}(N')$, $\rho_{\mathrm{s}} = \rho_{\kappa,e}$ (rank N), $\rho_{\mathrm{t}} = \rho_{\kappa,e'}$ (rank N'), $v^+ = v^+_l$, $v'^+ = v^+_{l'}$, $G = G_\kappa$ in the rank N', $I = I_{z,i}$, $I' = I_{z,i+1}$, $B_\kappa = \sum_\tau \sum_{1\le h<N'} h\,\kappa(\tau,h)$, and for a rank M and $0 \le p < M$ let $\lambda^{(M)}_\tau(p) = \sum_{p<h<M}\kappa(\tau,h)$; τ runs over the infinite places.

Step 0: weights and positions. By (1), (2) of Lemma 6.4.3: $N < N'$, $\kappa(\tau,h) = 0$ for $N < h < N'$, $e = \gamma' + \kappa(\cdot,N)$, and by (GP6) the edge characters α, β at the ends of ranks N, N' have the weights $\gamma_{z,i}(N) = e'$ and γ'. By (3) of Lemma 6.4.3, l and l' are doubled Harish-Chandra parameters of (κ,e) and (κ,e'). By (WM6), l_τ is injective with values in the set of the N numbers $2(e_\tau + \lambda^{(N)}_\tau(p)) + N - 1 - 2p$, $0 \le p < N$, which strictly decrease in p; so with $\pi_\tau(a) = \#\{a' \in n : l_\tau(a') > l_\tau(a)\}$, a bijection $n \to \{0,\dots,N-1\}$,

$$l_\tau(a) = 2\big(e_\tau + \lambda^{(N)}_\tau(\pi_\tau(a))\big) + N - 1 - 2\pi_\tau(a), \qquad l'_\tau(b) = 2\big(e'_\tau + \lambda^{(N')}_\tau(\pi'_\tau(b))\big) + N' - 1 - 2\pi'_\tau(b),$$

with π'_τ defined likewise for l'_τ. As $\kappa(\tau,h) = 0$ for $N < h < N'$: $\kappa(\tau,N) + \lambda^{(N)}_\tau(p) = \lambda^{(N')}_\tau(p)$ for $p < N$, and $\lambda^{(N')}_\tau(p) = 0$ for $N \le p < N'$.

Step 1: the block rule. All signs are $+1$ ((GP4)). For every τ the pair (l_τ, l'_τ) satisfies the block rule (WM8) for the weights (e'_τ,γ'_τ) and the signs $(\mathbf{1}_n,\mathbf{1}_{n'})$: (a) holds since the signs agree, l_τ, l'_τ are injective, and $l_\tau(a) \equiv N-1$, $l'_\tau(b) \equiv N'-1 \pmod 2$; for (b) and (c), the shifted entries are

so $\tilde{l}'_\tau(n') = \tilde{l}_\tau(n) \cup [N'-N]$, which is (b), and $\tilde{l}_\tau(a) \ge N+N'-1-2(N-1) = N'-N+1 > 0$ for every a, so (c) only requires $s(a) = t(b)$, which holds.

Step 2: the joint harmonic is a Schur coefficient. For $A \in \mathbb{C}^{I'}$ and $\xi \in \mathbb{C}^I$ let $\mathrm{Sc}(A,\xi) = A^{\mathsf{T}} S_\kappa(X)\,\xi$, a polynomial in the variables $X_{\tau,b,a}$ ($b \in n'$, $a \in n$), $S_\kappa(X)$ the rectangular Schur matrix from the rank N to the rank N'. We show $H = c_H\,\mathrm{Sc}(Gv'^+, v^+)$ with $c_H \in \mathbb{C}^\times$.

(i) *The polynomial of v'^+.* By (WM3), (WM4), $\mathrm{Pol}_\kappa(\mathrm{ev}_\kappa(y)) = \sum_{i,j}(F_\kappa^{\mathsf{T}}F_\kappa)_{ij}b_j(y)\,b_i = \sum_{T\in\mathcal{C}_\kappa}\Delta_T(y)\,\Delta_T$ in the rank N', because $\Delta_T = \sum_i (F_\kappa)_{Ti}b_i$. Let $y' = y^{l'}$ be the array of (WM6), with entry 1 at $(\tau,b,\pi'_\tau(b))$ and 0 elsewhere. For $T \in \mathcal{C}_\kappa$ and a set $T(\tau,h,j) = \{b_0,\dots,b_{h-1}\}$ listed increasingly, the matrix $(y'_{\tau,b_k,c})_{k,c<h}$ has in row k a single nonzero entry, in column $\pi'_\tau(b_k)$ if $\pi'_\tau(b_k) < h$; it is a permutation matrix if $T(\tau,h,j) = A'_{\tau,h} = \pi'^{-1}_\tau\{0,\dots,h-1\}$ and has a zero row otherwise. So $\Delta_T(y') \ne 0$ only for the filling with $T(\tau,h,j) = A'_{\tau,h}$ for all (τ,h,j),

and for it $\Delta_{\tau,A'_{\tau,h}}(y') = \epsilon_{\tau,h}$ is the sign of $k \mapsto \pi'_\tau(b_k)$. As $\epsilon_{\tau,h}\Delta_{\tau,A'_{\tau,h}} = M_{\tau,h}$, the minor $\det(y_{\tau,\pi'^{-1}_\tau(k),c})_{0\le k,c<h}$,

$$\mathrm{Pol}_\kappa(v'^+) = \prod_\tau \prod_{1\le h<N'} M_{\tau,h}^{\kappa(\tau,h)}.$$

(ii) *Evaluation at Xy^l.* Let y^l be the array of (WM6) for l (rows n). The rectangular Schur matrix of (WM2) satisfies $b'_{i'}(Xy) = \sum_i S_\kappa(X)_{i'i}\, b_i(y) \prod_\tau \Delta_\tau(y)^{\kappa(\tau,N)}$ as polynomials (the identity holds over the ring of polynomials in X). At $y = y^l$, $b_i(y^l) = v_i^+$, and $\Delta_\tau(y^l) = \pm 1$ is the determinant of a permutation matrix; so

$$\varepsilon\,(Gv'^+)^{\mathsf T} S_\kappa(X)\, v^+ = \sum_{i'} (Gv'^+)_{i'}\, b'_{i'}(Xy^l) = \mathrm{Pol}_\kappa(v'^+)(Xy^l), \qquad \varepsilon = \prod_\tau \Delta_\tau(y^l)^{\kappa(\tau,N)} = \pm 1.$$

Here $(Xy^l)_{\tau,b,c} = X_{\tau,b,\pi_\tau^{-1}(c)}$ for $c < N$ and $= 0$ for $c \ge N$. So $M_{\tau,h}(Xy^l) = \Delta^{\mathrm{st}}_{\tau,h} = \det\big(X_{\tau,\pi'^{-1}_\tau(k),\pi_\tau^{-1}(c)}\big)_{0\le k,c<h}$ for $h \le N$, while $\kappa(\tau,h) = 0$ for $N < h < N'$. With (i), $\mathrm{Sc}(Gv'^+, v^+) = \varepsilon \prod_\tau \Pi_\tau$, $\Pi_\tau = \prod_{1\le h\le N} (\Delta^{\mathrm{st}}_{\tau,h})^{\kappa(\tau,h)}$.

(iii) *Staircase and uniqueness.* Fix τ. Apply Lemma 6.4.48 with $R = n'$, $C = n$, $\sigma = \mathbf{1}_n$, the shifts $(\gamma_1,\gamma_2) = (e'_\tau, -\gamma'_\tau)$, l'_τ, l_τ, $k(1,p) = \kappa(\tau,p)$ and $k(-1,p) = 0$ ($1 \le p < N'$): $k(1,p) = 0$ for $p > N = \#\{c : \sigma(c) = 1\}$ by Step 0, $k(-1,\cdot) = 0$, and with these signs $x_1 = \pi'_\tau$, $o = \pi_\tau$, so its $D_{1,p}$ is $\Delta^{\mathrm{st}}_{\tau,p}$ and its P is Π_τ. Hence Π_τ is termwise harmonic, K-highest of weight ν for l'_τ for the row units of $(\mathbf{1}_n, \mathbf{1}_{n'})$ with the shifts $(e'_\tau, -\gamma'_\tau)$, with $\nu(b) = e'_\tau + \sum_{\pi'_\tau(b)<p<N'} \kappa(\tau,p) = e'_\tau + \lambda^{(N')}_\tau(\pi'_\tau(b))$, and killed by the column units $E^{\mathrm{col}}_{cc'}$ whenever $-l_\tau(c') < -l_\tau(c)$. For the signs $\mathbf{1}_{n'}$, $2\rho_{l'_\tau}(b) = 2\rho^{\mathrm c}_{l'_\tau}(b) = N' - 1 - 2\pi'_\tau(b)$, so by Step 0 $l'^{\rho}_b = e'_\tau + \lambda^{(N')}_\tau(\pi'_\tau(b)) + N' - 1 - 2\pi'_\tau(b)$ and $\mu_{l'_\tau} = \nu$ (Definition 3.2.3). By Step 1 and Lemma 6.4.19 a joint harmonic for the signs $(\mathbf{1}_n, \mathbf{1}_{n'})$, the weights (e'_τ, γ'_τ) and the parameters (l_τ, l'_τ) exists; these are the data of (GP7) for the lift move along $e_{z,i}$ ($\gamma = \gamma_{z,i}(N) = e'$, $\lambda = -p_{z,i,m} = l$, $\lambda' = l'$), so the factor H_τ of H is such a joint harmonic. By Lemma 6.4.49, $\Pi_\tau = u_\tau H_\tau$ with $u_\tau \in \mathbb{C}$, and $u_\tau \ne 0$ since Π_τ is a product of determinants of matrices of distinct variables. Hence $H = \prod_\tau H_\tau = c_H\, \mathrm{Sc}(Gv'^+, v^+)$ with $c_H = \varepsilon \prod_\tau u_\tau^{-1}$.

Step 3: real points on Schur coefficients. Let $x \in G_\infty(1)$ for n and $y \in G_\infty(1)$ for n'. By the second formula of (5) of Lemma 4.1.1 (all signs $+1$, so no block of sign -1), $\Omega_{e',\gamma'}(x,y)v_P = \prod_\tau \det(y_\tau)^{e'_\tau} \det(x_\tau)^{-\gamma'_\tau}\, v_{P'}$ with $P'(X) = P(y^{\mathsf T} X x^{-\mathsf T})$, the substitution acting on each $n' \times n$ matrix X_τ. For unitary matrices $y^{\mathsf T} = \overline{y^{-1}}$, $x^{-\mathsf T} = \bar x$, $\det y = \det \overline{y^{-1}}$ and $\det x^{-1} = \det \bar x$. By (WM2), for square z of size N' and w of size N, $S_\kappa(zXw) = S_\kappa(z)S_\kappa(X)S_\kappa(w)\prod_\tau \det(w_\tau)^{\kappa(\tau,N)}$: expand $b'_{i'}(zXwy)$ by the square case at z, the rectangular case at X, and the square case $b_j(wy) = \sum_k S_\kappa(w)_{jk} b_k(y)$ with $\Delta_\tau(wy) = \det(w_\tau)\Delta_\tau(y)$, and compare coefficients (they are unique by (WM2)). With $e = \gamma' + \kappa(\cdot, N)$ this gives

$$\Omega_{e',\gamma'}(x,y)\, v_{\mathrm{Sc}(A,\xi)} = v_{\mathrm{Sc}(\rho_{\mathrm t}(y^{-1})^{\mathsf T} A,\ \rho_{\mathrm s}(x)\xi)}. \tag{6.91}$$

Step 4: theta values are coefficients of jet matrices. For $g \in G_f(n)$, $h \in G_f(n')$ put $\mathcal J(g,h) = \mathrm{jm}(\omega_{\alpha,\beta}((1,g),(1,h))f)$. Each entry $P_{i'i}$ of $S_\kappa(X)$ is homogeneous of degree B_κ: in the identity of Step 2 (ii) the left side $b'_{i'}(Xy)$ is homogeneous of degree B_κ in X (a minor on h rows of Xy is homogeneous of degree h in X), and the coefficients are unique. So $\mathrm{jm}(F)_{i'i} = \eta_P^{B_\kappa}\theta_{n,n'}(v_{P_{i'i}} \otimes F)$, and $\eta_P \ne 0$ ((OL4): a quotient of nonzero numbers). By the first formula of (5) of Lemma 4.1.1 (α, β of weights e', γ'), (6.91), and the linearity of $P \mapsto v_P$, of $v \mapsto v \otimes F$ and of $\theta_{n,n'}$: for x, y as in Step 3,

$$\theta_{n,n'}\Big(\omega_{\alpha,\beta}((x,g),(y,h))\big(v_{\mathrm{Sc}(A,\xi)} \otimes f\big)\Big) = \eta_P^{-B_\kappa}\, A^{\mathsf T}\rho_{\mathrm t}(y^{-1})\, \mathcal J(g,h)\, \rho_{\mathrm s}(x)\, \xi. \tag{6.92}$$

(1) *Rational points.* By Lemma 6.3.77, $(\omega_{\alpha,\beta}, \theta_{n,n'})$ is a theta kernel for $(\mathrm{I}_n, \mathrm{I}_{n'})$: $\theta_{n,n'}(\omega(\gamma p, \delta q)\varphi) = \theta_{n,n'}(\omega(p,q)\varphi)$ for rational points γ of I_n, δ of $\mathrm{I}_{n'}$, and $(p,\varphi) \mapsto \omega(p)\varphi$ is continuous with $\theta_{n,n'}$ bounded. As $(1, \gamma_f g) = \gamma\,(\gamma_\infty^{-1}, g)$ and $(1, \delta_f h) = \delta\,(\delta_\infty^{-1}, h)$, (6.92) at $(1,1)$ and at $(\gamma_\infty^{-1}, \delta_\infty^{-1})$ gives $A^{\mathsf{T}} \mathcal{J}(\gamma_f g, \delta_f h)\xi = A^{\mathsf{T}} \rho_{\mathrm{t}}(\delta_\infty) \mathcal{J}(g,h) \rho_{\mathrm{s}}(\gamma_\infty^{-1})\xi$ for all A, ξ; by (c) of Lemma 4.6.10,

$$\mathcal{J}(\gamma_f g, \delta_f h) = \rho_{\mathrm{t}}(\delta_\infty)\, \mathcal{J}(g,h)\, \rho_{\mathrm{s}}(\gamma_\infty)^{-1}.$$

(2) *Open stabilisers.* Let U be the set of $k \in \mathbf{K}_0(n)$ with $\bar{k} = 1$ modulo $\mathfrak{m}$ and $\beta(\det k) = 1$, and U' the set of $k' \in \mathbf{K}_0(n')$ with $\bar{k}' = 1$ and $\alpha(\det k') = 1$. These are open subgroups: the kernels of the reductions are open, and a continuous homomorphism from the profinite group $\mathbf{K}_0(n)$ to the circle is trivial on an open subgroup (the preimage of a neighbourhood of 1 containing no nontrivial subgroup contains an open subgroup, whose image is then trivial). For $(k,k') \in U \times U'$ and every polynomial Q, (6) of Lemma 4.1.1 gives $\omega_{\alpha,\beta}((1,k),(1,k'))(v_Q \otimes f) = v_Q \otimes f$, the function $\Psi(\bar{k}'^{-1} X \bar{k}) = \Psi(X)$ being unchanged; with the first formula of (5) at real components 1 and the homomorphism property, $\mathcal{J}(gk, hk') = \mathcal{J}(g,h)$, and likewise $\omega_{\alpha,\beta}((x,gk),(y,hk'))(v_Q \otimes f) = \omega_{\alpha,\beta}((x,g),(y,h))(v_Q \otimes f)$.

Step 5: (a). Fix h and put $F_h(g) = \mathcal{J}(g,h)\eta(g)$. For a rational point γ of I_n, Step 4 (1) and (WM7) give $F_h(\gamma_f g) = \mathcal{J}(g,h)\rho_{\mathrm{s}}(\gamma_\infty)^{-1}\rho_{\mathrm{s}}(\gamma_\infty)\eta(g) = F_h(g)$; and $F_h(gk) = F_h(g)$ for k in U intersected with an open subgroup fixing η (Step 4 (2), (WM7)). So F_h satisfies the hypotheses of (LQ3) for every h. As $N < N'$ (Step 0), the algebraic theta lift along $e_{z,i}$ is the partial map of (LQ4), and by its definition it is defined at η exactly when every F_h satisfies the hypotheses of (LQ3); this uses only $\eta \in \mathcal{A}_{z,i}$ and the datum f, not the chains of (LQ5). Hence $\mathrm{Al}_{\vartheta,z,i}(f)$ is defined at η, and $\mathrm{Al}_{\vartheta,z,i}(f)\eta(h) = \mathrm{CI}_{\mathrm{K}}(F_h) \in \mathbb{V}_\kappa$ (rank N'), $\mathrm{K} = \mathrm{K}_\vartheta(z,i)$. By (1) of Lemma 6.4.74, $\mathrm{CI}_{\mathrm{K}}(F)$ is, for any open subgroup $\mathrm{K}'' \subseteq \mathrm{K}$ fixing F, an explicit finite linear combination of values of F; hence $\mathrm{CI}_{\mathrm{K}}(L \circ F) = L(\mathrm{CI}_{\mathrm{K}}(F))$ for every linear map L. For $k' \in U'$, $F_{hk'} = F_h$; for a rational point δ of $\mathrm{I}_{n'}$, $F_{\delta_f h} = \rho_{\mathrm{t}}(\delta_\infty)F_h$ by Step 4 (1). So $\mathrm{Al}_{\vartheta,z,i}(f)\eta$ is right invariant under U' and satisfies $\mathrm{Al}_{\vartheta,z,i}(f)\eta(\delta_f h) = \rho_{\mathrm{t}}(\delta_\infty)\,\mathrm{Al}_{\vartheta,z,i}(f)\eta(h)$: it is an algebraic form of weight (κ, e') on n', that is, it lies in $\mathcal{A}_{z,i+1}$.

Step 6: (b). Fix (y,h) with $y \in G_\infty(1)$ for n' and $h \in G_f(n')$, put $\varphi = v_H \otimes f$ and $\Theta(x,g) = \theta_{n,n'}(\omega_{\alpha,\beta}((x,g),(y,h))\varphi)$ on $\mathrm{G}(1_n) = G_\infty(1) \times G_f(n)$. By Step 4 (1), Θ is continuous and left invariant under the rational points of I_n; the form of I_n is definite, so by Theorem A.4.5 Θ is bounded and $[\mathrm{U}(\mathrm{I}_n)]$ has finite volume. By (b) of Lemma 6.4.72, F_η^l is continuous, bounded, left invariant under the rational points and right invariant under an open subgroup. So the integral $L_\varphi F_\eta^l(y,h) = \int_{[\mathrm{U}(\mathrm{I}_n)]} \Theta\, F_\eta^l$ converges absolutely. By Step 2 and (6.92) with $A = Gv'^+$, $\xi = v^+$, $\Theta(x,g) = c_H \eta_P^{-B_\kappa} A_y^{\mathsf{T}} \mathcal{J}(g,h)\rho_{\mathrm{s}}(x)v^+$ with $A_y = \rho_{\mathrm{t}}(y^{-1})^{\mathsf{T}} G v'^+$. The function ΘF_η^l is right invariant under an open subgroup $\mathrm{K}' \subseteq \mathbf{K}_0(n)$ in the finite variable (Step 4 (2)). By (2) of Lemma 6.4.74 (with $V = \mathbb{C}$ and $c_N > 0$ the total mass of the measure of $G_\infty(1)$ for n),

$$L_\varphi F_\eta^l(y,h) = \frac{c_N}{[\mathbf{K}_0(n) : \mathrm{K}]}\,\mathrm{CI}_{\mathrm{K}}(H_0), \qquad H_0(g) = \int_{G_\infty(1)} \Theta(x,g) F_\eta^l(x,g)\,dx,$$

dx the Haar probability measure. A linear map commutes with the integral of a continuous vector-valued function on the compact group $G_\infty(1)$, so by (WM7) and (b) of Lemma 6.4.72,

$$\begin{aligned} H_0(g) &= c_H \eta_P^{-B_\kappa} A_y^{\mathsf{T}} \mathcal{J}(g,h) \int_{G_\infty(1)} F_\eta^l(x,g)\rho_{\mathrm{s}}(x)v^+\,dx \\ &= \frac{c_H \eta_P^{-B_\kappa}}{\#I}\, A_y^{\mathsf{T}} \mathcal{J}(g,h)\,(F_\eta^l)_l^\flat(g) = \frac{c_H \eta_P^{-B_\kappa}}{\#I}\, A_y^{\mathsf{T}} F_h(g). \end{aligned}$$

By Step 5, $\mathrm{CI}_{\mathrm{K}}(H_0) = c_H \eta_P^{-B_\kappa} (\#I)^{-1} A_y^{\mathsf{T}}\,\mathrm{Al}_{\vartheta,z,i}(f)\eta(h)$. The entries of $v'^+ = \mathrm{ev}_\kappa(y^{l'})$ are values of integral polynomials at a 0/1 array, hence real, and G is symmetric; so $A_y^{\mathsf{T}} w =$

$\overline{v'^+}^{\mathsf{T}} G\rho_{\mathrm{t}}(y^{-1})w = \langle v'^+, \rho_{\mathrm{t}}(y^{-1})w\rangle_\kappa$, and with $w = \mathrm{Al}_{\vartheta,z,i}(f)\eta(h)$ this is $F^{l'}_{\mathrm{Al}_{\vartheta,z,i}(f)\eta}(y,h)$ (WM7). Hence (b) holds with

$$\beta^{\mathrm{h}}_{z,i} = \frac{c_N\, c_H\, \eta_P^{-B_\kappa}}{\#I_{z,i}},$$

which depends only on z and i (through H, N, κ and η_P), not on ϑ, $\mathfrak{m}$, Ψ or η. □

Lemma 6.4.70. *Let n be a nonempty finite index set and (κ, e) a weight for the ranks $\#n$ with a doubled Harish-Chandra parameter l. A function F on $\mathrm{G}(1_n)$ is a form of type $e_{l/2}$ on I_n if and only if $F = F^l_\eta$ for an algebraic form η of weight (κ, e) on n.*

Proof. Write $\rho = \rho_{\kappa,e}$, $\langle\,,\rangle = \langle\,,\rangle_\kappa$, $v = v_l^+$, $d = \#I_\kappa$ and $k_l = e_{l/2}$. By Lemma 6.4.71(b), $k_l(y) = d\langle v, \rho(y)v\rangle$ for $y \in G_\infty(1)$; so k_l is continuous on the compact group $G_\infty(1)$, hence integrable, and a form of type k_l is a continuous bounded function F, left invariant under Γ, with a level, such that

$$F(x,g) = \int_{G_\infty(1)} k_l(y)\, F(xy,g)\, dy = d\int_{G_\infty(1)} \langle v, \rho(y)v\rangle\, F(xy,g)\, dy \qquad (x \in G_\infty(1),\ g \in G_f(n)). \tag{6.93}$$

By Lemma 4.6.10(c), ρ is unitary, so $F^l_\eta(x,g) = \langle v, \rho(x^{-1})\eta(g)\rangle = \langle \rho(x)v, \eta(g)\rangle$.

The functions of algebraic forms are forms of type k_l. Let η be an algebraic form of weight (κ, e) on n. By Lemma 6.4.72(b), F^l_η is continuous, bounded, left invariant under Γ and has a level. For $x, y \in G_\infty(1)$ and $g \in G_f(n)$, $F^l_\eta(xy,g) = \langle \rho(y)v, \rho(x^{-1})\eta(g)\rangle = \overline{\langle \rho(x^{-1})\eta(g), \rho(y)v\rangle}$, and (6.94) of Lemma 6.4.71(a) with $a = b = b' = v$, $c = \rho(x^{-1})\eta(g)$, together with $\langle v, v\rangle = 1$ (Lemma 6.4.72(a)), gives

$$d\int \langle v, \rho(y)v\rangle\, \overline{\langle \rho(x^{-1})\eta(g), \rho(y)v\rangle}\, dy = \langle v, \rho(x^{-1})\eta(g)\rangle = F^l_\eta(x,g),$$

which is (6.93) for F^l_η.

Every form of type k_l is the function of an algebraic form. Let F be a form of type k_l, right invariant under an open subgroup K of $G_f(n)$, and $\eta = F^\flat_l$, $\eta(g) = d\int F(x,g)\rho(x)v\, dx$ (an integral of a continuous function on a compact group with values in a finite-dimensional space). Then $\eta(gk) = \eta(g)$ for $k \in K$. For $\gamma \in \Gamma$ and $x \in G_\infty(1)$, $F(x, \gamma_f g) = F(\gamma\,(\gamma_\infty^{-1}x, g)) = F(\gamma_\infty^{-1}x, g)$, and the substitution $x = \gamma_\infty x'$ (left invariance of the Haar measure) gives

$$\eta(\gamma_f g) = d\int F(x', g)\, \rho(\gamma_\infty x')v\, dx' = \rho(\gamma_\infty)\, \eta(g);$$

so η is an algebraic form of weight (κ, e) on n. Moving the linear form $\langle v, \rho(x^{-1})\cdot\rangle$ inside the integral and substituting $x' = xy$,

$$F^l_\eta(x,g) = d\int F(x', g)\, \langle v, \rho(x^{-1}x')v\rangle\, dx' = d\int F(xy, g)\, \langle v, \rho(y)v\rangle\, dy = F(x,g)$$

by (6.93). Hence $F = F^l_\eta$. □

Lemma 6.4.71. *Let n be a nonempty finite index set, $N = \#n$, and (κ, e) a weight for the ranks N.*

(a) For all $a, b, c, b' \in \mathbb{V}_\kappa$,

$$\int_{G_\infty(1)} \langle a, \rho_{\kappa,e}(y)b\rangle_\kappa\, \overline{\langle c, \rho_{\kappa,e}(y)b'\rangle_\kappa}\, dy = \frac{1}{\#I_\kappa}\, \langle a, c\rangle_\kappa\, \overline{\langle b, b'\rangle_\kappa}. \tag{6.94}$$

(b) If l is a doubled Harish-Chandra parameter of (κ, e), then for every $z \in G_\infty(1)$

$$e_{l/2}(z) = \#I_\kappa \cdot \langle v_l^+, \rho_{\kappa,e}(z)\, v_l^+ \rangle_\kappa.$$

Proof. For (b) write $v = v_l^+$, $\rho = \rho_{\kappa,e}$, $\langle\,,\rangle = \langle\,,\rangle_\kappa$ (conjugate-linear in the first argument), $d = \#I_\kappa$, and for $\tau \in \Phi$ let $\sigma_\tau(a) = \#\{a' \in n : l_\tau(a') > l_\tau(a)\}$, a bijection $n \to \{0, \dots, N-1\}$ as l_τ is injective, and $A_{\tau,h} = \sigma_\tau^{-1}(\{0, \dots, h-1\})$. For a set $P \subseteq n$ and a matrix g, $\Delta_P(g)$ is the principal minor on P.

Step 0: the representation. For any finite set Φ' of places and any weight on it, ρ is a continuous unitary representation of the compact group $\prod_{\tau\in\Phi'} \mathrm{U}(N)$ on $\mathbb{V}_\kappa$ for the positive definite form $\langle\,,\rangle$ (Lemma 4.6.10(b),(c); its matrix entries are polynomials in the entries of $\bar{h}$), and it is irreducible: over $\mathbb{C}$ the Weyl module spanned by the products of minors is the irreducible polynomial representation of $\prod \mathrm{GL}_N(\mathbb{C})$ of highest weight $(\lambda_\tau)_\tau$ [3], the matrices $S_\kappa(z)$ give its action in the dual coordinates, $\prod \mathrm{U}(N)$ is Zariski dense in $\prod \mathrm{GL}_N(\mathbb{C})$, $h \mapsto \bar{h}$ is an automorphism of $\prod \mathrm{U}(N)$, and the scalar factor $\prod \det(\bar{h}_\tau)^{e_\tau}$ does not change invariant subspaces. Hence the Schur orthogonality relations for an irreducible unitary representation of a compact group with its Haar probability measure, of dimension $\#I_\kappa$, give (6.94) for every such Φ', in particular (a) (the product of the Cayley measures is the Haar probability measure of $G_\infty(1)$).

Step 1: the coefficient of the highest vector is a product of principal minors. We show, for every family $z = (z_\tau)$ of complex $N \times N$ matrices,

$$\langle v, S_\kappa(z)\, v\rangle = \prod_{\tau\in\Phi} \prod_{1\le h<N} \Delta_{A_{\tau,h}}(z_\tau)^{\kappa(\tau,h)}. \tag{6.95}$$

By (WM2), $b_i(zy) = \sum_j S_\kappa(z)_{ij} b_j(y)$, so $S_\kappa(z)\,\mathrm{ev}_\kappa(y) = \mathrm{ev}_\kappa(zy)$ for every array y, and $S_\kappa(z)v = \mathrm{ev}_\kappa(zy^l)$. By (WM3), $\Delta_T = \sum_i (F_\kappa)_{Ti} b_i$ for $T \in \mathcal{C}_\kappa$, so $(F_\kappa\,\mathrm{ev}_\kappa(y))_T = \Delta_T(y)$; as F_κ and v are real, $\langle v, w\rangle = (F_\kappa v)^{\mathsf{T}} F_\kappa w$, hence

$$\langle v, S_\kappa(z)v\rangle = \sum_{T\in\mathcal{C}_\kappa} \Delta_T(y^l)\,\Delta_T(zy^l).$$

The entry (a,c) of y_τ^l is 1 if $c = \sigma_\tau(a)$ and 0 otherwise (WM6), so $(z_\tau y_\tau^l)_{a,c} = (z_\tau)_{a,\sigma_\tau^{-1}(c)}$ for $c < N$. For a set A of h rows, the matrix of the minor $\Delta_{\tau,A}(zy^l)$ is therefore the submatrix of z_τ on the rows A and the columns $\sigma_\tau^{-1}(0), \dots, \sigma_\tau^{-1}(h-1)$, in this order. For $z = 1$ it is 0 unless $A = A_{\tau,h}$, when it is a permutation matrix with determinant $\epsilon_{\tau,h} = \pm 1$; and for $A = A_{\tau,h}$, reordering the columns increasingly multiplies the determinant by the same sign, so $\Delta_{\tau,A_{\tau,h}}(zy^l) = \epsilon_{\tau,h}\Delta_{A_{\tau,h}}(z_\tau)$. Thus $\Delta_T(y^l)$ vanishes except for the filling T_0 with $T_0(\tau,h,j) = A_{\tau,h}$ for all (τ,h,j), for which $\Delta_{T_0}(y^l) = \epsilon := \prod_{\tau,h} \epsilon_{\tau,h}^{\kappa(\tau,h)}$ and $\Delta_{T_0}(zy^l) = \epsilon \prod_{\tau,h} \Delta_{A_{\tau,h}}(z_\tau)^{\kappa(\tau,h)}$. As $\epsilon^2 = 1$, (6.95) follows.

Step 2: the lowest K-type. Fix τ, write $\lambda(p) = \sum_{p<h<N} \kappa(\tau,h)$ (non-increasing in p, $\lambda(N-1) = 0$) and $\sigma = \sigma_\tau$. The values $2(e_\tau + \lambda(p)) + N - 1 - 2p$, $0 \le p < N$, are strictly decreasing in p; l_τ is injective with each value among them (WM6), so $l_\tau(a)$ is the value at $p = \sigma(a)$. All signs being $+1$, $2\rho_l^{\mathrm{c}} = 2\rho_l$ and $2\rho_l(a) = (N-1-\sigma(a)) - \sigma(a)$; so $l^\rho(a) = e_\tau + \lambda(\sigma(a)) + N - 1 - 2\sigma(a)$ and $\mu_l(a) = e_\tau + \lambda(\sigma(a))$. Hence $\min \mu_l = e_\tau$, and for an integer $s \ge 1$, $\{a : \mu_l(a) \ge e_\tau + s\} = \sigma^{-1}\{p : \lambda(p) \ge s\} = A_{\tau,h}$ with $h = \#\{p : \lambda(p) \ge s\}$; the value h $(1 \le h < N)$ occurs for exactly $\lambda(h-1) - \lambda(h) = \kappa(\tau,h)$ values of s. By (3.1),

$$m_{\mu_l}(g) = \det(g)^{e_\tau} \prod_{1\le h<N} \Delta_{A_{\tau,h}}(g)^{\kappa(\tau,h)}.$$

By (6.95) and $\rho(z) = \prod_\tau \det(\bar{z}_\tau)^{e_\tau} S_\kappa(\bar{z})$, for $z \in G_\infty(1)$ (minors and determinants being integral polynomials),

$$\langle v, \rho(z)v\rangle = \prod_{\tau\in\Phi} m_{\mu_{l_\tau}}(\bar{z}_\tau) = \prod_{\tau\in\Phi} \overline{m_{\mu_{l_\tau}}(z_\tau)}. \tag{6.96}$$

Step 3: Flensted-Jensen's function on $\mathrm{U}(N)$**.** Fix τ and apply Steps 0–2 to the one-point set $\{\tau\}$ with the weight $(\kappa(\tau,\cdot), e_\tau)$ and the parameter l_τ: with the module $\mathbb{V}_\tau$ of dimension d_τ, its representation ρ_τ and highest vector v_τ (a unit vector by Lemma 6.4.72(a)), (6.96) gives $m(k) = \overline{\langle v_\tau, \rho_\tau(k)v_\tau\rangle} = \langle v_\tau, \rho_\tau(k^{-1})v_\tau\rangle$ for $k \in \mathrm{U}(N)$, $m = m_{\mu_{l_\tau}}$, by unitarity. Substituting $y = k'^{-1}$ (Haar measure is inversion invariant) and using (6.94) for ρ_τ,

$$\int_{\mathrm{U}(N)} m(kk')\,\overline{m(k')}\,dk' = \int \langle v_\tau, \rho_\tau(y)\rho_\tau(k^{-1})v_\tau\rangle\,\overline{\langle v_\tau, \rho_\tau(y)v_\tau\rangle}\,dy = \frac{1}{d_\tau}\,\overline{\langle \rho_\tau(k^{-1})v_\tau, v_\tau\rangle} = \frac{m(k)}{d_\tau},$$

and in the same way $\|m\|^2 = \int |m|^2 = d_\tau^{-1}$. For $k \in \mathrm{U}(N)$ all signs agree, so $\mathrm{bd}(k) = k$ is invertible and $\mathrm{bd}(k)^{-1}k = 1$ is accretive; as $m_{-l^\rho}(k'^* k') = m_{-l^\rho}(1) = 1$, (3.2) gives $c_{l_\tau/2}(k) = \|m\|^{-2} \int m(kk')\overline{m(k')}\,dk' = m(k)$. Its formal degree for the Cayley measure, the Haar probability measure, is $(\int |m|^2)^{-1} = d_\tau$.

Step 4: conclusion. By (FT3), Step 3 and (6.96),

$$e_{l/2}(z) = \prod_\tau d_\tau\,\overline{m_{\mu_{l_\tau}}(z_\tau)} = (\prod_\tau d_\tau)\langle v, \rho(z)v\rangle.$$

Finally, by (6.94) with $a = b = c = b' = v$, Lemma 6.4.72(a), (6.96) and Fubini's theorem for the product of the Haar probability measures,

$$\frac{1}{d} = \int_{G_\infty(1)} |\langle v, \rho(z)v\rangle|^2\,dz = \prod_\tau \int_{\mathrm{U}(N)} |m_{\mu_{l_\tau}}|^2 = \prod_\tau \frac{1}{d_\tau},$$

so $\prod_\tau d_\tau = d$. □

Lemma 6.4.72. *Let n be a nonempty finite index set, (κ, e) a weight for the ranks $N = \#n$ with a doubled Harish-Chandra parameter l, $v^+ = v_l^+$ and $d = \#I_\kappa$. Then:*

(a) $\langle v^+, v^+\rangle_\kappa = 1$*;*

(b) for every algebraic form η of weight (κ, e) on n, the function F_η^l is continuous, bounded, left invariant under the rational points of I_n and right invariant under an open subgroup of $G_f(n)$, and its algebraic form for l is η: $(F_\eta^l)_l^\flat = \eta$;

(c) let $K \subseteq \mathbf{K}_0(n)$ be an open subgroup and χ a character of K with values of modulus 1, and let η, η' be algebraic forms of weight (κ, e) on n with $\eta(gk) = \chi(k)\eta(g)$ and $\eta'(gk) = \chi(k)\eta'(g)$ for $g \in G_f(n)$, $k \in K$. Then $g \mapsto \langle \eta(g), \eta'(g)\rangle_\kappa$ is constant on the double cosets of the rational points and K, and its class sum for K is

$$\sum_{[h]} \frac{1}{\#\Gamma_h} \langle \eta(h), \eta'(h)\rangle_\kappa = [\mathbf{K}_0(n) : K] \cdot d \cdot \langle F_\eta^l, F_{\eta'}^l\rangle_{\mathrm{I}_n}.$$

Proof. **Step 0: ρ is a continuous irreducible unitary representation.** By Lemma 4.6.10(c), ρ is a homomorphism on $G_\infty(1)$, unitary for $\langle\,,\rangle$, which is positive definite by Lemma 4.6.10(b); its matrix entries are polynomials in the entries of $\bar{h}$, so it is continuous. The $\mathbb{C}$-span of the products of minors $\nabla_\kappa \otimes \mathbb{C}$, with $\prod_\tau \mathrm{GL}_N(\mathbb{C})$ acting by $P \mapsto P(z\,\cdot)$, is the irreducible polynomial representation of highest weight $(\lambda_\tau)_\tau$ (the Weyl module, [3]); by (WM2) the matrices

$S_\kappa(z)$ describe this action in the basis (b_i), so $z \mapsto S_\kappa(z)$ on $\mathbb{V}_\kappa$ (the coordinates in the dual basis) is irreducible, an invariant subspace of $\mathbb{V}_\kappa$ having an invariant annihilator in $\nabla_\kappa \otimes \mathbb{C}$. The group $\prod_\tau \mathrm{U}(N)$ is Zariski dense in $\prod_\tau \mathrm{GL}_N(\mathbb{C})$, $h \mapsto \bar{h}$ is an automorphism of it, and the scalar $\prod_\tau \det(\bar{h}_\tau)^{e_\tau}$ does not change invariant subspaces; so ρ is irreducible.

Step 1: Schur orthogonality. For $a, b, c \in \mathbb{V}_\kappa$,

$$\int_{G_\infty(1)} \langle \rho(y)a, b\rangle \, \rho(y)c \, dy = d^{-1}\langle a, c\rangle \, b. \tag{6.97}$$

Indeed the operator $T\colon b \mapsto \int \langle \rho(y)a, b\rangle \rho(y)c \, dy$ satisfies $T\rho(z) = \rho(z)T$ (substitute $y \mapsto zy$, the measure being left invariant, and use unitarity), so it is a scalar λ by Schur's lemma (Step 0), and its trace is $d\lambda = \int \langle \rho(y)a, \rho(y)c\rangle \, dy = \langle a, c\rangle$, the trace of $b \mapsto \langle u, b\rangle w$ being $\langle u, w\rangle$ and the measure having mass 1.

Step 2: (a). For each τ, l_τ is injective, so $\sigma_\tau(a) = \#\{a' : l_\tau(a') > l_\tau(a)\}$ is a bijection $n \to \{0, \dots, N-1\}$, and $y^l_{\tau,a,c} = 1$ exactly when $c = \sigma_\tau(a)$ (WM6). For a set A of h rows, the columns $0, \dots, h-1$ of y^l_τ restricted to the rows A form a matrix with at most one entry 1 in each column, all others 0; its determinant $\Delta_{\tau,A}(y^l)$ is ± 1 if $A = A_{\tau,h} := \sigma_\tau^{-1}(\{0, \dots, h-1\})$ and 0 otherwise. By (WM3), $(F_\kappa v^+)_T = \sum_i (F_\kappa)_{Ti} b_i(y^l) = \Delta_T(y^l)$ for $T \in \mathcal{C}_\kappa$; so it is ± 1 for the one filling with $T(\tau, h, j) = A_{\tau,h}$ (as an increasing tuple) for all (τ, h, j), and 0 for the others. As v^+ is real, $\langle v^+, v^+\rangle = (v^+)^{\mathsf{T}} F_\kappa^{\mathsf{T}} F_\kappa v^+ = |F_\kappa v^+|^2 = 1$.

Step 3: (b). By unitarity $F^l_\eta(x, g) = \langle \rho(x)v^+, \eta(g)\rangle$. It is continuous ($\rho$ is continuous and η is locally constant, being right invariant under an open subgroup), bounded by $\sup_g \|\eta(g)\|$, which is finite because η takes finitely many values: $\Gamma\backslash G_f(n)/K'$ is finite for an open K' fixing η ([33]) and $\|\rho(\gamma_\infty)w\| = \|w\|$; right invariant under any open subgroup fixing η; and for $\gamma \in \Gamma$, $F^l_\eta(\gamma_\infty x, \gamma_f g) = \langle \rho(\gamma_\infty)\rho(x)v^+, \rho(\gamma_\infty)\eta(g)\rangle = F^l_\eta(x, g)$. By (WM7) and (6.97) with $a = c = v^+$, $b = \eta(g)$, and (a),

$$(F^l_\eta)^\flat_l(g) = d \int_{G_\infty(1)} \langle \rho(x)v^+, \eta(g)\rangle \, \rho(x)v^+ \, dx = \langle v^+, v^+\rangle \, \eta(g) = \eta(g).$$

Step 4: unfolding. The space I_n is definite, so $\mathrm{U}(\mathrm{I}_n)$ is anisotropic and $[\mathrm{U}(\mathrm{I}_n)]$ is compact [37]; as $G_\infty(1) \times K$ is open, $\Gamma\backslash \mathrm{G}(1_n)/(G_\infty(1) \times K)$ is finite, i.e. there are finitely many double cosets $\Gamma_f h K$ in $G_f(n)$, Γ_f the finite components of Γ. For a representative h, $\Gamma_h = \{\gamma \in \Gamma : \gamma_f \in hKh^{-1}\}$ is finite (discrete and contained in the compact $G_\infty(1) \times hKh^{-1}$), acts on $G_\infty(1) \times hK$ by left multiplication, freely, and its orbits are the intersections of $G_\infty(1) \times hK$ with the Γ-orbits; the sets $\Gamma(G_\infty(1) \times hK)$ partition $\mathrm{G}(1_n)$. As hK has measure $[\mathbf{K}_0(n) : K]^{-1}$, for Ψ bounded, measurable, left Γ-invariant and right K-invariant,

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} \Psi = [\mathbf{K}_0(n) : K]^{-1} \sum_{[h]} \frac{1}{\#\Gamma_h} \int_{G_\infty(1)} \Psi(y, h) \, dy. \tag{6.98}$$

Step 5: (c). For $\gamma \in \Gamma$ and $k \in K$, $\langle \eta(\gamma_f hk), \eta'(\gamma_f hk)\rangle = |\chi(k)|^2 \langle \rho(\gamma_\infty)\eta(h), \rho(\gamma_\infty)\eta'(h)\rangle = \langle \eta(h), \eta'(h)\rangle$, so the class sum is defined. The function $\Psi = \overline{F^l_\eta} F^l_{\eta'}$ is bounded, continuous, left Γ-invariant (Step 3) and right K-invariant ($F^l_\eta(x, gk) = \chi(k)F^l_\eta(x, g)$ and $|\chi(k)| = 1$). By (6.97) with $a = c = v^+$, $b = \eta'(h)$, paired with $\eta(h)$, and (a),

$$\int_{G_\infty(1)} \overline{\langle \rho(y)v^+, \eta(h)\rangle} \, \langle \rho(y)v^+, \eta'(h)\rangle \, dy = \Big\langle \eta(h), \int \langle \rho(y)v^+, \eta'(h)\rangle \rho(y)v^+ \, dy \Big\rangle = d^{-1}\langle \eta(h), \eta'(h)\rangle,$$

and (6.98) gives $\langle F^l_\eta, F^l_{\eta'}\rangle_{\mathrm{I}_n} = [\mathbf{K}_0(n) : K]^{-1} d^{-1} \sum_{[h]} (\#\Gamma_h)^{-1} \langle \eta(h), \eta'(h)\rangle$, which is (c). □

Lemma 6.4.73. *Let $E = E_s$ with its set Φ of infinite places and n a nonempty finite index set with $N = \#n$, the ranks being $N_\tau = N$ ($\tau \in \Phi$) as in (WM6). Let (κ, e) and (κ', e') be weights with doubled Harish-Chandra parameters l and $-l$ respectively. There is an integer matrix A with rows $I_{\kappa'}$ and columns I_κ such that:*

(1) for every field R of characteristic 0, every $g \in \prod_\tau \mathrm{GL}_n(R)$ and all $v' \in R^{I_{\kappa'}}$, $v \in R^{I_\kappa}$,

$$(P_{\kappa',e'}(g)v')^{\mathsf{T}} A\, P_{\kappa,e}(g)v = v'^{\mathsf{T}} A v, \qquad P_{\kappa,e}(g) = \prod_\tau \det(g_\tau)^{e_\tau}\, S_\kappa(g);$$

(2) for all signs s on n, every algebraic form η of weight (κ', e'), $x \in G_\infty(s)$ and $g \in G_f(n)$,

$$F_\eta^{-l}(x, g) = \eta(g)^{\mathsf{T}} A\, \rho_{\kappa,e}(x)\, v_l^+;$$

(3) there is an integer matrix C with $CA = h_{\kappa'} \cdot 1$, $h_{\kappa'}$ the hook number of κ' in the ranks N;

(4) for every $v \in \mathbb{V}_\kappa$ there is $v' \in \mathbb{V}_{\kappa'}$ with $\|v'\|_{\kappa'} = \|v\|_\kappa$ and $u^{\mathsf{T}} A v = \langle v', u\rangle_{\kappa'}$ for all $u \in \mathbb{V}_{\kappa'}$;

(5) if $L \in \mathbb{N}$ and $\sum_\tau \sum_{1 \le h < N} h\, \kappa'(\tau, h) \le L$, then $h_{\kappa'} \le (L^L)^3$.

Proof. Write $\mathrm{hc}_\tau(p) = 2(e_\tau + \lambda_\tau(p)) + N - 1 - 2p$ and $\mathrm{hc}'_\tau(p)$ for the same with (κ', e') ($0 \le p < N$), and $\beta'_\tau = \sum_{1 \le h < N} \kappa'(\tau, h)$.

Step 1: the parameters determine κ' and e'. Since $\lambda_\tau(p) - \lambda_\tau(p+1) = \kappa(\tau, p+1) \ge 0$, $\mathrm{hc}_\tau(p) - \mathrm{hc}_\tau(p+1) = 2\kappa(\tau, p+1) + 2 \ge 2$, so the N values $\mathrm{hc}_\tau(p)$ are distinct and decreasing; l_τ is injective with values among them, hence a bijection of n onto them, and $l_\tau(a) = \mathrm{hc}_\tau(p_\tau(a))$ with $p_\tau(a) = \#\{a' : l_\tau(a') > l_\tau(a)\}$. Likewise $-l_\tau$ is a bijection onto the decreasing values $\mathrm{hc}'_\tau(p)$; as the values of $-l_\tau$ in decreasing order are $-\mathrm{hc}_\tau(N-1) > \cdots > -\mathrm{hc}_\tau(0)$, we get $\mathrm{hc}'_\tau(p) = -\mathrm{hc}_\tau(N-1-p)$. Comparing consecutive differences gives $\kappa'(\tau, h) = \kappa(\tau, N-h)$ for $1 \le h < N$, and $p = 0$ gives (with $\lambda_\tau(N-1) = 0$, $\lambda'_\tau(0) = \beta'_\tau$)

$$e_\tau + e'_\tau = -\beta'_\tau.$$

Step 2: the duality matrix. Apply Lemma 3.2.20 with $\kappa^+ = 0$ and $\kappa^- = \kappa'$: the mixed weight is $\kappa'(\tau, N-i) = \kappa(\tau, i)$, $e^- = \beta'$, and the Weyl module of the zero weight is $\mathbb{C}$ with basis the empty product, Gram matrix 1, evaluation vector 1 and Schur matrices 1. We obtain $C_0 \in \mathrm{M}_{I_{\kappa'} \times I_\kappa}(\mathbb{Z})$ with an integral left inverse L ($LC_0 = 1$), with $C_0^{\mathsf{T}} G_{\kappa'} C_0 = G_\kappa$, $C_0 \,\mathrm{ev}_\kappa(y_{\mathrm{id}}) = \mathrm{ev}_{\kappa'}(y_{w_0})$, and, over every field R of characteristic 0 and for invertible g,

$$C_0 \prod_\tau \det(g_\tau)^{-\beta'_\tau} S_\kappa(g) = S_{\kappa'}((g^{-1})^{\mathsf{T}}) C_0. \tag{6.99}$$

Let π_τ be the permutation matrix with $(\pi_\tau)_{aa'} = 1$ if and only if $a' = p_\tau(a) + 1$. Then $\pi y_{\mathrm{id}} = y^l$ and $\pi y_{w_0} = y^{-l}$ (the frame of $-l$ has its 1 in the column $\#\{a' : -l_\tau(a') > -l_\tau(a)\} = N - 1 - p_\tau(a)$), so $v_l^+ = S_\kappa(\pi)\mathrm{ev}_\kappa(y_{\mathrm{id}})$ and $v_{-l}^+ = S_{\kappa'}(\pi)\mathrm{ev}_{\kappa'}(y_{w_0})$ by (WM2). With $g = \pi$ in (6.99), as $(\pi^{-1})^{\mathsf{T}} = \pi$,

$$C_0 v_l^+ = \epsilon\, v_{-l}^+, \qquad \epsilon = \prod_\tau \det(\pi_\tau)^{\beta'_\tau} \in \{\pm 1\}.$$

Put $A = \epsilon\, G_{\kappa'} C_0$, an integer matrix with rows $I_{\kappa'}$ and columns I_κ.

(1). By (a) of Lemma 4.6.10 (a polynomial identity, valid over R), $S_{\kappa'}(g)^{\mathsf{T}} G_{\kappa'} = G_{\kappa'} S_{\kappa'}(g^{\mathsf{T}})$. Hence, by (6.99) and Step 1,

$$P_{\kappa',e'}(g)^{\mathsf{T}} A\, P_{\kappa,e}(g) = \epsilon \prod_\tau \det(g_\tau)^{e_\tau + e'_\tau + \beta'_\tau}\, G_{\kappa'} S_{\kappa'}(g^{\mathsf{T}}) S_{\kappa'}((g^{-1})^{\mathsf{T}}) C_0 = A,$$

using the multiplicativity of $S_{\kappa'}$. This is (1).

(2). Since v_{-l}^{+} has integer coordinates and $G_{\kappa'}$ is symmetric, $\langle v_{-l}^{+}, w\rangle_{\kappa'} = w^{\mathsf{T}} G_{\kappa'} v_{-l}^{+} = w^{\mathsf{T}} A\, v_l^{+}$ for $w \in \mathbb{V}_{\kappa'}$. With $w = \rho_{\kappa',e'}(x^{-1})\eta(g)$ this is $F_\eta^{-l}(x,g)$. Applying (1) over $\mathbb{C}$ to $\bar{x}$ (invertible) and the vectors w, $\rho_{\kappa,e}(x)v_l^{+}$, and using $\rho_{\kappa',e'}(x)w = \eta(g)$ (multiplicativity of $P_{\kappa',e'}$ and of complex conjugation),

$$F_\eta^{-l}(x,g) = w^{\mathsf{T}} A\, v_l^{+} = \big(\rho_{\kappa',e'}(x)w\big)^{\mathsf{T}} A\, \rho_{\kappa,e}(x) v_l^{+} = \eta(g)^{\mathsf{T}} A\, \rho_{\kappa,e}(x)\, v_l^{+}.$$

(3). Let $H_{\kappa'}$ be the matrix of Fock pairings of Lemma 4.6.9, so $H_{\kappa'} G_{\kappa'} = h_{\kappa'} \cdot 1$. Put $C = \epsilon L H_{\kappa'}$, an integer matrix; then $CA = \epsilon^2 L\, H_{\kappa'} G_{\kappa'}\, C_0 = h_{\kappa'} L C_0 = h_{\kappa'} \cdot 1$.

(4). For $v \in \mathbb{V}_\kappa$ put $v' = \epsilon\, \overline{C_0 v}$. For $u \in \mathbb{V}_{\kappa'}$, $\langle v', u\rangle_{\kappa'} = \epsilon (C_0 v)^{\mathsf{T}} G_{\kappa'} u = u^{\mathsf{T}} A v$, as $G_{\kappa'}$ is symmetric. And $\|v'\|_{\kappa'}^2 = (C_0 v)^{\mathsf{T}} G_{\kappa'} \overline{C_0 v} = v^{\mathsf{T}} C_0^{\mathsf{T}} G_{\kappa'} C_0 \bar{v} = v^{\mathsf{T}} G_\kappa \bar{v} = \|v\|_\kappa^2$, the last since G_κ is real symmetric and $\|v\|_\kappa^2$ is real.

(5). The highest polynomial $f = f_{\kappa'}$ is the product of $\sum_{\tau,h} \kappa'(\tau,h)$ minors, one of size h for each column of height h; it is homogeneous of degree $\beta = \sum_\tau \sum_{1\le h<N} h\, \kappa'(\tau,h) \le L$. A minor of size h is a sum of $h! \le h^h$ monomials with coefficients ± 1, so expanding the product, $f = \sum_\alpha f_\alpha y^\alpha$ with $\sum_\alpha |f_\alpha| \le \prod_{\text{columns}} h^h \le L^\beta \le L^L$ (for $L \ge 1$; if $L = 0$ then $f = 1$ and $h_{\kappa'} = 1$). Every α occurring has $|\alpha| = \beta$, so $\alpha! \le \beta! \le L^L$, and

$$h_{\kappa'} = \langle f, f\rangle_{\mathrm{F}} = \sum_\alpha f_\alpha^2\, \alpha! \le L^L \Big(\sum_\alpha |f_\alpha|\Big)^2 \le (L^L)^3.$$

□

Lemma 6.4.74. *Let n be a nonempty finite index set, Γ the group of rational points of I_n, embedded in $\mathrm{G}(1)$ by $\gamma \mapsto (\gamma_\infty, \gamma_f)$, Γ_f the group of the γ_f, V a finite-dimensional complex vector space and $\mathrm{K} \subseteq \mathbf{K}_0(n)$ an open subgroup.*

(1) Let $F\colon G_f(n) \to V$ satisfy $F(\gamma_f g k) = F(g)$ for all $\gamma \in \Gamma$, $g \in G_f(n)$ and k in an open subgroup $\mathrm{K}' \subseteq \mathrm{K}$. Then $\Gamma_f \backslash G_f(n) / \mathrm{K}'$ is finite, the groups $\Gamma'_h = \Gamma_f \cap h\mathrm{K}'h^{-1}$ are finite, and

$$\mathrm{CI}_{\mathrm{K}}(F) = \frac{1}{[\mathrm{K} : \mathrm{K}']} \sum_{[h] \in \Gamma_f \backslash G_f(n)/\mathrm{K}'} \frac{F(h)}{\#\Gamma'_h}.$$

(2) Let $f\colon \mathrm{G}(1) \to V$ be continuous, with $f(\gamma x) = f(x)$ for $\gamma \in \Gamma$, $x \in \mathrm{G}(1)$, and $f(y, gk) = f(y,g)$ for $(y,g) \in \mathrm{G}(1)$ and k in an open subgroup $\mathrm{K}' \subseteq \mathbf{K}_0(n)$. Then

$$H(g) = \int_{G_\infty(1)} f(y,g)\, dy,$$

dy the Haar probability measure, satisfies the hypothesis of (1) with the open subgroup $\mathrm{K}' \cap \mathrm{K}$, and

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} f = \frac{c_n}{[\mathbf{K}_0(n) : \mathrm{K}]}\, \mathrm{CI}_{\mathrm{K}}(H),$$

where $c_n > 0$ is the total mass of the measure of $G_\infty(1)$.

Proof. **Finiteness.** $\mathrm{U}(\mathrm{I}_n)$ is a linear algebraic group over E^+ with group of finite adelic points $G_f(n)$ and group of rational points Γ_f, so $\Gamma_f \backslash G_f(n)/\mathrm{K}'$ is finite by the finiteness of class numbers [33]. An element $\gamma \in \Gamma$ with $\gamma_f \in \Gamma'_h$ lies in $G_\infty(1) \times h\mathrm{K}'h^{-1}$, a compact subset of $\mathrm{G}(1)$; as Γ is discrete in $\mathrm{G}(1)$ ((VP2) of Definition 3.2.2) and $\gamma \mapsto \gamma_f$ is injective (a matrix over E is

determined by its image in $\mathrm{GL}_n(\mathbb{A}_{E,f})$), Γ'_h is finite. Conjugate subgroups give $\#\Gamma'_{\delta hk} = \#\Gamma'_h$ for $\delta \in \Gamma_f$, $k \in \mathrm{K}'$, so the summand of (1) is constant on double cosets; the same holds for every open subgroup in place of K'.

Proof of (1). Let $\mathrm{K}_F = \{k \in \mathrm{K} : F(gk) = F(g) \text{ for all } g\}$, a subgroup of K containing K', hence open, and of finite index $[\mathrm{K} : \mathrm{K}_F]$ in the compact group K; by (LQ3), $\mathrm{CI}_{\mathrm{K}}(F) = [\mathrm{K} : \mathrm{K}_F]^{-1} \sum_{[h]} F(h)/\#\Gamma_h$, the sum over $\Gamma_f \backslash G_f(n)/\mathrm{K}_F$ with $\Gamma_h = \Gamma_f \cap h\mathrm{K}_F h^{-1}$. As $[\mathrm{K} : \mathrm{K}'] = [\mathrm{K} : \mathrm{K}_F][\mathrm{K}_F : \mathrm{K}']$, it suffices to show, for each $h \in G_f(n)$,

$$\sum_{[h']} \frac{1}{\#\Gamma'_{h'}} = \frac{[\mathrm{K}_F : \mathrm{K}']}{\#\Gamma_h}, \tag{6.100}$$

the sum over the double cosets $\Gamma_f h'\mathrm{K}'$ contained in $\Gamma_f h\mathrm{K}_F$; indeed F takes the value $F(h)$ on all of them, and summing (6.100) times $F(h)$ over $\Gamma_f \backslash G_f(n)/\mathrm{K}_F$ gives the sum of (1) with $[\mathrm{K}_F : \mathrm{K}']$ in front. The group Γ_h acts on the finite set $X = \mathrm{K}_F/\mathrm{K}'$ by $\delta \cdot k\mathrm{K}' = h^{-1}\delta hk\mathrm{K}'$. The map $X \to \Gamma_f \backslash \Gamma_f h\mathrm{K}_F/\mathrm{K}'$, $k\mathrm{K}' \mapsto \Gamma_f hk\mathrm{K}'$, is surjective, and $\Gamma_f hk\mathrm{K}' = \Gamma_f hk'\mathrm{K}'$ if and only if $hk' = \delta hkk''$ with $\delta \in \Gamma_f$, $k'' \in \mathrm{K}'$; then $\delta = hk'k''^{-1}k^{-1}h^{-1} \in \Gamma_h$ and $k'\mathrm{K}' = \delta \cdot k\mathrm{K}'$, and conversely. So the double cosets in question are the Γ_h-orbits on X. The stabiliser of $k\mathrm{K}'$ is $\{\delta \in \Gamma_h : h^{-1}\delta h \in k\mathrm{K}'k^{-1}\} = \Gamma_f \cap hk\mathrm{K}'(hk)^{-1} = \Gamma'_{hk}$ (an element of this intersection lies in $h\mathrm{K}_F h^{-1}$ as $k \in \mathrm{K}_F$). By the orbit-stabiliser theorem the orbit of $k\mathrm{K}'$ has $\#\Gamma_h/\#\Gamma'_{hk}$ elements, and summing over the orbits, $\#X = \#\Gamma_h \sum_{[h']} 1/\#\Gamma'_{h'}$, which is (6.100).

The function H. f is bounded on the compact sets $G_\infty(1) \times g\mathrm{K}'$, so H is defined. For $k \in \mathrm{K}'$, $H(gk) = H(g)$. For $\gamma \in \Gamma$, $f(y, \gamma_f g) = f(\gamma(\gamma_\infty^{-1} y, g)) = f(\gamma_\infty^{-1} y, g)$, and left invariance of dy gives $H(\gamma_f g) = H(g)$. So H satisfies the hypothesis of (1) with $\mathrm{K}'' = \mathrm{K}' \cap \mathrm{K}$, an open subgroup of K.

Proof of (2). The measure of $G_\infty(1)$ is a Haar measure (the Cayley measure, Definition 3.2.1) of total mass c_n, hence equal to $c_n\, dy$; the measure μ_f of $G_f(n)$ gives $\mathbf{K}_0(n)$ the mass 1, so $\mu_f(\mathrm{K}'') = [\mathbf{K}_0(n) : \mathrm{K}'']^{-1}$. Let $h_1, \ldots, h_s$ represent $\Gamma_f \backslash G_f(n)/\mathrm{K}''$ (finite by (1)) and put $\Gamma''_i = \Gamma_f \cap h_i \mathrm{K}'' h_i^{-1}$ and

$$\chi = \sum_{i=1}^{s} \frac{1}{\#\Gamma''_i} \mathbf{1}_{G_\infty(1) \times h_i \mathrm{K}''}.$$

For $x = (y, g) \in \mathrm{G}(1)$ write $g = \delta_f h_i k$ with $\delta \in \Gamma$, $k \in \mathrm{K}''$, for the unique i with $g \in \Gamma_f h_i \mathrm{K}''$. Then $\gamma x \in G_\infty(1) \times h_j \mathrm{K}''$ forces $j = i$, and holds if and only if $\gamma_f \delta_f \in \Gamma''_i$; as $\gamma \mapsto \gamma_f$ is injective, $\sum_{\gamma \in \Gamma} \chi(\gamma x) = 1$. The quotient measure satisfies $\int_{\mathrm{G}(1)} \phi = \int_{[\mathrm{U}(\mathrm{I}_n)]} \sum_{\gamma \in \Gamma} \phi(\gamma x)\, dx$ for integrable ϕ; applied to $\phi = \chi f$ (bounded, with compact support), whose sum over Γ is f, it gives, with Fubini's theorem and the right K''-invariance of f in the second variable,

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} f = \sum_{i=1}^{s} \frac{1}{\#\Gamma''_i} \int_{G_\infty(1)} \int_{h_i \mathrm{K}''} f(y, g)\, d\mu_f(g)\, c_n\, dy = \frac{c_n}{[\mathbf{K}_0(n) : \mathrm{K}'']} \sum_{i=1}^{s} \frac{H(h_i)}{\#\Gamma''_i}.$$

By (1) for H, K and K'', the last sum is $[\mathrm{K} : \mathrm{K}'']\, \mathrm{CI}_{\mathrm{K}}(H)$, and $[\mathbf{K}_0(n) : \mathrm{K}''] = [\mathbf{K}_0(n) : \mathrm{K}][\mathrm{K} : \mathrm{K}'']$ gives (2). □

Lemma 6.4.75. *Let D be admissible data, ϑ an m-fold twist, χ_ϑ the character of K_ϑ of Definition 3.2.31, and $y \in \mathcal{A}_M(Y_0)$. Put $F_0 = y$ and, for $1 \le j \le m$, let F_j be lift j applied to F_{j-1}.*

(a) For $0 \le j \le m$ the function F_j is defined (the integral defining lift j of F_{j-1} converges absolutely at every point) and is a form of type e_{λ_j} on Y_j, where $\lambda_j = \frac{1}{2} p_{\mathrm{I},0,j}$.

(b) $F_m(x, gk) = \chi_\vartheta(k)\, F_m(x, g)$ for all $x \in G_\infty(1)$, $g \in G_f(n)$ and $k \in \mathrm{K}_\vartheta$.

(c) $\Theta_\vartheta y$ is an algebraic form of weight (κ_Y, e_Y) on the index set of Y_m, and $\Theta_\vartheta y(gk) = \chi_\vartheta(k)\,\Theta_\vartheta y(g)$ for all $g \in G_f(n)$ and $k \in \mathrm{K}_\vartheta$. In particular Θ_ϑ maps $\mathcal{T}$ into $\mathcal{A}_\vartheta(Y_m)$.

Proof. **Notation.** Write n_j for the index set of Y_j, $N_j = \#n_j$, and $\mathcal{L}$ for the set of level places of ϑ (the chosen places and the places of Q): distinct prime ideals of $\mathcal{O}_E$ with product $\mathfrak{t}$, containing the conjugate of none of their members ($\mathfrak{t}$ is a modulus, (TV2) of Definition 3.2.30 and (XD4)). Through the point of Y_j the integral elements of $\mathrm{U}(Y_j)(\mathbb{A}_f)$ form $\mathbf{K}_0(n_j)$; for such k and $w \in \mathcal{L}$ let $\bar{k}_w \in \mathrm{GL}_{n_j}(k_w)$ be the reduction of its component at w. The indices of n_j form block 0 (the r indices of Y_0) and blocks $1, \dots, j$ of size t, and the blocks of n_{j-1} are the blocks $0, \dots, j-1$ of n_j. For $1 \le l \le m$ let $\psi_l = (\omega_w^l \psi_w^l)_w$, ψ the tame part of ϑ: multiplying the identities of Definition 3.1.1 for μ_l^0 and ϑ^l, ψ_l is a tame part of ν_l, the tame part used for the character of K_ϑ in Definition 3.2.31, and $\psi_{l,w} = 1$ for $w \notin \mathcal{L}$ ($\omega_w = 1$ off the chosen places, $\psi_w = 1$ off Q). For $0 \le j \le m$ let $\mathrm{K}^{(j)}$ be the set of $k \in \mathbf{K}_0(n_j)$ with $\bar{k}_w = 1$ at the chosen places and $\bar{k}_w$ block upper triangular at the places of Q, and

$$\chi^{(j)}(k) = \prod_{w \in \mathcal{L}} \prod_{l=1}^{j} \psi_{l,w}(\det \bar{k}_w^{(l)}) \qquad (k \in \mathrm{K}^{(j)}),$$

$\bar{k}_w^{(l)}$ the diagonal block l of $\bar{k}_w$. Then $\mathrm{K}^{(m)} = \mathrm{K}_\vartheta$ and $\chi^{(m)} = \chi_\vartheta$ by Definition 3.2.31, while $\mathrm{K}^{(0)} = \mathrm{K}_M$ (one block, so no condition at Q) and $\chi^{(0)} = 1$.

Step 1: the data of lift j, and (a). Let $1 \le j \le m$, $\alpha_j = \nu_j \upsilon^{\lfloor N_{j-1}/2 \rfloor}$, $\beta_j = \nu_j \upsilon^{\lfloor N_j/2 \rfloor}$, $\varphi_j = \mathrm{mod}_{\mathfrak{t}}(\mathbf{1}_j^{\mathrm{inc}}, v_{H_j})$, and let L_j be the lift of Lemma 6.3.76 for $S = Y_{j-1}$, $T = Y_j$, the characters (α_j, β_j), the modulus $\mathfrak{t}$, $\Psi = \mathbf{1}_j^{\mathrm{inc}}$ and $P = H_j$; so lift j is $c_j L_j$ with c_j the index of Definition 3.2.31. By (GP6), α_j and β_j are the edge characters for $c = \mathrm{II}$, $l = j-1$ at the ranks N_{j-1} and N_j, of weights $\gamma_j = \gamma_{\mathrm{II},j-1}(N_{j-1})$ and $\gamma_j' = \gamma_{\mathrm{II},j-1}(N_j)$. The polynomial H_j is the joint harmonic (GP7) for the signs of $(\mathrm{I}, 0, j-1)$ and $(\mathrm{I}, 0, j)$, the weights (γ_j, γ_j') and the doubled parameters $(p_{\mathrm{I},0,j-1}, p_{\mathrm{I},0,j})$; this is the case $z = \mathrm{I}$, $o = a = 0$, $b = j-1$ of Lemma 6.4.15(b) ($b+1 \le m$), so v_{H_j} is an edge for the weights (γ_j, γ_j') from $e_{\lambda_{j-1}}$ to e_{λ_j}. Now (a) follows by induction on j. For $j = 0$, $y \in \mathcal{F}_\triangle^0(\mathrm{I}, 0)$ by Lemma 6.4.67(a), and by (TV2) these are the forms of type e_{λ_0} on Y_0. If F_{j-1} is a form of type $e_{\lambda_{j-1}}$ on Y_{j-1}, then by Lemma 6.3.76(1) the integral $L_j F_{j-1}(h)$ converges absolutely for every h and $L_j F_{j-1}$ is a form of type e_{λ_j} on Y_j; so is $F_j = c_j L_j F_{j-1}$.

Step 2: the law of one lift. Let $1 \le j \le m$, let X_0 be the $n_j \times n_{j-1}$ matrix with entry 1 at (a, a) for $a \in n_{j-1}$ and 0 elsewhere, and let $k \in \mathbf{K}_0(n_j)$, $k' \in \mathbf{K}_0(n_{j-1})$ with $\bar{k}_w X_0 = X_0 \bar{k}_w'$ for every $w \in \mathcal{L}$. We show that for every form G of type $e_{\lambda_{j-1}}$ on Y_{j-1}

$$L_j G(h(1,k)) = c(k,k')\, L_j(G(\cdot\,(1,k')))(h), \qquad c(k,k') = \prod_{w \in \mathcal{L}} \frac{\psi_{j,w}(\det \bar{k}_w)}{\psi_{j,w}(\det \bar{k}_w')}. \tag{6.101}$$

By Lemma 4.1.1(6) for $S = Y_{j-1}$, $T = Y_j$, the modulus $\mathfrak{t}$ (so $A = \mathcal{L}$), $\Phi = \mathbf{1}_j^{\mathrm{inc}}$ and the elements k' of $\mathrm{U}(S)$, k of $\mathrm{U}(T)$,

$$\omega_{\alpha_j,\beta_j}((1,k'),(1,k))\varphi_j = \alpha_j(\det k)\,\beta_j(\det k')^{-1}\,\mathrm{mod}_{\mathfrak{t}}(\Phi_{k',k}, v_{H_j}), \qquad \Phi_{k',k}(X) = \Phi(\bar{k}^{-1} X \bar{k}'),$$

the reductions modulo $\mathfrak{t}$ being the families $(\bar{k}_w)_w$, $(\bar{k}_w')_w$ under $R_{\mathfrak{t}} \cong \prod_{w \in \mathcal{L}} k_w$ (Chinese remainder theorem). As Φ is the indicator function of X_0, $\Phi_{k',k}$ is that of $\bar{k} X_0 \bar{k}'^{-1} = X_0$; so $\Phi_{k',k} = \Phi$. The determinants $u = \det k$ and $u' = \det k'$ lie in $\mathrm{U}(1)(\mathbb{A})$ (k, k' are unitary), with infinite components 1 and finite components integral units ($k^{\pm 1}$, $k'^{\pm 1}$ are integral). For such u, $\upsilon(u) = 1$ (υ has trivial tame part) and $\nu_j(u) = \prod_w \psi_{j,w}(u \bmod w) = \prod_{w \in \mathcal{L}} \psi_{j,w}(u \bmod w)$ (Definition 3.1.1; the

weight factor is 1 as the infinite components are 1). Since $\det k \bmod w = \det \bar{k}_w$, this gives $\alpha_j(\det k)\beta_j(\det k')^{-1} = c(k,k')$, and Lemma 6.3.76(3), with $\Psi' = \Psi$, gives (6.101).

Step 3: (b). We show by induction on j that $F_j(h(1,k)) = \chi^{(j)}(k)F_j(h)$ for all h and all $k \in \mathrm{K}^{(j)}$; for $j = m$ this is (b). For $j = 0$ it is the right invariance of $y \in \mathcal{A}_M(Y_0)$ under $\mathrm{K}_M = \mathrm{K}^{(0)}$. Let $1 \le j \le m$ and $k \in \mathrm{K}^{(j)}$. For $w \in \mathcal{L}$ the matrix $\bar{k}_w$ is block upper triangular (at the chosen places it is 1), so

$$\bar{k}_w = \begin{pmatrix} A_w & B_w \\ 0 & D_w \end{pmatrix},$$

with A_w the block on $n_{j-1} \times n_{j-1}$ and $D_w = \bar{k}_w^{(j)}$; hence $\det \bar{k}_w = \det A_w \det D_w$, A_w is invertible, $A_w = 1$ at the chosen places, A_w is block upper triangular for the blocks of n_{j-1} at the places of Q, and its diagonal blocks are $\bar{k}_w^{(l)}$, $0 \le l \le j-1$. The reduction map $\mathbf{K}_0(n_{j-1}) \to \prod_{w\in\mathcal{L}} \mathrm{GL}_{n_{j-1}}(k_w)$ is a homomorphism whose kernel has index $\prod_{w\in\mathcal{L}} \#\mathrm{GL}_{n_{j-1}}(k_w)$ by Lemma 6.4.76, so it is onto; choose k' with $\bar{k}'_w = A_w$ for $w \in \mathcal{L}$. Then $k' \in \mathrm{K}^{(j-1)}$, and $\bar{k}_w X_0$, the matrix of the columns n_{j-1} of $\bar{k}_w$, is $X_0 A_w = X_0 \bar{k}'_w$. By (a), (6.101) with $G = F_{j-1}$, the hypothesis of the induction and the linearity of L_j,

$$F_j(h(1,k)) = c_j\, c(k,k')\, L_j(\chi^{(j-1)}(k')F_{j-1})(h) = c(k,k')\, \chi^{(j-1)}(k')\, F_j(h).$$

Finally $\psi_{j,w}(\det \bar{k}_w)/\psi_{j,w}(\det A_w) = \psi_{j,w}(\det D_w)$, and the diagonal blocks of A_w are those of $\bar{k}_w$, so $c(k,k')\chi^{(j-1)}(k') = \chi^{(j)}(k)$.

Step 4: (c). Let $F = F_m$, $\rho = \rho_{\kappa_Y, e_Y}$, $v^+ = v_l^+$, $d = \#I_{\kappa_Y}$ and $\eta = \Theta_\vartheta y = F_l^\flat$, so $\eta(g) = d \int_{G_\infty(1)} F(x,g)\rho(x)v^+\,dx$; the integral exists, $x \mapsto F(x,g)$ and ρ being continuous on the compact group $G_\infty(1)$. By (a) and (ED2), F is left invariant under the rational points Γ of I_n and right invariant under an open subgroup K' of $G_f(n)$. Hence $\eta(gk) = \eta(g)$ for $k \in K'$. For $\gamma \in \Gamma$, $F(x, \gamma_f g) = F(\gamma_\infty^{-1}x, g)$; substituting $x = \gamma_\infty x'$ (the Haar measure of $G_\infty(1)$ is invariant) and using $\rho(\gamma_\infty x') = \rho(\gamma_\infty)\rho(x')$ (Lemma 4.6.10(c)),

$$\eta(\gamma_f g) = d \int_{G_\infty(1)} F(x', g)\, \rho(\gamma_\infty)\rho(x')v^+\,dx' = \rho(\gamma_\infty)\, \eta(g).$$

So η is an algebraic form of weight (κ_Y, e_Y) (WM7). By (b), $\eta(gk) = d \int F(x,gk)\rho(x)v^+dx = \chi_\vartheta(k)\eta(g)$ for $k \in \mathrm{K}_\vartheta$. For $y \in \mathcal{T} \subseteq \mathcal{A}_M(Y_0)$ this says that η lies in $\mathcal{A}_\vartheta(Y_m)$. □

Lemma 6.4.76. *Let $n \ge 1$, let A be a finite set of finite places of E containing the conjugate of none of its members, $\mathfrak{m} = \prod_{w\in A} w$, and $\mathbf{K}_A = \{k \in \mathbf{K}_0(n) : k_w \equiv 1 \bmod w$ for every $w \in A\}$. Then*

$$[\mathbf{K}_0(n) : \mathbf{K}_A] = \prod_{w\in A} \#\mathrm{GL}_n(k_w) = \#\mathrm{GL}_n(\mathcal{O}_E/\mathfrak{m}).$$

Proof. Let $\mathrm{red}\colon \mathbf{K}_0(n) \to \prod_{w\in A} \mathrm{GL}_n(k_w)$, $k \mapsto (k_w \bmod w)_w$, a homomorphism with kernel $\mathbf{K}_A$; so the index is the order of its image, and it suffices to show that red is onto. Let $(a_w)_{w\in A}$ be given. For $w \in A$ lift a_w entrywise to $g_w \in \mathrm{M}_n(\mathcal{O}_{E,w})$; $\det g_w$ reduces to $\det a_w \ne 0$, so $g_w \in \mathrm{GL}_n(\mathcal{O}_{E,w})$. Define $k \in \mathrm{GL}_n(\mathbb{A}_{E,f})$ by $k_w = g_w$ and $k_{\bar{w}} = \overline{(g_w^{-1})^{\mathsf{T}}}$ for $w \in A$ (the bar being the conjugation $E_w \to E_{\bar{w}}$), and $k_u = 1$ at the other finite places u; this is consistent, as $\bar{w} \notin A$ and $\bar{w} \ne w$ for $w \in A$. At the places w and $\bar{w}$ the condition $k^*k = 1$ reads $\overline{k_{\bar{w}}}^{\mathsf{T}} k_w = 1$ and $\overline{k_w}^{\mathsf{T}} k_{\bar{w}} = 1$, which hold by construction (the second is the conjugate transpose of the inverse of the first); elsewhere it is trivial. All components of k and of k^{-1} are integral, so $k \in \mathbf{K}_0(n)$, and $\mathrm{red}(k) = (a_w)$. Finally the places of A are distinct prime ideals, so $\mathcal{O}_E/\mathfrak{m} \cong \prod_{w\in A} k_w$ by the Chinese remainder theorem, and GL_n of a finite product of rings is the product of the GL_n. □

Lemma 6.4.77. *Let D be admissible data. The space $\mathcal{F}_{\mathfrak{b}}(m)$ is spanned by the forms φ whose algebraic form $\varphi^\flat$ satisfies, for every $g \in G_f(n_{\mathrm{II},0})$,*

$$\begin{aligned} &\mathrm{gau}_\iota(g, \varphi^\flat(g)) \le 1 && \text{for every prime identification } \iota, \\ &\|\sigma(\varphi^\flat(g))\|^2_{\kappa_{\mathrm{II}}} \le \exp(x^{99}b^2) && \text{for every field automorphism } \sigma \text{ of } \mathbb{C}. \end{aligned}$$

Proof. Throughout, $\|z\|$ is the operator norm of a complex matrix. We use three facts on Schur matrices, read off from (WM1)–(WM3).

(S1) Integrality and multiplicativity. By the Cauchy–Binet formula, for a set A of h rows $\Delta_{\tau,A}(zy) = \sum_B \det(z_\tau[A,B])\Delta_{\tau,B}(y)$, the sum over the sets B of h rows, where $z_\tau[A,B]$ is the submatrix on the rows A and the columns B. Hence $\Delta_T(zy) = \sum_{T'} \Lambda(z)_{TT'}\Delta_{T'}(y)$ for $T, T' \in \mathcal{C}_\kappa$, where $\Lambda(z) = \bigotimes_{(\tau,h,j)} \wedge^h z_\tau$ is the tensor product of the compound matrices $(\wedge^h z_\tau)_{A,B} = \det z_\tau[A,B]$. Writing $b_i = \sum_T c_{iT}\Delta_T$ with $c_{iT} \in \mathbb{Z}$ (as $b_i \in \nabla_\kappa$) and $\Delta_{T'} = \sum_j (F_\kappa)_{T'j} b_j$, and using the linear independence of the b_j, we get $S_\kappa(z) = c\,\Lambda(z)\,F_\kappa$ and

$$F_\kappa S_\kappa(z) = \Lambda(z) F_\kappa.$$

So the entries of $S_\kappa(z)$ are polynomials with integer coefficients in the entries of z, S_κ commutes with ring homomorphisms applied entrywise, and $S_\kappa(zz') = S_\kappa(z)S_\kappa(z')$ for square z, z' (expand $b_i(z(z'y))$ twice and compare coefficients; a polynomial identity over $\mathbb{Z}$, so valid over every commutative ring); in particular $S_\kappa(1) = 1$ and $S_\kappa(z)$ is invertible for invertible z. Likewise $\mathrm{ev}_\kappa(y)$ has integer polynomial coordinates $b_i(y)$.

(S2) Norms. As F_κ is real, $\|\xi\|_\kappa = \|F_\kappa \xi\|$ (euclidean norm). For a complex matrix z_τ with singular value decomposition $u\,\mathrm{diag}(s)\,u'$, the Cauchy–Binet formula gives $\wedge^h z_\tau = \wedge^h u \wedge^h \mathrm{diag}(s) \wedge^h u'$ with $\wedge^h u$, $\wedge^h u'$ unitary (as $\wedge^h(u^*) \wedge^h (u) = \wedge^h(u^*u) = 1$ and $\wedge^h(u^*) = (\wedge^h u)^*$) and $\wedge^h\mathrm{diag}(s)$ diagonal with entries products of h singular values; so $\| \wedge^h z_\tau\| \le \|z_\tau\|^h$, with equality of norms of vectors if z_τ is unitary. With $\beta_\tau = \sum_{1 \le h < N_{\mathrm{t}}} h\,\kappa(\tau,h)$ this gives

$$\|S_\kappa(z)\xi\|_\kappa = \|\Lambda(z)F_\kappa\xi\| \le \prod_\tau \|z_\tau\|^{\beta_\tau}\,\|\xi\|_\kappa, \qquad \|S_\kappa(u)\xi\|_\kappa = \|\xi\|_\kappa \text{ if all } u_\tau \text{ are unitary.}$$

(S3) Evaluation vectors. Enumerate $n = \{a_0 < \cdots < a_{N_{\mathrm{t}}-1}\}$ and let $y^0_{\tau,a_j,c} = 1$ if $c = j$ and 0 otherwise. The polynomials b_i involve only the columns $c < N_{\mathrm{t}} - 1$. For an array y let $z(y)_\tau$ be the $n \times n$ matrix whose column a_j is $(y_{\tau,a,j})_{a \in n}$; then $(z(y)y^0)_{\tau,a,c} = y_{\tau,a,c}$ for $c < N_{\mathrm{t}}$, hence $\mathrm{ev}_\kappa(y) = S_\kappa(z(y))\,\mathrm{ev}_\kappa(y^0)$. Moreover $F_\kappa \mathrm{ev}_\kappa(y^0) = (\Delta_T(y^0))_T$, and $\Delta_{\tau,A}(y^0)$ is 1 if $A = \{a_0, \dots, a_{h-1}\}$ and 0 otherwise; so $F_\kappa \mathrm{ev}_\kappa(y^0)$ is a standard basis vector and $\|\mathrm{ev}_\kappa(y^0)\|_\kappa = 1$.

Step 1: transport to algebraic forms. By (4) of Lemma 6.4.68, $y \mapsto y^\flat$ is a linear bijection of $\mathcal{F}(\mathrm{II}, 0, m)$ onto $\mathcal{A}$ with $y = F^l_{y^\flat}$. If y is fixed by K, so is $y^\flat$ (its definition (WM7) integrates $y(x,g)$ in x only), and if $\eta \in \mathcal{A}$ is fixed by K, so is F^l_η. Hence $y \mapsto y^\flat$ maps $\mathcal{F}_{\mathfrak{b}}(m)$ bijectively onto the space $\mathcal{A}^{\mathrm{K}}$ of the K-invariant algebraic forms, and it suffices to show that $\mathcal{A}^{\mathrm{K}}$ is spanned by forms η with $\mathrm{gau}_\iota(g, \eta(g)) \le 1$ and $\|\sigma(\eta(g))\|^2_\kappa \le \exp(x^{99}b^2)$ for all ι, σ, g; call such η *short*.

Step 2: delta forms. K is an open subgroup of $\mathbf{K}_0(n)$, the stabiliser in $G_f(n)$ of the lattice $\widehat{\mathcal{O}}_E^n$ of I_n, and every element of K is $\equiv 1$ modulo π_M, hence acts trivially on $\mathcal{O}^n_{E_w}/w$ for each chosen place w (there is one, π_M being a non-unit of norm p_0p_1). By Lemma 4.6.2 (c), $\Gamma_f \cap g\mathrm{K}g^{-1} = \{1\}$ for every $g \in G_f(n)$. For $g \in G_f(n)$ and $\xi \in \mathbb{V}_\kappa$ define $\delta_{g,\xi}(\gamma_f g k) = \rho_{\kappa,e}(\gamma_\infty)\xi$ ($\gamma \in \Gamma$, $k \in \mathrm{K}$) and $\delta_{g,\xi} = 0$ off $\Gamma_f g \mathrm{K}$. It is well defined: $\gamma_f g k = \gamma'_f g k'$ gives $\gamma_f'^{-1}\gamma_f = gk'k^{-1}g^{-1} \in \Gamma_f \cap g\mathrm{K}g^{-1}$, so $\gamma = \gamma'$. Clearly $\delta_{g,\xi} \in \mathcal{A}^{\mathrm{K}}$ and $\xi \mapsto \delta_{g,\xi}$ is linear. By

the finiteness of class numbers, (K) of Theorem A.4.1 applied to $\mathrm{U}(\mathrm{I}_n)$ over E^+ and the open compact subgroup K, there are finitely many $g_1,\dots,g_h$ with $G_f(n)=\bigcup_i \Gamma_f g_i \mathrm{K}$, the double cosets disjoint after removing repetitions; then every $\eta\in\mathcal{A}^{\mathrm{K}}$ equals $\sum_i \delta_{g_i,\eta(g_i)}$. It therefore suffices to find, for each $g\in G_f(n)$, vectors ξ spanning $\mathbb{V}_\kappa$ with $\delta_{g,\xi}$ short.

Step 3: the vectors. Fix $g\in G_f(n)$ and put $B_{\mathrm{s}}=6q_1 d(fN_{\mathrm{t}})^2$. By Lemma 4.6.5, applied to g and to g^{e_τ} ($\tau\in\Phi$), there are $Y,Y'_\tau\in\mathrm{GL}_n(E)$ with $g^{-1}Y$ and $g^{-e_\tau}Y'_\tau$ in $\mathrm{M}_n(\widehat{\mathcal{O}}_E)$ and all entries of $\eta(Y)$, $\eta(Y'_\tau)$ of absolute value at most B_{s}, for every embedding $\eta\colon E\to\mathbb{C}$. For an array y whose entries are roots of unity put

$$\xi_y=\prod_{\tau\in\Phi}\det\epsilon_\tau(Y'_\tau)\cdot S_\kappa\big(\epsilon(Y)\big)\,\mathrm{ev}_\kappa(y).$$

They span $\mathbb{V}_\kappa$. The matrix $\prod_\tau\det\epsilon_\tau(Y'_\tau)\,S_\kappa(\epsilon(Y))$ is invertible by (S1). If a linear form λ on $\mathbb{V}_\kappa$ vanishes on all $\mathrm{ev}_\kappa(y)$, the polynomial $\sum_i\lambda_i b_i$ vanishes on all arrays with entries in the infinite set of roots of unity; a polynomial vanishing on a product of infinite sets is 0 (induction on the number of variables), and the b_i are linearly independent, so $\lambda=0$.

Gauge. Let ι be a prime identification. By (S1) and since $f_{\iota,\tau}$ is a ring homomorphism agreeing with $\iota\circ\epsilon_\tau$ on E,

$$\prod_\tau\det f_{\iota,\tau}(g^{-1})^{e_\tau}\,S_\kappa\big(f_\iota(g^{-1})\big)\,\iota(\xi_y)=\prod_\tau\det f_{\iota,\tau}\big(g^{-e_\tau}Y'_\tau\big)\cdot S_\kappa\big(f_\iota(g^{-1}Y)\big)\,\mathrm{ev}_\kappa(\iota(y)).$$

The matrices $f_{\iota,\tau}(g^{-e_\tau}Y'_\tau)$ and $f_{\iota,\tau}(g^{-1}Y)$ have integral entries (the projection of an integral finite adele to E_w lies in $\mathcal{O}_{E_w}$, and the continuous extension of the embedding maps $\mathcal{O}_{E_w}$ into the valuation ring of $\overline{\mathbb{Q}}_\ell$), $\iota(y)$ has root-of-unity entries, and S_κ, ev_κ are integer polynomials; so $\mathrm{gau}_\iota(g,\xi_y)\le 1$.

Norm. Let σ be a field automorphism of $\mathbb{C}$. Each $\sigma\circ\epsilon_\tau$ is an embedding of E, and $\sigma(\mathrm{ev}_\kappa(y))=\mathrm{ev}_\kappa(\sigma(y))=S_\kappa(z(\sigma(y)))\mathrm{ev}_\kappa(y^0)$ by (S3), where $z(\sigma(y))_\tau$ has root-of-unity entries, so $\|z(\sigma(y))_\tau\|\le N_{\mathrm{t}}$; also $\|\sigma\epsilon_\tau(Y)\|\le N_{\mathrm{t}}B_{\mathrm{s}}$ and $|\det\sigma\epsilon_\tau(Y'_\tau)|\le\|\sigma\epsilon_\tau(Y'_\tau)\|^{N_{\mathrm{t}}}\le(N_{\mathrm{t}}B_{\mathrm{s}})^{N_{\mathrm{t}}}$. By (S1)–(S3), with $\beta=\sum_\tau\beta_\tau$ and $N_{\mathrm{t}}\le B_{\mathrm{s}}$,

$$\|\sigma(\xi_y)\|_\kappa^2\le\prod_\tau(N_{\mathrm{t}}B_{\mathrm{s}})^{2N_{\mathrm{t}}}\big(N_{\mathrm{t}}^2B_{\mathrm{s}}\big)^{2\beta_\tau}\le B_{\mathrm{s}}^{4fN_{\mathrm{t}}+6\beta}\le B_{\mathrm{s}}^{6(fN_{\mathrm{t}}+\beta)}.$$

Step 4: the delta forms are short. Let $h=\gamma_f g k$ with $\gamma\in\Gamma$, $k\in\mathrm{K}$; the value of δ_{g,ξ_y} at h is $\rho_{\kappa,e}(\gamma_\infty)\xi_y$ (at the other points it is 0). As $\iota(\rho_{\kappa,e}(\gamma_\infty)\xi)=\prod_\tau\det f_{\iota,\tau}(\gamma)^{e_\tau}S_\kappa(f_\iota(\gamma))\iota(\xi)$ and $h^{-1}=k^{-1}g^{-1}\gamma_f^{-1}$, (S1) gives

$$\prod_\tau\det f_{\iota,\tau}(h^{-1})^{e_\tau}S_\kappa\big(f_\iota(h^{-1})\big)\iota\big(\rho_{\kappa,e}(\gamma_\infty)\xi_y\big)=\prod_\tau\det f_{\iota,\tau}(k^{-1})^{e_\tau}S_\kappa\big(f_\iota(k^{-1})\big)\cdot\zeta,$$

ζ the vector of the gauge of ξ_y at g, whose coordinates are integral by Step 3. Since $k^{-1}\in\mathrm{GL}_n(\widehat{\mathcal{O}}_E)$, $f_{\iota,\tau}(k^{-1})$ is integral with unit determinant, so the gauge at h is at most 1. For σ, $\sigma(\rho_{\kappa,e}(\gamma_\infty)\xi_y)=\prod_\tau\det u_\tau^{e_\tau}S_\kappa(u)\sigma(\xi_y)$ with $u_\tau=(\sigma\epsilon_\tau)(\gamma)$; as E is a CM field, every embedding commutes with complex conjugation, so $\bar\gamma^{\mathsf{T}}\gamma=1$ gives $u_\tau^*u_\tau=1$, and by (S2) the norm equals $\|\sigma(\xi_y)\|_\kappa$. So δ_{g,ξ_y} is short once $B_{\mathrm{s}}^{6(fN_{\mathrm{t}}+\beta)}\le\exp(x^{99}b^2)$.

Step 5: the numerical bound. By (4) of Lemma 6.4.3 at $i=0$, $\beta\le br^4$. By (AD2), $r\le 3x^5$, $m\le 2\sqrt{x}$, $t=5r$, so $N_{\mathrm{t}}=r+1+m(t-1)\le 34x^{11/2}$ (Definition 3.2.28). By Lemma 4.3.5, $f\le q_1\le b$, $d\le b$, $\log b\ge 10$ and $x\ge 4$; by (AD1), $\log x\le 10^{-12}\log b$, so $x^{11/2}\le b^{1/2}$ and $34\le b^{1/2}$, whence $N_{\mathrm{t}}\le b$. Hence $B_{\mathrm{s}}\le 6b^6\le b^7$, $fN_{\mathrm{t}}+\beta\le 34bx^{11/2}+81bx^{20}\le 115\,bx^{20}$, and

$$6(fN_{\mathrm{t}}+\beta)\log B_{\mathrm{s}}\le 6\cdot 115\cdot 7\,x^{20}b\log b\le 4830\,x^{20}b^2\le x^{99}b^2,$$

as $x^{79}\ge 4^{79}$. By Steps 2–4 the short forms span $\mathcal{A}^{\mathrm{K}}$, and by Step 1 the lemma follows. □

Lemma 6.4.78. *Let D be admissible data. There is a branching pairing at (m,m).*

Proof. Write $n = n_{\mathrm{I},m}$, $n' = n_{\mathrm{II},m} = n \sqcup \{*\}$, $N = \#n$ (so $\#n' = N+1$), $(\kappa, e) = (\kappa_{\mathrm{II}}, e^{\det}_{\mathrm{II},m})$ in the rank $N+1$ with highest weights $\lambda_\tau(q) = \sum_{q<h\le N} \kappa(\tau,h)$ $(0 \le q \le N)$, $l' = l_{\mathrm{II},m} = -p_{\mathrm{II},m,m}$, and $(\kappa', e') = (\kappa_{\mathrm{I}}, e^{\det}_{\mathrm{I},m})$ in the rank N with highest weights λ'_τ, $l = l_{\mathrm{I},m} = -p_{\mathrm{I},m,m}$. By (3) of Lemma 6.4.3, l' and l are doubled Harish-Chandra parameters of (κ, e) and (κ', e'). For a doubled parameter l'' of a weight (κ'', e'') in a rank N'', the numbers $c_q = 2(e''_\tau + \lambda''_\tau(q)) + N'' - 1 - 2q$ $(0 \le q < N'')$ strictly decrease, so the injective map l''_τ is a bijection onto them and its q-th largest value (counted from $q=0$) is c_q.

Step 1: a weight with the parameter $p = -l$. Put $\kappa_0(\tau,h) = \kappa'(\tau, N-h)$ $(1 \le h < N)$ and $e_{0,\tau} = -e'_\tau - \lambda'_\tau(0)$. Its highest weight is $\lambda_{0,\tau}(q) = \sum_{0<h<N-q} \kappa'(\tau,h) = \lambda'_\tau(0) - \lambda'_\tau(N-1-q)$. The q-th largest value of $p_\tau = -l_\tau$ is minus the $(N-1-q)$-th largest value of l_τ:

$$-2(e'_\tau + \lambda'_\tau(N-1-q)) - (N-1) + 2(N-1-q) = 2(e_{0,\tau} + \lambda_{0,\tau}(q)) + N - 1 - 2q.$$

As p_τ is injective, p is a doubled Harish-Chandra parameter of (κ_0, e_0) on n.

Step 2: the values of p and l' alternate. Fix τ and put $K_j = 2\kappa_j(\tau)$ $(j \ge 1)$, so that $K_{j+1} = K_j + 2\Delta$ and $K_1 > 3\Delta$; moreover $\Delta > t + k + 1$. By (GP5) at the node (I, m, m) (here $\delta = 0$, as $2m$ is even; the base indices are $0, \dots, r-1 = k-2+m$), the values of p_τ are the $k-1$ numbers of $S = \{k-2, k-4, \dots, 2-k\}$ (base $i < k-1$, $2\varsigma_i = k-2-2i$), the numbers $b_0 = \Delta - K_1$ and $b_j = \Delta - K_{j+1} = -(K_j + \Delta)$ $(1 \le j < m$; base $i = k-1+j)$, and for $1 \le l \le m$ the blocks $U_l = \{K_l + t - 1 - 2c : 0 \le c < t\}$ (up) and $D_l = \{-K_l + t - 1 - 2c : 0 \le c < t\}$ (down). At (II, m, m) (base indices $0, \dots, r$, up blocks of length $t+1$, down blocks of length $t-1$), the values of $l'_\tau = -p_{\mathrm{II},m,m}(\tau)$ are the $k-1$ numbers of $S' = \{k-3, k-5, \dots, 1-k\}$, the numbers $b'_0 = K_1 - \Delta - 1$ and $b'_j = K_{j+1} - \Delta - 1 = K_j + \Delta - 1$ $(1 \le j \le m)$, and the blocks $U'_l = \{K_l + t - 2 - 2c : 0 \le c \le t-2\}$ and $D'_l = \{-K_l + t - 2c : 0 \le c \le t\}$. Writing $X > Y$ when every element of X exceeds every element of Y,

$$\begin{aligned} b'_m > U_m \cup U'_m > b'_{m-1} > \cdots > b'_1 > U_1 \cup U'_1 > b'_0 > S \cup S' \\ > b_0 > D'_1 \cup D_1 > b_1 > D'_2 \cup D_2 > \cdots > b_{m-1} > D'_m \cup D_m, \end{aligned}$$

and inside each group the two blocks are arithmetic progressions of step 2 whose lengths differ by one and whose centres agree (U_l, U'_l centred at K_l, of lengths t, $t-1$; S, S' of length $k-1$ with $S' = S - 1$; D'_l, D_l centred at $-K_l$, of lengths $t+1$, t), so their elements alternate, the group starting with U_l, S, D'_l and ending with U_l, S', D'_l respectively. Hence the $2N+1$ values, listed in decreasing order, alternate between l'_τ and p_τ, beginning and ending with a value of l'_τ: if $L'_0 > \cdots > L'_N$ are the values of l'_τ and $P_0 > \cdots > P_{N-1}$ those of p_τ, then $L'_q > P_q > L'_{q+1}$ for $0 \le q < N$.

Step 3: a horizontal strip. Put $\mu_q = e_\tau + \lambda_\tau(q)$ $(0 \le q \le N)$ and $\nu_q = e_{0,\tau} + \lambda_{0,\tau}(q)$ $(0 \le q < N)$, so $L'_q = 2\mu_q + N - 2q$ and $P_q = 2\nu_q + N - 1 - 2q$. The integer inequalities $L'_q > P_q$ and $P_q > L'_{q+1}$ give $2\mu_q > 2\nu_q - 1$ and $2\nu_q > 2\mu_{q+1} - 1$, so $\mu_{q+1} \le \nu_q \le \mu_q$. Put $d_q = \nu_q - \mu_{q+1}$, so $0 \le d_q \le \mu_q - \mu_{q+1} = \kappa(\tau, q+1)$, and define weights for the rank $N+1$ by $\kappa_1(\tau,h) = d_{h-1}$ and $\gamma(\tau,h) = \kappa(\tau,h) - d_{h-1}$ $(1 \le h \le N)$. Then $\kappa = \gamma + \kappa_1$; for $1 \le h < N$, $\kappa_1(\tau,h) + \gamma(\tau,h+1) = d_{h-1} + \mu_h - \mu_{h+1} - d_h = \nu_{h-1} - \nu_h = \kappa_0(\tau,h)$; and $e_\tau + \kappa_1(\tau,N) = e_\tau + \nu_{N-1} - \mu_N = \nu_{N-1} = e_{0,\tau}$, as $\mu_N = e_\tau$ and $\lambda_{0,\tau}(N-1) = 0$. Moreover $e_{0,\tau} = \nu_{N-1} \ge \mu_N = e_\tau$, so $\lambda_{0,\tau}(q) = \nu_q - e_{0,\tau} \le \mu_q - e_\tau = \lambda_\tau(q)$, and summing over q and τ (the number of boxes is the sum of the row lengths),

$$\sum_\tau \sum_{1 \le h < N} h\,\kappa_0(\tau,h) = \sum_\tau \sum_{q<N} \lambda_{0,\tau}(q) \le \sum_\tau \sum_{q \le N} \lambda_\tau(q) = B_0. \tag{6.102}$$

Step 4: the matrix. Number n as $1,\dots,N$ and $*$ as $N+1$. This identifies the polynomial rings of (WM1) for the rows n, n' with those for the rows $1,\dots,N$ and $1,\dots,N+1$; it carries products of minors to products of minors up to sign, hence ∇_κ to itself and the products Δ_T of increasing fillings to themselves up to sign. So it identifies the Weyl modules up to an integral change of coordinates with integral inverse (both standard bases being $\mathbb{Z}$-bases of ∇_κ), compatibly with the Schur matrices, and isometrically, since $\|\xi\|_\kappa^2 = \sum_T |\xi(\Delta_T)|^2$ by (WM3) with ξ read as a linear form on ∇_κ. Through it, Lemma 4.6.7 for Φ, $n_\tau = N$, (κ, e), γ, κ_1 and $q = 1$ (its κ_0, e_0 are ours by Step 3) gives an integer matrix br with rows I_{κ_0} and columns $I_{\mathrm{II},m}$ satisfying its (a)–(c), with $w_\tau = \sum_{1\le h\le N} \gamma(\tau, h)$. Lemma 6.4.73 for n, the weight (κ_0, e_0) with parameter p (Step 1) and (κ', e') with parameter $-p = l$ gives an integer matrix A with rows $I_{\mathrm{I},m}$ and columns I_{κ_0} satisfying its (1)–(4). Put $B = A\,\mathrm{br}$, an integer matrix with rows $I_{\mathrm{I},m}$ and columns $I_{\mathrm{II},m}$.

Step 5: (a) of (LQ6). Put $N_\tau = w_\tau + e_\tau$; as $0 \le w_\tau \le \sum_h \kappa(\tau, h) \le B_0$, $|N_\tau| \le B_0 + |e_\tau|$. Let S, R, f_τ, g, u, v', v be as in (a). From $\bar g^{\mathsf T} g = 1$, $\det g$ is a unit of S, so $f(g) \in \prod_\tau \mathrm{GL}_n(R)$, $s = (f_\tau(u))_\tau \in \prod_\tau R^\times$ and $f(\mathrm{diag}(g, u)) = \mathrm{diag}(f(g), s)$. By (a) of Lemma 4.6.7 and (1) of Lemma 6.4.73 over R,

$$\begin{aligned}\big(P_{\kappa',e'}(f(g))v'\big)^{\mathsf T} A\,\mathrm{br}\,P_{\kappa,e}(\mathrm{diag}(f(g),s))v &= \prod_\tau s_\tau^{N_\tau}\big(P_{\kappa',e'}(f(g))v'\big)^{\mathsf T} A\,P_{\kappa_0,e_0}(f(g))\,\mathrm{br}\,v\\ &= \prod_\tau f_\tau(u)^{N_\tau}\, v'^{\mathsf T} B v.\end{aligned}$$

Step 6: (b) of (LQ6). Let $M \ge B_0$, $v' \in \mathbb{V}_{\mathrm{I},m}$, $v \in \mathbb{V}_{\mathrm{II},m}$. By (4) of Lemma 6.4.73 for $\mathrm{br}\,v$ there is w' with $\|w'\|_{\kappa'} = \|\mathrm{br}\,v\|_{\kappa_0}$ and $v'^{\mathsf T} A\,\mathrm{br}\,v = \langle w', v'\rangle_{\kappa'}$. The form $\langle\cdot,\cdot\rangle_{\kappa'}$ is positive semidefinite ($G_{\kappa'} = F_{\kappa'}^{\mathsf T} F_{\kappa'}$), so by the Cauchy–Schwarz inequality and (b) of Lemma 4.6.7 with $L = M$ (admissible by (6.102)),

$$|v'^{\mathsf T} B v| \le \|w'\|_{\kappa'}\|v'\|_{\kappa'} = \|\mathrm{br}\,v\|_{\kappa_0}\|v'\|_{\kappa'} \le (M^M)^3 \|v'\|_{\kappa'}\|v\|_\kappa.$$

Step 7: (c) of (LQ6). Fix $g \in G_\infty(1)$ for n', let dy be the Haar probability measure of $G_\infty(1)$ for n, and define the linear map T from $\mathbb{V}_{\mathrm{II},m}$ to $\mathbb{V}_{\kappa_0}$ by

$$T\xi = \int_{G_\infty(1)} \rho_{\kappa_0,e_0}(y)\, v_p^+ \,\big\langle v_{l'}^+, \rho_{\kappa,e}(g^{-1})\,\rho_{\kappa,e}(\mathrm{diag}(y,1))^{-1}\xi\big\rangle_\kappa\, dy.$$

For $y_0 \in G_\infty(1)$, substituting $y = y_0 y'$ and using the left invariance of dy and the multiplicativity of ρ gives $T\rho_{\kappa,e}(\mathrm{diag}(y_0,1)) = \rho_{\kappa_0,e_0}(y_0)T$, that is $T\,P_{\kappa,e}(\mathrm{diag}(h,1)) = P_{\kappa_0,e_0}(h)\,T$ for every $h = \bar y_0 \in \prod_\tau \mathrm{U}(N)$; by (c) of Lemma 4.6.7, $T = c_g\,\mathrm{br}$ for some $c_g \in \mathbb{C}$, which depends only on g. Now let η, φ, u, K be as in (c), and $f(x) = F_\eta^l(x)\,F_\varphi^{l'}(\mathrm{diag}(x,(1,u))(g,1))$ on $\mathrm{G}(1)$ for n. For $x = (y,h)$, (2) of Lemma 6.4.73 (signs 1) gives $F_\eta^l(y,h) = \eta(h)^{\mathsf T} A \rho_{\kappa_0,e_0}(y) v_p^+$, and (WM7) gives $F_\varphi^{l'}(\mathrm{diag}(y,1)g, \mathrm{diag}(h,u)) = \langle v_{l'}^+, \rho_{\kappa,e}(g^{-1})\rho_{\kappa,e}(\mathrm{diag}(y,1))^{-1}\varphi(\mathrm{diag}(h,u))\rangle_\kappa$; hence

$$\int_{G_\infty(1)} f(y,h)\,dy = \eta(h)^{\mathsf T} A\,T\,\varphi(\mathrm{diag}(h,u)) = c_g\,F_u(h), \qquad F_u(h) = \eta(h)^{\mathsf T} B\,\varphi(\mathrm{diag}(h,u)).$$

The function f is continuous, and right invariant in the finite variable under the open subgroup of the $k \in \mathbf{K}_0(n)$ with $\eta(hk) = \eta(h)$ and $\varphi(h'\mathrm{diag}(k,1)) = \varphi(h')$ for all h, h' (as $\mathrm{diag}(hk,u) = \mathrm{diag}(h,u)\mathrm{diag}(k,1)$). It is left invariant under the rational points γ of I_n: $\mathrm{diag}(\gamma,1)$ is a rational point of $\mathrm{I}_{n'}$, so by (WM7)

$$\eta(\gamma_f h) = \rho_{\kappa',e'}(\gamma_\infty)\eta(h) \quad\text{and}\quad \varphi(\mathrm{diag}(\gamma_f h, u)) = \rho_{\kappa,e}(\mathrm{diag}(\gamma_\infty,1))\varphi(\mathrm{diag}(h,u)),$$

and $F^{l''}_{\eta''}(\gamma x) = F^{l''}_{\eta''}(x)$ for both forms, as $\rho((\gamma_\infty x_\infty)^{-1})\rho(\gamma_\infty) = \rho(x_\infty^{-1})$ by multiplicativity. By Step 5, B is an integer matrix with (a) of (LQ6), so by (i) of Lemma 6.4.79 F_u satisfies the hypotheses of (LQ3), and so does $c_g F_u$. By (2) of Lemma 6.4.74,

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} F^l_\eta(x)\, F^{l'}_\varphi\big(\mathrm{diag}(x,(1,u))(g,1)\big)\,dx = \frac{c_n}{[\mathbf{K}_0(n):\mathrm{K}]}\,\mathrm{CI}_\mathrm{K}(c_g F_u) = \frac{c_n c_g}{[\mathbf{K}_0(n):\mathrm{K}]}\,\mathrm{CI}_\mathrm{K}(F_u),$$

the last step by (1) of Lemma 6.4.74 (both class integrals computed at one level fixing F_u). This is (c) with $c = c_n c_g$, which depends only on g. So B is a branching pairing at (m,m). □

Lemma 6.4.79. *Let D be admissible data, $n = n_{\mathrm{I},m}$ and $n' = n_{\mathrm{II},m} = n \sqcup \{*\}$, as in (LQ6).*

(i) Let B be an integer matrix with rows $I_{\mathrm{I},m}$ and columns $I_{\mathrm{II},m}$ that satisfies (a) of (LQ6), let $\eta \in \mathcal{A}_{\mathrm{I},m}$, $\varphi \in \mathcal{A}_{\mathrm{II},m}$ and $u \in \mathrm{U}(1)(\mathbb{A}_f)$, and put

$$F(h) = \eta(h)^{\mathsf{T}} B\, \varphi(\mathrm{diag}(h,u)) \qquad (h \in G_f(n)).$$

Then $F(\gamma_f h) = F(h)$ for every rational point γ of I_n and every h; if η is right invariant under an open subgroup K_1 of $G_f(n)$ and φ under an open subgroup K_2 of $G_f(n')$, then F is right invariant under the open subgroup $\{k \in \mathrm{K}_1 : \mathrm{diag}(k,1) \in \mathrm{K}_2\}$. In particular F satisfies the hypotheses of (LQ3) for the index set n.

(ii) Let B be a branching pairing at (m,m), ϑ an m-fold twist, Ψ a function on the edge tuples for ϑ and $y = (y_\mathrm{I}, y_\mathrm{II})$ a pair with $y_z \in \mathcal{F}(z,0,m)$. For every edge tuple S for ϑ, both chains $\mathrm{Lc}^S_{z,m}(y_z)$ are defined and lie in $\mathcal{A}_{z,m}$, and the integrand $h \mapsto \mathrm{Lc}^S_{\mathrm{I},m}(y_\mathrm{I})(h)^{\mathsf{T}} B\, \mathrm{Lc}^S_{\mathrm{II},m}(y_\mathrm{II})(\mathrm{bp}_\vartheta(h))$ of (LQ7) is the function F of (i) for $\eta = \mathrm{Lc}^S_{\mathrm{I},m}(y_\mathrm{I})$, $\varphi = \mathrm{Lc}^S_{\mathrm{II},m}(y_\mathrm{II})$ and $u = \mathrm{flp}^m_\vartheta$. In particular $\mathrm{Ls}_B(\Psi, y)$ is defined.

Proof. Write $(\kappa', e') = (\kappa_\mathrm{I}, e^{\det}_{\mathrm{I},m})$ and $(\kappa, e) = (\kappa_\mathrm{II}, e^{\det}_{\mathrm{II},m})$ for the weights at (I,m,m) and (II,m,m), so that $\mathcal{A}_{\mathrm{I},m} = \mathcal{A}_n(\kappa', e')$ and $\mathcal{A}_{\mathrm{II},m} = \mathcal{A}_{n'}(\kappa, e)$ ((LQ1)). For $h \in G_f(n)$ the block matrix $\mathrm{diag}(h,u)$ lies in $G_f(n')$, as h is unitary for I_n and $u\bar{u} = 1$.

Step 1: left invariance in (i). Let γ be a rational point of I_n, that is $\gamma \in \mathrm{M}_n(E)$ with $\bar{\gamma}^{\mathsf{T}}\gamma = 1$; its infinite component γ_∞ is the family of the matrices $\tau(\gamma)$ ($\tau \in \Phi$). Then $\mathrm{diag}(\gamma, 1)$ is a rational point of $\mathrm{I}_{n'}$, with infinite component $\mathrm{diag}(\gamma_\infty, 1)$ and finite component $\mathrm{diag}(\gamma_f, 1)$, and $\mathrm{diag}(\gamma_f h, u) = \mathrm{diag}(\gamma_f, 1)\,\mathrm{diag}(h,u)$. By the transformation law (WM7) for η and for φ,

$$\eta(\gamma_f h) = P_{\kappa',e'}(\bar{\gamma}_\infty)\,\eta(h), \qquad \varphi(\mathrm{diag}(\gamma_f h, u)) = P_{\kappa,e}\big(\mathrm{diag}(\bar{\gamma}_\infty, 1)\big)\,\varphi(\mathrm{diag}(h,u)),$$

with $\bar{\gamma}_\infty$ the entrywise complex conjugate. Apply (a) of (LQ6) to the ring E with its complex conjugation, the field $R = \mathbb{C}$, the ring homomorphisms $f_\tau(x) = \overline{\tau(x)}$ ($\tau \in \Phi$), the matrix $g = \gamma$, the scalar $1 \in E$ in place of its unit, and $v' = \eta(h)$, $v = \varphi(\mathrm{diag}(h,u))$. As E is a CM field, $\overline{\tau(x)} = \tau(\bar{x})$, so $f(\gamma) = \bar{\gamma}_\infty$, $f(\mathrm{diag}(\gamma,1)) = \mathrm{diag}(\bar{\gamma}_\infty, 1)$ and $f_\tau(1) = 1$; hence

$$F(\gamma_f h) = \big(P_{\kappa',e'}(f(\gamma))\,v'\big)^{\mathsf{T}} B\, P_{\kappa,e}(f(\mathrm{diag}(\gamma,1)))\,v = v'^{\mathsf{T}} B v = F(h).$$

Step 2: right invariance in (i). By (WM7) every algebraic form is right invariant under some open subgroup, so open subgroups K_1, K_2 as in (i) exist. The map $k \mapsto \mathrm{diag}(k,1)$ is a continuous homomorphism $G_f(n) \to G_f(n')$, so $\mathrm{K}_3 = \{k \in \mathrm{K}_1 : \mathrm{diag}(k,1) \in \mathrm{K}_2\}$ is an open subgroup of $G_f(n)$. For $k \in \mathrm{K}_3$, $\eta(hk) = \eta(h)$ and $\mathrm{diag}(hk,u) = \mathrm{diag}(h,u)\,\mathrm{diag}(k,1)$, so $\varphi(\mathrm{diag}(hk,u)) = \varphi(\mathrm{diag}(h,u))$ and $F(hk) = F(h)$. With Step 1, F is a complex-valued function on $G_f(n)$, left invariant under the finite components of the rational points of I_n and right invariant under an open subgroup: these are the hypotheses of (LQ3).

Step 3: (ii). By (2) of Lemma 6.4.68, for both z and every edge tuple S for ϑ the chain $\mathrm{Lc}^{S}_{z,m}(y_z)$ is defined and lies in $\mathcal{A}_{z,m}$. The flip idele has norm one, $\mathrm{flp}_\vartheta\,\overline{\mathrm{flp}_\vartheta}=1$, so $u=\mathrm{flp}_\vartheta^{m}\in \mathrm{U}(1)(\mathbb{A}_f)$, and $\mathrm{bp}_\vartheta(h)=\mathrm{diag}(h,u)$ by (LQ7). So the integrand of (LQ7) for S is the function F of (i) for these η, φ, u, and a branching pairing is an integer matrix with rows $I_{\mathrm{I},m}$ and columns $I_{\mathrm{II},m}$ satisfying (a) of (LQ6). By (i) the integrand satisfies the hypotheses of (LQ3). Both conditions of (LQ7) hold for every S, so $\mathrm{Ls}_B(\Psi,y)$ is defined. □

On the last path every space is definite and every move is an algebraic theta lift. This group proves the three properties of the last path that the spanning statement uses (Proposition 6.4.80): every family of functions of edge tuples is proportional to the last-path sum; one integer $N\le\exp(x^{99}b^2)$ clears its denominators at every prime identification $\iota\mid\ell$ with ℓ prime to the modulus of the twist; and at the trivial twist its conjugates are at most $\exp(x^{200}b^2)$ times the geometric mean of the bounds for the two forms. The arithmetic input is the jet matrices of the algebraic lifts along the edges (Lemma 6.4.82), whose denominators are bounded by one integer and whose conjugates at the trivial twist are bounded as operators; in our reading their entries are homogeneous jets of theta functions with complex multiplication, of the kind studied in Section 4. For the integrality of theta functions with complex multiplication see [272, 159]. The class numbers of the levels are bounded in Lemma 6.4.86; for mass formulas of definite hermitian lattices see [97].

Proposition 6.4.80. *Let D be admissible data and B a branching pairing at (m,m); put $V_1=\#(\mathcal{O}_E/\mathfrak{b})^{2n_m}$ and $T_1=\#(\mathcal{O}_E/\mathfrak{b})^{8mn_m^2}$. On the last path:*

(a) for every family (Ψ_ϑ) of functions on edge tuples and every real point g of $\mathrm{U}(\tilde{Y}^m_m)$ there is $\beta\in\mathbb{C}$ such that for every m-fold twist ϑ, every conjugate form y on Y_m and every form φ on $\tilde{Y}^m_0$, the period of (Ψ_ϑ) at g on (ϑ,y,φ) is $\beta\,\mathrm{Ls}_B(\Psi_\vartheta,(y,\varphi))/[\mathbf{K}_0(n_{\mathrm{I},0}):\mathrm{K}_\vartheta(\mathrm{I},0)]$, $n_{\mathrm{I},0}$ the index set of Y_m and $\mathbf{K}_0$ the standard level of (VP1) of Definition 3.2.2;

(b) there is an integer $1\le N\le\exp(x^{99}b^2)$ such that $N\,\mathrm{Ls}_B(\Psi,y)$ is integral at ι for every m-fold twist ϑ, every prime identification $\iota\mid\ell$ with $\ell\nmid \mathrm{N}Q$, every path datum Ψ on the edges of the last path and every pair y fixed by the outer levels whose algebraic forms $y^\flat_z$ are integral at ι;

(c) for the trivial twist, every path datum Ψ of bound $V_1^{m^2}T_1$ on the edges of the last path, every pair y fixed by the outer levels, every field automorphism σ of $\mathbb{C}$ and all $M_z\ge0$ with $\|\sigma(y^\flat_z(g))\|^2_{\kappa_z}\le M_z$ for all g and $z\in\{\mathrm{I},\mathrm{II}\}$, $|\sigma(\mathrm{Ls}_B(\Psi,y))|\le\exp(x^{200}b^2)\sqrt{M_\mathrm{I}M_\mathrm{II}}$; and the class numbers of the levels $\mathrm{K}_1(z,i)$ $(0\le i\le m)$ of the trivial twist are at most $\exp(x^{99}b^2)$.

Proof. Write $n_z=n_{z,0}$, $n=n_{\mathrm{I},m}$ and, for an m-fold twist ϑ, $\mathrm{ix}_\vartheta(z,i)=[\mathbf{K}_0(n_{z,i}):\mathrm{K}_\vartheta(z,i)]$. For $(z,i)\ne(\mathrm{I},0)$ the level $\mathrm{K}_\vartheta(z,i)$ is the kernel of the reduction modulo $\mathfrak{b}$, so it and $\mathrm{ix}_\vartheta(z,i)$ do not depend on ϑ.

Step 1: the class integrals are defined. Let $y=(y_\mathrm{I},y_\mathrm{II})$ with $y_z\in\mathcal{F}(z,0,m)$ and let S be an edge tuple for ϑ supported on the edges of the last path. By (ii) of Lemma 6.4.79, for both z the chain $\mathrm{Lc}^{S}_{z,m}(y_z)$ is defined and lies in $\mathcal{A}_{z,m}$, the integrand

$$F_S(h)=\mathrm{Lc}^{S}_{\mathrm{I},m}(y_\mathrm{I})(h)^{\mathsf{T}}\,B\,\mathrm{Lc}^{S}_{\mathrm{II},m}(y_\mathrm{II})(\mathrm{bp}_\vartheta(h))$$

of (LQ7) on $G_f(n)$ satisfies the hypotheses of (LQ3), and $\mathrm{Ls}_B(\Psi,y)$ is defined for every function Ψ on the edge tuples. By (1) of Lemma 6.4.74, CI_K is linear on such functions (compute at a common open subgroup).

Step 2: the period is a multiple of Ls_B. Fix (Ψ_ϑ) and g. By (3) of Lemma 6.4.68 for $j = m$ and both z, the images of y and φ under the composites of the lift moves along the two halves of the last path with the matrices of S are $F_{\eta_S}^{l_{\mathrm{I},m}}$ and $F_{\eta'_S}^{l_{\mathrm{II},m}}$ with

$$\eta_S = \frac{c_{\mathrm{I},m}}{\mathrm{ix}_\vartheta(\mathrm{I},0)}\,\mathrm{Lc}^S_{\mathrm{I},m}(y), \qquad \eta'_S = \frac{c_{\mathrm{II},m}}{\mathrm{ix}_1(\mathrm{II},0)}\,\mathrm{Lc}^S_{\mathrm{II},m}(\varphi),$$

algebraic forms by Step 1. By (3.3) and (1) of Lemma 6.4.68, the period is the sum over the edge tuples S supported on the edges of the last path of $\Psi_\vartheta(S)$ times

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} F_{\eta_S}^{l_{\mathrm{I},m}}(x)\, F_{\eta'_S}^{l_{\mathrm{II},m}}(\mathrm{diag}(x,(1,\mathrm{flp}_\vartheta^m))(g,1))\,dx.$$

The flip idele has norm one, so $\mathrm{flp}_\vartheta^m \in \mathrm{U}(1)(\mathbb{A}_f)$; by (c) of (LQ6) for g (with its constant c_g, which depends only on g), $u = \mathrm{flp}_\vartheta^m$ and $\mathrm{K} = \mathrm{K}_\vartheta(\mathrm{I},m)$, and as $\mathrm{diag}(h,\mathrm{flp}_\vartheta^m) = \mathrm{bp}_\vartheta(h)$, the integral is

$$c_g\,\mathrm{ix}_1(\mathrm{I},m)^{-1}\mathrm{CI}_\mathrm{K}(h \mapsto \eta_S(h)^\mathsf{T} B\,\eta'_S(\mathrm{bp}_\vartheta(h))).$$

By linearity (Step 1),

$$\mathrm{per}_{\mathrm{last}}(\Psi;g)(\vartheta,y,\varphi) = \frac{\gamma_g}{\mathrm{ix}_\vartheta(\mathrm{I},0)}\,\mathrm{Ls}_B(\Psi_\vartheta,(y,\varphi)), \qquad \gamma_g = \frac{c_g\,c_{\mathrm{I},m}\,c_{\mathrm{II},m}}{\mathrm{ix}_1(\mathrm{II},0)\,\mathrm{ix}_1(\mathrm{I},m)}, \tag{6.103}$$

with γ_g independent of ϑ, y, φ; this is (a) with $\beta = \gamma_g$.

Step 3: the flip idele at a prime prime to $\mathrm{N}Q$. Let $\iota \mid \ell$ with $\ell \nmid \mathrm{N}Q$ and $\tau \in \Phi$. We show $\|f_{\iota,\tau}(M^b q_1\,\mathrm{flp}_\vartheta^{\pm 1})\| \le 1$. As $x^{q_1}-1 = \mathrm{cyc}_{q_1}(x)g(x)$ with $g \in \mathbb{Z}[x]$, $q_1\zeta^{q_1-1} = \mathrm{cyc}'_{q_1}(\zeta)g(\zeta)$, so by (WD2) $q_1\alpha$ and α^{-1} are integral finite adeles. The idele t_ϑ is integral and generates $\mathfrak{m}_\vartheta\widehat{\mathcal{O}}_E$, and $\mathrm{N}Q\,M^b \in \mathfrak{m}_\vartheta$ ($M^b = (\pi_M\bar\pi_M)^b$, and $\mathrm{N}w \in w$ for $w \in Q$); so $t_\vartheta^{-1}\mathrm{N}Q\,M^b$ is integral, and so is

$$a_+ = \mathrm{N}Q\,M^b q_1\,\mathrm{flp}_\vartheta = \left(t_\vartheta^{-1}\mathrm{N}Q\,M^b\right)\bar t_\vartheta\,\overline{q_1\alpha}\,\alpha^{-1},$$

and $a_- = \bar a_+ = \mathrm{N}Q\,M^b q_1\,\mathrm{flp}_\vartheta^{-1}$. The map $f_{\iota,\tau}$ carries integral adeles into the valuation ring of $\overline{\mathbb{Q}}_\ell$ and $\|f_{\iota,\tau}(\mathrm{N}Q)\| = 1$; this gives the claim.

Step 4: (b). Let N_τ ($\tau \in \Phi$) be the exponents of (a) of (LQ6) for B, so $|N_\tau| \le B_0 + |e^{\det}_{\mathrm{II},m}(\tau)| \le br^4 + 20b$ by (4), (5) of Lemma 6.4.3 for $z = \mathrm{II}$, $i = m$ (B_0 is the sum of (4) in the rank of (II,m,m)). Put $N = d_J^{2m}\prod_\tau (M^b q_1)^{m|N_\tau|}$, a positive integer depending only on D and B, with $\log N \le 2m\log d_J + fm(br^4+20b)\log(M^b q_1) \le x^{99}b^2$ by (2) of Lemma 6.4.86. Let ϑ, ι, Ψ, y be as in (b), and write $\mathrm{Ls}_B(\Psi,y) = \sum_t \zeta_t\, v_t'^\mathsf{T} B v_t$ as in Lemma 6.4.81, with $v'_t = \Pi^{\mathrm{I}}_t y^\flat_{\mathrm{I}}(g^{\mathrm{I}}_{t,0})$ and $v_t = \Pi^{\mathrm{II}}_t y^\flat_{\mathrm{II}}(g^{\mathrm{II}}_{t,0})$. By the integrality of $y^\flat_z$ and (1) of Lemma 6.4.82, applied m times along each half, $\mathrm{gau}_\iota(h, d_J^m v'_t) \le 1$ and $\mathrm{gau}_\iota(\mathrm{bp}_\vartheta(h), d_J^m v_t) \le 1$, $h = g^{\mathrm{I}}_{t,m}$. Put $w' = P_{\kappa_{\mathrm{I}},e^{\det}_{\mathrm{I},m}}(f_\iota(h^{-1}))\iota(d_J^m v'_t)$ and $w = P_{\kappa_{\mathrm{II}},e^{\det}_{\mathrm{II},m}}(f_\iota(\mathrm{bp}_\vartheta(h)^{-1}))\iota(d_J^m v_t)$, vectors with coordinates in the valuation ring. Apply (a) of (LQ6) with $S = \mathbb{A}_{E,f}$ (complex conjugation), $R = \overline{\mathbb{Q}}_\ell$, $f_\tau = f_{\iota,\tau}$, the unitary h^{-1} and $u = \mathrm{flp}_\vartheta^{-m}$; as $\mathrm{diag}(h^{-1},\mathrm{flp}_\vartheta^{-m}) = \mathrm{bp}_\vartheta(h)^{-1}$ and B has integer entries,

$$w'^\mathsf{T} B w = \prod_\tau f_{\iota,\tau}(\mathrm{flp}_\vartheta)^{-mN_\tau}\,\iota(d_J^{2m}\,v_t'^\mathsf{T} B v_t).$$

The left side has absolute value at most 1 (ultrametric inequality), and by Step 3

$$\|(M^b q_1)^{m|N_\tau|} f_{\iota,\tau}(\mathrm{flp}_\vartheta)^{mN_\tau}\| \le 1 \qquad \text{for every } \tau.$$

Hence $\|\iota(N\,v_t'^\mathsf{T} B v_t)\| \le 1$; as $\|\iota(\zeta_t)\| = 1$, the ultrametric inequality gives $\|\iota(N\,\mathrm{Ls}_B(\Psi,y))\| \le 1$.

Step 5: (c). Let $\vartheta = 1$ and Ψ, y, σ, M_z be as in (c). The space at a node (z,j,m) has rank at most $r+1+2mt < 2n_m$ ((GP), $n_m = r+1+mt$), so each of the $2m$ edges of the last path carries a matrix over $\mathcal{O}_E/\mathfrak{b}$ with at most $(2n_m)^2$ entries and T_1 bounds the number of edge tuples on the last path. By Lemma 6.4.81 with the bound $V_1^{m^2}T_1$ and (1) of Lemma 6.4.86, $\mathrm{Ls}_B(\Psi,y) = \sum_{t=1}^{T} \zeta_t v_t'^{\mathsf{T}} B v_t$ with $T \le V_1^{m^2}T_1^2 G^{2m+1}$. By (2) of Lemma 6.4.82, applied m times, $\|\sigma(v_t')\|_{\kappa_{\mathrm{I}}} \le G^m\sqrt{M_{\mathrm{I}}}$ and $\|\sigma(v_t)\|_{\kappa_{\mathrm{II}}} \le G^m\sqrt{M_{\mathrm{II}}}$. As B has integer entries, $\sigma(v_t'^{\mathsf{T}}Bv_t) = \sigma(v_t')^{\mathsf{T}}B\,\sigma(v_t)$, and by (b) of (LQ6) with $M = br^4 \ge B_0$ this has absolute value at most $((br^4)^{br^4})^3 G^{2m}\sqrt{M_{\mathrm{I}}M_{\mathrm{II}}}$. Summing, with $|\sigma(\zeta_t)| = 1$,

$$|\sigma(\mathrm{Ls}_B(\Psi,y))| \le V_1^{m^2}T_1^2G^{4m+1}\big((br^4)^{br^4}\big)^3\sqrt{M_{\mathrm{I}}M_{\mathrm{II}}} \le \exp(x^{200}b^2)\sqrt{M_{\mathrm{I}}M_{\mathrm{II}}}$$

by (3) of Lemma 6.4.86. The bound for the class numbers is (1) of Lemma 6.4.86. □

Lemma 6.4.81. *Let D be admissible data and B a branching pairing at (m,m). For an m-fold twist ϑ, an edge tuple S, $z \in \{\mathrm{I},\mathrm{II}\}$, $0 \le j < m$, $g \in G_f(n_{z,j})$ and $h \in G_f(n_{z,j+1})$ put*

$$J^S_{z,j}(g,h) = \mathrm{jm}\big(\omega_{\alpha,\beta}((1,g),(1,h))\,\varphi^S_{z,j}\big) \in \mathrm{M}_{I_{z,j+1}\times I_{z,j}}(\mathbb{C}).$$

Let ϑ be an m-fold twist, T_ϑ the number of edge tuples for ϑ supported on the edges of the last path, and

$$C_\vartheta = \prod_{j=0}^{m} \#(\Gamma_{n_{\mathrm{I},j}}\backslash G_f(n_{\mathrm{I},j})/\mathrm{K}_\vartheta(\mathrm{I},j)) \cdot \prod_{j=0}^{m-1} \#(\Gamma_{n_{\mathrm{II},j}}\backslash G_f(n_{\mathrm{II},j})/\mathrm{K}_\vartheta(\mathrm{II},j)),$$

the class sets taken for the finite components of the rational points. Let $N \in \mathbb{N}$, Ψ a path datum of bound N on the edges of the last path and y a pair fixed by the outer levels. Then there are an integer $0 \le T \le N\,T_\vartheta\,C_\vartheta$ and, for $1 \le t \le T$, a root of unity ζ_t, an edge tuple S_t and points $g^z_{t,j} \in G_f(n_{z,j})$ ($z \in \{\mathrm{I},\mathrm{II}\}$, $0 \le j \le m$) with $g^{\mathrm{II}}_{t,m} = \mathrm{bp}_\vartheta(g^{\mathrm{I}}_{t,m})$, such that

$$\mathrm{Ls}_B(\Psi,y) = \sum_{t=1}^{T} \zeta_t\,\big(\Pi^{\mathrm{I}}_t\,y^\flat_{\mathrm{I}}(g^{\mathrm{I}}_{t,0})\big)^{\mathsf{T}} B\,\big(\Pi^{\mathrm{II}}_t\,y^\flat_{\mathrm{II}}(g^{\mathrm{II}}_{t,0})\big),$$
$$\Pi^z_t = J^{S_t}_{z,m-1}(g^z_{t,m-1},g^z_{t,m})\cdots J^{S_t}_{z,0}(g^z_{t,0},g^z_{t,1}).$$

Proof. Fix ϑ, N, Ψ, y and write $\mathfrak{m} = \mathfrak{m}_\vartheta$, $\mathcal{S}$ for the set of edge tuples supported on the edges of the last path ($\#\mathcal{S} = T_\vartheta$), $G_{z,j} = G_f(n_{z,j})$, $\mathrm{K}_{z,j} = \mathrm{K}_\vartheta(z,j)$ and Γ for the finite components of the rational points. For $S \in \mathcal{S}$ write $L^S_{z,j} = \mathrm{Lc}^S_{z,j}(y_z)$; by (2) of Lemma 6.4.68 these chains are defined (so $\mathrm{Al}_{\vartheta,z,j}(\varphi^S_{z,j})$ is defined at $L^S_{z,j}$ for $j < m$, (LQ4), (LQ5)), they are algebraic forms, and $L^S_{z,j+1}(h) = \mathrm{CI}_{\mathrm{K}_{z,j}}(g \mapsto J^S_{z,j}(g,h)L^S_{z,j}(g))$, the integrands satisfying the hypothesis of (LQ3).

Step 1: free levels. Every $\mathrm{K}_{z,j}$ consists of elements of $\mathbf{K}_0(n_{z,j})$ that are $\equiv 1$ modulo $\mathfrak{b}$ (for $(\mathrm{I},0)$ modulo $\mathfrak{m}$, a multiple of $\mathfrak{b}$), hence act trivially on $\mathcal{O}^n_{E_w}/w$ for every chosen place w; by Lemma 4.6.2 (c), $\Gamma \cap g\mathrm{K}'g^{-1} = \{1\}$ for every open $\mathrm{K}' \subseteq \mathrm{K}_{z,j}$ and every g. So in (1) of Lemma 6.4.74 all $\#\Gamma'_h = 1$, and for such K' each double coset $\Gamma x\mathrm{K}_{z,j}$ is the disjoint union of the $[\mathrm{K}_{z,j} : \mathrm{K}']$ cosets $\Gamma xk\mathrm{K}'$, $k \in \mathrm{K}_{z,j}/\mathrm{K}'$.

Step 2: expansion. Choose open subgroups $\mathrm{K}'_{z,j} \subseteq \mathrm{K}_{z,j}$ and finite sets $\mathcal{R}'_{z,j}$ of representatives of $\Gamma\backslash G_{z,j}/\mathrm{K}'_{z,j}$ from the top down: first $\mathrm{K}'_{\mathrm{I},m}$ fixing the integrands

$$F_S(h) = L^S_{\mathrm{I},m}(h)^{\mathsf{T}} B L^S_{\mathrm{II},m}(\mathrm{bp}_\vartheta(h))$$

of Ls_B for all $S \in \mathcal{S}$, and $\mathcal{R}'_{\mathrm{I},m}$. (By (ii) of Lemma 6.4.79 each F_S satisfies the hypotheses of (LQ3) and $\mathrm{Ls}_B(\Psi,y)$ is defined; by (i) there, an open subgroup fixing F_S exists.) Then, for

$j = m-1, \dots, 0$, $\mathrm{K}'_{z,j}$ fixing the finitely many integrands $g \mapsto J^S_{z,j}(g,h)L^S_{z,j}(g)$ with $S \in \mathcal{S}$ and h in $\mathcal{R}'_{z,j+1}$ (with $h \in \mathrm{bp}_\vartheta(\mathcal{R}'_{\mathrm{I},m})$ for $(z,j+1) = (\mathrm{II},m)$), and $\mathcal{R}'_{z,j}$. Applying (1) of Lemma 6.4.74 at every class integral, with weights $w_{z,j} = [\mathrm{K}_{z,j} : \mathrm{K}'_{z,j}]^{-1}$,

$$\mathrm{Ls}_B(\Psi, y) = W \sum_{S \in \mathcal{S}} \Psi(S) \sum_x \Theta(S; x), \qquad W = \prod_{j=0}^{m} w_{\mathrm{I},j} \prod_{j=0}^{m-1} w_{\mathrm{II},j},$$

the inner sum over the tuples $x = (x_0, \dots, x_m; x''_0, \dots, x''_{m-1})$ with $x_j \in \mathcal{R}'_{\mathrm{I},j}$, $x''_j \in \mathcal{R}'_{\mathrm{II},j}$, where, with $x''_m = \mathrm{bp}_\vartheta(x_m)$,

$$\Theta(S;x) = \big(J^S_{\mathrm{I},m-1}(x_{m-1}, x_m) \cdots J^S_{\mathrm{I},0}(x_0, x_1)\, y^\flat_{\mathrm{I}}(x_0)\big)^{\mathsf{T}} \\ B\, \big(J^S_{\mathrm{II},m-1}(x''_{m-1}, x''_m) \cdots J^S_{\mathrm{II},0}(x''_0, x''_1)\, y^\flat_{\mathrm{II}}(x''_0)\big).$$

The same identity holds for every other choice of the representative sets (the summands are invariant functions of the classes).

Step 3: the jet matrices under integral points. Let $e = ((z,j,m),(z,j+1,m))$, $k \in \mathbf{K}_0(n_{z,j})$, $k' \in \mathbf{K}_0(n_{z,j+1})$ with infinite components 1, and $\bar{k}, \bar{k}'$ their reductions modulo $\mathfrak{m}$. For a finite point x and $f \in \mathcal{M}_{n,n'}$, $\omega_{\alpha,\beta}(x)(v \otimes f) = v \otimes \omega_{\alpha,\beta}(x)f$: by (WR6) $v \otimes f = \int v(y)\rho(\mathrm{rv}(y))f\, d\nu(y)$, $\omega_{\alpha,\beta}(x)$ is bounded and $\omega_{\alpha,\beta}(x)\rho(z)\omega_{\alpha,\beta}(x)^{-1} = \rho(xz) = \rho(z)$ for z with finite components 0 ((2) of Lemma 4.1.1). By (6) of Lemma 4.1.1 with Φ the indicator of $S(e)$, for which $\Phi_{k,k'}$ is the indicator of $\bar{k}'S(e)\bar{k}^{-1}$, and by the representation property,

$$J^S_{z,j}(gk, hk') = \alpha(\det k')\beta(\det k)^{-1}\, J^{S_{k,k'}}_{z,j}(g,h),$$

$S_{k,k'}$ the tuple with the matrix $\bar{k}'S(e)\bar{k}^{-1}$ on e and that of S elsewhere; indeed each entry of the jet matrix is $\theta(v_P \otimes \omega_{\alpha,\beta}((1,g),(1,h))\omega_{\alpha,\beta}((1,k),(1,k'))\varphi^S_{z,j})$. The characters α, β of (GP6) are $\mu^0_{j+1}(\vartheta^{j+1})^{[z]}$ times powers of v; their common tame part is trivial at the places not dividing $\mathfrak{m}$ (the tame part of v is trivial, that of μ^0_{j+1} lives at the chosen places, that of ϑ at Q), and for the row m the edge e is vertical exactly in the grid $\tilde{Y}$. So, for integral u with infinite components 1, $\alpha(u) = \beta(u) = \chi_e(u \bmod \mathfrak{m})$, and the factor is $\chi_e(\det \bar{k}'/\det \bar{k})$.

Step 4: invariance of the sum over S. Let $k_j \in \mathrm{K}_{\mathrm{I},j}$ $(0 \le j \le m)$, $k''_j \in \mathrm{K}_{\mathrm{II},j}$ $(0 \le j < m)$, and put $k''_m = \mathrm{diag}(k_m, 1)$, so that $\mathrm{bp}_\vartheta(x_m k_m) = \mathrm{bp}_\vartheta(x_m)k''_m$, and $k''_m \in \mathrm{K}_{\mathrm{II},m}$. Let g be the family of the reductions modulo $\mathfrak{m}$ of k_j at (I,j,m) and of k''_j at (II,j,m), and 1 at the other nodes. It is admissible in the sense of (PD2): at $(\mathrm{I},0,m)$ it is 1, at $(\mathrm{II},0,m)$ it lies in the reduction of the level, at (II,m,m) it is $g_{(\mathrm{I},m,m)} \oplus 1$, and at the other diagonal nodes it is $1 = 1 \oplus 1$. Write xk for the tuple $(x_j k_j; x''_j k''_j)$. As y is fixed by the outer levels, $y^\flat_z(x_0k_0) = y^\flat_z(x_0)$, and Step 3 on each of the $2m$ edges gives $\Theta(S; xk) = \epsilon_w(g)\,\Theta(g^{-1} \cdot S; x)$, w the edges of the last path. As $S \mapsto g \cdot S$ is a bijection of $\mathcal{S}$, (PD2) gives

$$\sum_{S \in \mathcal{S}} \Psi(S)\Theta(S; xk) = \sum_S \epsilon_w(g)\Psi(g \cdot S)\,\Theta(S; x) = \sum_{S \in \mathcal{S}} \Psi(S)\Theta(S; x).$$

Step 5: sum over classes. Let $\mathcal{R}_{z,j}$ be representatives of $\Gamma\backslash G_{z,j}/\mathrm{K}_{z,j}$ and, by Step 1, take $\mathcal{R}'_{z,j} = \{xk : x \in \mathcal{R}_{z,j},\ k \in \mathrm{K}_{z,j}/\mathrm{K}'_{z,j}\}$ (for a fixed set of coset representatives k). By Steps 2 and 4,

$$\mathrm{Ls}_B(\Psi, y) = W \sum_x \sum_k \sum_S \Psi(S)\Theta(S; xk) = \sum_x \sum_{S \in \mathcal{S}} \Psi(S)\,\Theta(S; x),$$

x now over the tuples with $x_j \in \mathcal{R}_{\mathrm{I},j}$, $x''_j \in \mathcal{R}_{\mathrm{II},j}$: there are C_ϑ of them, and the number of the k is W^{-1}. By (PD4), each $\Psi(S)$ is a sum of at most N roots of unity. Expanding gives at most $N\,T_\vartheta C_\vartheta$ terms $\zeta_t\Theta(S_t; x_t)$, and with $g^{\mathrm{I}}_{t,j} = (x_t)_j$, $g^{\mathrm{II}}_{t,j} = (x_t)''_j$ $(j < m)$, $g^{\mathrm{II}}_{t,m} = \mathrm{bp}_\vartheta(g^{\mathrm{I}}_{t,m})$, each $\Theta(S_t; x_t)$ is the asserted pairing. □

Lemma 6.4.82. *Let D be admissible data, $f = \#\Phi$ and $d_J = (M^b)^{2f(2n_m)^2}(M^b d)^{br^4}$. At the node (z,j,m) the gauge of (WM9) is taken for the weight $(\kappa_z, e^{\det}_{z,j})$. Then for every $z \in \{\mathrm{I},\mathrm{II}\}$, $0 \le j < m$, edge tuple S, $g \in G_f(n_{z,j})$, $h \in G_f(n_{z,j+1})$ and $v \in \mathbb{V}_{z,j}$:*

(1) for every m-fold twist ϑ and every prime identification $\iota \mid \ell$ with $\ell \nmid \mathrm{N}Q$,

$$\mathrm{gau}_\iota(h, d_J\, J^S_{z,j}(g,h)v) \le \mathrm{gau}_\iota(g,v);$$

(2) for the trivial twist and every field automorphism σ of $\mathbb{C}$,

$$\|\sigma(J^S_{z,j}(g,h)v)\|_{\kappa_z} \le \exp(x^{99}b^2)\,\|\sigma(v)\|_{\kappa_z},$$

σ applied coordinatewise.

Proof. Fix z, $0 \le j < m$, an m-fold twist ϑ, an edge tuple S, $g \in G_f(n_{z,j})$ and $h \in G_f(n_{z,j+1})$; all symbols below are local. Put $n = n_{z,j}$, $n' = n_{z,j+1}$, $N = \#n$, $N' = \#n'$, $R = NN'$, $\mathcal{C} = n' \times n$, $D = fR$, $N_0 = M^b$, $\kappa = \kappa_z$, and $B = \sum_\tau \sum_{1 \le h' < N'} h' \kappa(\tau, h')$. By (1) and (4) of Lemma 6.4.3 (at j and $j+1$), $N < N'$, $\kappa(\tau,h') = 0$ for $N < h' < N'$ and $B \le br^4$. The ranks of the nodes of the row m are at most $r + 1 + 2mt \le 2n_m$, so $R \le (2n_m)^2$.

Step 0: the space, the matrix and the datum. By (GP8) both I_n and $\mathrm{I}_{n'}$ have Gram matrix and point the identity, so $\mathrm{Hom}(\mathrm{I}_n, \mathrm{I}_{n'})$ is the standard definite space with index set $\mathcal{C}$ ((WR3)), the identity is a transporting element, and by Lemma 4.1.17 its theta functional $\theta_{n,n'}$ is the canonical functional θ_R of the standard definite space I_R on $\mathcal{C}$. Let α, β be the edge characters of (LQ4), of weights $e^{\det}_{z,j+1}$ and $\gamma' = \gamma_{z,j}(N')$ ((GP6) and (2) of Lemma 6.4.3), and put $F = \omega_{\alpha,\beta}((1,g),(1,h))\varphi^S_{z,j}$ and $J = J^S_{z,j}(g,h) = \mathrm{jm}(F)$. By (LQ4), $J_{i'i} = \theta_R(v_{P_{i'i}(\eta_P\,\cdot)} \otimes F)$, where $P_{i'i}$ is the entry of the rectangular Schur matrix $S_\kappa(X)$; by (WM2) each $P_{i'i}$ is homogeneous of degree B with integer coefficients (a minor on h' rows of Xy is homogeneous of degree h' in X). By (XD2), $F = c\,\omega(k)\varphi^S_{z,j}$ with $c = \alpha(\det h)\beta(\det g)^{-1}$ and $\omega(k)$ the Weil operator of the finite unitary transformation $k\colon X \mapsto hXg^{-1}$ of I_R; by Proposition 4.1.14, $\omega(k)\rho(\mathrm{fv}(u)) = \rho(\mathrm{fv}(ku))\omega(k)$ (the finite coordinates being $\mathbb{A}_{E,f}$-linear) and $\omega(k)$ preserves the span $\mathcal{M}_{\mathrm{fin}}$ of the finite Heisenberg translates of the vacuum v^0. Products of finite Heisenberg operators satisfy $\rho(\mathrm{fv}(x))\rho(\mathrm{fv}(x')) = \zeta\,\rho(\mathrm{fv}(x+x'))$ with ζ a value of the finite additive character (WD8), hence a root of unity (a continuous character of $\mathbb{A}_{E,f}$ is trivial on an open subgroup of each compact open subgroup, and every element lies in one).

Step 1: the edge datum as a combination of translated vacua. Let $A = \Sigma(\mathfrak{m}_\vartheta)$ and $T = R^{\mathcal{C}}_{\mathfrak{m}_\vartheta}$. By (LQ4), $\varphi^S_{z,j} = c_s^R\,\Phi^\natural \otimes_A v^0$, where Φ is the indicator function of the point $S(e_{z,j})$ of T. Apply Lemma 6.4.84 to the modulus $\mathfrak{m}_\vartheta$, the index set $\mathcal{C}$ with the signs $+1$ of I_R (Step 0) and $\Psi = \Phi$: it gives $\#T = \mathrm{N}(\mathfrak{m}_\vartheta)^R = (M^{fb}\,\mathrm{N}Q)^R$, as $\mathrm{N}_{E/\mathbb{Q}}(\pi_M) = M^f$, and vectors $u_\chi \in \mathbb{A}^{\mathcal{C}}_{E,f}$, one for each character χ of T, such that, after multiplication by c_s^R,

$$\varphi^S_{z,j} = \sum_\chi a_\chi\,\rho(\mathrm{fv}(u_\chi))\,c_s^R v^0, \qquad a_\chi = \frac{\overline{\chi(S(e_{z,j}))}}{\#T}, \qquad \sum_\chi |\sigma(a_\chi)| \le 1 \quad (\sigma \in \mathrm{Aut}(\mathbb{C})),$$

the bound by (a) of Lemma 6.4.84, as Φ takes the value 1 at one point and 0 elsewhere. Let $\ell \nmid \mathrm{N}Q$ be a prime and $w \in A$ a place above ℓ. Then $w \notin Q$ (each $w' \in Q$ lies above a prime dividing $\mathrm{N}w'$, hence $\mathrm{N}Q$), so $\mathfrak{m}_\vartheta \mathcal{O}_{E,w} = \pi_M^b \mathcal{O}_{E,w}$, and it contains $N_0 = M^b = \pi_M^b \bar\pi_M^b$ (Definition 2.3.4). By (b) of Lemma 6.4.84 with $\delta = N_0$, every $N_0 u_\chi$ is integral at every place above ℓ. With Step 0, $F = c\sum_\chi a_\chi\,\rho(\mathrm{fv}(ku_\chi))\,\omega(k)c_s^R v^0 \in \mathcal{M}_{\mathrm{fin}}$.

Step 2: values of characters of $\mathrm{U}(1)$**.** The edge characters α, β and the splitting character υ (Definition 3.2.12) are automorphic characters of $\mathrm{U}(1)$, and $\det g$, $\det h$, $\det k$ are finite adelic

elements u with $u\bar{u}=1$. By (a) of Lemma 6.4.85, $|\sigma(c)|=1$ for every $\sigma\in\mathrm{Aut}(\mathbb{C})$. If g,h and their inverses are integral at the places above ℓ, then so are k and k^{-1}, so $\det g$, $\det h$, $\det k$ are units at those places; as a prime identification $\iota\mid\ell$ followed by the inclusion of $\overline{\mathbb{Q}}_\ell$ in $\mathbb{C}_\ell$ is a ring homomorphism $\mathbb{C}\to\mathbb{C}_\ell$ (Definition 3.1.3), (b) of Lemma 6.4.85 shows that $\iota(c)$ and $\iota(\upsilon(1,\det k))$ are ℓ-adic units.

Step 3: a contraction bound for rectangular Schur matrices. For a family Y of complex $N'\times N$ matrices and $w\in\mathbb{V}_\kappa$ (rank N), $\|S_\kappa(Y)w\|_\kappa\le|Y|^B\|w\|_\kappa$, the left norm in the rank N' and $|Y|^2=\sum_\tau\sum_{b,a}|Y_{\tau,b,a}|^2$. Indeed, for a filling T' of κ in the rank N', the Cauchy–Binet formula expands each factor $\Delta_{A'}(Yy)$ of height $h'\le N$ as $\sum_A Y_\tau[A';A]\Delta_A(y)$ (A running over the h'-subsets of n; for $h'=N$ only $A=n$); comparing with (WM2) and using the uniqueness of coordinates there gives $F'_\kappa S_\kappa(Y)=\mathfrak{L}(Y)F_\kappa$, where F'_κ, F_κ are the filling matrices in the ranks N', N and $\mathfrak{L}(Y)$ is the matrix, in the standard wedge bases, of the tensor product over the factors of $\Lambda^{h'}Y_\tau$ ($h'<N$) and of $1\mapsto y_1\wedge\cdots\wedge y_N$ (the columns of Y_τ; $h'=N$). Each factor has operator norm at most $\|Y_\tau\|_{\mathrm{op}}^{h'}\le|Y|^{h'}$, so $\|S_\kappa(Y)w\|_\kappa=\|\mathfrak{L}(Y)F_\kappa w\|\le|Y|^B\|F_\kappa w\|=|Y|^B\|w\|_\kappa$ ((WM3)).

Step 4: clause (2). Let ϑ be the trivial twist and $\sigma\in\mathrm{Aut}(\mathbb{C})$. If σ does not fix K, replace it by $\mathrm{conj}\circ\sigma$, which fixes K; as the Gram matrices are real, complex conjugation of coordinates preserves both norms, so the claim for $\mathrm{conj}\circ\sigma$ gives it for σ. So let σ fix K, and let $t_\sigma,\epsilon_\sigma,\Theta=\Theta_{\sigma,F}$ be as in Lemma 4.2.16 for I_R and $F\in\mathcal{M}_{\mathrm{fin}}$ (Step 1). For $x\in\mathbb{A}^{\mathcal{C}}_{E,f}$, Steps 0 and 1 give $\theta_R(\rho(\mathrm{fv}(x))F)=c\sum_\chi a_\chi\zeta_\chi\,\mathcal{V}_R(x+ku_\chi,k)$ with roots of unity ζ_χ and $\mathcal{V}_R(x,k)=\theta_R(\rho(\mathrm{fv}(x))\omega(1,k)(c_s^Rv^0))$; by Steps 1, 2 and (2) of Lemma 6.4.83, $|\sigma\theta_R(\rho(\mathrm{fv}(x))F)|\le e^{A_R}$, $A_R=C_P(q_1(s)R+1)^2$. Since the $P_{i'i}$ have integer coefficients, (b) of Lemma 4.2.16 gives, coordinatewise,

$$\sigma(Jv)=\pi^{-D}\int_{\mathbb{C}^{\Phi\times\mathcal{C}}}S_\kappa\big(t_\sigma\bar{y}^{\epsilon_\sigma}\big)\,\sigma(v)\,e^{-|y|^2/2}\,\Theta(y)\,dy,$$

with $(t_\sigma\bar{y}^{\epsilon_\sigma})_\tau$ the $n'\times n$ matrix with entries $t_\sigma\overline{y_{\epsilon_\sigma(\tau),(b,a)}}$. By (a) of Lemma 4.2.16, $|\Theta|\le e^{A_R}$; by Step 3 and $|t_\sigma\bar{y}^{\epsilon_\sigma}|=|t_\sigma||y|=\Omega_s^{-1/2}|y|$,

$$\|\sigma(Jv)\|_\kappa\le e^{A_R}\,\Omega_s^{-B/2}\,\pi^{-D}\int|y|^Be^{-|y|^2/2}\,dy\;\|\sigma(v)\|_\kappa\le e^{A_R}\,4^D\,\big(2(B+1)\big)^{B/2}\,\Omega_s^{-B/2}\,\|\sigma(v)\|_\kappa,$$

using $|y|^Be^{-|y|^2/4}\le(2B)^{B/2}e^{-B/2}$ and $\pi^{-D}\int e^{-|y|^2/4}dy=4^D$ (for $B=0$, $\pi^{-D}\int e^{-|y|^2/2}dy=2^D$). Numerically, by Lemma 4.2.67: $n_m\le r+m(t+1)\le 3x^5+2\sqrt{x}(15x^5+1)\le x^6$, so $R\le 4x^{12}$; $f\le q_1\le x^{12}$, so $D\le 4x^{24}$ and $q_1R+1\le x^{25}$; $C_P\le\log x$ by (AD1), so $A_R\le x^{51}$ and $D\log 4\le x^{25}$. If $B\ge 1$: $B\le br^4\le 81bx^{20}$, $\log(2(B+1))\le\log(324\,b\,x^{20})\le 2\log b$ (as $x^{10^{12}}\le b$), $\log\Omega_s^{-1}\le\log\mathrm{rd}_E+|c_{\mathrm{CS}}(d)|\le 13\log x\le\log b$ (Lemma 5.2.13), and $\log b\le x^{1/100}$; so $\frac{B}{2}\big(\log(2(B+1))+\log\Omega_s^{-1}\big)\le\frac{3}{2}\cdot 81\,b\,x^{20}x^{1/100}\le bx^{22}$. Altogether the factor is at most $\exp(x^{51}+x^{25}+bx^{22})\le\exp(x^{99}b^2)$, which is (2).

Step 5: integrality at integral points. Let $\ell\nmid\mathrm{N}Q$, $\iota\mid\ell$, $\mathcal{O}_\iota=\{w\in\mathbb{C}:|\iota(w)|_\ell\le 1\}$, and suppose that g,h and their inverses are integral at the places above ℓ; then so are k and k^{-1}. Let $\mathcal{S}$ be the $\mathcal{O}_\iota$-span of the vectors $\rho(\mathrm{fv}(x))c_s^Rv^0$ with N_0x_w integral at every $w\mid\ell$, and $A_0=N_0^{fR}$. By Proposition 4.1.14 the finite operator $\omega(k)$ is the normalised finite kernel of Lemma 4.1.13, so $\omega(k)v^0=\frac{\Delta(k)}{I(k)}\sum_{y\in L/(L\cap k^{-1}L)}\zeta_y\,\rho(\mathrm{fv}(ky))v^0$ with $L=\widehat{\mathcal{O}}^{\mathcal{C}}_E$ and roots of unity ζ_y (the factor $s_L(ky)$ of Lemma 4.1.13 and the phase between its Heisenberg operator and $\rho(\mathrm{fv}(\cdot))$, both values of the finite additive character). Here $I(k)=[L:L\cap k^{-1}L]$ is prime to ℓ (the local indices above ℓ are 1), ky is integral above ℓ, and $\Delta(k)^2=I(k)\,\upsilon(1,\det k)^{-1}$ by $|\Delta(k)|^2=I(k)$ and $\overline{\Delta(k)}=\upsilon(1,\det k)\Delta(k)$, so $\iota(\Delta(k))$ is an ℓ-adic unit by Step 2. Together with Steps 0–2 ($\iota(c)$ a unit; $\iota(A_0a_\chi)\in\mathcal{O}_\iota$, as the ℓ-part of $\#T$ divides $M^{fbR}=A_0$; N_0ku_χ integral above ℓ; products of Heisenberg operators adding roots of unity) this gives $A_0F\in\mathcal{S}$.

By Proposition 4.2.35 with $C = \mathcal{C}$, $\delta = N_0$, $\zeta_{\mathrm{sc}} = 1$ and $Q = P_{i'i}$, and by Lemma 4.2.2 $(Q(\partial_Z)\mathscr{E}_{A_0F}(\eta_P Z)|_{Z=0} = \theta_R(v_{Q(\eta_P\cdot)} \otimes A_0F))$,

$$\left|\iota(A_0\,(N_0d)^B J_{i'i})\right|_\ell \le 1 \qquad \text{for all } i', i. \tag{M}$$

At such points the gauge is the maximum of the ℓ-adic absolute values of the coordinates, since $P_{\kappa,e}(f_\iota(g^{-1}))$ is an integral matrix with integral inverse $P_{\kappa,e}(f_\iota(g))$ (S_κ has integral polynomial entries, $\det f_\iota(g)$ is a unit). Hence, with $D_0 = A_0(N_0d)^B$, $\mathrm{gau}_\iota(h, D_0Jv) \le \mathrm{gau}_\iota(g, v)$.

Step 6: clause (1). Let g, h be arbitrary and $\ell \nmid \mathrm{N}Q$. Let R_ℓ be the set of places of E above ℓ, which is stable under conjugation. The set of $x \in \mathrm{G}(s_+) = G_\infty(s_+) \times G_f(n)$, s_+ the signs $+1$ on n (those of I_n), with $x_w \in g_w\mathrm{GL}_N(\mathcal{O}_{E,w})$ for every $w \in R_\ell$ is open and contains $(1, g)$; by Lemma 4.2.64 for I_n and R_ℓ it contains a product $\gamma'(1, k)$ with γ' a rational point of I_n and $k_w = 1$ for $w \in R_\ell$, so $\gamma'_w \in g_w\mathrm{GL}_N(\mathcal{O}_{E,w})$ there, and $\gamma = \gamma'^{-1}$ is a rational point of $\mathrm{U}(\mathrm{I}_n)$ with $\gamma_f g$ and its inverse integral above ℓ. In the same way there is a rational point δ of $\mathrm{U}(\mathrm{I}_{n'})$ with $\delta_f h$ and its inverse integral above ℓ. Write $J(g, h) = J^S_{z,j}(g, h)$. Since $(1, \gamma_f g) = \gamma \cdot (\gamma_\infty^{-1}, g)$, the rational invariance and the real part of the theta kernel $(\omega_{\alpha,\beta}, \theta_R)$ (Lemma 6.3.77) and (5) of Lemma 4.1.1 (all signs $+1$) give $J(\gamma_f g, \delta_f h)_{i'i} = \det(\bar\delta_\infty)^{e^{\det}_{z,j+1}} \det(\bar\gamma_\infty)^{-\gamma'} \theta_R(v_{Q'} \otimes F)$ with $Q'(X) = P_{i'i}(\eta_P\, \bar\delta_\infty X \bar\gamma_\infty^{-1})$ (for unitary x, $x^{-\mathsf{T}} = \bar x$ and $\det x^{-1} = \det \bar x$). By (WM2), $S_\kappa(zXw) = S_\kappa(z)S_\kappa(X)S_\kappa(w)\prod_\tau \det(w_\tau)^{\kappa(\tau,N)}$ for square z, w of sizes N', N (expand $b'_{i'}(zXwy)$ in three steps), and $e^{\det}_{z,j} = \gamma' + \kappa(\cdot, N)$ by (1) of Lemma 6.4.3; with $\rho_{\mathrm{s}} = \rho_{\kappa, e^{\det}_{z,j}}$, $\rho_{\mathrm{t}} = \rho_{\kappa, e^{\det}_{z,j+1}}$,

$$J(\gamma_f g, \delta_f h) = \rho_{\mathrm{t}}(\delta_\infty)\, J(g, h)\, \rho_{\mathrm{s}}(\gamma_\infty)^{-1}. \tag{N}$$

For a rational point γ, $f_{\iota,\tau}(\gamma_f) = \iota(\tau(\bar\gamma))$ and $\overline{\tau(\gamma)} = \tau(\bar\gamma)$ (E is CM), so $\iota(\rho_{\mathrm{s}}(\gamma_\infty)\xi) = P_{\kappa, e^{\det}_{z,j}}(f_\iota(\gamma_f))\iota(\xi)$, and multiplicativity of $P_{\kappa,e}$ and of f_ι gives $\mathrm{gau}_\iota(\gamma_f g, \rho_{\mathrm{s}}(\gamma_\infty)\xi) = \mathrm{gau}_\iota(g, \xi)$; likewise at the target. By these identities, (N) and Step 5 at the integral points $\gamma_f g$, $\delta_f h$,

$$\mathrm{gau}_\iota(h, D_0J(g, h)v) = \mathrm{gau}_\iota\big(\delta_f h, D_0J(\gamma_f g, \delta_f h)\rho_{\mathrm{s}}(\gamma_\infty)v\big) \le \mathrm{gau}_\iota(\gamma_f g, \rho_{\mathrm{s}}(\gamma_\infty)v) = \mathrm{gau}_\iota(g, v).$$

Finally $d_J/D_0 = N_0^{2f(2n_m)^2 - fR}(N_0d)^{br^4 - B}$ is a positive integer, as $R \le (2n_m)^2$ and $B \le br^4$, and multiplication by an integer does not increase an ℓ-adic gauge. This is (1). □

Lemma 6.4.83. *For every integer $N \ge 0$, let I_N be the standard positive hermitian space over E_s, with Gram matrix and framed point both the identity, and let $v_{0,N}$ be its unit vacuum. For arbitrary $x \in \mathbb{A}^N_{E_s,f}$ and $k \in G_f(N)$ put*

$$\mathcal{V}_N(x, k) = \theta_{I_N}(\rho(\mathrm{fv}\, x)\omega(1, k)(c_s^N v_{0,N})).$$

Then:

1. *For every prime ℓ and every ring homomorphism $\iota : \mathbb{C} \to \mathbb{C}_\ell$, $\|\iota(\mathcal{V}_N(x, k))\|_\ell \le 1$.*
2. *For every field automorphism σ of $\mathbb{C}$,*

$$|\sigma(\mathcal{V}_N(x, k))| \le \exp\big(C_P(q_1(s)N + 1)^2\big).$$

The choices of c_s, theta functional and vacuum are fixed before $N, x, k, \ell, \iota, \sigma$. Rank zero uses the empty tensor and has value 1. These assertions concern theta values in standard positive spaces; they assert neither a bound on raw coherent coefficients nor an arbitrary-signature statement.

Proof. Put $R = N$ and $q = q_1(s)$. Rank zero is immediate, so assume $R > 0$. Throughout the proof all spaces are standard positive and all theta values use the functional in the statement. The coherent action, its unitarity and covariance are Lemma 4.1.1; tensor theta factorization and rational invariance are Lemma 4.1.17. In normalized finite coordinates write $\mathsf{R}_f(x) = \rho(\mathrm{fv}\, x)$, $L = \widehat{\mathcal{O}}_E^R$. These coordinates are the canonical finite coordinates, with the fixed theta functional and vacuum scale.

We first identify its finite factor without any scalar ambiguity. A finite-part vector has coefficient $e^{-|x_\infty|^2/2}\varphi(x_f/\alpha)$ in the normalized coordinates of (WD6). For a finite unitary g, (WR3) gives a product integral. The real factor is

$$e^{-|x_\infty|^2/2}\pi^{-fR}\int_{\mathbb{C}^{fR}} e^{-|u|^2+\bar{x}_\infty\cdot u}\,du = e^{-|x_\infty|^2/2}.$$

The equality follows coordinatewise by angular integration of the exponential; its nonconstant holomorphic monomials have integral zero, and Gaussian domination justifies the expansion. The finite factor, with $L = \widehat{\mathcal{O}}_E^R$, is therefore exactly

$$(\Omega(g)\varphi)(z) = \Delta(g)\int \varphi(y)\psi_0(z^*gy/2)v_0(z-gy)\,d\mu(y), \qquad \mu(L) = 1,$$

where $\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$, $v_0(z) = \psi_0(c_0 z^* z/2)\mathbf{1}_L(z)$ and $\Delta(g) = \upsilon_f(\det u_g)|\det u_g|_f^{-1/2}$, $u_g L = L \cap g^{-1}L$. This is the literal operator of Lemma 4.1.13. Thus its vacuum expansion and scalar identities apply to the selected action itself.

The hypotheses of Lemma 4.1.13 hold in these coordinates. Indeed $c_0 = (1-\jmath^{-1})/2 = (1+\sqrt{-d})/2 \in \mathcal{O}_K$. The prescribed ideal $\alpha\bar{\alpha} = \mathfrak{D}_{E/K}^{-1}$ and $\mathfrak{D}_{E/\mathbb{Q}} = \sqrt{-d}\mathfrak{D}_{E/K}$ show that $\jmath\alpha\bar{\alpha}$ generates the absolute inverse different locally, so the trace character has integral self-dual lattice $\widehat{\mathcal{O}}_E$. The splitting character is unitary and trivial on finite integral units. Also $\upsilon_f(\bar{t}) = \upsilon_f(t)^{-1}$: their product is its restriction to the norm idele $t\bar{t}$, and the quadratic character of E/F is trivial on local norms. These are precisely the character and lattice hypotheses of that lemma.

Finite integrality and positive rational transport. The rank-one property of the SAME c_s supplied by Proposition 4.2.1, with its fixed-scale specification, and exact tensor factorization give

$$\theta(\mathsf{R}_f(x)c_s^R v^0) = \prod_i c_s\theta_+(\mathsf{R}_f(x_i)v_+^0).$$

Every factor is integral at every prime under every embedding. For $k \in G_f(R)$ put $J = L\cap k^{-1}L$, $I = [L:J]$. The finite expansion just identified gives

$$\theta(\mathsf{R}_f(x)\omega(k)c_s^R v^0) = \frac{\Delta(k)}{I}\sum_{y\in L/J}\xi_{x,y}\theta(\mathsf{R}_f(x+ky)c_s^R v^0),$$

where every $\xi_{x,y}$ is a value of the finite additive character, hence a root of unity. In particular multiplying this expression by $I/\Delta(k)$ makes it integral at every prime. If k and its inverse are integral at the places above ℓ, then $\ell \nmid I$. The splitting-character value $\upsilon_f(\det k)$ is an ℓ-unit under every embedding: principalize a power of the determinant ideal, then divide its generator by its conjugate. The resulting norm-one element is a unit above ℓ, and the principal-idele character relation expresses a positive power of the character value as a root of unity times integral powers of its algebraic conjugates. Those conjugates are ℓ-units. Since $\upsilon_f(\det k)\Delta(k)^2 = I$, the factor $\Delta(k)/I$ is also an ℓ-unit. The ultrametric inequality proves integrality for such k.

For a general k, we justify the required rational approximation explicitly. Put $F = E^+$. At each $v \mid \ell$ choose a unitary u_v arbitrarily close to k_v^{-1} with $1+u_v$ invertible. This is possible

by multiplying k_v^{-1} by the scalar $(1+t_v\jmath)/(1-t_v\jmath)$ for sufficiently small $t_v \in F_v$, avoiding the finitely many zeros of its determinant. The Cayley matrix $B_v = (1-u_v)(1+u_v)^{-1}$ is skew hermitian. Such matrices form an F-linear space. Number-field weak approximation therefore supplies a single rational skew-hermitian B close to all B_v. Arrange that $1+B$ is invertible and put $\gamma = (1-B)(1+B)^{-1} \in U(I_R)(F)$. Then $\gamma_v k_v$ and its inverse are integral at all $v \mid \ell$.

There is no archimedean multiplier in this step. In (WR3) the negative block is empty, so its real multiplier is 1. Changing variables by the unitary γ_∞ preserves the Gaussian, the measure and the reproducing kernel, and gives $\omega(\gamma_\infty)v^0 = v^0$ exactly. It also fixes every finite translate of v^0. Rational theta invariance and Heisenberg covariance consequently give the exact identity

$$\theta(\mathsf{R}_f(x)\omega(1,k)c_s^R v^0) = \theta(\mathsf{R}_f(\gamma_f x)\omega(1,\gamma_f k)c_s^R v^0).$$

The integral-operator case applies to its right side at ℓ. Since ℓ and the embedding were arbitrary, this proves (1). This argument is why the assertion is restricted to the standard positive space.

Conjugate theta values. Apply Lemma 4.2.14(CT10) in rank $R = N$, with the SAME parameter pair P, field E_s, standard positive space I_R, fixed theta functional, finite coordinates, scale c_s and constant C_P as in the statement. Its hypotheses are exactly those already fixed here: $R \geq 1$, x is an arbitrary finite vector, and k is an arbitrary finite unitary element. Its conclusion applies to every automorphism σ of $\mathbb{C}$ and reads

$$\left|\sigma\theta_{\mathrm{I}_R}(\mathsf{R}_f(x)\omega(1,k)c_s^R v_{0,R})\right| \leq \exp\!\big(C_P(q_1(s)R+1)^2\big).$$

This theta value is precisely $\mathcal{V}_N(x,k)$ in the statement. In particular, this application retains the fixed scaled vacuum: the scalar bounded in (CT8) of that lemma is $\sigma(c_s^R)C_{\sigma,R,\mathcal{A}}$, rather than the unscaled transport scalar. The map and all its factors in that lemma are chosen before x and k; neither this application nor its bound reselects c_s or the theta normalization. The all-automorphism scope is expressly (CT10), so no extension of its semilinear-map assertion beyond automorphisms fixing K is needed. This proves (2). Rank zero was the empty tensor case at the start. □

Lemma 6.4.84. *Let $\mathfrak{m}$ be a modulus, $A = \Sigma(\mathfrak{m})$, C a finite index set, s signs on C, $T = R_{\mathfrak{m}}^C$ and $\widehat{T}$ the group of characters of T. Then $\#T = \mathrm{N}(\mathfrak{m})^{\#C}$, and there are vectors $u_\chi \in \mathbb{A}_{E,f}^C$ ($\chi \in \widehat{T}$), with components 0 at every place outside $\bar{A}$, such that for every function Ψ on T, in $\mathcal{M}_s$,*

$$\Psi^\natural \otimes_A v^0 = \sum_{\chi\in\widehat{T}} a_\chi(\Psi)\,\rho(\mathrm{fv}(u_\chi))\,v^0, \qquad a_\chi(\Psi) = \frac{1}{\#T}\sum_{X\in T}\Psi(X)\,\overline{\chi(X)}.$$

Moreover:

- *(a) $\sum_\chi |\sigma(a_\chi(\Psi))| \leq \sum_{X\in T}|\sigma(\Psi(X))|$ for every automorphism σ of $\mathbb{C}$;*
- *(b) if p is a prime and $\delta \geq 1$ an integer with $\delta \in \mathfrak{m}\mathcal{O}_{E,w}$ for every $w \in A$ above p, then every δu_χ has integral components at every place of E above p.*

Proof. Put $\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$, $Q_A = \prod_{w\in A} E_w^C$ and $L_A = \prod_{w\in A}\mathcal{O}_{E,w}^C$. By Lemma 4.1.3, $\widehat{\mathcal{O}}_E^C$ is its own annihilator under $(u,x) \mapsto \psi_0(\bar{u}^{\mathsf{T}}x)$. Since A is the set of places of $\mathfrak{m}$, the reduction map identifies $L_A/\mathfrak{m}L_A$ with T (Chinese remainder theorem), and $\#T = \#(\mathcal{O}_E/\mathfrak{m})^{\#C} = \mathrm{N}(\mathfrak{m})^{\#C}$.

Step 1: the characters of T. Let $\bar{\mathfrak{m}}$ be the conjugate ideal and $U = \bar{\mathfrak{m}}^{-1}\widehat{\mathcal{O}}_E^C$. For $u \in U$ and $x \in L_A$ (read as a finite adelic vector with components 0 off A) put $\chi_u(x) = \psi_0(\bar{u}^{\mathsf{T}}x)$, a character of L_A. For $\mu \in \mathfrak{m}$ and $x \in L_A$, $\bar{u}^{\mathsf{T}}(\mu x) = \overline{(\bar{\mu}u)}^{\mathsf{T}}x$ with $\bar{\mu}u \in \widehat{\mathcal{O}}_E^C$, so $\chi_u(\mu x) = 1$ by

self-duality; as $\mathfrak{m}L_A$ is generated by such μx, χ_u is trivial on $\mathfrak{m}L_A$, and χ_u is a character of T; it is trivial when $u \in \widehat{\mathcal{O}}_E^C$. So $u \mapsto \chi_u$ is a homomorphism $U/\widehat{\mathcal{O}}_E^C \to \widehat{T}$. It is injective: as $\bar{\mathfrak{m}}^{-1}\mathcal{O}_{E,w'} = \mathcal{O}_{E,w'}$ for $w' \notin \bar{A}$, every class has a representative u with components 0 off $\bar{A}$; then $\bar{u}$ vanishes off A, so $\psi_0(\bar{u}^{\mathsf{T}}x) = \chi_u(x_A)$ for every $x \in \widehat{\mathcal{O}}_E^C$, and if $\chi_u = 1$ then u annihilates $\widehat{\mathcal{O}}_E^C$, so $u \in \widehat{\mathcal{O}}_E^C$. Both groups have $\mathrm{N}(\mathfrak{m})^{\#C}$ elements ($\#(\bar{\mathfrak{m}}^{-1}\widehat{\mathcal{O}}_E/\widehat{\mathcal{O}}_E) = \mathrm{N}(\bar{\mathfrak{m}}) = \mathrm{N}(\mathfrak{m})$, and $\widehat{T} \cong T$), so the map is bijective. For each $\chi \in \widehat{T}$ let $u_\chi \in U$ be the representative with components 0 off $\bar{A}$ of the class with $\chi_{u_\chi} = \chi$.

Step 2: the expansion. By Fourier inversion on the finite abelian group T, $\Psi = \sum_\chi a_\chi(\Psi)\chi$. Read as a function on Q_A, $\Psi^\natural$ is $x \mapsto \Psi(x \bmod \mathfrak{m})$ on L_A and 0 off L_A, so $\Psi^\natural = \sum_\chi a_\chi(\Psi)\,(\chi \circ \mathrm{red})\,\mathbf{1}_A$, with $\mathbf{1}_A$ the indicator of L_A. In the notation of (2) of Lemma 4.1.3, $(R_A(u_\chi)\mathbf{1}_A)(x) = \psi_0(\bar{u}_\chi^{\mathsf{T}}(2x + u_\chi)_A/2)\mathbf{1}_A(x + (u_\chi)_A) = \psi_0(\bar{u}_\chi^{\mathsf{T}}x_A)\mathbf{1}_A(x) = \chi(x \bmod \mathfrak{m})\mathbf{1}_A(x)$, because $(u_\chi)_A = 0$ (A contains no conjugate of its members). The map $\Psi' \mapsto \Psi' \otimes_A v^0$ is linear ((WR6): the section is linear), and by (2) and (3) of Lemma 4.1.3, $(R_A(u_\chi)\mathbf{1}_A) \otimes_A v^0 = \rho(\mathrm{fv}(u_\chi))(\mathbf{1}_A \otimes_A v^0) = \rho(\mathrm{fv}(u_\chi))v^0$. This gives the displayed identity.

Step 3: (a). The values of χ are roots of unity, so $|\sigma(\overline{\chi(X)})| = 1$ and $|\sigma(a_\chi(\Psi))| \le (\#T)^{-1}\sum_X |\sigma(\Psi(X))|$; summing over the $\#\widehat{T} = \#T$ characters gives (a).

Step 4: (b). The components of u_χ vanish except at the places $\bar{w}$, $w \in A$. A place of E above p at which u_χ does not vanish is therefore $\bar{w}$ with $w \in A$ above p; there $\delta \in \mathfrak{m}\mathcal{O}_{E,w}$ gives $\delta = \bar{\delta} \in \bar{\mathfrak{m}}\mathcal{O}_{E,\bar{w}}$, so $\delta(u_\chi)_{\bar{w}} \in \delta\bar{\mathfrak{m}}^{-1}\mathcal{O}_{E,\bar{w}}^C \subseteq \mathcal{O}_{E,\bar{w}}^C$. □

Lemma 6.4.85. *Let χ be an automorphic character of $\mathrm{U}(1)$ and let $u \in \mathbb{A}_{E,f}$ with $u\bar{u} = 1$, read as the element of $\mathrm{U}(1)(\mathbb{A})$ with components 1 at the infinite places. Then:*

(a) $|\sigma(\chi(u))| = 1$ for every automorphism σ of $\mathbb{C}$;

(b) if p is a prime such that the component of u at every place of E above p is a unit, then $|\iota(\chi(u))|_p = 1$ for every ring homomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_p$.

Proof. **Step 1: the infinite part and an open subgroup in the kernel.** For $v \in \Phi$ the component group $\{z \in \mathbb{C} : z\bar{z} = 1\}$ is the circle, and the restriction of χ to it is a continuous character, so it is $z \mapsto z^{n(v)}$ for an integer $n(v)$. The restriction of χ to the finite part $\mathrm{U}(1)(\mathbb{A}_f) = \{u \in \mathbb{A}_{E,f} : u\bar{u} = 1\}$ is continuous, so some open subgroup U_1 of the compact open group $\mathrm{U}(1)(\widehat{\mathcal{O}}_F) = \mathrm{U}(1)(\mathbb{A}_f) \cap \widehat{\mathcal{O}}_E^\times$ is mapped into $\{z : |z| = 1,\ \mathrm{Re}\, z > 0\}$; its image is a subgroup of the circle contained in that arc, hence trivial. So χ is trivial on U_1.

Step 2: a power of u is rational up to U_1. By (K) of Theorem A.4.1 for the torus $G = \mathrm{U}(1)$ over F and $K_f = U_1$, the group $\mathrm{U}(1)(\mathbb{A})$ is the union of finitely many sets

$$\mathrm{U}(1)(F)x_i\mathrm{U}(1)(F_\infty)U_1;$$

projecting to the finite part, the abelian group $\mathrm{U}(1)(F)\backslash\mathrm{U}(1)(\mathbb{A}_f)/U_1$ is finite. Hence there are an integer $q \ge 1$, $\gamma \in E$ with $\gamma\bar{\gamma} = 1$ and $u_1 \in U_1$ with $u^q = \gamma_f u_1$, where γ_f is the finite part of the diagonal image of γ.

Step 3: the value. As χ is trivial on the diagonal image $(\gamma_\infty, \gamma_f)$ of γ and on U_1,

$$\chi(u)^q = \chi((1, \gamma_f)) = \chi((\gamma_\infty, 1))^{-1} = \prod_{v \in \Phi} v(\gamma)^{-n(v)},$$

$v(\gamma)$ the image of γ under the embedding chosen at v. So $\chi(u)$ is algebraic.

(a) For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $v \in \Phi$, $\sigma \circ v$ is an embedding of the CM field E into $\mathbb{C}$, so it commutes with complex conjugation and $|\sigma(v(\gamma))|^2 = (\sigma \circ v)(\gamma\bar{\gamma}) = 1$. Hence $|\sigma(\chi(u))|^q = |\sigma(\chi(u)^q)| = \prod_v |\sigma(v(\gamma))|^{-n(v)} = 1$, and $|\sigma(\chi(u))| = 1$.

(b) At every place w of E above p, $\gamma = u^q u_1^{-1}$ has a unit component (u_w is a unit by hypothesis, $u_{1,w}$ as $u_1 \in \widehat{\mathcal{O}}_E^\times$). For $v \in \Phi$, $\iota \circ v$ is an embedding of E into $\mathbb{C}_p$, and $|\iota(v(y))|_p$ is the absolute value of y at a place of E above p; so $|\iota(v(\gamma))|_p = 1$, and $|\iota(\chi(u))|_p^q = \prod_v |\iota(v(\gamma))|_p^{-n(v)} = 1$. □

Lemma 6.4.86. *Let D be admissible data, $f = \#\Phi$, $G = \exp(x^{99}b^2)$, $V_1 = \#(\mathcal{O}_E/\mathfrak{b})^{2n_m}$ and $T_1 = \#(\mathcal{O}_E/\mathfrak{b})^{8mn_m^2}$. Then:*

(1) for $z \in \{\mathrm{I}, \mathrm{II}\}$ and $0 \le j \le m$, the number of double cosets $\Gamma_{n_{z,j}} \backslash G_f(n_{z,j}) / \mathrm{K}_1(z,j)$ (for the finite components of the rational points) is at most G;

(2) $2m \log d_J + fm\,(br^4 + 20b) \log(M^b q_1) \le x^{99}b^2$;

(3) $V_1^{m^2}\, T_1^2\, G^{4m+1}\, \left((br^4)^{br^4}\right)^3 \le \exp(x^{200}b^2)$.

Proof. We use, from Lemma 4.2.67 and Lemma 2.3.6: $x \ge e^{6.4\cdot 10^{15}}$, $(\log b)^{100} \le x$, $x^{10^{12}} \le b$, $r \le 3x^5$, $m \le 2\sqrt{x}$, $t \le 15x^5$, $f \le q_1 \le x^{12}$, $\log M \le 22 \log b$, and $d \le C_P \le \log x$. Hence $n_m = r + 1 + mt \le 34x^{11/2}$ and every rank $N_{z,j}$ ($j \le m$) satisfies $N_{z,j} \le r + 1 + 2mt \le 2n_m \le 68x^{11/2}$. As $[E : K] = f$, the norm of π_M from E to $\mathbb{Q}$ is M^f, so $\#(\mathcal{O}_E/\mathfrak{b}) = M^{fb}$ and $\log \#(\mathcal{O}_E/\mathfrak{b}) = fb \log M \le 22x^{12} b \log b$. Below, $\log b \le x^{1/100}$ and $b \ge x^{10^{12}}$ absorb all factors x^c with $c \le 10$.

(1). Let $n = n_{z,j}$. The level $\mathrm{K}_1(z,j)$ is the kernel of the reduction $\mathbf{K}_0(n) \to \mathrm{GL}_n(\mathcal{O}_E/\mathfrak{b})$ ((LQ2), $\mathfrak{m}_1 = \mathfrak{b}$), so $[\mathbf{K}_0(n) : \mathrm{K}_1(z,j)] \le \#(\mathcal{O}_E/\mathfrak{b})^{(\#n)^2}$, and every double coset $\Gamma x \mathbf{K}_0(n)$ is the union of the at most $[\mathbf{K}_0(n) : \mathrm{K}_1(z,j)]$ double cosets $\Gamma x k \mathrm{K}_1(z,j)$, $k \in \mathbf{K}_0(n)/\mathrm{K}_1(z,j)$. By Lemma 4.6.4, applied with $1 \le \#n \le 2n_m$ to the standard definite space I_n and $\mathrm{K} = \mathbf{K}_0(n)$, the stabiliser of $\widehat{\mathcal{O}}_E^n$, which contains its principal congruence subgroup of level $\pi_M \pi_N$, the number of double cosets $\Gamma x \mathbf{K}_0(n)$ is at most $\exp(x^{30})$; so the logarithm of the number of double cosets for $\mathrm{K}_1(z,j)$ is at most

$$x^{30} + (68x^{11/2})^2 \cdot 22x^{12} b \log b \le x^{30} + 1.1 \cdot 10^5 x^{23} b \log b \le x^{99} b^2.$$

(2). With $\log d \le \log x$,

$$\begin{aligned} \log d_J &= 2f(2n_m)^2 b \log M + br^4(b \log M + \log d) \\ &\le 2.1 \cdot 10^5 x^{23} b \log b + 1782\, x^{20} b^2 \log b + 81 x^{20} b \log x \le 2000\, x^{20} b^2 \log b, \end{aligned}$$

so $2m \log d_J \le 8000\, x^{41/2} b^2 \log b \le x^{21} b^2$. Further $fm \le 2x^{25/2}$, $br^4 + 20b \le 82x^{20}b$ and $\log(M^b q_1) \le 22 b \log b + 12 \log x \le 23 b \log b$, so $fm(br^4 + 20b) \log(M^b q_1) \le 3772\, x^{65/2} b^2 \log b \le x^{33} b^2$. The sum is at most $2x^{33}b^2 \le x^{99}b^2$.

(3). The logarithm of the left side is $m^2 \log V_1 + 2 \log T_1 + (4m+1)x^{99}b^2 + 3br^4 \log(br^4)$. Here $m^2 \log V_1 = 2m^2 n_m f b \log M \le 4x \cdot 68x^{11/2} \cdot 22x^{12} b \log b \le 6000\, x^{37/2} b \log b$, $2 \log T_1 = 16 m n_m^2 f b \log M \le 16 \cdot 2\sqrt{x} \cdot 1156 x^{11} \cdot 22 x^{12} b \log b \le 9 \cdot 10^5 x^{47/2} b \log b$, $(4m+1)x^{99}b^2 \le 9x^{199/2}b^2$, and, as $\log(br^4) \le \log b + \log 81 + 20 \log x \le 2 \log b$, $3br^4 \log(br^4) \le 486\, x^{20} b \log b$. The total is at most $10\, x^{199/2} b^2 \le x^{200} b^2$. □

The next group moves periods from the first path to the last (Proposition 6.4.87), in m^2 turns, one at each position (i,j), with bound V^{m^2}, where $V(\vartheta) = \#(\mathcal{O}_E/\mathfrak{m}_\vartheta)^{2n_m}$, and with a constant whose absolute value is $\prod_{0 \le i,j < m} \tau_{ij}$ up to a norm power. A turn off the diagonal is the square law of a unit square (Lemma 6.4.89), an instance of the one-square identity, and costs one turn ratio. A turn on the diagonal also moves the meeting node of the two halves: two seesaws

in the sense of Kudla [171] compare the periods with periods over the subdiagonal space, and between them the line added at the diagonal is reversed by a Fourier transform (Lemma 6.4.94; see [322]).

Proposition 6.4.87. *Let D be admissible data and $V(\vartheta) = \#(\mathcal{O}_E/\mathfrak{m}_\vartheta)^{2n_m}$. There is a function c from the m-fold twists to $\mathbb{C}^\times$, of absolute value $\prod_{0\le i,j<m}\tau_{ij}$ up to a norm power, such that the first path transfers periods to the last path with bound V^{m^2} and constant c.*

Proof. **Step 1: transfers compose.** (i) $\Gamma \xrightarrow{1,1} \Gamma$ for every path Γ: given path data Ψ' of bound N on the edges of Γ, take $\Psi = \Psi'$. (ii) If $\Gamma' \xrightarrow{k,c} \Gamma$ and $k \le k'$ pointwise, then $\Gamma' \xrightarrow{k',c} \Gamma$, since a sum of at most kN roots of unity is a sum of at most $k'N$ of them (add zeros: 0 is the empty sum), so (PD4) passes to the larger bound and (PD1)–(PD3) do not involve it. (iii) If $\Gamma_1 \xrightarrow{k,c} \Gamma_2$ and $\Gamma_2 \xrightarrow{k',c'} \Gamma_3$, then $\Gamma_1 \xrightarrow{kk',cc'} \Gamma_3$: path data Ψ_1 of bound N on Γ_1 give Ψ_2 of bound kN on Γ_2 and then Ψ_3 of bound $k'kN$ on Γ_3; if $\Psi_3 \propto_{\Gamma_3} \Xi$, then $\Psi_2 \propto_{\Gamma_2} \Xi/c'$, and applying the first transfer to the function Ξ/c' gives $\Psi_1 \propto_{\Gamma_1} \Xi/(cc')$.

Step 2: one turn. For $s = (i,j)$ with $0 \le i,j < m$ let c_s be the function of Lemma 6.4.88 if $i \ne j$ and that of Lemma 6.4.93 if $i = j$; it has absolute value τ_{ij} up to a norm power. Every turn at s from a path Γ' to a path Γ gives $\Gamma' \xrightarrow{V,c_s} \Gamma$: for $i = j$ this is Lemma 6.4.93; for $i \ne j$, Lemma 6.4.88 gives bound 1, and $1 \le V$ because $V(\vartheta)$ is a positive integer, so Step 1 (ii) applies. The function c_s depends only on s, not on the path.

Step 3: the squares of a path. For a half p (a monotone path from $(0,m)$ to (l,l)) let $\mathrm{sq}(p)$ be the union, over the steps down $(a,b+1) \to (a,b)$ of p, of the sets $\{(i,b) : 0 \le i < a\}$. For a path Γ with $l = l(\Gamma)$ put

$$\Sigma(\Gamma) = [0,l)^2 \ \cup\ \mathrm{sq}(\Gamma_{\mathrm{I}}) \ \cup\ \{(j,i) : (i,j) \in \mathrm{sq}(\Gamma_{\mathrm{II}})\},$$

the unit squares between the first path and Γ, named by their least corner. Along a half the first index rises from 0 to l and the second falls from m to l, so every step down $(a,b+1) \to (a,b)$ has $a \le l \le b$ and $b+1 \le m$; hence

$$\mathrm{sq}(p) \subseteq [0,l) \times [l,m) \quad \text{for every half } p, \qquad \text{and} \qquad \Sigma(\Gamma) \subseteq [0,m)^2. \tag{B}$$

We prove three facts.

(C1) $\Sigma(\text{first}) = \emptyset$ and $\Sigma(\text{last}) = [0,m)^2$. The first path has $l = 0$ and halves whose steps down all lie in the column $a = 0$, so all sets $\{(i,b) : i < 0\}$ are empty. The last path has $l = m$ and halves without steps down.

(C2) If Γ has a turn at s to Γ_1, then $s \in [0,m)^2$, $s \notin \Sigma(\Gamma)$ and $\Sigma(\Gamma_1) = \Sigma(\Gamma) \cup \{s\}$. Suppose first that the two coordinates of s differ. By Lemma 6.4.92 (a), applied to the turn from Γ to Γ_1, there are z and $a < b < m$ (the (z,i',j') there), with $s = (a,b)$ if $z = \mathrm{I}$ and $s = (b,a)$ if $z = \mathrm{II}$, such that Γ_z is a path q to $(a,b+1)$, the steps $(a,b+1) \to (a,b) \to (a+1,b)$, and a path r from $(a+1,b)$ to (l,l); the half $(\Gamma_1)_z$ has the steps $(a,b+1) \to (a+1,b+1) \to (a+1,b)$ instead, $l(\Gamma_1) = l$, and the other half is unchanged. As $(a+1,b)$ is a node of Γ_z and along a half the first index rises from 0 to l while the second falls from m to l, $a+1 \le l \le b$. The steps down of q and r are unchanged; the old step down contributes $\{(i,b) : i < a\}$ and the new one $\{(i,b) : i < a+1\}$, so $\mathrm{sq}((\Gamma_1)_z) = \mathrm{sq}(\Gamma_z) \cup \{(a,b)\}$. Moreover $(a,b) \notin \mathrm{sq}(\Gamma_z)$: the steps down of q end in rows $\ge b+1$, those of r in rows $< b$, and the old corner gives only $i < a$. The other two parts of $\Sigma(\Gamma)$ do not contain s: an element of $[0,l)^2$ has both coordinates $< l \le b$, and by (B) an element of sq of the other half has first coordinate

$< l \le b$ (so after the exchange of coordinates for $z = \mathrm{I}$ its second coordinate is $< b$). This gives $s \notin \Sigma(\Gamma)$ and $\Sigma(\Gamma_1) = \Sigma(\Gamma) \cup \{s\}$. Suppose next that $s = (j, j)$. By Lemma 6.4.92 (b), $l(\Gamma) = j$, both halves are paths q_z (the u_z there) to $(j, j+1)$ followed by the step down to (j, j), and Γ_1 has $l = j + 1$ and halves q_z followed by the step right to $(j + 1, j + 1)$. Then $\mathrm{sq}(\Gamma_z) = \mathrm{sq}(q_z) \cup \{(i, j) : i < j\}$ and $\mathrm{sq}((\Gamma_1)_z) = \mathrm{sq}(q_z)$, and

$$[0, j + 1)^2 = [0, j)^2 \cup \{(i, j) : i < j\} \cup \{(j, i) : i < j\} \cup \{(j, j)\},$$

so $\Sigma(\Gamma_1) = \Sigma(\Gamma) \cup \{(j, j)\}$. Every step down of q_z lies in a column $a \le j$, so $\mathrm{sq}(q_z)$ has first coordinates $< j$; hence $(j, j) \notin \Sigma(\Gamma)$. In both cases $s \in \Sigma(\Gamma_1) \subseteq [0, m)^2$ by (B).

(C3) Every path $\Gamma \neq \mathrm{last}$ has a turn. If some half has a step down followed at once by a step right, that corner is a turn. Otherwise in each half all steps right precede all steps down, so each half is l steps right followed by $m - l$ steps down, $l = l(\Gamma)$. If $l = m$, both halves run along the row m and $\Gamma = \mathrm{last}$. If $l < m$, both halves end with the step down $(l, l + 1) \to (l, l)$, and the turn on the diagonal at (l, l) applies.

Step 4: from every path to the last. For a path Γ put $T(\Gamma) = [0, m)^2 \setminus \Sigma(\Gamma)$ and $c_T = \prod_{s \in T} c_s$. We show by strong induction on $k = \#T(\Gamma)$ that

$$\Gamma \xrightarrow{V^{\#T(\Gamma)},\, c_{T(\Gamma)}} \mathrm{last}$$

for every path Γ. If $\Gamma = \mathrm{last}$, then $T(\Gamma) = \emptyset$ by (C1), and Step 1 (i) gives the claim with bound $V^0 = 1$ and the empty product 1. Otherwise (C3) gives a turn at some s from Γ to a path Γ_1, and by (C2) $s \in T(\Gamma)$ and $T(\Gamma) = T(\Gamma_1) \cup \{s\}$ with $s \notin T(\Gamma_1)$; so $\#T(\Gamma_1) = k - 1$. By Step 2, $\Gamma \xrightarrow{V, c_s} \Gamma_1$; by the induction hypothesis, $\Gamma_1 \xrightarrow{V^{k-1}, c_{T(\Gamma_1)}} \mathrm{last}$; by Step 1 (iii), $\Gamma \xrightarrow{V^k,\, c_s c_{T(\Gamma_1)}} \mathrm{last}$, and $c_s c_{T(\Gamma_1)} = c_{T(\Gamma)}$. For $\Gamma = \mathrm{first}$, (C1) gives $T(\mathrm{first}) = [0, m)^2$, with m^2 elements: the first path transfers periods to the last with bound V^{m^2} and constant $c = \prod_{0 \le i, j < m} c_{ij}$, which takes values in $\mathbb{C}^\times$.

Step 5: the absolute value. For each $s = (i, j)$ there are $C_s \in \mathbb{R}$ and $k_s \in \mathbb{Z}$ with $|c_s(\vartheta)|^2 = C_s \, \mathrm{N}Q(\vartheta)^{k_s} \, \tau_{ij}(\vartheta)^2$ for every ϑ. Multiplying over the m^2 pairs,

$$|c(\vartheta)|^2 = \Big(\prod_s C_s\Big) \, \mathrm{N}Q(\vartheta)^{\sum_s k_s} \Big(\prod_{0 \le i, j < m} \tau_{ij}(\vartheta) \Big)^2,$$

so c has absolute value $\prod_{0 \le i, j < m} \tau_{ij}$ up to a norm power. □

Lemma 6.4.88. *Let D be admissible data and $0 \le i, j < m$ with $i \neq j$. There is a function c_{ij} from the m-fold twists to $\mathbb{C}^\times$, of absolute value τ_{ij} up to a norm power, such that for every turn at (i, j) from a path Γ' through the two grids to a path Γ, the path Γ' transfers periods to Γ with bound 1 and constant c_{ij}.*

Proof. For a monotone path u in the grid z and an edge tuple S let $\mathrm{fm}_u(S)$ be the composite, along u, of the moves by the matrices of S: the transposed move

$$\mathrm{mv}_{((z,a',b'),(z,a',b'+1))}(S((z, a', b'), (z, a', b' + 1)))$$

for a step down from $(a', b' + 1)$ to (a', b'), the lift move

$$\mathrm{mv}_{((z,a',b'),(z,a'+1,b'))}(S((z, a', b'), (z, a' + 1, b')))$$

for a step right from (a',b'); so $\mathrm{fm}_u(S)$ depends only on the matrices of S on the edges of u, and the map along a concatenation of two paths is the composite of the two maps.

Step 1: the square and the constant. Put $(z,i',j') = (\mathrm{I},i,j)$ if $i<j$ and $(z,i',j') = (\mathrm{II},j,i)$ if $i>j$; then $i'<j'<m$. Apply Lemma 6.4.89 to the grid z and the square at (i',j'). It gives $\kappa\in\mathbb{C}$, a bijection $\sigma\colon n_{(z,i',j')}\sqcup n_{(z,i'+1,j'+1)}\to n_{(z,i',j'+1)}\sqcup n_{(z,i'+1,j')}$ and a function c from the m-fold twists to $\mathbb{C}^\times$ of absolute value τ up to a norm power, where $\tau=\tau_{i'j'}=\tau_{ij}$ if $z=\mathrm{I}$ and $\tau=\tau_{j'i'}=\tau_{ij}$ if $z=\mathrm{II}$. Put $c_{ij}=c$; it depends only on (i,j).

Step 2: the shape of a turn at (i,j). Let Γ arise from Γ' by a turn at (i,j), and put $l=l(\Gamma')$. By Lemma 6.4.92 (a) there are monotone paths q from $(0,m)$ to $(i',j'+1)$ and r from $(i'+1,j')$ to (l,l) in the grid z with

$$\Gamma'_z = q,\ (i',j'{+}1)\to(i',j')\to(i'{+}1,j'),\ r;\qquad \Gamma_z = q,\ (i',j'{+}1)\to(i'{+}1,j'{+}1)\to(i'{+}1,j'),\ r,$$

while $\Gamma_{\neg z}=\Gamma'_{\neg z}$ and $l(\Gamma)=l$. In the notation of Lemma 6.4.91 (case $i\neq j$) this is $p=(z,i',j')$, $p_{\mathrm{a}}=(z,i',j'+1)$, $p_{\mathrm{b}}=(z,i'+1,j')$, $p^+=(z,i'+1,j'+1)$, $T=\emptyset$, $B=n_{p_{\mathrm{b}}}$; so σ is a bijection $n_p\sqcup n_{p^+}\to n_{p_{\mathrm{a}}}\sqcup B$, and the character $\chi_\square$ of Lemma 6.4.89 is the character $\chi_{\mathrm{c}}=(\chi_{(p,p_{\mathrm{a}})}\chi_{(p,p_{\mathrm{b}})})^{-1}$ of Lemma 6.4.91. By Lemma 6.4.92 (a), the edges of q, of r and of $\Gamma'_{\neg z}$ are the edges of both paths other than the four edges of the square, so they form the set $\mathcal{E}_0$ of Lemma 6.4.91 (1). Since $T=\emptyset$, the matrices $v(S)$ and v' are the empty matrix, $\mathrm{tr}(v'v)=0$, $\psi(0)=1$, and the sums over v and over v' in Lemma 6.4.91 (2), (3) have one term; the matrix $y(S)$ has the row blocks $y_1=S(p,p_{\mathrm{a}})$, $y_2=S(p,p_{\mathrm{b}})$, and $x(S)$ the column blocks $x_1=S(p_{\mathrm{a}},p^+)$, $x_2=S(p_{\mathrm{b}},p^+)$.

Step 3: the new path data. Let N be a function from the m-fold twists to $\mathbb{N}$ and $\Psi'=(\Psi'_\vartheta)$ a family of path data of bound $N(\vartheta)$ on the edges of Γ'. For each ϑ, Lemma 6.4.91, applied with σ, the trivial additive character and Ψ'_ϑ, gives a path datum Ψ_ϑ of bound $\#(R_\vartheta)^0N(\vartheta)=1\cdot N(\vartheta)$ on the edges of Γ satisfying (3).

Step 4: the two periods differ by κ/c_{ij}. Fix a real point g of $\mathrm{U}(\tilde{Y}_l^l)$ and a period triple (ϑ,y',φ). For an edge tuple S supported on $\mathcal{E}_0$ define a form $f_S\in\mathcal{F}(p_{\mathrm{a}})$ and a linear functional l_S on $\mathcal{F}(p_{\mathrm{b}})$ by

$$\begin{aligned} z=\mathrm{I}:&\quad f_S=\mathrm{fm}_q(S)\,y',\qquad l_S(w)=\langle \mathrm{fm}_r(S)\,w,\ \mathrm{fm}_{\Gamma'_{\mathrm{II}}}(S)\,\varphi\rangle_{l,g},\\ z=\mathrm{II}:&\quad f_S=\mathrm{fm}_q(S)\,\varphi,\qquad l_S(w)=\langle \mathrm{fm}_{\Gamma'_{\mathrm{I}}}(S)\,y',\ \mathrm{fm}_r(S)\,w\rangle_{l,g};\end{aligned}$$

l_S is linear because the moves are linear maps and $\langle\cdot,\cdot\rangle_{l,g}$, the integral of the product of its two arguments over $[\mathrm{U}(Y_l^l)]$, is linear in each argument. Let J be the set of pairs (f,l) of a form $f\in\mathcal{F}(p_{\mathrm{a}})$ and a linear functional l on $\mathcal{F}(p_{\mathrm{b}})$, let $L_{(f,l)}$, $L'_{(f,l)}$ be the functions L, L' of Lemma 6.4.89 for ϑ, f and l, and $F(S,v')=(f_S,l_S)$. Since f_S and l_S involve only the matrices of S on the edges of q, r and $\Gamma'_{\neg z}$, which lie in $\mathcal{E}_0$, we have $(f_S,l_S)=(f_{S_0},l_{S_0})$ for every edge tuple S. For S supported on the edges of Γ', the map along Γ'_z is $\mathrm{fm}_r(S)\,\mathrm{mv}_{(p,p_{\mathrm{b}})}(y_2)\,\mathrm{mv}_{(p,p_{\mathrm{a}})}(y_1)\,\mathrm{fm}_q(S)$, so the term of S in (3.3) for Γ' is $\Psi'_\vartheta(S)\,L'_{F(S_0,v')}(y(S))$; for S supported on the edges of Γ, the map along Γ_z is $\mathrm{fm}_r(S)\,\mathrm{mv}_{(p_{\mathrm{b}},p^+)}(x_2)\,\mathrm{mv}_{(p_{\mathrm{a}},p^+)}(x_1)\,\mathrm{fm}_q(S)$, so the term of S for Γ is $\Psi_\vartheta(S)\,L_{F(S_0,v'(S))}(x(S))$. By Lemma 6.4.89, for every $(f,l)\in J$, $\sum_y\Psi_1(y)L'_{(f,l)}(y)=\frac{\kappa}{c_{ij}(\vartheta)}\sum_x\Phi_1(x)L_{(f,l)}(x)$ whenever Φ_1, Ψ_1 are matched for $(\sigma,\chi_{\mathrm{c}})$ over R_ϑ. So Lemma 6.4.91 (3), with $C=\kappa/c_{ij}(\vartheta)$, gives

$$\mathrm{per}_{\Gamma'}(\Psi';g)(\vartheta,y',\varphi)=\frac{\kappa}{c_{ij}(\vartheta)}\,\mathrm{per}_{\Gamma}(\Psi;g)(\vartheta,y',\varphi).\tag{6.104}$$

Step 5: the transfer. Let Ξ be a function of period triples with $\Psi\propto_\Gamma\Xi$. The paths Γ' and Γ have the same index l, so their periods are taken at the same real points g of $\mathrm{U}(\tilde{Y}_l^l)$. For such g there is $\beta\in\mathbb{C}$ with $\mathrm{per}_\Gamma(\Psi;g)(\vartheta,y',\varphi)=\beta\,\Xi(\vartheta,y',\varphi)$ for all period triples, and by (6.104)

$$\mathrm{per}_{\Gamma'}(\Psi';g)(\vartheta,y',\varphi)=\kappa\beta\cdot\Xi(\vartheta,y',\varphi)/c_{ij}(\vartheta)$$

for all period triples, with $\kappa\beta$ independent of the triple. Hence

$$\Psi' \propto_{\Gamma'} \big((\vartheta, y', \varphi) \mapsto \Xi(\vartheta, y', \varphi)/c_{ij}(\vartheta)\big).$$

Together with Step 3 this says that Γ' transfers periods to Γ with bound 1 and constant c_{ij}. $\square$

Lemma 6.4.89. *For a node p let n_p be the index set of its basis; for an m-fold twist ϑ put $R_\vartheta = \mathcal{O}_E/\mathfrak{m}_\vartheta$; for an edge $e = (p, q)$ and a matrix X over R_ϑ with rows n_q and columns n_p let $\mathrm{mv}_e(X)$ be the move along e with matrix X: the lift move $\mathcal{F}(p) \to \mathcal{F}(q)$ if the first index changes along e, the transposed move $\mathcal{F}(q) \to \mathcal{F}(p)$ if the second index changes.*

Let D be admissible data, $z \in \{\mathrm{I}, \mathrm{II}\}$ and $0 \le i, j < m$. Write $p_{00} = (z, i, j)$, $p_{01} = (z, i, j+1)$, $p_{10} = (z, i+1, j)$, $p_{11} = (z, i+1, j+1)$, and put $\tau = \tau_{ij}$ if $z = \mathrm{I}$ and $\tau = \tau_{ji}$ if $z = \mathrm{II}$. There are $\kappa \in \mathbb{C}$ (possibly 0), a bijection $\sigma\colon n_{p_{00}} \sqcup n_{p_{11}} \to n_{p_{01}} \sqcup n_{p_{10}}$ and a function c from the m-fold twists to $\mathbb{C}^\times$, of absolute value τ up to a norm power, with the following property. Let ϑ be an m-fold twist, $f \in \mathcal{F}(p_{01})$ and l a linear functional on $\mathcal{F}(p_{10})$. For a matrix x over R_ϑ with rows $n_{p_{11}}$ and columns $n_{p_{01}} \sqcup n_{p_{10}}$, with column blocks x_1, x_2, and a matrix y over R_ϑ with rows $n_{p_{01}} \sqcup n_{p_{10}}$ and columns $n_{p_{00}}$, with row blocks y_1, y_2, put

$$L(x) = l\big(\mathrm{mv}_{(p_{10},p_{11})}(x_2)\,\mathrm{mv}_{(p_{01},p_{11})}(x_1)\,f\big), \qquad L'(y) = l\big(\mathrm{mv}_{(p_{00},p_{10})}(y_2)\,\mathrm{mv}_{(p_{00},p_{01})}(y_1)\,f\big).$$

Then

$$\sum_y \Psi(y)\,L'(y) = \frac{\kappa}{c(\vartheta)} \sum_x \Phi(x)\,L(x)$$

for all functions Φ of the matrices x and Ψ of the matrices y that are matched for $(\sigma, \chi_\square)$ over R_ϑ, where $\chi_\square = \big(\chi_{(p_{00},p_{01})}\,\chi_{(p_{00},p_{10})}\big)^{-1}$.

Proof. Fix D, z, i, j, and write $[z] = 0$ if $z = \mathrm{I}$, $[z] = 1$ if $z = \mathrm{II}$. By (GP5) the basis of the node (z, a, b) has $r + [z] + a(t + [z]) + b(t - [z])$ elements. Put $N_{ab} = \mathrm{rk}\, p_{ab}$; then

$$N_{11} = N_{00} + 2t, \qquad N_{00} + N_{11} = N_{01} + N_{10} = N_\square := 2r + 2[z] + (2i+1)(t+[z]) + (2j+1)(t-[z]),$$

and $N_\square$ is even, as $(2i+1)(t+[z]) + (2j+1)(t-[z]) \equiv 2t \pmod 2$. For a flag c, $l \ge 0$ and a rank N write $\mathrm{ec}_{c,l}(N)$ for the edge character of (GP6); its weight $\gamma_{c,l}(N)$ does not depend on the twist. Put $h = j + 1$ if $z = \mathrm{I}$ and $h = i + 1$ if $z = \mathrm{II}$, $\mu^\square = \mu^0_{i+1}\mu^0_{j+1}$, $\nu_\vartheta = (\mu^\square \vartheta^h)^{-1}$ for an m-fold twist ϑ, and $\mu_\square(v) = \kappa_{i+1}(v) + \kappa_{j+1}(v)$ for $v \in \Phi$.

Step 1: numerical facts. By Lemma 2.3.6, $\log x > 0$, so the integer x is at least 2. By (AD2), $m \le 2\sqrt{x}$ and $r \ge x^5$; hence $m + 1 \le 3\sqrt{x} \le x^5 \le r$ and $26m \le 52\sqrt{x} \le 12x^{35} \le 12r^7$. By (AD4), $\kappa(v) > 0$, so $\kappa_l(v) = \kappa(v) + (2l+1)r^8 \ge 3r^8$ for $l \ge 1$. By Lemma 2.3.6, $q_0 \ge 18480$, and $d \ge 1$, $s = 22(e+1)$, $m = 2^{e+1}$ by (AD2); so $q_2 \ge 2^{s+6} > 2^{e+2} = 2m$.

Step 2: the grid modulus. By Lemma 6.4.90, $\mathfrak{m}_\vartheta$ is a modulus with $\Sigma(\mathfrak{m}_\vartheta) = S_M \sqcup Q$, and so $\prod_{w \in \Sigma(\mathfrak{m}_\vartheta)} \mathrm{N}w = \mathrm{N}S_M \cdot \mathrm{N}Q$ with $\mathrm{N}S_M = \prod_{w \in S_M} \mathrm{N}w$.

Step 3: tameness and the character of the square. If two automorphic characters of $\mathrm{U}(1)$ have weights e, e' and tame parts ψ, ψ', their product has weight $e + e'$ and tame part $\psi\psi'$, and the inverse of the first has weight $-e$ and tame part ψ^{-1} (the defining formula of Definition 3.1.2). Let ψ be the tame part of ϑ. Then ν_ϑ has weight $\mu_\square$ (ϑ has weight 0) and tame part $\psi_\nu = (\omega^{i+j+2}\psi^h)^{-1}$. At a chosen place w, $\psi_w = 1$ (as $w \notin Q$) and $\omega_w^{i+j+2} \ne 1$, since ω_w has order q_2 and $2 \le i + j + 2 \le 2m < q_2$ (Step 1). At $w \in Q$, $\omega_w = 1$ and $\psi_w^h \ne 1$, since $1 \le h \le m$ and ϑ is an m-fold twist. At every other place both are trivial. So ν_ϑ is tame of weight $\mu_\square$ for $(\mathfrak{m}_\vartheta, \chi_\vartheta)$, where $\chi_\vartheta(u) = \prod_{w \in \Sigma(\mathfrak{m}_\vartheta)} \psi_{\nu,w}(u \bmod w)$ for $u \in R_\vartheta^\times$. By (GP6), of the two edges (p_{00}, p_{01}) (the second index changes, $l = j$) and (p_{00}, p_{10}) (the first

index changes, $l = i$), exactly one is vertical: the first if $z = \mathrm{I}$, the second if $z = \mathrm{II}$. So for $u \in R_\vartheta^\times$, $\chi_{(p_{00},p_{01})}(u)\chi_{(p_{00},p_{10})}(u) = \prod_w(\omega^{i+j+2}\psi^h)_w(u \bmod w)$, and $\chi_\square(u) = \chi_\vartheta(u)$. Matching (XD6) evaluates the character only at determinants of invertible matrices; hence functions are matched for $(\sigma, \chi_\square)$ if and only if they are matched for (σ, χ_ϑ).

Step 4: the data of the one-square identity. Apply Proposition 6.3.72 with: I the set of m-fold twists; $A' = p_{00}$, $A = p_{11}$, $U_1 = p_{01}$, $U_2 = p_{10}$, with index sets $n = n_{p_{00}}$, $n' = n_{p_{11}}$, $n_1 = n_{p_{01}}$, $n_2 = n_{p_{10}}$; $\sigma\colon n_{p_{00}} \sqcup n_{p_{11}} \to n_{p_{01}} \sqcup n_{p_{10}}$ any bijection (both sides have $N_\square$ elements); $t = 5r$; $\mu = \mu_\square$; for $\vartheta \in I$ the modulus $\mathfrak{m}_\vartheta$, the character χ_ϑ and ν_ϑ (Steps 2, 3); the characters

$$(a_1, b_1) = \big(\mathrm{ec}_{z,i}(N_{01}), \mathrm{ec}_{z,i}(N_{11})\big),\ (a_2, b_2) = \big(\mathrm{ec}_{\neg z,j}(N_{10}), \mathrm{ec}_{\neg z,j}(N_{11})\big),$$

$$(b_1', a_1') = \big(\mathrm{ec}_{\neg z,j}(N_{00}), \mathrm{ec}_{\neg z,j}(N_{01})\big),\ (b_2', a_2') = \big(\mathrm{ec}_{z,i}(N_{00}), \mathrm{ec}_{z,i}(N_{10})\big)$$

on the pairs (U_1, A), (U_2, A), (A', U_1), (A', U_2) (the first character at the first space); P_1, P_2, P_1', P_2' the joint harmonics (GP7) of the moves along (p_{01}, p_{11}), (p_{10}, p_{11}), (p_{00}, p_{01}), (p_{00}, p_{10}); and the types $e = e_{p_{11}}$, $e' = e_{p_{00}}$, $e_1 = e_{p_{01}}$, $e_2 = e_{p_{10}}$. The move along (p_{01}, p_{11}) is a lift move with $c = z$, $l = i$, along (p_{10}, p_{11}) a transposed move from p_{11} to p_{10} with $c = \neg z$, $l = j$, along (p_{00}, p_{01}) a transposed move from p_{01} to p_{00} with $c = \neg z$, $l = j$, and along (p_{00}, p_{10}) a lift move with $c = z$, $l = i$; by (GP6), (GP7) each uses at its two ends the characters and the joint harmonic listed above, and the indicator of its matrix modulo $\mathfrak{m}_\vartheta$. Hence the maps $\Theta_\vartheta(x)$, $\Theta'_\vartheta(y)$ of Proposition 6.3.72 are $\mathrm{mv}_{(p_{10},p_{11})}(x_2)\,\mathrm{mv}_{(p_{01},p_{11})}(x_1)$ and $\mathrm{mv}_{(p_{00},p_{10})}(y_2)\,\mathrm{mv}_{(p_{00},p_{01})}(y_1)$ from $\mathcal{F}(p_{01})$ to $\mathcal{F}(p_{10})$, and the functions L, L' there are those of the statement.

Step 5: (OS1). $\mathrm{rk}\,A = N_{11} = N_{00} + 2t = \mathrm{rk}\,A' + 2t$. By (GP4), with $M_z = m - j + [z]$, $\mathrm{neg}(U_1) = \mathrm{neg}(A) = \lfloor (M_z - 1)/2 \rfloor$ and $\mathrm{neg}(U_2) = \mathrm{neg}(A') = \lfloor M_z/2 \rfloor$; so $\mathrm{neg}(W) = \mathrm{neg}(A) + \mathrm{neg}(A') = \lfloor (M_z - 1)/2 \rfloor + \lfloor M_z/2 \rfloor = M_z - 1 \le m$. Hence $2\,\mathrm{neg}(W) + 2 \le 2m + 2 \le 2r \le N_{01} + N_{10} = \mathrm{rk}\,W$ by Step 1. Finally, as $i, j \le m - 1$,

$$\begin{aligned}\mathrm{rk}\,W = N_\square &\le 2r + 2 + (2m-1)(2t+1) \le 2r + 2 + 20mr + 2m \le 26mr \\ &\le 12r^8 \le 2\mu_\square(\upsilon) \le 2\mu_\square(\upsilon) + 2\,\mathrm{neg}(W)\end{aligned}$$

for every $\upsilon \in \Phi$, by Step 1.

Step 6: (OS2). The weights of the characters of each pair are the weights $\gamma_{c,l}(N)$ of (GP6), independent of ϑ. By Lemma 6.4.15 (d) for $(a, b) = (i, j+1)$, (b) with offset $o = 0$ for $(a, b) = (i+1, j)$, (b) with $o = 0$ for $(a, b) = (i, j)$ and (d) for $(a, b) = (i, j)$ (the ranges hold as $i, j < m$), the Fock vectors of P_1, P_2, P_1', P_2' are edges for the weights of the characters of their pairs, the first first, from $e_{p_{01}}$ to $e_{p_{11}}$, from $\bar{e}_{p_{10}}$ to $\bar{e}_{p_{11}}$, from $\bar{e}_{p_{00}}$ to $\bar{e}_{p_{01}}$ and from $e_{p_{00}}$ to $e_{p_{10}}$, that is from e_1 to e, from $\bar{e}_2$ to $\bar{e}$, from $\bar{e}'$ to $\bar{e}_1$ and from e' to e_2.

Step 7: (OS3). Let $N + N' = N_\square$, so $N \equiv N' \pmod 2$. If $z = \mathrm{I}$, then

$$\mathrm{ec}_{z,i}(N)\,\mathrm{ec}_{\neg z,j}(N') = \mu_{i+1}^0 \upsilon^{\lceil N/2 \rceil} \cdot \mu_{j+1}^0 \vartheta^{j+1} \upsilon^{\lfloor N'/2 \rfloor};$$

if $z = \mathrm{II}$, it is

$$\mu_{i+1}^0 \vartheta^{i+1} \upsilon^{\lfloor N/2 \rfloor} \cdot \mu_{j+1}^0 \upsilon^{\lceil N'/2 \rceil}.$$

In both cases $\lceil N/2 \rceil + \lfloor N'/2 \rfloor = \lfloor N/2 \rfloor + \lceil N'/2 \rceil = N_\square/2$, so the product is $\mu^\square \vartheta^h \upsilon^{N_\square/2}$. The base change of υ on $\mathrm{U}(1)(\mathbb{A})$ is $x \mapsto \upsilon(x/\bar{x}) = \upsilon(x)\upsilon(\bar{x})^{-1} = \upsilon(x)^2$ by (WD3). Hence, for characters α, β with $\alpha\beta = \mathrm{ec}_{z,i}(N)\,\mathrm{ec}_{\neg z,j}(N')$,

$$\alpha[N]\,\beta[N'] = \widetilde{\alpha\beta}\,\upsilon^{-N_\square} = \widetilde{\mu^\square \vartheta^h} = \tilde{\nu}_\vartheta^{-1}.$$

The four products of (OS3) are of this form, with (N, N') equal to (N_{01}, N_{10}), (N_{10}, N_{01}), (N_{11}, N_{00}) and (N_{00}, N_{11}) respectively, the character $\mathrm{ec}_{z,i}$ taken at N and $\mathrm{ec}_{\neg z,j}$ at N'.

Step 8: the square law. By Steps 4 to 7 the hypotheses of Proposition 6.3.72 hold. It gives $C > 0$, $\kappa \in \mathbb{C}$ (possibly 0), both independent of ϑ, and for each ϑ a number $c(\vartheta)$ with $|c(\vartheta)|^2 = C\, L_*(\mathfrak{m}_\vartheta, \nu_\vartheta; t) \neq 0$, so $c(\vartheta) \in \mathbb{C}^\times$, such that for every $f \in \mathcal{F}(p_{01})$ and every linear functional l on $\mathcal{F}(p_{10})$, $\sum_y \Psi(y)L'(y) = \frac{\kappa}{c(\vartheta)} \sum_x \Phi(x)L(x)$ whenever Φ, Ψ are matched for (σ, χ_ϑ), that is, by Step 3, for $(\sigma, \chi_\square)$.

Step 9: the absolute value of c. By (XD7) and Step 2, in the notation of Definition 4.2.4,

$$|c(\vartheta)|^2 = C\,(\mathrm{N}S_M\,\mathrm{N}Q)^{t^2} \prod_{\iota=1}^{t} \bigl|L^{S_M \cup Q}\bigl(\iota, (\mu^\square \vartheta^h)^{-1}\bigr)\bigr|^2.$$

For the trivial twist ($Q = \emptyset$, $\vartheta^h = 1$) this reads $0 \neq |c(1)|^2 = C\,(\mathrm{N}S_M)^{t^2} \prod_{\iota=1}^{t} |L^{S_M}(\iota, (\mu^\square)^{-1})|^2$; so $L^{S_M}(\iota, (\mu^\square)^{-1}) \neq 0$ for $1 \leq \iota \leq t$. As ϑ^h has modulus Q, the definition of the ratio gives $|L^{S_M \cup Q}(\iota, (\mu^\square \vartheta^h)^{-1})| = r_{(\mu^\square, \iota)}(\vartheta^h)\, |L^{S_M}(\iota, (\mu^\square)^{-1})|$, and therefore

$$|c(\vartheta)|^2 = |c(1)|^2\,(\mathrm{N}Q)^{t^2}\, F_{\mu^\square, t}(\vartheta^h)^2.$$

If $z = \mathrm{I}$, $F_{\mu^\square, t}(\vartheta^h) = F_{\mu^0_{i+1}\mu^0_{j+1}, t}(\vartheta^{\,j+1}) = \tau_{ij}(\vartheta)$; if $z = \mathrm{II}$, $F_{\mu^\square, t}(\vartheta^h) = F_{\mu^0_{j+1}\mu^0_{i+1}, t}(\vartheta^{\,i+1}) = \tau_{ji}(\vartheta)$. So c has absolute value τ up to a norm power, with the constant $|c(1)|^2$ and the exponent t^2. □

Lemma 6.4.90. *Let D be admissible data and ϑ an m-fold twist with modulus Q. Then $\mathfrak{m}_\vartheta$ is a modulus with $\Sigma(\mathfrak{m}_\vartheta) = S_M \sqcup Q$, and $\#R_{\mathfrak{m}_\vartheta} = \#(\mathcal{O}_E/\mathfrak{b}) \cdot \mathrm{N}Q$.*

Proof. **The places of $\mathfrak{m}_\vartheta$.** The prime ideals containing $\mathfrak{b} = \pi_M^b$ ($b \geq 1$) are those containing π_M, that is the chosen places, which lie above p_0 or p_1 since $\pi_M \bar{\pi}_M = p_0 p_1$ (AD5). The places of Q have norms prime to M (Definition 3.1.2), so they lie above no prime dividing M; hence $S_M \cap Q = \emptyset$, and the prime ideals dividing $\mathfrak{m}_\vartheta$ form the set $S_M \sqcup Q$, nonempty because π_M is not a unit ($\mathrm{N}_{K/\mathbb{Q}}(\pi_M) = p_0 p_1 > 1$). So $\mathfrak{m}_\vartheta = \prod_{w \in S_M \sqcup Q} w^{e_w}$ with all $e_w \geq 1$.

No two places are conjugate. The conjugate of a place above p_0 or p_1 lies above the same prime, so no place of Q is conjugate to a chosen place; and Q contains the conjugate of none of its members (Definition 3.1.2). Suppose that a chosen place w, above p_0 say, has its conjugate $\bar{w}$ (possibly w itself) among the chosen places. Then $\pi_M \in \bar{w}$, so $\bar{\pi}_M \in w$; thus π_M and $\bar{\pi}_M$, which lie in $\mathcal{O}_K$, lie in the prime $\mathfrak{p} = w \cap \mathcal{O}_K$ of K above p_0, and $\mathfrak{p} \nmid p_1$ as $p_0 \neq p_1$. Hence $v_\mathfrak{p}(p_0) = v_\mathfrak{p}(p_0 p_1) = v_\mathfrak{p}(\pi_M) + v_\mathfrak{p}(\bar{\pi}_M) \geq 2$, so p_0 ramifies in K. The discriminant of $K = \mathbb{Q}(\sqrt{-d})$ divides $4d$, so $p_0 \mid 4d$. But $q_2 = (q_0 d)^{s+6} \geq q_0 \geq 18480$ by Lemma 2.3.6, and $p_0 \equiv 1 \pmod{q_2^2}$ (AD5); so the prime p_0 is $\geq q_2^2 + 1 > 2$, hence odd, and $p_0 \equiv 1 \pmod{d}$ as $d \mid q_2$, hence $p_0 \nmid d$; so $p_0 \nmid 4d$. This contradiction (and the same argument for p_1) shows that $\Sigma(\mathfrak{m}_\vartheta) = S_M \sqcup Q$ contains the conjugate of none of its members. So $\mathfrak{m}_\vartheta$ is a modulus (XD4).

The order of the residue ring. The ideals $\mathfrak{b}$ and $w \in Q$ are pairwise coprime (their prime divisors are disjoint, and the $w \in Q$ are distinct primes), so by the Chinese remainder theorem $R_{\mathfrak{m}_\vartheta} = \mathcal{O}_E/\mathfrak{m}_\vartheta \cong \mathcal{O}_E/\mathfrak{b} \times \prod_{w \in Q} \mathcal{O}_E/w$, of order $\#(\mathcal{O}_E/\mathfrak{b}) \prod_{w \in Q} \mathrm{N}w = \#(\mathcal{O}_E/\mathfrak{b})\,\mathrm{N}Q$. □

Lemma 6.4.91. *For a node p let n_p be the index set of its basis, and for an m-fold twist ϑ put $R_\vartheta = \mathcal{O}_E/\mathfrak{m}_\vartheta$. Let D be admissible data, ϑ an m-fold twist, and let the path Γ through the two grids arise from the path Γ' by a turn at (i, j). Define nodes $p, \tilde{p}, p_\mathrm{a}, p_\mathrm{b}, p^+, \tilde{p}^+$, a finite set T and an index set B for the basis of p_b as follows.*

- *If $i \neq j$, let $(z, i', j') = (\mathrm{I}, i, j)$ if $i < j$ and $(z, i', j') = (\mathrm{II}, j, i)$ if $i > j$; put $p = \tilde{p} = (z, i', j')$, $p_\mathrm{a} = (z, i', j' + 1)$, $p_\mathrm{b} = (z, i' + 1, j')$, $p^+ = \tilde{p}^+ = (z, i' + 1, j' + 1)$, $T = \emptyset$ and $B = n_{p_\mathrm{b}}$.*

- *If $i = j$, put $p = (\mathrm{I}, j, j)$, $\tilde{p} = (\mathrm{II}, j, j)$, $p_\mathrm{a} = (\mathrm{I}, j, j+1)$, $p_\mathrm{b} = (\mathrm{II}, j, j+1)$, $p^+ = (\mathrm{I}, j+1, j+1)$, $\tilde{p}^+ = (\mathrm{II}, j+1, j+1)$; let T have one element, the added line, so that $n_{\tilde{p}} = n_p \sqcup T$ and $n_{\tilde{p}^+} = n_{p^+} \sqcup T$ through the decompositions of (GP2); and let $B = n_{(\mathrm{I},j+1,j)}$, numbering the basis of p_b through the renumbering of (GP3).*

In both cases $n_{\tilde{p}} = n_p \sqcup T$ and $n_{\tilde{p}^+} = n_{p^+} \sqcup T$. For an edge tuple S let $y(S)$ be the matrix with rows $n_{p_\mathrm{a}} \sqcup B$ and columns n_p whose row blocks are $S(p, p_\mathrm{a})$ and the columns n_p of $S(\tilde{p}, p_\mathrm{b})$; $v(S)$ the columns T of $S(\tilde{p}, p_\mathrm{b})$ (a $B \times T$ matrix); $x(S)$ the matrix with rows n_{p^+} and columns $n_{p_\mathrm{a}} \sqcup B$ whose column blocks are $S(p_\mathrm{a}, p^+)$ and the rows n_{p^+} of $S(p_\mathrm{b}, \tilde{p}^+)$; and $v'(S)$ the rows T of $S(p_\mathrm{b}, \tilde{p}^+)$ (a $T \times B$ matrix). Then:

(1) there is a set $\mathcal{E}_0$ of edges, containing none of the four edges (p, p_a), $(\tilde{p}, p_\mathrm{b})$, (p_a, p^+), $(p_\mathrm{b}, \tilde{p}^+)$, such that the edges of Γ' are $\mathcal{E}_0 \cup \{(p, p_\mathrm{a}), (\tilde{p}, p_\mathrm{b})\}$ and those of Γ are $\mathcal{E}_0 \cup \{(p_\mathrm{a}, p^+), (p_\mathrm{b}, \tilde{p}^+)\}$.

For an edge tuple S supported on $\mathcal{E}_0$ and matrices y, v, x, v' of the shapes above, let $S[y, v]$ be the tuple that agrees with S except at the two edges (p, p_a), $(\tilde{p}, p_\mathrm{b})$, where it is given by $y(S[y, v]) = y$ and $v(S[y, v]) = v$, and $S\langle x, v'\rangle$ the tuple that agrees with S except at the two edges (p_a, p^+), $(p_\mathrm{b}, \tilde{p}^+)$, where it is given by $x(S\langle x, v'\rangle) = x$ and $v'(S\langle x, v'\rangle) = v'$; for any tuple S let S_0 be the tuple equal to S on $\mathcal{E}_0$ and to 0 elsewhere. Let ψ be an additive character of R_ϑ, $\sigma\colon n_p \sqcup n_{p^+} \to n_{p_\mathrm{a}} \sqcup B$ a bijection, $\chi_\mathrm{c} = (\chi_{(p,p_\mathrm{a})}\chi_{(\tilde{p},p_\mathrm{b})})^{-1}$, $N \in \mathbb{N}$, and Ψ' a path datum of bound N on the edges of Γ'. Then there is a path datum Ψ of bound $\#(R_\vartheta)^{\#B \cdot \#T} N$ on the edges of Γ such that:

(2) for every edge tuple S supported on $\mathcal{E}_0$ and every $T \times B$ matrix v', the functions

$$x \longmapsto \Psi\big(S\langle x, v'\rangle\big) \qquad \text{and} \qquad y \longmapsto \sum_v \psi\big(\mathrm{tr}(v'v)\big)\, \Psi'\big(S[y, v]\big)$$

are matched for $(\sigma, \chi_\mathrm{c})$ over R_ϑ, the sum running over the $B \times T$ matrices v;

(3) for every set J, every $C \in \mathbb{C}$, every family $(L_\iota, L'_\iota)_{\iota \in J}$ of functions of the matrices x, resp. y, such that $\sum_y \Psi_1(y) L'_\iota(y) = C \sum_x \Phi_1(x) L_\iota(x)$ for all Φ_1, Ψ_1 matched for $(\sigma, \chi_\mathrm{c})$ over R_ϑ, and every function F from the pairs (S, v') (S supported on $\mathcal{E}_0$, v' a $T \times B$ matrix) to J,

$$\sum_S \Psi'(S) \sum_{v'} \psi\big(\mathrm{tr}(v'v(S))\big)\, L'_{F(S_0, v')}\big(y(S)\big) = C \sum_S \Psi(S)\, L_{F(S_0, v'(S))}\big(x(S)\big),$$

the sum on the left over the edge tuples supported on the edges of Γ', that on the right over those supported on the edges of Γ.

Proof. Write $R = R_\vartheta$, $n = n_{p_\mathrm{a}} \sqcup B$, and identify n with $n_p \sqcup n_{p^+}$ through σ; for $k \in \mathrm{GL}_n(R)$ let $k_\flat$ be the matrix of the columns n_p of k and $k^\sharp$ that of the rows n_{p^+} of k^{-1}. Thus y ranges over the $n \times n_p$ and x over the $n_{p^+} \times n$ matrices, and Φ, Ψ are matched for (σ, χ) if Φ vanishes off the matrices $k^\sharp$, Ψ off the matrices $k_\flat$, and $\Phi(k^\sharp) = \chi(\det k)\Psi(k_\flat)$ for all $k \in \mathrm{GL}_n(R)$. Values of multiplicative characters of R at units are roots of unity, as $R^\times$ is finite. Put $e_1 = (p, p_\mathrm{a})$, $e_2 = (\tilde{p}, p_\mathrm{b})$, $e_3 = (p_\mathrm{a}, p^+)$, $e_4 = (p_\mathrm{b}, \tilde{p}^+)$.

Step 1: the edges, clause (1). By Lemma 6.4.92 (a) (case $i \neq j$, with the nodes named there $(z, i', j') = p$, $(z, i', j'+1) = p_\mathrm{a}$, $(z, i'+1, j') = p_\mathrm{b}$, $(z, i'+1, j'+1) = p^+$) and (b) (case $i = j$), the edges of Γ' are $\mathcal{E}_0 \cup \{e_1, e_2\}$ and those of Γ are $\mathcal{E}_0 \cup \{e_3, e_4\}$, where $\mathcal{E}_0$ is the set of edges of the paths q, r, $\Gamma'_{\neg z}$, resp. u_I, u_II of that lemma, and contains none of $e_1, \ldots, e_4$; this is

(1), and $e_1 \neq e_2$, $e_3 \neq e_4$. No edge of $\mathcal{E}_0$ has p or $\tilde{p}$ as an endpoint: by monotonicity the nodes (a, b) of q have $b \geq j' + 1$, those of r have $a \geq i' + 1$, those of u_z have $b \geq j + 1$, and $\Gamma'_{\neg z}$ lies in the other grid. The nodes $p, \tilde{p}$ are not the starting nodes $(z, 0, m)$ (their second index is $< m$), and off the diagonal p is not a diagonal node.

Step 2: characters and vertical edges of the square. By (GP6) the character χ_e depends only on whether e is vertical and on the value at the source of the index that changes along e. Off the diagonal, e_1 and e_4 both change the second index, from j', and e_2, e_3 the first, from i', in the same grid; on the diagonal, e_1 (in the grid Y, second index) and e_4 (in the grid $\tilde{Y}$, first index) are vertical and change an index from j, while e_2 (grid $\tilde{Y}$, second index) and e_3 (grid Y, first index) are not vertical and change an index from j. Hence $\chi_{e_3} = \chi_{e_2}$, $\chi_{e_4} = \chi_{e_1}$, and in both cases exactly one of e_1, e_2 is vertical, the opposite edge e_4, resp. e_3, is vertical, and the other new edge is not (off the diagonal e_1, e_4 are vertical if $z = \mathrm{I}$ and e_2, e_3 if $z = \mathrm{II}$).

Step 3: two facts on matrices over R. (i) *Completion.* If y has a left inverse, then $y = k_\flat$ for some $k \in \mathrm{GL}_n(R)$. By the Chinese remainder theorem $R = \prod_w \mathcal{O}_E/w^{e_w}$ is a finite product of local rings, and a square matrix over R is invertible if and only if each component is; so let R be local with maximal ideal $\mathfrak{q}$. From $Ly = 1$ the reduction of y modulo $\mathfrak{q}$ has rank $\#n_p$; complete its columns by $\#n - \#n_p = \#n_{p^+}$ columns (the sizes match because σ is a bijection) to a basis of the residue space, lift them to R, and let k have the columns y at n_p and the lifts at n_{p^+}; its determinant is a unit, as its reduction is nonzero. (ii) *Transfer of left inverses.* Let $k \in \mathrm{GL}_n(R)$ and write $k_\flat$ in row blocks Y_1, Y_2 (rows n_{p_a}, B) and $k^\sharp$ in column blocks X_1, X_2 (columns n_{p_a}, B). If Y_1 has a left inverse, so has X_2; if Y_2 has a left inverse, so has X_1. For the first, let $LY_1 = 1$ and let s be the $n_p \times n$ matrix equal to L on the columns n_{p_a} and to 0 on the columns B; then $sk_\flat = 1$. The square matrix P with the rows s (at n_p) and $k^\sharp$ (at n_{p^+}) satisfies $Pk = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ in the blocks n_p, n_{p^+}, since $k^\sharp k$ is the rows n_{p^+} of 1; so $\det P \det k = 1$ and P is invertible. Reading $P^{-1}P = 1$ at the rows and columns B, where the rows s of P vanish, gives $MX_2 = 1$ for the block M of P^{-1} with rows B and columns n_{p^+}. The second is the same with the roles of the blocks exchanged.

Step 4: the Fourier transform of Ψ'. For S supported on $\mathcal{E}_0$ and a $T \times B$ matrix v' put $\hat{\Psi}'_{S,v'}(y) = \sum_v \psi(\mathrm{tr}(v'v))\, \Psi'(S[y, v])$. By Step 1 the map $(S, y, v) \mapsto S[y, v]$ is a bijection from the triples (S supported on $\mathcal{E}_0$) onto the tuples supported on the edges of Γ', with inverse $S \mapsto (S_0, y(S), v(S))$, and similarly $(S, x, v') \mapsto S\langle x, v' \rangle$ onto the tuples supported on the edges of Γ. We show:

(F1) if $\hat{\Psi}'_{S,v'}(y) \neq 0$, then $y = k_\flat$ for some $k \in \mathrm{GL}_n(R)$, and for such k and some v with $\Psi'(S[k_\flat, v]) \neq 0$ the vertical edges in $\mathcal{E}_0$ carry matrices of S with left inverses, and Y_1 has a left inverse if e_1 is vertical, Y_2 if e_2 is vertical;

(F2) $\chi_\mathrm{c}(\det h)\, \hat{\Psi}'_{S,v'}(yh) = \hat{\Psi}'_{S,v'}(y)$ for every $h \in \mathrm{GL}_{n_p}(R)$;

(F3) for every admissible family g, with $g_+ = g_{p^+}$, $\hat{g}$ the element $g_{p_\flat}$ written in the numbering B, $g_n = \mathrm{diag}(g_{p_\mathrm{a}}, \hat{g}) \in \mathrm{GL}_n(R)$ and the root of unity $A_g = \epsilon_{\mathcal{E}_0}(g)\, \chi_{e_1}(\det g_{p_\mathrm{a}})\, \chi_{e_2}(\det g_{p_\flat})$, one has $\hat{\Psi}'_{g \cdot S,\, v'\hat{g}}(y) = A_g^{-1}\, \hat{\Psi}'_{S,v'}(g_n y)$.

(F1): some v has $\Psi'(S[y, v]) \neq 0$; by (PD3) for Ψ' the vertical edges of Γ' carry left invertible matrices. These include the edges of $\mathcal{E}_0$ that are vertical and, by Step 2, e_1 or e_2. If e_1 is vertical, $Y_1 = S[y, v](e_1)$ has a left inverse; if e_2 is vertical, $(Y_2 \,|\, v) = S[y, v](e_2)$ has a left inverse L, and the rows n_p of L form a left inverse of Y_2. In both cases y has a left inverse (extend a left inverse of the block by 0), and Step 3 (i) applies. (F2): let g' be the family equal to h at p, to $h \oplus 1_T$ at $\tilde{p}$ (that is, h at $\tilde{p} = p$ off the diagonal) and to 1 elsewhere; it is admissible by Step 1. As no edge of $\mathcal{E}_0$

meets $p, \tilde{p}$, $g' \cdot S[y,v] = S[yh,v]$ and $\epsilon_{E(\Gamma')}(g') = \chi_{e_1}(\det h)^{-1}\chi_{e_2}(\det h)^{-1} = \chi_{\mathrm{c}}(\det h)$; (PD2) for Ψ' gives $\chi_{\mathrm{c}}(\det h)\Psi'(S[yh,v]) = \Psi'(S[y,v])$, and summing over v gives (F2). (F3): let g' be equal to g except at p and $\tilde{p}$, where it is 1; it is admissible by Step 1. Then $g' \cdot S[y,v] = (g \cdot S)[g_n^{-1}y, \hat{g}^{-1}v]$ and $\epsilon_{E(\Gamma')}(g') = A_g$, so (PD2) gives $\Psi'((g \cdot S)[g_n^{-1}y, \hat{g}^{-1}v]) = A_g^{-1}\Psi'(S[y,v])$. Substituting $v = \hat{g}^{-1}u$ in the sum defining $\hat{\Psi}'_{g\cdot S, v'\hat{g}}(g_n^{-1}y)$, $\mathrm{tr}(v'\hat{g}\hat{g}^{-1}u) = \mathrm{tr}(v'u)$, which gives (F3) after replacing y by $g_n y$.

Step 5: the match and the new datum. For S supported on $\mathcal{E}_0$ and v' define $\Phi_{S,v'}(x) = \chi_{\mathrm{c}}(\det k)\,\hat{\Psi}'_{S,v'}(k_\flat)$ if $x = k^\sharp$ for some $k \in \mathrm{GL}_n(R)$, and $\Phi_{S,v'}(x) = 0$ otherwise. This does not depend on k: if $k^\sharp = k'^\sharp$, the rows n_{p^+} of $M = k^{-1}k'$ are those of $k^\sharp k' = k'^\sharp k'$, that is $(0\ 1)$, so $M = \begin{pmatrix} h & * \\ 0 & 1 \end{pmatrix}$ with $\det h = \det M \in R^\times$, $k'_\flat = k_\flat h$ and $\det k' = \det k \det h$; by (F2), $\chi_{\mathrm{c}}(\det k')\hat{\Psi}'_{S,v'}(k'_\flat) = \chi_{\mathrm{c}}(\det k)\hat{\Psi}'_{S,v'}(k_\flat)$. By construction and (F1), $\Phi_{S,v'}$ and $\hat{\Psi}'_{S,v'}$ are matched for $(\sigma, \chi_{\mathrm{c}})$. Put

$$\Psi(S) = \Phi_{S_0,\, v'(S)}\big(x(S)\big)$$

for every edge tuple S. For S supported on $\mathcal{E}_0$, $(S\langle x, v'\rangle)_0 = S$, $v'(S\langle x, v'\rangle) = v'$ and $x(S\langle x, v'\rangle) = x$, so $x \mapsto \Psi(S\langle x, v'\rangle)$ is $\Phi_{S,v'}$; this is (2).

Step 6: Ψ is a path datum of bound $\#(R)^{\#B\cdot\#T}N$ on the edges of Γ. (PD1): $\Psi(S)$ depends only on S_0, $v'(S)$ and $x(S)$, that is on the matrices of S on $\mathcal{E}_0 \cup \{e_3, e_4\}$. (PD4): if $\Psi(S) \neq 0$, then $\Psi(S) = \chi_{\mathrm{c}}(\det k)\sum_v \psi(\mathrm{tr}(v'v))\Psi'(S_0[k_\flat, v])$ with $\#(R)^{\#B\cdot\#T}$ terms v; $\chi_{\mathrm{c}}(\det k)$ and the values of ψ are roots of unity (R is finite), each $\Psi'(\cdot)$ is a sum of at most N roots of unity, and a root of unity times such a sum is again one; so $\Psi(S)$ is a sum of at most $\#(R)^{\#B\cdot\#T}N$ roots of unity, and 0 is the empty sum. (PD3): let $\Psi(S) \neq 0$, so $x(S) = k^\sharp$ with $\hat{\Psi}'_{S_0,v'(S)}(k_\flat) \neq 0$. By (F1) the vertical edges of $\mathcal{E}_0$ carry left invertible matrices of S_0, hence of S, and Y_1 (if e_1 is vertical) or Y_2 (if e_2 is vertical) has a left inverse. By Step 3 (ii), X_2, resp. X_1, has a left inverse. By Step 2 the vertical new edge is e_4 in the first case and e_3 in the second. In the second, $S(e_3) = X_1$. In the first, $S(e_4)$ has the rows X_2 (at n_{p^+}) and $v'(S)$ (at T), and $(M \mid 0)$ is a left inverse of it if $MX_2 = 1$. (PD2): let g be admissible. At $\tilde{p}^+$, $g_{\tilde{p}^+} = g_{p^+} \oplus 1_T$ (admissibility on the diagonal; trivially off it), so $x(g \cdot S) = g_+^{-1}x(S)g_n$, $v'(g \cdot S) = v'(S)\hat{g}$ and $(g \cdot S)_0 = g \cdot S_0$. From (F3) and the definition of Φ: for $x = k^\sharp$, $\Phi_{g\cdot S_0, v'\hat{g}}(x) = \chi_{\mathrm{c}}(\det k)A_g^{-1}\hat{\Psi}'_{S_0,v'}((g_n k)_\flat) = A_g^{-1}\chi_{\mathrm{c}}(\det g_n)^{-1}\Phi_{S_0,v'}(xg_n^{-1})$, since $xg_n^{-1} = (g_n k)^\sharp$; and both sides vanish when x is not of the form $k^\sharp$, since then neither is xg_n^{-1}. Next, $g_+^{-1}k^\sharp = (kD^{-1})^\sharp$ for $D = \mathrm{diag}(1_{n_p}, g_+^{-1})$, with $(kD^{-1})_\flat = k_\flat$ and $\det(kD^{-1}) = \det k \det g_+$; so $\Phi_{S_0,v'}(g_+^{-1}x) = \chi_{\mathrm{c}}(\det g_+)\Phi_{S_0,v'}(x)$ for all x (both sides vanish off the matrices $k^\sharp$). Hence

$$\Psi(g \cdot S) = A_g^{-1}\,\chi_{\mathrm{c}}(\det g_n)^{-1}\chi_{\mathrm{c}}(\det g_+)\,\Psi(S).$$

On the other hand, by Step 1, $\det g_{\tilde{p}^+} = \det g_+$ and Step 2,

$$\epsilon_{E(\Gamma)}(g) = \epsilon_{\mathcal{E}_0}(g)\,\frac{\chi_{e_2}(\det g_+)}{\chi_{e_2}(\det g_{p_\mathrm{a}})}\,\frac{\chi_{e_1}(\det g_+)}{\chi_{e_1}(\det g_{p_\mathrm{b}})} = A_g\,\chi_{\mathrm{c}}(\det g_n)\,\chi_{\mathrm{c}}(\det g_+)^{-1},$$

using $\det g_n = \det g_{p_\mathrm{a}} \det g_{p_\mathrm{b}}$ and $\chi_{\mathrm{c}} = (\chi_{e_1}\chi_{e_2})^{-1}$. So $\epsilon_{E(\Gamma)}(g)\Psi(g \cdot S) = \Psi(S)$.

Step 7: clause (3). By the bijections of Step 4, the left side of (3) equals

$$\sum_S \sum_{v'} \sum_y \Big(\sum_v \psi(\mathrm{tr}(v'v))\Psi'(S[y,v])\Big) L'_{F(S,v')}(y) = \sum_S \sum_{v'} \sum_y \hat{\Psi}'_{S,v'}(y)\,L'_{F(S,v')}(y),$$

the sums over the tuples S supported on $\mathcal{E}_0$, the $T \times B$ matrices v' and the matrices y. For fixed S, v' the functions $\Phi_{S,v'}$, $\hat{\Psi}'_{S,v'}$ are matched (Step 5), so by hypothesis the inner sum is

$C\sum_x \Phi_{S,v'}(x)L_{F(S,v')}(x)$. As $\Phi_{S,v'}(x) = \Psi(S\langle x,v'\rangle)$, the bijection $(S,x,v') \mapsto S\langle x,v'\rangle$ turns $C\sum_S\sum_{v'}\sum_x \Phi_{S,v'}(x)L_{F(S,v')}(x)$ into the right side of (3). □

Lemma 6.4.92. *Let D be admissible data and let the path Γ through the two grids arise from the path Γ' by a turn at (i,j), $0 \le i,j < m$; put $l = l(\Gamma')$.*

(a) Let $i \ne j$, and let $(z,i',j') = (\mathrm{I},i,j)$ if $i < j$ and $(z,i',j') = (\mathrm{II},j,i)$ if $i > j$. Then $i' < j' < m$, $l(\Gamma) = l$, $\Gamma_{\neg z} = \Gamma'_{\neg z}$, and there are monotone paths q from $(0,m)$ to $(i',j'+1)$ and r from $(i'+1,j')$ to (l,l) in the grid z such that Γ'_z is q followed by the steps $(i',j'+1) \to (i',j') \to (i'+1,j')$ and by r, and Γ_z is q followed by the steps $(i',j'+1) \to (i'+1,j'+1) \to (i'+1,j')$ and by r. The edges of Γ' are those of $\mathcal{E}_0$ and $((z,i',j'),(z,i',j'+1))$, $((z,i',j'),(z,i'+1,j'))$; the edges of Γ are those of $\mathcal{E}_0$ and $((z,i',j'+1),(z,i'+1,j'+1))$, $((z,i'+1,j'),(z,i'+1,j'+1))$; here $\mathcal{E}_0$ is the set of edges of q, of r and of $\Gamma'_{\neg z}$, and contains none of these four edges.

(b) Let $i = j$. Then $l = j$, $l(\Gamma) = j+1$, and for each z there is a monotone path u_z from $(0,m)$ to $(j,j+1)$ in the grid z such that Γ'_z is u_z followed by the step down to (j,j) and Γ_z is u_z followed by the step right to $(j+1,j+1)$. The edges of Γ' are those of $\mathcal{E}_0$ and $((\mathrm{I},j,j),(\mathrm{I},j,j+1))$, $((\mathrm{II},j,j),(\mathrm{II},j,j+1))$; the edges of Γ are those of $\mathcal{E}_0$ and $((\mathrm{I},j,j+1),(\mathrm{I},j+1,j+1))$, $((\mathrm{II},j,j+1),(\mathrm{II},j+1,j+1))$; here $\mathcal{E}_0$ is the set of edges of u_{I} and u_{II}, and contains none of these four edges.

Proof. The edge of a step of a monotone path in the grid z is the pair of its two nodes with the smaller one first: a step down from $(a,b+1)$ to (a,b) has the edge $((z,a,b),(z,a,b+1))$, a step right from (a,b) to $(a+1,b)$ the edge $((z,a,b),(z,a+1,b))$; the edges of a monotone path are those of its steps, and the edges of a path through the two grids are those of its halves.

Nodes of a half. Along a half of a path with index l, from $(0,m)$ to (l,l), the first index never decreases and the second never increases; so every node (a,b) of the half has $0 \le a \le l \le b \le m$. If the half comes down to a node (a,b) and leaves it to the right, then $(a+1,b)$ is also a node of the half, so $a+1 \le l \le b$ and in particular $a < b$.

(a). By the definition of a turn, a turn at (i,j) either replaces a corner of a half (at the node (i,j) of the half in the grid Y, or at the node (j,i) of the half in the grid $\tilde{Y}$), or it is the turn on the diagonal, which requires $i = j$. So for $i \ne j$ it replaces a corner at which a half comes down to a node and leaves it to the right; by the preceding paragraph the node (a,b) of such a corner has $a < b$. The node (i,j) in the grid Y has this property only if $i < j$, the node (j,i) in the grid $\tilde{Y}$ only if $j < i$. Hence the corner is at the node (i',j') of the half Γ'_z, with $i' < j'$; the node $(i',j'+1)$ before it lies on the half, so $j'+1 \le m$. Write Γ'_z as q, the steps $(i',j'+1) \to (i',j') \to (i'+1,j')$, and r; the turn replaces the two middle steps by $(i',j'+1) \to (i'+1,j'+1) \to (i'+1,j')$ and changes nothing else, which gives the description of Γ_z, $\Gamma_{\neg z} = \Gamma'_{\neg z}$ and $l(\Gamma) = l$. The two replaced steps have the edges $((z,i',j'),(z,i',j'+1))$ and $((z,i',j'),(z,i'+1,j'))$, the two new steps the edges $((z,i',j'+1),(z,i'+1,j'+1))$ and $((z,i'+1,j'),(z,i'+1,j'+1))$. Every node (a,b) of q has $a \le i'$ and $b \ge j'+1$, every node of r has $a \ge i'+1$ and $b \le j'$. Each of the four edges has a node outside q (the node (i',j'), (i',j'), $(i'+1,j'+1)$, $(i'+1,j')$ respectively) and a node outside r (the node $(i',j'+1)$, (i',j'), $(i'+1,j'+1)$, $(i'+1,j'+1)$ respectively), so it is an edge neither of q nor of r; nor of $\Gamma'_{\neg z}$, whose edges lie in the other grid.

(b). For $i = j$ the turn is not at a corner, since the node (j,j) of a corner would have $j < j$; so it is the turn on the diagonal: $j = l$, both halves of Γ' end with a step down, necessarily from $(j,j+1)$ to (j,j), and Γ has index $j+1$ and both final steps replaced by the steps right

from $(j,j+1)$ to $(j+1,j+1)$. Let u_z be Γ'_z without its last step. The edges of the replaced and of the new steps are the four edges listed. Every node (a,b) of u_z has $a \le j$ and $b \ge j+1$, while each of the four edges has a node with $b = j$ (the node (j,j)) or with $a = j+1$ (the node $(j+1,j+1)$); so none of them is an edge of u_{I} or of u_{II}. □

Lemma 6.4.93. *Let D be admissible data, $0 \le j < m$, and $V(\vartheta) = \#(\mathcal{O}_E/\mathfrak{m}_\vartheta)^{2n_m}$ for every m-fold twist ϑ. There is a function c_{jj} from the m-fold twists to $\mathbb{C}^\times$, of absolute value τ_{jj} up to a norm power, such that for every turn at (j,j) from a path Γ' through the two grids to a path Γ, the path Γ' transfers periods to Γ with bound V and constant c_{jj}.*

Proof. **Step 1: the constant.** Apply Lemma 6.4.89 to the grid Y ($z = \mathrm{I}$) and the square at (j,j), with corners (I,j,j), $(\mathrm{I},j,j+1)$, $(\mathrm{I},j+1,j)$, $(\mathrm{I},j+1,j+1)$. It gives $\kappa \in \mathbb{C}$, a bijection $\sigma\colon n_{(\mathrm{I},j,j)} \sqcup n_{(\mathrm{I},j+1,j+1)} \to n_{(\mathrm{I},j,j+1)} \sqcup n_Z$ and a function c from the m-fold twists to $\mathbb{C}^\times$ of absolute value τ_{jj} up to a norm power. For each m-fold twist ϑ apply Lemma 6.4.94 to the space Z with its signs s, $\epsilon = (-1)^{j+1}$, $\beta = \chi^{\mathrm{h}}$, $\alpha' = \chi^{\mathrm{v}}_\vartheta$ and $k = f^j_\vartheta$; it gives an additive character ψ_ϑ of R_ϑ and $c_{\mathrm{ln}}(\vartheta) \in \mathbb{C}^\times$ with $|c_{\mathrm{ln}}(\vartheta)|^2 \,\#(R_\vartheta)^{\mathrm{rk}\,Z} = 1$. Take $D = \delta_w$ there, for an $n_Z \times 1$ matrix w, and $h = \rho(g)$. Then $\hat{D}(\varrho) = \psi_\vartheta(\mathrm{tr}(\varrho w)) = \sum_{w'} \psi_\vartheta(\mathrm{tr}(w'w))\,\delta_{w'}(\varrho)$, the sum over the $1 \times n_Z$ matrices w'; the datum $\mathrm{mod}_{\mathfrak{m}_\vartheta}(\cdot, v_{P'})$ and the theta kernel are linear in the function on matrices, and $-\epsilon = (-1)^j$, $k\,\mathrm{flp}_\vartheta = f^{j+1}_\vartheta$ since $f_\vartheta = \mathrm{flp}_\vartheta$ (Lemma 6.4.95). So, with P' the polynomial obtained from P by the substitution of Lemma 6.4.94 (it depends on P and on the signs of Z, not on ϑ),

$$K^{\mathrm{col}}_\vartheta[w,P](g) = c_{\mathrm{ln}}(\vartheta) \sum_{w'} \psi_\vartheta(\mathrm{tr}(w'w))\, K^{\mathrm{row}}_\vartheta[w',P'](g) \tag{6.105}$$

for all w, P and $g \in \mathrm{U}(\tilde{Y}^{j+1}_j)(\mathbb{A}_F)$. Put $c_{jj}(\vartheta) = c(\vartheta)/c_{\mathrm{ln}}(\vartheta) \in \mathbb{C}^\times$.

Step 2: the absolute value of c_{jj}. Let ϑ have modulus Q. By Lemma 6.4.90, $\#R_\vartheta = \#(\mathcal{O}_E/\mathfrak{b})\,\mathrm{N}Q$ with $\mathfrak{b} = \pi_M^b$. There are $C_1 \in \mathbb{R}$ and $n_1 \in \mathbb{Z}$ with $|c(\vartheta)|^2 = C_1(\mathrm{N}Q)^{n_1}\tau_{jj}(\vartheta)^2$ for all ϑ; hence

$$|c_{jj}(\vartheta)|^2 = |c(\vartheta)|^2\,\#(R_\vartheta)^{\mathrm{rk}\,Z} = C_1\,\#(\mathcal{O}_E/\mathfrak{b})^{\mathrm{rk}\,Z}\,(\mathrm{N}Q)^{n_1+\mathrm{rk}\,Z}\,\tau_{jj}(\vartheta)^2,$$

and c_{jj} has absolute value τ_{jj} up to a norm power.

Step 3: the shape of a turn at (j,j). Let Γ arise from Γ' by a turn at (j,j). By Lemma 6.4.92 (b), $l(\Gamma') = j$, the halves of Γ' are monotone paths u_z from $(0,m)$ to $(j,j+1)$ followed by the step down to (j,j), and Γ has index $j+1$ and the halves u_z followed by the step right to $(j+1,j+1)$. These are the paths Γ', Γ of Lemma 6.4.95, and in Lemma 6.4.91 (case $i = j$) $p = (\mathrm{I},j,j)$, $p_{\mathrm{a}} = (\mathrm{I},j,j+1)$, $p^+ = (\mathrm{I},j+1,j+1)$, $B = n_Z$, so σ is a bijection $n_p \sqcup n_{p^+} \to n_{p_{\mathrm{a}}} \sqcup B$. By Lemma 6.4.92 (b), the edges of u_{I} and u_{II} are the edges of both paths other than the four edges of the turn, so they form the set $\mathcal{E}_0$ of Lemma 6.4.91 (1). Finally the character $\chi_{\mathrm{c}} = (\chi_{(p,p_{\mathrm{a}})}\chi_{(\tilde{p},p_{\mathrm{b}})})^{-1}$ of Lemma 6.4.91 is the character $\chi_\square = (\chi_{((\mathrm{I},j,j),(\mathrm{I},j,j+1))}\chi_{((\mathrm{I},j,j),(\mathrm{I},j+1,j))})^{-1}$ of Step 1: by (GP6) the edges $(\tilde{p},p_{\mathrm{b}}) = ((\mathrm{II},j,j),(\mathrm{II},j,j+1))$ and $((\mathrm{I},j,j),(\mathrm{I},j+1,j))$ are both not vertical (in the grid $\tilde{Y}$ the first index stays, in the grid Y the second index stays) and the index that changes along each starts at j, so both have the character of the tame part of μ^0_{j+1}.

Step 4: the new path data. Let N be a function from the m-fold twists to $\mathbb{N}$ and $\Psi' = (\Psi'_\vartheta)$ a family of path data of bound $N(\vartheta)$ on the edges of Γ'. For each ϑ, Lemma 6.4.91, with ψ_ϑ, σ and Ψ'_ϑ, gives a path datum Ψ_ϑ of bound $\#(R_\vartheta)^{\mathrm{rk}\,Z}N(\vartheta)$ on the edges of Γ (here $\#B = \mathrm{rk}\,Z$, $\#T = 1$) satisfying (3). The node $(\mathrm{I},j+1,j)$ carries a space of rank $r+(2j+1)t$ (Definition 3.2.28) and $n_m = k+m(t+1) = r+1+mt$ (Definition 2.3.4); so $\mathrm{rk}\,Z = r+(2j+1)t \le r+(2m-1)t < 2(r+1+mt) = 2n_m$, and $\#R_\vartheta \ge 1$; so $\#(R_\vartheta)^{\mathrm{rk}\,Z} \le V(\vartheta)$. A sum of at most $\#(R_\vartheta)^{\mathrm{rk}\,Z}N(\vartheta)$ roots of unity is a sum of at most $V(\vartheta)N(\vartheta)$ roots of unity, and (PD1)–(PD3) do not involve the bound; hence Ψ_ϑ is a path datum of bound $V(\vartheta)N(\vartheta)$ on the edges of Γ.

Step 5: column periods are row periods. Fix a real point g_0 of $\mathrm{U}(Z)$, a polynomial P and a period triple (ϑ, y', φ). For an edge tuple S supported on $\mathcal{E}_0$ and a $1 \times n_Z$ matrix w' let $l_{S,w'}$ be the linear functional

$$l_{S,w'}(y'') = \int_{[\mathrm{U}(\tilde{Y}_j^{j+1})]} K_\vartheta^{\mathrm{row}}[w', P'](g)\, \varphi_S(g)\, y''(\rho(g)g_0)\, dg \qquad (y'' \in \mathcal{F}(\mathrm{I}, j+1, j)),$$

defined by Lemma 6.4.95 (1), and put $F(S, w') = (\mathrm{fm}_{u_\mathrm{I}}(S)y',\, l_{S,w'})$, an element of the set J of pairs (f, l) with $f \in \mathcal{F}(\mathrm{I}, j, j+1)$ and l a linear functional on $\mathcal{F}(\mathrm{I}, j+1, j)$. Since φ_S and $\mathrm{fm}_{u_\mathrm{I}}(S)$ involve only the matrices of S on the edges of u_II, u_I, which lie in $\mathcal{E}_0$, $F(S, w') = F(S_0, w')$ for every tuple S. For $(f, l) \in J$ let $L_{(f,l)}$, $L'_{(f,l)}$ be the functions L, L' of Lemma 6.4.89 for the square of Step 1, ϑ, f and l; by the definitions of $y_S^\pm$,

$$L'_{F(S_0,w')}(y(S)) = l_{S,w'}(y_S^-), \qquad L_{F(S_0,w')}(x(S)) = l_{S,w'}(y_S^+).$$

Insert (6.105) with $w = v(S)$ into the column period; the sum over w' is finite and each of its integrals converges absolutely by Lemma 6.4.95 (1), so it may be taken outside the integral:

$$C(\Psi'; g_0, P)(\vartheta, y', \varphi) = c_\mathrm{ln}(\vartheta) \sum_S \Psi'_\vartheta(S) \sum_{w'} \psi_\vartheta(\mathrm{tr}(w'v(S)))\, L'_{F(S_0,w')}(y(S)),$$

the sum over the edge tuples supported on the edges of Γ'. By Lemma 6.4.89 and Step 3, for every $(f, l) \in J$,

$$\sum_y \Psi_1(y) L'_{(f,l)}(y) = \frac{\kappa}{c(\vartheta)} \sum_x \Phi_1(x) L_{(f,l)}(x)$$

whenever Φ_1, Ψ_1 are matched for $(\sigma, \chi_\mathrm{c})$ over R_ϑ. So Lemma 6.4.91 (3), with $C = \kappa/c(\vartheta)$, turns the right side into

$$c_\mathrm{ln}(\vartheta) \frac{\kappa}{c(\vartheta)} \sum_S \Psi_\vartheta(S)\, l_{S,v'(S)}(y_S^+),$$

the sum over the tuples supported on the edges of Γ, which is $c_\mathrm{ln}(\vartheta)\frac{\kappa}{c(\vartheta)} R(\Psi; g_0, P')(\vartheta, y', \varphi)$. Thus

$$C(\Psi'; g_0, P)(\vartheta, y', \varphi) = \frac{\kappa}{c_{jj}(\vartheta)}\, R(\Psi; g_0, P')(\vartheta, y', \varphi). \tag{6.106}$$

Step 6: the transfer. Let Ξ be a function of period triples with $\Psi \propto_\Gamma \Xi$. Fix g_0 and P. By Lemma 6.4.95 (3), applied to g_0 and P', there is $\beta \in \mathbb{C}$ with $R(\Psi; g_0, P') = \beta\, \Xi$ on all period triples; by (6.106), $C(\Psi'; g_0, P)(\vartheta, y', \varphi) = \kappa\beta \cdot \Xi(\vartheta, y', \varphi)/c_{jj}(\vartheta)$ for all period triples, with $\kappa\beta$ independent of the triple. By Lemma 6.4.95 (2), applied to the function $(\vartheta, y', \varphi) \mapsto \Xi(\vartheta, y', \varphi)/c_{jj}(\vartheta)$, $\Psi' \propto_{\Gamma'} \Xi/c_{jj}$. With Step 4, Γ' transfers periods to Γ with bound V and constant c_{jj}, which by Step 2 has absolute value τ_{jj} up to a norm power. □

Lemma 6.4.94. *For $\epsilon \in \{\pm 1\}$ let ℓ_ϵ be the standard line of sign ϵ, a space with a point; by (GP8) these are the lines added in (GP2). Let S be a space with a point with signs s on an index set n, $\epsilon \in \{\pm 1\}$, ϑ an m-fold twist, β and α' automorphic characters of $\mathrm{U}(1)$, and $k \in G_f(1)$. There are an additive character ψ of $R_{\mathfrak{m}_\vartheta}$ and $C \in \mathbb{C}^\times$ with*

$$|C|^2 \cdot \#(R_{\mathfrak{m}_\vartheta})^{\#n} = 1$$

such that for every function D on the $n \times 1$ matrices over $R_{\mathfrak{m}_\vartheta}$, every polynomial P in the entries $X_{v,i,1}$ ($v \in \Phi$, $i \in n$) of such matrices and every $h \in \mathrm{G}(s)$,

$$\theta_{1,\beta}^{\ell_{-\epsilon},S}(\mathrm{mod}_{\mathfrak{m}_\vartheta}(D, v_P))((1,k),h) = C\, \theta_{\alpha',\upsilon}^{S,\ell_\epsilon}(\mathrm{mod}_{\mathfrak{m}_\vartheta}(\hat{D}, v_{P'}))(h, (1, k\,\mathrm{flp}_\vartheta)),$$

where $\hat{D}(\varrho) = \sum_y D(y)\,\psi(\mathrm{tr}(\varrho y))$ *for* $1 \times n$ *matrices* ϱ *(the sum over the* $n \times 1$ *matrices* y*),* P' *is the polynomial obtained from* P *by the substitution* $X_{v,i,1} \mapsto -\epsilon s_i\, X_{v,1,i}$*, and* $(1,k)$*,* $(1, k\,\mathrm{flp}_\vartheta)$ *are the points of* $\mathrm{G}(\pm\epsilon)$ *with infinite part* 1 *and finite parts* k*,* $k\,\mathrm{flp}_\vartheta$*.*

Proof. Throughout, ϵ, S (signs s on n), ϑ, β, α', k are fixed. Put $\mathfrak{m} = \mathfrak{m}_\vartheta$, $\Sigma = \Sigma(\mathfrak{m})$, $R = R_\mathfrak{m}$, $t = t_\vartheta$ (local in this proof), $\mathcal{O}_\Sigma = \prod_{w\in\Sigma} \mathcal{O}_{E,w}$ and $\psi_0(x) = \psi_{E,f}(\jmath\alpha\bar{\alpha}x)$. By Lemma 6.4.90, $\mathfrak{m}$ is a modulus, so Σ contains the conjugate of none of its members; hence for $w \in \Sigma$ the place $\bar{w}$ does not divide $\mathfrak{m}$ and $t_{\bar{w}} = 1$. As t generates $\mathfrak{m}\widehat{\mathcal{O}}_E$, $t_w\mathcal{O}_{E,w} = \mathfrak{m}\mathcal{O}_{E,w}$ for $w \in \Sigma$, so $R \cong \mathcal{O}_\Sigma/t\mathcal{O}_\Sigma$ (Chinese remainder theorem) and $\#R = \prod_{w\in\Sigma} |t_w|_w^{-1}$. By Lemma 4.1.3, $\widehat{\mathcal{O}}_E$ is its own annihilator under $(u,v) \mapsto \psi_0(\bar{u}v)$; in particular ψ_0 is trivial on $\widehat{\mathcal{O}}_E$. Put $c_\epsilon = c^\epsilon$, a unit of $\widehat{\mathcal{O}}_E$ with $c_\epsilon\bar{c}_\epsilon = (c\bar{c})^\epsilon = -1$, so $\bar{c}_\epsilon^{-1} = -c_\epsilon$. Write $p = x_{\ell_{-\epsilon}}$ and $p' = x_{\ell_\epsilon}$ for the standard points (ideles: 1, or the idele with components 1 at the infinite places and c at the finite places, for the signs $+1$, -1); then $p = (1, c_\epsilon)\,p'$ in both cases, $(1,y)$ denoting the idele with infinite components 1 and finite part y. Put

$$\gamma_\alpha = (1, \bar{\alpha}/\alpha), \qquad \gamma_t = (1, \bar{t}/t), \qquad \gamma_\alpha\,(1,k)\,\gamma_t = (1, k\,\mathrm{flp}_\vartheta),$$

elements of norm one, that is points of the unitary group $\mathrm{G}(\pm\epsilon) = \mathrm{U}(1)(\mathbb{A}_F)$ of a line. Let $\ell' = (\ell_\epsilon)_{\gamma_\alpha} = (\epsilon, p'\gamma_\alpha)$ be the moved line, and

$$\mathrm{H}_1 = \mathrm{Hom}(\ell_{-\epsilon}, S), \qquad \mathrm{H}_2 = \mathrm{Hom}(S, \ell'), \qquad \mathrm{H}_3 = \mathrm{Hom}(S, \ell_\epsilon).$$

The vectors of H_1 are $n \times 1$ matrices, those of H_2, H_3 are $1 \times n$ matrices; we index both by n. The sign of i in H_1 is $\sigma_i = -\epsilon s_i$, in H_2 and H_3 it is $-\sigma_i$; H_2 and H_3 have the same theta model $\mathcal{M}'$, and $\mathcal{M}$ denotes the theta model of H_1. Let $\hat{\sigma}_i \in \mathbb{A}_E$ be σ_i at the infinite places and 1 at the finite places.

Step 1: the coordinate flip. Let $\lambda = (1, -c_\epsilon\alpha/\bar{\alpha}) \in \mathbb{A}_E^\times$, so $\lambda\bar{\lambda}$ is 1 at the infinite and $c_\epsilon\bar{c}_\epsilon = -1$ at the finite places. For a $1 \times n$ adelic matrix Y let $\mathrm{fl}(Y)$ be the $n \times 1$ matrix with entries $\mathrm{fl}(Y)_i = \lambda\hat{\sigma}_i\bar{Y}_i$; in matrix form $\mathrm{fl}(Y) = \lambda\,\varepsilon_s\,Y^*\,\varepsilon_{-\epsilon}$, $\varepsilon_{-\epsilon}$ the sign idele of $\ell_{-\epsilon}$. It is additive, a homeomorphism, and has the following four properties.

(i) *Bicharacters.* $\overline{\mathrm{fl}(Y')_i}\,\hat{\sigma}_i\,\mathrm{fl}(Y)_i = \lambda\bar{\lambda}\,\hat{\sigma}_i\,Y_i'\bar{Y}_i = -\hat{\sigma}_i'Y_i'\bar{Y}_i$, where $\hat{\sigma}_i'$ is the sign of i in H_2 at the infinite places and 1 at the finite places. Hence the argument of ψ_E in $\beta_{\mathrm{H}_1}(\mathrm{fl}Y, \mathrm{fl}Y')$ is $-\frac{\jmath}{2}\sum_i \hat{\sigma}_i'Y_i'\bar{Y}_i = \overline{\frac{\jmath}{2}\sum_i \hat{\sigma}_i'\bar{Y}_i'Y_i}$, and $\beta_{\mathrm{H}_1}(\mathrm{fl}Y, \mathrm{fl}Y') = \beta_{\mathrm{H}_2}(Y, Y')$.

(ii) *Vacua.* At $v \in \Phi$, $|\mathrm{fl}(Y)_{i,v}| = |Y_{i,v}|$. At the finite places, for $Y = \alpha Z$, $\mathrm{fl}(\alpha Z)_i = -c_\epsilon(\alpha/\bar{\alpha})\bar{\alpha}\bar{Z}_i = \alpha(-c_\epsilon\bar{Z}_i)$; as c_ϵ is a unit, $\mathrm{fl}(Y)_f \in \alpha\widehat{\mathcal{O}}_E^n$ if and only if $Y_f \in \alpha\widehat{\mathcal{O}}_E^n$. On this set $\sum_i \overline{\mathrm{fl}(Y)_i}\mathrm{fl}(Y)_i = -\sum_i \bar{Y}_iY_i = -\alpha\bar{\alpha}q$ at the finite places, with $q = \sum_i \bar{Z}_iZ_i \in \widehat{\mathcal{O}}_E$, and the two finite vacuum phases of (WD5) differ by $\psi_{E,f}(c_0\jmath\alpha\bar{\alpha}q) = \psi_0(c_0q) = 1$, as c_0q is integral. So $e_n(\mathrm{fl}(Y)) = e_n(Y)$.

(iii) *Measures.* In the coordinates (u,z) of (WD6), fl is $u_{v,i} \mapsto \sigma_i\bar{u}_{v,i}$ at the infinite places and $z_i \mapsto -c_\epsilon\bar{z}_i$ at the finite places (by (ii)); these preserve Lebesgue measure, respectively $\widehat{\mathcal{O}}_E^n$ and its Haar measure. So fl carries μ_n to μ_n.

(iv) *Rational vectors.* Those of H_1 are $x_S^{-1}\xi p$ ($\xi \in E^{n\times 1}$), those of H_2 are $(p'\gamma_\alpha)^{-1}\eta x_S$ ($\eta \in E^{1\times n}$). From $x_S^*J_Sx_S = \varepsilon_s$ we get $\varepsilon_s x_S^* = x_S^{-1}J_S^{-1}$, and from the point of $\ell_{-\epsilon}$, $\varepsilon_{-\epsilon} = -\epsilon\bar{p}p$. Hence

$$\mathrm{fl}((p'\gamma_\alpha)^{-1}\eta x_S) = \lambda\,x_S^{-1}J_S^{-1}\eta^*\,\overline{p'\gamma_\alpha}^{-1}(-\epsilon)\bar{p}p = x_S^{-1}(-\epsilon J_S^{-1}\eta^*)p\cdot\lambda\,(1,\bar{c}_\epsilon)\,\gamma_\alpha,$$

using $\bar{p}/\bar{p}' = (1,\bar{c}_\epsilon)$ and $\bar{\gamma}_\alpha^{-1} = \gamma_\alpha$. The finite part of $\lambda(1,\bar{c}_\epsilon)\gamma_\alpha$ is $-c_\epsilon\bar{c}_\epsilon = 1$, its infinite part 1. As $\eta \mapsto -\epsilon J_S^{-1}\eta^*$ is a bijection of $E^{1\times n}$ onto $E^{n\times 1}$, fl maps the rational vectors of H_2 onto those of H_1.

Let $\mathrm{Fl}f = f \circ \mathrm{fl}$. By (i), (ii) and (iii), applied to the images under fl^{-1} of vectors of H_1, the additive homeomorphism fl^{-1} from the vectors of H_1 to those of H_2 satisfies the hypotheses of (1) of Lemma 6.3.65 for the signs of H_1 and of H_2, and $\mathrm{Fl}f = f \circ (\mathrm{fl}^{-1})^{-1}$. So Fl is a linear bijection of $\mathcal{M}$ onto $\mathcal{M}'$ preserving norm and inner product, $\mathrm{Fl}v^0 = v^0$, and $\mathrm{Fl}\,\rho(X) = \rho(\mathrm{fl}^{-1}X)\,\mathrm{Fl}$, that is $\mathrm{Fl}\,\rho(\mathrm{fl}(Y)) = \rho(Y)\,\mathrm{Fl}$. By (iv), fl^{-1} maps the rational vectors of H_1 onto those of H_2, so (3) of Lemma 6.3.65 (for $V = \mathrm{H}_1$, $V' = \mathrm{H}_2$, $\tau = \mathrm{fl}^{-1}$ and $\Pi = \mathrm{Fl}$) gives $z_0 \in \mathbb{C}$, $|z_0| = 1$, with

$$\theta_{\mathrm{H}_2}(\mathrm{Fl}f) = z_0\,\theta_{\mathrm{H}_1}(f) \qquad (f \in \mathcal{M}). \tag{6.107}$$

Step 2: the Weil operators. Let $g \in \mathrm{G}(-\epsilon)$ and $h \in \mathrm{G}(s)$. The pair map gives the operator $A\colon X \mapsto hXg^{-1} = g^{-1}hX$ of H_1 and, for the pair (h, g) of (S, ℓ'), the operator $B\colon Y \mapsto gYh^{-1}$ of H_2; on coordinate vectors indexed by n, $A = g^{-1}h$ and $B = gh^{-\mathsf{T}} = (A^{-1})^{\mathsf{T}}$. Since $h^*\varepsilon_s h = \varepsilon_s$ gives $\varepsilon_s h^{-*} = h\varepsilon_s$, and $\bar{g} = g^{-1}$,

$$\mathrm{fl}(BY) = \lambda\,\varepsilon_s h^{-*} Y^* \bar{g}\,\varepsilon_{-\epsilon} = g^{-1}h\,\mathrm{fl}(Y) = A\,\mathrm{fl}(Y).$$

The multipliers. Let $d = \det A = g^{-\#n} \det h$. The operator A lies in the group G of the signs $(\sigma_i)_{i\in n}$ of H_1 (it is a pair image), the signs of H_2 are $(-\sigma_i)_{i\in n}$, and $B = (A^{-1})^{\mathsf{T}}$; so (4) of Lemma 6.3.65 for $\delta = (\sigma_i)_{i\in n}$ gives $\Delta_{\mathrm{H}_2}(B) = \upsilon(d)\,\Delta_{\mathrm{H}_1}(A)$, and

$$\frac{\Delta_{\mathrm{H}_1}(A)}{\Delta_{\mathrm{H}_2}(B)} = \upsilon(d)^{-1} = \upsilon(g)^{\#n}\,\upsilon(\det h)^{-1}.$$

Intertwining. As $\mathrm{fl}(BY) = A\,\mathrm{fl}(Y)$ for every vector Y of H_2, $\mathrm{fl}^{-1}(AX) = B\,\mathrm{fl}^{-1}(X)$ for every vector X of H_1, and (2) of Lemma 6.3.65 gives

$$\mathrm{Fl}\,\omega(A) = \frac{\Delta_{\mathrm{H}_1}(A)}{\Delta_{\mathrm{H}_2}(B)}\,\omega(B)\,\mathrm{Fl}.$$

With (WR4) this reads, for the pair operators with trivial characters,

$$\mathrm{Fl}\,\omega_{1,1}^{\ell_{-\epsilon},S}(g,h) = \omega_{\upsilon^{\#n},\upsilon}^{S,\ell'}(h,g)\,\mathrm{Fl}. \tag{6.108}$$

Step 3: the reversed kernel on the standard line. The point of H_2 is that of H_3 composed with $\Lambda\colon Y \mapsto \gamma_\alpha Y$, the image of $(1, \gamma_\alpha)$ under the pair map of (S, ℓ_ϵ); so $\mathrm{H}_2 = (\mathrm{H}_3)_\Lambda$ and its rational vectors are Λ^{-1} of those of H_3. If k_3 is a transporting element of H_3 (WR3), then $k_3\Lambda$ is one of H_2, so by Lemma 4.1.17 (independence of the transporting element) and the multiplicativity of ω (Proposition 4.1.14), $\theta_{\mathrm{H}_2} = \theta_{\mathrm{H}_3} \circ \omega(\Lambda)$. Hence for automorphic characters a, a' and $F \in \mathcal{M}'$, by (WR4) and $\omega(\Lambda)\omega_{1,1}^{S,\ell_\epsilon}(h,g) = \omega_{1,1}^{S,\ell_\epsilon}(h,\gamma_\alpha g)$,

$$\theta_{a,a'}^{S,\ell'}(F)(h,g) = a(g)\,a'(\det h)^{-1}\,\theta_{\mathrm{H}_3}\big(\omega_{1,1}^{S,\ell_\epsilon}(h,\gamma_\alpha g)F\big) = a(\gamma_\alpha)^{-1}\,\theta_{a,a'}^{S,\ell_\epsilon}(F)(h,\gamma_\alpha g).$$

Combining with (6.107) and (6.108): for $f \in \mathcal{M}$, $g \in \mathrm{G}(-\epsilon)$, $h \in \mathrm{G}(s)$,

$$\theta_{1,1}^{\ell_{-\epsilon},S}(f)(g,h) = z_0^{-1}\,\theta_{\mathrm{H}_2}\big(\mathrm{Fl}\,\omega_{1,1}^{\ell_{-\epsilon},S}(g,h)f\big) = z_0^{-1}\,\upsilon(\gamma_\alpha)^{-\#n}\,\theta_{\upsilon^{\#n},\upsilon}^{S,\ell_\epsilon}(\mathrm{Fl}f)(h,\gamma_\alpha g). \tag{6.109}$$

Step 4: Fock vectors. At the infinite places fl^{-1} sends X to Y with $Y_i = \sigma_i \bar{X}_i$; as κ_v is real, $\mathrm{fl}^{-1}(\mathrm{rv}(y)) = \mathrm{rv}(y^\sharp)$ with $y^\sharp_{v,1,i} = \sigma_i \bar{y}_{v,i,1}$, and $y \mapsto y^\sharp$ preserves ν. By Step 1, $\mathrm{Fl}\rho(\mathrm{rv}(y)) = \rho(\mathrm{rv}(y^\sharp))\mathrm{Fl}$, and Fl is a bounded linear map, so it passes under the integral of (WR6): for a Fock vector v of H_1 and $f \in \mathcal{M}$, substituting $y' = y^\sharp$,

$$\mathrm{Fl}(v \otimes f) = \int v(y)\,\rho(\mathrm{rv}(y^\sharp))\,\mathrm{Fl}f\,d\nu(y) = v^\flat \otimes \mathrm{Fl}f, \qquad v^\flat(y') = v(y),\ y_{v,i,1} = \sigma_i \bar{y}'_{v,1,i}.$$

For $v = v_P$: $e^\infty(y) = e^\infty(y')$; if $\sigma_i = 1$, the coordinate $\zeta(y)_{v,i,1} = \bar{y}_{v,i,1} = y'_{v,1,i} = \zeta(y')_{v,1,i}$ (the sign of $(1,i)$ in H_2 is -1); if $\sigma_i = -1$, $\zeta(y)_{v,i,1} = y_{v,i,1} = -\bar{y}'_{v,1,i} = -\zeta(y')_{v,1,i}$. So $\zeta(y)_{v,i,1} = \sigma_i\,\zeta(y')_{v,1,i} = -\epsilon s_i\,\zeta(y')_{v,1,i}$, and $v_P^\flat = v_{P'}$. By Proposition 4.1.14, $\omega(\Lambda')\rho(z)\omega(\Lambda')^{-1} = \rho(\Lambda' z)$; for Λ' the pair image of $((1,\gamma'),1)$, γ' finite, $\Lambda'\mathrm{rv}(y) = \mathrm{rv}(y)$, so likewise

$$\omega_{1,1}^{\ell_{-\epsilon},S}((1,\gamma'),1)(v\otimes f) = v\otimes \omega_{1,1}^{\ell_{-\epsilon},S}((1,\gamma'),1)f. \tag{6.110}$$

Step 5: class indicators. For $y\in R^{n\times 1}$ fix a lift $\tilde{y}\in\mathcal{O}_\Sigma^n$, let q_y be the finite vector with components $-t_w^{-1}\tilde{y}_w$ at $w\in\Sigma$ and 0 elsewhere, and $\varphi_y = \rho(\mathrm{fv}(q_y))v^0\in\mathcal{M}$; let δ_y be the indicator of $\{y\}$.

(a) By (2) and (3) of Lemma 4.1.3 (with $A=\Sigma$, $\mathbf{1}_\Sigma\otimes_\Sigma v^0 = v^0$), $\varphi_y = \Psi_y\otimes_\Sigma v^0$ with $\Psi_y(x) = \psi_0(\sum_i \overline{(q_y)_i}(2x+q_y)_{\Sigma,i}/2)\mathbf{1}_\Sigma(x+q_y) = \mathbf{1}_\Sigma(x - t^{-1}\tilde{y})$, the phase being 1 because $\bar{q}_y$ is supported on $\bar{\Sigma}$, which is disjoint from Σ. Apply (1) of Lemma 4.1.16 to the pair $(\ell_{-\epsilon},S)$, $A=\Sigma$, trivial characters, the finite point $\bar{t}/t$ of $\ell_{-\epsilon}$ (unitary, 1 outside $\Sigma\cup\bar{\Sigma}$) and 1 of S. At $w\in\Sigma$, $(\bar{t}/t)_w = \overline{t_{\bar{w}}}/t_w = t_w^{-1}$; so $\Psi'_y(X) = \Psi_y(X\bar{t}/t) = 1$ if and only if $t_w^{-1}(X_w - \tilde{y}_w)$ is integral for all $w\in\Sigma$, that is $\Psi'_y = \delta_y^\natural$. The scalar is $\upsilon(\delta_t)^{\#n}|\delta_t|^{\#n/2}$, δ_t the idele with components t_w^{-1} at $w\in\Sigma$ and 1 elsewhere ($1[p] = \upsilon^{-p}$), and $|\delta_t| = \#R$. With $A_\vartheta = \upsilon(\delta_t)^{\#n}(\#R)^{\#n/2}$, $|A_\vartheta|^2 = (\#R)^{\#n}$,

$$\omega_{1,1}^{\ell_{-\epsilon},S}(\gamma_t,1)\,\varphi_y = A_\vartheta\,\delta_y^\natural\otimes_\Sigma v^0.$$

(b) By (ii), $\mathrm{fl}^{-1}(\mathrm{fv}(q)) = \mathrm{fv}(c_\epsilon\bar{q}^{\mathsf{T}})$ ($\alpha q = \alpha(-c_\epsilon)\bar{Z}$ gives $Z = -\bar{c}_\epsilon^{-1}\bar{q} = c_\epsilon\bar{q}$). So, by Step 1, $\mathrm{Fl}\varphi_y = \rho(\mathrm{fv}(r_y))v^0$ with $r_y = c_\epsilon\bar{q}_y^{\mathsf{T}}$, supported on $\bar{\Sigma}$, and $\bar{r}_y = \bar{c}_\epsilon q_y^{\mathsf{T}}$. By (2), (3) of Lemma 4.1.3 for H_3 (whose theta model is $\mathcal{M}'$), $\mathrm{Fl}\varphi_y = \hat{\Psi}_y\otimes_\Sigma v^0$ with, as $(r_y)_\Sigma = 0$,

$$\hat{\Psi}_y(x) = \psi_0\Big(\sum_i(\bar{r}_y)_i x_{\Sigma,i}\Big)\mathbf{1}_\Sigma(x) = \psi_0\Big(-\bar{c}_\epsilon t^{-1}\sum_i \tilde{y}_i x_{\Sigma,i}\Big)\mathbf{1}_\Sigma(x).$$

Define $\psi(a \bmod t\mathcal{O}_\Sigma) = \psi_0(-\bar{c}_\epsilon t^{-1}a)$ for $a\in\mathcal{O}_\Sigma$, read as the finite adele with components a_w at Σ and 0 elsewhere. It is well defined, since for $a\in t\mathcal{O}_\Sigma$ the argument is integral and ψ_0 is trivial on $\widehat{\mathcal{O}}_E$, and it is an additive character of R; it depends only on ϵ and ϑ. For x integral at Σ with $x_\Sigma \bmod \mathfrak{m} = \varrho$, $\sum_i \tilde{y}_i x_{\Sigma,i}$ lifts $\mathrm{tr}(\varrho y)$. Hence $\hat{\Psi}_y = \hat{\delta}_y^\natural$ with $\hat{\delta}_y(\varrho) = \psi(\mathrm{tr}(\varrho y))$.

Step 6: conclusion. Fix y. By (XD5), Step 5 (a) and (6.110),

$$\mathrm{mod}_{\mathfrak{m}}(\delta_y, v_P) = v_P\otimes(\delta_y^\natural\otimes_\Sigma v^0) = A_\vartheta^{-1}\,\omega_{1,1}^{\ell_{-\epsilon},S}(\gamma_t,1)\,(v_P\otimes\varphi_y).$$

By (WR4),

$$\omega_{1,\beta}((1,k),h) = \beta((1,k))^{-1}\omega_{1,1}((1,k),h),$$

and $\omega_{1,1}((1,k),h)\omega_{1,1}(\gamma_t,1) = \omega_{1,1}((1,k)\gamma_t,h)$ (the pair map is a homomorphism and ω is multiplicative, Proposition 4.1.14). Hence, by (6.109), Step 4 and Step 5 (b),

$$\theta_{1,\beta}^{\ell_{-\epsilon},S}(\mathrm{mod}_{\mathfrak{m}}(\delta_y,v_P))((1,k),h) = \frac{\theta_{1,1}^{\ell_{-\epsilon},S}(v_P\otimes\varphi_y)((1,k)\gamma_t,h)}{A_\vartheta\,\beta((1,k))} = \frac{\theta^{S,\ell_\epsilon}_{\upsilon^{\#n},\upsilon}(\mathrm{mod}_{\mathfrak{m}}(\hat{\delta}_y,v_{P'}))(h,g_1)}{A_\vartheta\,\beta((1,k))\,z_0\,\upsilon(\gamma_\alpha)^{\#n}},$$

with $g_1 = \gamma_\alpha(1,k)\gamma_t = (1, k\,\mathrm{flp}_\vartheta)$, since $\mathrm{Fl}(v_P\otimes\varphi_y) = v_{P'}\otimes(\hat{\delta}_y^\natural\otimes_\Sigma v^0)$. By (WR4),

$$\theta^{S,\ell_\epsilon}_{\upsilon^{\#n},\upsilon}(F)(h,g_1) = \upsilon(g_1)^{\#n}\alpha'(g_1)^{-1}\theta^{S,\ell_\epsilon}_{\alpha',\upsilon}(F)(h,g_1).$$

Put

$$C = \frac{\upsilon(g_1)^{\#n}}{A_\vartheta\,\beta((1,k))\,z_0\,\upsilon(\gamma_\alpha)^{\#n}\,\alpha'(g_1)},$$

which does not depend on y, P, h and has $|C|^2 = |A_\vartheta|^{-2} = (\#R)^{-\#n}$, all other factors having modulus one; so $C \neq 0$ and $|C|^2\,\#(R)^{\#n} = 1$. Then the identity of the statement holds for $D = \delta_y$. A general D is $\sum_y D(y)\delta_y$, and $\hat{D} = \sum_y D(y)\hat{\delta}_y$; the maps $D \mapsto D^\natural \mapsto D^\natural \otimes_\Sigma v^0$ ((1) of Lemma 4.1.3: the datum does not depend on the section), $v \otimes \cdot$, the Weil operators and the theta functionals are linear, so both sides are linear in D, and the identity holds for every D, every P and every h, with the same ψ and C. □

Lemma 6.4.95. *Let D be admissible data and $0 \leq j < m$. Let $Z = Y_j^{j+1}$, the space at the node $(\mathrm{I}, j+1, j)$, with index set n_Z, and let $\rho\colon \mathrm{U}(\tilde{Y}_j^{j+1})(\mathbb{A}_F) \to \mathrm{U}(Z)(\mathbb{A}_F)$ be the isomorphism given by the renumbering of (GP3). For an m-fold twist ϑ let $f_\vartheta = \mathrm{flp}_\vartheta$, and let $\chi^{\mathrm{h}} = \mu_{j+1}^0 \upsilon^{\lceil \mathrm{rk}\, Z/2 \rceil}$ and $\chi_\vartheta^{\mathrm{v}} = \mu_{j+1}^0 \vartheta^{j+1} \upsilon^{\lfloor \mathrm{rk}\, Z/2 \rfloor}$ be the edge characters of (GP6) for $l = j$ and $N = \mathrm{rk}\, Z$. For an $n_Z \times 1$ matrix w and a $1 \times n_Z$ matrix w' over R_ϑ, polynomials P in the entries $X_{v,i,1}$ and P' in the entries $X_{v,1,i}$ ($v \in \Phi$, $i \in n_Z$), and $g \in \mathrm{U}(\tilde{Y}_j^{j+1})(\mathbb{A}_F)$ put*

$$K_\vartheta^{\mathrm{col}}[w,P](g) = \theta_{1,\chi^{\mathrm{h}}}^{\ell_{(-1)^j},Z}\big(\mathrm{mod}_{\mathfrak{m}_\vartheta}(\delta_w, v_P)\big)\big((1, f_\vartheta^j), \rho(g)\big),$$
$$K_\vartheta^{\mathrm{row}}[w',P'](g) = \theta_{\chi_\vartheta^{\mathrm{v}},\upsilon}^{Z,\ell_{(-1)^{j+1}}}\big(\mathrm{mod}_{\mathfrak{m}_\vartheta}(\delta_{w'}, v_{P'})\big)\big(\rho(g), (1, f_\vartheta^{j+1})\big),$$

δ denoting the indicator of one matrix. For a monotone path u in the grid z and an edge tuple S let $\mathrm{fm}_u(S)$ be the composite along u of the moves by the matrices of S (the transposed move for a step down, the lift move for a step right). Let u_{I}, u_{II} be monotone paths from $(0,m)$ to $(j, j+1)$ in the grids Y and $\tilde{Y}$; let Γ' be the path with index j whose halves are u_z followed by the step down to (j,j), and Γ the path with index $j+1$ whose halves are u_z followed by the step right to $(j+1, j+1)$. For a period triple (ϑ, y', φ) and an edge tuple S put $\varphi_S = \mathrm{fm}_{u_{\mathrm{II}}}(S)\varphi \in \mathcal{F}(\mathrm{II}, j, j+1)$ and

$$y_S^- = \mathrm{mv}_{((\mathrm{I},j,j),(\mathrm{I},j+1,j))}\big(y_2(S)\big)\,\mathrm{mv}_{((\mathrm{I},j,j),(\mathrm{I},j,j+1))}\big(y_1(S)\big)\,\mathrm{fm}_{u_{\mathrm{I}}}(S)\,y',$$
$$y_S^+ = \mathrm{mv}_{((\mathrm{I},j+1,j),(\mathrm{I},j+1,j+1))}\big(x_2(S)\big)\,\mathrm{mv}_{((\mathrm{I},j,j+1),(\mathrm{I},j+1,j+1))}\big(x_1(S)\big)\,\mathrm{fm}_{u_{\mathrm{I}}}(S)\,y',$$

both in $\mathcal{F}(\mathrm{I}, j+1, j)$. For families $\Psi' = (\Psi'_\vartheta)$, $\Psi = (\Psi_\vartheta)$ of functions of edge tuples, a real point g_0 of $\mathrm{U}(Z)$ (read in $\mathrm{U}(Z)(\mathbb{A}_F)$ with finite part 1) and polynomials P, P' as above define the column period *and the* row period

$$C(\Psi'; g_0, P)(\vartheta, y', \varphi) = \sum_S \Psi'_\vartheta(S) \int_{[\mathrm{U}(\tilde{Y}_j^{j+1})]} K_\vartheta^{\mathrm{col}}[v(S), P](g)\,\varphi_S(g)\,y_S^-\big(\rho(g)g_0\big)\,dg,$$

$$R(\Psi; g_0, P')(\vartheta, y', \varphi) = \sum_S \Psi_\vartheta(S) \int_{[\mathrm{U}(\tilde{Y}_j^{j+1})]} K_\vartheta^{\mathrm{row}}[v'(S), P'](g)\,\varphi_S(g)\,y_S^+\big(\rho(g)g_0\big)\,dg,$$

the sums over the edge tuples supported on the edges of Γ', resp. of Γ. Then:

(1) for every m-fold twist ϑ, all w, w', P, P' as above, every $\varphi'' \in \mathcal{F}(\mathrm{II}, j, j+1)$ and every $y'' \in \mathcal{F}(\mathrm{I}, j+1, j)$, the functions

$$g \mapsto K_\vartheta^{\mathrm{col}}[w,P](g)\,\varphi''(g)\,y''(\rho(g)g_0),$$
$$g \mapsto K_\vartheta^{\mathrm{row}}[w',P'](g)\,\varphi''(g)\,y''(\rho(g)g_0)$$

are functions on $[\mathrm{U}(\tilde{Y}_j^{j+1})]$ whose integrals converge absolutely; in particular the column and row periods are defined;

(2) if for every g_0 and every P there is $\beta \in \mathbb{C}$ with $C(\Psi'; g_0, P)(\vartheta, y', \varphi) = \beta\,\Xi(\vartheta, y', \varphi)$ for all period triples, then $\Psi' \propto_{\Gamma'} \Xi$;

(3) if $\Psi \propto_\Gamma \Xi$, then for every g_0 and every P' there is $\beta \in \mathbb{C}$ with $R(\Psi; g_0, P')(\vartheta, y', \varphi) = \beta\, \Xi(\vartheta, y', \varphi)$ for all period triples.

Proof. Fix D and $j < m$. Write $p = (\mathrm{I}, j, j)$, $p_Z = (\mathrm{I}, j+1, j)$, $p^+ = (\mathrm{I}, j+1, j+1)$, $\tilde{p} = (\mathrm{II}, j, j)$, $q = (\mathrm{II}, j, j+1)$, $\tilde{p}^+ = (\mathrm{II}, j+1, j+1)$, N_x for the rank of the space at a node x, $\ell_- = \ell_{(-1)^j}$, $\ell_+ = \ell_{(-1)^{j+1}}$, and $\mathrm{ec}_{c,l}(N)$ for the edge character of (GP6). For a monotone path, an edge tuple S and a period triple (ϑ, y', φ) put $u_S = \mathrm{mv}_{(p,(\mathrm{I},j,j+1))}(y_1(S))\,\mathrm{fm}_{u_\mathrm{I}}(S) y' \in \mathcal{F}(p)$ and $U_S = \mathrm{mv}_{((\mathrm{I},j,j+1),p^+)}(x_1(S))\,\mathrm{fm}_{u_\mathrm{I}}(S) y' \in \mathcal{F}(p^+)$, so that y_S^- is the lift move of u_S along (p, p_Z) with $y_2(S)$ and y_S^+ the transposed move of U_S along (p_Z, p^+) with $x_2(S)$.

Step 0: spaces, kernels, heights, types.

(a) *Identifications.* By (GP2) and (GP8) there is a sign-preserving bijection $e_-\colon n_{\tilde{p}} \to n_p \sqcup \{*\}$ (the decomposition $n_{\tilde{p}} = n_p \sqcup T$) with $\tilde{Y}_j^j = (Y_j^j \oplus \ell_-)^{e_-}$: for $j = 0$ this is (GP8) with e_- the identity, the added line being the base index r; for $j \geq 1$ both $\tilde{Y}_j^j$ and $Y_j^j \oplus \ell_-$ are standard spaces (the orthogonal sum of standard spaces is standard), and a standard space renumbered by a sign-preserving bijection is the standard space of the renumbered signs. Likewise $\tilde{Y}_{j+1}^{j+1} = (Y_{j+1}^{j+1} \oplus \ell_+)^{e_+}$ $(j+1 \geq 1)$, and $\tilde{Y}_j^{j+1} = Z^{e_0}$ with e_0 the renumbering of (GP3), $\rho(g^{e_0}) = g$. Write $\iota_\pm(h,k) = \mathrm{diag}(h,k)^{e_\pm}$; this is the inclusion of (GP2) used in (3.3), and, real and finite components commuting, the integrand of $\langle u, v\rangle_{l,g}$ there is $u(h)\, v(\iota(h, (1, \mathrm{flp}_\vartheta^l))(g,1))$.

(b) *Characters.* By (GP6), for the pair (x, x') of an edge in the notation of (XD2) (Hom of the end with the smaller indices into the other), the character on x' is the edge character at the rank N_x and the character on x the one at $N_{x'}$. The ranks are $N_p = r + 2jt$, $N_{\tilde{p}} = N_p + 1$, $N_q = N_{p_Z} = r + (2j+1)t$, $N_{p^+} = r + 2(j+1)t$, $N_{\tilde{p}^+} = N_{p^+} + 1$, with N_p, N_{p^+} odd. Hence: the edge $(\tilde{p}, q)$ (transposed move, $c = \mathrm{I}$, $l = j$) has $\mathrm{ec}_{\mathrm{I},j}(N_{\tilde{p}}) = \mu_{j+1}^0 \upsilon^{(N_p+1)/2} = \mathrm{ec}_{\mathrm{I},j}(N_p)$ on q and χ^h on $\tilde{p}$; the edge (p, p_Z) (lift move) has $\mathrm{ec}_{\mathrm{I},j}(N_p)$ on Z and χ^h on p; the edge $(q, \tilde{p}^+)$ (lift move, $c = \mathrm{II}$) has $\chi_\vartheta^\mathrm{v}$ on $\tilde{p}^+$ and $\mathrm{ec}_{\mathrm{II},j}(N_{\tilde{p}^+}) = \mathrm{ec}_{\mathrm{II},j}(N_{p^+})\,\upsilon$ on q (as $\lfloor N_{\tilde{p}^+}/2 \rfloor = \lfloor N_{p^+}/2 \rfloor + 1$); the edge (p_Z, p^+) (transposed move) has $\chi_\vartheta^\mathrm{v}$ on p^+ and $\mathrm{ec}_{\mathrm{II},j}(N_{p^+})$ on Z.

(c) *Splitting.* Read matrices and polynomials on $q \times \tilde{p}$ and $\tilde{p}^+ \times q$ entries through $e_0, e_\pm$ as in Lemma 6.4.91; so $S(\tilde{p}, q) = (y_2(S) \,|\, v(S))$ and $S(q, \tilde{p}^+) = \binom{x_2(S)}{v'(S)}$. For polynomials A in the variables of $n_Z \times n_p$ (resp. $n_{p^+} \times n_Z$) entries and B in those of $n_Z \times \{*\}$ (resp. $\{*\} \times n_Z$) entries, let Q_{AB} be the polynomial of the $q \times \tilde{p}$ (resp. $\tilde{p}^+ \times q$) entries obtained from AB by (4) of Lemma 6.4.96. Then (3), (4) of Lemma 6.4.96, Lemma 6.4.97 (a), (b) and (b) above give, with $\kappa_{X,Q}^e$ the theta kernel of the moves along an edge e with the datum $\mathrm{mod}_{\mathfrak{m}_\vartheta}(\delta_X, v_Q)$ and the characters of (b), for $h \in \mathrm{G}$ of p, resp. of p^+, $k \in G_f(1)$ and $g \in \mathrm{U}(\tilde{Y}_j^{j+1})(\mathbb{A}_F)$:

$$\kappa^{(\tilde{p},q)}_{(X_1|X_2),Q_{AB}}(\iota_-(h,(1,k)), g) = \kappa^{(p,p_Z)}_{X_1,A}(h, \rho(g)) \cdot \theta^{\ell_-,Z}_{1,\chi^\mathrm{h}}(\mathrm{mod}_{\mathfrak{m}_\vartheta}(\delta_{X_2}, v_B))((1,k), \rho(g)), \tag{6.111}$$

$$\kappa^{(q,\tilde{p}^+)}_{\binom{X_1}{X_2},Q_{AB}}(g, \iota_+(h,(1,k))) = \theta^{Z,\ell_+}_{\chi_\vartheta^\mathrm{v},\upsilon}(\mathrm{mod}_{\mathfrak{m}_\vartheta}(\delta_{X_2}, v_B))(\rho(g), (1,k)) \cdot \kappa^{(p_Z,p^+)}_{X_1,A}(\rho(g), h). \tag{6.112}$$

In (6.111) the target characters multiply, $\mathrm{ec}_{\mathrm{I},j}(N_p) \cdot 1$, and the source character χ^h is common; in (6.112) the target character $\chi_\vartheta^\mathrm{v}$ is common and the source characters multiply, $\mathrm{ec}_{\mathrm{II},j}(N_{p^+})\,\upsilon$. The indicator $\delta_{(X_1|X_2)}$ is $\delta_{X_1} \otimes \delta_{X_2}$. For $k = \mathrm{flp}_\vartheta^j$, resp. $\mathrm{flp}_\vartheta^{j+1}$, and $(X_2, B) = (v(S), P)$, resp. $(v'(S), P')$, the line factors are $K_\vartheta^\mathrm{col}[v(S), P](g)$ and $K_\vartheta^\mathrm{row}[v'(S), P'](g)$.

(d) *Heights.* For spaces with points V, V' and $h \in \mathrm{G}$ of V, h' of V', the lattice of $(V \oplus V')_{\mathrm{diag}(h,h')}$ is the product of the lattices of V_h, $V'_{h'}$ and the euclidean norm of a pair dominates that of each entry, so

$$\lambda_{V\oplus V'}(\mathrm{diag}(h,h')) = \min(\lambda_V(h), \lambda_{V'}(h')) \quad \text{and} \quad \mathrm{ht}_{V\oplus V'}(\mathrm{diag}(h,h')) \le \mathrm{ht}_V(h)\,\mathrm{ht}_{V'}(h').$$

With (1) of Lemma 6.4.96:

$$\mathrm{ht}_{\tilde{Y}^j_j}(\iota_-(h,(1,k))) \le c_k\,\mathrm{ht}_{Y^j_j}(h), \qquad \mathrm{ht}_{\tilde{Y}^{j+1}_{j+1}}(\iota_+(h,(1,k))) \le c_k\,\mathrm{ht}_{Y^{j+1}_{j+1}}(h)$$

with $c_k = \mathrm{ht}_{\ell_\pm}((1,k))$, and $\mathrm{ht}_Z(\rho(g)) = \mathrm{ht}_{\tilde{Y}^{j+1}_j}(g)$; ρ carries rational points to rational points and the measure to the measure.

(e) *Types.* By Lemma 6.4.15 (d) the Fock vectors of the joint harmonics $H_{(p,p_Z)}$ and $H_{(q,\tilde{p}^+)}$ of the lift moves along (p,p_Z) and $(q,\tilde{p}^+)$ ($j<m$) are edges for the weights of their characters from e_p to e_{p_Z} and from e_q to $e_{\tilde{p}^+}$; by Lemma 6.4.15 (b) ($o=0$, $a=j+1$, $b=j$) that of the transposed move along (p_Z,p^+) is an edge from $\bar{e}_{p_Z}$ to $\bar{e}_{p^+}$, hence a half edge for the reversed real part from e_{p^+} to e_{p_Z}. By (a) of (ED4) and the invariance of (ED1) under complex conjugation, e_p, e_q, e_{p^+} are cusp kernels of order 0; so by Lemma 6.3.75 the grid forms at p, q, p^+ decrease rapidly: $\sup|F|\,\mathrm{ht}^A < \infty$ for every A. Grid forms are bounded (ED2).

(f) *Bounds.* By Proposition 4.1.14 the Heisenberg operators are isometric and strongly continuous for the L^1 norm, so the integral of (WR6) gives $\|v\otimes f\| \le \|v\|\,\|f\|$ ($\|v\|$ the $L^1(\nu)$ norm). By Lemma 6.2.28, every theta kernel above satisfies $|\theta(v\otimes f)(g,h)| \le Y\|v\|\|f\|(\mathrm{ht}(g)\mathrm{ht}(h))^A$ with Y, A depending only on the two spaces. By Theorem 4.2.18 the quotients $[\mathrm{U}(V)]$ have finite volume; by Lemma 6.3.77 the kernels are invariant under rational points in each variable, continuous, and have real part $\Omega_{\gamma,\gamma'}$, the real Weil representation for the weights (γ,γ') of their characters, which acts by bounded operators of the Fock model (Proposition 4.1.14).

Step 1: (1). Let $\varphi'' \in \mathcal{F}(q)$, $y'' \in \mathcal{F}(p_Z)$. The function $F(g) = K^{\mathrm{col}}_\vartheta[w,P](g)\varphi''(g)y''(\rho(g)g_0)$ is continuous and left invariant under the rational points of $\tilde{Y}^{j+1}_j$ (by (d), (f) and the invariance of φ'', y''). By (f) and (d), $|K^{\mathrm{col}}_\vartheta[w,P](g)| \le C\,\mathrm{ht}_{\tilde{Y}^{j+1}_j}(g)^A$; by (e), $|\varphi''(g)|\mathrm{ht}(g)^A$ is bounded; y'' is bounded. So F is bounded on the quotient, which has finite volume, and its integral converges absolutely. The same holds for K^{row}. This is (1); in particular every summand of C and R is defined.

Step 2: (2). Let Ξ satisfy the hypothesis, and fix a real point g of $\mathrm{U}(\tilde{Y}^j_j)$ and Ψ'. For an edge tuple S on Γ', a period triple $x=(\vartheta,y',\varphi)$ and v in the Fock model $\mathcal{F}_1$ of $\mathrm{Hom}(\tilde{Y}^j_j,\tilde{Y}^{j+1}_j)$ let $f_S = \delta^{\natural}_{S(\tilde{p},q)} \otimes_{\Sigma(\mathfrak{m}_\vartheta)} v^0$ and

$$\Lambda_{S,x}(v) = \int_{[\mathrm{U}(Y^j_j)]} u_S(h) \int_{[\mathrm{U}(\tilde{Y}^{j+1}_j)]} \theta^{\tilde{p},q}(v\otimes f_S)\big(\iota_-(h,(1,\mathrm{flp}^j_\vartheta)),g'\big)\,\varphi_S(g')\,dg'\,dh,$$

the kernel with the characters of $(\tilde{p},q)$. By (d), (e), (f) the integrand is bounded by $C_S\|v\|$ on the product of the two quotients, of finite volume; so the double integral converges absolutely, Fubini's theorem applies, and $\Lambda_{S,x}$ is a continuous linear functional on $\mathcal{F}_1$ with $|\Lambda_{S,x}(v)| \le C'_S\|v\|$. Put $L_x = \sum_S \Psi'_\vartheta(S)\Lambda_{S,x}$, the sum over the tuples supported on the edges of Γ'.

(i) *The period.* The half Γ'_{I} carries y' to u_S, and Γ'_{II} carries φ to the transposed move of φ_S along $(\tilde{p},q)$, whose value at $\tilde{x}$ is $\int \kappa^{(\tilde{p},q)}_{S(\tilde{p},q),H}(\tilde{x},g')\varphi_S(g')\,dg'$, H its joint harmonic (Lemma 6.3.76

(1)). At $\tilde{x} = \iota_-(h,(1,\mathrm{flp}_\vartheta^j))(g,1)$, the homomorphism property and the real part (f) give $\kappa(\tilde{x},g') = \theta^{\tilde{p},q}(w\otimes f_S)(\iota_-(h,(1,\mathrm{flp}_\vartheta^j)),g')$ with $w=\Omega(g,1)v_H\in\mathcal{F}_1$. So by (a) and (3.3), $\mathrm{per}_{\Gamma'}(\Psi';g)(x)=L_x(w)$ for every x.

(ii) *Products.* Let A, B be as in (c) (column case) and $Q=Q_{AB}$. By (6.111) and Fubini,

$$\Lambda_{S,x}(v_Q)=\int_{[\mathrm{U}(\tilde{Y}_j^{j+1})]} K_\vartheta^{\mathrm{col}}[v(S),B](g')\,\varphi_S(g')\,(L_Au_S)(\rho(g'))\,dg',$$

where L_A is the lift of Lemma 6.3.76 for the pair (Y_j^j,Z) with the characters of (p,p_Z), the datum $\delta_{y_2(S)}$ and the polynomial A. By (e) and (4) of Lemma 6.3.76 for $P=H_{(p,p_Z)}$, there is a continuous compactly supported a_A on G_∞ of Z, depending only on the weights, the types and A, with $L_Au_S=R(a_A)y_S^-$, i.e. $(L_Au_S)(\rho(g'))=\int a_A(z)\,y_S^-(\rho(g')(z,1))\,dz$. As y_S^- is bounded and $K^{\mathrm{col}}\varphi_S$ integrable (Step 1), Fubini gives, after summing over S,

$$L_x(v_Q)=\int_{G_\infty} a_A(z)\,C(\Psi';z,B)(x)\,dz \qquad \text{for every period triple } x,$$

$z\mapsto a_A(z)C(\Psi';z,B)(x)$ being integrable. By hypothesis $C(\Psi';z,B)=\beta_z\Xi$ for every z. If $\Xi=0$, then $L_x(v_Q)=0$ for all x. Otherwise fix x_0 with $\Xi(x_0)\neq0$; then $C(\Psi';z,B)(x)=\Xi(x)\,C(\Psi';z,B)(x_0)/\Xi(x_0)$, and $L_x(v_Q)=\beta_{AB}\Xi(x)$ with $\beta_{AB}=\Xi(x_0)^{-1}\int a_AC(\Psi';z,B)(x_0)\,dz$.

(iii) *Closure.* Let $\mathcal{V}$ be the set of $v\in\mathcal{F}_1$ for which some β satisfies $L_x(v)=\beta\,\Xi(x)$ for all x. If $\Xi=0$ it is $\bigcap_x\ker L_x$; otherwise it is $\bigcap_x\ker\bigl(L_x-\Xi(x)\Xi(x_0)^{-1}L_{x_0}\bigr)$. In both cases it is a closed linear subspace. By (ii) it contains $v_{Q_{AB}}$ for all A, B; every monomial in the variables of the $n_Z\times(n_p\sqcup\{*\})$ entries is such a product, so by linearity of $Q\mapsto v_Q$ it contains the Fock vectors of all polynomials, which are dense by Lemma 6.3.29. So $\mathcal{V}=\mathcal{F}_1\ni w$, and by (i) $\mathrm{per}_{\Gamma'}(\Psi';g)=\beta\,\Xi$. As g was arbitrary, $\Psi'\propto_{\Gamma'}\Xi$.

Step 3: (3). Let $\Psi\propto_\Gamma\Xi$, and fix g_0, P'. For a tuple S on Γ, a triple x and v in the Fock model $\mathcal{F}_2$ of $\mathrm{Hom}(Z,Y_{j+1}^{j+1})$ put $f'_S=\delta^{\natural}_{x_2(S)}\otimes_{\Sigma(\mathfrak{m}_\vartheta)}v^0$ and

$$M_{S,x}(v)=\int_{[\mathrm{U}(\tilde{Y}_j^{j+1})]} K_\vartheta^{\mathrm{row}}[v'(S),P'](g')\,\varphi_S(g')\int_{[\mathrm{U}(Y_{j+1}^{j+1})]}\theta^{Z,p^+}(v\otimes f'_S)(\rho(g'),h)\,U_S(h)\,dh\,dg',$$

the kernel with the characters of (p_Z,p^+). As in Step 2, by (d), (e), (f), the integrand is bounded by $C_S\|v\|$ on a product of quotients of finite volume, so $M_{S,x}$ is a continuous linear functional and Fubini applies; put $M_x=\sum_S\Psi_\vartheta(S)M_{S,x}$, the sum over the tuples supported on the edges of Γ.

(i) *The row period.* $y_S^+(\rho(g')g_0)=\int\kappa^{(p_Z,p^+)}_{x_2(S),H'}(\rho(g')(g_0,1),h)U_S(h)\,dh$, H' the joint harmonic of the transposed move; by the real part (f), $\kappa(\rho(g')(g_0,1),h)=\theta^{Z,p^+}(w'\otimes f'_S)(\rho(g'),h)$ with $w'=\Omega(g_0,1)v_{H'}\in\mathcal{F}_2$. So $R(\Psi;g_0,P')(x)=M_x(w')$.

(ii) *Polynomials.* Let A be a polynomial in the variables of the $n_{p^+}\times n_Z$ entries and $Q=Q_{AP'}$ (row case of (c)). By (6.112) with $k=\mathrm{flp}_\vartheta^{j+1}$ and Fubini,

$$M_{S,x}(v_A)=\int_{[\mathrm{U}(Y_{j+1}^{j+1})]}U_S(h)\,(L_Q\varphi_S)\bigl(\iota_+(h,(1,\mathrm{flp}_\vartheta^{j+1}))\bigr)\,dh,$$

L_Q the lift of Lemma 6.3.76 for the pair $(\tilde{Y}_j^{j+1},\tilde{Y}_{j+1}^{j+1})$ with the characters of $(q,\tilde{p}^+)$, the datum $\delta_{S(q,\tilde{p}^+)}$ and the polynomial Q. By (e) and (4) of Lemma 6.3.76 for $P=H_{(q,\tilde{p}^+)}$, $L_Q\varphi_S=R(a_Q)V_S$ with a_Q continuous, compactly supported and independent of S and ϑ, V_S the lift move of φ_S along $(q,\tilde{p}^+)$ with $S(q,\tilde{p}^+)$, which is the image of φ along Γ_{II}; U_S is the image of y' along Γ_{I}. By Fubini (U_S rapidly decreasing, V_S bounded) and (3.3) with (a),

$$M_x(v_A)=\int_{G_\infty}a_Q(z)\,\mathrm{per}_\Gamma(\Psi;z)(x)\,dz,$$

z running over the real points of $\mathrm{U}(\tilde{Y}_{j+1}^{j+1})$. By hypothesis $\mathrm{per}_\Gamma(\Psi;z)=\beta_z\Xi$ for every z, and as in Step 2 (ii), $M_x(v_A)=\beta_A\Xi(x)$ for all x, with β_A independent of x.

(iii) *Closure.* As in Step 2 (iii), the set of $v\in\mathcal{F}_2$ with $M_x(v)=\beta\Xi(x)$ for all x and some β is a closed subspace; it contains every v_A, hence $\mathcal{F}_2$ by Lemma 6.3.29, hence w'. By (i), $R(\Psi;g_0,P')=\beta\,\Xi$ on all period triples. □

Lemma 6.4.96. *Let $e\colon n'\to n$ be a bijection of finite index sets and P_e the $n'\times n$ matrix with entries $(P_e)_{i'i}=1$ if $i=e(i')$ and 0 otherwise, so that $(P_eu)_{i'}=u_{e(i')}$ and $P_e^{-1}=P_e^{\mathsf T}$. For a space with a point $V=(J_V,x_V)$ with signs s on n put $V^e=(P_eJ_VP_e^{\mathsf T},P_ex_VP_e^{\mathsf T})$; for $g\in\mathrm{G}(s)$ put $g^e=P_egP_e^{\mathsf T}$; for a function f on $\mathbb{A}_E^n$ put $f^e=f\circ P_e^{\mathsf T}$, a function on $\mathbb{A}_E^{n'}$.*

(1) V^e is a space with a point with signs $s\circ e$, and $g\mapsto g^e$ is an isomorphism of topological groups $\mathrm{G}(s)\to\mathrm{G}(s\circ e)$ which carries the measure of $\mathrm{G}(s)$ to that of $\mathrm{G}(s\circ e)$ and Γ_V onto Γ_{V^e}, with $\det g^e=\det g$ and $\mathrm{ht}_{V^e}(g^e)=\mathrm{ht}_V(g)$.

(2) $f\mapsto f^e$ is a linear isometry of $\mathcal{M}_s$ onto $\mathcal{M}_{s\circ e}$ with $(v^0)^e=v^0$, $\rho(P_ez)f^e=(\rho(z)f)^e$ $(z\in\mathbb{A}_E^n)$, $\omega_{s\circ e}(g^e)f^e=(\omega_s(g)f)^e$ and $\theta_{V^e}(f^e)=\theta_V(f)$.

(3) Let S, T be spaces with points with signs s on n_S and s' on n_T, and $e_S\colon n_S'\to n_S$, $e_T\colon n_T'\to n_T$ bijections; write $e_T\times e_S$ for the bijection $(a',b')\mapsto(e_T(a'),e_S(b'))$ of $n_T'\times n_S'$ onto $n_T\times n_S$. Then $\mathrm{Hom}(S^{e_S},T^{e_T})=\mathrm{Hom}(S,T)^{e_T\times e_S}$, and for all automorphic characters α,β of $\mathrm{U}(1)$, $\varphi\in\mathcal{M}_{S,T}$, $g\in\mathrm{G}(s)$ and $h\in\mathrm{G}(s')$,

$$\theta_{\alpha,\beta}^{S^{e_S},T^{e_T}}\big(\varphi^{e_T\times e_S}\big)\big(g^{e_S},h^{e_T}\big)=\theta_{\alpha,\beta}^{S,T}(\varphi)(g,h).$$

(4) In the situation of (3), for a modulus $\mathfrak{m}$, a function Φ on the $n_T\times n_S$ matrices over $R_{\mathfrak{m}}$ and a polynomial Q in the variables $X_{v,a,b}$ $(v\in\Phi,\ a\in n_T,\ b\in n_S)$,

$$\mathrm{mod}_{\mathfrak{m}}(\Phi,v_Q)^{e_T\times e_S}=\mathrm{mod}_{\mathfrak{m}}(\Phi',v_{Q'}),\qquad \Phi'(X')=\Phi(P_{e_T}^{\mathsf T}X'P_{e_S}),$$

where Q' is obtained from Q by the substitution $X_{v,e_T(a'),e_S(b')}\mapsto X_{v,a',b'}$.

Proof. **(1).** With $P=P_e$, $(Px_VP^{\mathsf T})^*(PJ_VP^{\mathsf T})(Px_VP^{\mathsf T})=Px_V^*J_Vx_VP^{\mathsf T}=P\varepsilon_sP^{\mathsf T}=\varepsilon_{s\circ e}$, and $PJ_VP^{\mathsf T}$ is hermitian over E; so V^e is a space with a point with signs $s\circ e$. Conjugation by P is a homeomorphic group isomorphism of GL carrying $\{g:g^*\varepsilon_sg=\varepsilon_s\}$ onto $\{g:g^*\varepsilon_{s\circ e}g=\varepsilon_{s\circ e}\}$, preserving determinants, the integral finite points and the rational points: $\Gamma_{V^e}=(Px_VP^{\mathsf T})^{-1}P\,\mathrm{U}(V)(F)\,P^{\mathsf T}(Px_VP^{\mathsf T})=P\Gamma_VP^{\mathsf T}$. Measures: at a real place, $S(PyP^{\mathsf T})=PS(y)P^{\mathsf T}$ entrywise in Definition 3.2.1 (both entries are indexed by the pair $(e(a),e(b))$), so $D(PyP^{\mathsf T})=PD(y)P^{\mathsf T}$ for $PJP^{\mathsf T}$, the Cayley charts are intertwined by conjugation, and $y\mapsto PyP^{\mathsf T}$ preserves Lebesgue measure and $|\det D|$; at the finite places conjugation is an automorphism of the unimodular group $G_f(n)\cong G_f(n')$ carrying $G_f(n)\cap\mathrm{GL}_n(\widehat{\mathcal{O}}_E)$ onto $G_f(n')\cap\mathrm{GL}_{n'}(\widehat{\mathcal{O}}_E)$, hence Haar measure of mass one there to the same. Heights: $(V^e)_{g^e}=(PJ_VP^{\mathsf T},P(x_Vg)P^{\mathsf T})$, its lattice is P applied to the lattice of V_g, and $((Px_VgP^{\mathsf T})^{-1}Pu)_\infty=P((x_Vg)^{-1}u)_\infty$ has the same euclidean norm as $((x_Vg)^{-1}u)_\infty$. So the infima λ of (VP4) of Definition 3.2.2 agree, and $\mathrm{ht}_{V^e}(g^e)=\mathrm{ht}_V(g)$.

(2). The bicharacter, the vacuum and the vacuum measure of (WD5), (WD6) are sums, products and product measures over the coordinates, so $\beta_{s\circ e}(Px,Py)=\beta_s(x,y)$, $e_{n'}(Px)=e_n(x)$ and P carries μ_n to $\mu_{n'}$. Substituting $y=P^{\mathsf T}y'$ in the identity (WD7) shows that $f^e\in\mathcal{M}_{s\circ e}$ when $f\in\mathcal{M}_s$, with the same L^1 norm; the inverse is $f'\mapsto f'\circ P$. Directly from the definitions, $(v^0)^e=e_{n'}=v^0$ and $(\rho(z)f)^e(x')=\beta_s(z,P^{\mathsf T}x')f(P^{\mathsf T}x'-z)=\beta_{s\circ e}(Pz,x')f^e(x'-$

Pz). For $g \in \mathrm{G}(s)$ the negative block of g_v^e is a permutation conjugate of that of g_v, and $\widehat{\mathcal{O}}_E^{n'} \cap (g_f^e)^{-1}\widehat{\mathcal{O}}_E^{n'} = P(\widehat{\mathcal{O}}_E^n \cap g_f^{-1}\widehat{\mathcal{O}}_E^n)$, so PuP^{T} serves for g^e in (WR3) when u serves for g; hence $\Delta_{s\circ e}(g^e) = \Delta_s(g)$, and substituting $y = P^{\mathsf{T}}y'$ in the integral (WR3) gives $\omega_{s\circ e}(g^e)f^e = (\omega_s(g)f)^e$. *Standard spaces.* Let V_s be the standard space of signs s; then V_s^e is the standard space $V_{s\circ e}$ (diagonal Gram matrix and point, permuted). The functional $\Theta(f') = \theta_{V_s}(f' \circ P)$ on $\mathcal{M}_{s\circ e}$ is bounded, and invariant under $\rho(\gamma')$ for the rational vectors γ' of $V_{s\circ e}$, which are the $P\gamma$ with γ rational for V_s. By (WC1), for $x \in \mathbb{A}_E^n$, $\Theta(\rho(Px)v^0) = \theta_{V_s}(\rho(x)v^0) = \prod_{i\in n}\theta_{s_i}(\rho(x_i)v^0) = \prod_{i'\in n'}\theta_{s_{e(i')}}(\rho((Px)_{i'})v^0) = \theta_{V_{s\circ e}}(\rho(Px)v^0)$. By Lemma 4.1.7 the bounded rational-Heisenberg-invariant functionals on $\mathcal{M}_{s\circ e}$ form a line, so $\Theta = c\,\theta_{V_{s\circ e}}$; as $\theta_{V_{s\circ e}}$ is normalized, $\theta_{V_{s\circ e}}(\rho(x')v^0) \neq 0$ for some x', and comparing there gives $c = 1$. So $\theta_{V_{s\circ e}}(f^e) = \theta_{V_s}(f)$. *General spaces.* Let k be a transporting element of V (WR3). For $u \in E^n$ with $kx_V^{-1}u = x_{V_s}^{-1}u_1$, $k^e(Px_V P^{\mathsf{T}})^{-1}(Pu) = P\,kx_V^{-1}u = (Px_{V_s}P^{\mathsf{T}})^{-1}(Pu_1)$; so k^e is a transporting element of V^e for $V_{s\circ e}$. By Lemma 4.1.17 (independence of the transporting element) and what precedes,

$$\theta_{V^e}(f^e) = \theta_{V_{s\circ e}}\big(\omega_{s\circ e}(k^e)f^e\big) = \theta_{V_{s\circ e}}\big((\omega_s(k)f)^e\big) = \theta_{V_s}(\omega_s(k)f) = \theta_V(f).$$

(3). Write $P_S = P_{e_S}$, $P_T = P_{e_T}$. The matrix of $e_T \times e_S$ on $n'_T \times n'_S$ matrices is $X \mapsto P_T X P_S^{\mathsf{T}}$, that is $P_T \otimes P_S$ on coordinates. By (VP5) of Definition 3.2.2, $\mathrm{Hom}(S^{e_S}, T^{e_T})$ has matrix $(P_T J_T P_T^{\mathsf{T}}) \otimes ((P_S J_S P_S^{\mathsf{T}})^{-1})^{\mathsf{T}} = (P_T \otimes P_S)(J_T \otimes (J_S^{-1})^{\mathsf{T}})(P_T \otimes P_S)^{\mathsf{T}}$, the same for the point, and the signs $s'(e_T(a'))s(e_S(b'))$; this is $\mathrm{Hom}(S,T)^{e_T\times e_S}$. The pair map sends (g^{e_S}, h^{e_T}) to $X \mapsto P_T h P_T^{\mathsf{T}} X P_S g^{-1} P_S^{\mathsf{T}}$, which is the conjugate $(h\otimes(g^{-1})^{\mathsf{T}})^{e_T\times e_S}$ of the pair image of (g,h), and $\det g^{e_S} = \det g$, $\det h^{e_T} = \det h$. So by (2) for $\mathrm{Hom}(S,T)$ and $e_T \times e_S$, $\omega_{\alpha,\beta}(g^{e_S}, h^{e_T})\varphi^{e_T\times e_S} = (\omega_{\alpha,\beta}(g,h)\varphi)^{e_T\times e_S}$, and applying $\theta_{\mathrm{Hom}(S,T)^{e_T\times e_S}}$ gives (3) by the last identity of (2).

(4). Write $\pi(X') = P_T^{\mathsf{T}} X' P_S$ for $n'_T \times n'_S$ matrices X' and put $\iota = n_T \times n_S$, $\iota' = n'_T \times n'_S$. By (4) of Lemma 4.1.3, for $A = \Sigma(\mathfrak{m})$ the datum $\mathrm{mod}_{\mathfrak{m}}(\Phi, v_Q) = v_Q \otimes (\Phi^\natural \otimes_A v^0)$ is the function whose value is the product of v_Q at the normalized real coordinates, of $\mathcal{I}_A(\Phi^\natural)$ at the scaled finite coordinates at A, and of the finite vacuum at the other finite coordinates. Each factor is compatible with the permutation π of the coordinates: the finite vacuum and $\mathbf{1}_A$ are products over the coordinates; $R_A(\pi^{-1}z)\mathbf{1}_A = (R_A(z)\mathbf{1}_A)\circ\pi$ by the formula of Lemma 4.1.3 (the sum $\bar{y}^{\mathsf{T}}(2x+y)_A$ is over the coordinates), and π preserves the Haar measure of Q_A, so $\mathcal{I}_A(\Phi^\natural \circ \pi) = \mathcal{I}_A(\Phi^\natural)\circ\pi$, where $\Phi^\natural \circ \pi = (\Phi')^\natural$ as reduction modulo $\mathfrak{m}$ and integrality are entrywise; and $v_Q \circ \pi = v_{Q'}$, since e^∞ is invariant and the coordinate ζ of (WR5) at (v, a', b') is the coordinate at $(v, e_T(a'), e_S(b'))$, whose sign is the same. Hence $\mathrm{mod}_{\mathfrak{m}}(\Phi, v_Q)\circ\pi = \mathrm{mod}_{\mathfrak{m}}(\Phi', v_{Q'})$, which is (4), as $f^{e_T\times e_S} = f\circ\pi$. □

Lemma 6.4.97. *Let $\mathfrak{m}$ be a modulus and α, β, β' automorphic characters of* $\mathrm{U}(1)$*; write* $\mathrm{diag}(g_1, g_2)$ *for block diagonal elements.*

(a) (Columns.) Let S_1, S_2, T be spaces with points with signs s_1 on n_1, s_2 on n_2, s' on n_T. Let Φ_i be a function on the $n_T \times n_i$ matrices over $R_{\mathfrak{m}}$ and A, B polynomials in the variables $X_{v,a,b}$ with $a \in n_T$ and $b \in n_1$, resp. $b \in n_2$; put $(\Phi_1 \otimes \Phi_2)(X_1 \mid X_2) = \Phi_1(X_1)\Phi_2(X_2)$ on $n_T \times (n_1 \sqcup n_2)$ matrices. Then

$$\mathrm{mod}_{\mathfrak{m}}(\Phi_1 \otimes \Phi_2, v_{AB}) = \mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_A) \boxtimes^{\mathrm{sc}} \mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_B) \in \mathcal{M}_{S_1\oplus S_2, T},$$

and for $g_i \in \mathrm{G}(s_i)$, $h \in \mathrm{G}(s')$,

$$\theta^{S_1\oplus S_2,T}_{\beta\beta',\alpha}\big(\mathrm{mod}_{\mathfrak{m}}(\Phi_1\otimes\Phi_2, v_{AB})\big)\big(\mathrm{diag}(g_1,g_2), h\big) = \theta^{S_1,T}_{\beta,\alpha}\big(\mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_A)\big)(g_1,h)\,\theta^{S_2,T}_{\beta',\alpha}\big(\mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_B)\big)(g_2,h).$$

(b) (Rows.) Let S, T_1, T_2 be spaces with points with signs s on n_S, s_1 on n_1, s_2 on n_2. Let Φ_i be a function on the $n_i \times n_S$ matrices over $R_{\mathfrak{m}}$ and A, B polynomials in the variables $X_{v,a,b}$ with $b \in n_S$ and $a \in n_1$, resp. $a \in n_2$; put $(\Phi_1 \otimes \Phi_2)\binom{X_1}{X_2} = \Phi_1(X_1)\Phi_2(X_2)$. Then $\mathrm{mod}_{\mathfrak{m}}(\Phi_1 \otimes \Phi_2, v_{AB}) = \mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_A) \boxtimes^{\mathrm{tg}} \mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_B)$, and for $g \in \mathrm{G}(s)$, $h_i \in \mathrm{G}(s_i)$,

$$\theta^{S,T_1\oplus T_2}_{\alpha,\beta\beta'}(\mathrm{mod}_{\mathfrak{m}}(\Phi_1 \otimes \Phi_2, v_{AB}))(g, \mathrm{diag}(h_1, h_2)) = \theta^{S,T_1}_{\alpha,\beta}(\mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_A))(g, h_1)\, \theta^{S,T_2}_{\alpha,\beta'}(\mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_B))(g, h_2).$$

Proof. **The data.** We prove (a); (b) is the same with rows in place of columns. Put $A_0 = \Sigma(\mathfrak{m})$, $\iota_i = n_T \times n_i$ and $\iota = n_T \times (n_1 \sqcup n_2) = \iota_1 \sqcup \iota_2$. By (4) of Lemma 4.1.3, for a function Φ on the matrices over $R_{\mathfrak{m}}$ with these index sets and a polynomial Q, the datum $\mathrm{mod}_{\mathfrak{m}}(\Phi, v_Q) = v_Q \otimes (\Phi^\natural \otimes_{A_0} v^0)$ is the function whose value at a vector is the product of three factors: v_Q at its normalized real coordinates, $\mathcal{I}_{A_0}(\Phi^\natural)$ at its scaled finite coordinates at A_0, and the finite vacuum at its other finite coordinates. Each factor for $(\Phi_1 \otimes \Phi_2, AB)$ on ι is the product of the factors for (Φ_1, A) on ι_1 and for (Φ_2, B) on ι_2:

- the Fock coordinates ζ of (WR5) are coordinatewise and the signs of $\mathrm{Hom}(S_1 \oplus S_2, T)$ restrict to those of $\mathrm{Hom}(S_i, T)$ on ι_i ((WR6), context), and e^∞ is a product over the coordinates, so $v_{AB} = v_A \cdot v_B$;
- integrality at A_0 and reduction modulo $\mathfrak{m}$ are entrywise, so $(\Phi_1 \otimes \Phi_2)^\natural = \Phi_1^\natural \otimes \Phi_2^\natural$ on $Q^\iota_{A_0} = Q^{\iota_1}_{A_0} \times Q^{\iota_2}_{A_0}$; the function $R_{A_0}(z)\mathbf{1}_{A_0}$ of Lemma 4.1.3 is the product of the corresponding functions for the two blocks of z (its phase is a sum over the coordinates and $\mathbf{1}_{A_0}$ a product), and the Haar measure of $Q^\iota_{A_0}$ is the product measure; so by Fubini (all functions are locally constant with compact support) $\mathcal{I}_{A_0}(\Psi_1 \otimes \Psi_2)(z_1, z_2) = \mathcal{I}_{A_0}(\Psi_1)(z_1)\mathcal{I}_{A_0}(\Psi_2)(z_2)$;
- the finite vacuum (WD5) is a product over the coordinates.

Hence $\mathrm{mod}_{\mathfrak{m}}(\Phi_1 \otimes \Phi_2, v_{AB})(x) = \mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_A)(x|_{\iota_1})\,\mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_B)(x|_{\iota_2})$, which is the source product of (WR6).

The kernels. By (d) of Lemma 6.3.52 for $f = \mathrm{mod}_{\mathfrak{m}}(\Phi_1, v_A)$ and $f' = \mathrm{mod}_{\mathfrak{m}}(\Phi_2, v_B)$, the theta kernel of the source product at $(\mathrm{diag}(g_1, g_2), h)$ is the product of the two theta kernels, with the characters as displayed; for (b), the target product and the target part of (d) of Lemma 6.3.52. □

The last group proves deep-level vanishing (Lemma 6.4.98), which makes the flat restriction onto: at a suitable real point ω, the restrictions of discrete series forms of very deep level from $Y_0 \oplus \langle 1 \rangle$ to Y_0 detect every form of principal level. It is the case $V = Y_0$ of a criterion for framed hermitian spaces (Proposition 6.4.99). Its hypothesis at ω, that the pairing of the Flensted-Jensen coefficients [81] of the two parameters is not small against the volume of a ball, holds by Lemma 6.4.109, which rests on Laplace's method for periods of two discrete series coefficients (Theorem 5.7.21, Section 5).

The proof of the criterion is a halving argument for the supremum S of a form y orthogonal to the restrictions. High in the cusp, a bounded form of large-gap type is at most $S/2$, by a cusp estimate (Lemma 6.4.100). Away from the cusp, Poincaré series of the discrete series kernel at deep level [234] pair with the form through their identity-coset term, and Schur orthogonality for square-integrable representations [163, 119], with the lower bound at ω, gives again $|y| \le S/2$ (Lemma 6.4.103). Hence $S \le S/2$ and $y = 0$. For the reduction theory behind the division into the cusp and its complement see [37, 33].

Lemma 6.4.98. *Let D be admissible data, let s_v be the signs of Y_0 at $v \in \Phi$, give $\tilde{Y}_0^0 = Y_0 \oplus \langle 1 \rangle$ the frame of Y_0 followed by the added basis vector, and let $\lambda'(v) = -\frac{1}{2} p_{\mathrm{II},0,0}(v)$ be the Harish-Chandra parameter of $\tilde{Y}_0^0$ at v. There is a real point $\omega \in \prod_{v\in\Phi} \mathrm{U}(\mathrm{diag}(s_v, 1))$ of $\mathrm{U}(\tilde{Y}_0^0)$ with the following property: if $y \in \mathcal{A}_M(Y_0)$ satisfies*

$$\int_{\mathrm{U}(Y_0)(E^+)\backslash \mathrm{U}(Y_0)(\mathbb{A})} y(x)\, \overline{\varphi\big((x \oplus 1)\,\omega\big)}\, dx = 0$$

for every form φ on $\tilde{Y}_0^0$ of type λ' and level $\mathrm{K}(\pi_M^b)$ (the grid forms at the node $(\mathrm{II}, 0, 0)$ of the deep level), then $y = 0$.

Proof. Put $V = Y_0$, so $n = r$ and $Z = \tilde{Y}_0^0 = V \oplus \langle 1 \rangle$, and $\lambda(v) = \varsigma(v) = \lambda_{\mathrm{I}}(v)$, $\lambda'(v) = -\lambda_{\mathrm{II}}(v)$. Choose

$$A = r^4, \quad L = 20b, \quad a_{\mathrm{p}} = 20(2r+1)^2, \quad \beta = M^{\lceil x \rceil/2}, \quad R = \beta^{20A(f+1)}, \quad h = r^7.$$

We verify the hypotheses of Proposition 6.4.99, finding ω with Lemma 6.4.109.

Sizes used. By Lemma 4.2.67, Lemma 5.2.13, Proposition 2.3.5 and Lemma 2.3.6: $x \geq e^{6.4\cdot 10^{15}}$, $x^5 \leq r \leq 3x^5$, $m \leq 2\sqrt{x}$, $f \leq q_1 \leq x^{12}$, $q_1 d \leq x^{13}$, $\mathrm{rd}_E \leq x^{12}$, $\log M \leq 22 \log b$, $(\log b)^{100} \leq x$, $x^{10^{12}} \leq b$ and $x^{10^6} \leq x_1 \leq b^{10^{-6}}$; by (AD5), $M \geq b^{20}$; by (AD4), $b/x_1 \leq \kappa(v) \leq b$. By (AD2), $\Delta^2 = 4r^{16} \leq 4 \cdot 3^{16} x^{80} \leq x^{81} \leq x_1$; since $x_1^2 \leq b$, this gives $\Delta^2 \leq x_1 \leq b/x_1 \leq \kappa(v)$. So $r < \Delta \leq \Delta^2 \leq \kappa(v)$ and $\kappa_0(v) = \kappa(v) + \frac{\Delta}{2} \leq b + \frac{b}{2} \leq 2b$. Also $\log \beta \leq x \log M \leq 22x \log b \leq 22x^{1.01}$ and $\beta \geq M^{x/2} \geq b^{10x}$.

Gaps and entries. By (GP5), $2\lambda(v)$ has the entries $2\varsigma_i(v)$, $0 \leq i \leq r-1$, and $2\lambda'(v)$ the entries $-(2\varsigma_i(v)+1)$, $0 \leq i \leq r$; they are distinct (the sequence $\varsigma_i(v)$ is strictly decreasing), the first are even and $r-1$ is even, the second are odd and $(r+1)-1 = r$ is odd (AD2). The base index i has the sign $+1$ for $i \leq k-1$ and $(-1)^a$ for $i = k-1+a$. If two indices have different signs, then either both are $\geq k-1$, or one is $\leq k-2$ and the other is $k-1+a$ with $a \geq 1$. In the first case $|\varsigma_i - \varsigma_j|$ is a positive multiple of Δ; in the second $\varsigma_i - \varsigma_{k-1+a} \geq 1 - \frac{k}{2} - \frac{\Delta}{2} + \kappa_{a+1} \geq \kappa + \Delta - k \geq \Delta$. So opposite-sign entries of $\lambda(v)$, and of $\lambda'(v)$, differ by at least $\Delta = 2r^8$, which exceeds $500(r+1)^3 + \max(4A, h) = 500(r+1)^3 + \max(4r^4, r^7)$. Hence $2\lambda(v)$ has large gap $4A$ and large gap h for s_v, and $2\lambda'(v)$ has large gap $2A$ and large gap h for $(s_v, +1)$, the signs of Z (the added index r has the sign $(-1)^m = +1$ by (GP5)). Every entry has absolute value at most $\max(k,\ 2\kappa_{m+1}(v) - \Delta + 1) = 2\kappa(v) + (2m+2)\Delta + 1 \leq 5b \leq L$, since $(2m+2)\Delta \leq 2\Delta^2 \leq 2\kappa(v) \leq 2b$.

Deep-level sizes for rank r. So $(A, L, a_{\mathrm{p}}, \beta, R)$ satisfy the deep-level size conditions for rank r.

The real point. Fix $v \in \Phi$ and apply Lemma 6.4.109 with $n = r$, $\varepsilon = s_v$, $l = 2\lambda(v)$, $l' = 2\lambda'(v)$, $X = \{0, \ldots, k-2\}$, $M = m$ (even), $e = e_{\mathrm{I}}$, $e' = e_{\mathrm{II}}$, $B = \kappa_0(v) \geq 1$, β, h and L. Hypothesis 1 is Lemma 5.7.20 for $z = \mathrm{I}$ (for $\frac{1}{2}l = \varsigma(v)$) and for $z = \mathrm{II}$ (for $-\frac{1}{2}l' = \lambda_{\mathrm{II}}(v)$), the domains being $X \sqcup \{0, \ldots, m-1\}$ and $X \sqcup \{0, \ldots, m\}$ since $2\varrho + 2 = m$. Hypotheses 2 and 3 were checked above. Hypothesis 4: $\beta \geq 10^{12}$; $2r + 1 + L \leq 21b + 1 \leq \beta$; $(r+1)^4 (6\kappa_0(v))^{2m+2} \leq b\,(12b)^{2m+2} \leq b^{4m+5} \leq b^{10x} \leq \beta$; and $10^{30}\big((2r+1)^2 \log \beta\big)^3 \leq 10^{30}(9r^2 \cdot 22x^{1.01})^3 \leq 10^{37} r^6 x^{3.03} \leq r^7 = h$ since $r \geq x^5$. So there is $\omega_v \in \mathrm{U}(\mathrm{diag}(s_v, 1))$ with $\|\omega_v\|^2 \leq \beta$ and

$$\mathrm{vol}\{r' \in \mathrm{U}(\mathrm{diag}\, s_v) : \|r' - 1\| \leq 1\} \leq \beta^{a_{\mathrm{p}}} \Big| \int c_{\lambda(v)}(r')\, \overline{c_{\lambda'(v)}\big(\omega_v^{-1}(r' \oplus 1)\omega_v\big)}\, dr' \Big|,$$

as $20(2n+1)^2 = a_{\mathrm{p}}$. Put $\omega = (\omega_v)_{v\in\Phi}$.

Conclusion. The forms of principal level $\mathcal{A}_M(Y_0)$ are the forms on V of type $\varsigma = \lambda$ and level $\mathrm{K}(\pi_M)$: the type condition of Definition 3.2.16 is (DZ2) for $\lambda = \varsigma$, and $\mathrm{K}_M = \mathrm{K}(\pi_M)$ because

$\pi_M\mathcal{O}_E$ is the product of the chosen places, each with exponent one (by Lemma 4.2.66(3), the $2f$ chosen places divide π_M and the product of their norms is $M^f = \mathrm{N}_{E/\mathbb{Q}}(\pi_M)$). All hypotheses of Proposition 6.4.99 hold for V, λ, λ', $(A, L, a_{\mathrm{p}}, \beta, R)$ and ω, so every $y \in \mathcal{A}_M(Y_0)$ with the stated orthogonality vanishes. □

Proposition 6.4.99. *Let D be admissible data, V a hermitian space over E of rank $n \geq 1$ with a frame and signs s_v, and $Z = V \oplus \langle 1 \rangle$ with the frame of V followed by the added basis vector. Let $A, L, a_{\mathrm{p}} \in \mathbb{N}$ and $\beta, R > 0$ satisfy the deep-level size conditions for rank n. Let λ, λ' be such that, for every $v \in \Phi$, $2\lambda(v)$ has large gap $4A$ for s_v with entries of absolute value at most L, and $2\lambda'(v)$ has large gap $2A$ for $(s_v, +1)$. Let $\omega \in \prod_{v\in\Phi} \mathrm{U}(\mathrm{diag}(s_v, 1))$ satisfy, for every v, $\|\omega_v\|^2 \leq \beta$ and*

$$\mathrm{vol}\{r \in \mathrm{U}(\mathrm{diag}\, s_v) : \|r - 1\| \leq 1\} \leq \beta^{a_{\mathrm{p}}} \Big| \int_{\mathrm{U}(\mathrm{diag}\, s_v)} c_{\lambda(v)}(r)\, \overline{c_{\lambda'(v)}(\omega_v^{-1}(r \oplus 1)\omega_v)}\, dr \Big|.$$

If y is a form on V of type λ and level $K(\pi_M)$ such that

$$\int_{\mathrm{U}(V)(E^+)\backslash \mathrm{U}(V)(\mathbb{A})} y(x)\, \overline{\varphi\big((x \oplus 1)\,\omega\big)}\, dx = 0$$

for every form φ on Z of type λ' and level $K(\pi_M^b)$, then $y = 0$.

Proof. **Levels.** Since π_M divides $M = \pi_M\bar{\pi}_M$ and π_M^b in $\mathcal{O}_E$, every element of $K(M)$ or of $K(\pi_M^b)$ is congruent to 1 modulo π_M, so $K(M) \subseteq K(\pi_M)$ and $K(\pi_M^b) \subseteq K(\pi_M)$. Hence y has level $K(M)$ and level $K(\pi_M^b)$.

Halving. Let $S \geq 0$ with $|y(g)| \leq S$ for all $g \in \mathrm{U}(V)(\mathbb{A})$, and let $x \in \mathrm{U}(V)(\mathbb{A})$. If $\lambda_V(x) < \theta_{\mathrm{cs}}$, then $|y(x)| \leq S/2$ by Lemma 6.4.100, applied to y with level $K(M)$; its hypotheses are among ours. If $\lambda_V(x) \geq \theta_{\mathrm{cs}}$, then $|y(x)| \leq S/2$ by Lemma 6.4.103, applied to y with level $K(\pi_M^b)$ and the same ω, whose hypotheses are ours (the orthogonality being the displayed one). So every bound S of $|y|$ on $\mathrm{U}(V)(\mathbb{A})$ gives the bound $S/2$.

Conclusion. A form is bounded (DZ2); let $S_0 = \sup|y| < \infty$. By induction, $2^{-j}S_0$ bounds $|y|$ for every $j \geq 0$, so $y = 0$. □

Lemma 6.4.100. *Let D be admissible data, V a hermitian space over E of rank $n \geq 1$ with a frame and signs s_v ($v \in \Phi$), and let $A, L, a_{\mathrm{p}} \in \mathbb{N}$ and $\beta, R > 0$ satisfy the deep-level size conditions for rank n. Let $\lambda = (\lambda(v))_{v\in\Phi}$ be such that $2\lambda(v)$ has large gap $4A$ for s_v and entries of absolute value at most L, for every v. Let y be a form on V of type λ and level $K(M)$, and $S \geq 0$ with $|y(g)| \leq S$ for all $g \in \mathrm{U}(V)(\mathbb{A})$. If $x \in \mathrm{U}(V)(\mathbb{A})$ and $\lambda_V(x) < \theta_{\mathrm{cs}}$, then $|y(x)| \leq S/2$.*

Proof. Write $G_\infty = \prod_v G_v$, $G_v = \mathrm{U}(\mathrm{diag}\, s_v)$, $c_v = c_{\lambda(v)}$, $d_v = d_{\lambda(v)}$, $D_0 = \prod_v d_v$, $\psi(z) = \prod_v \overline{c_v(z_v)}$, so $e_\lambda = D_0\psi$, and

$$\mathrm{dc}(z) = \prod_v \Big(\frac{2}{\|z_v\|}\Big)^{2A}, \qquad \mathrm{mj}(z) = D_0\,\mathrm{dc}(z) \qquad (z \in G_\infty).$$

From the size conditions $\beta \geq 2000$, $R \geq \beta^{20A(f+1)} \geq \beta$, $f \leq \beta$ and $100(n + L + 1)^5 \leq \beta$. Put $\delta = R^{-2} \in (0, 1]$.

Step 1: local bounds. By Lemma 5.7.38(1) (gap $500n^3 + 4A \geq 4A$), $|c_v| \leq 1$, $|c_v(z)| \leq (2/\|z\|)^{4A}$ and $\overline{c_v(z)} = c_v(z^{-1})$; hence $|\psi(z)| \leq \mathrm{dc}(z)^2$. By Lemma 5.7.38(3) with $c = 2A > 8n^2$ and entries at most L, mj is integrable with

$$\int_{G_\infty} \mathrm{mj} = \prod_v \int_{G_v} d_v\Big(\frac{2}{\|z\|}\Big)^{2A} dz \leq T^f, \qquad T = 8\big(2000(n + L + 1)^6\big)^{2n^2} 2^{2A};$$

in particular e_λ is integrable ($|e_\lambda| \le D_0 \,\mathrm{dc}^2 \le 2^{2Af}\mathrm{mj}$).

Step 2: the kernel varies slowly. Let $u \in G_\infty$ with $\max_v \|u_v - 1\| \le \delta$ and $z \in G_\infty$. At each v, $\overline{c_v(u_v^{-1}z_v)} - \overline{c_v(z_v)} = c_v(z_v^{-1}u_v) - c_v(z_v^{-1})$, of modulus at most $100(n+L+1)^5\|u_v - 1\| \le \beta\delta$ by Lemma 5.7.38(2). As all factors have modulus at most 1, $|\prod_v a_v - \prod_v b_v| \le \sum_v |a_v - b_v|$ gives $|\psi(u^{-1}z) - \psi(z)| \le f\beta\delta \le \beta^2\delta$. Combining with $|\psi| \le \mathrm{dc}^2$,

$$|\psi(u^{-1}z) - \psi(z)|^2 \le \beta^2\delta\big(\mathrm{dc}(u^{-1}z)^2 + \mathrm{dc}(z)^2\big) \le \beta^2 R^{-2}\big(\mathrm{dc}(u^{-1}z) + \mathrm{dc}(z)\big)^2,$$

so $|e_\lambda(u^{-1}z) - e_\lambda(z)| \le (\mathrm{mj}(u^{-1}z) + \mathrm{mj}(z))\,\beta/R$.

Step 3: the oscillation. By the type condition (DZ2), $y(g') = \int e_\lambda(z) y(g'(z,1))\,dz$ for every g'. For u as in Step 2, substituting $z \mapsto u^{-1}z$ (left invariance) in the expression for $y(x(u,1))$,

$$y(x(u,1)) - y(x) = \int_{G_\infty} \big(e_\lambda(u^{-1}z) - e_\lambda(z)\big)\, y(x(z,1))\,dz,$$

hence, by Steps 1 and 2 and the left invariance of dz,

$$|y(x(u,1)) - y(x)| \le S\,\frac{\beta}{R}\int (\mathrm{mj}(u^{-1}z) + \mathrm{mj}(z))dz \le 2S\,\frac{\beta}{R}\,T^f \le \frac{S}{2},$$

the last step because $T \le 8\beta^{2n^2}2^{2A} \le \beta^{1+2n^2+2A} \le \beta^{3A}$ ($2n^2+1 \le A$) and $4\beta\cdot\beta^{3Af} \le \beta^{3Af+2} \le \beta^{20A(f+1)} \le R$.

Step 4: the cusp. Apply Lemma 6.4.101 with $M_0 = M$, the given y (its large gap $4A$ implies large gap 0), $\delta = R^{-2}$, $g = x$ and $C = S/2$: the hypothesis $\lambda_V(x) < \theta_{\mathrm{cs}} = (\delta/(Mc_{\mathrm{cs}}(n)^2))^n$ is ours, and the oscillation bound is Step 3. So $|y(x)| \le S/2$. □

Lemma 6.4.101. *Let D be admissible data, V a hermitian space over E of rank $n \ge 1$ with a frame and signs s_v, and λ such that $2\lambda(v)$ has large gap 0 for s_v (gap $500n^3$) for every $v \in \Phi$. Let $M_0 \ge 1$ be an integer, y a form on V of type λ and level $K(M_0)$, $0 < \delta \le 1$, $g \in \mathrm{U}(V)(\mathbb{A})$ and $C \ge 0$, and assume*

$$\lambda_V(g) < \Big(\frac{\delta}{M_0\, c_{\mathrm{cs}}(n)^2}\Big)^n \quad \text{and} \quad |y(g(u,1)) - y(g)| \le C \ \text{ for all } u \in G_\infty \text{ with } \max_{v\in\Phi}\|u_v - 1\| \le \delta.$$

Then $|y(g)| \le C$.

Proof. With its frame, V is a framed space of rank n with signs s, the same at every $v \in \Phi$ (Definition 5.6.3), $\mathrm{U}(V)(\mathbb{A}) = G_\infty \times G_f$, and the principal level $\mathrm{K}(M_0)$ of (DZ2) is the principal level $\mathrm{K}_f(M_0)$ of (FT5); $\lambda_V(g)$ is the first minimum of (VP4) of Definition 3.2.2 of this framed space at g.

Step 1: the type kernel is a cusp kernel. By hypothesis each $2\lambda(v)$ has gap $500n^3 = 500n^3 + 0$ for s, so by Lemma 5.6.21 with $N = n$ and $B = 0$ the type kernel e_λ is a cusp kernel of order 0 on G_∞. The form y is continuous, bounded, left invariant under Γ_V, right invariant under $\mathrm{K}_f(M_0)$, and $R(e_\lambda)y = y$ by (FT2), (FT3) and (DZ2).

Step 2: an integer in the different. Put $N_D = q_1 d$, a positive rational integer; by Lemma 6.4.102, N_D lies in the different $\mathfrak{D}_E$ of E. With $N = n$ and this N_D, the constant $\kappa = 8fn\,\mathrm{rd}_E + 48n^6f^2N_D + 1$ of (i) of Proposition 5.8.13 is $c_{\mathrm{cs}}(n)$ of (DZ4) ($f = \#\Phi = [E^+ : \mathbb{Q}]$).

Step 3: a cusp frame. As $\lambda_V(g) < (\delta/(M_0c_{\mathrm{cs}}(n)^2))^n$, $M_0 \ge 1$ and $0 < \delta \le 1$, (i) of Proposition 5.8.13 gives a nonzero isotropic subspace $A \subseteq V$ with $N_A(\mathbb{A}_{E^+})\, g \subseteq N_A(E^+)\, g\, Z_\delta\, \mathrm{K}_f(M_0)$.

Step 4: the oscillation bound. Apply (ii) of Lemma 5.8.9 to $U = V$, the cusp kernel $e = e_\lambda$ of order 0, $F = y$ (left invariant under $N_A(E^+) \subseteq \Gamma_V$), $J = \mathrm{K}_f(M_0)$, this δ and g, the point $1 \in G_\infty$ and $D = C$: the hypothesis $|y(g(z,1)) - y(g)| \le C$ for $z \in Z_\delta$ is the assumed oscillation bound (with $u = z$). Hence $|y(g)| \le C$. □

Lemma 6.4.102. *For admissible data D, the positive integer $q_1 d$ lies in $\mathfrak{D}_E$.*

Proof. Put $N = q_1 d$ and let ζ_N be a primitive N-th root of unity. By (1) of Lemma 4.2.66, $E \subseteq L = \mathbb{Q}(\zeta_N)$. The ring of integers of L is $\mathbb{Z}[\zeta_N]$, so the different $\mathfrak{D}_L$ of $L/\mathbb{Q}$ is generated by $\mathrm{cyc}_N'(\zeta_N)$, cyc_N the cyclotomic polynomial. Differentiating $x^N - 1 = \mathrm{cyc}_N(x)h(x)$ with $h \in \mathbb{Z}[x]$ at ζ_N gives $N\zeta_N^{N-1} = \mathrm{cyc}_N'(\zeta_N)h(\zeta_N)$, so $N \in \mathfrak{D}_L$. By the transitivity of differents, $\mathfrak{D}_L = \mathfrak{D}_{L/E}\,\mathfrak{D}_E\mathcal{O}_L \subseteq \mathfrak{D}_E\mathcal{O}_L$, and as $\mathcal{O}_L$ is faithfully flat over $\mathcal{O}_E$, $\mathfrak{D}_E\mathcal{O}_L \cap \mathcal{O}_E = \mathfrak{D}_E$. Hence $N \in \mathfrak{D}_E\mathcal{O}_L \cap E = \mathfrak{D}_E$ (classical facts on differents and cyclotomic fields, cited in passing). $\square$

Lemma 6.4.103. *Let D be admissible data, V a hermitian space over E of rank $n \geq 1$ with a frame and signs s_v, and $Z = V \oplus \langle 1\rangle$ with the frame of V followed by the added basis vector (signs $(s_v, +1)$); write $x \mapsto x \oplus 1$ for the embedding $\mathrm{U}(V) \to \mathrm{U}(Z)$. Let $A, L, a_{\mathrm{p}} \in \mathbb{N}$ and $\beta, R > 0$ satisfy the deep-level size conditions for rank n. Let $\lambda = (\lambda(v))$ and $\lambda' = (\lambda'(v))$ be such that, for every $v \in \Phi$, $2\lambda(v)$ has large gap $4A$ for s_v with entries of absolute value at most L, and $2\lambda'(v)$ has large gap $2A$ for $(s_v, +1)$. Let $\omega = (\omega_v) \in \prod_{v\in\Phi} \mathrm{U}(\mathrm{diag}(s_v, 1))$ satisfy, for every v, $\|\omega_v\|^2 \leq \beta$ and*

$$\mathrm{vol}\{r \in \mathrm{U}(\mathrm{diag}\, s_v) : \|r - 1\| \leq 1\} \leq \beta^{a_{\mathrm{p}}} \Big| \int_{\mathrm{U}(\mathrm{diag}\, s_v)} c_{\lambda(v)}(r)\, \overline{c_{\lambda'(v)}(\omega_v^{-1}(r \oplus 1)\omega_v)}\, dr \Big|.$$

Let y be a form on V of type λ and level $K(\pi_M^b)$, $S \geq 0$ with $|y| \leq S$ on $\mathrm{U}(V)(\mathbb{A})$, and assume

$$\int_{\mathrm{U}(V)(E^+)\backslash \mathrm{U}(V)(\mathbb{A})} y(x)\, \overline{\varphi((x \oplus 1)\,\omega)}\, dx = 0$$

for every form φ on Z of type λ' and level $K(\pi_M^b)$. If $x \in \mathrm{U}(V)(\mathbb{A})$ and $\lambda_V(x) \geq \theta_{\mathrm{cs}}$, then $|y(x)| \leq S/2$.

Proof. Write $G_\infty = \prod_v \mathrm{U}(\mathrm{diag}\, s_v)$, $G'_\infty = \prod_v \mathrm{U}(\mathrm{diag}(s_v, 1))$, $\tilde{x} = x \oplus 1$, $\mathfrak{b} = \pi_M^b$, $\Gamma_Z = \mathrm{U}(Z)(E^+)$, $\mathrm{vol}_v = \mathrm{vol}\{r \in \mathrm{U}(\mathrm{diag}\, s_v) : \|r - 1\| \leq 1\} > 0$ (the set contains a neighbourhood of 1), $c_\lambda(u) = \prod_v c_{\lambda(v)}(u_v)$, $d_v = d_{\lambda(v)}$ and $D' = \prod_v d_{\lambda'(v)} > 0$ (formal degrees of discrete series are positive, Theorem A.5.18). From the size conditions, $\beta \geq 2000 \geq 16$ and $R \geq \beta^{20A(f+1)} \geq \beta$. For $\alpha \in G'_\infty$ put $\psi_\alpha(r) = \phi_{\alpha,\omega}(r \oplus 1)$ $(r \in G_\infty)$ and

$$Y_\alpha(x') = \int_{G_\infty} \psi_\alpha(r)\, y(x'(r, 1))\, dr \qquad (x' \in \mathrm{U}(V)(\mathbb{A})).$$

Step 1: the Poincaré series are orthogonal to y. Fix $\alpha \in G'_\infty$ and put

$$P_\alpha(g) = \phi_{\alpha,\omega}(g_\infty)\mathbf{1}_{K_Z(\mathfrak{b})}(g_f)$$

on $\mathrm{U}(Z)(\mathbb{A})$. Since $e_{\lambda'}(u) = \prod_v d_{\lambda'(v)}\overline{c_{\lambda'(v)}(u_v)}$, we have $\phi_{\alpha,\omega}(u) = D'^{-1} e_{\lambda'}(\alpha^{-1}u\omega)$. Apply Lemma 6.4.107 to Z $(N = n + 1)$, $B = 2A > 8(n+1)^2$, λ', $a = \mathfrak{b}$ and $c = (\alpha, 1)^{-1}\tilde{x}^{-1}$: there is a form Q on Z of type λ' and level $K_Z(\mathfrak{b})$ with

$$\overline{Q(g')} = \sum_\gamma e_{\lambda'}((c\gamma g')_\infty)\mathbf{1}_{K_Z(\mathfrak{b})}((c\gamma g')_f).$$

For $g' = (h \oplus 1)(\omega, 1)$ one has $(c\gamma g')_\infty = \alpha^{-1}(\tilde{x}^{-1}\gamma(h \oplus 1))_\infty\omega$ and $(c\gamma g')_f = (\tilde{x}^{-1}\gamma(h \oplus 1))_f$, so

$$\sum_{\gamma\in\Gamma_Z} P_\alpha(\tilde{x}^{-1}\gamma(h \oplus 1)) = D'^{-1}\,\overline{Q((h \oplus 1)\omega)}.$$

Multiplying by $y(h)$ and integrating over $\Gamma_V\backslash \mathrm{U}(V)(\mathbb{A})$ gives 0, by the orthogonality hypothesis for $\varphi = Q$.

Step 2: the smoothing is small. Apply Lemma 6.4.104 with $F = y$ (bounded and measurable, left invariant under Γ_V, of level $K_V(\mathfrak{b})$, $|y| \le S$), $a = \mathfrak{b}$, $R_0 = M^{b/2}$ in place of its R (indeed $|\mathfrak{b}|_v = M^{b/2}$ at every v), the given A, λ', α, ω, x, and $d = R$. Its hypotheses hold: $8(n+1)^2 < A$ and $2f(n+1)+2 \le A$ are size conditions; $d = R \ge \beta \ge \max(10, f)$; and, as $\lambda_V(x) \ge \theta_{\mathrm{cs}}$, $M^{b/2} \ge R^{3n+2}$ and $Mc_{\mathrm{cs}}(n)^2 \le \beta \le R$,

$$R_0\lambda_V(x) \ge R^{3n+2}\, R^{-2n}\big(Mc_{\mathrm{cs}}(n)^2\big)^{-n} \ge R^{n+2}R^{-n} = d^2.$$

By Step 1 the period in that lemma is 0; dividing by $\mathrm{vol}(K_V(\mathfrak{b})) > 0$,

$$|Y_\alpha(x)| \le C_{\mathrm{sl}}(\alpha,\omega)\, S\, \frac{10}{R} \qquad (\alpha \in G'_\infty). \tag{6.113}$$

Step 3: moving a real translation into α. Let $u \in G_\infty$. Substituting $r' = ur$ (left invariance of dr) and using $\omega^{-1}(u^{-1} \oplus 1) = ((u \oplus 1)\omega)^{-1}$,

$$Y_\omega(x(u,1)) = \int \phi_{\omega,\omega}\big((u^{-1}r' \oplus 1)\big) y\big(x(r',1)\big)\, dr' = Y_{(u\oplus 1)\omega}(x),$$

so $|Y_\omega(x(u,1))| \le C_{\mathrm{sl}}((u\oplus 1)\omega,\omega) S \cdot 10/R$ by (6.113). Since $\|u_v \oplus 1\| = \|u_v\|$ (as $\|u_v\| \ge 1$) and $\|\omega_v\|^2 \le \beta$,

$$C_{\mathrm{sl}}((u\oplus 1)\omega,\omega) \le \prod_{v\in\Phi} \big(2\|u_v\|^2\beta^2\big)^A 16^A \mathrm{vol}_v.$$

Step 4: the average. By Lemma 5.7.38(1) with $c = 4A$ and the gap $500n^3 + 4A$ of $2\lambda(v)$, $|e_\lambda(u)| \le \prod_v d_v (2/\|u_v\|)^{4A}$, which is integrable by Lemma 5.7.38(3) ($4A > 8n^2$, entries at most L); so e_λ is integrable, and ψ_ω is integrable by Lemma 6.4.104. By Fubini (y bounded) and Lemma 6.4.108 (with $g = x$),

$$\begin{aligned}\int_{G_\infty} e_\lambda(u)\, Y_\omega\big(x(u,1)\big)\, du &= \int_{G_\infty} \psi_\omega(r)\Big(\int_{G_\infty} e_\lambda(u)\, y\big(x(u,1)(r,1)\big)\, du\Big) dr \\ &= \eta\, y(x), \qquad \eta = \int_{G_\infty} \psi_\omega(r) c_\lambda(r)\, dr.\end{aligned}$$

By Fubini on $G_\infty = \prod_v G_v$, $\eta = \prod_v \int c_{\lambda(v)}(r)\overline{c_{\lambda'(v)}(\omega_v^{-1}(r\oplus 1)\omega_v)}\, dr$, so $|\eta| \ge \beta^{-a_{\mathrm{p}} f} \prod_v \mathrm{vol}_v$ by the hypothesis on ω.

Step 5: the bound. By Steps 3 and 4, and $2^{2A}\cdot 2^A \cdot 16^A\beta^{2A} \le (2\beta)^{3A}$ (as $16 \le \beta$),

$$|e_\lambda(u)|\, |Y_\omega(x(u,1))| \le S\, \frac{10}{R}\, (2\beta)^{3Af} \prod_v \mathrm{vol}_v \cdot \prod_v d_v\Big(\frac{2}{\|u_v\|}\Big)^{2A}.$$

Integrating, with Lemma 5.7.38(3) for $c = 2A > 8n^2$ at each place and $T = 8(2000(n+L+1)^6)^{2n^2}2^{2A}$,

$$\beta^{-a_{\mathrm{p}} f}\Big(\prod_v \mathrm{vol}_v\Big)|y(x)| \le |\eta\, y(x)| \le S\, \frac{10}{R}\, (2\beta)^{3Af}\, T^f \prod_v \mathrm{vol}_v.$$

Now $T \le 8\beta^{2n^2}2^{2A} \le \beta^{1+2n^2+2A}$ and $(2\beta)^{3A} \le \beta^{6A}$, and $a_{\mathrm{p}} \le A-1$, $2n^2 \le A-1$, so $\beta^{a_{\mathrm{p}}}(2\beta)^{3A}T \le \beta^{10A}$ and $16\beta\,(\beta^{a_{\mathrm{p}}}(2\beta)^{3A}T)^f \le \beta^{10Af+2} \le \beta^{20A(f+1)} \le R$. Hence $|y(x)| \le S\cdot 10/(16\beta) \le S/2$. □

Lemma 6.4.104. *Let V be a hermitian space over E of rank $n \geq 1$ with a frame and signs s_v, and $Z = V \oplus Ee$ with $\langle e, e\rangle = 1$ and the frame of V followed by e; write $h \mapsto h \oplus 1$ for $\mathrm{U}(V) \to \mathrm{U}(Z)$, $\Gamma_V = \mathrm{U}(V)(E^+)$, $\Gamma_Z = \mathrm{U}(Z)(E^+)$, $G_\infty(V) \times G_f(V)$ and $G_\infty(Z) \times G_f(Z)$ the frame decompositions, and $K_V(a)$, $K_Z(a)$ the principal levels. Fix Haar measures dr on $G_\infty(V)$ (a product over $v \in \Phi$) and dk on $G_f(V)$, put $dh = dr\,dk$ on $\mathrm{U}(V)(\mathbb{A})$, and let $\dot{dh}$ be the quotient measure on $\Gamma_V\backslash\mathrm{U}(V)(\mathbb{A})$ for which $\int_{\mathrm{U}(V)(\mathbb{A})} u\,dh = \int \sum_{\delta\in\Gamma_V} u(\delta h)\,\dot{dh}$ for integrable u. Let $0 \neq a \in \mathcal{O}_E$ and $R \geq 0$ with $R \leq |a|_v$ for all $v \in \Phi$; let $A \in \mathbb{N}$ with $8(n+1)^2 < A$ and $2f(n+1) + 2 \leq A$; let λ' be such that $2\lambda'(v)$ has large gap $2A$ for $(s_v, +1)$; let $\alpha, \omega \in G_\infty(Z)$. For $g \in \mathrm{U}(Z)(\mathbb{A})$ put $P(g) = \phi_{\alpha,\omega}(g_\infty)\,\mathbf{1}_{K_Z(a)}(g_f)$. Let F be a bounded measurable function on $\mathrm{U}(V)(\mathbb{A})$, left invariant under Γ_V and right invariant under $K_V(a)$, with $|F| \leq S$. Let $x \in \mathrm{U}(V)(\mathbb{A})$ and $d \geq \max(10, f)$ with $d^2 \leq R\lambda_V(x)$. Then $\psi(r) = \phi_{\alpha,\omega}(r \oplus 1)$ is integrable on $G_\infty(V)$, the function $h \mapsto \sum_{\gamma\in\Gamma_Z} P\big((x\oplus 1)^{-1}\gamma(h\oplus 1)\big)F(h)$ is integrable on $\Gamma_V\backslash\mathrm{U}(V)(\mathbb{A})$, and*

$$\Big|\int \sum_{\gamma\in\Gamma_Z} P\big((x\oplus 1)^{-1}\gamma(h\oplus 1)\big)F(h)\,\dot{dh} - \mathrm{vol}(K_V(a))\int_{G_\infty(V)} \psi(r)F\big(x(r,1)\big)\,dr\Big| \leq \mathrm{vol}(K_V(a))\,C_{\mathrm{sl}}(\alpha,\omega)\,S\,\frac{10}{d}.$$

Proof. Write $\tilde{x} = x \oplus 1$, $X = \prod_{v\in\Phi} \mathbb{C}^{n+1}$ with its Euclidean norm (a real space of dimension $2f(n+1)$), and for $u \in Z$ let $c_x(u) = \tilde{x}^{-1}c(u) \in \mathbb{A}_E^{n+1}$ be its coordinates at $\tilde{x}$, $c_x(u)_\infty \in X$ their infinite part. For $g \in G_\infty(Z)$ every factor satisfies $\|g_v\| \geq 1$, since $g_v^{-1} = J_v g_v^* J_v$ with $J_v = \mathrm{diag}(s_v, 1)$ gives $\|g_v^{-1}\| = \|g_v\|$ and $1 \leq \|g_v\|\|g_v^{-1}\|$.

Step 1: integrability of ψ and of the kernels. By Lemma 6.4.105 with $g = 1$ and $t = 1$ (each $\|r_v \oplus 1\| \geq 1$), ψ is integrable. Let $g \in \mathrm{U}(Z)(\mathbb{A})$ with $g_f \in K_Z(a)$. For $h \in \mathrm{U}(V)(\mathbb{A})$, $g_f(h_f \oplus 1) \in K_Z(a)$ if and only if $h_f \oplus 1 \in K_Z(a)$, if and only if $h_f \in K_V(a)$; so $P(g(h\oplus 1)) = \phi_{\alpha,\omega}(g_\infty(h_\infty \oplus 1))\mathbf{1}_{K_V(a)}(h_f)$ and, by Fubini,

$$\int_{\mathrm{U}(V)(\mathbb{A})} \big|P\big(g(h\oplus 1)\big)\big|\,dh = \mathrm{vol}(K_V(a))\int_{G_\infty(V)} \big|\phi_{\alpha,\omega}\big(g_\infty(r\oplus 1)\big)\big|\,dr \leq \mathrm{vol}(K_V(a))\,C_{\mathrm{sl}}(\alpha,\omega)\,t^{-A} \tag{6.114}$$

whenever $0 < t \leq \prod_v \|g_v(r_v \oplus 1)\|$ for all r (Lemma 6.4.105); $t = 1$ is always allowed.

Step 2: the cosets that contribute. The map $h \mapsto h\oplus 1$ identifies Γ_V with the stabiliser of e in Γ_Z: an isometry fixing e preserves $V = e^\perp$. Choose a set Σ of representatives of $\Gamma_Z/(\Gamma_V \oplus 1)$ containing 1, so that $\gamma = \sigma(\delta \oplus 1)$ with unique $\sigma \in \Sigma$, $\delta \in \Gamma_V$. Let $\Sigma' \subseteq \Sigma$ be the set of σ for which some $k_\sigma \in \mathrm{U}(V)(\mathbb{A})$ has $(\tilde{x}^{-1}\sigma(k_\sigma \oplus 1))_f \in K_Z(a)$; fix such k_σ, with $k_1 = x$. If $\sigma \notin \Sigma'$, every term $P(\tilde{x}^{-1}\sigma(\delta\oplus 1)(h\oplus 1))$ vanishes. For $\sigma \in \Sigma$ let $y_\sigma = \sigma e \in Z$; then $y_\sigma = y_{\sigma'}$ implies $\sigma^{-1}\sigma' e = e$, so $\sigma^{-1}\sigma' \in \Gamma_V \oplus 1$ and $\sigma = \sigma'$. Put $p_\sigma = c_x(y_\sigma)_\infty \in X$.

Step 3: separation. Let $\sigma \neq \sigma'$ in Σ' and $g = \tilde{x}^{-1}\sigma(k_\sigma \oplus 1)$, $g' = \tilde{x}^{-1}\sigma'(k_{\sigma'} \oplus 1)$. As $(k \oplus 1)$ fixes the last coordinate vector, $c_x(y_\sigma) = g\,c(e)$ is the last column of the matrix of g, and likewise for σ'; since $g_f, g'_f \in K_Z(a)$, the finite parts of $c_x(y_\sigma)$ and $c_x(y_{\sigma'})$ are congruent to the last coordinate vector modulo $a\widehat{\mathcal{O}}_E$. Hence $u = a^{-1}(y_\sigma - y_{\sigma'}) \in Z$ is nonzero with $c_x(u)_f \in \widehat{\mathcal{O}}_E^{n+1}$. Write $u = u_V + ce$ with $u_V \in V$, $c \in E$; then $c_x(u) = (x^{-1}c(u_V), c)$, so $c \in \mathcal{O}_E$ and $x^{-1}c(u_V)$ has integral finite part. If $c \neq 0$, some $|c|_v \geq 1$ by the product formula, so $\|c_x(u)_\infty\| \geq 1$; if $c = 0$, then $u_V \neq 0$ and $\|c_x(u)_\infty\| \geq \lambda_V(x)$. In both cases $\|c_x(u)_\infty\| \geq \lambda_V(x)$, and since $|a|_v \geq R$ at every v,

$$\|p_\sigma - p_{\sigma'}\| = \|a\,c_x(u)_\infty\| \geq R\lambda_V(x) \geq d^2.$$

Also $p_1 = c(e)_\infty$ has norm $\sqrt{f} \leq d$.

Step 4: the terms. Let $\sigma \in \Sigma'$, $\sigma \neq 1$; then $\|p_\sigma\| \geq d^2 - d \geq d^2/2 > 0$. For $h \in \mathrm{U}(V)(\mathbb{A})$ let $g_h = \tilde{x}^{-1}\sigma(k_\sigma h \oplus 1)$. Its last column at the infinite places is p_σ, so $\|p_\sigma\| \leq \sum_v \|(g_h)_v\| \leq f \prod_v \|(g_h)_v\| \leq d \prod_v \|(g_h)_v\|$, using $\|(g_h)_v\| \geq 1$. Apply (6.114) to $g = g_1 = \tilde{x}^{-1}\sigma(k_\sigma \oplus 1)$ with $t = \|p_\sigma\|/d$, and substitute $h \mapsto k_\sigma h$ (left invariance of dh):

$$\int_{\mathrm{U}(V)(\mathbb{A})} \bigl|P(\tilde{x}^{-1}\sigma(h \oplus 1))F(h)\bigr|\, dh \leq \mathrm{vol}(K_V(a))\, C_{\mathrm{sl}}(\alpha,\omega)\, S\Bigl(\frac{d}{\|p_\sigma\|}\Bigr)^A.$$

For $\sigma = 1$ the same integral is finite by (6.114) with $t = 1$.

Step 5: summing the terms. The points p_σ, $\sigma \in \Sigma' \setminus \{1\}$, have norms and mutual distances at least $d^2/2$, and $\dim_{\mathbb{R}} X + 2 = 2f(n+1) + 2 \leq A$; by Lemma 6.4.106 with $\delta = d^2/2$,

$$\sum_{\sigma \in \Sigma' \setminus \{1\}} \Bigl(\frac{d}{\|p_\sigma\|}\Bigr)^A \leq d^A \Bigl(\frac{10}{d^2}\Bigr)^A = \Bigl(\frac{10}{d}\Bigr)^A \leq \frac{10}{d},$$

as $d \geq 10$ and $A \geq 1$.

Step 6: unfolding. By Steps 4 and 5, $\sum_{\sigma \in \Sigma'} \int |P(\tilde{x}^{-1}\sigma(h \oplus 1))F(h)|\, dh < \infty$. Since $F(\delta h) = F(h)$ for $\delta \in \Gamma_V$, the defining property of $d\dot{h}$ and Tonelli's theorem give that

$$\sum_{\sigma \in \Sigma'} \sum_{\delta \in \Gamma_V} |P(\tilde{x}^{-1}\sigma(\delta h \oplus 1))F(h)|$$

is integrable on the quotient, and, with Step 2 and Fubini,

$$\int \sum_{\gamma \in \Gamma_Z} P(\tilde{x}^{-1}\gamma(h \oplus 1))F(h)\, d\dot{h} = \sum_{\sigma \in \Sigma'} \int_{\mathrm{U}(V)(\mathbb{A})} P(\tilde{x}^{-1}\sigma(h \oplus 1))F(h)\, dh.$$

This proves the integrability claim.

Step 7: the identity coset. For $\sigma = 1$ substitute $h = xh'$:

$$P(h' \oplus 1)F(xh') = \psi(h'_\infty)\mathbf{1}_{K_V(a)}(h'_f)F(x(h'_\infty, h'_f)),$$

and $F(x(h'_\infty, h'_f)) = F(x(h'_\infty, 1))$ for $h'_f \in K_V(a)$ by right invariance; Fubini gives the term

$$\mathrm{vol}(K_V(a)) \int_{G_\infty(V)} \psi(r)F(x(r,1))\, dr.$$

The difference in the statement is the sum over $\sigma \in \Sigma' \setminus \{1\}$, bounded by Steps 4 and 5 by $\mathrm{vol}(K_V(a))C_{\mathrm{sl}}(\alpha,\omega)S \cdot 10/d$. □

Lemma 6.4.105. *Let $n \geq 1$, $s_v \in \{\pm 1\}^n$ for $v \in \Phi$, $G_\infty = \prod_{v\in\Phi} \mathrm{U}(\mathrm{diag}\, s_v)$ and $G'_\infty = \prod_{v\in\Phi} \mathrm{U}(\mathrm{diag}(s_v, 1))$, each factor with a Haar measure dr_v, $r \mapsto r \oplus 1$ the block embedding, and $\|\cdot\|$ the operator norm. Let $A \in \mathbb{N}$ with $8(n+1)^2 < A$, and let $\lambda' = (\lambda'(v))$ be such that $2\lambda'(v)$ has large gap $2A$ for $(s_v, +1)$. For $\alpha, \omega \in G'_\infty$ put*

$$\phi_{\alpha,\omega}(g) = \prod_{v\in\Phi} \overline{c_{\lambda'(v)}(\alpha_v^{-1} g_v \omega_v)},$$

$$C_{\mathrm{sl}}(\alpha,\omega) = \prod_{v\in\Phi} \bigl(2(\|\alpha_v\|\,\|\omega_v\|)^2\bigr)^A\, 16^A\, \mathrm{vol}\{r \in \mathrm{U}(\mathrm{diag}\, s_v) : \|r - 1\| \leq 1\}.$$

Let $g \in G'_\infty$ and $t > 0$ with $t \leq \prod_{v\in\Phi} \|g_v(r_v \oplus 1)\|$ for every $r \in G_\infty$. Then $r \mapsto \phi_{\alpha,\omega}(g(r \oplus 1))$ is integrable on G_∞ and

$$\int_{G_\infty} \bigl|\phi_{\alpha,\omega}(g(r \oplus 1))\bigr|\, dr \leq C_{\mathrm{sl}}(\alpha,\omega)\, t^{-A}.$$

Proof. **Step 1: two norm facts.** For $x \in \mathrm{U}(\mathrm{diag}\,\epsilon)$ (any signs ϵ), $x^{-1} = \mathrm{diag}(\epsilon)\,\bar{x}^{\mathsf{T}}\mathrm{diag}(\epsilon)$, so $\|x^{-1}\| = \|x\|$. Hence, with $X_v = \alpha_v^{-1} g_v(r_v \oplus 1)\omega_v$, the identity $g_v(r_v \oplus 1) = \alpha_v X_v \omega_v^{-1}$ gives $\|g_v(r_v \oplus 1)\| \le \|\alpha_v\|\,\|X_v\|\,\|\omega_v\|$.

Step 2: pointwise decay. For each v, $2\lambda'(v) \in \mathbb{Z}^{n+1}$ has large gap $2A$ for $(s_v, +1)$, i.e. gap $500(n+1)^3 + 2A \ge 2A$, so (1) of Lemma 5.7.38 (rank $n+1$, $c = 2A$) gives $|c_{\lambda'(v)}(u)| \le (2/\|u\|)^{2A}$ for all u. With Step 1,

$$\begin{aligned}|\phi_{\alpha,\omega}(g(r \oplus 1))| &\le \prod_{v\in\Phi} \Big(\frac{2\|\alpha_v\|\|\omega_v\|}{\|g_v(r_v \oplus 1)\|}\Big)^{2A} \\ &= \prod_{v\in\Phi} (2(\|\alpha_v\|\|\omega_v\|)^2)^A \Big(\frac{2}{\|g_v(r_v \oplus 1)\|}\Big)^{A} \cdot \Big(\prod_{v\in\Phi} \|g_v(r_v \oplus 1)\|\Big)^{-A}.\end{aligned}$$

By hypothesis the last factor is at most t^{-A}.

Step 3: the slice integral at one place. Fix v and let $H = \mathrm{U}(\mathrm{diag}\, s_v)$ with its Haar measure dr_v (right invariant; H is unimodular, so the measure is also inversion invariant), and $j(r) = r \oplus 1 \in \mathrm{U}(\mathrm{diag}(s_v, 1))$, a continuous homomorphism which is a closed embedding, hence proper. By Step 1, $\|g_v(r \oplus 1)\| = \|(r^{-1} \oplus 1)g_v^{-1}\|$; substituting $r \mapsto r^{-1}$ and applying (ii) of Lemma 5.6.32 with $S = H$, $N = n+1$, $y = g_v^{-1}$ and $c = A > 8(n+1)^2$,

$$\int_H \Big(\frac{2}{\|g_v(r \oplus 1)\|}\Big)^A dr_v = \int_H \Big(\frac{2}{\|j(r)g_v^{-1}\|}\Big)^A dr_v \le 16^A \,\mathrm{vol}\{r \in H : \|r - 1\| \le 1\},$$

the volume being finite by the same item.

Step 4: conclusion. The function $r \mapsto \phi_{\alpha,\omega}(g(r \oplus 1))$ is continuous ($c_{\lambda'(v)}$ is continuous) and, by Step 2, dominated by t^{-A} times a product over v of nonnegative functions of r_v alone; by Tonelli's theorem for the product measure and Step 3 the dominating function is integrable, with integral at most $t^{-A}\prod_v (2(\|\alpha_v\|\|\omega_v\|)^2)^A 16^A \mathrm{vol}\{\|r-1\| \le 1\} = C_{\mathrm{sl}}(\alpha,\omega)\,t^{-A}$. Hence the slice is integrable with the stated bound. □

Lemma 6.4.106. *Let X be a real normed space of finite dimension D, let $\delta > 0$, and let $(p_i)_{i\in I}$ be a family in X with $\|p_i - p_j\| \ge \delta$ for $i \ne j$ and $\|p_i\| \ge \delta$ for all i. For every integer $A \ge D+2$ the series $\sum_{i\in I} \|p_i\|^{-A}$ converges and*

$$\sum_{i\in I} \|p_i\|^{-A} \le \Big(\frac{5}{\delta}\Big)^A.$$

Proof. **Counting.** Fix a Haar measure μ on X (a Lebesgue measure in a basis); $\mu(B(c,\rho)) = \rho^D \mu(B(0,1))$ for every closed ball, by translation invariance and homogeneity, and $0 < \mu(B(0,1)) < \infty$. For $t \ge 0$ let $N(t) = \#\{i : \|p_i\| \le t\}$. The balls $B(p_i, \delta/2)$ are pairwise disjoint up to boundaries, of measure zero, by the triangle inequality, and for $\|p_i\| \le t$ they lie in $B(0, t + \delta/2)$. Comparing measures, $N(t)(\delta/2)^D \le (t + \delta/2)^D$, so $N(t)$ is finite and

$$N(t) = 0 \quad (t < \delta), \qquad N(t) \le \Big(\frac{2t+\delta}{\delta}\Big)^D \le \Big(\frac{3t}{\delta}\Big)^D \quad (t \ge \delta).$$

In particular I is countable.

Layer cake. For each i, $\|p_i\|^{-A} = \int_{\|p_i\|}^{\infty} At^{-A-1}\,dt$. Summing over i and interchanging sum and integral (Tonelli; all terms are nonnegative),

$$\sum_i \|p_i\|^{-A} = \int_0^\infty At^{-A-1} N(t)\,dt \le A\Big(\frac{3}{\delta}\Big)^D \int_\delta^\infty t^{D-A-1}\,dt = \frac{A}{A-D}\,3^D\delta^{-A},$$

which is finite since $A - D \geq 2$.

The constant. As $A-D \geq 2$ and $D \leq A$, $\frac{A}{A-D}3^D \leq \frac{A}{2}\,3^A \leq 5^A$, the last inequality because $(5/3)^A \geq 1 + \frac{2A}{3} \geq \frac{A}{2}$ (Bernoulli's inequality). □

Lemma 6.4.107. *Let Z be a hermitian space over E of rank $N \geq 1$ with a frame and signs t_v ($v \in \Phi$), $\Gamma_Z = \mathrm{U}(Z)(E^+)$, and let $B \in \mathbb{N}$ with $8N^2 < B$. Let λ' be such that $2\lambda'(v)$ has large gap B for t_v for every v. Let $0 \neq a \in \mathcal{O}_E$ and $c \in \mathrm{U}(Z)(\mathbb{A})$. Then for every $g \in \mathrm{U}(Z)(\mathbb{A})$ the series*

$$\sum_{\gamma \in \Gamma_Z} e_{\lambda'}((c\gamma g)_\infty)\, \mathbf{1}_{K(a)}((c\gamma g)_f)$$

converges absolutely, and there is a form Q on Z of type λ' and level $K(a)$ whose complex conjugate is this series, as a function of g.

Proof. With its frame Z is a framed space of rank N whose signs are the same at every place (Definition 5.6.3), $\mathrm{U}(Z)(\mathbb{A}) = G_\infty \times G_f$, and Γ_Z is read through the frames. Write $e = e_{\lambda'}$ and $K = K(a)$, an open compact subgroup of G_f; put $\kappa(y) = e(y_\infty)\mathbf{1}_K(y_f)$ for $y \in G_\infty \times G_f$.

Step 1: the kernel. Each $2\lambda'(v)$ has gap $500N^3 + B$ for the signs, so by Lemma 5.6.21 e is a cusp kernel of order B: an idempotent kernel with $e(y^{-1}) = \overline{e(y)}$ and $\sup_y |e(y)|\,\|y\|_\infty^B < \infty$ (EK4 (b)).

Step 2: absolute convergence, uniformly in g. As K is a group and $e(y^{-1}) = \overline{e(y)}$, $|\kappa(c\gamma g)| = |\kappa(g^{-1}\gamma^{-1}c^{-1})|$. Apply (ii) of Proposition 5.8.22 with $\varphi = \mathbf{1}_K$ (locally constant, compactly supported), $k = e$, $j = 0$, $B \geq 8N^2 + 1$, $r = 1$ (admissible since $\|y_v\| \geq 1$ for every $y_v \in \mathrm{U}(\operatorname{diag} t)$, as $\|y_v^{-1}\| = \|y_v\|$, so $\|y\|_\infty \geq 1$), the points g and $x = c^{-1}$, and $Z = \Gamma_Z$; after the bijection $\gamma \mapsto \gamma^{-1}$ of Γ_Z it gives

$$\sum_{\gamma \in \Gamma_Z} |\kappa(c\gamma g)| = \sum_{\gamma \in \Gamma_Z} |\kappa(g^{-1}\gamma c^{-1})| \leq C' \lambda_Z(c^{-1})^{-4fN^2} =: C_0 < \infty$$

for every g, with C_0 independent of g. So the series $P(g) = \sum_\gamma \kappa(c\gamma g)$ converges absolutely, $|P| \leq C_0$, and, since $\overline{\kappa(y)} = \kappa(y^{-1})$, $\overline{P(g)} = \sum_\gamma \kappa(g^{-1}\gamma^{-1}c^{-1}) = \sum_\gamma \kappa(g^{-1}\gamma c^{-1}) =: F(g)$, an absolutely convergent sum (reindexing by $\gamma \mapsto \gamma^{-1}$).

Step 3: invariance. Each term of F is continuous in g, so F is measurable, and $|F| \leq C_0$. For $\delta \in \Gamma_Z$, $F(\delta g) = \sum_\gamma \kappa(g^{-1}\delta^{-1}\gamma c^{-1}) = F(g)$ by the bijection $\gamma \mapsto \delta\gamma$. For $u \in K$, $\kappa(u^{-1}g^{-1}\gamma c^{-1}) = \kappa(g^{-1}\gamma c^{-1})$ since $\mathbf{1}_K(u^{-1}y_f) = \mathbf{1}_K(y_f)$; so $F(g(1,u)) = F(g)$.

Step 4: reproduction by e. For g fixed, by Tonelli's theorem and Step 2 at the points $g(h,1)$,

$$\int_{G_\infty} \sum_\gamma |e(h)\,\kappa((h,1)^{-1}g^{-1}\gamma c^{-1})|\,dh \leq C_0 \int_{G_\infty} |e| < \infty,$$

so sum and integral may be exchanged in $R(e)F(g) = \int e(h)F(g(h,1))\,dh$. For each γ, with $y = g^{-1}\gamma c^{-1}$, the identity of (EK3) at $x = 1$ gives $\int e(h)\,e(h^{-1}y_\infty)\,dh = e(y_\infty)$, hence $R(e)F = F$.

Step 5: continuity and conclusion. By Step 3, Lemma 5.8.29 applies to the bounded measurable, left Γ_Z-invariant, right K-invariant function F: $R(e)F$ is continuous; by Step 4 it is F. Hence $Q = F$ is continuous, bounded, left invariant under Γ_Z, right invariant under $K(a)$ and satisfies $R(e_{\lambda'})Q = Q$ with $e_{\lambda'}$ integrable: a form of type λ' and level $K(a)$ ((FT2), (DZ2)), and $\overline{Q(g)} = P(g)$ is the series of the statement. □

Lemma 6.4.108. *Let V be a hermitian space over E with a frame and signs s_v, $\lambda = (\lambda(v))_{v\in\Phi}$ Harish-Chandra parameters for the groups $\mathrm{U}(\operatorname{diag} s_v)$ such that e_λ is integrable on G_∞, and put*

$c_\lambda(x) = \prod_{v\in\Phi} c_{\lambda(v)}(x_v)$ *for* $x \in G_\infty$. *If* y *is a form on* V *of type* λ, *then for all* $g \in \mathrm{U}(V)(\mathbb{A})$ *and* $x \in G_\infty$

$$\int_{G_\infty} e_\lambda(z)\, y(g\,(z,1)(x,1))\, dz = c_\lambda(x)\, y(g).$$

Proof. **The local kernels.** Fix $v \in \Phi$ and write $G_v = \mathrm{U}(\mathrm{diag}\, s_v)$, $c_v = c_{\lambda(v)}$, $d_v = d_{\lambda(v)}$ and $e_v = d_v\overline{c_v}$, so that $e_\lambda(z) = \prod_v e_v(z_v)$. By definition c_v is a diagonal matrix coefficient of a unit vector w of the discrete series $(\sigma, \mathcal{H})$ of G_v with parameter $\lambda(v)$, and $d_v^{-1} = \int_{G_v} |c_v|^2$. With inner products linear in the second argument, $c_v(g) = \langle w, \sigma(g)w\rangle$ or $c_v(g) = \langle \sigma(g)w, w\rangle$; we treat the first case, the second being the same computation with all coefficients conjugated. Schur's orthogonality relations (Theorem A.5.18, applicable since σ is irreducible, unitary and has a square-integrable coefficient) give, with $a = b = a' = b' = w$, that the formal degree of σ is d_v, and in general

$$\int_{G_v} \langle b, \sigma(z)a\rangle\, \overline{\langle b', \sigma(z)a'\rangle}\, dz = d_v^{-1}\langle a', a\rangle\langle b, b'\rangle.$$

For $x, u \in G_v$, unitarity gives $\overline{c_v(x^{-1}z^{-1}u)} = \langle \sigma(u)w, \sigma(z)\sigma(x)w\rangle$, so, with $b = \sigma(u)w$, $a = \sigma(x)w$, $b' = a' = w$,

$$\begin{aligned} K_{v,x}(u) := \int_{G_v} e_v(z)\, e_v(x^{-1}z^{-1}u)\, dz &= d_v^2 \int_{G_v} \langle \sigma(u)w, \sigma(z)\sigma(x)w\rangle\, \overline{\langle w, \sigma(z)w\rangle}\, dz \\ &= d_v\langle w, \sigma(x)w\rangle\langle \sigma(u)w, w\rangle = c_v(x)\, e_v(u). \end{aligned}$$

Each e_v is integrable, since $\int |e_\lambda| = \prod_v \int |e_v| < \infty$ and every factor is positive. By Fubini on the product $G_\infty = \prod_v G_v$, for $x \in G_\infty$

$$\int_{G_\infty} e_\lambda(z)\, e_\lambda(x^{-1}z^{-1}u)\, dz = \prod_v K_{v,x_v}(u_v) = c_\lambda(x)\, e_\lambda(u) \qquad (u \in G_\infty).$$

The form. Let $f(u) = y(g(u,1))$ for $u \in G_\infty$; it is bounded and measurable, and the type condition (DZ2) says $\int e_\lambda(h) f(uh)\, dh = f(u)$ for every u. Hence $f(zx) = \int e_\lambda(h) f(zxh)\, dh$, and

$$\begin{aligned} \int e_\lambda(z) f(zx)\, dz &= \iint e_\lambda(z) e_\lambda(h) f(zxh)\, dh\, dz \\ &= \int f(u)\Big(\int e_\lambda(z) e_\lambda(x^{-1}z^{-1}u)\, dz\Big) du = c_\lambda(x) \int e_\lambda(u) f(u)\, du, \end{aligned}$$

where the interchange and the substitution $u = zxh$ (left invariance of dh) are justified by $\iint |e_\lambda(z)||e_\lambda(h)| \sup|f|\, dh\, dz < \infty$. The last integral is $f(1) = y(g)$ by the type condition at $u = 1$, and $f(zx) = y(g(z,1)(x,1))$. □

Lemma 6.4.109. *Let* $n \geq 1$, $\varepsilon \in \{\pm 1\}^n$ *and* $\varepsilon' = (\varepsilon, +1) \in \{\pm 1\}^{n+1}$; *let* $l \in \mathbb{Z}^n$ *and* $l' \in \mathbb{Z}^{n+1}$, *let* X *be a finite set,* $M \geq 0$ *an even integer, and* $e\colon X \sqcup \{0, \dots, M-1\} \to \{1, \dots, n\}$, $e'\colon X \sqcup \{0, \dots, M\} \to \{1, \dots, n+1\}$ *bijections. Let* $B \geq 1$ *and* $\beta \geq 10^{12}$ *be reals and* $h, L \in \mathbb{N}$. *Assume:*

1. l *is steep for* (ε, e, B), *and* $-l'$ *is steep for* (ε', e', B);
2. l *has large gap* h *for* ε *and* l' *has large gap* h *for* ε';
3. $|l_a| \leq L$ *and* $|l'_{a'}| \leq L$ *for all entries;*
4. $2n + 1 + L \leq \beta$, $(n+1)^4(6B)^{2M+2} \leq \beta$ *and* $10^{30}\big((2n+1)^2 \log\beta\big)^3 \leq h$.

Then there is $y \in \mathrm{U}(\operatorname{diag}\varepsilon')$ with $\|y\|^2 \le \beta$ such that, for any Haar measure dg on $\mathrm{U}(\operatorname{diag}\varepsilon)$,

$$\operatorname{vol}\{g \in \mathrm{U}(\operatorname{diag}\varepsilon) : \|g-1\| \le 1\} \le \beta^{20(2n+1)^2}\Big|\int_{\mathrm{U}(\operatorname{diag}\varepsilon)} c_{l/2}(g)\,\overline{c_{l'/2}(y^{-1}(g\oplus 1)y)}\,dg\Big|,$$

where $\|\cdot\|$ is the operator norm and $g \oplus 1 = \operatorname{diag}(g,1)$.

Proof. Write $\Omega = \{1,\dots,n\}$, $* = n+1$, $\Omega' = \Omega \sqcup \{*\}$, $J = \operatorname{diag}(\varepsilon)$, $J' = \operatorname{diag}(\varepsilon')$, $(b_i)_{i\in\Omega'}$ for the standard basis of $\mathbb{C}^{n+1}$, $\mu = \mu_l$ and $\nu = \mu_{l'}$. As l' has a gap, $\mu_{-l'} = -\nu$ (Lemma 5.6.27 (iii)). So the hypotheses (1) say: $\varepsilon_{e(x)} = 1$ and $\mu(e(x)) = 0$ for $x \in X$, $\varepsilon_{e(i)} = (-1)^i$ and $i \mapsto \mu(e(i))$ is strictly decreasing on $\{0,\dots,M-1\}$ with values in $[-3B,-B]$; $\varepsilon'_{e'(x)} = 1$ and $\nu(e'(x)) = 0$ for $x \in X$, $\varepsilon'_{e'(a)} = (-1)^a$ and $a \mapsto \nu(e'(a))$ is strictly increasing on $\{0,\dots,M\}$ with values in $[B,3B]$.

Nodes and Lagrange weights. Put $\tau_k = \nu(e'(M-k))$ for $0 \le k \le M$ and $\tau_{M+1+k} = \mu(e(k))$ for $0 \le k < M$. The first $M+1$ decrease strictly in $[B,3B]$, the last M decrease strictly in $[-3B,-B]$; so $\tau_0 > \tau_1 > \cdots > \tau_{2M}$ are integers with $|\tau_k - \tau_{k'}| \le 6B$. For $0 \le k \le 2M$ the product $\prod_{k'\ne k}(\tau_k - \tau_{k'})$ has exactly k negative factors, so $c_k = \big|\prod_{k'\ne k}(\tau_k-\tau_{k'})\big|^{1/2} \ge 1$ satisfies $c_k^2 = (-1)^k \prod_{k'\ne k}(\tau_k-\tau_{k'})$, and Lagrange interpolation at the $2M+1$ nodes gives, for every polynomial π of degree at most $2M$,

$$[\text{coefficient of } \tau^{2M} \text{ in } \pi] = \sum_{k\le 2M} \frac{\pi(\tau_k)}{\prod_{k'\ne k}(\tau_k-\tau_{k'})} = \sum_{k\le 2M} (-1)^k \frac{\pi(\tau_k)}{c_k^2}. \tag{6.115}$$

For $0 \le a \le M$ put

$$d_a = \frac{(-1)^a}{c_a}\prod_{k<M}(\tau_a - \tau_{M+1+k}), \qquad \pi_a(\tau) = d_a \prod_{k\le M,\,k\ne a}(\tau-\tau_k), \qquad \rho_{ka} = \frac{\pi_a(\tau_{M+1+k})}{c_{M+1+k}} \quad (k<M).$$

By the formula for c_a^2, $\pi_a(\tau_{a'}) = \delta_{aa'}c_a$ for $a,a' \le M$. The polynomial $\pi_a\pi_{a'}$ has degree $2M$ and leading coefficient $d_ad_{a'}$, so (6.115) gives $\sum_{k\le 2M}(-1)^k\pi_a(\tau_k)\pi_{a'}(\tau_k)/c_k^2 = d_ad_{a'}$; the terms with $k \le M$ add up to $(-1)^a\delta_{aa'}$, and those with $k = M+1+k'$ are $-(-1)^{k'}\rho_{k'a}\rho_{k'a'}$ because M is even. Hence

$$\sum_{k<M}(-1)^k\rho_{ka}\rho_{ka'} + d_ad_{a'} = (-1)^a\delta_{aa'} \qquad (a,a' \le M). \tag{6.116}$$

Sizes: $d_a^2 = \prod_{k<M}|\tau_a-\tau_{M+1+k}| / \prod_{k\le M,k\ne a}|\tau_a-\tau_k| \le (6B)^M$, as the nodes are distinct integers; and since $c_{M+1+k}^2 \ge |\tau_{M+1+k}-\tau_a|\prod_{k'\le M,k'\ne a}|\tau_{M+1+k}-\tau_{k'}|$, $\rho_{ka}^2 \le d_a^2\prod_{k'\le M,k'\ne a}|\tau_{M+1+k}-\tau_{k'}| \le (6B)^{2M}$.

The point y. Let y be the real $\Omega'\times\Omega'$ matrix with columns

$$y\,b_{e'(x)} = b_{e(x)} \quad (x\in X), \qquad y\,b_{e'(a)} = \sum_{k<M}\rho_{k,M-a}\,b_{e(k)} + d_{M-a}\,b_* \quad (0\le a\le M);$$

this defines y because e' is a bijection onto Ω'. The columns $b_{e(x)}$ are distinct basis vectors of sign $+1$ that do not occur in the other columns; and by (6.116), with $\varepsilon_{e(k)} = (-1)^k$ and $\varepsilon'_* = 1$, the J'-product of the columns at $e'(a)$ and $e'(a')$ is $(-1)^{M-a}\delta_{aa'} = (-1)^a\delta_{aa'} = \varepsilon'_{e'(a)}\delta_{aa'}$. So $y^*J'y = J'$ and $y \in \mathrm{U}(\operatorname{diag}\varepsilon')$; in particular $\|y^{-1}\| = \|y\|$ (Lemma 5.6.30 (i)). Every entry of y has absolute value at most $(6B)^M$, so $\|y\|^2 \le \|y\|_{\mathrm{HS}}^2 \le (n+1)^2(6B)^{2M} \le (n+1)^4(6B)^{2M+2} \le \beta$.

The weights are conjugate. We show that the rows with index in Ω of $y\operatorname{diag}(\nu)\,y^{-1}$ are those of the $\Omega\times\Omega'$ matrix $(\operatorname{diag}(\mu) \mid v)$, for the column v at $*$ with $v_{e(x)} = 0$ and $v_{e(k)} = -\prod_{k'\le M}(\tau_{M+1+k}-\tau_{k'})/c_{M+1+k}$; equivalently, row $r \in \Omega$ of $y\operatorname{diag}(\nu)$ equals $\mu(r)$ times row r

of y plus v_r times row $*$ of y. In a column $e'(x)$ both sides vanish, as $\nu(e'(x))=0$, $\mu(e(x))=0$ and $y_{*,e'(x)}=0$. In a column $e'(a)$ and a row $e(x)$ both sides vanish. In a column $e'(a)$ and a row $e(k)$, where $\nu(e'(a))=\tau_{M-a}$ and $\mu(e(k))=\tau_{M+1+k}$, the difference of the two sides is, with $a'=M-a$,

$$(\tau_{a'}-\tau_{M+1+k})\rho_{ka'}-v_{e(k)}d_{a'}=0,$$

because $(\tau_{a'}-\tau)\pi_{a'}(\tau)=-d_{a'}\prod_{k'\le M}(\tau-\tau_{k'})$. Consequently, for every $\Omega\times\Omega$ matrix Z,

$$\operatorname{tr}\bigl(\operatorname{diag}(\nu)\,y^{-1}(Z\oplus 0)\,y\bigr)=\operatorname{tr}\bigl(y\operatorname{diag}(\nu)\,y^{-1}(Z\oplus 0)\bigr)=\operatorname{tr}\bigl(\operatorname{diag}(\mu)Z\bigr). \tag{6.117}$$

Laplace's method. Let dg be a Haar measure on $S=\mathrm{U}(\operatorname{diag}\varepsilon)$; it is right invariant (the group is unimodular) and finite on compact sets. Let $j=(j_1,j_2)$, $j_1(g)=g$, $j_2(g)=y^{-1}(g\oplus 1)y$, a homomorphism to $\mathrm{U}(\operatorname{diag}\varepsilon)\times\mathrm{U}(\operatorname{diag}\varepsilon')$ with j_1 proper and j_2 continuous, and $\varphi=0$. Note $j_2(g)-1=y^{-1}((g-1)\oplus 0)y$. We check that these data form a Laplace datum, that is (LA1)–(LA4) of Definition 5.7.2, for $\mathfrak{n}=\Omega$, $\mathfrak{m}=\Omega'$, the signs ε,ε', the vectors l and $k=l'$, the bijection e, the map $\tau=e'|_X$, and B,β,L,h.

(LA1) is hypothesis (1) for l.

(LA2) $\varphi=0$ is additive with real part 0; by (6.117), $\operatorname{tr}(\operatorname{diag}(\mu)(j_1(g)-1))-\operatorname{tr}(\operatorname{diag}(\nu)(j_2(g)-1))=0$; and $\operatorname{tr}(j_2(g)-1)=\operatorname{tr}((g-1)\oplus 0)=\operatorname{tr}(j_1(g)-1)$.

(LA3) $\varepsilon'_{e'(x)}=1$ and $\nu(e'(x))=0$; $\nu\ge 0$ on Ω', with values 0 or in $[B,3B]$, so $\nu\le 3B$ everywhere. For a unitary $X\times X$ matrix u let $g=1+\iota_e(u-1)\iota_e^*$; it is unitary and equals 1 off the block $e(X)\times e(X)$, where $J=1$, so $g\in\mathrm{U}(\operatorname{diag}\varepsilon)$. Since $yb_{e'(x)}=b_{e(x)}$ and the row of y at $e(x)$ is $b^{\mathsf T}_{e'(x)}$ (the other columns vanish at $e(x)$), $y^{-1}b_{e(x)}=b_{e'(x)}$ and $b^{\mathsf T}_{e(x')}y=b^{\mathsf T}_{e'(x')}$; hence $j_2(g)-1=\iota_\tau(u-1)\iota_\tau^*$ and $j_1(g)-1=\iota_e(u-1)\iota_e^*$.

(LA4) is hypotheses (2), (3), (4), with $\#\mathfrak{n}+\#\mathfrak{m}=2n+1$.

Theorem 5.7.21 gives that $g\mapsto c_{l/2}(g)\overline{c_{l'/2}(j_2(g))}$ is integrable and

$$\operatorname{vol}\{g:\|g\|\le\beta+1,\ \|j_2(g)\|\le\beta+1\}\le\beta^{20(2n+1)^2}\Bigl|\int_{\mathrm{U}(\operatorname{diag}\varepsilon)}c_{l/2}(g)\,\overline{c_{l'/2}(y^{-1}(g\oplus 1)y)}\,dg\Bigr|.$$

If $\|g-1\|\le 1$, then $\|g\|\le 2\le\beta+1$ and $\|j_2(g)-1\|\le\|y^{-1}\|\,\|g-1\|\,\|y\|\le\|y\|^2\le\beta$, so $\|j_2(g)\|\le\beta+1$. Hence the set on the left contains $\{g:\|g-1\|\le 1\}$, which proves the lemma. □

6.5 The norm of the chain of lifts

The proof of Theorem C compares $Z(\vartheta)=\det(\Theta_\vartheta^*\Theta_\vartheta)$ with $Z(1)$ along a regular family, and this section factors their ratio into pieces whose heights can be bounded one by one (Proposition 6.5.1). The Gram operator of the chain is, up to a scalar, the product of the Gram operators of the m single lifts, the j-th taken at ϑ^j, and each of these is a Rallis series; the scalar has modulus a fixed constant times a power of $\mathrm{N}Q$ times the product of the chain ratios $q_{ij}(\vartheta)$. Taking determinants on the target forms gives

$$\frac{Z(\vartheta)}{Z(1)}=\Bigl((\mathrm{N}Q)^{e/2}\prod_{1\le i<j\le m}q_{ij}(\vartheta)\Bigr)^{d_\mathcal{T}}\ \prod_{j=1}^{m}\frac{Z_j(\vartheta)}{Z_j(1)}.$$

The chain ratios have small height along a family with small ratios (Lemma 6.1.5), and the volumes of the single lifts are bounded one lift at a time in Section 6.6 (Theorem 6.6.6). The

injectivity of the chain, and so $Z(\vartheta) > 0$, comes out of the same computation. The organisation follows [259].

The tool throughout is the Rallis inner product formula [237, 238], the basis of the doubling method [231]; for theta lifts along Witt towers see [236]. In Weil's convergent range [323] it was proved by Li for cusp forms [187], and beyond that range it rests on the regularised Siegel–Weil formula [176, 139, 94, 331]. The pairs here are in the convergent range, but the forms are rapidly decreasing rather than cuspidal, so the form used is proved at the end of the section (Theorem 6.5.13).

Proposition 6.5.1. *There are $C > 0$, $e \in \mathbb{Z}$ and a function c from the m-fold twists to $\mathbb{C}$ such that for every m-fold twist ϑ with modulus Q:*

1. $|c(\vartheta)| = C\,(\mathrm{N}Q)^{e/2} \prod_{1\le i<j\le m} q_{ij}(\vartheta)$;
2. $\Theta_\vartheta^*\Theta_\vartheta = c(\vartheta)\, c_Q^m\, R^{\mathcal{T}}_{1,\vartheta} R^{\mathcal{T}}_{2,\vartheta^2} \cdots R^{\mathcal{T}}_{m,\vartheta^m}$ *on* $\mathcal{T}$;
3. Θ_ϑ *is injective,* $Z(\vartheta) > 0$, *and* $Z_j(\vartheta) > 0$ *for* $1 \le j \le m$;
4. *with the same* e,
$$\frac{Z(\vartheta)}{Z(1)} = \Big((\mathrm{N}Q)^{e/2} \prod_{1\le i<j\le m} q_{ij}(\vartheta)\Big)^d \prod_{j=1}^m \frac{Z_j(\vartheta)}{Z_j(1)}.$$

Proof. **Step 1: the operator identity.** By Lemma 6.5.2 there are c, $C_0 \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $|c(\vartheta)|^2 = C_0(\mathrm{N}Q)^k f(\vartheta)^2$ and $\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c(\vartheta) c_Q^m \langle x, R_\vartheta y\rangle$ for all $x, y \in \mathcal{T}$. By the definition of the adjoint, $\langle x, \Theta_\vartheta^*\Theta_\vartheta y\rangle = \langle x, c(\vartheta) c_Q^m R_\vartheta y\rangle$ for all x, which is (2) for this c.

Step 2: the modulus of c. At the trivial twist $Q = \emptyset$, $\mathrm{N}Q = 1$ and $f(1) = 1$, since $q_{ij}(1) = 1$ for all $i < j$ by Lemma 6.1.7(c). If $c(1) = 0$, then $\langle \Theta_1 y, \Theta_1 y\rangle = 0$, so $\Theta_1 y = 0$ and $y = 0$ by the injectivity in Lemma 6.5.2: $\mathcal{T} = 0$. Then both sides of (2) are 0 for every function c, and (1)–(4) hold with $C = 1$, $e = 0$, $c = f$ ((4) reads $1 = 1$, the empty determinants being 1). If $c(1) \neq 0$, then $C_0 = |c(1)|^2 > 0$, and as $f \ge 0$,
$$|c(\vartheta)| = |c(1)|\,(\mathrm{N}Q)^{k/2} f(\vartheta);$$
this is (1) with $C = |c(1)|$ and $e = k$.

Step 3: positivity. Θ_ϑ is injective by Lemma 6.5.2, so $Z(\vartheta) = \det(\Theta_\vartheta^*\Theta_\vartheta)$, the Gram determinant of the images of an orthonormal basis, is > 0. $Z_j(\vartheta) > 0$ by Proposition 6.5.9(1). This is (3).

Step 4: the factorisation. Taking determinants in (2) on the d-dimensional space $\mathcal{T}$ and using Proposition 6.5.9(3) for each j (the powers ϑ^j have the modulus Q, so the same c_Q),
$$Z(\vartheta) = \big(c(\vartheta)c_Q^m\big)^d \prod_{j=1}^m \det R^{\mathcal{T}}_{j,\vartheta^j} = \Big(\frac{c(\vartheta)}{A_1\cdots A_m}\Big)^d \prod_{j=1}^m (A_j c_Q)^d \det R^{\mathcal{T}}_{j,\vartheta^j} = \Big(\frac{c(\vartheta)}{A_1\cdots A_m}\Big)^d \prod_{j=1}^m Z_j(\vartheta).$$
$Z(\vartheta)$ and the $Z_j(\vartheta)$ are positive reals, so taking absolute values and (1),
$$Z(\vartheta) = \big(C'(\mathrm{N}Q)^{e/2} f(\vartheta)\big)^d \prod_{j=1}^m Z_j(\vartheta), \qquad C' = \frac{C}{|A_1\cdots A_m|} > 0.$$
At $\vartheta = 1$ this reads $Z(1) = C'^{\,d} \prod_j Z_j(1)$, by Step 2. Dividing gives (4). □

The factorisation is the Gram form of the chain (Lemma 6.5.2). It is not computed lift by lift. The lifts and descents form the twisted triangle of Definition 3.2.30, and the abstract triangle lemma (Lemma 6.4.59) reduces the Gram form of a weighted chain to the product of the Gram operators of the first arrows, given that the squares commute up to the chain ratios (Proposition 6.4.11) and that the first lift of each row is injective with a Rallis series as Gram form (Lemma 6.4.58). The ratio of group orders that the lemma produces cancels against the index of the last level (Lemma 6.5.8).

The single lift is the subject of Proposition 6.5.9: it is injective, its Gram operator is a nonzero constant independent of the twist times c_Q times a Rallis series, and the series preserves the good forms, so that $Z_j(\vartheta)$ is a determinant of a Rallis series on the target forms. The passage from the good forms on W_j to the target forms is the comparison of Section 5 (Proposition 5.8.55, Proposition 5.8.4). The block datum is sharp: its matrix coefficient is its norm on a principal congruence subgroup and 0 off it (Lemma 6.5.10). Sharpness turns the Rallis inner product formula (Theorem 6.5.13) into a series (Lemma 6.5.12); for cusp forms the formula is Li's theorem [187], and the form used here is proved with the Siegel–Weil formula of the convergent range [280, 323] (see [231, 190]). The Rallis series converges absolutely, preserves automorphy and level, and is injective on bounded forms (Proposition 6.5.4); its summability uses neighbour steps of lattices, as in Kneser's method of neighbours [166]. For injectivity of theta liftings in general see [237].

Lemma 6.5.2. *For an m-fold twist ϑ let $R_\vartheta = R^{\mathcal{T}}_{1,\vartheta} R^{\mathcal{T}}_{2,\vartheta^2} \cdots R^{\mathcal{T}}_{m,\vartheta^m}$. There is a function c from the m-fold twists to $\mathbb{C}$ with absolute value $\prod_{1\le i<j\le m} q_{ij}$ up to a norm power such that*

$$\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c(\vartheta)\, c_Q^m \,\langle x, R_\vartheta y\rangle \qquad (x, y \in \mathcal{T})$$

for every m-fold twist ϑ with modulus Q. Moreover the chain of lifts is injective on the forms of level J_M on Y_0; in particular Θ_ϑ is injective.

Proof. Write $N_n(\vartheta) = \#\mathrm{GL}_{n_n}(R_\vartheta)$ and $X_\vartheta = \mathrm{Wc}^0_{\vartheta,\mathrm{I},m}$.

Step 0: the relation and the base forms. $\mathcal{M}(1,1)$ holds ($C=1$, $n=0$), $\mathcal{M}(c,g)$ and $\mathcal{M}(c',g')$ give $\mathcal{M}(cc', gg')$ (multiply the two identities), and $\mathcal{M}(c,1)$ holds for every $c(\vartheta) = C'(\mathrm{N}Q)^{k'}$ with $C' \in \mathbb{C}$, $k' \in \mathbb{Z}$ (take $C = |C'|^2$, $n = 2k'$). By (a) of Lemma 6.4.67, $\mathcal{A}_M(Y_0) \subseteq \mathcal{F}^0_\triangle(\mathrm{I}, 0)$; its forms are right invariant under K_M, the integral elements reducing to 1 at the chosen places, which in the coordinates of the point of Y_0 (the space with a point of (VP3) of Definition 3.2.2, in the basis of Y_0, given by its fixed frame) is the base level $\mathrm{K}^{\mathrm{bs}}_{\mathrm{I}}$ (the fixed frame of Y_0, (XF1) of Definition 3.2.15, is $L_0 g$ with g integral and $g \equiv 1$ at the chosen places, (HP5) of Definition 3.2.13, so integrality and the reductions at the chosen places are the same in both bases). Hence

$$\mathcal{T} \subseteq \mathcal{A}_M(Y_0) \subseteq \mathcal{F}^{\mathrm{bs}}_{\mathrm{I}}.$$

If $\mathcal{F}^{\mathrm{bs}}_{\mathrm{I}} = 0$, then $\mathcal{T} = 0$ and $\mathcal{A}_M(Y_0) = 0$, both claims hold with $c = 0$ (which has absolute value $\prod q_{ij}$ up to a norm power with $C = 0$), and nothing remains. Assume from now on $\mathcal{F}^{\mathrm{bs}}_{\mathrm{I}} \neq 0$.

Step 1: the abstract triangle. Fix left pairs $\Phi_{i,n}$ at the positions (i,n) with $i+n<m$ ((c) of Lemma 6.4.15) and take the data of Proposition 6.4.11 for $z = \mathrm{I}$: $\mathcal{H}_{i,n} = \mathcal{F}^i_\triangle(\mathrm{I}, n)$, $a^\vartheta_{i,n} = \mathrm{Tl}^i_{\vartheta,\mathrm{I},n}$, the descents $b^\vartheta_{i,n}$, $w_\vartheta(n) = N_n(\vartheta)$, $V_0 = \mathcal{F}^{\mathrm{bs}}_{\mathrm{I}}$ and the inclusions u_i. By (a) and (b) of Proposition 6.4.11, (Qu1)–(Qu3), (Qx1) and (Qx2) hold with $f_{r,k} = q_{rk}$. The weighted chain of row 0 of the abstract triangle is the weighted chain of (TV3): $X^\vartheta_{0,n} = \mathrm{Wc}^0_{\vartheta,\mathrm{I},n}$, both being defined by $X_{0,0} = \mathrm{id}$ and $X_{0,n+1} = N_n(\vartheta)\,\mathrm{Tl}^0_{\vartheta,\mathrm{I},n} \circ X_{0,n}$.

Step 2: (Qx3). Let $U_0 = \mathcal{T}$, $s\colon \mathcal{T} \to V_0$ the inclusion (Step 0) and $S_\vartheta(i) = R^{\mathcal{T}}_{i+1,\vartheta^{\,i+1}}$ for $i < m$, an endomorphism of $\mathcal{T}$ by (3) of Proposition 6.5.9. For $i < m$ the map $a^\vartheta_{i,0} \circ u_i$ is the

first lift a_ϑ of row i of Lemma 6.4.58, since $u_i x$ is x read on the adelic group of $V^i_{\mathrm{I},0}$, the copy of x (TV5). By (a) of Lemma 6.4.58 (with $z = \mathrm{I}$; its $y \in \mathcal{F}^{\mathrm{bs}}_{\mathrm{I}}$ is our $s(y)$, its x our $x \in V_0$) and Lemma 6.5.3,

$$w_\vartheta(0)\langle a^\vartheta_{i,0} u_i x,\ a^\vartheta_{i,0} u_i\, s(y)\rangle = \frac{A_{\mathrm{I},i}}{\#\mathrm{M}_{n_1\times n_0}(R_\vartheta)}\,\langle x,\ R_{\Sigma_\vartheta}\, y\rangle = C_i(\vartheta)\,\langle x,\ s(S_\vartheta(i)y)\rangle,$$

with $C_i(\vartheta) = A_{\mathrm{I},i}/\#\mathrm{M}_{n_1\times n_0}(R_\vartheta)$, the Petersson product being that of $V^0_{\mathrm{I},0} = Y_0$. As $\mathrm{N}Q$ is prime to M (Definition 3.1.2), the ideals $\pi_M\mathcal{O}_E$ and $\prod_{w\in Q} w$ are coprime, so $\#R_\vartheta = \#(\mathcal{O}_E/\pi_M\mathcal{O}_E)\cdot \mathrm{N}Q$ and $C_i(\vartheta) = A_{\mathrm{I},i}\,\#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^{-r(r+t)}(\mathrm{N}Q)^{-r(r+t)}$; as $A_{\mathrm{I},i}$ does not depend on ϑ, $\mathcal{M}(C_i, 1)$ by Step 0. So (Qx3) holds.

Step 3: the Gram form of the weighted chain. By (2) of Lemma 6.4.59 for $i = 0$, $n = m$, there is κ_0 with $\mathcal{M}(\kappa_0, \prod_{0<r<k\le m} q_{rk})$ such that, since $S_\vartheta(0)\cdots S_\vartheta(m-1) = R^{\mathcal{T}}_{1,\vartheta}R^{\mathcal{T}}_{2,\vartheta^2}\cdots R^{\mathcal{T}}_{m,\vartheta^m} = R_\vartheta$,

$$\langle X_\vartheta x,\ X_\vartheta y\rangle_{Y_m} = \kappa_0(\vartheta)\,\frac{N_0(\vartheta)\,N_1(\vartheta)^m}{N_m(\vartheta)}\,\langle x, R_\vartheta y\rangle \qquad (x, y \in \mathcal{T}),$$

the product on the left being the Petersson product of $V^0_{\mathrm{I},m} = Y_m$.

Step 4: the chain. By Lemma 6.5.7 and Lemma 6.5.8, with $C > 0$, $k \in \mathbb{Z}$ independent of ϑ and $d = \dim W_{\lambda^\circ}$,

$$\begin{aligned}\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle &= [\mathbf{K}_0 : \mathrm{K}_\vartheta]\, d\,\langle X_\vartheta x, X_\vartheta y\rangle_{Y_m}\\ &= C(\mathrm{N}Q)^k c_Q^m \frac{N_m(\vartheta)}{N_0(\vartheta)N_1(\vartheta)^m}\cdot d\cdot \kappa_0(\vartheta)\frac{N_0(\vartheta)N_1(\vartheta)^m}{N_m(\vartheta)}\langle x, R_\vartheta y\rangle.\end{aligned}$$

The group orders cancel (they are positive integers). Put $c(\vartheta) = C\,d\,(\mathrm{N}Q)^k\kappa_0(\vartheta)$. Then

$$\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c(\vartheta)\, c_Q^m\,\langle x, R_\vartheta y\rangle,$$

and $\mathcal{M}(c, \prod_{1\le i<j\le m} q_{ij})$ by Step 0 (multiplicativity, with $\mathcal{M}(Cd(\mathrm{N}Q)^k, 1)$).

Step 5: injectivity. By (1) of Lemma 6.4.59 (for $i = 0$, $n = m$), X_ϑ is injective on $V_0 \supseteq \mathcal{A}_M(Y_0)$. Let $y \in \mathcal{A}_M(Y_0)$ with $\Theta_\vartheta y = 0$, Θ_ϑ the chain on all of $\mathcal{A}_M(Y_0)$. By (b) of Lemma 6.4.67, $X_\vartheta y = F^{p_{\mathrm{I},0,m}}_{\Theta_\vartheta y} = F^{p_{\mathrm{I},0,m}}_0 = 0$, as F^l_η is linear in η; so $y = 0$. Hence the chain is injective on the forms of level $\mathrm{K}_M = J_M$, and in particular on $\mathcal{T}$. □

Lemma 6.5.3. *Let D be admissible data, $0 \le i < m$ and ϑ an m-fold twist, and let R_{Σ_ϑ} be the Rallis series of Lemma 6.4.58 for $z = \mathrm{I}$ and the row i. Then $R_{\Sigma_\vartheta} y = R^{\mathcal{T}}_{i+1,\vartheta^{i+1}}\, y$ for every $y \in \mathcal{T}$.*

Proof. Write $j = i+1$, $\vartheta' = \vartheta^j$ and $O = O_{j,\vartheta'}$; ϑ' has the modulus Q of ϑ (Definition 3.1.2).

Step 1: the data of the two series agree. The series R_{Σ_ϑ} is formed for $(S, T, \beta, \Sigma) = (V^i_{\mathrm{I},0}, V^i_{\mathrm{I},1}, b_\vartheta, \Sigma_\vartheta)$ and $R^{\mathcal{T}}_{j,\vartheta'}$ for $(W_j, \mathrm{I}_{r+t}, \chi'_O, \Sigma)$. (a) Ranks: S and W_j have rank r, and T and I_{r+t} have rank $N_1 = r+t$, since n_1 is n_0 with one down block of length t (GP5). (b) Characters: b_ϑ is the edge character with $c = \mathrm{II}$, $l = i$ at the rank N_1, that is $\mu^0_{i+1}\vartheta^{i+1}\upsilon^{\lfloor (r+t)/2\rfloor}$ (GP6), and $\chi'_O = \mu^0_j\vartheta'\upsilon^{\lfloor (r+t)/2\rfloor}$; they are equal. (c) Places: the places dividing $\mathfrak{t}_\vartheta(\mathrm{I}) = \pi_M\prod_{w\in Q} w$ are the chosen places and the places of Q, which are the places $M_O \cup Q_O$ of the level system of O (LF4); so $\Sigma_\vartheta = \Sigma$. (d) Groups and representatives: through the points of the spaces, $\mathrm{U}(S)(\mathbb{A}_F)$, $\mathrm{U}(W_j)(\mathbb{A}_F)$ and $\mathrm{U}(Y_0)(\mathbb{A}_F)$ are written as groups $\mathrm{G}(\cdot) = G_\infty(\cdot)\times G_f(r)$ with the same finite part $G_f(r)$ ((VP1), (VP2) of Definition 3.2.2; S and Y_0 have the same adelic group of matrices

by (TV1)), and the groups K_Σ of Proposition 6.5.4 are formed in $G_f(r)$ from the components K_v of $\mathbf{K}_0(r)$ (Lemma 6.5.6). So the cosets q and their fixed representatives u_q, which have components 1 off Σ, are the same for both series, and both act on functions y on $\mathrm{U}(Y_0)(\mathbb{A}_F)$ by $y \mapsto \sum_q \mathrm{mc}(u_q)\, y(\cdot\, u_q)$, with the vacuum coefficient of the respective pair.

Step 2: the vacuum coefficient depends only on the ranks, the character and Σ. Let (S,T) be spaces with points with signs s on n and s' on n', β a character and $\gamma \in G_f(n)$ with components 1 off the places above Σ. By (WR4) with $\alpha = 1$ and $g = (1,\gamma)$, $h = 1$,

$$\mathrm{mc}_{\beta,\Sigma}(\gamma) = \beta(\det\gamma)^{-1}\langle v^0,\ \omega_\varepsilon(k_\gamma)\, v^0\rangle, \qquad k_\gamma = 1_{n'} \otimes (\gamma^{-1})^{\mathsf{T}},$$

with $\varepsilon = s' \otimes s$ on $\iota = n' \times n$ and $k_\gamma \in \mathrm{G}(\varepsilon)$ the finite element $X \mapsto X\gamma^{-1}$ (infinite components 1). By (WR3) and (WD7), with $N = \#\iota$,

$$\langle v^0, \omega_\varepsilon(k) v^0\rangle = \Delta_\varepsilon(k) \iint \overline{e_N(x)}\, e_N(y)\, \beta_\varepsilon(ky,x)\, e_N(x-ky)\, d\mu_N(y)\, d\mu_N(x) \qquad (k = k_\gamma).$$

Here $\Delta_\varepsilon(k) = v(\det u)|\det u|^{-1/2}$ does not depend on ε, since $k_v = 1$ at the infinite places. In the integrand ε enters only through the infinite part of $\beta_\varepsilon(ky,x) = \psi_E(\frac{\jmath}{2}\bar{x}^{\mathsf{T}} S\, ky)$ (S is 1 at the finite places, (WD6)), which, as $(ky)_\infty = y_\infty$, is $\prod_{v\in\Phi,\, a\in\iota} \exp(-2\pi i \,\mathrm{Re}(\jmath_v \varepsilon_a \bar{x}_{a,v} y_{a,v}))$ by (WD4). Substitute $x_{a,v} \mapsto \bar{x}_{a,v}$, $y_{a,v} \mapsto \bar{y}_{a,v}$ in the infinite coordinates (a,v) with $\varepsilon_a = -1$. This preserves μ_N (Lebesgue measure on these coordinates, scaled by the real κ_v, (WD6)), e_N and $e_N(x-ky)$ (they depend on the infinite coordinates through $|x_{a,v}|^2$, $|x_{a,v}-y_{a,v}|^2$, (WD5)), and the finite coordinates, on which k acts; and it replaces $\mathrm{Re}(\jmath_v\varepsilon_a\bar{x}_{a,v}y_{a,v})$ by $\mathrm{Re}(\jmath_v\varepsilon_a x_{a,v}\bar{y}_{a,v}) = \mathrm{Re}(\bar{\jmath}_v\varepsilon_a\bar{x}_{a,v}y_{a,v}) = \mathrm{Re}(\jmath_v\cdot(-\varepsilon_a)\bar{x}_{a,v}y_{a,v})$, as $\bar{\jmath} = -\jmath$ (WD1). So the integral for ε equals the integral for the constant sign $+1$ on ι. The integrand is integrable (a Gaussian in the infinite coordinates times a bounded function of compact support in the finite ones), so the substitution is legitimate. Finally a bijection $\iota \to \iota'$ of index sets with the constant sign transports e_N, μ_N, β_{+1} and k_γ to the same objects for ι'. Hence $\mathrm{mc}_{\beta,\Sigma}(\gamma)$ depends only on $\#n$, $\#n'$, β and γ.

Step 3: conclusion. By Step 1 (a)–(c) and Step 2, $\mathrm{mc}_{b_\vartheta,\Sigma_\vartheta}(u_q)$ for the pair $(V^i_{\mathrm{I},0}, V^i_{\mathrm{I},1})$ equals $\mathrm{mc}_{\chi'_O,\Sigma}(u_q)$ for the pair (W_j, I_{r+t}), for every representative u_q. With Step 1 (d) the two series agree term by term on every function y on $\mathrm{U}(Y_0)(\mathbb{A}_F)$; by (1) of Proposition 6.5.4 (applicable as $2r < r+t$) both converge absolutely for bounded y, in particular for $y \in \mathcal{T}$, and so $R_{\Sigma_\vartheta} y = R^{\mathcal{T}}_{i+1,\vartheta^{\,i+1}} y$. □

Proposition 6.5.4. *Let S, T be spaces with $2\,\mathrm{rk}\,S < \mathrm{rk}\,T$, β an automorphic character of $\mathrm{U}(1)$ and Σ a set of finite places of F. Let K_Σ be the group of $g \in \mathrm{U}(S)(\mathbb{A}_{F,f})$ with $g_v \in K_v$ for $v \in \Sigma$, and K'_Σ the group of $k \in \prod_v K_v$ with $k_v = 1$ for $v \notin \Sigma$. For each coset $q \in \mathrm{U}(S)(\mathbb{A}_{F,f})/K_\Sigma$ fix a representative u_q with $(u_q)_v = 1$ for $v \notin \Sigma$. For a function y on $\mathrm{U}(S)(\mathbb{A}_F)$ the **Rallis series** is*

$$(R_\Sigma y)(g) = \sum_q \mathrm{mc}_{\beta,\Sigma}(u_q)\, y(g u_q) \qquad (g \in \mathrm{U}(S)(\mathbb{A}_F)).$$

Then:

1. *$\sum_q |\mathrm{mc}_{\beta,\Sigma}(u_q)| < \infty$. Hence for y bounded and measurable the series converges absolutely and uniformly, $\sup|R_\Sigma y| \le (\sum_q |\mathrm{mc}_{\beta,\Sigma}(u_q)|)\sup|y|$, and for y moreover left invariant under $\mathrm{U}(S)(F)$, y' integrable on $[\mathrm{U}(S)]$ and $x \in \mathrm{U}(S)(\mathbb{A}_F)$,*

$$\sum_q \mathrm{mc}_{\beta,\Sigma}(u_q)\, \langle y',\, y(\cdot\, x\, u_q)\rangle_S = \langle y',\, (R_\Sigma y)(\cdot\, x)\rangle_S,$$

the series converging absolutely.

2. *Suppose* $\beta(\det k) = 1$ *for all* $k \in K'_\Sigma$. *Let* y *be bounded, continuous, left invariant under* $\mathrm{U}(S)(F)$, *right invariant under* K'_Σ *and under some open subgroup of* $\mathrm{U}(S)(\mathbb{A}_{F,f})$. *Then* $R_\Sigma y$ *has the same four properties;* R_Σ *commutes with right translation by* $\mathrm{U}(S)(F \otimes \mathbb{R})$ *and with convolution on the right by integrable functions on* $\mathrm{U}(S)(F \otimes \mathbb{R})$; *and if* y *is right invariant under a subgroup* J *with* $K'_\Sigma \le J \le K_\Sigma$, *so is* $R_\Sigma y$.

3. *Under the hypothesis on* β *of (2), a bounded function* y, *right invariant under* K'_Σ, *with* $R_\Sigma y = 0$ *is* 0.

Proof. **Step 1: the coefficients.** $\langle v^0, v^0\rangle = 1$: For $\gamma \in G_f(n)$ supported on a set A of finite places, by the definition in Lemma 6.5.6

$$\mathrm{mc}_{\beta,A}(\gamma) = \mathrm{mc}(\gamma) := \langle v^0, \omega((1,\gamma),1)v^0\rangle,$$

which does not depend on A. By (5) of Lemma 6.2.33 (with $A = \emptyset$), v^0 is a vacuum above every set of places. Three properties:

(i) $|\mathrm{mc}(\gamma)| = I(\gamma)^{-m/2}$. By (WR4),

$$\omega((1,\gamma),1) = \beta(\det\gamma)^{-1}\omega_{s'\otimes s}(g') \quad \text{with} \quad g' = 1 \otimes (\gamma^{-1})^{\mathsf{T}},$$

the map $X \mapsto X\gamma^{-1}$ of $m\times n$ matrices, whose infinite components are 1. By Lemma 6.2.13(1),(2) applied to g'^{-1}, and the inner-product preservation and inverse law of Lemma 4.1.1(2), the coefficients of $\omega_{s'\otimes s}(g')$ and $\omega_{s'\otimes s}(g'^{-1})$ against v^0 are complex conjugates. Hence $|\mathrm{mc}(\gamma)| = |\Delta_{s'\otimes s}(g'^{-1})|^{-1}$, and by (WR3)

$$|\Delta_{s'\otimes s}(g'^{-1})| = |\det u|^{-1/2} = [\widehat{\mathcal{O}}_E^{mn} : \widehat{\mathcal{O}}_E^{mn} \cap g'\widehat{\mathcal{O}}_E^{mn}]^{1/2}$$

(the other factors have modulus 1, and $|\det u| = [\widehat{\mathcal{O}}_E^{mn} : u\widehat{\mathcal{O}}_E^{mn}]^{-1}$). As g' acts on each of the m rows by $x^{\mathsf{T}} \mapsto (\gamma^{-1})^{\mathsf{T}}x^{\mathsf{T}}$ and $(\gamma^{-1})^{\mathsf{T}} = \bar\gamma$ ($\gamma^*\gamma = 1$), this index is $[L : L \cap \bar\gamma L]^m = I(\gamma)^m$, complex conjugation preserving $\widehat{\mathcal{O}}_E$.

(ii) If γ is supported on A_1 and γ' on A_2, $A_1 \cap A_2 = \emptyset$, then $\mathrm{mc}(\gamma\gamma') = \mathrm{mc}(\gamma)\mathrm{mc}(\gamma')$: Lemma 6.5.6 with $\Sigma = A_1$, $\varphi = v^0$ and $g = (1,\gamma')$ (whose components above A_1 are $1 \in K_v$).

(iii) If $\beta(\det k) = 1$ for $k \in K'_\Sigma$, then $\mathrm{mc}(k\gamma k') = \mathrm{mc}(\gamma)$ for $k, k' \in K'_\Sigma$: by (4) of Lemma 6.2.33 (with $A = \emptyset$), $v^0 = v_1 \otimes v^0$, so by (6) of Lemma 4.1.1 $\omega((1,k),1)v^0 = \beta(\det k)^{-1}v^0 = v^0$, and ω is unitary by (2) there.

Step 2: one place. Let v be a finite place of F and $s_v = \#\mathrm{St}_v \cdot \mathrm{N}w^{-m/2}$. By Lemma 6.5.5(c) and $m \ge 2n+1$: if v splits, $s_v \le 2\frac{\mathrm{N}w}{\mathrm{N}w-1}\mathrm{N}w^{n-1}\mathrm{N}w^{-n-1/2} \le 3\,\mathrm{N}v^{-3/2}$; if v is inert, $s_v \le \mathrm{N}v\frac{\mathrm{N}w}{\mathrm{N}w-1}\mathrm{N}w^{n-1}\mathrm{N}w^{-n-1/2} \le 2\,\mathrm{N}v^{-2}$. As $\mathrm{N}w \ge 316$, $s_v \le 1/50$ in both cases. Since at most $[F:\mathbb{Q}]$ places lie above a prime p, each with $\mathrm{N}v \ge p$, $\sum_v s_v \le 3[F:\mathbb{Q}]\sum_p p^{-3/2} < \infty$. By Lemma 6.5.5(b), I is constant on the cosets $c \in G_v/K_v$ (here $K_v = G_v \cap \mathbf{K}_0(n)$), $I(c) = 1$ only for the trivial coset c_0, and $\rho \in \mathrm{St}_v$ acts on G_v/K_v by left multiplication, injectively. Put $S_v = \sum_{c\in G_v/K_v} I(c)^{-m/2}$ and $S_v(N) = \sum_{I(c)\le N} I(c)^{-m/2}$ (sums of nonnegative terms). For $c \ne c_0$ choose $\rho_c \in \mathrm{St}_v$ with $\mathrm{N}w\, I(\rho_c c) \le I(c)$; then $I(c)^{-m/2} \le \mathrm{N}w^{-m/2}I(\rho_c c)^{-m/2}$, $I(\rho_c c) < I(c)$, and $c \mapsto (\rho_c, \rho_c c)$ is injective. Hence

$$S_v(N) \le 1 + \mathrm{N}w^{-m/2}\sum_{\rho\in\mathrm{St}_v}\sum_{I(c')\le N-1} I(c')^{-m/2} = 1 + s_v S_v(N-1),$$

and by induction on N (with $S_v(0) = 0$), $S_v(N) \le (1 - s_v)^{-1}$. So $S_v \le (1 - s_v)^{-1}$ and $S_v - 1 \le s_v/(1 - s_v) \le 2s_v$.

Step 3: (1). Let $A \subseteq \Sigma$ and K_A be defined as K_Σ. The map $gK_A \mapsto (g_v K_v)_{v\in A}$ is a bijection from $G_f(n)/K_A$ onto the families $(c_v)_{v\in A}$, $c_v \in G_v/K_v$, with $c_v = c_0$ for almost all v ($G_f(n)$ is the restricted product of the G_v with respect to the K_v). For a representative u_q supported on A, $I(u_q) = \prod_v I(c_v)$ by Lemma 6.5.5(b), so by Step 1 (i), with finite partial products and monotone limits,

$$\sum_q |\mathrm{mc}(u_q)| = \prod_{v\in A} S_v \le \exp\Big(2\sum_{v\in A} s_v\Big) < \infty, \qquad \sum_{q\neq q_0} |\mathrm{mc}(u_q)| \le \exp\Big(2\sum_{v\in A} s_v\Big) - 1, \qquad (*)$$

q_0 the trivial coset. For $A = \Sigma$ this is the summability in (1). For y bounded and measurable, $|\mathrm{mc}(u_q)y(gu_q)| \le |\mathrm{mc}(u_q)| \sup|y|$, so the series converges absolutely and uniformly (Weierstrass), with the stated bound. If moreover y is left invariant under $\mathrm{U}(S)(F)$ and y' is integrable on $[\mathrm{U}(S)]$, each $g \mapsto y(gxu_q)$ is bounded, measurable and left invariant, and $\sum_q \int_{[\mathrm{U}(S)]} |y'(g)|\,|\mathrm{mc}(u_q)|\,|y(gxu_q)|\,dg \le \sup|y| \sum_q |\mathrm{mc}(u_q)| \int |y'| < \infty$; Fubini gives the identity of (1).

Step 4: (2). Assume $\beta(\det k) = 1$ for $k \in K'_\Sigma$, and let y be bounded and right invariant under K'_Σ. *Independence of the representatives:* two representatives of q supported on Σ differ by $k \in K_\Sigma$ supported on Σ, i.e. $k \in K'_\Sigma$, and both mc (Step 1 (iii)) and $y(g\,\cdot)$ are unchanged; the same holds for every $A \subseteq \Sigma$. *Right invariance under K'_Σ:* for $k \in K'_\Sigma$, $q \mapsto kq$ permutes $G_f(n)/K_\Sigma$ and ku_q is a representative of kq supported on Σ, so $(R_\Sigma y)(gk) = \sum_q \mathrm{mc}(ku_q)y(gku_q) = (R_\Sigma y)(g)$ by Step 1 (iii) and the independence. *A level:* if y is right invariant under an open subgroup U, it contains a product $\prod_v U_v$ of open subgroups with $U_v = K_v$ for almost all v; let $U' = \prod_{v\in\Sigma} K_v \times \prod_{v\notin\Sigma} U_v$, open. For $k \in U'$ write $k = k_\Sigma k^\Sigma$ (components above Σ, resp. off Σ); $k^\Sigma \in U$ commutes with every u_q, so $(R_\Sigma y)(gk) = \sum_q \mathrm{mc}(u_q)y(gk_\Sigma u_q k^\Sigma) = (R_\Sigma y)(gk_\Sigma) = (R_\Sigma y)(g)$, as $k_\Sigma \in K'_\Sigma$. *A level J with $K'_\Sigma \le J \le K_\Sigma$:* for $j \in J$, $j_\Sigma \in K'_\Sigma \le J$, so $j^\Sigma = j_\Sigma^{-1} j \in J$, and the same computation applies. *The other properties:* each term is continuous and left invariant under $\mathrm{U}(S)(F)$ when y is, and the convergence is uniform, so $R_\Sigma y$ is bounded, continuous and left invariant. Right translation by $(x, 1)$, $x \in \mathrm{U}(S)(F \otimes \mathbb{R})$, commutes with every $(1, u_q)$, hence with R_Σ; for integrable k on $\mathrm{U}(S)(F\otimes\mathbb{R})$, $\int k(x) \sum_q \mathrm{mc}(u_q)y(g(x,1)u_q)\,dx = \sum_q \mathrm{mc}(u_q) \int k(x)y(gu_q(x,1))\,dx$ by Fubini, the double series–integral of absolute values being at most $\sup|y| \sum_q |\mathrm{mc}(u_q)| \int |k|$.

Step 5: (3). Let $\mathcal{B}$ be the Banach space of bounded functions on $\mathrm{U}(S)(\mathbb{A}_F)$ right invariant under K'_Σ, with the supremum norm. For $A \subseteq \Sigma$, R_A maps $\mathcal{B}$ to $\mathcal{B}$: it is right invariant under K'_A by Step 4 (the hypothesis on β holds on $K'_A \subseteq K'_\Sigma$), and under $K'_{\Sigma\setminus A}$, whose elements commute with the u_q supported on A; and $K'_\Sigma = K'_A K'_{\Sigma\setminus A}$. The trivial coset may be represented by 1, with $\mathrm{mc}(1) = 1$, so $R_A = \mathrm{id} + N_A$ with $\|N_A\| \le \sum_{q\neq q_0} |\mathrm{mc}(u_q)|$. *Multiplicativity:* if $A = A_1 \sqcup A_2 \subseteq \Sigma$, then $u_{q_1}u_{q_2}$ (q_i cosets of K_{A_i}) runs once through a system of representatives of $G_f(n)/K_A$ supported on A (Step 3), $\mathrm{mc}(u_{q_1}u_{q_2}) = \mathrm{mc}(u_{q_1})\mathrm{mc}(u_{q_2})$ by Step 1 (ii), and by absolute convergence and the independence of representatives, $R_{A_1}(R_{A_2}y) = R_A y$ for $y \in \mathcal{B}$. *Conclusion:* choose a finite $T \subseteq \Sigma$ with $\sum_{v\in\Sigma\setminus T} s_v \le 1/10$ (Step 2). By $(*)$, $\|N_{\Sigma\setminus T}\| \le e^{1/5} - 1 < 1/4$, and for $v \in T$, $\|N_{\{v\}}\| \le S_v - 1 \le 2s_v \le 1/25$. An operator $\mathrm{id} + N$ with $\|N\| < 1$ is injective, since $\|y + Ny\| \ge (1 - \|N\|)\|y\|$. By multiplicativity, $R_\Sigma = R_{\{v_1\}} \circ \cdots \circ R_{\{v_k\}} \circ R_{\Sigma\setminus T}$ for $T = \{v_1, \ldots, v_k\}$, a composition of injective operators of $\mathcal{B}$; so $R_\Sigma y = 0$ implies $y = 0$. □

Lemma 6.5.5. *Let $n \ge 1$, $L = \widehat{\mathcal{O}}_E^n$, and for $\gamma \in G_f(n)$ put $I(\gamma) = [L : L \cap \gamma L]$. For a finite place v of F let G_v be the group of $\gamma \in G_f(n)$ whose components at the finite places of E not above v are 1. Then:*

(a) every finite place of F is unramified in E, and $\mathrm{N}w \geq 316$ for every finite place w of E;

(b) $I(\gamma)$ is a positive integer, $I(\gamma) = 1$ if and only if $\gamma \in \mathbf{K}_0(n)$, $I(\kappa\gamma\kappa') = I(\gamma)$ for $\kappa, \kappa' \in \mathbf{K}_0(n)$, and $I(\gamma\gamma') = I(\gamma)I(\gamma')$ whenever at every finite place of E one of γ, γ' has component 1;

(c) for every finite place v of F there is a finite set $\mathrm{St}_v \subseteq G_v$ such that for every $\gamma \in G_v \setminus \mathbf{K}_0(n)$ some $\rho \in \mathrm{St}_v$ satisfies $\mathrm{N}w \cdot I(\rho\gamma) \leq I(\gamma)$ (w a place above v), and

$$\#\mathrm{St}_v \leq 2\,\frac{\mathrm{N}w^n - 1}{\mathrm{N}w - 1} \quad \text{if } v \text{ splits in } E,$$
$$\#\mathrm{St}_v \leq \mathrm{N}v\,\frac{\mathrm{N}w^n - 1}{\mathrm{N}w - 1} \quad \text{if } v \text{ is inert in } E \text{ (then } \mathrm{N}w = \mathrm{N}v^2\text{).}$$

Proof. **Step 1: (a).** The primitive 5040-th root of unity ζ gives $i = \zeta^{1260}$ and $\zeta_3 = \zeta^{1680}$ in E; as F is totally real, $i, \sqrt{-3} \notin F$, so $E = F(i) = F(\sqrt{-3})$. The extension $F(i)/F$ is unramified at the finite places not above 2, and $F(\sqrt{-3})/F$ at those not above 3 (the discriminants of X^2+1 and X^2+X+1 are -4 and -3), so E/F is unramified at every finite place. If w has residue characteristic p, the group $\mu_{N'}$ of N'-th roots of unity, N' the prime-to-p part of $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, injects into $k_w^\times$ ($X^{N'} - 1$ is separable mod p), so $\mathrm{N}w - 1 \geq N' \geq 315$.

Step 2: (b). $L \cap \gamma L$ is a lattice, so the index is a positive integer, and $I(\gamma) = \prod_w [L_w : L_w \cap \gamma_w L_w]$. If $\kappa, \kappa' \in \mathbf{K}_0(n)$ then $\kappa' L = L$ and $\kappa L = L$, so $L \cap \kappa\gamma\kappa' L = \kappa(L \cap \gamma L)$ and $I(\kappa\gamma\kappa') = I(\gamma)$. The product rule follows from the product formula over the places. If $\gamma \in \mathbf{K}_0(n)$ then $I(\gamma) = 1$. Conversely, if $I(\gamma) = 1$ then $L \subseteq \gamma L$, i.e. $\gamma^{-1} = \bar\gamma^{\mathsf T}$ has integral entries; complex conjugation preserves $\widehat{\mathcal{O}}_E$, so γ has integral entries, $\gamma L \subseteq L$, hence $\gamma L = L$ and $\gamma \in \mathbf{K}_0(n)$.

Step 3: v split, $v = w\bar w$. Here G_v is identified with $\mathrm{GL}_n(E_w)$ by $\gamma \mapsto \gamma_w$: given $a \in \mathrm{GL}_n(E_w)$, the element with components a at w, $\overline{(a^{-1})^{\mathsf T}}$ at $\bar w$ and 1 elsewhere lies in G_v, and every element of G_v is of this form. For a lattice $M \subset E_w^n$ let $M^\vee = \{y : y^{\mathsf T} x \in \mathcal{O}\ \forall x \in M\}$; then $L_w^\vee = L_w$, $(aL_w)^\vee = (a^{-1})^{\mathsf T} L_w$, $M^{\vee\vee} = M$, $(M \cap M')^\vee = M^\vee + M'^\vee$, and $[M : M'] = [M'^\vee : M^\vee]$ for $M' \subseteq M$. For $\gamma \in G_v$ and $M = \gamma_w L_w$, the conjugate of $\gamma_{\bar w} L_{\bar w}$ is $(\gamma_w^{-1})^{\mathsf T} L_w = M^\vee$, and

$$[L_w : L_w \cap M^\vee] = [L_w + M : L_w] = [M : L_w \cap M], \qquad \text{so} \qquad I(\gamma) = [L_w : L_w \cap M]\,[M : L_w \cap M],$$

and symmetrically $I(\gamma) = [L_{\bar w} : L_{\bar w} \cap M']\,[M' : L_{\bar w} \cap M']$ for $M' = \gamma_{\bar w} L_{\bar w}$. *The steps.* For each line $\ell \subset L_w/\varpi L_w = k_w^n$ (ϖ a uniformiser of E_w) let $L_\ell = L_w + \varpi^{-1}\mathcal{O}\tilde\ell$, $\tilde\ell \in L_w$ a lift of a generator (independent of the lift), choose $a_\ell \in \mathrm{GL}_n(E_w)$ with $a_\ell L_\ell = L_w$, and let $\rho_\ell \in G_v$ have component a_ℓ at w; define ρ'_ℓ in the same way at $\bar w$. St_v is the set of these $2(\mathrm{N}w^n - 1)/(\mathrm{N}w - 1)$ elements. *The step lowers I.* Let $\gamma \in G_v \setminus \mathbf{K}_0(n)$ and $M = \gamma_w L_w$; then $M \neq L_w$ (otherwise γ_w and $\gamma_{\bar w} = \overline{(\gamma_w^{-1})^{\mathsf T}}$ are integral). Suppose first $M \not\subseteq L_w$. Take $y \in M \setminus L_w$ and the least $k \geq 1$ with $\varpi^k y \in L_w$; then $x = \varpi^{k-1} y \in M \setminus L_w$ and $\varpi x \in L_w \setminus \varpi L_w$. Let ℓ be the line of ϖx; then $L_\ell = L_w + \mathcal{O}x$ and $[L_\ell : L_w] = \mathrm{N}w$. Put $D = L_\ell \cap M \supseteq (L_w \cap M) + \mathcal{O}x$; as $x \notin L_w$ and $\varpi x \in L_w \cap M$, $[D : L_w \cap M] \geq \mathrm{N}w$. Applying a_ℓ^{-1}, which preserves indices,

$$I(\rho_\ell\gamma) = [L_\ell : D]\,[M : D] = \frac{[L_\ell : L_w][L_w : L_w \cap M]}{[D : L_w \cap M]} \cdot \frac{[M : L_w \cap M]}{[D : L_w \cap M]} \leq \frac{I(\gamma)}{\mathrm{N}w}.$$

If $M \subseteq L_w$, then $M \subsetneq L_w$, so $M^\vee \supsetneq L_w$ and $M' = \overline{M^\vee} \not\subseteq L_{\bar w}$; the same argument at $\bar w$, with the symmetric formula for I, gives ℓ with $I(\rho'_\ell\gamma) \leq I(\gamma)/\mathrm{N}w$.

Step 4: v inert, w the place above v. E_w/F_v is an unramified quadratic extension, $\mathrm{N}w = \mathrm{N}v^2$; fix a uniformiser $\varpi \in F_v$ (also one of E_w). Let $h(x,y) = y^* x$ on E_w^n and $M^\# =$

$\{y : h(x,y) \in \mathcal{O}\ \forall x \in M\}$; $L_w^{\#} = L_w$, $M^{\#\#} = M$, $(M \cap M')^{\#} = M^{\#} + M'^{\#}$ and $[M : M'] = [M'^{\#} : M^{\#}]$. G_v is the unitary group of h (component at w). For $\gamma \in G_v$, $M = \gamma L_w$ satisfies $M^{\#} = M$, and $I(\gamma) = [L_w : L_w \cap M]$. Call a lattice L_1 a *neighbour* if $L_1^{\#} = L_1$ and $[L_w : L_w \cap L_1] = \mathrm{N}w$. *Neighbours are moved to* L_w. Let $L_1^{\#} = L_1$. The form $h \bmod \varpi$ on $L_1/\varpi L_1$ is a nondegenerate hermitian form over k_w. A nondegenerate hermitian form $\bar{h}$ over k_w has an anisotropic vector: otherwise polarisation gives $\bar{h}(a,b) + \bar{h}(b,a) = 0$, and with ca in place of a, $(c - \bar{c})\bar{h}(a,b) = 0$ for all $c \in k_w$, so $\bar{h} = 0$. Lift it to $e \in L_1$; $h(e,e) \in \mathcal{O}_{F_v}^{\times}$, and as E_w/F_v is unramified the norm $\mathcal{O}^{\times} \to \mathcal{O}_{F_v}^{\times}$ is surjective, so some $c \in \mathcal{O}^{\times}$ gives $h(ce,ce) = 1$. Then $L_1 = \mathcal{O}ce \oplus \{z \in L_1 : h(z,ce) = 0\}$ (write $z = h(z,ce)ce + (z - h(z,ce)ce)$), the second summand is again self-dual in the orthogonal complement, and by induction L_1 has an h-orthonormal basis $f_1, \dots, f_n$. The matrix ρ with $\rho f_i = e_i$ (the standard basis) preserves h, so $\rho \in G_v$ and $\rho L_1 = L_w$. For every neighbour $L_1 \neq L_w$ fix such ρ_{L_1}; St_v is the set of these. *Counting.* Let $L_1 \neq L_w$ be a neighbour and $L_0 = L_w \cap L_1$. L_w/L_0 has order $\mathrm{N}w$, so it is $\cong k_w$ and $\varpi L_w \subseteq L_0$: L_0 corresponds to a hyperplane of k_w^n, at most $(\mathrm{N}w^n - 1)/(\mathrm{N}w - 1)$ choices. Given L_0: $L_0^{\#} = L_w + L_1$, $[L_0^{\#} : L_w] = [L_w : L_0] = \mathrm{N}w$, so $V = L_0^{\#}/L_0$ has order $\mathrm{N}w^2$; $\varpi L_1 \subseteq L_0$ ($[L_1 : L_0] = [L_w : L_0]$, as L_1, L_w are self-dual), so ϖ kills V and V is a k_w-plane. As $L_0^{\#} \subseteq \varpi^{-1} L_0$, $h(L_0^{\#}, L_0^{\#}) \subseteq \varpi^{-1}\mathcal{O}$, and $\bar{h}(a,b) = \varpi h(a,b) \bmod \varpi$ is a well-defined hermitian form on V; it is nondegenerate, since $h(a, L_0^{\#}) \subseteq \mathcal{O}$ forces $a \in L_0^{\#\#} = L_0$. The self-dual lattices L' with $L_0 \subset L' \subset L_0^{\#}$ and $[L' : L_0] = \mathrm{N}w$ give isotropic lines L'/L_0 of V, distinct lattices giving distinct lines. In a basis e, f of V with $\bar{h}(e,f) = 0$ (an anisotropic e and its orthogonal), $\bar{h}(e,e) = \alpha$, $\bar{h}(f,f) = \beta \in k_v^{\times}$, the isotropic lines are those of $xe + f$ with $x\bar{x} = -\beta/\alpha$; the norm $k_w^{\times} \to k_v^{\times}$ is surjective with fibres of size $\mathrm{N}v + 1$, so there are $\mathrm{N}v + 1$ isotropic lines, one of which is L_w/L_0. So at most $\mathrm{N}v$ neighbours $L_1 \neq L_w$ contain L_0, which gives the bound for $\#\mathrm{St}_v$. *The step lowers* I. Let $\gamma \in G_v \setminus \mathbf{K}_0(n)$ and $M = \gamma L_w$. Then $M \not\subseteq L_w$: otherwise $M = M^{\#} \supseteq L_w^{\#} = L_w$, so $M = L_w$. As in Step 3 pick $x \in M \setminus L_w$ with $\varpi x \in L_w$. The map $L_w \to \varpi^{-1}\mathcal{O}/\mathcal{O}$, $z \mapsto h(z,x) \bmod \mathcal{O}$, is well defined ($h(z, \varpi x) = \varpi h(z,x) \in \mathcal{O}$) and $\mathcal{O}$-linear; it is nonzero, since $\varpi x \notin \varpi L_w = \varpi L_w^{\#}$ gives $z \in L_w$ with $h(z, \varpi x) \notin \varpi\mathcal{O}$; so its kernel L_0 has index $\mathrm{N}w$. Let $L_1 = L_0 + \mathcal{O}x$. Then $h(x,x) \in \mathcal{O}$ ($x \in M = M^{\#}$), $h(L_0, x) \subseteq \mathcal{O}$ and $h(L_0, L_0) \subseteq \mathcal{O}$, so $L_1 \subseteq L_1^{\#}$; $x \notin L_0$ and $\varpi x \in L_0$, so $[L_1 : L_0] = \mathrm{N}w$; then $[L_1^{\#} : L_0] = [L_0^{\#} : L_0]/[L_0^{\#} : L_1^{\#}] = \mathrm{N}w^2/\mathrm{N}w = [L_1 : L_0]$, whence $L_1 = L_1^{\#}$. As $x \in L_1 \setminus L_w$ and $[L_1 : L_0] = \mathrm{N}w$ has no intermediate $\mathcal{O}$-lattices, $L_w \cap L_1 = L_0$; so $L_1 \neq L_w$ is a neighbour. Moreover $L_w \cap M \subseteq L_0$ ($h(z,x) \in \mathcal{O}$ for $z, x \in M$), and $L_1 \cap M \supseteq (L_w \cap M) + \mathcal{O}x$ with $[(L_w \cap M) + \mathcal{O}x : L_w \cap M] = \mathrm{N}w$. With $\rho = \rho_{L_1}$,

$$I(\rho\gamma) = [L_1 : L_1 \cap M] \leq \frac{[L_1 : L_0]\,[L_0 : L_w \cap M]}{\mathrm{N}w} = \frac{[L_w : L_w \cap M]}{\mathrm{N}w} = \frac{I(\gamma)}{\mathrm{N}w},$$

using $[L_0 : L_w \cap M] = [L_w : L_w \cap M]/\mathrm{N}w$. By Step 1 every finite place of F is split or inert, which completes (c). □

Lemma 6.5.6. *Let S, T be spaces, $n = \mathrm{rk}\, S$, β an automorphic character of $\mathrm{U}(1)$ and $\omega = \omega_{1,\beta}$. Let Σ be a set of finite places of F and, for $v \in \Sigma$, K_v the component at v of $\mathbf{K}_0(n)$, the stabiliser of the standard lattice. For $\gamma \in G_f(n)$ whose components at the finite places not above Σ are 1 put*

$$\mathrm{mc}_{\beta,\Sigma}(\gamma) = \langle v^0,\ \omega((1,\gamma),1)\, v^0 \rangle,$$

the ***vacuum coefficient****. Let $\varphi \in \mathcal{M}_{S,T}$ be a vacuum above Σ. Then for every such γ and every $g \in \mathrm{U}(S)(\mathbb{A}_F)$ with $g_v \in K_v$ for all $v \in \Sigma$,*

$$\langle \varphi, \omega((1,\gamma)g,1)\varphi \rangle = \mathrm{mc}_{\beta,\Sigma}(\gamma)\, \langle \varphi, \omega(g,1)\varphi \rangle.$$

Proof. **Step 1: $u = \omega(g,1)\varphi$ is a vacuum above Σ.** Let $y \in \widehat{\mathcal{O}}_E^N$ have components 0 at the finite places not above Σ. By (WD8), $\rho_f(y) = c(y)\rho(\mathrm{fv}(y))$ with $c(y) = \psi_{E,f}(\jmath\alpha\bar{\alpha}\frac{c_0}{2}y^*y)$ and $\mathrm{fv}(y) = (0, \alpha y)$. By (2) of Lemma 4.1.1, $\omega_{s'\otimes s}(g')^{-1}\rho(z)\,\omega_{s'\otimes s}(g') = \rho(g'^{-1}z)$, and the scalar $\beta(\det g)^{-1}$ cancels in the conjugation. As $\mathrm{fv}(y)$ has archimedean part 0 and the finite part of g'^{-1} is linear, $g'^{-1}\mathrm{fv}(y) = \mathrm{fv}(y_1)$ with $y_1 = g_f'^{-1}y$; g_f' is unitary for the standard form, so $y_1^*y_1 = y^*y$ and $c(y_1) = c(y)$. At the places above Σ, $g_v \in K_v$, so $g_f'^{-1}$ is integral there, and y_1 is again integral with components 0 off the places above Σ. Hence $\omega(g,1)^{-1}\rho_f(y)\,\omega(g,1) = \rho_f(y_1)$ and $\rho_f(y)u = \omega(g,1)\rho_f(y_1)\varphi = u$.

Step 2: the factorisation. By (2) of Lemma 4.1.1, $\omega((1,\gamma)g,1)\varphi = \omega((1,\gamma),1)u = \beta(\det\gamma)^{-1}\omega_{s'\otimes s}(\gamma')u$ with $\gamma' = 1\otimes(\gamma^{-1})^{\mathsf{T}}$, whose components are 1 at the infinite places and at the finite places not above Σ. As φ and u are vacua above Σ, (7) of Lemma 6.2.33 gives

$$\langle\varphi, \omega((1,\gamma)g,1)\varphi\rangle = \beta(\det\gamma)^{-1}\langle v^0, \omega_{s'\otimes s}(\gamma')v^0\rangle\langle\varphi,u\rangle = \mathrm{mc}_{\beta,\Sigma}(\gamma)\,\langle\varphi,\omega(g,1)\varphi\rangle.$$

□

Lemma 6.5.7. *Let D be admissible data and ϑ an m-fold twist, and write $X_\vartheta = \mathrm{Wc}^0_{\vartheta,\mathrm{I},m}$. For all $x, y \in \mathcal{T}$,*

$$\langle\Theta_\vartheta x, \Theta_\vartheta y\rangle = [\mathbf{K}_0 : \mathrm{K}_\vartheta]\cdot\dim W_{\lambda^\circ}\cdot\langle X_\vartheta x,\ X_\vartheta y\rangle_{Y_m}.$$

Proof. Let $p = p_{\mathrm{I},0,m}$, which is the doubled Harish-Chandra parameter $l = 2\lambda(Y_m)$ of (κ_Y, e_Y) used by the chain in Definition 3.2.31 , and $d = \#I_{\kappa_Y} = \dim W_{\lambda^\circ}$. Let $x, y \in \mathcal{T}$ and $\eta = \Theta_\vartheta x$, $\eta' = \Theta_\vartheta y$, elements of $\mathcal{A}_\vartheta(Y_m)$ by Lemma 6.4.75(c): algebraic forms of weight (κ_Y, e_Y) with $\eta(gk) = \chi(k)\eta(g)$ and $\eta'(gk) = \chi(k)\eta'(g)$ for $k \in \mathrm{K}_\vartheta$, χ the character of K_ϑ of Definition 3.2.31, whose values are values of characters of finite groups and so have modulus 1. The group K_ϑ is an open subgroup of $\mathbf{K}_0(n)$ (it contains the elements congruent to 1 modulo the level places). By (WM3), the summand of (TL1) is $\eta(g_c)^*G_W\eta'(g_c) = \langle\eta(g_c),\eta'(g_c)\rangle_{\kappa_Y}$, so (TL1) is the class sum for K_ϑ of $h \mapsto \langle\eta(h),\eta'(h)\rangle_{\kappa_Y}$, which is constant on the double cosets by (c) of Lemma 6.4.72. By that lemma,

$$\langle\Theta_\vartheta x, \Theta_\vartheta y\rangle = [\mathbf{K}_0(n) : \mathrm{K}_\vartheta]\cdot d\cdot\langle F^p_\eta, F^p_{\eta'}\rangle_{\mathrm{I}_n}.$$

Since $x, y \in \mathcal{A}_M(Y_0)$, Lemma 6.4.67(b) gives $F^p_\eta = X_\vartheta x$ and $F^p_{\eta'} = X_\vartheta y$, and $\mathrm{I}_n = Y_m$; this is the assertion. □

Lemma 6.5.8. *Let D be admissible data. For an m-fold twist ϑ put $N_n(\vartheta) = \#\mathrm{GL}_{n_n}(R_\vartheta)$ for the triangle $z = \mathrm{I}$ ($\#n_n = r + nt$). There are $C > 0$ and $k \in \mathbb{Z}$, independent of ϑ, such that for every m-fold twist ϑ with modulus Q*

$$[\mathbf{K}_0 : \mathrm{K}_\vartheta] = C\,(\mathrm{N}Q)^k\,c_Q^m\,\frac{N_m(\vartheta)}{N_0(\vartheta)\,N_1(\vartheta)^m}.$$

Proof. **Step 1: the residue ring.** The primes $p_0 \neq p_1$ are $\equiv 1 \pmod{q_2^2}$ ((AD5)), hence prime to d and to every prime factor of q_1 (all of which divide q_0, as $2, 3 \mid q_0$, and so divide $q_2 = (q_0 d)^{s+6}$; Definition 2.3.1); so they are unramified in $E = K(\zeta_{q_1})$. As $\pi_M\bar{\pi}_M = p_0p_1$ in $\mathcal{O}_K$, neither p_ϵ is inert in K, so $\pi_M\mathcal{O}_K$, and hence $\pi_M\mathcal{O}_E$, is a product of distinct primes; these are the chosen places, each with exponent 1. With (TV2) and $\mathrm{N}Q$ prime to M, $R_\vartheta = \prod_{w\in P} k_w$ for P the set of chosen places and places of Q, and $N_n(\vartheta) = \prod_{w\in P}\#\mathrm{GL}_{r+nt}(k_w)$.

Step 2: the index as a product. The places of P are distinct and none is the conjugate of itself or of another ((DP1) of Definition 3.2.23, for the level system of (LF4) of Definition 3.2.26);

so each lies over a place of F split in E, the component of the unitary group at that place is $\mathrm{GL}_N(E_w)$, $N = r+mt$, that of $\mathbf{K}_0$ is $\mathrm{GL}_N(\mathcal{O}_{E,w})$, and reduction maps $\mathbf{K}_0$ onto $\prod_{w\in P}\mathrm{GL}_N(k_w)$. As K_ϑ is the inverse image of $\prod_{w\ \mathrm{chosen}}\{1\}\times\prod_{w\in Q}P_w$, $P_w\subseteq\mathrm{GL}_N(k_w)$ the block upper triangular group,

$$[\mathbf{K}_0:\mathrm{K}_\vartheta]=\prod_{w\ \mathrm{chosen}}\#\mathrm{GL}_N(k_w)\cdot\prod_{w\in Q}[\mathrm{GL}_N(k_w):P_w].$$

Step 3: the local factors. Let $q=\mathrm{N}w$ and write $\#\mathrm{GL}_a$ for $\#\mathrm{GL}_a(\mathbb{F}_q)$. The group P_w is the semidirect product of the block diagonal group with the block unipotent group, so $\#P_w=\#\mathrm{GL}_r\,(\#\mathrm{GL}_t)^m q^e$ with $e=rmt+t^2m(m-1)/2$ (the number of entries above the diagonal blocks). The matrices of GL_{r+t} fixing the first r basis vectors form a group of order $\prod_{i=r}^{r+t-1}(q^{r+t}-q^i)=q^{rt}\#\mathrm{GL}_t$, whose index is the number $\prod_{i=0}^{r-1}(q^{r+t}-q^i)=q^{(r+t)r}f(q)$ of $(r+t)\times r$ matrices of rank r; so $\#\mathrm{GL}_{r+t}=q^{(r+t)r+rt}f(q)\,\#\mathrm{GL}_t$. Hence, for $w\in Q$,

$$[\mathrm{GL}_N:P_w]=\frac{\#\mathrm{GL}_N}{\#\mathrm{GL}_r(\#\mathrm{GL}_t)^m q^e}=q^k\left(q^{rt}f(q)\right)^m\frac{\#\mathrm{GL}_N}{\#\mathrm{GL}_r\,(\#\mathrm{GL}_{r+t})^m},$$
$$k=m(r+t)r-e=mr^2-t^2m(m-1)/2,$$

and for a chosen place w, $\#\mathrm{GL}_N(k_w)=C_w\,\#\mathrm{GL}_N(k_w)/(\#\mathrm{GL}_r(k_w)\#\mathrm{GL}_{r+t}(k_w)^m)$ with $C_w=\#\mathrm{GL}_r(k_w)\#\mathrm{GL}_{r+t}(k_w)^m$.

Step 4: conclusion. Multiplying over P and using Step 1 ($N_0=\prod_P\#\mathrm{GL}_r(k_w)$, $N_1=\prod_P\#\mathrm{GL}_{r+t}(k_w)$, $N_m=\prod_P\#\mathrm{GL}_N(k_w)$), $\prod_{w\in Q}q_w^k=(\mathrm{N}Q)^k$ and $\prod_{w\in Q}(q_w^{rt}f(q_w))^m=c_Q^m$, the formula holds with $C=\prod_{w\ \mathrm{chosen}}C_w>0$ and k as above, both independent of ϑ. □

Proposition 6.5.9. *Let $1\le j\le m$, and for a tame character ϑ let $\Theta^{\mathrm{pr}}_{j,\vartheta}$ be the theta lift of the descent pair at ϑ on $\mathcal{A}^{\mathrm{pr}}(W_j)$, so that $\Theta_{j,\vartheta}$ is its restriction to the good forms. Let $R_{j,\vartheta}$ be the* ***Rallis series*** *on $\mathcal{A}^{\mathrm{pr}}(W_j)$,*

$$R_{j,\vartheta}\,y=\sum_q \mathrm{mc}_{j,\vartheta}(q)\,\rho(q)y,$$

q over $\mathrm{U}(W_j)(\mathbb{A}_f)$ modulo the points integral at the places outside $S_M\cup Q$ and their conjugates, $\mathrm{mc}_{j,\vartheta}(q)$ the matrix coefficient of the unramified vector of the Weil representation for the second character of the pair (the one twisting the action of $\mathrm{U}(W_j)$, read with the rank $r+t$ of V^{def}, as in (XD2) of Definition 3.2.22). Put

$$c_Q=(\mathrm{N}Q)^{rt}\prod_{w\in Q}\prod_{i=t+1}^{t+r}(1-(\mathrm{N}w)^{-i}).$$

Then:

1. *for every tame ϑ, $\Theta^{\mathrm{pr}}_{j,\vartheta}$ is injective; hence $\Theta_{j,\vartheta}$ is injective and $Z_j(\vartheta)>0$ for every m-fold twist ϑ;*
2. *there is $A_j\in\mathbb{C}\setminus\{0\}$, independent of ϑ, with $(\Theta^{\mathrm{pr}}_{j,\vartheta})^*\Theta^{\mathrm{pr}}_{j,\vartheta}=A_j\,c_Q\,R_{j,\vartheta}$ for every tame ϑ with modulus Q;*
3. *$R_{j,\vartheta}$ preserves the good forms, and for every m-fold twist ϑ with modulus Q,*

$$Z_j(\vartheta)=(A_j\,c_Q)^{d_\mathcal{T}}\,\det(R^{\mathcal{T}}_{j,\vartheta^j}),$$

where $R^{\mathcal{T}}_{j,\vartheta}$ is the Rallis series of the j-th pair acting on the target forms $\mathcal{T}=\mathcal{A}^{\mathrm{good}}(Y_0)$.

Proof. **Step 1: the block datum is sharp.** The finite datum F of the descent pair (block function at $S_M \cup Q$, vacuum elsewhere) is sharp for J_M off $(S_M \cup Q) \cup \overline{(S_M \cup Q)}$ in the sense of Lemma 6.5.12, with mass

$$c_\Psi = \frac{\#\prod_{w\in Q} \mathrm{GL}_r(k_w) \cdot (\mathrm{N}Q)^{tr}}{\#\{(r+t)\times r \text{ matrices mod } \pi_M Q\} \cdot \#\prod_{w\in S_M} \mathrm{GL}_r(k_w)}.$$

This is Lemma 6.5.10 for $O = O_{j,\vartheta}$ (ranks $r < r+t$, $S_O = S_M \cup Q$, $J_O = J_M$, $c_\Psi = c_O$): (a), (b′) and (4) of Lemma 6.2.33 give (SH1) of Lemma 6.5.12 with S_0 the places of F below $S_M \cup Q$, (c) gives (SH2) for the character $b = \chi'_O$, (d) gives (SH3) ($S_M \neq \emptyset$), and (e) gives (SH4).

Step 2: Rallis' formula and the algebraic forms. Let $O = O_{j,\vartheta}$, $n = r+t$, $\xi_O = v_{l_O}^+$ ((LF5), (DP11)), $v = v_{\mathrm{Har}_O(\xi_O)}$, a Schwartz vector ((XD3) of Definition 3.2.22), so that $\mathrm{J}_O(\xi_O) = v \otimes \Upsilon(O)$ (DP8). By Lemma 6.5.11, O fits the parameter $l = 2\varsigma$ on W_j, and by Lemma 6.4.15(e), v is an edge, for the weights $\gamma = r_-(O)$ and $\gamma' = \lfloor n/2 \rfloor + \mathrm{wt}_O$, from $e_\varsigma = e_{l/2}$, the type of the forms of $\mathcal{A}^{\mathrm{pr}}(W_j)$ (Definition 3.2.16), to $e_{l_O/2}$. These are the weights of $\chi_O = \upsilon^{r_-(O)}$ and $\chi'_O = \mu_O \upsilon^{\lfloor n/2 \rfloor}$ (υ has weight 1, $\mu_O = \mu_j^0 \vartheta$ has weight $\mathrm{wt}_O = -\kappa_j$), and they do not depend on ϑ, nor do O's weight (κ_O, e_O), l_O and Har_O ((LF4) and (DP3), (DP8), (DP11): the twist enters only through μ_O and the level system). By Lemma 6.5.12, applied to $S = W_j$, $T = \mathrm{I}_n$, the edge v, the characters $(a,b) = (\chi_O, \chi'_O)$, $J = J_M$, the places S_0 of F below $S_M \cup Q$ and $F = \Upsilon(O)$, sharp by Step 1 with mass c_Ψ, there is $A \neq 0$ depending only on W_j, I_n, v, the weights and e_ς (so not on ϑ), such that the lift $L = L_{v\otimes\Upsilon(O)}$ maps $\mathcal{A}^{\mathrm{pr}}(W_j)$ to forms of type $e_{l_O/2}$ on I_n, is injective on $\mathcal{A}^{\mathrm{pr}}(W_j)$ (its forms are right invariant under J_M), and $\langle Lx, Ly\rangle_{\mathrm{I}_n} = A\, c_\Psi \langle x, R_{j,\vartheta} y\rangle$ for $x, y \in \mathcal{A}^{\mathrm{pr}}(W_j)$, $R_{j,\vartheta}$ being the Rallis series of $(W_j, \mathrm{I}_n, \chi'_O, \Sigma)$, Σ the finite places of F off S_0. By (LF5), $\Theta^{\mathrm{pr}}_{j,\vartheta} y$ is the algebraic form $(Ly)^\flat_{l_O}$ attached to Ly, an element of $\mathrm{Form}(O)$. By Lemma 6.4.70, $Ly = F^{l_O}_\eta$ for an algebraic form η of weight (κ_O, e_O), and by Lemma 6.4.72(b), $\eta = (Ly)^\flat_{l_O} = \Theta^{\mathrm{pr}}_{j,\vartheta} y$; so $Ly = F^{l_O}_{\Theta^{\mathrm{pr}}_{j,\vartheta} y}$. In particular $\Theta^{\mathrm{pr}}_{j,\vartheta} y = 0$ forces $Ly = 0$ and $y = 0$, which is (1); and since $Z_j(\vartheta)$ is the Gram determinant of the images of an orthonormal basis under an injective map, $Z_j(\vartheta) > 0$. The elements of $\mathrm{Form}(O)$ transform under $\mathbf{K}(O)$ by the character $\tilde{\eta}_O$, whose values have modulus 1 (values of characters of finite groups), and the inner product of $\mathrm{Form}(O)$ is the class sum for $\mathbf{K}(O)$ of the Gram pairings (DP5); so Lemma 6.4.72(c) gives

$$\langle \Theta^{\mathrm{pr}}_{j,\vartheta} x, \Theta^{\mathrm{pr}}_{j,\vartheta} y\rangle = [\mathbf{K}_0(n) : \mathbf{K}(O)] \cdot \delta \cdot \langle Lx, Ly\rangle_{\mathrm{I}_n} = [\mathbf{K}_0(n) : \mathbf{K}(O)] \cdot \delta \cdot A\, c_\Psi \langle x, R_{j,\vartheta} y\rangle,$$

$\delta = \dim \mathbb{V}_O$, independent of ϑ.

Step 3: the constant. For a place w, the number of $(r+t)\times r$ matrices of rank r over k_w is

$$\prod_{i=0}^{r-1} (q_w^{r+t} - q_w^i) = q_w^{(r+t)r} f(q_w), \qquad f(q) = \prod_{i=t+1}^{t+r} (1 - q^{-i}), \quad q_w = \mathrm{N}w.$$

Reduction maps $\mathbf{K}_0(n)$ onto $\prod_{w\in S_M\cup Q} \mathrm{GL}_n(k_w)$ (the places are split and pairwise non-conjugate, (DP1)), so $[\mathbf{K}_0(n) : \mathbf{K}(O)] = \prod_w [\mathrm{GL}_n(k_w) : \mathbf{L}(O)_w]$ (DP4). At $w \in S_M$, $\mathbf{L}(O)_w$ is the stabiliser S'_w of the r basis vectors in the image of inc, of index the number above. At $w \in Q$, $\mathbf{L}(O)_w$ is the stabiliser P_w of the span of the other t basis vectors, of order $\#\mathrm{GL}_r(k_w)\#\mathrm{GL}_t(k_w) q_w^{rt}$, while $\#S'_w = \prod_{i=r}^{n-1}(q_w^n - q_w^i) = q_w^{rt}\#\mathrm{GL}_t(k_w)$; so $[\mathrm{GL}_n(k_w) : P_w]\,\#\mathrm{GL}_r(k_w) = [\mathrm{GL}_n(k_w) : S'_w]$. Hence $[\mathbf{K}_0(n) : \mathbf{K}(O)] \cdot \#\prod_{w\in Q} \mathrm{GL}_r(k_w) = \prod_{w\in S_M\cup Q} q_w^{(r+t)r} f(q_w)$. Dividing by the number $\prod_{w\in S_M\cup Q} q_w^{(r+t)r}$ of matrices modulo $\pi_M Q$,

$$[\mathbf{K}_0(n) : \mathbf{K}(O)]\, c_\Psi = \Big(\prod_{w\in S_M} \frac{f(q_w)}{\#\mathrm{GL}_r(k_w)} \Big) \cdot (\mathrm{N}Q)^{rt} \prod_{w\in Q} f(q_w) = c_M\, c_Q,$$

with $c_M \neq 0$ (as $q_w^i \neq 1$ for $i \geq 1$) depending only on the chosen places. So $A_j = A\,\delta\,c_M \neq 0$ does not depend on ϑ, and $\langle \Theta x, \Theta y\rangle = A_j c_Q \langle x, R_{j,\vartheta} y\rangle$ for all x, y, which is (2) by the definition of the adjoint.

Step 4: determinants. If $u\colon U \to U'$ is linear between finite-dimensional inner product spaces with $\langle ux, uy\rangle = a\langle x, Sy\rangle$ for an endomorphism S of U, then $u^*u = aS$ and $\det(u^*u) = a^{\dim U} \det S$. The target parameter has gap Δ for Y_0 and for W_j, and $\dim \mathcal{A}^{\mathrm{good}}(W_j) = d_{\mathcal{T}}$, by Proposition 5.8.4(iii). The series $R_{j,\vartheta}$ is given on W_j and on Y_0 by the same summable family of Hecke translates (Step 2), so by Proposition 5.8.55 it is a spherical Hecke operator T_a away from the level on both spaces, preserves the good forms, and has the same determinant on the good forms of W_j and on $\mathcal{T} = \mathcal{A}^{\mathrm{good}}(Y_0)$. The powers ϑ^j have the modulus Q of ϑ, so c_Q is the same; apply the rule to $u = \Theta_{j,\vartheta^j}$ on the good forms of W_j with $a = A_j c_Q$ and $S = R_{j,\vartheta^j}$, which gives (3). □

Lemma 6.5.10. *Let O be a descent pair over E for ranks $r < n$, $S_O = M_O \cup Q_O$, $\bar{S}_O$ the set of conjugates of its places, $F = \Upsilon(O)$, and*

$$J_O = \{k \in \mathbf{K}_0(r) :\ k_w \equiv 1 \pmod{w} \text{ for all } w \in M_O\},$$
$$c_O = \frac{\prod_{w\in Q_O} \#\mathrm{GL}_r(k_w) \cdot (\mathrm{N}Q_O)^{(n-r)r}}{\prod_{w\in S_O} (\mathrm{N}w)^{nr} \cdot \prod_{w\in M_O} \#\mathrm{GL}_r(k_w)},$$

with $\mathrm{N}Q_O = \prod_{w\in Q_O} \mathrm{N}w$. Then:

(a) $[\mathbf{K}_0(r) : J_O] = \prod_{w\in M_O} \#\mathrm{GL}_r(k_w)$ *and* $\langle F, F\rangle = [\mathbf{K}_0(r) : J_O]\, c_O \neq 0$*;*

(b) $P^{A'}F = F$ *for every finite set A' of finite places containing the conjugate of none of its members with $(A' \cup \bar{A}') \cap (S_O \cup \bar{S}_O) = \emptyset$;*

(b′) *F is a vacuum above the set of finite places of F below no place of S_O, in the sense of (7) of Lemma 6.2.33;*

(c) *for every $k \in G_f(r)$ with $k_w \in \mathrm{GL}_r(\mathcal{O}_{E,w})$ at all finite places $w \notin S_O \cup \bar{S}_O$,*

$$\langle F,\ \omega_O((1,k),1)\,F\rangle = \begin{cases} \langle F, F\rangle & \text{if } k \in J_O, \\ 0 & \text{otherwise;} \end{cases}$$

(d) *for every $w \in S_O$ and every finite adelic $n \times r$ matrix x with $\Phi_O(x) \neq 0$, the matrix x_w over E_w is integral and has a left inverse over $\mathcal{O}_{E,w}$; in particular, if $S_O \neq \emptyset$, every datum $v \otimes F$ (v a Fock vector) is Φ_O at the places of S_O times a datum at the other places, with $\Phi_O(x) \neq 0$ only if x_w has a left inverse;*

(e) $\omega_O((1,k),(1,k'))F = F$ *for all $k \in \mathbf{K}_0(r)$ and $k' \in \mathbf{K}_0(n)$ whose components at the places of S_O reduce to 1; these form open subgroups.*

Proof. **Step 0: reductions and the vacuum.** The places of S_O are distinct and none is the conjugate of itself or of another (DP1); so each $w \in S_O$ lies over a place v of F split in E, distinct w lie over distinct v, and the component of $G_f(r)$ at v is identified, by projection to w, with $\mathrm{GL}_r(E_w)$ (the condition $g^*g = 1$ determines the component at $\bar{w}$), the component of $\mathbf{K}_0(r)$ with $\mathrm{GL}_r(\mathcal{O}_{E,w})$. Hence $(\mathrm{red}_w)_{w\in M_O}\colon \mathbf{K}_0(r) \to \prod_{w\in M_O} \mathrm{GL}_r(k_w)$ is surjective with kernel J_O, which gives the index in (a). If $S_O \neq \emptyset$ let $\mathfrak{m} = \prod_{w\in S_O} w$, a modulus with $\Sigma(\mathfrak{m}) = S_O$; by the Chinese remainder theorem $R_{\mathfrak{m}} = \prod_{w\in S_O} k_w$, Ψ_O is a function on $n \times r$ matrices over $R_{\mathfrak{m}}$, and

$\Phi_O = \Psi_O^\natural$ in the notation of (6) of Lemma 4.1.1. If $S_O = \emptyset$, Ψ_O is the empty product 1, $\Phi_O = 1$ and $F = 1 \otimes_\emptyset v^0 = v^0$, since $\mathcal{T}_\emptyset$ is spanned by $\tau_0 = 1$ and $P^\emptyset$ is the identity (WR6). Finally $\langle v^0, v^0\rangle = 1$:

Step 1: inner products of data modulo $\mathfrak{m}$. Let $S_O \neq \emptyset$ and Ψ_1, Ψ_2 be functions on $n \times r$ matrices over $R_\mathfrak{m}$, and $N_\mathfrak{m} = \#\mathrm{M}_{n\times r}(R_\mathfrak{m}) = \prod_{w\in S_O}(\mathrm{N}w)^{nr}$. Then

$$\langle \Psi_1^\natural \otimes_{S_O} v^0,\ \Psi_2^\natural \otimes_{S_O} v^0\rangle = \frac{1}{N_\mathfrak{m}} \sum_X \overline{\Psi_1(X)}\, \Psi_2(X).$$

Indeed, by (1) and (2) of Lemma 6.2.33 and Step 0 the left side is $\langle \Psi_1^\natural, \Psi_2^\natural\rangle_{S_O}$. The integrand vanishes unless x is integral at S_O, so the integral is over $\widehat{\mathcal{O}}_E^{nr}$, of mass 1, of a function of x mod $\mathfrak{m}$; reduction is a surjective continuous homomorphism of compact groups $\widehat{\mathcal{O}}_E^{nr} \to \mathrm{M}_{n\times r}(R_\mathfrak{m})$, so it carries the Haar probability measure to the uniform measure, which gives the right side.

Step 2: (a). Let $S_O \neq \emptyset$. For $X = (X_w)_{w\in S_O}$, $|\Psi_O(X)|^2$ is 1 if $X_w = \mathrm{Inc}_O$ for all $w \in M_O$ and the inc-rows of X_w are invertible for all $w \in Q_O$, and 0 otherwise: the values of a character of the finite group $k_w^\times$ have modulus 1, and its value at 0 is 0. An $n \times r$ matrix over k_w with invertible inc-rows is an element of $\mathrm{GL}_r(k_w)$ on the inc-rows and arbitrary on the other $n - r$ rows. So

$$\sum_X |\Psi_O(X)|^2 = \prod_{w\in Q_O} \#\mathrm{GL}_r(k_w)\,(\mathrm{N}w)^{(n-r)r},$$

and by Step 1, $\langle F, F\rangle$ is this number divided by $N_\mathfrak{m}$, which is $[\mathbf{K}_0(r) : J_O]\, c_O$; it is positive. If $S_O = \emptyset$, $\langle F, F\rangle = \langle v^0, v^0\rangle = 1 = c_O$ and $J_O = \mathbf{K}_0(r)$.

Step 3: (b) and (b′). If $S_O \neq \emptyset$, (b) is (3) of Lemma 6.2.33 with $A = S_O$; if $S_O = \emptyset$ it is (2) there. A vector $y \in \widehat{\mathcal{O}}_E^{nr}$ has components 0 at the places not above the places of F below no place of S_O exactly when its components at $S_O \cup \bar{S}_O$ are 0; so (b′) is (5) of Lemma 6.2.33 with $A = S_O$ (with $A = \emptyset$, $\Psi = 1$ if $S_O = \emptyset$).

Step 4: the integral points. Let $k \in \mathbf{K}_0(r)$ and write $\bar{k} = (\mathrm{red}_w k)_{w\in S_O}$. Then $u = \det k$ lies in $\mathrm{U}(1)(\mathbb{A})$ (as $k^*k = 1$), has infinite components 1 and unit finite components, and $u \bmod w = \det(\mathrm{red}_w k)$. As μ_O has tame part $(\psi_{O,w})$ with $\psi_{O,w} = 1$ off S_O (DP2), $\mu_O(u) = \prod_{w\in S_O} \psi_{O,w}(\det \mathrm{red}_w k)$ (the weight factor is 1), and $v(u) = \prod_{v\in\Phi} u_v/|u_v| = 1$ by (WD3). So

$$\chi'_O(\det k) = \prod_{w\in S_O} \psi_{O,w}(\det \mathrm{red}_w k).$$

By (4) of Lemma 6.2.33, $F = v_1 \otimes F$ for the real vacuum v_1; so (6) of Lemma 4.1.1 (with $v = v_1$, $\alpha = \chi_O$, $\beta = \chi'_O$, $k' = 1$; for $S_O = \emptyset$ its last sentence) and (4) of Lemma 6.2.33 again give

$$\omega_O((1,k),1)F = \chi'_O(\det k)^{-1}\, \Psi_{O,k}^\natural \otimes_{S_O} v^0, \qquad \Psi_{O,k}(X) = \Psi_O(X\bar{k}).$$

If $k \in J_O$: at $w \in M_O$, $X_w \,\mathrm{red}_w k = X_w$ and $\psi_{O,w}(\det \mathrm{red}_w k) = 1$; at $w \in Q_O$ the inc-rows of $X_w \,\mathrm{red}_w k$ are those of X_w times $\mathrm{red}_w k$, so $\psi_{O,w}$ of their determinant is

$$\psi_{O,w}(\det \text{inc-rows}(X_w))\, \psi_{O,w}(\det \mathrm{red}_w k)$$

(both sides 0 if the inc-rows are singular). Hence $\Psi_{O,k} = \chi'_O(\det k)\Psi_O$, so $\omega_O((1,k),1)F = F$ and the coefficient is $\langle F, F\rangle$. *If $k \notin J_O$* (then $S_O \supseteq M_O \neq \emptyset$): by Step 1 the coefficient is $\chi'_O(\det k)^{-1} N_\mathfrak{m}^{-1} \sum_X \overline{\Psi_O(X)}\Psi_O(X\bar{k})$, and every term vanishes: if $\Psi_O(X) \neq 0 \neq \Psi_O(X\bar{k})$, then for $w \in M_O$ both X_w and $X_w \,\mathrm{red}_w k$ equal Inc_O, and since $\mathrm{Inc}_O^\mathsf{T}\mathrm{Inc}_O = 1_r$ (inc_O is injective), $\mathrm{red}_w k = \mathrm{Inc}_O^\mathsf{T} X_w \,\mathrm{red}_w k = 1$; so $k \in J_O$, a contradiction.

Step 5: (d). Let $w \in S_O$ and $\Phi_O(x) \neq 0$. Then x is integral at w, and $X_w = x_w \bmod w$ satisfies $\Psi_O(X) \neq 0$. If $w \in M_O$, $X_w = \mathrm{Inc}_O$, so $B = \mathrm{Inc}_O^{\mathsf{T}} x_w \in \mathrm{M}_r(\mathcal{O}_{E,w})$ reduces to 1_r and is invertible over $\mathcal{O}_{E,w}$. If $w \in Q_O$, $\psi_{O,w}(\det \text{inc-rows}(X_w)) \neq 0$, so the inc-rows B of x_w have unit determinant and $B \in \mathrm{GL}_r(\mathcal{O}_{E,w})$. In both cases $B^{-1}\mathrm{Inc}_O^{\mathsf{T}}$ is an integral left inverse of x_w (the inc-rows of x_w are $\mathrm{Inc}_O^{\mathsf{T}} x_w$). The last assertion is the definition of the datum $v \otimes (\Phi_O \otimes_{S_O} v^0)$ in (WR6).

Step 6: (c) for $k \notin \mathbf{K}_0(r)$. Let $k \in G_f(r) \setminus \mathbf{K}_0(r)$ with $k_w \in \mathrm{GL}_r(\mathcal{O}_{E,w})$ for $w \notin S_O \cup \bar{S}_O$; as $k \notin \mathbf{K}_0(r)$, $S_O \neq \emptyset$. Write $k = k_1 k_2$ in $G_f(r)$, where k_1 has the components of k at the places of F below S_O and 1 elsewhere, and k_2 has components 1 there and those of k elsewhere. Then $k_2 \in J_O$, so $\omega_O((1,k_2),1)F = F$ by Step 4, and by (2) of Lemma 4.1.1, $\omega_O((1,k),1)F = \omega_O((1,k_1),1)F$. As k_1 is 1 outside $S_O \cup \bar{S}_O$, (4) of Lemma 4.1.1 (with $A = S_O$, $h = 1$) gives a scalar c with $\omega_O((1,k_1),1)F = c\,\Phi' \otimes_{S_O} v^0$, $\Phi'(x) = \Phi_O(xk_1) \in \mathcal{T}_{S_O}$. By (1), (2) of Lemma 6.2.33 and Step 0, $\langle F, \omega_O((1,k),1)F\rangle = c\,\langle \Phi_O, \Phi'\rangle_{S_O}$, and the integrand $\overline{\Phi_O(x)}\Phi_O(xk_1)$ vanishes everywhere. Otherwise, for each $w \in S_O$ both x_w and $(xk_1)_w = x_w k_w$ are integral with integral left inverses L, L' (Step 5); then $k_w = L\,x_w k_w$ and $k_w^{-1} = L' x_w$ are integral, so $k_w \in \mathrm{GL}_r(\mathcal{O}_{E,w})$, and the component at $\bar{w}$, determined by k_w through $k^*k = 1$, is integral too. With the hypothesis off $S_O \cup \bar{S}_O$, k would be integral at every finite place, i.e. $k \in \mathbf{K}_0(r)$, a contradiction. So the coefficient is 0. Together with Step 4 this proves (c).

Step 7: (e). Let k, k' be as in (e). As in Step 4 ($F = v_1 \otimes F$ and (6) of Lemma 4.1.1, now with k'), $\omega_O((1,k),(1,k'))F = \chi_O(\det k')\chi'_O(\det k)^{-1}\,\Psi^{\natural}_{O,k,k'} \otimes_{S_O} v^0$ with $\Psi_{O,k,k'}(X) = \Psi_O(\bar{k'}^{-1} X \bar{k}) = \Psi_O(X)$, the reductions being 1. By the formula of Step 4, $\chi'_O(\det k) = \prod_{w \in S_O} \psi_{O,w}(1) = 1$, and $\chi_O(\det k') = \upsilon(\det k')^{r_-(O)} = 1$ by (WD3), $\det k'$ having infinite components 1 and unit finite components. So F is fixed. The two groups contain the kernels of the reductions $\mathbf{K}_0 \to \prod_{w \in S_O} \mathrm{GL}(k_w)$, which are open. □

Lemma 6.5.11. *Let D be admissible data, $1 \leq j \leq m$ and ϑ a tame character. The descent pair $O_{j,\vartheta}$ fits the parameter $l = (l_\tau)_{\tau \in \Phi}$ given by $l_\tau(c) = 2\varsigma(\tau)_{\mathrm{num}(c)}$ for the basis vectors c of W_j.*

Proof. Fix $\tau \in \Phi$ and write $O = O_{j,\vartheta}$, $n = r + t$, $r_+ = r_+(O)$, $r_- = r_-(O)$, $t' = \frac{t-1}{2}$ and $\varsigma(i) = \varsigma_\tau(i)$ for $i \geq 0$ (with the extension of (LF4) for $i \geq r$). By (LF4) and (HP4), $r_+ = k + j - 1$ and $r_- = r - r_+ = m - j$ (recall $r = k + m - 1$), and $\mathrm{wt}_O(\tau) = -\kappa_j(\tau)$, the weight of $\mu_j^0\vartheta$. Since $r + t = 6r$ is even, $\lfloor n/2 \rfloor = \frac{r+t}{2}$; r and $t = 5r$ are odd. For $i \geq 0$ put

$$\sigma(i) = \varsigma(i) + \kappa_j(\tau).$$

Sizes. By (AD1) and Definition 2.3.3, $C_P \geq 6$, so $\log x \geq 100 \log\log b \geq 6 \cdot 10^{14}$; hence $r \geq x^5 \geq 10$ by (AD2), and $\frac{\Delta}{2} = r^8$ exceeds k, t and $t' + 1$, and $4r^8 \geq 343000\, r^3$. Since $\kappa_j - \kappa_{j'} = (j - j')\Delta$, the values of σ are

$$\sigma(i) = \kappa_j(\tau) + \tfrac{k}{2} - 1 - i \quad (0 \leq i \leq k-2), \qquad \sigma(k - 2 + j') = \tfrac{\Delta}{2} + (j - j')\Delta \quad (j' \geq 1),$$

the second formula covering both the entries $\frac{\Delta}{2} - \kappa_{j'}$ of $\varsigma(\tau)$ ($1 \leq j' \leq m$) and the extension ($j' = i + 2 - k \geq m + 1$). Hence:

(1) σ is a strictly decreasing sequence of integers: the steps are 1 among the first $k - 1$ values, $\sigma(k-2) - \sigma(k-1) = \kappa(\tau) + \Delta - \frac{k}{2} + 1 > 0$, and Δ afterwards;

(2) $\sigma(i) \geq \frac{\Delta}{2} > t' + 1$ for $i < r_+$ and $\sigma(i) \leq -\frac{\Delta}{2} < -t' - 1$ for $i \geq r_+$ (here $j' \leq j$, respectively $j' \geq j + 1$, and $j \geq 1$).

Note that $\varsigma(h-1)-\max(\varsigma(h),t'-\kappa_j)=\sigma(h-1)-\max(\sigma(h),t')$ and $\min(\varsigma(r-h-1),-\kappa_j-t')-\varsigma(r-h)=\min(\sigma(r-h-1),-t')-\sigma(r-h)$, so (LF4) reads $\kappa^+(\tau,h)=(\sigma(h-1)-\max(\sigma(h),t')-1)^+$ and $\kappa^-(\tau,h)=(\min(\sigma(r-h-1),-t')-\sigma(r-h)-1)^+$ for $1\le h<n$.

The column counts. Let $1\le h<n$. By (1) and (2):

(P1) for $h<r_+$, $\sigma(h)>t'$, so $\kappa^+(\tau,h)=\sigma(h-1)-\sigma(h)-1\ge 0$;

(P2) for $h=r_+$, $\sigma(r_+)<t'$ and $\sigma(r_+-1)>t'+1$, so $\kappa^+(\tau,r_+)=\sigma(r_+-1)-t'-1\ge 0$;

(P3) for $h>r_+$, $\sigma(h-1)<t'$, so $\kappa^+(\tau,h)=0$.

For κ^- put $i=r-h$. If $h\ge r$, both arguments are read as 0 and $\kappa^-(\tau,h)=(\min(\sigma(0),-t')-\sigma(0)-1)^+=0$ since $\sigma(0)>0$. If $1\le i\le r_+-1$, then $\sigma(i)>0\ge\min(\sigma(i-1),-t')$, so $\kappa^-(\tau,h)=0$. The remaining cases have $i\ge r_+$, that is $h\le r-r_+=r_-$:

(M1) for $h=r_-$ (so $i=r_+$; only if $r_-\ge 1$), $\sigma(r_+-1)>0$, so $\kappa^-(\tau,r_-)=-t'-\sigma(r_+)-1\ge 0$;

(M2) for $1\le h\le r_--1$ (so $i\ge r_++1$), $\sigma(i-1)<-t'$, so $\kappa^-(\tau,h)=\sigma(r-h-1)-\sigma(r-h)-1\ge 0$.

In particular $\kappa^+(\tau,h)=0$ for $r_+<h<n$ (P3) and $\kappa^-(\tau,h)=0$ for $r_-<h<n$; this is (iii).

(ii). By (1), $\varsigma(\tau)$ is strictly decreasing in the number, so $l_\tau(c)<l_\tau(c')$ if and only if $\mathrm{num}(c')<\mathrm{num}(c)$. For c of sign $+1$ the vectors c' of sign $+1$ with $l_\tau(c)<l_\tau(c')$ are those with $\mathrm{num}(c')<\mathrm{num}(c)$; as the numbers of sign $+1$ are $0,\dots,r_+-1$, there are $\mathrm{num}(c)=\mathrm{ord}(c)$ of them. For c of sign -1 the vectors c' of sign -1 with $-l_\tau(c)<-l_\tau(c')$ are those with $\mathrm{num}(c')>\mathrm{num}(c)$; the numbers of sign -1 being $r_+,\dots,r-1$, there are $r-1-\mathrm{num}(c)=\mathrm{ord}(c)$ of them.

The parameter l_O. Define $g(a)$ for $1\le a\le n$ by

$$\begin{aligned}
&g(a)=2\sigma(a-1)\ (a\le r_+),\\
&g(a)=2(t'-(a-1-r_+))\ (r_+<a\le r_++t),\\
&g(a)=2\sigma(a-1-t)\ (a>r_++t).
\end{aligned}$$

As $a-1-t$ runs over $r_+,\dots,r-1$ in the last range and $\{2(t'-b):0\le b<t\}=[t]$, the set of values of g is $\{2\sigma(i):0\le i<r\}\cup[t]$. We show

$$l_{O,\tau}(a)-2r_-+r=g(a)\qquad(1\le a\le n).\tag{6.118}$$

By (DP11), $l_{O,\tau}(a)=2(e_O(\tau)+\lambda_{O,\tau}(a-1))+n-1-2(a-1)$, and $\lambda_{O,\tau}(a-1)-\lambda_{O,\tau}(a)=\kappa_O(\tau,a)=\kappa^+(\tau,a)+\kappa^-(\tau,n-a)$ for $1\le a<n$ (head of Definition 3.2.19, (DP3)). Hence

$$l_{O,\tau}(a)-l_{O,\tau}(a+1)=2\kappa^+(\tau,a)+2\kappa^-(\tau,n-a)+2\qquad(1\le a<n).$$

We compare with $g(a)-g(a+1)$, using (iii) for the vanishing terms:

- $a<r_+$: $g(a)-g(a+1)=2(\sigma(a-1)-\sigma(a))=2\kappa^+(\tau,a)+2$ by (P1), and $n-a>r_-$;
- $a=r_+$: $g(a)-g(a+1)=2\sigma(r_+-1)-2t'=2\kappa^+(\tau,r_+)+2$ by (P2), and $n-a=r_-+t>r_-$;
- $r_+<a<r_++t$: $g(a)-g(a+1)=2$, $\kappa^+(\tau,a)=0$ and $n-a>r_-$;
- $a=r_++t<n$ (so $r_-\ge 1$ and $n-a=r_-$): $g(a)-g(a+1)=-2t'-2\sigma(r_+)=2\kappa^-(\tau,r_-)+2$ by (M1), and $\kappa^+(\tau,a)=0$;

- $r_+ + t < a < n$: with $i = a - t \in [r_+ + 1, r - 1]$ and $h = n - a = r - i \le r_- - 1$, $g(a) - g(a+1) = 2(\sigma(i-1) - \sigma(i)) = 2\kappa^-(\tau, h) + 2$ by (M2), and $\kappa^+(\tau, a) = 0$.

So both sides of (6.118) have the same steps, and it suffices to check $a = n$. As $\lambda_{O,\tau}(n-1) = 0$ (an empty sum), $l_{O,\tau}(n) = 2e_O(\tau) - (n-1)$ with $e_O(\tau) = -\sum_{1 \le h < n} \kappa^-(\tau, h)$. If $r_- = 0$, then $\kappa^-(\tau, \cdot) = 0$, $r_+ = r$, and $l_{O,\tau}(n) + r = 1 - t = g(n)$. If $r_- \ge 1$, then by (M1), (M2) and telescoping,

$$\sum_{1 \le h < n} \kappa^-(\tau, h) = (\sigma(r_+) - \sigma(r-1) - (r_- - 1)) + (-t' - \sigma(r_+) - 1) = -\sigma(r-1) - r_- - t',$$

so $l_{O,\tau}(n) - 2r_- + r = 2\sigma(r-1) + 2r_- + (t-1) - (r+t-1) - 2r_- + r = 2\sigma(r-1) = g(n)$. This proves (6.118).

(i). Apply (WM8) with the index sets $\{1, \dots, r\}$ (signs ε_O, $N = r$) and $\{1, \dots, n\}$ (signs $\mathbf{1}_n$, $N' = n$), and the weights $\gamma = r_-$, $\gamma' = \frac{r+t}{2} - \kappa_j(\tau)$. The shifted entries are

$$\tilde{l}_\tau(c) = l_\tau(c) - 2\gamma' + n = 2\sigma(\mathrm{num}(c)), \qquad \tilde{l}_{O,\tau}(a) = l_{O,\tau}(a) - 2r_- + r = g(a).$$

(a) l_τ is injective by (1); its values are even, and $r - 1$ is even. If c, c' have the signs $+1$, -1, then $\mathrm{num}(c) < r_+ \le \mathrm{num}(c')$ and, by (2), $l_\tau(c) - l_\tau(c') = 2(\sigma(\mathrm{num}(c)) - \sigma(\mathrm{num}(c'))) \ge 2\Delta = 4r^8 \ge 2 \cdot 500(r+n)^3$, as $(r+n)^3 = 343r^3$. So l_τ has gap $500(r+n)^3$ for ε_O. The map $l_{O,\tau}$ is strictly decreasing, hence injective, with $l_{O,\tau}(a) \equiv n - 1 \pmod 2$, and $\mathbf{1}_n$ takes one value; so it has every gap for $\mathbf{1}_n$. (b) $N = r \le N'$, and by (6.118) the set $\tilde{l}_{O,\tau}(\{1, \dots, n\})$ is the set of values of g, which is $\{2\sigma(i) : 0 \le i < r\} \cup [t] = \tilde{l}_\tau(\{1, \dots, r\}) \cup [n - r]$, since num is a bijection onto $\{0, \dots, r-1\}$. (c) The signs $\mathbf{1}_n$ are all $+1$. By (2), $\tilde{l}_\tau(c) = 2\sigma(\mathrm{num}(c))$ is positive when $\mathrm{num}(c) < r_+$, that is when $\varepsilon_O(c) = +1$, and negative otherwise, when $\varepsilon_O(c) = -1$. So (c) holds. Hence $(l_\tau, l_{O,\tau})$ satisfies the block rule for the weights $(r_-, \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau))$ and the signs $(\varepsilon_O, \mathbf{1}_n)$ at every $\tau \in \Phi$, which is (i). □

Lemma 6.5.12. *Let S, T be spaces with signs s on n and s' on n', $2\#n < \#n'$, let $\gamma, \gamma' \colon \Phi \to \mathbb{Z}$, e a type on S, e' a type on T, and $v \in \mathcal{F}_{S,T}$ a Schwartz vector that is an edge for the weights (γ, γ') from e to e'. Let dx be the Haar measure of (ED1) on $G_\infty(s)$, dk the Haar measure on $G_f(n)$ giving $\mathbf{K}_0(n)$ mass 1, $\lambda_S > 0$ the constant with $dg = \lambda_S \, dx \, dk$ for the Haar measure dg on $\mathrm{U}(S)(\mathbb{A}_F) = \mathrm{G}(s)$ inducing the quotient measure of $[\mathrm{U}(S)]$, and*

$$A = \mathrm{vol}([\mathrm{U}(T)]) \, \lambda_S \int_{G_\infty(s)} \langle v, \Omega_{\gamma,\gamma'}(x, 1) v \rangle \, \frac{\overline{e(x)}}{e(1)} \, dx.$$

*Let a, b be automorphic characters of $\mathrm{U}(1)$ of weights γ, γ', S_0 a finite set of finite places of F, Σ the set of the other finite places of F, $J \subseteq G_f(n)$ an open subgroup and $c \in \mathbb{C}$. Let $F \in \mathcal{M}_{S,T}$ be **sharp for J off S_0 with mass** c:*

(SH1) $\langle F, F \rangle \ne 0$, $c = \langle F, F \rangle / [\mathbf{K}_0(n) : J \cap \mathbf{K}_0(n)]$, $v_1 \otimes F = F$, *and F is a vacuum above Σ;*

(SH2) for every $k \in G_f(n)$ with $k_v \in K_v$ for all $v \in \Sigma$, $\langle F, \omega_{1,b}((1,k),1) F \rangle$ is $\langle F, F \rangle$ if $k \in J \cap \mathbf{K}_0(n)$ and 0 otherwise;

(SH3) either all signs of S agree, or $F = \Psi \otimes_A v^0$ for a finite set A of finite places of E containing the conjugate of none of its members, a place $w \in A$ and some $\Psi \in \mathcal{T}_A$ with $\Psi(x) \ne 0$ only if the $\#n' \times \#n$ matrix x_w has a left inverse;

(SH4) $\omega_{a,b}((1,k),(1,k')) F = F$ for all k, k' in some open subgroups of $G_f(n)$, $G_f(n')$.

Let $L = L_{v\otimes F}$ be the lift of (ED3) for the theta kernel $(\omega_{a,b}, \theta_{\mathrm{Hom}(S,T)})$. Then $A \neq 0$; L maps forms of type e on S to forms of type e' on T; L is injective on the forms of type e that are right invariant under J; and for every such form y and every form y' of type e on S,

$$\langle Ly', Ly\rangle_T = A\, c\, \langle y', R_\Sigma y\rangle_S = \sum_q A\, c\, \mathrm{mc}_{b,\Sigma}(u_q)\, \langle y',\, y(\,\cdot\, u_q)\rangle_S,$$

the series absolutely convergent, R_Σ the Rallis series of (S, T, b, Σ) and q running over

$$\mathrm{U}(S)(\mathbb{A}_{F,f})/K_\Sigma.$$

Proof. **Step 0: preliminaries.**

(i) By Lemma 6.3.77, $(\omega_{a,b}, \theta_{\mathrm{Hom}(S,T)})$ is a theta kernel for (S, T) on $\mathrm{Hom}(S, T)$ with real part Ω, and L is the theta lift Θ_φ of Theorem 6.5.13 for $(\alpha, \beta) = (a, b)$.

(ii) By (ED4)(a), e is a cusp kernel of order 0, in particular an integrable idempotent kernel. Forms of type e are continuous, bounded, left invariant under Γ_S, have a level (ED2), and are rapidly decreasing in the height: $\sup |F|\, \mathrm{ht}_S^N < \infty$ for every N, by Lemma 6.3.75 for the cusp kernel e of order 0; so they satisfy the decay hypothesis of Theorem 6.5.13, which is stated in this height form. They are integrable on $[\mathrm{U}(S)]$, which has finite volume.

(iii) φ is a datum whose Fock vector is a Schwartz vector, and it is clean: under the first alternative of (SH3) this is (DC1) of Theorem 6.5.13; under the second, $\varphi = v \otimes (\Psi \otimes_A v^0)$ is the datum of (v, Ψ) at A ((WR6)), and Ψ, a combination of the functions τ_y, is a Schwartz–Bruhat function of the components at A; this is (DC2).

(iv) For $x \in G_\infty(s)$ and $k \in G_f(n)$, $\langle \varphi, \omega_{a,b}((x,k),1)\varphi\rangle = \langle v, \Omega(x,1)v\rangle\, \langle F, \omega_{1,b}((1,k),1)F\rangle$. Indeed $a(\det 1) = 1$, so $\omega_{a,b}(g,1) = \omega_{1,b}(g,1)$; by (5) of Lemma 4.1.1, $\omega_{a,b}((x,k),1)\varphi = (\Omega(x,1)v)\otimes(\omega_{1,b}((1,k),1)F)$; by (6) of Lemma 6.2.33 the inner product is $\langle v, \Omega(x,1)v\rangle\langle v_1 \otimes F, v_1 \otimes \omega_{1,b}((1,k),1)F\rangle$; and $v_1 \otimes \omega_{1,b}((1,k),1)F = \omega_{1,b}((1,k),1)(v_1 \otimes F) = \omega_{1,b}((1,k),1)F$ by (5) of Lemma 4.1.1 at $x = 1$ and (SH1).

(v) For $x \in G_\infty(s)$, $|\langle v, \Omega(x,1)v\rangle| \leq \langle v, v\rangle$: $\Omega(x,1)$ is a scalar of modulus 1 (determinants of the unitary matrices x_v) times the unitary operator $\omega_\varepsilon^\infty(\cdot)$ ((2) of Lemma 4.1.1). As e is integrable, the integral defining A converges absolutely; it is not 0 by (ED4)(b), and $\mathrm{vol}([\mathrm{U}(T)]) > 0$, $\lambda_S > 0$; so $A \neq 0$.

Step 1: the character b is unramified on K'_Σ, and $K'_\Sigma \subseteq J$. Let $k \in K'_\Sigma$ (integral, with components 1 off Σ). By Lemma 6.5.6 applied to F (a vacuum above Σ by (SH1)) with $\gamma = k$ and $g = 1$, $\langle F, \omega_{1,b}((1,k),1)F\rangle = \mathrm{mc}(k)\langle F, F\rangle$. By (WR4), $\mathrm{mc}(k) = b(\det k)^{-1}\langle v^0, \omega_{s'\otimes s}(k')v^0\rangle$ with $k' = 1 \otimes (k^{-1})^{\mathsf{T}}$, and $\omega_{s'\otimes s}(k')v^0 = \omega_{1,1}((1,k),1)(v_1 \otimes v^0) = v^0$ by (6) of Lemma 4.1.1 and (4) of Lemma 6.2.33; so $\mathrm{mc}(k) = b(\det k)^{-1}$ (as $\langle v^0, v^0\rangle = 1$), of modulus 1. Hence the coefficient is not 0, and (SH2) gives $k \in J \cap \mathbf{K}_0(n)$ and $\mathrm{mc}(k)\langle F, F\rangle = \langle F, F\rangle$, i.e. $b(\det k) = 1$. So the hypotheses of (2) and (3) of Proposition 6.5.4 hold for $\beta = b$, and every J-invariant function is K'_Σ-invariant.

Step 2: L maps forms of type e to forms of type e'. By (SH4) and (5) of Lemma 4.1.1 at real points 1, $\omega_{a,b}((1,k),(1,k'))\varphi = v \otimes \omega_{a,b}((1,k),(1,k'))F = \varphi$ for k' in an open subgroup U' of $G_f(n')$, so for a form y of type e and $h \in \mathrm{G}(s')$,

$$Ly(h(1,k')) = \int \theta(\omega_{a,b}(g,h)\omega_{a,b}(1,(1,k'))\varphi)y(g)\, dg = Ly(h)\,:$$

Ly has the level U'. By (ED4)(c), applied to the half edge v, the theta kernel of Step 0 (i), $f = F$ and y, Ly is a form of type e' on T.

Step 3: the Rallis integral, cut into cosets. Let y be a J-invariant form of type e and y' a form of type e. By Step 0 (ii), (iii) and Theorem 6.5.13,

$$\langle Ly', Ly\rangle_T = \mathrm{vol}([\mathrm{U}(T)]) \int_{\mathrm{G}(s)} \langle \varphi, \omega_{a,b}(g,1)\varphi\rangle \, \langle y', R(g)y\rangle_S \, dg,$$

absolutely convergent. With $dg = \lambda_S \, dx \, dk$ and Fubini, for almost every x the function

$$h_x(k) = \langle \varphi, \omega_{a,b}((x,k),1)\varphi\rangle \langle y', R((x,k))y\rangle_S$$

is integrable on $G_f(n)$, and $\langle Ly', Ly\rangle_T = \mathrm{vol}([\mathrm{U}(T)])\lambda_S \iint h_x(k)\, dk\, dx$. The group K_Σ is open, so $G_f(n) = \bigsqcup_q u_q K_\Sigma$ is a countable disjoint union of open cosets, and by left invariance of dk, $\int h_x = \sum_q \int_{K_\Sigma} h_x(u_q\kappa)\, d\kappa$, absolutely convergent. For $\kappa \in K_\Sigma$, Step 0 (iv), Lemma 6.5.6 applied to F (with $\gamma = u_q$, supported on Σ, and $g = (1,\kappa)$, $\kappa_v \in K_v$ for $v \in \Sigma$) and (SH2) give

$$\langle \varphi, \omega_{a,b}((x,u_q\kappa),1)\varphi\rangle = \langle v, \Omega(x,1)v\rangle \, \mathrm{mc}(u_q) \, \langle F, F\rangle \, \mathbf{1}_{J\cap \mathbf{K}_0(n)}(\kappa),$$

and for $\kappa \in J$, $y(g(x,u_q\kappa)) = y(g(x,1)(1,u_q))$ by the J-invariance of y. As $dk(J \cap \mathbf{K}_0(n)) = [\mathbf{K}_0(n) : J \cap \mathbf{K}_0(n)]^{-1}$,

$$\int_{K_\Sigma} h_x(u_q\kappa)\, d\kappa = c \, \langle v, \Omega(x,1)v\rangle \, \mathrm{mc}(u_q) \, \big\langle y', y(\,\cdot\,(x,1)u_q)\big\rangle_S,$$

and summing over q with (1) of Proposition 6.5.4 (y bounded, measurable and left invariant, y' integrable), $\int h_x = c \, \langle v, \Omega(x,1)v\rangle \, \langle y', z(\,\cdot\,(x,1))\rangle_S$ with $z = R_\Sigma y$.

Step 4: the real integral. By Step 1, (2) of Proposition 6.5.4 applies to y (bounded, continuous, left invariant, right invariant under $K'_\Sigma \subseteq J$ and under an open subgroup): z is bounded, continuous, left invariant under Γ_S, has a level, and $R(e)z = R_\Sigma(R(e)y) = z$, as R_Σ commutes with convolution by the integrable e and $R(e)y = y$. So z is a form of type e, and by Lemma 6.3.78 (e an idempotent kernel), $\langle y', z(\,\cdot\,(x,1))\rangle_S = \frac{\overline{e(x)}}{e(1)}\langle y', z\rangle_S$. Hence, by Step 0 (v),

$$\langle Ly', Ly\rangle_T = \mathrm{vol}([\mathrm{U}(T)])\, \lambda_S \, c \, \langle y', z\rangle_S \int \langle v, \Omega(x,1)v\rangle \frac{\overline{e(x)}}{e(1)}\, dx = A \, c \, \langle y', R_\Sigma y\rangle_S.$$

By (1) of Proposition 6.5.4 at $x = 1$, $\langle y', R_\Sigma y\rangle_S = \sum_q \mathrm{mc}(u_q)\langle y', y(\,\cdot\,u_q)\rangle_S$, absolutely convergent; this is the displayed formula.

Step 5: injectivity. Let y be a J-invariant form of type e with $Ly = 0$, and $z = R_\Sigma y$, a form of type e by Step 4. The formula with $y' = z$ gives $0 = \langle Lz, Ly\rangle_T = A \, c \, \langle z, z\rangle_S$. Here $A \neq 0$ (Step 0 (v)) and $c \neq 0$ (SH1), so $\int_{[\mathrm{U}(S)]} |z|^2 = 0$; z is continuous and the quotient measure gives positive mass to the image of every nonempty open set, so $z = 0$. As y is bounded and right invariant under K'_Σ, (3) of Proposition 6.5.4 gives $y = 0$. □

This group proves the Rallis inner product formula in the form used for the single lifts (Theorem 6.5.13): for spaces with $2\,\mathrm{rk}\, S < \mathrm{rk}\, T$, rapidly decreasing functions y, y' on $[\mathrm{U}(S)]$ and Schwartz data φ, φ' with φ clean,

$$\big\langle \Theta_{\varphi'} y', \Theta_\varphi y\big\rangle_T = \mathrm{vol}([\mathrm{U}(T)]) \int_{\mathrm{U}(S)(\mathbb{A}_F)} \langle \varphi', \omega(g,1)\varphi\rangle \, \langle y', R(g)y\rangle_S \, dg,$$

both sides converging absolutely. The formula goes back to Rallis [237, 238] and the doubling method [231], through the Siegel–Weil formula [280, 323]; for unitary groups see [187, 121, 139,

190, 331, 94]. In Li's theorem the forms are cusp forms; here they are rapidly decreasing, and the cleanness of the datum takes the place of cuspidality.

The proof doubles the space, computes the integral over $[\mathrm{U}(T)]$ by the Siegel–Weil formula on the theta model (Proposition 6.5.15), and uses cleanness to select the Lagrangians: a Lagrangian with a nonzero term is the graph of a rational isometry (Lemma 6.5.14), whose term is a matrix coefficient of the Weil representation, so that the sum unfolds to the integral over $\mathrm{U}(S)(\mathbb{A}_F)$. The Siegel–Weil formula is proved in the group: the terms are summable by the convergence of Siegel Eisenstein series [184, 208], a uniform majorant comes from Weil's convergence criterion [323, 139], and Weil's identity in the convergent range (Theorem A.4.28; see also [121]) is carried to the theta model by the comparison in every rational polarisation (Lemma 6.5.22). That comparison is derived from Weil's theta transform and metaplectic group [322], Kudla's splitting [172] and the Stone–von Neumann theorem in Mackey's form [195].

Theorem 6.5.13. *Let S, T be spaces with $2\operatorname{rk} S < \operatorname{rk} T$, let α, β be automorphic characters of $\mathrm{U}(1)$, and write $\omega = \omega_{\alpha,\beta}$, $\theta = \theta^{S,T}_{\alpha,\beta}$. Let dg be the Haar measure on $\mathrm{U}(S)(\mathbb{A}_F)$ whose quotient by the counting measure on $\mathrm{U}(S)(F)$ is the measure of $[\mathrm{U}(S)]$; let $\langle\cdot,\cdot\rangle$ be the L^2 inner product on data, $\langle F, F'\rangle_V = \int_{[\mathrm{U}(V)]} \overline{F(x)}\, F'(x)\, dx$ for functions on $[\mathrm{U}(V)]$ (both conjugate-linear in the first argument), and R right translation. For a datum φ and a function y on $[\mathrm{U}(S)]$ the theta lift is*

$$(\Theta_\varphi y)(h) = \int_{[\mathrm{U}(S)]} \theta(\varphi)(g,h)\, y(g)\, dg \qquad (h \in \mathrm{U}(T)(\mathbb{A}_F)).$$

Let φ, φ' be finite sums of data whose Fock vectors are Schwartz vectors, and suppose that φ is clean*: either*

(DC1) *all signs of S agree; or*

(DC2) *there are a set A of finite places of E containing no two conjugate places, a place $w \in A$ and a Schwartz–Bruhat function Ψ of the components above A of $x \in \operatorname{Hom}(S,T)(\mathbb{A}_E)$, with $\Psi(x) \neq 0$ only if the $\operatorname{rk} T \times \operatorname{rk} S$ matrix x_w over E_w has a left inverse, such that φ is Ψ at the places of A times a datum at the other places.*

Let y, y' be continuous functions on $\mathrm{U}(S)(\mathbb{A}_F)$, left invariant under $\mathrm{U}(S)(F)$, right invariant under an open subgroup of $\mathrm{U}(S)(\mathbb{A}_{F,f})$, and rapidly decreasing in the height: for every $N \geq 0$, $|y|\,\mathrm{ht}_S^N$ and $|y'|\,\mathrm{ht}_S^N$ are bounded on $\mathrm{U}(S)(\mathbb{A}_F)$. Then:

1. *the function $(g', g, h) \mapsto \overline{y'(g')\,\theta(\varphi')(g',h)}\; y(g)\,\theta(\varphi)(g,h)$ is integrable on $[\mathrm{U}(S)] \times [\mathrm{U}(S)] \times [\mathrm{U}(T)]$, and $g \mapsto \langle \varphi', \omega(g,1)\varphi\rangle\, \langle y', R(g)y\rangle_S$ is integrable on $\mathrm{U}(S)(\mathbb{A}_F)$;*
2. $$\big\langle \Theta_{\varphi'} y',\, \Theta_\varphi y \big\rangle_T = \operatorname{vol}([\mathrm{U}(T)]) \int_{\mathrm{U}(S)(\mathbb{A}_F)} \big\langle \varphi', \omega(g,1)\varphi \big\rangle\, \big\langle y', R(g)y \big\rangle_S\, dg.$$

Proof. The argument is the doubling method of [238] in the form of [187], Section 2, with cleanness in place of cuspidality. Let s on n and s' on n' be the signs of S and T, so $\#n' > 2\#n$, and write $\omega = \omega_{\alpha,\beta}$, $\theta = \theta^{S,T}_{\alpha,\beta}$, $\Theta = \Theta_S$. Let $W = S \oplus S^-$, with signs $s \sqcup (-s)$ on $n \sqcup n$ and matrix $J_W = \operatorname{diag}(J_S, -J_S)$, let $c = c_{S,T}$ be as in Lemma 6.4.56 (stated with the conjugate cj of (WR7)), and $\mathrm{dbl} = \mathrm{dbl}_c$ the doubled datum of (WR7): $\mathrm{dbl}(\varphi_1, \varphi_1') = \varphi_1 \boxtimes^{\mathrm{sc}} c\,\mathrm{cj}\,\varphi_1' \in \mathcal{M}_{W,T}$ for $\varphi_1, \varphi_1' \in \mathcal{M}_{S,T}$ ((1) of Lemma 6.4.56). Under (DC2), Ψ is read as the function $x \mapsto \Psi(x_A)$ of finite adelic matrices, an element of $\mathcal{T}_A$ by Lemma 6.2.22, and "φ is Ψ at the places of A times a datum at the other places" means $\varphi = \Psi \otimes_A \varphi_0$ for some $\varphi_0 \in \mathcal{M}_{S,T}$.

Step 1: the doubling identity for theta kernels. For $\varphi_1, \varphi_1' \in \mathcal{M}_{S,T}$ and $h \in \mathrm{G}(s')$,

$$\theta^{W,T}_{\upsilon^{\#n},1}(\mathrm{dbl}(\varphi_1,\varphi_1'))(1,h) = \Theta(\omega(1,h)\varphi_1)\,\overline{\Theta(\omega(1,h)\varphi_1')}. \tag{6.119}$$

Put $\alpha' = \alpha^{-1}\upsilon^{\#n}$ and $\beta' = \beta^{-1}\upsilon^{\#n'}$, so $\alpha\alpha' = \upsilon^{\#n}$. By (a) of Lemma 6.3.52 for $S_1 = S$, $S_2 = S^-$, the characters α, α' on the side of T and 1 on the side of the source, and $g_1 = g_2 = 1$,

$$\omega^{W,T}_{\upsilon^{\#n},1}(1,h)\,\mathrm{dbl}(\varphi_1,\varphi_1') = \omega^{S,T}_{\alpha,1}(1,h)\varphi_1 \boxtimes^{\mathrm{sc}} \omega^{S^-,T}_{\alpha',1}(1,h)\,c\,\mathrm{cj}\,\varphi_1',$$

and at the point $(1,h)$ the character of the source is irrelevant (WR4), so the factors are $\omega(1,h)\varphi_1$ and $c\,\omega^{S^-,T}_{\alpha',\beta'}(1,h)\,\mathrm{cj}\,\varphi_1' = c\,\mathrm{cj}\,\omega(1,h)\varphi_1'$ ((2) of Lemma 6.4.56). By (c) of Lemma 6.3.52 the theta functional of $\mathrm{Hom}(W,T)$ at this product is $\Theta(\omega(1,h)\varphi_1)\,\Theta_{S^-}(c\,\mathrm{cj}\,\omega(1,h)\varphi_1')$, and the second factor is $\overline{\Theta(\omega(1,h)\varphi_1')}$ by (3) of Lemma 6.4.56. This is (6.119). For $g, g' \in \mathrm{G}(s)$ put $\varphi^{\mathrm{d}}_{g,g'} = \mathrm{dbl}(\omega(g,1)\varphi, \omega(g',1)\varphi')$; as ω is a representation of $\mathrm{G}(s)\times\mathrm{G}(s')$ ((2) of Lemma 4.1.1), $\omega(1,h)\omega(g,1) = \omega(g,h)$, and (6.119) becomes

$$\theta^{W,T}_{\upsilon^{\#n},1}(\varphi^{\mathrm{d}}_{g,g'})(1,h) = \theta(\varphi)(g,h)\,\overline{\theta(\varphi')(g',h)}. \tag{6.120}$$

Step 2: the Siegel–Weil formula and the graphs. W has the splitting $(k,l,\emptyset)$ over the zero space $V_\emptyset$ ((MX13) of Definition 4.1.18) along a copy p of n with $k = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $l = \frac{1}{2}\begin{pmatrix} J_S^{-1} \\ J_S^{-1} \end{pmatrix}$: $k^* J_W k = J_S - J_S = 0$, $l^* J_W l = \frac{1}{4}(J_S^{-1} - J_S^{-1}) = 0$, $l^* J_W k = \frac{1}{2} + \frac{1}{2} = 1$ (J_S is hermitian), and $\#(n \sqcup n) = 2\#p$. As $\#n' > 2\#p$, Proposition 6.5.15 applies to (W,T): $M = \sum_L \|Z_L\| < \infty$, and for every g, g' the function $h \mapsto \theta^{W,T}_{\upsilon^{\#n},1}(\varphi^{\mathrm{d}}_{g,g'})(1,h)$ is integrable on $[\mathrm{U}(T)]$ with integral $\mathrm{vol}([\mathrm{U}(T)])\sum_L Z_L(\varphi^{\mathrm{d}}_{g,g'})$. By Lemma 6.5.14, applied to φ (whose hypotheses are (DC1), resp. (DC2) in the form above), g and the datum $\omega(g',1)\varphi'$, every rational Lagrangian L with $Z_L(\varphi^{\mathrm{d}}_{g,g'}) \neq 0$ is $\{(\delta_0 y, y) : y \in E^n\}$ for a unique $\delta_0 \in \mathrm{U}(S)(F)$; substituting $y = -u$, it is $L_\delta = \{(\delta^{-1}u, -u)\}$ for $\delta = -\delta_0^{-1} \in \mathrm{U}(S)(F)$. Conversely each L_δ is a graph of this kind, and $\delta \mapsto L_\delta$ is injective (L_δ determines the linear map $u \mapsto \delta^{-1}u$). Hence $\sum_L Z_L(\varphi^{\mathrm{d}}_{g,g'}) = \sum_{\delta\in\mathrm{U}(S)(F)} Z_{L_\delta}(\varphi^{\mathrm{d}}_{g,g'})$, and by Lemma 6.5.23 and the invariance of the inner product under $\omega(g',1)$ ((2) of Lemma 4.1.1; the characters have modulus one),

$$Z_{L_\delta}(\varphi^{\mathrm{d}}_{g,g'}) = \langle \omega(g',1)\varphi',\, \omega(x_S^{-1}\delta x_S, 1)\,\omega(g,1)\varphi\rangle = \langle \varphi',\, \omega(g'^{-1}\gamma g, 1)\varphi\rangle, \qquad \gamma = x_S^{-1}\delta x_S.$$

As δ runs over $\mathrm{U}(S)(F)$, γ runs over Γ_S once. With (6.120), for all $g, g' \in \mathrm{G}(s)$,

$$\int_{[\mathrm{U}(T)]} \theta(\varphi)(g,h)\,\overline{\theta(\varphi')(g',h)}\,dh = \mathrm{vol}([\mathrm{U}(T)])\sum_{\gamma\in\Gamma_S}\langle \varphi', \omega(g'^{-1}\gamma g,1)\varphi\rangle, \tag{6.121}$$

the integrand integrable and the series absolutely convergent, $|Z_{L_\delta}(\varphi^{\mathrm{d}}_{g,g'})| \le \|Z_{L_\delta}\|\,\|\varphi^{\mathrm{d}}_{g,g'}\|$.

Step 3: bounds in the heights. The L^1 norm of a source product is the product of the norms (the measure of $\mathrm{Hom}(W,T)$ is the product of those of the two factors, (WR6), and Tonelli), cj is isometric ((1) of Lemma 6.4.56) and $|c| = 1$; so $\|\varphi^{\mathrm{d}}_{g,g'}\| = \|\omega(g,1)\varphi\|\,\|\omega(g',1)\varphi'\|$. With N of (2) of Proposition 6.5.15 for (W,T) and (6.120), and with Step 2,

$$\int_{[\mathrm{U}(T)]} |\theta(\varphi)(g,h)|\,|\theta(\varphi')(g',h)|\,dh \le \|N\|_1\,P(g,g'), \qquad \sum_{\gamma\in\Gamma_S}\big|\langle \varphi', \omega(g'^{-1}\gamma g,1)\varphi\rangle\big| \le M\,P(g,g'),$$

where $P(g,g') = \|\omega(g,1)\varphi\|\,\|\omega(g',1)\varphi'\|$ and $\|N\|_1 = \int_{[\mathrm{U}(T)]} N < \infty$. Both left sides are unchanged when g and g' are replaced by $\gamma_1 g$ and $\gamma_2 g'$ with $\gamma_1, \gamma_2 \in \Gamma_S$: the first because

$\theta(\varphi)(\gamma_1 g, h) = \theta(\varphi)(g,h)$ (Lemma 6.2.5), the second by the substitution $\gamma \mapsto \gamma_2\gamma\gamma_1^{-1}$ of Γ_S. Let Y, A be the constants of Lemma 6.2.28 for (S,T), and by its clause (2) choose $\gamma_1, \gamma_2 \in \Gamma_S$ with

$$\|\omega(\gamma_1 g, 1)\varphi\| \le Y \mathrm{ht}_S(g)^A \|\varphi\| \quad \text{and} \quad \|\omega(\gamma_2 g', 1)\varphi'\| \le Y \mathrm{ht}_S(g')^A \|\varphi'\|.$$

With $K = \max(\|N\|_1, M)\, Y^2 \|\varphi\|\, \|\varphi'\|$ this gives, for all $g, g' \in \mathrm{G}(s)$,

$$\begin{aligned} \int_{[\mathrm{U}(T)]} |\theta(\varphi)(g,h)\, \theta(\varphi')(g',h)|\, dh &\le K\, \mathrm{ht}_S(g)^A \mathrm{ht}_S(g')^A, \\ \sum_{\gamma \in \Gamma_S} |\langle \varphi', \omega(g'^{-1}\gamma g, 1)\varphi\rangle| &\le K\, \mathrm{ht}_S(g)^A \mathrm{ht}_S(g')^A. \end{aligned} \tag{6.122}$$

Step 4: integration; proof of the first half of (1). By the decay hypothesis there are B, B' with $|y|\, \mathrm{ht}_S^A \le B$ and $|y'|\, \mathrm{ht}_S^A \le B'$ on $\mathrm{G}(s)$. The functions

$$(g', g, h) \mapsto \overline{y'(g')\theta(\varphi')(g',h)}\, y(g)\theta(\varphi)(g,h)$$

are continuous ((2) of Lemma 4.1.1, Θ bounded) and left invariant under $\Gamma_S \times \Gamma_S \times \Gamma_T$ (the invariance of y, y' and Lemma 6.2.5), so they are functions on $[\mathrm{U}(S)] \times [\mathrm{U}(S)] \times [\mathrm{U}(T)]$. By Tonelli and (6.122) the integral of their absolute value is at most $K\, B\, B'\, \mathrm{vol}([\mathrm{U}(S)])^2$, which is finite since $[\mathrm{U}(S)]$ has finite volume (Theorem 4.2.18; for $\#n = 0$ the group is trivial). This is the first half of (1). For every h the integral defining $(\Theta_\varphi y)(h)$ converges absolutely, as $|\theta(\varphi)(g,h)| \le Y\|\varphi\|(\mathrm{ht}_S(g)\mathrm{ht}_T(h))^A$ ((1) of Lemma 6.2.28) and $|y|\mathrm{ht}_S^A \le B$; by Fubini and (6.121),

$$\begin{aligned} \langle \Theta_{\varphi'} y', \Theta_\varphi y\rangle_T &= \int_{[\mathrm{U}(S)]^2} \overline{y'(g')}\, y(g) \int_{[\mathrm{U}(T)]} \overline{\theta(\varphi')(g',h)}\, \theta(\varphi)(g,h)\, dh\, dg\, dg' \\ &= \mathrm{vol}([\mathrm{U}(T)]) \int_{[\mathrm{U}(S)]^2} \overline{y'(g')}\, y(g) \sum_{\gamma \in \Gamma_S} \langle \varphi', \omega(g'^{-1}\gamma g, 1)\varphi\rangle\, dg\, dg'. \end{aligned}$$

Step 5: unfolding; proof of the second half of (1) and of (2). The quotient measure of $[\mathrm{U}(S)]$ is defined by $\int_{[\mathrm{U}(S)]} \sum_{\gamma\in\Gamma_S} \Phi(\gamma g)\, dg = \int_{\mathrm{G}(s)} \Phi(x)\, dx$ for nonnegative measurable Φ, hence for integrable Φ, and dx is left invariant. Fix g' and apply this to $\Phi(x) = y(x)\langle \varphi', \omega(g'^{-1}x, 1)\varphi\rangle$, then substitute $x \mapsto g'x$: as $y(\gamma g) = y(g)$,

$$\int_{[\mathrm{U}(S)]} y(g) \sum_{\gamma\in\Gamma_S} \langle \varphi', \omega(g'^{-1}\gamma g, 1)\varphi\rangle\, dg = \int_{\mathrm{G}(s)} y(g'x)\, \langle \varphi', \omega(x,1)\varphi\rangle\, dx,$$

first for the absolute values, which by (6.122) and $|y|\mathrm{ht}_S^A \le B$ gives a finite value at most $K\, B\, \mathrm{vol}([\mathrm{U}(S)])\, \mathrm{ht}_S(g')^A$, then as stated. Multiplying by $\overline{y'(g')}$, the function

$$(g', x) \mapsto \overline{y'(g')}\, y(g'x) \langle \varphi', \omega(x,1)\varphi\rangle$$

is integrable on $[\mathrm{U}(S)] \times \mathrm{G}(s)$ (its integral in absolute value is at most $K\, B\, B'\, \mathrm{vol}([\mathrm{U}(S)])^2$; it is invariant under $g' \mapsto \gamma g'$), and Fubini gives

$$\langle \Theta_{\varphi'} y', \Theta_\varphi y\rangle_T = \mathrm{vol}([\mathrm{U}(T)]) \int_{\mathrm{G}(s)} \langle \varphi', \omega(x,1)\varphi\rangle \int_{[\mathrm{U}(S)]} \overline{y'(g')}\, y(g'x)\, dg'\, dx.$$

The inner integral is $\langle y', R(x)y\rangle_S$, and the integrability of the function of (g', x) shows that $x \mapsto \langle \varphi', \omega(x,1)\varphi\rangle \langle y', R(x)y\rangle_S$ is integrable on $\mathrm{G}(s) = \mathrm{U}(S)(\mathbb{A}_F)$, with dx the measure dg of the statement. This is the second half of (1), and the display is (2). $\square$

Lemma 6.5.14. *Let S, T be spaces with signs s on n and s' on n', α, β automorphic characters of $\mathrm{U}(1)$, $\omega = \omega_{\alpha,\beta}$, and $W = S \oplus S^-$, with matrix $\mathrm{diag}(J_S, -J_S)$, so that $W(E) = S(E) \oplus S^-(E) = E^n \oplus E^n$. Let $\varphi, \varphi' \in \mathcal{M}_{S,T}$ and suppose that either all signs of S agree, or there are a finite set A of finite places of E containing the conjugate of none of its members, a place $w \in A$, a function $\Psi \in \mathcal{T}_A$ with $\Psi(x) \neq 0$ only if the $n' \times n$ matrix x_w over E_w has a left inverse, and $\varphi_0 \in \mathcal{M}_{S,T}$ with $\varphi = \Psi \otimes_A \varphi_0$. Let $g \in \mathrm{U}(S)(\mathbb{A}_F)$ and $\varphi_{\mathrm{d}} = \mathrm{dbl}_{c_{S,T}}(\omega(g,1)\varphi, \varphi')$. If L is a rational Lagrangian of W with $Z_L(\varphi_{\mathrm{d}}) \neq 0$, then*

$$L \cap (S \oplus 0) = 0 = L \cap (0 \oplus S^-),$$

and there is a unique $\delta \in \mathrm{U}(S)(F)$ with $L = \{(\delta y, y) : y \in S(E)\}$.

Proof. **Step 1: standard properties used.** The following are properties of the Weil and Heisenberg representations ([322]; for unitary dual pairs [121], Section 1):

(a) for $g \in \mathrm{U}(S)(\mathbb{A}_F)$ and $h \in \mathrm{Hom}(S,T)(\mathbb{A}_E)$, $\omega(g,1)\,\varrho(h)\,\omega(g,1)^{-1} = \varrho(h \circ g^{-1})$ (the splitting characters only multiply $\omega(g,1)$ by a scalar);

(b) for $h \in \mathrm{Hom}(S,T)(\mathbb{A}_E)$, the Heisenberg operator of $(h,0) \in \mathrm{Hom}(W,T)(\mathbb{A}_E)$ acts on a doubled datum through its first factor: $\varrho_W((h,0))$ applied to the doubled datum of (φ_1, φ_1') is the doubled datum of $(\varrho(h)\varphi_1, \varphi_1')$;

(c) the term Z_L is invariant under $\varrho_W(k)$ for every $k \in \mathrm{Hom}(W,T)(\mathbb{A}_E)$ vanishing on L: write $Z_L(d) = (\omega_W(\gamma)d)(0)$ in the Schrödinger model on $\mathrm{Hom}(W^+,T)(\mathbb{A}_E)$ of a polarisation $W = W^+ \oplus W^-$, with $\gamma \in \mathrm{U}(W)(F)$ and $\gamma^{-1}W^+ = L$; then $\omega_W(\gamma)\varrho_W(k)\omega_W(\gamma)^{-1} = \varrho_W(k \circ \gamma^{-1})$, and $k \circ \gamma^{-1}$ vanishes on W^+, so it acts in this model by multiplication by a character of value 1 at 0;

(d) at a finite place v split as $w\bar{w}$, the operator $\varrho(h)$ of an h whose only nonzero component is at $\bar{w}$ acts in the Schrödinger model by multiplication by $x_w \mapsto \psi_{\mathrm{add},v}(B_v(h_{\bar{w}}, x_w))$, where B_v is the restriction of the symplectic form to $\mathrm{Hom}(S,T)(E_{\bar{w}}) \times \mathrm{Hom}(S,T)(E_w)$, a perfect pairing. For h of rank one, $h(x) = z\langle x, u\rangle_S$ ($z \in T(E_{\bar{w}})$, $u \in S(E_w)$), one has $B_v(h, x_w) = \lambda_v(z, x_w u)$ with $\lambda_v \colon T(E_{\bar{w}}) \times T(E_w) \to F_v$ a perfect F_v-bilinear pairing (the trace of the form of T multiplied by the fixed purely imaginary element defining the symplectic form).

Step 2: $L \cap (S \oplus 0) = 0$. Suppose $(u,0) \in L$ with $u \in S(E)$, $u \neq 0$. Since L is isotropic, $\langle u, u\rangle_S = 0$. If all signs of S agree, then at every real place the form of S is definite; as $S(E)$ embeds in its completions, S is anisotropic and $u = 0$, a contradiction. Otherwise $\varphi = \Psi \otimes_A \varphi_0$ as in the statement; every $w' \in A$ lies over a split place, A containing no two conjugate places; Ψ, a finite combination of the functions τ_y of (WR6), is a locally constant, compactly supported function of $x_A = (x_{w'})_{w' \in A}$; and $\Psi(x_A) \neq 0$ only if x_{w_0} has a left inverse, $w_0 \in A$ the place of the statement, over v_0. Define the linear functional $\ell(\varphi_1) = Z_L(\varphi_{\mathrm{d},\varphi_1})$, $\varphi_{\mathrm{d},\varphi_1}$ the doubled datum of $(\omega(g,1)\varphi_1, \varphi')$. For $z \in T(\mathbb{A}_E)$ let $k_z \in \mathrm{Hom}(S,T)(\mathbb{A}_E)$ be $k_z(x) = z\langle x, u\rangle_S$, and $h_z = k_z \circ g$, i.e. $h_z(x) = z\langle x, u_g\rangle_S$ with $u_g = g^{-1}u$. The map $(x,y) \mapsto k_z(x) = z\langle (x,y), (u,0)\rangle_W$ vanishes on L, because $(u,0) \in L$ and L is isotropic. By (a), (b), (c),

$$\ell(\varrho(h_z)\varphi_1) = Z_L\big(\varrho_W((k_z,0))\,\varphi_{\mathrm{d},\varphi_1}\big) = \ell(\varphi_1) \qquad \text{for all } z \text{ and } \varphi_1.$$

Now take z with only one nonzero component, at $\bar{w}_0$. The component u_{g,w_0} is nonzero, since g is invertible and $u \neq 0$. For x_A in the support of Ψ, x_{w_0} is injective, so $x_{w_0}u_{g,w_0} \neq 0$, and as λ_{v_0} is perfect and ψ_{add,v_0} is nontrivial there is z with $\psi_{\mathrm{add},v_0}(\lambda_{v_0}(z, x_{w_0}u_{g,w_0})) \neq 1$. By (d), $\varrho(h_z)$

multiplies data by the locally constant function $\chi_z(x_{w_0}) = \psi_{\mathrm{add},v_0}(\lambda_{v_0}(z, x_{w_0}u_{g,w_0}))$; it acts at the place v_0 only, and commutes with multiplication by functions of x_A. The support of Ψ is compact and Ψ is locally constant, so $\Psi = \sum_j c_j \mathbf{1}_{B_j}$, a finite sum over pairwise disjoint compact open boxes B_j, each so small that χ_{z_j} is constant on B_j with value $\epsilon_j \neq 1$ for a suitable z_j; the functions $x \mapsto \mathbf{1}_{B_j}(x_A)$ lie in $\mathcal{T}_A$ by Lemma 6.2.22, and $\Psi \mapsto \Psi \otimes_A \varphi_0$ is linear (WR6), so $\varphi = \sum_j c_j\, \mathbf{1}_{B_j} \otimes_A \varphi_0$. Then $\varrho(h_{z_j})(\mathbf{1}_{B_j} \otimes_A \varphi_0) = \epsilon_j\, \mathbf{1}_{B_j} \otimes_A \varphi_0$, and the invariance gives $\ell(\mathbf{1}_{B_j} \otimes_A \varphi_0) = \epsilon_j\, \ell(\mathbf{1}_{B_j} \otimes_A \varphi_0)$, so $\ell(\mathbf{1}_{B_j} \otimes_A \varphi_0) = 0$ for every j, and $Z_L(\varphi_{\mathrm{d}}) = \ell(\varphi) = 0$, a contradiction. Hence $L \cap (S \oplus 0) = 0$.

Step 3: L is a graph. By Step 2 the projection $L \to S$, $(x, y) \mapsto y$ onto the second summand is injective; as $\dim L = \operatorname{rk} S$ it is bijective, so $L = \{(\delta y, y) : y \in S(E)\}$ for a unique E-linear map $\delta\colon S(E) \to S(E)$. The form of W vanishes on L: $\langle \delta y, \delta y' \rangle_S = \langle y, y' \rangle_S$ for all y, y', so δ is an isometry of the nondegenerate space S, hence invertible and $\delta \in \mathrm{U}(S)(F)$. Finally $L \cap (0 \oplus S^-) = \{(\delta y, y) : \delta y = 0\} = 0$. □

Proposition 6.5.15. *Let W be a space with a point with signs on an index set m_W, p an index set with $\#m_W = 2\#p$ such that W has a splitting over the zero space $V_\emptyset$ along p, and T a space with a point with signs t on n_T and $\#n_T > 2\#p$. Then every rational Lagrangian L of W is split along p, so that its term $Z_L\colon \mathcal{M}_{W,T} \to \mathbb{C}$ is defined, and:*

1. *the operator norms satisfy $\sum_L \|Z_L\| < \infty$, the sum over the rational Lagrangians of W;*
2. *there is a measurable function $N\colon [\mathrm{U}(T)] \to [0, \infty]$ with finite integral such that*

$$|\theta^{W,T}_{\upsilon^{\#p},1}(d)(1, h)| \leq N(\Gamma_T h)\, \|d\|$$

   for all $h \in \mathrm{G}(t)$ and $d \in \mathcal{M}_{W,T}$;
3. *for every $d \in \mathcal{M}_{W,T}$ the function $h \mapsto \theta^{W,T}_{\upsilon^{\#p},1}(d)(1, h)$ is integrable on $[\mathrm{U}(T)]$ and*

$$\int_{[\mathrm{U}(T)]} \theta^{W,T}_{\upsilon^{\#p},1}(d)(1, h)\, dh = \mathrm{vol}([\mathrm{U}(T)]) \sum_L Z_L(d).$$

Proof. **The terms.** By hypothesis W has a splitting D_0 over $V_\emptyset$ along p. By (1) of Lemma 6.5.16, every rational Lagrangian L of W is split along p, so Z_L is defined (MX13); by (3) of Lemma 6.5.16, $Z_D = Z_L$ for every splitting $D = (k, l, \emptyset)$ of W over $V_\emptyset$ along p with $kE^p = L$.

Clause (1) is Lemma 6.5.17, and **clause (2)** is Lemma 6.5.18.

Clause (3). Let $d \in \mathcal{M}_{W,T}$. The function $h \mapsto \theta^{W,T}_{\upsilon^{\#p},1}(d)(1, h) = \theta_{\mathrm{Hom}(W,T)}(\omega_{\upsilon^{\#p},1}(1, h)d)$ is continuous on $\mathrm{G}(t)$ ((2) of Lemma 4.1.1; the theta functional is bounded) and invariant under left translation by Γ_T (Lemma 6.2.5), hence a measurable function on $[\mathrm{U}(T)]$; by clause (2) it is dominated by $N\, \|d\|$, so it is integrable. The series $\sum_L Z_L(d)$ converges absolutely, since $|Z_L(d)| \leq \|Z_L\|\, \|d\|$ and clause (1). The identity of clause (3) is Lemma 6.5.21. □

Lemma 6.5.16. *Let W be a space with a point with signs on an index set m_W, p an index set with $\#m_W = 2\#p$, $D_0 = (k_0, l_0, i_0)$ a splitting of W over the zero space $V_\emptyset$ along p (i_0 empty), $L_0 = k_0E^p$, and $\mathrm{Stab}(L_0) = \{\delta \in \mathrm{U}(W)(F) : \delta L_0 = L_0\}$. Let T be a space with a point.*

1. *For every rational Lagrangian L of W there is $\delta \in \mathrm{U}(W)(F)$ with $\delta L_0 = L$; for every such δ, $(\delta k_0, \delta l_0, i_0)$ is a splitting of W over $V_\emptyset$ along p with $\delta k_0 E^p = L$; in particular every rational Lagrangian of W is split along p. The map $\delta \mapsto \delta L_0$ induces a bijection from $\mathrm{U}(W)(F)/\mathrm{Stab}(L_0)$ onto the set of rational Lagrangians of W.*

2. *For $\delta \in \mathrm{U}(W)(F)$, all automorphic characters α, β of $\mathrm{U}(1)$ and $d \in \mathcal{M}_{W,T}$,*

$$Z_{\delta L_0}(d) = Z_{L_0}(\omega_{\alpha,\beta}(x_W^{-1}\delta^{-1}x_W, 1)\, d).$$

3. *For every splitting $D = (k, l, i_0)$ of W over $V_\emptyset$ along p, $Z_D = Z_{kE^p}$.*

Proof. **(1).** The form h_W is nondegenerate (J_W is invertible). Let L be a rational Lagrangian; L and L_0 have dimension $\#p$ (for L_0: k_0 has a left inverse $l_0^* J_W$, as $l_0^* J_W k_0 = 1$) and h_W vanishes on both ($k_0^* J_W k_0 = 0$). So every E-linear bijection $\sigma\colon L_0 \to L$ is an isometry between subspaces of (E^{m_W}, h_W), and by Witt's extension theorem (Theorem A.6.7) it extends to some $\delta \in \mathrm{U}(W)(F)$; then $\delta L_0 = L$. For any $\delta \in \mathrm{U}(W)(F)$ with $\delta L_0 = L$, the matrices $k = \delta k_0$, $l = \delta l_0$ satisfy $k^* J_W k = k_0^* J_W k_0 = 0$, $l^* J_W l = 0$, $l^* J_W k = 1$, the conditions on i_0 are empty and $\#m_W = 0 + 2\#p$; so (k, l, i_0) is a splitting of W over $V_\emptyset$ along p, with $kE^p = \delta L_0 = L$. Conversely every δL_0, $\delta \in \mathrm{U}(W)(F)$, is a rational Lagrangian (δ preserves h_W and dimensions). The map $\delta \mapsto \delta L_0$ is constant exactly on the cosets $\delta\mathrm{Stab}(L_0)$, by the definition of $\mathrm{Stab}(L_0)$, and it is onto by the first sentence; this is the bijection.

(3). Let $L = kE^p$, split along p by D, and $D_L = (k', l', i_0)$ the splitting fixed in (MX13), so $k'E^p = L$ and $k^* J_W k' = 0$ (L is isotropic). By (2) of Lemma 6.2.25 it suffices that the splittings of $\mathrm{Hom}(W, T)$ induced by D and D_L satisfy the same condition (their third frames are empty). Their k-frames are $\tilde{k} = J_T^{-1} \otimes (k^* J_W)^{\mathsf{T}}$, $\tilde{k}' = J_T^{-1} \otimes (k'^* J_W)^{\mathsf{T}}$ and the matrix of $\mathrm{Hom}(W, T)$ is $J_T \otimes (J_W^{-1})^{\mathsf{T}}$ (VP5); with $J_T^* = J_T$ and $J_W^{\mathsf{T}} = \overline{J_W}$,

$$\tilde{k}^*(J_T \otimes (J_W^{-1})^{\mathsf{T}})\tilde{k}' = J_T^{-1} \otimes \overline{k^* J_W}\, \overline{J_W^{-1}}\, J_W^{\mathsf{T}}\, \overline{k'} = J_T^{-1} \otimes \overline{k^* J_W k'} = 0.$$

So $Z_D = Z_{D_L} = Z_L$.

(2). Let $g = x_W^{-1}\delta x_W$, an element of Γ_W (VP2). By (1) and (3), $Z_{\delta L_0}$ and Z_{L_0} are the zero terms along the splittings of $\mathrm{Hom}(W, T)$ over $\mathrm{Hom}(V_\emptyset, T)$ induced by $(\delta k_0, \delta l_0, i_0)$ and by D_0. Their adelic k-frames are $b \mapsto x_T^{-1} J_T^{-1}\, b\, k^* J_W\, x_W$ for $k = \delta k_0$ and $k = k_0$ (the point of $\mathrm{Hom}(W, T)$ is $x_T \otimes (x_W^{-1})^{\mathsf{T}}$). As $\delta^* J_W = J_W \delta^{-1}$,

$$x_T^{-1} J_T^{-1}\, b\, (\delta k_0)^* J_W\, x_W = X_b\, g^{-1}, \qquad X_b = x_T^{-1} J_T^{-1}\, b\, k_0^* J_W\, x_W.$$

For $g' \in \mathrm{G}(s)$ the pair element of $(g', 1)$ is $Y \mapsto Y g'^{-1}$ (VP5) and the scalar $\beta(\det g')^{-1}$ cancels under conjugation, so (2) of Lemma 4.1.1 gives $\omega(g', 1)\rho(Y)\omega(g', 1)^{-1} = \rho(Y g'^{-1})$. With $g' = g$ and Lemma 6.2.5 ($g \in \Gamma_W$),

$$\theta(\rho(X_b g^{-1})d) = \theta(\omega(g, 1)\rho(X_b)\omega(g^{-1}, 1)d) = \theta(\rho(X_b)\, \omega(g^{-1}, 1)d).$$

The canonical theta functional of $\mathrm{Hom}(V_\emptyset, T)$ is the identity (WC1, rank 0), so by (MX4) both terms are the means of these functions over $b \in \mathrm{M}_{n' \times p}(\mathbb{A}_E)/\mathrm{M}_{n' \times p}(E)$, and $Z_{\delta L_0}(d) = Z_{L_0}(\omega(g^{-1}, 1)d)$ with $g^{-1} = x_W^{-1}\delta^{-1}x_W$. □

Lemma 6.5.17. *Let W be a space with a point with signs on an index set m_W, p an index set with $\#m_W = 2\#p$ such that W has a splitting over the zero space $V_\emptyset$ along p, and T a space with a point with signs t on n_T and $\#n_T > 2\#p$. Then every rational Lagrangian of W is split along p, and $\sum_L \|Z_L\| < \infty$, the sum over the rational Lagrangians L of W of the operator norms.*

Proof. If $p = \emptyset$, then W has rank 0, its only rational Lagrangian is 0, and its term is a bounded functional on $\mathcal{M}_{W,T} = \mathbb{C}$; the sum has one term. Let $p \neq \emptyset$. Fix a splitting $D_0 = (k_0, l_0, i_0)$ of W over $V_\emptyset$ along p, $L_0 = k_0 E^p$, adelic frames $k_\mathbb{A} = x_W^{-1} k_0$, $l_\mathbb{A} = x_W^{-1} l_0$, and $\mathrm{Stab}(L_0)$ as

in Lemma 6.5.16. By (1) and (3) of Lemma 6.5.16 every rational Lagrangian of W is split along p, and $Z_{L_0} = Z_{D_0}$ is the zero term along the splitting of $\mathrm{Hom}(W,T)$ induced by D_0. Let $\mathrm{G}_{L_0} = \{u \in \mathrm{G}(s) : uk_{\mathbb{A}} = k_{\mathbb{A}} m_u \text{ for some } m_u \in \mathrm{GL}_p(\mathbb{A}_E)\}$; then $m_u = \mathrm{pr}_k\, u\, k_{\mathbb{A}}$ ($\mathrm{pr}_k k_{\mathbb{A}} = l_0^* J_W k_0 = 1$), so $u \mapsto |\det m_u|$ is a continuous homomorphism $\mathrm{G}_{L_0} \to (0,\infty)$, and $\iota(\mathrm{G}_{L_0})$ is the group $\mathrm{Stab}(L_0)(\mathbb{A}_F)$ of adelic points of the stabiliser $\mathrm{Stab}(L_0)$ of L_0 in $\mathrm{U}(W)$ (an algebraic group over F whose rational points form the group $\mathrm{Stab}(L_0)$ of Lemma 6.5.16); m_u is the matrix, in the basis k_0, of the action of $\iota(u)$ on $L_0 \otimes \mathbb{A}_E$.

Step 1: an Iwasawa decomposition. By (1) of Lemma 6.2.46 there is $g_D \in \mathrm{G}(s)$ at which D_0 is adapted in the basis 1: $g_D^{-1}(l_{\mathbb{A}} \,|\, k_{\mathbb{A}})$ has orthonormal columns at the infinite places and entries in $\widehat{\mathcal{O}}_E$ at the finite places. So D_0 is a base-adapted splitting of the moved space $W_{g_D} = (J_W, x_W g_D)$ (same signs), whose adelic frames are $g_D^{-1} l_{\mathbb{A}}$, $g_D^{-1} k_{\mathbb{A}}$. Let $K = K_\infty(s) \times \mathbf{K}_0(m_W)$, a compact subgroup of $\mathrm{G}(s)$. For $g \in \mathrm{G}(s)$, Lemma 6.6.88 for W_{g_D} and D_0 writes $g_D^{-1} g = \mathrm{rad}'(c,b)\,\mathrm{lev}'(m,g_0)\,\kappa$ with $\kappa \in K$, the elements formed with the frames of W_{g_D} (here c and g_0 are empty). By (MX2), $\mathrm{lev}'(m,g_0)$ sends $g_D^{-1} k_{\mathbb{A}}$ to $g_D^{-1} k_{\mathbb{A}} m$, and $\mathrm{rad}'(c,b)$ fixes it (pr_l and pr_i of W_{g_D} vanish on it, by the relations of (MX1)). Hence

$$g = u\, g_D \kappa, \qquad u = g_D\, \mathrm{rad}'(c,b)\,\mathrm{lev}'(m,g_0)\, g_D^{-1} \in \mathrm{G}_{L_0}, \quad m_u = m, \quad \kappa \in K. \tag{6.123}$$

Put $H(g) = |\det m_u|$ for any such decomposition: if $u_1 g_D \kappa_1 = u_2 g_D \kappa_2$, then $u_2^{-1} u_1 \in \mathrm{G}_{L_0} \cap g_D K g_D^{-1}$, a compact group, on which the continuous homomorphism $|\det m.|$ is 1; so H is well defined, and $H(ug) = |\det m_u|\, H(g)$ for $u \in \mathrm{G}_{L_0}$.

Step 2: the norm of a translated term. Let $C_K = \sup_{\kappa \in K} \|\omega(g_D\kappa, 1)\|$. It is finite: $\mathcal{M}_{W,T}$ is a Banach space ((1) of Lemma 4.1.10), for each d the map $\kappa \mapsto \omega(g_D\kappa,1)d$ is continuous on the compact set K ((2) of Lemma 4.1.1), hence bounded, and the uniform boundedness principle applies. The zero term Z_{L_0} along the induced splitting is bounded ((7) of Lemma 4.1.1). By the law of the zero term in (2) of Lemma 6.2.45 (its case $x = 0$, with $\alpha = \beta = 1$, $h = 1$; g_0 is trivial and $\omega_{1,1}(g_0,1)$ is the identity of $\mathcal{M}_{V_\emptyset,T} = \mathbb{C}$, and the character there has modulus one), $|Z_{L_0}(\omega(u,1)F)| = |\det m_u|^{n'/2} |Z_{L_0}(F)|$ for $u \in \mathrm{G}_{L_0}$, $F \in \mathcal{M}_{W,T}$. With (6.123) and as ω is a representation,

$$\big\|Z_{L_0} \circ \omega(g,1)\big\| \le C_K\, \|Z_{L_0}\|\, H(g)^{n'/2} \qquad (g \in \mathrm{G}(s)). \tag{6.124}$$

Step 3: from Lagrangians to cosets. By Lemma 6.5.16, $\delta \mapsto \delta L_0$ is a bijection from $\mathrm{U}(W)(F)/\mathrm{Stab}(L_0)$ onto the rational Lagrangians, and $Z_{\delta L_0} = Z_{L_0} \circ \omega(x_W^{-1}\delta^{-1} x_W, 1)$. As $\delta \mapsto \delta^{-1}$ carries left cosets to right cosets, with (6.124),

$$\begin{aligned}\sum_L \|Z_L\| &= \sum_{\gamma \in \mathrm{Stab}(L_0)\backslash \mathrm{U}(W)(F)} \big\|Z_{L_0} \circ \omega(x_W^{-1}\gamma x_W, 1)\big\| \\ &\le C_K\, \|Z_{L_0}\| \sum_{\gamma \in \mathrm{Stab}(L_0)\backslash \mathrm{U}(W)(F)} H(x_W^{-1}\gamma x_W)^{n'/2}.\end{aligned}$$

Step 4: convergence of the Siegel Eisenstein series. The space (E^{m_W}, h_W) is hermitian of dimension $2n$ and Witt index n (L_0 is isotropic of dimension n), and $\mathrm{Stab}(L_0)$ is the stabiliser of the maximal isotropic subspace L_0. Let K_1 be a maximal compact subgroup of $\mathrm{U}(W)(\mathbb{A}_F)$ with $\mathrm{U}(W)(\mathbb{A}_F) = \mathrm{Stab}(L_0)(\mathbb{A}_F) K_1$ (it exists, cited in passing as the standard Iwasawa decomposition), and $H_1(pk) = |\det a(p)|$ for $p \in \mathrm{Stab}(L_0)(\mathbb{A}_F)$, $k \in K_1$, $a(p)$ the action of p on $L_0 \otimes \mathbb{A}_E$; it is well defined because $|\det a|$ is 1 on the compact group $\mathrm{Stab}(L_0)(\mathbb{A}_F) \cap K_1$. Then $f_z(pk) = H_1(pk)^{z+n/2}$ is a K_1-finite flat section of $\mathrm{Ind}(|\det a|^z)$ with $\mu = 1$ and $f_0 = 1$ on K_1, and Theorem A.4.29 gives that $\sum_{\gamma \in \mathrm{Stab}(L_0)\backslash \mathrm{U}(W)(F)} H_1(\gamma)^{z+n/2}$ converges for real $z > n/2$. Take $z = (n'-n)/2$, which exceeds $n/2$ as $n' > 2n$: $\sum_\gamma H_1(\gamma)^{n'/2} < \infty$. Finally compare H and H_1. By Step 1, $\mathrm{U}(W)(\mathbb{A}_F) = \mathrm{Stab}(L_0)(\mathbb{A}_F) K_2$ with $K_2 = \iota(g_D K g_D^{-1})$, and $H(x_W^{-1} y x_W) = |\det a(p)|$

for $y = pk$, $p \in \mathrm{Stab}(L_0)(\mathbb{A}_F)$, $k \in K_2$, since m_u is the matrix of $a(\iota(u))$. If $y = pk = p_1k_1$ with $k \in K_2$, $k_1 \in K_1$, then $p_1^{-1}p = k_1k^{-1} \in \mathrm{Stab}(L_0)(\mathbb{A}_F) \cap K_1K_2$, a compact set ($K_1K_2$ is compact, $\mathrm{Stab}(L_0)(\mathbb{A}_F)$ closed), on which the continuous function $|\det a|$ is at most some $c_1 \geq 1$. Hence $H(x_W^{-1}yx_W) \leq c_1H_1(y)$, and

$$\sum_\gamma H(x_W^{-1}\gamma x_W)^{n'/2} \leq c_1^{n'/2} \sum_\gamma H_1(\gamma)^{n'/2} < \infty.$$

With Step 3 this proves $\sum_L \|Z_L\| < \infty$. □

Lemma 6.5.18. *Let W be a space with a point with signs on an index set m_W, p an index set with $\#m_W = 2\#p$ such that W has a splitting over the zero space $V_\emptyset$ ((MX13) of Definition 4.1.18) along p, and T a space with a point with signs t on n_T and $\#n_T > 2\#p$. There is a measurable function $N\colon [\mathrm{U}(T)] \to [0,\infty]$ with finite integral such that $|\theta^{W,T}_{\upsilon^{\#p},1}(d)(1,h)| \leq N(\Gamma_T h)\,\|d\|$ for all $h \in \mathrm{G}(t)$ and $d \in \mathcal{M}_{W,T}$.*

Proof. **Step 1: the case $p = \emptyset$.** Then $\#m_W = 0$, so $\mathrm{Hom}(W,T)$ has the empty index set, $\mathcal{M}_{W,T} = \mathbb{C}$ with $\|d\| = |d|$, its canonical theta functional is the identity (WC1), and $\omega(1,h)$ is the identity ($\upsilon^0 = 1$, and the Weil representation of the empty sign vector is trivial, (WR3)). So $|\theta^{W,T}_{1,1}(d)(1,h)| = \|d\|$ and $N = 1$ works, $\mathrm{vol}([\mathrm{U}(T)])$ being finite by Theorem 4.2.18 (trivial for $n' = \emptyset$). From now on $p \neq \emptyset$, and then $n' \neq \emptyset$.

Step 2: the theta kernel as a sum of slices. Fix a splitting $D = (k,l,i)$ of W over $V_\emptyset$ along p (i empty), with Hom slices $d[x] \in \mathcal{M}_{V_\emptyset,T} = \mathbb{C}$ for adelic $n' \times p$ matrices x. The canonical theta functional of $\mathrm{Hom}(V_\emptyset, T)$ is the identity (WC1, rank 0), so by (5) of Lemma 6.2.44, for every $F \in \mathcal{M}_{W,T}$,

$$\theta_{\mathrm{Hom}(W,T)}(F) = \sum_{\xi \in \mathrm{M}_{n' \times p}(E)} F[x_T^{-1}\xi],$$

absolutely convergent. By (2) of Lemma 6.2.45 with $g = 1$, $m = 1$, $g_0 = 1$, $\alpha = \beta = 1$ (there the matrix W is empty, $B = \mathrm{pr}_k l_{\mathbb{A}} = l^* J_W l = 0$, and the character is 1 at $\det m = 1$), and since $\omega_{1,1}(1,h)$ is the identity on the rank-0 model $\mathcal{M}_{V_\emptyset,T}$,

$$(\omega(1,h)d)[x] = d[h^{-1}x] \qquad (h \in \mathrm{G}(t)).$$

With (WC4), for all h and d,

$$|\theta^{W,T}_{\upsilon^{\#p},1}(d)(1,h)| = \Big|\sum_\xi d[h^{-1}x_T^{-1}\xi]\Big| \leq \sum_{\xi \in \mathrm{M}_{n' \times p}(E)} |d[h^{-1}x_T^{-1}\xi]|. \tag{6.125}$$

Step 3: Hom slices are slices of the induced splitting. Let $D_T = (k', l', i')$ be the splitting of $\mathrm{Hom}(W,T)$ over $\mathrm{Hom}(V_\emptyset, T)$ along $n' \times p$ induced by D, a splitting of a space with a point along a nonempty index set. Its adelic frame is $l'_{\mathbb{A}} = (x_T^{-1} \otimes x_W^{\mathsf{T}})(1 \otimes (l^* J_W)^{\mathsf{T}})$, which sends an adelic $n' \times p$ matrix X to $x_T^{-1} X\, l^* J_W x_W = x_T^{-1} X\, \mathrm{pr}_k$, because $\mathrm{pr}_k = l^*_{\mathbb{A}}\varepsilon = l^* x_W^{-*}\varepsilon = l^* J_W x_W$ ($x_W^* J_W x_W = \varepsilon$, ε the adelic sign matrix of W). Hence the slice of d along D_T at X, $Z_{D_T}(\rho(l'_{\mathbb{A}}X)d)$, is the Hom slice $d[x_T^{-1}X]$ (MX4).

Step 4: the majorant. Let $B \subseteq \mathbb{A}_E^{n' \times p}$ and M be the compact set and the constant of Lemma 6.5.19 for D_T. For $h \in \mathrm{G}(t)$ the map $\phi_h(\xi) = x_T h^{-1} x_T^{-1}\xi$ is a homomorphism $\mathrm{M}_{n' \times p}(E) \to \mathbb{A}_E^{n' \times p}$, and $d[h^{-1}x_T^{-1}\xi]$ is the slice of d along D_T at $\phi_h(\xi)$ by Step 3. By (6.125) and Lemma 6.5.19,

$$|\theta^{W,T}_{\upsilon^{\#p},1}(d)(1,h)| \leq M \cdot \#\phi_h^{-1}(B) \cdot \|d\| = M\, N_B(h)\, \|d\|,$$

with N_B the count of Lemma 6.5.20. By that lemma ($\#n' > 2\#p$) N_B is invariant under Γ_T, measurable, and has finite integral over $[\mathrm{U}(T)]$. So $N(\Gamma_T h) = M\,N_B(h)$ is the required function. □

Lemma 6.5.19. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set, and (k,l,i) a splitting of V over V_0 along p, with slices $f[x] \in \mathcal{M}_{s_0}$ ($f \in \mathcal{M}_s$, $x \in \mathbb{A}_E^p$). There are a compact set $B \subseteq \mathbb{A}_E^p$ and $M \geq 0$ such that for every abelian group I, every group homomorphism $\phi\colon I \to \mathbb{A}_E^p$ and every $f \in \mathcal{M}_s$*

$$\sum_{\xi\in I} \|f[\phi(\xi)]\| \leq M \cdot \#\phi^{-1}(B) \cdot \|f\|.$$

Proof. **Step 1: a pointwise bound for the slices.** By (1) of Lemma 6.2.46 there is $g \in \mathrm{G}(s)$ at which the splitting is adapted in the basis 1. The slice maps $\Pi_x\colon f \mapsto f[x] = Z(\rho(l_{\mathbb{A}}x)f)$ are bounded and linear (Z is bounded by (7) of Lemma 4.1.1 and $\rho(l_{\mathbb{A}}x)$ is an isometry), and by the second display of (1) of Lemma 6.2.44 they satisfy the hypothesis of (2) of Lemma 6.2.46. Hence, with $u = \omega_s(g)v^0$, there is $C \in \mathbb{C}$ with $u[x] = C\,|e_p(x)|^2 v^0$ for all $x \in \mathbb{A}_E^p$. Let $f \in \mathcal{M}_s$ and $h = \omega_s(g^{-1})f$. By (1) of Lemma 6.2.12, $h = \int h(z)\rho(z)v^0\,d\mu_n(z)$ as a Bochner integral; applying the bounded operator $\omega_s(g)$ and $\omega_s(g)\rho(z) = \rho(gz)\omega_s(g)$ ((2) of Lemma 4.1.1), and then the bounded map Π_y,

$$f[y] = \int h(z)\,(\rho(gz)u)[y]\,d\mu_n(z) \qquad (y \in \mathbb{A}_E^p).$$

Write $gz = l_{\mathbb{A}}a_z + i_{\mathbb{A}}w_z + k_{\mathbb{A}}b_z$ with $a_z = \mathrm{pr}_l(gz)$, $w_z = \mathrm{pr}_i(gz)$, $b_z = \mathrm{pr}_k(gz)$. By (1) of Lemma 6.2.44, $(\rho(gz)u)[y]$ is a number of modulus one times $\rho(w_z)u[y+a_z]$, and $\rho(w_z)$ is an isometry; so $\|(\rho(gz)u)[y]\| = |C|\,|e_p(y+a_z)|^2\,\|v^0\|$ and

$$\|f[y]\| \leq |C|\,\|v^0\| \int |h(z)|\,|e_p(y+a_z)|^2\,d\mu_n(z) \qquad (y \in \mathbb{A}_E^p). \tag{6.126}$$

Step 2: summation. Let I, ϕ be as in the statement and $K(\phi) = \sup_{a\in\mathbb{A}_E^p} \sum_{\xi\in I} |e_p(\phi(\xi)+a)|^2 \in [0,\infty]$. Summing (6.126) over $y = \phi(\xi)$ and exchanging sum and integral (Tonelli),

$$\begin{aligned}\sum_{\xi\in I} \|f[\phi(\xi)]\| &\leq |C|\,\|v^0\| \int |h(z)| \sum_{\xi\in I} |e_p(\phi(\xi)+a_z)|^2\,d\mu_n(z)\\ &\leq |C|\,\|v^0\|\,K(\phi)\,\|h\| \leq |C|\,\|v^0\|\,\|\omega_s(g^{-1})\|\,K(\phi)\,\|f\|.\end{aligned}$$

Step 3: $K(\phi)$ is at most a constant times a count. By (WD5),

$$|e_p(x)|^2 = \exp(-Q(x_\infty))\,\mathbf{1}[x_f \in \alpha\widehat{\mathcal{O}}_E^p] \quad \text{with} \quad Q(y) = 2\pi \sum_{j\in p}\sum_{v\in\Phi} \mathrm{Im}(\jmath_v)|y_{j,v}|^2,$$

a positive definite quadratic form on $\mathbb{C}^{\Phi\times p} = \mathbb{R}^{2r\#p}$ ($\mathrm{Im}\,\jmath_v > 0$, (WD1)); so $Q(y) \geq c_Q|y|^2$ for some $c_Q > 0$. Let $D = [0,1]^{2r\#p}$ and, for $t \in \mathbb{Z}^{2r\#p}$, $c_t = \sup_{y\in t+D} e^{-Q(y)}$. On $t+D$, $|y| \geq |t| - \sqrt{2r\#p}$, so $c_t \leq \exp(-c_Q \max(0, |t| - \sqrt{2r\#p})^2)$ and $K_0 = \sum_t c_t < \infty$. Put $B_0 = \{x : x_\infty \in D,\ x_f \in \alpha\widehat{\mathcal{O}}_E^p\}$ and

$$B = B_0 - B_0 = \{x : x_\infty \in [-1,1]^{2r\#p},\ x_f \in \alpha\widehat{\mathcal{O}}_E^p\},$$

compact ($\alpha\widehat{\mathcal{O}}_E^p$ is a compact group). As the translates $t+D$ cover $\mathbb{R}^{2r\#p}$, $|e_p(x)|^2 \le \sum_t c_t \mathbf{1}_{t+B_0}(x)$, where t translates the archimedean coordinates. Hence for every a

$$\sum_{\xi\in I} |e_p(\phi(\xi)+a)|^2 \le \sum_t c_t \,\#\{\xi\in I : \phi(\xi)\in t+B_0-a\} \le K_0\,\#\phi^{-1}(B) :$$

if ξ_0 lies in the set counted, then $\xi \mapsto \xi-\xi_0$ maps it injectively into $\phi^{-1}(B)$, as $\phi(\xi-\xi_0) = \phi(\xi)-\phi(\xi_0) \in B_0 - B_0$. So $K(\phi) \le K_0\,\#\phi^{-1}(B)$, and the statement holds with this B and $M = |C|\,\|v^0\|\,\|\omega_s(g^{-1})\|\,K_0$, which depend only on the splitting. □

Lemma 6.5.20. *Let T be a space with a point with signs t on n', p an index set with $\#n' > 2\#p$, and $B \subseteq \mathrm{M}_{n'\times p}(\mathbb{A}_E)$ a compact set. For $h \in \mathrm{G}(t)$ put*

$$N_B(h) = \#\{\xi \in \mathrm{M}_{n'\times p}(E) :\ x_T h^{-1} x_T^{-1}\,\xi \in B\}.$$

Then $N_B(\gamma h) = N_B(h)$ for $\gamma \in \Gamma_T$, N_B is measurable, and $\int_{[\mathrm{U}(T)]} N_B(h)\,dh < \infty$.

Proof. **Step 1: invariance and measurability.** For $\gamma\in\Gamma_T$, $\gamma' = x_T\gamma x_T^{-1} \in \mathrm{U}(T)(F) \subseteq \mathrm{GL}_{n'}(E)$ and $x_T(\gamma h)^{-1}x_T^{-1}\xi = h'^{-1}\gamma'^{-1}\xi$; as $\xi\mapsto\gamma'^{-1}\xi$ is a bijection of $\mathrm{M}_{n'\times p}(E)$, $N_B(\gamma h) = N_B(h)$. For each ξ the set $\{h : h'^{-1}\xi \in B\}$ is closed (B is compact, hence closed, and $h\mapsto h'^{-1}\xi$ is continuous), so $N_B = \sum_\xi \mathbf{1}_{\{h:h'^{-1}\xi\in B\}}$ is a countable sum of measurable functions.

Step 2: the case $p = \emptyset$. Then $\mathrm{M}_{n'\times p}$ has one element, $N_B \le 1$, and $\int_{[\mathrm{U}(T)]} N_B \le \mathrm{vol}([\mathrm{U}(T)]) < \infty$ by Theorem 4.2.18 (for $n' = \emptyset$ the group is trivial). From now on $\#p \ge 1$.

Step 3: domination by a Schwartz–Bruhat function. The projections of B to the archimedean and to the finite components are compact. So there is $R>0$ with $|y_{v,ij}| \le R$ for all $y\in B$, $v\in\Phi$, $i\in n'$, $j\in p$, and, as $\mathbb{A}_{E,f}$ is the increasing union of the open subgroups $u^{-1}\widehat{\mathcal{O}}_E$ ($u\ge 1$ an integer), there is u_0 with $y_f \in u_0^{-1}\widehat{\mathcal{O}}_E^{n'\times p}$ for all $y\in B$. Read $\mathrm{M}_{n'\times p}(\mathbb{A}_E)$ as $T(\mathbb{A}_E)^p = T^p(\mathbb{A}_F)$, the space of p-tuples of vectors of the hermitian space $T = (E^{n'}, J_T)$. For the Schrödinger model of the Weil representation of the pair $(\mathrm{U}(W'),\mathrm{U}(T))$, W' a skew-hermitian space of dimension $2\#p$ and Witt index $\#p$ with a polarisation, on $\mathcal{S}(T^p(\mathbb{A}_F))$, let $\varphi_0 = \bigotimes_v \varphi_{0,v}$ with $\varphi_{0,v}$ the vector of the constant polynomial of the Fock model at $v\in\Phi$ and $\varphi_{0,f} = \mathbf{1}[y_f \in u_0^{-1}\widehat{\mathcal{O}}_E^{n'\times p}]$. Each $\varphi_{0,v}$ is a Gaussian $\exp(-q_v(y_v))$ with q_v a positive definite real quadratic form on $\mathrm{M}_{n'\times p}(\mathbb{C})$ (the Fock vacuum in a Schrödinger model, cited in passing: [322]; for unitary dual pairs [121], Section 1), and $\varphi_{0,f}$ is fixed by an open subgroup of the compact group $K_{G,f}$, hence K_G-finite. With $C_B = \exp(\sum_{v\in\Phi}\max_{|y_{ij}|\le R} q_v(y))$ we get $\mathbf{1}_B \le C_B\,\varphi_0$ pointwise, and $\varphi_0 \ge 0$.

Step 4: Weil's criterion. In this model $\mathrm{U}(T)(\mathbb{A}_F)$ acts linearly: $|(\omega(1,h')\varphi_0)(x)| = \varphi_0(h'^{-1}x)$ (the factor is a unitary character of $\det h'$). Hence, by Step 3,

$$N_B(h) \le C_B \sum_{\xi\in T^p(F)} \varphi_0(h'^{-1}\xi) = C_B \sum_{\xi\in T^p(F)} |(\omega(1,h')\varphi_0)(\xi)|.$$

The hypotheses of Theorem A.4.27 hold for $V = T$ of dimension $m = \#n'$ and Witt index r, and $n = \#p \ge 1$: $m > 2n > n$, and $r \le m/2$ (an isotropic subspace has dimension at most $m/2$), so $m - r \ge m/2 > n$; a Hecke character χ of E with $\chi|_{\mathbb{A}_F^\times} = \eta^m$ exists (classical, cited in passing), and φ_0 is K_G-finite with archimedean component a polynomial vector of the Fock model. Its item (1) at $g=1$, absolute convergence of the theta integral, gives $\int_{[\mathrm{U}(T)]}\sum_\xi |(\omega(1,h')\varphi_0)(\xi)|\,dh' < \infty$ for the Haar measure of total mass 1 on $\mathrm{U}(T)(F)\backslash\mathrm{U}(T)(\mathbb{A}_F)$. The measure of $\mathrm{G}(t)$ is a Haar measure (a product of Cayley measures, which are Haar measures, and a Haar measure of $G_f(n')$, (VP1) of Definition 3.2.2), so its image under $h\mapsto h'$ is a Haar measure of $\mathrm{U}(T)(\mathbb{A}_F)$, and the quotient measure of $[\mathrm{U}(T)]$ is a positive multiple of the measure used there. Therefore $\int_{[\mathrm{U}(T)]} N_B(h)\,dh < \infty$. □

Lemma 6.5.21. *Let W be a space with a point with signs on an index set m_W, p an index set with $\#m_W = 2\#p$ such that W has a splitting over the zero space $V_\emptyset$ along p, and T a space with a point with signs t on n_T and $\#n_T > 2\#p$. For every $d \in \mathcal{M}_{W,T}$ the function $h \mapsto \theta^{W,T}_{\upsilon^{\#p},1}(d)(1,h)$ is integrable on $[\mathrm{U}(T)]$, the family $(Z_L(d))_L$ is absolutely summable, and*

$$\int_{[\mathrm{U}(T)]} \theta^{W,T}_{\upsilon^{\#p},1}(d)(1,h)\,dh = \mathrm{vol}([\mathrm{U}(T)]) \sum_L Z_L(d),$$

the sum over the rational Lagrangians L of W.

Proof. Write $\mathcal{M} = \mathcal{M}_{W,T}$, $\theta = \theta_{\mathrm{Hom}(W,T)}$, $\omega = \omega_{\upsilon^{\#p},1}$, so that $\theta^{W,T}_{\upsilon^{\#p},1}(d)(1,h) = \theta(\omega(1,h)d)$, and $m = \#n_T$.

Reduction to smooth vectors. By (1) of Lemma 6.5.16 every rational Lagrangian is split along p, so each Z_L is defined. The function $h \mapsto \theta(\omega(1,h)d)$ is continuous on $\mathrm{G}(t)$ ((2) of Lemma 4.1.1, θ bounded) and left invariant under Γ_T (Lemma 6.2.5), hence a measurable function on $[\mathrm{U}(T)]$; by Lemma 6.5.18 it is dominated by $N\|d\|$ with N integrable, so it is integrable. By Lemma 6.5.17, $\sum_L |Z_L(d)| \le \big(\sum_L \|Z_L\|\big)\|d\| < \infty$. The volume $v = \mathrm{vol}([\mathrm{U}(T)])$ is finite and positive by Theorem 4.2.18 ($[\mathrm{U}(T)]$ is $\mathrm{U}(T)(F)\backslash\mathrm{U}(T)(\mathbb{A}_F)$ through the point, (VP2) of Definition 3.2.2). Hence

$$\Lambda_1(d) = \int_{[\mathrm{U}(T)]} \theta(\omega(1,h)d)\,dh, \qquad \Lambda_2(d) = v\sum_L Z_L(d)$$

are linear functionals on $\mathcal{M}$ with $|\Lambda_1(d)| \le \|N\|_1\|d\|$ and $|\Lambda_2(d)| \le v\sum_L \|Z_L\|\,\|d\|$, so both are continuous. By (1) of Lemma 6.5.22 (for $S = W$) the smooth vectors $\mathcal{M}^\infty$ are dense in $\mathcal{M}$; it therefore suffices to prove $\Lambda_1(d) = \Lambda_2(d)$ for $d \in \mathcal{M}^\infty$.

The polarisation adapted to the splitting. Let $D_0 = (k_0, l_0, \emptyset)$ be the splitting of W over $V_\emptyset$ along p of the hypothesis, $L_0 = k_0E^p$, and $\mathrm{Stab}(L_0) \subset \mathrm{U}(W)(F)$ its stabiliser. By (MX1), $k_0^*J_Wk_0 = l_0^*J_Wl_0 = 0$, $l_0^*J_Wk_0 = 1$, the matrix $(l_0 \mid k_0)$ is invertible, and

$$J_W^{-1} = k_0l_0^* + l_0k_0^*.$$

The rational vectors of $\mathrm{Hom}(W,T)$ are $x_T^{-1}ux_W$ with $u \in \mathrm{M}_{n_T\times m_W}(E)$, and

$$B(x_T^{-1}ux_W, x_T^{-1}u'x_W) = \mathrm{Tr}_{E/F}(\jmath\,\mathrm{tr}(J_W^{-1}u'^*J_Tu))$$

($x_W\varepsilon x_W^* = J_W^{-1}$, $x_T^{-*}\varepsilon x_T^{-1} = J_T$ by (VP2) of Definition 3.2.2). Put

$$\mathbb{Y} = \{x_T^{-1}ux_W : uk_0 = 0\}, \qquad \mathbb{X} = \{x_T^{-1}ux_W : ul_0 = 0\}.$$

By the display, $\mathrm{tr}(J_W^{-1}u'^*J_Tu) = \mathrm{tr}(k_0^*u'^*J_Tul_0) + \mathrm{tr}(l_0^*u'^*J_Tuk_0)$, which vanishes if $uk_0 = u'k_0 = 0$ or if $ul_0 = u'l_0 = 0$; and $u \mapsto (ul_0, uk_0)$ is a bijection onto pairs of $n_T \times p$ matrices, under which $\mathbb{Y}$ and $\mathbb{X}$ are the pairs $(c,0)$ and $(0,c)$. So $(\mathbb{X},\mathbb{Y})$ is a polarisation in the sense of Lemma 6.5.22; let $\Lambda = \Lambda_{\mathbb{X}}\colon \mathcal{M}^\infty \to \mathcal{S}(\mathbb{X}_\mathbb{A})$ be the bijection of (2) of Lemma 6.5.22 for $S = W$, $\alpha = \upsilon^{\#p}$, $\beta = 1$.

The classical data. The splitting characters of $\omega^\mathbb{X}$ are $\beta[m] = \upsilon^{-m}$ on $\mathrm{U}(W)$ and $\alpha[\#m_W] = \alpha[2\#p]$ on $\mathrm{U}(T)$. Since $\upsilon(\bar{x}) = \upsilon(x)^{-1}$ (WD3), the base change of $\alpha = \upsilon^{\#p}$ is $\tilde{\alpha}(x) = \upsilon(x/\bar{x})^{\#p} = \upsilon(x)^{2\#p}$, so $\alpha[2\#p] = 1$. On $\mathbb{A}_F^\times$, υ is a character of $\mathbb{A}_F^\times/F^\times$ with $\upsilon^2 = 1$, trivial on the norms ($\upsilon(y\bar{y}) = 1$), and nontrivial (the idele equal to -1 at one infinite place and to 1 elsewhere has value -1 by (WD3)); by class field theory $\mathbb{A}_F^\times/F^\times\mathrm{N}(\mathbb{A}_E^\times)$ has order 2 with nontrivial character η, so $\upsilon|_{\mathbb{A}_F^\times} = \eta$ and $\chi = \upsilon^{-m}$ satisfies $\chi|_{\mathbb{A}_F^\times} = \eta^m$. In the dictionary of Lemma 4.1.2 (the tensor map $\mathcal{A}(v\otimes z) = \bar{v}\,\bar{z}^\mathsf{T}$ from the hermitian space $V = (E^{n_T}, h_V)$ and the

skew-hermitian space $W' = (E^{m_W}, h_W)$ of its statement, the pair (g,h) acting as $(\overline{g^{-\mathsf{T}}}, \bar{h})$), a pure tensor gives $\mathcal{A}(v\otimes z)k_0 = \bar{v}\,(\bar{z}^{\mathsf{T}}k_0)$, so $\mathcal{A}$ maps $V\otimes L$ into $\{u : uk_0 = 0\}$ and $V\otimes L'$ into $\{u : ul_0 = 0\}$, and the dimensions agree ($\#n_T \cdot \#p$); hence $\mathbb{Y} = \mathcal{A}(V\otimes L)$ and $\mathbb{X} = \mathcal{A}(V\otimes L')$ with

$$L = J_W k_0 E^p, \qquad L' = J_W l_0 E^p,$$

complementary maximal isotropic subspaces of W', and $g \mapsto \overline{g^{-\mathsf{T}}}$ maps $\mathrm{Stab}(L_0)$ onto the stabiliser $P(F)$ of L and $\mathrm{U}(W)(F)$ onto $\mathrm{U}(W')(F)$.

Thus the data of Theorem A.4.28 are: the hermitian space V of dimension m, the skew-hermitian space W' of dimension $2\#p$ and Witt index $\#p$, the Lagrangians $\mathbb{X} = V\otimes L'$, $\mathbb{Y} = V\otimes L$, the character χ, the trivial character on $\mathrm{U}(V)$, and $\omega = \omega^{\mathbb{X}}$; and $m > 2\#p$ by hypothesis.

The terms as values of a section. Let $d \in \mathcal{M}^\infty$. By (3) of Lemma 6.5.16, $Z_{L_0} = Z_{D_0}$. The splitting of $\mathrm{Hom}(W,T)$ induced by D_0 has k-frame $J_T^{-1}\otimes(k_0^* J_W)^{\mathsf{T}}$ (MX1), so its adelic frame sends $b \in \mathbb{A}_E^{n_T\times p}$ to $y_b = x_T^{-1}J_T^{-1}bk_0^* J_W x_W$; the space over which it splits has rank 0, its canonical functional is the identity of $\mathbb{C}$ (WC1), and (MX4) reads

$$Z_{L_0}(d) = \int_{\mathbb{A}_E^{n_T\times p}/E^{n_T\times p}} \theta(\rho(y_b)d)\,db.$$

Since $u = J_T^{-1}bk_0^*J_W$ satisfies $uk_0 = 0$ and $ul_0 = J_T^{-1}b$, the map $b\mapsto y_b$ is an $\mathbb{A}_F$-linear bijection of $\mathbb{A}_E^{n_T\times p}$ onto $\mathbb{Y}_{\mathbb{A}}$ carrying $E^{n_T\times p}$ onto $\mathbb{Y}$, hence the invariant probability measure of the first quotient to that of $\mathbb{Y}_{\mathbb{A}}/\mathbb{Y}$. By (4) of Lemma 6.5.22,

$$Z_{L_0}(d) = (\Lambda d)(0).$$

For $\delta \in \mathrm{U}(W)(F)$ put $g_\delta = x_W^{-1}\delta^{-1}x_W \in \Gamma_W$. By (2) of Lemma 6.5.16, (1) and (2) of Lemma 6.5.22,

$$Z_{\delta L_0}(d) = Z_{L_0}\big(\omega(g_\delta,1)d\big) = \big(\Lambda\,\omega(g_\delta,1)d\big)(0) = \big(\omega^{\mathbb{X}}(g_\delta,1)\Lambda d\big)(0).$$

By (1) of Lemma 6.5.16, $\delta\mapsto \delta L_0$ is a bijection of $\mathrm{U}(W)(F)/\mathrm{Stab}(L_0)$ onto the rational Lagrangians; with $\gamma = \delta^{-1}$ and the identification of the previous paragraph,

$$\sum_L Z_L(d) = \sum_{\gamma\in \mathrm{Stab}(L_0)\backslash \mathrm{U}(W)(F)} \big(\omega^{\mathbb{X}}(x_W^{-1}\gamma x_W,1)\Lambda d\big)(0) = E(1,\Lambda d),$$

the Eisenstein series of (1) of Theorem A.4.28 for $\Phi = \Lambda d$ (both series converge absolutely, by Lemma 6.5.17 and by (1) of Theorem A.4.28, and they have the same terms).

The theta integral. For $h\in \mathrm{G}(t)$, $\omega(1,h)d \in \mathcal{M}^\infty$ and, by (3) and (2) of Lemma 6.5.22,

$$\theta(\omega(1,h)d) = \theta_{\mathbb{X}}\big(\Lambda\omega(1,h)d\big) = \sum_{x\in\mathbb{X}} \big(\omega^{\mathbb{X}}(1,h)\Lambda d\big)(x).$$

Through the point and $h\mapsto \bar{h}$, $[\mathrm{U}(T)]$ is $\mathrm{U}(V)(F)\backslash\mathrm{U}(V)(\mathbb{A}_F)$ and $v^{-1}dh$ is the invariant probability measure. Hence (3) of Theorem A.4.28 at $g=1$ gives

$$\sum_L Z_L(d) = E(1,\Lambda d) = v^{-1}\int_{[\mathrm{U}(T)]} \theta(\omega(1,h)d)\,dh,$$

that is $\Lambda_2(d) = \Lambda_1(d)$ for $d\in\mathcal{M}^\infty$. By the reduction, $\Lambda_1 = \Lambda_2$ on $\mathcal{M}$, which is the identity of the statement. □

Lemma 6.5.22. *Let S, T be spaces with points with signs s on n and s' on n', α, β automorphic characters of $\mathrm{U}(1)$, $\mathcal{M} = \mathcal{M}_{S,T}$ and $\theta = \theta_{\mathrm{Hom}(S,T)}$. Read the vectors of $\mathbb{A}_E^{n'\times n}$ as $n' \times n$ matrices, so that the rational vectors of $\mathrm{Hom}(S,T)$ are $H_E = \{x_T^{-1} u x_S : u \in \mathrm{M}_{n'\times n}(E)\}$. Let ψ_F be the character of $\mathbb{A}_F/F$ with $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$, and*

$$B(z,w) = \mathrm{Tr}_{E/F}\big(\jmath\,\mathrm{tr}(\varepsilon_s w^* \varepsilon_{s'} z)\big) \qquad (z, w \in \mathbb{A}_E^{n'\times n}).$$

*A **polarisation** is a pair $(\mathbb{X}, \mathbb{Y})$ of F-subspaces of H_E with $H_E = \mathbb{X} \oplus \mathbb{Y}$ and $B(\mathbb{X},\mathbb{X}) = B(\mathbb{Y},\mathbb{Y}) = 0$; $\mathbb{X}_\mathbb{A}$, $\mathbb{Y}_\mathbb{A}$ denote the $\mathbb{A}_F$-submodules of $\mathbb{A}_E^{n'\times n}$ they span, $\mathbb{X}_\mathbb{A}$ carries the Haar measure with $\mathrm{vol}(\mathbb{X}_\mathbb{A}/\mathbb{X}) = 1$, and $\mathbb{Y}_\mathbb{A}/\mathbb{Y}$ its invariant probability measure. For a polarisation, $\rho^\mathbb{X}$ is the Schrödinger realisation on $L^2(\mathbb{X}_\mathbb{A})$ of Theorem A.5.3 for the character ψ_F and the pairing $B|_{\mathbb{X}_\mathbb{A}\times\mathbb{Y}_\mathbb{A}}$, $\theta_\mathbb{X}(\varphi) = \sum_{q\in\mathbb{X}} \varphi(q)$ for $\varphi \in \mathcal{S}(\mathbb{X}_\mathbb{A})$, and $\omega^\mathbb{X}(g,h)$ ($g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$) is the Weil representation $\omega_{\psi_F,\chi}$ of Theorem A.5.4 on $\mathcal{S}(\mathbb{X}_\mathbb{A})$ of the dual pair $(\mathrm{U}(S),\mathrm{U}(T))$ in the symplectic group of (H_E, B), with the splitting characters $\beta[\#n']$ on $\mathrm{U}(S)$ and $\alpha[\#n]$ on $\mathrm{U}(T)$, as in Lemma 4.1.2; here $\mathrm{G}(s)$, $\mathrm{G}(s')$ are identified with $\mathrm{U}(S)(\mathbb{A}_F)$, $\mathrm{U}(T)(\mathbb{A}_F)$ through the points. Let $\mathcal{M}^\infty$ be the set of $f \in \mathcal{M}$ for which $z \mapsto \rho(z)f$ is locally constant in the finite components of z and infinitely differentiable in its archimedean components. Then:*

1. *$\mathcal{M}^\infty$ is dense in $\mathcal{M}$, and $\rho(z)\mathcal{M}^\infty \subset \mathcal{M}^\infty$, $\omega_{\alpha,\beta}(g,h)\mathcal{M}^\infty \subset \mathcal{M}^\infty$ for all $z \in \mathbb{A}_E^{n'\times n}$, $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$;*
2. *for every polarisation $(\mathbb{X},\mathbb{Y})$ there is a linear bijection $\Lambda_\mathbb{X}\colon \mathcal{M}^\infty \to \mathcal{S}(\mathbb{X}_\mathbb{A})$ with*

$$\Lambda_\mathbb{X}\,\rho(z) = \rho^\mathbb{X}(z)\,\Lambda_\mathbb{X}, \qquad \Lambda_\mathbb{X}\,\omega_{\alpha,\beta}(g,h) = \omega^\mathbb{X}(g,h)\,\Lambda_\mathbb{X}$$

   for all z, g, h;

3. *$\theta(\rho(z)f) = \theta_\mathbb{X}(\rho^\mathbb{X}(z)\Lambda_\mathbb{X} f)$ for all z and $f \in \mathcal{M}^\infty$; in particular $\theta = \theta_\mathbb{X} \circ \Lambda_\mathbb{X}$ on $\mathcal{M}^\infty$;*
4. *$\displaystyle\int_{\mathbb{Y}_\mathbb{A}/\mathbb{Y}} \theta(\rho(y)f)\,dy = (\Lambda_\mathbb{X} f)(0)$ for every $f \in \mathcal{M}^\infty$.*

Proof. Write $\psi = \psi_F$, $\omega = \omega_{\alpha,\beta}$, $\mathbb{A}^N = \mathbb{A}_E^{n'\times n}$. Throughout, θ is a theta functional of $\mathrm{Hom}(S,T)$ in the sense of (WD9) by Lemma 4.1.17: bounded, and $\theta(\rho(q)f) = \theta(f)$ for $q \in H_E$.

The form. B is $\mathbb{A}_F$-bilinear and alternating, and $\beta_{s'\otimes s}(z,w) = \psi(\frac{1}{2}B(z,w))$. Indeed $\bar{w}^\mathsf{T} S z = \mathrm{tr}(\varepsilon_s w^* \varepsilon_{s'} z)$ for the sign matrix S of $\mathrm{Hom}(S,T)$ in (WD6), so

$$\beta_{s'\otimes s}(z,w) = \psi_E(\tfrac{\jmath}{2}\,\mathrm{tr}(\varepsilon_s w^* \varepsilon_{s'} z)) = \psi(\tfrac{1}{2}B(z,w));$$

and $B(w,z) = \mathrm{Tr}_{E/F}(\jmath\,\overline{\mathrm{tr}(\varepsilon_s w^* \varepsilon_{s'} z)}) = -B(z,w)$ because $\bar{\jmath} = -\jmath$. For $z = x_T^{-1} u x_S$, $w = x_T^{-1} u' x_S$ with u, u' rational,

$$B(z,w) = \mathrm{Tr}_{E/F}\big(\jmath\,\mathrm{tr}(J_S^{-1} u'^* J_T u)\big) \in F,$$

by $x_S \varepsilon_s x_S^* = J_S^{-1}$ and $x_T^{-*} \varepsilon_{s'} x_T^{-1} = J_T$ ((VP2) of Definition 3.2.2); so B is an F-valued nondegenerate alternating form on the F-space H_E, and $\mathbb{A}^N = H_E \otimes_F \mathbb{A}_F$ since the point is invertible. For a polarisation, B restricts to a perfect pairing $\mathbb{X} \times \mathbb{Y} \to F$, and $\mathbb{A}^N = \mathbb{X}_\mathbb{A} \oplus \mathbb{Y}_\mathbb{A}$. By (1) of Lemma 4.1.10 and the formula of Theorem A.5.3 the two families of operators have the same cocycle:

$$\rho(z)\rho(w) = \psi\big(\tfrac{1}{2}B(z,w)\big)\rho(z+w), \qquad \rho^\mathbb{X}(z)\rho^\mathbb{X}(w) = \psi\big(\tfrac{1}{2}B(z,w)\big)\rho^\mathbb{X}(z+w). \tag{1}$$

Consequently $\rho(w)\rho(z) = \psi(B(w,z))\rho(z)\rho(w)$.

Item (1). If $f \in \mathcal{M}^\infty$, then $\rho(w)\rho(z)f = \psi(B(w,z))\rho(z)\rho(w)f$ is smooth in the archimedean and locally constant in the finite components of w ($\rho(z)$ is bounded, and $w \mapsto \psi(B(w,z))$ is a character of this kind), so $\rho(z)f \in \mathcal{M}^\infty$. For $q = (g,h)$ with pair map P, (2) of Lemma 4.1.1 gives $\rho(z)\omega(q)f = \omega(q)\rho(P^{-1}z)f$; P^{-1} is $\mathbb{A}_E$-linear and preserves components, and $\omega(q)$ is bounded, so $\omega(q)f \in \mathcal{M}^\infty$. Density (Gårding): let dz be a Haar measure on $\mathbb{A}^N$ and $\kappa = \kappa_\infty \otimes \mathrm{vol}(K)^{-1}\mathbf{1}_K$ with $\kappa_\infty \geq 0$ smooth, compactly supported, of integral 1 on the archimedean part, and K a compact open subgroup of the finite part. For $f \in \mathcal{M}$ put $\rho(\kappa)f = \int \kappa(z)\rho(z)f\,dz$, a Bochner integral in $\mathcal{M}$ (ρ is strongly continuous and isometric by (1) of Lemma 4.1.10). By (1) and $B(z',z') = 0$,

$$\rho(z')\rho(\kappa)f = \int \kappa(z-z')\,\psi(\tfrac{1}{2}B(z',z))\rho(z)f\,dz.$$

The integrand is smooth in the archimedean part of z', with derivatives bounded uniformly for z' in a compact set and z in the compact support, so differentiation under the integral is allowed; and for z' in a compact open subgroup $K' \subset K$ of the finite part so small that $\psi(\frac{1}{2}B(K',C)) = 1$, C the finite projection of the support, the integrand does not depend on the finite part of z'. Hence $\rho(\kappa)f \in \mathcal{M}^\infty$. Since $\|\rho(\kappa)f - f\| \leq \int \kappa(z)\|\rho(z)f - f\|\,dz$, choosing κ_∞ and K with supports shrinking to 0 gives $\rho(\kappa)f \to f$ by strong continuity.

Theta functions. Let $\mathcal{H}_B$ be the space of measurable functions Θ on $\mathbb{A}^N$, modulo null functions, with $\Theta(w+q) = \psi(\frac{1}{2}B(w,q))\Theta(w)$ for $q \in H_E$ and almost all w, and with $\|\Theta\|^2 = \int_{\mathbb{A}^N/H_E} |\Theta|^2$ finite (the invariant probability measure; $|\Theta|$ is H_E-periodic). For $w' \in \mathbb{A}^N$ put $(R(w')\Theta)(w) = \psi(\frac{1}{2}B(w,w'))\Theta(w+w')$; $R(w')$ is a unitary operator of $\mathcal{H}_B$.

Let π be a family of linear operators on a vector space with the cocycle (1) and λ a linear functional with $\lambda \circ \pi(q) = \lambda$ for $q \in H_E$. Then $\Theta_v(w) = \lambda(\pi(w)v)$ satisfies

$$\Theta_v(w+q) = \psi(\tfrac{1}{2}B(w,q))\Theta_v(w), \qquad \Theta_{\pi(w')v} = R(w')\Theta_v, \tag{2}$$

because $\pi(w+q) = \psi(-\frac{1}{2}B(q,w))\pi(q)\pi(w)$ and $\pi(w)\pi(w') = \psi(\frac{1}{2}B(w,w'))\pi(w+w')$.

The theta model. For $f \in \mathcal{M}$ let $\vartheta_{\mathcal{M}}f(w) = \theta(\rho(w)f)$, a continuous function (ρ strongly continuous, θ bounded). By (2) and (2) of Lemma 4.1.10, $\vartheta_{\mathcal{M}}$ is a linear map $\mathcal{M} \to \mathcal{H}_B$ with $\langle \vartheta_{\mathcal{M}}f, \vartheta_{\mathcal{M}}f' \rangle = \langle f, f' \rangle$ and $\vartheta_{\mathcal{M}}\rho(w') = R(w')\vartheta_{\mathcal{M}}$.

A Schrödinger model. Let $(\mathbb{X},\mathbb{Y})$ be a polarisation. For $q = q_1 + q_2 \in H_E$ ($q_1 \in \mathbb{X}$, $q_2 \in \mathbb{Y}$) and $\varphi \in \mathcal{S}(\mathbb{X}_\mathbb{A})$, $\theta_\mathbb{X}(\rho^\mathbb{X}(q)\varphi) = \sum_{a\in\mathbb{X}} \psi(B(a,q_2) + \frac{1}{2}B(q_1,q_2))\varphi(a+q_1) = \theta_\mathbb{X}(\varphi)$, as $B(a,q_2), \frac{1}{2}B(q_1,q_2) \in F$. For $u \in \mathbb{X}_\mathbb{A}$, $v \in \mathbb{Y}_\mathbb{A}$,

$$\vartheta_\mathbb{X}\varphi(u+v) := \theta_\mathbb{X}(\rho^\mathbb{X}(u+v)\varphi) = \psi(\tfrac{1}{2}B(u,v)) \sum_{a\in\mathbb{X}} \psi(B(a,v))\varphi(u+a) = \psi(\tfrac{1}{2}B(u,v))\, Z_\psi\varphi(u,v),$$

with Weil's theta transform Z_ψ of Theorem A.5.5 for $X = \mathbb{X}$, $Y = \mathbb{Y}$ and the pairing B. The map $G \mapsto [\, u+v \mapsto \psi(\frac{1}{2}B(u,v))G(u,v)\,]$ is a unitary isomorphism of the space $\mathcal{H}_\beta$ of Theorem A.5.5 onto $\mathcal{H}_B$: it preserves $|G|$, the product of the probability measures on $\mathbb{X}_\mathbb{A}/\mathbb{X}$ and $\mathbb{Y}_\mathbb{A}/\mathbb{Y}$ is the invariant probability measure of $\mathbb{A}^N/H_E$, and the two quasi-periodicities correspond.

By Theorem A.5.5, $\vartheta_\mathbb{X}$ therefore extends from $\mathcal{S}(\mathbb{X}_\mathbb{A})$ to a unitary isomorphism $\vartheta_\mathbb{X}\colon L^2(\mathbb{X}_\mathbb{A}) \to \mathcal{H}_B$, and for Schwartz φ the displayed function is its continuous representative. By (2) for $\pi = \rho^\mathbb{X}$ on $\mathcal{S}(\mathbb{X}_\mathbb{A})$ (which $\rho^\mathbb{X}$ preserves) and by continuity, $\vartheta_\mathbb{X}\rho^\mathbb{X}(w') = R(w')\vartheta_\mathbb{X}$ on $L^2(\mathbb{X}_\mathbb{A})$. Put

$$\Lambda^0_\mathbb{X} = \vartheta_\mathbb{X}^{-1} \circ \vartheta_{\mathcal{M}}\colon \mathcal{M} \to L^2(\mathbb{X}_\mathbb{A});$$

it is linear, preserves inner products and satisfies $\Lambda^0_\mathbb{X}\rho(w') = \rho^\mathbb{X}(w')\Lambda^0_\mathbb{X}$.

The polarisation of the comparison. Apply Lemma 4.1.2 to S, T, α, β and the theta functional θ. Read in the coordinates of the point (under which its form is B by the display

above), it supplies a polarisation $(\mathbb{X}_1, \mathbb{Y}_1)$, a linear bijection $\Lambda_1 \colon \mathcal{M}^\infty \to \mathcal{S}(\mathbb{X}_{1,\mathbb{A}})$ with $\Lambda_1\rho(z) = \rho^{\mathbb{X}_1}(z)\Lambda_1$ and $\Lambda_1\omega(q) = \omega^{\mathbb{X}_1}(q)\Lambda_1$, and a constant $c_1 \neq 0$ with $\theta = c_1\,\theta_{\mathbb{X}_1} \circ \Lambda_1$ on $\mathcal{M}^\infty$. For $f \in \mathcal{M}^\infty$ and every w, since $\rho(w)f \in \mathcal{M}^\infty$ by (1),

$$\vartheta_{\mathcal{M}} f(w) = c_1\,\theta_{\mathbb{X}_1}(\Lambda_1\rho(w)f) = c_1\,\theta_{\mathbb{X}_1}(\rho^{\mathbb{X}_1}(w)\Lambda_1 f) = c_1\,\vartheta_{\mathbb{X}_1}(\Lambda_1 f)(w),$$

hence $\Lambda^0_{\mathbb{X}_1} = c_1\Lambda_1$ on $\mathcal{M}^\infty$.

Change of polarisation. Let $(\mathbb{X}, \mathbb{Y})$ be any polarisation and $\Psi = \vartheta_{\mathbb{X}}^{-1}\vartheta_{\mathbb{X}_1}$, a unitary isomorphism $L^2(\mathbb{X}_{1,\mathbb{A}}) \to L^2(\mathbb{X}_{\mathbb{A}})$ with $\Psi\rho^{\mathbb{X}_1}(w) = \rho^{\mathbb{X}}(w)\Psi$ and $\Lambda^0_{\mathbb{X}} = \Psi\Lambda^0_{\mathbb{X}_1}$. We show $\Psi\mathcal{S}(\mathbb{X}_{1,\mathbb{A}}) = \mathcal{S}(\mathbb{X}_{\mathbb{A}})$. Choose an F-basis (e_i) of $\mathbb{X}_1$ and the basis (f_i) of $\mathbb{Y}_1$ with $B(e_i, f_j) = \delta_{ij}$, and likewise (e_i'), (f_i') for $(\mathbb{X}, \mathbb{Y})$; the F-linear map σ with $\sigma e_i = e_i'$, $\sigma f_i = f_i'$ preserves B and H_E, and we extend it $\mathbb{A}_F$-linearly. Then:

- $\Psi_0\varphi = \varphi \circ \sigma^{-1}|_{\mathbb{X}_{\mathbb{A}}}$ maps $\mathcal{S}(\mathbb{X}_{1,\mathbb{A}})$ onto $\mathcal{S}(\mathbb{X}_{\mathbb{A}})$ (σ is defined over F), is unitary (σ carries the Haar measure of $\mathbb{X}_{1,\mathbb{A}}$ with quotient volume 1 to that of $\mathbb{X}_{\mathbb{A}}$), satisfies $\theta_{\mathbb{X}} \circ \Psi_0 = \theta_{\mathbb{X}_1}$ ($\sigma\mathbb{X}_1 = \mathbb{X}$), and $\Psi_0\rho^{\mathbb{X}_1}(w) = \rho^{\mathbb{X}}(\sigma w)\Psi_0$ (insert $B(\sigma\cdot, \sigma\cdot) = B$ in the formula of Theorem A.5.3).
- With $(C_\sigma\Theta)(w) = \Theta(\sigma^{-1}w)$, a unitary operator of $\mathcal{H}_B$ (σ preserves B, H_E and the invariant probability measure of the compact group $\mathbb{A}^N/H_E$), the first item gives $\vartheta_{\mathbb{X}}(\Psi_0\varphi)(w) = \theta_{\mathbb{X}}(\Psi_0\rho^{\mathbb{X}_1}(\sigma^{-1}w)\varphi) = \vartheta_{\mathbb{X}_1}\varphi(\sigma^{-1}w)$, that is $\vartheta_{\mathbb{X}}\Psi_0 = C_\sigma\vartheta_{\mathbb{X}_1}$ on $\mathcal{S}(\mathbb{X}_{1,\mathbb{A}})$, hence on L^2.
- $U = \vartheta_{\mathbb{X}}^{-1}C_\sigma^{-1}\vartheta_{\mathbb{X}}$ is unitary with $U\rho^{\mathbb{X}}(w)U^{-1} = \rho^{\mathbb{X}}(\sigma^{-1}w)$, because $C_\sigma^{-1}R(w)C_\sigma = R(\sigma^{-1}w)$. So (σ^{-1}, U) belongs to the metaplectic group of Theorem A.5.3, and U maps $\mathcal{S}(\mathbb{X}_{\mathbb{A}})$ onto itself.

Hence $\Psi = \vartheta_{\mathbb{X}}^{-1}C_\sigma^{-1}\vartheta_{\mathbb{X}}\Psi_0 = U\Psi_0$ maps $\mathcal{S}(\mathbb{X}_{1,\mathbb{A}})$ onto $\mathcal{S}(\mathbb{X}_{\mathbb{A}})$.

Item (2). Put $\Lambda_{\mathbb{X}} = \Lambda^0_{\mathbb{X}}|_{\mathcal{M}^\infty} = c_1\Psi\Lambda_1$. It is a linear bijection of $\mathcal{M}^\infty$ onto $\mathcal{S}(\mathbb{X}_{\mathbb{A}})$, and it intertwines ρ with $\rho^{\mathbb{X}}$ (both sides on $\mathcal{M}^\infty$, which ρ preserves by (1)). For the Weil representations, $\Lambda_{\mathbb{X}}\omega(q) = c_1\Psi\omega^{\mathbb{X}_1}(q)\Lambda_1 = \Psi\omega^{\mathbb{X}_1}(q)\Psi^{-1}\Lambda_{\mathbb{X}}$ on $\mathcal{M}^\infty$ (by (1), $\omega(q)$ preserves $\mathcal{M}^\infty$). By Theorem A.5.3 for the two realisations, $(\sigma', U') \mapsto (\sigma', \Psi U'\Psi^{-1})$ identifies the metaplectic group of the realisation on $L^2(\mathbb{X}_{1,\mathbb{A}})$ with that of the realisation on $L^2(\mathbb{X}_{\mathbb{A}})$ over the identity of the symplectic group; it does not depend on the unitary equivalence Ψ, which is unique up to a scalar by (i) of Theorem A.5.9 ($\rho^{\mathbb{X}_1}$ is generated by the translations and the characters $\psi(B(\cdot, v))$ of $\mathbb{X}_{1,\mathbb{A}}$, and $v \mapsto \psi(B(\cdot, v))$ identifies $\mathbb{Y}_{1,\mathbb{A}}$ with the dual group). Under this identification the metaplectic group, Kudla's splitting (i) of Theorem A.5.4 and the Weil representation (ii) of Theorem A.5.4 on the Schwartz space of an arbitrary Lagrangian are one and the same; hence $\Psi\omega^{\mathbb{X}_1}(q)\Psi^{-1} = \omega^{\mathbb{X}}(q)$, and $\Lambda_{\mathbb{X}}\omega(q) = \omega^{\mathbb{X}}(q)\Lambda_{\mathbb{X}}$.

Item (3). For $f \in \mathcal{M}^\infty$, $\vartheta_{\mathbb{X}}(\Lambda_{\mathbb{X}}f) = \vartheta_{\mathcal{M}}f$ in $\mathcal{H}_B$. Both $w \mapsto \theta_{\mathbb{X}}(\rho^{\mathbb{X}}(w)\Lambda_{\mathbb{X}}f)$ ($\Lambda_{\mathbb{X}}f$ is Schwartz, so this is $\psi(\frac{1}{2}B(u,v))Z_\psi(\Lambda_{\mathbb{X}}f)(u,v)$, continuous by Theorem A.5.5) and $w \mapsto \theta(\rho(w)f)$ are continuous; they agree almost everywhere, hence everywhere.

Item (4). For $y \in \mathbb{Y}_{\mathbb{A}}$, item (3) and the display above give $\theta(\rho(y)f) = Z_\psi(\Lambda_{\mathbb{X}}f)(0, y)$. The inverse formula of Theorem A.5.5, which holds pointwise for Schwartz functions, gives $\int_{\mathbb{Y}_{\mathbb{A}}/\mathbb{Y}} Z_\psi(\Lambda_{\mathbb{X}}f)(0,y)\,dy = (\Lambda_{\mathbb{X}}f)(0)$. □

Lemma 6.5.23. *Let S, T be spaces with points with signs s on n and s' on n', and $W = S \oplus S^-$, with signs $s \sqcup (-s)$ on $n \sqcup n$ and matrix* $\mathrm{diag}(J_S, -J_S)$. *For $\delta \in \mathrm{GL}_n(E)$ with $\delta^* J_S \delta = J_S$ let*

$$L_\delta = \{(\delta^{-1}u, -u) : u \in E^n\},$$

a rational Lagrangian of W. Then for all $f, f' \in \mathcal{M}_{S,T}$ and all automorphic characters α, β of $\mathrm{U}(1)$,

$$Z_{L_\delta}(\mathrm{dbl}_{c_{S,T}}(f, f')) = \langle f',\, \omega_{\alpha,\beta}(x_S^{-1}\delta x_S, 1)\, f\rangle.$$

Proof. Fix δ, f, f', α, β as in the statement and put $g_\delta = x_S^{-1}\delta x_S \in \Gamma_S$ and $\varphi_{\mathrm{d}} = \mathrm{dbl}_{c_{S,T}}(f, f') = f \boxtimes^{\mathrm{sc}} c_{S,T}\,\mathrm{cj}f'$ (WR7), a vector of $\mathcal{M}_{W,T}$ since $\mathrm{cj}f' \in \mathcal{M}_{S^-,T}$ ((1) of Lemma 6.4.56).

Step 1: L_δ is the Lagrangian of a splitting of W over the zero space. Let p be a copy of n and

$$k = \begin{pmatrix} \delta^{-1} \\ -1 \end{pmatrix}, \qquad l = \frac{1}{2}\begin{pmatrix} \delta^{-1} \\ 1 \end{pmatrix} J_S^{-1},$$

$(n \sqcup n) \times p$ matrices over E, and i the empty $(n \sqcup n) \times \emptyset$ matrix. With $J_W = \mathrm{diag}(J_S, -J_S)$, $J_S^* = J_S$ and $(\delta^{-1})^* J_S \delta^{-1} = J_S$ (which follows from $\delta^* J_S \delta = J_S$):

$$\begin{aligned} k^* J_W k &= (\delta^{-1})^* J_S \delta^{-1} - J_S = 0, \\ l^* J_W l &= \tfrac{1}{4} J_S^{-1}\big((\delta^{-1})^* J_S \delta^{-1} - J_S\big) J_S^{-1} = 0, \\ l^* J_W k &= \tfrac{1}{2} J_S^{-1}\big((\delta^{-1})^* J_S \delta^{-1} + J_S\big) = 1, \end{aligned}$$

the conditions on i are empty, and $\#(n \sqcup n) = 0 + 2\#p$. So $D_\delta = (k, l, i)$ is a splitting of W over the zero space $V_\emptyset$ along p, and $kE^p = \{(\delta^{-1}u, -u) : u \in E^n\} = L_\delta$. In particular L_δ has dimension $\#p$ and h_W vanishes on it ($k^* J_W k = 0$), so it is a rational Lagrangian of W, split along p, and by (3) of Lemma 6.5.16, Z_{L_δ} is the zero term Z along the splitting of $\mathrm{Hom}(W, T)$ over $\mathrm{Hom}(V_\emptyset, T)$ induced by D_δ.

Step 2: the term as a mean over adelic matrices. The space $\mathrm{Hom}(V_\emptyset, T)$ has the empty index set $n' \times \emptyset$; so $\mathcal{M}_{V_\emptyset,T} = \mathbb{C}$, $\rho(0) = 1$, and its canonical theta functional is the identity (WC1, rank 0). The induced splitting has k-frame $J_T^{-1} \otimes (k^* J_W)^{\mathsf{T}}$ (MX1), and the point of $\mathrm{Hom}(W, T)$ is $x_T \otimes (x_W^{-1})^{\mathsf{T}}$; so its adelic k-frame sends an adelic $n' \times p$ matrix b to $\tilde{k}_{\mathbb{A}} b = x_T^{-1} J_T^{-1}\, b\, k^* J_W\, x_W$. By (MX4), as $n_0 = \emptyset$,

$$Z_{L_\delta}(\varphi_{\mathrm{d}}) = \int_{\mathrm{M}_{n' \times p}(\mathbb{A}_E)/\mathrm{M}_{n' \times p}(E)} \theta_{\mathrm{Hom}(W,T)}\big(\rho(\tilde{k}_{\mathbb{A}} b)\, \varphi_{\mathrm{d}}\big)\, db.$$

Now $k^* J_W = \big((\delta^{-1})^* J_S \mid J_S\big)$ in column blocks indexed by the two copies of n, and $(\delta^{-1})^* J_S = J_S \delta$ because $\delta^* J_S \delta = J_S$. With $X_b = x_T^{-1} J_T^{-1}\, b\, J_S\, x_S$, an adelic $n' \times n$ matrix, and since the scalar idele c commutes with matrices,

$$\tilde{k}_{\mathbb{A}} b = \big(x_T^{-1} J_T^{-1} b J_S \delta x_S \mid x_T^{-1} J_T^{-1} b J_S\, c\, x_S\big) = \big(X_b\, g_\delta \mid c\, X_b\big). \tag{6.127}$$

Step 3: the integrand. For adelic $n' \times n$ matrices z_1, z_2, $F_1 \in \mathcal{M}_{S,T}$ and $F_2 \in \mathcal{M}_{S^-,T}$, $\rho\big((z_1 \mid z_2)\big)(F_1 \boxtimes^{\mathrm{sc}} F_2) = \rho(z_1)F_1 \boxtimes^{\mathrm{sc}} \rho(z_2)F_2$. By (6.127) and (c) of Lemma 6.3.52 (source sums $S \oplus S^-$, constant 1),

$$\theta_{\mathrm{Hom}(W,T)}\big(\rho(\tilde{k}_{\mathbb{A}} b)\, \varphi_{\mathrm{d}}\big) = \theta\big(\rho(X_b g_\delta) f\big) \cdot c_{S,T}\, \theta_{\mathrm{Hom}(S^-,T)}\big(\rho(cX_b)\, \mathrm{cj}f'\big).$$

By (1) of Lemma 6.4.56 with $z = X_b$,

$$\mathrm{cj}\, \rho(X_b) = \rho(\sigma_c^{-1} X_b)\, \mathrm{cj} = \rho(cX_b)\, \mathrm{cj},$$

since $\sigma_c^{-1} x = (x_\infty, c\, x_f) = c\, x$; so by (3) of Lemma 6.4.56 the second factor is

$$c_{S,T}\theta_{\mathrm{Hom}(S^-,T)}(\mathrm{cj}\, \rho(X_b) f') = \overline{\theta(\rho(X_b) f')}.$$

For the first factor: for $g \in \mathrm{G}(s)$ the pair element of $(g, 1)$ is $Y \mapsto Yg^{-1}$ (VP5), and the scalars $\alpha(\det 1)\beta(\det g)^{-1}$ cancel under conjugation, so (2) of Lemma 4.1.1 gives $\omega(g, 1)\rho(Y)\omega(g, 1)^{-1} =$

$\rho(Yg^{-1})$ for every adelic $n' \times n$ matrix Y. With $g = g_\delta^{-1}$, $\rho(X_b g_\delta) = \omega(g_\delta^{-1}, 1)\rho(X_b)\omega(g_\delta, 1)$, and as $g_\delta^{-1} \in \Gamma_S$, Lemma 6.2.5 gives

$$\theta\big(\rho(X_b g_\delta) f\big) = \theta\big(\rho(X_b)\,\omega(g_\delta, 1) f\big).$$

Hence

$$\theta_{\mathrm{Hom}(W,T)}\big(\rho(\tilde{k}_{\mathbb{A}} b)\,\varphi_{\mathrm{d}}\big) = \overline{\theta(\rho(X_b) f')}\;\theta\big(\rho(X_b)\,\omega(g_\delta, 1) f\big). \tag{6.128}$$

Step 4: change of variables and the Plancherel identity. The map $b \mapsto X_b$ is a continuous additive automorphism of $\mathrm{M}_{n' \times n}(\mathbb{A}_E)$, with inverse $X \mapsto J_T x_T\, X\, x_S^{-1} J_S^{-1}$, and it maps $\mathrm{M}_{n' \times n}(E)$ onto $\{x_T^{-1} u\, x_S : u \in \mathrm{M}_{n' \times n}(E)\}$ (as $b \mapsto J_T^{-1} b J_S$ is a bijection of $\mathrm{M}_{n' \times n}(E)$), which is the set $\mathrm{Hom}(S,T)(E)$ of rational vectors of $\mathrm{Hom}(S,T)$, its point being $x_T \otimes (x_S^{-1})^{\mathsf{T}}$. So it induces an isomorphism of compact groups $\mathrm{M}_{n' \times p}(\mathbb{A}_E)/\mathrm{M}_{n' \times p}(E) \to \mathbb{A}_E^{n' \times n}/\mathrm{Hom}(S,T)(E)$, which carries the invariant probability measure to the invariant probability measure (uniqueness of normalised Haar measure). The canonical functional θ is a theta functional of $\mathrm{Hom}(S,T)$ in the sense of (WD9) by Lemma 4.1.17; by (2) of Lemma 4.1.10 the function $x \mapsto \overline{\theta(\rho(x) f')}\,\theta(\rho(x)\omega(g_\delta, 1) f)$ is a continuous function on $\mathbb{A}_E^{n' \times n}/\mathrm{Hom}(S,T)(E)$ with mean $\langle f', \omega(g_\delta, 1) f\rangle$. With Step 2 and (6.128),

$$Z_{L_\delta}(\varphi_{\mathrm{d}}) = \int_{\mathbb{A}_E^{n' \times n}/\mathrm{Hom}(S,T)(E)} \overline{\theta(\rho(x) f')}\;\theta\big(\rho(x)\,\omega(g_\delta, 1) f\big)\,dx = \big\langle f',\, \omega_{\alpha,\beta}(x_S^{-1}\delta x_S, 1)\, f\big\rangle.$$

□

6.6 The cost of one lift

This section bounds the volumes of the single lifts, one lift at a time (Theorem 6.6.6): along a regular family θ, the family $\iota \mapsto Z_j(1)/Z_j(\theta(\iota))$ has height at most $d_{\mathcal{T}}\, k\,(\log b \cdot \beta_{\mathrm{den}} - h_{\mathrm{dir}})$, with h_{dir} the direction amount (Definition 2.4.2) and β_{den} the denominator budget (Definition 3.2.34).

The volume at a twist is compared with its value at the trivial twist through a comparison operator (Definition 6.6.1). This is a family U_ϑ that, on the lifts, is the Hecke ratio $\rho_j(\vartheta) = \Gamma_j(\vartheta)/\Gamma_j(1)$ of products of partial Hecke L-values [128] times one positive operator independent of ϑ; it maps lifts to forms orthogonal to the false lifts, keeps integrality away from $\mathrm{N}Q$, and changes conjugate norms at the trivial twist by a bounded factor. Its compression Δ to the space of lifts satisfies $Z_j(\vartheta)\,\Delta(\vartheta) = C\rho_j(\vartheta)^d$ (Lemma 6.6.7), so that

$$\frac{Z_j(1)}{Z_j(\theta(\iota))} = \frac{\Delta(\theta(\iota)^j)}{\Delta(1)} \cdot \rho_j\big(\theta(\iota)^j\big)^{-d}, \qquad d = d_{\mathcal{T}}.$$

The first factor is bounded by the clearing condition of the regular family and the size of $\Delta(1)$; the second by the heights of the Hecke ratios (Proposition 6.7.1, Section 6.7), where L-values are realised as squared norms of lifts from $\mathrm{U}(1)$. The rest of the section proves the abstract comparison, constructs comparison operators (Proposition 6.6.13) and proves that the periods of false lifts vanish (Proposition 6.6.81). The integrality that a regular family carries is established in Section 7. The architecture follows [259]; for the arithmetic of automorphic forms on unitary groups, the background of the comparison, see [273, 274].

Definition 6.6.1. Fix $1 \le j \le m$ and let S_0 be the set of chosen places.

The exchange pairs. The three **exchange pairs** of the j-th lift are the critical pairs

$$p_j^* = (\mu_j^0 \upsilon^{n_*},\, z_*), \qquad p_j^{\pm} = (\upsilon^{-z_\pm} \chi_M^{\pm j},\, z_\pm),$$

where $n_* = j + (t - 11m - 1)/2$, $z_* = 7\lfloor m/2 \rfloor + k/2$, $z_+ = m$ and $z_- = \lfloor m/2 \rfloor$; their tame parts are ω^j, ω^j and ω^{-j}. Put $s_* = s_+ = 1$ and $s_- = -1$.

The Hecke value product. For a twist ϑ with modulus Q the **Hecke value product** of the j-th lift is

$$\Gamma_j(\vartheta) = \prod_{i \in \{*,+,-\}} L_*(S_0 \cup Q,\ (\nu_i\, \vartheta^{s_i})^{-1};\ z_i), \qquad p_j^i = (\nu_i, z_i), \tag{6.129}$$

and $\rho_j(\vartheta) = \Gamma_j(\vartheta)/\Gamma_j(1)$ is its **Hecke ratio**.

Comparison operators. A **comparison operator** for the j-th lift is a family $U = (U_\vartheta)$, one $\mathbb{C}$-linear operator U_ϑ on $\mathcal{A}_{j,\vartheta}(V^{\mathrm{def}})$ for each twist ϑ, such that:

(CO1) $\rho_j(\vartheta) \neq 0$ for every twist ϑ;

(CO2) there is an operator U° on $\mathcal{A}^{\mathrm{good}}(W_j)$, independent of ϑ, with $\langle y, U^\circ y\rangle > 0$ for $y \neq 0$ and

$$\langle \Theta_{j,\vartheta} y',\, U_\vartheta\, \Theta_{j,\vartheta} y\rangle = \rho_j(\vartheta)\, \langle y', U^\circ y\rangle \qquad \text{for all twists } \vartheta \text{ and } y, y' \in \mathcal{A}^{\mathrm{good}}(W_j);$$

(CO3) for every twist ϑ and every $z \in \mathcal{L}_{j,\vartheta}$, the form $U_\vartheta z$ is orthogonal to the false lifts for $(e_0, \mathcal{L}_{j,\vartheta}^{\perp})$;

(CO4) for every twist ϑ, every prime $\ell \nmid \mathrm{N}Q$ and every embedding $\mathbb{C} \to \mathbb{C}_\ell$, U_ϑ maps the forms integral at that embedding to forms integral at it;

(CO5) $\|U_1\varphi\|_\sigma^2 \leq \exp(6\, f\, m\, b\, (\log b)^2)\, \|\varphi\|_\sigma^2$ for every $\varphi \in \mathcal{A}_{j,1}(V^{\mathrm{def}})$ and every automorphism σ of $\mathbb{C}$.

For a comparison operator U, the **compression** $\mathrm{pr}\, U_\vartheta$ is the operator $z \mapsto \mathrm{pr}_\vartheta(U_\vartheta z)$ on $\mathcal{L}_{j,\vartheta}$, with pr_ϑ the orthogonal projection onto $\mathcal{L}_{j,\vartheta}$.

Definition 6.6.2. Let O be a descent pair for ranks $r < n$. Let $\varpi_O \in \mathbb{A}_{E,f}^\times$ be a fixed idele whose component at each chosen place w generates $w\mathcal{O}_{E,w}$ and which is 1 elsewhere; $R_2 = \mathcal{O}_E/\mathfrak{M}_O^2$ with $\mathfrak{M}_O = \prod_{w \in M_O} w$, and $\bar{\varpi}$ the class of ϖ_O in R_2. For an embedding u of index sets, I_u is the inclusion matrix, whose column j is the basis vector $e_{u(j)}$.

(GU1) Divided splittings. For finite index sets N, m, a and an embedding $z\colon \{1,\dots,r\} \hookrightarrow m$, a **divided splitting** $\mathcal{V}$ consists of a bijection $\varepsilon\colon N \sqcup m \to \{1,\dots,n\}$, an embedding $\phi_{\mathcal{V}}\colon \{1,\dots,r\} \hookrightarrow N$ with $\varepsilon(\phi_{\mathcal{V}}(i)) = \mathrm{inc}_O(i)$ for all i, and an embedding $a \hookrightarrow N$ (the **divided embedding**), such that $\kappa_O^+(\tau, h) = 0$ for $|N| \leq h < n$ and $\kappa_O^-(\tau, h) = 0$ for $|m| \leq h < n$. For an $n \times r$ matrix x, x_N and x_m are its rows $\varepsilon(N)$ and $\varepsilon(m)$. Its **loss** is $\mathrm{loss}(\mathcal{V}) = \sum_{\tau \in \Phi}(|a| \sum_{h<|N|} \kappa_O^+(\tau,h) + |m| \sum_{h<|m|} \kappa_O^-(\tau,h))$.

(GU2) Divided lattices. A **divided lattice** L is a point $\gamma_L \in G_f(n)$ with components 1 outside the chosen places and their conjugates such that every entry of $\varpi_O \gamma_L^{-1}$ is integral. Its **reduced inverse** $G_L \in \mathrm{M}_n(R_2)$ is the reduction of $\varpi_O \gamma_L^{-1}$, and its **Weil scalar** is $\beta_L = \chi_O[r](\delta)/|\delta|^{r/2}$, where δ is the idele equal to $\det \gamma_L$ at the chosen places and to 1 elsewhere.

(GU3) The congruence group and block normalisation. For an $a \times N$ matrix s over R_2, $[s; \bar{\varpi}]$ is the $(a \sqcup N) \times N$ matrix with s stacked over $\bar{\varpi}\cdot 1$. $\mathrm{C}(\mathcal{V}) = \{A \in \mathrm{GL}_r(R_2) : A\bar{\varpi} = \bar{\varpi}\}$. An $n \times r$ matrix x over R_2 is **block normalised for** s if $x_m = I_z$ and $[s; \bar{\varpi}]\, x_N = [s; \bar{\varpi}]\, I_{\phi_{\mathcal{V}}}$.

(GU4) Glued lattices. A divided lattice L is **glued for** s if (1) for every $n \times r$ matrix x over R_2 with $G_L x = \bar{\varpi}\,\mathrm{Inc}_O$ there is exactly one $A \in \mathrm{C}(\mathcal{V})$ with xA block normalised for s; (2) if xA is block normalised for s for some $A \in \mathrm{C}(\mathcal{V})$, then $G_L x = \bar{\varpi}\,\mathrm{Inc}_O$; and (3) there is $J \in \mathrm{GL}_N(R_2)$ such that $\bar{\varpi}$ times the product of the columns $\varepsilon(N)$ of G_L with J vanishes in every column outside the image of the divided embedding.

(GU5) Delta forms. For $x \in G_f(n)$ and $v \in \mathbb{V}_{\kappa_O}$, the **delta form** $\delta_x(v) \in \mathrm{Form}(O)$ is

$$\delta_x(v)(y) = \sum_{\gamma} \tilde{\eta}_O(x^{-1}\gamma y)\,\rho_O(\gamma)^{-1} v \qquad (y \in G_f(n)),$$

the sum over the rational points γ of $\mathrm{U}(\mathrm{I}_n)$ with $x^{-1}\gamma y \in \mathbf{K}(O)$.

(GU6) Cartan product and product forms. Fix an integral left inverse C'_O of the matrix C_O of (DP8). For y of weight κ_O^+ in the rows N and η of weight κ_O^- in the rows m, the **Cartan product** is $y \cdot \eta = C'_O(S_{\kappa_O^+}(\iota_N)\,y \otimes S_{\kappa_O^-}(\iota_m)\,\eta)$, of weight κ_O, where ι_N, ι_m are the $n \times N$ and $n \times m$ matrices formed by the columns $\varepsilon(N)$, $\varepsilon(m)$ of the identity. For a divided lattice L, $h = (h_1, h_2) \in G_f(N) \times G_f(m)$ and such y, the **product form** is the linear map $\eta \mapsto \mathrm{pr}_{\mathcal{V},L}(h; y)(\eta) = \delta_{\mathrm{diag}_\varepsilon(h_1,h_2)\gamma_L}(y \cdot \eta)$, $\mathrm{diag}_\varepsilon$ the block diagonal embedding along ε.

Definition 6.6.3. Throughout, c, N and m are finite index sets, and $W(O)$ of a descent pair O of rank r has the index set $\{1, \dots, r\}$.

(KN1) Reduction modulo a modulus; levels of subgroups. Let $\mathfrak{m} = \prod_{w \in \Sigma(\mathfrak{m})} w^{e_w}$ be a modulus, so that $R_\mathfrak{m} = \prod_{w \in \Sigma(\mathfrak{m})} \mathcal{O}_{E,w}/w^{e_w}\mathcal{O}_{E,w}$ by the Chinese remainder theorem. For $k \in \mathbf{K}_0(N)$, $\mathrm{red}_\mathfrak{m}(k) \in \mathrm{GL}_N(R_\mathfrak{m})$ is the matrix whose image in $\mathrm{GL}_N(\mathcal{O}_{E,w}/w^{e_w}\mathcal{O}_{E,w})$ is the reduction of the component of k at w (read in $\mathrm{GL}_N(\mathcal{O}_{E,w})$ as for red_w), for every $w \in \Sigma(\mathfrak{m})$. For a subgroup H of $\mathrm{GL}_N(R_\mathfrak{m})$ the **level of** H is

$$\mathbf{K}_\mathfrak{m}[H] = \{k \in \mathbf{K}_0(N) :\ \mathrm{red}_\mathfrak{m}(k) \in H\}.$$

(KN2) Row stabilisers and column data. Let $\phi\colon R \to S$ be a homomorphism of commutative rings, $\Lambda \in \mathrm{M}_{c \times N}(S)$ and $u\colon \{1, \dots, r\} \hookrightarrow N$ an embedding; $\phi(A)$ is the entrywise image of a matrix A. The **row stabiliser** $\mathrm{Stab}_\phi(\Lambda, u)$ is the subgroup of the $A \in \mathrm{GL}_N(R)$ whose columns of index outside $u(\{1, \dots, r\})$ are those of the identity matrix and which satisfy $\Lambda\,\phi(A) = \Lambda$. For a multiplicative character χ of R (a homomorphism of multiplicative monoids $R \to \mathbb{C}$ that vanishes on the nonunits), the **column datum** is the function on $\mathrm{M}_{N \times r}(R)$

$$\mathrm{cd}_{\chi,\phi,\Lambda,u}(\sigma) = \chi(\det \sigma_u) \ \text{ if } \ \Lambda\,\phi(\sigma) = \Lambda\, I_u, \qquad \mathrm{cd}_{\chi,\phi,\Lambda,u}(\sigma) = 0 \ \text{ otherwise,}$$

where σ_u is the $r \times r$ matrix whose row c' is the row $u(c')$ of σ.

(KN3) Row-fixing levels. Let O be a descent pair of rank r with $M_O \neq \emptyset$, and $\mathfrak{Q}_O = \prod_{w \in Q_O} w$; then $\mathfrak{M}_O^2$ and $\mathfrak{M}_O^2\mathfrak{Q}_O$ are moduli, and $\mathrm{red}_0\colon R_{\mathfrak{M}_O^2\mathfrak{Q}_O} \to R_{\mathfrak{M}_O^2}$ is the reduction. The **residue character** of O is the multiplicative character ψ_O^{res} of $R_{\mathfrak{M}_O^2\mathfrak{Q}_O}$ with $\psi_O^{\mathrm{res}}(x) = \prod_{w \in M_O \cup Q_O} \psi_{O,w}(x \bmod w)$ for units x and $\psi_O^{\mathrm{res}}(x) = 0$ for nonunits. For $\Lambda \in \mathrm{M}_{c \times m}(R_{\mathfrak{M}_O^2})$ and an embedding $u\colon \{1, \dots, r\} \hookrightarrow m$ the **row-fixing level** is

$$\mathbf{K}_O[\Lambda, u] = \mathbf{K}_{\mathfrak{M}_O^2\mathfrak{Q}_O}[\mathrm{Stab}_{\mathrm{red}_0}(\Lambda, u)] \subseteq \mathbf{K}_0(m).$$

(KN4) Kernel vectors. Let O be a descent pair of rank r, b an automorphic character of $\mathrm{U}(1)$, $\mathfrak{m}$ a modulus, Ψ a function on $\mathrm{M}_{m\times r}(R_{\mathfrak{m}})$ and κ a weight in rank $|m|$ with rows m, with standard basis $(b_i)_{i\in I_\kappa}$; the letter substitution Let_{-1} of (DP8) is read with the rows in m, so that it takes values in $\mathbb{C}[x]_{m\times r}$. For $g\in \mathrm{U}(W(O))(\mathbb{A})$ and $h\in G_f(m)$ the **kernel vector** $\mathrm{kv}_{O,b;\mathfrak{m}}[\Psi;\kappa](g,h)\in\mathbb{V}_\kappa$ has the coordinates

$$\mathrm{kv}_{O,b;\mathfrak{m}}[\Psi;\kappa](g,h)_i=\overline{\theta^{W(O),\mathrm{I}_m}_{\chi_O,b;\mathfrak{m}}\big[\Psi,\mathrm{Let}_{-1}(b_i)\big](g,h)}\qquad(i\in I_\kappa).$$

(KN5) Coset kernel sums. Let O, $\mathfrak{Q}_O$, red_0 and ψ^{res}_O be as in (KN3), b an automorphic character of $\mathrm{U}(1)$, $\Lambda\in\mathrm{M}_{c\times m}(R_{\mathfrak{M}^2_O})$, $u\colon\{1,\dots,r\}\hookrightarrow m$ an embedding, and y a function from $G_f(m)$ to the Weyl module $\mathbb{V}_{\kappa^+_O}$ of the weight κ^+_O in rank $|m|$ with rows m (its values $\kappa^+_O(\tau,h')$, $1\le h'<|m|$); Let_1 is read with the rows in m. For $g\in\mathrm{U}(W(O))(\mathbb{A})$ the **coset kernel sum** is the sum (6.22) over the double cosets of $\mathrm{U}(\mathrm{I}_m)(F)\backslash G_f(m)/\mathbf{K}_O[\Lambda,u]$, evaluated at their fixed representatives h,

$$\mathrm{cks}_{O,b}(\Lambda,u;y)(g)=\sum_{h\bmod \mathbf{K}_O[\Lambda,u]}^{\mathrm{cl}}\theta^{W(O),\mathrm{I}_m}_{\chi_O,b;\mathfrak{M}^2_O\mathfrak{Q}_O}\Big[\mathrm{cd}_{\psi^{\mathrm{res}}_O,\mathrm{red}_0,\Lambda,u},\ \mathrm{Let}_1\big(\mathrm{Pol}_{\kappa^+_O}(y(h))\big)\Big](g,h).$$

(KN6) Point lifts and point forms. Let O be a descent pair for ranks $r<n$, $Y=(Y_w)_{w\in M_O}$ with $Y_w\in\mathrm{M}_{n\times r}(k_w)$, and $u\colon\{1,\dots,r\}\hookrightarrow\{1,\dots,n\}$ an embedding. A **point lift** for (Y,u) is an element $k\in\mathbf{K}_0(n)$ such that: its component is 1 at every finite place w with $w\notin M_O\cup Q_O$ and $\bar w\notin M_O\cup Q_O$; $\mathrm{red}_w(k)\,\mathrm{Inc}_O=Y_w$ for every $w\in M_O$; and, for every $w\in Q_O$ and $1\le c'\le r$, the row $u(c')$ of $\mathrm{red}_w(k)$ is the row $\mathrm{inc}_O(c')$ of the identity matrix. For $h\in G_f(n)$, $v\in\mathbb{V}_O$ and a point lift k for (Y,u), the delta form $\delta_{hk}(v)\in\mathrm{Form}(O)$ of (GU5) is the **point form** of v at h for k.

Definition 6.6.4. Let $1\le j\le m$, let ϑ be a twist, $O=O_{j,\vartheta}$ and $n=r+t$. The weights $\kappa^\pm_O$ do not involve ϑ ((LF4)), and $\mathfrak{M}=\mathfrak{M}_O=\pi_M\mathcal{O}_E$, the product of the chosen places, does not involve ϑ either. The basis of W_j is indexed by $\mathcal{R}=\{1,\dots,r\}$, read in $\{1,\dots,n\}$ through inc_O as the first r indices.

(TG1) Index sets and spaces. Put $\mathcal{A}^-_j=\{1,\dots,j\}$, $\mathcal{A}^+_j=\{1,\dots,3m-j\}$, $\mathcal{B}^\flat=\{1,\dots,6m+1\}$ and

$$\mathcal{B}_j=\mathcal{R}\sqcup\mathcal{A}^-_j\sqcup\mathcal{A}^+_j\sqcup\mathcal{B}^\flat,\qquad\mathcal{B}^\circ=\mathcal{R}\sqcup\{1,\dots,t-9m-1-r\},\qquad\mathcal{B}^{\mathrm{c}}=\{1,\dots,2m\}\sqcup\{1,\dots,m\},$$

so that $|\mathcal{B}_j|=r+9m+1$, $|\mathcal{B}^\circ|=t-9m-1$ and $|\mathcal{B}^{\mathrm{c}}|=3m$. The **first factor space** is $V''_j=\mathrm{I}_{\mathcal{B}_j}$, the **second factor space** is $V^\circ=\mathrm{I}_{\mathcal{B}^\circ}$, and $W^\flat=\mathrm{I}_{\mathcal{B}^\flat}$.

(TG2) The gluing splitting. Fix, independently of ϑ, a bijection of $(\mathcal{A}^-_j\sqcup\mathcal{A}^+_j\sqcup\mathcal{B}^\flat)\sqcup\mathcal{B}^\circ$ onto $\{r+1,\dots,n\}$ and an embedding $d_j\colon\mathcal{B}^{\mathrm{c}}\hookrightarrow\mathcal{B}_j$. Let $\varepsilon_j\colon\mathcal{B}_j\sqcup\mathcal{B}^\circ\to\{1,\dots,n\}$ be the identity on $\mathcal{R}\subseteq\mathcal{B}_j$ and the fixed bijection on the other indices, let $\phi_j\colon\mathcal{R}\hookrightarrow\mathcal{B}_j$ and $z_j\colon\mathcal{R}\hookrightarrow\mathcal{B}^\circ$ be the inclusions of $\mathcal{R}$. The **gluing splitting** of the j-th lift is $\mathcal{V}_j=(\varepsilon_j,\phi_j,d_j)$; with $\mathcal{B}_j$, $\mathcal{B}^\circ$, $\mathcal{B}^{\mathrm{c}}$ and z_j in the roles of N, m, a and z of (GU1), it is a divided splitting for $O_{j,\vartheta}$ exactly when $\kappa^+_O(\tau,h)=0$ for $|\mathcal{B}_j|\le h<n$ and $\kappa^-_O(\tau,h)=0$ for $|\mathcal{B}^\circ|\le h<n$, for all $\tau\in\Phi$.

(TG3) Classes and choices. For a finite index set $\mathcal{B}$, the **classes** of $\mathrm{I}_{\mathcal{B}}$ at the principal level are the double cosets $\mathrm{U}(\mathrm{I}_{\mathcal{B}})(F)\backslash G_f(\mathcal{B})/\mathbf{K}_{\mathfrak{M}^2}[\{1\}]$; $\mathrm{Cl}^\flat$ and Cl° are those of $W^\flat$ and of

V°, with fixed representatives $x_{\mathfrak{c}}$. Put $\Delta' = \frac{\Delta}{2} - \frac{t+1}{2}$. A **choice** for the j-th lift is a triple $i = (\Lambda_i, \mathfrak{c}_i, \lambda_i)$ of a matrix $\Lambda_i \in \mathrm{M}_{\mathcal{B}^{\mathrm{c}} \times \mathcal{B}_j}(\mathcal{O}_E/\mathfrak{M}^2)$, a class $\mathfrak{c}_i \in \mathrm{Cl}^\flat$ and a family $\lambda_i = (\lambda_{i,\tau})_{\tau \in \Phi}$ of maps $\lambda_{i,\tau} \colon \{1, \dots, \Delta'\} \times \{1, \dots, 2m\} \to \mathcal{B}^\flat$ (lists of rows). Mem_j is the finite set of choices.

(TG4) Tables. Let $\mathbb{V}_j^-$ be the Weyl module of the weight κ_O^- in rank $|\mathcal{B}^\circ|$ with rows $\mathcal{B}^\circ$, with the Gram form $\langle , \rangle_{\kappa_O^-}$. The **space of tables** Tab_j is the orthogonal sum of copies of $\mathbb{V}_j^-$, one for each pair $(i, \mathfrak{c}) \in \mathrm{Mem}_j \times \mathrm{Cl}^\circ$; its elements are the families $\zeta = (\zeta_{i,\mathfrak{c}})$, with $\langle \zeta, \zeta' \rangle = \sum_{i,\mathfrak{c}} \langle \zeta_{i,\mathfrak{c}}, \zeta'_{i,\mathfrak{c}} \rangle_{\kappa_O^-}$, and for $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\|\zeta\|_\sigma^2 = \sum_{i,\mathfrak{c}} \|\sigma(\zeta_{i,\mathfrak{c}})\|_{\kappa_O^-}^2$, σ applied to the coordinates.

(TG5) Gluing levels. For $\Lambda \in \mathrm{M}_{\mathcal{B}^{\mathrm{c}} \times \mathcal{B}_j}(\mathcal{O}_E/\mathfrak{M}^2)$ the **gluing level** is the row-fixing level of $O_{j,\vartheta}$

$$\mathbf{J}_\vartheta(\Lambda) = \mathbf{K}_O\big[[\Lambda; \bar{\varpi}],\, \phi_j\big] \subseteq \mathbf{K}_0(\mathcal{B}_j),$$

with $[\Lambda; \bar{\varpi}]$ the $(\mathcal{B}^{\mathrm{c}} \sqcup \mathcal{B}_j) \times \mathcal{B}_j$ matrix of (GU3).

(TG6) Gluing maps. Suppose that $\mathcal{V}_j$ is a divided splitting for $O_{j,\vartheta}$. Let $L = (L_i)_{i \in \mathrm{Mem}_j}$ be divided lattices of $O_{j,1}$ (they are divided lattices of $O_{j,\vartheta}$, since (GU2) reads only the chosen places), and let $\varphi'' = (\varphi''_i)_{i \in \mathrm{Mem}_j}$ be functions $\varphi''_i \colon G_f(\mathcal{B}_j) \to \mathbb{V}_j^+$, the Weyl module of κ_O^+ in rank $|\mathcal{B}_j|$ with rows $\mathcal{B}_j$. The **gluing map** $\mathrm{gl}_\vartheta[L, \varphi''] \colon \mathrm{Tab}_j \to \mathrm{Form}(O_{j,\vartheta})$ is

$$\mathrm{gl}_\vartheta[L, \varphi''](\zeta) = M^{\mathrm{loss}(\mathcal{V}_j)} \sum_{(i,\mathfrak{c}) \in \mathrm{Mem}_j \times \mathrm{Cl}^\circ} \sum_{x \bmod \mathbf{J}_\vartheta(\Lambda_i)}^{\mathrm{cl}} \mathrm{pr}_{\mathcal{V}_j, L_i}\big((x, x_{\mathfrak{c}}); \varphi''_i(x)\big)(\zeta_{i,\mathfrak{c}}),$$

the inner sum (6.22) over the double cosets $\mathrm{U}(V_j'')(F) \backslash G_f(\mathcal{B}_j) / \mathbf{J}_\vartheta(\Lambda_i)$ at fixed representatives x; it is linear in ζ.

Definition 6.6.5. Let $1 \le j \le m$, ϑ a tame character, $O = O_{j,\vartheta}$ and $O^1 = O_{j,1}$ (the weights $\kappa_O^\pm = \kappa_{O^1}^\pm$ do not involve ϑ), and $r_+ = k + j - 1$. For a finite index set $\mathcal{B}$, $\mathrm{I}_{\mathcal{B}}$ is the standard definite space and $\mathrm{I}_{\mathcal{B}}^-$ the standard space of signs -1 on $\mathcal{B}$; a finite point of a definite group is read as the adelic point with infinite components 1. For $\tau \in \Phi$ and $z \in E$, $\tau(z)$ is the image of z under the embedding chosen at τ, applied entrywise to matrices. The two blocks of $\mathcal{B}^{\mathrm{c}} = \{1, \dots, 2m\} \sqcup \{1, \dots, m\}$ are written $(1, a)$ and $(2, a)$; S^{c} is the standard space on $\mathcal{B}^{\mathrm{c}}$ with signs $+1$ on the first and -1 on the second block.

(XE1) Characters. Put $e_j = j + \lfloor (t + k - 1)/2 \rfloor$. The **twisted level character**, the **first-factor character** and the **untwisted carrier character** are

$$\chi'_{j,\vartheta} = \chi_M^j\, \vartheta, \qquad \nu''_{j,\vartheta} = \mu_j^0 \vartheta\, \upsilon^{e_j}, \qquad \nu_j^\flat = \mu^\flat\, \upsilon^{e_j - j\Delta};$$

the **carrier character** is $\nu_j^\flat \upsilon^m = \mu^\flat \upsilon^{m + e_j - j\Delta}$.

(XE2) Trace characters. Let $\varpi_{\mathfrak{M}^2}$ be the finite idele with component π_M^2 at the places dividing $\mathfrak{M}$ and 1 at the other finite places. The **primitive character** of $R_{\mathfrak{M}^2}$ is $\psi_{\mathfrak{M}^2}(x \bmod \mathfrak{M}^2) = \psi_{E,f}\big(\jmath \alpha \bar{\alpha}\, \varpi_{\mathfrak{M}^2}^{-1} x\big)$ for $x \in \widehat{\mathcal{O}}_E$, and for $\Lambda \in \mathrm{M}_{\mathcal{B}^{\mathrm{c}} \times \mathcal{B}_j}(R_{\mathfrak{M}^2})$ the **trace character** is the function $\psi_\Lambda(y) = \psi_{\mathfrak{M}^2}(\mathrm{tr}(y\Lambda))$ on $\mathrm{M}_{\mathcal{B}_j \times \mathcal{B}^{\mathrm{c}}}(R_{\mathfrak{M}^2})$.

(XE3) Padding, determinant index and the padded minor product. The **padded space** is $W_j \oplus \mathrm{I}_{\mathcal{A}_j^-}^- \oplus \mathrm{I}_{\mathcal{A}_j^+}$ on $\mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{A}_j^+$, and $W'_j = W_j \oplus \mathrm{I}_{\mathcal{A}_j^-}^- \oplus \mathrm{I}_{\mathcal{A}_j^+} \oplus \mathrm{I}_{\mathcal{B}^\flat}$, on $\mathcal{B}_j$ (orthogonal

sums). The **negative rows** are $\mathrm{ng}_j(i) = i \in \mathcal{A}_j^-$ for $1 \le i \le j$ and $\mathrm{ng}_j(i)$ the basis vector of W_j of number $k + i - 2$ for $j < i \le m$. For $1 \le h < r_+$ and $1 \le a \le 2m$ the **positive row** $\mathrm{ps}_j(h,a)$ is the basis vector of W_j of number $h + a - 1$ if $h + a - 1 < r_+$, and the element a of $\mathcal{A}_j^+$ otherwise. The **determinant index** at $\tau \in \Phi$ is

$$D_j(\tau) = \{(h,c) : 1 \le h < r_+,\ 1 \le c \le \kappa_O^+(\tau,h)\} \sqcup \{1,\dots,\Delta'\}.$$

In the variables $x_{\tau,a,b}$ ($a \in \mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{A}_j^+$, $b \in \mathcal{B}^{\mathrm{c}}$) let $y_\tau^- = (x_{\tau,\mathrm{ng}_j(i),(2,i')})_{1\le i,i'\le m}$ and $y_{\tau,h} = (x_{\tau,\mathrm{ps}_j(h,a),(1,a')})_{1\le a,a'\le 2m}$. The **padded minor product** is

$$P_j = \prod_{\tau\in\Phi} \det(y_\tau^-)^{\#D_j(\tau)+|\mathcal{B}_j|+m} \prod_{\tau\in\Phi} \prod_{1\le h<r_+} \det(y_{\tau,h})^{\kappa_O^+(\tau,h)}.$$

(XE4) The carrier and its coefficients. The **carrier polynomial** is

$$P^\circ = \prod_{\tau\in\Phi} \det\big((x_{\tau,(2,i),i'})_{1\le i,i'\le m}\big)^{2m} \in \mathbb{C}[x]_{\mathcal{B}^{\mathrm{c}}\times\{1,\dots,m\}},$$

and the **carrier** is the function on the adelic group of S^{c}

$$\varphi^\circ(x) = \int_{[\mathrm{U}(\mathrm{I}^-_{\{1,\dots,m\}})]} \theta^{\mathrm{I}^-_{\{1,\dots,m\}},S^{\mathrm{c}}}_{v^m,1}\big(v_{P^\circ}\otimes v^0\big)(g,x)\,dg,$$

with the quotient measure. For an automorphic character β of $\mathrm{U}(1)$ the **carrier coefficient** is the function on $G_f(\mathcal{B}^{\mathrm{c}})$

$$\mathsf{a}_\beta(h) = \sum_{g \bmod \mathbf{K}_0(\{1,\dots,m\})}^{\mathrm{cl}} \theta^{\mathrm{I}_{\{1,\dots,m\}},\mathrm{I}_{\mathcal{B}^{\mathrm{c}}}}_{\beta,1}\big(v_1\otimes v^0\big)(g,h).$$

(XE5) Short row families and choice functions. A **short row family** is a family $Y = (Y_{\mathfrak{c}})_{\mathfrak{c}\in\mathrm{Cl}^\flat}$ of matrices $Y_{\mathfrak{c}} \in \mathrm{GL}_{\mathcal{B}^\flat}(E)$ such that $x_{\mathfrak{c}}^{-1}Y_{\mathfrak{c}}^{\mathsf{T}}$ has entries in $\widehat{\mathcal{O}}_E$ and $|\psi(Y_{\mathfrak{c},ab})| \le M^4$ for every ring homomorphism $\psi\colon E \to \mathbb{C}$ and all a,b. In (XE5)–(XE7) a short row family Y is fixed and the dependence on it is not written. For a choice i the **flat determinant product** is

$$P_i^\flat = \prod_{\tau\in\Phi}\prod_{k=1}^{\Delta'} \det\Big(\tau(Y_{\mathfrak{c}_i})[\lambda_{i,\tau}(k,\cdot)]\ (x_{\tau,b,(1,a)})_{b\in\mathcal{B}^\flat,\,1\le a\le 2m}\Big) \in \mathbb{C}[x]_{\mathcal{B}_j\times\mathcal{B}^{\mathrm{c}}},$$

where $\tau(Y_{\mathfrak{c}_i})[\lambda_{i,\tau}(k,\cdot)]$ is the $2m \times \mathcal{B}^\flat$ matrix whose row a is the row $\lambda_{i,\tau}(k,a)$ of $\tau(Y_{\mathfrak{c}_i})$, and P_j is read in $\mathbb{C}[x]_{\mathcal{B}_j\times\mathcal{B}^{\mathrm{c}}}$ through the inclusion of its rows in $\mathcal{B}_j$. The **choice function** of i is the function on $\mathrm{U}(W_j)(\mathbb{A})$

$$\mathsf{F}_i(g) = \int_{[\mathrm{U}(S^{\mathrm{c}})]} \varphi^\circ(x)\ \theta^{S^{\mathrm{c}},W_j'}_{v^m,\nu_j^\flat v^m;\mathfrak{M}^2}\big[\psi_{\Lambda_i},\ P_jP_i^\flat\big]\big(x,(g,1,x_{\mathfrak{c}_i})\big)\,dx,$$

with the quotient measure, $(g,1,x_{\mathfrak{c}_i})$ the block diagonal point with components g on $\mathcal{R}$, 1 on $\mathcal{A}_j^- \sqcup \mathcal{A}_j^+$ and $x_{\mathfrak{c}_i}$ on $\mathcal{B}^\flat$.

(XE6) Line series. Put $\mathcal{B}_j^{\mathrm{add}} = \mathcal{A}_j^+ \sqcup \mathcal{B}^\flat \sqcup \mathcal{B}^{\mathrm{c}}$, $W_j^{\mathrm{add}} = \mathrm{I}_{\mathcal{B}_j^{\mathrm{add}}}$ and $\mathcal{L}_j = \{0,\dots,r_+-1\} \sqcup \mathcal{B}_j^{\mathrm{add}}$, and fix, independently of ϑ, a bijection $\epsilon_j\colon \{1,\dots,2m\} \sqcup \mathcal{B}_j \to \mathcal{L}_j$. For a choice i, $\tau \in \Phi$ and $d \in D_j(\tau)$ the **constant rows** $\mathsf{c}_{i,\tau,d}$ form the complex $2m \times \mathcal{L}_j$ matrix whose row a is:

for $d = (h, c)$, the unit vector at the column $h + a - 1 \in \{0, \dots, r_+ - 1\}$ if $h + a - 1 < r_+$ and at the column $a \in \mathcal{A}_j^+$ otherwise; for $d \in \{1, \dots, \Delta'\}$, the row $\lambda_{i,\tau}(d, a)$ of $\tau(Y_{\mathfrak{c}_i})$ in the columns $\mathcal{B}^\flat$ and 0 in the other columns. Let $\Pi_i = \Delta^{\mathrm{id}}(\epsilon_j; \mathsf{c}_i) \in \mathbb{C}[x]_{\mathcal{B}_j \times \mathcal{L}_j}$ be the product of stacked determinants (6.23) whose index sets p, a, m, n, l there are $\{1, \dots, 2m\}$, $\mathcal{B}_j$, $\mathcal{L}_j$, $\mathcal{L}_j$, $\emptyset$, with the identity of $\mathcal{L}_j$ and ϵ_j as its bijections, $\iota_\tau = D_j(\tau)$ and the matrices $\mathsf{c}_{i,\tau,d}$. For $l \in I_{\kappa_O^+}$ (the standard basis of $\mathbb{V}_j^+$, rows $\mathcal{B}_j$) the **line coordinate** is

$$\mathrm{lc}_{i,l} = h_{\kappa_O^+}^{-1} \langle b_l, \Pi_i \rangle_{\mathrm{F}} \in \mathbb{C}[x]_{\mathcal{B}_j \times \mathcal{B}_j^{\mathrm{add}}},$$

the Fock pairing taken in the variables $x_{\tau,b,c}$ with $c \in \{0, \dots, r_+ - 1\}$ (in which b_l is written, $y_{\tau,b,c} \mapsto x_{\tau,b,c}$), the variables of the columns $\mathcal{B}_j^{\mathrm{add}}$ being coefficients, and $h_{\kappa_O^+}$ the hook number in rank $|\mathcal{B}_j|$. For $\Lambda \in \mathrm{M}_{\mathcal{B}^{\mathrm{c}} \times \mathcal{B}_j}(R_{\mathfrak{M}^2})$ the **added datum** is the function on $\mathrm{M}_{\mathcal{B}_j \times \mathcal{B}_j^{\mathrm{add}}}(R_{\mathfrak{M}^2 \mathfrak{Q}_O})$

$$\Psi_\Lambda^{\mathrm{add}}(z) = \sum_{\gamma \in \mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(R_{\mathfrak{M}^2})} \psi_\Lambda\big(\mathrm{red}_0(z^{\mathrm{c}})\, \gamma\big)$$

if the columns of z on $\mathcal{A}_j^+ \sqcup \mathcal{B}^\flat$ are those of the inclusion matrix of $\mathcal{A}_j^+ \sqcup \mathcal{B}^\flat$ in $\mathcal{B}_j$, and $\Psi_\Lambda^{\mathrm{add}}(z) = 0$ otherwise; z^{c} is the $\mathcal{B}_j \times \mathcal{B}^{\mathrm{c}}$ block of z. Let $\mathbf{1}_j^-$ be the indicator function of the inclusion matrix of $\mathcal{A}_j^-$ in $\mathcal{B}_j$ over $R_{\mathfrak{M}^2 \mathfrak{Q}_O}$. The **line series** $\Upsilon_{\vartheta,i} \colon G_f(\mathcal{B}_j) \to \mathbb{V}_j^+$ has the coordinates

$$\Upsilon_{\vartheta,i}(h)_l = \theta^{\mathrm{I}_{\mathcal{A}_j^-}^-, \mathrm{I}_{\mathcal{B}_j}}_{\upsilon^j, \nu''_{j,\vartheta}; \mathfrak{M}^2 \mathfrak{Q}_O} [\mathbf{1}_j^-, 1](1, h) \sum_{c \bmod \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})}^{\mathrm{cl}} \mathsf{a}_{\chi'_{j,\vartheta}}(c)\, \theta^{W_j^{\mathrm{add}}, \mathrm{I}_{\mathcal{B}_j}}_{\nu''_{j,\vartheta}, \nu''_{j,\vartheta}; \mathfrak{M}^2 \mathfrak{Q}_O} [\Psi_{\Lambda_i}^{\mathrm{add}}, \mathrm{lc}_{i,l}]\big((1, x_{\mathfrak{c}_i}, c), h\big)$$

($l \in I_{\kappa_O^+}$), with $(1, x_{\mathfrak{c}_i}, c)$ the block diagonal point on $\mathcal{A}_j^+ \sqcup \mathcal{B}^\flat \sqcup \mathcal{B}^{\mathrm{c}}$.

(XE7) The theta kernel pairing and the comparison form. For a function $\zeta \colon \mathrm{Cl}^\circ \to \mathbb{V}_j^-$ the **theta kernel pairing** is the function on $\mathrm{U}(W_j)(\mathbb{A})$

$$\mathrm{tt}_j(\zeta)(g) = \sum_{\mathfrak{c} \in \mathrm{Cl}^\circ} \theta^{W_j, V^\circ}_{\chi_{O^1}, \upsilon^{\lfloor m/2 \rfloor - j}; \mathfrak{M}^2} \Big[\mathbf{1}_{z_j},\ \mathrm{Let}_{-1}\big(\mathrm{Pol}_{\kappa_O^-}(\overline{\zeta(\mathfrak{c})})\big)\Big](g, x_{\mathfrak{c}}),$$

where $\mathbf{1}_{z_j}$ is the indicator function of the inclusion matrix of z_j over $R_{\mathfrak{M}^2}$, $\overline{\zeta(\mathfrak{c})}$ has the conjugate coordinates and Let_{-1} is read with the rows in $\mathcal{B}^\circ$. For $c'' \in \mathbb{C}^\times$ the **first factors with scale** c'' are $\varphi''_{\vartheta,i}(h) = \overline{c''\, \Upsilon_{\vartheta,i}(h)}$ (conjugate coordinates), $i \in \mathrm{Mem}_j$. When $\mathcal{V}_j$ is a divided splitting for O and $L = (L_i)_{i \in \mathrm{Mem}_j}$ are divided lattices of O^1, the **comparison form** of the j-th lift is the gluing map

$$\mathrm{gl}_{j,\vartheta}[L; c''] = \mathrm{gl}_\vartheta\big[L, (\varphi''_{\vartheta,i})_{i \in \mathrm{Mem}_j}\big] \colon \mathrm{Tab}_j \to \mathrm{Form}(O).$$

Theorem 6.6.6. *Let D be admissible data, θ a regular family of m-fold twists, and $1 \le j \le m$. Then the family $\iota \mapsto Z_j(1)/Z_j(\theta(\iota))$ has height at most*

$$d_{\mathcal{T}}\, k \big(\log b \cdot \beta_{\mathrm{den}} - h_{\mathrm{dir}}\big).$$

Proof. Let $d = \dim \mathcal{A}^{\mathrm{good}}(W_j)$; then $d = d_{\mathcal{T}}$ by Proposition 5.8.4(iii). Put $A = f\,m\,b\,(\log b)^2$ and $\xi_\iota = \theta(\iota)^j$, a twist with the modulus of $\theta(\iota)$.

The ratio as a quotient of compressions. By Proposition 6.6.13 there is a comparison operator U for the j-th lift; put $\Delta(\vartheta) = \det(\mathrm{pr}\, U_\vartheta)$. By Lemma 6.6.7(a) there is $C \neq 0$ with $\det(\Theta^*_{j,\vartheta}\Theta_{j,\vartheta})\Delta(\vartheta) = C\rho_j(\vartheta)^d$ for every twist ϑ, and $\Delta(\vartheta) \neq 0$, $\rho_j(\vartheta) \neq 0$. At $\vartheta = \xi_\iota$ the left side is $Z_j(\theta(\iota))\Delta(\xi_\iota)$; at $\vartheta = 1$, where $\rho_j(1) = 1$ and $1^j = 1$, it is $Z_j(1)\Delta(1) = C$. Dividing,

$$\frac{Z_j(1)}{Z_j(\theta(\iota))} = \frac{\Delta(\xi_\iota)}{\Delta(1)} \cdot \rho_j(\xi_\iota)^{-d}.$$

The first factor. Since θ is regular, (RW) gives a denominator bound $(\beta(\ell))_\ell$ for (θ, j): by (DB1), $\sum_{\ell\in T}\beta(\ell) \leq \beta_{\mathrm{den}}$ for every finite set T of primes, and by (DB2) and (DL1), for every $\iota \mid \ell$ the clearing condition (DL1) holds for ξ_ι at ι for j with $\beta(\ell)$; by (SR1), ℓ is prime to the norm of the modulus of ξ_ι. By Lemma 6.6.7(b), $\|\iota(\Delta(\xi_\iota))\| \leq e^{dr\beta(\ell)}$, and by Lemma 6.6.7(c), $|\sigma(\Delta(1))| \leq e^{3dA}$ for every automorphism σ of $\mathbb{C}$. So (HG1)–(HG3) hold for the family $\Delta(\xi_\iota)/\Delta(1)$ with $H = \Delta(1)$, $\alpha = 3dA$ and $h_\ell = dr\beta(\ell)/\log\ell$, and this family has height at most $3dA + dr\beta_{\mathrm{den}}$.

The second factor. By Proposition 6.7.1(c) the family $\rho_j(\xi_\iota)^{-1}$ has height at most $10mx_1 + k(6\beta_{\mathrm{den}} - h_{\mathrm{dir}})$, so its d-th power has height at most d times this (Lemma 3.1.8).

Conclusion. Heights add under products (Lemma 3.1.8), so by the displayed quotient the family $Z_j(1)/Z_j(\theta(\iota))$ has height at most

$$d\Big(3A + r\beta_{\mathrm{den}} + 10mx_1 + k\big(6\beta_{\mathrm{den}} - h_{\mathrm{dir}}\big)\Big) \leq d\,k\,\big(\log b \cdot \beta_{\mathrm{den}} - h_{\mathrm{dir}}\big),$$

the inequality being the scale inequality Lemma 6.1.6(v). Monotonicity of heights finishes the proof. □

The abstract comparison (Lemma 6.6.7) is linear algebra on the space of lifts, except for one point: the projection to the lifts does not increase any conjugate norm on forms orthogonal to the false lifts, although the conjugate norms are not the norms for which it is orthogonal. This comes from a lemma on idempotents that are self-adjoint for a family of pairings (Lemma 6.6.8), applied with the good idempotent and the Hecke polynomial at the lift root at a split place; in the relevant space, the forms orthogonal to the lifts are exactly the false lifts, which is why orthogonality to them is the hypothesis. That a single lift is killed by this Hecke polynomial (Lemma 6.6.10) follows by moving the polynomial through the theta integral (Lemma 6.6.11), where a factor vanishes at the lift root. The Hecke polynomials are those of the Satake isomorphism [248, 109]; for the compatibility of the theta correspondence with the unramified Hecke algebras see [134, 236, 173, 205].

Lemma 6.6.7. *Let $1 \leq j \leq m$, let $d = \dim \mathcal{A}^{\mathrm{good}}(W_j)$, let U be a comparison operator for the j-th lift, and put $\Delta(\vartheta) = \det(\mathrm{pr}\, U_\vartheta)$ for every twist ϑ.*

(a) There is $C \in \mathbb{C}\setminus\{0\}$ such that $\det(\Theta^*_{j,\vartheta}\Theta_{j,\vartheta})\,\Delta(\vartheta) = C\,\rho_j(\vartheta)^d$ for every twist ϑ. In particular $\Delta(\vartheta) \neq 0$.

(b) Let ϑ be a twist, $\iota \mid \ell$ a prime identification with $\ell \nmid \mathrm{N}Q$, and $\beta \in \mathbb{R}$ such that (DL1) holds for ϑ at ι for j with β. Then $\|\iota(\Delta(\vartheta))\| \leq e^{dr\beta}$.

(c) $|\sigma(\Delta(1))| \leq \exp\big(3\,d\,f\,m\,b\,(\log b)^2\big)$ for every automorphism σ of $\mathbb{C}$.

(d) If $y \in \mathcal{A}_{j,\vartheta}(V^{\mathrm{def}})$ is orthogonal to the false lifts for $(e_0, \mathcal{L}^\perp_{j,\vartheta})$, then $\|\mathrm{pr}_\vartheta y\|_\sigma \leq \|y\|_\sigma$ for every automorphism σ of $\mathbb{C}$.

Proof. Write $\Theta = \Theta_{j,\vartheta}$, $R = \mathcal{L}_{j,\vartheta}$ and $\mathrm{pr} = \mathrm{pr}_\vartheta$. The lift Θ is injective (single lift, Proposition 6.5.9), so $\dim R = d$.

Part (a). Since Θ^* vanishes on $R^\perp$, we have $\Theta^* U_\vartheta \Theta = \Theta^* \,\mathrm{pr}\, U_\vartheta \Theta = (\Theta^*\Theta) \circ (\Theta^{-1} \circ \mathrm{pr}\, U_\vartheta \circ \Theta)$, where $\Theta^{-1}\colon R \to \mathcal{A}^{\mathrm{good}}(W_j)$ is the inverse of the isomorphism onto R. The second factor is conjugate to $\mathrm{pr}\, U_\vartheta$ on R, so

$$\det(\Theta^* U_\vartheta \Theta) = \det(\Theta^*\Theta)\, \Delta(\vartheta).$$

By (CO2), $\langle y', \Theta^* U_\vartheta \Theta y\rangle = \langle \Theta y', U_\vartheta \Theta y\rangle = \rho_j(\vartheta)\langle y', U^\circ y\rangle$ for all y, y', so $\Theta^* U_\vartheta \Theta = \rho_j(\vartheta) U^\circ$ and its determinant is $\rho_j(\vartheta)^d \det U^\circ$. The operator U° has trivial kernel, since $\langle y, U^\circ y\rangle > 0$ for $y \neq 0$; so $C = \det U^\circ \neq 0$. With (CO1), $\Delta(\vartheta) \neq 0$.

Part (b). Let Λ be the operator given by (DL1); here $\dim \mathcal{L}_{j,\vartheta} = d$, so $\|\iota(\det \Lambda)\| \geq e^{-dr\beta}$, and $T = \Lambda \circ \mathrm{pr}\, U_\vartheta$, an operator on R. Let α be an eigenvalue of T with eigenvector $z \neq 0$. There is $c \in \mathbb{C}^\times$ such that cz is integral at ι and $\|\iota(\mu)\| \leq 1$ whenever μcz is integral at ι: the values of z at the finitely many class representatives have finitely many coordinates, and c is the inverse of a coordinate of largest ι-adic size. By (CO4) the form $U_\vartheta(cz)$ is integral at ι, and by (CO3) it is orthogonal to the false lifts. Hence $T(cz) = \Lambda(\mathrm{pr}\, U_\vartheta(cz)) = \alpha cz$ is integral at ι, and $\|\iota(\alpha)\| \leq 1$. As $\det T$ is the product of the eigenvalues of T with multiplicity and ι is a ring isomorphism, $\|\iota(\det \Lambda)\|\, \|\iota(\Delta(\vartheta))\| = \|\iota(\det T)\| \leq 1$, and the bound on $\det \Lambda$ gives $\|\iota(\Delta(\vartheta))\| \leq e^{dr\beta}$.

Part (c). Let α be an eigenvalue of $\mathrm{pr}\, U_1$ with eigenvector $z \in R \setminus \{0\}$. By (CO3) the form $U_1 z$ is orthogonal to the false lifts, so by part (d) and (CO5)

$$|\sigma(\alpha)|^2\, \|z\|_\sigma^2 = \|\mathrm{pr}\, U_1 z\|_\sigma^2 \leq \|U_1 z\|_\sigma^2 \leq \exp\bigl(6fmb(\log b)^2\bigr)\|z\|_\sigma^2.$$

Here $\|z\|_\sigma > 0$, since the Gram matrix of (TW4) of Definition 3.1.4 is positive definite and σ maps the nonzero coordinate vector of z to a nonzero vector. So $|\sigma(\alpha)| \leq \exp(3fmb(\log b)^2)$, and $|\sigma(\Delta(1))|$, the product of the d numbers $|\sigma(\alpha)|$ over the eigenvalues with multiplicity, is at most $\exp(3dfmb(\log b)^2)$.

Part (d). Let $\mathcal{A} = \mathcal{A}_{j,\vartheta}(V^{\mathrm{def}})$, and let w, the pairings $\langle\cdot,\cdot\rangle_\sigma$ with $\|y\|_\sigma^2 = \langle y, y\rangle_\sigma$ and the operator $B = H_w(\lambda_w)$ be those of Lemma 6.6.9. Each $\langle\cdot,\cdot\rangle_\sigma$ is biadditive, since σ is additive; it satisfies $\langle v, u\rangle_\sigma = \overline{\langle u, v\rangle_\sigma}$ and $\langle u, u\rangle_\sigma \geq 0$, since G is rational, hermitian and positive definite; and $\langle\cdot,\cdot\rangle_1$ is the Petersson product, a positive definite hermitian form. As y is orthogonal to the false lifts for $(e_0, R^\perp)$, it is orthogonal to $\mathcal{F}_w(e_0, R^\perp) = e_0\mathcal{A} \cap \ker B \cap R^\perp$. By Lemma 6.6.9(a), $R \subseteq e_0\mathcal{A} \cap \ker B$, and (b), (c) of the same lemma are the hypotheses (i), (ii) of Lemma 6.6.8 for $\Pi = e_0$ and B. So Lemma 6.6.8, with $\mathcal{H} = \mathcal{A}$, $Z = R$ and the automorphisms of $\mathbb{C}$ as index set, gives $\|\mathrm{pr}\, y\|_\sigma \leq \|y\|_\sigma$ for every σ, as pr is the orthogonal projection onto R for the Petersson product. □

Lemma 6.6.8. *Let $\mathcal{H}$ be a finite-dimensional complex vector space and $(\langle\cdot,\cdot\rangle_\sigma)_{\sigma\in I}$ a family of biadditive maps $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, indexed by a set I, with $\langle v, u\rangle_\sigma = \overline{\langle u, v\rangle_\sigma}$ and $\langle u, u\rangle_\sigma \geq 0$ for all u, v (they need not be $\mathbb{C}$-sesquilinear), such that I contains an element 1 for which $\langle\cdot,\cdot\rangle_1$ is a positive definite hermitian form, linear in the second variable; write $\|y\|_\sigma^2 = \langle y, y\rangle_\sigma$, and let* pr *be the orthogonal projection onto a subspace for $\langle\cdot,\cdot\rangle_1$. Let Π and B be linear operators on $\mathcal{H}$ such that for every $\sigma \in I$:*

(i) $\Pi^2 = \Pi$ *and* $\langle \Pi u, v\rangle_\sigma = \langle u, \Pi v\rangle_\sigma$ *for all* $u, v \in \mathcal{H}$;

(ii) $\langle Bv, z\rangle_\sigma = 0$ *for all* $v \in \mathcal{H}$ *and all* $z \in \ker B$.

Let Z be a subspace of $\Pi\mathcal{H} \cap \ker B$, let $Z^\perp$ be its orthogonal complement for $\langle\cdot,\cdot\rangle_1$, and let $y \in \mathcal{H}$ satisfy $\langle y, f\rangle_1 = 0$ for every $f \in \Pi\mathcal{H} \cap \ker B \cap Z^\perp$. Then $\|\mathrm{pr}_Z\, y\|_\sigma \leq \|y\|_\sigma$ for every $\sigma \in I$.

Proof. Write $\perp$ for orthogonal complements for the positive definite form $\langle\cdot,\cdot\rangle_1$, and put $K = \Pi\mathcal{H} \cap \ker B$.

The complement of K is $\ker\Pi + B\mathcal{H}$. If $\Pi u = 0$, then $\langle u, \Pi v\rangle_1 = \langle \Pi u, v\rangle_1 = 0$ by (i), so $\ker\Pi \subseteq (\Pi\mathcal{H})^\perp$; as Π is idempotent, $\mathcal{H} = \Pi\mathcal{H} \oplus \ker\Pi$, so both spaces have dimension $\dim\mathcal{H} - \dim\Pi\mathcal{H}$ and $(\Pi\mathcal{H})^\perp = \ker\Pi$. By (ii) at $\sigma = 1$, $B\mathcal{H} \subseteq (\ker B)^\perp$, and $\dim B\mathcal{H} = \dim\mathcal{H} - \dim\ker B = \dim(\ker B)^\perp$, so $(\ker B)^\perp = B\mathcal{H}$. For subspaces of a finite-dimensional space with a positive definite hermitian form, $(U \cap U')^\perp = U^\perp + U'^\perp$; hence $K^\perp = \ker\Pi + B\mathcal{H}$.

The part of y off Z lies in $K^\perp$. Put $z = \mathrm{pr}_Z\, y \in Z$ and $x = y - z \in Z^\perp$. Every $f \in K$ is $f = \mathrm{pr}_Z f + (f - \mathrm{pr}_Z f)$ with $\mathrm{pr}_Z f \in Z \subseteq K$, so $f - \mathrm{pr}_Z f \in K \cap Z^\perp$. Now $\langle x, \mathrm{pr}_Z f\rangle_1 = 0$ as $x \in Z^\perp$, and $\langle x, f - \mathrm{pr}_Z f\rangle_1 = \langle y, f - \mathrm{pr}_Z f\rangle_1 - \langle z, f - \mathrm{pr}_Z f\rangle_1 = 0$, by the hypothesis on y and because $z \in Z$. So $x \in K^\perp$, and $x = u + Bv$ with $\Pi u = 0$ and $v \in \mathcal{H}$.

x is orthogonal to Z for every form. Let $\sigma \in I$ and $z' \in Z$. Since $z' \in \Pi\mathcal{H}$ and Π is idempotent, $\Pi z' = z'$, so $\langle u, z'\rangle_\sigma = \langle u, \Pi z'\rangle_\sigma = \langle \Pi u, z'\rangle_\sigma = 0$ by (i); and $\langle Bv, z'\rangle_\sigma = 0$ by (ii), since $z' \in \ker B$. Hence $\langle x, z\rangle_\sigma = 0$, and $\langle z, x\rangle_\sigma = \overline{\langle x, z\rangle_\sigma} = 0$.

Pythagoras. For every $\sigma \in I$,

$$\|y\|_\sigma^2 = \langle z + x, z + x\rangle_\sigma = \|z\|_\sigma^2 + \|x\|_\sigma^2 \geq \|z\|_\sigma^2,$$

by biadditivity, $\langle z, x\rangle_\sigma = \langle x, z\rangle_\sigma = 0$ and $\langle x, x\rangle_\sigma \geq 0$. So $\|\mathrm{pr}_Z\, y\|_\sigma \leq \|y\|_\sigma$. □

Lemma 6.6.9. *Let $1 \leq j \leq m$, let ϑ be a tame character with modulus Q, $O = O_{j,\vartheta}$, $\mathcal{A} = \mathrm{Form}(O)$ and $\mathcal{L} = \mathcal{L}_{j,\vartheta}$. Fix representatives $h_1, \ldots, h_C$ of the double cosets*

$$\mathrm{U}(\mathrm{I}_n)(E)\backslash G_f(n)/\mathbf{K}(O),$$

let $\mathrm{co}(y) = (y(h_1), \ldots, y(h_C))$ for $y \in \mathcal{A}$, and G the block diagonal matrix with blocks $(\#\Gamma_{h_c})^{-1} G_{\kappa_O}$, so that $\langle y, y'\rangle = \mathrm{co}(y)^* G\, \mathrm{co}(y')$. For an automorphism σ of $\mathbb{C}$ put $\langle y, y'\rangle_\sigma = \sigma(\mathrm{co}(y))^*\, G\, \sigma(\mathrm{co}(y'))$, σ applied to the coordinates, and $\|y\|_\sigma^2 = \langle y, y\rangle_\sigma$. There are infinitely many finite places w of E with $w \neq \bar{w}$ and $w, \bar{w} \notin S_O$, and for every such w:

(a) $\mathcal{L} \subseteq [e_0]\mathcal{A} \cap \ker H_w(X_*(w))$;

(b) $[e_0]$ *is an idempotent of* $\mathcal{A}$, *and* $\langle [e_0]u, v\rangle_\sigma = \langle u, [e_0]v\rangle_\sigma$ *for all* $u, v \in \mathcal{A}$ *and every* σ;

(c) $\langle H_w(X_*(w))v, z\rangle_\sigma = 0$ *for all* $v \in \mathcal{A}$, *all* $z \in \ker H_w(X_*(w))$ *and every* σ.

Proof. Write $K = \mathbf{K}(O)$, $\Gamma = \mathrm{U}(\mathrm{I}_n)(E)$ (a discrete subgroup of $G_f(n)$, the infinite part of $\mathrm{U}(\mathrm{I}_n)$ being compact), $R(x)y(h) = y(hx)$, and $\rho = \rho_{\kappa_O, e_O}$. A form is determined by its values at the h_c, as $y(\gamma h_c k) = \tilde{\eta}_O(k)\rho(\gamma)y(h_c)$; so co is injective, and G is rational and positive definite because $G_{\kappa_O} = F^{\mathsf{T}} F$ with F an integer matrix of full column rank. Hence $\langle u, u\rangle_\sigma > 0$ for $u \neq 0$, since $\sigma(\mathrm{co}(u)) \neq 0$.

Split places. The rational primes that split completely in the Galois closure of E are infinite in number (Chebotarev's density theorem). Above such a p every place w of E has trivial decomposition group, so complex conjugation, which is nontrivial on E, moves w: $w \neq \bar{w}$. Only finitely many of these places meet $S_O \cup \bar{S}_O$.

A coefficient field. Let $\mathcal{K} \subset \mathbb{C}$ be the compositum of the Galois closure of E and of $\mathbb{Q}(\mu_N)$, $N = \#\bar{G} \cdot \prod_{v \in S_O}(\mathrm{N}v - 1)$. It is a CM field, Galois over $\mathbb{Q}$, so complex conjugation is central in its Galois group: $\sigma(\bar{x}) = \overline{\sigma(x)}$ for $x \in \mathcal{K}$ and every σ; hence $\sigma(M^*) = \sigma(M)^*$ for matrices M over $\mathcal{K}$. Let $T = \sum_x a_x R(x)$ be a finite sum with $a_x \in \mathcal{K}$ and $x \in G_f(n)$ that maps $\mathcal{A}$ to itself. Writing $h_c x = \gamma h_{c'} k'$ with $\gamma \in \Gamma$, $k' \in K$, we get $y(h_c x) = \tilde{\eta}_O(k')\rho(\gamma)y(h_{c'})$; the values of $\tilde{\eta}_O$ are

roots of unity of order dividing $\prod_v(\mathrm{N}v-1)$, and the entries of $\rho(\gamma)=\prod_\tau \det(\bar\gamma_\tau)^{e_O(\tau)}S_{\kappa_O}(\bar\gamma)$ are integral polynomials in the conjugates of the entries of the $\tau(\gamma)$, all in $\mathcal{K}$. So $\mathrm{co}(Ty)=M_T\,\mathrm{co}(y)$ with a matrix M_T over $\mathcal{K}$. This applies to $[g]=\prod_{v\in M_O}\psi_{O,v}(\det g_v)\,R(k_g)$ $(g\in\bar G)$, to the $T_{w,i}$, and to $R(z)$ for central $z\in G_f(n)$.

Part (a). By (LF5), $\mathcal{L}=\Theta_{j,\vartheta}(e_0\,\mathcal{A}_M(W_j))$. By Lemma 6.6.10(i), extended linearly to $\mathbb{C}[\bar G]$, $\Theta_{j,\vartheta}(e_0y)=[e_0]\Theta_{j,\vartheta}(y)\in[e_0]\mathcal{A}$, and by Lemma 6.6.10(ii) it lies in $\ker H_w(X_*(w))$.

Part (b). The action $[\cdot]$ of $\bar G$ on $\mathcal{A}$ is a representation, so $[e_0]$ is an idempotent, e_0 being one. Each $[g]$ preserves $\langle\,,\rangle$: $|\psi_{O,v}|=1$, and k_g normalises K with $\tilde\eta_O(k_g^{-1}kk_g)=\tilde\eta_O(k)$ (at $v\in M_O$ the reduction $\mathrm{diag}(g_v,1)$ conjugates the stabiliser of Inc_O into itself, at Q_O the reduction is 1), so $h\mapsto hk_g$ permutes the double cosets with $\#\Gamma_{hk_g}=\#\Gamma_h$. Hence $M_g^*GM_g=G$, i.e. $M_g^*G=GM_{g^{-1}}$ for $M_g=M_{[g]}$, and applying σ entrywise, $\sigma(M_g)^*G=G\,\sigma(M_{g^{-1}})$. Write $e_0=\sum_g c_g g$. By Definition 3.2.16, c_g is a sum of products of the numbers $\frac{\Xi_c(1)}{|\mathrm{GL}_r(k_v)|}\overline{\Xi_c(g_v)}$; the Ξ_c are virtual characters, so their values lie in $\mathbb{Q}(\mu_{\#\bar G})\subseteq\mathcal{K}$ and $\Xi_c(g^{-1})=\overline{\Xi_c(g)}$; hence $c_{g^{-1}}=\overline{c_g}$ and $\overline{\sigma(c_{g^{-1}})}=\sigma(c_g)$. Therefore

$$\sigma(M_{e_0})^*G=\sum_g\overline{\sigma(c_g)}\,G\,\sigma(M_{g^{-1}})=G\sum_g\overline{\sigma(c_{g^{-1}})}\,\sigma(M_g)=G\,\sigma(M_{e_0}),$$

which is $\langle[e_0]u,v\rangle_\sigma=\langle u,[e_0]v\rangle_\sigma$.

The adjoints of the Hecke operators. Let μ be the right invariant measure on $\Gamma\backslash G_f(n)$ with μ the quotient of the Haar measure giving K volume 1 by the counting measure. For left Γ-invariant functions f right invariant under an open compact K',

$$\int f\,d\mu=\mathrm{vol}(K')\sum_{[h]\in\Gamma\backslash G_f(n)/K'}f(h)/\#(\Gamma\cap hK'h^{-1});$$

so $\langle y,y'\rangle=\int\langle y(h),y'(h)\rangle_{\kappa_O}\,d\mu(h)$, and the right hand side defines a pairing B of all left Γ-equivariant, locally constant functions $G_f(n)\to\mathbb{V}_O$ with $B(R(x)f,f')=B(f,R(x^{-1})f')$. Let $K_w=\iota_w(\mathrm{GL}_n(\mathcal{O}_{E,w}))\subseteq K$ (its elements have reductions 1 at S_O and $\tilde\eta_O=1$ on them), so the forms are K_w-invariant. For $u,v\in\mathcal{A}$ the number $B(u,R(x)v)$ depends only on K_wxK_w. Since $\iota_w(\delta_i)^{-1}=z\,\iota_w(\delta'_{n-i})$ with $z=\iota_w(\varpi_w^{-1}1_n)$ central and $\delta'_{n-i}=\mathrm{diag}(1_i,\varpi_w1_{n-i})$ in the double coset of δ_{n-i}, and the numbers of left cosets in the double cosets of δ_i and δ_{n-i} are equal (both are the number of i-dimensional subspaces of k_w^n), we get

$$\langle T_{w,i}u,v\rangle=\#R_i\,B(u,R(\iota_w(\delta_i)^{-1})v)=\langle u,T_i'v\rangle,\qquad T_i'=R(z)\,T_{w,n-i}.$$

$R(z)$ maps $\mathcal{A}$ to itself, z being central. The operators $T_{w,i}$ and T_i' commute with each other on $\mathcal{A}$: on K_w-invariant vectors they are the action of the spherical Hecke algebra of $\mathrm{GL}_n(E_w)$, which is commutative (Satake), and $R(z)$ is central.

Part (c). Put $H=H_w(X)$ with $X=X_*(w)$, $q=\mathrm{N}w$, $X'=\sigma^{-1}(\overline{\sigma(X)})$ and $H'=\sum_i(-X')^iq^{i(i-1)/2}T_i'$. From $\langle T_{w,i}u,v\rangle=\langle u,T_i'v\rangle$ we get $M^*_{T_{w,i}}G=GM_{T_i'}$, and applying σ, $\sigma(M_{T_{w,i}})^*G=G\sigma(M_{T_i'})$. As $\sigma(M_H)=\sum_i(-\sigma X)^iq^{i(i-1)/2}\sigma(M_{T_{w,i}})$ and $\sigma(X')=\overline{\sigma(X)}$,

$$\sigma(M_H)^*G=\sum_i\big(-\overline{\sigma(X)}\big)^iq^{i(i-1)/2}G\,\sigma(M_{T_i'})=G\,\sigma(M_{H'}),$$

that is, $\langle Hx,y\rangle_\sigma=\langle x,H'y\rangle_\sigma$ for all $x,y\in\mathcal{A}$. H and H' commute. Let $Hz=0$. Then $\langle H'z,H'z\rangle_\sigma=\langle HH'z,z\rangle_\sigma=\langle H'Hz,z\rangle_\sigma=0$, so $H'z=0$, and $\langle Hv,z\rangle_\sigma=\langle v,H'z\rangle_\sigma=0$. □

Lemma 6.6.10. *Let $1\le j\le m$, let ϑ be a tame character with modulus Q, and $O=O_{j,\vartheta}$; its chosen places are those of the data, so $\mathrm{GL}_r^{\mathrm{ch}}(O)=\bar G$. Then:*

(i) $\Theta_{j,\vartheta}(g\cdot y)=[g]\,\Theta_{j,\vartheta}(y)$ *for every* $g\in\bar G$ *and* $y\in\mathcal{A}_M(W_j)$*;*

(ii) $H_w(X_*(w))\,\Theta_{j,\vartheta}(y)=0$ *for every* $y\in\mathcal{A}_M(W_j)$ *and every finite place* w *of* E *with* $w\neq\bar w$ *and* $w,\bar w\notin S_O$*.*

Proof. Write $n=r+t$, $W=W_j$, $\mathrm{J}=\mathrm{J}_O(\xi_O)$, and let $\mathfrak{m}=\prod_{w\in S_O}w$, a modulus by (DP1) ($S_O$ contains the chosen places), so that $R_{\mathfrak{m}}=\prod_{w\in S_O}k_w$. By (DP7), (DP8) and (XD5), $\mathrm{J}=\mathrm{mod}_{\mathfrak{m}}(\Psi_O,v_{\mathrm{Har}_O(\xi_O)})$. Let $\gamma=r_-(O)$ and $\gamma'=\lfloor n/2\rfloor+\mathrm{wt}_O$ be the weights of χ_O and χ'_O. By Lemma 6.5.11, O fits the parameter $l=2\varsigma$ (read on the basis of W), and $r\geq 1$; by Lemma 6.4.15(e), $v_{\mathrm{Har}_O(\xi_O)}$ is an edge for (γ,γ') from $e=e_\varsigma$ to $e'=e_{l_O/2}$. So Lemma 6.3.76 applies to $(S,T)=(W,\mathrm{I}_n)$, $(\alpha,\beta)=(\chi_O,\chi'_O)$, $\mathfrak{m}$, Ψ_O and $P=\mathrm{Har}_O(\xi_O)$, with $\kappa(u,h)=\theta^{W,\mathrm{I}_n}_{\chi_O,\chi'_O}(\mathrm{J})(u,h)=\theta_O(\omega_O(u,h)\mathrm{J})$; its lift L satisfies $\Theta_{j,\vartheta}(y)=(Ly)^\flat$ by (LF5). For $k'\in G_f(n)$ write $R(k')F=F(\cdot\,(1,k'))$; by (WM7), $(R(k')F)^\flat=R(k')F^\flat$.

The block datum at integral points. Let $k\in\mathbf{K}_0(r)$ and $k'\in\mathbf{K}_0(n)$ have reductions $\bar k_w$, $\bar k'_w$ at $w\in S_O$. By (6) of Lemma 4.1.1 for the modulus $\mathfrak{m}$,

$$\omega_O((1,k),(1,k'))\mathrm{J}=\chi_O(\det k')\,\chi'_O(\det k)^{-1}\,\mathrm{mod}_{\mathfrak{m}}(\Psi_O(\bar k'^{-1}\cdot\bar k),\,v_{\mathrm{Har}_O(\xi_O)}).\tag{1}$$

Here $\chi_O(\det k')=1$: $\det k'$ has unit finite components and infinite components 1, so $\upsilon(\det k')=1$ by (WD3) of Definition 3.2.12; likewise $\upsilon(\det k)=1$, so $\chi'_O(\det k)=\mu_O(\det k)=\prod_w\psi_{O,w}(\det k\bmod w)$ by the definition of the tame part, the factors with $w\notin S_O$ being 1.

(i). Let $g=(g_w)_{w\in M_O}\in\bar G$. Choose $k\in\mathbf{K}_0(r)$ with $\bar k_w=g_w$ for $w\in M_O$ and $\bar k_w=1$ for $w\in Q_O$, and $k'\in\mathbf{K}_0(n)$ with $\bar k'_w=1$ for $w\in Q_O$ and, for $w\in M_O$, $\bar k'_w$ acting as g_w on the span of the basis vectors in the image of inc_O and as the identity on the others (reduction at finitely many split places is surjective onto the product of the groups $\mathrm{GL}(k_w)$). Then $\bar k'_w\mathrm{Inc}_O=\mathrm{Inc}_O\,g_w$ for $w\in M_O$, so $\bar k'^{-1}_wX\bar k_w=\mathrm{Inc}_O$ if and only if $X=\mathrm{Inc}_O$, and the families at Q_O are unchanged: $\Psi_O(\bar k'^{-1}\cdot\bar k)=\Psi_O$. With $c_g=\prod_{w\in M_O}\psi_{O,w}(\det g_w)$, (1) reads $\omega_O((1,k),(1,k'))\mathrm{J}=c_g^{-1}\mathrm{J}$. By (3) of Lemma 6.3.76 (with $\Psi'=\Psi_O$, $c=c_g^{-1}$), $Ly(h(1,k'))=c_g^{-1}L(y(\cdot\,(1,k)))(h)$. As $g\cdot y=y(\cdot\,(1,k))$ (Definition 3.2.16),

$$\Theta_{j,\vartheta}(g\cdot y)=\big(L(g\cdot y)\big)^\flat=c_g\,R(k')(Ly)^\flat=c_g\,R(k')\,\Theta_{j,\vartheta}(y)=[g]\,\Theta_{j,\vartheta}(y)$$

by (DP5).

(ii). Let w be a finite place with $w\neq\bar w$, $w,\bar w\notin S_O$, $q=\mathrm{N}w=\mathrm{N}\bar w$, $X_*=X_*(w)$, and $y\in\mathcal{A}_M(W)$. The group K_w consists of elements of $\mathbf{K}_0(n)$ with components 1 outside $\{w,\bar w\}$; it is also the group $K_{\bar w}$ of the place $\bar w$. For $k'\in K_w$, (1) with $k=1$ and trivial reductions at S_O gives $\omega_O(1,(1,k'))\mathrm{J}=\mathrm{J}$, so Ly is right invariant under K_w. Put $G=H^{\mathrm{I}_n}_w(X_*)\,Ly$. By (DP10), $H_w(X_*)$ acts on $\mathrm{Form}(O)$ by the same combination of right translations, so $H_w(X_*)\Theta_{j,\vartheta}(y)=G^\flat$, and it suffices to show $G=0$.

By (1) of Lemma 6.3.76, Ly is a form of type e' on I_n; G is a finite combination of right translates of Ly by elements of $G_f(n)$, hence a form of type e' ((3) of Lemma 6.3.76), and right invariant under K_w (a double coset $D=K\delta K$ is stable under left multiplication by $K=\mathrm{GL}_n(\mathcal{O}_{E,w})$). So $y'=\bar G$ is a form of type $\bar e'$, continuous, left invariant under the rational points, with a level and right invariant under $K_{\bar w}=K_w$, and $F'=H^{\mathrm{I}_n}_{\bar w}(X_*)y'$ is a form of type $\bar e'$ as well.

Transposition of the Hecke operators. Let μ be the Haar measure of $\mathrm{GL}_n(E_w)$ with $\mu(K)=1$ and $D_i=K\mathrm{diag}(\varpi_w1_i,1_{n-i})K$. For right K_w-invariant f, $(T_{w,i}f)(h)=\int_{D_i}f(h\iota_w(x))\,d\mu(x)$, ι_w the embedding at w. As D_i is compact, $[\mathrm{U}(\mathrm{I}_n)]$ compact (Theorem A.4.5) and y', Ly continuous

and bounded, Fubini's theorem, the right invariance of the quotient measure and the invariance of μ under $x \mapsto x^{-1}$ ($\mathrm{GL}_n(E_w)$ is unimodular) give

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} y'\,(T_{w,i}Ly) = \int_{[\mathrm{U}(\mathrm{I}_n)]} Ly(h) \int_{D_i^{-1}} y'(h\iota_w(x))\,d\mu(x)\,dh = \int_{[\mathrm{U}(\mathrm{I}_n)]} (T_{\bar{w},i}y')\,Ly.$$

Summing with the coefficients $(-X_*)^i q^{\binom{i}{2}}$: $\int y'\,G = \int F' \cdot Ly$.

Adjointness and the vanishing factor. By (2) of Lemma 6.3.76 (forms y of type e on W and F' of type $\bar{e}'$ on I_n), $\int_{[\mathrm{U}(\mathrm{I}_n)]} F' \cdot Ly = \int_{[\mathrm{U}(W)]} y \cdot L^{\mathsf{t}}F'$. By (WR4) of Definition 3.2.18, $\kappa(u,h) = \chi_O'(\det u)^{-1}\theta^{W,\mathrm{I}_n}_{\chi_O,1}(\mathrm{J})(u,h)$, so $L^{\mathsf{t}}F' = \chi_O'(\det\cdot)^{-1}F_{F'}$ with $F_{y''}(u) = \int \theta^{W,\mathrm{I}_n}_{\chi_O,1}(\mathrm{J})(u,h)y''(h)\,dh$ the function of Lemma 6.6.11. Apply Lemma 6.6.11 at the place $\bar{w}$ (whose conjugate is w), to $S = W$ of rank r, $T = \mathrm{I}_n$ of rank $r+t$, $a = \chi_O$, the datum J and y'. Its hypotheses hold: the base change of $\chi_O = \upsilon^{r-(O)}$ is unramified at w (for a unit x of E_w read as an idele, $x/\bar{x}$ has unit finite components and infinite components 1, so $\upsilon(x/\bar{x}) = 1$); J is the vacuum at w and $\bar{w}$, since for an integral z supported at $\{w,\bar{w}\}$, disjoint from $S_O \cup \bar{S}_O$, $\rho_f(z)$ fixes $\Phi_O \otimes_{S_O} v^0$ by (5) of Lemma 6.2.33 and commutes with the real Heisenberg operators $\rho(\mathrm{rv}(\cdot))$ of (WR6) (the bicharacter of vectors with disjoint supports is 1, by (WD6)), hence with $v \otimes (\cdot)$; and y' is as required. Its constant is $\alpha = \chi_O[r](\varpi_w)q^{r/2} = \chi_{O,w}\{r\}$ (DP10), and

$$F_{F'} = \prod_{j=0}^{t-1}(1 - q^j\chi_{O,w}\{r\}X_*) \cdot H^W_{\bar{w}}(q^t\alpha\beta^{-1}X_*)F_{y'} = 0,$$

since $t - 1 = n - r - 1$ and $X_* = (\chi_{O,w}\{r\}q^{n-r-1})^{-1}$ make the factor $j = t-1$ vanish. Hence $L^{\mathsf{t}}F' = 0$ and

$$\int_{[\mathrm{U}(\mathrm{I}_n)]} |G|^2 = \int y'\,G = \int_{[\mathrm{U}(W)]} y \cdot L^{\mathsf{t}}F' = 0.$$

As $|G|^2$ is continuous and nonnegative on the compact quotient, whose measure charges every nonempty open set, $G = 0$. □

Lemma 6.6.11. *For a space V of rank n, a finite place $w \neq \bar{w}$ and $X \in \mathbb{C}$, let $H^V_w(X) = \sum_{i=0}^n (-X)^i(\mathrm{N}w)^{\binom{i}{2}}T_{w,i}$, acting on functions on $\mathrm{U}(V)(\mathbb{A}_F)$ that are right invariant under K_w. Let S be a space of rank n_S, T a standard definite space of rank $n_T = n_S + k$ with $k \geq 0$, w a finite place of E with $w \neq \bar{w}$, $q = \mathrm{N}w$, and a an automorphic character of $\mathrm{U}(1)$ whose base change $\tilde{a}$ is unramified at $\bar{w}$. Let $\varphi \in \mathcal{M}_{S,T}$ be a datum that is the* vacuum at w and $\bar{w}$*: $\rho_f(z)\varphi = \varphi$ for every integral finite adelic $n_T \times n_S$ matrix z whose components outside $\{w,\bar{w}\}$ are 0 (ρ_f the finite Heisenberg operators of (WD8) of Definition 3.2.12). Let y be a continuous function on $\mathrm{U}(T)(\mathbb{A}_F)$, left invariant under $\mathrm{U}(T)(F)$, right invariant under an open subgroup of $\mathrm{U}(T)(\mathbb{A}_{F,f})$ and under K_w. Put*

$$F_y(g) = \int_{[\mathrm{U}(T)]} \theta^{S,T}_{a,1}(\varphi)(g,h)\,y(h)\,dh \qquad (g \in \mathrm{U}(S)(\mathbb{A}_F)).$$

Let $\varpi_{\bar{w}}$ be an idele with a uniformiser of $E_{\bar{w}}$ at $\bar{w}$ and 1 elsewhere, $a[n_S] = \tilde{a}\,\upsilon^{-n_S}$ the space character of a in rank n_S, and

$$\alpha = a[n_S](\varpi_{\bar{w}})\,q^{n_S/2}, \qquad \beta = \upsilon(\varpi_{\bar{w}})^{-n_T}\,q^{n_T/2}$$

(independent of the uniformiser; $|\alpha| = q^{n_S/2}$, $|\beta| = q^{n_T/2}$). Then F_y is right invariant under K_w, and for every $X \in \mathbb{C}$

$$F_{H^T_w(X)y} = \prod_{j=0}^{k-1}(1 - q^j\alpha X) \cdot H^S_w(q^k\alpha\beta^{-1}X)F_y.$$

Proof. Throughout, $A = \{\bar{w}\}$; it contains the conjugate of none of its members. Write $\omega = \omega_{a,1}$, so $\theta_{a,1}^{S,T}(\varphi)(g,h) = \theta(\omega(g,h)\varphi)$ with $\theta = \theta_{\mathrm{Hom}(S,T)}$, and $\omega(g,h)\omega(g',h') = \omega(gg',hh')$ by (2) of Lemma 4.1.1; for finite elements we write $\omega(g,1)$, $\omega(1,h)$ for $\omega((1,g),(1,1))$, $\omega((1,1),(1,h))$. Complex conjugation is a ring isomorphism $E_w \to E_{\bar{w}}$ carrying $\mathcal{O}_w$ onto $\mathcal{O}_{\bar{w}}$; for a matrix γ over E_w, $\bar{\gamma}$ is the matrix over $E_{\bar{w}}$. The residue field k of $\bar{w}$ has q elements; for an integral matrix x over $E_{\bar{w}}$, $\bar{x}$ (in this proof) denotes its reduction modulo $\bar{w}$ and $r(x)$ the rank of the reduction. For $m \geq 0$ let $\mathrm{K} = \mathrm{GL}_m(\mathcal{O}_w)$, $D_i^{(m)} = \mathrm{K}\delta_i\mathrm{K}$, and recall that $\iota_w(\gamma)$ has component γ at w and $(\bar{\gamma}^{\mathsf{T}})^{-1}$ at $\bar{w}$. We use three classical facts: (C1) by the elementary divisor theorem, $\gamma\mathrm{K} \mapsto \gamma\mathcal{O}_w^m$ is a bijection from $D_i^{(m)}/\mathrm{K}$ onto the lattices L with $\varpi_w\mathcal{O}_w^m \subseteq L \subseteq \mathcal{O}_w^m$ and $[\mathcal{O}_w^m : L] = q^i$, and from $(D_i^{(m)})^{-1}/\mathrm{K}$ onto the lattices with $\mathcal{O}_w^m \subseteq L \subseteq \varpi_w^{-1}\mathcal{O}_w^m$ and $[L : \mathcal{O}_w^m] = q^i$; (C2) a vector space of dimension N over k has $\binom{N}{i}_q$ subspaces of dimension i (Gaussian binomial coefficient, 0 for $i > N$), and $\prod_{l=0}^{N-1}(1+q^l t) = \sum_{i=0}^{N} q^{\binom{i}{2}}\binom{N}{i}_q t^i$ (the q-binomial theorem); (C3) $\mathrm{GL}_m(E_w)$ is unimodular, and $[\mathrm{U}(T)]$ is compact because T is definite.

Step 1: the vacuum and the action at $\bar{w}$. For an integral $y \in \mathbb{A}_{E,f}^{n_T \times n_S}$, $y_{A\cup\bar{A}}$ is integral with components 0 outside $\{w,\bar{w}\}$, so $\rho_f(y_{A\cup\bar{A}})\varphi = \varphi$ and, averaging, $P^A\varphi = \varphi$ ((WR6)). By Lemma 6.6.12, $\varphi = \mathbf{1}_A \otimes_A \varphi$. For a function Ψ on $\mathrm{M}_{n_T\times n_S}(E_{\bar{w}})$ we read it as the function $x \mapsto \Psi(x_{\bar{w}})$ of finite adelic matrices. Let $g \in G_f^{(n_S)}$, $h \in G_f^{(n_T)}$ have components 1 outside $\{w,\bar{w}\}$. By (4) of Lemma 4.1.1 (for $\alpha = a$, $\beta = 1$, so $\beta[n_T] = \upsilon^{-n_T}$),

$$\begin{aligned} \omega(g,1)\omega(1,h)\varphi = a[n_S](\det_A h)\,\upsilon^{n_T}(\det_A g)\,|\det_A g|^{n_T/2}\,|\det_A h|^{-n_S/2}\;\Psi'\otimes_A\varphi,\\ \Psi'(x) = \mathbf{1}_A(h_{\bar{w}}^{-1}x\,g_{\bar{w}}), \end{aligned} \tag{1}$$

where $\mathbf{1}_A$ is the indicator of the integral matrices. The characters $\tilde{a}$ (by hypothesis) and υ (by (WD3)) are trivial on the ideles with a unit at $\bar{w}$ and 1 elsewhere; so for an idele d with component of valuation j at $\bar{w}$ and 1 elsewhere, $a[n_S](d) = a[n_S](\varpi_{\bar{w}})^j$ and $\upsilon(d) = \upsilon(\varpi_{\bar{w}})^j$, $\varpi_{\bar{w}}$ the idele with a uniformiser at $\bar{w}$ and 1 elsewhere. Put

$$\alpha = a[n_S](\varpi_{\bar{w}})\,q^{n_S/2}, \qquad \beta = \upsilon(\varpi_{\bar{w}})^{-n_T}\,q^{n_T/2};$$

$|\alpha| = q^{n_S/2}$ and $|\beta| = q^{n_T/2}$, as automorphic characters of $\mathrm{U}(1)$ and υ take values of modulus 1, and they depend only on a, w, n_S, n_T. For $k_1 \in \mathrm{GL}_{n_S}(\mathcal{O}_w)$, $k_2 \in \mathrm{GL}_{n_T}(\mathcal{O}_w)$ the determinants at $\bar{w}$ of $\iota_w(k_1)$, $\iota_w(k_2)$ are units, the scalar in (1) is 1 and $\Psi' = \mathbf{1}_A$; hence

$$\omega(\iota_w(k_1),1)\varphi = \varphi, \qquad \omega(1,\iota_w(k_2))\varphi = \varphi. \tag{2}$$

Step 2: invariance of F_y and the T-side. By (2),

$$F_y(g\iota_w(k_1)) = \int \theta(\omega(g,h)\omega(\iota_w(k_1),1)\varphi)y(h)\,dh = F_y(g)\,:$$

F_y is right invariant under K_w. Fix g and put $\Phi(h) = \theta(\omega(g,h)\varphi)$, a continuous function on $\mathrm{U}(T)(\mathbb{A}_F)$ (bounded functional, continuous representation), left invariant under $\mathrm{U}(T)(F)$ by Lemma 6.2.5 and, by (2), right invariant under K_w; right translates of Φ and of y keep the left invariance, so all integrals below are over $[\mathrm{U}(T)]$. Let μ be the Haar measure of $\mathrm{GL}_{n_T}(E_w)$ with $\mu(\mathrm{K}) = 1$. For a right K-invariant y and a set R_i of representatives of D_i/K ($D_i = D_i^{(n_T)}$), $\sum_{\gamma\in R_i} y(h\iota_w(\gamma)) = \int_{D_i} y(h\iota_w(x))\,d\mu(x)$. As D_i is compact, $[\mathrm{U}(T)]$ compact and all functions continuous, Fubini's theorem, the right invariance of the quotient measure and the invariance of

μ under $x \mapsto x^{-1}$ (C3) give

$$F_{T_{w,i}y}(g) = \int_{D_i}\int \Phi(h)\, y(h\iota_w(x))\, dh\, d\mu(x) = \int y(h) \int_{D_i^{-1}} \Phi(h\iota_w(x))\, d\mu(x)\, dh$$
$$= \int \theta\Big(\omega(g,h) \sum_{\varepsilon\in R_i'} \omega(1,\iota_w(\varepsilon))\varphi\Big) y(h)\, dh,$$

with R_i' a set of representatives of D_i^{-1}/K, using the right K-invariance of Φ and linearity of θ.

For $\varepsilon \in R_i'$, $h = \iota_w(\varepsilon)$ has $h_{\bar w}^{-1} = \bar\varepsilon^{\mathsf T}$ and $\det_A h = \overline{\det \varepsilon}^{\,-1}$, of valuation i at $\bar w$; by (1) $\omega(1,\iota_w(\varepsilon))\varphi = \alpha^i\, \mathbf{1}_A(\bar\varepsilon^{\mathsf T}\cdot) \otimes_A \varphi$. A matrix x over $E_{\bar w}$ has $\bar\varepsilon^{\mathsf T} x$ integral exactly when its columns lie in the dual lattice $L^* = \{c : c^{\mathsf T} L \subseteq \mathcal{O}_{\bar w}\}$ of $L = \bar\varepsilon \mathcal{O}_{\bar w}^{n_T}$. By (C1), transported by conjugation, L runs once over the lattices with $\mathcal{O} \subseteq L \subseteq \varpi^{-1}\mathcal{O}$, $[L:\mathcal{O}] = q^i$, so L^* runs once over the lattices with $\varpi\mathcal{O} \subseteq L^* \subseteq \mathcal{O}$, $[\mathcal{O}:L^*] = q^i$, that is over the subspaces $V = L^*/\varpi\mathcal{O}$ of k^{n_T} of dimension $n_T - i$. The columns of x lie in L^* iff x is integral and $\operatorname{im}\bar x \subseteq V$. By (C2), applied to $k^{n_T}/\operatorname{im}\bar x$ (dimension $n_T - r(x)$),

$$\sum_{\varepsilon\in R_i'} \mathbf{1}_A(\bar\varepsilon^{\mathsf T} x) = \mathbf{1}_A(x) \begin{pmatrix} n_T - r(x) \\ i \end{pmatrix}_q.$$

By linearity of $\Psi \mapsto \Psi \otimes_A \varphi$ and (C2) with $t = -\alpha X$,

$$\sum_{i=0}^{n_T} (-X)^i q^{\binom{i}{2}} \sum_{\varepsilon\in R_i'} \omega(1,\iota_w(\varepsilon))\varphi = P_T \otimes_A \varphi, \qquad P_T(x) = \mathbf{1}_A(x) \prod_{l=0}^{n_T - r(x) - 1} (1 - q^l \alpha X). \tag{3}$$

Step 3: the S-side. Let R_i be a set of representatives of $D_i^{(n_S)}/\mathrm{GL}_{n_S}(\mathcal{O}_w)$. Then $(T_{w,i}F_y)(g) = \sum_{\gamma\in R_i} F_y(g\iota_w(\gamma)) = \int \theta(\omega(g,h)\sum_\gamma \omega(\iota_w(\gamma),1)\varphi) y(h)\, dh$. For $\gamma \in R_i$, $g' = \iota_w(\gamma)$ has $g'_{\bar w} = (\bar\gamma^{\mathsf T})^{-1}$ and $\det_A g'$ of valuation $-i$, so by (1) $\omega(\iota_w(\gamma),1)\varphi = \beta^i\, \mathbf{1}_A(\cdot\,(\bar\gamma^{\mathsf T})^{-1}) \otimes_A \varphi$. Now $x(\bar\gamma^{\mathsf T})^{-1}$ is integral iff $\bar\gamma^{-1}x^{\mathsf T}$ is, iff the columns of $x^{\mathsf T}$ lie in $M = \bar\gamma \mathcal{O}_{\bar w}^{n_S}$; by (C1) M runs once over the lattices with $\varpi\mathcal{O} \subseteq M \subseteq \mathcal{O}$ of index q^i, i.e. over the subspaces $M/\varpi\mathcal{O}$ of k^{n_S} of dimension $n_S - i$, and the condition says that x is integral and $\operatorname{im}\bar x^{\mathsf T} \subseteq M/\varpi\mathcal{O}$. As $\operatorname{rank}\bar x^{\mathsf T} = r(x)$, (C2) gives, for every $Y \in \mathbb{C}$,

$$\sum_{i=0}^{n_S} (-Y)^i q^{\binom{i}{2}} \sum_{\gamma\in R_i} \omega(\iota_w(\gamma),1)\varphi = P_S \otimes_A \varphi, \qquad P_S(x) = \mathbf{1}_A(x) \prod_{l=0}^{n_S - r(x) - 1} (1 - q^l \beta Y). \tag{4}$$

Step 4: comparison. Take $Y = q^k \alpha\beta^{-1} X$, so $q^l \beta Y = q^{k+l}\alpha X$. Since $r(x) \le n_S$ and $n_T - r(x) = k + (n_S - r(x))$, splitting the product in P_T after its first k factors gives $P_T = \prod_{j=0}^{k-1}(1 - q^j\alpha X)\, P_S$. Summing the formula of Step 2 over i with the coefficients $(-X)^i q^{\binom{i}{2}}$ and using (3), then (4) and Step 3,

$$F_{H_w^T(X)y}(g) = \int \theta(\omega(g,h)(P_T \otimes_A \varphi)) y(h)\, dh$$
$$= \prod_{j=0}^{k-1} (1 - q^j\alpha X) \int \theta(\omega(g,h)(P_S \otimes_A \varphi)) y(h)\, dh$$
$$= \prod_{j=0}^{k-1} (1 - q^j\alpha X)\, (H_w^S(Y)F_y)(g).$$

□

Lemma 6.6.12. *Let s be signs on n and A a finite set of finite places of E containing the conjugate of none of its members. Then $\mathbf{1}_A \in \mathcal{T}_A$ and $\mathbf{1}_A \otimes_A f = P^A f$ for every $f \in \mathcal{M}_s$.*

Proof. The function $\mathbf{1}_A$ depends only on the components at A, where it is the indicator of $\prod_{w\in A} \mathcal{O}_{E,w}^n$, a compact open subset of Q_A; so $\mathbf{1}_A$ is locally constant with compact support on Q_A, and $\mathbf{1}_A \in \mathcal{T}_A$ by (1) of Lemma 4.1.3. (Directly: $\mathbf{1}_A = \tau_0$ in (WR6).) The identity $\mathbf{1}_A \otimes_A f = P^A f$ is the last assertion of (1) of Lemma 4.1.3; it rests on the independence of $\Psi \otimes_A f$ from the chosen section proved there, which gives $\mathbf{1}_A \otimes_A f = \tau_0 \otimes_A f = \rho(\mathrm{fv}(0)) P^A f = P^A f$. □

The comparison operator is built from a seesaw: $U_\vartheta = n_0 \,\mathrm{gl}_\vartheta \mathrm{gl}_\vartheta^*$, with gl_ϑ the comparison form of the j-th lift, a gluing map from tables to forms (Proposition 6.6.13). The seesaw identity of the gluing (Lemma 6.6.29) gives numbers c'_ϑ, with $\Gamma_j(\vartheta)|c'_\vartheta|^2$ independent of ϑ, and one map Θ° independent of the twist such that $c'_\vartheta \,\mathrm{gl}_\vartheta^* \circ \Theta_{j,\vartheta} = \Theta^\circ$. So on the lifts U_ϑ is $\rho_j(\vartheta)$ times a fixed operator, positive as soon as Θ° is injective. Orthogonality to the false lifts, integrality and size follow from Lemma 6.6.39 and Lemma 6.6.16.

Integrality is measured here by the chosen group order N_{ch}, one of the quantities of the clearing bound (Definition 6.6.15), which is printed at this point together with the defect bounds it rests on (Definition 6.6.14). The clearing bound also contains an odd-class integer X_{odd}, of which only a size bound is assumed. Section 7 fixes it through the odd Sha bound (Theorem 7.2.8) and uses the clearing bound for the denominators of Ribet's element (Theorem 7.6.6).

The L-values enter through the third exchange (Proposition 6.6.36), which rests on the explicit matching of Section 6.3 and on the carrier, a theta integral over a negative definite unitary group of rank m; the carrier is not zero, by the Rallis inner product formula and the injectivity of the Rallis series (Lemma 6.6.64). For seesaw dual pairs see [171]; identities that move a theta integral across a seesaw are the mechanism of the Rallis inner product formula and the doubling method [231]. Orthogonality to the false lifts uses Green's theory of the characters of the finite general linear groups [106] (Lemma 6.6.42).

The injectivity of Θ° is the statement that products span (Proposition 6.6.44), proved by isolation at a peak. A peak of a holomorphic form is not deep in a cusp (Lemma 6.6.58); away from the cusps a combination of the choice functions is close to a fixed function that does not vanish at 1 (Lemma 6.6.61); and the isolation estimate (Lemma 6.6.80) then forces the form to vanish. Its inputs are the holomorphic model functions, which are Flensted-Jensen's functions of the holomorphic chamber [163, 81], reduction theory to locate the cusps [37, 33], and the Fock model [14]. That the good forms on W_j are not zero is Theorem B (Theorem 5.1.2), for the Poincaré map fixed in Section 5, through the equality of dimensions (Proposition 5.8.4).

Proposition 6.6.13. *Let D be admissible data. For every $1 \le j \le m$ there is a comparison operator for the j-th lift.*

Proof. Fix $1 \le j \le m$. Let Y, c'', L, n_0 be as in Lemma 6.6.16; for a twist ϑ let $\mathrm{gl}_\vartheta = \mathrm{gl}_{j,\vartheta}[L; c'']$, formed with Y, and

$$U_\vartheta = n_0 \,\mathrm{gl}_\vartheta \,\mathrm{gl}_\vartheta^* \quad \text{on } \mathcal{A}_{j,\vartheta}(V^{\mathrm{def}}) = \mathrm{Form}(O_{j,\vartheta}).$$

The hypotheses of Lemma 6.6.29 and Lemma 6.6.39 hold for (Y, c'', L); let C, c'_ϑ, c_i° and Θ° be as in Lemma 6.6.29. For $\zeta \in \mathrm{Tab}_j$ write $\zeta_i \colon \mathfrak{c} \mapsto \zeta_{i,\mathfrak{c}}$. The pairing $\mathrm{tt}_j(\zeta')$ is conjugate-linear in ζ' (each kernel of (XE7) is linear in its polynomial, and $\mathrm{Pol}_{\kappa_O^-}$, Let_{-1} are linear), so $\mathrm{tt}_j(0) = 0$.

Step 1: Θ° is injective. Let $\Theta^\circ y = 0$. Fix $i \in \mathrm{Mem}_j$ and $\zeta' \colon \mathrm{Cl}^\circ \to \mathbb{V}_j^-$, and let $\zeta \in \mathrm{Tab}_j$ have $\zeta_i = \zeta'$ and $\zeta_{i'} = 0$ for $i' \ne i$. By (iii) of Lemma 6.6.29,

$$0 = \langle \zeta, \Theta^\circ y \rangle = c_i^\circ \int_{[\mathrm{U}(W_j)]} y \,\mathsf{F}_i \,\mathrm{tt}_j(\zeta'),$$

and $c_i^\circ \neq 0$. As i and ζ' are arbitrary, Proposition 6.6.44 (for the short row family Y) gives $y = 0$.

Step 2: (CO1). By (iii) of Proposition 5.8.4 and Theorem 5.1.2, applied to the parameters fixed above (the Poincaré map, the seesaw constant and the divisibility errors), whose map satisfies the hypothesis of that theorem (the two moment bounds) by Proposition 5.6.48, the hypothesis of which it satisfies by Lemma 5.6.33 (ii), $\dim \mathcal{A}^{\mathrm{good}}(W_j) = d_{\mathcal{T}} \geq 1$; take $y_0 \neq 0$, so $\Theta^\circ y_0 \neq 0$ by Step 1. If $c'_\vartheta = 0$ for some twist ϑ, (ii) of Lemma 6.6.29 gives $\Theta^\circ = 0$; so $c'_\vartheta \neq 0$ for every ϑ. By (i) there, $C = \Gamma_j(1)|c'_1|^2 \neq 0$ and $\Gamma_j(\vartheta) = C/|c'_\vartheta|^2$, so

$$\rho_j(\vartheta) = \frac{\Gamma_j(\vartheta)}{\Gamma_j(1)} = \frac{|c'_1|^2}{|c'_\vartheta|^2} \neq 0.$$

Step 3: (CO2). Let $U^\circ = n_0|c'_1|^{-2}\,\Theta^{\circ *}\Theta^\circ$ on $\mathcal{A}^{\mathrm{good}}(W_j)$ (the adjoint for the Petersson product and the inner product of Tab_j, both spaces being finite-dimensional); it does not depend on ϑ, and $\langle y, U^\circ y\rangle = n_0|c'_1|^{-2}\|\Theta^\circ y\|^2 > 0$ for $y \neq 0$ by Step 1. By (ii) of Lemma 6.6.29, $\mathrm{gl}^*_\vartheta \Theta_{j,\vartheta} y = c'^{-1}_\vartheta \Theta^\circ y$, so for $y, y' \in \mathcal{A}^{\mathrm{good}}(W_j)$

$$\langle \Theta_{j,\vartheta} y', U_\vartheta \Theta_{j,\vartheta} y\rangle = n_0 \langle \mathrm{gl}^*_\vartheta \Theta_{j,\vartheta} y', \mathrm{gl}^*_\vartheta \Theta_{j,\vartheta} y\rangle = \frac{n_0}{|c'_\vartheta|^2}\langle \Theta^\circ y', \Theta^\circ y\rangle = \rho_j(\vartheta)\,\langle y', U^\circ y\rangle,$$

by Step 2.

Step 4: (CO3). Let $z = \Theta_{j,\vartheta} y \in \mathcal{L}_{j,\vartheta}$; then $U_\vartheta z = n_0 c'^{-1}_\vartheta\, \mathrm{gl}_\vartheta(\Theta^\circ y)$. Let w be a finite place with $w \neq \bar{w}$, neither w nor $\bar{w}$ a place of $O_{j,\vartheta}$, and $\varphi \in \mathcal{F}_w(e_0, \mathcal{L}^\perp_{j,\vartheta})$. With $\zeta = \mathrm{gl}^*_\vartheta \varphi$, (iii) of Lemma 6.6.29 and Lemma 6.6.39 give

$$\langle \varphi, \mathrm{gl}_\vartheta(\Theta^\circ y)\rangle = \langle \zeta, \Theta^\circ y\rangle = \sum_i c_i^\circ \int_{[\mathrm{U}(W_j)]} y\, \mathsf{F}_i\, \mathrm{tt}_j(\zeta_i) = 0.$$

So $U_\vartheta z$ is orthogonal to $\mathcal{F}_w(e_0, \mathcal{L}^\perp_{j,\vartheta})$ for every such w, i.e. orthogonal to the false lifts for $(e_0, \mathcal{L}^\perp_{j,\vartheta})$.

Step 5: (CO4), (CO5). These are (a) and (b) of Lemma 6.6.16.

Hence $U = (U_\vartheta)$ is a comparison operator for the j-th lift. □

Definition 6.6.14. Let D be admissible data and put $h = \Delta/2 = r^8$.

CM sign. For a complex embedding $\sigma\colon E \to \mathbb{C}$, $\mathrm{sg}(\sigma) = 1$ if $\sigma = \iota_0 \circ g$ for some $g \in \mathrm{Gal}(E/K)$ (the embedding chosen at its place) and $\mathrm{sg}(\sigma) = -1$ otherwise. For a complex embedding σ, $\kappa_j(\sigma)$ means κ_j at the place of σ. For a prime ℓ, an isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, an embedding $e\colon E \to \overline{\mathbb{Q}}_\ell$ and $g \in G_\ell$, put $\sigma_{e,g} = \iota^{-1} \circ g^{-1} \circ e$.

Main-summand weight systems. Let $1 \leq r' \leq r$ and $t' \leq t$. A family $J = (J(\sigma))_\sigma$ of finite multisets of integers, indexed by the complex embeddings σ of E, is a **main-summand weight system** of rank r' and length t' if

(DW1) there is $j \leq m$ such that for every σ and every $u \in J(\sigma)$: either $\Delta \leq 4|u|$ and $|u| \leq 2m\Delta$ (a *low* jump), or $|\mathrm{sg}(\sigma)\, u + \kappa_j(\sigma)| \leq t$ (a *high* jump);

(DW2) for every σ, $k \cdot \#\{u \in J(\sigma) : |u| \leq 2h^2\} \leq r'm$;

(DW3) for every σ, $\#\{u \in J(\sigma) : u < 0\} + \#\{u \in J(\bar{\sigma}) : u < 0\} \leq r'$,

multisets counted with multiplicity and $\bar{\sigma}$ the complex conjugate of σ. The **far window start** at σ for $j \leq m$ is $S_j(\sigma) = \kappa_j(\sigma) - 2t$ (a non-negative integer, as $\kappa_j(\sigma) \geq h = r^8 > 2t$).

Loss and defect bounds. For a prime ℓ, the **crystalline loss** is

$$\lambda_\ell = v_\ell\Big((2b)!\cdot\Big(\prod_{\max(b-30h^3,0)<a\le b+30h^3} a\Big)^{h^3}\Big),$$

the **defect bound** and the **local defect bound** are

$$\mathrm{db}_\ell = 10^6\,\frac{m}{k}\,(b\,v_\ell(M)+\lambda_\ell), \qquad \mathrm{loc}_\ell = 4f\,\mathrm{db}_\ell, \tag{6.130}$$

non-negative reals.

Definition 6.6.15. Let D be admissible data and ℓ a prime. Write $h=\Delta/2=r^8$ (the **half gap**).

(CX1) Chosen group order. $N_{\mathrm{ch}}=\prod_w \#\mathrm{GL}_{rt}(k_w)$, the product over the chosen places w of E, with k_w the residue field at w.

(CX2) Twist error. $e'_\ell=(4f+3)\,r\,v_\ell(N_{\mathrm{ch}})+2^{-\ell}$.

(CX3) Odd-class integer. The **odd-class integer** of D is an integer X_{odd} depending on D alone with

$$1\le X_{\mathrm{odd}}\le 2^{4(q_1d+t+3)^{27}}\,\#\mathrm{Cl}(E),$$

where $\#\mathrm{Cl}(E)$ is the class number of E; this definition uses no other property of it.

(CX4) Denominator error.

$$e_\ell = t\,v_\ell(X_{\mathrm{odd}})+(8ft+2+(4t+1)h^{22}+h)\,v_\ell(N_{\mathrm{ch}}).$$

(CX5) Clearing bound.

$$\beta_\ell=\mathrm{loc}_\ell+(1+2t)\,e'_\ell+e_\ell+v_\ell(N_{\mathrm{ch}}).$$

In (CX4) the term $t\,v_\ell(X_{\mathrm{odd}})$ pays for the odd classes (and, at $e=0$, the class group), $((4t+1)h^{22}+h)v_\ell(N_{\mathrm{ch}})$ for the triangle, and the rest for the principal level, the idempotent and the automorphisms.

Lemma 6.6.16. *Let $1\le j\le m$. There are a short row family Y, a number $c''\in\mathbb{C}^\times$, divided lattices $L=(L_i)_{i\in\mathrm{Mem}_j}$ of $O_{j,1}$ with L_i glued for Λ_i, and an integer $n_0\ge 1$, such that $\mathcal{V}_j$ is a divided splitting for $O_{j,\vartheta}$ for every twist ϑ and, with $\mathrm{gl}_\vartheta=\mathrm{gl}_{j,\vartheta}[L;c'']$ formed with Y and $U_\vartheta=n_0\,\mathrm{gl}_\vartheta\mathrm{gl}_\vartheta^*$ on $\mathrm{Form}(O_{j,\vartheta})$:*

(a) for every twist ϑ, every prime $\ell\nmid \mathrm{N}Q$ and every embedding $\mathbb{C}\to\mathbb{C}_\ell$, U_ϑ maps the forms integral at that embedding to forms integral at it;

(b) $\|U_1\varphi\|_\sigma^2\le\exp\big(6fmb(\log b)^2\big)\|\varphi\|_\sigma^2$ for every $\varphi\in\mathrm{Form}(O_{j,1})$ and every automorphism σ of $\mathbb{C}$.

Proof. Fix j and write $\kappa^-=\kappa^-_{O_{j,1}}$ in rank $|\mathcal{B}^\circ|$, $h=h_{\kappa^-}\ge 1$, $E=fmb(\log b)^2$.

Step 1: the divided splitting. By (TG2) we must show, for every τ and every twist (the weights do not involve it), $\kappa^+_O(\tau,h')=0$ for $|\mathcal{B}_j|\le h'<n$ and $\kappa^-_O(\tau,h')=0$ for $|\mathcal{B}^\circ|\le h'<n$. First, $18m+5<2r$:

So $|\mathcal{B}^\circ| = 5r - 9m - 1 > r$. For $i \geq r$ the extended entry is $\varsigma_\tau(i) = \frac{\Delta}{2} - \kappa_{i+2-k}(\tau)$ with $i + 2 - k \geq m + 1 > j$, and $\kappa_{j'} = \kappa_j + (j' - j)\Delta$, so $\varsigma_\tau(i) \leq -\frac{\Delta}{2} - \kappa_j(\tau) < \frac{t-1}{2} - \kappa_j(\tau)$. If $h' \geq |\mathcal{B}_j| = r + 9m + 1$, then $h' - 1 \geq r$, so $\varsigma_\tau(h' - 1) - \max(\varsigma_\tau(h'), \frac{t-1}{2} - \kappa_j(\tau)) - 1 < 0$ and $\kappa_O^+(\tau, h') = 0$. If $h' \geq |\mathcal{B}^\circ| > r$, both arguments $r - h' - 1$ and $r - h'$ are negative and read as 0, so $\kappa_O^-(\tau, h') = (\min(\varsigma_\tau(0), -\kappa_j(\tau) - \frac{t-1}{2}) - \varsigma_\tau(0) - 1)^+ = 0$. Hence $\mathcal{V}_j$ is a divided splitting for every $O_{j,\vartheta}$.

Step 2: the choices. Let Y and c'' be as in Lemma 6.6.17. For each $i \in \mathrm{Mem}_j$, Lemma 6.6.24 (for $O_{j,1}$, the divided splitting $\mathcal{V}_j$ of Step 1 and $s = \Lambda_i$) gives a divided lattice L_i glued for Λ_i. Put $n_0 = N_{\mathrm{ch}} h^2$, a positive integer, $\mathrm{gl}_\vartheta = \mathrm{gl}_{j,\vartheta}[L; c'']$ and $U_\vartheta = n_0 \mathrm{gl}_\vartheta \mathrm{gl}_\vartheta^*$. By Steps 1 and 2 the hypotheses of Lemma 6.6.25 hold.

Step 3: frames. Let $(f_a)_{a \in A}$ be tables with $\sum_a \langle f_a, \zeta \rangle f_a = h^2 \zeta$ for all $\zeta \in \mathrm{Tab}_j$. For $\Xi \in \mathrm{Form}(O_{j,\vartheta})$, applied to $\zeta = \mathrm{gl}_\vartheta^* \Xi$ and since $\langle f_a, \mathrm{gl}_\vartheta^* \Xi \rangle = \langle \mathrm{gl}_\vartheta f_a, \Xi \rangle$,

$$U_\vartheta \Xi = N_{\mathrm{ch}} h^2 \, \mathrm{gl}_\vartheta (h^{-2} \textstyle\sum_a \langle \mathrm{gl}_\vartheta f_a, \Xi \rangle f_a) = \sum_{a \in A} c_a \, \mathrm{gl}_\vartheta f_a, \qquad c_a = N_{\mathrm{ch}} \langle \mathrm{gl}_\vartheta f_a, \Xi \rangle. \tag{6.131}$$

Step 4: (a). Let ϑ, $\ell \nmid \mathrm{N}Q$ and ι be given and let Ξ be integral at ι. Take the frame of (a) of Lemma 6.6.27 for ℓ. For every ring homomorphism $\iota' \colon \mathbb{C} \to \mathbb{C}_\ell$ the conjugated values $\overline{(f_a)}_{i,\mathfrak{c}}$ have gauge at most 1 at $x_\mathfrak{c}$ for $\iota' \circ c$; applied with $\iota' = \iota$ and with $\iota' = \iota \circ c$ (for which $\iota' \circ c = \iota$), (a) of Lemma 6.6.25 shows that every $\mathrm{gl}_\vartheta f_a$ is integral at ι and at $\iota \circ c$. By Lemma 6.6.28 (with Ξ there $\mathrm{gl}_\vartheta f_a$ and Ξ' there Ξ), $|\iota(c_a)|_\ell \leq 1$. The conditions defining integrality at ι are bounds $|\cdot|_\ell \leq 1$ on coordinates that depend linearly on the form, so by the ultrametric inequality the forms integral at ι are closed under sums and under multiplication by numbers c with $|\iota(c)|_\ell \leq 1$. By (6.131), $U_\vartheta \Xi$ is integral at ι.

Step 5: (b). Take $\vartheta = 1$ and the frame of (b) of Lemma 6.6.27. For an automorphism σ of $\mathbb{C}$ let $\sigma' = c \circ \sigma \circ c$; then $\sigma(\bar{z}) = \overline{\sigma'(z)}$ for $z \in \mathbb{C}$. The weights $1/\#\Gamma_h$ and the Gram matrix are rational, so for $u, v \in \mathrm{Form}(O_{j,1})$, $\sigma(\langle u, v \rangle)$ is the class sum of $\overline{\sigma'(u(h))}^{\mathsf{T}} G_{\kappa_O} \sigma(v(h))$, and the Cauchy–Schwarz inequality for this positive semidefinite hermitian form gives $|\sigma(\langle u, v \rangle)| \leq \|u\|_{\sigma'} \|v\|_\sigma$. With (b) of Lemma 6.6.25 for σ' and $\|f_a\|_{\sigma'} \leq h$,

$$|\sigma(c_a)| \leq N_{\mathrm{ch}} \, \|\mathrm{gl}_1 f_a\|_{\sigma'} \, \|\Xi\|_\sigma \leq N_{\mathrm{ch}} \, e^E h \, \|\Xi\|_\sigma.$$

By (6.131), $U_1 \Xi = \mathrm{gl}_1 \zeta$ with $\zeta = \sum_a c_a f_a$, and as σ is additive and multiplicative on coordinates, the triangle inequality gives $\|\zeta\|_\sigma \leq \sum_a |\sigma(c_a)| \, \|f_a\|_\sigma \leq \#A \, N_{\mathrm{ch}} h^2 e^E \|\Xi\|_\sigma$. By (b) of Lemma 6.6.25 and (c) of Lemma 6.6.27 (with $\#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ) \geq 1$),

$$\|U_1 \Xi\|_\sigma^2 \leq e^{2E} \|\zeta\|_\sigma^2 \leq e^{4E} \big(\#A \, N_{\mathrm{ch}} h^2\big)^2 \|\Xi\|_\sigma^2 \leq e^{6E} \|\Xi\|_\sigma^2 = \exp\big(6 f m b (\log b)^2\big) \|\Xi\|_\sigma^2.$$

□

Lemma 6.6.17. *Let $1 \leq j \leq m$. There are a short row family Y and $c'' \in \mathbb{C}^\times$ such that the first factors $\varphi''_{\vartheta,i}$ with scale c'', formed with Y, satisfy:*

(i) for every twist ϑ with modulus Q, every $i \in \mathrm{Mem}_j$, every $h \in G_f(\mathcal{B}_j)$, every prime $\ell \nmid \mathrm{N}Q$ and every ring homomorphism $\iota \colon \mathbb{C} \to \mathbb{C}_\ell$: $\mathrm{gau}_\iota\big(h, \varphi''_{\vartheta,i}(h)\big) \leq 1$;

(ii) for every $i \in \mathrm{Mem}_j$, every $h \in G_f(\mathcal{B}_j)$ and every automorphism σ of $\mathbb{C}$: $\big\|\sigma\big(\varphi''_{1,i}(h)\big)\big\|_{\kappa_O^+}^2 \leq \exp\big(f m b (\log b)^2\big)$, σ applied to the coordinates.

Proof. Fix j, write $\kappa = \kappa^+_{O_{j,1}}$ (the weight of $\mathbb{V}^+_j$, free of the twist), $\mathfrak{M} = \pi_M\mathcal{O}_E$, $\#D_j = \sum_\tau \#D_j(\tau)$, and for a twist ϑ with modulus Q put $\mathfrak{n}_\vartheta = \mathfrak{M}^2\prod_{w\in Q} w$ (so $\mathfrak{n}_1 = \mathfrak{M}^2$). For an element or matrix z write $\bar z$ for its complex conjugate (entrywise).

Step 0: the choices. Let Y be a short row family (Lemma 6.6.18), and d, H as in (a) of Lemma 6.6.21. By Lemma 6.6.20 choose $c_0, c_1, c_2 \in \mathbb{C}^\times$ for the pairs of index sets $(\{1,\dots,m\}, \mathcal{B}^{\mathrm{c}})$, $(\mathcal{A}^-_j, \mathcal{B}_j)$, $(\mathcal{B}^{\mathrm{add}}_j, \mathcal{B}_j)$ and the degrees 0, 0, d, and by Lemma 6.4.55 let c_3, $|c_3| = 1$, be the constant for $(S,T) = (\mathrm{I}_{\mathcal{A}^-_j}, \mathrm{I}_{\mathcal{B}_j})$. Put

$$c'' = N^2_{\mathrm{ch}}\, c_0\, \bar c_1\, c_2\, H\, c_3 \neq 0.$$

These choices depend only on j and D.

Step 1: the first factor as a double class sum. Fix a twist ϑ with modulus Q, a choice i and $h \in G_f(\mathcal{B}_j)$; write $\chi' = \chi'_{j,\vartheta}$, $\nu'' = \nu''_{j,\vartheta}$, $b' = \nu''^{-1}\upsilon^{|\mathcal{B}_j|}$, $\mathfrak{n} = \mathfrak{n}_\vartheta$ (the modulus $\mathfrak{M}^2\mathfrak{O}_O$ of (XE6)), $Y = Y_{\mathfrak{c}_i}$, $x = x_{\mathfrak{c}_i}$. Let $u \in R^\times_{\mathfrak{n}}$ be the unit of Lemma 6.4.55 and $\Psi_1(X) = \mathbf{1}^-_j(uX)$, a function with values 0, 1. By Lemma 6.4.55 for $S = \mathrm{I}_{\mathcal{A}^-_j}$ (rank j), $T = \mathrm{I}_{\mathcal{B}_j}$, the characters 1, b', the function Ψ_1 and $P = 1$, whose right side has the characters υ^j, $b'^{-1}\upsilon^{|\mathcal{B}_j|} = \nu''$ and the function $\overline{\Psi_1(u^{-1}X)} = \mathbf{1}^-_j(X)$,

$$\theta^{\mathrm{I}^-_{\mathcal{A}^-_j},\mathrm{I}_{\mathcal{B}_j}}_{\upsilon^j,\nu'';\mathfrak{n}}\big[\mathbf{1}^-_j, 1\big](1,h) = \overline{c_3\,\kappa_1(h)}, \qquad \kappa_1(h) = \theta^{\mathrm{I}_{\mathcal{A}^-_j},\mathrm{I}_{\mathcal{B}_j}}_{1,b';\mathfrak{n}}[\Psi_1, 1](1,h).$$

By (XD5), $\mathrm{mod}_{\mathfrak{M}^2}(1,v_1) = v_1 \otimes (\mathbf{1}_A \otimes_A v^0) = v_1 \otimes v^0$ ($A = \Sigma(\mathfrak{M}^2)$; Lemma 6.6.12 and (2) of Lemma 6.2.33), so by (XE4) $\mathsf{a}_{\chi'}(g_2) = \sum^{\mathrm{cl}}_{g_1} k_0(g_1,g_2)$ with $k_0(g_1,g_2) = \theta^{\mathrm{I}_{\{1,\dots,m\}},\mathrm{I}_{\mathcal{B}^{\mathrm{c}}}}_{\chi',1;\mathfrak{M}^2}[1,1](g_1,g_2)$, the class sum over $g_1 \bmod \mathbf{K}_0(\{1,\dots,m\})$. For a polynomial $P \in \mathbb{C}[x]_{\mathcal{B}_j\times\mathcal{B}^{\mathrm{add}}_j}$ and $g \in G_f(\mathcal{B}^{\mathrm{add}}_j)$ write $\theta_g(P) = \theta^{W^{\mathrm{add}}_j,\mathrm{I}_{\mathcal{B}_j}}_{\nu'',\nu'';\mathfrak{n}}[\Psi^{\mathrm{add}}_{\Lambda_i}, P](g,h)$, linear in P, and for $g_2 \in G_f(\mathcal{B}^{\mathrm{c}})$ let $\bar g_2 = (1,x,g_2)$ be the block diagonal point on $\mathcal{A}^+_j \sqcup \mathcal{B}^\flat \sqcup \mathcal{B}^{\mathrm{c}}$ and $u_{g_2} = (c_2\,\theta_{\bar g_2}(H\mathrm{lc}_{i,l}))_{l\in I_\kappa}$. With the class-sum weights ω_{g_1}, ω_{g_2} of (6.22) (positive rationals) and $s(g_1,g_2) = c_0 k_0(g_1,g_2)\,\overline{c_1\kappa_1(h)}$, (XE6) and $c_3\bar c_3 = 1$ give

$$\varphi''_{\vartheta,i}(h) = \overline{c''\,\Upsilon_{\vartheta,i}(h)} = \sum_{g_2}\sum_{g_1}(N_{\mathrm{ch}}\omega_{g_2})(N_{\mathrm{ch}}\omega_{g_1})\,\overline{s(g_1,g_2)}\;\overline{u_{g_2}}, \tag{6.132}$$

the sums over the fixed representatives g_2 of the classes of $\mathrm{I}_{\mathcal{B}^{\mathrm{c}}}$ at $\mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$ and g_1 of those of $\mathrm{I}_{\{1,\dots,m\}}$ at $\mathbf{K}_0(\{1,\dots,m\})$. By Lemma 6.7.9 ($m$, $3m \le rt$ by Lemma 6.6.22) the numbers $N_{\mathrm{ch}}\omega$ are positive integers.

Step 2: three facts on the gauge. Let ℓ be a prime, $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism and $\mathrm{gau} = \mathrm{gau}_\iota$ for the weight $(\kappa,0)$ on $\mathcal{B}_j$. (G1) $\mathrm{gau}(h,a\xi) = |\iota(a)|\,\mathrm{gau}(h,\xi)$ and $\mathrm{gau}(h,\sum_k n_k\xi_k) \le \max_k \mathrm{gau}(h,\xi_k)$ for integers n_k: $\iota(a\xi) = \iota(a)\iota(\xi)$, the map is linear, and $|\cdot|_\ell$ is ultrametric with $|n|_\ell \le 1$. (G2) For a rational point γ of $\mathrm{I}_{\mathcal{B}_j}$, $\mathrm{gau}(\gamma_f h, \rho_{\kappa,0}(\gamma)\xi) = \mathrm{gau}(h,\xi)$: for $y \in E$, $f_{\iota,\tau}(y) = \iota(\tau(\bar y)) = \iota(\overline{\tau(y)})$ (the embedding of the CM type commutes with conjugation), so $f_\iota(\gamma_f) = \iota(\overline{\tau(\gamma)})_\tau$, and as S_κ has integral polynomial entries, $\iota(\rho_{\kappa,0}(\gamma)\xi) = \iota(S_\kappa(\overline{\tau(\gamma)})\xi) = S_\kappa(f_\iota(\gamma_f))\iota(\xi)$; then $S_\kappa(f_\iota((\gamma_f h)^{-1}))S_\kappa(f_\iota(\gamma_f)) = S_\kappa(f_\iota(h^{-1}))$ by the multiplicativity of f_ι and of S_κ ((a) of Lemma 4.6.10). (G3) If the components of h at the places above ℓ are integral invertible, then $\mathrm{gau}(h,\xi) \le \max_k|\iota(\xi_k)|$: the place inducing $f_{\iota,\tau}$ lies above ℓ, so $f_\iota(h^{-1})$ has entries of absolute value at most 1, and so has $S_\kappa(f_\iota(h^{-1}))$.

Step 3: rational translation of definite kernels. Let $S = \mathrm{I}_L$, $T = \mathrm{I}_N$ be standard definite spaces (for instance $W^{\mathrm{add}}_j = \mathrm{I}_{\mathcal{B}^{\mathrm{add}}_j}$), α, β automorphic characters of weights e_α, e_β, γ_S, γ_T rational points of S, T, g, h' finite points and P a polynomial. Then

$$\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi,P](\gamma_{S,f}g, \gamma_{T,f}h') = \prod_{\tau\in\Phi}\det\tau(\gamma_T)^{-e_\alpha(\tau)}\det\tau(\gamma_S)^{e_\beta(\tau)}\;\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Psi, P^{A,B}](g,h'), \tag{6.133}$$

with $A_\tau = \tau(\gamma_T)^{-1}$, $B_\tau = \tau(\gamma_S)$ and $P^{A,B}$ the substitution $x_\tau \mapsto A_\tau^{\mathsf{T}} x_\tau B_\tau^{\mathsf{T}}$. Indeed, by Lemma 6.2.5 the left side is the kernel at the points with infinite components $x = \tau(\gamma_S)^{-1}$, $y = \tau(\gamma_T)^{-1}$ and finite components g, h'; by (1) of Lemma 6.3.71 it is the kernel at (g, h') of $\mathrm{mod}_{\mathfrak{m}}(\Psi, \Omega_{e_\alpha, e_\beta}(x,y)v_P)$; by (5) of Lemma 4.1.1, all signs being $+1$,

$$\Omega_{e_\alpha,e_\beta}(x,y)v_P = \prod_\tau \det(y_\tau)^{e_\alpha(\tau)} \det(x_\tau)^{-e_\beta(\tau)} v_{P'}$$

with P' the substitution $x_\tau \mapsto y_\tau^{\mathsf{T}} x_\tau (x_\tau^{-1})^{\mathsf{T}}$, which is $P^{A,B}$.

Step 4: integrality of the three factors. Let ϑ, i, h be as in Step 1, $\ell \nmid \mathrm{N}Q$ a prime, $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism, $\iota' = \iota \circ \bar{\cdot}$, also a ring homomorphism, and R the set of places of E above ℓ. By Lemma 4.2.64 (with the set R), for every finite point h_0 of a standard space there is a rational point δ with $\delta_f h_0$ integral invertible at R: the points whose components at the places $w \in R$ lie in $(h_0)_w$ times the integral invertible matrices form an open neighbourhood of $(1, h_0)$, so it contains some $\gamma(1,k)$ with γ rational and k trivial at R, and $\delta = \gamma^{-1}$ works. The number $q = \mathrm{N}Q$ satisfies $qM^2 \in \mathfrak{n}$ ($M \in \mathfrak{M}$, and $\mathrm{N}w \in w$). The characters 1, χ' (weight 0), b' and ν'' have weights and tame parts, and ν'' has the weight $-\#D_j(\tau)$: by (XE1), $\nu'' \upsilon^m = \chi' \nu_j^\flat \upsilon^m$, which has the weight $m - \#D_j(\tau)$ by (b) of Lemma 6.6.23, and υ has weight 1 ((ii) of Lemma 3.2.11). (a) $|\iota'(c_0 k_0(g_1,g_2))| \le 1$. Choose rational δ_1, δ_2 with $\delta_{1,f} g_1$, $\delta_{2,f} g_2$ integral invertible at R. By (6.133) with $P = 1$ and both characters of weight 0, $k_0(g_1,g_2) = k_0(\delta_{1,f} g_1, \delta_{2,f} g_2)$, and (i) of Lemma 6.6.20 with $q = 1$ ($M^2 \in \mathfrak{M}^2$), $\Psi = 1$ and ι' gives the claim. (b) $|\iota(c_1 \kappa_1(h))| \le 1$. Choose a rational δ' of $\mathrm{I}_{\mathcal{B}_j}$ with $\delta'_f h$ integral invertible at R; by (6.133) with $\gamma_S = 1$, $P = 1$ and the target character 1, $\kappa_1(h) = \kappa_1(\delta'_f h)$, and (i) of Lemma 6.6.20 with $q = \mathrm{N}Q$, Ψ_1 (values 0, 1) and ι gives the claim. (c) $\mathrm{gau}_\iota(h, \overline{u_{g_2}}) \le 1$. Choose rational points γ_1 of $W^\flat$, γ_2 of $\mathrm{I}_{\mathcal{B}^c}$ and γ' of $\mathrm{I}_{\mathcal{B}_j}$ with $\gamma_{1,f} x$, $\gamma_{2,f} g_2$, $\gamma'_f h$ integral invertible at R, put $\Gamma = \mathrm{diag}(1, \gamma_1, \gamma_2)$, a rational point of W_j^{add} with $\Gamma_f \bar{g}_2$ integral invertible at R, and $Y' = Y\gamma_1^{\mathsf{T}} \in \mathrm{M}_{\mathcal{B}^\flat}(E)$. Let $A_\tau = \tau(\gamma')^{-1}$, $U_\tau = \tau(\gamma_1)$, $V_\tau = \tau(\gamma_2)$, $B_\tau = \mathrm{diag}(1, U_\tau, V_\tau) = \tau(\Gamma)$; then $\tau(Y') = \tau(Y) U_\tau^{\mathsf{T}}$. Let $u'_k = c_2\, \theta^{W_j^{\mathrm{add}}, \mathrm{I}_{\mathcal{B}_j}}_{\nu'', \nu''; \mathfrak{n}}[\Psi^{\mathrm{add}}_{\Lambda_i}, H\mathrm{lc}_{i,k}[Y']](\Gamma_f \bar{g}_2, \gamma'_f h)$. By (6.133) (both weights $-\#D_j(\tau)$), linearity in the polynomial and (c) of Lemma 6.6.21,

$$\begin{aligned}\sum_k S_\kappa(A)_{lk} u'_k &= \prod_\tau \det \tau(\gamma')^{\#D_j(\tau)} \det \tau(\Gamma)^{-\#D_j(\tau)} \cdot c_2\, \theta_{\bar{g}_2}\Big(\big(\textstyle\sum_k S_\kappa(A)_{lk} H\mathrm{lc}_{i,k}[Y']\big)^{A,B} \Big) \\ &= c_2\, \theta_{\bar{g}_2}(H\mathrm{lc}_{i,l}[Y]),\end{aligned}$$

as $\det A_\tau = \det \tau(\gamma')^{-1}$ and $\det B_\tau = \det \tau(\Gamma)$; that is, $u_{g_2} = S_\kappa(A) u'$. Hence $\overline{u_{g_2}} = S_\kappa(\bar{A})\overline{u'} = \rho_{\kappa,0}(\gamma'^{-1})\overline{u'}$, and by (G2) and (G3), $\mathrm{gau}(h, \overline{u_{g_2}}) = \mathrm{gau}(\gamma'_f h, \overline{u'}) \le \max_k |\iota'(u'_k)|$. Now apply (i) of Lemma 6.6.20 for $(\mathcal{B}_j^{\mathrm{add}}, \mathcal{B}_j, d)$ with c_2, $q = \mathrm{N}Q$ and ι': the values of $\Psi^{\mathrm{add}}_{\Lambda_i}$ are 0 or sums of values of the additive character $\psi_{\mathfrak{M}^2}$, roots of unity, so they are integral at every ring homomorphism into every $\mathbb{C}_{\ell'}$; the points are integral invertible at R; $H\mathrm{lc}_{i,k}[Y']$ is homogeneous of degree d with coefficients in the subring $\{z : |\iota'(z)| \le 1\}$ of $\mathbb{C}$ by (a) of Lemma 6.6.21, because $Y'^{\mathsf{T}} = (\gamma_{1,f} x)(x^{-1} Y^{\mathsf{T}})$ has entries in E integral at every place above ℓ ((XE5): $x^{-1}Y^{\mathsf{T}}$ is integral), and $|\iota'(\tau(y))| \le 1$ for such y, $\iota' \circ \tau$ being an embedding of E into $\mathbb{C}_\ell$ that induces a place above ℓ. So $|\iota'(u'_k)| \le 1$ for all k.

Step 5: (i). By (6.132), (G1) and Step 4,

$$\mathrm{gau}_\iota(h, \varphi''_{\vartheta,i}(h)) \le \max_{g_1,g_2} |\iota(\overline{c_0 k_0})|\, |\iota(c_1\kappa_1(h))|\, \mathrm{gau}_\iota(h, \overline{u_{g_2}}) \le 1,$$

since $\iota(\bar{z}) = \iota'(z)$.

Step 6: (ii). Let $\vartheta = 1$ (so $Q = \emptyset$, $\mathfrak{n} = \mathfrak{M}^2$), $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $\sigma' = \sigma \circ \bar{\cdot} \in \mathrm{Aut}(\mathbb{C})$. For $w \in \mathbb{C}^{I_\kappa}$, $\|w\|_\kappa = \max\{|f^{\mathsf{T}} w| : f \in \mathbb{C}^{I_\kappa},\ \bar{f}^{\mathsf{T}} G_\kappa^{-1} f \le 1\}$: G_κ is real symmetric and positive

definite ((b) of Lemma 4.6.10), $f^{\mathsf T}w = \langle G_\kappa^{-1}\bar f, w\rangle_\kappa$ and $\|G_\kappa^{-1}\bar f\|_\kappa^2 = \bar f^{\mathsf T}G_\kappa^{-1}f$, so this is the Cauchy–Schwarz inequality, with equality for $f = \overline{G_\kappa w}/\|w\|_\kappa$. By the first two assertions of Lemma 6.6.22, the ranks m, $3m$, j, $|\mathcal B_j|$, $|\mathcal B_j^{\mathrm{add}}| = 12m+1-j$ are at most Δ and $d \le M$; and $M^2 \in \mathfrak M^2$. By (ii) of Lemma 6.6.20 with $A=1$ and $B=1$ ($\Psi = 1$ or Ψ_1, $P=1$), $|\sigma'(c_0k_0)| \le e^{x_1}$ and $|\sigma'(\overline{c_1\kappa_1(h)})| = |\sigma(c_1\kappa_1(h))| \le e^{x_1}$. For f with $\bar f^{\mathsf T}G_\kappa^{-1}f \le 1$ let $Q_f = \sum_l \sigma'^{-1}(f_l)H\mathrm{lc}_{i,l}$, homogeneous of degree d; then $f^{\mathsf T}\sigma'(u_{g_2}) = \sigma'(c_2\theta_{\bar g_2}(Q_f))$, and for z with $|\sigma'(z_u)| \le M^4$ for all u, $|\sigma'(Q_f(z))| = |f^{\mathsf T}(\sigma'(H\mathrm{lc}_{i,l}(z)))_l| \le H\,M^{9(12m+1)\#D_j/2}$ by the duality and (b) of Lemma 6.6.21 ($|\sigma'(\tau(Y_{ab}))| \le M^4$ by (XE5)). The values of $\Psi_{\Lambda_i}^{\mathrm{add}}$ have all conjugates of absolute value at most $\#\mathrm{GL}_{3m}(\mathcal O_E/\mathfrak M^2)$. So (ii) of Lemma 6.6.20 and the duality give $\|\sigma'(u_{g_2})\|_\kappa \le e^{x_1}\,\#\mathrm{GL}_{3m}(\mathcal O_E/\mathfrak M^2)\,H\,M^{9(12m+1)\#D_j/2}$. The number of classes of $\mathrm I_{\mathcal B^{\mathrm c}}$ at $\mathbf K_0(\mathcal B^{\mathrm c})$ and of $\mathrm I_{\{1,\dots,m\}}$ at $\mathbf K_0(\{1,\dots,m\})$ is at most $e^{x^{30}}$ each (Lemma 4.6.4, m, $3m \le 2n_m$), and the weights ω are at most 1. By (6.132), applying σ coordinatewise ($\sigma(\bar z) = \sigma'(z)$) and the triangle inequality,

$$\|\sigma(\varphi''_{1,i}(h))\|_\kappa \le N_{\mathrm{ch}}^2\, e^{2x^{30}}\, e^{3x_1}\,\#\mathrm{GL}_{3m}(\mathcal O_E/\mathfrak M^2)\,H\,M^{9(12m+1)\#D_j/2},$$

and with $H \le d^{3d}$, $d \le (10m+1)\#D_j$, the last assertion of Lemma 6.6.22 gives (ii). □

Lemma 6.6.18. *For admissible data D there is a short row family $Y = (Y_{\mathfrak c})_{\mathfrak c\in \mathrm{Cl}^\flat}$.*

Proof. Write $N = \#\mathcal B^\flat = 6m+1$ and fix a class $\mathfrak c \in \mathrm{Cl}^\flat$, with representative $x = x_{\mathfrak c} \in G_f(\mathcal B^\flat)$.

Step 1: the framed space and its lattice. By (VP3) of Definition 3.2.2 the standard definite space $W^\flat = \mathrm I_{\mathcal B^\flat}$ (matrix 1, point 1, signs $s = +1$ on $\mathcal B^\flat$) is a framed hermitian space $U = (E^{\mathcal B^\flat}, h)$, $h(u,u') = u'^*u$, with the standard basis as frame at every $v \in \Phi$ and the identity of $\mathbb A_{E,f}^{\mathcal B^\flat}$ as isometry (a framed space of rank N in the sense of (FT1)), and $\lambda_U(g)$ is the first minimum of (VP4) for both readings. Its group is $\mathrm G(s)$, $g = (1,x) \in \mathrm G(s)$, and its coordinate map is $c_U(u) = u$, that is $(\tau(u))_{\tau\in\Phi}$ at the infinite places and u at the finite places; so, by (VP4),

$$L_g = \{u \in E^{\mathcal B^\flat} : x^{-1}u \in \widehat{\mathcal O}_E^{\mathcal B^\flat}\}, \qquad |u|_g^2 = \sum_{\tau\in\Phi}\sum_{b\in\mathcal B^\flat} |\tau(u_b)|^2.$$

Step 2: $\lambda_U(g) = 1$. Let $0 \ne u \in L_g$ and $y = x^{-1}u \in \widehat{\mathcal O}_E^{\mathcal B^\flat}$. As $x^*x = 1$ and x is square, $xx^* = 1$, so $a := u^*u = y^*x^*xy = y^*y$ in $\mathbb A_{E,f}$; the entries of y and of $\bar y$ are integral, so $a \in E \cap \widehat{\mathcal O}_E = \mathcal O_E$. Since $\bar a = a$, $a \in \mathcal O_{E^+}$, and $\tau(a) = \sum_b |\tau(u_b)|^2 > 0$ for every $\tau \in \Phi$ ($u \ne 0$ and τ is injective). The restrictions of the $\tau \in \Phi$ to E^+ are its f real embeddings, so $\prod_{\tau\in\Phi}\tau(a) = N_{E^+/\mathbb Q}(a)$ is a positive integer, hence at least 1. By the inequality of arithmetic and geometric means,

$$|u|_g^2 = \sum_{\tau\in\Phi}\tau(a) \ge f\Big(\prod_{\tau\in\Phi}\tau(a)\Big)^{1/f} \ge f \ge 1.$$

So the infimum in (VP4) is at least 1, and $\lambda_U(g) = 1$.

Step 3: a short basis. By Lemma 6.4.102, $N_{\mathfrak D} = q_1 d$ is a positive integer in the different $\mathfrak D_E$. By Lemma 6.6.19 for U, this $N_{\mathfrak D}$ and the point g, there is a basis $u_1,\dots,u_N$ of U over E with $u_k \in L_g$ and $|u_k|_g \le fNq_1d$ (Step 2). Number the basis by $\mathcal B^\flat$ and let $Y_{\mathfrak c} \in \mathrm M_{\mathcal B^\flat}(E)$ be the matrix whose row k is $u_k^{\mathsf T}$. Then $Y_{\mathfrak c} \in \mathrm{GL}_{\mathcal B^\flat}(E)$, and the columns of $x^{-1}Y_{\mathfrak c}^{\mathsf T}$ are the vectors $x^{-1}u_k \in \widehat{\mathcal O}_E^{\mathcal B^\flat}$, so $x^{-1}Y_{\mathfrak c}^{\mathsf T}$ has entries in $\widehat{\mathcal O}_E$.

Step 4: the entries are small. Every ring homomorphism $\psi\colon E \to \mathbb C$ is τ or $\bar\tau$ for some $\tau \in \Phi$ (E is totally complex and Φ contains one embedding of each conjugate pair), and

$|\bar\tau(z)| = |\tau(z)|$. So for all k, b, $|\psi(Y_{\mathfrak{c},kb})| = |\tau(u_{k,b})| \le |u_k|_g \le fNq_1d$. By Lemma 4.2.67 (items 2–5: $f \le x^{12}$, $q_1d \le x^{13}$, $m \le 2\sqrt{x}$, $x \ge e^{6.4\cdot 10^{15}}$, $x^{10^{12}} \le b$) and (AD5),

$$fNq_1d \le x^{12} \cdot 13\sqrt{x} \cdot x^{13} \le x^{26} \le b \le p_0p_1 = M \le M^4.$$

Doing this for every $\mathfrak{c} \in \mathrm{Cl}^\flat$ (a finite set) gives the short row family $Y = (Y_\mathfrak{c})_\mathfrak{c}$. □

Lemma 6.6.19. *Let E be a CM field with totally real subfield E^+, $f = [E^+ : \mathbb{Q}]$ and different $\mathfrak{D}_E$, and let $N_\mathfrak{D}$ be a positive integer with $N_\mathfrak{D} \in \mathfrak{D}_E$. Let $N \ge 1$ and let U be a framed space of rank N over E, with real group G_∞^U and $G^U = G_\infty^U \times G_f$. For $g \in G^U$ let*

$$L_g = \{u \in U : \ (g^{-1}c(u))_f \in \widehat{\mathcal{O}}_E^N\}, \qquad |u|_g = \|(g^{-1}c(u))_\infty\|,$$

$c(u) \in \mathbb{A}_E^N$ the coordinate vector of u and $\|\cdot\|$ the Euclidean norm on $\prod_v \mathbb{C}^N$. Then for every $g \in G^U$ there is a basis $u_1, \dots, u_N$ of U over E with $u_j \in L_g$ and

$$|u_j|_g \le f\, N\, N_\mathfrak{D}\ \lambda_U(g)^{-1} \qquad (1 \le j \le N).$$

Proof. Let $L_{g,\infty}$, $\langle\,,\rangle$ and σ be as in Lemma 5.8.15, and let β be the $\mathbb{R}$-linear extension of $u \mapsto (g^{-1}c(u))_\infty$ to $U \otimes_\mathbb{Q} \mathbb{R}$, so $|u|_g = \|\beta(u)\|$. Put $n = 2fN$. By (iii) of Lemma 5.8.15, $L_{g,\infty}$ is a lattice of rank n in $\prod_v \mathbb{C}^N$, so $\lambda_1(L_{g,\infty}) = \min\{|u|_g : 0 \ne u \in L_g\} > 0$ and $\lambda_U(g) = \min(1, \lambda_1(L_{g,\infty}))$.

Step 1: $\lambda_1(L_{g,\infty}^*) \ge 2\lambda_1(L_{g,\infty})/N_\mathfrak{D}$. By (iv) of Lemma 5.8.15 a nonzero vector of $L_{g,\infty}^*$ is $\sigma\beta(w)$ with $0 \ne w \in 2\mathfrak{D}_E^{-1}L_g$. As $N_\mathfrak{D} \in \mathfrak{D}_E$, $N_\mathfrak{D}\mathfrak{D}_E^{-1} \subseteq \mathcal{O}_E$, so $\frac{N_\mathfrak{D}}{2}w \in N_\mathfrak{D}\mathfrak{D}_E^{-1}L_g \subseteq L_g$ (L_g is an $\mathcal{O}_E$-module by (i) of Lemma 5.8.15), and $\frac{N_\mathfrak{D}}{2}w \ne 0$. Hence $\lambda_1(L_{g,\infty}) \le |\frac{N_\mathfrak{D}}{2}w|_g = \frac{N_\mathfrak{D}}{2}\|\beta(w)\| = \frac{N_\mathfrak{D}}{2}\|\sigma\beta(w)\|$, σ being an isometry.

Step 2: transference. By Banaszczyk's transference theorem [12], $\lambda_n(L_{g,\infty})\lambda_1(L_{g,\infty}^*) \le n$; with Step 1,

$$\lambda_n(L_{g,\infty}) \le \frac{n\,N_\mathfrak{D}}{2\lambda_1(L_{g,\infty})} = \frac{fNN_\mathfrak{D}}{\lambda_1(L_{g,\infty})} \le \frac{fNN_\mathfrak{D}}{\lambda_U(g)}.$$

So there are $u_1', \dots, u_n' \in L_g$ with $|u_k'|_g \le fNN_\mathfrak{D}\lambda_U(g)^{-1}$ and $\beta(u_1'), \dots, \beta(u_n')$ linearly independent over $\mathbb{R}$. Their E-span W is U: otherwise $\beta(W \otimes_\mathbb{Q} \mathbb{R})$ would contain them and have real dimension $2f \dim_E W < n$. Choose $u_1, \dots, u_N$ among the u_k' forming an E-basis. □

Lemma 6.6.20. *Let D be admissible data, L, N finite index sets and $\delta \ge 0$ an integer. There is $c \in \mathbb{C}^\times$ such that for all automorphic characters α, β of $\mathrm{U}(1)$ that have a weight and a tame part, every modulus $\mathfrak{m}$, every function Ψ on $\mathrm{M}_{N\times L}(R_\mathfrak{m})$, every $Q \in \mathbb{C}[x]_{N\times L}$ homogeneous of degree δ and all $g \in G_f(L)$, $h \in G_f(N)$, the number $\kappa = c\,\theta_{\alpha,\beta;\mathfrak{m}}^{\mathrm{I}_L,\mathrm{I}_N}[\Psi, Q](g,h)$ satisfies:*

- *(i) let $q \ge 1$ be an integer with $qM^2 \in \mathfrak{m}$, and suppose $|\iota'(\Psi(X))| \le 1$ for every X, every prime $\ell \nmid q$ and every ring homomorphism $\iota' \colon \mathbb{C} \to \mathbb{C}_\ell$; then for every prime $p \nmid q$ and every ring homomorphism $\iota \colon \mathbb{C} \to \mathbb{C}_p$: if the components of g and of h at every place of E above p are integral invertible matrices and every coefficient a of Q has $|\iota(a)| \le 1$, then $|\iota(\kappa)| \le 1$;*
- *(ii) suppose $|L| \le \Delta$, $|N| \le \Delta$, $\delta \le M$ and $M^2 \in \mathfrak{m}$; let $A \in \mathbb{N}$ with $|\sigma'(\Psi(X))| \le A$ for all $\sigma' \in \mathrm{Aut}(\mathbb{C})$ and all X, let $\sigma \in \mathrm{Aut}(\mathbb{C})$, and let $B \in \mathbb{R}$ with $|\sigma(Q(z))| \le B$ for every $z \in \mathbb{C}^{\Phi\times N\times L}$ with $|\sigma(z_i)| \le M^4$ for all i; then $|\sigma(\kappa)| \le e^{x_1}AB$.*

Proof. Write $C = N \times L$, $R = \#C = |N||L|$, and define

$$c = M^{4fR}\, c_s^R\, (M^2 d\, \eta_P)^\delta \in \mathbb{C}^\times,$$

which depends only on L, N, δ and D. Fix $\alpha, \beta, \mathfrak{m}, \Psi, Q, g, h$ as in the statement, put $A_0 = \Sigma(\mathfrak{m})$ and $\theta_0 = \theta^{\mathrm{I}_L,\mathrm{I}_N}_{\alpha,\beta;\mathfrak{m}}[\Psi, Q](g,h)$, so $\kappa = c\,\theta_0$.

Step 0: the kernel as a theta value of one finite datum. By (VP5), $\mathrm{Hom}(\mathrm{I}_L, \mathrm{I}_N)$ has matrix 1, point 1 and signs $+1$ on C: it is the standard definite space I_C, the identity is a transporting element, and by Lemma 4.1.17 its theta functional is the canonical functional θ_C of I_C. By (WR4), $\omega_{\alpha,\beta}((1,g),(1,h)) = c'\,\omega_+(k)$ with $c' = \alpha(\det h)\beta(\det g)^{-1}$, $\omega_+ = \omega_s$ for $s = +1$ on C, and $k \in G_f(C)$ the map $X \mapsto hXg^{-1}$. By (5) of Lemma 4.1.1 at the real points 1 (and $\Omega_{\gamma,\gamma'}(1,1) = 1$),

$$\theta_0 = \theta_C(v_Q \otimes F), \qquad F = c'\,\omega_+(k)\big(\Psi^\natural \otimes_{A_0} v^0\big).$$

By Lemma 6.4.84 for $\mathfrak{m}$, C, the signs $+1$ and the function Ψ on $T = R_{\mathfrak{m}}^C = \mathrm{M}_{N\times L}(R_{\mathfrak{m}})$, $\Psi^\natural \otimes_{A_0} v^0 = \sum_\chi a_\chi \rho(\mathrm{fv}(u_\chi))v^0$. By Proposition 4.1.14, $\omega_+(k)\rho(\mathrm{fv}(u)) = \rho(\mathrm{fv}(ku))\omega_+(k)$ (the finite coordinates are $\mathbb{A}_{E,f}$-linear and k has trivial infinite part), and $\omega_+(k)$ preserves the span $\mathcal{M}_{\mathrm{fin}}$ of the vectors $\rho(\mathrm{fv}(t))v^0$. Hence

$$c_s^R F = c' \sum_\chi a_\chi\, \rho(\mathrm{fv}(ku_\chi))\, \omega_+(k)\big(c_s^R v^0\big) \in \mathcal{M}_{\mathrm{fin}}. \tag{6.134}$$

Products of finite Heisenberg operators satisfy $\rho(\mathrm{fv}(x))\rho(\mathrm{fv}(x')) = \zeta\rho(\mathrm{fv}(x+x'))$ with ζ a value of the finite additive character of (WD8), hence a root of unity (a continuous character of $\mathbb{A}_{E,f}$ has finite image on every compact open subgroup, and every element lies in one). The finite points $\det g$, $\det h$ satisfy $u\bar{u} = 1$; by (a) of Lemma 6.4.85, $|\sigma(c')| = 1$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Step 1: (i). Let q, $p \nmid q$, ι be as in (i), suppose the components of g, h above p are integral invertible and the coefficients of Q lie in $\mathcal{O}_\iota = \{z \in \mathbb{C} : |\iota(z)|_p \le 1\}$. Let $\mathcal{S}$ be the $\mathcal{O}_\iota$-span of the vectors $\rho(\mathrm{fv}(x))c_s^R v^0$ with $M^2 x_w \in \mathcal{O}_{E,w}^C$ for every place $w \mid p$, and put $F_0 = M^{4fR} c_s^R F$. We show $F_0 \in \mathcal{S}$.

- The components of k and k^{-1} above p are integral, and $\det g$, $\det h$ are units above p; by (b) of Lemma 6.4.85, $\iota(c')$ is a p-adic unit.
- By Proposition 4.1.14 the restriction of $\omega_+(k)$ to $\mathcal{M}_{\mathrm{fin}}$ is the normalised finite kernel of Lemma 4.1.13 in the scaled finite coordinates, so $\omega_+(k)v^0 = \frac{\Delta(k)}{I(k)} \sum_{y \in L_0/J(k)} \zeta_y\, \rho(\mathrm{fv}(ky))v^0$, with $L_0 = \widehat{\mathcal{O}}_E^C$, $J(k) = L_0 \cap k^{-1}L_0$ and roots of unity ζ_y (the factor $s_L(ky)$ of Lemma 4.1.13 and the phase between its Heisenberg operator and $\rho(\mathrm{fv}(\cdot))$ are values of continuous characters of $\mathbb{A}_{E,f}$). The index $I(k) = [L_0 : J(k)]$ is the product of the local indices, the one at w a power of the residue characteristic of w and equal to 1 at $w \mid p$; so $p \nmid I(k)$. By Lemma 4.1.13, $\Delta(k)\overline{\Delta(k)} = I(k)$ and $\overline{\Delta(k)} = \upsilon_f(\det k)\Delta(k)$, so $\Delta(k)^2 = I(k)\,\upsilon_f(\det k)^{-1}$, where $\upsilon_f(\det k)$ is the value of the splitting character, an automorphic character of U(1) ((LF2) of Definition 3.2.26), at the finite point $\det k$, a unit above p; by (b) of Lemma 6.4.85 $\iota(\Delta(k))$ is a p-adic unit, and so is $\iota(\Delta(k)/I(k))$.
- $a_\chi = (\#T)^{-1}\sum_X \Psi(X)\overline{\chi(X)}$, where $|\iota(\Psi(X))|_p \le 1$ ($p \nmid q$) and the $\chi(X)$ are roots of unity. Since $qM^2 \in \mathfrak{m}$, $\mathfrak{m}$ divides $qM^2\mathcal{O}_E$, so $\mathrm{N}(\mathfrak{m})$ divides $\mathrm{N}(qM^2\mathcal{O}_E) = (qM^2)^{2f}$ and $\#T = \mathrm{N}(\mathfrak{m})^R$ divides $q^{2fR}M^{4fR}$; as $p \nmid q$, $|\iota(M^{4fR}/\#T)|_p \le 1$. So $M^{4fR}a_\chi \in \mathcal{O}_\iota$.
- For $w \in A_0$ above p, q is a unit at w, so $M^2 \in \mathfrak{m}\mathcal{O}_{E,w}$; by (b) of Lemma 6.4.84 with $\delta = M^2$, $M^2 u_\chi$ is integral above p. Hence $M^2 k(u_\chi + y)$ is integral above p for $y \in L_0$.

Inserting $\omega_+(k)v^0$ into (6.134) and combining the Heisenberg operators gives

$$F_0 = \sum_{\chi,y} z_{\chi,y}\, \rho(\mathrm{fv}(k(u_\chi + y)))\, c_s^R v^0$$

with $z_{\chi,y} \in \mathcal{O}_\iota$, so $F_0 \in \mathcal{S}$. Put $\zeta_{\mathrm{sc}} = M^2 d$ if $\delta \geq 1$ and $\zeta_{\mathrm{sc}} = 1$ if $\delta = 0$. As Q is homogeneous of degree δ, $Q(\zeta_{\mathrm{sc}}\eta_P\,\cdot) = (M^2 d\eta_P)^\delta Q$; by linearity of the datum in both arguments and Lemma 4.2.2 for I_C,

$$\kappa = M^{4fR}(M^2 d\eta_P)^\delta\, \theta_C(v_Q \otimes c_s^R F) = \theta_C(v_{Q(\zeta_{\mathrm{sc}}\eta_P\cdot)} \otimes F_0) = Q(\partial_Z)\mathscr{E}_{F_0}(\zeta_{\mathrm{sc}}\eta_P Z)\big|_{Z=0}.$$

By Proposition 4.2.35 for the coordinate set C, the element $M^2 \in \mathcal{O}_K$ in the role of its δ, the module $\mathcal{S}$ and the homogeneous $Q \in \mathcal{O}_\iota[Z]$, $|\iota(\kappa)|_p \leq |\iota(\zeta_{\mathrm{sc}}/(M^2 d))|_p^\delta = 1$ (for $\delta \geq 1$ the ratio is 1; for $\delta = 0$ the exponent is 0). This is (i).

Step 2: reduction of (ii) to automorphisms over K. Let the hypotheses of (ii) hold for σ. As K is imaginary quadratic, σ or $\sigma'' = \bar{\cdot} \circ \sigma$ fixes K; since $|\sigma''(z)| = |\sigma(z)|$ for all z, the hypotheses of (ii) hold for σ'' with the same A, B, and $|\sigma''(\kappa)| = |\sigma(\kappa)|$. So we may assume that σ fixes K.

Step 3: the Gaussian integral. Put $Q'(Z) = Q(M^2 dZ)$ and $F' = c_s^R F \in \mathcal{M}_{\mathrm{fin}}$, so that $\kappa = M^{4fR}\theta_C(v_{Q'(\eta_P\cdot)} \otimes F')$. Let t_σ, ϵ_σ and $\Theta = \Theta_{\sigma,F'}$ be as in Lemma 4.2.16 for I_C; with $D_0 = fR$,

$$\sigma(\kappa) = M^{4fR}\,\pi^{-D_0} \int_{\mathbb{C}^{\Phi\times C}} Q'^\sigma\big(t_\sigma \bar{y}^{\epsilon_\sigma}\big)\, e^{-|y|^2/2}\, \Theta(y)\, dy.$$

The polynomial. $Q'^\sigma(w) = Q^\sigma(M^2 dw)$, as σ fixes $\mathbb{Q}$. If $|w_i| \leq M^4$ for all i, then $z = \sigma^{-1}(w)$ (coordinatewise) has $|\sigma(z_i)| \leq M^4$, so $|Q^\sigma(w)| = |\sigma(Q(z))| \leq B$; in particular $B \geq 0$. As Q^σ is homogeneous of degree δ, $Q^\sigma(w) = (\rho_w/M^4)^\delta Q^\sigma(M^4 w/\rho_w)$ for $w \neq 0$, $\rho_w = \max_i |w_i|$, and $Q^\sigma(0) = 0$ if $\delta \geq 1$; so $|Q^\sigma(w)| \leq B(\max_i |w_i|/M^4)^\delta$ for all w ($0^0 = 1$). For $w = M^2 dt_\sigma \bar{y}^{\epsilon_\sigma}$, $\max_i |w_i| \leq M^2 d|t_\sigma||y| = M^2 d\,\Omega_s^{-1/2}|y|$, so $|Q'^\sigma(t_\sigma \bar{y}^{\epsilon_\sigma})| \leq B\big(d\,\Omega_s^{-1/2}|y|/M^2\big)^\delta$. *The theta values.* By (a) of Lemma 4.2.16, $|\Theta(y)| \leq \sup_x |\sigma(\theta_C(\rho(\mathrm{fv}(x))F'))|$. By (6.134) and Step 0, $\theta_C(\rho(\mathrm{fv}(x))F') = c' \sum_\chi a_\chi \zeta_{x,\chi}\, \mathcal{V}_R(x + ku_\chi, k)$ with roots of unity $\zeta_{x,\chi}$ and $\mathcal{V}_R(x,k) = \theta_C(\rho(\mathrm{fv}(x))\omega_+(k)(c_s^R v^0))$, the value of Lemma 6.4.83 for I_C. By (2) of Lemma 6.4.83, $|\sigma(c')| = 1$ and (a) of Lemma 6.4.84,

$$|\Theta(y)| \leq \sum_\chi |\sigma(a_\chi)|\, e^{A_R} \leq \#T \cdot A \cdot e^{A_R}, \qquad A_R = C_P(q_1 R + 1)^2.$$

The moment. For $\delta \geq 1$,

$$|y|^\delta e^{-|y|^2/4} \leq (2\delta)^{\delta/2} e^{-\delta/2} \qquad (\text{maximum at } |y|^2 = 2\delta),$$

and $\pi^{-D_0} \int e^{-|y|^2/4} dy = 4^{D_0}$; so

$$\pi^{-D_0} \int |y|^\delta e^{-|y|^2/2} dy \leq (2\delta)^{\delta/2} 4^{D_0},$$

which also holds for $\delta = 0$ ($2^{D_0} \leq 4^{D_0}$). Altogether

$$|\sigma(\kappa)| \leq M^{4fR}\, \#T\, 4^{fR}\, e^{A_R}\, A\, B\, \Big(\frac{\sqrt{2\delta}\, d\,\Omega_s^{-1/2}}{M^2}\Big)^\delta.$$

Step 4: the sizes. By Lemma 4.2.67 (items 1–5): $d \leq C_P \leq \log\log b \leq \log x$ (with (AD1)), $x \geq e^{6.4\cdot 10^{15}}$, $(\log b)^{100} \leq x$, $x^{10^{12}} \leq b$, $r \leq 3x^5$, $f \leq q_1 \leq x^{12}$, $\log M \leq 22 \log b$;

by (2) of Lemma 5.2.13, $\Omega_s^{-1} = \mathrm{rd}_E e^{-c_{\mathrm{CS}}(d)}/(2\pi) \le x^{13}$; by (AD5), $M \ge b^{20}$. As $\delta \le M$, $\sqrt{2\delta}\, d\, \Omega_s^{-1/2} \le \sqrt{2M}\, x^7 \le M^2$, so the last factor is at most 1. Since $M^2 \in \mathfrak{m}$, $\mathrm{N}(\mathfrak{m})$ divides $\mathrm{N}(M^2\mathcal{O}_E) = M^{4f}$, so $\#T \le M^{4fR}$. As $|L|, |N| \le \Delta = 2r^8$, $R \le 4r^{16} \le 4 \cdot 3^{16} x^{80} \le x^{81}$, so $fR \le x^{93}$ and $q_1 R + 1 \le 2x^{93}$. Hence

$$\log(M^{8fR} 4^{fR} e^{A_R}) \le 8fR \cdot 22\, x^{1/100} + 2fR + 4x^{186} \log x \le x^{95} + x^{187} \le x^{10^6} \le x_1,$$

the last step by (AD1). So $|\sigma(\kappa)| \le e^{x_1} AB$, which is (ii). □

Lemma 6.6.21. *Let $1 \le j \le m$ and $\kappa^+ = \kappa^+_{O_{j,1}}$. For a choice i, a matrix $Y \in \mathrm{M}_{\mathcal{B}^\flat}(E)$ and $l \in I_{\kappa^+}$, let $\mathrm{lc}_{i,l}[Y] \in \mathbb{C}[x]_{\mathcal{B}_j \times \mathcal{B}_j^{\mathrm{add}}}$ be the line coordinate of (XE6) formed with Y in place of $Y_{\mathfrak{c}_i}$ in the constant rows, and put $\#D_j = \sum_{\tau \in \Phi} \#D_j(\tau)$, the number of determinants. Then:*

(a) there are integers d and H with $0 \le d \le (10m+1)\#D_j$ and $0 < H \le d^{3d}$ ($0^0 = 1$) such that for every choice i, every $Y \in \mathrm{M}_{\mathcal{B}^\flat}(E)$ and every l, the polynomial $H\,\mathrm{lc}_{i,l}[Y]$ is homogeneous of degree d and its coefficients lie in every subring of $\mathbb{C}$ that contains all the numbers $\tau(Y_{ab})$ ($\tau \in \Phi$, $a, b \in \mathcal{B}^\flat$);

(b) for every ring endomorphism φ of $\mathbb{C}$, every choice i, every Y with $|\varphi(\tau(Y_{ab}))| \le M^4$ for all τ, a, b and every $z \in \mathbb{C}^{\Phi \times \mathcal{B}_j \times \mathcal{B}_j^{\mathrm{add}}}$ with $|\varphi(z_u)| \le M^4$ for all u,

$$\big\| \big(\varphi(\mathrm{lc}_{i,l}[Y](z))\big)_{l \in I_{\kappa^+}} \big\|_{\kappa^+}^2 \le \big(M^9\big)^{(12m+1)\#D_j};$$

(c) let $A_\tau \in \mathrm{M}_{\mathcal{B}_j}(\mathbb{C})$, $U_\tau \in \mathrm{M}_{\mathcal{B}^\flat}(\mathbb{C})$ and $V_\tau \in \mathrm{M}_{\mathcal{B}^\mathrm{c}}(\mathbb{C})$ ($\tau \in \Phi$), let $B_\tau = \mathrm{diag}(1, U_\tau, V_\tau)$ on $\mathcal{B}_j^{\mathrm{add}} = \mathcal{A}_j^+ \sqcup \mathcal{B}^\flat \sqcup \mathcal{B}^\mathrm{c}$, and let $Y, Y' \in \mathrm{M}_{\mathcal{B}^\flat}(E)$ satisfy $\tau(Y') = \tau(Y)\, U_\tau^\mathsf{T}$ for every τ. For a polynomial P let $P^{A,B}$ be P after the substitution $x_\tau \mapsto A_\tau^\mathsf{T} x_\tau B_\tau^\mathsf{T}$ of the $\mathcal{B}_j \times \mathcal{B}_j^{\mathrm{add}}$ matrices of variables $x_\tau = (x_{\tau,b,c})$. Then for every choice i and every l

$$\Big(\sum_{k \in I_{\kappa^+}} S_{\kappa^+}(A)_{lk}\, \mathrm{lc}_{i,k}[Y'] \Big)^{A,B} = \prod_{\tau \in \Phi} \big(\det A_\tau\, \det B_\tau\big)^{\#D_j(\tau)}\, \mathrm{lc}_{i,l}[Y].$$

Proof. **The coordinates reconstruct the determinant product over its coefficient ring.** Fix j. Put $n_{\mathrm{row}} = |\mathcal{B}_j|$, $r_+ = k + j - 1$, and $\kappa = \kappa^+$. The weight formulas (LF4) give

$$\kappa(\tau, h) = 0 \quad (h < k-1 \text{ or } h > r_+), \qquad \kappa(\tau, r_+) = \Delta'. \tag{1}$$

Indeed consecutive entries before position $k-1$ differ by one. At r_+ the difference in (LF4) is $\Delta/2 - (t+1)/2 = \Delta'$; afterwards the preceding target entry is already below the truncation threshold, since $\kappa_{j+1} = \kappa_j + \Delta$. In particular

$$n_{\mathrm{row}} - h \le 10m + 1 \quad \text{whenever } \kappa(\tau, h) > 0. \tag{2}$$

Here $n_{\mathrm{row}} = r + 9m + 1$ and $k - 1 = r - m$.

We use the following coefficient-ring description, valid for arbitrary Y, not just a short row family. For a determinant of height $h < r_+$, its constant rows are zero in columns $0, \dots, h-1$, and row $a+1$, for $0 \le a < r_+ - h$, is the unit row in column $h + a$. These rows exist since $r_+ - h \le j \le m \le 2m$. Subtracting the appropriate multiples of these rows from each variable row clears its entries in columns $h, \dots, r_+ - 1$. This operation leaves all added-column entries unchanged. Expansion along the first h columns now expresses this determinant as a combination of their height-h minors with coefficients involving only added variables and entries

of $\tau(Y)$. The extra determinants have height r_+; all their constant entries in these first columns vanish, so the same expansion needs no clearing. All signs are the original Laplace signs.

Let $R \subseteq \mathbb{C}$ be any subring containing the entries of every $\tau(Y)$, and put

$$R_{\mathrm{add}} = R[x_{\tau,b,c} : c \in \mathcal{B}_j^{\mathrm{add}}].$$

Products of the preceding expansions have exactly $\kappa(\tau, h)$ minors of each height. The integral standard-basis property (WM1) therefore gives

$$\Pi_i[Y] = \sum_l a_l \, \widehat{b}_l, \qquad a_l \in R_{\mathrm{add}}, \tag{3}$$

where $\widehat{b}_l$ is b_l renamed to the first r_+ columns. No column of greater height occurs, so no b_l involves a later column.

Write G_κ for the filling Gram matrix and H_κ for the matrix of ordinary bilinear Fock pairings of the b_l. By Lemma 4.6.9, $H_\kappa G_\kappa = G_\kappa H_\kappa = h_\kappa I$. The scalar is nonzero. The definition of the line coordinates, pairing only the first-column variables, gives

$$\mathbf{lc}_i[Y] = h_\kappa^{-1} H_\kappa (a_l)_l = G_\kappa^{-1} (a_l)_l, \qquad \mathrm{Pol}_\kappa(\mathbf{lc}_i[Y]) = \Pi_i[Y]. \tag{4}$$

These equalities hold over the complex polynomial coefficient ring. The pairing used here has $\langle X^\alpha, X^\beta \rangle_{\mathrm{F}} = \mathbf{1}_{\alpha=\beta} \alpha!$, with no coefficient conjugation. Thus (4) uses precisely the normalization in (XE6).

Homogeneity and a denominator depending on the complementary degree. Set

$$d_{\mathrm{add}} = \sum_\tau \sum_{1 \le h < n_{\mathrm{row}}} (n_{\mathrm{row}} - h) \kappa(\tau, h). \tag{5}$$

In a cleared determinant of height h, every nonzero term uses exactly h variable entries from the first h columns and $n_{\mathrm{row}} - h$ variable entries from the added columns. The other entries are constant. Its added degree is therefore $n_{\mathrm{row}} - h$. Formula (3) is homogeneous of added degree d_{add}: decomposition by added degree and linear independence of the $\widehat{b}_l$ show that each a_l is homogeneous of that degree. Equation (4) proves the same for the line coordinates. This includes the zero polynomial. By (1)–(2),

$$0 \le d_{\mathrm{add}} \le (10m + 1) \# D_j. \tag{6}$$

For the required denominator use the complementary weight $\kappa^\vee(\tau, h) = \kappa(\tau, n_{\mathrm{row}} - h)$. Its number of boxes is exactly d_{add}. We give the integral Gram comparison explicitly. In each square block of variables let w_0 reverse the columns and let $c_\tau = \sum_h \kappa(\tau, h)$. On the span of the products of minors of weight κ consider

$$\mathcal{D}_\kappa P(X) = \prod_\tau \det(X_\tau)^{c_\tau} \, P(X_\tau^{-\mathsf{T}} w_0). \tag{7}$$

Although written on the invertible-matrix open set, this is polynomial on that span: the complementary-minor identity for the inverse transpose sends each flag minor of height h, after multiplication by $\det X_\tau$, to a signed flag minor on the complementary rows and the first $n_{\mathrm{row}} - h$ columns. This identity follows directly from the adjugate formula, or by expanding a determinant in its specified rows. Reversing the columns only supplies a sign. Consequently (7) bijects the products of minors of κ and of $\kappa^\vee$, up to signs. Applying the complementary operation again gives a fixed sign times the original polynomial, since $(X^{-\mathsf{T}} w_0)^{-\mathsf{T}} w_0 = X$. It is therefore an isomorphism of their integral spans.

Let T be its matrix in the two integral standard bases; T and T^{-1} have integer entries. Complementing every column filling gives a signed permutation matrix P of the filling tensors. If the rows of T give the coefficients of $\mathcal{D}_\kappa b_l$, comparison of the expansions of every product of minors gives

$$F_\kappa T = P F_{\kappa^\vee}, \qquad T^{\mathsf{T}} G_\kappa T = G_{\kappa^\vee}. \tag{8}$$

Put $H_{\mathrm{dual}} = h_{\kappa^\vee}$. Applying Lemma 4.6.9 to the complementary weight in (8) yields

$$H_{\mathrm{dual}} G_\kappa^{-1} = T H_{\kappa^\vee} T^{\mathsf{T}} \in \mathrm{M}_{I_\kappa}(\mathbb{Z}). \tag{9}$$

Here the $H_{\kappa^\vee}$ on the right is the Fock-pairing matrix, not the scalar H_{dual}. Equations (3)–(4) and (9) prove that every coefficient of $H_{\mathrm{dual}} \mathrm{lc}_{i,l}[Y]$ belongs to R. The same integer works for every i, Y and every such subring.

We check its numerical bound without suppressing factorials. The highest polynomial of $\kappa^\vee$ has total degree d_{add}, and its coefficient absolute sum is at most $\prod_{\tau,h} (h!)^{\kappa^\vee(\tau,h)} \le d_{\mathrm{add}}^{d_{\mathrm{add}}}$ when $d_{\mathrm{add}} \ge 1$. Every monomial factorial is at most $d_{\mathrm{add}}! \le d_{\mathrm{add}}^{d_{\mathrm{add}}}$. Thus its Fock self-pairing satisfies

$$0 < H_{\mathrm{dual}} \le d_{\mathrm{add}}^{d_{\mathrm{add}}} \left(d_{\mathrm{add}}^{d_{\mathrm{add}}} \right)^2 = d_{\mathrm{add}}^{3 d_{\mathrm{add}}}.$$

The self-pairing is a positive integer because the highest polynomial has integer coefficients and is nonzero. When $d_{\mathrm{add}} = 0$, it is the constant polynomial one and $H_{\mathrm{dual}} = 1$. Taking $d = d_{\mathrm{add}}$, $H = H_{\mathrm{dual}}$ proves (a), including $0^0 = 1$.

Products do not increase the product of the Gram norms. We record the precise tensor argument needed for (b). For two weights η, η', let K be the integer multiplication matrix $b_i^\eta b_j^{\eta'} = \sum_l K_{(i,j),l} b_l^{\eta+\eta'}$. Concatenating the column fillings, with only a permutation of tensor factors, gives

$$F_{\eta+\eta'} = (F_\eta \otimes F_{\eta'}) K, \qquad G_{\eta+\eta'} = K^{\mathsf{T}} (G_\eta \otimes G_{\eta'}) K. \tag{10}$$

Thus K is an isometric embedding for the indicated positive Gram forms. The forms are positive definite: each standard-basis product is one of the filling rows, so its filling matrix has full column rank. For vectors v, w, the polynomial product corresponds, by (WM4), to

$$u = G_{\eta+\eta'}^{-1} K^{\mathsf{T}} (G_\eta \otimes G_{\eta'})(v \otimes w).$$

This is the adjoint of the isometry K, and has norm at most $\|v\|_\eta \|w\|_{\eta'}$. Indeed $KK^\dagger$ is the orthogonal projection onto the image of K, so its adjoint is a contraction. Iterating proves the product assertion for any finite number of factors, with coefficient one. The empty product has norm one.

A single cleared determinant satisfies the stated bound. Apply the field endomorphism φ after specializing the added variables at z. It fixes the integer Gram matrices and their rational inverses, so (4) commutes with this operation. In the cleared determinant of height h, denote the matrix of remaining columns by Z_{rest}. It has $2m + n_{\mathrm{row}}$ rows and $2m + n_{\mathrm{row}} - h$ columns. Each entry is zero, one, an image of an entry of $\tau(Y)$, or an image of an added variable. Thus every entry has absolute value at most M^4.

Laplace expansion in the first h columns gives its coefficient vector in the one-column minor basis. The coordinates are, with their signs, complementary maximal minors of Z_{rest}, using all the constant rows and the complementary variable rows. In a one-column weight the filling matrix is the identity, so the squared coordinate norm is the sum of their squared absolute values. It is bounded by the sum over all maximal minors of Z_{rest}. Cauchy–Binet and Hadamard therefore give

$$\|\text{single-factor vector}\|^2 \le \det(Z_{\mathrm{rest}}^* Z_{\mathrm{rest}}) \le \prod_{\text{columns } c} \sum_a |(Z_{\mathrm{rest}})_{ac}|^2. \tag{11}$$

No division by a minor or assumption of full rank is needed; a rank-deficient matrix has the same nonnegative identity.

By Lemma 4.2.67, the admissible scales imply $r + 11m + 1 \le M$. Explicitly $r \le 3x^5$, $m \le 2\sqrt{x}$, $b \ge x^{10^{12}}$, and $M = p_0 p_1 \ge b^{20}$; these give $r + 11m + 1 \le 3x^5 + 22\sqrt{x} + 1 \le M$. The same inequalities ensure $M \ge 1$. Hence each column sum in (11) is at most M^9. By (1),

$$2m + n_{\mathrm{row}} - h \le 12m + 1.$$

Each determinant factor consequently has squared Gram norm at most $(M^9)^{12m+1}$. Apply (10) to the $\#D_j$ factors. Their product has weight κ by (1), and its coordinate vector is exactly $(\varphi(\mathrm{lc}_{i,l}[Y](z)))_l$ by (4). This proves

$$\big\|(\varphi(\mathrm{lc}_{i,l}[Y](z)))_l\big\|_\kappa^2 \le (M^9)^{(12m+1)\#D_j},$$

as in (b). The proof did not require φ to be surjective or continuous.

The two-sided substitution has exactly the displayed Schur factor. Write $\widetilde{B}_\tau = \mathrm{diag}(1_{r_+}, B_\tau)$ on $\mathcal{L}_j$. The hypothesis on Y, Y' and the explicit constant rows give

$$\mathsf{c}_{i,\tau,d}[Y'] = \mathsf{c}_{i,\tau,d}[Y]\,\widetilde{B}_\tau^{\mathsf{T}}. \tag{12}$$

For the unit constant rows this uses that the positive and $\mathcal{A}_j^+$ blocks are fixed. For the other rows it is precisely $\tau(Y') = \tau(Y)U_\tau^{\mathsf{T}}$; the carrier block has zero entries in every constant row. After substituting $x_\tau \mapsto A_\tau^{\mathsf{T}} x_\tau \widetilde{B}_\tau^{\mathsf{T}}$, the full stacked matrix for Y' is the original one multiplied on the left by $\mathrm{diag}(1_{2m}, A_\tau^{\mathsf{T}})$ and on the right by $\widetilde{B}_\tau^{\mathsf{T}}$. Multiplicativity of determinants proves the polynomial identity

$$\Pi_i[Y']^{A,\widetilde{B}} = s\,\Pi_i[Y], \qquad s = \prod_\tau (\det A_\tau \det B_\tau)^{\#D_j(\tau)}. \tag{13}$$

This holds for singular matrices as well.

Regard the polynomials as polynomials in the first columns with added-variable coefficients. Let $\mathcal{U}$ substitute $z_\tau \mapsto A_\tau^{\mathsf{T}} z_\tau B_\tau^{\mathsf{T}}$ on these coefficients. The full substitution in (13) acts on the first-column variables by $X_\tau \mapsto A_\tau^{\mathsf{T}} X_\tau$. The ordinary bilinear Fock pairing has the transpose-adjoint identity

$$\langle p(X), q(A^{\mathsf{T}}X)\rangle_{\mathrm{F}} = \langle p(AX), q(X)\rangle_{\mathrm{F}}. \tag{14}$$

The chain rule proves this for every matrix A: substitute the commuting constant-coefficient derivatives in $p(\partial)$ and evaluate at zero. Thus no inverse matrix or nonsingularity is required. Coefficient substitution commutes with this pairing, since the b_l have integer coefficients.

Pair (13) with b_l in the first columns, divide by the same nonzero h_κ, and use (14). By (WM2), $b_l(AX) = \sum_k S_\kappa(A)_{lk} b_k(X)$. The result is

$$\mathcal{U}\left(\sum_k S_\kappa(A)_{lk}\mathrm{lc}_{i,k}[Y']\right) = s\,\mathrm{lc}_{i,l}[Y].$$

The substitution $\mathcal{U}$ fixes the complex coefficients $S_\kappa(A)_{lk}$; on the remaining variables it is exactly the operation denoted $P^{A,B}$ in the statement. This is (c), with precisely its transpose convention and without any complex conjugation. □

Lemma 6.6.22. *Let $1 \le j \le m$, put $\#D_j = \sum_{\tau\in\Phi} \#D_j(\tau)$, and let d be an integer with $0 \le d \le (10m+1)\#D_j$. Then $d \le M$, $|\mathcal{B}_j| \le \Delta$, $12m + 1 \le \Delta$, $3m \le rt$, and*

$$N_{\mathrm{ch}}^4\Big(e^{2x^{30}+3x_1}\,\#\mathrm{GL}_{3m}(\mathcal{O}_E/\mathfrak{M}^2)\,d^{3d}\,M^{9(12m+1)\#D_j/2}\Big)^2 \le \exp\big(f\,m\,b\,(\log b)^2\big) \qquad (0^0 = 1).$$

Proof. By Lemma 4.2.67 we use: $x \geq e^{6.4\cdot 10^{15}}$ and $\log\log b \geq 6.4\cdot 10^{13}$ (item 1), $(\log b)^{100} \leq x$ and $x^{10^{12}} \leq b$ (item 2), $r \leq 3x^5$, $2 \leq m \leq 2\sqrt{x}$ (item 3, and $m = 2^{e+1}$), $f \leq x^{12}$ (item 4) and $\log M \leq 22\log b$ (item 5). By (AD1), $\log x \leq 10^6 \log x \leq \log x_1 \leq 10^6 \log x_1 \leq \log b$, so $x \leq x_1 \leq b$; by (AD2), $r \geq x^5$; by (AD5), $M \geq p_0 \geq b^{10}$.

Step 1: $\#D_j(\tau) \leq 2b$ for every $\tau \in \Phi$, hence $\#D_j \leq 2fb$. Let $\vartheta = 1$ be the trivial twist and $\alpha_e = \chi'_{j,1}\nu_j^\flat \upsilon^m = \chi_M^j \mu^\flat \upsilon^{m+e_j-j\Delta}$. If characters μ_1, μ_2 have weights e_1, e_2 and tame parts ψ_1, ψ_2, then $\mu_1\mu_2$ has weight e_1+e_2 and tame part $\psi_1\psi_2$ (multiply the defining formulas of Definition 3.1.1); so α_e has the weight $\tau \mapsto m + e_j - j\Delta - \kappa_0(\tau)$. By (b) of Lemma 6.6.23, α_e also has the weight $\tau \mapsto m - \#D_j(\tau)$. A weight is determined by the character: if μ has weight e, then for $v \in \Phi$ and $z \in \mathbb{C}$ with $|z| = 1$ the element $u \in \mathrm{U}(1)(\mathbb{A})$ with component z at v and 1 at all other places has integral finite components with reduction 1, so $\mu(u) = z^{e(v)}$, and $z \mapsto z^{e(v)}$ on the unit circle determines $e(v)$. Hence

$$\#D_j(\tau) = \kappa_0(\tau) + j\Delta - e_j = \kappa(\tau) + \tfrac{\Delta}{2} + j\Delta - e_j \leq \kappa(\tau) + (m+1)\Delta,$$

as $e_j \geq 0$ and $j \leq m$. Here $\kappa(\tau)$ is b or $a/x_1 = xb/x_1 \leq b$, and $(m+1)\Delta = 2(m+1)r^8 \leq 2\cdot 3\sqrt{x}\cdot 3^8 x^{40} \leq x^{41} \leq b$. So $\#D_j(\tau) \leq 2b$ and $\#D_j = \sum_{\tau\in\Phi} \#D_j(\tau) \leq 2fb$.

Step 2: the first four assertions. As $9m+1 \leq 19\sqrt{x} \leq x^5 \leq r$, we have $|\mathcal{B}_j| = r + 9m + 1 \leq 2r \leq 2r^8 = \Delta$, $12m+1 \leq r \leq \Delta$ and $3m \leq r \leq 5r^2 = rt$. By Step 1, $d \leq (10m+1)\#D_j \leq 11m\cdot 2fb \leq 44\sqrt{x}\, x^{12} b \leq x^{13} b \leq b^2 \leq M$.

Step 3: the logarithm of the left side. Write $G = \#\mathrm{GL}_{3m}(\mathcal{O}_E/\mathfrak{M}^2)$ and Λ for the left side of the inequality. All its factors are at least 1 ($N_{\mathrm{ch}} \geq 1$ by Lemma 6.7.10), and

$$\log\Lambda = 4\log N_{\mathrm{ch}} + 4x^{30} + 6x_1 + 2\log G + 6\,d\log d + 9(12m+1)\#D_j \log M,$$

with $d\log d = 0$ for $d = 0$ (the convention $0^0 = 1$). We bound the terms.

- $\log N_{\mathrm{ch}} \leq x^{33}$ by Lemma 6.7.10.
- $G \leq \#\mathrm{M}_{3m}(\mathcal{O}_E/\mathfrak{M}^2) = \#(\mathcal{O}_E/\mathfrak{M}^2)^{9m^2}$. Since $M = \pi_M\bar{\pi}_M \in \pi_M\mathcal{O}_E = \mathfrak{M}$, we have $M^2\mathcal{O}_E \subseteq \mathfrak{M}^2$, so $\#(\mathcal{O}_E/\mathfrak{M}^2) \leq \#(\mathcal{O}_E/M^2\mathcal{O}_E) = M^{2[E:\mathbb{Q}]} = M^{4f}$. Hence $\log G \leq 36fm^2\log M \leq 36\cdot x^{12}\cdot 4x\cdot 22\log b \leq 3168\, x^{13}\log b$.
- For $d \geq 1$, $\log d \leq \log M$ by Step 2, so $6\,d\log d \leq 6(10m+1)\#D_j\log M$.

The terms with $\#D_j$ add up to at most $(6(10m+1) + 9(12m+1))\#D_j\log M = (168m+15)\#D_j\log M \leq 183m\cdot 2fb\cdot 22\log b = 8052\, fmb\log b$, by Step 1, $m \geq 1$ and item 5. The remaining terms are at most $4x^{33} + 4x^{30} + 6x_1 + 6336\, x^{13}\log b \leq 6350\, b \leq 6350\, fmb\log b$, because $x^{33} \leq b$, $x_1 \leq b$ and $x^{13}\log b \leq x^{14} \leq b$ (as $\log b \leq x^{1/100}$), and $f, m, \log b \geq 1$. So

$$\log\Lambda \leq 14402\, fmb\log b \leq fmb(\log b)^2,$$

since $\log b \geq e^{6.4\cdot 10^{13}} \geq 14402$. Exponentiating gives the inequality. □

Lemma 6.6.23. *Let $1 \leq j \leq m$, $r_+ = k + j - 1$, ϑ a twist with modulus Q, $O = O_{j,\vartheta}$ and $\mathfrak{n} = \mathfrak{M}^2\prod_{w\in Q} w$. Let χ_ϑ be the multiplicative character of $R_\mathfrak{n}$ equal on units to $u \mapsto \prod_{w\in S_0\cup Q}\psi'_w(u \bmod w)$, $(\psi'_w)_w$ the tame part of $\chi'_{j,\vartheta}$, and to 0 on nonunits, and put $\alpha_e = \chi'_{j,\vartheta}\nu_j^\flat\upsilon^m$. Let $W = \mathrm{I}_{\mathcal{B}^{\mathrm{c}}} \oplus W'_j$, with index set $n = \mathcal{B}^{\mathrm{c}} \sqcup \mathcal{B}_j$; let $\pi_j\colon \mathcal{L}_j \to n$ send $c \in \{0,\dots,r_+-1\}$ to the basis vector of W_j of number c and be the inclusion on $\mathcal{A}_j^+$, $\mathcal{B}^\flat$ and $\mathcal{B}^{\mathrm{c}}$; let $\jmath_W\colon \mathcal{L}_j \sqcup \{1,\dots,m\} \to n$ be π_j on $\mathcal{L}_j$ and ng_j on $\{1,\dots,m\}$; and for $i \in \mathrm{Mem}_j$ let $\Pi_i^W \in \mathbb{C}[x]_{\mathcal{B}_j\times n}$ be Π_i with every variable $x_{\tau,b,c}$ renamed $x_{\tau,b,\pi_j(c)}$. Then:*

(a) $\mathfrak{n}$ *is a modulus with* $\Sigma(\mathfrak{n}) = S_0 \cup Q$*, and* $\chi_\vartheta = \psi_O^{\mathrm{res}}$*;*

(b) $\chi'_{j,\vartheta}$ *is tame of weight* 0*, and* α_e *is tame of weight* $\tau \mapsto m - \#D_j(\tau)$*, for* $(\mathfrak{n}, \chi_\vartheta)$*;* $\nu_j^\flat \upsilon^m$ *has trivial tame part;*

(c) $\jmath_W$ *is a bijection,* W *has the signs* $\mathrm{sgn}_{\jmath_W}$*, and* $\Pi_i^W = \Delta^{\jmath_W}(\epsilon_j; \mathsf{c}_i)$*, the product of stacked determinants with the index sets* $p = \{1, \dots, 2m\}$*,* $a = \mathcal{B}_j$*,* $m = \mathcal{L}_j$*,* n*,* $l = \{1, \dots, m\}$ *and* $\iota_\tau = D_j(\tau)$*;*

(d) for the pair $(W, \mathrm{I}_{\mathcal{B}_j})$*,* $\gamma(\tau) = m - \#D_j(\tau)$ *and every* $\gamma' \colon \Phi \to \mathbb{Z}$*,* $\Omega_{\gamma,\gamma'}(1, y)\, v_{\Pi_i^W} = v_{\Pi_i^W}$ *for all* $y \in G_\infty(1_{\mathcal{B}_j})$*.*

Proof. **(a)** The level system of O consists of the chosen places S_0 and the places of Q ((LF4)), all distinct and none conjugate to itself or another ((DP1)); so $\mathfrak{n} = \prod_{w \in S_0} w^2 \prod_{w \in Q} w$ is a modulus with $\Sigma(\mathfrak{n}) = S_0 \cup Q$. The character of O is $\mu_j^0 \vartheta$, whose tame part $\omega^j \psi_\vartheta$ (ψ_ϑ the tame part of ϑ) is that of $\chi'_{j,\vartheta} = \chi_M^j \vartheta$; hence χ_ϑ and ψ_O^{res} ((KN3), the product of the tame components of O over $M_O \cup Q_O = S_0 \cup Q$ on units, 0 on nonunits) coincide.

(b) $\chi'_{j,\vartheta}$ has weight 0 (χ_M and ϑ have weight 0) and tame part $\psi' = \omega^j \psi_\vartheta$. At a chosen place w, $\psi_{\vartheta,w} = 1$ (the places of Q are not chosen places) and $\omega_w^j \neq 1$, because ω_w has order $q_2 = (q_0 d)^{s+6} \geq 3^s > 2^{e+1} = m \geq j$ ($d \geq 3$ as $-d \equiv 1 \pmod 4$). At $w \in Q$, $\omega_w = 1$ and $\psi_{\vartheta,w} \neq 1$, ϑ being a twist. At all other places both are trivial. So $\psi'_w \neq 1$ exactly for $w \in \Sigma(\mathfrak{n})$, and $\chi'_{j,\vartheta}$ is tame of weight 0 for $(\mathfrak{n}, \chi_\vartheta)$. The characters $\mu^\flat$ and υ have trivial tame parts ((LF1), (LF2)), so $\nu_j^\flat \upsilon^m = \mu^\flat \upsilon^{m + e_j - j\Delta}$ has trivial tame part and weight $-\kappa_0 + m + e_j - j\Delta$. Hence α_e has the tame part ψ' and the weight $\tau \mapsto m + e_j - j\Delta - \kappa_0(\tau)$. It remains to show $\#D_j(\tau) = \kappa_0(\tau) + j\Delta - e_j$. By (LF4) and Definition 3.2.16, $\varsigma_\tau(c)$ is strictly decreasing in c and $\varsigma_\tau(r_+ - 1) = \frac{\Delta}{2} - \kappa_j(\tau)$ (the entry $k - 1 + (j-1)$ of the target parameter). As $\frac{\Delta}{2} = r^8 > \frac{t-1}{2}$, for $1 \leq h < r_+$ we have $\varsigma_\tau(h) \geq \varsigma_\tau(r_+ - 1) > \frac{t-1}{2} - \kappa_j(\tau)$, so the maximum in (LF4) is $\varsigma_\tau(h)$ and $\kappa_O^+(\tau, h) = \varsigma_\tau(h-1) - \varsigma_\tau(h) - 1 \geq 0$. Telescoping,

$\#D_j(\tau) = \sum_{1 \leq h < r_+} \kappa_O^+(\tau, h) + \Delta' = \kappa_0(\tau) + j\Delta - e_j$, so the weight of α_e is $m - \#D_j(\tau)$ and α_e is tame of that weight for $(\mathfrak{n}, \chi_\vartheta)$.

(c) The signs of W are $+1$ on $\mathcal{B}^{\mathrm{c}}$, $\mathcal{A}_j^+$ and $\mathcal{B}^\flat$, -1 on $\mathcal{A}_j^-$, and on $\mathcal{R}$ those of W_j: $+1$ on the basis vectors of numbers $0, \dots, r_+ - 1$ and -1 on those of numbers $r_+, \dots, r-1$ ((HP4)). The map π_j is injective and its image is the set of indices of sign $+1$; ng_j is injective ($\mathrm{ng}_j(i) = i \in \mathcal{A}_j^-$ for $i \leq j$, the basis vector of number $k + i - 2 \in \{r_+, \dots, r-1\}$ for $j < i \leq m$, as $r = k + m - 1$) and its image is the set of indices of sign -1. So $\jmath_W$ is a bijection and the sign of $\jmath_W(x)$ is $+1$ for $x \in \mathcal{L}_j$ and -1 for $x \in \{1, \dots, m\}$: W has the signs $\mathrm{sgn}_{\jmath_W}$. By (6.23), $\Delta^{\jmath_W}(\epsilon_j; \mathsf{c}_i)$ is the product over τ and $d \in D_j(\tau)$ of the determinants of the $\mathcal{L}_j \times \mathcal{L}_j$ matrices whose row $\epsilon_j(a)$ is the row a of $\mathsf{c}_{i,\tau,d}$ ($a \leq 2m$) and whose row $\epsilon_j(b)$ is $(x_{\tau,b,\jmath_W(k')})_{k' \in \mathcal{L}_j} = (x_{\tau,b,\pi_j(k')})_{k'}$ ($b \in \mathcal{B}_j$); $\Pi_i = \Delta^{\mathrm{id}}(\epsilon_j; \mathsf{c}_i)$ is the same product with $x_{\tau,b,k'}$ in place of $x_{\tau,b,\pi_j(k')}$. So $\Pi_i^W = \Delta^{\jmath_W}(\epsilon_j; \mathsf{c}_i)$.

(d) By (c), W has the signs $\mathrm{sgn}_{\jmath_W}$ and $\Pi_i^W = \Delta^{\jmath_W}(\epsilon_j; \mathsf{c}_i)$ with $\gamma(\tau) = \#l - \#\iota_\tau = m - \#D_j(\tau)$; (5) of Lemma 6.3.71 for the pair $(W, \mathrm{I}_{\mathcal{B}_j})$ gives the invariance. □

Lemma 6.6.24. *Let* O *be a descent pair for ranks* $r < n$*,* $\mathcal{V}$ *a divided splitting for finite index sets* N*,* m*,* a *and an embedding* $z \colon \{1, \dots, r\} \hookrightarrow m$*, and* s *an* $a \times N$ *matrix over* R_2*. Then there is a divided lattice* L *that is glued for* s*.*

Proof. Write $M = M_O$, $\hat{\mathcal{O}}_M = \prod_{w \in M} \mathcal{O}_{E,w}$, ϖ for the family of components of ϖ_O at M, $\phi = \phi_{\mathcal{V}}$, $d \colon a \hookrightarrow N$ for the divided embedding, and $\mathrm{red} \colon \hat{\mathcal{O}}_M \to R_2$ for the reduction; it is surjective with kernel $\varpi^2 \hat{\mathcal{O}}_M$, since $R_2 \cong \prod_{w \in M} \mathcal{O}_{E,w}/w^2$. Matrices over $\hat{\mathcal{O}}_M$ are reduced entrywise, written

$\bar{X}$. Two facts on R_2: an element y with $\bar{\varpi}y = 0$ is the reduction of an element of $\varpi\hat{\mathcal{O}}_M$, so the product of two such elements is 0; and $\bar{\varpi}^2 = 0$. We index the rows and columns of $(N \sqcup m)$-square matrices by $N \sqcup m$, write ι_z for the $(N \sqcup m) \times r$ matrix with entry 1 at $(z(i), i)$ in the block m and 0 elsewhere, and for an $n \times r$ matrix x write $x^\varepsilon = \left(\begin{smallmatrix} x_N \\ x_m \end{smallmatrix}\right)$ for its rows reordered by ε.

A Hermite form of the stacked matrix. Lift s to an $a \times N$ matrix t over $\hat{\mathcal{O}}_M$ and let $T = \left(\begin{smallmatrix} t \\ \varpi 1_N \end{smallmatrix}\right)$. At each $w \in M$, T_w has rank $\#N$ (its lower block is invertible over E_w), so by the elementary divisor theorem over the discrete valuation ring $\mathcal{O}_{E,w}$, $T_w = P\left(\begin{smallmatrix} \Lambda \\ 0 \end{smallmatrix}\right)Q$ with P, Q invertible and Λ diagonal; moving the rows of Λ to the bottom by a permutation, $T = U\left(\begin{smallmatrix} 0 \\ D \end{smallmatrix}\right)$ with $U \in \mathrm{GL}_{a\sqcup N}(\hat{\mathcal{O}}_M)$ and $D \in \mathrm{M}_N(\hat{\mathcal{O}}_M)$. Writing U in blocks U_{ij} along $a \sqcup N$, this says $t = U_{12}D$ and $\varpi 1_N = U_{22}D$; put $B = U_{22}$, so $BD = \varpi 1_N$. Moreover D is the block of the rows N of $U^{-1}T$. Reducing, $w := [s; \bar{\varpi}] = \bar{T} = \overline{U_N}\,\bar{D}$ and $\bar{D} = \overline{(U^{-1})_N}\,w$, where U_N are the columns N of U and $(U^{-1})_N$ the rows N of U^{-1}. Hence, for $N \times r$ matrices X, X' over R_2,

$$wX = wX' \iff \bar{D}X = \bar{D}X'. \tag{6.135}$$

The lattice. Let Δ' (resp. Δ) be the diagonal $m \times m$ matrix with entries ϖ (resp. 1) at the indices $z(i)$ and 1 (resp. ϖ) elsewhere, so $\Delta\Delta' = \varpi 1_m$ and $I_z^{\mathsf{T}}\Delta' = \varpi I_z^{\mathsf{T}}$. Put

$$G = \begin{pmatrix} D & -DI_\phi I_z^{\mathsf{T}} \\ 0 & \Delta' \end{pmatrix}, \qquad C = \begin{pmatrix} B & I_\phi I_z^{\mathsf{T}} \\ 0 & \Delta \end{pmatrix},$$

integral over $\hat{\mathcal{O}}_M$; then $CG = \varpi 1$, the upper right block being $-BDI_\phi I_z^{\mathsf{T}} + I_\phi I_z^{\mathsf{T}}\Delta' = -\varpi I_\phi I_z^{\mathsf{T}} + \varpi I_\phi I_z^{\mathsf{T}} = 0$. Choose a bijection $\varepsilon'\colon N \sqcup m \to \{1, \dots, n\}$ with $\varepsilon'(z(i)) = \mathrm{inc}_O(i)$ for all i (extend the injective partial map $z(i) \mapsto \mathrm{inc}_O(i)$ between sets of the same size n). Let $G^\star$, $C^\star$ be the $n \times n$ matrices with $G^\star_{\varepsilon'(p),\varepsilon(q)} = G_{pq}$ and $C^\star_{\varepsilon(p),\varepsilon'(q)} = C_{pq}$; then $C^\star G^\star = \varpi 1_n$. Let $c \in \mathrm{GL}_n(\mathbb{A}_{E,f})$ be $C^\star$ at the places of M (invertible over E_w as $C^\star G^\star = \varpi 1$) and 1 elsewhere, c^* its conjugate transpose (supported on $\bar{M}$, disjoint from M by (DP1), so c and c^* commute), and $\gamma = (c^*)^{-1}c$. Then $\gamma^*\gamma = c^*c^{-1}(c^*)^{-1}c = 1$, so $\gamma \in G_f(n)$. At $w \in M$, $\gamma_w = C^\star_w$ and $\varpi_O\gamma_w^{-1} = G^\star_w$; at $\bar{w}$, ϖ_O is 1 and $\gamma_{\bar{w}}^{-1}$ is the conjugate transpose of $C^\star_w$, integral; elsewhere $\gamma = 1$. So $L = \gamma$ is a divided lattice with $G_L = \overline{G^\star}$.

The congruence for L. For an $n \times r$ matrix x over R_2, the row $\varepsilon'(p)$ of $G_L x$ is the row p of $\bar{G}x^\varepsilon$, and the row $\varepsilon'(p)$ of $\bar{\varpi}\,\mathrm{Inc}_O$ is the row p of $\bar{\varpi}\iota_z$, as $\varepsilon'(p) = \mathrm{inc}_O(i)$ exactly when $p = z(i)$. So $G_L x = \bar{\varpi}\mathrm{Inc}_O$ if and only if $\bar{\Delta}'x_m = \bar{\varpi}I_z$ and $\bar{D}x_N = \bar{D}I_\phi I_z^{\mathsf{T}}x_m$. The first equation says that the rows of x_m outside the image of z vanish and that $Z = I_z^{\mathsf{T}}x_m$ (the rows $z(i)$) satisfies $\bar{\varpi}(1 - Z) = 0$; so

$$G_L x = \bar{\varpi}\,\mathrm{Inc}_O \iff x_m = I_z Z \text{ with } \bar{\varpi}(1 - Z) = 0, \text{ and } \bar{D}x_N = \bar{D}I_\phi Z. \tag{6.136}$$

For $A \in \mathrm{C}(\mathcal{V})$, by (6.135) the matrix xA is block normalised for s if and only if $x_m A = I_z$ and $\bar{D}x_N A = \bar{D}I_\phi$.

Condition (1). Let $G_L x = \bar{\varpi}\mathrm{Inc}_O$, with Z as in (6.136). Then $Y = 1 - Z$ has $\bar{\varpi}Y = 0$, so $Y^2 = 0$ by the first fact on R_2, Z is invertible with $Z^{-1} = 1 + Y$, and $\bar{\varpi}Z^{-1} = \bar{\varpi}$: $A_0 = Z^{-1} \in \mathrm{C}(\mathcal{V})$. Now $x_m A_0 = I_z$ and $\bar{D}x_N A_0 = \bar{D}I_\phi Z Z^{-1} = \bar{D}I_\phi$, so xA_0 is block normalised. If xA is block normalised for some $A \in \mathrm{C}(\mathcal{V})$, then $I_z Z A = I_z$, and multiplying by I_z^{T} on the left, $ZA = 1$, so $A = A_0$.

Condition (2). If xA is block normalised for some $A \in \mathrm{C}(\mathcal{V})$, put $Z = A^{-1}$; then $\bar{\varpi}Z = \bar{\varpi}$, $x_m = I_z Z$ and $\bar{D}x_N = \bar{D}I_\phi Z$, so $G_L x = \bar{\varpi}\mathrm{Inc}_O$ by (6.136).

Condition (3). The columns $\varepsilon(N)$ of G_L have the rows $\varepsilon'(N)$ equal to $\bar{D}$ and the rows $\varepsilon'(m)$ equal to 0. Reduce s modulo the chosen places to an $a \times N$ matrix over $\prod_{w\in M} k_w$; at each

w, column operations bring it to a matrix whose nonzero columns are among the a columns of $d(a)$ (its rank is at most $\#a$). Lift this invertible matrix to $J \in \mathrm{M}_N(R_2)$; J is invertible, its determinant reducing to a unit at each place. Then sJ has entries in the image of $\mathfrak{M}_O$ in the columns outside $d(a)$, so $\bar{\varpi} sJ$ vanishes there, and $\bar{\varpi} \cdot \bar{\varpi} J = 0$; hence $\bar{\varpi} wJ$ vanishes in those columns, and so does $\bar{\varpi} \bar{D} J = \overline{(U^{-1})_N}\, \bar{\varpi} wJ$. This is condition (3). □

Lemma 6.6.25. *Let $1 \le j \le m$, let Y and c'' be a short row family and a scale with (i) and (ii) of Lemma 6.6.17, let $L = (L_i)_{i \in \mathrm{Mem}_j}$ be divided lattices of $O_{j,1}$ with L_i glued for Λ_i, suppose that $\mathcal{V}_j$ is a divided splitting for $O_{j,\vartheta}$ for every twist ϑ, and let $\mathrm{gl}_\vartheta = \mathrm{gl}_{j,\vartheta}[L; c'']$, formed with Y. For $\zeta \in \mathrm{Tab}_j$ let $\bar{\zeta}_{i,\mathfrak{c}}$ have the conjugate coordinates of $\zeta_{i,\mathfrak{c}}$.*

(a) Let ϑ be a twist with modulus Q, $\ell \nmid \mathrm{N}Q$ a prime and $\iota \colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism, and let $\zeta \in \mathrm{Tab}_j$ satisfy $\mathrm{gau}_{\iota \circ c}(x_\mathfrak{c}, \bar{\zeta}_{i,\mathfrak{c}}) \le 1$ for all $(i, \mathfrak{c})$, for the weight $(\kappa_O^-, 0)$ in rank $|\mathcal{B}^\circ|$ and $\iota \circ c$ the composite of ι with complex conjugation. Then $\mathrm{gl}_\vartheta \zeta$ is integral at ι.

(b) For every $\zeta \in \mathrm{Tab}_j$ and every automorphism σ of $\mathbb{C}$, $\|\mathrm{gl}_1 \zeta\|_\sigma^2 \le \exp(2fmb(\log b)^2) \|\zeta\|_\sigma^2$.

Proof. The weights, chosen modulus and conditions (GU2)–(GU4) do not depend on the twist, so the given L_i are also glued lattices for each $O_{j,\vartheta}$. All class sums at an open finite level are finite: definiteness and Theorem A.4.5 make the adelic quotient compact.

We first record the two elementary bounds for a point form. If v is a weight vector and z, g are finite points, then

$$\mathrm{gau}_\iota(g, \delta_z(v)(g)) \le \mathrm{gau}_\iota(z, v).$$

Indeed, in a nonzero summand of (GU5) write $z^{-1}\gamma g = u \in \mathbf{K}(O)$. In the gauge, rational covariance changes $g^{-1}\gamma^{-1}$ into $u^{-1}z^{-1}$. The Schur matrices of u and u^{-1} have integral entries and their determinant characters are units, so they preserve the coordinate maximum norm. The level character is a root of unity, hence has absolute value one under every ι. The ultrametric inequality proves the assertion after summing over γ.

For every automorphism σ of $\mathbb{C}$, the other bound is

$$\|\sigma(\delta_z(v)(g))\|_{\kappa_O} \le \#\Gamma_z \, \|\sigma(v)\|_{\kappa_O}, \qquad \Gamma_z = \mathrm{U}(\mathrm{I}_n)(E^+) \cap z\mathbf{K}(O)z^{-1}.$$

If the summation set is nonempty it is a coset of Γ_z. Each summand has the same Gram norm as $\sigma(v)$: the level character has absolute value one, and a rational unitary matrix remains unitary at every complex embedding after applying σ. This last assertion uses the actual CM field: complex conjugation commutes with every embedding of E, so the conjugate of its unitary equation is again the unitary equation. The Gram model (WM3) is the restriction of the ordinary exterior-tensor norm. Thus its representation preserves that norm. The ordinary triangle inequality gives the displayed bound.

We also have $\#\Gamma_z \le N_{\mathrm{ch}}$ for all the point forms used below. The group is finite by definiteness. After conjugating by z, reduce it at the chosen places. The kernel is trivial by Lemma 4.6.2(c), since it lies in the principal level at a chosen place. Hence it embeds in the product of $\mathrm{GL}_n(k_w)$. Here $n = r + t = 6r \le rt$ as $r \ge 3$, and block diagonal inclusion embeds every such finite group into $\mathrm{GL}_{rt}(k_w)$. Lagrange's theorem and (CX1) give the asserted bound.

We next prove the loss estimate for the Cartan product, retaining its conjugate embedding. Write $h_+ \in G_f(\mathcal{B}_j)$, $h_- \in G_f(\mathcal{B}^\circ)$, let L_i be one of the given glued lattices, and put

$$g = \mathrm{diag}_{\varepsilon_j}(h_+, h_-)\gamma_{L_i}.$$

For vectors y and η in the two block Weyl modules we claim

$$\mathrm{gau}_{\iota}(g, y \cdot \eta) \le |M|_{\ell}^{-\mathrm{loss}(\mathcal{V}_j)} \mathrm{gau}^{+}_{\iota}(h_{+}, y)\, \mathrm{gau}^{-}_{\iota \circ c}(h_{-}, \bar{\eta}). \tag{1}$$

Here is the matrix estimate underlying this claim. For a block of rows and a subset of its columns, suppose that after an invertible change of those columns the columns of $\gamma_{L_i}^{-1}$ outside that subset are integral, while multiplication by M makes every remaining column integral. Lift the invertible change modulo $\mathfrak{M}^2$ to an integral invertible finite matrix. This is possible locally at each chosen split place by lifting entries and a unit determinant; use its inverse conjugate transpose at the conjugate place and the identity elsewhere. Absorb that matrix into h_{+} or h_{-}. Such a change preserves the gauge, because its Schur matrix and inverse are integral. The relevant inverse columns then factor as

$$A_{\mathrm{int}}\ \mathrm{diag}(M^{-1} \text{ on the subset}, 1 \text{ elsewhere})\, h_{\pm}^{-1},$$

where A_{int} is integral. Schur matrices of integral rectangular matrices contract the coordinate maximum norm: this follows from Cauchy–Binet and the integral standard basis in (WM1)–(WM2). The middle diagonal matrix contributes at most

$$|M|_{\ell}^{-\sum_{\tau} (\#\ \mathrm{subset}) \sum_{h} \kappa_{O}^{\pm}(\tau, h)}.$$

To check the exponent, in a filling each column has distinct row entries, and so contains at most the size of the subset from that subset. Multiplication over all columns gives precisely this upper bound. The divided-splitting hypothesis excludes full-height determinant factors.

For the positive block use the subset $\mathrm{im}(d_j)$, of size $3m$. Condition (3) of (GU4) supplies the required change of columns; denote its invertible residue matrix by J. Indeed G_{L_i} is the reduction of $\varpi_O \gamma_{L_i}^{-1}$; outside the divided columns, $\bar{\varpi} G_{L_i} J = 0$ modulo $\mathfrak{M}^2$ says that $\varpi_O \gamma_{L_i}^{-1} J$ is divisible by ϖ_O, hence that $\gamma_{L_i}^{-1} J$ is integral there. On the divided columns use (GU2) and $M = \pi_M \bar{\pi}_M \in \mathfrak{M}$ to clear the denominator. Away from the chosen places (GU2) already gives integrality. For the negative block use the entire block as subset, and the same denominator clearing, without a change of columns.

Finally apply the mixed-action identity (d) of Lemma 3.2.20 to $C_O(y \cdot \eta)$, whose value is the tensor of the included vectors by Lemma 6.6.26(a). The positive part is evaluated on the indicated columns of g^{-1}. The negative part is evaluated on the columns of the inverse transpose of that matrix. Since g is unitary, these are the conjugates of the corresponding columns of g^{-1}. Thus that block is read at $\iota \circ c$ with $\bar{\eta}$, because $(\iota \circ c)(\bar{\eta}) = \iota(\eta)$. Applying the integral left inverse C'_O does not increase the maximum norm. The two block exponents add to exactly (GU1)'s $\mathrm{loss}(\mathcal{V}_j)$, proving (1).

We can now prove (a). The gluing level is contained in the principal level at every chosen place. In fact fixing the stacked matrix $[\Lambda_i; \bar{\varpi}]$ forces $\bar{\varpi} A = \bar{\varpi} I$ modulo $\mathfrak{M}^2$, hence $A = I$ modulo $\mathfrak{M}$. Its class stabilizers are therefore trivial by Lemma 4.6.2(c). Consequently the class sums in (TG6) are plain finite sums, with no stabilizer denominators. For their term indexed by $(i, \mathfrak{c}, h_{+})$, apply (1) with $h_{-} = x_{\mathfrak{c}}$, $y = \varphi''_{\vartheta, i}(h_{+})$ and $\eta = \zeta_{i, \mathfrak{c}}$. The first gauge is at most one by the assumed first-factor property (i), and the second is at most one by the hypothesis of (a). The point-form bound and multiplication by $M^{\mathrm{loss}(\mathcal{V}_j)}$ cancel the remaining factor exactly. Every term has gauge at most one at every point, including at primes dividing M. The ultrametric inequality for the finite sum proves integrality at ι.

For (b) we use the same finite expansion at the trivial twist. First,

$$\|\sigma(y \cdot \eta)\|_{\kappa_O}^2 = \|\sigma(y)\|_{\kappa_O^{+}}^2 \|\sigma(\eta)\|_{\kappa_O^{-}}^2. \tag{2}$$

Indeed the Cartan matrices and coordinate inclusions are integral, so commute with σ; by (a) of Lemma 6.6.26, their mixed image is the tensor of the included vectors. By (b) of Lemma 3.2.20 the mixed embedding is an isometry, and the exterior coordinate inclusions are isometries by (WM3). This proves (2). The assumed property (ii), the bound for Γ_z, and the fact that each squared table-entry norm is at most $\|\zeta\|_\sigma^2$ now show that the squared norm of every value of every point-form term is at most

$$N_{\mathrm{ch}}^2 \exp\bigl(fmb(\log b)^2\bigr)\|\zeta\|_\sigma^2. \tag{3}$$

It remains to bound explicitly the finite counts and the factor $M^{2\,\mathrm{loss}(\mathcal{V}_j)}$. We give the estimates, since a finiteness assertion alone would not imply the stated bound.

By Lemma 2.3.6 and (AD1),

$$x \ge 10^6, \qquad \log b \ge 319, \qquad \log b \le x, \qquad b \ge x^{10^{12}}. \tag{4}$$

For example $10^{14} q_0 \le \log x$ proves the first inequality, the two 10^6 inequalities in (AD1) give the last one, and $100 \log\log b \le \log x$ gives $\log b \le x$. Also $q_0 \le 10^{-14} \log x$. From the explicit tower formula, $l_0(s) \le 81 s$, $s \le 32 \log x$, and $2^s = m^{22} \le 2^{22} x^{11}$,

$$f \le q_1 = 3 q_0 l_0(s) 2^s \le (\log x)^2 x^{11} \le x^{13}. \tag{5}$$

The numerical coefficient used here is $3 \cdot 81 \cdot 32 \cdot 2^{22} \cdot 10^{-14} < 1$. Furthermore

$$n_m = r + 1 + 5mr \le 34 x^{11/2} \le x^6, \quad m \le x, \quad r \le x^6, \quad t \le x^7, \quad \Delta \le x^{49}, \quad \log M \le 22 \log b. \tag{6}$$

For every rank $1 \le n' \le 2n_m$, a level containing the principal level π_M^2 has at most e^b classes. To see this from the exact level in Lemma 4.6.4, let K_1 be the principal level $\pi_M \pi_N$ and K_2 the principal level π_M^2. Refining classes from K_1 to $\mathrm{K}_1 \cap \mathrm{K}_2$ increases their number by at most $[\mathrm{K}_1 : \mathrm{K}_1 \cap \mathrm{K}_2]$, and passing from the intersection to K_2 cannot increase it. Since K_1 is the identity modulo π_M, reduction of $u - I$ modulo π_M^2 embeds that quotient into the additive matrices over $\pi_M \mathcal{O}_E / \pi_M^2 \mathcal{O}_E$. Cross products vanish modulo π_M^2, so this reduction is a homomorphism with precisely that kernel. In particular its index is at most $M^{2f(n')^2}$ (in fact $M^{f(n')^2}$ suffices). Thus the class number is at most

$$\exp(x^{30}) M^{2f(n')^2} \le \exp(x^{30} + 176 x^{26}) \le \exp(x^{31}) \le e^b \tag{7}$$

by (4)–(6). This argument needs no coprimality assumption involving π_N.

The first-factor gluing levels at the trivial twist contain the principal level π_M^2 by (KN3); so does the output level by (DP4). The flat and second-factor class sets are defined at that principal level. All four ranks are at most $2n_m$: the first rank is $r + 9m + 1$, the second at most t, the flat rank $6m + 1$, and the output rank $6r$; (4), (AD2) and $m \ge 2$ give the claim. Therefore (7) bounds each of these class counts by e^b.

We now count tables. In the notation of (TG1)–(TG4), the residue ring has cardinality $\#R_2 = M^{2f} \le M^{4f}$, because the absolute norm of $\pi_M \mathcal{O}_E$ is M^f. Choosing Λ_i, the flat class, its row lists, and the second class gives

$$\#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ) \le M^{\,4f(3m)(r+9m+1)} e^{2b} (6m+1)^{\,2mf\Delta'}. \tag{8}$$

Here $\Delta' \le \Delta/2$. The inequalities $r + 9m + 1 \le x^7$, $3m \le x^2$ and $6m + 1 \le x^2$ follow from (4)–(6). The logarithm of the first factor in (8) is at most $88 f x^9 \log b \le x^{24}$. The logarithm of its row-list factor is at most $4 f x^{50} \log x \le x^{65}$. Since $x^{24} + x^{65} \le b$ by (4), we obtain

$$\log \#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ) \le 3b. \tag{9}$$

Every set being counted is nonempty.

The chosen group order has the equally explicit bound

$$\log N_{\mathrm{ch}} \le (rt)^2 \sum_{w \text{ chosen}} \log \mathrm{N}w \le f(rt)^2 \log M \le 22 f x^{26} \log b \le x^{42} \le b. \tag{10}$$

Indeed $\#\mathrm{GL}_{rt}(k_w) \le (\mathrm{N}w)^{(rt)^2}$, and the product of the chosen prime ideals divides $\pi_M \mathcal{O}_E$, which gives the second inequality.

For completeness the weight formulas give the required bound on the loss without estimating individual hooks or dimensions. At a fixed τ, the extended target sequence in (LF4) is decreasing: successive drops are 1 in its first block and Δ in its tail; the junction drop is positive because $\Delta \ge k$. The positive column counts are its successive drops clipped at $(t-1)/2 - \kappa_j(\tau)$, with one subtracted at each step. More explicitly, if $B_\tau = (t-1)/2 - \kappa_j(\tau)$, each positive count is at most $\max(\varsigma_\tau(h-1), B_\tau) - \max(\varsigma_\tau(h), B_\tau)$. Summing these nonnegative differences telescopes. Their sum is therefore at most

$$\big(\varsigma_\tau(0) - (t-1)/2 + \kappa_j(\tau)\big)^+ \le \kappa_j(\tau) \le b + (m+1)\Delta.$$

Here $\varsigma_\tau(0) = k/2 - 1$, $k \le t+1$, and $\kappa(\tau) \le b$ since $x_1 \ge x$. The negative column counts are at most Δ. In the first block their adjacent drop is 1; in the tail it is Δ. At the junction the minimum with $-\kappa_j(\tau) - (t-1)/2$ makes the count zero, since $j \ge 1$. When both arguments have been truncated to zero it is zero as well. This checks the endpoint convention in (LF4). Hence (GU1), $|\mathcal{B}^{\mathrm{c}}| = 3m$ and $|\mathcal{B}^{\circ}| \le t$ imply

$$\mathrm{loss}(\mathcal{V}_j) \le f\big[3m(b + xb/x_1 + (m+1)\Delta) + t^2\Delta\big] \le 7fmb. \tag{11}$$

For the last inequality $xb/x_1 \le b$, and (6) gives $3m(m+1)\Delta + t^2\Delta \le 6x^{51} + x^{63} \le x^{65} \le mb$.

Combining (9)–(11) gives exactly the remaining scalar budget:

$$\begin{aligned} &\log\Big[M^{2\,\mathrm{loss}(\mathcal{V}_j)} e^b \big(\#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ) e^b N_{\mathrm{ch}}\big)^2\Big] \\ &\quad = 2\,\mathrm{loss}(\mathcal{V}_j)\log M + 3b + 2\log\#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ) + 2\log N_{\mathrm{ch}} \\ &\quad \le 308 fmb \log b + 11b \le 319 fmb \log b \le fmb(\log b)^2. \end{aligned} \tag{12}$$

Finally apply the ordinary triangle inequality at a fixed output point to the expansion (TG6), using (3). There are $\#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ)$ outer indices and at most e^b first-factor classes for each. Thus the norm at that point is at most

$$M^{\mathrm{loss}(\mathcal{V}_j)} \#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ) e^b N_{\mathrm{ch}} \exp\big(\tfrac{1}{2} fmb(\log b)^2\big) \|\zeta\|_\sigma.$$

The conjugate norm of a form is the class sum of the squared norms of these values. There are at most e^b output classes and each class weight $1/\#\Gamma$ is at most one. Squaring the last display, summing with those weights, and applying (12), we conclude

$$\|\mathrm{gl}_1\zeta\|_\sigma^2 \le \exp\big(2fmb(\log b)^2\big)\|\zeta\|_\sigma^2.$$

Every estimate holds for every σ, with the given Y, c'' and glued lattices unchanged. This proves (b). □

Lemma 6.6.26. *Let O be a descent pair for ranks $r < n$, let N, m be finite index sets and $\varepsilon\colon N \sqcup m \to \{1,\dots,n\}$ a bijection such that $\kappa_O^+(\tau,h) = 0$ for $|N| \le h < n$ and $\kappa_O^-(\tau,h) = 0$ for $|m| \le h < n$ ($\tau \in \Phi$). For a polynomial P in variables $X_{\tau,b,c}$ with b in N (or in m) let P^ε be P with every $X_{\tau,b,c}$ renamed $X_{\tau,\varepsilon(b),c}$; $\mathrm{Let}_{\pm 1}$ on polynomials in the variables $y_{\tau,b,a}$ with b in N (or m) is read with these rows. Then for all $y \in \mathbb{V}_{\kappa_O^+}$ (rows N) and $\eta \in \mathbb{V}_{\kappa_O^-}$ (rows m):*

(a) $C_O(y \cdot \eta) = S_{\kappa_O^+}(\iota_N)\, y \otimes S_{\kappa_O^-}(\iota_m)\, \eta$;

(b) $\overline{y \cdot \eta} = \bar{y} \cdot \bar{\eta}$, *the bars denoting the conjugate coordinates;*

(c) $\mathrm{Har}_O(y \cdot \eta) = \mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(y))^{\varepsilon} \cdot \mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}(\eta))^{\varepsilon}$.

Proof. Write $\kappa^{\pm} = \kappa_O^{\pm}$, $\kappa = \kappa_O$. By the hypothesis the rectangular Schur matrices $S_{\kappa^+}(z)$ for $z_\tau \in \mathrm{M}_{n\times|N|}$ and $S_{\kappa^-}(z)$ for $z_\tau \in \mathrm{M}_{n\times|m|}$ of (WM2) are defined with $d = 0$: $b'_i(zY) = \sum_j S(z)_{ij} b_j(Y)$, b'_i the standard basis in rank n; equivalently $S(z)\,\mathrm{ev}(Y) = \mathrm{ev}(zY)$ for every array Y, and likewise for square z. Every b_i for a weight λ is a product of minors $\Delta_{\tau,A}$ with $\lambda(\tau,|A|) \neq 0$, which reads only the columns $c < |A|$; for κ^+ these are columns $c < |N|$, for κ^- columns $c < |m|$.

Step 1: polynomials of rectangular Schur images. Let λ be κ^+ and $N' = |N|$, or $\lambda = \kappa^-$ and $N' = |m|$, and let $z_\tau \in \mathrm{M}_{n\times N'}(\mathbb{C})$. We show $\mathrm{Pol}_\lambda(S_\lambda(z)\xi)(Y) = \mathrm{Pol}_\lambda(\xi)(z^{\mathsf{T}}Y)$ for ξ in rank N' and arrays Y with n rows, the left side in rank n. As $G = F^{\mathsf{T}}F$ and $\Delta_T = \sum_i F_{Ti} b_i$,

$$\mathrm{Pol}_\lambda(\xi) = \sum_i (F^{\mathsf{T}}F\xi)_i b_i = \sum_{T\in\mathcal{C}_\lambda} (F\xi)_T\, \Delta_T$$

in either rank; the fillings in the two ranks have the same slots (τ,h,j), as $\lambda(\tau,h) = 0$ for $h \geq N'$. For fillings T (rank n) and T' (rank N') let $\Lambda(z)_{TT'} = \prod_{\tau,h,j} \det(z_\tau)_{T(\tau,h,j),T'(\tau,h,j)}$, the product of the minors of z_τ on the rows $T(\tau,h,j)$ and the columns $T'(\tau,h,j)$, both listed increasingly. By the Cauchy–Binet formula, $\Delta_{\tau,A}(zY') = \sum_{A'} \det(z_\tau)_{A,A'} \Delta_{\tau,A'}(Y')$ for arrays Y' with N' rows, the sum over the $|A|$-subsets A'; multiplying over the slots, $\Delta_T(zY') = \sum_{T'} \Lambda(z)_{TT'} \Delta_{T'}(Y')$. Expanding both sides in the $b_j(Y')$ (the left side by $\Delta_T = \sum_i F'_{Ti} b'_i$ and (WM2)) and using the linear independence of the b_j, $F'S_\lambda(z) = \Lambda(z)F$, F' the filling matrix in rank n. Hence

$$\mathrm{Pol}_\lambda(S_\lambda(z)\xi)(Y) = \sum_T (\Lambda(z)F\xi)_T \Delta_T(Y) = \sum_{T'} (F\xi)_{T'} \sum_T \Lambda(z)_{TT'} \Delta_T(Y) = \sum_{T'} (F\xi)_{T'} \Delta_{T'}(z^{\mathsf{T}}Y),$$

the last step by the Cauchy–Binet formula $\Delta_{\tau,A'}(z^{\mathsf{T}}Y) = \sum_A \det(z_\tau)_{A,A'} \Delta_{\tau,A}(Y)$, multiplied over the slots. This is the claim.

Step 2: (a). Let $U \subseteq \mathbb{V}_{\kappa^+} \otimes \mathbb{V}_{\kappa^-}$ (rank n) be the image of C_O. We show that $w(y,\eta) = S_{\kappa^+}(\iota_N)y \otimes S_{\kappa^-}(\iota_m)\eta$ lies in U. It is bilinear in (y,η), and the evaluation vectors span each Weyl module: a linear form (λ_i) vanishing on all $\mathrm{ev}(Y)$ gives the polynomial identity $\sum_i \lambda_i b_i = 0$, and the b_i, a $\mathbb{Z}$-basis of polynomials with integer coefficients, are linearly independent over $\mathbb{Q}$, hence over $\mathbb{C}$. So it suffices that $w(\mathrm{ev}(Y_1),\mathrm{ev}(Y_2)) = \mathrm{ev}_{\kappa^+}(\iota_N Y_1) \otimes \mathrm{ev}_{\kappa^-}(\iota_m Y_2)$ lies in U for all arrays Y_1 (rows N) and Y_2 (rows m). By the column remark we may take $Y_{1,\tau} \in \mathrm{M}_{N\times|N|}(\mathbb{C})$ (columns $0,\dots,|N|-1$) and $Y_{2,\tau} \in \mathrm{M}_{m\times|m|}(\mathbb{C})$. Membership in U is the vanishing of the linear forms that vanish on U, which are polynomials in the entries of (Y_1,Y_2); so it suffices to treat all $Y_{1,\tau}$, $Y_{2,\tau}$ invertible, a nonempty Zariski open subset of the affine space of these entries. Then for each τ let $g_\tau \in \mathrm{M}_n(\mathbb{C})$ have the column $c+1$ equal to $\iota_N Y_{1,\tau} e_c$ $(0 \leq c < |N|)$ and the column $n-c$ equal to $\iota_m (Y_{2,\tau}^{-1})^{\mathsf{T}} e_c$ $(0 \leq c < |m|)$; as $|N| + |m| = n$ these are all columns, and g_τ is invertible, its columns being bases of the two complementary coordinate subspaces $\iota_N\mathbb{C}^N$, $\iota_m\mathbb{C}^m$. Let h_τ have the column $c+1$ equal to $\iota_N (Y_{1,\tau}^{-1})^{\mathsf{T}} e_c$ and the column $n-c$ equal to $\iota_m Y_{2,\tau} e_c$. The entry (p,q) of $g_\tau^{\mathsf{T}} h_\tau$ is the dot product of the columns p of g_τ and q of h_τ: it is 0 if one lies in $\iota_N\mathbb{C}^N$ and the other in $\iota_m\mathbb{C}^m$, and it is $(Y_1^{\mathsf{T}}(Y_1^{-1})^{\mathsf{T}})_{cc'} = \delta_{cc'}$, respectively $((Y_2^{-1})Y_2)_{cc'} = \delta_{cc'}$, otherwise. So $h_\tau = (g_\tau^{-1})^{\mathsf{T}}$. With the arrays y_{id}, y_{w_0} of (c) of Lemma 3.2.20, the column c of

gy_{id} is $ge_{c+1} = \iota_N Y_1 e_c$ for $c < |N|$, and the column c of $(g^{-1})^{\mathsf{T}} y_{w_0}$ is $he_{n-c} = \iota_m Y_2 e_c$ for $c < |m|$. By the column remark,

$$\mathrm{ev}_{\kappa^+}(\iota_N Y_1) = \mathrm{ev}_{\kappa^+}(gy_{\mathrm{id}}) = S_{\kappa^+}(g)\,\mathrm{ev}_{\kappa^+}(y_{\mathrm{id}}), \qquad \mathrm{ev}_{\kappa^-}(\iota_m Y_2) = S_{\kappa^-}((g^{-1})^{\mathsf{T}})\,\mathrm{ev}_{\kappa^-}(y_{w_0}),$$

and by (d) and (c) of Lemma 3.2.20, with $e^-(\tau) = \sum_{1\le i<n} \kappa^-(\tau, i)$,

$$\begin{aligned} C_O\Big(\prod_\tau \det(g_\tau)^{-e^-(\tau)} S_\kappa(g)\,\mathrm{ev}_\kappa(y_{\mathrm{id}})\Big) &= \big(S_{\kappa^+}(g) \otimes S_{\kappa^-}((g^{-1})^{\mathsf{T}})\big)\big(\mathrm{ev}_{\kappa^+}(y_{\mathrm{id}}) \otimes \mathrm{ev}_{\kappa^-}(y_{w_0})\big) \\ &= \mathrm{ev}_{\kappa^+}(\iota_N Y_1) \otimes \mathrm{ev}_{\kappa^-}(\iota_m Y_2). \end{aligned}$$

So $w(y,\eta) \in U$ for all y, η: $w(y,\eta) = C_O x$ for some x, and as $C_O' C_O = 1$, $C_O(y \cdot \eta) = C_O C_O' C_O x = C_O x = w(y,\eta)$.

Step 3: (b). The entries of C_O' are integers, and those of $S_{\kappa^\pm}(\iota_N)$, $S_{\kappa^\pm}(\iota_m)$ are integers (by the Cauchy–Binet formula a minor of ιY is an integral combination of minors of Y, and the b_j form a $\mathbb{Z}$-basis). So complex conjugation of the coordinates commutes with $y \cdot \eta = C_O'(S_{\kappa^+}(\iota_N)y \otimes S_{\kappa^-}(\iota_m)\eta)$.

Step 4: (c). By (DP8), $\mathrm{Har}_O(\xi) = L(C_O\xi)$ for the linear map

$$L(\alpha \otimes \beta) = \mathrm{Let}_1(\mathrm{Pol}_{\kappa^+}\alpha)\,\mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa^-}\beta),$$

so by (a)

$$\mathrm{Har}_O(y \cdot \eta) = \mathrm{Let}_1(\mathrm{Pol}_{\kappa^+}(S_{\kappa^+}(\iota_N)y))\,\mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa^-}(S_{\kappa^-}(\iota_m)\eta)).$$

By Step 1, $\mathrm{Pol}_{\kappa^+}(S_{\kappa^+}(\iota_N)y)(Y) = \mathrm{Pol}_{\kappa^+}(y)(\iota_N^{\mathsf{T}} Y)$, and $(\iota_N^{\mathsf{T}} Y)_{\tau,b,a} = Y_{\tau,\varepsilon(b),a}$ ($b \in N$); so this polynomial is $\mathrm{Pol}_{\kappa^+}(y)$ with $y_{\tau,b,a}$ renamed $y_{\tau,\varepsilon(b),a}$. The letter substitution keeps the row index ($y_{\tau,i,a} \mapsto \sum_c X_{\tau,i,c}$), so Let_1 of it is $\mathrm{Let}_1(\mathrm{Pol}_{\kappa^+}(y))^\varepsilon$; likewise for η with ι_m and Let_{-1}. □

Lemma 6.6.27. *Let $1 \le j \le m$, $\kappa = \kappa^-_{O_{j,1}}$ in rank $|\mathcal{B}^\circ|$, $h = h_\kappa$, and $A = \mathrm{Mem}_j \times \mathrm{Cl}^\circ \times \mathcal{C}_\kappa$. For $\zeta \in \mathrm{Tab}_j$ let $\bar\zeta_{i,\mathfrak{c}}$ have the conjugate coordinates of $\zeta_{i,\mathfrak{c}}$.*

(a) For every prime ℓ there are tables $f_a \in \mathrm{Tab}_j$ ($a \in A$) with $\sum_{a\in A}\langle f_a, \zeta\rangle f_a = h^2\zeta$ for every $\zeta \in \mathrm{Tab}_j$, and with $\mathrm{gau}_{\iota\circ c}(x_{\mathfrak{c}}, \overline{(f_a)}_{i,\mathfrak{c}}) \le 1$ for the weight $(\kappa, 0)$, every ring homomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ and all a, i, $\mathfrak{c}$ ($\iota \circ c$ the composite with complex conjugation).

(b) There are tables $f_a \in \mathrm{Tab}_j$ ($a \in A$) with $\sum_{a\in A}\langle f_a, \zeta\rangle f_a = h^2\zeta$ for every ζ and $\|f_a\|_\sigma^2 \le h^2$ for every a and every automorphism σ of $\mathbb{C}$.

(c) $\#(\mathrm{Mem}_j \times \mathrm{Cl}^\circ) \cdot \big(\#A \cdot N_{\mathrm{ch}}\, h^2\big)^2 \le \exp\big(2fmb(\log b)^2\big)$.

Proof. Write $F = F_\kappa$, $G = F^{\mathsf{T}}F$, and let H be the integer matrix of the Fock pairing. Thus $GH = HG = hI$. All inner products below are conjugate linear in the first argument.

An integral frame. For each increasing-column filling T, let e_T be its coordinate vector in the row space of F, and put

$$\eta_T = HF^{\mathsf{T}} e_T = hG^{-1}F^{\mathsf{T}} e_T.$$

These vectors have integer coordinates. Since G, H are real symmetric,

$$\sum_T \eta_T \eta_T^* G = HF^{\mathsf{T}}FHG = HGHG = h^2 I.$$

This is exactly $\sum_T \langle \eta_T, \xi\rangle_\kappa \eta_T = h^2\xi$. Moreover

$$\|\eta_T\|_\kappa^2 = h^2 e_T^{\mathsf{T}} F G^{-1} F^{\mathsf{T}} e_T \le h^2.$$

Indeed $FG^{-1}F^{\mathsf{T}}$ is a real symmetric idempotent, hence an orthogonal projection. Every automorphism of $\mathbb{C}$ fixes η_T. Placing η_T in just the $(i,\mathfrak{c})$-slot and zero in the other slots proves (b), including its specified indexing set and the original Gram norm.

Integral frames at a finite prime. Fix ℓ. For each $\mathfrak{c}$, choose a rational unitary matrix $\delta_{\mathfrak{c}}$ such that $\delta_{\mathfrak{c}}x_{\mathfrak{c}}$ and its inverse are integral at every place above ℓ. For this choice apply Lemma 4.2.64 to the actual standard definite space underlying Cl°, with the finite set of all places of E above ℓ. The auxiliary finite factor in the dense products is 1 at these places, so the rational unitary components are dense in their product. At every prescribed place the inverse of $x_{\mathfrak{c}}$ belongs to the nonempty open set of movers δ for which $\delta x_{\mathfrak{c}}$ and its inverse are integral. Thus the lemma supplies the required $\delta_{\mathfrak{c}}$.

Let ρ be the rational unitary representation of weight $(\kappa,0)$, and put

$$v_{\mathfrak{c},T}=\overline{\rho(\delta_{\mathfrak{c}})^{-1}\eta_T}.$$

Because G is real and $\rho(\delta_{\mathfrak{c}})$ is unitary by Lemma 4.6.10, the map $\xi\mapsto\overline{\rho(\delta_{\mathfrak{c}})^{-1}\bar{\xi}}$ is a complex-linear unitary map. Consequently the vectors $v_{\mathfrak{c},T}$ have the same tight-frame identity. Define $f_{(i,\mathfrak{c},T)}$ to have value $v_{\mathfrak{c},T}$ in that slot and zero elsewhere.

We check the gauge with its stated conjugation. The identity $\rho(\delta_{\mathfrak{c}})\bar{v}_{\mathfrak{c},T}=\eta_T$ and multiplicativity of the Schur matrices give directly from (WM9)

$$\mathrm{gau}_{\iota oc}(x_{\mathfrak{c}},\bar{v}_{\mathfrak{c},T})=\mathrm{gau}_{\iota oc}(\delta_{\mathfrak{c}}x_{\mathfrak{c}},\rho(\delta_{\mathfrak{c}})\bar{v}_{\mathfrak{c},T})=\mathrm{gau}_{\iota oc}(\delta_{\mathfrak{c}}x_{\mathfrak{c}},\eta_T)\le 1.$$

For the last inequality, the inverse local matrix is integral, its Schur matrix is polynomial with integer coefficients, and η_T is an integer vector. There is no determinant factor since the determinant weight is zero. The approximation was made at every place above ℓ, so this argument applies to every ι, not merely a chosen embedding. Zero slots have gauge zero. The orthogonal-sum frame identity now proves (a).

Cardinalities in the prescribed scales. All estimates in this paragraph concern the fixed sets in the statement. Put $C=\#\mathcal{C}_{\kappa}$ and $P=\#(\mathrm{Mem}_j\times\mathrm{Cl}^{\circ})$. We give deliberately coarse polynomial bounds for their logarithms. By Lemma 4.2.67,

$$f\le x^{12},\quad t\le 15x^{5},\quad m\le 2\sqrt{x},\quad \Delta=2r^{8}\le 2\cdot 3^{8}x^{40},$$
$$\log M\le 22\log b,\quad \log b\le x^{1/100},\quad b\ge x^{10^{12}},$$

and $x\ge\exp(6.4\cdot 10^{15})$.

First, $0\le\kappa(\tau,h)\le\Delta$. To see the upper bound from (LF4), consecutive entries in the initial block of the target parameter differ by one, giving zero column count there. Consecutive entries in its tail differ by Δ, giving a count at most $\Delta-1$. At the interface with the first tail entry, the minimum with $-\kappa_j-(t-1)/2$ makes the positive part zero, since $\kappa_j\ge\kappa_1$. Truncated arguments equal to zero give a nonpositive difference. Taking the minimum in any remaining case can only decrease the difference. These cases exhaust the displayed formula in (LF4). Since the rank $n^{\circ}=|\mathcal{B}^{\circ}|$ is at most t, the total number of boxes is at most

$$L_0=ft^{2}\Delta\le 2\,952\,450x^{62}\le x^{63}.$$

A filling assigns a row to each box; forgetting the increasing-column condition bounds C by $(t+1)^{L_0}$. Thus

$$\log C\le L_0\log(t+1)\le x^{70}.$$

The hook estimate in Lemma 6.4.73, applied with the integer L_0, gives

$$\log h\le 3L_0\log L_0\le 3L_0^{2}\le x^{127}.$$

(The empty diagram has $C = h = 1$ and satisfies these bounds as well.)

Both standard definite ranks $|\mathcal{B}^\flat|$ and $|\mathcal{B}^\circ|$ lie between one and t, and hence are at most $2n_m$. The maximal integral level contains the principal level of $\pi_M\pi_N$. Therefore Lemma 4.6.4 bounds each maximal-level class set by $\exp(x^{30})$. To pass to the principal level $\mathfrak{M}^2$, use the rational integer M^2: its principal subgroup is contained in that of π_M^2, since $\pi_M\bar{\pi}_M = M$. Reduction modulo M^2 bounds the index by

$$\#\mathrm{M}_n(\mathcal{O}_E/M^2\mathcal{O}_E) = M^{4fn^2}.$$

Each maximal-level class splits into at most this many finer classes. For either of the two ranks, their logarithmic cardinality is consequently at most

$$x^{30} + 4ft^2 \log M \le x^{30} + 19\,800x^{22+1/100} \le x^{31}.$$

There are $3m|\mathcal{B}_j|$ matrix entries in a choice, and $\#(\mathcal{O}_E/\mathfrak{M}^2) \le M^{4f}$. Since $|\mathcal{B}_j| \le t$, their logarithmic contribution is at most

$$12fmt \log M \le 7\,920x^{17+1/2+1/100} \le x^{19}.$$

The lists of rows contribute exactly $f\Delta'(2m)\log(6m+1)$ to the logarithm. Here $0 \le \Delta' \le r^8$, so this is at most

$$x^{12} \cdot 3^8 x^{40} \cdot 4\sqrt{x}\log(12\sqrt{x}+1) \le x^{60}.$$

Including the two class sets proves

$$\log P \le x^{19} + 2x^{31} + x^{60} \le x^{61}.$$

All sets are nonempty: the identity gives a class, and zero matrices and constant lists give choices. The preceding logarithms are therefore legitimate. Also $\log N_{\mathrm{ch}} \le x^{33}$ by Lemma 6.7.10. Since $\#A = PC$, the logarithm of the left side of (c) equals

$$3\log P + 2\log C + 2\log N_{\mathrm{ch}} + 4\log h \le 3x^{61} + 2x^{70} + 2x^{33} + 4x^{127} \le x^{129}.$$

Finally $f, m, \log b \ge 1$ and $b \ge x^{10^{12}}$, so $x^{129} \le 2fmb(\log b)^2$. Exponentiating proves (c). □

Lemma 6.6.28. *Let $1 \le j \le m$, ϑ a twist, $O = O_{j,\vartheta}$, ℓ a prime and $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism, and $\iota \circ c$ its composite with complex conjugation. If $\Xi \in \mathrm{Form}(O)$ is integral at $\iota \circ c$ and $\Xi' \in \mathrm{Form}(O)$ is integral at ι, then $|\iota(N_{\mathrm{ch}}\langle \Xi, \Xi'\rangle)|_\ell \le 1$.*

Proof. Put $n = r + t$, $\kappa = \kappa_O$, $e = e_O$, and $G = G_\kappa = F_\kappa^{\mathsf{T}} F_\kappa$. In particular G has integer entries. By admissibility $r \ge 3$ and $t = 5r$, so

$$1 \le n = 6r \le 5r^2 = rt.$$

The level $\mathbf{K}(O)$ is an open subgroup of $\mathbf{K}_0(n)$: it is the inverse image of a subgroup of a finite residue group under reduction. Thus Lemma 6.7.9 applies to each $h \in G_f(n)$ and gives

$$d_h = \#(\mathrm{U}(\mathrm{I}_n)(E) \cap h\mathbf{K}(O)h^{-1}) \mid N_{\mathrm{ch}}. \tag{1}$$

These stabilizers are the original ones in (DP5). There are finitely many double cosets. Indeed I_n is positive definite, so its adelic quotient is compact by Theorem A.4.5; passage to the quotient by the open subgroup $\mathbf{K}(O)$ gives a compact discrete space.

We first establish the exact bilinear identity for the two gauges. For a tuple of invertible matrices $A = (A_\tau)_\tau$ over $\mathbb{C}_\ell$, write

$$\mathscr{R}(A) = \left(\prod_\tau \det(A_\tau)^{e_\tau} \right) S_\kappa(A).$$

This denotes the matrix appearing in the gauge, without an additional conjugation. The substitution definition of the Schur matrices gives $\mathscr{R}(AB) = \mathscr{R}(A)\mathscr{R}(B)$ and $\mathscr{R}(1) = 1$.

The integer filling Gram satisfies

$$GS_\kappa(A) = S_\kappa(A^{\mathsf{T}})^{\mathsf{T}} G. \tag{2}$$

Here is a direct verification that fixes the transpose. For two arrays X, Y, let $\mathrm{ev}_\kappa(X)$ and $\mathrm{ev}_\kappa(Y)$ be their evaluation vectors. Then

$$\mathrm{ev}_\kappa(X)^{\mathsf{T}} G \, \mathrm{ev}_\kappa(Y) = \sum_T \Delta_T(X)\Delta_T(Y).$$

For each column of height h, Cauchy–Binet identifies its factor in this sum with $\det(X_h^{\mathsf{T}} Y_h)$, where X_h, Y_h are the first h columns of the appropriate arrays. Therefore replacing (X, Y) by (AX, Y) has the same effect as replacing it by $(X, A^{\mathsf{T}}Y)$. Since $\mathrm{ev}_\kappa(AX) = S_\kappa(A)\mathrm{ev}_\kappa(X)$, this gives

$$S_\kappa(A)^{\mathsf{T}} G = GS_\kappa(A^{\mathsf{T}}).$$

The evaluation vectors span: a linear functional vanishing on all of them is a polynomial relation among the independent standard polynomials b_i, hence is zero. Thus this is a matrix identity; replacing A by A^{T} proves (2). Determinants are unchanged by transpose, so (2) also gives

$$G\mathscr{R}(A) = \mathscr{R}(A^{\mathsf{T}})^{\mathsf{T}} G, \qquad \mathscr{R}(A^{\mathsf{T}})^{\mathsf{T}} G \mathscr{R}(A^{-1}) = G. \tag{3}$$

No inverse Gram matrix occurs.

Fix a double-coset representative h. For $\tau \in \Phi$, let $A_\tau = f_{\iota,\tau}(h)$, where $f_{\iota,\tau}$ is projection to the completion selected by $x \mapsto \iota(\tau(\bar{x}))$ followed by its continuous extension to $\mathbb{C}_\ell$, as in (WM9). The same notation applies to the arbitrary embedding $\iota : \mathbb{C} \to \mathbb{C}_\ell$, exactly as in the definition of integral forms. The standard finite unitary equation is $h^{-1} = \bar{h}^{\mathsf{T}}$. Consequently

$$f_{\iota\circ c,\tau}(h^{-1}) = A_\tau^{\mathsf{T}}. \tag{4}$$

To check the embedding in this equation, for $a \in E$ one has $\overline{\tau(\bar{a})} = \tau(a)$, since τ commutes with the CM involution. Thus the conjugate embedding in the left side, applied to the entry $\bar{h}_{ba}$ of h^{-1}, is the original embedding applied to h_{ba}. The corresponding identity on completions follows by continuity. Also $f_{\iota,\tau}(h^{-1}) = A_\tau^{-1}$.

Put

$$u = (\iota \circ c)(\Xi(h)), \qquad v = \iota(\Xi'(h)), \qquad x = \mathscr{R}(A^{\mathsf{T}})u, \qquad y = \mathscr{R}(A^{-1})v.$$

The two integrality hypotheses, at this very h, assert precisely that every coordinate of x and y belongs to the valuation ring of $\mathbb{C}_\ell$. By (3),

$$x^{\mathsf{T}} G y = u^{\mathsf{T}} G v = \iota(\overline{\Xi(h)}^{\mathsf{T}} G \Xi'(h)) = \iota(\langle \Xi(h), \Xi'(h) \rangle_\kappa). \tag{5}$$

The middle equality uses that G is integral and hence fixed by the embedding. The left side is integral because x, y and G have integral entries. This proves integrality of each actual pointwise

Gram pairing. It makes no assertion that either change-of-gauge matrix is integral, or that it individually preserves the Gram form over $\mathbb{C}_\ell$.

Finally the Petersson product is exactly

$$\langle \Xi, \Xi' \rangle = \sum_{[h]} \frac{1}{d_h} \langle \Xi(h), \Xi'(h) \rangle_\kappa .$$

Multiply by N_{ch}. By (1), every coefficient N_{ch}/d_h is an ordinary integer. By (5), each remaining pointwise factor is integral under ι. The finite sum is therefore integral by the ultrametric inequality, giving the stated bound. No class representative, level or stabilizer weight has been replaced. □

Lemma 6.6.29. *Let $1 \le j \le m$, let Y be a short row family, $c'' \in \mathbb{C}^\times$, and $L = (L_i)_{i \in \mathrm{Mem}_j}$ divided lattices of $O_{j,1}$ with L_i glued for Λ_i, and suppose that $\mathcal{V}_j$ is a divided splitting for $O_{j,\vartheta}$ for every twist ϑ. For a twist ϑ let $\mathrm{gl}_\vartheta = \mathrm{gl}_{j,\vartheta}[L; c'']$, formed with Y, and gl_ϑ^* its adjoint; let F_i be the choice functions for Y, and for $\zeta \in \mathrm{Tab}_j$ and $i \in \mathrm{Mem}_j$ let $\zeta_i \colon \mathrm{Cl}^\circ \to \mathbb{V}_j^-$ be $\mathfrak{c} \mapsto \zeta_{i,\mathfrak{c}}$. Then there are $C \in \mathbb{C}$, numbers $c'_\vartheta \in \mathbb{C}$ (one for each twist ϑ), nonzero numbers $c_i^\circ \in \mathbb{C}$ ($i \in \mathrm{Mem}_j$) and a linear map $\Theta^\circ \colon \mathcal{A}^{\mathrm{good}}(W_j) \to \mathrm{Tab}_j$ such that for every twist ϑ:*

(i) $\Gamma_j(\vartheta) \neq 0$ *and* $\Gamma_j(\vartheta)\,|c'_\vartheta|^2 = C$;

(ii) $c'_\vartheta\, \mathrm{gl}_\vartheta^* \circ \Theta_{j,\vartheta} = \Theta^\circ$;

(iii) for all $y \in \mathcal{A}^{\mathrm{good}}(W_j)$ and $\zeta \in \mathrm{Tab}_j$, the functions $y\,\mathsf{F}_i\,\mathrm{tt}_j(\zeta_i)$ are integrable on $[\mathrm{U}(W_j)]$ and

$$\langle \zeta, \Theta^\circ y \rangle = \sum_{i \in \mathrm{Mem}_j} c_i^\circ \int_{[\mathrm{U}(W_j)]} y(g)\, \mathsf{F}_i(g)\, \mathrm{tt}_j(\zeta_i)(g)\, dg .$$

Proof. Fix j, Y, c'' and L. Write $\mathcal{V} = \mathcal{V}_j$, $\mathrm{C} = \mathrm{C}(\mathcal{V})$, $\ell = \mathrm{loss}(\mathcal{V})$ and $\beta_i = \beta_{L_i}$. These do not depend on the twist: $\mathcal{V}$, R_2, $\bar\varpi$, C and the conditions (GU1)–(GU4) read of a descent pair only its chosen places, inc, its ranks and $\kappa_O^\pm$, which are those of $O_{j,1}$ for every $O_{j,\vartheta}$ ((LF4)); β_i reads $\chi_O = v^{r_-(O)}$, free of ϑ. So each L_i is a divided lattice of $O_{j,\vartheta}$ glued for Λ_i. For every $A \in \mathrm{C}$ fix $k_A \in \mathbf{K}_0(\mathcal{R})$ with $\mathrm{red}_{\mathfrak{M}^2}(k_A) = A$ and with component 1 at every finite place w such that neither w nor $\bar w$ is a chosen place; it exists because the chosen places are split and pairwise not conjugate, and at such a pair $(w, \bar w)$ an element of $G_f(\mathcal{R})$ is given by an arbitrary component in $\mathrm{GL}_{\mathcal{R}}(\mathcal{O}_{E,w})$ (the component at $\bar w$ being determined by $g^* g = 1$). Since $A\bar\varpi = \bar\varpi$ and $\bar\varpi$ generates $\mathfrak{M}/\mathfrak{M}^2$, $A \equiv 1$ modulo every chosen place, so k_A is integral and reduces to 1 at every chosen place: $k_A \in \mathrm{K}_M$ (the finite point of W_j is a diagonal matrix of units, so integrality and reductions are read in $G_f(\mathcal{R})$).

Step 1: adjointness. Let ϑ be a twist, $O = O_{j,\vartheta}$, $y \in \mathcal{A}^{\mathrm{good}}(W_j)$ and $\zeta \in \mathrm{Tab}_j$. By the definition of the adjoint and (a) of Lemma 6.6.30 ($y \in \mathcal{A}_M(W_j)$),

$$\langle \zeta, \mathrm{gl}_\vartheta^* \Theta_{j,\vartheta} y \rangle = \langle \mathrm{gl}_\vartheta \zeta, \Theta_{j,\vartheta} y \rangle = \int_{[\mathrm{U}(W_j)]} y(u)\, \overline{\mathrm{Th}_O(\mathrm{gl}_\vartheta \zeta)(u)}\, du .$$

Step 2: the descent of the comparison form. By (TG6) and linearity of Th_O, $\mathrm{Th}_O(\mathrm{gl}_\vartheta \zeta) = M^\ell \sum_{(i,\mathfrak{c})} \mathrm{Th}_O(\Xi_{i,\mathfrak{c}})$ with

$$\Xi_{i,\mathfrak{c}} = \sum_{x \bmod \mathbf{J}_\vartheta(\Lambda_i)}^{\mathrm{cl}} \mathrm{pr}_{\mathcal{V},L_i}((x, x_{\mathfrak{c}}); \varphi''_{\vartheta,i}(x))(\zeta_{i,\mathfrak{c}})$$

and $\mathbf{J}_\vartheta(\Lambda_i) = \mathbf{K}_O[[\Lambda_i;\bar{\varpi}],\phi_j]$ (TG5). We apply Lemma 6.6.32 to O, $(N,m,a,z) = (\mathcal{B}_j,\mathcal{B}^\circ,\mathcal{B}^{\mathrm{c}},z_j)$, $\mathcal{V}$, $s=\Lambda_i$, $L=L_i$, $b=\nu''_{j,\vartheta}$, $b'=\upsilon^{\lfloor m/2\rfloor-j}$, the elements k_A, $h^\circ = x_{\mathfrak{c}}$, $\varphi=\varphi''_{\vartheta,i}$ and $\eta=\zeta_{i,\mathfrak{c}}$. The characters satisfy $bb' = \chi'_O$:

As $\bar{\varphi}''_{\vartheta,i} = c''\Upsilon_{\vartheta,i}$ and, by (XE7), the sum over $\mathfrak{c}\in\mathrm{Cl}^\circ$ of the second kernels of Lemma 6.6.32 (characters $\chi_O=\chi_{O_{j,1}}$, $\upsilon^{\lfloor m/2\rfloor-j}$, modulus $\mathfrak{M}^2$, datum $\mathbf{1}_{z_j}$, polynomial $\mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}(\bar{\zeta}_{i,\mathfrak{c}}))$, point $x_{\mathfrak{c}}$) is $\mathrm{tt}_j(\zeta_i)$, we get for $u\in\mathrm{U}(W_j)(\mathbb{A}_F)$

$$\overline{\mathrm{Th}_O(\mathrm{gl}_\vartheta\zeta)(u)} = M^\ell \sum_{i\in\mathrm{Mem}_j}\beta_i\sum_{A\in\mathrm{C}}\mathrm{cks}_{O,\nu''_{j,\vartheta}}([\Lambda_i;\bar{\varpi}],\phi_j;c''\Upsilon_{\vartheta,i})(uk_A)\,\mathrm{tt}_j(\zeta_i)(uk_A). \tag{6.137}$$

Step 3: the first-factor pairing. The coset kernel sum is linear in its function (each kernel is linear in its polynomial, and $\mathrm{Pol}_{\kappa_O^+}$, Let_1 are linear), so the coset kernel sum in (6.137) is $c''\,\mathrm{cks}_{O,\nu''_{j,\vartheta}}([\Lambda_i;\bar{\varpi}],\phi_j;\Upsilon_{\vartheta,i})$. Apply Proposition 6.6.33 with $a=\mathcal{B}^{\mathrm{c}}\sqcup\mathcal{B}_j$, $w=[\Lambda_i;\bar{\varpi}]$ and $u_0=(1\,|\,0)\in\mathrm{M}_{\mathcal{B}^{\mathrm{c}}\times a}(R_2)$, for which $u_0w=\Lambda_i$: the coset kernel sum is the left side $L_{\vartheta,i}$ of the third exchange. By Proposition 6.6.36 (for the short row family Y) there are $C\in\mathbb{R}$ and numbers c_ϑ with $\Gamma_j(\vartheta)\,|c_\vartheta|^2=C$ and $c_\vartheta L_{\vartheta,i}=\mathsf{F}_i$ for all i; the function $L_{\vartheta,i}$ there is the one of Proposition 6.6.33. Hence

$$y(u)\,\overline{\mathrm{Th}_O(\mathrm{gl}_\vartheta\zeta)(u)} = \sum_{i\in\mathrm{Mem}_j}\sum_{A\in\mathrm{C}} a_i\,f_{\vartheta,i}(uk_A),\qquad a_i = c''M^\ell\beta_i,\quad f_{\vartheta,i}=y\,L_{\vartheta,i}\,\mathrm{tt}_j(\zeta_i),$$

using $y(uk_A)=y(u)$ ($k_A\in\mathrm{K}_M$ and y is right invariant under K_M).

Step 4: integrability and integration. $L_{\vartheta,i}=\mathrm{cks}_{O,\nu''}(\dots)$ is a finite combination of kernels $\theta^{W_j,V''_j}_{\chi_O,\nu''_{j,\vartheta};\mathfrak{M}^2\mathfrak{Q}_O}[\Phi,P](\cdot,h)$ and $\mathrm{tt}_j(\zeta_i)$ a finite combination of kernels $\theta^{W_j,V^\circ}_{\chi_O,\upsilon^{\lfloor m/2\rfloor-j};\mathfrak{M}^2}[\Phi',P'](\cdot,x_{\mathfrak{c}})$. The characters $\chi_O=\upsilon^{m-j}$ and $\upsilon^{\lfloor m/2\rfloor-j}$ have trivial tame parts, and $\nu''_{j,\vartheta}=\mu_j^0\vartheta\upsilon^{e_j}$ has the tame part $\omega^j\psi_\vartheta$, trivial off $S_0\cup Q=\Sigma(\mathfrak{M}^2\mathfrak{Q}_O)$. By Lemma 6.6.37, $f_{\vartheta,i}$ is a finite sum of continuous, left invariant functions integrable on $[\mathrm{U}(W_j)]$, hence integrable. The quotient measure of $[\mathrm{U}(W_j)]$ is invariant under right translation ($\mathrm{U}(W_j)(\mathbb{A}_F)$ is unimodular), so $u\mapsto f_{\vartheta,i}(uk_A)$ is integrable with the same integral. By Steps 1 and 3,

$$\langle\zeta,\mathrm{gl}^*_\vartheta\Theta_{j,\vartheta}y\rangle = \#\mathrm{C}\sum_{i\in\mathrm{Mem}_j}a_i\int_{[\mathrm{U}(W_j)]}y\,L_{\vartheta,i}\,\mathrm{tt}_j(\zeta_i)\,du. \tag{6.138}$$

Step 5: (ii) and (iii). Put $c'_\vartheta=c_\vartheta$ and $c_i^\circ=\#\mathrm{C}\,a_i=c''M^\ell\,\#\mathrm{C}\,\beta_i$, independent of ϑ. It is nonzero: $c''\neq0$, $M\geq1$, $\#\mathrm{C}\geq1$, and $\beta_i=\chi_O[r](\delta)/|\delta|^{r/2}$ is a value of a unitary character divided by a positive number. Since $c_\vartheta L_{\vartheta,i}=\mathsf{F}_i$, the function $y\mathsf{F}_i\mathrm{tt}_j(\zeta_i)=c_\vartheta f_{\vartheta,i}$ is integrable, and multiplying (6.138) by c_ϑ (the inner product is linear in its second argument)

$$\langle\zeta,c'_\vartheta\,\mathrm{gl}^*_\vartheta\Theta_{j,\vartheta}y\rangle = \sum_{i\in\mathrm{Mem}_j}c_i^\circ\int_{[\mathrm{U}(W_j)]}y(g)\,\mathsf{F}_i(g)\,\mathrm{tt}_j(\zeta_i)(g)\,dg =: R(\zeta,y),$$

for every twist ϑ, the right side not involving ϑ. Let $\Theta^\circ=c'_1\,\mathrm{gl}_1^*\circ\Theta_{j,1}$, a linear map $\mathcal{A}^{\mathrm{good}}(W_j)\to\mathrm{Tab}_j$. The case $\vartheta=1$ is (iii). For any ϑ, $\langle\zeta,c'_\vartheta\mathrm{gl}^*_\vartheta\Theta_{j,\vartheta}y\rangle=\langle\zeta,\Theta^\circ y\rangle$ for all ζ, so $c'_\vartheta\mathrm{gl}^*_\vartheta\Theta_{j,\vartheta}y=\Theta^\circ y$: this is (ii).

Step 6: (i). By Lemma 6.6.38, $\Gamma_j(\vartheta)>0$, and $\Gamma_j(\vartheta)\,|c'_\vartheta|^2=\Gamma_j(\vartheta)\,|c_\vartheta|^2=C$ by Proposition 6.6.36. This is (i). □

Lemma 6.6.30. *Let $1\leq j\leq m$, ϑ a twist, $O=O_{j,\vartheta}$ and $\Xi\in\mathrm{Form}(O)$. Then:*

(a) for every $y \in \mathcal{A}_M(W_j)$ the function $u \mapsto y(u)\,\overline{\mathrm{Th}_O(\Xi)(u)}$ is integrable on $[\mathrm{U}(W_j)]$ and

$$\langle \Xi, \Theta_{j,\vartheta}(y)\rangle = \int_{[\mathrm{U}(W_j)]} y(u)\,\overline{\mathrm{Th}_O(\Xi)(u)}\,du;$$

(b) $\mathrm{Th}_O(\Xi) \in \mathcal{A}_M(W_j)$.

Proof. Fix j, ϑ and put $O = O_{j,\vartheta}$. The rank r is positive and $n = r + t > r$. We first prove (b), then construct the vector-valued integral that gives (a).

The transposed lift gives the required archimedean type. By Lemma 6.5.11, O fits $l = 2\varsigma$. By Lemma 6.4.15(e), with

$$\gamma(\tau) = r_-(O), \qquad \gamma'(\tau) = \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau),$$

the vector $v_{\mathrm{Har}_O(\xi_O)}$ is an edge from e_ς to $e_{l_O/2}$ for these weights. They are exactly the weights of the actual pair characters $\chi_O = \upsilon^{r-(O)}$ and $\chi'_O = \mu_O \upsilon^{\lfloor n/2 \rfloor}$. The modulus is the product of the chosen and twist places. It is nonempty because the chosen places of D are nonempty; its places are pairwise nonconjugate. The actual datum is

$$\mathrm{J}_O(\xi_O) = \mathrm{mod}_{\mathfrak{m}}(\Psi_O, v_{\mathrm{Har}_O(\xi_O)}).$$

Here Ψ_O is (DP7)'s block function, so this is a datum of precisely the form required by Lemma 6.3.76; its Fock vector is a polynomial vector.

For $\Xi \in \mathrm{Form}(O)$ put $Y_\Xi = \overline{F_\Xi}$. By Lemma 6.6.83(1), Y_Ξ is a form of type $\overline{e_{l_O/2}}$ on I_n. Apply the reverse assertion of Lemma 6.3.76(1) to $S = W$, $T = \mathrm{I}_n$, characters (χ_O, χ'_O), the modulus and datum just specified, and $F' = Y_\Xi$. It proves absolute convergence, at every u, of

$$L^{\mathsf{t}} Y_\Xi(u) = \int_{[\mathrm{U}(\mathrm{I}_n)]} \theta_O(\omega_O(u,h)\mathrm{J}_O(\xi_O)) Y_\Xi(h)\,dh,$$

and proves that this function is a form of type $\overline{e_\varsigma}$ on W. Thus all hypotheses of this convergence assertion, including the conjugate target type and the edge in both directions, have been supplied.

By Lemma 6.6.83(2), $\mathrm{Th}_O(\Xi) = c_O \overline{L^{\mathsf{t}} Y_\Xi}$, with $c_O > 0$ depending only on O. Complex conjugation takes a form of type $\overline{e_\varsigma}$ to a form of type e_ς: it preserves continuity, boundedness, rational invariance and an open finite level, and $R(e_\varsigma)\overline{H} = \overline{R(\overline{e_\varsigma})H}$. Consequently $\mathrm{Th}_O(\Xi)$ has the required archimedean type.

The finite level is the prescribed principal level. Let $k \in \mathrm{K}_M$ in the actual finite frame of W_j. Then k is integral at every finite place and reduces to 1 at every chosen place. In the notation of Lemma 6.5.10, this says $k \in J_O$; it also supplies the integrality hypothesis away from $S_O \cup \bar{S}_O$ in its part (c). With $F = \Upsilon(O)$, that part gives

$$\langle F, \omega_O((1,k),1)F\rangle = \langle F, F\rangle.$$

The actual pair action preserves the positive inner product by Lemma 4.1.1(2). Therefore

$$\|\omega_O((1,k),1)F - F\|^2 = 2\|F\|^2 - 2\,\mathrm{Re}\langle F, \omega_O((1,k),1)F\rangle = 0,$$

and $\omega_O((1,k),1)F = F$. By the real/finite factorization Lemma 4.1.1(5), the same finite action fixes $\mathrm{J}_O(\xi) = v_{\mathrm{Har}_O(\xi)} \otimes F$ for every ξ. This uses the full pair action in Lemma 6.5.10, including its characters, so there is no omitted finite character factor at the twist places.

The representation law now gives $\mathcal{C}_O(u(1,k),h)(\xi) = \mathcal{C}_O(u,h)(\xi)$. Taking the finite class sum in (DP9) shows

$$\mathrm{Th}_O(\Xi)(u(1,k)) = \mathrm{Th}_O(\Xi)(u) \qquad (k \in \mathrm{K}_M).$$

Together with the preceding archimedean-type conclusion, this is exactly $\mathrm{Th}_O(\Xi) \in \mathcal{A}_M(W_j)$, proving (b).

The vector kernel and its pairing direction. For fixed u and finite h, define a vector $\mathsf{Kern}_O(u,h) \in \mathbb{V}_O$ by

$$\langle \xi, \mathsf{Kern}_O(u,h)\rangle_{\kappa_O} = \theta_O(\omega_O(u,(1,h))\mathrm{J}_O(\bar{\xi})) \qquad (\xi \in \mathbb{V}_O). \tag{1}$$

The right side is conjugate-linear in ξ. The positive Gram matrix is invertible by Lemma 4.6.10(b), so (1) defines a unique vector. Taking conjugates gives

$$\langle \mathsf{Kern}_O(u,h), \xi\rangle_{\kappa_O} = \mathcal{C}_O(u,h)(\xi). \tag{2}$$

We check that $h \mapsto \mathsf{Kern}_O(u,h)$ belongs to $\mathrm{Form}(O)$. For $k' \in \mathbf{K}(O)$, Lemma 6.6.31(b) and the representation law give, after pairing with every ξ,

$$\mathsf{Kern}_O(u,hk') = \tilde{\eta}_O(k')\,\mathsf{Kern}_O(u,h).$$

For a rational point δ of $\mathrm{U}(\mathrm{I}_n)$, rational theta invariance Lemma 6.2.5 first changes its finite component δ_f to the inverse real component. Part (a) of Lemma 6.6.31 therefore gives

$$\begin{aligned}\langle \xi, \mathsf{Kern}_O(u,\delta_f h)\rangle &= \theta_O(\omega_O(u,(1,h))\mathrm{J}_O(\overline{\rho_O(\delta_\infty^{-1})\xi}))\\ &= \langle \rho_O(\delta_\infty^{-1})\xi, \mathsf{Kern}_O(u,h)\rangle\\ &= \langle \xi, \rho_O(\delta_\infty)\mathsf{Kern}_O(u,h)\rangle,\end{aligned}$$

where the last equality is Lemma 4.6.10(c). These are precisely (DP5)'s transformation laws. Denote this form by $\mathsf{Kern}_O(u)$.

The space $\mathrm{Form}(O)$ is finite dimensional: Lemma 6.4.74(1), for the nonempty standard definite rank n and the open subgroup $\mathbf{K}(O) \subseteq \mathbf{K}_0(n)$, gives finitely many finite double cosets and finite stabilizers. A form is determined by its values on those representatives in the finite-dimensional space $\mathbb{V}_O$. Its class-sum pairing is positive definite, since its summands have positive weights and the Gram pairing is positive definite. Continuity of $u \mapsto \mathsf{Kern}_O(u)$ follows from (1), the continuous actual action and bounded theta functional, evaluated at those finitely many representatives.

Equations (2), (DP5) and (DP9) give the exact identity

$$\mathrm{Th}_O(\Xi)(u) = \langle \mathsf{Kern}_O(u), \Xi\rangle_{\mathrm{Form}(O)}. \tag{3}$$

The class weights are the SAME $1/\#\Gamma_h$ on both sides. No level index or real-dimension factor has been introduced.

Integrability in the finite-dimensional space of forms. Choose an orthonormal basis $(\Xi_a)_{a\in A}$ of $\mathrm{Form}(O)$, with A finite. By (3) and the conjugate symmetry of the inner product,

$$y(u)\mathsf{Kern}_O(u) = \sum_{a\in A} y(u)\,\overline{\mathrm{Th}_O(\Xi_a)(u)}\,\Xi_a. \tag{4}$$

Every descent on the right belongs to $\mathcal{A}_M(W_j)$ by (b), already proved. Both it and y are continuous bounded forms, left invariant under rational points. The quotient $[\mathrm{U}(W_j)]$ has finite

volume by Theorem 4.2.18: here E/E^+ is a CM quadratic extension, W_j is positive dimensional with nondegenerate hermitian form, and the measure in (XD1) is induced from its nonzero Haar measure by counting measure on rational points. These are exactly the hypotheses of that theorem, regardless of signature. Each scalar product in (4) is therefore absolutely integrable. The finite sum proves Bochner integrability of $y(u)\mathsf{Kern}_O(u)$ in $\mathrm{Form}(O)$. It also proves directly that $y\,\overline{\mathrm{Th}_O(\Xi)}$ is integrable for every Ξ, as asserted in (a).

Identification of the vector integral with the single lift. The highest vector ξ_O is real: it is the evaluation vector at the 0-1 array in (LF5), whose products of minors have integer values. For a real target point $x \in G_\infty(1_n)$, Lemma 6.6.31(a), (1) and unitarity give

$$\begin{aligned}\theta_O(\omega_O(u,(x,h))\mathrm{J}_O(\xi_O)) &= \langle \rho_O(x)\xi_O, \mathsf{Kern}_O(u,h)\rangle \\ &= \langle \xi_O, \rho_O(x)^{-1}\mathsf{Kern}_O(u,h)\rangle \\ &= F^{l_O}_{\mathsf{Kern}_O(u)}(x,h).\end{aligned} \tag{5}$$

Thus the actual full theta kernel of (LF5), with the actual pair characters, is the attached function of this vector kernel.

Put

$$I_y = \int_{[\mathrm{U}(W_j)]} y(u)\mathsf{Kern}_O(u)\,du \in \mathrm{Form}(O).$$

For fixed (x,h) the map $\eta \mapsto F^{l_O}_\eta(x,h)$ is a linear functional on a finite-dimensional space, hence is continuous. It commutes with the Bochner integral just proved to exist. By (5),

$$F^{l_O}_{I_y}(x,h) = \int_{[\mathrm{U}(W_j)]} \theta_O(\omega_O(u,(x,h))\mathrm{J}_O(\xi_O))y(u)\,du. \tag{6}$$

This equality also proves absolute convergence of each scalar integral in (6). By (LF5), $\Theta_{j,\vartheta}(y)$ is the algebraic form attached to its right side. By Lemma 6.4.72(b), applied to the algebraic form I_y of weight (κ_O, e_O) and its doubled parameter l_O, the algebraic form attached to $F^{l_O}_{I_y}$ is I_y itself. Consequently

$$\Theta_{j,\vartheta}(y) = I_y. \tag{7}$$

Only continuous linear functionals of an integrable finite-dimensional vector have been moved through integrals; there is no unproved interchange of two noncompact integrals.

Finally apply the continuous linear functional $\eta \mapsto \langle \Xi, \eta\rangle$ to (7). By (3),

$$\begin{aligned}\langle \Xi, \Theta_{j,\vartheta}(y)\rangle &= \int_{[\mathrm{U}(W_j)]} y(u)\langle \Xi, \mathsf{Kern}_O(u)\rangle\,du \\ &= \int_{[\mathrm{U}(W_j)]} y(u)\,\overline{\mathrm{Th}_O(\Xi)(u)}\,du.\end{aligned}$$

This is (a), with the stated conjugate order and the original quotient measure. If $\mathrm{Form}(O) = 0$, its orthonormal basis is empty, all vector kernels and descents vanish, and the same argument gives the identity with both sides zero. □

Lemma 6.6.31. *Let O be a descent pair for ranks $r < n$, $\rho = \rho_{\kappa_O, e_O}$, and $u \in \mathrm{G}(\varepsilon_O)$.*

(a) For $y \in G_\infty(1_n)$ and $\xi \in \mathbb{V}_O$, $\omega_O\big(1,(y,1)\big)\,\mathrm{J}_O(\bar\xi) = \mathrm{J}_O\big(\overline{\rho(y)\xi}\big)$.

(b) For $k \in \mathbf{K}(O)$ and $\xi \in \mathbb{V}_O$, $\omega_O\big(1,(1,k)\big)\,\mathrm{J}_O(\xi) = \tilde\eta_O(k)\,\mathrm{J}_O(\xi)$.

(c) For every rational point $\gamma = (\gamma_\infty, \gamma_f)$ of $\mathrm{U}(\mathrm{I}_n)$, $h \in G_f(n)$, $k \in \mathbf{K}(O)$ and $\xi \in \mathbb{V}_O$,

$$\mathcal{C}_O(u, \gamma_f h k)\big(\tilde\eta_O(k)\,\rho_O(\gamma)\xi\big) = \mathcal{C}_O(u,h)(\xi);$$

in particular, for $\Xi \in \mathrm{Form}(O)$ the function $h \mapsto \mathcal{C}_O(u,h)(\Xi(h))$ is constant on the double cosets $\mathrm{U}(\mathrm{I}_n)(E)\,h\,\mathbf{K}(O)$.

(d) For $x \in G_f(n)$ and $v \in \mathbb{V}_O$, $\mathrm{Th}_O\big(\delta_x(v)\big)(u) = \mathcal{C}_O(u,x)(v)$.

(e) For $x \in G_f(n)$, $v \in \mathbb{V}_O$ and $\Xi \in \mathrm{Form}(O)$, $\langle \delta_x(v), \Xi\rangle = \langle v, \Xi(x)\rangle_{\kappa_O}$, the inner product of (DP5) on the left.

Proof. Put $\kappa = \kappa_O$, $e = e_O$, and let $\gamma_0(\tau) = r_-(O)$, $\gamma_0'(\tau) = \lfloor n/2\rfloor + \mathrm{wt}_O(\tau)$; by (ii) of Lemma 3.2.11, υ has weight 1 at every $\tau \in \Phi$, so $\chi_O = \upsilon^{r_-(O)}$ and $\chi_O' = \mu_O \upsilon^{\lfloor n/2\rfloor}$ have the weights γ_0 and γ_0'. The matrices $S_\lambda(z)$ have entries in $\mathbb{Z}[z]$ for every weight λ (the minors of zY are integral combinations of those of Y by the Cauchy–Binet formula, and (b_j) is a $\mathbb{Z}$-basis of ∇_λ), so $S_\lambda(\bar z) = \overline{S_\lambda(z)}$. The map J_O is $\mathbb{C}$-linear (Har_O is linear, and so are $P \mapsto v_P$ and $v \mapsto v \otimes \Upsilon(O)$), and so is $\mathcal{C}_O(u,h)$ (two complex conjugations).

(a). By (5) of Lemma 4.1.1 for the pair $(W(O), \mathrm{I}_n)$, the characters χ_O, χ_O' and $x = 1$, $k = k' = 1$, $\omega_O(1,(y,1))(v \otimes \Upsilon(O)) = (\Omega_{\gamma_0,\gamma_0'}(1,y)v) \otimes \Upsilon(O)$ for every v in the Fock model of $\mathrm{Hom}(W(O), \mathrm{I}_n)$. Every y_τ is block diagonal for the signs of I_n (all $+1$), so the second part of (5) applies to $v = v_P$: the scalar is $\prod_\tau \det(y_\tau)^{\gamma_0(\tau)-r_-(O)} = 1$ (the block of sign -1 of y_τ is empty and $x = 1$), and P is replaced by $P'(X) = P(X')$ with

$$X'_{\tau,\cdot,c} = y_\tau^{\mathsf T} X_{\tau,\cdot,c} \ \text{ if } \varepsilon_O(c) = 1, \qquad X'_{\tau,\cdot,c} = y_\tau^{-1} X_{\tau,\cdot,c} \ \text{ if } \varepsilon_O(c) = -1,$$

$X_{\tau,\cdot,c}$ the column $(X_{\tau,i,c})_{1\le i\le n}$. For a sign σ let $Y^\sigma(X)$ be the array with entries $Y^\sigma(X)_{\tau,i,a} = \sum X_{\tau,i,c}$ over c with $\varepsilon_O(c) = \sigma$, $\mathrm{ord}_O(c) = a$, so that $\mathrm{Let}_\sigma(Q)(X) = Q(Y^\sigma(X))$. Then $Y^1(X') = y^{\mathsf T} Y^1(X)$ and $Y^{-1}(X') = y^{-1} Y^{-1}(X)$, the matrices acting on the rows at every τ as in (WM2). For a weight λ of rank n, $\alpha \in \mathbb{V}_\lambda$ and a family z of $n \times n$ matrices,

$$\mathrm{Pol}_\lambda(\alpha)(zY) = \sum_i (G_\lambda \alpha)_i \sum_j S_\lambda(z)_{ij} b_j(Y) = \sum_j \big(S_\lambda(z)^{\mathsf T} G_\lambda \alpha\big)_j b_j(Y) = \mathrm{Pol}_\lambda\big(S_\lambda(z^{\mathsf T})\alpha\big)(Y)$$

by (a) of Lemma 4.6.10. Applied with $z = y^{\mathsf T}$ for κ_O^+ and $z = y^{-1}$ for κ_O^-, this shows that the substitution $X \mapsto X'$ carries $\mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}\alpha)\,\mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}\beta)$ to the polynomial of $S_{\kappa_O^+}(y)\alpha \otimes S_{\kappa_O^-}((y^{-1})^{\mathsf T})\beta$. Let L be the linear map $\alpha \otimes \beta \mapsto \mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}\alpha)\,\mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}\beta)$, so $\mathrm{Har}_O(w) = L(C_O w)$. By (d) of Lemma 3.2.20 with $g = y$ and $e^-(\tau) = \sum_{1\le i<n} \kappa_O^-(\tau,i) = -e(\tau)$ (DP3),

$$\begin{aligned}\mathrm{Har}_O(w)' &= L\big((S_{\kappa_O^+}(y) \otimes S_{\kappa_O^-}((y^{-1})^{\mathsf T}))C_O w\big) = L\Big(C_O \prod_\tau \det(y_\tau)^{e(\tau)} S_\kappa(y) w\Big)\\ &= \mathrm{Har}_O\Big(\prod_\tau \det(y_\tau)^{e(\tau)} S_\kappa(y) w\Big).\end{aligned}$$

For $w = \bar\xi$, by (WM2),

$$\prod_\tau \det(y_\tau)^{e(\tau)} S_\kappa(y)\bar\xi = \overline{\prod_\tau \det(\bar y_\tau)^{e(\tau)} S_\kappa(\bar y)\xi} = \overline{\rho(y)\xi}.$$

Hence $\Omega_{\gamma_0,\gamma_0'}(1,y) v_{\mathrm{Har}_O(\bar\xi)} = v_{\mathrm{Har}_O(\overline{\rho(y)\xi})}$, and (DP8) gives (a).

(b). If $S_O = \emptyset$, then $\Upsilon(O) = v^0$ ($\Phi_O = 1$), $\tilde\eta_O = 1$, and the last sentence of (6) of Lemma 4.1.1 gives $\omega_O(1,(1,k))(v \otimes v^0) = \chi_O(\det k)\, v \otimes v^0$; here $\chi_O(\det k) = \upsilon(\det k)^{r_-(O)} = 1$

by (ii) of Lemma 3.2.11, $\det k$ having unit finite components and infinite components 1. If $S_O \neq \emptyset$, let $\mathfrak{m} = \prod_{w\in S_O} w$, a modulus with $\Sigma(\mathfrak{m}) = S_O$ by (DP1), and identify $R_\mathfrak{m} = \prod_{w\in S_O} k_w$ by the Chinese remainder theorem; reduction modulo $\mathfrak{m}$ of $k \in \mathbf{K}_0(n)$ is then $(\mathrm{red}_w k)_w$. Reading Ψ_O of (DP7) as a function on $n \times r$ matrices over $R_\mathfrak{m}$, its function $\Psi_O^\natural$ of (6) is Φ_O, so $\mathrm{J}_O(\xi) = v \otimes (\Psi_O^\natural \otimes_{S_O} v^0)$ with $v = v_{\mathrm{Har}_O(\xi)}$, and (6) gives

$$\omega_O(1,(1,k))\mathrm{J}_O(\xi) = \chi_O(\det k)\, v\otimes(\Psi_k^\natural \otimes_{S_O} v^0), \qquad \Psi_k(X) = \Psi_O(j^{-1}X),\ j = (\mathrm{red}_w k)_w \in \mathbf{L}(O),$$

with $\chi_O(\det k) = 1$ as before. For $w \in M_O$, $j_w \mathrm{Inc}_O = \mathrm{Inc}_O$ gives $j_w^{-1}\mathrm{Inc}_O = \mathrm{Inc}_O$, so $j_w^{-1}X_w = \mathrm{Inc}_O$ if and only if $X_w = \mathrm{Inc}_O$. For $w \in Q_O$, ordering the indices as the image of inc_O followed by its complement, $j_w = \begin{pmatrix} A & 0 \\ B & D \end{pmatrix}$ with A its inc-block, so $j_w^{-1} = \begin{pmatrix} A^{-1} & 0 \\ * & D^{-1} \end{pmatrix}$ and the inc-rows of $j_w^{-1}X_w$ are A^{-1} times the inc-rows of X_w. Hence $\Psi_O(j^{-1}X) = \prod_{w\in Q_O} \psi_{O,w}(\det A_w)^{-1}\, \Psi_O(X) = \eta_O(j)\Psi_O(X)$, i.e. $\Psi_k = \tilde\eta_O(k)\Psi_O$, which gives (b) by linearity of $\Psi \mapsto \Psi^\natural \otimes_{S_O} v^0$.

(c). In $\mathrm{G}(\varepsilon_O) \times \mathrm{G}(1_n)$ we have $(u,(1,\gamma_f hk)) = (1,\gamma)\,(u,(1,h))\,(1,(\gamma_\infty^{-1},1))\,(1,(1,k))$, the second components multiplying to $(\gamma_\infty\gamma_\infty^{-1}, \gamma_f hk)$. Put $\zeta = \overline{\rho_O(\gamma)}\xi$, so $\tilde\eta_O(k)\rho_O(\gamma)\xi = \overline{\tilde\eta_O(k)}\,\zeta$. Since ω_O is a representation ((2) of Lemma 4.1.1) and J_O is linear,

$$\omega_O(u,(1,\gamma_f hk))\mathrm{J}_O(\overline{\tilde\eta_O(k)}\zeta) = \overline{\tilde\eta_O(k)}\,\tilde\eta_O(k)\,\omega_O(1,\gamma)\,\omega_O(u,(1,h))\,\omega_O(1,(\gamma_\infty^{-1},1))\mathrm{J}_O(\zeta)$$

by (b); here $\overline{\tilde\eta_O(k)}\tilde\eta_O(k) = 1$, as $\tilde\eta_O(k)$ is a value of characters of finite groups. By (a) with $y = \gamma_\infty^{-1}$ and (c) of Lemma 4.6.10 ($\gamma_\infty \in \prod_\tau \mathrm{U}(n)$, $\rho_O(\gamma) = \rho(\gamma_\infty)$ by (DP3)), $\omega_O(1,(\gamma_\infty^{-1},1))\mathrm{J}_O(\zeta) = \mathrm{J}_O(\overline{\rho(\gamma_\infty)^{-1}\rho(\gamma_\infty)\xi}) = \mathrm{J}_O(\bar\xi)$. Finally $\theta_O \circ \omega_O(1,\gamma) = \theta_O$ by Lemma 6.2.5 (1 and γ are rational points). Applying θ_O and complex conjugation gives the identity of (c). For $\Xi \in \mathrm{Form}(O)$, (DP5) gives $\Xi(\gamma_f hk) = \tilde\eta_O(k)\rho_O(\gamma)\Xi(h)$, so $\mathcal{C}_O(u,\gamma_f hk)(\Xi(\gamma_f hk)) = \mathcal{C}_O(u,h)(\Xi(h))$.

(d). For $y \in G_f(n)$ the set of $\gamma \in \Gamma$ with $x^{-1}\gamma y \in \mathbf{K}(O)$ is finite (Γ_f is discrete in $G_f(n)$, as I_n is definite, and $x\mathbf{K}(O)y^{-1}$ is compact), so $\delta_x(v)(y)$ is a finite sum, and $\delta_x(v)(y) = 0$ unless $y \in \Gamma_f x\mathbf{K}(O)$. By (c) and (DP9), only the double coset $D = \Gamma_f x\mathbf{K}(O)$ contributes to the class sum, with its representative $h_D = \gamma_1 x k_1$ ($\gamma_1 \in \Gamma$, $k_1 \in \mathbf{K}(O)$), and the summand is constant on D:

$$\mathrm{Th}_O(\delta_x(v))(u) = \frac{1}{\#\Gamma_{h_D}}\,\mathcal{C}_O(u,x)(\delta_x(v)(x)), \qquad \delta_x(v)(x) = \sum_{\gamma\in\Gamma_x} \tilde\eta_O(x^{-1}\gamma x)\,\rho_O(\gamma)^{-1}v,$$

where $\Gamma_x = \Gamma \cap x\mathbf{K}(O)x^{-1}$. For $\gamma \in \Gamma_x$ put $k = x^{-1}\gamma_f x \in \mathbf{K}(O)$; then $x = \gamma_f^{-1}xk$, and (c) for the rational point γ^{-1}, the point x and k gives

$$\mathcal{C}_O(u,x)(\tilde\eta_O(k)\rho_O(\gamma)^{-1}v) = \mathcal{C}_O(u,\gamma_f^{-1}xk)(\tilde\eta_O(k)\rho_O(\gamma^{-1})v) = \mathcal{C}_O(u,x)(v).$$

So, by linearity of $\mathcal{C}_O(u,x)$,

$$\mathcal{C}_O(u,x)(\delta_x(v)(x)) = \#\Gamma_x \cdot \mathcal{C}_O(u,x)(v).$$

Finally $\Gamma_{h_D} = \Gamma \cap h_D\mathbf{K}(O)h_D^{-1} = \gamma_1\Gamma_x\gamma_1^{-1}$, so $\#\Gamma_{h_D} = \#\Gamma_x$, and (d) follows.

(e). For $\Xi_1, \Xi_2 \in \mathrm{Form}(O)$ the summand $h \mapsto \langle \Xi_1(h), \Xi_2(h)\rangle_{\kappa_O}$ of the inner product (DP5) is constant on the double cosets: by (DP5), (c) of Lemma 4.6.10 and $|\tilde\eta_O(k)| = 1$, $\langle \Xi_1(\gamma_f hk), \Xi_2(\gamma_f hk)\rangle_{\kappa_O} = |\tilde\eta_O(k)|^2\langle \rho_O(\gamma)\Xi_1(h), \rho_O(\gamma)\Xi_2(h)\rangle_{\kappa_O} = \langle \Xi_1(h), \Xi_2(h)\rangle_{\kappa_O}$. As in (d), only the double coset D of x contributes, and $\langle \delta_x(v), \Xi\rangle = \#\Gamma_{h_D}^{-1}\langle \delta_x(v)(x), \Xi(x)\rangle_{\kappa_O}$. For $\gamma \in \Gamma_x$ with $k = x^{-1}\gamma_f x$, (DP5) gives $\rho_O(\gamma)\Xi(x) = \Xi(\gamma_f x) = \Xi(xk) = \tilde\eta_O(k)\Xi(x)$, so, the form being conjugate-linear in its first argument and $\rho_O(\gamma)$ unitary, $\langle \tilde\eta_O(k)\rho_O(\gamma)^{-1}v, \Xi(x)\rangle_{\kappa_O} = \overline{\tilde\eta_O(k)}\langle v, \rho_O(\gamma)\Xi(x)\rangle_{\kappa_O} = |\tilde\eta_O(k)|^2\langle v, \Xi(x)\rangle_{\kappa_O} = \langle v, \Xi(x)\rangle_{\kappa_O}$. Summing over the $\#\Gamma_x = \#\Gamma_{h_D}$ elements of Γ_x gives (e). □

Lemma 6.6.32. *Let O be a descent pair for ranks $r < n$ with $M_O \neq \emptyset$, let N, m, a be finite index sets, $z\colon \{1,\dots,r\} \hookrightarrow m$ an embedding, $\mathcal{V} = (\varepsilon, \phi_{\mathcal{V}}, d)$ a divided splitting for (N, m, a, z), $s \in \mathrm{M}_{a\times N}(R_2)$, L a divided lattice glued for s, and b, b' automorphic characters of $\mathrm{U}(1)$ with $bb' = \chi'_O$. For every $A \in \mathrm{C}(\mathcal{V})$ let $k_A \in \mathbf{K}_0(r)$ satisfy $\mathrm{red}_{\mathfrak{M}_O^2}(k_A) = A$ and have component 1 at every finite place w such that neither w nor $\bar{w}$ is a chosen place of O. Let $h^\circ \in G_f(m)$, let $\varphi\colon G_f(N) \to \mathbb{V}_{\kappa_O^+}$ (weight κ_O^+ in rank $|N|$, rows N) be any function, $\bar{\varphi}$ the function with the conjugate coordinates, and $\eta \in \mathbb{V}_{\kappa_O^-}$ (rank $|m|$, rows m). Let*

$$\Xi = \sum_{h \bmod \mathbf{K}_O[[s;\bar{\varpi}],\phi_{\mathcal{V}}]}^{\mathrm{cl}} \mathrm{pr}_{\mathcal{V},L}\big((h,h^\circ);\varphi(h)\big)(\eta) \ \in \mathrm{Form}(O),$$

the sum (6.22) *over the double cosets $\mathrm{U}(\mathrm{I}_N)(F)\backslash G_f(N)/\mathbf{K}_O[[s;\bar{\varpi}],\phi_{\mathcal{V}}]$ at their fixed representatives. Then for every $g \in \mathrm{U}(W(O))(\mathbb{A}_F)$*

$$\mathrm{Th}_O(\Xi)(g) = \overline{\beta_L \sum_{A\in\mathrm{C}(\mathcal{V})} \mathrm{cks}_{O,b}([s;\bar{\varpi}],\phi_{\mathcal{V}};\bar{\varphi})(gk_A)\cdot \theta^{W(O),\mathrm{I}_m}_{\chi_O,b';\mathfrak{M}_O^2}\Big[\mathbf{1}_{I_z}, \mathrm{Let}_{-1}\big(\mathrm{Pol}_{\kappa_O^-}(\bar{\eta})\big)\Big](gk_A,h^\circ)},$$

where $\mathbf{1}_{I_z}$ is the indicator function of the inclusion matrix of z over R_2, $\bar{\eta}$ has the conjugate coordinates, and Let_{-1} is read with the rows in m.

Proof. Write $\gamma = \gamma_L$, $\mathfrak{n} = \mathfrak{M}_O^2\mathfrak{Q}_O$ (a modulus with $\Sigma(\mathfrak{n}) = S_O$, (KN3)), $\bar{T} = \mathrm{red}_0(T)$ for matrices T over $R_{\mathfrak{n}}$, and $\mathrm{cd} = \mathrm{cd}_{\psi_O^{\mathrm{res}},\mathrm{red}_0,[s;\bar{\varpi}],\phi_{\mathcal{V}}}$. For $A \in \mathrm{C}(\mathcal{V})$ let $\bar{k}_A = \mathrm{red}_{\mathfrak{n}}(k_A) \in \mathrm{GL}_r(R_{\mathfrak{n}})$; its reduction modulo $\mathfrak{M}_O^2$ is A, and its components at the places of Q_O are 1, since such a place w is neither chosen nor the conjugate of a chosen place (DP1), so k_A has component 1 there.

Step 1: reduction to one product form. The class sum defining Ξ is a finite sum $\sum_c \omega_c \,\mathrm{pr}_{\mathcal{V},L}((h_c,h^\circ);\varphi(h_c))(\eta)$ at the fixed representatives h_c with rational weights ω_c, and the coset kernel sum $\mathrm{cks}_{O,b}([s;\bar{\varpi}],\phi_{\mathcal{V}};\bar{\varphi})$ of (KN5) is the class sum over the same double cosets $\mathrm{U}(\mathrm{I}_N)(F)\backslash G_f(N)/\mathbf{K}_O[[s;\bar{\varpi}],\phi_{\mathcal{V}}]$, at the same representatives with the same weights. Th_O is linear and complex conjugation fixes the ω_c. So it suffices to show, for every $h \in G_f(N)$, $y \in \mathbb{V}_{\kappa_O^+}$ (rows N) and $g \in \mathrm{U}(W(O))(\mathbb{A}_F)$,

$$\mathrm{Th}_O\big(\mathrm{pr}_{\mathcal{V},L}((h,h^\circ);y)(\eta)\big)(g) = \overline{\beta_L \sum_{A\in\mathrm{C}(\mathcal{V})} K_1(gk_A,h)\,K_2(gk_A,h^\circ)}, \tag{6.139}$$

with $K_1 = \theta^{W(O),\mathrm{I}_N}_{\chi_O,b;\mathfrak{n}}\big[\mathrm{cd},\mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(\bar{y}))\big]$ and $K_2 = \theta^{W(O),\mathrm{I}_m}_{\chi_O,b';\mathfrak{M}_O^2}\big[\mathbf{1}_{I_z},\mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}(\bar{\eta}))\big]$; applied with $y = \varphi(h_c)$, $\bar{y} = \bar{\varphi}(h_c)$, this gives the claim. Fix h, y, g and put $x' = \mathrm{diag}_\varepsilon(h,h^\circ)$, $x = x'\gamma$.

Step 2: the descent of the delta form. By (GU6) and (d) of Lemma 6.6.31, the left side of (6.139) is $\mathcal{C}_O(g,x)(y\cdot\eta) = \overline{\theta_O\big(\omega_O(g,(1,x))\,\mathrm{J}_O(\overline{y\cdot\eta})\big)}$ (DP9). By (b) and (c) of Lemma 6.6.26 (its hypotheses are those of (GU1) on $\kappa_O^{\pm}$),

$$\mathrm{J}_O(\overline{y\cdot\eta}) = v_P \otimes \Upsilon(O), \qquad P = \mathrm{Har}_O(\bar{y}\cdot\bar{\eta}) = P_1^\varepsilon P_2^\varepsilon,$$
$$P_1 = \mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(\bar{y})), \qquad P_2 = \mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}(\bar{\eta})),$$

P_i^ε the polynomials with the row b of their variables renamed $\varepsilon(b)$.

Step 3: the point γ moves into the datum. As ω_O is a representation ((2) of Lemma 4.1.1), $\omega_O(g,(1,x)) = \omega_O(g,(1,x'))\,\omega_O(1,(1,\gamma))$. By (5) of Lemma 4.1.1 (real components 1), $\omega_O(1,(1,\gamma))(v_P\otimes f) = v_P \otimes \omega_O(1,(1,\gamma))f$, and by (4) there, for the set S_O (which contains the conjugate of none of its members, (DP1)), the function $\Phi_O \in \mathcal{T}_{S_O}$ (DP7) and the

point γ of $G_f(n)$ (its components are 1 outside the chosen places and their conjugates, (GU2)), with the pair $(W(O),\mathrm{I}_n)$ of index sets $\{1,\dots,r\}$ and $\{1,\dots,n\}$,

$$\omega_O(1,(1,\gamma))(\Phi_O\otimes_{S_O}v^0)=\frac{\chi_O[r](\det_{S_O}\gamma)}{|\det_{S_O}\gamma|^{r/2}}\,\Phi'\otimes_{S_O}v^0,\qquad \Phi'(X)=\Phi_O(\gamma^{-1}X).$$

Since γ has components 1 at the places of Q_O, $\det_{S_O}\gamma=\delta$, and the scalar is β_L (GU2).

Step 4: the moved datum. Let Ψ_L be the function on $\mathrm{M}_{n\times r}(R_\mathfrak{n})$ with $\Psi_L(T)=\prod_{w\in Q_O}\psi_{O,w}(\det(\text{inc-rows of } T \bmod w))$ if $G_L\bar{T}=\bar{\varpi}\,\mathrm{Inc}_O$ and $\Psi_L(T)=0$ otherwise. We show $\Phi'=\Psi_L^\natural$ ((XD5): $\Psi_L^\natural(X)=\Psi_L(X_{S_O}\bmod\mathfrak{n})$ if X is integral at S_O, 0 otherwise), so that $\omega_O(1,(1,\gamma))\mathrm{J}_O(\overline{y\cdot\eta})=\beta_L\,\mathrm{mod}_\mathfrak{n}(\Psi_L,v_P)$. Let X be a finite adelic $n\times r$ matrix. (i) If $\gamma^{-1}X$ is integral at S_O, so is X: the matrix $\varpi_O\gamma^{-1}$ is integral, hence so is $(\varpi_O\gamma^{-1})^*=\bar{\varpi}_O\gamma$ (using $(\gamma^{-1})^*=\gamma$); at $w\in S_O$ the component of $\bar{\varpi}_O$ is the conjugate of that of ϖ_O at $\bar{w}$, which is 1 because $\bar{w}$ is not a chosen place (DP1); so γ_w is integral and $X_w=\gamma_w(\gamma^{-1}X)_w$ is integral. (ii) Let X be integral at S_O. At $w\in Q_O$, $\gamma_w=1$. At a chosen place w, with ϖ_w the component of ϖ_O (a generator of $w\mathcal{O}_{E,w}$): if $\gamma_w^{-1}X_w=\mathrm{Inc}_O+\varpi_wZ$ with Z integral, then $\varpi_w\gamma_w^{-1}X_w\equiv\varpi_w\mathrm{Inc}_O\pmod{w^2}$, and the left side modulo w^2 is the component at w of $G_L\bar{X}$; conversely, if $G_L\bar{X}=\bar{\varpi}\mathrm{Inc}_O$, then $\varpi_w\gamma_w^{-1}X_w=\varpi_w\mathrm{Inc}_O+\varpi_w^2Z$ with Z integral, so $\gamma_w^{-1}X_w=\mathrm{Inc}_O+\varpi_wZ$. Hence, for X integral at S_O: $\gamma^{-1}X$ is integral at S_O with $(\gamma^{-1}X)_w\equiv\mathrm{Inc}_O\pmod w$ for all $w\in M_O$ if and only if $G_L\bar{X}=\bar{\varpi}\mathrm{Inc}_O$, and then $(\gamma^{-1}X)_w=X_w$ at $w\in Q_O$. Comparing with (DP7), $\Phi'(X)=\Phi_O(\gamma^{-1}X)=\Psi_L^\natural(X)$ when X is integral at S_O; when it is not, both vanish by (i).

Step 5: the glued decomposition. For $T\in\mathrm{M}_{n\times r}(R_\mathfrak{n})$ write T_N, T_m for the matrices of its rows $\varepsilon(N)$ and $\varepsilon(m)$, and put $\Psi^\times(T)=\mathrm{cd}(T_N)\,\mathbf{1}_{I_z}(\overline{T_m})$. We show

$$\Psi_L(T)=\sum_{A\in\mathrm{C}(\mathcal{V})}\Psi^\times(T\bar{k}_A).\tag{6.140}$$

The reduction of $T\bar{k}_A$ modulo $\mathfrak{M}_O^2$ is $\bar{T}A$. By (GU3), $\Psi^\times(T\bar{k}_A)\neq0$ forces $\bar{T}A$ to be block normalised for s. If $G_L\bar{T}\neq\bar{\varpi}\mathrm{Inc}_O$, no $\bar{T}A$ is block normalised by (2) of (GU4), and both sides vanish. Otherwise, by (1) of (GU4), there is exactly one A with $\bar{T}A$ block normalised, and it remains to see that $\Psi^\times(T\bar{k}_A)=\Psi_L(T)$ for it. Here $\mathbf{1}_{I_z}((\bar{T}A)_m)=1$ and $[s;\bar{\varpi}](\bar{T}A)_N=[s;\bar{\varpi}]I_{\phi_\mathcal{V}}$, so $\Psi^\times(T\bar{k}_A)=\psi_O^{\mathrm{res}}(\det\sigma)$ with σ the $r\times r$ matrix of the rows $\varepsilon(\phi_\mathcal{V}(c))=\mathrm{inc}_O(c)$ of $T\bar{k}_A$ ((GU1), (KN2)). The rows $\bar{\varpi}\cdot1$ of $[s;\bar{\varpi}]$ give $\bar{\varpi}(\bar{T}A)_N=\bar{\varpi}I_{\phi_\mathcal{V}}$, and as $\bar{\varpi}u=0$ in R_2 exactly for $u\in\mathfrak{M}_O/\mathfrak{M}_O^2$, $(\bar{T}A)_N\equiv I_{\phi_\mathcal{V}}$ modulo every chosen place; so $\det\sigma\equiv1$ modulo every $w\in M_O$, and $\psi_{O,w}(\det\sigma\bmod w)=1$ there. At $w\in Q_O$, $\bar{k}_A$ is 1, so σ mod w is the inc-rows of T mod w. Hence $\psi_O^{\mathrm{res}}(\det\sigma)=\prod_{w\in Q_O}\psi_{O,w}(\det(\text{inc-rows of } T\bmod w))=\Psi_L(T)$ (both sides 0 when some of these determinants vanish), proving (6.140).

Step 6: the translates move to the first variable. By (6) of Lemma 4.1.1 for the modulus $\mathfrak{n}$ and $k=k_A\in\mathbf{K}_0(r)$, $k'=1$,

$$\omega_O((1,k_A),1)\,\mathrm{mod}_\mathfrak{n}(\Psi^\times,v_P)=\chi_O'(\det k_A)^{-1}\,\mathrm{mod}_\mathfrak{n}(\Psi^\times(\cdot\,\bar{k}_A),v_P).$$

Here $\chi_O'(\det k_A)=1$: $\chi_O'=\mu_O\upsilon^{\lfloor n/2\rfloor}$, υ has trivial tame part ((ii) of Lemma 3.2.11), $\det k_A$ has unit finite components and infinite components 1, and the tame part of μ_O is trivial off S_O (DP2), while $\det k_A$ is 1 at the places of Q_O and $\equiv1$ modulo every chosen place ($A\bar{\varpi}=\bar{\varpi}$ gives $A\equiv1$ modulo $\mathfrak{M}_O$). By Steps 3–5, linearity of $\mathrm{mod}_\mathfrak{n}$ in the function, and (2) of Lemma 4.1.1,

$$\begin{aligned}\theta_O(\omega_O(g,(1,x))\mathrm{J}_O(\overline{y\cdot\eta}))&=\beta_L\sum_{A\in\mathrm{C}(\mathcal{V})}\theta_O(\omega_O(gk_A,(1,x'))\,\mathrm{mod}_\mathfrak{n}(\Psi^\times,v_P))\\&=\beta_L\sum_A\theta^{W(O),\mathrm{I}_n}_{\chi_O,\chi_O';\mathfrak{n}}[\Psi^\times,P_1^\varepsilon P_2^\varepsilon](gk_A,x').\end{aligned}$$

Step 7: the kernel factors over $\mathrm{I}_n = \mathrm{I}_N \oplus \mathrm{I}_m$. Apply (3) and (4) of Lemma 6.4.96 with $S = W(O)$, e_S the identity, $T = \mathrm{I}_n$ and $e_T = \varepsilon$: $\mathrm{I}_n^{\varepsilon}$ is the standard space of signs $+1$ on $N \sqcup m$, i.e. $\mathrm{I}_N \oplus \mathrm{I}_m$; $x'^{\varepsilon} = \mathrm{diag}(h, h^{\circ})$ by the definition of $\mathrm{diag}_{\varepsilon}$; the function $\Psi^{\times}(P_{\varepsilon}^{\mathsf{T}} X')$ is $X' \mapsto \mathrm{cd}(X'_N)\mathbf{1}_{I_z}(\overline{X'_m})$, X'_N, X'_m the rows N, m of X' (the row $\varepsilon(a)$ of $P_{\varepsilon}^{\mathsf{T}} X'$ is the row a of X'); and the renaming of (4) turns $P_1^{\varepsilon} P_2^{\varepsilon}$ into $P_1 P_2$, read in the rows N and m. By (a) of Lemma 6.3.80 for the modulus $\mathfrak{n}$ and Lemma 6.4.66 ($\mathfrak{n} \subseteq \mathfrak{M}_O^2$, the function $\mathbf{1}_{I_z} \circ \mathrm{red}_0$),

$$\mathrm{mod}_{\mathfrak{n}}(\mathrm{cd} \otimes (\mathbf{1}_{I_z} \circ \mathrm{red}_0), v_{P_1 P_2}) = \mathrm{mod}_{\mathfrak{n}}(\mathrm{cd}, v_{P_1}) \boxtimes^{\mathrm{tg}} \mathrm{mod}_{\mathfrak{M}_O^2}(\mathbf{1}_{I_z}, v_{P_2}),$$

and by (d) of Lemma 6.3.52 (target sums, $\alpha = \chi_O$, $\beta = b$, $\beta' = b'$, $bb' = \chi'_O$)

$$\theta^{W(O),\mathrm{I}_n}_{\chi_O,\chi'_O;\mathfrak{n}}\big[\Psi^{\times}, P_1^{\varepsilon} P_2^{\varepsilon}\big](g k_A, x') = K_1(g k_A, h)\, K_2(g k_A, h^{\circ}).$$

With Steps 2 and 6 this is (6.139). □

Proposition 6.6.33. *Let $1 \le j \le m$, ϑ a twist with modulus Q, $O = O_{j,\vartheta}$, $\chi' = \chi'_{j,\vartheta}$, $\nu'' = \nu''_{j,\vartheta}$, and let $\mathfrak{n}$, χ_ϑ, α_e, W and Π_i^W be as in Lemma 6.6.23. For $\Lambda \in \mathrm{M}_{\mathcal{B}^{\mathrm{c}} \times \mathcal{B}_j}(R_{\mathfrak{M}^2})$ and $y \in \mathrm{M}_{\mathcal{B}_j \times \mathcal{B}^{\mathrm{c}}}(R_{\mathfrak{n}})$ put $\psi^{\mathrm{inv}}_{\Lambda}(y) = \sum_{\gamma \in \mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(R_{\mathfrak{M}^2})} \psi_{\Lambda}(\mathrm{red}_0(y)\,\gamma)$, and for $X = (X' \,|\, X'') \in \mathrm{M}_{\mathcal{B}_j \times n}(R_{\mathfrak{n}})$ (X' the columns $\mathcal{B}^{\mathrm{c}}$) put $\Phi_{\vartheta,i}(X) = \chi_\vartheta(\det X'')\, \psi^{\mathrm{inv}}_{\Lambda_i}(X''^{-1} X')$ if X'' is invertible and $\Phi_{\vartheta,i}(X) = 0$ otherwise. The **left side of the third exchange** is the function on $\mathrm{U}(W_j)(\mathbb{A}_F)$*

$$L_{\vartheta,i}(g) = \sum_{h' \bmod \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})}^{\mathrm{cl}} \mathsf{a}_{\chi'}(h') \sum_{h \bmod \mathbf{K}_0(\mathcal{B}_j)}^{\mathrm{cl}} \theta^{W,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}\big[\Phi_{\vartheta,i}, \Pi_i^W\big]\big(\mathrm{diag}(h', g, 1, x_{\mathfrak{c}_i}), h\big),$$

with the block diagonal point of components h' on $\mathcal{B}^{\mathrm{c}}$, g on $\mathcal{R}$, 1 on $\mathcal{A}_j^- \sqcup \mathcal{A}_j^+$ and $x_{\mathfrak{c}_i}$ on $\mathcal{B}^{\flat}$ (the function of the same name in Proposition 6.6.36). Let a be a finite index set, $w \in \mathrm{M}_{a \times \mathcal{B}_j}(R_{\mathfrak{M}^2})$ and $u \in \mathrm{M}_{\mathcal{B}^{\mathrm{c}} \times a}(R_{\mathfrak{M}^2})$ with $uw = \Lambda_i$, and let $\phi_j \colon \mathcal{R} \hookrightarrow \mathcal{B}_j$ be the inclusion. Then for every $g \in \mathrm{U}(W_j)(\mathbb{A}_F)$

$$L_{\vartheta,i}(g) = \mathrm{cks}_{O,\nu''}(w, \phi_j; \Upsilon_{\vartheta,i})(g).$$

Proof. Fix i, g, w and u. By (XE1), $\nu''/\chi' = \nu_j^{\flat}$, so $\alpha_e = \chi' \nu_j^{\flat} \upsilon^m = \nu'' \upsilon^m$. The level system of O has the chosen places S_0 and the places of Q ((LF4)), so $\mathfrak{M}_O^2 \mathfrak{Q}_O = \mathfrak{n}$ and $\mathrm{red}_0 \colon R_{\mathfrak{n}} \to R_{\mathfrak{M}^2}$ is the reduction of (KN3). Let $H = \mathrm{Stab}_{\mathrm{red}_0}(w, \phi_j) \subseteq \mathrm{GL}_{\mathcal{B}_j}(R_{\mathfrak{n}})$; by (KN3), $\mathbf{K}_{\mathfrak{n}}[H] = \mathbf{K}_O[w, \phi_j]$. For $h' \in G_f(\mathcal{B}^{\mathrm{c}})$ write $p(h') = \mathrm{diag}(h', g, 1, x_{\mathfrak{c}_i})$.

Step 1: the inner class sum at the row-fixing level. We apply Lemma 6.6.34 with the index sets $\mathcal{B}^{\mathrm{c}}$, $\mathcal{B}_j$, a, $\mathcal{R}$ in the roles of n, b, c, r, the space W, the modulus $\mathfrak{n}$, $\chi = \chi_\vartheta$, $\alpha = \alpha_e$, $\beta = \nu''$, $P = \Pi_i^W$, $\phi = \mathrm{red}_0$, $\Lambda = w$, $e = \phi_j$ and $F = \psi^{\mathrm{inv}}_{\Lambda_i}$. Its hypotheses: α_e is tame of weight $\gamma(\tau) = m - \#D_j(\tau)$ for $(\mathfrak{n}, \chi_\vartheta)$ by (b) of Lemma 6.6.23, and $\Omega_{\gamma,0}(1, y) v_{\Pi_i^W} = v_{\Pi_i^W}$ for $y \in G_\infty(1_{\mathcal{B}_j})$ by (d) there. For $A \in H$ we have $w\,\mathrm{red}_0(A) = w$, hence $\Lambda_i\,\mathrm{red}_0(A) = uw\,\mathrm{red}_0(A) = \Lambda_i$, and for $y \in \mathrm{M}_{\mathcal{B}_j \times \mathcal{B}^{\mathrm{c}}}(R_{\mathfrak{n}})$ and $\gamma \in \mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(R_{\mathfrak{M}^2})$

$$\mathrm{tr}\big(\mathrm{red}_0(Ay)\,\gamma\,\Lambda_i\big) = \mathrm{tr}\big(\mathrm{red}_0(y)\,\gamma\,\Lambda_i\,\mathrm{red}_0(A)\big) = \mathrm{tr}\big(\mathrm{red}_0(y)\,\gamma\,\Lambda_i\big),$$

so $\psi_{\Lambda_i}(\mathrm{red}_0(Ay)\gamma) = \psi_{\Lambda_i}(\mathrm{red}_0(y)\gamma)$ and $F(Ay) = F(y)$. The function Φ of Lemma 6.6.34 is $\Phi_{\vartheta,i}$; let Φ_0 be its function Φ_0:

$$\Phi_0(X) = \mathrm{cd}_{\chi_\vartheta,\mathrm{red}_0,w,\phi_j}(X''_{\phi_j})\, \psi^{\mathrm{inv}}_{\Lambda_i}(X') \quad \text{if the columns of } X'' \text{ off } \mathcal{R} \text{ are those of } 1,$$
$$\Phi_0(X) = 0 \text{ otherwise.}$$

Lemma 6.6.34 gives, for every h',

$$\sum_{h \bmod \mathbf{K}_0(\mathcal{B}_j)}^{\mathrm{cl}} \theta^{W,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}[\Phi_{\vartheta,i},\Pi_i^W](p(h'),h) = \sum_{h \bmod \mathbf{K}_O[w,\phi_j]}^{\mathrm{cl}} \theta^{W,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}[\Phi_0,\Pi_i^W](p(h'),h), \qquad (6.141)$$

and the summands on the right are invariant under $\mathrm{U}(V_j'')(F)$ on the left and $\mathbf{K}_O[w,\phi_j]$ on the right.

Step 2: exchange of the two class sums. Multiply (6.141) by $\mathsf{a}_{\chi'}(h')$ and take the class sum over h' modulo $\mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$. Both class sums are finite sums at fixed representatives with fixed weights, so they may be interchanged:

$$L_{\vartheta,i}(g) = \sum_{h \bmod \mathbf{K}_O[w,\phi_j]}^{\mathrm{cl}} K(h), \qquad K(h) = \sum_{h' \bmod \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})}^{\mathrm{cl}} \mathsf{a}_{\chi'}(h')\, \theta^{W,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}[\Phi_0,\Pi_i^W](p(h'),h).$$

Step 3: regrouping W. Let $n_1 = \mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{B}_j^{\mathrm{add}}$, $\mathcal{B}_j^{\mathrm{add}} = \mathcal{A}_j^+ \sqcup \mathcal{B}^\flat \sqcup \mathcal{B}^{\mathrm{c}}$, and let $e\colon n_1 \to n$ be the identity on each of the blocks $\mathcal{R}, \mathcal{A}_j^-, \mathcal{A}_j^+, \mathcal{B}^\flat$ (into $\mathcal{B}_j$) and $\mathcal{B}^{\mathrm{c}}$. Each summand of W is a standard space of signs (W_j by (HP4)), and an orthogonal sum of standard spaces of signs is the standard space of the concatenated signs (block diagonal J and point); so W is the standard space of its signs s on n; since J_W and the point of W are diagonal with entries that depend only on the sign, the space W^e of (1) of Lemma 6.4.96 is the standard space of signs $s \circ e$ on n_1, $W^e = W_j \oplus \mathrm{I}^-_{\mathcal{A}_j^-} \oplus W_j^{\mathrm{add}}$. For a matrix X with columns n_1 let $X^{(e)} = XP_e$, the matrix with columns n whose column $e(b)$ is the column b of X (P_e the permutation matrix of e, as in Lemma 6.4.96); for a function Φ on $\mathrm{M}_{\mathcal{B}_j \times n}(R_{\mathfrak{n}})$ put $\Phi^e(X) = \Phi(X^{(e)})$, and for $P \in \mathbb{C}[x]_{\mathcal{B}_j \times n}$ let P^e be P with $x_{\tau,b,c}$ substituted for $x_{\tau,b,e(c)}$ ($b \in \mathcal{B}_j$, $c \in n_1$). We apply (3) and (4) of Lemma 6.4.96 with $S = W$, $e_S = e$, $T = V_j''$ and e_T the identity of $\mathcal{B}_j$ (so $T^{e_T} = T$ and $h^{e_T} = h$). By (4), $\mathrm{mod}_{\mathfrak{n}}(\Phi_0, v_{\Pi_i^W})^{e_T \times e} = \mathrm{mod}_{\mathfrak{n}}(\Phi_0^e, v_{(\Pi_i^W)^e})$; and $p(h')^e = P_e\, p(h')\, P_e^{\mathsf{T}}$ is the block diagonal point $\mathrm{diag}(g, 1, (1, x_{\mathfrak{c}_i}, h'))$ on $\mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{B}_j^{\mathrm{add}}$. So by (3), for every h',

$$\theta^{W,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}[\Phi_0,\Pi_i^W](p(h'),h) = \theta^{W^e,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}[\Phi_0^e,(\Pi_i^W)^e]\big(\mathrm{diag}(g,1,(1,x_{\mathfrak{c}_i},h')),h\big),$$

where $(1, x_{\mathfrak{c}_i}, h')$ is the block diagonal point on $\mathcal{A}_j^+ \sqcup \mathcal{B}^\flat \sqcup \mathcal{B}^{\mathrm{c}}$. For $X \in \mathrm{M}_{\mathcal{B}_j \times n_1}(R_{\mathfrak{n}})$ with the column blocks $X_{\mathcal{R}}$, X_- and X_{add}, the matrix $X^{(e)}$ has $X''_{\phi_j} = X_{\mathcal{R}}$, its columns $\mathcal{A}_j^-$ are X_-, its columns $\mathcal{A}_j^+ \sqcup \mathcal{B}^\flat$ are those of X_{add}, and X' is the block $X^{\mathrm{c}}_{\mathrm{add}}$ of X_{add} on $\mathcal{B}^{\mathrm{c}}$. The columns of X'' off $\mathcal{R}$ are those of 1 exactly when X_- is the inclusion matrix of $\mathcal{A}_j^-$ in $\mathcal{B}_j$ and the columns $\mathcal{A}_j^+ \sqcup \mathcal{B}^\flat$ of X_{add} are those of the inclusion matrix of $\mathcal{A}_j^+ \sqcup \mathcal{B}^\flat$ in $\mathcal{B}_j$; in that case $\psi^{\mathrm{inv}}_{\Lambda_i}(X') = \Psi^{\mathrm{add}}_{\Lambda_i}(X_{\mathrm{add}})$ by (XE6). As $\chi_\vartheta = \psi^{\mathrm{res}}_O$ ((a) of Lemma 6.6.23),

$$\Phi_0^e(X) = \mathrm{cd}(X_{\mathcal{R}}) \cdot \mathbf{1}_j^-(X_-) \cdot \Psi^{\mathrm{add}}_{\Lambda_i}(X_{\mathrm{add}}), \qquad \mathrm{cd} = \mathrm{cd}_{\psi^{\mathrm{res}}_O,\mathrm{red}_0,w,\phi_j}.$$

Step 4: the polynomial. $(\Pi_i^W)^e$ is Π_i with $x_{\tau,b,c}$ renamed $x_{\tau,b,e^{-1}\pi_j(c)}$; $e^{-1}\pi_j$ sends $c < r_+$ to the basis vector of W_j of number c in $\mathcal{R}$ and is the identity on $\mathcal{B}_j^{\mathrm{add}}$. For $l \in I_{\kappa_O^+}$ let P_l be as in Lemma 6.6.35 and $P_l^{\mathcal{R}}$ the polynomial P_l with $x_{\tau,b,c}$ renamed $x_{\tau,b,\pi_j(c)}$, in the columns $\mathcal{R}$. By Lemma 6.6.35,

$$(\Pi_i^W)^e = \sum_{l \in I_{\kappa_O^+}} P_l^{\mathcal{R}} \cdot \mathrm{lc}_{i,l},$$

with $\mathrm{lc}_{i,l}$ in the columns $\mathcal{B}_j^{\mathrm{add}}$. Moreover $P_l^{\mathcal{R}} = \mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(e_l))$, rows in $\mathcal{B}_j$: Let_1 substitutes for $y_{\tau,b,c}$ the sum of $x_{\tau,b,c'}$ over the basis vectors c' of W_j of sign $+1$ with $\mathrm{ord}(c') = c$; for these

$\mathrm{ord}(c')$ is the number of c' ((LF4)), and the numbers of the vectors of sign $+1$ are $0,\dots,r_+-1$ ((HP4)); so the sum is $x_{\tau,b,\pi_j(c)}$ for $c<r_+$ and empty for $c\ge r_+$, which is the substitution defining P_l, renamed by π_j.

Step 5: the kernel factors. The kernels are linear in the polynomial (the Fock vector, the datum modulo $\mathfrak{n}$ and the theta functional are linear). For each l, by (b) of Lemma 6.3.80 applied twice (to $W_j\oplus(\mathrm{I}^-_{\mathcal{A}_j^-}\oplus W_j^{\mathrm{add}})$ and to $\mathrm{I}^-_{\mathcal{A}_j^-}\oplus W_j^{\mathrm{add}}$; the second block carries the polynomial 1),

$$\mathrm{mod}_{\mathfrak{n}}(\Phi_0^e, v_{P_l^{\mathcal{R}}\mathrm{lc}_{i,l}}) = \mathrm{mod}_{\mathfrak{n}}(\mathrm{cd}, v_{P_l^{\mathcal{R}}})\boxtimes^{\mathrm{sc}}\Big(\mathrm{mod}_{\mathfrak{n}}(\mathbf{1}_j^-, v_1)\boxtimes^{\mathrm{sc}}\mathrm{mod}_{\mathfrak{n}}(\Psi^{\mathrm{add}}_{\Lambda_i}, v_{\mathrm{lc}_{i,l}})\Big).$$

The signs of W_j have $r_-(O)=r-r_+=m-j$ entries -1 ($r=k+m-1$), so $\chi_O=\upsilon^{m-j}$ and $\alpha_e=\nu''\upsilon^m=\chi_O\cdot\upsilon^j\cdot\nu''$. By (d) of Lemma 6.3.52 (source sums, applied twice, second character ν''),

$$\begin{aligned}&\theta^{W^e,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}[\Phi_0^e, P_l^{\mathcal{R}}\mathrm{lc}_{i,l}]\,(\mathrm{diag}(g,1,(1,x_{\mathfrak{c}_i},h')),h)\\ &\quad=\theta^{W_j,V_j''}_{\chi_O,\nu'';\mathfrak{n}}[\mathrm{cd},P_l^{\mathcal{R}}](g,h)\;\theta^{\mathrm{I}^-_{\mathcal{A}_j^-},V_j''}_{\upsilon^j,\nu'';\mathfrak{n}}[\mathbf{1}_j^-,1](1,h)\;\theta^{W_j^{\mathrm{add}},V_j''}_{\nu'',\nu'';\mathfrak{n}}[\Psi^{\mathrm{add}}_{\Lambda_i},\mathrm{lc}_{i,l}]((1,x_{\mathfrak{c}_i},h'),h).\end{aligned}$$

Only the last factor depends on h'. Multiplying by $\mathsf{a}_{\chi'}(h')$ and taking the class sum over h', by Steps 3 and 4, linearity in the polynomial and the definition (XE6) of $\Upsilon_{\vartheta,i}(h)_l$ (with $\mathfrak{n}=\mathfrak{M}^2\mathfrak{Q}_O$),

$$K(h)=\sum_{l\in I_{\kappa_O^+}}\Upsilon_{\vartheta,i}(h)_l\,\theta^{W_j,V_j''}_{\chi_O,\nu'';\mathfrak{n}}[\mathrm{cd},P_l^{\mathcal{R}}](g,h)=\theta^{W_j,V_j''}_{\chi_O,\nu'';\mathfrak{n}}\Big[\mathrm{cd},\mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(\Upsilon_{\vartheta,i}(h)))\Big](g,h),$$

since $\sum_l\Upsilon_{\vartheta,i}(h)_l\,\mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(e_l))=\mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(\Upsilon_{\vartheta,i}(h)))$ by linearity.

Step 6: conclusion. By Step 2 and Step 5, $L_{\vartheta,i}(g)$ is the class sum over h modulo $\mathbf{K}_O[w,\phi_j]$ of the kernels of $(W(O),\mathrm{I}_{\mathcal{B}_j})$ with characters χ_O, ν'', modulus $\mathfrak{M}_O^2\mathfrak{Q}_O$, datum $\mathrm{cd}_{\psi_O^{\mathrm{res}},\mathrm{red}_0,w,\phi_j}$ and polynomial $\mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}(\Upsilon_{\vartheta,i}(h)))$ at (g,h), which is $\mathrm{cks}_{O,\nu''}(w,\phi_j;\Upsilon_{\vartheta,i})(g)$ by (KN5). □

Lemma 6.6.34. *Let n, b, c, r be finite index sets, W a space with index set $n\sqcup b$, and write the $b\times(n\sqcup b)$ matrices as $x=(x'\,|\,x'')$ with x' of size $b\times n$ and x'' of size $b\times b$. Let $\mathfrak{m}$ be a modulus, χ a multiplicative character of $R_{\mathfrak{m}}$, α an automorphic character of $\mathrm{U}(1)$ tame of weight γ for $(\mathfrak{m},\chi)$, and β an automorphic character of $\mathrm{U}(1)$. Let $P\in\mathbb{C}[x]_{b\times(n\sqcup b)}$ with $\Omega_{\gamma,0}(1,u)\,v_P=v_P$ for every $u\in G_\infty(1_b)$. Let $\phi\colon R_{\mathfrak{m}}\to S$ be a homomorphism of commutative rings, $\Lambda\in\mathrm{M}_{c\times b}(S)$, $e\colon r\hookrightarrow b$ an embedding, $H=\mathrm{Stab}_\phi(\Lambda,e)$, and F a function on $\mathrm{M}_{b\times n}(R_{\mathfrak{m}})$ with $F(Ay)=F(y)$ for all $A\in H$. Put*

$$\Phi(x)=\begin{cases}\chi(\det x'')\,F(x''^{-1}x') & x''\text{ invertible},\\ 0 & \text{otherwise},\end{cases}$$

$$\Phi_0(x)=\begin{cases}\mathrm{cd}_{\chi,\phi,\Lambda,e}(x''_e)\,F(x') & \text{the columns of } x''\text{ off } e(r)\text{ are those of } 1,\\ 0 & \text{otherwise},\end{cases}$$

x''_e the $b\times r$ matrix of the columns $e(r)$ of x''. Then for every $g\in\mathrm{U}(W)(\mathbb{A}_F)$ the functions $h\mapsto\theta^{W,\mathrm{I}_b}_{\alpha,\beta;\mathfrak{m}}[\Phi,P](g,h)$ and $h\mapsto\theta^{W,\mathrm{I}_b}_{\alpha,\beta;\mathfrak{m}}[\Phi_0,P](g,h)$ on $\mathrm{U}(\mathrm{I}_b)(\mathbb{A}_{F,f})$ are left invariant under $\mathrm{U}(\mathrm{I}_b)(F)$ and right invariant under $\mathbf{K}_b$ and under $\mathbf{K}_{\mathfrak{m}}[H]$ respectively, and

$$\sum_{h\bmod\mathbf{K}_{\mathfrak{m}}[H]}^{\mathrm{cl}}\theta^{W,\mathrm{I}_b}_{\alpha,\beta;\mathfrak{m}}[\Phi_0,P](g,h)=\sum_{h\bmod\mathbf{K}_b}^{\mathrm{cl}}\theta^{W,\mathrm{I}_b}_{\alpha,\beta;\mathfrak{m}}[\Phi,P](g,h).$$

Proof. Write $R = R_{\mathfrak{m}}$, $A = \Sigma(\mathfrak{m})$, $\Gamma = \mathrm{U}(\mathrm{I}_b)(F)$, $\mathrm{K} = \mathbf{K}_b$, $\mathrm{K}_H = \mathbf{K}_{\mathfrak{m}}[H]$, and, for g fixed and a function Ψ on $\mathrm{M}_{b\times(n\sqcup b)}(R)$, $\vartheta_\Psi(h) = \theta^{W,\mathrm{I}_b}_{\alpha,\beta;\mathfrak{m}}[\Psi, P](g,h)$; $\Psi \mapsto \vartheta_\Psi$ is linear, since $\Psi \mapsto \mathrm{mod}_{\mathfrak{m}}(\Psi, v_P)$ is ((XD5), (WR6)). For $B \in \mathrm{GL}_b(R)$ put $(\mathrm{tr}_B\Psi)(x) = \chi(\det B)\,\Psi(B^{-1}x)$.

Step 1: Φ_0 is Φ cut down to $x'' \in H$. Let $z \in \mathrm{M}_b(R)$ have the columns of 1 off $e(r)$, and $I_e \in \mathrm{M}_{b\times r}(R)$ the matrix of the columns $e(r)$ of 1. Then $z = 1 + uI_e^{\mathsf{T}}$ with $u = (z-1)I_e$, $z_e = I_e + u$, and the $r \times r$ matrix of the rows $e(r)$ of z_e is $1_r + I_e^{\mathsf{T}}u$, whose determinant is $\det(1_b + uI_e^{\mathsf{T}}) = \det z$ (Sylvester's identity). Moreover $\Lambda\phi(z_e) = \Lambda I_e$ iff $\Lambda\phi(u) = 0$ iff $\Lambda\phi(z) = \Lambda$ (as $\phi(I_e) = I_e$ and $I_e^{\mathsf{T}}I_e = 1_r$). Hence $\mathrm{cd}_{\chi,\phi,\Lambda,e}(z_e) = \chi(\det z)$ if $\Lambda\phi(z) = \Lambda$, and 0 otherwise. If z is invertible and $\Lambda\phi(z) = \Lambda$, then $z \in H$ and $F(z^{-1}y) = F(y)$; if z is not invertible, $\det z$ is not a unit and $\chi(\det z) = 0$. A z whose columns off $e(r)$ are not those of 1 is not in H. Therefore

$$\Phi_0(x) = \Phi(x) \ \text{ if } x'' \in H, \qquad \Phi_0(x) = 0 \ \text{ otherwise.} \tag{1}$$

Step 2: translates. For $B \in \mathrm{GL}_b(R)$, $\mathrm{tr}_B\Phi = \Phi$: if x'' is invertible,

$$\chi(\det B)\chi(\det B^{-1}x'')F((B^{-1}x'')^{-1}B^{-1}x') = \chi(\det x'')F(x''^{-1}x'),$$

and otherwise both sides vanish. By (1), $(\mathrm{tr}_B\Phi_0)(x) = (\mathrm{tr}_B\Phi)(x)$ if $B^{-1}x'' \in H$ and 0 otherwise, that is $\mathrm{tr}_B\Phi_0 = \Phi \cdot \mathbf{1}[x'' \in BH]$. If $\mathcal{B}$ is a set of representatives of $\mathrm{GL}_b(R)/H$, the cosets BH partition the invertible matrices, and Φ vanishes at the others; so

$$\Phi = \sum_{B\in\mathcal{B}} \mathrm{tr}_B\Phi_0, \qquad \mathrm{tr}_k\Phi_0 = \Phi_0 \ \ (k \in H). \tag{2}$$

Step 3: right translation of the kernels. Let $k \in \mathrm{K}$ and $\bar{k} = \mathrm{red}_{\mathfrak{m}}(k)$. By (6) of Lemma 4.1.1 (with the element 1 of the group of W and k of that of I_b), $\omega_{\alpha,\beta}(1,k)\,\mathrm{mod}_{\mathfrak{m}}(\Psi, v_P) = \alpha(\det k)\,\mathrm{mod}_{\mathfrak{m}}(\Psi(\bar{k}^{-1}\,\cdot\,), v_P)$. The finite components of $\det k$ are units and its infinite components are 1, so by the definition of the weight and tame part (Definition 3.1.2) and (XD4), $\alpha(\det k) = \prod_{w\in A}\psi_w(\det k_w \bmod w) = \chi(\det \bar{k})$. As $\omega_{\alpha,\beta}$ is a representation ((2) of Lemma 4.1.1),

$$\vartheta_\Psi(hk) = \vartheta_{\mathrm{tr}_{\bar{k}}\Psi}(h) \qquad (h \in \mathrm{U}(\mathrm{I}_b)(\mathbb{A}_{F,f}),\ k \in \mathrm{K}). \tag{3}$$

By (2) and (3), ϑ_Φ is right invariant under K and ϑ_{Φ_0} under K_H. The places of A satisfy $w \neq \bar{w}$, so the reduction $\mathrm{K} \to \mathrm{GL}_b(R)$ is onto (at each $w \in A$ the component of $\mathbf{K}_b$ above the place of F below w is $\mathrm{GL}_b(\mathcal{O}_{E,w})$ through ι_w); choose $k_B \in \mathrm{K}$ with $\mathrm{red}_{\mathfrak{m}}(k_B) = B$. By (2), linearity and (3),

$$\vartheta_\Phi(h) = \sum_{B\in\mathcal{B}} \vartheta_{\Phi_0}(hk_B). \tag{4}$$

Step 4: left invariance. Let $\delta \in \Gamma$, with archimedean and finite components δ_∞, δ_f. By Lemma 6.2.5, $\theta^{W,\mathrm{I}_b}_{\alpha,\beta}(\varphi)(g,(1,\delta_f h)) = \theta^{W,\mathrm{I}_b}_{\alpha,\beta}(\varphi)(g,(\delta_\infty^{-1},h))$ for every datum φ, and by (5) of Lemma 4.1.1, $\omega_{\alpha,\beta}(1,(\delta_\infty^{-1},1))\,\mathrm{mod}_{\mathfrak{m}}(\Psi, v_P) = \mathrm{mod}_{\mathfrak{m}}(\Psi, \Omega_{\gamma,0}(1,\delta_\infty^{-1})v_P) = \mathrm{mod}_{\mathfrak{m}}(\Psi, v_P)$ by the hypothesis on P (the second weight does not enter at the point 1 of the group of W). Hence $\vartheta_\Psi(\delta_f h) = \vartheta_\Psi(h)$ for every Ψ.

Step 5: the tower of class sums. Γ is discrete in $G = \mathrm{U}(\mathrm{I}_b)(\mathbb{A}_{F,f})$ (I_b is definite), and for an open compact K' and a function f left invariant under Γ and right invariant under K', with a Haar measure μ of G and the quotient measure on $\Gamma\backslash G$,

$$\int_{\Gamma\backslash G} f\,d\mu = \mu(K') \sum_{h \bmod K'}^{\mathrm{cl}} f(h) :$$

$\Gamma\backslash G$ is the finite disjoint union of the $\Gamma\backslash\Gamma h_{\mathfrak{c}}K'$, on which f is constant, and $\Gamma\backslash\Gamma h_{\mathfrak{c}}K'$ is the quotient of $h_{\mathfrak{c}}K'$ by the finite group $\Gamma\cap h_{\mathfrak{c}}K'h_{\mathfrak{c}}^{-1}$, of measure $\mu(K')/\#(\Gamma\cap h_{\mathfrak{c}}K'h_{\mathfrak{c}}^{-1})$. Apply this to $f=\vartheta_\Phi$, $K'=\mathrm{K}$ and to $f=\vartheta_{\Phi_0}$, $K'=\mathrm{K}_H$; K_H is open in K of index $\#\mathcal{B}$, as the preimage of H under the surjection onto $\mathrm{GL}_b(R)$. By (4) and the right invariance of μ,

$$\mu(\mathrm{K})\sum_{h\bmod \mathrm{K}}^{\mathrm{cl}}\vartheta_\Phi=\sum_{B\in\mathcal{B}}\int\vartheta_{\Phi_0}(hk_B)\,d\mu(h)=\#\mathcal{B}\,\mu(\mathrm{K}_H)\sum_{h\bmod \mathrm{K}_H}^{\mathrm{cl}}\vartheta_{\Phi_0},$$

and $\#\mathcal{B}\,\mu(\mathrm{K}_H)=\mu(\mathrm{K})$. □

Lemma 6.6.35. *Let $1\le j\le m$, let $i\in\mathrm{Mem}_j$, and write $\kappa=\kappa^+_{O_{j,1}}$. For $l\in I_\kappa$ let $e_l\in\mathbb{V}_j^+$ be the l-th unit vector and let $P_l\in\mathbb{C}[x]_{\mathcal{B}_j\times\{0,\dots,r_+-1\}}$ be the polynomial obtained from $\mathrm{Pol}_\kappa(e_l)$ by substituting $x_{\tau,b,c}$ for $y_{\tau,b,c}$ when $0\le c<r_+$ and 0 for $y_{\tau,b,c}$ when $c\ge r_+$ ($\tau\in\Phi$, $b\in\mathcal{B}_j$). Then, in $\mathbb{C}[x]_{\mathcal{B}_j\times\mathcal{L}_j}$,*

$$\Pi_i=\sum_{l\in I_\kappa}P_l\cdot\mathrm{lc}_{i,l},$$

both factors of each term read in $\mathbb{C}[x]_{\mathcal{B}_j\times\mathcal{L}_j}$ through the inclusions of $\{0,\dots,r_+-1\}$ and of $\mathcal{B}_j^{\mathrm{add}}$ in $\mathcal{L}_j$.

Proof. **The determinant indices have exactly the required heights.** Fix j and i, and put $\kappa=\kappa^+_{O_{j,1}}$. The formulas for the target parameter and (LF4) give

$$\kappa(\tau,h)=0\quad(1\le h<k-1\text{ or }h>r_+),\qquad\kappa(\tau,r_+)=\Delta',\quad r_+=k+j-1.\tag{1}$$

Here the weight is read in rank $|\mathcal{B}_j|$. We check the endpoints explicitly. Before position $k-1$, consecutive entries of the target parameter differ by one, so the positive part in (LF4) is zero. At $h=r_+$, the preceding entry is $\Delta/2-\kappa_j(\tau)$, and the maximum in (LF4) is $(t-1)/2-\kappa_j(\tau)$. Their difference minus one is $\Delta/2-(t+1)/2=\Delta'$. For $h>r_+$, the preceding entry has the form $\Delta/2-\kappa_{h+1-k}(\tau)$, with $h+1-k\ge j+1$. Since $\kappa_{j+1}=\kappa_j+\Delta$, it is at most $-\kappa_j-\Delta/2$, hence at most the threshold $(t-1)/2-\kappa_j$; the positive part is again zero. The gap inequalities used here follow from the admissible ranks and $\Delta=2r^8$, $t=5r$; in particular $\Delta>t+1$. Thus the index set $D_j(\tau)$ has one determinant of height h for each of the $\kappa(\tau,h)$ columns: the indices (h,c) provide those below r_+, and the Δ' extra indices provide those at r_+.

Clearing columns makes each factor a linear combination of minors. Let

$$R_{\mathrm{add}}=\mathbb{C}[x_{\tau,b,c}:\tau\in\Phi,\ b\in\mathcal{B}_j,\ c\in\mathcal{B}_j^{\mathrm{add}}].$$

The disjoint union defining $\mathcal{L}_j$ identifies the full polynomial ring with the polynomial ring over R_{add} in the first r_+ columns. This is a literal rearrangement of independent indeterminates.

Consider a determinant indexed by $d=(h,c)$. Its $2m$ constant rows are those of (XE6). In particular their entries in columns $0,\dots,h-1$ are zero. For each $0\le a<r_+-h$, constant row $a+1$ is exactly the unit row in column $h+a$. These rows exist: by (1), an occurring h satisfies $h\ge k-1$, and consequently

$$r_+-h\le j\le m\le 2m.$$

For each variable row b, subtract $x_{\tau,b,h+a}$ times constant row $a+1$, for all such a. These elementary row operations have determinant one. They leave the first h columns unchanged and set all variable entries in columns $h,\dots,r_+-1$ equal to zero. They leave the added columns unchanged, because each row used for clearing is a pure unit row in its indicated positive column.

Now expand along columns $0, \dots, h-1$. Only variable rows can occur there. For each set of h such rows, its contribution is the minor in those rows and the first h columns, times its complementary determinant and its Laplace sign. After clearing, that complementary determinant has entries in R_{add}. Hence this factor is an R_{add}-linear combination of the height-h minors of (WM1). For an extra index $d \in \{1, \dots, \Delta'\}$, all constant entries in the first r_+ columns vanish. The identical Laplace expansion, without a clearing operation, gives an R_{add}-linear combination of the height-r_+ minors. Both expansions use the original row and column orders and therefore retain every determinant sign, including the fixed bijection ϵ_j.

Multiplying over all determinants and using (1) proves that Π_i, as a polynomial over R_{add} in the first columns, is an R_{add}-linear combination of products of minors with exactly the column counts of κ. By (WM1), these products have the standard basis (b_l). Every such basis polynomial uses only columns $c < r_+$, since every occurring height is at most r_+. Write $\widehat{b_l}$ for its renaming to the first-column variables x. Extension of the standard basis from $\mathbb{C}$ to the polynomial coefficient ring therefore gives coefficients $a_{\mathrm{bas},l} \in R_{\mathrm{add}}$ with

$$\Pi_i = \sum_l a_{\mathrm{bas},l} \widehat{b_l}. \tag{2}$$

No evaluation of the added variables, choice of theta functional, or passage to a fraction field is involved.

The Fock pairing gives exactly the line coordinates. Let H_κ be the matrix with entries $(H_\kappa)_{ls} = \langle b_l, b_s \rangle_{\mathrm{F}}$. The pairing is bilinear, not hermitian, and extends R_{add}-bilinearly when only the first columns are differentiated. Explicitly,

$$\langle X^\alpha, X^\beta \rangle_{\mathrm{F}} = \mathbf{1}_{\alpha=\beta} \alpha!, \qquad \alpha! = \prod_{\tau,b,c} \alpha_{\tau,b,c}!. \tag{3}$$

Thus the coefficients in R_{add} are not differentiated. Renaming the first-column variables does not change (3). By Lemma 4.6.9,

$$G_\kappa H_\kappa = h_\kappa I. \tag{4}$$

Its hypotheses are precisely the products-of-minors basis, filling Gram matrix and ordinary Fock pairing of (WM1)–(WM5), in the same rank $|\mathcal{B}_j|$ used here. The scalar h_κ is nonzero: the highest polynomial is a nonzero polynomial with integer coefficients, so its self-pairing is the positive sum of its squared coefficients times the monomial factorials in (3).

Apply (3) to (2). The definition (XE6) then reads, as an equality of vectors over R_{add},

$$(\mathrm{lc}_{i,l})_l = h_\kappa^{-1} H_\kappa (a_{\mathrm{bas},l})_l. \tag{5}$$

This uses exactly the hook number of (WM5); no product of row or column factorials is cancelled or inserted.

The polynomial reconstruction is the claimed identity. By (WM4) and the definition of P_l,

$$P_l = \sum_s (G_\kappa)_{sl} \widehat{b_s}. \tag{6}$$

The substitution of zero for $c \geq r_+$ makes no further change to these basis polynomials, as their column support was checked above. Substituting (5) and (6), and then using (4), gives

$$\begin{aligned} \sum_l P_l \, \mathrm{lc}_{i,l} &= \sum_s \widehat{b_s} \big(G_\kappa h_\kappa^{-1} H_\kappa (a_{\mathrm{bas},l})_l \big)_s \\ &= \sum_s \widehat{b_s} \, a_{\mathrm{bas},s} = \Pi_i. \end{aligned}$$

Every equality is in $R_{\mathrm{add}}[x_{\tau,b,c} : c < r_+]$, and its canonical identification with the ring on $\mathcal{L}_j$ is the two inclusions in the statement. This proves the assertion for the fixed choice and hence for every j and i in the stated ranges. □

Proposition 6.6.36. *Let $1 \le j \le m$ and $r_+ = k + j - 1$. For a twist ϑ with modulus Q let $\mathfrak{n} = \mathfrak{M}^2 \prod_{w \in Q} w$, let χ_ϑ be the character of $R_\mathfrak{n}$ given on units by the tame part of $\chi'_{j,\vartheta}$, $u \mapsto \prod_{w \in S_0 \cup Q} \psi'_w(u \bmod w)$, and by 0 on nonunits, and for $\Lambda \in \mathrm{M}_{\mathcal{B}^{\mathrm{c}} \times \mathcal{B}_j}(\mathcal{O}_E/\mathfrak{M}^2)$ let $\psi^{\mathrm{inv}}_\Lambda(y) = \sum_{\gamma \in \mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(\mathcal{O}_E/\mathfrak{M}^2)} \psi_\Lambda(\bar{y}\gamma)$ for $y \in \mathrm{M}_{\mathcal{B}_j \times \mathcal{B}^{\mathrm{c}}}(R_\mathfrak{n})$, $\bar{y}$ its reduction modulo $\mathfrak{M}^2$. Let $W = \mathrm{I}_{\mathcal{B}^{\mathrm{c}}} \oplus W'_j$, with index set $\mathcal{B}^{\mathrm{c}} \sqcup \mathcal{B}_j$; for $X = (X' \mid X'') \in \mathrm{M}_{\mathcal{B}_j \times (\mathcal{B}^{\mathrm{c}} \sqcup \mathcal{B}_j)}(R_\mathfrak{n})$ put $\Phi_{\vartheta,i}(X) = \chi_\vartheta(\det X'')\, \psi^{\mathrm{inv}}_{\Lambda_i}(X''^{-1}X')$ if X'' is invertible and 0 otherwise. Let $\pi_j \colon \mathcal{L}_j \to \mathcal{B}^{\mathrm{c}} \sqcup \mathcal{B}_j$ send $c \in \{0, \dots, r_+ - 1\}$ to the basis vector of W_j of number c and be the inclusion on $\mathcal{A}^+_j$, $\mathcal{B}^\flat$ and $\mathcal{B}^{\mathrm{c}}$, and let Π^W_i be Π_i with every variable $x_{\tau,b,c}$ renamed $x_{\tau,b,\pi_j(c)}$. The **left side** of the third exchange is the function on $\mathrm{U}(W_j)(\mathbb{A}_F)$*

$$L_{\vartheta,i}(g) = \sum_{h' \bmod \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})}^{\mathrm{cl}} \mathsf{a}_{\chi'_{j,\vartheta}}(h') \sum_{h \bmod \mathbf{K}_0(\mathcal{B}_j)}^{\mathrm{cl}} \theta^{W,V''_j}_{\chi'_{j,\vartheta}\nu^\flat_j \upsilon^m,\, \nu''_{j,\vartheta};\,\mathfrak{n}}\big[\Phi_{\vartheta,i}, \Pi^W_i\big]\big(\mathrm{diag}(h', g, 1, x_{\mathfrak{c}_i}), h\big),$$

$\mathrm{diag}(h', g, 1, x_{\mathfrak{c}_i})$ the block diagonal point with components h' on $\mathcal{B}^{\mathrm{c}}$, g on $\mathcal{R}$, 1 on $\mathcal{A}^-_j \sqcup \mathcal{A}^+_j$ and $x_{\mathfrak{c}_i}$ on $\mathcal{B}^\flat$. Then there are $C \in \mathbb{R}$ and, for every twist ϑ, a number $c_\vartheta \in \mathbb{C}$ with

$$\Gamma_j(\vartheta)\, |c_\vartheta|^2 = C \qquad \text{and} \qquad c_\vartheta\, L_{\vartheta,i}(g) = \mathsf{F}_i(g) \quad \text{for all } i \in \mathrm{Mem}_j,\ g \in \mathrm{U}(W_j)(\mathbb{A}_F).$$

Proof. Fix ϑ and write $\chi' = \chi'_{j,\vartheta}$, $\nu'' = \nu''_{j,\vartheta}$, $\beta^{\mathrm{c}} = \nu^\flat_j \upsilon^m$ (the carrier character) and $\alpha_e = \chi'\beta^{\mathrm{c}}$; by the sidecalc of (XE1), $\nu'' = \chi'\nu^\flat_j$, so $\alpha_e = \nu''\upsilon^m$. Let $\mathbf{K}^{\mathrm{c}} = \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$ and $N = \#\mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(\mathcal{O}_E/\mathfrak{M}^2)$.

Step 0: characters. By (a) and (b) of Lemma 6.6.23 (whose $\mathfrak{n}$, χ_ϑ, α_e, W, π_j and Π^W_i are those of the statement): (i) $\mathfrak{n}$ is a modulus with $\Sigma(\mathfrak{n}) = S_0 \cup Q$; (ii) χ' is tame of weight 0 for $(\mathfrak{n}, \chi_\vartheta)$; (iv) α_e is tame of weight $\tau \mapsto m - \#D_j(\tau)$ for $(\mathfrak{n}, \chi_\vartheta)$; and β^{c} has trivial tame part. (iii) Hence $\beta^{\mathrm{c}}(\det k) = 1$ for $k \in \mathbf{K}_0(n)$, while $\chi'(\det k) = \nu''(\det k) = \alpha_e(\det k) = \chi_\vartheta(\det \bar{k})$, $\bar{k}$ the reduction modulo $\mathfrak{n}$ ((ii) of Lemma 3.2.11, Definition 3.1.1).

Step 1: the matched polynomial of the constant rows. Let $n = \mathcal{B}^{\mathrm{c}} \sqcup \mathcal{B}_j$ (index set of W), $l = \{1, \dots, m\}$, $p = \{1, \dots, 2m\}$, and let $\jmath_W \colon \mathcal{L}_j \sqcup l \to n$ be π_j on $\mathcal{L}_j$ and ng_j on l. By (c) of Lemma 6.6.23, $\jmath_W$ is a bijection, W has the signs $\mathrm{sgn}_{\jmath_W}$, and $\Delta^{\jmath_W}(\epsilon_j; \mathsf{c}_i) = \Pi^W_i$ (with $e = \epsilon_j$, $\iota_\tau = D_j(\tau)$ and $\eta = \mathsf{c}_i$). Let $f \colon p \sqcup l \to \mathcal{B}^{\mathrm{c}}$ be $a \mapsto (1, a)$, $i \mapsto (2, i)$; S^{c} has the signs sgn_f. In the matched polynomial $Q_{f,\jmath_W}(\mathsf{c}_i)$, the exponent of $\det y^{--}_\tau$ is $\#D_j(\tau) + |n| - 2m = \#D_j(\tau) + |\mathcal{B}_j| + m$, and $y^{--}_\tau = (x_{\tau,\mathrm{ng}_j(i),(2,i')})_{i,i'} = y^-_\tau$. For $d = (h, c) \in D_j(\tau)$, row a of $\mathsf{c}_{i,\tau,d}$ is the unit vector at the column $h + a - 1$ or at $a \in \mathcal{A}^+_j$, so row a of $\mathsf{c}_{i,\tau,d}\, y^{++}_\tau$ is $(x_{\tau,\mathrm{ps}_j(h,a),(1,a')})_{a'}$ and $\det(\mathsf{c}_{i,\tau,d} y^{++}_\tau) = \det y_{\tau,h}$; for $d \in \{1, \dots, \Delta'\}$, $\mathsf{c}_{i,\tau,d} y^{++}_\tau = \tau(Y_{\mathfrak{c}_i})[\lambda_{i,\tau}(d, \cdot)](x_{\tau,b,(1,a)})_{b \in \mathcal{B}^\flat, a}$, as the constant rows vanish in the columns $\mathcal{B}^{\mathrm{c}}$. Hence $Q_{f,\jmath_W}(\mathsf{c}_i) = P_j P^\flat_i$, read in the variables of the rows $\mathcal{B}_j$ of n.

Step 2: the exchange identity for $(W, V''_j, S^{\mathrm{c}})$. Fix i and g. For $h' \in G_f(\mathcal{B}^{\mathrm{c}})$ and a function Φ on $\mathrm{M}_{\mathcal{B}_j \times n}(R_\mathfrak{n})$ let $K_{h'}(\Phi)(h)$ be the mean over $k \in \mathbf{K}_0(\mathcal{B}_j)$ (Haar probability) of $\theta^{W,V''_j}_{\alpha_e,\nu'';\mathfrak{n}}[\Phi, \Pi^W_i](\mathrm{diag}(h', g, 1, x_{\mathfrak{c}_i}), hk)$; it is right $\mathbf{K}_0(\mathcal{B}_j)$-invariant and, by Lemma 6.2.5, left $\mathrm{U}(V''_j)(F)$-invariant. Let Φ, Ψ be matched for $(\mathrm{id}_n, \chi_\vartheta^{-1})$: Φ vanishes off the matrices $(k^{-1})_{\mathcal{B}_j}$ (rows $\mathcal{B}_j$ of k^{-1}, $k \in \mathrm{GL}_n(R_\mathfrak{n})$) and $\Phi((k^{-1})_{\mathcal{B}_j}) = \chi_\vartheta^{-1}(\det k)\Psi(k_{\mathcal{B}^{\mathrm{c}}})$. For $\kappa \in \mathbf{K}_0(\mathcal{B}_j)$ and $\kappa' = \mathrm{diag}(1_{\mathcal{B}^{\mathrm{c}}}, \bar{\kappa})$, $\bar{\kappa}^{-1}(k^{-1})_{\mathcal{B}_j} = ((k\kappa')^{-1})_{\mathcal{B}_j}$ and $(k\kappa')_{\mathcal{B}^{\mathrm{c}}} = k_{\mathcal{B}^{\mathrm{c}}}$, so $\Phi(\bar{\kappa}^{-1}X) = \chi_\vartheta(\det \bar{\kappa})^{-1}\Phi(X)$ for all X; the same law holds for $\Phi_{\vartheta,i}$ directly from its definition. Hence $\alpha_e(\det \kappa)\Phi(\bar{\kappa}^{-1}X) =$

$\Phi(X)$ (Step 0 (iii)); and $v_{\Pi_i^W}$ is fixed by $\Omega(1,y)$ for $y \in G_\infty(1_{\mathcal{B}_j})$ by (d) of Lemma 6.6.23 (for the weights of α_e, which is $m - \#D_j(\tau) = |l| - |\iota_\tau|$ by Step 0 (iv), and of ν''). By (4) of Lemma 6.3.71, $K_{h'}(\Phi)$ is the kernel itself, for these Φ and for $\Phi_{\vartheta,i}$. Now Proposition 6.3.67 applies with W, $V' = S^{\mathrm{c}}$ (index set $\mathcal{B}^{\mathrm{c}}$), $\mathrm{I}_a = V_j''$, the bijections $\jmath_W$, f, $e = \epsilon_j$ and $\sigma = \mathrm{id}_n$, $d = z_* = \frac{7m}{2} + \frac{k}{2}$ ($|\mathcal{B}_j| = r + 9m + 1 = 3m + 2z_*$, and $2|l| + 2 \le |n|$), $\iota_\tau = D_j(\tau)$, the modulus $\mathfrak{n}$, the character χ_ϑ, $\alpha = \alpha_e$ (Step 0 (iv)), $\beta = \nu''$ and $\eta = \mathsf{c}_i$, at the point $\mathrm{diag}(h', g, 1, x_{\mathsf{c}_i})$: there are $C_3 \in \mathbb{R}$, independent of ϑ (the index data do not involve ϑ), and $c_3 = c_3(\vartheta)$ with $L_*(\mathfrak{n}, \alpha_e^{-1}\upsilon^{3m+z_*}; z_*)\,|c_3|^2 = C_3$ and, by Step 1 and $\alpha_e/\nu'' = \upsilon^m$,

$$c_3 \sum_{h \bmod \mathbf{K}_0(\mathcal{B}_j)}^{\mathrm{cl}} K_{h'}(\Phi)(h) = \int_{[\mathrm{U}(S^{\mathrm{c}})]} \theta^{S^{\mathrm{c}},W}_{\upsilon^m,\alpha_e;\mathfrak{n}}\big[\Psi, \widetilde{P_j P_i^\flat}\big]\big(x, \mathrm{diag}(h', g, 1, x_{\mathsf{c}_i})\big)\,dx.$$

Here Ψ vanishes off the matrices $k_{\mathcal{B}^{\mathrm{c}}}$, which have left inverses, so the integrand is integrable by (2) of Lemma 6.3.71. This is the exchange identity of Lemma 6.6.63 for $V' = S^{\mathrm{c}}$, $b = \mathcal{B}^{\mathrm{c}}$, $W' = W_j'$, $a = \mathcal{B}_j$, the point $(g, 1, x_{\mathsf{c}_i})$ of $\mathrm{U}(W_j')$, $c = c_3$, $\alpha = \upsilon^m$, $\beta = \chi'$, $\beta' = \beta^{\mathrm{c}}$ and $Q = P_j P_i^\flat$.

Step 3: the one exchange against the carrier. By (a) of Lemma 6.6.62 for $\mathfrak{n}$, χ_ϑ and $c = \chi'$ (Step 0 (ii)) there is $\gamma = \gamma(\vartheta)$ with $L_*(\mathfrak{n}, \chi'^{-1}\upsilon^m; m)\,L_*(\mathfrak{n}, \chi'\upsilon^{m/2}; m/2)\,|\gamma|^2 = C_{12}$, C_{12} independent of ϑ, and $\gamma \sum_{h'} \omega_{h'}\kappa_{h'} = \varphi^\circ$, where h' runs over fixed representatives of the classes of $\mathrm{I}_{\mathcal{B}^{\mathrm{c}}}$ at the level $\mathbf{K}^{\mathrm{c}}$, $\omega_{h'} = \mathsf{a}_{\chi'}(h')/\#(\mathrm{U}(\mathrm{I}_{\mathcal{B}^{\mathrm{c}}})(F) \cap h'\mathbf{K}^{\mathrm{c}}h'^{-1})$ and $\kappa_{h'}(x) = \theta^{S^{\mathrm{c}},\mathrm{I}_{\mathcal{B}^{\mathrm{c}}}}_{\upsilon^m,\chi';\mathfrak{n}}[\chi_\vartheta \circ \det, 1](x, h')$, the function $\kappa_{h'}$ of Lemma 6.6.63. The function $\Psi' = \psi^{\mathrm{inv}}_{\Lambda_i}$ satisfies $\Psi'(y\gamma') = \Psi'(y)$ for $\gamma' \in \mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(R_{\mathfrak{n}})$ (substitute $\bar\gamma'\gamma$ for γ), and $\Phi_{\Psi'} = \Phi_{\vartheta,i}$. So (ii) of Lemma 6.6.63, with $c_0 = \gamma$ and $F = \varphi^\circ$, gives

$$\begin{aligned} &c_3\gamma \sum_{h'} \omega_{h'} \sum_{h}^{\mathrm{cl}} \theta^{W,V_j''}_{\alpha_e,\nu'';\mathfrak{n}}\big[\Phi_{\vartheta,i}, \Pi_i^W\big]\big(\mathrm{diag}(h', g, 1, x_{\mathsf{c}_i}), h\big) \\ &\qquad = \int_{[\mathrm{U}(S^{\mathrm{c}})]} \varphi^\circ(x)\,\theta^{S^{\mathrm{c}},W_j'}_{\upsilon^m,\beta^{\mathrm{c}};\mathfrak{n}}\big[\psi^{\mathrm{inv}}_{\Lambda_i}, P_j P_i^\flat\big]\big(x, (g, 1, x_{\mathsf{c}_i})\big)\,dx. \end{aligned} \tag{6.142}$$

The left side is $c_3\gamma\, L_{\vartheta,i}(g)$: the summand of the outer class sum of $L_{\vartheta,i}$ is constant on the classes, since for $k \in \mathbf{K}^{\mathrm{c}}$, (6) of Lemma 4.1.1 gives $\mathsf{a}_{\chi'}(h'k) = \chi'(\det k)\mathsf{a}_{\chi'}(h')$ (datum $v_1 \otimes v^0$, first character χ') and multiplies the inner kernel by $\nu''(\det k)^{-1}$ ($\Phi_{\vartheta,i}(X'\bar k \mid X'') = \Phi_{\vartheta,i}(X)$ by the invariance of ψ^{inv}), and $\chi'(\det k) = \nu''(\det k)$ (Step 0 (iii)).

Step 4: the right side is $N\,\mathsf{F}_i(g)$. By Lemma 6.4.66 the datum of $\psi^{\mathrm{inv}}_{\Lambda_i}$ modulo $\mathfrak{n}$ is that of $\sum_\gamma \psi_{\Lambda_i}(\cdot\,\gamma)$ modulo $\mathfrak{M}^2$, and the kernel is linear in the function. For $\gamma \in \mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(\mathcal{O}_E/\mathfrak{M}^2)$ choose $k_\gamma \in \mathbf{K}^{\mathrm{c}}$ reducing to γ: the chosen places are split ($w \ne \bar w$) and pairwise not conjugate, and at such a pair $(w, \bar w)$ an element of $G_f(\mathcal{B}^{\mathrm{c}})$ is given by an arbitrary component in $\mathrm{GL}_{\mathcal{B}^{\mathrm{c}}}(\mathcal{O}_{E,w})$, the component at $\bar w$ being determined by $g^*g = 1$; take the components at the chosen places lifting γ and 1 elsewhere. By (6) of Lemma 4.1.1 and Step 0 (iii), $\theta^{S^{\mathrm{c}},W_j'}_{\upsilon^m,\beta^{\mathrm{c}};\mathfrak{M}^2}[\psi_{\Lambda_i}(\cdot\,\gamma), P](x, \cdot) = \theta^{S^{\mathrm{c}},W_j'}_{\upsilon^m,\beta^{\mathrm{c}};\mathfrak{M}^2}[\psi_{\Lambda_i}, P](xk_\gamma, \cdot)$. The quotient measure of $[\mathrm{U}(S^{\mathrm{c}})]$ is right invariant, and $\varphi^\circ(xk_\gamma^{-1}) = \varphi^\circ(x)$ by (b) of Lemma 6.6.62 (χ' is tame of weight 0); so each of the N translates contributes $\mathsf{F}_i(g)$ (XE5), and the right side of (6.142) is $N\,\mathsf{F}_i(g)$.

Step 5: the constant. Put $c_\vartheta = c_3\gamma/N$; then $c_\vartheta L_{\vartheta,i}(g) = \mathsf{F}_i(g)$ for all i and g. Let $p_j^* = (\nu_*, z_*)$, $p_j^\pm = (\nu_\pm, z_\pm)$. The characters agree:

$\alpha_e^{-1}\upsilon^{3m+z_*} = (\nu_*\vartheta)^{-1}$, $\chi'^{-1}\upsilon^m = (\nu_+\vartheta)^{-1}$, $\chi'\upsilon^{m/2} = (\nu_-\vartheta^{-1})^{-1}$, and $\Sigma(\mathfrak{n}) = S_0 \cup Q$. Hence, by (6.129) and $|c_\vartheta|^2 = |c_3|^2|\gamma|^2N^{-2}$,

$$\begin{aligned} \Gamma_j(\vartheta)\,|c_\vartheta|^2 = &\Big(L_*(\mathfrak{n}, \alpha_e^{-1}\upsilon^{3m+z_*}; z_*)\,|c_3|^2\Big) \\ &\cdot \Big(L_*(\mathfrak{n}, \chi'^{-1}\upsilon^m; m)\,L_*(\mathfrak{n}, \chi'\upsilon^{m/2}; m/2)\,|\gamma|^2\Big)N^{-2} = C_3C_{12}N^{-2}, \end{aligned}$$

which is the claim with $C = C_3 C_{12} N^{-2}$. □

Lemma 6.6.37. *Let $1 \le j \le m$ and $y \in \mathcal{A}_M(W_j)$. For $k = 1, 2$ let a_k be a finite index set, $\mathfrak{m}_k$ a modulus, α_k, β_k automorphic characters of $\mathrm{U}(1)$ whose tame parts are trivial at every finite place outside $\Sigma(\mathfrak{m}_k)$, Φ_k a function on $\mathrm{M}_{a_k \times \mathcal{R}}(R_{\mathfrak{m}_k})$, $P_k \in \mathbb{C}[x]_{a_k \times \mathcal{R}}$ and $h_k \in G_f(a_k)$, and put $\kappa_k(u) = \theta^{W_j, \mathrm{I}_{a_k}}_{\alpha_k, \beta_k; \mathfrak{m}_k}[\Phi_k, P_k](u, h_k)$. Then the function $u \mapsto y(u)\,\kappa_1(u)\,\kappa_2(u)$ on $\mathrm{U}(W_j)(\mathbb{A}_F)$ is continuous, left invariant under the rational points and integrable on $[\mathrm{U}(W_j)]$.*

Proof. **The principal-level form has rapid decay.** The space W_j has rank r and the diagonal signs of (HP4). Its target parameter ς has gap Δ for these signs by Proposition 5.8.4(iii). We check that the type kernel is a cusp kernel of order zero. The rank r is odd and m is even by (AD2), so $k = r + 1 - m$ is even. The displayed formula for ς in Definition 3.2.16 therefore has integral entries. Consequently the entries of 2ς are congruent to $r - 1$ modulo 2. They are distinct, and entries belonging to opposite signs differ by at least 2Δ. By Lemma 4.2.67(i) and $r \ge x^5$ one has $r \ge 4$, whence

$$\Delta = 2r^8 \ge 500 r^3.$$

Thus 2ς has gap $500r^3$ in (DZ1) of Definition 5.6.3. Applying Lemma 5.6.21 with $B = 0$ shows that e_ς is a cusp kernel of order zero, for the same product of Cayley measures used here.

By the definition of $\mathcal{A}_M(W_j)$, the function y is continuous, bounded, invariant under the rational points, invariant under its principal open level, and reproduced by e_ς. It is therefore a form of this type in (ED2). By Lemma 6.3.75, for every integer $A \ge 0$ there is a finite constant $C_{y,A} \ge 0$ such that

$$|y(u)|\,\mathrm{ht}_{W_j}(u)^A \le C_{y,A} \qquad (u \in \mathrm{U}(W_j)(\mathbb{A}_F)).$$

This argument uses no projection onto good forms.

The two kernels have polynomial growth. For each $k \in \{1, 2\}$ put

$$\varphi_k = \mathrm{mod}_{\mathfrak{m}_k}(\Phi_k, v_{P_k}) \in \mathcal{M}_{W_j, \mathrm{I}_{a_k}}.$$

This is the datum specified by (XD5) and Definition 6.3.1; in particular its theta-model norm is finite. Read the finite point h_k as $(1, h_k)$ at the adelic places. By Lemma 6.2.28 there are $Y_k \ge 0$ and $A_k \in \mathbb{N}$ such that

$$|\kappa_k(u)| \le Y_k \|\varphi_k\|\,\mathrm{ht}_{W_j}(u)^{A_k} \mathrm{ht}_{\mathrm{I}_{a_k}}((1, h_k))^{A_k} = C_{\mathrm{ker},k}\,\mathrm{ht}_{W_j}(u)^{A_k},$$

where the last equality defines a finite nonnegative constant $C_{\mathrm{ker},k}$. The target point is fixed, so its height is a finite number. This also covers $a_k = \emptyset$: its target group is trivial and its height is 1.

The joint continuity of the Weil action in Lemma 4.1.1(2), continuity of the pair map and the automorphic characters, and boundedness of the canonical theta functional show that each κ_k is continuous. By Lemma 6.2.5 it is left invariant under the rational points of W_j. Hence the product with y is continuous and rationally left invariant, and descends to a continuous, therefore measurable, function on $[\mathrm{U}(W_j)]$.

A bounded majorant proves absolute integrability. Take $A = A_1 + A_2$ in the rapid-decay bound. For every u,

$$|y(u)\kappa_1(u)\kappa_2(u)| \le C_{y,A_1+A_2}\, C_{\mathrm{ker},1} C_{\mathrm{ker},2}.$$

The CM extension is E/F and W_j is a nondegenerate hermitian space of positive rank r. The adelic finite-volume theorem applies to this unitary group with the product Haar measure fixed

in (XD1), and to its quotient by counting measure on the rational points; see [37] and [233], Chapter 5. Thus this quotient has finite measure. Integrating the displayed constant majorant proves

$$\int_{[\mathrm{U}(W_j)]} |y(u)\kappa_1(u)\kappa_2(u)|\,du < \infty.$$

This is the asserted integrability, with the continuity and invariance already established. □

Lemma 6.6.38. *Let $1 \le j \le m$. Then $\Gamma_j(\vartheta) > 0$ for every twist ϑ.*

Proof. Fix a twist ϑ with modulus Q and tame part ψ_ϑ, fix $i \in \{*, +, -\}$, and put $\nu = (\nu_i \vartheta^{s_i})^{-1}$, $z = z_i$, $S = S_0 \cup Q$. It suffices to show $L_*(S, \nu; z) > 0$. Since p_j^i is a critical pair, z is an integer with $z \ge 1$, so by (XD7)

$$L_*(S, \nu; z) = \Big(\prod_{w \in S} \mathrm{N}w\Big)^{z^2} \prod_{z'=1}^{z} |L^S(z', \tilde{\nu})|^2,$$

and it suffices to show $L^S(z', \tilde{\nu}) \ne 0$ for every integer $z' \ge 1$.

The base change is a unitary Hecke character that is ramified at a chosen place. The map $x \mapsto x/\bar{x}$ is a continuous homomorphism $\mathbb{A}_E^\times \to \mathrm{U}(1)(\mathbb{A})$ carrying $E^\times$ into $\mathrm{U}(1)(E)$, so $\tilde{\nu}(x) = \nu(x/\bar{x})$ is a unitary Hecke character of E. If μ, μ' have weights e, e' and tame parts ψ, ψ', then $\mu\mu'$ has weight $e + e'$ and tame part $\psi\psi'$, and μ^{-1} has weight $-e$ and tame part ψ^{-1} (the defining identity is multiplicative). The tame parts of ν_*, ν_+, ν_- are $\omega^j, \omega^j, \omega^{-j}$, and that of ϑ^{s_i} is $\psi_\vartheta^{s_i}$; hence ν has the tame part $\psi = \omega^{-\epsilon j} \psi_\vartheta^{-s_i}$ with $\epsilon = 1$ for $i \in \{*, +\}$ and $\epsilon = -1$ for $i = -$. Since $M = \pi_M \bar{\pi}_M$ is not a unit, π_M lies in some prime ideal, so there is a chosen place w. By (a) of Lemma 4.6.2, $w \ne \bar{w}$, $\bar{w}$ is not a chosen place, and w lies above p_0 or p_1; so $\mathrm{N}w = \mathrm{N}\bar{w}$ is a power of a prime dividing M, and as $\mathrm{N}Q$ is prime to M, neither w nor $\bar{w}$ lies in Q. Therefore $\psi_{\vartheta,w} = \psi_{\vartheta,\bar{w}} = 1$, $\omega_{\bar{w}} = 1$, and $\psi_w = \omega_w^{-\epsilon j} \ne 1$ because ω_w has order q_2 and $1 \le j \le m < q_2$ ($q_2 = (q_0 d)^{s+6} \ge 3^s > 2^{e+1} = m$ with $s = 22(e+1)$ and $m = 2^{e+1}$ by (AD2) of Definition 2.3.4 and Definition 2.3.1, $d \ge 3$ as $-d \equiv 1 \pmod 4$). Choose $u \in \mathcal{O}_{E,w}^\times$ with $\psi_w(u \bmod w) \ne 1$, and let $x \in \mathbb{A}_E^\times$ have the component u at w and 1 at every other place. Conjugation maps E_w onto $E_{\bar{w}}$, so $\bar{x}$ has the component $\bar{u} \in \mathcal{O}_{E,\bar{w}}^\times$ at $\bar{w}$ and 1 elsewhere, and $x/\bar{x} \in \mathrm{U}(1)(\mathbb{A})$ has the components u at w, $\bar{u}^{-1}$ at $\bar{w}$ and 1 elsewhere; its finite components are integral and its infinite components are 1. By the defining identity of weight and tame part,

$$\tilde{\nu}(x) = \nu(x/\bar{x}) = \psi_w(u \bmod w)\, \psi_{\bar{w}}(\bar{u}^{-1} \bmod \bar{w}) = \psi_w(u \bmod w) \ne 1.$$

As $|x|_{\mathbb{A}} = 1$, $\tilde{\nu}$ is not of the form $|\cdot|_{\mathbb{A}}^{i\tau}$ for any $\tau \in \mathbb{R}$.

Nonvanishing. By (3) of Theorem A.1.12, applied to the number field E and the unitary Hecke character $\tilde{\nu}$ (Hecke [224], VII.8), the continuation of $L(s, \tilde{\nu})$ is holomorphic and has no zeros in $\mathrm{Re}\, s \ge 1$. For $\mathrm{Re}\, s > 1$,

$$L^S(s, \tilde{\nu}) = L(s, \tilde{\nu}) \prod_{w'} (1 - \tilde{\nu}(\varpi_{w'})\, \mathrm{N}w'^{-s}),$$

the product over the places $w' \in S \cup \bar{S}$ at which $\tilde{\nu}$ is unramified; the right side is holomorphic on $\mathrm{Re}\, s \ge 1$, so it is the continuation of $L^S(s, \tilde{\nu})$ there. At an integer $z' \ge 1$ each factor satisfies $|\tilde{\nu}(\varpi_{w'})\mathrm{N}w'^{-z'}| = \mathrm{N}w'^{-z'} < 1$, hence is nonzero, and $L(z', \tilde{\nu}) \ne 0$. So $L^S(z', \tilde{\nu}) \ne 0$, and $L_*(S, \nu; z) > 0$. The product of the three positive numbers is $\Gamma_j(\vartheta) > 0$. □

Lemma 6.6.39. *Let $1 \le j \le m$, Y a short row family, $c'' \in \mathbb{C}^\times$, and $L = (L_i)_{i \in \mathrm{Mem}_j}$ divided lattices of $O_{j,1}$ with L_i glued for Λ_i, and suppose that $\mathcal{V}_j$ is a divided splitting for $O_{j,\vartheta}$ for every twist ϑ. Let ϑ be a twist, $\mathrm{gl}_\vartheta = \mathrm{gl}_{j,\vartheta}[L; c'']$ (formed with Y), and w a finite place of E with $w \ne \bar{w}$ such that neither w nor $\bar{w}$ is a place of $O_{j,\vartheta}$. Then for every $\varphi \in \mathcal{F}_w(e_0, \mathcal{L}_{j,\vartheta}^\perp)$ and every $i \in \mathrm{Mem}_j$, the table $\zeta = \mathrm{gl}_\vartheta^* \varphi$ satisfies*

$$\mathrm{tt}_j(\zeta_i)(g) = 0 \qquad (g \in \mathrm{U}(W_j)(\mathbb{A})), \qquad \zeta_i(\mathfrak{c}) = \zeta_{i,\mathfrak{c}} \ (\mathfrak{c} \in \mathrm{Cl}^\circ).$$

Proof. Write $O = O_{j,\vartheta}$, $\mathcal{V} = \mathcal{V}_j$, $\ell = \mathrm{loss}(\mathcal{V})$, $b = \upsilon^{\lfloor m/2 \rfloor - j}$, and $\kappa^- = \kappa_O^-$ (free of ϑ). As in (TG6), each L_i is a divided lattice of O. Fix φ, i and g, and for $\mathfrak{c} \in \mathrm{Cl}^\circ$ let $\bar{k}_\mathfrak{c} \in \mathbb{V}_j^-$ have the conjugate coordinates of $\mathrm{kv}_\mathfrak{c} = \mathrm{kv}_{O,b;\mathfrak{M}^2}[\mathbf{1}_{z_j}; \kappa^-](g, x_\mathfrak{c})$; it does not depend on ϑ, since $W(O) = W_j$, $\chi_O = \upsilon^{m-j}$, Let_{-1} and κ^- do not.

Step 1: the theta kernel pairing as an inner product with a table. For $\xi \in \mathbb{V}_j^-$, $\mathrm{Pol}_{\kappa^-}(\bar{\xi}) = \sum_l (G_{\kappa^-}\bar{\xi})_l \, b_l$, and the kernels are linear in the polynomial; by (KN4),

$$\theta^{W_j, V^\circ}_{\chi_O, b; \mathfrak{M}^2}[\mathbf{1}_{z_j}, \mathrm{Let}_{-1}(b_l)](g, x_\mathfrak{c}) = \overline{(\mathrm{kv}_\mathfrak{c})_l}.$$

As G_{κ^-} is a real symmetric matrix,

$$\theta^{W_j, V^\circ}_{\chi_O, b; \mathfrak{M}^2}\big[\mathbf{1}_{z_j}, \mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa^-}(\bar{\xi}))\big](g, x_\mathfrak{c}) = \overline{\textstyle\sum_l (G_{\kappa^-}\xi)_l (\mathrm{kv}_\mathfrak{c})_l} = \overline{\langle \bar{k}_\mathfrak{c}, \xi \rangle_{\kappa^-}}.$$

Let $\eta \in \mathrm{Tab}_j$ have $\eta_{i,\mathfrak{c}} = \bar{k}_\mathfrak{c}$ and $\eta_{i',\mathfrak{c}} = 0$ for $i' \ne i$. Summing over $\mathfrak{c}$ ((XE7)) gives

$$\overline{\mathrm{tt}_j(\zeta_i)(g)} = \sum_{\mathfrak{c} \in \mathrm{Cl}^\circ} \langle \bar{k}_\mathfrak{c}, \zeta_{i,\mathfrak{c}} \rangle_{\kappa^-} = \langle \eta, \zeta \rangle = \langle \eta, \mathrm{gl}_\vartheta^* \varphi \rangle = \langle \mathrm{gl}_\vartheta \eta, \varphi \rangle.$$

Step 2: unfolding the comparison form. By (TG6), $\mathrm{gl}_\vartheta \eta$ is M^ℓ times a combination with positive real weights of the product forms $\mathrm{pr}_{\mathcal{V}, L_{i'}}((x, x_\mathfrak{c}); \varphi''_{\vartheta,i'}(x))(\eta_{i',\mathfrak{c}})$, which vanish for $i' \ne i$ (linearity in $\eta_{i',\mathfrak{c}}$). Exchanging the finite sums,

$$\langle \mathrm{gl}_\vartheta \eta, \varphi \rangle = M^\ell \sum_{x \bmod \mathbf{J}_\vartheta(\Lambda_i)}^{\mathrm{cl}} S(x), \qquad S(x) = \sum_{\mathfrak{c} \in \mathrm{Cl}^\circ} \langle \mathrm{pr}_{\mathcal{V}, L_i}((x, x_\mathfrak{c}); \varphi''_{\vartheta,i}(x))(\bar{k}_\mathfrak{c}), \ \varphi \rangle.$$

It suffices to show $S(x) = 0$ for every $x \in G_f(\mathcal{B}_j)$.

Step 3: the hypotheses of the period vanishing. We check the hypotheses of Proposition 6.6.40 for O and $\Xi = \varphi$. O has ranks $1 \le r < r + t$ and fits a parameter by Lemma 6.6.41; $M_O \ne \emptyset$. At $w \in Q_O = Q$ the tame part of the character of O is $\omega_w^j \psi_{\vartheta,w} = \psi_{\vartheta,w} \ne 1$ (ϑ is a twist, $\omega_w = 1$ off the chosen places). The hyperplane sums kill $\varphi \in [e_0]\mathrm{Form}(O)$ by Lemma 6.6.42. As $\varphi \in [e_0]\mathrm{Form}(O)$ is orthogonal to $\mathcal{L}_{j,\vartheta}$, $\mathrm{Th}_O(\varphi) = 0$ by Lemma 6.6.43. $H_w(X_*(w))\varphi = 0$ for the given w, with $w \ne \bar{w}$ and $w, \bar{w} \notin S_O$. For $N = \mathcal{B}_j$, $l = \mathcal{B}^\circ$ and $n = r + t = |\mathcal{B}_j| + |\mathcal{B}^\circ|$, the rank condition $2r + 2\#N + 3 < n$ reads $2r + |\mathcal{B}_j| + 3 < |\mathcal{B}^\circ|$, that is $18m + 5 < 2r$:

Step 4: conclusion. Let $\mathbf{K}^\circ = \mathbf{K}_{\mathfrak{M}^2}[\{1\}] \subseteq \mathbf{K}_0(\mathcal{B}^\circ)$, the level of the classes Cl° (TG3). Its elements act trivially on the reduction of the standard lattice of V° at every chosen place, so by (c) of Lemma 4.6.2 the stabilisers $\mathrm{U}(V^\circ)(F) \cap h\mathbf{K}^\circ h^{-1}$ are trivial, and the class sum modulo $\mathbf{K}^\circ$ at the representatives $x_\mathfrak{c}$ is the plain sum over $\mathfrak{c} \in \mathrm{Cl}^\circ$. Hence $S(x)$ is the class sum of Proposition 6.6.40 for $(N, l, a, z) = (\mathcal{B}_j, \mathcal{B}^\circ, \mathcal{B}^\mathrm{c}, z_j)$, the divided splitting $\mathcal{V}$, the divided lattice L_i, the point x, $y = \varphi''_{\vartheta,i}(x)$, the character b, $\Psi = \mathbf{1}_{z_j}$ and g, and it vanishes. By Steps 1 and 2, $\mathrm{tt}_j(\zeta_i)(g) = 0$; as g is arbitrary, the claim follows (the second display of the statement is the notation $\zeta_i(\mathfrak{c}) = \zeta_{i,\mathfrak{c}}$). □

Proposition 6.6.40. *Let O be a descent pair for ranks $1 \le r < n$ that fits a parameter, with $M_O \neq \emptyset$, and let $\Xi \in \mathrm{Form}(O)$ satisfy: $\psi_{O,w} \neq 1$ for every $w \in Q_O$; $\sum_{s \in H(\lambda)} \psi_{O,w}(\det s)^{-1}[s]_w \Xi = 0$ for every $w \in M_O$ and every nonzero row $\lambda \in k_w^r$; $\mathrm{Th}_O(\Xi) = 0$; and $H_w(X_*(w))\,\Xi = 0$ for some finite place w with $w \neq \bar{w}$ and $w, \bar{w} \notin S_O$. Let N, l, a be finite index sets with $2r + 2\#N + 3 < n$, $z\colon \{1, \dots, r\} \hookrightarrow l$ an embedding, $\mathcal{V}$ a divided splitting for (N, l, a, z), L a divided lattice, $x \in G_f(N)$, $y \in \mathbb{V}_{\kappa_O^+}$ (rank $\#N$, rows N), b an automorphic character of $\mathrm{U}(1)$, Ψ a function on $\mathrm{M}_{l\times r}(R_2)$ and $g \in \mathrm{U}(W(O))(\mathbb{A}_F)$. Let $\mathbf{K}^\circ = \mathbf{K}_{\mathfrak{M}_O^2}[\{1\}] \subseteq \mathbf{K}_0(l)$, and for $h \in G_f(l)$ let $\bar{k}(h) \in \mathbb{V}_{\kappa_O^-}$ (rank $\#l$, rows l) have the conjugate coordinates of $\mathrm{kv}_{O,b;\mathfrak{M}_O^2}[\Psi; \kappa_O^-](g, h)$. Then, the class sum being taken over the double cosets $\mathrm{U}(\mathrm{I}_l)(F)\backslash G_f(l)/\mathbf{K}^\circ$ at their fixed representatives,*

$$\sum_{h \bmod \mathbf{K}^\circ}^{\mathrm{cl}} \langle \mathrm{pr}_{\mathcal{V},L}((x,h);y)(\bar{k}(h)),\ \Xi\rangle = 0.$$

Proof. Write $W = W(O)$, $\kappa = \kappa_O$, $\kappa^\pm = \kappa_O^\pm$, $\rho = \rho_{\kappa,e_O}$, $\langle\,,\rangle = \langle\,,\rangle_\kappa$, and $\varepsilon\colon N \sqcup l \to \{1, \dots, n\}$ for the bijection of $\mathcal{V}$. For $h \in G_f(l)$ let $x'(h) = \mathrm{diag}_\varepsilon(x, h)\gamma_L$ and $\mathrm{kv}(h) = \mathrm{kv}_{O,b;\mathfrak{M}_O^2}[\Psi; \kappa^-](g, h)$, so that $\bar{k}(h) = \overline{\mathrm{kv}(h)}$; let $f(h)$ be the summand of the statement. For $u \in G_\infty(1_l)$ put $u^\varepsilon = \mathrm{diag}_\varepsilon(1, u) \in G_\infty(1_n)$. Let $C(\zeta) = y \cdot \bar{\zeta}$ for $\zeta \in \mathbb{V}_{\kappa^-}$ (rows l), a conjugate-linear map into $\mathbb{V}_O$. Let $(e_i)_{i \in I_{\kappa^-}}$ be the coordinate vectors of $\mathbb{V}_{\kappa^-}$ and $\varphi_i = \mathrm{mod}_{\mathfrak{M}_O^2}(\Psi, v_{\mathrm{Let}_{-1}(b_i)}) \in \mathcal{M}_{W,\mathrm{I}_l}$, so that $\mathrm{kv}(h)_i = \overline{\theta^{W,\mathrm{I}_l}_{\chi_O,b}(\varphi_i)(g, (1,h))}$ (KN4).

Step 1: the summand is a Gram pairing. By (GU6) and (e) of Lemma 6.6.31, $f(h) = \langle y \cdot \bar{k}(h), \Xi(x'(h))\rangle = \langle C(\mathrm{kv}(h)), \Xi(x'(h))\rangle$.

Step 2: C is equivariant for the real points of the second block. Let $u \in G_\infty(1_l)$ and $\zeta \in \mathbb{V}_{\kappa^-}$; we show $\rho(u^\varepsilon)C(\zeta) = C(\rho_{\kappa^-,0}(u)\zeta)$. For a square z and a rectangular z', $S(z)S(z') = S(zz')$ (apply the defining identity of (WM2) twice). By (WM2), $e_O = -e^-$ ((DP3), $e^-(\tau) = \sum_{1 \le i < n} \kappa^-(\tau, i)$), (d) of Lemma 3.2.20 for $g = \overline{u^\varepsilon}$ and (a) of Lemma 6.6.26 (its hypotheses are those of (GU1)),

$$C_O\,\rho(u^\varepsilon)(y \cdot \bar{\zeta}) = \big(S_{\kappa^+}(\overline{u^\varepsilon}) \otimes S_{\kappa^-}((\overline{u^\varepsilon})^{-\mathsf{T}})\big) C_O(y \cdot \bar{\zeta}) = S_{\kappa^+}(\overline{u^\varepsilon}\iota_N)\, y \otimes S_{\kappa^-}\big((\overline{u^\varepsilon})^{-\mathsf{T}}\iota_l\big)\bar{\zeta}.$$

Here $\overline{u^\varepsilon}\iota_N = \iota_N$, and u unitary gives $\bar{u}^{-\mathsf{T}} = u$, so $(\overline{u^\varepsilon})^{-\mathsf{T}}\iota_l = u^\varepsilon\iota_l = \iota_l u$. Hence the right side is $S_{\kappa^+}(\iota_N)y \otimes S_{\kappa^-}(\iota_l)(S_{\kappa^-}(u)\bar{\zeta}) = C_O(y \cdot S_{\kappa^-}(u)\bar{\zeta})$ by (a) of Lemma 6.6.26 again. As C_O has a left inverse, $\rho(u^\varepsilon)C(\zeta) = y \cdot S_{\kappa^-}(u)\bar{\zeta}$, and $S_{\kappa^-}(u)\bar{\zeta} = \overline{S_{\kappa^-}(\bar{u})\zeta} = \overline{\rho_{\kappa^-,0}(u)\zeta}$ (S has integral polynomial entries).

Step 3: real points act on the kernel vector. For $u \in G_\infty(1_l)$ and $h \in G_f(l)$ we show $(\rho_{\kappa^-,0}(u^{-1})\,\mathrm{kv}(h))_i = \overline{\theta^{W,\mathrm{I}_l}_{\chi_O,b}(\varphi_i)(g, (u,h))}$. By (1) of Lemma 6.3.71,

$$\theta(\varphi_i)(g, (u,h)) = \theta(\mathrm{mod}_{\mathfrak{M}_O^2}(\Psi, \Omega_{\gamma,\gamma'}(1,u)v_{\mathrm{Let}_{-1}(b_i)}))(g, (1,h)),$$

γ, γ' the weights of χ_O, b. By (5) of Lemma 4.1.1 for the pair (W, I_l) (u is block diagonal, I_l having only the sign $+1$): the scalar is $\prod_\tau \det(u_\tau)^{\gamma(\tau) - r_-(O)} = 1$, as υ has weight 1 ((ii) of Lemma 3.2.11) and $\chi_O = \upsilon^{r_-(O)}$; and the variables $X_{\tau,\cdot,c}$ of the columns c with $\varepsilon_O(c) = -1$, the only ones in $\mathrm{Let}_{-1}(b_i)$, are replaced by $u_\tau^{-1}X_{\tau,\cdot,c}$. So $\mathrm{Let}_{-1}(b_i)$ becomes $\mathrm{Let}_{-1}(b_i(u^{-1}\cdot)) = \sum_k S_{\kappa^-}(u^{-1})_{ik}\mathrm{Let}_{-1}(b_k)$ (WM2), and by linearity $\theta(\varphi_i)(g, (u,h)) = \sum_k S_{\kappa^-}(u^{-1})_{ik}\overline{\mathrm{kv}(h)_k}$. Conjugating, and using $\overline{S(z)} = S(\bar{z})$, gives the claim.

Step 4: the summand is the conjugate of a sum of period integrands. Let ι be the block diagonal embedding $\mathrm{G}(1_N) \times \mathrm{G}(1_l) \to \mathrm{G}(1_n)$ along ε, and for $h_2 \in \mathrm{G}(1_l)$ put

$$F(h_2) = \sum_{i \in I_{\kappa^-}} \theta^{W,\mathrm{I}_l}_{\chi_O,b}(\varphi_i)(g, h_2)\, Y_{C(e_i)}(\iota((1,x), h_2)\,(1, \gamma_L)).$$

Let $u \in G_\infty(1_l)$, $h \in G_f(l)$ and $z = \rho_{\kappa^-,0}(u^{-1})\mathrm{kv}(h)$. By Step 2, $C(z) = \rho((u^{-1})^\varepsilon)C(\mathrm{kv}(h))$, so by Step 1 and the unitarity of ρ ((c) of Lemma 4.6.10), $f(h) = \langle C(z), \rho(u^\varepsilon)^{-1}\Xi(x'(h))\rangle$. As $C(z) = \sum_i \bar{z}_i C(e_i)$ and the form is conjugate-linear in its first argument,

$$f(h) = \sum_i z_i \langle C(e_i), \rho(u^\varepsilon)^{-1}\Xi(x'(h))\rangle.$$

The point $\iota((1,x),(u,h))(1,\gamma_L)$ has infinite component u^ε and finite component $x'(h)$, so the last pairing is

$$\overline{Y_{C(e_i)}(\iota((1,x),(u,h))(1,\gamma_L))};$$

with Step 3,

$$f(h) = \overline{F(u,h)} \qquad (u \in G_\infty(1_l),\ h \in G_f(l)). \tag{6.143}$$

Step 5: invariance of the summand. (a) For a rational point δ of $\mathrm{U}(\mathrm{I}_l)$, $F(\delta h_2) = F(h_2)$: the kernels are left invariant by Lemma 6.2.5, $\iota((1,x),\delta h_2) = \iota(1,\delta)\,\iota((1,x),h_2)$ with $\iota(1,\delta)$ a rational point of $\mathrm{U}(\mathrm{I}_n)$, and $Y_\zeta(\delta' X) = Y_\zeta(X)$ for rational δ' by (DP5) and the multiplicativity of ρ. With (6.143) (taking $u = \delta_\infty u'$), $f(\delta_f h) = f(h)$. (b) For $k \in \mathbf{K}^\circ$, $f(hk) = f(h)$. First, $\mathrm{kv}(hk) = \mathrm{kv}(h)$: by (6) of Lemma 4.1.1 for the modulus $\mathfrak{M}_O^2$, $\omega_{\chi_O,b}((1,1),(1,k))\varphi_i = \chi_O(\det k)\,\mathrm{mod}_{\mathfrak{M}_O^2}(\Psi(\bar{k}^{-1}\,\cdot), v) = \varphi_i$, as $\bar{k} = 1$ and $\chi_O(\det k) = v(\det k)^{r-(O)} = 1$ ((ii) of Lemma 3.2.11; $\det k$ has unit finite and trivial infinite components). Second, $x'(hk) = x'(h)c$ with $c = \gamma_L^{-1} d \gamma_L$, $d = \mathrm{diag}_\varepsilon(1,k) \in \mathbf{K}_0(n)$, and $c \in \mathbf{K}(O)$ with $\tilde{\eta}_O(c) = 1$: outside the chosen places and their conjugates $\gamma_L = 1$ and $c = d$. At a chosen place w, $\varpi_w\gamma_w^{-1}$ is integral (GU2) and so is γ_w (the matrix $(\varpi_O\gamma_L^{-1})^* = \bar{\varpi}_O\gamma_L$ is integral, and the component of $\bar{\varpi}_O$ at w is 1, $\bar{w}$ not being chosen (DP1)); as $k \equiv 1$ modulo w^2, $d_w = 1 + \varpi_w^2 Z$ with Z integral, so $c_w = 1 + (\varpi_w\gamma_w^{-1})Z(\varpi_w\gamma_w)$ is integral and $\equiv 1 \pmod{w}$, hence in $\mathrm{GL}_n(\mathcal{O}_{E,w})$, and then $c_{\bar{w}}$, determined by $c^*c = 1$, is integral and invertible too. So $c \in \mathbf{K}_0(n)$, $\mathrm{red}_w c = 1$ for $w \in M_O$, and for $w \in Q_O$, $\mathrm{red}_w c = \mathrm{red}_w d$ has the rows $\mathrm{inc}_O(i) = \varepsilon(\phi_{\mathcal{V}}(i)) \in \varepsilon(N)$ ((GU1)) equal to those of the identity. By (DP4), $c \in \mathbf{K}(O)$ with inc-blocks 1 at Q_O, so $\tilde{\eta}_O(c) = 1$ and $\Xi(x'(hk)) = \Xi(x'(h))$ (DP5). With Step 1, $f(hk) = f(h)$.

Step 6: the class sum is a multiple of the period integral. By (6.143) and Step 5, $F(u,h) = \overline{f(h)}$ does not depend on u, is left invariant under the rational points of $\mathrm{U}(\mathrm{I}_l)$ and right invariant under the open subgroup $\mathbf{K}^\circ$ in h, hence continuous (locally constant in h). By (2) and then (1) of Lemma 6.4.74 with $\mathrm{K} = \mathrm{K}' = \mathbf{K}^\circ$ (the inner integral over $G_\infty(1_l)$ is $\overline{f(h)}$, the Haar measure there being a probability measure),

$$\int_{[\mathrm{U}(\mathrm{I}_l)]} F = \frac{c_l}{[\mathbf{K}_0(l) : \mathbf{K}^\circ]} \sum_{[h]} \frac{\overline{f(h)}}{\#(\mathrm{U}(\mathrm{I}_l)(F) \cap h\mathbf{K}^\circ h^{-1})} = \frac{c_l}{[\mathbf{K}_0(l) : \mathbf{K}^\circ]} \overline{\sum_{h \bmod \mathbf{K}^\circ}^{\mathrm{cl}} f(h)},$$

$c_l > 0$, the class sum (6.22) having rational weights.

Step 7: the periods vanish. Each summand of F is the integrand of Proposition 6.6.81 for O, Ξ, $\zeta = C(e_i)$, the character b, the datum φ_i, the points g, $h_1 = (1,x) \in \mathrm{G}(1_N)$ and $\gamma = \gamma_L$, and the partition $\{1,\dots,n\} = \varepsilon(N) \sqcup \varepsilon(l)$ (identified with $N \sqcup l$ through ε, so that $\mathrm{I}_n = \mathrm{I}_N \oplus \mathrm{I}_l$ and the block diagonal embedding is ι). Its hypotheses on O and Ξ and $2r + 2\#N + 3 < n$ are those of the statement. The components of γ_L outside the chosen places and their conjugates, hence outside Σ, are 1 (GU2). The datum $\varphi_i = v \otimes (\Psi^\natural \otimes_{M_O} v^0)$ ($v = v_{\mathrm{Let}_{-1}(b_i)}$, (XD5), $\Sigma(\mathfrak{M}_O^2) = M_O$) is the vacuum at the finite places outside Σ: for an integral finite adelic z with $z_w = 0$ at the places of $\Sigma \supseteq M_O \cup \bar{M}_O$, $\rho_f(z)$ is a scalar times $\rho(\mathrm{fv}(z))$ (WD8), which commutes with the operators $\rho(\mathrm{rv}(\cdot))$ of (WR6), the two vectors having no common nonzero component (WD7), so $\rho_f(z)(v \otimes f') = v \otimes \rho_f(z)f'$; and $\rho_f(z)(\Psi^\natural \otimes_{M_O} v^0) = \Psi^\natural \otimes_{M_O} v^0$ by (5) of Lemma 6.2.33. So $\int_{[\mathrm{U}(\mathrm{I}_l)]} F = 0$, and by Step 6 the class sum of the statement vanishes. □

Lemma 6.6.41. *Let $1 \le j \le m$ and ϑ a tame character with modulus Q. For $\tau \in \Phi$ and a basis index c of W_j put $l_\tau(c) = 2\varsigma_\tau(\mathrm{num}(c))$, $\varsigma_\tau(i)$ the entry of number i of $\varsigma(\tau)$. Then $O_{j,\vartheta}$ fits the parameter $l = (l_\tau)_{\tau\in\Phi}$.*

Proof. Fix $\tau \in \Phi$ and drop it from the notation: $\varsigma(i) = \varsigma_\tau(i)$ ($i \ge 0$, extended as in (LF4)), $w = \kappa_j(\tau)$, $l = l_\tau$, $\kappa^\pm(h) = \kappa^\pm_O(\tau, h)$ for $O = O_{j,\vartheta}$, $n = r + t$. Put $a = k + j - 1$, $b = m - j$ (so $a + b = r$), and $T = \frac{t-1}{2}$.

Step 0: arithmetic of the data. By Definition 2.3.3, $C_P \ge 6$, so (AD1) gives $\log\log b > 0$ and $\log x \ge 100 \log\log b > 0$; hence $x \ge 2$ and $r \ge x^5 \ge 32$ by (AD2). The integers r and $t = 5r$ are odd and $k = r+1-m$ is even. The extended target parameter is a strictly decreasing sequence of integers: $\varsigma(i) = \frac{k}{2} - 1 - i$ for $0 \le i \le k-2$, and $\varsigma(k-1+i) = \frac{\Delta}{2} - \kappa_{i+1} = -\kappa - (i+1)\Delta$ for $i \ge 0$, which is below $1 - \frac{k}{2}$. In particular

$$\varsigma(a-1) = \tfrac{\Delta}{2} - w, \qquad \varsigma(a) = -\tfrac{\Delta}{2} - w, \qquad \tfrac{\Delta}{2} = r^8 > T + 1, \qquad w \ge \tfrac{\Delta}{2} > k. \tag{6.144}$$

By (HP4) read in the numbering num, a basis vector c of W_j has sign $\varepsilon(c) = 1$ if and only if $\mathrm{num}(c) < a$; so $r_+(O) = a$, $r_-(O) = b$, and num is a bijection from the basis indices onto $\{0, \dots, r-1\}$.

Step 1: the weight of O. From (LF4) and (6.144), for $1 \le h < n$:

- if $h < a$, then $\varsigma(h) \ge \varsigma(a-1) > T - w$, so $\kappa^+(h) = \varsigma(h-1) - \varsigma(h) - 1 \ge 0$; $\kappa^+(a) = \varsigma(a-1) - (T-w) - 1 = \frac{\Delta}{2} - T - 1$, as $\varsigma(a) < -T - w < T - w$; and if $h > a$, then $\varsigma(h-1) \le \varsigma(a) < T - w$, so $\kappa^+(h) = 0$;
- if $h < b$, then $r - h - 1 \ge a$ and $\varsigma(r-h-1) \le \varsigma(a) < -w - T$, so $\kappa^-(h) = \varsigma(r-h-1) - \varsigma(r-h) - 1 \ge 0$; if $h = b$, then $r - h - 1 = a - 1$ and $\kappa^-(b) = -w - T - \varsigma(a) - 1 = \frac{\Delta}{2} - T - 1$; and if $h > b$, both arguments $r - h - 1$, $r - h$ (read as 0 when negative) are at most $a - 1$, so both values of ς exceed $-w - T$ and $\kappa^-(h) = 0$.

In particular $\kappa^+(h) = 0$ for $r_+(O) = a < h < n$ and $\kappa^-(h) = 0$ for $r_-(O) = b < h < n$, which is (iii) of (DP11).

Step 2: the parameter l_O. For $0 \le p < n$ the highest weight of $\kappa_O(h) = \kappa^+(h) + \kappa^-(n-h)$ is $\lambda_O(p) = \sum_{p<h<n} \kappa^+(h) + \sum_{0<h'<n-p} \kappa^-(h')$, and $e_O = -\sum_{0<h'<n} \kappa^-(h')$, so

$$e_O + \lambda_O(p) = A(p) - B(p), \qquad A(p) = \sum_{p<h<n} \kappa^+(h), \quad B(p) = \sum_{n-p\le h'<n} \kappa^-(h').$$

By Step 1, $A(p) = 0$ for $p \ge a$, and for $p < a$ the sum telescopes:

$$\begin{aligned} A(p) &= \sum_{h=p+1}^{a-1} \big(\varsigma(h-1) - \varsigma(h) - 1\big) + \tfrac{\Delta}{2} - T - 1 \\ &= \varsigma(p) - \varsigma(a-1) - (a - 1 - p) + \tfrac{\Delta}{2} - T - 1 = \varsigma(p) + w - T - (a - p). \end{aligned}$$

Likewise $B(p) = 0$ unless $n - p \le b$, i.e. $p \ge a + t$, and then, with $q = r - h'$,

$$\begin{aligned} B(p) &= \sum_{q=a+1}^{p-t} \big(\varsigma(q-1) - \varsigma(q) - 1\big) + \tfrac{\Delta}{2} - T - 1 \\ &= \varsigma(a) - \varsigma(p-t) - (p - t - a) + \tfrac{\Delta}{2} - T - 1 = -w - \varsigma(p-t) - p + t + a - T - 1. \end{aligned}$$

With $l_O(p+1) = 2(e_O + \lambda_O(p)) + n - 1 - 2p$ (DP11) and $2T = t - 1$ this gives

$$l_O(p+1) = \begin{cases} 2\varsigma(p) + 2w - 2a + r, & 0 \le p < a, \\ r + t - 1 - 2p, & a \le p < a + t, \\ 2\varsigma(p-t) + 2w - 2a + r, & a + t \le p < n. \end{cases}$$

Step 3: (i), the block rule. We check (WM8) for (l, l_O) with the index sets of W_j and $\{1, \dots, n\}$, $N = r \le N' = n$, the weights $\gamma = r_-(O) = b$ and $\gamma' = \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau) = \frac{r+t}{2} - w$ (the character $\mu_j^0 \vartheta$ has weight $-\kappa_j$, ϑ having weight 0), and the signs $(\varepsilon, \mathbf{1}_n)$.

(a) $l(c) = 2\varsigma(\mathrm{num}(c))$ is injective and even, and $N - 1 = r - 1$ is even. If $\varepsilon(c) \ne \varepsilon(c')$, say $\mathrm{num}(c) < a \le \mathrm{num}(c')$, then $|l(c) - l(c')| \ge 2(\varsigma(a-1) - \varsigma(a)) = 2\Delta = 4r^8 \ge 2 \cdot 500(7r)^3$, as $r^5 \ge 32^5 > 85750$. So l has gap $500(N + N')^3 = 500(7r)^3$ for ε. The map l_O is strictly decreasing by its definition in (DP11), and $l_O(p+1) \equiv n - 1 \pmod 2$; as all signs of $\mathbf{1}_n$ agree, it has the same gap for $\mathbf{1}_n$.

(b) The shifted entries are $\tilde{l}(c) = l(c) - 2\gamma' + n = 2\varsigma(\mathrm{num}(c)) + 2w$ and $\tilde{l}_O(p+1) = l_O(p+1) - 2b + r$. As $2a + 2b = 2r$, Step 2 gives $\tilde{l}_O(p+1) = 2\varsigma(p) + 2w$ for $p < a$, $t - 1 - 2(p-a)$ for $a \le p < a+t$, and $2\varsigma(p-t) + 2w$ for $p \ge a+t$. Since num is a bijection onto $\{0, \dots, r-1\}$, $\tilde{l}_O(\{1, \dots, n\}) = \tilde{l}(\{c\}) \cup [t] = \tilde{l}(\{c\}) \cup [N' - N]$.

(c) For $\mathrm{num}(c) < k - 1$, $\tilde{l}(c) \ge 2 - k + 2w > 0$ by (6.144); for $\mathrm{num}(c) = k - 1 + i$, $\tilde{l}(c) = \Delta - 2\kappa_{i+1} + 2\kappa_j = (2(j-i) - 1)\Delta$, positive exactly when $i < j$. So $\tilde{l}(c) > 0$ if and only if $\mathrm{num}(c) < a$, i.e. $\varepsilon(c) = 1$, and $\tilde{l}(c) < 0$ otherwise. As every sign of $\mathbf{1}_n$ is 1, (c) of (WM8) holds.

Step 4: (ii), the numbering. If $\varepsilon(c) = 1$, the c' of sign 1 with $l(c) < l(c')$ are those with $\mathrm{num}(c') < \mathrm{num}(c)$ (ς is strictly decreasing, and $\mathrm{num}(c') < \mathrm{num}(c) < a$ forces sign 1); there are $\mathrm{num}(c) = \mathrm{ord}_O(c)$ of them. If $\varepsilon(c) = -1$, the c' of sign -1 with $-l(c) < -l(c')$ are those with $\mathrm{num}(c) < \mathrm{num}(c') \le r - 1$; there are $r - 1 - \mathrm{num}(c) = \mathrm{ord}_O(c)$ of them (LF4).

Steps 3, 4 and 1 are (i), (ii) and (iii) of (DP11) at every $\tau \in \Phi$; so $O_{j,\vartheta}$ fits l. □

Lemma 6.6.42. *Let $1 \le j \le m$, ϑ a twist and $O = O_{j,\vartheta}$. Then for every chosen place w, every nonzero row $\lambda \in k_w^r$ and every $\Xi \in [e_0]\mathrm{Form}(O)$,*

$$\sum_{s \in H(\lambda)} \psi_{O,w}(\det s)^{-1} \, [s]_w \, \Xi = 0.$$

Proof. Fix a chosen place w, put $k_0 = k_w$, $G = \mathrm{GL}_r(k_0)$ and $\chi = \psi_{O,w}$. The tame component of a twist is trivial at every chosen place, since its modulus is prime to MN. Thus (LF2)–(LF4) give $\chi = \omega_w^j$, hence $\chi^{q_2} = 1$.

We first prove a finite-group assertion. Let c be a label allowed for q_2 and let $\lambda \ne 0$. Put $K_\lambda = \ker \lambda$. Choose $b \in k_0^r$ with $\lambda(b) = 1$. Relative to $k_0^r = K_\lambda \oplus k_0 b$, the elements of $H(\lambda)$ are precisely

$$\begin{pmatrix} 1 & u \\ 0 & a \end{pmatrix}, \qquad u \in K_\lambda, \quad a \in k_0^\times.$$

Indeed $1 + v\lambda$ has this matrix with $u = v - \lambda(v)b$ and $a = 1 + \lambda(v)$, and it is invertible exactly when $a \ne 0$. In particular its determinant is a. For $r > 1$, let P be the stabilizer of K_λ and

U its unipotent radical. Then $H(\lambda) = U \rtimes k_0^\times$, where the latter factor is the GL_1 factor of the Levi subgroup $\mathrm{GL}_{r-1}(k_0) \times \mathrm{GL}_1(k_0)$.

Let V_c be the irreducible representation with character $\pm\Xi_c$, which exists by Theorem A.5.27. Its central idempotent is e_c: in the usual irreducible-character idempotent formula, changing both factors $\Xi_c(1)$ and $\overline{\Xi_c(g)}$ by the same sign leaves their product unchanged. For $r > 1$, suppose that a nonzero vector in V_c transforms under $H(\lambda)$ by $\chi \circ \det$. It lies in V_c^U, and its GL_1 character is χ. Complete reducibility for the other Levi factor therefore gives an irreducible constituent $\tau \boxtimes \chi$ of the Harish-Chandra restriction V_c^U. We verify the required restriction by the label factorization in Theorem A.5.29. Factor $c = \prod_{i=1}^{t} f_i$ into its distinct monic irreducible factors, put $d_i = \deg f_i$, and let ρ_i be the irreducible cuspidal representation with character $\pm\Xi_{f_i}$, as supplied by Theorem A.5.28. Thus

$$V_c \simeq \mathrm{Ind}_Q^G(\rho_1 \boxtimes \cdots \boxtimes \rho_t),$$

where Q is the block parabolic of type $(d_1, \ldots, d_t)$ and the representation on its Levi subgroup is inflated across its unipotent radical.

Here is the specialization of finite-group Mackey decomposition that we need. The relative position of a hyperplane and a flag of type $(d_1, \ldots, d_t)$ is specified by the unique block i in which the one-dimensional quotient occurs. Indeed, intersecting the successive flag spaces with the hyperplane reduces the successive quotient dimensions by 1 in that block and by 0 in the others. Choosing bases in the successive quotients shows both that these are all the double cosets $P\backslash G/Q$ and that each such position is one double coset. In position i, the stabilizer on the i-th block is the parabolic of type $(d_i - 1, 1)$. Mackey decomposition, followed by averaging over U, consequently replaces ρ_i by its invariants under the unipotent radical of that parabolic; the other blocks remain unchanged, and induction on the hyperplane combines their dimensions d_h with $d_i - 1$. This averaging assertion also follows directly from the induced-function model: an invariant function is constant on the left U-orbits, while its values must be invariant under the image of U in the block stabilizer. To see the image explicitly, choose the one quotient vector in the i-th block and a basis for its intersection with the hyperplane. The maps in U from that quotient vector to the intersection basis project onto all matrices

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad u \in k_0^{d_i - 1},$$

on that block. These are exactly the required unipotent radical; their parameters are unrestricted.

If $d_i > 1$, these invariants are zero by cuspidality of ρ_i. If $d_i = 1$, the radical is trivial, and the GL_1 factor acts by ρ_i. Consequently this Mackey decomposition gives

$$V_c^U \simeq \bigoplus_{\{i : d_i = 1\}} \left(\mathrm{Ind}_{Q_i}^{\mathrm{GL}_{r-1}(k_0)} \boxtimes_{h \neq i} \rho_h \right) \boxtimes \rho_i,$$

where Q_i is the block parabolic with the i-th block omitted and the factors are inflated to it. In particular every GL_1 character occurring in V_c^U is one of these ρ_i. If the character χ is the factor ρ_i corresponding to $f_i = X - a_i$, the degree-one clause of Theorem A.5.29 gives $\mathrm{ord}(\chi) = \mathrm{ord}(a_i)$. Since $\chi^{q_2} = 1$, it follows that $a_i^{q_2} = 1$. This contradicts the allowance condition on c, since $a_i \in k_0$ is a root of c. For $r = 1$ the representation V_c itself is ρ_{X-a} for the unique root a of c; the same order argument excludes a χ-isotypic vector directly.

In a complex representation of G, the element

$$T_{\chi,\lambda} = \sum_{s \in H(\lambda)} \chi(\det s)^{-1} s$$

acts as $|H(\lambda)|$ times the projector onto its $\chi \circ \det$ isotypic subspace. This follows directly by multiplying the sum on the left by an element of $H(\lambda)$ and by observing that the sum acts as $|H(\lambda)|$ on that subspace. The subspace just excluded is zero in V_c, so $T_{\chi,\lambda}$ acts as zero there. It follows that $T_{\chi,\lambda} e_c = 0$ in $\mathbb{C}[G]$: on the simple summand selected by e_c the first factor is zero, and on every other simple summand e_c is zero; the regular representation is faithful.

Embed $T_{\chi,\lambda}$ in the w factor of $\mathbb{C}[\bar{G}]$, with identity elements in every other factor. Every good family has its label c_w allowed for q_2, so multiplication of tensor factors gives

$$T_{\chi,\lambda} e_0 = \sum_{c \text{ good}} (T_{\chi,\lambda} e_{c_w}) \otimes \bigotimes_{u \neq w} e_{c_u} = 0.$$

Finally, write $\Xi = [e_0]Y$. The action in (DP5) extends to an algebra homomorphism, and hence

$$\sum_{s \in H(\lambda)} \psi_{O,w}(\det s)^{-1} [s]_w \Xi = [T_{\chi,\lambda}][e_0]Y = [T_{\chi,\lambda} e_0]Y = 0.$$

This is the required one-place assertion. □

Lemma 6.6.43. *Let $1 \leq j \leq m$, ϑ a twist and $O = O_{j,\vartheta}$. If $\varphi \in [e_0]\mathrm{Form}(O)$ is orthogonal to $\mathcal{L}_{j,\vartheta}$, then $\mathrm{Th}_O(\varphi) = 0$.*

Proof. Write $\Theta = \Theta_{j,\vartheta}$ and $\mathcal{L} = \mathcal{L}_{j,\vartheta} = \Theta(\mathcal{A}^{\mathrm{good}}(W_j))$. Let $y \in \mathcal{A}_M(W_j)$. By (i) of Lemma 6.6.10 and linearity of Θ, $\Theta(a \cdot y) = [a]\Theta(y)$ for every $a \in \mathbb{C}[\bar{G}]$; in particular $[e_0]\Theta(y) = \Theta(e_0 y) \in \mathcal{L}$, as $e_0 y$ is a good form. By (b) of Lemma 6.6.9 (for σ the identity, where $\langle\,,\rangle_\sigma$ is the inner product of $\mathrm{Form}(O)$), $[e_0]$ is an idempotent, so $[e_0]\varphi = \varphi$, and it is self-adjoint; hence

$$\langle \varphi, \Theta(y) \rangle = \langle [e_0]\varphi, \Theta(y) \rangle = \langle \varphi, [e_0]\Theta(y) \rangle = \langle \varphi, \Theta(e_0 y) \rangle = 0.$$

By (a) of Lemma 6.6.30, $\int_{[\mathrm{U}(W_j)]} y\, \overline{\mathrm{Th}_O(\varphi)} = \langle \varphi, \Theta(y) \rangle = 0$ for every $y \in \mathcal{A}_M(W_j)$. By (b) there, $y = \mathrm{Th}_O(\varphi)$ is admissible, so $\int_{[\mathrm{U}(W_j)]} |\mathrm{Th}_O(\varphi)|^2 = 0$. The function $\mathrm{Th}_O(\varphi)$ is continuous and left invariant under the rational points, and the quotient measure gives positive mass to every nonempty open subset of $[\mathrm{U}(W_j)]$ (the image of an open set of positive Haar measure); so $\mathrm{Th}_O(\varphi) = 0$. □

Proposition 6.6.44. *Let $1 \leq j \leq m$ and $y \in \mathcal{A}^{\mathrm{good}}(W_j)$. If*

$$\int_{[\mathrm{U}(W_j)]} y(g)\, \mathsf{F}_i(g)\, \mathrm{tt}_j(\zeta)(g)\, dg = 0$$

for every choice $i \in \mathrm{Mem}_j$ and every $\zeta \colon \mathrm{Cl}^\circ \to \mathbb{V}_j^-$, then $y = 0$.

Proof. Let s_j be the signs of W_j, $n_* = 2ft^3$, and let o, u, u° be as in Lemma 6.6.45. Suppose $y \neq 0$ and put $F = \bar{y}$ on $\mathrm{G}(s_j)$: it is continuous and bounded (y is a form of type e_ς, Definition 3.2.16), adelically holomorphic of weight u for o ((b) of Lemma 6.6.45), and $0 < \sup|F| < \infty$; by the definition of the supremum there is z with $|F(z)| > \frac{1}{2}\sup|F|$, a peak of F.

Let F_0 be as in Lemma 6.6.61, $C = \sup|F_0| < \infty$ and $\delta = 2b^{-19n_*}|F_0(1)|$. Let x be a peak of F. By Lemma 6.6.58, $\lambda_{W_j}(x) \geq (5b^{24})^{-1/2}$, so Lemma 6.6.61 gives (c_i) with $|K(h) - F_0(h)| \leq \delta |h|^{6n_*}$ for $K(h) = \sum_i c_i \mathsf{F}_i(x(h,1))$. Each $h \mapsto \mathsf{F}_i(x(h,1))$ is holomorphic of weight $u - u^\circ$ for o ((c) of Lemma 6.6.45), hence so is K (the sum of the extensions is an extension of the sum). As $\overline{F(x(h,1))} = y(x(h,1))$, (d) of Lemma 6.6.45 gives $\int_{G_\infty(s_j)} \overline{F(x(h,1))}\, K(h)\, m^{\mathrm{hol}}_{s_j,u^\circ}(h)\, dh = 0$.

These are the hypotheses of Lemma 6.6.80 for the index set of W_j (of size $r \le t$), s_j, o, the weights u, $u' = u^\circ$ ((a) of Lemma 6.6.45), F, z, $K_0 = F_0$, C and δ. Hence, with $B = b^{n_*}$,

$$|F_0(1)| \le (4B+1) \cdot 2B^{-19} \, |F_0(1)|.$$

By (AD1) and $C_P \ge 6$, $\log\log b \ge 6 \cdot 10^{12}$, so $b > 10$, and $n_* \ge 1$, so $B \ge b > 10$ and $(4B+1) \cdot 2 = 8B + 2 < B^2 \le B^{19}$; so $(4B+1) \cdot 2B^{-19} < 1$, which contradicts $F_0(1) \ne 0$. Therefore $y = 0$. □

Lemma 6.6.45. *Let $1 \le j \le m$ and $n_* = 2ft^3$. There are an order o on the basis of W_j, positive-first for s_j, and weights $u = (u_v)_{v \in \Phi}$, $u^\circ = (u_v^\circ)_{v \in \Phi}$ on that basis such that:*

(a) every u_v and every u_v° is K-dominant for (s_j, o) and has holomorphic gap $300n_*$ for s_j, and $|u_v(i)| \le 5b$ for all v and i;

(b) for every $y \in \mathcal{A}^{\mathrm{good}}(W_j)$, $\bar{y}$ is adelically holomorphic of weight u for o;

(c) for every choice $i \in \mathrm{Mem}_j$, F_i is adelically holomorphic of weight $u - u^\circ$ for o;

(d) if $y \in \mathcal{A}^{\mathrm{good}}(W_j)$ satisfies $\int_{[\mathrm{U}(W_j)]} y \, \mathsf{F}_i \, \mathrm{tt}_j(\zeta) = 0$ for all $i \in \mathrm{Mem}_j$ and all $\zeta \colon \mathrm{Cl}^\circ \to \mathbb{V}_j^-$, then for every $x \in \mathrm{G}(s_j)$ and all $(c_i)_{i \in \mathrm{Mem}_j}$ in $\mathbb{C}$ the function

$$h \mapsto y(x(h,1)) \sum_i c_i \mathsf{F}_i(x(h,1)) \, m^{\mathrm{hol}}_{s_j, u^\circ}(h)$$

is integrable on $G_\infty(s_j)$ with integral 0.

Proof. Write $\varsigma = \varsigma_\tau$, $\kappa_{j'} = \kappa_{j'}(\tau)$ where τ is fixed, $r_+ = k + j - 1$ and $q = \mathrm{num}(c)$, the number of a basis vector c; so $s_j(c) = 1$ if and only if $q < r_+$. By Lemma 4.2.67 and (AD2), $x \ge e^{6.4 \cdot 10^{15}}$, $x^{10^{12}} \le b$, $\sqrt{x} \le m \le 2\sqrt{x}$, $x^5 \le r \le 3x^5$, $t \le 15x^5$, $f \le x^{12}$; in particular $r > 9m + 1$, $\Delta > k$, and, as $t + k - 1$ is even, $e_j = j + \frac{t+k-1}{2}$. As $n_* = 2ft^3 \le 2x^{12}(15x^5)^3$, we have $300n_* + \frac{t+1}{2} \le 600x^{12}(15x^5)^3 + 8x^5 \le x^{40} \le r^8$, i.e. $300n_* \le \Delta'$; and $\kappa \le b$ (as $a/x_1 = xb/x_1 \le b$), so $\kappa_m \le b + (2m+1)r^8 \le 2b$.

Step 1: the target parameter and the order. By Definition 3.2.16, $\varsigma(i) = \frac{k}{2} - 1 - i$ for $0 \le i \le k-2$ and $\varsigma(k - 2 + j') = \frac{\Delta}{2} - \kappa_{j'}$ for $1 \le j' \le m$; these are integers, and the sequence is strictly decreasing, since $\varsigma(k-1) = -\kappa - \Delta < 1 - \frac{k}{2} = \varsigma(k-2)$ and $\kappa_{j'+1} - \kappa_{j'} = \Delta$. Put $l_\tau(c) = 2\varsigma(\mathrm{num}(c))$ and define the order

$$o(c) = -1 - \mathrm{num}(c) \ \ (s_j(c) = 1), \qquad o(c) = r - 1 - \mathrm{num}(c) \ \ (s_j(c) = -1).$$

It is injective, with values in $\{-r_+, \dots, -1\}$ on the positive and in $\{0, \dots, m - j - 1\}$ on the negative vectors, so it is positive-first for s_j. If $s_j(a) = s_j(b)$ and $o(a) < o(b)$, then $\mathrm{num}(a) > \mathrm{num}(b)$, hence $l_\tau(a) < l_\tau(b)$.

Step 2: the weight u. Let $u_\tau = \mu_{-l_\tau}$, the lowest K-type of $-l_\tau$ for s_j. As $-l_\tau(c) = -2\varsigma(q)$ is strictly increasing in q, $2\rho_{-l_\tau}(c) = q - (r - 1 - q)$, and $l_c^\rho = -\varsigma(q) + q - \frac{r-1}{2}$ (an integer, r being odd). Counting within the signs, $2\rho^{\mathrm{c}}(c) = q - (r_+ - 1 - q)$ for $s_j(c) = 1$ and $(q - r_+) - (r - 1 - q)$ for $s_j(c) = -1$. Hence

$$u_\tau(c) = -\varsigma(q) - q + r_+ - \tfrac{r+1}{2} \quad (s_j(c) = 1), \qquad u_\tau(c) = -\varsigma(q) - q + r_+ + \tfrac{r-1}{2} \quad (s_j(c) = -1).$$

Explicitly, with $k = r + 1 - m$: $u_\tau(c) = j - \frac{m}{2}$ for $q \le k - 2$, and $u_\tau(c) = \kappa_{j'} - \frac{\Delta}{2} + j - j' + 1 - \frac{r+1}{2}$ resp. $\kappa_{j'} - \frac{\Delta}{2} + j - j' + 1 + \frac{r-1}{2}$ for $q = k - 2 + j'$ of sign $+1$ resp. -1.

Step 3: the column counts. From (LF4) and Step 1 (with the extension $\varsigma(i) = \frac{\Delta}{2} - \kappa_{i+2-k}$ for $i \ge r$, and $\frac{\Delta}{2} > \frac{t-1}{2}$):

- for $1 \le h \le r_+ - 1$: $\varsigma(h) \ge \frac{\Delta}{2} - \kappa_j \ge \frac{t-1}{2} - \kappa_j$, so $\kappa_O^+(\tau, h) = \varsigma(h-1) - \varsigma(h) - 1 \ge 0$; $\kappa_O^+(\tau, r_+) = \frac{\Delta}{2} - \kappa_j - \frac{t-1}{2} + \kappa_j - 1 = \Delta'$ (as $\varsigma(r_+) = -\frac{\Delta}{2} - \kappa_j < \frac{t-1}{2} - \kappa_j$); $\kappa_O^+(\tau, h) = 0$ for $h > r_+$ (then $\varsigma(h-1) \le \varsigma(r_+)$);
- for $1 \le h < m - j$: $\kappa_O^-(\tau, h) = \Delta - 1$ (the minimum is $\varsigma(r-h-1) = \frac{\Delta}{2} - \kappa_{m-h}$, as $m - h > j$, and $\varsigma(r-h) = \frac{\Delta}{2} - \kappa_{m+1-h}$); $\kappa_O^-(\tau, m-j) = -\kappa_j - \frac{t-1}{2} - \frac{\Delta}{2} + \kappa_{j+1} - 1 = \Delta'$; $\kappa_O^-(\tau, h) = 0$ for $h > m - j$ (then $\varsigma(r-h) \ge \frac{\Delta}{2} - \kappa_j > -\kappa_j - \frac{t-1}{2}$).

For a positive c ($\mathrm{ord}_O(c) = q$), as $r_+ < |\mathcal{B}_j|$ the sum telescopes:

$$\sum_{q < p < |\mathcal{B}_j|} \kappa_O^+(\tau, p) = \varsigma(q) - \varsigma(r_+ - 1) - (r_+ - 1 - q) + \Delta'.$$

For a negative c with $q = k - 2 + j'$ ($j < j' \le m$), $\mathrm{ord}_O(c) = r - 1 - q = m - j'$, and as $m - j < |\mathcal{B}^\circ|$,

$$\sum_{m - j' < p < |\mathcal{B}^\circ|} \kappa_O^-(\tau, p) = (j' - j - 1)(\Delta - 1) + \Delta'.$$

Step 4: the weights u° and $u - u^\circ$. Let u° be the weights of Lemma 6.6.52 for o ($\lfloor m/2 \rfloor = \frac{m}{2}$): $u_\tau^\circ(c) = j - \frac{m}{2}$ for $s_j(c) = 1$ and, by Step 3, $u_\tau^\circ(c) = j - \frac{m}{2} + |\mathcal{B}^\circ| + (j' - j - 1)(\Delta - 1) + \Delta'$ for $q = k - 2 + j'$ negative. Let w be the weights of Lemma 6.6.49. For positive c, by Step 3 and $\varsigma(r_+ - 1) = \frac{\Delta}{2} - \kappa_j$, $\frac{\Delta}{2} - \Delta' = \frac{t+1}{2}$, $e_j = j + \frac{t+k-1}{2}$,

$$w_\tau(c) = \kappa_j - e_j - \varsigma(q) + \tfrac{\Delta}{2} - \kappa_j + (r_+ - 1 - q) - \Delta' = -\varsigma(q) - q + \tfrac{k}{2} - 1,$$

and $u_\tau(c) - u_\tau^\circ(c) = -\varsigma(q) - q + r_+ - \frac{r+1}{2} - j + \frac{m}{2} = -\varsigma(q) - q + \frac{k}{2} - 1$, as $r_+ - j = k - 1$ and $\frac{r+1-m}{2} = \frac{k}{2}$. For negative c with $q = k - 2 + j'$, $w_\tau(c) = \kappa_j - e_j + |\mathcal{B}_j|$, and with $-\varsigma(q) = \kappa_j + (j' - j)\Delta - \frac{\Delta}{2}$ and $-q + r_+ = j - j' + 1$,

$$u_\tau(c) - u_\tau^\circ(c) = \kappa_j + \tfrac{\Delta}{2} - \Delta' - j + \tfrac{r-1}{2} + \tfrac{m}{2} - |\mathcal{B}^\circ| = \kappa_j + \tfrac{t+1}{2} - j + \tfrac{r-1}{2} + \tfrac{m}{2} - |\mathcal{B}^\circ|;$$

its difference with $w_\tau(c)$ is $\frac{t+1}{2} + \frac{r-1}{2} + \frac{m}{2} + \frac{t+k-1}{2} - (|\mathcal{B}^\circ| + |\mathcal{B}_j|) = \frac{2t+r+m+k-1}{2} - (r+t) = 0$, as $|\mathcal{B}^\circ| + |\mathcal{B}_j| = r + t$ and $k = r + 1 - m$. So $u - u^\circ = w$.

Step 5: (a). *K-dominance.* Within each sign, $o(a) < o(b)$ means $\mathrm{num}(a) > \mathrm{num}(b)$, so K-dominance means that the weight does not decrease when q increases within a sign. For u_τ, by Step 2, the increment from q to $q+1$ (same sign) is $\varsigma(q) - \varsigma(q+1) - 1 \ge 0$, the $\varsigma(i)$ being strictly decreasing integers. For u_τ° it is 0 on the positive vectors and $\Delta - 1 \ge 0$ on the negative ones. *Holomorphic gap.* If $j = m$ there are no negative vectors and nothing is to be shown. Otherwise, u_τ being nondecreasing in q within each sign, its largest positive value is at $q = r_+ - 1$, namely $\kappa_j - \frac{\Delta}{2} + 1 - \frac{r+1}{2}$, and its smallest negative value at $q = r_+$, namely $\kappa_{j+1} - \frac{\Delta}{2} + \frac{r-1}{2}$; their difference is $\Delta + r - 1 \ge \Delta' \ge 300 n_*$. For u_τ° the smallest negative value minus the positive value is $|\mathcal{B}^\circ| + \Delta' \ge 300 n_*$. So both have holomorphic gap $300 n_*$ for s_j. *Size.* For $q \le k - 2$, $|u_\tau(c)| = |j - \frac{m}{2}| \le m$. Otherwise $\kappa_{j'} - \frac{\Delta}{2} = \kappa + j'\Delta$ lies in $[\kappa + \Delta, \kappa_m]$, so $u_\tau(c)$ lies between $\kappa + \Delta - m - r > 0$ and $\kappa_m + m + r \le 2b + 3m + 3x^5 \le 5b$. So $|u_\tau(c)| \le 5b$.

Step 6: (b), (c), (d). By Step 1, o satisfies the hypotheses of Lemma 6.6.46, Lemma 6.6.49 and Lemma 6.6.52. (b) is Lemma 6.6.46 (good forms lie in $\mathcal{A}_M(W_j)$). (c) is (b) of Lemma 6.6.49, as $w = u - u^\circ$ by Step 4. For (d): $\#\mathcal{R} = r \le t$, o is positive-first, and u, u° satisfy (a); so Lemma 6.6.68 (with $u' = u^\circ$) gives $\int_{G_\infty(s_j)} |m_{s_j,u^\circ}^{\mathrm{hol}}(h)|\,|h|^{6n_*}\,dh < \infty$ with $|h| \ge 1$, hence $m_{s_j,u^\circ}^{\mathrm{hol}}$ (a continuous function) is integrable, and the u_τ° are K-dominant by (a). Then (d) is Lemma 6.6.52 for $y \in \mathcal{A}^{\mathrm{good}}(W_j) \subseteq \mathcal{A}_M(W_j)$. □

Lemma 6.6.46. *Let $1 \le j \le m$, and for $\tau \in \Phi$ let $l_\tau \colon \mathcal{R} \to \mathbb{Z}$, $l_\tau(c) = 2\varsigma_\tau(\mathrm{num}(c))$, $\varsigma_\tau(i)$ the entry of number i of $\varsigma(\tau)$. Let o be an order on $\mathcal{R}$, positive-first for s_j, such that $l_\tau(a) < l_\tau(b)$ whenever $s_j(a) = s_j(b)$ and $o(a) < o(b)$, and let $u = (u_\tau)_{\tau\in\Phi}$ with $u_\tau = \mu_{-l_\tau}$ the lowest K-type of $-l_\tau$ for the signs s_j. Then for every $y \in \mathcal{A}_M(W_j)$, in particular for every good form y on W_j, the conjugate $\bar y$ is adelically holomorphic of weight u for o.*

Proof. **Step 1: every form of principal level is a normalised descent.** By Proposition 5.8.4 (with $V = V' = W_j$, for which ς has gap Δ by its (iii)), $\mathcal{A}_M(W_j)$ is finite-dimensional. Its Petersson product $(y_1, y_2) = \int_{[\mathrm{U}(W_j)]} y_1 \bar y_2$ is positive definite: a nonzero continuous function has $|y|^2 > 0$ on a nonempty open set, which has positive quotient measure. By (b) of Lemma 6.6.30, $U = \mathrm{Th}_O(\mathrm{Form}(O))$ is a subspace of $\mathcal{A}_M(W_j)$. Let $y \in \mathcal{A}_M(W_j)$ be orthogonal to U. By (a) of Lemma 6.6.30, $\langle \Xi, \Theta_{j,1}(y)\rangle = (y, \mathrm{Th}_O\Xi) = 0$ for every $\Xi \in \mathrm{Form}(O)$; for $\Xi = \Theta_{j,1}(y)$ this gives $\langle \Xi, \Xi\rangle = 0$. The inner product of $\mathrm{Form}(O)$ is positive definite: it is a class sum with positive weights of $\langle \Xi(h), \Xi(h)\rangle_{\kappa_O}$, the Gram form is positive definite ((b) of Lemma 4.6.10), and by (DP5) a form vanishing at the class representatives vanishes. So $\Theta_{j,1}(y) = 0$, and $y = 0$ by (1) of Proposition 6.5.9 (the lift $\Theta^{\mathrm{pr}}_{j,1}$ there is $\Theta_{j,1}$ on $\mathcal{A}_M(W_j)$). Hence $U^\perp = 0$ and, the dimension being finite, $U = \mathcal{A}_M(W_j)$: every $y \in \mathcal{A}_M(W_j)$ is $\mathrm{Th}_O(\Xi)$ for some $\Xi \in \mathrm{Form}(O)$.

Step 2: the theta datum of the highest vector is holomorphic. By Lemma 6.6.41, O fits l, and $1 \le r < n$. By Lemma 6.4.47, $\mathrm{Har}_O(\xi_O) = \prod_\tau P_\tau$ with P_τ in the variables of τ a joint harmonic for the signs $(s_j, \mathbf{1}_n)$, the weights $(r_-(O), \lfloor n/2\rfloor + \mathrm{wt}_O(\tau))$ and the parameters $(l_\tau, l_{O,\tau})$; by (b) of (HV7), P_τ is jointly highest of weights $(\mu_{l_{O,\tau}}, \mu_{-l_\tau})$ for $(l_{O,\tau}, -l_\tau)$ for the row and column units of $(s_j, \mathbf{1}_n)$ with the shifts $(r_-(O), -(\lfloor n/2\rfloor + \mathrm{wt}_O(\tau)))$. Put $\gamma = r_-(O)$ and $\gamma'(\tau) = \lfloor n/2\rfloor + \mathrm{wt}_O(\tau)$. By the hypothesis on o and Lemma 6.6.47 (with $l_v = l_\tau$, $l'_v = l_{O,\tau}$, $u_v = \mu_{-l_\tau}$, $e = \gamma'$), $\mathrm{Har}_O(\xi_O) \ne 0$ and $v = v_{\mathrm{Har}_O(\xi_O)}$ is holomorphic of weight u for the pair (γ, γ'). In particular $\xi_O \ne 0$. The characters $\chi_O = \upsilon^{r_-(O)}$ and $\chi'_O = \mu_O \upsilon^{\lfloor n/2\rfloor}$ of $\omega_O = \omega_{\chi_O,\chi'_O}$ (DP6) have the weights γ and γ', since υ has weight 1 ((ii) of Lemma 3.2.11) and μ_O has weight wt_O. By (3) of Lemma 6.6.48 with $S = W_j$, $T = \mathrm{I}_n$, $f = \Upsilon(O)$, for every $h' \in \mathrm{G}(1_n)$ the function

$$g \longmapsto \theta^{W_j,\mathrm{I}_n}_{\chi_O,\chi'_O}(\mathrm{J}_O(\xi_O))(g, h') = \theta_O\big(\omega_O(g, h')\,\mathrm{J}_O(\xi_O)\big)$$

is adelically holomorphic of weight u for o.

Step 3: all theta data. The highest vector has integer coordinates (y_{id} has entries 0 and 1 and the b_i have integer coefficients), so $\bar\xi_O = \xi_O$. The span of the $\rho(y)\xi_O$, $y \in G_\infty(1_n)$, is a nonzero ρ-invariant subspace of $\mathbb{V}_O$, hence $\mathbb{V}_O$ (the Weyl module is irreducible under $\rho(G_\infty(1_n))$: it is the irreducible polynomial representation of $\prod_\tau \mathrm{GL}_n(\mathbb{C})$ of its highest weight, read in the dual basis and twisted by a power of the determinant, and $\prod_\tau \mathrm{U}(n)$ is Zariski dense; classical). Conjugation of coordinates is a conjugate-linear bijection of $\mathbb{V}_O$, so the $\overline{\rho(y)\xi_O}$ span $\mathbb{V}_O$ as well: every $\zeta \in \mathbb{V}_O$ is $\sum_k c_k \overline{\rho(y_k)\xi_O}$ for finitely many $c_k \in \mathbb{C}$, $y_k \in G_\infty(1_n)$. As J_O is linear, (a) of Lemma 6.6.31 (with $\xi = \xi_O = \bar\xi_O$) gives

$$\mathrm{J}_O(\zeta) = \sum_k c_k\,\mathrm{J}_O\big(\overline{\rho(y_k)\xi_O}\big) = \sum_k c_k\,\omega_O(1, (y_k, 1))\mathrm{J}_O(\xi_O),$$

and for $h \in G_f(n)$, as ω_O is a representation ((2) of Lemma 4.1.1) and $(1, h)\cdot(y_k, 1) = (y_k, h)$ in $\mathrm{G}(1_n)$,

$$\theta_O\big(\omega_O(g, (1, h))\,\mathrm{J}_O(\zeta)\big) = \sum_k c_k\,\theta^{W_j,\mathrm{I}_n}_{\chi_O,\chi'_O}(\mathrm{J}_O(\xi_O))(g, (y_k, h)).$$

By Step 2 and (1) of Lemma 6.6.48, $g \mapsto \theta_O(\omega_O(g, (1, h))\mathrm{J}_O(\zeta))$ is adelically holomorphic of weight u for o, for every $\zeta \in \mathbb{V}_O$ and $h \in G_f(n)$.

Step 4: conclusion. Let $y \in \mathcal{A}_M(W_j)$ and Ξ as in Step 1. By (DP9), $\bar{y}(g) = \overline{\mathrm{Th}_O(\Xi)(g)} = \sum_{[h]} \frac{1}{\#\Gamma_h} \theta_O(\omega_O(g,(1,h))\, \mathrm{J}_O(\overline{\Xi(h)}))$, a finite sum over the classes at fixed representatives h, with $\zeta = \overline{\Xi(h)} \in \mathbb{V}_O$. By Step 3 and (1) of Lemma 6.6.48, $\bar{y}$ is adelically holomorphic of weight u for o. Good forms lie in $\mathcal{A}_M(W_j)$ (Definition 3.2.16). □

Lemma 6.6.47. *Let n be a nonempty finite index set with signs s, n' a finite index set, o an order on n that is positive-first for s, and $\gamma, e\colon \Phi \to \mathbb{Z}$. For every $v \in \Phi$ let $l_v\colon n \to \mathbb{Z}$ with $l_v(a) < l_v(b)$ whenever $s(a) = s(b)$ and $o(a) < o(b)$, let $l'_v\colon n' \to \mathbb{Z}$, $\nu_v\colon n' \to \mathbb{Z}$ and $u_v\colon n \to \mathbb{Z}$, and let $P_v \in \mathbb{C}[X_{ac} : a \in n', c \in n]$ be jointly highest of weights (ν_v, u_v) for $(l'_v, -l_v)$, for the row and column units of the signs $(s, \mathbf{1}_{n'})$ with the shifts $(\gamma(v), -e(v))$. Let $Q = \prod_{v\in\Phi} P_v(X_{v,\cdot,\cdot})$, the polynomial P_v being written in the variables $X_{v,a,c}$. Then $Q \neq 0$, and for all spaces with points S, T with signs s on n and $+1$ on n' and every bounded linear functional ℓ on $\mathcal{F}_{S,T}$, the function*

$$x \longmapsto \ell(\Omega_{\gamma,e}(x,1)\, v_Q) \qquad (x \in G_\infty(s))$$

is holomorphic of weight $u = (u_v)_{v\in\Phi}$ for o.

Proof. Fix S, T and ℓ, and write $f(x) = \ell(\Omega_{\gamma,e}(x,1)v_Q)$ $(x \in G)$, $G = G_\infty(s)$, $M = \mathrm{M}_n(\mathbb{C})^\Phi$, $\mathfrak{g} = \{X \in M : X_v^* \mathrm{diag}(s) + \mathrm{diag}(s) X_v = 0\ \forall v\}$ the Lie algebra of G, $\mathfrak{b} \subseteq M$ the families of matrices Z with $(Z_v)_{ij} = 0$ whenever $o(j) < o(i)$, and $B \subseteq \mathfrak{b}$ the families of o-triangular matrices (the invertible elements of $\mathfrak{b}$). For $Z \in M$ put $\lambda(Z) = \sum_{v\in\Phi}\sum_{c\in n} u_v(c)(Z_v)_{cc}$ and, for $b \in B$, $b^u = \prod_v \prod_c b_{v,cc}^{u_v(c)}$ as in (HV3). Products and inverses of elements of B lie in B, and the diagonal of a product of elements of $\mathfrak{b}$ is the product of the diagonals, o being injective; so $(bb')^u = b^u b'^u$ and $\lambda(bZb^{-1}) = \lambda(Z)$ for $b \in B$, $Z \in \mathfrak{b}$, and $(\exp Z)^u = e^{\lambda(Z)}$ for $Z \in \mathfrak{b}$.

Step 1: $Q \neq 0$, and the column action. Each P_v is nonzero by (HV7), and the P_v are written in disjoint sets of variables, so $Q \neq 0$. For $Z \in M$ let $\rho(Z)$ be the operator $\sum_{v\in\Phi}\sum_{c,c'\in n}(Z_v)_{cc'} E^{\mathrm{col}}_{v,cc'}$ on $\mathbb{C}[X_\iota]$, where $E^{\mathrm{col}}_{v,cc'}$ is the column unit of $(s, \mathbf{1}_{n'})$ with the shifts $(\gamma(v), -e(v))$ in the variables $X_{v,a,c}$; it is $\mathbb{C}$-linear in Z. We show

$$\rho(Z)Q = \lambda(Z)\, Q \qquad (Z \in \mathfrak{b}). \tag{6.145}$$

As $E^{\mathrm{col}}_{v,cc'}$ involves only the variables at v, $E^{\mathrm{col}}_{v,cc'}Q = (\prod_{w\neq v} P_w) E^{\mathrm{col}}_{v,cc'} P_v$, so it suffices that $E^{\mathrm{col}}_{v,cc} P_v = u_v(c) P_v$ and $E^{\mathrm{col}}_{v,cc'} P_v = 0$ whenever $o(c) < o(c')$. The first is the weight condition of P_v ((HV6) for the column units, order $-l_v$). Let $o(c) < o(c')$. If $s(c) = s(c')$, then $l_v(c) < l_v(c')$ by hypothesis, i.e. $-l_v(c') < -l_v(c)$, and $E^{\mathrm{col}}_{v,cc'} P_v = 0$ by (HV6). If $s(c) \neq s(c')$, then $s(c) = 1$ and $s(c') = -1$, o being positive-first; the variables $X_{v,a,c}$ and $X_{v,a,c'}$ have the signs 1 and -1, so by (HV5) $E^{\mathrm{col}}_{v,cc'} = -\sum_a \mathrm{lo}_{ac}\mathrm{ra}_{ac'} = \sum_{a\in n'} \partial^2/\partial X_{v,a,c}\partial X_{v,a,c'}$, and this kills P_v because P_v is termwise harmonic (two variables of one row from columns of opposite signs).

Step 2: derivatives along one-parameter subgroups. By Lemma 6.4.24 (signs $(s, \mathbf{1}_{n'})$, weights (γ, e)), for $X \in \mathfrak{g}$ and every polynomial P the map $t \mapsto \Omega_{\gamma,e}(\exp(tX), 1)v_P$ is differentiable at 0 with derivative $v_{\rho(X)P}$.

By (2) of Lemma 4.1.1, $\Omega_{\gamma,e}$ is a representation by bounded operators of $\mathcal{F}_\varepsilon$, and $x \mapsto \Omega_{\gamma,e}(x,1)w$ is continuous for every w (continuity of $(g,w) \mapsto \omega_\varepsilon^\infty(g)w$, of the pair map and of the determinants). Hence, writing $\Omega_{\gamma,e}(x\exp((t+h)X),1) = \Omega_{\gamma,e}(x\exp(tX),1)\Omega_{\gamma,e}(\exp(hX),1)$ and applying the bounded functional $\ell \circ \Omega_{\gamma,e}(x\exp(tX),1)$, for $x \in G$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$:

$$\frac{d}{dt} f(x\exp(tX)) = \Lambda_{x\exp(tX)}(X), \qquad \Lambda_x(Z) = \ell(\Omega_{\gamma,e}(x,1)v_{\rho(Z)Q}) \quad (Z \in M). \tag{6.146}$$

For fixed x, $Z \mapsto \Lambda_x(Z)$ is $\mathbb{C}$-linear, and for each Z the function $x \mapsto \Lambda_x(Z)$ is continuous. By (6.145), $\Lambda_x(Z) = \lambda(Z)f(x)$ for $Z \in \mathfrak{b}$.

Step 3: the torus. Let $d \in G \cap B$. Then $d_v^{-1} = \mathrm{diag}(s) d_v^* \mathrm{diag}(s)$ lies in $\mathfrak{b}$ and its (i,j) entry $s_i s_j \overline{(d_v)_{ji}}$ vanishes when $o(i) < o(j)$; so d_v^{-1}, hence d_v, is diagonal, and its entries have modulus 1. Write $d = \exp D$ with $D_v = \mathrm{diag}(i\theta_{v,c})_c$, $\theta_{v,c} \in \mathbb{R}$; then $D \in \mathfrak{g} \cap \mathfrak{b}$. By (6.146) and Step 2, $\phi(t) = e^{-t\lambda(D)} f(x \exp(tD))$ has derivative $e^{-t\lambda(D)}(-\lambda(D) f(x\exp(tD)) + \lambda(D) f(x \exp(tD))) = 0$, so $\phi(1) = \phi(0)$:

$$f(xd) = e^{\lambda(D)} f(x) = d^u f(x) \qquad (x \in G,\ d \in G \cap B). \tag{6.147}$$

Step 4: the extension. Let $\mathcal{U} = \{xb : x \in G,\ b \in B\} \subseteq M$ (products taken place by place). If $xb = x'b'$ with $x, x' \in G$, $b, b' \in B$, then $d = x'^{-1}x = b'b^{-1} \in G \cap B$, $b' = db$ and, by (6.147), $b'^u f(x') = b^u d^u f(x') = b^u f(x'd) = b^u f(x)$. So $\tilde{F}(xb) = b^u f(x)$ defines a function on $\mathcal{U}$; put $\tilde{F} = 0$ on $M \setminus \mathcal{U}$. It remains to show that every point of $\mathcal{U}$ has a neighbourhood in $\mathcal{U}$ on which $\tilde{F}$ is complex differentiable; then $\tilde{F}$ is holomorphic, hence complex analytic, near every point xb, which is the assertion.

Step 5: the strictly lower part. For $Z \in M$ let $N(Z) \in M$ keep the entries $(Z_v)_{ij}$ with $o(j) < o(i)$ and replace the others by 0, $J = \mathrm{diag}(s)$, and $\mathrm{v}(Z) = N(Z) - JN(Z)^*J$ (place by place); N is $\mathbb{C}$-linear and v is $\mathbb{R}$-linear. Then $\mathrm{v}(Z) \in \mathfrak{g}$ (as $\mathrm{v}^*J = N^*J - JN = -J\mathrm{v}$), $JN(Z)^*J \in \mathfrak{b}$ has zero diagonal, and $Z - \mathrm{v}(Z) = (Z - N(Z)) + JN(Z)^*J \in \mathfrak{b}$ has the diagonal of Z. By Step 2, $\Lambda_x(JN(Y)^*J) = \lambda(JN(Y)^*J) f(x) = 0$, so

$$\Lambda_x(\mathrm{v}(Y)) = \Lambda_x(N(Y)) \qquad (x \in G,\ Y \in M). \tag{6.148}$$

Step 6: complex differentiability at the identity of a chart. Fix $x_0 \in G$ and put $h(Z) = f(x_0 \exp \mathrm{v}(Z))$ for $Z \in M$. For $Y \in M$, $\mathrm{v}(tY) = t\mathrm{v}(Y)$, so by (6.146) and (6.148) the function $t \mapsto h(tY)$ is differentiable with derivative $D(tY)Y$, where $D(Z)Y = \Lambda_{x_0 \exp \mathrm{v}(Z)}(N(Y))$. Each $D(Z)$ is $\mathbb{C}$-linear, and $Z \mapsto D(Z)$ is continuous for the operator norm: in the basis of matrix units E_k of M, $D(Z)Y = \sum_k N(Y)_k \Lambda_{x_0 \exp \mathrm{v}(Z)}(E_k)$ with coefficients continuous in Z (Step 2). The derivative $t \mapsto D(tY)Y$ is continuous, so $h(Y) - h(0) = \int_0^1 D(tY)Y\, dt$ and

$$|h(Y) - h(0) - D(0)Y| \le \sup_{0 \le t \le 1} \|D(tY) - D(0)\|\, |Y| = o(|Y|) \qquad (Y \to 0).$$

Thus h is complex differentiable at 0 with derivative $D(0)$.

Step 7: complex differentiability on $\mathcal{U}$. Let $\Theta(Z) = \exp(\mathrm{v}(Z))\,(1 + Z - \mathrm{v}(Z))$, a smooth map $M \to M$ with $\Theta(0) = 1$ and real derivative the identity at 0. By the inverse function theorem there are open neighbourhoods W of 0 and V of 1, with $1 + Z - \mathrm{v}(Z)$ invertible for $Z \in W$, such that $\Theta\colon W \to V$ is a diffeomorphism; its inverse ψ has real derivative the identity at 1. Let $g_0 = x_0 b_0 \in \mathcal{U}$ and $V_0 = \{x_0 y b_0 : y \in V\}$, an open neighbourhood of g_0. For $g \in V_0$ put $Z = \psi(x_0^{-1} g b_0^{-1})$; then $g = (x_0 \exp \mathrm{v}(Z))\,((1 + Z - \mathrm{v}(Z)) b_0)$ with the first factor in G and the second in B, so $V_0 \subseteq \mathcal{U}$ and, as the diagonal of $1 + Z - \mathrm{v}(Z)$ is that of $1 + Z$,

$$\tilde{F}(g) = k(\psi(x_0^{-1} g b_0^{-1})), \qquad k(Z) = \prod_{v \in \Phi} \prod_{c \in n} (1 + Z_{v,cc})^{u_v(c)}\ b_0^u\ h(Z).$$

The product is holomorphic near $Z = 0$ and h is complex differentiable at 0 (Step 6), so k is complex differentiable at $0 = \psi(1)$. The map $g \mapsto x_0^{-1} g b_0^{-1}$ is $\mathbb{C}$-linear and ψ has the $\mathbb{C}$-linear derivative id at 1; by the chain rule $\tilde{F}$ is real differentiable at g_0 with $\mathbb{C}$-linear derivative, i.e. complex differentiable at g_0. As $g_0 \in \mathcal{U}$ was arbitrary and $V_0 \subseteq \mathcal{U}$, $\mathcal{U}$ is open and $\tilde{F}$ is complex differentiable at every point of $\mathcal{U}$; so $\tilde{F}$ is holomorphic on $\mathcal{U}$ (complex differentiable at every point of an open subset of $\mathbb{C}^N$, hence continuous and holomorphic in each variable, hence holomorphic by Osgood's lemma), and therefore complex analytic at every point of $\mathcal{U}$. By Step 4, $\tilde{F}(xb) = b^u f(x)$ for $x \in G$, $b \in B$: f is holomorphic of weight u for o. □

Lemma 6.6.48. *Let S, T be spaces with points with signs s on a nonempty index set n and s' on n', let α, β be automorphic characters of $\mathrm{U}(1)$ of weights γ, $\gamma' \colon \Phi \to \mathbb{Z}$, let o be an order on n and $u = (u_v)_{v\in\Phi}$ weights on n. Call $v \in \mathcal{F}_{S,T}$ **holomorphic of weight** u if for every bounded linear functional ℓ on $\mathcal{F}_{S,T}$ the function $x \mapsto \ell(\Omega_{\gamma,\gamma'}(x,1)v)$ on $G_\infty(s)$ is holomorphic of weight u for o. Then:*

(1) a finite linear combination of functions on $G_\infty(s)$ (resp. on $\mathrm{G}(s)$) that are holomorphic (resp. adelically holomorphic) of weight u for o is holomorphic (resp. adelically holomorphic) of weight u for o;

(2) the vectors holomorphic of weight u form a linear subspace of $\mathcal{F}_{S,T}$ that contains $\Omega_{\gamma,\gamma'}(1,y)v$ for every $y \in G_\infty(s')$ and every v in it;

(3) if v is holomorphic of weight u, then for every $f \in \mathcal{M}_{S,T}$ and every $h \in \mathrm{G}(s')$ the function $g \mapsto \theta^{S,T}_{\alpha,\beta}(v\otimes f)(g,h)$ on $\mathrm{G}(s)$ is adelically holomorphic of weight u for o; in particular, if v_P is holomorphic of weight u, so is $g \mapsto \theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Phi,P](g,h)$ for every modulus $\mathfrak{m}$ and every function Φ on $n' \times n$ matrices over $R_\mathfrak{m}$.

Proof. **(1).** Let $F_1,\dots,F_p$ be holomorphic of weight u for o on $G_\infty(s)$, with functions $\tilde F_k$ as in (HV3), and $c_1,\dots,c_p \in \mathbb{C}$. Then $\tilde F = \sum_k c_k\tilde F_k$ is complex analytic at every point xb ($x \in G_\infty(s)$, b a family of o-triangular matrices) at which all $\tilde F_k$ are, and $\tilde F(xb) = \bigl(\prod_v b_v^{u_v}\bigr)\sum_k c_kF_k(x)$; so $\sum_k c_kF_k$ is holomorphic of weight u for o. For functions on $\mathrm{G}(s)$ apply this, for each $g \in \mathrm{G}(s)$, to the functions $x \mapsto F_k(g(x,1))$.

(2). By (1), sums and scalar multiples of vectors holomorphic of weight u are holomorphic of weight u, and 0 is. Let v be holomorphic of weight u, $y \in G_\infty(s')$ and ℓ a bounded linear functional on $\mathcal{F}_{S,T}$. As $\Omega_{\gamma,\gamma'}$ is a representation of $G_\infty(s)\times G_\infty(s')$ ((2) of Lemma 4.1.1) and $(x,1)(1,y) = (1,y)(x,1)$,

$$\ell(\Omega_{\gamma,\gamma'}(x,1)\,\Omega_{\gamma,\gamma'}(1,y)v) = \ell'(\Omega_{\gamma,\gamma'}(x,1)v), \qquad \ell' = \ell\circ\Omega_{\gamma,\gamma'}(1,y),$$

and ℓ' is a bounded linear functional, $\Omega_{\gamma,\gamma'}(1,y)$ being a scalar times the bounded operator $\omega^\infty_\varepsilon$ of an element of the unitary group ((2) of Lemma 4.1.1). So $\Omega_{\gamma,\gamma'}(1,y)v$ is holomorphic of weight u.

(3). Fix f, h and $g \in \mathrm{G}(s)$, and let $\ell(w) = \theta_{\mathrm{Hom}(S,T)}(\omega_{\alpha,\beta}(g,h)(w\otimes f))$ for $w \in \mathcal{F}_{S,T}$. The map $w \mapsto w\otimes f$ is linear and $\|w\otimes f\| \le \|w\|\,\|f\|$ by (1) of Lemma 6.2.21; $\omega_{\alpha,\beta}(g,h)$ is a scalar of modulus 1 times a bounded operator ((2) of Lemma 4.1.1), and the theta functional is bounded. So ℓ is a bounded linear functional on $\mathcal{F}_{S,T}$. For $x \in G_\infty(s)$, $(g(x,1),h) = (g,h)\cdot((x,1),1)$, so by (2) and (5) of Lemma 4.1.1 (with $k=1$, $y=1$, $k'=1$; $\omega_{\alpha,\beta}(1,1)f = f$)

$$\theta^{S,T}_{\alpha,\beta}(v\otimes f)(g(x,1),h) = \theta_{\mathrm{Hom}(S,T)}\Bigl(\omega_{\alpha,\beta}(g,h)\bigl((\Omega_{\gamma,\gamma'}(x,1)v)\otimes f\bigr)\Bigr) = \ell(\Omega_{\gamma,\gamma'}(x,1)v),$$

which is holomorphic of weight u for o as a function of x. As g was arbitrary, $g \mapsto \theta^{S,T}_{\alpha,\beta}(v\otimes f)(g,h)$ is adelically holomorphic of weight u for o. The last assertion is the case $v = v_P$, $f = \Phi^\natural\otimes_{\Sigma(\mathfrak{m})}v^0$, since $\theta^{S,T}_{\alpha,\beta;\mathfrak{m}}[\Phi,P] = \theta^{S,T}_{\alpha,\beta}(\mathrm{mod}_\mathfrak{m}(\Phi,v_P))$ and $\mathrm{mod}_\mathfrak{m}(\Phi,v_P) = v_P\otimes(\Phi^\natural\otimes_{\Sigma(\mathfrak{m})}v^0)$ (Definition 6.3.1, (XD5)). □

Lemma 6.6.49. *Let $1 \le j \le m$, $O = O_{j,1}$, and for $\tau \in \Phi$ let $l_\tau(c) = 2\varsigma_\tau(\mathrm{num}(c))$ ($c \in \mathcal{R}$). Let o be an order on $\mathcal{R}$, positive-first for s_j, such that $l_\tau(a) < l_\tau(b)$ whenever $s_j(a) = s_j(b)$ and $o(a) < o(b)$. Let $w = (w_\tau)_{\tau\in\Phi}$ be the weights on $\mathcal{R}$*

$$w_\tau(c) = \kappa_j(\tau) - e_j + |\mathcal{B}_j|\,[s_j(c) = -1] - [s_j(c) = 1]\sum_{\mathrm{ord}_O(c)<p<|\mathcal{B}_j|}\kappa^+_O(\tau,p),$$

$[\cdot]$ *being* 1 *if the condition holds and* 0 *otherwise. Let* $\chi_O = \upsilon^{r_-(O)}$ *and* $\nu'' = \nu''_{j,1}$. *Then for every choice* $i \in \mathrm{Mem}_j$*:*

(a) there are finitely many $c_k \in \mathbb{C}$, $h_k \in G_f(\mathcal{B}_j)$, *functions* Φ_k *on* $\mathcal{B}_j \times \mathcal{R}$ *matrices over* $R_{\mathfrak{M}^2}$ *and vectors* ξ_k *of the Weyl module* $\mathbb{V}_j^+$ *of* κ_O^+ *in rank* $|\mathcal{B}_j|$ *with rows* $\mathcal{B}_j$ *such that, with the letter substitution* Let_1 *of* O *read with the rows in* $\mathcal{B}_j$,

$$\mathsf{F}_i(g) = \sum_k c_k\, \theta^{W_j,\mathrm{I}_{\mathcal{B}_j}}_{\chi_O,\nu'';\mathfrak{M}^2}\big[\Phi_k, \mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}\xi_k)\big](g,h_k) \qquad (g \in \mathrm{U}(W_j)(\mathbb{A}_F));$$

(b) the choice function F_i *is adelically holomorphic of weight* w *for* o.

Proof. **Step 1: the choice function is a multiple of a coset kernel sum.** By Proposition 6.6.36 for the trivial twist there is $c_1 \in \mathbb{C}$ with $\mathsf{F}_i = c_1 L_{1,i}$ for all $i \in \mathrm{Mem}_j$. By Proposition 6.6.33 for the trivial twist, with $a = \mathcal{B}^{\mathrm{c}}$, u the identity matrix and $w = \Lambda_i$ (so $uw = \Lambda_i$), $L_{1,i} = \mathrm{cks}_{O,\nu''}(\Lambda_i, \phi_j; \Upsilon_{1,i})$, $\nu'' = \nu''_{j,1}$. For the trivial twist $Q_O = \emptyset$, so $\mathfrak{Q}_O = \mathcal{O}_E$, and by (KN5)

$$\mathsf{F}_i(g) = c_1 \sum_{h \bmod \mathbf{K}_O[\Lambda_i,\phi_j]}^{\mathrm{cl}} \theta^{W_j,\mathrm{I}_{\mathcal{B}_j}}_{\chi_O,\nu'';\mathfrak{M}^2}\Big[\mathrm{cd},\ \mathrm{Let}_1\big(\mathrm{Pol}_{\kappa_O^+}(\Upsilon_{1,i}(h))\big)\Big](g,h),$$

a finite sum at fixed representatives $h \in G_f(\mathcal{B}_j)$, with $\mathrm{cd} = \mathrm{cd}^{\mathrm{res}}_{\psi_O,\mathrm{red}_0,\Lambda_i,\phi_j}$, a function on $\mathcal{B}_j \times \mathcal{R}$ matrices over $R_{\mathfrak{M}^2}$ (red_0 being the identity of $R_{\mathfrak{M}^2}$). This is (a), with $c_k = c_1/\#\Gamma_h$, $h_k = h$, $\Phi_k = \mathrm{cd}$ and $\xi_k = \Upsilon_{1,i}(h)$.

Step 2: the letter polynomials are holomorphic. Apply Lemma 6.6.50 with $S = W_j$, $T = \mathrm{I}_{\mathcal{B}_j}$ (rows $\mathcal{B}_j$, as for the Weyl module $\mathbb{V}_j^+$ of κ_O^+ in rank $|\mathcal{B}_j|$), $\sigma = 1$, $\mathrm{ord} = \mathrm{ord}_O$ on the positive basis vectors, $\kappa = \kappa_O^+$, the order o, the maps l_τ, $\gamma = r_-(O)$ and $e(\tau) = e_j - \kappa_j(\tau)$. Its hypotheses: by (LF4), $\mathrm{ord}_O(c) = \mathrm{num}(c)$ on the positive basis vectors, which are those of numbers $0, \dots, r_+ - 1$ with $r_+ = k + j - 1$ ((HP4) of Definition 3.2.13), so ord_O is a bijection of them onto $\{0, \dots, r_+ - 1\}$; by Lemma 6.6.41, O fits l, so (ii) of (DP11) is (ii) of Lemma 6.6.50 and (iii) of (DP11) gives $\kappa_O^+(\tau, p) = 0$ for $r_+ < p < n$, in particular for $r_+ < p < |\mathcal{B}_j|$; the l_τ are injective (by (i) of (DP11), (a) of the block rule); (i) is the hypothesis on o. With $N = |\mathcal{B}_j|$ the weight of Lemma 6.6.50 is w. So for every $\xi \in \mathbb{V}_j^+$ the vector $v_{\mathrm{Let}_1(\mathrm{Pol}_{\kappa_O^+}\xi)}$ is holomorphic of weight w for the pair $(r_-(O), e_j - \kappa_j)$.

Step 3: conclusion. The character $\chi_O = \upsilon^{r_-(O)}$ has weight $r_-(O)$ and $\nu'' = \mu_j^0 \upsilon^{e_j}$ has weight $-\kappa_j + e_j$, υ having weight 1 ((ii) of Lemma 3.2.11). By Step 2 and (3) of Lemma 6.6.48 (with $S = W_j$, $T = \mathrm{I}_{\mathcal{B}_j}$, $\alpha = \chi_O$, $\beta = \nu''$, the modulus $\mathfrak{M}^2$ and $\Phi = \mathrm{cd}$), every summand in Step 1 is adelically holomorphic of weight w for o as a function of g, and by (1) of Lemma 6.6.48 so is F_i, which is (b). □

Lemma 6.6.50. *Let* S *be a space with a point with signs* s *on a nonempty index set* n, T *a space with a point with signs* $+1$ *on a nonempty totally ordered index set* n', $N = \#n'$, σ *a sign,* $n_\sigma = \{c \in n : s(c) = \sigma\}$ *and* $\mathrm{ord}\colon n_\sigma \to \{0, \dots, \#n_\sigma - 1\}$ *a bijection; write* $\mathrm{Let}_\sigma = \mathrm{Let}^{\mathrm{ord}}_\sigma$. *Let* κ *be a weight in rank* N *with rows* n' *such that* $\kappa(\tau, p) = 0$ *for* $\#n_\sigma < p < N$ *and all* $\tau \in \Phi$. *Let* o *be an order on* n *that is positive-first for* s, *and* $l = (l_\tau)_{\tau\in\Phi}$ *injective maps* $l_\tau\colon n \to \mathbb{Z}$ *such that for every* τ*:*

(i) $l_\tau(a) < l_\tau(b)$ *whenever* $s(a) = s(b)$ *and* $o(a) < o(b)$*;*

(ii) $\mathrm{ord}(c) = \#\{d \in n_\sigma : \sigma\, l_\tau(c) < \sigma\, l_\tau(d)\}$ *for every* $c \in n_\sigma$.

Let $\gamma, e\colon \Phi \to \mathbb{Z}$ and let $w = (w_\tau)_{\tau\in\Phi}$ be the weights on n

$$w_\tau(c) = -e(\tau) + N\,[s(c) = -1] - \sigma\,[s(c) = \sigma] \sum_{\mathrm{ord}(c)<p<N} \kappa(\tau, p),$$

$[\cdot]$ being 1 if the condition holds and 0 otherwise. Then the linear map $\xi \mapsto \mathrm{Let}_\sigma(\mathrm{Pol}_\kappa\xi)$ on $\mathbb{V}_\kappa$ is injective, and for every $\xi \in \mathbb{V}_\kappa$ and every bounded linear functional ℓ on $\mathcal{F}_{S,T}$ the function $x \mapsto \ell\big(\Omega_{\gamma,e}(x, 1)\, v_{\mathrm{Let}_\sigma(\mathrm{Pol}_\kappa\xi)}\big)$ on $G_\infty(s)$ is holomorphic of weight w for o.

Proof. **Step 1: the highest vector.** Let $\pi\colon n' \to \{0, \dots, N-1\}$ be a bijection and y^π the array with $y^\pi_{\tau,a,c} = 1$ if $c = \pi(a)$ and 0 otherwise, and put $\xi_0 = \mathrm{ev}_\kappa(y^\pi)$. By Lemma 6.4.50 (with $\pi_\tau = \pi$ for all τ),

$$\mathrm{Pol}_\kappa(\xi_0) = \prod_{\tau\in\Phi} \prod_{1\le p<N} M_{\tau,p}^{\kappa(\tau,p)}, \qquad M_{\tau,p} = \det\big(y_{\tau,\pi^{-1}(i),j}\big)_{0\le i,j<p}.$$

Let $c_j = \mathrm{ord}^{-1}(j)$ for $0 \le j < \#n_\sigma$. By definition $\mathrm{Let}_\sigma(y_{\tau,a,j}) = X_{\tau,a,c_j}$ for $j < \#n_\sigma$ and $= 0$ for $j \ge \#n_\sigma$. For $p \le \#n_\sigma$, $\mathrm{Let}_\sigma(M_{\tau,p}) = \det(X_{\tau,\pi^{-1}(i),c_j})_{0\le i,j<p}$; for $p > \#n_\sigma$ the exponent $\kappa(\tau, p)$ is 0. As Let_σ is a homomorphism of algebras, $Q_0 = \mathrm{Let}_\sigma(\mathrm{Pol}_\kappa\xi_0) = \prod_\tau P_\tau(X_{\tau,\cdot,\cdot})$ with

$$P_\tau = \prod_{1\le p\le \min(\#n_\sigma, N-1)} D_p^{\kappa(\tau,p)}, \qquad D_p = \det\big(X_{\pi^{-1}(i),c_j}\big)_{0\le i,j<p},$$

in the variables X_{ac} ($a \in n'$, $c \in n$).

Step 2: the staircase is jointly highest. Fix τ and apply Lemma 6.4.48 with $R = n'$, $C = n$, the signs s on C, the shifts $(\gamma_1, \gamma_2) = (\gamma(\tau), -e(\tau))$, $l'(a) = -\sigma\pi(a)$ and $l = l_\tau$ (both injective), $k(\sigma, p) = \kappa(\tau, p)$ and $k(-\sigma, p) = 0$ ($1 \le p < N$). Then $x_\sigma(a) = \#\{a' : -\pi(a') > -\pi(a)\} = \pi(a)$, so the row a_i of that lemma with $x_\sigma(a_i) = i$ is $\pi^{-1}(i)$; by (ii) the number $o(c)$ of that lemma is $\mathrm{ord}(c)$ for $c \in n_\sigma$, so its column of sign σ and number j is c_j; the condition $k(\eta, p) = 0$ for $p > \#\{c : s(c) = \eta\}$ holds by the hypothesis on κ for $\eta = \sigma$ and trivially for $\eta = -\sigma$; and since $k(-\sigma, \cdot) = 0$, $k(1, i) = 0$ or $k(-1, j) = 0$ for all i, j. Its polynomial is P_τ (the factors of sign $-\sigma$ have exponent 0), so P_τ is jointly highest of weights (ν_τ, w_τ) for $(l', -l_\tau)$ for the row and column units of the signs $(s, \mathbf{1}_{n'})$ with the shifts $(\gamma(\tau), -e(\tau))$, with some ν_τ and with the column weight of that lemma, which is

$$-e(\tau) + N\,[s(c) = -1] - s(c) \sum_{o(c)<p<N} k(s(c), p) = w_\tau(c),$$

the sum being 0 unless $s(c) = \sigma$.

Step 3: holomorphy of the highest vector. By (i), Step 2 and Lemma 6.6.47 (with $l_v = l_\tau$, $l'_v = l'$, $\nu_v = \nu_\tau$, $u_v = w_\tau$, $P_v = P_\tau$), $Q_0 \ne 0$ and v_{Q_0} is holomorphic of weight w: for every bounded linear functional ℓ on $\mathcal{F}_{S,T}$, $x \mapsto \ell(\Omega_{\gamma,e}(x, 1)v_{Q_0})$ is holomorphic of weight w for o. In particular $\xi_0 \ne 0$, as Pol_κ and Let_σ are linear.

Step 4: all vectors. Let H be the set of $\xi \in \mathbb{V}_\kappa$ for which $v_{\mathrm{Let}_\sigma(\mathrm{Pol}_\kappa\xi)}$ is holomorphic of weight w. As $\xi \mapsto v_{\mathrm{Let}_\sigma(\mathrm{Pol}_\kappa\xi)}$ is linear, H is a linear subspace by (2) of Lemma 6.6.48, and $\xi_0 \in H$ by Step 3. For $\xi \in H$ and $y \in G_\infty(1_{n'})$, Lemma 6.6.51 gives

$$v_{\mathrm{Let}_\sigma(\mathrm{Pol}_\kappa(S_\kappa(z_\sigma)\xi))} = \prod_v \det(y_v)^{r_- - \gamma(v)}\; \Omega_{\gamma,e}(1, y)\, v_{\mathrm{Let}_\sigma(\mathrm{Pol}_\kappa\xi)},$$

with $z_{+1} = y$, $z_{-1} = \bar{y}$, which is holomorphic of weight w by (2) of Lemma 6.6.48. As y runs over $G_\infty(1_{n'}) = \prod_\tau \mathrm{U}(N)$, so does z_σ (complex conjugation is a bijection of $\mathrm{U}(N)$). Hence H

is a nonzero subspace invariant under the operators $S_\kappa(z)$, $z \in \prod_\tau \mathrm{U}(N)$, equivalently under $\rho_{\kappa,0}(h) = S_\kappa(\bar{h})$, $h \in \prod_\tau \mathrm{U}(N)$ ((WM2)). The Weyl module is irreducible under these operators (it is the irreducible polynomial representation of $\prod_\tau \mathrm{GL}_N(\mathbb{C})$ of highest weight $(\lambda_\tau)_\tau$, read in the dual basis, and $\prod_\tau \mathrm{U}(N)$ is Zariski dense; classical), so $H = \mathbb{V}_\kappa$, which is the second assertion.

Step 5: injectivity. Pol_κ is injective, as the b_i are linearly independent and G_κ is invertible ((b) of Lemma 4.6.10). Each b_i is a product of minors $\Delta_{\tau,A}$ with $\#A = h$ and $\kappa(\tau,h) \neq 0$ ((WM1)), which involve the columns $0,\dots,h-1$ only; by the hypothesis on κ, $h \leq \#n_\sigma$. So every $\mathrm{Pol}_\kappa\xi$ is a polynomial in the variables $y_{\tau,a,j}$ with $j < \#n_\sigma$, on which Let_σ is the injective renaming $y_{\tau,a,j} \mapsto X_{\tau,a,c_j}$ of variables. Hence $\xi \mapsto \mathrm{Let}_\sigma(\mathrm{Pol}_\kappa\xi)$ is injective. □

Lemma 6.6.51. *Let S be a space with a point with signs s on a nonempty index set n, T a space with a point with signs $+1$ on a nonempty totally ordered index set n', $r_- = \#\{c \in n : s(c) = -1\}$ and $\gamma,\gamma' \colon \Phi \to \mathbb{Z}$. For a sign σ and a map $\mathrm{ord} \colon \{c \in n : s(c) = \sigma\} \to \mathbb{N}$ let $\mathrm{Let}^{\mathrm{ord}}_\sigma$ be the homomorphism of algebras sending $y_{\tau,a,i}$ $(a \in n')$ to $\sum X_{\tau,a,c}$ over the $c \in n$ with $s(c) = \sigma$ and $\mathrm{ord}(c) = i$, the $X_{\tau,a,c}$ being the Fock variables of $\mathrm{Hom}(S,T)$. Then for every weight λ in rank $\#n'$ with rows n', every $\alpha \in \mathbb{V}_\lambda$ and every $y \in G_\infty(1_{n'})$,*

$$\Omega_{\gamma,\gamma'}(1,y)\, v_{\mathrm{Let}^{\mathrm{ord}}_\sigma(\mathrm{Pol}_\lambda\alpha)} = \prod_{v\in\Phi} \det(y_v)^{\gamma(v)-r_-}\; v_{\mathrm{Let}^{\mathrm{ord}}_\sigma(\mathrm{Pol}_\lambda(S_\lambda(z_\sigma)\alpha))}, \qquad z_{+1} = y, \quad z_{-1} = \bar{y},$$

$\bar{y}$ the entrywise complex conjugate.

Proof. As T has only the sign $+1$, y_v is block diagonal for the signs of T (its block of sign -1 is empty) and so is $1 \in G_\infty(s)$; so the second part of (5) of Lemma 4.1.1 applies to $(x,y) = (1,y)$: the scalar is $\prod_v \det(y_v)^{\gamma(v)-r_-}$ (the factors of the empty blocks and of $x = 1$ are 1), and v_P goes to $v_{P'}$ with $P'(X) = P(X')$, where, writing $X_{\tau,\cdot,c} = (X_{\tau,a,c})_{a\in n'}$ for the columns,

$$X'_{\tau,\cdot,c} = y_\tau^{\mathsf{T}} X_{\tau,\cdot,c} \;\text{ if } s(c) = 1, \qquad X'_{\tau,\cdot,c} = y_\tau^{-1} X_{\tau,\cdot,c} \;\text{ if } s(c) = -1.$$

Let $Y^\sigma(X)$ be the array with entries $Y^\sigma(X)_{\tau,a,i} = \sum X_{\tau,a,c}$ over c with $s(c) = \sigma$, $\mathrm{ord}(c) = i$, so that $\mathrm{Let}^{\mathrm{ord}}_\sigma(Q)(X) = Q(Y^\sigma(X))$. Then $Y^1(X') = y^{\mathsf{T}} Y^1(X)$ and $Y^{-1}(X') = y^{-1} Y^{-1}(X)$, the matrices acting on the rows at every τ as in (WM2). For a family z of $n' \times n'$ matrices, by (WM2) and (a) of Lemma 4.6.10,

$$\mathrm{Pol}_\lambda(\alpha)(zY) = \sum_i (G_\lambda\alpha)_i \sum_j S_\lambda(z)_{ij}\, b_j(Y) = \sum_j \big(S_\lambda(z)^{\mathsf{T}} G_\lambda\alpha\big)_j b_j(Y) = \mathrm{Pol}_\lambda(S_\lambda(z^{\mathsf{T}})\alpha)(Y).$$

With $z = y^{\mathsf{T}}$ for $\sigma = 1$ and $z = y^{-1}$ for $\sigma = -1$, and $(y^{-1})^{\mathsf{T}} = \bar{y}$ as y_v is unitary, $\mathrm{Let}^{\mathrm{ord}}_\sigma(\mathrm{Pol}_\lambda\alpha)' = \mathrm{Let}^{\mathrm{ord}}_\sigma(\mathrm{Pol}_\lambda(S_\lambda(z_\sigma)\alpha))$, which is the claim. □

Lemma 6.6.52. *Let $1 \leq j \leq m$, $O = O_{j,1}$, and for $\tau \in \Phi$ let $l_\tau(c) = 2\varsigma_\tau(\mathrm{num}(c))$ $(c \in \mathcal{R})$. Let o be an order on $\mathcal{R}$, positive-first for s_j, such that $l_\tau(a) < l_\tau(b)$ whenever $s_j(a) = s_j(b)$ and $o(a) < o(b)$, and let $u^\circ = (u^\circ_\tau)_{\tau\in\Phi}$ be the weights on $\mathcal{R}$*

$$u^\circ_\tau(c) = j - \lfloor m/2 \rfloor + [s_j(c) = -1]\Big(|\mathcal{B}^\circ| + \sum_{\mathrm{ord}_O(c)<p<|\mathcal{B}^\circ|} \kappa^-_O(\tau,p)\Big),$$

$[\cdot]$ being 1 if the condition holds and 0 otherwise. Suppose that every u°_τ is K-dominant for (s_j,o) and that $m^{\mathrm{hol}}_{s_j,u^\circ}$ is integrable on $G_\infty(s_j)$. Let $y \in \mathcal{A}_M(W_j)$ satisfy $\int_{[\mathrm{U}(W_j)]} y\, \mathsf{F}_i\, \mathrm{tt}_j(\zeta) = 0$ for all $i \in \mathrm{Mem}_j$ and all $\zeta \colon \mathrm{Cl}^\circ \to \mathbb{V}^-_j$. Then for every $x \in \mathrm{G}(s_j)$ and all $(c_i)_{i\in\mathrm{Mem}_j}$ in $\mathbb{C}$ the function $h \mapsto y(x(h,1)) \sum_i c_i \mathsf{F}_i(x(h,1))\, m^{\mathrm{hol}}_{s_j,u^\circ}(h)$ is integrable on $G_\infty(s_j)$ with integral 0.

Proof. Put $\mathfrak{m} = \mathfrak{M}^2 = \pi_M^2\mathcal{O}_E$, $\mathrm{I} = V^\circ = \mathrm{I}_{\mathcal{B}^\circ}$, $\alpha = \chi_O = \upsilon^{r_-(O)}$, $\beta = \upsilon^{\lfloor m/2\rfloor - j}$; they have trivial tame parts and the weights $\gamma' = r_-(O)$ and $\gamma = \lfloor m/2\rfloor - j$, so that $\mathrm{tt}_j(\zeta)(g) = \sum_{\mathfrak{c}} \theta^{W_j,\mathrm{I}}_{\alpha,\beta;\mathfrak{m}}[\mathbf{1}_{z_j}, \mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}\overline{\zeta(\mathfrak{c})})](g, x_{\mathfrak{c}})$.

Step 1: the kernel polynomials. Apply Lemma 6.6.50 with $S = W_j$, $T = \mathrm{I}$ (rows $\mathcal{B}^\circ$), $\sigma = -1$, $\mathrm{ord} = \mathrm{ord}_O$ on the negative basis vectors, $\kappa = \kappa_O^-$ in rank $|\mathcal{B}^\circ|$, the order o, the maps l_τ and the shifts (γ', γ) in the roles of (γ, e). Its hypotheses: by (LF4), $\mathrm{ord}_O(c) = r - 1 - \mathrm{num}(c)$ on the negative basis vectors, which are those of numbers $r_+, \dots, r-1$ ($r_+ = r - r_-(O)$; (HP4) of Definition 3.2.13), so ord_O is a bijection of them onto $\{0, \dots, r_-(O)-1\}$; by Lemma 6.6.41, O fits l, so (ii) of (DP11) gives (ii) of Lemma 6.6.50, (iii) of (DP11) gives $\kappa_O^-(\tau, p) = 0$ for $r_-(O) < p < |\mathcal{B}^\circ|$ (as $|\mathcal{B}^\circ| < r + t$), and (i) of (DP11) makes l_τ injective; (i) is the hypothesis on o. The weight of Lemma 6.6.50 is $-\gamma + |\mathcal{B}^\circ|[s_j(c) = -1] + [s_j(c) = -1]\sum_{\mathrm{ord}_O(c) < p < |\mathcal{B}^\circ|} \kappa_O^-(\tau, p) = u^\circ_\tau(c)$. Let $(e_l)_{l\in I}$ be the coordinate basis of $\mathbb{V}_j^-$, $Q_l = \mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-} e_l)$, and fix $\xi_0 \in \mathbb{V}_j^- \setminus \{0\}$ and $Q_0 = \mathrm{Let}_{-1}(\mathrm{Pol}_{\kappa_O^-}\xi_0)$. By Lemma 6.6.50, $Q_0 \neq 0$ and v_{Q_0} is holomorphic of weight u° for the pair (γ', γ) and the order o. By Lemma 6.6.51 ($\sigma = -1$), for $y' \in G_\infty(1_{\mathcal{B}^\circ})$,

$$\Omega_{\gamma',\gamma}(1, y')\, v_{Q_0} = \prod_v \det(y'_v)^{\gamma' - r_-(O)} \sum_{l\in I} \big(S_{\kappa_O^-}(\bar{y}')\xi_0\big)_l\, v_{Q_l} \in \mathrm{span}\{v_{Q_l} : l \in I\}.$$

Step 2: the reproducing formula. We apply Lemma 6.6.53 with $S = W_j$ (signs s_j on the nonempty $\mathcal{R}$), $n' = \mathcal{B}^\circ$, $\mathfrak{m} = \mathfrak{M}^2$ (so $N = \pi_M^2$), α, β as above, the order o, the weights u°, $e = z_j$ (so $\delta_e = \mathbf{1}_{z_j}$), Q_0, the representatives $h_{\mathfrak{c}} = x_{\mathfrak{c}}$, $\iota = I$ and $Q_{\mathfrak{c},l} = Q_l$. Its hypotheses: $2\#\mathcal{R} = 2r < t - 9m - 1 = |\mathcal{B}^\circ|$, as $3r > 9m + 1$ by Lemma 4.2.67 ($r \geq x^5$, $m \leq 2\sqrt{x}$); o is positive-first, the u°_τ are K-dominant and $m^{\mathrm{hol}}_{s_j,u^\circ}$ is integrable by hypothesis; Q_0 and the span condition are Step 1; the group $\mathbf{K}_{n'}(\mathfrak{m})$ of that lemma is $\mathbf{K}_{\mathfrak{M}^2}[\{1\}]$, so its classes are Cl° (TG3). Fix $x \in \mathrm{G}(s_j)$ and let $a_{\mathfrak{c},l}$ be the numbers it provides.

Step 3: $\psi_i = y\,\mathsf{F}_i$ is of level $\mathfrak{m}$. By (a) of Lemma 6.6.49, F_i is a finite combination of functions $\kappa(g) = \theta^{W_j,\mathrm{I}_{\mathcal{B}_j}}_{\chi_O,\nu'';\mathfrak{M}^2}[\Phi, P](g, h')$ with $\nu'' = \nu''_{j,1}$, $h' \in G_f(\mathcal{B}_j)$. Each such κ is continuous (joint continuity of the Weil representation, (2) of Lemma 4.1.1, continuity of the characters and of the pair map, boundedness of the theta functional), left invariant under Γ_{W_j} (Lemma 6.2.5) and bounded by $C\,\mathrm{ht}_{W_j}(g)^A$ for some C, A (Lemma 6.2.28, h' being fixed). It is right invariant under $\mathbf{K}_{\mathcal{R}}(\mathfrak{m})$: for k in that group, (6) of Lemma 4.1.1 (with $k' = 1$; the reduction of k modulo $\mathfrak{m}$ is 1, so $\Phi_{k,1} = \Phi$) gives $\omega_{\chi_O,\nu''}((1,k),1)\mathrm{mod}_{\mathfrak{m}}(\Phi, \upsilon_P) = \nu''(\det k)^{-1}\mathrm{mod}_{\mathfrak{m}}(\Phi, \upsilon_P)$, and $\nu''(\det k) = \prod_w \psi_w(\det k \bmod w) = 1$ by Definition 3.1.1, since the tame part ψ of ν'' is trivial off the chosen places and $\det k \equiv 1$ modulo every chosen place; then $\kappa(gk) = \kappa(g)$ by (2) of Lemma 4.1.1. The form y is continuous, left invariant under Γ_{W_j}, right invariant under $\mathrm{K}_M \supseteq \mathbf{K}_{\mathcal{R}}(\mathfrak{m})$ (Definition 3.2.16), and $|y|\,\mathrm{ht}^{M'}_{W_j}$ is bounded for every M' (Lemma 6.6.57). Hence ψ_i is continuous, invariant as required, and $|\psi_i|\,\mathrm{ht}^{M'}_{W_j} \leq C'|y|\,\mathrm{ht}^{M'+A}_{W_j}$ is bounded for every M': ψ_i is of level $\mathfrak{m}$.

Step 4: conclusion. By Step 2 for ψ_i,

$$h \mapsto \psi_i(x(h,1))m^{\mathrm{hol}}_{s_j,u^\circ}(h) \quad \text{and} \quad g \mapsto \psi_i(g)\theta^{W_j,\mathrm{I}}_{\alpha,\beta;\mathfrak{m}}[\mathbf{1}_{z_j}, Q_l](g, x_{\mathfrak{c}})$$

are integrable, and

$$\int_{G_\infty(s_j)} \psi_i(x(h,1))\, m^{\mathrm{hol}}_{s_j,u^\circ}(h)\, dh = \sum_{\mathfrak{c},l} a_{\mathfrak{c},l} \int_{[\mathrm{U}(W_j)]} y(g)\, \mathsf{F}_i(g)\, \theta^{W_j,\mathrm{I}}_{\alpha,\beta;\mathfrak{m}}[\mathbf{1}_{z_j}, Q_l](g, x_{\mathfrak{c}})\, dg.$$

Let $\zeta_{\mathfrak{c},l}\colon \mathrm{Cl}^\circ \to \mathbb{V}_j^-$ be e_l at $\mathfrak{c}$ and 0 elsewhere. As e_l has real coordinates and the kernels are linear in the polynomial ($v_0 = 0$ and data are linear in the Fock vector), $\mathrm{tt}_j(\zeta_{\mathfrak{c},l})(g) =$

$\theta^{W_j,\mathrm{I}}_{\alpha,\beta;\mathfrak{m}}[\mathbf{1}_{z_j},Q_l](g,x_{\mathfrak{c}})$, so each integral on the right is $\int y\,\mathsf{F}_i\,\mathrm{tt}_j(\zeta_{\mathfrak{c},l})=0$ by hypothesis. Hence the left side vanishes for every i; the integrand of the statement is $\sum_i c_i\psi_i(x(h,1))m^{\mathrm{hol}}_{s_j,u^\circ}(h)$, a finite combination of integrable functions with integral 0. □

Lemma 6.6.53. *Let S be a space with a point with signs s on a nonempty index set n, let n' be an index set with $2\#n<\#n'$ and $\mathrm{I}=\mathrm{I}_{n'}$. Let $\mathfrak{m}$ be a modulus with $\mathfrak{m}=N\mathcal{O}_E$ for some $N\in\mathcal{O}_E$, and for an index set m let*

$$\mathbf{K}_m(\mathfrak{m})=\{k\in\mathbf{K}_0(m):\ k_w\equiv 1\pmod{\mathfrak{m}\mathcal{O}_{E,w}}\ \text{for every } w\in\Sigma(\mathfrak{m})\}.$$

*A function $\psi\colon \mathrm{G}(s)\to\mathbb{C}$ is **of level** $\mathfrak{m}$ if it is continuous, left invariant under Γ_S, right invariant under $\mathbf{K}_n(\mathfrak{m})$, and $|\psi|\,\mathrm{ht}_S^M$ is bounded on $\mathrm{G}(s)$ for every $M\in\mathbb{N}$. Let α, β be automorphic characters of $\mathrm{U}(1)$ of weights γ', $\gamma\colon\Phi\to\mathbb{Z}$, both with trivial tame part. Let o be an order on n that is positive-first for s, and $u=(u_v)_{v\in\Phi}$ weights on n with every u_v K-dominant for (s,o) and $m^{\mathrm{hol}}_{s,u}$ integrable on $G_\infty(s)$. Let $e\colon n\to n'$ be injective, $\iota_e\in\mathrm{M}_{n'\times n}(R_{\mathfrak{m}})$ the inclusion matrix ($(\iota_e)_{a,k}=1$ if $a=e(k)$ and 0 otherwise) and δ_e the function on $\mathrm{M}_{n'\times n}(R_{\mathfrak{m}})$ equal to 1 at ι_e and 0 elsewhere. Let $Q_0\neq 0$ be a polynomial in the variables $X_{v,a,k}$ ($v\in\Phi$, $a\in n'$, $k\in n$) such that for every bounded linear functional ℓ on $\mathcal{F}_{S,\mathrm{I}}$ the function $x\mapsto\ell(\Omega_{\gamma',\gamma}(x,1)v_{Q_0})$ on $G_\infty(s)$ is holomorphic of weight u for o. Let $(h_{\mathfrak{c}})_{\mathfrak{c}}$ be representatives of the classes of I modulo $\mathbf{K}_{n'}(\mathfrak{m})$, ι a finite set and $(Q_{\mathfrak{c},i})_{\mathfrak{c},i\in\iota}$ polynomials in the same variables such that*

$$\Omega_{\gamma',\gamma}(1,y)\,v_{Q_0}\in\mathrm{span}_{\mathbb{C}}\{v_{Q_{\mathfrak{c},i}}:i\in\iota\}\qquad\text{for every class }\mathfrak{c}\text{ and every }y\in G_\infty(1_{n'}).$$

Then for every $x\in\mathrm{G}(s)$ there are numbers $a_{\mathfrak{c},i}\in\mathbb{C}$ such that for every function ψ of level $\mathfrak{m}$ the functions $h\mapsto\psi(x(h,1))\,m^{\mathrm{hol}}_{s,u}(h)$ on $G_\infty(s)$ and $g\mapsto\psi(g)\,\theta^{S,\mathrm{I}}_{\alpha,\beta;\mathfrak{m}}[\delta_e,Q_{\mathfrak{c},i}](g,h_{\mathfrak{c}})$ on $[\mathrm{U}(S)]$ are integrable and

$$\int_{G_\infty(s)}\psi(x(h,1))\,m^{\mathrm{hol}}_{s,u}(h)\,dh=\sum_{\mathfrak{c}}\sum_{i\in\iota}a_{\mathfrak{c},i}\int_{[\mathrm{U}(S)]}\psi(g)\,\theta^{S,\mathrm{I}}_{\alpha,\beta;\mathfrak{m}}[\delta_e,Q_{\mathfrak{c},i}](g,h_{\mathfrak{c}})\,dg.$$

Proof. Fix $x\in\mathrm{G}(s)$ and let $\mathcal{V}$ be the complex vector space of functions of level $\mathfrak{m}$.

Step 0: the linear forms. Every $\psi\in\mathcal{V}$ is bounded, as $\mathrm{ht}_S\geq 1$. Since m is integrable and ψ is bounded and continuous, $h\mapsto\psi(x(h,1))m(h)$ is integrable, and $\ell(\psi)=\int_{G_\infty(s)}\psi(x(h,1))m(h)\,dh$ is a linear form on $\mathcal{V}$. For a polynomial P and $h'\in\mathrm{G}(1_{n'})$, Lemma 6.2.28 gives $Y\geq 0$ and $A\in\mathbb{N}$ with $|\theta_P(g,h')|\leq Y\|v_P\otimes F\|\,(\mathrm{ht}_S(g)\mathrm{ht}_{\mathrm{I}}(h'))^A$, and $|\psi|\,\mathrm{ht}_S^A$ is bounded; the function $g\mapsto\psi(g)\theta_P(g,h')$ is continuous ((2) of Lemma 4.1.1, the theta functional being bounded), left invariant under Γ_S (Lemma 6.2.5) and bounded, hence integrable on $[\mathrm{U}(S)]$, which has finite volume (Theorem 4.2.18). This is the integrability assertion of the lemma, and $L_{\mathfrak{c},i}(\psi)=\int_{[\mathrm{U}(S)]}\psi(g)\theta_{Q_{\mathfrak{c},i}}(g,h_{\mathfrak{c}})\,dg=\Theta_{v_{Q_{\mathfrak{c},i}}\otimes F}\psi(h_{\mathfrak{c}})$ is a linear form on $\mathcal{V}$; there are finitely many, the classes being finite in number.

Step 1: reduction by linear duality. If ℓ vanishes on the common kernel of the finitely many $L_{\mathfrak{c},i}$, then ℓ is a linear combination $\sum a_{\mathfrak{c},i}L_{\mathfrak{c},i}$ (the map $\mathcal{V}\to\mathbb{C}^r$, $\psi\mapsto(L_{\mathfrak{c},i}(\psi))$, has kernel inside $\ker\ell$, so ℓ factors through its image, and a linear form on a subspace of $\mathbb{C}^r$ extends to $\mathbb{C}^r$). The numbers $a_{\mathfrak{c},i}$ depend on x only, which is the assertion. So let $\psi\in\mathcal{V}$ with $L_{\mathfrak{c},i}(\psi)=0$ for all $\mathfrak{c}$ and i; we show $\ell(\psi)=0$.

Step 2: the lift by Q_0 vanishes. For a class $\mathfrak{c}$ let $\mathcal{S}_{\mathfrak{c}}=\mathrm{span}\{v_{Q_{\mathfrak{c},i}}:i\in\iota\}$ and, for $w\in\mathcal{S}_{\mathfrak{c}}$, $\Lambda_{\mathfrak{c}}(w)=\Theta_{w\otimes F}\psi(h_{\mathfrak{c}})$; as $w\mapsto w\otimes F$ is linear, $w\otimes F$ is a finite combination of the data $v_{Q_{\mathfrak{c},i}}\otimes F$, so the integral converges by Step 0, $\Lambda_{\mathfrak{c}}$ is linear, and it vanishes on the spanning vectors, hence on $\mathcal{S}_{\mathfrak{c}}$. Let $h'\in\mathrm{G}(1_{n'})$. Its finite component lies in a double coset: $h'_f=\gamma_f h_{\mathfrak{c}}k$ with $\gamma\in\Gamma_{\mathrm{I}}$

and $k \in \mathbf{K}_{n'}(\mathfrak{m})$; with $y = \gamma_\infty^{-1} h'_\infty \in G_\infty(1_{n'})$, $h' = \gamma\,(1,h_{\mathfrak{c}})\,(1,k)\,(y,1)$. By Lemma 6.2.5, $\theta(\varphi)(g,h') = \theta(\varphi)(g,(1,h_{\mathfrak{c}})(1,k)(y,1))$. By (5) of Lemma 4.1.1 (α of weight γ', β of weight γ), $\omega_{\alpha,\beta}(1,(y,1))(v_{Q_0}\otimes F) = (\Omega_{\gamma',\gamma}(1,y)v_{Q_0})\otimes F$, and by (5) there and (SH4) of Lemma 6.6.56 (α, β have trivial tame parts) $\omega_{\alpha,\beta}(1,(1,k))(w\otimes F) = w\otimes F$ for every Fock vector w. Hence

$$\Theta_{v_{Q_0}\otimes F}\psi(h') = \Lambda_{\mathfrak{c}}(\Omega_{\gamma',\gamma}(1,y)v_{Q_0}) = 0,$$

since $\Omega_{\gamma',\gamma}(1,y)v_{Q_0} \in \mathcal{S}_{\mathfrak{c}}$ by hypothesis. So $\Theta_{v_{Q_0}\otimes F}\psi = 0$.

Step 3: the matrix coefficient of v_{Q_0}. Put $v_0 = v_{Q_0}$ and $\kappa(h) = \langle v_0, \Omega_{\gamma',\gamma}(h,1)v_0\rangle$ for $h \in G_\infty(s)$. The functional $w \mapsto \langle v_0, w\rangle$ is bounded on $\mathcal{F}_{S,\mathrm{I}}$ (its norm is that of L^1, and v_0 is bounded), so κ is holomorphic of weight u for o by hypothesis. $\Omega_{\gamma',\gamma}(h,1)$ is a scalar of modulus 1 times an operator preserving the inner product ((2) of Lemma 4.1.1), and $\Omega_{\gamma',\gamma}$ is a representation, so $\kappa(h^{-1}) = \langle \Omega_{\gamma',\gamma}(h,1)v_0, v_0\rangle = \overline{\kappa(h)}$. By Lemma 6.6.54, $\kappa = \kappa(1)\,m = \langle v_0, v_0\rangle\, m$; in particular κ is integrable, and $\langle v_0, v_0\rangle > 0$ because $Q_0 \neq 0$ makes v_0 a nonzero continuous function.

Step 4: the convolution vanishes. Apply Lemma 6.6.55 to S, $T = \mathrm{I}$ ($2\#n < \#n'$), the characters α, β of weights γ', γ, the Schwartz vector v_0 (the Fock vector of a polynomial, (XD3) of Definition 3.2.22) with the integrable coefficient κ, the datum F, which is sharp for $J = \mathbf{K}_n(\mathfrak{m})$ off the places below $\Sigma(\mathfrak{m})$ for β (Lemma 6.6.56), and $y = \psi$, which is continuous, left invariant under Γ_S, right invariant under J and rapidly decreasing. By Step 2 its lift vanishes, so $\int \kappa(h)\psi(x(h,1))\,dh = 0$, that is $\langle v_0, v_0\rangle\,\ell(\psi) = 0$ by Step 3, and $\ell(\psi) = 0$. □

Lemma 6.6.54. *Let n be a nonempty finite index set, s signs on n, o an order that is positive-first for s, $u = (u_v)_{v\in\Phi}$ weights with every u_v K-dominant for (s,o), and $F\colon G_\infty(s) \to \mathbb{C}$ holomorphic of weight u for o with $F(g^{-1}) = \overline{F(g)}$ for every $g \in G_\infty(s)$. Then*

$$F(h) = m^{\mathrm{hol}}_{s,u}(h)\,F(1) \qquad (h \in G_\infty(s)).$$

Proof. Let $\tilde{F}$ be an extension of F as in (HV3). Call $t = (t_v)_{v\in\Phi}$ a *torus element* if every t_v is a diagonal matrix with entries of modulus 1; then $t \in G_\infty(s)$, every t_v is o-triangular, and $\overline{t^{\,u}} = (t^{-1})^u$, where $t^u = \prod_{v\in\Phi} t_v^{u_v}$.

Right translation by the torus. Let $x \in G_\infty(s)$ and t a torus element. By (HV3) at the point xt with the o-triangular family $b = 1$, $\tilde{F}(xt) = F(xt)$; at the point x with $b = t$, $\tilde{F}(xt) = t^u F(x)$. Hence $F(xt) = t^u F(x)$.

Left translation by the torus. Let $y \in G_\infty(s)$ and t a torus element. The hypothesis applied to $g = y^{-1}$ gives $F(y) = \overline{F(y^{-1})}$. Applying the hypothesis to $g = ty$ and then the right law to $x = y^{-1}$ and the torus element t^{-1},

$$F(ty) = \overline{F(y^{-1}t^{-1})} = \overline{(t^{-1})^u}\;\overline{F(y^{-1})} = t^u\,F(y).$$

Conclusion. Let $h \in G_\infty(s)$ and $d(\theta)$ ($\theta \in \mathbb{R}/\mathbb{Z}$) the torus element with component $\mathrm{diag}(e^{2\pi\mathrm{i}\theta o(i)})_{i\in n}$ at every v, so $d(\theta)^u = \prod_{v\in\Phi}\prod_{i\in n} e^{2\pi\mathrm{i}\theta o(i)u_v(i)}$. By the left law the integrand of the circle mean of Lemma 6.6.72 at the point h is

$$F(d(\theta)h)\prod_{v\in\Phi}\prod_{i\in n} e^{-2\pi\mathrm{i}\theta\,o(i)u_v(i)} = F(h)$$

for every θ, so the mean is $F(h)$ ($d\theta$ is a probability measure). By Lemma 6.6.72 the mean is $m^{\mathrm{hol}}_{s,u}(h)\,F(1)$. □

Lemma 6.6.55. *Let S, T be spaces with points with signs s on n and s' on n', $2\#n < \#n'$, let a, b be automorphic characters of $\mathrm{U}(1)$ of weights γ, $\gamma' \colon \Phi \to \mathbb{Z}$, and let $v \in \mathcal{F}_{S,T}$ be a Schwartz vector such that*

$$\kappa(x) = \langle v, \Omega_{\gamma,\gamma'}(x,1)\, v\rangle \qquad (x \in G_\infty(s))$$

is integrable on $G_\infty(s)$. Let S_0 be a finite set of finite places of F, $J \subseteq G_f(n)$ an open subgroup, $c \in \mathbb{C}$, and $F \in \mathcal{M}_{S,T}$ sharp for J off S_0 with mass c for the character b ((SH1)–(SH3)). Let $y \colon \mathrm{G}(s) \to \mathbb{C}$ be continuous, left invariant under Γ_S, right invariant under J, with $|y|\,\mathrm{ht}_S^M$ bounded for every $M \in \mathbb{N}$, and let $\Theta = \Theta_{v\otimes F}$ be the theta lift for the kernel $\theta_{a,b}^{S,T}$. If $\Theta y = 0$, then

$$\int_{G_\infty(s)} \kappa(x)\, y(g(x,1))\, dx = 0 \qquad \text{for every } g \in \mathrm{G}(s).$$

Proof. **Step 0: preliminaries.**

(i) φ is clean in the sense of Theorem 6.5.13: under the first alternative of (SH3) this is (DC1); under the second, $\varphi = v \otimes (\Psi \otimes_A v^0)$ is the datum of (v, Ψ) at A ((WR6)), and $\Psi \in \mathcal{T}_A$, a combination of the functions τ_y, is a Schwartz–Bruhat function of the components at A that vanishes unless x_w has a left inverse; this is (DC2). Its Fock vector v is a Schwartz vector.

(ii) For $x \in G_\infty(s)$ and $k \in G_f(n)$, $\langle \varphi, \omega((x,k),1)\varphi\rangle = \kappa(x)\,\langle F, \omega_{1,b}((1,k),1)F\rangle$: as $a(\det 1) = 1$, $\omega(g,1) = \omega_{1,b}(g,1)$; by (5) of Lemma 4.1.1, $\omega((x,k),1)\varphi = (\Omega_{\gamma,\gamma'}(x,1)v)\otimes(\omega_{1,b}((1,k),1)F)$; by (6) of Lemma 6.2.33 the inner product is $\kappa(x)\langle v_1 \otimes F, v_1 \otimes \omega_{1,b}((1,k),1)F\rangle$; and $v_1 \otimes \omega_{1,b}((1,k),1)F = \omega_{1,b}((1,k),1)(v_1 \otimes F) = \omega_{1,b}((1,k),1)F$ by (5) of Lemma 4.1.1 at the real point 1 and $v_1 \otimes F = F$ (SH1).

(iii) $K'_\Sigma \subseteq J$ and $b(\det k) = 1$ for $k \in K'_\Sigma$. Let $k \in K'_\Sigma$. By Lemma 6.5.6 for the vacuum F above Σ (SH1), $\gamma = k$ and $g = 1$, $\langle F, \omega_{1,b}((1,k),1)F\rangle = \mathrm{mc}(k)\langle F, F\rangle$. By (WR4), $\mathrm{mc}(k) = b(\det k)^{-1}\langle v^0, \omega_{1,1}((1,k),1)v^0\rangle$, and $\omega_{1,1}((1,k),1)v^0 = v^0$ by the last sentence of (6) of Lemma 4.1.1 with $v = v_1$ and $v_1 \otimes v^0 = v^0$ ((4) of Lemma 6.2.33); so $\mathrm{mc}(k) = b(\det k)^{-1}$, of modulus 1, as $\langle v^0, v^0\rangle = 1$. Thus the coefficient is not 0, and (SH2) gives $k \in J \cap \mathbf{K}_0(n)$ and $b(\det k)^{-1}\langle F, F\rangle = \langle F, F\rangle$, i.e. $b(\det k) = 1$. So the hypotheses of (2) and (3) of Proposition 6.5.4 hold for $\beta = b$, and every right J-invariant function is right K'_Σ-invariant.

Step 1: for every admissible y', $\int_{G_\infty(s)} \kappa(x)\langle y', z_0(\cdot\,(x,1))\rangle_S\, dx = 0$, where $z_0 = R_\Sigma y$. The function y is admissible. By Step 0 (i) and Theorem 6.5.13 with $\varphi' = \varphi$, the function $g \mapsto \langle \varphi, \omega(g,1)\varphi\rangle\langle y', R(g)y\rangle_S$ is integrable on $\mathrm{G}(s)$ and its integral times $\mathrm{vol}([\mathrm{U}(T)])$ is $\langle \Theta y', \Theta y\rangle_T = 0$; $\mathrm{vol}([\mathrm{U}(T)]) > 0$. The measure of $\mathrm{G}(s)$ is $dx\,dk$ ((VP1) of Definition 3.2.2), so by Fubini for almost every x the function $h_x(k) = \langle \varphi, \omega((x,k),1)\varphi\rangle\langle y', R((x,k))y\rangle_S$ is integrable on $G_f(n)$, and $\int\!\int h_x(k)\,dk\,dx = 0$. The group K_Σ is open, so $G_f(n) = \bigsqcup_q u_q K_\Sigma$ is a countable disjoint union of open cosets and, dk being left invariant, $\int h_x = \sum_q \int_{K_\Sigma} h_x(u_q\kappa')\,d\kappa'$, absolutely convergent. For $\kappa' \in K_\Sigma$, Step 0 (ii), Lemma 6.5.6 (with $\gamma = u_q$ and $g = (1,\kappa')$) and (SH2) give

$$\langle \varphi, \omega((x,u_q\kappa'),1)\varphi\rangle = \kappa(x)\,\mathrm{mc}(u_q)\,\langle F, F\rangle\,\mathbf{1}_{J\cap\mathbf{K}_0(n)}(\kappa'),$$

and for $\kappa' \in J$, $y(g(x,u_q\kappa')) = y(g(x,1)u_q)$ by the right J-invariance of y. As $dk(J\cap\mathbf{K}_0(n)) = [\mathbf{K}_0(n) : J\cap\mathbf{K}_0(n)]^{-1}$, $\int_{K_\Sigma} h_x(u_q\kappa')\,d\kappa' = c\,\kappa(x)\,\mathrm{mc}(u_q)\,\langle y', y(\cdot\,(x,1)u_q)\rangle_S$. Summing over q with (1) of Proposition 6.5.4 (y bounded, measurable and left invariant, y' integrable on $[\mathrm{U}(S)]$), $\int h_x = c\,\kappa(x)\,\langle y', z_0(\cdot\,(x,1))\rangle_S$ for almost every x. Since $c \neq 0$ (SH1), the claim follows; the

integrand is integrable, as $|\langle y', z_0(\cdot\,(x,1))\rangle_S| \le \|y'\|_{L^1} \sup|z_0|$ with $\sup|z_0| < \infty$ by (1) of Proposition 6.5.4.

Step 2: the convolution and its series. Put $y_1(g) = \int_{G_\infty(s)} \kappa(x)\, y(g(x,1))\, dx$. As y is bounded and continuous and κ integrable, y_1 is defined, $|y_1| \le \|\kappa\|_{L^1} \sup|y|$, and y_1 is continuous (dominated convergence); it is left invariant under Γ_S, and right invariant under J because $(1,k)$ commutes with $(x,1)$; hence also under K'_Σ (Step 0 (iii)). By (2) of Proposition 6.5.4 applied to y (bounded, continuous, left invariant, right invariant under K'_Σ and under the open group J), R_Σ commutes with this convolution: $z = R_\Sigma y_1$ satisfies $z(g) = \int \kappa(x)\, z_0(g(x,1))\, dx$. By Fubini ($y'$ integrable on $[\mathrm{U}(S)]$, z_0 bounded, κ integrable) and Step 1, $\langle y', z\rangle_S = \int \kappa(x)\langle y', z_0(\cdot\,(x,1))\rangle_S\, dx = 0$ for every admissible y'.

Step 3: $z = 0$. For $w \in \mathrm{G}(s)$ the translate $y_w = y(\cdot\, w)$ is admissible: it is continuous and left invariant; for $k \in w_f J w_f^{-1}$, $(1,k)w = w(1, w_f^{-1} k w_f)$, so $y_w(g(1,k)) = y_w(g)$, and $w_f J w_f^{-1}$ is open; and by (2) of Lemma 6.2.6 for the compact set $\{w^{-1}\}$ there is $c_w \ge 1$ with $\mathrm{ht}_S(g) \le c_w\, \mathrm{ht}_S(gw)$, so $|y_w|\,\mathrm{ht}_S^M \le c_w^M \sup|y|\,\mathrm{ht}_S^M$ is bounded. By Step 2, $\langle z, y_w\rangle_S = \overline{\langle y_w, z\rangle_S} = 0$ for every w. For $u \in G_f(n)$, Fubini as in Step 2 gives $\langle z, y_1(\cdot\,(1,u))\rangle_S = \int \kappa(x)\langle z, y_{(x,u)}\rangle_S\, dx = 0$, since $(1,u)(x,1) = (x,u)$. By (2) of Proposition 6.5.4 applied to y_1, z is bounded, continuous and left invariant, hence integrable on $[\mathrm{U}(S)]$, and (1) there for y_1 and $y' = z$ at the point 1 gives

$$\langle z, z\rangle_S = \langle z, R_\Sigma y_1\rangle_S = \sum_q \mathrm{mc}(u_q)\, \langle z, y_1(\cdot\, u_q)\rangle_S = 0.$$

So $\int_{[\mathrm{U}(S)]} |z|^2 = 0$; z is continuous and the quotient measure gives positive mass to the image of every nonempty open set, so $z = 0$.

Step 4: conclusion. y_1 is bounded and right invariant under K'_Σ, and $R_\Sigma y_1 = z = 0$; by (3) of Proposition 6.5.4 (hypothesis on b from Step 0 (iii)), $y_1 = 0$, which is the assertion. □

Lemma 6.6.56. *Let S, T be spaces with points with signs s on n and s' on n', $e\colon n \to n'$ injective, $\mathfrak{m}$ a modulus, $A = \Sigma(\mathfrak{m})$, S_0 the set of finite places of F below the places of A, and*

$$F = \delta_e^\natural \otimes_A v^0 \in \mathcal{M}_{S,T}.$$

Then $J = \mathbf{K}_n(\mathfrak{m})$ is an open subgroup of $\mathbf{K}_0(n)$ of finite index, $\langle F, F\rangle \ne 0$, and for every automorphic character b of $\mathrm{U}(1)$ with trivial tame part, F is sharp for J off S_0 with mass $c = \langle F, F\rangle/[\mathbf{K}_0(n) : J]$: (SH1)–(SH3) hold for b, and (SH4) holds for every pair a, b of automorphic characters of $\mathrm{U}(1)$ with trivial tame parts, with the open subgroups $\mathbf{K}_n(\mathfrak{m})$ and $\mathbf{K}_{n'}(\mathfrak{m})$.

Proof. **Step 0: the places.** A place $v \in S_0$ lies below some $w \in A$; the places of E above v are w and $\bar w$, and $w \ne \bar w$ because A contains the conjugate of none of its members. So the places of E above S_0 are exactly those of $A \cup \bar A$, and each $v \in S_0$ is split. For $k \in G_f(n)$ and $w \in A$, unitarity $k^* k = 1$ gives $k_{\bar w} = \left(\bar k_w^{\mathsf{T}}\right)^{-1}$ under the identification of $E_{\bar w}$ with E_w by conjugation; so k is integral at $\bar w$ if and only if $k_w \in \mathrm{GL}_n(\mathcal{O}_{E,w})$. Hence J is the set of $k \in \mathbf{K}_0(n)$ with $k_w \equiv 1$ modulo $\mathfrak{m}\mathcal{O}_{E,w}$ at the finitely many $w \in A$; it is an open subgroup of the compact group $\mathbf{K}_0(n)$, so its index is finite.

Step 1: left inverses. Let $w \in A$ and $x \in \mathrm{M}_{n'\times n}(\mathcal{O}_{E,w})$ with $x \equiv \iota_e$ modulo $\mathfrak{m}\mathcal{O}_{E,w}$. Read ι_e^{T} as the 0/1 matrix over $\mathcal{O}_{E,w}$; then $\iota_e^{\mathsf{T}} x \equiv \iota_e^{\mathsf{T}} \iota_e = 1_n$ modulo the maximal ideal, so $\det(\iota_e^{\mathsf{T}} x)$ is a unit and $L = (\iota_e^{\mathsf{T}} x)^{-1} \iota_e^{\mathsf{T}}$ is a left inverse of x over $\mathcal{O}_{E,w}$. Consequently: if $\delta_e^\natural(x) \ne 0$ then x_w has a left inverse over $\mathcal{O}_{E,w}$ for every $w \in A$.

Step 2: (SH1) and (SH3). By (1) and (2) of Lemma 6.2.33, $\langle F, F\rangle = \langle \delta_e^\natural, \delta_e^\natural\rangle_A \langle v^0, v^0\rangle$, and $\langle \delta_e^\natural, \delta_e^\natural\rangle_A$ is the measure of the set of $x \in X_A$ whose components at A are integral and reduce to ι_e; this set contains a nonempty open set, so $\langle F, F\rangle \ne 0$. Put $c = \langle F, F\rangle/[\mathbf{K}_0(n) : J]$. By (4)

of Lemma 6.2.33, $v_1 \otimes F = F$; by (5) there, $\rho_f(y)F = F$ for every $y \in \widehat{\mathcal{O}}_E^{n' \times n}$ with components 0 at $A \cup \bar{A}$, which by Step 0 are the y whose components at the places not above Σ vanish: F is a vacuum above Σ. This is (SH1). For (SH3), $A \neq \emptyset$ (a modulus has a nonempty set of places), $F = \Psi \otimes_A v^0$ with $\Psi = \delta_e^\natural \in \mathcal{T}_A$ ((6) of Lemma 4.1.1), and for any $w \in A$, $\Psi(x) \neq 0$ only if x_w has a left inverse (Step 1): the second alternative of (SH3) holds.

Step 3: (SH4). Let a, b have trivial tame parts, $k \in \mathbf{K}_n(\mathfrak{m})$, $k' \in \mathbf{K}_{n'}(\mathfrak{m})$. As $F = v_1 \otimes (\delta_e^\natural \otimes_A v^0)$ (Step 2), (6) of Lemma 4.1.1 with $v = v_1$ and then (4) of Lemma 6.2.33 give $\omega_{a,b}((1,k),(1,k'))F = a(\det k')\, b(\det k)^{-1}\, (\delta_e)^\natural_{k,k'} \otimes_A v^0$ with $(\delta_e)_{k,k'}(X) = \delta_e(\bar{k}'^{-1} X \bar{k}) = \delta_e(X)$, the reductions being 1. The determinants $\det k$, $\det k'$ lie in $\mathrm{U}(1)(\mathbb{A})$, have infinite components 1 and integral finite components, so by the definition of weight and tame part (Definition 3.1.1) $a(\det k') = b(\det k) = 1$, the tame parts being trivial. So $\omega_{a,b}((1,k),(1,k'))F = F$. Both groups are open (Step 0).

Step 4: (SH2) on J. Let b have trivial tame part. For $k \in J$, Step 3 with $a = 1$, $k' = 1$ gives $\omega_{1,b}((1,k),1)F = F$, so $\langle F, \omega_{1,b}((1,k),1)F\rangle = \langle F, F\rangle$.

Step 5: (SH2) off J. Let $k \in G_f(n)$ with $k_v \in K_v$ for all $v \in \Sigma$ and $k \notin J$. Write $k = k_2 k_1$, where k_1 has the components of k above S_0 and 1 elsewhere and k_2 the components above Σ and 1 elsewhere (components at different places commute). By Lemma 6.5.6 for the vacuum F above Σ (Step 2), $\gamma = k_2$ and $g = (1, k_1)$ (whose components at Σ are $1 \in K_v$),

$$\langle F, \omega_{1,b}((1,k),1)F\rangle = \mathrm{mc}_{b,\Sigma}(k_2)\, \langle F, \omega_{1,b}((1,k_1),1)F\rangle.$$

By (4) of Lemma 4.1.1 (k_1 has components 1 outside $A \cup \bar{A}$, $h = 1$), $\omega_{1,b}((1,k_1),1)F = \lambda\, \Psi' \otimes_A v^0$ for a scalar λ and $\Psi'(x) = \delta_e^\natural(xk_1)$, $\Psi' \in \mathcal{T}_A$. By (1), (2) of Lemma 6.2.33 the inner product on the right is $\lambda \langle \delta_e^\natural, \Psi'\rangle_A$, the integral over X_A of $\overline{\delta_e^\natural(x)}\, \delta_e^\natural(xk_1)$. Suppose this integrand is not 0 at some x. For $w \in A$, x_w and $x_w k_w$ are integral and congruent to ι_e modulo $\mathfrak{m}\mathcal{O}_{E,w}$; by Step 1 they have left inverses L, L' over $\mathcal{O}_{E,w}$, so $k_w = L(x_w k_w)$ and $k_w^{-1} = L' x_w$ are integral, $k_w \in \mathrm{GL}_n(\mathcal{O}_{E,w})$, and by Step 0 k is integral at $\bar{w}$ too. Then $\iota_e \bar{k}_w \equiv x_w k_w \equiv \iota_e$, and since the rows $e(n)$ of $\iota_e \bar{k}_w$ are the rows of $\bar{k}_w$, $\bar{k}_w = 1$. With $k_v \in K_v$ for $v \in \Sigma$ this gives $k \in J$, a contradiction. Hence the integrand vanishes identically and $\langle F, \omega_{1,b}((1,k),1)F\rangle = 0$. Together with Step 4 (and $J \subseteq \mathbf{K}_0(n)$) this is (SH2). □

Lemma 6.6.57. *Let $1 \leq j \leq m$ and $y \in \mathcal{A}_M(W_j)$. Then $\sup_{g \in \mathrm{G}(s_j)} |y(g)|\, \mathrm{ht}_{W_j}(g)^A < \infty$ for every $A \in \mathbb{N}$.*

Proof. **The type kernel is a cusp kernel.** As r is odd and m even, k is even and the entries of $\varsigma(v)$ in Definition 3.2.16 are integers; so the entries of $2\varsigma(v)$ are even, i.e. congruent to $r - 1$ modulo 2. They are distinct, and by (iii) of Proposition 5.8.4 ς has gap Δ for the signs of W_j, so entries attached to basis vectors of opposite signs differ by at least 2Δ. By Lemma 4.2.67, $r \geq x^5 \geq 4$, so $\Delta = 2r^8 \geq 500 r^3$. Hence $2\varsigma(v)$ has gap $500r^3$ for s_j at every $v \in \Phi$, and by Lemma 5.6.21 with $N = r$ and $B = 0$ the type kernel e_ς is a cusp kernel of order 0 on $G_\infty(s_j)$.

Rapid decay. By Definition 3.2.16, y is continuous, bounded, left invariant under the rational points, right invariant under the open subgroup K_M and reproduced by convolution with e_ς, i.e. it is a form of type e_ς in the sense of (ED2). By Lemma 6.3.75 for $V = W_j$, $B = 0$ and $e = e_\varsigma$, $\sup_g |y(g)|\, \mathrm{ht}_{W_j}(g)^A < \infty$ for every $A \in \mathbb{N}$. □

Lemma 6.6.58. *Let $1 \leq j \leq m$ and $y \in \mathcal{A}^{\mathrm{good}}(W_j)$. Every peak $g \in \mathrm{G}(s_j)$ of $\bar{y}$ satisfies $\lambda_{W_j}(g) \geq (5b^{24})^{-1/2}$.*

Proof. Let s_j be the signs of W_j (on the nonempty basis $\mathcal{R}$, $r \geq 1$), and let o, u be the order and the weight of Lemma 6.6.45. Put $F = \bar{y}$, $A = 5b$, $A' = b^{23}$ and $p = M\sqrt{-d} \in \mathcal{O}_E$. We check the hypotheses of Lemma 6.6.59 for $V = W_j$.

The weight. By (a) of Lemma 6.6.45 every u_v has holomorphic gap $300n_* \geq 1$ for s_j, hence gap 1 ($u_v(i)+1 \leq u_v(i)+300n_* \leq u_v(i')$ whenever $s_j(i) = +1$, $s_j(i') = -1$), and $|u_v(i)| \leq 5b = A$. By (AD1) and $C_P \geq 6$, $\log\log b \geq 6 \cdot 10^{12}$, so $b > 10$ and $A \geq 2$.

The level. $\bar{p} = M\overline{\sqrt{-d}} = -p$, and $p_v = \mathrm{i}M\sqrt{d}$ for every $v \in \Phi$, so $\operatorname{Im} p_v = M\sqrt{d} > 0$. As $10^{12}(2d+6) \leq 10^{12}C_P \leq \log\log b$, we have $d \leq b^2$, and $M \leq b^{22}$ by (AD5); so $\operatorname{Im} p_v \leq b^{23} = A'$, and $A' \geq 1$. Let $k \in \mathbf{K}_0(\mathcal{R})$ with $k-1 \in p\,\widehat{\mathcal{O}}_E$. Every chosen place w contains π_M, which divides M and hence p in $\mathcal{O}_E$; so k is integral and reduces to 1 at every chosen place, i.e. $k \in \mathrm{K}_M$. As y is a form of principal level (Definition 3.2.16), $y(xk) = y(x)$, hence $F(xk) = F(x)$.

Boundedness, automorphy, holomorphy. y is a form of type e_ς, so by (FT2) it is bounded and left invariant under Γ_{W_j}, and so is $F = \bar{y}$. By (b) of Lemma 6.6.45, F is adelically holomorphic of weight u for o, and o is positive-first for s_j.

Hence Lemma 6.6.59 applies: every peak g of $\bar{y}$ satisfies $\lambda_{W_j}(g) \geq (AA')^{-1/2} = (5b\cdot b^{23})^{-1/2} = (5b^{24})^{-1/2}$. □

Lemma 6.6.59. *Let V be a space with a point with signs s on a nonempty finite index set n, o an order on n that is positive-first for s, and $u = (u_v)_{v\in\Phi}$ weights such that every u_v has holomorphic gap 1 for s and $|u_v(i)| \leq A$ for all v and i, where $A \geq 2$. Let $p \in \mathcal{O}_E$ with $\bar{p} = -p$ and $0 < \operatorname{Im} p_v \leq A'$ for every $v \in \Phi$, where $A' \geq 1$. Let $F\colon \mathrm{G}(s) \to \mathbb{C}$ be bounded, left invariant under Γ_V, right invariant under every $k \in \mathbf{K}_0(n)$ whose entries satisfy $k - 1 \in p\,\widehat{\mathcal{O}}_E$, and adelically holomorphic of weight u for o. Then every peak g of F satisfies*

$$\lambda_V(g) \geq (AA')^{-1/2}.$$

Proof. Let $C = \sup|F|$. As g is a peak, $|F(g)| > \frac{C}{2}$, so $C > 0$. Suppose, to the contrary, that $\lambda_V(g) < (AA')^{-1/2}$. Since $(AA')^{-1/2} \leq 2^{-1/2} < 1$, there is a nonzero u in the lattice of V_g with $\|w_\infty\| < (AA')^{-1/2}$, where $w = (x_V g)^{-1}u$; w_f is integral.

Step 1: u is isotropic. As $g \in \mathrm{G}(s)$, $(x_V g)^* J_V (x_V g) = \varepsilon_s$, so $c = u^* J_V u = w^* \varepsilon_s w$. Here $c \in F$ (J_V is hermitian over E); its finite part is $w_f^* w_f$, integral, so $c \in \mathcal{O}_F$; at the real place of F below $v \in \Phi$, $|c_v| = |w_v^* \operatorname{diag}(s) w_v| \leq \|w_v\|^2 < 1$. Thus $|\mathrm{N}_{F/\mathbb{Q}}(c)| < 1$, and $c = 0$. In particular $w^*\varepsilon_s w = 0$ at every place, and $w_v \neq 0$ for every $v \in \Phi$ ($u \neq 0$ and $x_V g$ is invertible).

Step 2: the transvection. Let $m = 1 - p\,ww^*\varepsilon_s$ (adelic matrix, p read diagonally). By $\bar{p} = -p$ and $w^*\varepsilon_s w = 0$, $m^*\varepsilon_s m = \varepsilon_s - (\bar{p}+p)\varepsilon_s ww^*\varepsilon_s = \varepsilon_s$, so $m \in \mathrm{G}(s)$; and $(1 - p\,ww^*\varepsilon_s)(1 + p\,ww^*\varepsilon_s) = 1$. Since $(x_V g)^{-*}\varepsilon_s (x_V g)^{-1} = J_V$, $(x_V g) m (x_V g)^{-1} = 1 - p\,uu^* J_V \in \mathrm{U}(V)(F)$, so $gmg^{-1} \in \Gamma_V$. The finite part $m_f = 1 - p\,w_f w_f^*$ is integral with $m_f - 1 \in p\widehat{\mathcal{O}}_E$, and so is $m_f^{-1} = 1 + p\,w_f w_f^*$; hence $m_f^{-1} \in \mathbf{K}_0(n)$ fixes F from the right. Put $\phi(h) = F(g(h,1))$ for $h \in G_\infty(s)$. As $(m_\infty h, 1) = m(h,1)(1, m_f^{-1})$,

$$\phi(m_\infty h) = F\big(gmg^{-1} \cdot g(h,1)(1, m_f^{-1})\big) = F\big(g(h,1)\big) = \phi(h).$$

Step 3: the holomorphic function on the upper half plane. ϕ is holomorphic of weight u for o (adelic holomorphy at g); let $\tilde{\phi}$ be an extension as in (HV3). For $\zeta \in \mathbb{C}$ let $X(\zeta) \in \mathrm{M}_n(\mathbb{C})^\Phi$ have the components $1 - p_v(\zeta - \mathrm{i})w_v w_v^* \operatorname{diag}(s)$. Then $X(\mathrm{i}) = 1$, $X(1+\mathrm{i}) = m_\infty$, and, w_v being isotropic, $(1 - a\,w_v w_v^* \operatorname{diag}(s))(1 - a'\,w_v w_v^* \operatorname{diag}(s)) = 1 - (a+a')w_v w_v^* \operatorname{diag}(s)$, so $m_\infty X(\zeta) = X(\zeta+1)$. Let $\eta_v = \operatorname{Im}(p_v)\|w_v\|^2 > 0$; then $\sum_v \eta_v \leq A'\|w_\infty\|^2 < A^{-1} \leq \frac{1}{2}$. As $\operatorname{Re} p_v = 0$ ($\bar{p} = -p$), $p_v = \mathrm{i}\operatorname{Im} p_v$, and Lemma 6.6.60 (with $B = A \geq 1$, $y = w_v$, $\alpha = \operatorname{Im} p_v$) gives for every z with $\operatorname{Im} z > 0$ and every v a factorisation $X(z)_v = h_v(z)b_v(z)$, $h_v(z) \in \mathrm{U}(\operatorname{diag} s)$, $b_v(z)$ o-triangular, with $|b_v(z)^{u_v}| \leq e^{2A\eta_v}(1 + \eta_v \operatorname{Im} z)^{-1/2}$. Put $f(z) = \tilde{\phi}(X(z))$. By (HV3)

at the point $h(z)$ with $b(z)$, $\tilde\phi$ is complex analytic at $X(z)$ and $f(z) = \big(\prod_v b_v(z)^{u_v}\big)\phi(h(z))$; as $z \mapsto X(z)$ is affine, f is holomorphic on the upper half plane. Moreover:

(i) $f(\mathrm{i}) = \tilde\phi(1) = \phi(1) = F(g)$ ((HV3) at the point 1 with $b = 1$);

(ii) $|f(z)| \le C\prod_v e^{2A\eta_v}(1+\eta_v \operatorname{Im} z)^{-1/2} \le Ce^{2A\sum_v \eta_v} \le Ce^2$, and $f(z) \to 0$ as $\operatorname{Im} z \to \infty$;

(iii) $f(z+1) = \tilde\phi(m_\infty h(z)b(z)) = \big(\prod_v b_v(z)^{u_v}\big)\phi(m_\infty h(z)) = f(z)$, by (HV3) at the point $m_\infty h(z)$ and Step 2.

Step 4: Schwarz's lemma at the cusp. By (iii), $f(z) = \varphi(e^{2\pi \mathrm{i} z})$ for a holomorphic φ on the punctured unit disc, bounded by Ce^2 by (ii); its singularity at 0 is removable and $\varphi(0) = 0$ by (ii). Schwarz's lemma gives $|\varphi(q)| \le Ce^2|q|$ for $|q| < 1$, so by (i), at $q = e^{-2\pi}$, $|F(g)| \le Ce^{2-2\pi} < \frac{C}{2}$, as $2\pi - 2 > \log 2$. This contradicts $|F(g)| > \frac{C}{2}$. Hence $\lambda_V(g) \ge (AA')^{-1/2}$. □

Lemma 6.6.60. *Let n be a nonempty finite index set, s signs on n, $J = \operatorname{diag}(s)$, o an order on n that is positive-first for s, and w a weight on n with holomorphic gap 1 for s and $|w(i)| \le B$ for all i, where $B \ge 1$. Let $y \in \mathbb{C}^n$ with $y^*Jy = 0$, $\alpha > 0$, and $\eta = \alpha\, y^*y$ with $0 < \eta \le \frac12$. Then for every $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ there are $h \in \mathrm{U}(\operatorname{diag} s)$ and an o-triangular matrix b with*

$$1 - \mathrm{i}\alpha(z - \mathrm{i})\, yy^*J = h\, b, \qquad |b^w| \le \frac{e^{2B\eta}}{\sqrt{1+\eta \operatorname{Im} z}}.$$

Proof. Put $t = \mathrm{i}\alpha(z - \mathrm{i})$, so $\operatorname{Re} t = \alpha(1 - \operatorname{Im} z)$, and $M = 1 - t\, yy^*J$.

Step 1: the Gram matrix. As $J^* = J$ and $y^*Jy = 0$, $M^*JM = (1-\bar t\, Jyy^*)J(1-t\, yy^*J) = J - \tau\, Jyy^*J$ with $\tau = 2\operatorname{Re} t$, the term in $|t|^2$ containing y^*Jy. List n in increasing order of o, and for $a \in \mathbb{Z}$ let $T_a = \{l : o(l) < a\}$ and $x_l = s(l)|y_l|^2$. By the matrix determinant lemma, the principal minor of M^*JM on T_a is $\det(J_{T_a})\, p_a$ with

$$p_a = 1 - \tau \sum_{l \in T_a} x_l.$$

All p_a are positive. Since $\sum_l x_l = y^*Jy = 0$ and o lists the signs $+1$ first, the partial sums $S_a = \sum_{l\in T_a} x_l$ satisfy $0 \le S_a \le \frac12 y^*y$. If $\operatorname{Im} z \ge 1$ then $\tau \le 0$ and $p_a \ge 1$; otherwise $0 < \tau S_a \le \alpha y^*y = \eta \le \frac12$ and $p_a \ge \frac12$.

Step 2: the factorisation. A hermitian matrix whose leading principal minors (in the order o) are nonzero has a factorisation $U^*\Delta U$ with U unipotent and upper triangular for o and Δ real diagonal, Δ_{kk} the quotient of consecutive leading minors (Gaussian elimination). Here $\Delta_{kk} = s(k)\, p_{o(k)+1}/p_{o(k)}$, as $T_{o(k)+1} = T_{o(k)} \sqcup \{k\}$. Put $b = \operatorname{diag}\big(\sqrt{p_{o(k)+1}/p_{o(k)}}\big)_k U$; it is o-triangular, and $b^*Jb = U^*\Delta U = M^*JM$ (J and the diagonal factor commute). So $h = Mb^{-1}$ satisfies $h^*Jh = J$, i.e. $h \in \mathrm{U}(\operatorname{diag} s)$, and $M = hb$. With $L_k = \log p_{o(k)+1} - \log p_{o(k)}$, $|b^w| = \exp\big(\frac12 \sum_k L_k w(k)\big)$.

Step 3: the sums of the L_k. The L_k telescope along o: $\sum_k L_k = \log p_{\max} - \log p_{\min} = 0$ (both extreme values of p are 1), and $\sum_{s(k)=+1} L_k = \log p_*$ with $p_* = 1 - \tau\frac{y^*y}{2} = 1 - \eta(1 - \operatorname{Im} z) = Y - \eta$, $Y = 1 + \eta \operatorname{Im} z$. As $p_* \ge 1 - \eta \ge \frac12$, $Y/p_* = 1 + \eta/p_* \le 1 + 2\eta$, so $\log Y \le \log p_* + 2\eta$. Moreover $s(k)(p_{o(k)+1} - p_{o(k)}) = -\tau|y_k|^2$, so $s(k)L_k$ has the sign of $-\tau$ (or vanishes).

Step 4: the bound. As w has holomorphic gap 1, there is an integer c with $w(k) + 1 \le c$ for $s(k) = +1$ and $c \le w(k)$ for $s(k) = -1$ (take $c = 1 + \max_{s(k)=+1} w(k)$, or any integer at most $\min_{s(k)=-1} w(k)$ if there is no sign $+1$). If $\tau \le 0$, then $L_k \ge 0$ for $s(k) = +1$ and $L_k \le 0$

for $s(k)=-1$, and $\sum_k L_k w(k) = \sum_k L_k(w(k)-c) \le -\sum_{s(k)=+1} L_k = -\log p_*$. If $\tau>0$, the signs are reversed, and $\sum_k L_k w(k) = \sum_k L_k(w(k)+B) \le 2B\sum_{s(k)=-1} L_k = -2B\log p_*$. Use $-\log p_* \le -\log Y + 2\eta$, $\log Y \ge 0$ and $B \ge 1$. For $\tau \le 0$: $\frac12\sum_k L_k w(k) \le -\frac12 \log p_* \le \eta - \frac12 \log Y \le 2B\eta - \frac12 \log Y$. For $\tau > 0$: $\frac12 \sum_k L_k w(k) \le -B\log p_* \le -B\log Y + 2B\eta \le -\frac12 \log Y + 2B\eta$. So in both cases

$$\tfrac12 \sum_k L_k w(k) \le 2B\eta - \tfrac12 \log Y,$$

and exponentiating gives $|b^w| \le e^{2B\eta} Y^{-1/2}$. □

Lemma 6.6.61. *Let $1 \le j \le m$, $n_* = 2ft^3$, and $|h| = \max_{v\in\Phi} \|h_v\|$ for $h \in G_\infty(s_j)$. There are a bounded function $F_0 \colon G_\infty(s_j) \to \mathbb{C}$ with $F_0(1) \ne 0$ and, for every $g \in \mathrm{G}(s_j)$ with $\lambda_{W_j}(g) \ge (5b^{24})^{-1/2}$, numbers $(c_i)_{i\in \mathrm{Mem}_j}$ in $\mathbb{C}$ with*

$$\Big| \sum_{i\in \mathrm{Mem}_j} c_i\, \mathsf{F}_i(g(h,1)) - F_0(h) \Big| \le 2\, b^{-19n_*}\, |F_0(1)|\, |h|^{6n_*} \qquad (h \in G_\infty(s_j)).$$

Proof. Let $\tilde W_j$ be the padded space, with signs $\tilde s_j$ on $\mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{A}_j^+$; let o be the order on $\mathcal{B}^{\mathrm{c}}$ of Lemma 6.6.64, u^{c} the weight $(0;3m)$ there; let γ_j', u_1, u_2, u_3 be as in Lemma 6.6.66, $c_P = c_m$ the coefficient there for $\gamma = m$, $m_i = m^{\mathrm{hol}}_{s^{\mathrm{c}},u_i}$, and $J = \int_{G_\infty(s^{\mathrm{c}})} |m_3|^2$. For $h \in G_\infty(s_j)$ let $\iota(h) \in G_\infty(\tilde s_j)$ be h on $\mathcal{R}$ and 1 on the padding; then $|\iota(h)| = \max(|h|,1) = |h|$, since every $x \in \mathrm{U}(\mathrm{diag}\, s)$ has $\|x\| \ge 1$ (its inverse $\mathrm{diag}(s)x^*\mathrm{diag}(s)$ has the same norm and $1 \le \|x\|\|x^{-1}\|$).

Step 0: numbers. By (AD1) and $C_P \ge 6$, $\log\log b \ge 6\cdot 10^{12}$, so $b > 5^4$; $10^{12}(2d+6) \le \log\log b$ gives $d \le b$; by (AD2), $m \le 2\sqrt{x} \le b$, $3m \le t$ and $r + 3m \le t$ ($r \ge x^5$). Both index sets are nonempty.

Step 1: the carrier and a peak off the cusp. χ_M is tame of weight 0 for $(\mathfrak{M},\chi)$, χ the character of $\mathcal{O}_E/\mathfrak{M}$ given by ω on units. So by (b) of Lemma 6.6.62 (with $B=0$) φ° is continuous, bounded, left $\Gamma_{S^{\mathrm{c}}}$-invariant and right $\mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$-invariant; by (i) of Lemma 6.6.64 it is adelically holomorphic of weight u^{c} for o; by (ii) there it is not 0. Hence $0 < \sup|\varphi^\circ| < \infty$ and φ° has a peak g_1. Apply Lemma 6.6.59 with $V = S^{\mathrm{c}}$, $u = u^{\mathrm{c}}$ (holomorphic gap $3m \ge 1$, entries at most $3m \le 5b = A$, $A \ge 2$), $p = \sqrt{-d}$ ($\bar p = -p$, $\mathrm{Im}\, p_v = \sqrt{d} \in (0,\sqrt{b}]$, $A' = \sqrt{b} \ge 1$); an element $k \in \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$ with $k \equiv 1$ modulo p fixes φ°. So $\lambda_{S^{\mathrm{c}}}(g_1) \ge (5b\sqrt{b})^{-1/2}$. As g_1 is a peak, $\varphi^\circ(g_1) \ne 0$ and $|\varphi^\circ(x)| \le 2|\varphi^\circ(g_1)|$ for all x.

Step 2: the coefficient. Let $v = v_{P_j}$. For $h \in G_\infty(s^{\mathrm{c}})$, $h' \in G_\infty(\tilde s_j)$, the determinants in $\Omega_{m,\gamma_j'}(h,h')$ have modulus 1 (from $g^*\mathrm{diag}(s)g = \mathrm{diag}(s)$), and ω^∞ preserves the inner product ((2) of Lemma 4.1.1); by Cauchy–Schwarz $|c_P(h,h')| \le c_P(1,1) = \langle v, v\rangle$, and c_P is continuous ((2) there). $P_j \ne 0$: it is a product of determinants of square matrices of distinct variables (the maps ng_j and $\mathrm{ps}_j(h,\cdot)$ of (XE3) are injective), so $\langle v,v\rangle > 0$.

Step 3: a bound for real integrals against the carrier. By (c) of Lemma 6.6.66, Lemma 6.6.68 applies with $n = \mathcal{B}^{\mathrm{c}}$ ($3m \le t$), $s = s^{\mathrm{c}}$, o, $u = u_3$ and $u' = u_1$: $0 < J < \infty$ and $\int |m_1(h')|\,|h'|^{6n_*} dh' \le b^{n_*} J$. Hence, if A is a measurable function on $G_\infty(s^{\mathrm{c}})$ with $|A(h')| \le c\,|h'|^{6n_*}$, then $h' \mapsto \varphi^\circ(g_1(h',1))A(h')m_1(h')$ is integrable and

$$\Big| \int_{G_\infty(s^{\mathrm{c}})} \varphi^\circ(g_1(h',1))A(h')\, m_1(h')\, dh' \Big| \le 2\,|\varphi^\circ(g_1)|\, c\, b^{n_*} J. \tag{6.149}$$

Step 4: the function F_0. Put

$$F_0(h) = \int_{G_\infty(s^{\mathrm{c}})} \varphi^\circ(g_1(h',1))\, c_P(h', \iota(h))\, m_1(h')\, dh' \qquad (h \in G_\infty(s_j)).$$

As $|c_P| \le c_P(1,1) \le c_P(1,1)|h'|^{6n_*}$ ($|h'| \ge 1$), (6.149) shows that the integral exists and $|F_0(h)| \le 2|\varphi^\circ(g_1)|c_P(1,1)b^{n_*}J$: F_0 is bounded. At $h=1$, (b) of Lemma 6.6.66 gives $c_P(h',1) = c_P(1,1)m_2(h')\overline{m_3(h')}$, so $F_0(1) = \int f\,\overline{m_3}$ with $f(h') = c_P(1,1)\,\varphi^\circ(g_1(h',1))\,m_1(h')\,m_2(h')$, and $f\,\overline{m_3}$ is integrable. The weights u_1, $u_2(\tau)$ are constant on each block, hence K-dominant for (s^{c}, o), so m_1, m_2 are holomorphic of weights u_1, u_2 for o with value 1 at 1 ((i) of Lemma 6.6.70); $h' \mapsto \varphi^\circ(g_1(h',1))$ is holomorphic of weight u^{c}; and a product of functions holomorphic of weights w, w' is holomorphic of weight $w+w'$ (the product of the extensions is analytic at xb with value $b^w b^{w'}$ times the product). Here

$$u^{\mathrm{c}} + u_1 + u_2(\tau) = \big(0 - \Delta' + \Delta';\ 3m+6m+1+d_j(\tau)+r+2m\big) = \big(0;\ d_j(\tau)+|\mathcal{B}_j|+2m\big) = u_3(\tau),$$

as $|\mathcal{B}_j| = r+9m+1$. So f is holomorphic of weight u_3, $u_3(\tau)$ is K-dominant, and Lemma 6.6.71 gives

$$F_0(1) = f(1)\,J = c_P(1,1)\,\varphi^\circ(g_1)\,J \neq 0, \qquad |F_0(1)| = J\,|\varphi^\circ(g_1)|\,c_P(1,1).$$

Step 5: approximation off the cusps. Let $g \in \mathrm{G}(s_j)$ with $\lambda_{W_j}(g) \ge (5b^{24})^{-1/2}$. The lattice of $\tilde{W}_j$ moved by $(g,1)$ is the product of that of W_j moved by g and those of the two definite summands; a nonzero vector of a definite standard summand has, by the trace and the arithmetic and geometric means as for a definite space ($\pm h(u,u)$ is a nonzero totally positive integer of F), euclidean length at least 1. So $\lambda_{\tilde{W}_j}((g,1)) \ge \min(\lambda_{W_j}(g),1) = \lambda_{W_j}(g)$, and

$$\lambda_{S^{\mathrm{c}}}(g_1)\,\lambda_{\tilde{W}_j}((g,1)) \ge (5b\sqrt{b})^{-1/2}(5b^{24})^{-1/2} = \tfrac{1}{5}\,b^{-51/4} \ge b^{-13}$$

($b^{1/4} \ge 5$). By (a) and (d) of Lemma 6.6.66, Lemma 6.6.73 applies to $V = S^{\mathrm{c}}$, $V' = \tilde{W}_j$, $\alpha = \upsilon^m$ (weight m), $\beta = \nu_j^\flat \upsilon^m$ (weight γ_j'), $Q = P_j$ and the points g_1, $(g,1)$: there is Ψ with

$$\big|\theta_\Psi(g_1(h',1), g(h,1)) - c_P(h', \iota(h))\big| \le b^{-20n_*}\big(|h'|\,|h|\big)^{6n_*} c_P(1,1)$$

for all h', h, where θ_Ψ is the kernel of Lemma 6.6.79 (note $(g,1)(\iota(h),1) = (g(h,1),1)$). By Lemma 6.6.79 for g_1 and Ψ there are (c_i) with

$$\sum_i c_i \mathsf{F}_i(g') = \int \varphi^\circ(g_1(h',1))\,\theta_\Psi(g_1(h',1), g')\,m_1(h')\,dh'$$

for all g', the integrand being integrable. Fix h and take $g' = g(h,1)$. Both this integral and $F_0(h)$ exist, so their difference is the integral of $\varphi^\circ(g_1(h',1))A(h')m_1(h')$ with $A(h') = \theta_\Psi(g_1(h',1), g(h,1)) - c_P(h', \iota(h))$, $|A(h')| \le \varepsilon|h'|^{6n_*}$, $\varepsilon = b^{-20n_*}|h|^{6n_*}c_P(1,1)$. By (6.149) and Step 4,

$$\Big|\sum_i c_i \mathsf{F}_i(g(h,1)) - F_0(h)\Big| \le 2|\varphi^\circ(g_1)|\,b^{-20n_*}|h|^{6n_*}c_P(1,1)\,b^{n_*}J = 2\,b^{-19n_*}\,|F_0(1)|\,|h|^{6n_*}.$$

□

Lemma 6.6.62. *Let s^{c} be the signs of S^{c} and $\mathrm{I} = \mathrm{I}_{\mathcal{B}^{\mathrm{c}}}$. For a character χ of $R_{\mathfrak{m}}$ (extended by 0 to the nonunits) write $\chi \circ \det$ for the function $y \mapsto \chi(\det y)$ on $\mathcal{B}^{\mathrm{c}} \times \mathcal{B}^{\mathrm{c}}$ matrices over $R_{\mathfrak{m}}$.*

(a) There is $C \in \mathbb{R}$ such that for every modulus $\mathfrak{m}$, every character χ of $R_{\mathfrak{m}}$ and every automorphic character c of $\mathrm{U}(1)$ tame of weight 0 for $(\mathfrak{m}, \chi)$ there is $\gamma \in \mathbb{C}$ with

$$L_*(\mathfrak{m}, c^{-1}\upsilon^m; m)\,L_*(\mathfrak{m}, c\,\upsilon^{m/2}; m/2)\,|\gamma|^2 = C$$

and, for every $x \in \mathrm{G}(s^{\mathrm{c}})$,

$$\gamma \sum_{h \bmod \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})}^{\mathrm{cl}} \mathsf{a}_c(h)\,\theta^{S^{\mathrm{c}},\mathrm{I}}_{\upsilon^m, c; \mathfrak{m}}[\chi \circ \det, 1](x,h) = \varphi^\circ(x).$$

(b) If some automorphic character of $\mathrm{U}(1)$ *of weight* 0 *is tame for some* $(\mathfrak{m},\chi)$*, then* φ° *is continuous, left invariant under* $\Gamma_{S^{\mathrm{c}}}$ *and right invariant under* $\mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$*, and*

$$\sup_x |\varphi^\circ(x)|\,\mathrm{ht}_{S^{\mathrm{c}}}(x)^B < \infty$$

for every $B \in \mathbb{N}$.

Proof. Write I_m, I_m^- for the standard spaces on $b = \{1,\dots,m\}$ with signs $+1$, -1, $\mathbf{K}_m = \mathbf{K}_0(b)$, $a = \mathcal{B}^{\mathrm{c}}$, $\mathbf{K}_a = \mathbf{K}_0(a)$, and $A = \Sigma(\mathfrak{m})$. Fix $\mathfrak{m}$, χ and c as in (a), with tame part ψ.

Step 0: three facts on characters. (i) For $k \in \mathbf{K}_0(n)$ (n any index set), $\det k \in \mathrm{U}(1)(\mathbb{A})$ has integral finite components and infinite components 1, so $c(\det k) = \prod_{w\in A}\psi_w(\det k \bmod w) = \chi(\det \bar{k})$, $\bar{k}$ the reduction modulo $\mathfrak{m}$, by Definition 3.1.1 and (XD4), and $\upsilon(\det k) = 1$ by (ii) of Lemma 3.2.11. (ii) υ has weight 1 and trivial tame part ((ii) of Lemma 3.2.11), so $c\upsilon^m$ is tame of weight m for $(\mathfrak{m},\chi)$, and c^{-1} is tame of weight 0 for $(\mathfrak{m},\chi^{-1})$, χ^{-1} being $u \mapsto \chi(u)^{-1}$ on units and 0 on nonunits. (iii) $\mathrm{mod}_{\mathfrak{m}}(1,v) = v \otimes v^0$ for every Fock vector v: $1^\natural = \mathbf{1}_A$, and $\mathbf{1}_A \otimes_A v^0 = P^A v^0 = v^0$ by Lemma 6.6.12 and (2) of Lemma 6.2.33.

Step 1: the first exchange. Apply Proposition 6.3.67 with $n = b$, $W = \mathrm{I}_m$, $l = p = \emptyset$, the identity maps j, e, σ, the zero space V' (index set $\emptyset$), $d = m/2$ ($|b| = 0+2d$, and $2|l|+2 = 2 \le m$), $\iota_\tau = \emptyset$, the character χ^{-1} and $\alpha = \beta = c^{-1}$ (tame of weight $0 = |l| - |\iota_\tau|$, Step 0 (ii)). Then $\Delta^j(e;\eta) = 1$ and $Q_{f,j}(\eta) = 1$ (empty products and the determinant of the empty matrix). Matched functions for (σ,χ) are a constant Ψ on the empty matrix and $\Phi(y) = \Psi\,\chi(\det y)^{-1}$ on invertible y, 0 elsewhere; take $\Psi = 1$, so $\Phi = \chi^{-1}\circ\det$. On the right side, $\mathrm{Hom}(V',\mathrm{I}_m)$ has the empty index set: its theta model is $\mathbb{C}$ with $v^0 = 1$, its canonical theta functional is the identity ($\mathrm{G}(\emptyset)$ is trivial, so the transporting element is 1), the Fock model on the empty index set is $\mathbb{C}$ with $v_1 = 1$ and $v \otimes f = vf$, and $\omega_{1,c^{-1}}$ is the identity; with Step 0 (iii) the kernel $\theta^{V',\mathrm{I}_m}_{1,c^{-1};\mathfrak{m}}[1,1]$ is the constant 1. By (VP1) of Definition 3.2.2 the measure of $G_\infty(\emptyset) = \{1\}$ is the unit point mass and $G_f(\emptyset) = \mathbf{K}_0(\emptyset) = \{1\}$ has mass 1, so $[\mathrm{U}(V')] = \mathrm{G}(\emptyset) = \{1\}$ has mass $v_0 = 1$. So there are $C_1 \in \mathbb{R}$, independent of $(\mathfrak{m},\chi,c)$, and γ_1 with $L_*(\mathfrak{m}, c\,\upsilon^{m/2}; m/2)\,|\gamma_1|^2 = C_1$ and

$$\gamma_1 \sum_{h \bmod \mathbf{K}_m}^{\mathrm{cl}} \theta^{\mathrm{I}_m,\mathrm{I}_m}_{c^{-1},c^{-1};\mathfrak{m}}[\chi^{-1}\circ\det,1](g,h) = v_0 \qquad (g \in \mathrm{G}(1_b)). \tag{6.150}$$

Step 2: conjugation to the negative space. By Lemma 6.4.55 for $S = T = \mathrm{I}_m$, $\alpha = \beta = c^{-1}$, $\Psi = \chi^{-1}\circ\det$, $P = 1$, there are $c_{\mathrm{I,I}}$ of modulus 1 and the unit $u \in R_{\mathfrak{m}}^\times$ (reduction of the unit of (1) of Lemma 4.1.1) with $\overline{\theta^{\mathrm{I}_m,\mathrm{I}_m}_{c^{-1},c^{-1};\mathfrak{m}}[\chi^{-1}\circ\det,1](g,h)} = c_{\mathrm{I,I}}\,\theta^{\mathrm{I}_m^-,\mathrm{I}_m}_{c\upsilon^m,c\upsilon^m;\mathfrak{m}}[\Psi',1](g,h)$, where $\Psi'(X) = \overline{\chi^{-1}(\det(u^{-1}X))} = \chi(u)^{-m}\chi(\det X)$ ($|\chi| = 1$ on units, both sides 0 on noninvertible X). Conjugating (6.150) (the class-sum weights are rational) gives, with $c_0 = \overline{\gamma_1}\,c_{\mathrm{I,I}}\,\chi(u)^{-m}/v_0$, the class representatives h_i' of $\mathrm{U}(\mathrm{I}_m)(F)\backslash G_f(b)/\mathbf{K}_m$ and their weights $\omega_i = 1/\#(\mathrm{U}(\mathrm{I}_m)(F) \cap h_i'\mathbf{K}_m h_i'^{-1})$,

$$c_0 \sum_i \omega_i\,\kappa_{h_i'}(y) = 1, \qquad \kappa_{h'}(y) = \theta^{\mathrm{I}_m^-,\mathrm{I}_m}_{c\upsilon^m,c\upsilon^m;\mathfrak{m}}[\chi\circ\det,1](y,h') \qquad (y \in \mathrm{G}(-1_b) = \mathrm{G}(1_b)). \tag{6.151}$$

Step 3: the second exchange as the hypothesis of the one exchange. Let $W' = S^{\mathrm{c}}$ (index set a) and $W = \mathrm{I}_m \oplus S^{\mathrm{c}}$ (index set $n = b \sqcup a$), so W has the signs sgn_j for j the identification of $m' \sqcup l$ with n, $m' = b \sqcup \{1,\dots,2m\}$, l the second block of a. Fix $x \in \mathrm{G}(s^{\mathrm{c}})$. For $h' \in G_f(b)$ and a function Φ on $\mathrm{M}_{a\times n}(R_{\mathfrak{m}})$ let $K_{h'}(\Phi)(h) = \int_{\mathbf{K}_a} \theta^{W,\mathrm{I}}_{c\upsilon^m,1;\mathfrak{m}}[\Phi,1](\mathrm{diag}(h',x),hk)\,dk$ (Haar probability measure; the kernel is continuous by (2) of Lemma 4.1.1); it is right $\mathbf{K}_a$-invariant, and left $\mathrm{U}(\mathrm{I})(F)$-invariant by Lemma 6.2.5. Let Φ, Ψ be matched for $(\mathrm{id}_{b\sqcup a},\chi^{-1})$.

Then Φ vanishes off the matrices $(k^{-1})_a$, $k \in \mathrm{GL}_n(R_\mathfrak{m})$, with $\Phi((k^{-1})_a) = \chi^{-1}(\det k)\Psi(k_b)$; for $\kappa \in \mathbf{K}_a$ and $\kappa' = \mathrm{diag}(1_b, \bar\kappa)$ we have $\bar\kappa^{-1}(k^{-1})_a = ((k\kappa')^{-1})_a$ and $(k\kappa')_b = k_b$, so $\Phi(\bar\kappa^{-1}X) = \chi(\det\bar\kappa)^{-1}\Phi(X)$ for all X, and $(c\upsilon^m)(\det\kappa)\Phi(\bar\kappa^{-1}X) = \Phi(X)$ by Step 0 (i). The vector v_1 is fixed by $\Omega_{m,0}(1,y)$, $y \in G_\infty(1_a)$ (m and 0 the weights of $c\upsilon^m$ and 1), by (5) of Lemma 4.1.1 (exponent $m - \#\{\text{signs } -1 \text{ of } W\} = 0$). So (4) of Lemma 6.3.71 makes $h \mapsto \theta^{W,\mathrm{I}}_{c\upsilon^m,1;\mathfrak{m}}[\Phi,1](\mathrm{diag}(h',x),h)$ right $\mathbf{K}_a$-invariant, hence equal to $K_{h'}(\Phi)$. Now apply Proposition 6.3.67 with W, $V' = \mathrm{I}_m^-$ (index set b, $f\colon l \to b$ the identification, $p = \emptyset$), $e\colon a \to m'$ any bijection, $\sigma = \mathrm{id}_{b\sqcup a}$, $d = m$ ($|a| = 3m = |b| + 2d$; $2|l| + 2 = 2m + 2 \le 4m$), $\iota_\tau = \emptyset$, the character χ, $\alpha = c\upsilon^m$ (tame of weight $m = |l|$, Step 0 (ii)) and $\beta = 1$, at the point $\mathrm{diag}(h',x)$. Here $\Delta^j(e;\eta) = 1$ and $Q_{f,j}(\eta) = \prod_\tau \det\big((x_{\tau,(2,i),i'})_{i,i'}\big)^{2m} = \tilde P^\circ$, the carrier polynomial P° in the variables of the rows a. So there are $C_2 \in \mathbb{R}$, independent of $(\mathfrak{m},\chi,c)$, and c_2 with $L_*(\mathfrak{m},(c\upsilon^m)^{-1}\upsilon^{m+m};m)\,|c_2|^2 = C_2$, where $(c\upsilon^m)^{-1}\upsilon^{m+m} = c^{-1}\upsilon^m$, and

$$c_2 \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} K_{h'}(\Phi)(h) = \int_{[\mathrm{U}(\mathrm{I}_m^-)]} \theta^{\mathrm{I}_m^-,W}_{c\upsilon^m,c\upsilon^m;\mathfrak{m}}\big[\Psi,\tilde P^\circ\big](y,\mathrm{diag}(h',x))\,dy;$$

the integrand is continuous ((2) of Lemma 4.1.1) on the compact $[\mathrm{U}(\mathrm{I}_m^-)]$, hence integrable. This is the exchange identity of Lemma 6.6.63 for $V' = \mathrm{I}_m^-$, $W' = S^{\mathrm{c}}$, $g = x$, $c = c_2$, $\alpha = \beta = c\upsilon^m$, $\beta' = 1$ and $Q = P^\circ$.

Step 4: the one exchange. By (ii) of Lemma 6.6.63 with $\Psi' = 1$, the family (h_i',ω_i) and c_0 of (6.151) (so $F = 1$),

$$c_2c_0\sum_i \omega_i \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} K_{h_i'}(\Phi_1)(h) = \int_{[\mathrm{U}(\mathrm{I}_m^-)]} \theta^{\mathrm{I}_m^-,S^{\mathrm{c}}}_{c\upsilon^m,1;\mathfrak{m}}\big[1,P^\circ\big](y,x)\,dy, \tag{6.152}$$

with $\Phi_1(X) = \chi(\det X'')$, X'' the columns a of X. On the right, $\omega_{c\upsilon^m,1}(y,x) = c(\det x)\,\omega_{\upsilon^m,1}(y,x)$ (WR4) and $\mathrm{mod}_\mathfrak{m}(1,v_{P^\circ}) = v_{P^\circ}\otimes v^0$ (Step 0 (iii)), so the right side is $c(\det x)\,\varphi^\circ(x)$ by (XE4).

Step 5: splitting the kernel on W. $\Phi_1 = 1\otimes(\chi\circ\det)$ on the columns b and a, and $v_1 = v_{1\cdot1}$, so by (b) of Lemma 6.3.80 and Step 0 (iii), $\mathrm{mod}_\mathfrak{m}(\Phi_1,v_1) = (v_1\otimes v^0)\boxtimes^{\mathrm{sc}}\mathrm{mod}_\mathfrak{m}(\chi\circ\det,v_1)$. By (d) of Lemma 6.3.52 with $\beta = c$, $\beta' = \upsilon^m$ and the second character 1, and by (WR4),

$$\begin{aligned} K_{h'}(\Phi_1)(h) &= \theta^{\mathrm{I}_m,\mathrm{I}}_{c,1}(v_1\otimes v^0)(h',h)\;\theta^{S^{\mathrm{c}},\mathrm{I}}_{\upsilon^m,1;\mathfrak{m}}[\chi\circ\det,1](x,h)\\ &= \theta^{\mathrm{I}_m,\mathrm{I}}_{c,1}(v_1\otimes v^0)(h',h)\;c(\det x)\,\theta^{S^{\mathrm{c}},\mathrm{I}}_{\upsilon^m,c;\mathfrak{m}}[\chi\circ\det,1](x,h).\end{aligned}$$

Summing over i with the weights ω_i gives $\mathsf{a}_c(h)$ (XE4) in the first factor; the class sums are finite, so they may be interchanged, and the left side of (6.152) is

$$c_2c_0\,c(\det x)\sum_h^{\mathrm{cl}} \mathsf{a}_c(h)\,\theta^{S^{\mathrm{c}},\mathrm{I}}_{\upsilon^m,c;\mathfrak{m}}[\chi\circ\det,1](x,h).$$

Dividing by $c(\det x)$, of modulus 1, gives (a) with $\gamma = c_2c_0$: as $|c_0| = |\gamma_1|/v_0$, $|\gamma|^2 = |c_2|^2|\gamma_1|^2v_0^{-2}$, and multiplying by the two L-value products gives

$$L_*(\mathfrak{m},c^{-1}\upsilon^m;m)\,L_*(\mathfrak{m},c\upsilon^{m/2};m/2)\,|\gamma|^2 = C_2C_1v_0^{-2} = C$$

for $C = C_1C_2v_0^{-2}$.

Step 6: (b). Let c be tame of weight 0 for $(\mathfrak{m},\chi)$. By (a), φ° is a finite linear combination of the functions $\kappa_h(x) = \theta^{S^{\mathrm{c}},\mathrm{I}}_{\upsilon^m,c;\mathfrak{m}}[\chi\circ\det,1](x,h)$. The function $\chi\circ\det$ vanishes at every matrix without a (left) inverse over $R_\mathfrak{m}$, so by (2) of Lemma 6.3.71 every κ_h is continuous,

left $\Gamma_{S^{\mathrm{c}}}$-invariant and satisfies $\sup_x |\kappa_h(x)|\,\mathrm{ht}_{S^{\mathrm{c}}}(x)^B < \infty$ for every B. For $k \in \mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$, (6) of Lemma 4.1.1 gives $\omega_{\mathcal{V}^m,c}((1,k),1)\,\mathrm{mod}_{\mathfrak{m}}(\chi\circ\det, v_1) = c(\det k)^{-1}\mathrm{mod}_{\mathfrak{m}}(\Phi_k, v_1)$ with $\Phi_k(X) = \chi(\det(X\bar{k})) = \chi(\det\bar{k})\chi(\det X)$, so by Step 0 (i) the datum is fixed and $\kappa_h(xk) = \kappa_h(x)$. Hence φ° has the properties of (b). □

Lemma 6.6.63. *Let V' be a space with index set b, W' a space with index set a, and $W = \mathrm{I}_b \oplus W'$, with index set $b \sqcup a$; write $\mathrm{diag}(h', g)$ for the point of $\mathrm{U}(W)(\mathbb{A}_F)$ with blocks $h' \in \mathrm{U}(\mathrm{I}_b)(\mathbb{A}_F)$ and $g \in \mathrm{U}(W')(\mathbb{A}_F)$. Let $\mathfrak{m}$ be a modulus, χ a multiplicative character of $R = R_{\mathfrak{m}}$, χ^{-1} the character $u \mapsto \chi(u)^{-1}$ on units and 0 on nonunits, α, β, β' automorphic characters of $\mathrm{U}(1)$, $Q \in \mathbb{C}[x]_{a\times b}$, and $\tilde{Q} \in \mathbb{C}[x]_{(b\sqcup a)\times b}$ the polynomial Q in the variables of the rows a. Let $g \in \mathrm{U}(W')(\mathbb{A}_F)$, $c \in \mathbb{C}$, and for every $h' \in \mathrm{U}(\mathrm{I}_b)(\mathbb{A}_{F,f})$ and every function Φ on $\mathrm{M}_{a\times(b\sqcup a)}(R)$ let $K_{h'}(\Phi)$ be a function on $\mathrm{U}(\mathrm{I}_a)(\mathbb{A}_{F,f})$, left invariant under $\mathrm{U}(\mathrm{I}_a)(F)$ and right invariant under $\mathbf{K}_a$. Assume the* exchange identity*: for every h' and all Φ, Ψ matched for $(\mathrm{id}_{b\sqcup a}, \chi^{-1})$ modulo $\mathfrak{m}$, the function $x \mapsto \theta^{V',W}_{\alpha,\beta\beta';\mathfrak{m}}[\Psi, \tilde{Q}](x, \mathrm{diag}(h', g))$ is integrable on $[\mathrm{U}(V')]$ and*

$$c \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} K_{h'}(\Phi)(h) = \int_{[\mathrm{U}(V')]} \theta^{V',W}_{\alpha,\beta\beta';\mathfrak{m}}[\Psi, \tilde{Q}](x, \mathrm{diag}(h', g))\, dx.$$

Let Ψ' be a function on $\mathrm{M}_{a\times b}(R)$ with $\Psi'(y\gamma) = \Psi'(y)$ for all $\gamma \in \mathrm{GL}_b(R)$, and for $x = (x' \,|\, x'') \in \mathrm{M}_{a\times(b\sqcup a)}(R)$ (x' the columns b, x'' the columns a) let $\Phi_{\Psi'}(x) = \chi(\det x'')\Psi'(x''^{-1}x')$ if x'' is invertible and $\Phi_{\Psi'}(x) = 0$ otherwise. Write $\kappa_{h'}(x) = \theta^{V',\mathrm{I}_b}_{\alpha,\beta;\mathfrak{m}}[\chi\circ\det, 1](x, h')$ and $\lambda(x) = \theta^{V',W'}_{\alpha,\beta';\mathfrak{m}}[\Psi', Q](x, g)$ for $x \in \mathrm{U}(V')(\mathbb{A}_F)$.

(i) For every h', $\kappa_{h'}\lambda$ is integrable on $[\mathrm{U}(V')]$ and

$$c \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} K_{h'}(\Phi_{\Psi'})(h) = \int_{[\mathrm{U}(V')]} \kappa_{h'}(x)\lambda(x)\, dx.$$

(ii) If $c_0 \in \mathbb{C}$, $(h'_j, \omega_j)_{j\in J}$ is a finite family of finite points and complex numbers and F is the function $F(x) = c_0 \sum_j \omega_j \kappa_{h'_j}(x)$, then $F\lambda$ is integrable and

$$c\, c_0 \sum_{j\in J} \omega_j \sum_{h \bmod \mathbf{K}_a}^{\mathrm{cl}} K_{h'_j}(\Phi_{\Psi'})(h) = \int_{[\mathrm{U}(V')]} F(x)\lambda(x)\, dx.$$

Proof. **Step 1: the matched partner of $\Phi_{\Psi'}$.** Put $\Psi(y) = \chi(\det y_1)\Psi'(y_2)$ for $y \in \mathrm{M}_{(b\sqcup a)\times b}(R)$, y_1 the rows b and y_2 the rows a (with $\chi(\det y_1) = 0$ unless y_1 is invertible). We show that $\Phi_{\Psi'}$ and Ψ are matched for (id, χ^{-1}). If $\Phi_{\Psi'}(x) \neq 0$, then x'' is invertible and x is the matrix of the rows a of the invertible matrix $\begin{pmatrix} 1_b & 0 \\ x' & x'' \end{pmatrix}$, which is k^{-1} for k its inverse. If $\Psi(y) \neq 0$, then $\det y_1$ is a unit and y is the matrix of the columns b of the invertible $\begin{pmatrix} y_1 & 0 \\ y_2 & 1_a \end{pmatrix}$. Now let $k \in \mathrm{GL}_{b\sqcup a}(R)$, in blocks $k = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $k^{-1} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ (rows and columns b, a); the rows a of k^{-1} are $(C' \,|\, D')$ and the columns b of k are $\binom{A}{C}$. By Jacobi's theorem on complementary minors, $\det D' = \det A\,(\det k)^{-1}$, so D' is invertible iff A is; if they are not, both $\Phi_{\Psi'}(C' \,|\, D')$

and $\chi^{-1}(\det k)\Psi\binom{A}{C}$ vanish. If they are, the block (a,b) of $k^{-1}k=1$ gives $C'A+D'C=0$, so $D'^{-1}C'=C(-A^{-1})$ and $\Psi'(D'^{-1}C')=\Psi'(C)$ by the invariance of Ψ'; hence

$$\Phi_{\Psi'}(C'\mid D')=\chi(\det A)\chi(\det k)^{-1}\Psi'(C)=\chi^{-1}(\det k)\,\Psi\begin{pmatrix}A\\C\end{pmatrix}.$$

Step 2: the kernel of the partner factors. $\Psi=(\chi\circ\det)\otimes\Psi'$ and $\tilde{Q}=1\cdot Q$ in the notation of (a) of Lemma 6.3.80 (blocks $T_1=\mathrm{I}_b$, $T_2=W'$, source V'), so $\mathrm{mod}_{\mathfrak{m}}(\Psi,v_{\tilde{Q}})=\mathrm{mod}_{\mathfrak{m}}(\chi\circ\det,v_1)\boxtimes^{\mathrm{tg}}\mathrm{mod}_{\mathfrak{m}}(\Psi',v_Q)$. By the exact factorisation (d) of Lemma 6.3.52 (target sums, characters α and $\beta\beta'$),

$$\theta^{V',W}_{\alpha,\beta\beta';\mathfrak{m}}[\Psi,\tilde{Q}](x,\mathrm{diag}(h',g))=\kappa_{h'}(x)\,\lambda(x)\qquad(x\in\mathrm{U}(V')(\mathbb{A}_F)),$$

the finite point h' being read with infinite components 1.

Step 3: conclusion. (i) By Step 1 the exchange identity applies to $(\Phi_{\Psi'},\Psi)$: the left side of Step 2 is integrable on $[\mathrm{U}(V')]$ with integral $c\sum^{\mathrm{cl}}K_{h'}(\Phi_{\Psi'})$. So $\kappa_{h'}\lambda$ is integrable, and the identity follows. (ii) $F\lambda=c_0\sum_j\omega_j\kappa_{h'_j}\lambda$ is a finite combination of integrable functions; integrate term by term and use (i) for each h'_j. □

Lemma 6.6.64. *Let s^{c} be the signs of S^{c}, o the order on $\mathcal{B}^{\mathrm{c}}$ with $o((1,a))=-a$ $(1\le a\le 2m)$ and $o((2,a))=a-1$ $(1\le a\le m)$, and $u=(u_v)_{v\in\Phi}$ the weights with $u_v((1,a))=0$ and $u_v((2,a))=3m$.*

(i) If some automorphic character of $\mathrm{U}(1)$ *of weight* 0 *is tame for some* $(\mathfrak{m},\chi)$*, then φ° is adelically holomorphic of weight u for o.*

(ii) $\varphi^\circ\neq 0$.

Proof. **(i).** Fix $(\mathfrak{m},\chi)$ and an automorphic character c of $\mathrm{U}(1)$ of weight 0 tame for $(\mathfrak{m},\chi)$, and write $\mathrm{I}=\mathrm{I}_{\mathcal{B}^{\mathrm{c}}}$. By (a) of Lemma 6.6.62 and the definition of class sums there are finitely many $h_1,\dots,h_r\in G_f(\mathcal{B}^{\mathrm{c}})$ (read as adelic points with infinite components 1) and $a_1,\dots,a_r\in\mathbb{C}$ with $\varphi^\circ=\sum_k a_k\kappa_{h_k}$, where

$$\kappa_h(x)=\theta^{S^{\mathrm{c}},\mathrm{I}}_{\upsilon^m,c}(v_1\otimes f)(x,h),\qquad f=(\chi\circ\det)^\natural\otimes_{\Sigma(\mathfrak{m})}v^0\in\mathcal{M}_{S^{\mathrm{c}},\mathrm{I}},$$

since $\theta_{\upsilon^m,c;\mathfrak{m}}[\chi\circ\det,1]$ is the kernel of $\mathrm{mod}_{\mathfrak{m}}(\chi\circ\det,v_1)=v_1\otimes f$. A finite linear combination of functions holomorphic of weight u for o is holomorphic of weight u for o (the same combination of the extensions is an extension). So it suffices to show that each κ_h is adelically holomorphic of weight u for o.

Fix h and $g\in\mathrm{G}(s^{\mathrm{c}})$, and put $\ell(w)=\theta_{\mathrm{Hom}(S^{\mathrm{c}},\mathrm{I})}(\omega_{\upsilon^m,c}(g,h)(w\otimes f))$ for $w\in\mathcal{F}_{S^{\mathrm{c}},\mathrm{I}}$. The map $w\mapsto w\otimes f=\int w(y)\rho(\mathrm{rv}(y))f\,d\nu_\iota(y)$ is linear, and $\|w\otimes f\|\le\int|w(y)|\,\|\rho(\mathrm{rv}(y))f\|\,d\nu_\iota(y)=\|w\|\,\|f\|$, because each $\rho(z)$ is an isometry of $L^1(\mu_n)$ ($|\beta_s|=1$ and μ_n is translation invariant). As $\omega_{\upsilon^m,c}(g,h)$ is bounded ((2) of Lemma 4.1.1) and the theta functional is bounded, ℓ is a bounded linear functional on $\mathcal{F}_{S^{\mathrm{c}},\mathrm{I}}$. For $z\in G_\infty(s^{\mathrm{c}})$, $\omega_{\upsilon^m,c}(g(z,1),h)=\omega_{\upsilon^m,c}(g,h)\,\omega_{\upsilon^m,c}((z,1),1)$ ((2) of Lemma 4.1.1), and by (5) there (υ^m has weight m, c weight 0)

$$\omega_{\upsilon^m,c}((z,1),1)(v_1\otimes f)=\big(\Omega_{m,0}(z,1)v_1\big)\otimes f=\big(\Omega_{0,0}(z,1)v_1\big)\otimes f,$$

the factor $\det(1)^m$ of (WR5) being 1. Hence $\kappa_h(g(z,1))=\ell(\Omega_{0,0}(z,1)v_1)$. Apply Lemma 6.6.65 with p the first block, q the second block of $\mathcal{B}^{\mathrm{c}}$, $n'=\mathcal{B}^{\mathrm{c}}$, i the inclusion of the first block, $d=0$

(so $Q_0 = 1$ and $v_{Q_0} = v_1$), $S = S^{\mathrm{c}}$, $T = \mathrm{I}$ and the order o, which is injective and positive-first for s^{c} ($o((1,a)) = -a \le -1 < 0 \le a' - 1 = o((2,a'))$). Its weight u_0 is 0 on the first and $\#\mathcal{B}^{\mathrm{c}} = 3m$ on the second block, that is u. So $z \mapsto \kappa_h(g(z,1))$ is holomorphic of weight u for o for every g, which is (i).

(ii). Let $b = \{1,\dots,m\}$, $S = \mathrm{I}_b^-$ (signs -1), $T = S^{\mathrm{c}}$, $v = v_{P^\circ} \in \mathcal{F}_{S,T}$ (variables $X_{\tau,a,c}$, $a \in \mathcal{B}^{\mathrm{c}}$, $c \in b$) and $\varphi = v \otimes v^0$, the datum $v \otimes \Phi \otimes \Phi^\circ$ with $A = \emptyset$, $\Phi = 1$. By (XE4), $\varphi^\circ(x) = \int_{[\mathrm{U}(S)]} \theta^{S,T}_{v^m,1}(\varphi)(g,x)\,dg$ is the theta lift $\Theta_\varphi \mathbf{1}$ of the constant function $\mathbf{1}$ in the notation of Theorem 6.5.13.

Step 1: the hypotheses of Theorem 6.5.13. $2\,\mathrm{rk}\,S = 2m < 3m = \mathrm{rk}\,T$; v is a Schwartz vector; φ is clean by (DC1), all signs of S being -1. The function $\mathbf{1}$ is continuous and invariant on both sides, and $\mathrm{ht}_S \equiv 1$: let x_S be the point of S, $g \in \mathrm{G}(-1_b)$, $u \ne 0$ in the lattice of S_g and $w = (x_S g)^{-1}u$, so $w_f \in \widehat{\mathcal{O}}_E^m$. Since $(x_S g)^* J_S (x_S g) = \varepsilon_{-1_b}$, the number $a = -u^* J_S u \in F$ has the component $|w_v|^2 > 0$ at each $v \in \Phi$ (the infinite part of ε_{-1_b} is -1) and the finite part $-w_f^* w_f \in \widehat{\mathcal{O}}_F$ (its finite part is 1); so a is a nonzero totally positive integer of F and $\|w_\infty\|^2 = \mathrm{Tr}_{F/\mathbb{Q}}\, a \ge \#\Phi\, \mathrm{N}_{F/\mathbb{Q}}(a)^{1/\#\Phi} \ge 1$ (arithmetic and geometric means). Thus $\lambda_S \equiv 1$, and $\mathbf{1}\cdot \mathrm{ht}_S^N$ is bounded. Finally $V_S = \mathrm{vol}([\mathrm{U}(S)]) \in (0,\infty)$: $G_\infty(-1_b)$ is compact and $\Gamma_S \backslash G_f(b)/\mathbf{K}_0(b)$ is finite (finiteness of class numbers, Borel), so $[\mathrm{U}(S)]$ is compact. Hence $\langle \mathbf{1}, R(g)\mathbf{1}\rangle_S = V_S$ for every g.

Step 2: the coefficient of the datum. Let $g = (x,k)$, $x \in G_\infty(-1_b)$, $k \in G_f(b)$. By (5) of Lemma 4.1.1 (v^m of weight m, 1 of weight 0), $\omega_{v^m,1}(g,1)\varphi = (\Omega_{m,0}(x,1)v) \otimes f'_k$ with $f'_k = \omega_{v^m,1}((1,k),1)v^0 = \omega_{1,1}((1,k),1)v^0$ (WR4). The element x is block diagonal for the signs of b (one block) and so is $1 \in G_\infty(1_{\mathcal{B}^{\mathrm{c}}})$, so the second formula of (5) applies: the factor is $\prod_{v\in\Phi} \det(x_v)^{\#\{a : s^{\mathrm{c}}(a) = +1\}} = \prod_v \det(x_v)^{2m}$, and the substitution sends $X_{v,(2,i),c}$ (where $s^{\mathrm{c}}((2,i)) = -1 = s(c)$) to $\sum_{j\in b}(x_v^{-1})_{cj} X_{v,(2,i),j}$, so the matrix $(X_{v,(2,i),c})_{i,c}$ goes to itself times $(x_v^{-1})^{\mathsf{T}}$ and P° to $\prod_v \det(x_v)^{-2m} P^\circ$. Hence

$$\Omega_{m,0}(x,1)v = v \qquad (x \in G_\infty(-1_b)).$$

By (4) of Lemma 6.2.33 ($A = \emptyset$), $v_1 \otimes v^0 = v^0$, and by (5) of Lemma 4.1.1 for v_1, $f'_k = \omega_{1,1}((1,k),1)(v_1 \otimes v^0) = v_1 \otimes f'_k$. So (6) of Lemma 6.2.33 gives

$$\langle \varphi, \omega_{v^m,1}(g,1)\varphi\rangle = \langle v,v\rangle\, \langle v_1 \otimes v^0, v_1 \otimes f'_k\rangle = \langle v,v\rangle\, \mathrm{mc}(k), \qquad \mathrm{mc}(k) = \langle v^0, \omega_{1,1}((1,k),1)v^0\rangle.$$

mc is the vacuum coefficient $\mathrm{mc}_{1,\Sigma}$ for $\beta = 1$ and Σ the set of all finite places of F. It is right invariant under $\mathbf{K}_0(b)$: for $\kappa \in \mathbf{K}_0(b)$, $\omega_{1,1}((1,\kappa),1)v^0 = v^0$ by (6) of Lemma 4.1.1 (with $v_1 \otimes v^0 = v^0$).

Step 3: the Rallis inner product formula. By (1) of Theorem 6.5.13 for $y = y' = \mathbf{1}$, $(x,k) \mapsto V_S \langle v,v\rangle\, \mathrm{mc}(k)$ is integrable on $\mathrm{G}(-1_b)$ with the product of the measures of $G_\infty(-1_b)$ and $G_f(b)$. Here $\langle v,v\rangle > 0$ ($P^\circ \ne 0$, so v_{P° is a nonzero continuous function) and $V_S > 0$, so mc is integrable on $G_f(b)$ (Fubini, $0 < \mathrm{vol}(G_\infty(-1_b)) < \infty$), and (2) there gives

$$\langle \varphi^\circ, \varphi^\circ\rangle_T = \mathrm{vol}([\mathrm{U}(T)])\, V_S\, \langle v,v\rangle\, \mathrm{vol}(G_\infty(-1_b)) \int_{G_f(b)} \mathrm{mc}(k)\,dk.$$

Step 4: the finite integral is not zero. With Σ all finite places of F, the group K_Σ of Proposition 6.5.4 is $\mathbf{K}_0(b)$; let u_q be representatives of $G_f(b)/\mathbf{K}_0(b)$. By (1) there, $\sum_q |\mathrm{mc}(u_q)| < \infty$; as mc is right $\mathbf{K}_0(b)$-invariant and $\mathbf{K}_0(b)$ has mass 1, $\int_{G_f(b)} \mathrm{mc} = \sum_q \mathrm{mc}(u_q)$. The hypothesis of (2), (3) there holds for $\beta = 1$, and the Rallis series of $\mathbf{1}$ is the constant $\sum_q \mathrm{mc}(u_q)$. By (3), $R_\Sigma \mathbf{1} = 0$ would give $\mathbf{1} = 0$; so $\sum_q \mathrm{mc}(u_q) \ne 0$.

As $\mathrm{vol}([\mathrm{U}(T)]) > 0$, Steps 3 and 4 give $\langle \varphi^\circ, \varphi^\circ\rangle_T \ne 0$, so $\varphi^\circ \ne 0$. □

Lemma 6.6.65. *Let p, q and n' be finite index sets, $n = p \sqcup q$ with the signs $s = +1$ on p and $s = -1$ on q, $i\colon p \to n'$ an injective map and $d \in \mathbb{N}$. Let S, T be spaces with points with signs s on n and $+1$ on n', so that $\mathcal{F}_{S,T} = \mathcal{F}_\varepsilon$ with $\iota = \Phi \times n' \times n$ and $\varepsilon(v,a,b) = s(b)$, and put*

$$Q_d = \prod_{v\in\Phi} \det\bigl((X_{v,i(a),b})_{a,b\in p}\bigr)^d \in \mathbb{C}[X_\iota].$$

Then $Q_d \neq 0$, and for every order o on n that is positive-first for s and every bounded linear functional ℓ on $\mathcal{F}_{S,T}$ the function $x \mapsto \ell(\Omega_{0,0}(x,1)\, v_{Q_d})$ on $G_\infty(s)$ is holomorphic of weight u_d for o, where $u_d(v)(b) = -d$ for $b \in p$ and $u_d(v)(b) = \#n'$ for $b \in q$ ($v \in \Phi$).

Proof. Fix $v \in \Phi$ and write $X_{ac} = X_{v,a,c}$ ($a \in n'$, $c \in n$), $D = \det\bigl((X_{i(a),b})_{a,b\in p}\bigr)$ and $P = D^d$, so that $Q_d = \prod_v P(X_{v,\cdot,\cdot})$. The variable X_{ac} has the sign $s(c)$ for the signs $(s, \mathbf{1}_{n'})$ of (HV5); P involves only variables X_{ac} with $c \in p$ and $a \in i(p)$, all of sign $+1$.

$Q_d \neq 0$. As i is injective, the monomial $\prod_{a\in p} X_{i(a),a}$ occurs in D with coefficient 1, so $D \neq 0$ ($D = 1$ if $p = \emptyset$); the polynomial ring is an integral domain, so $Q_d \neq 0$.

P is termwise harmonic. The rows all have sign $+1$, so the first condition of (HV6) is empty; in the second, one of c, c' lies in q, and P does not involve the variables of that column, so the mixed derivative vanishes.

Column units, shifts $(0,0)$. By (HV5), $E^{\mathrm{col}}_{cc'} = -\sum_{a\in n'} \mathrm{lo}_{ac}\,\mathrm{ra}_{ac'} + \#n'\,\delta_{cc'}$. For $c, c' \in p$ the variables have sign $+1$, $\mathrm{lo}_{ac}\mathrm{ra}_{ac'} = \partial_{ac}X_{ac'} = X_{ac'}\partial_{ac} + \delta_{cc'}$, and so $E^{\mathrm{col}}_{cc'} = -\sum_a X_{ac'}\partial_{ac}$. By Laplace expansion along the column c, $\sum_a X_{ac'}\partial D/\partial X_{ac}$ is the determinant obtained from D by replacing the column c by the column c': it is D for $c' = c$ and 0 for $c' \neq c$ (two equal columns). Hence $E^{\mathrm{col}}_{cc}P = -dP$ and $E^{\mathrm{col}}_{cc'}P = dD^{d-1}\cdot 0 = 0$ for $c \neq c'$ in p. For $c, c' \in q$ the variables have sign -1, $\mathrm{lo}_{ac}\mathrm{ra}_{ac'} = -X_{ac}\partial_{ac'}$ and $\partial_{ac'}P = 0$, so $E^{\mathrm{col}}_{cc'}P = \#n'\,\delta_{cc'}P$. Thus P is K-highest of weight $u_d(v)$ (equal to $-d$ on p and $\#n'$ on q) for $-o$: every $E^{\mathrm{col}}_{cc'}$ with $c \neq c'$ of the same sign kills P.

Row units, shift 0. Let $l'\colon n' \to \mathbb{Z}$ be 1 on $i(p)$ and 0 elsewhere. By (HV5), $E^{\mathrm{row}}_{aa'} = \sum_{c\in p} X_{ac}\partial_{a'c} - \sum_{c\in q}\partial_{ac}X_{a'c}$. The second sum acts on P as $-\#q\,\delta_{aa'}$. By Laplace expansion along the row a', $\sum_{c\in p} X_{ac}\partial D/\partial X_{a'c}$ is 0 if $a' \notin i(p)$, and otherwise the determinant obtained by replacing the row a' by the row a: D for $a = a'$ and 0 for $a \neq a'$ in $i(p)$. Hence $E^{\mathrm{row}}_{aa}P = \nu(a)P$ with $\nu(a) = d\,[a \in i(p)] - \#q$, and $E^{\mathrm{row}}_{aa'}P \neq 0$ with $a \neq a'$ is possible only for $a' \in i(p)$, $a \notin i(p)$, where $l'(a') = 1 > 0 = l'(a)$. So $E^{\mathrm{row}}_{aa'}P = 0$ whenever $l'(a') < l'(a)$: P is K-highest of weight ν for l'.

Conclusion. P is jointly highest of weights $(\nu, u_d(v))$ for $(l', -o)$, for the row and column units of the signs $(s, \mathbf{1}_{n'})$ with the shifts $(0,0)$. With $l_v = o$ (so $l_v(a) < l_v(b)$ whenever $o(a) < o(b)$), $\gamma = e = 0$ and $P_v = P$ for every v, Lemma 6.6.47 shows that $x \mapsto \ell(\Omega_{0,0}(x,1)v_{Q_d})$ is holomorphic of weight u_d for o, for every bounded linear functional ℓ on $\mathcal{F}_{S,T}$. □

Lemma 6.6.66. *Let $1 \le j \le m$, $n_* = 2ft^3$, $\tilde{W}_j$ the padded space of the j-th lift, with signs $\tilde{s}_j$ on $\mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{A}_j^+$, and for $\tau \in \Phi$ let $d_j(\tau) = \#D_j(\tau)$ and $\gamma'_j(\tau) = m - d_j(\tau)$. On $\mathcal{B}^{\mathrm{c}}$ let u_1, $u_2(\tau)$, $u_3(\tau)$ be the weights with the values, on the first and on the second block,*

$$u_1 = (-\Delta';\ 6m+1), \qquad u_2(\tau) = (\Delta';\ d_j(\tau) + r + 2m), \qquad u_3(\tau) = (0;\ d_j(\tau) + |\mathcal{B}_j| + 2m),$$

and write u_i also for the families $(u_i(\tau))_{\tau\in\Phi}$ ($u_1(\tau) = u_1$). For $\gamma\colon \Phi \to \mathbb{Z}$, $h \in G_\infty(s^{\mathrm{c}})$ and $h' \in G_\infty(\tilde{s}_j)$ put

$$c_\gamma(h,h') = \bigl\langle v_{P_j},\, \Omega_{\gamma,\gamma'_j}(h,h')\, v_{P_j}\bigr\rangle,$$

in the Fock model of $\mathrm{Hom}(S^{\mathrm{c}}, \tilde{W}_j)$. *Then:*

(a) the carrier character $\nu_j^\flat \upsilon^m$ has weight γ'_j and trivial tame part;

(b) for every γ and every $h \in G_\infty(s^{\mathrm{c}})$,

$$c_\gamma(h,1) = c_\gamma(1,1)\, m^{\mathrm{hol}}_{s^{\mathrm{c}},u_2}(h)\, \overline{m^{\mathrm{hol}}_{s^{\mathrm{c}},u_3}(h)};$$

(c) for every order o on $\mathcal{B}^{\mathrm{c}}$ and every $\tau \in \Phi$, the weights u_1 and $u_3(\tau)$ are K-dominant for (s^{c}, o) and have holomorphic gap $300n_*$ for s^{c}, and $|u_3(\tau)(b')| \le 5b$ for all $b' \in \mathcal{B}^{\mathrm{c}}$;

(d) P_j is homogeneous of a degree at most $b^{3/2}$.

Proof. Fix $\tau \in \Phi$ where it is not displayed and write $\varsigma = \varsigma_\tau$, $\kappa^+(h) = \kappa^+_O(\tau, h)$. By Lemma 4.2.67, $x \ge e^{6.4\cdot 10^{15}}$, $x^{10^{12}} \le b$, $m \le 2\sqrt{x}$, $x^5 \le r \le 3x^5$, $t \le 15x^5$, $r^8 \le 3^8 x^{40}$ and $f \le x^{12}$; in particular $k = r+1-m \ge 2$ and $r \ge 2(11m+2)$. As t is odd and k even, $e_j = j + \frac{t+k-1}{2}$ and $\Delta' = r^8 - \frac{t+1}{2}$ are integers, and $\Delta' \ge 0$.

Step 1: the determinant index. We show $d_j(\tau) = \kappa_j(\tau) - e_j$. By Definition 3.2.16, $\varsigma(i) = \frac{k}{2} - 1 - i$ for $0 \le i \le k-2$ and $\varsigma(k-2+j') = \frac{\Delta}{2} - \kappa_{j'}(\tau)$ for $1 \le j' \le m$; this sequence is strictly decreasing, since $\frac{\Delta}{2} - \kappa_1(\tau) = -\kappa(\tau) - \Delta < 1 - \frac{k}{2}$ and $\kappa_{j'}$ increases with j'. For $1 \le h \le r_+ - 1 = k + j - 2$ we have $\varsigma(h) \ge \varsigma(k+j-2) = \frac{\Delta}{2} - \kappa_j(\tau) \ge \frac{t-1}{2} - \kappa_j(\tau)$ (as $r^8 \ge \frac{t-1}{2}$), so by (LF4) $\kappa^+(h) = (\varsigma(h-1) - \varsigma(h) - 1)^+$. The difference $\varsigma(h-1) - \varsigma(h) - 1$ is 0 for $1 \le h \le k-2$, equals $\kappa_1(\tau) - \frac{\Delta}{2} - \frac{k}{2} = \kappa(\tau) + \Delta - \frac{k}{2} \ge 0$ for $h = k-1$, and equals $\kappa_{j'}(\tau) - \kappa_{j'-1}(\tau) - 1 = \Delta - 1 \ge 0$ for $h = k-2+j'$, $2 \le j' \le j$. So all terms are nonnegative and the sum telescopes:

$$\sum_{h=1}^{r_+-1} \kappa^+(h) = \varsigma(0) - \varsigma(k+j-2) - (k+j-2) = \kappa_j(\tau) - \tfrac{\Delta}{2} - \tfrac{k}{2} - j + 1.$$

As $D_j(\tau)$ is the disjoint union of these column sets and $\{1, \dots, \Delta'\}$, $d_j(\tau) = \kappa_j(\tau) - \frac{\Delta}{2} - \frac{k}{2} - j + 1 + \frac{\Delta}{2} - \frac{t+1}{2} = \kappa_j(\tau) - e_j$. In particular $d_j(\tau) \ge \Delta'$.

(a). By the defining formula of Definition 3.1.1, a product of characters with weights w, w' and tame parts ψ, ψ' has weight $w + w'$ and tame part $\psi\psi'$, and an integer power of a character with trivial tame part has trivial tame part. So $\nu_j^\flat \upsilon^m = \mu^\flat \upsilon^{m+e_j - j\Delta}$ has trivial tame part and weight $-\kappa_0(\tau) + m + e_j - j\Delta = m + e_j - \kappa_j(\tau) = m - d_j(\tau) = \gamma'_j(\tau)$, as $\kappa_j - \kappa_0 = j\Delta$.

(b). The rows $\mathrm{ng}_j(i)$ $(1 \le i \le m)$ are distinct and have sign -1 in the padded space (the elements of $\mathcal{A}_j^-$, and the basis vectors of W_j of numbers $k + i - 2 \ge k + j - 1$); the rows $\mathrm{ps}_j(h,a)$ $(1 \le a \le 2m)$ are distinct for fixed h and have sign $+1$ (basis vectors of numbers $h + a - 1 < r_+$, or elements of $\mathcal{A}_j^+$). The columns $(2, i')$ have sign -1 and $(1, a')$ sign $+1$. So every variable of P_j has sign $+1$, the padded space has $r_- = (m-j) + j = m$ negative indices, and $\#(\mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{A}_j^+) = r + 3m$. For $u \in G_\infty(s^{\mathrm{c}})$ the substitution of Lemma 6.6.67 replaces y_τ^- by $y_\tau^-(u_\tau^-)^{\mathsf{T}}$, u_τ^- the block of u_τ on the second block (all m columns of sign -1 occur), and $y_{\tau,h}$ by $y_{\tau,h}(u_\tau^+)^{\mathsf{T}}$; so it multiplies P_j by $\prod_\tau \Delta_{n_+}(u_\tau)^{\nu(\tau)} \Delta_{n_-}(u_\tau)^{c(\tau)}$ with $\nu(\tau) = \sum_h \kappa^+(h)$ and $c(\tau) = d_j(\tau) + |\mathcal{B}_j| + m$. Lemma 6.6.67 with $e = \gamma'_j$ gives (b) with the weights $w_1(\tau) = (-\nu(\tau) + m - m + d_j(\tau);\ r + 3m - m + d_j(\tau)) = (\Delta';\ d_j(\tau) + r + 2m) = u_2(\tau)$ (Step 1: $d_j - \nu = \Delta'$) and $w_2(\tau) = (0;\ d_j(\tau) + |\mathcal{B}_j| + 2m) = u_3(\tau)$, and $\langle v_{P_j}, v_{P_j} \rangle = c_\gamma(1,1)$.

(c). u_1 and $u_3(\tau)$ are constant on each sign block, hence K-dominant for (s^{c}, o) for every order o. We have $300n_* + \frac{t+1}{2} \le 600x^{12}(15x^5)^3 + 8x^5 \le x^{40} \le r^8$, i.e. $300n_* \le \Delta'$. Hence u_1 has holomorphic gap $300n_*$ ($-\Delta' + 300n_* \le 0 < 6m+1$), and so has $u_3(\tau)$ ($300n_* \le \Delta' \le d_j(\tau)$). For the bound, $e_j \ge \frac{t-1}{2} \ge r + 11m + 1$, so by Step 1 $0 \le d_j(\tau) + |\mathcal{B}_j| + 2m = \kappa_j(\tau) - e_j + r + 11m + 1 \le \kappa_j(\tau)$; and $\kappa(\tau) \le b$ (as $a/x_1 = xb/x_1 \le b$), so $\kappa_j(\tau) \le b + (2m+1)r^8 \le b + 5\sqrt{x}\cdot 3^8 x^{40} \le 2b \le 5b$.

(d). $\det(y_\tau^-)$ and $\det(y_{\tau,h})$ are homogeneous of degrees m and $2m$, so P_j is homogeneous of degree $\sum_\tau (m\,c(\tau)+2m\,\nu(\tau)) \le \sum_\tau 3m(d_j(\tau)+|\mathcal{B}_j|+2m) \le 3fm\cdot 2b \le 12x^{25/2}b \le b^{3/2}$, using $\nu(\tau)\le d_j(\tau)$, (c) and $x^{25}\le b/144$. □

Lemma 6.6.67. *Let S, T be spaces with points with signs s on a nonempty index set n and s' on n', $n_\pm = \{i\in n : s(i)=\pm1\}$ and $r_- = \#\{a\in n' : s'(a)=-1\}$. Let P be a polynomial in the variables $X_{v,a,k}$ ($v\in\Phi$, $a\in n'$, $k\in n$) that involves only variables of sign $s'(a)s(k)=+1$. Let $\nu, c\colon \Phi\to\mathbb{N}$ and suppose that for every $u\in G_\infty(s)$ the substitution*

$$X_{v,a,k} \longmapsto \sum_{k'\in n:\ s(k')=s(k)} (u_v)_{k,k'}\, X_{v,a,k'} \qquad (s'(a)s(k)=+1)$$

multiplies P by $\prod_{v\in\Phi} \Delta_{n_+}(u_v)^{\nu(v)}\,\Delta_{n_-}(u_v)^{c(v)}$. Let $\gamma, e\colon \Phi\to\mathbb{Z}$ and let w_1, w_2 be the families of weights on n with

$$w_1(v)(i) = \begin{cases} -\nu(v)+r_- - e(v) & s(i)=+1,\\ \#n' - e(v) & s(i)=-1,\end{cases} \qquad w_2(v)(i) = \begin{cases} 0 & s(i)=+1,\\ c(v)+r_- & s(i)=-1.\end{cases}$$

Then for every $u\in G_\infty(s)$

$$\langle v_P, \Omega_{\gamma,e}(u,1)\,v_P\rangle = \langle v_P, v_P\rangle\; m^{\mathrm{hol}}_{s,w_1}(u)\,\overline{m^{\mathrm{hol}}_{s,w_2}(u)}.$$

Proof. For signs ε on a finite set ι put $\iota_\pm = \{i\in\iota : \varepsilon_i=\pm1\}$, write $x=(x_+,x_-)$ for $x\in\mathbb{C}^\iota$, $\mathrm{pr}_\pm = \frac12(1\pm\operatorname{diag}\varepsilon)$, and for a matrix g on ι let $g_{++},g_{+-},g_{-+},g_{--}$ be its blocks (rows first) and $\mathrm{bd}(g)$ the block-diagonal matrix with blocks g_{++},g_{--}. For a polynomial P in the variables X_i ($i\in\iota_+$) and an $\iota_+\times\iota_+$ matrix M let P^M be the polynomial $X\mapsto P(M^{\mathsf T}X)$, i.e. P after the substitution $X_i\mapsto\sum_{k\in\iota_+} M_{ki}X_k$; then $(P^M)^N = P^{NM}$. If P involves only these variables, $v_P(x) = P(\bar x_+)\,e^{-|x|^2/2}$.

Step 1: block identities in $\mathrm{U}(\operatorname{diag}\varepsilon)$. Let $g\in\mathrm{U}(\operatorname{diag}\varepsilon)$. From $g^*\operatorname{diag}(\varepsilon)g=\operatorname{diag}(\varepsilon)$ we get $g^{-1}=\operatorname{diag}(\varepsilon)g^*\operatorname{diag}(\varepsilon)$, hence $g\operatorname{diag}(\varepsilon)g^*=\operatorname{diag}(\varepsilon)$, $(g^{-1})_{++}=g^*_{++}$, $(g^{-1})_{--}=g^*_{--}$, and $|\det g|=1$. The blocks of $g^*\operatorname{diag}(\varepsilon)g=\operatorname{diag}(\varepsilon)$ on $\iota_-\times\iota_-$ and on $\iota_+\times\iota_+$ read

$$g^*_{--}g_{--} = 1+g^*_{+-}g_{+-}, \qquad g^*_{++}g_{++} = 1+g^*_{-+}g_{-+},$$

so g_{--} and g_{++} are invertible, and $Z=g_{+-}g_{--}^{-1}$ satisfies $Z^*Z = 1-(g_{--}g^*_{--})^{-1}$, so its operator norm is <1. Jacobi's theorem on complementary minors, $\det((g^{-1})_{--}) = \det(g_{++})/\det g$, gives

$$\overline{\det g_{--}} = \det(g_{++})/\det(g), \tag{6.153}$$

and the Schur complement formula, $(g^{-1})_{++} = (g_{++}-g_{+-}g_{--}^{-1}g_{-+})^{-1}$, gives $g_{++}-Zg_{-+}=M$ with $M=(g^*_{++})^{-1}$. Applied to $u_v\in\mathrm{U}(\operatorname{diag}s)$ (signs s on n), the same identities give $\Delta_{n_\pm}(u_v)\neq0$, $|\det u_v|=1$ and, from $(u_v^{-1})_{n_+n_+} = (u_v)^*_{n_+n_+}$ and Jacobi,

$$\overline{\Delta_{n_+}(u_v)} = \Delta_{n_-}(u_v)/\det(u_v). \tag{6.154}$$

Step 2: the Gaussian matrix. Let $H=\frac12(1+g^*g)$, a positive definite hermitian matrix. Since $(g\operatorname{diag}(\varepsilon)+\operatorname{diag}(\varepsilon)\,g)_{ij} = (\varepsilon_i+\varepsilon_j)g_{ij}$, we have $g\operatorname{diag}(\varepsilon)+\operatorname{diag}(\varepsilon)\,g = 2\operatorname{diag}(\varepsilon)\operatorname{bd}(g)$, and with $g^*=\operatorname{diag}(\varepsilon)g^{-1}\operatorname{diag}(\varepsilon)$,

$$1+g^*g = \operatorname{diag}(\varepsilon)g^{-1}\big(g\operatorname{diag}(\varepsilon)+\operatorname{diag}(\varepsilon)\,g\big) = 2\operatorname{diag}(\varepsilon)g^{-1}\operatorname{diag}(\varepsilon)\operatorname{bd}(g).$$

Hence $\det H = \det(g_{++})\det(g_{--})/\det g = |\det g_{--}|^2$ by (6.153), $H^{-1} = \mathrm{bd}(g)^{-1}\mathrm{diag}(\varepsilon)\, g\, \mathrm{diag}(\varepsilon)$, and, by $g\,\mathrm{diag}(\varepsilon)g^* = \mathrm{diag}(\varepsilon)$,

$$gH^{-1}g^* = g\,\mathrm{bd}(g)^{-1},$$

whose block on $\iota_+ \times \iota_-$ is Z. For $y = (y_+, 0)$ one computes $\mathrm{diag}(\varepsilon)y = y$, $\mathrm{diag}(\varepsilon)gy = (g_{++}y_+, -g_{-+}y_+)$, $\mathrm{bd}(g)^{-1}$ of this is $(y_+, -g_{--}^{-1}g_{-+}y_+)$, and g of this is $\big((g_{++} - Zg_{-+})y_+, 0\big) = (My_+, 0)$; so gH^{-1} maps $(y_+, 0)$ to $(My_+, 0)$.

Step 3: Gaussian integrals. Let H be positive definite hermitian on $\mathbb{C}^\iota$, $u, w \in \mathbb{C}^\iota$ and R a polynomial in the X_i, $i \in \iota_+$. Then

$$\int R(\bar y_+)\, \exp(-y^*Hy + u^{\mathsf T}y + y^*w)\, d\nu_\iota(y) = \det(H)^{-1}\, R(\lambda_+)\, \exp(u^{\mathsf T}H^{-1}w), \qquad \lambda^{\mathsf T} = u^{\mathsf T}H^{-1}. \tag{6.155}$$

Indeed, the integrand of $I(u,w) = \int \exp(-y^*Hy + u^{\mathsf T}y + y^*w)\, d\nu_\iota(y)$ is bounded by $\exp(-h|y|^2 + (|u|+|w|)|y|)$, $h > 0$ the least eigenvalue of H, so I is entire in (u,w) and may be differentiated under the integral sign. For $u = \bar w$ the substitution $y = z + H^{-1}w$ turns the exponent into $-z^*Hz + w^*H^{-1}w$, and $\int \exp(-z^*Hz)\, d\nu_\iota(z) = \det(H)^{-1}$ (diagonalise H by a unitary matrix, which preserves ν_ι, and use $\pi^{-1}\int_{\mathbb{C}} e^{-h|z|^2}\, dz = h^{-1}$); so $I(u,w) = \det(H)^{-1}\exp(u^{\mathsf T}H^{-1}w)$ on the totally real subspace $u = \bar w$ of maximal dimension, hence everywhere. As $\partial_{w_i}\exp(y^*w) = \bar y_i \exp(y^*w)$, the left side of (6.155) is $R(\partial_{w_+})I(u,w)$, and $R(\partial_{w_+})\exp(u^{\mathsf T}H^{-1}w) = R(\lambda_+)\exp(u^{\mathsf T}H^{-1}w)$.

Step 4: the Fock coefficient. Let $g \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$ and P a polynomial in the X_i, $i \in \iota_+$. We show

$$\langle v_P, \omega_\varepsilon^\infty(g)v_P\rangle = \det(g_{--})^{-1}\,\langle v_P, v_{P^M}\rangle, \qquad M = (g_{++}^*)^{-1}. \tag{6.156}$$

By the definitions of β_ε^∞ and e^∞,

$$-\tfrac12|y|^2 + \tfrac12(x^*\mathrm{diag}(\varepsilon)gy - y^*g^*\mathrm{diag}(\varepsilon)x) - \tfrac12|x - gy|^2 = -\tfrac12|x|^2 - y^*Hy + x^*\mathrm{pr}_+gy + y^*g^*\mathrm{pr}_-x,$$

so the integral defining $(\omega_\varepsilon^\infty(g)v_P)(x)$ is (6.155) with $R = P$, $u^{\mathsf T} = x^*\mathrm{pr}_+g$, $w = g^*\mathrm{pr}_-x$, times $\overline{\det g_{--}}\, e^{-|x|^2/2}$. By Step 2, $u^{\mathsf T}H^{-1}w = x^*\mathrm{pr}_+(gH^{-1}g^*)\mathrm{pr}_-x = \bar x_+^{\mathsf T}Zx_-$, and for $i \in \iota_+$, $\lambda_i = x^*\mathrm{pr}_+gH^{-1}e_i = \sum_{k\in\iota_+}\bar x_k M_{ki}$, so $P(\lambda_+) = P^M(\bar x_+)$. With $\overline{\det g_{--}}/\det H = \det(g_{--})^{-1}$ (Step 2):

$$(\omega_\varepsilon^\infty(g)v_P)(x) = \det(g_{--})^{-1}\, P^M(\bar x_+)\, \exp(\bar x_+^{\mathsf T}Zx_-)\, e^{-|x|^2/2}.$$

Multiply by $\overline{v_P(x)} = \overline{P(\bar x_+)}e^{-|x|^2/2}$. As $|\bar x_+^{\mathsf T}Zx_-| \le \|Z\||x_+||x_-| \le \frac12\|Z\||x|^2$ and $\|Z\| < 1$ (Step 1), the product is bounded by a polynomial times $\exp(-\frac12|x|^2)$, hence ν_ι-integrable, and Fubini over $\nu_\iota = \nu_{\iota_+}\otimes\nu_{\iota_-}$ applies. For fixed x_+, (6.155) on $\mathbb{C}^{\iota_-}$ with $H = 1$, $R = 1$, $u = Z^{\mathsf T}\bar x_+$, $w = 0$ gives $\int \exp(\bar x_+^{\mathsf T}Zx_-)e^{-|x_-|^2}\, d\nu_{\iota_-}(x_-) = 1$, and with $u = 0$ also $\int e^{-|x_-|^2}\, d\nu_{\iota_-} = 1$. So both sides of (6.156) equal $\det(g_{--})^{-1}\int \overline{P(\bar x_+)}\, P^M(\bar x_+)\, e^{-|x_+|^2}\, d\nu_{\iota_+}(x_+)$.

Step 5: the pair. Now $\iota = \Phi \times n' \times n$ and $\varepsilon(v,a,k) = s'(a)s(k)$. Fix $u \in G_\infty(s)$ and let $g = \bigoplus_{v\in\Phi} 1 \otimes (u_v^{-1})^{\mathsf T}$, the map $X \mapsto Xu_v^{-1}$ at each place; it lies in $\mathrm{U}(\mathrm{diag}\,\varepsilon)$ by (VP5) of Definition 3.2.2, and

$$g_{(v,a,k),(v',a',k')} = \delta_{vv'}\delta_{aa'}\,(u_v^{-1})_{k'k}.$$

By (WR5), $\Omega_{\gamma,e}(u,1) = \prod_{v\in\Phi}\det(u_v)^{-e(v)}\,\omega_\varepsilon^\infty(g)$. The block g_{++} lives on the (v,a,k) with $s(k) = s'(a)$; for such k,k' the relation $u_v^{-1} = \mathrm{diag}(s)u_v^*\mathrm{diag}(s)$ gives $\overline{(u_v^{-1})_{kk'}} = (u_v)_{k'k}$, so $(g_{++}^*)_{(v,a,k'),(v,a,k)} = (u_v)_{kk'}$ and the substitution $X \mapsto g_{++}^{\mathsf T}X$ is exactly the substitution of the statement. By hypothesis, $P^{g_{++}^*} = \chi(u)P$ with $\chi(u) = \prod_v \Delta_{n_+}(u_v)^{\nu(v)}\Delta_{n_-}(u_v)^{c(v)} \neq 0$ (Step 1). Applying $(\cdot)^M$ with $M = (g_{++}^*)^{-1}$ gives $P = (P^{g_{++}^*})^M = \chi(u)P^M$, so $P^M = \chi(u)^{-1}P$. The block g_{--} lives on the (v,a,k) with $s(k) = -s'(a)$ and is block diagonal over (v,a), the block

at (v,a) being the transpose of the principal block of u_v^{-1} on $n_{-s'(a)}$; by Jacobi's theorem its determinant is $\Delta_{n_{s'(a)}}(u_v)/\det(u_v)$. With $r_+ = \#n' - r_-$,

$$\det g_{--} = \prod_{v\in\Phi} \Big(\frac{\Delta_{n_+}(u_v)}{\det u_v}\Big)^{r_+} \Big(\frac{\Delta_{n_-}(u_v)}{\det u_v}\Big)^{r_-}.$$

By (6.156),

$$\langle v_P, \Omega_{\gamma,e}(u,1)v_P\rangle = \langle v_P, v_P\rangle \prod_{v\in\Phi} \det(u_v)^{\#n'-e(v)}\, \Delta_{n_+}(u_v)^{-\nu(v)-r_+}\, \Delta_{n_-}(u_v)^{-c(v)-r_-}. \quad (6.157)$$

Step 6: the model functions. Let w be a weight with the value a on n_+ and b on n_-, and $C = 2\sum_j |w(j)|$. If both n_+ and n_- are nonempty, $C \ge 2|a| + 2|b|$, so $a + C \ge b$, and (3.1) gives $m_{w+w^\sharp} = (\det)^b \Delta_{n_+}^{a+C-b}$ and $m_{w^\sharp} = \Delta_{n_+}^C$; if $n_- = \emptyset$, $m_{w+w^\sharp} = (\det)^{a+C}$ and $m_{w^\sharp} = (\det)^C$; if $n_+ = \emptyset$, $w^\sharp = 0$ and $m_w = (\det)^b$. In all cases, as $\Delta_{n_+} = \det$ if $n_- = \emptyset$ and $\Delta_\emptyset = 1$,

$$m^{\mathrm{hol}}_{s,w}(x) = \det(x)^b\, \Delta_{n_+}(x)^{a-b} \qquad (x \in \mathrm{U}(\mathrm{diag}\, s)).$$

For $w_1(v)$ ($a = -\nu(v) + r_- - e(v)$, $b = \#n' - e(v)$, $a - b = -\nu(v) - r_+$) this is

$$\det(u_v)^{\#n'-e(v)} \Delta_{n_+}(u_v)^{-\nu(v)-r_+}.$$

For $w_2(v)$ ($a = 0$, $b = c(v) + r_-$) it is

$$\det(u_v)^{c(v)+r_-} \Delta_{n_+}(u_v)^{-c(v)-r_-},$$

whose conjugate is $\Delta_{n_-}(u_v)^{-c(v)-r_-}$ by $\overline{\det u_v} = \det(u_v)^{-1}$ and (6.154). The product over $v \in \Phi$ of these two factors is the product in (6.157), which proves the lemma. □

Lemma 6.6.68. *Let D be admissible data and $n_* = 2ft^3$. Let n be a nonempty finite index set with $\#n \le t$, s signs on n, o a positive-first order for s, and $u = (u_v)_{v\in\Phi}$, $u' = (u'_v)_{v\in\Phi}$ weights such that every u_v and every u'_v is K-dominant for (s,o) and has holomorphic gap $300n_*$ for s, and $|u_v(i)| \le 5b$ for all v, i. For $h \in G_\infty(s)$ let $|h| = \max_{v\in\Phi} \|h_v\|$ (operator norms). Then $|h| \ge 1$ for every h, and*

$$0 < I := \int_{G_\infty(s)} |m^{\mathrm{hol}}_{s,u}|^2 < \infty, \qquad \int_{G_\infty(s)} |m^{\mathrm{hol}}_{s,u'}(h)|\, |h|^{6n_*}\, dh \le b^{n_*} I.$$

Proof. Write $N = \#n$.

Step 0: numbers. $f \ge 1$ and $N \le t$, so $n_* \ge 2N^3$. By Definition 2.3.3 and (AD1), $\log\log b \ge 6 \cdot 10^{12}$, so $b \ge 2^{40}$, and $\log b \ge 10^{12} \log x$; by (AD2) $t \le 15x^5$, so $2t + 1 \le 31x^5 \le b$; and $\log x \ge 100 \log\log b > 0$, so the integer x is at least 2, $r \ge x^5 \ge 32$ and $t = 5r \ge 160$. Put $L = 10b + N$ and $T = 8\big(2000(N + L + 1)^6\big)^{2N^2} 2^{10N^3}$. As $N + L + 1 \le 11b$ and $2000 \cdot 11^6 \le b$, $T \le 2^{3+10t^3} b^{14t^2}$, and

$$2^{6n_*} T^f \le 2^{12ft^3+3f+10ft^3} b^{14ft^2} \le 2^{25ft^3} b^{14ft^2} \le b^{ft^3} b^{ft^3} = b^{n_*} \quad (6.158)$$

($b \ge 2^{25}$, $t \ge 14$).

Step 1: the model functions as discrete series coefficients. Let w be one of the weights u_v, u'_v, and $l = 2w + 2\rho$ as in Lemma 6.6.69. As $300n_* > 500N^3 + 8n_* + N + 1$ (by $n_* \ge 2N^3$), Lemma 6.6.69 with $A = 8n_*$ and $\gamma = 300n_*$ gives: l has large gap $8n_*$, i.e. gap

$500N^3 + 8n_*$, and $c_{l/2} = m^{\mathrm{hol}}_{s,w}$ on $\mathrm{U}(\mathrm{diag}\, s)$. By (1) of Lemma 5.7.38, $|m^{\mathrm{hol}}_{s,w}(g)| \le (2/\|g\|)^c$ for $0 \le c \le 500N^3+8n_*$. For $w = u_v$, $|l_i| \le 10b+N = L$ ($|2\rho(i)| \le N-1$), and $10N^3 > 8N^2$, so (3) of Lemma 5.7.38 gives $d_v := d_{l/2} > 0$ and $\int d_v(2/\|g\|)^{10N^3} dg \le T$. Hence $|m^{\mathrm{hol}}_{s,u_v}|^2$ is integrable with integral d_v^{-1}, and by Tonelli over the factors of $G_\infty(s)$, $I = \prod_v d_v^{-1} \in (0,\infty)$.

Step 2: the second integral. For $h \in G_\infty(s)$ choose $a \in \Phi$ with $\|h_a\| = |h|$, and use $c = 10N^3+6n_*$ at a and $c = 10N^3$ at the other places for the factors of $m^{\mathrm{hol}}_{s,u'}$: $|m^{\mathrm{hol}}_{s,u'}(h)|\,|h|^{6n_*} \le 2^{6n_*}\prod_v (2/\|h_v\|)^{10N^3}$. So, by Tonelli and (6.158),

$$\int_{G_\infty(s)} |m^{\mathrm{hol}}_{s,u'}(h)|\,|h|^{6n_*} dh \le 2^{6n_*}\prod_v \Big(d_v^{-1}\int d_v(2/\|g\|)^{10N^3} dg\Big) \le 2^{6n_*}T^f I \le b^{n_*} I.$$

Finally every $h \in G_\infty(s)$ has $|h| \ge 1$: $h_v^{-1} = \mathrm{diag}(s) h_v^* \mathrm{diag}(s)$ has the norm of h_v, and $1 \le \|h_v\|\|h_v^{-1}\| = \|h_v\|^2$. □

Lemma 6.6.69. *Let n be a nonempty finite index set, $N = \#n$, s signs on n, o a positive-first order for s, $A \in \mathbb{N}$, and w a weight that is K-dominant for (s,o) and has holomorphic gap γ for s, where $\gamma > 500N^3 + A + N + 1$. Let $2\rho(a) = \#\{j : o(a) < o(j)\} - \#\{j : o(j) < o(a)\}$ and $l = 2w + 2\rho$. Then:*

(a) l has large gap A for s;

(b) $c_{l/2}(g) = m^{\mathrm{hol}}_{s,w}(g)$ for every $g \in \mathrm{U}(\mathrm{diag}\, s)$.

Proof. **(a).** $l(a) = 2w(a) + N - 1 - 2r(a) \equiv N-1$ modulo 2. If $s(a) = s(b)$ and $o(a) < o(b)$, then $w(b) \le w(a)$ (K-dominance) and $r(b) > r(a)$, so $l(b) \le l(a) - 2$: along o, l strictly decreases within each sign block. If $s(a) = +1$, $s(b) = -1$ (so $o(a) < o(b)$), then $w(a) + \gamma \le w(b)$ and $r(b) - r(a) \le N-1$, so $l(b) - l(a) \ge 2\gamma - 2(N-1) \ge 2(500N^3 + A)$, as $\gamma - N \ge 500N^3 + A + 2$. Hence l is injective and has gap $500N^3 + A$, i.e. large gap A; and $l(a) < l(b)$ whenever $s(a) = +1$, $s(b) = -1$.

Step 1: $\mu_l = w$. By (a), $l_j < l_i$ holds exactly when either $s(j) = s(i)$ and $o(j) > o(i)$, or $s(j) = +1$, $s(i) = -1$. Counting, with $P = \#n_+$, $Q = \#n_-$: for $s(i) = +1$, $2\rho_l(i) = 2\rho_l^{\mathrm{c}}(i) - Q$ and $2\rho(i) = 2\rho_l^{\mathrm{c}}(i) + Q$; for $s(i) = -1$, $2\rho_l(i) = 2\rho_l^{\mathrm{c}}(i) + P$ and $2\rho(i) = 2\rho_l^{\mathrm{c}}(i) - P$. As the entries of l have the parity of $N-1$, (3) of Lemma 5.6.27 gives $2l^\rho = l + 2\rho_l$, so $2\mu_l(i) = l_i + 2\rho_l(i) - 2\cdot 2\rho_l^{\mathrm{c}}(i) = 2w(i) + 2\rho(i) + 2\rho_l(i) - 2\cdot 2\rho_l^{\mathrm{c}}(i) = 2w(i)$ in both cases.

Step 2: both sides are analytic on G. By (ii) of Lemma 5.6.24 (l has gap 0), $c_{l/2}$ is the restriction to G of holomorphic functions on open neighbourhoods in $\mathrm{GL}_n(\mathbb{C})$ of the points of G; $m_{w+w^\sharp}$ and $m_{w^\sharp}$ are polynomials in the entries and the inverse determinant, and $m_{w^\sharp}$ does not vanish on G ((HV2)); so the same holds for $m^{\mathrm{hol}}_{s,w}$. Hence both restrictions are real analytic on the connected real Lie group G, and by the identity theorem it suffices to show that they agree on a neighbourhood of 1 in G.

Step 3: a factorisation near 1. Write matrices in blocks for $n = n_+ \sqcup n_-$. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ near 1, A and $S = D - CA^{-1}B$ are invertible and

$$g = x'\,a\,x, \qquad x' = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix}, \quad a = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}, \quad x = \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix};$$

for g near 1, g and a are accretive and the leading principal minors of A and S in the order o are nonzero, so (Gaussian elimination in each block) $a = c^* b$ with c, b block-diagonal and upper triangular in the order o inside each block, c with unit diagonal.

Step 4: $c_{l/2}(g) = b^w$. By (1) of Lemma 5.6.26 with the block-diagonal factor 1, $c_{l/2}(g) = \mathrm{FJ}_l(1, g)$, whose integrand at $k \in K_s$ has the denominator $m_{-l^\rho}(k^*gk)$. Put $X = k^*xk$ and $X' = k^*x'k$; k is block-diagonal, so $X - 1$ has nonzero entries only in rows n_+ and columns n_-, $X' - 1$ only in rows n_- and columns n_+, and $(X-1)^2 = (X'-1)^2 = 0$. Let ε' be the constant sign $+1$ on n, and B'_l the set of invertible matrices y with $y_{ij} = 0$ unless $l_i \geq l_j$ (the set B_l of Lemma 5.6.27 for ε'). Since $l_i < l_j$ for $i \in n_+$, $j \in n_-$, the matrices $X^{-*} = (2-X)^*$ and $X'^{-1} = 2 - X'$ lie in B'_l, with unit diagonals. By (1) and then (2) of Lemma 5.6.27 (for ε', with the weight l^ρ, which satisfies $l_i^\rho \geq l_j^\rho$ whenever $l_i > l_j$ by (3) there), for $y = k^*ak$,

$$m_{-l^\rho}(X'yX) = m_{l^\rho}(X^{-1}y^{-1}X'^{-1}) = \overline{(X^{-*})^{l^\rho}}\,(X'^{-1})^{l^\rho}\, m_{l^\rho}(y^{-1}) = m_{-l^\rho}(y).$$

As $k^*gk = X'(k^*ak)X$ and a is accretive, $\mathrm{FJ}_l(1, g) = \mathrm{FJ}_l(1, a)$. Now $a = c^* \cdot 1 \cdot b$ with b, c block-diagonal and $b_{ij} = c_{ij} = 0$ whenever $l_i < l_j$ (inside a block, $l_i < l_j$ means $o(i) > o(j)$), and 1 and a are accretive; by (3) of Lemma 5.6.26 and Step 1, $\mathrm{FJ}_l(1, a) = \overline{c^w}\, b^w\, \mathrm{FJ}_l(1, 1) = b^w$, since c has unit diagonal and $\mathrm{FJ}_l(1, 1) = \|m_{\mu_l}\|^{-2} \int |m_{\mu_l}(k)|^2 dk = 1$ ($m_v(1) = 1$).

Step 5: $m^{\mathrm{hol}}_{s,w}(g) = b^w$. Let $C = 2\sum_j |w(j)|$. For ε' and the injective vector $-o$, the set B'_{-o} consists of the invertible matrices that are upper triangular in the order o; it contains x and x'^* (o lists n_+ first), both with unit diagonal. The weights $v = w + w^\sharp$ and $v = w^\sharp$ satisfy $v_i \geq v_j$ whenever $o(i) < o(j)$: within a block by K-dominance, and for $i \in n_+$, $j \in n_-$ because $w(j) - w(i) \leq C$. So (2) of Lemma 5.6.27 gives $m_v(x'ax) = m_v(a)$ for both, and $m^{\mathrm{hol}}_{s,w}(g) = m_{w+w^\sharp}(a)/m_{w^\sharp}(a)$. Next, for the signs s and l, b and c lie in B_l (block-diagonal, $b_{ij} \neq 0$ only if $o(i) \leq o(j)$, i.e. $l_i \geq l_j$), and $v = w + w^\sharp$, $v = w^\sharp$ satisfy $v_i \geq v_j$ whenever $s(i) = s(j)$ and $l_i > l_j$ ($w^\sharp$ is constant on each block); so (2) of Lemma 5.6.27 gives $m_v(a) = m_v(c^*b) = b^v$, and $m^{\mathrm{hol}}_{s,w}(g) = b^{w+w^\sharp}/b^{w^\sharp} = b^w$, the diagonal entries of b being nonzero. With Step 4, $c_{l/2} = m^{\mathrm{hol}}_{s,w}$ near 1 in G, and Step 2 concludes. □

Lemma 6.6.70. *Let n be a nonempty finite index set, s signs on n, o an order that is positive-first for s, and $u = (u_v)_{v\in\Phi}$ weights with u_v K-dominant for (s, o) for every v. Call a complex $n \times n$ matrix a o-*lower unipotent *if $a_{ii} = 1$ for all i and $a_{ij} = 0$ whenever $o(i) < o(j)$. Then:*

(i) $m^{\mathrm{hol}}_{s,u}$ is holomorphic of weight u for o, and $m^{\mathrm{hol}}_{s,u}(1) = 1$;

(ii) if $x \in G_\infty(s)$ and $x_v = a_v b_v$ for every v, with a_v o-lower unipotent and b_v o-triangular, then $m^{\mathrm{hol}}_{s,u}(x) = \prod_{v\in\Phi} b_v^{u_v}$;

(iii) if $d = (d_v)_{v\in\Phi}$ with every d_v diagonal with entries of modulus 1, then

$$m^{\mathrm{hol}}_{s,u}(dx) = \Big(\prod_v d_v^{u_v}\Big)\, m^{\mathrm{hol}}_{s,u}(x)$$

for every $x \in G_\infty(s)$.

Proof. Call a weight w *dominant for o* if $w(j) \leq w(i)$ whenever $o(i) < o(j)$, and a set $P \subseteq n$ an *initial segment* if $i \in P$ and $o(i') < o(i)$ imply $i' \in P$.

Step 1: minors of initial segments. Let P be an initial segment, y a complex $n \times n$ matrix, a o-lower unipotent, b o-triangular and d diagonal. For $i \in P$ and $k \notin P$ we have $o(i) < o(k)$, so $a_{ik} = 0$ and $b_{ki} = 0$. Hence $(ayb)_{P,P} = a_{P,P}\, y_{P,P}\, b_{P,P}$, where $a_{P,P}$ and $b_{P,P}$ are triangular for the order o on P, with diagonals 1 and $(b_{ii})_{i\in P}$, and

$$\Delta_P(ayb) = \Delta_P(y) \prod_{i\in P} b_{ii}, \qquad \Delta_P(dy) = \Delta_P(y) \prod_{i\in P} d_{ii}. \tag{6.159}$$

Step 2: the extension of m_w. Let w be dominant for o, $w_0 = \min w$, $w_1 = \max w$. For $w_0 < t \le w_1$ the set $P_t = \{i : w(i) \ge t\}$ is an initial segment (if $w(i) \ge t$ and $o(i') < o(i)$ then $w(i') \ge w(i)$). On the open set Ω_w of the $y \in \mathrm{GL}_n(\mathbb{C})$ put $\tilde m_w(y) = (\det y)^{w_0} \prod_{w_0 < t \le w_1} \Delta_{P_t}(y)$, a holomorphic function equal to m_w on $\mathrm{U}(\operatorname{diag} s)$ by (3.1). By (6.159) and $\det(ayb) = \det y \prod_i b_{ii}$, for $y \in \mathrm{GL}_n(\mathbb{C})$

$$\tilde m_w(ayb) = \tilde m_w(y) \prod_{i \in n} b_{ii}^{w_0 + \#\{t:\, w_0 < t \le w(i)\}} = \tilde m_w(y)\, b^w, \qquad \tilde m_w(dy) = d^w\, \tilde m_w(y) \ (d \text{ invertible}),$$

as $w_0 + \#\{t : w_0 < t \le w(i)\} = w(i)$.

Step 3: the weights $u_v + u_v^\sharp$ and $u_v^\sharp$ are dominant for o. Fix v, $w = u_v$ and $C = 2\sum_j |w(j)|$. Let $o(i) < o(j)$. If $s(i) = s(j)$, then $w(j) \le w(i)$ (K-dominance) and $w^\sharp(i) = w^\sharp(j)$. Otherwise $s(i) = +1$, $s(j) = -1$ (o is positive-first), so $w^\sharp(i) = C \ge 0 = w^\sharp(j)$ and $w(j) \le w(i) + C$. In both cases $w^\sharp(j) \le w^\sharp(i)$ and $(w + w^\sharp)(j) \le (w + w^\sharp)(i)$.

Step 4: (i). Put $\tilde F(y) = \prod_v \tilde m_{u_v + u_v^\sharp}(y_v) / \tilde m_{u_v^\sharp}(y_v)$ where all y_v are invertible and the denominators do not vanish, and $\tilde F = 0$ elsewhere on $\mathrm{M}_n(\mathbb{C})^\Phi$; it is holomorphic on the open set where the formula applies. Let $x \in G_\infty(s)$ and b a family of o-triangular matrices. By Step 2 (with $a = 1$), $\tilde m_{u_v^\sharp}(x_v b_v) = m_{u_v^\sharp}(x_v)\, b_v^{u_v^\sharp} \ne 0$ by (HV2) and $b_{v,ii} \ne 0$; so $\tilde F$ is complex analytic at xb and

$$\tilde F(xb) = \prod_v \frac{m_{u_v + u_v^\sharp}(x_v)\, b_v^{u_v + u_v^\sharp}}{m_{u_v^\sharp}(x_v)\, b_v^{u_v^\sharp}} = \Big(\prod_v b_v^{u_v}\Big) m_{s,u}^{\mathrm{hol}}(x).$$

So $m_{s,u}^{\mathrm{hol}}$ is holomorphic of weight u for o; and $m_w(1) = 1$ for every w by (3.1), so $m_{s,u}^{\mathrm{hol}}(1) = 1$.

Step 5: (ii) and (iii). In (ii), Step 2 with $y = 1$ gives $m_w(x_v) = \tilde m_w(a_v \cdot 1 \cdot b_v) = b_v^w$ for $w = u_v + u_v^\sharp$ and $w = u_v^\sharp$, hence $m_{s,u_v}^{\mathrm{hol}}(x_v) = b_v^{u_v + u_v^\sharp} / b_v^{u_v^\sharp} = b_v^{u_v}$. In (iii), $d_v x_v \in \mathrm{U}(\operatorname{diag} s)$ and Step 2 gives $m_w(d_v x_v) = d_v^w m_w(x_v)$, hence $m_{s,u_v}^{\mathrm{hol}}(d_v x_v) = d_v^{u_v} m_{s,u_v}^{\mathrm{hol}}(x_v)$; take the product over v. □

Lemma 6.6.71. *Let n be a nonempty finite index set, s signs on n, o a positive-first order for s, $u = (u_v)_{v \in \Phi}$ weights with every u_v K-dominant for (s, o), $m = m_{s,u}^{\mathrm{hol}}$, and let $F \colon G_\infty(s) \to \mathbb{C}$ be holomorphic of weight u for o with $F\,\overline{m}$ integrable on $G_\infty(s)$. Then $x \mapsto F(1)\,|m(x)|^2$ is integrable and*

$$\int_{G_\infty(s)} F(x)\, \overline{m(x)}\, dx = F(1) \int_{G_\infty(s)} |m(x)|^2\, dx.$$

Proof. Let $d(\theta)$ be as in Lemma 6.6.72 and $\chi(\theta) = \prod_{v,i} e^{2\pi \mathrm{i} \theta o(i) u_v(i)}$, of modulus 1. By (iii) of Lemma 6.6.70, $m(d(\theta)x) = \chi(\theta) m(x)$, so $\overline{m(d(\theta)x)} = \chi(\theta)^{-1} \overline{m(x)}$. The measure of $G_\infty(s)$ is invariant under left translation by $d(\theta) \in G_\infty(s)$ (Lemma 5.2.41 at every v). Let $f = F\overline{m}$ and $g(\theta, x) = f(d(\theta)x) = F(d(\theta)x)\chi(\theta)^{-1}\overline{m(x)}$, a continuous function on $\mathbb{R}/\mathbb{Z} \times G_\infty(s)$ (F is continuous, being the restriction of a function analytic at every point of $G_\infty(s)$, and m is continuous by (i) of Lemma 6.6.70). By the invariance, $\int |g(\theta, x)|\, dx = \int |f|$ for every θ, so g is integrable on the product (Tonelli) and, by Fubini and the invariance again,

$$\int_{G_\infty(s)} f\, dx = \int_{\mathbb{R}/\mathbb{Z}} \int_{G_\infty(s)} g(\theta, x)\, dx\, d\theta = \int_{G_\infty(s)} \overline{m(x)} \int_{\mathbb{R}/\mathbb{Z}} F(d(\theta)x) \chi(\theta)^{-1} d\theta\, dx.$$

By Lemma 6.6.72 the inner integral is $m(x)F(1)$; Fubini also shows that $x \mapsto \int g(\theta, x)\, d\theta = F(1)|m(x)|^2$ is integrable. This is the claim. □

Lemma 6.6.72. *Let n be a nonempty finite index set, s signs on n, o a positive-first order for s, $u = (u_v)_{v\in\Phi}$ weights with every u_v K-dominant for (s,o), and $F\colon G_\infty(s)\to\mathbb{C}$ holomorphic of weight u for o. For $\theta\in\mathbb{R}/\mathbb{Z}$ let $d(\theta)\in G_\infty(s)$ have the component $\operatorname{diag}(e^{2\pi\mathrm{i}\theta o(i)})_{i\in n}$ at every $v\in\Phi$. Then for every $x\in G_\infty(s)$*

$$\int_{\mathbb{R}/\mathbb{Z}} F\big(d(\theta)x\big)\prod_{v\in\Phi}\prod_{i\in n} e^{-2\pi\mathrm{i}\theta\, o(i)u_v(i)}\,d\theta = m^{\mathrm{hol}}_{s,u}(x)\,F(1),$$

$d\theta$ the invariant probability measure.

Proof. Let $\tilde F$ be an extension of F as in (HV3). For $x\in G_\infty(s)$ write $L(x)$ for the left side and $R(x) = m^{\mathrm{hol}}_{s,u}(x)F(1)$; for $s'\in\mathbb{C}^\times$ let $d(s')$ be the family with components $\operatorname{diag}(s'^{\,o(i)})_i$, so $d(\theta) = d(e^{2\pi\mathrm{i}\theta})$, and put $\chi(s') = \prod_{v,i} s'^{\,o(i)u_v(i)}$. Applying (HV3) at a point $y\in G_\infty(s)$ with the family $b = 1$ gives $\tilde F(y) = F(y)$; so $\tilde F$ is complex analytic at every point of $G_\infty(s)$, and $\tilde F(1) = F(1)$.

Step 1: near 1**.** Order n by o. A complex $n\times n$ matrix y whose leading principal minors (on the initial segments of o) do not vanish has a unique factorisation $y = ab$ with a o-lower unipotent and b o-triangular, the entries of a and b being rational functions of y regular on that set (Gaussian elimination); at $y = 1$, $a = b = 1$. Since $\tilde F$ is analytic at the family 1, there is $\epsilon > 0$ such that $\tilde F$ is holomorphic on the polydisc N of the families y with $|y_{v,ij} - \delta_{ij}| < \epsilon$ for all v, i, j. Let $K = \max o - \min o$, and let U_0 be the set of $x\in G_\infty(s)$ whose components have nonvanishing leading principal minors and whose factors a_v satisfy $|a_{v,ij}| < \epsilon\, 2^{-K}$ for $i\neq j$; it is an open neighbourhood of 1 in $G_\infty(s)$. Let $x\in U_0$, $x_v = a_v b_v$. For $s'\in\mathbb{C}$ let $A(s')$ be the family $d(s')\,a\,d(s')^{-1}$, with entries $s'^{\,o(i)-o(j)}a_{v,ij}$; these are zero unless $i = j$ (entry 1) or $o(i) > o(j)$, so A is a polynomial family in s' with $A(0) = 1$, and $A(s')\in N$ for $|s'| < 2$. Let $|s'| = 1$ and $y = d(s')x\in G_\infty(s)$. The family $\beta = (d(s')b)^{-1}$ is o-triangular (inverse of an o-triangular matrix) with diagonal entries $(s'^{\,o(i)}b_{v,ii})^{-1}$, and $y\beta = A(s')$. By (HV3) at y with β,

$$\tilde F\big(A(s')\big) = \beta^u F(y) = \chi(s')^{-1}\Big(\prod_v b_v^{u_v}\Big)^{-1} F\big(d(s')x\big).$$

The function $s'\mapsto \tilde F(A(s'))$ is holomorphic on $|s'| < 2$, so by the mean value property on the unit circle and $A(0) = 1$,

$$L(x) = \Big(\prod_v b_v^{u_v}\Big)\int_{\mathbb{R}/\mathbb{Z}} \tilde F\big(A(e^{2\pi\mathrm{i}\theta})\big)d\theta = \Big(\prod_v b_v^{u_v}\Big)\tilde F(1) = m^{\mathrm{hol}}_{s,u}(x)\,F(1)$$

by (ii) of Lemma 6.6.70. So $L = R$ on U_0.

Step 2: everywhere. $G_\infty(s) = \prod_{v\in\Phi}\mathrm{U}(\operatorname{diag} s)$ is a connected real Lie group (each $\mathrm{U}(p,q)$ is connected), a real-analytic submanifold of $\mathrm{M}_n(\mathbb{C})^\Phi$. R is the restriction of a function holomorphic near $G_\infty(s)$ ((i) of Lemma 6.6.70). For L, let $x_0\in G_\infty(s)$: the compact set $\{d(\theta)x_0\}$ lies in $G_\infty(s)$, where $\tilde F$ is analytic, so there is an open neighbourhood V of x_0 in $\mathrm{M}_n(\mathbb{C})^\Phi$ with $\tilde F$ holomorphic near $d(\theta)y$ for all θ and $y\in V$; then $(\theta,y)\mapsto\tilde F(d(\theta)y)$ is continuous on $\mathbb{R}/\mathbb{Z}\times V$ and holomorphic in y, so $y\mapsto\int\tilde F(d(\theta)y)\chi(e^{2\pi\mathrm{i}\theta})^{-1}d\theta$ is holomorphic on V (differentiation under the integral sign), and its restriction to $V\cap G_\infty(s)$ is L. So L and R are real analytic on $G_\infty(s)$, agree on the nonempty open set U_0, and therefore agree on the connected $G_\infty(s)$. □

Lemma 6.6.73. *Let $n_* = 2ft^3$. Let V, V' be spaces with points with signs s on a nonempty n and s' on a nonempty n', $\#n\le t$, $\#n'\le t$, and let α, β be automorphic characters of $\mathrm{U}(1)$ of weights γ, γ'. Let Q be a homogeneous polynomial of degree $d\le b^{3/2}$ in the variables indexed by $\Phi\times n'\times n$, and $c_Q(h,h') = \langle v_Q, \Omega_{\gamma,\gamma'}(h,h')v_Q\rangle$. For $h\in G_\infty(s)$ write $|h| = \max_{v\in\Phi}\|h_v\|$. Let $g\in$*

$\mathrm{G}(s)$ *and* $g' \in \mathrm{G}(s')$ *with* $\lambda_V(g)\,\lambda_{V'}(g') \geq b^{-13}$. *Then there is a function* Ψ *on* $\mathrm{M}_{n'\times n}(\mathcal{O}_E/\mathfrak{M}^2)$ *such that for all* $h \in G_\infty(s)$ *and* $h' \in G_\infty(s')$

$$\left|\theta^{V,V'}_{\alpha,\beta;\mathfrak{M}^2}[\Psi,Q](g(h,1),g'(h',1)) - c_Q(h,h')\right| \leq b^{-20n_*}\left(|h|\,|h'|\right)^{6n_*}|c_Q(1,1)|.$$

Proof. By Lemma 4.2.67, $x \geq e^{6.4\cdot 10^{15}}$, $x^{10^{12}} \leq b$, $t \leq 15x^5$, $f \leq q_1 \leq x^{12}$ and $q_1 d \leq x^{13}$. Note $\lambda_{V_g}(1) = \lambda_V(g)$ by (VP4) of Definition 3.2.2.

Step 1: an integer in the different. Put $N_{\mathfrak{D}} = q_1 d$; by Lemma 6.4.102, $N_{\mathfrak{D}} \in \mathfrak{D}_E$.

Step 2: the first minimum of Hom**.** By (VP3) of Definition 3.2.2 V_g is a framed hermitian space (a framed space of rank $\#n$ in the sense of (FT1)) and $\lambda_{V_g}(1) = \lambda_V(g)$ is the first minimum of (VP4) for both readings; its coordinate map is $c_{V_g}(u) = (x_V g)^{-1}u$, so L_1 is the lattice of V_g and $|u|_1 = \|((x_V g)^{-1}u)_\infty\|$. By Lemma 6.6.19 (at $g = 1$, with $N_{\mathfrak{D}}$ of Step 1) the lattice of V_g contains a basis of E^n of vectors of length at most $B = f\,\#n\,N_{\mathfrak{D}}\,\lambda_V(g)^{-1} \geq 1$. By (i) of Lemma 6.6.78 for $S = V_g$, $T = V'_{g'}$,

$$\lambda_{\mathrm{Hom}(V_g,V'_{g'})}(1) \geq \frac{\lambda_{V'}(g')}{B} = \frac{\lambda_V(g)\lambda_{V'}(g')}{f\,\#n\,q_1 d} \geq \frac{b^{-13}}{15x^{30}}.$$

Step 3: the scale c**.** Put $c = q_1/\pi_M^2 \in E^\times$. From $x^{q_1} - 1 = \mathrm{cyc}_{q_1}(x)h(x)$, $h \in \mathbb{Z}[x]$, $q_1\zeta^{q_1-1} = \mathrm{cyc}'_{q_1}(\zeta)h(\zeta)$, so $q_1\alpha_w$ is integral at every finite place w. If z is an integral finite adele whose components at the chosen places lie in $\mathfrak{M}^2 = \pi_M^2\mathcal{O}_E$, then $\pi_M^{-2}z_w$ is integral at the chosen places, and also at the other finite places, where π_M is a unit. So $c\,\alpha z = (q_1\alpha)(\pi_M^{-2}z)$ is integral. At every $v \in \Phi$, $|(\pi_M)_v|^2 = \pi_M\bar{\pi}_M = M \geq b^{20}$, so $|c_v| = q_1/M \leq x^{12}b^{-20}$.

Step 4: the separation and the conclusion. Let $r = b^6$ and $q = 2n_*$. Then $\delta = \#(\Phi \times n' \times n) \leq ft^2 < q$, and, with $\sqrt{2\pi\,\mathrm{Im}\,\jmath_v} = \sqrt{2\pi}\,d^{-1/4} \geq d^{-1}$ and Step 2,

$$r\,|c_v| \leq x^{12}b^{-14} \leq \frac{b^{-13}}{d\cdot 15x^{30}} \leq \sqrt{2\pi\,\mathrm{Im}\,\jmath_v}\;\lambda_{\mathrm{Hom}(V_g,V'_{g'})}(1),$$

since $15x^{42}d \leq 15x^{55} \leq b$. So Lemma 6.6.74 applies to $S = V$, $T = V'$, the points g, g', $\alpha' = \alpha$, $\beta' = \beta$, the modulus $\mathfrak{M}^2$, c, r, q and $P = Q$ of degree $d \leq b^{3/2}$, and gives Ψ with

$$\left|\theta^{V,V'}_{\alpha,\beta;\mathfrak{M}^2}[\Psi,Q](g(h,1),g'(h',1)) - c_Q(h,h')\right| \leq \Big(\frac{100(d+4n_*)}{b^{12}}\Big)^{2n_*}\left(|h|\,|h'|\right)^{6n_*}\langle v_Q, v_Q\rangle.$$

Here $n_* = 2ft^3 \leq 2x^{12}(15x^5)^3 \leq b^{3/2}$, so $100(d+4n_*) \leq 500\,b^{3/2} \leq b^2$ and the first factor is at most $(b^{-10})^{2n_*} = b^{-20n_*}$; and $\langle v_Q, v_Q\rangle = c_Q(1,1) = |c_Q(1,1)|$, as $\Omega_{\gamma,\gamma'}(1,1)$ is the identity. □

Lemma 6.6.74. *Let* S, T *be spaces with points with signs* s *on a nonempty* n *and* s' *on a nonempty* n', *let* $\delta = \#(\Phi \times n' \times n)$, *and let* α', β' *be automorphic characters of* $\mathrm{U}(1)$ *of weights* γ, γ'. *Let* $\mathfrak{m}$ *be a modulus,* $g \in \mathrm{G}(s)$, $h \in \mathrm{G}(s')$, *and* $\lambda = \lambda_{\mathrm{Hom}(S_g,T_h)}(1)$. *Let* $c \in E^\times$ *be such that* $c\,\alpha z$ *is integral for every integral finite adele* z *whose components at the places of* $\Sigma(\mathfrak{m})$ *lie in* $\mathfrak{m}$, *let* $r > 0$ *satisfy* $r\,|c_v| \leq \sqrt{2\pi\,\mathrm{Im}\,\jmath_v}\;\lambda$ *for every* $v \in \Phi$, *and let* $q > \delta$ *be an integer. Let* P *be a homogeneous polynomial of degree* d *in the variables indexed by* $\Phi \times n' \times n$. *For* $x \in G_\infty(s)$ *write* $\|x\| = \max_{v\in\Phi}\|x_v\|$, *the operator norm. Then there is a function* Ψ *on* $\mathrm{M}_{n'\times n}(R_{\mathfrak{m}})$ *such that for all* $x \in G_\infty(s)$ *and* $y \in G_\infty(s')$

$$\left|\theta^{S,T}_{\alpha',\beta'}(\mathrm{mod}_{\mathfrak{m}}(\Psi,v_P))(g(x,1),h(y,1)) - \langle v_P, \Omega_{\gamma,\gamma'}(x,y)\,v_P\rangle\right| \leq \Big(\frac{100(d+2q)}{r^2}\Big)^q\left(\|x\|\,\|y\|\right)^{3q}\langle v_P, v_P\rangle.$$

Proof. Write $v = v_P$, $H = \mathrm{Hom}(S,T)$ with point $x_H = x_T \otimes (x_S^{-1})^{\mathsf T}$, $p = h \otimes (g^{-1})^{\mathsf T} \in \mathrm{G}(s' \otimes s)$, $\omega = \omega_{s' \otimes s}$, $\chi = \alpha'(\det h)\,\beta'(\det g)^{-1}$ (of modulus 1), and for a function Φ on $\mathrm{M}_{n' \times n}(R_{\mathfrak m})$ and $f \in \mathcal{F}_{S,T}$ put $\varphi(\Phi, f) = \mathrm{mod}_{\mathfrak m}(\Phi, f)$, linear in Φ and in f. By (WR4) of Definition 3.2.18 and (WC4) of Definition 4.1.15, $\theta^{S,T}_{\alpha',\beta'}(\varphi)(g,h) = \chi\,\theta_H(\omega(p)\varphi)$. Since $x_H p = x_T h \otimes ((x_S g)^{-1})^{\mathsf T}$, the moved space H_p is $\mathrm{Hom}(S_g, T_h)$, so λ is the first minimum of the lattice $\Lambda = \{u \in \mathrm{M}_{n' \times n}(E) : ((x_H p)^{-1}u)_f \text{ integral}\}$ of H_p for the lengths $\|((x_H p)^{-1}u)_\infty\|$. For $u \in \mathrm{M}_{n' \times n}(E)$ put $z(u) = (x_H p)^{-1}u \in \mathbb{A}_E^{n' \times n}$ and $\xi(u) = (\kappa_v^{-1} z(u)_{v,a,b})_{v,a,b} \in \mathbb{C}^{\Phi \times n' \times n}$, with $\kappa_v^{-1} = \sqrt{2\pi\,\mathrm{Im}\,\jmath_v}$ as in (WD6); $u \mapsto \xi(u)$ is additive and injective.

The datum. For $X \in \mathrm{M}_{n' \times n}(R_{\mathfrak m})$ let $\mathbf{1}_X$ be the indicator of $\{X\}$ and $K_X = \theta^{S,T}_{\alpha',\beta'}(\varphi(\mathbf{1}_X, v))$. Put $\Psi = \sum_X \overline{K_X(g,h)}\,\mathbf{1}_X$. By linearity, $\theta^{S,T}_{\alpha',\beta'}(\varphi(\Psi, v)) = \sum_X \overline{K_X(g,h)}\,K_X$.

The kernel as a lattice sum. Fix $x \in G_\infty(s)$, $y \in G_\infty(s')$ and put $w = \Omega_{\gamma,\gamma'}(x,y)v$. By (5) of Lemma 4.1.1 (with finite components 1), $\omega_{\alpha',\beta'}((x,1),(y,1))\varphi(\mathbf{1}_X, v) = \varphi(\mathbf{1}_X, w)$, so by (2) of Lemma 4.1.1 $K_X(g(x,1), h(y,1)) = \chi\,\theta_H(\omega(p)\varphi(\mathbf{1}_X, w))$, while $K_X(g,h) = \chi\,\theta_H(\omega(p)\varphi(\mathbf{1}_X, v))$. As $|\chi| = 1$,

$$\overline{K_X(g,h)}\,K_X(g(x,1),h(y,1)) = \theta_H\big(\omega(p)\varphi(\mathbf{1}_X,w)\big)\,\overline{\theta_H\big(\omega(p)\varphi(\mathbf{1}_X,v)\big)}.$$

By Lemma 6.6.75 for H this is the absolutely convergent sum over $u \in \mathrm{M}_{n' \times n}(E)$ of

$$\langle \omega(p)\varphi(\mathbf{1}_X, v), \rho(x_H^{-1}u)\,\omega(p)\varphi(\mathbf{1}_X, w)\rangle.$$

Since $\omega(p)$ is unitary and $\omega(p)^{-1}\rho(z)\omega(p) = \rho(p^{-1}z)$ ((2) of Lemma 4.1.1), the term of u is $\langle \varphi(\mathbf{1}_X, v), \rho(z(u))\,\varphi(\mathbf{1}_X, w)\rangle$. Summing over the finitely many X and applying Lemma 6.6.76 to each u,

$$\theta^{S,T}_{\alpha',\beta'}(\varphi(\Psi,v))(g(x,1),h(y,1)) = \sum_u t(u), \qquad t(u) = c_{\mathfrak m}(z(u)_f)\,\langle v, \rho^\infty(\xi(u))\,w\rangle,$$

absolutely convergent. As $c_{\mathfrak m}(0) = 1$ and $\xi(0) = 0$, $t(0) = \langle v, w\rangle = \langle v, \Omega_{\gamma,\gamma'}(x,y)v\rangle$; so the left side of the assertion is at most $\sum_{u \neq 0} |t(u)|$.

Support and separation. Let $u \neq 0$ with $t(u) \neq 0$. Then $c_{\mathfrak m}(z(u)_f) \neq 0$, so by Lemma 6.6.76 $\alpha^{-1}z(u)_f$ is integral with components in $\mathfrak m$ at $\Sigma(\mathfrak m)$, and by the hypothesis on c the finite part $c\,z(u)_f = c\alpha \cdot \alpha^{-1}z(u)_f$ of $z(cu)$ is integral: cu is a nonzero vector of Λ. Let L be the set of $u \neq 0$ with $cu \in \Lambda$. For $u \in L$, by the definition of λ and the hypothesis on r,

$$r\lambda \le r\Big(\sum_{v,a,b} |c_v|^2 |z(u)_{v,a,b}|^2\Big)^{1/2} \le \lambda\Big(\sum_{v,a,b} \kappa_v^{-2} |z(u)_{v,a,b}|^2\Big)^{1/2} = \lambda\,\|\xi(u)\|,$$

so $\|\xi(u)\| \ge r$, as $\lambda > 0$. For distinct $u, u' \in L$, $u - u' \neq 0$ and $c(u-u') \in \Lambda$, so $\|\xi(u) - \xi(u')\| = \|\xi(u-u')\| \ge r$.

Packing. We show $\sum_{u \in L} \|\xi(u)\|^{-2q} \le (5/r)^{2q}$. The points $\xi(u)$, $u \in L$, lie in $\mathbb{C}^\delta = \mathbb{R}^{2\delta}$ and the open balls of radius $r/2$ around them are disjoint. For $j \ge 0$ let L_j be the set of $u \in L$ with $2^j r \le \|\xi(u)\| < 2^{j+1}r$; the balls of radius $r/2$ around $\xi(u)$, $u \in L_j$, lie in the ball of radius $(2^{j+1} + \frac12)r$ about 0, so comparing volumes, $\#L_j \le (2^{j+2}+1)^{2\delta} \le (5 \cdot 2^j)^{2\delta}$. As $L = \bigcup_j L_j$ and $2q \ge 2\delta + 2$,

$$\sum_{u \in L} \|\xi(u)\|^{-2q} \le \sum_{j \ge 0} (5 \cdot 2^j)^{2\delta} (2^j r)^{-2q} \le 5^{2\delta} r^{-2q} \sum_{j \ge 0} 2^{-2j} = \tfrac43\, 5^{2\delta} r^{-2q} \le (5/r)^{2q}.$$

Decay of each term. Let ε be the signs of $\mathcal{F}_{S,T}$ on $\iota = \Phi \times n' \times n$ and $\sigma = \bigoplus_{v \in \Phi} y_v \otimes (x_v^{-1})^{\mathsf T} \in \mathrm{U}(\mathrm{diag}\,\varepsilon)$. By (WR5), $w = \prod_v \det(y_v)^{\gamma(v)} \det(x_v)^{-\gamma'(v)}\,\omega_\varepsilon^\infty(\sigma)v$, and the determinants

have modulus 1 (from $x_v^*\mathrm{diag}(s)x_v = \mathrm{diag}(s)$ and likewise for y_v). For $u \in L$, $|c_{\mathfrak{m}}| \le 1$ and Lemma 6.6.77 (with $A = d+\delta$ and $n = q$) give

$$|t(u)| \le \big|\langle v, \rho^\infty(\xi(u))\,\omega_\varepsilon^\infty(\sigma)v\rangle\big| \le \big(4(d+\delta+q)\big)^q\,\|\sigma\|^{\delta+2q}\,\|\xi(u)\|^{-2q}\,\langle v,v\rangle.$$

The operator norm of a Kronecker product is the product of the norms, a transpose has the norm of the matrix, and $x_v^{-1} = \mathrm{diag}(s)x_v^*\mathrm{diag}(s)$ has the norm of x_v; hence $\|\sigma\| = \max_v \|y_v\|\|x_v\| \le \|x\|\|y\|$. Also $\|\sigma\| \ge 1$, since $\sigma^{-1} = \mathrm{diag}(\varepsilon)\sigma^*\mathrm{diag}(\varepsilon)$ has the norm of σ; with $\delta < q$ this gives $\|\sigma\|^{\delta+2q} \le (\|x\|\|y\|)^{3q}$ and $4(d+\delta+q) \le 4(d+2q)$.

Conclusion. Summing over L and using the packing bound,

$$\sum_{u\neq 0}|t(u)| \le \big(4(d+2q)\big)^q(\|x\|\|y\|)^{3q}\langle v,v\rangle\Big(\frac{5}{r}\Big)^{2q} = \Big(\frac{100(d+2q)}{r^2}\Big)^q(\|x\|\|y\|)^{3q}\langle v,v\rangle.$$

□

Lemma 6.6.75. *Let V be a space with a point with signs s on n, and θ_V its canonical theta functional. For all $f, f' \in \mathcal{M}_s$ the family $\big(\langle f', \rho(\gamma)f\rangle\big)_{\gamma\in V(E)}$ is absolutely summable, and*

$$\sum_{\gamma\in V(E)}\langle f', \rho(\gamma)f\rangle = \theta_V(f)\,\overline{\theta_V(f')}.$$

Proof. Write $W = \mathbb{A}_E^n$, $L = V(E)$, $K = W/L$ with its invariant probability measure dx, $\beta = \beta_s$, $\theta = \theta_V$; only the properties of a theta functional are used. Since $x_V^* J_V x_V = \varepsilon_s$, $\bar\gamma^{\mathsf T}\varepsilon_s\gamma' = u^* J_V u'$ for $\gamma = x_V^{-1}u$, $\gamma' = x_V^{-1}u'$; and $\psi_E(\bar a) = \psi_E(a)$ for $a \in \mathbb{A}_E$ by (WD4).

Step 1: a periodic function. For $\gamma \in L$, $\theta(\rho(x+\gamma)f) = \beta(\gamma,x)^{-1}\theta(\rho(\gamma)\rho(x)f) = \beta(\gamma,x)^{-1}\theta(\rho(x)f)$, and $|\beta| = 1$. So $F(x) = \theta(\rho(x)f)\,\overline{\theta(\rho(x)f')}$ is L-periodic, and it is continuous (θ bounded, ρ strongly continuous); we view it on K.

Step 2: its Fourier coefficients. For $\gamma \in L$ put $\chi_\gamma(x) = \beta(x,\gamma)\beta(\gamma,x)^{-1}$. By (WD6) and $\bar\jmath = -\jmath$, $\beta(\gamma,x)^{-1} = \psi_E(\bar a)$ with $a = \frac{\jmath}{2}\bar\gamma^{\mathsf T}\varepsilon_s x$, so $\chi_\gamma(x) = \psi_E(a)\psi_E(\bar a) = \psi_E(\jmath\,\bar\gamma^{\mathsf T}\varepsilon_s x)$, a continuous character of W; it is trivial on L, since $\bar\gamma^{\mathsf T}\varepsilon_s\gamma' \in E$ for $\gamma' \in L$ and ψ_E is trivial on E. From $\rho(x)\rho(\gamma) = \beta(x,\gamma)\rho(x+\gamma)$ and Step 1, $\theta(\rho(x)\rho(\gamma)f) = \chi_\gamma(x)\theta(\rho(x)f)$. By orthogonality of theta functions with $u = f'$, $v = \rho(\gamma)f$,

$$\int_K F(x)\,\chi_\gamma(x)\,dx = \int_K \overline{\theta(\rho(x)f')}\,\theta(\rho(x)\rho(\gamma)f)\,dx = \langle f', \rho(\gamma)f\rangle.$$

Step 3: these are all the characters of K. The map $y \mapsto x_V^{-1}y$ is an isomorphism of topological groups $\mathbb{A}_E^n \to W$ carrying E^n onto L, so it identifies $(\mathbb{A}_E/E)^n$ with K. For $\gamma = x_V^{-1}u$, $\chi_\gamma(x_V^{-1}y) = \psi_E(\jmath\,u^*J_V y) = \psi_E(\xi\cdot y)$ with $\xi = \jmath\,\overline{J_V u} \in E^n$ (J_V is hermitian). As u runs over E^n, ξ runs bijectively over E^n ($\jmath \neq 0$, J_V invertible). By (a) of Theorem A.5.2 and since the characters of a finite product are the products of characters of the factors, $\xi \mapsto (y \mapsto \psi_E(\xi\cdot y))$ is a bijection of E^n onto the characters of $(\mathbb{A}_E/E)^n$. Hence $\gamma \mapsto \chi_\gamma$ is a bijection of L onto the group of characters of K.

Step 4: absolute summability. By superposition and boundedness of $\langle\cdot,\cdot\rangle$ on Bochner integrals,

$$\langle f', \rho(\gamma)f\rangle = \iint \overline{f'(z')}\,f(z)\,\langle\rho(z')v^0, \rho(\gamma)\rho(z)v^0\rangle\,d\mu_n(z)\,d\mu_n(z').$$

Since the ρ preserve the inner product and compose up to phases of modulus 1,

$$|\langle\rho(z')v^0, \rho(\gamma)\rho(z)v^0\rangle| = |\langle v^0, \rho(\gamma+z-z')v^0\rangle| = |e_n(\gamma+z-z')|.$$

Let $K_0 = \sup_{y\in W}\sum_{\gamma\in L}|e_n(\gamma+y)|$. It is finite: the γ with $e_n(\gamma+y)\neq 0$ have $(\gamma+y)_f \in \alpha\widehat{\mathcal{O}}_E^n$ and so form (if any) a coset γ_0+L_0 of $L_0=\{x_V^{-1}u : u\in M\}$, $M=\{u\in E^n : (x_V^{-1}u)_f\in\alpha\widehat{\mathcal{O}}_E^n\}$ an $\mathcal{O}_E$-lattice of E^n, whose archimedean image Λ_0 is a lattice of $\mathbb{C}^{\Phi\times n}$; with $c=\pi\min_v \operatorname{Im}\jmath_v>0$ the sum is at most $\sup_t\sum_{\lambda\in\Lambda_0}\exp(-c\|\lambda+t\|^2)$, a continuous Λ_0-periodic function of t, hence bounded. By Tonelli, $\sum_{\gamma\in L}|\langle f', \rho(\gamma)f\rangle| \le K_0\,\|f\|\,\|f'\|<\infty$.

Step 5: Fourier inversion. F is continuous on the compact abelian group K and, by Steps 2–4, its Fourier coefficients $\int F\chi$ over all characters χ of K form the absolutely summable family $(\langle f', \rho(\gamma)f\rangle)_{\gamma\in L}$. By Fourier inversion on compact abelian groups (the continuous function $\sum_\chi(\int F\chi)\bar\chi$ has the same coefficients as F, so equals F), $F(0)=\sum_{\gamma\in L}\langle f', \rho(\gamma)f\rangle$, and $F(0)=\theta(f)\overline{\theta(f')}$. □

Lemma 6.6.76. *Let S, T be spaces with points with signs on n and n', and $\mathfrak{m}$ a modulus. For $X\in \mathrm{M}_{n'\times n}(R_\mathfrak{m})$ let $\mathbf{1}_X$ be the indicator function of $\{X\}$ on $\mathrm{M}_{n'\times n}(R_\mathfrak{m})$. For $z\in W$ let $\xi(z)\in\mathbb{C}^{\Phi\times n'\times n}$ have the components $\kappa_v^{-1}z_{v,a,b}$. There is a function $c_\mathfrak{m}$ on $\mathrm{M}_{n'\times n}(\mathbb{A}_{E,f})$ with $|c_\mathfrak{m}|\le 1$ and $c_\mathfrak{m}(0)=1$, vanishing at every z_f for which $\alpha^{-1}z_f$ is not integral or has an entry whose component at some $w\in\Sigma(\mathfrak{m})$ does not lie in $\mathfrak{m}\mathcal{O}_{E,w}$, such that for all $v,w\in\mathcal{F}_{S,T}$ and $z\in W$*

$$\sum_{X\in \mathrm{M}_{n'\times n}(R_\mathfrak{m})} \langle \mathrm{mod}_\mathfrak{m}(\mathbf{1}_X, v),\, \rho(z)\,\mathrm{mod}_\mathfrak{m}(\mathbf{1}_X, w)\rangle = c_\mathfrak{m}(z_f)\,\langle v, \rho^\infty(\xi(z))\,w\rangle.$$

Proof. Let $A=\Sigma(\mathfrak{m})$, $N=n'\times n$, $Q_A=\prod_{w\in A}E_w^N$ with $L_A=\prod_{w\in A}\mathcal{O}_{E,w}^N$ of volume 1, and $f_X=\mathbf{1}_X^\natural\otimes_A v^0$, so $\mathrm{mod}_\mathfrak{m}(\mathbf{1}_X,v)=v\otimes f_X$. As functions on Q_A, the $\mathbf{1}_X^\natural$ are the indicators of the cosets B_X of $\mathfrak{m}L_A$ in L_A, $X\in\mathrm{M}_{n'\times n}(R_\mathfrak{m})$, each of volume $(\#R_\mathfrak{m})^{-\#N}$. Write $z=z_\infty+z'+z''$, with z' the finite components at $A\cup\bar A$ and z'' the other finite components.

Step 1: separation of the archimedean part. By (WD6) $z_\infty=\mathrm{rv}(\xi(z))$, and z_∞, z', z'' have disjoint supports, so $\rho(z)=\rho(z_\infty)\rho(z')\rho(z'')$. By Lemma 6.2.21, $\rho(z)(w\otimes f_X)=(\rho^\infty(\xi(z))w)\otimes\rho(z'+z'')f_X$, and by (6), (4) of Lemma 6.2.33 and (3) of Lemma 6.2.21,

$$\langle v\otimes f_X, \rho(z)(w\otimes f_X)\rangle = \langle v, \rho^\infty(\xi(z))w\rangle\,\langle f_X, \rho(z'+z'')f_X\rangle.$$

So the claim holds with $c_\mathfrak{m}(z_f)=\sum_X\langle f_X,\rho(z'+z'')f_X\rangle$, which depends only on z_f; it remains to show the properties of $c_\mathfrak{m}$.

Step 2: the two finite parts. The operator $\rho(z'')$ commutes with P^A and with every $\rho(\mathrm{fv}(y))$, y supported on $A\cup\bar A$ (disjoint supports), so $\rho(z'')f_X=\mathbf{1}_X^\natural\otimes_A\rho(z'')v^0$. With $y=\alpha^{-1}z'$ (so $\mathrm{fv}(y)=z'$) and Lemma 4.1.3,

$$\langle f_X,\rho(z')\rho(z'')f_X\rangle = \langle\mathbf{1}_X^\natural, R_A(y)\mathbf{1}_X^\natural\rangle_{Q_A}\,\langle v^0, P^A\rho(z'')v^0\rangle = \langle\mathbf{1}_{B_X}, R_A(y)\mathbf{1}_{B_X}\rangle_{Q_A}\,\langle v^0,\rho(z'')v^0\rangle,$$

using $P^A\rho(z'')v^0=\rho(z'')P^Av^0=\rho(z'')v^0$. Here $\langle v^0,\rho(z'')v^0\rangle=\overline{e(z'')}$ (e the vacuum of (WD5)), of modulus at most 1, equal to 1 for $z''=0$, and 0 unless $z''\in\alpha\widehat{\mathcal{O}}_E^N$.

Step 3: the sum over the classes. Let

$$S(y)=\sum_X\langle\mathbf{1}_{B_X}, R_A(y)\mathbf{1}_{B_X}\rangle_{Q_A}=\sum_X\int_{B_X}\psi_0(\bar y^{\mathsf T}(2x+y)_A/2)\,\mathbf{1}_{B_X}(x+y_A)\,dx.$$

If $y_A\notin\mathfrak{m}L_A$, then $x+y_A\notin B_X$ for $x\in B_X$, and $S(y)=0$. If $y_A\in\mathfrak{m}L_A$, the cosets B_X partition L_A and

$$S(y)=\psi_0(\tfrac12\bar y^{\mathsf T}y_A)\int_{L_A}\psi_0(\bar y^{\mathsf T}x_A)\,dx;$$

the integrand is a character of the compact group L_A (the components of $\bar{y}$ at A are those of y at $\bar{A}$, conjugated), trivial exactly when these components are integral, by self-duality; so $S(y) = 0$ unless y is integral at $\bar{A}$, and $|S(y)| \le 1$, $S(0) = 1$.

Conclusion. $c_{\mathfrak{m}}(z_f) = S(\alpha^{-1}z')\,\overline{e(z'')}$, so $|c_{\mathfrak{m}}| \le 1$ and $c_{\mathfrak{m}}(0) = 1$. If $\alpha^{-1}z_f$ is not integral, it fails at a place outside $A \cup \bar{A}$ (then $e(z'') = 0$), at a place of $\bar{A}$ or at a place of A (then $S = 0$); if $\alpha^{-1}z_f$ has an entry whose component at some $w \in A$ is not in $\mathfrak{m}\mathcal{O}_{E,w}$, then $y_A \notin \mathfrak{m}L_A$ and $S = 0$. In all these cases $c_{\mathfrak{m}}(z_f) = 0$. □

Lemma 6.6.77. *Let ι be a nonempty finite index set, ε signs on ι, $P \in \mathbb{C}[X_i : i \in \iota]$ homogeneous of degree d, $v = v_P$, $\sigma \in \mathrm{U}(\operatorname{diag}\varepsilon)$, $y \in \mathbb{C}^\iota \setminus \{0\}$, and $A \ge d + \#\iota$, $n \ge 0$ integers. Write $|y|$ for the euclidean norm on $\mathbb{C}^\iota$, $\|\sigma\|$ for the operator norm of σ for it, and $(\varrho(y)c)(x) = \beta_\varepsilon^\infty(y,x)\,c(x-y)$ for functions c on $\mathbb{C}^\iota$. Then $\overline{v}\cdot\varrho(y)\omega_\varepsilon^\infty(\sigma)v$ is ν_ι-integrable, and*

$$\Big|\int \overline{v}\;\varrho(y)\,\omega_\varepsilon^\infty(\sigma)v\;d\nu_\iota\Big| \le (4(A+n))^n\,\|\sigma\|^{\#\iota+2n}\,|y|^{-2n}\,\langle v,v\rangle.$$

Proof. Write $\beta = \beta_\varepsilon^\infty$, $e = e^\infty$, $\nu = \nu_\iota$, $\mathrm{D} = \operatorname{diag}\varepsilon$ and $k = \#\iota$. Recall $\beta(a,x) = \exp((x^*\mathrm{D}a - a^*\mathrm{D}x)/2)$ and $e(x) = \exp(-|x|^2/2)$, and $v(x) = P(\zeta(x))e(x)$ with $\zeta_i(x) \in \{x_i, \bar{x}_i\}$, a Fock vector, so $v \in \mathcal{F}_\varepsilon$.

Elementary facts. The exponent of $\beta(a,x)$ is purely imaginary and real-linear in each argument, so $|\beta| = 1$, $\beta(a,x)\beta(b,x) = \beta(a+b,x)$, $\beta(a,x-y) = \beta(a,x)\beta(a,y)^{-1}$, and $\overline{\beta(t,z)} = \beta(z,t)$. As $\sigma^*\mathrm{D}\sigma = \mathrm{D}$, $|\det\sigma| = 1$, so $x \mapsto \sigma x + y$ preserves ν (its real Jacobian is $|\det\sigma|^2$); and $\|\sigma\| \ge 1$, since $\sigma^{-1} = \mathrm{D}\sigma^*\mathrm{D}$ has the same norm as σ. The block σ_- of σ on the indices of sign -1 is a compression of σ, so $|\det\sigma_-| \le \|\sigma\|^k$. Finally $|v(x)| \le C(1+|x|)^d\exp(-|x|^2/2)$ for some C, so v is bounded and integrable.

Pairing with a translated vacuum. For $w \in \mathcal{F}_\varepsilon$ and $z \in \mathbb{C}^\iota$, $\int \overline{w(t)}\,\beta(z,t)\,e(t-z)\,d\nu(t) = \overline{w(z)}$: this is the complex conjugate of the reproducing identity $w(z) = \int w(t)\beta(t,z)e(z-t)\,d\nu(t)$ of (WR5), as $\overline{\beta(t,z)} = \beta(z,t)$ and e is real and even.

Reduction to a convolution of $|v|$ with itself. By (WR5),

$$\omega_\varepsilon^\infty(\sigma)v(x) = \overline{\det\sigma_-}\int v(t)\beta(\sigma t,x)e(x-\sigma t)\,d\nu(t),$$

so

$$\overline{v(x)}\,(\varrho(y)\omega_\varepsilon^\infty(\sigma)v)(x) = \overline{\det\sigma_-}\int \overline{v(x)}\,v(t)\,\beta(y,x)\beta(\sigma t,x-y)\,e\big(x-(\sigma t+y)\big)\,d\nu(t).$$

The absolute value of the integrand is at most $\sup|v|\cdot|v(t)|\,e(x-\sigma t-y)$, integrable on $\mathbb{C}^\iota\times\mathbb{C}^\iota$ (integrate first in x); so the left side is integrable in x and Fubini's theorem applies. By the elementary facts, $\beta(y,x)\beta(\sigma t,x-y) = \beta(\sigma t,y)^{-1}\beta(\sigma t+y,x)$, and by the pairing identity with $w = v$, $z = \sigma t + y$,

$$\int \overline{v}\;\varrho(y)\omega_\varepsilon^\infty(\sigma)v\;d\nu = \overline{\det\sigma_-}\int v(t)\,\beta(\sigma t,y)^{-1}\,\overline{v(\sigma t+y)}\,d\nu(t),$$

whose absolute value is at most $\|\sigma\|^k\int|v(t)|\,|v(\sigma t+y)|\,d\nu(t)$.

A pointwise bound by dilation. Put $c = \sqrt{1+n/A} \ge 1$ and

$$G(x) = |P(\zeta(x))|^2\exp(-|x|^2/c^2).$$

Since $1-c^{-2}=n/(n+A)$,

$$|v(x)| = G(x)^{1/2}\exp\Big(-\frac{n\,|x|^2}{2(n+A)}\Big).$$

From $y=(\sigma t+y)-\sigma t$ and $\|\sigma\|\ge 1$, $|y|\le\|\sigma\|(|\sigma t+y|+|t|)$, so $|y|^2\le 2\|\sigma\|^2(|t|^2+|\sigma t+y|^2)$. Hence $u=(|t|^2+|\sigma t+y|^2)/(2(n+A))$ satisfies $u\ge u_0=|y|^2/(4\|\sigma\|^2(n+A))$, and $\exp(u)\ge\exp(1)\,u\ge\exp(1)\,u_0$ (as $\exp(u-1)\ge u$), so

$$\exp(-u)\le K_1=4(n+A)\|\sigma\|^2/(\exp(1)\,|y|^2).$$

With $\sqrt{ab}\le(a+b)/2$,

$$|v(t)|\,|v(\sigma t+y)| = \sqrt{G(t)G(\sigma t+y)}\;\exp(-nu)\le \tfrac12 K_1^n\big(G(t)+G(\sigma t+y)\big).$$

Integration. Substituting $x=cx'$ ($d\nu(x)=c^{2k}d\nu(x')$) and using $P(\zeta(cx'))=c^dP(\zeta(x'))$,

$$\int G\,d\nu = c^{2(k+d)}\int |P(\zeta(x'))|^2\exp(-|x'|^2)\,d\nu(x') = c^{2(k+d)}\langle v,v\rangle\le\exp(n)\,\langle v,v\rangle,$$

because $c^{2(k+d)}=(1+n/A)^{k+d}\le\exp(n(k+d)/A)\le\exp(n)$ by $k+d\le A$. By the invariance of ν, $\int G(\sigma t+y)\,d\nu(t)=\int G\,d\nu$. Therefore $\int|v(t)||v(\sigma t+y)|\,d\nu(t)\le K_1^n\exp(n)\langle v,v\rangle=\big(4(n+A)\big)^n\|\sigma\|^{2n}|y|^{-2n}\langle v,v\rangle$, and with the factor $\|\sigma\|^k$ this is the assertion. □

Lemma 6.6.78. *Let S be a space with a point with signs on n, T a space with a point with signs on n'. Let f be the number of infinite places of E^+ and d_E the discriminant of E.*

(i) If $B\ge 1$ and the lattice of S contains a basis $u_1,\dots,u_{\#n}$ of E^n over E with $\|(x_S^{-1}u_j)_\infty\|\le B$ for every j, then $\lambda_{\mathrm{Hom}(S,T)}(1)\ge\lambda_T(1)/B$.

(ii) $\lambda_{\mathrm{Hom}(S,T)}(1)\ge\lambda_S(1)\lambda_T(1)/(f\,\#n\,|d_E|)$.

Proof. **(i).** By (VP5) of Definition 3.2.2 the inverse of the point of $\mathrm{Hom}(S,T)$ is $Z\mapsto x_T^{-1}Zx_S$ on $n'\times n$ matrices. Let $Z\in\mathrm{M}_{n'\times n}(E)$ be a nonzero vector of the lattice of $\mathrm{Hom}(S,T)$, and $k=x_T^{-1}Zx_S\in\mathrm{M}_{n'\times n}(\mathbb{A}_E)$; so k_f is integral, and the length of Z is $\|k_\infty\|=\big(\sum_{v\in\Phi}\|k_v\|_{\mathrm F}^2\big)^{1/2}$, $\|\cdot\|_{\mathrm F}$ the Frobenius norm. As the u_j form a basis and $Z\ne0$, there is j with $Zu_j\ne0$. The vector $Zu_j\in E^{n'}$ satisfies $x_T^{-1}(Zu_j)=k\,y$ with $y=x_S^{-1}u_j$, whose finite part k_fy_f is integral because k_f and y_f are; so Zu_j is a nonzero vector of the lattice of T, and by (VP4) of Definition 3.2.2 $\lambda_T(1)\le\|(ky)_\infty\|$. By the Cauchy–Schwarz inequality $\|k_vy_v\|\le\|k_v\|_{\mathrm F}\|y_v\|$ at each $v\in\Phi$, hence

$$\|(ky)_\infty\|^2=\sum_{v\in\Phi}\|k_vy_v\|^2\le\Big(\max_{v\in\Phi}\|k_v\|_{\mathrm F}^2\Big)\sum_{v\in\Phi}\|y_v\|^2\le\|k_\infty\|^2\,\|y_\infty\|^2\le\|k_\infty\|^2B^2.$$

So every nonzero lattice vector of $\mathrm{Hom}(S,T)$ has length at least $\lambda_T(1)/B$, and $\lambda_{\mathrm{Hom}(S,T)}(1)\ge\min(1,\lambda_T(1)/B)=\lambda_T(1)/B$, since $\lambda_T(1)\le1\le B$.

(ii). Enumerate n. By (VP3) of Definition 3.2.2 S is a framed hermitian space (a framed space of rank $\#n$ in the sense of (FT1)) and $\lambda_S(1)$ is the first minimum of (VP4) for both readings; its coordinate map is $c_S(u)=x_S^{-1}u$, so at $g=1$ the set L_1 of Lemma 5.8.21 is the lattice of S and $|u|_1=\|(x_S^{-1}u)_\infty\|$. By Lemma 5.8.21 the lattice of S contains a basis of E^n of vectors of length at most $B=f\,\#n\,|d_E|\,\lambda_S(1)^{-1}$, and $B\ge1$ since $\lambda_S(1)\le1$. Part (i) gives $\lambda_{\mathrm{Hom}(S,T)}(1)\ge\lambda_T(1)/B=\lambda_S(1)\lambda_T(1)/(f\,\#n\,|d_E|)$. □

Lemma 6.6.79. *Let $1 \le j \le m$, $\tilde{W}_j$ the padded space (signs $\tilde{s}_j$ on $\mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{A}_j^+$), and u_1 the weight on $\mathcal{B}^{\mathrm{c}}$ equal to $-\Delta'$ on the first and $6m+1$ on the second block, the same at every $\tau \in \Phi$. For $g \in \mathrm{G}(s_j)$ let $(g,1) \in \mathrm{G}(\tilde{s}_j)$ be the block diagonal point with g on $\mathcal{R}$ and 1 on $\mathcal{A}_j^- \sqcup \mathcal{A}_j^+$. Let $g_1 \in \mathrm{G}(s^{\mathrm{c}})$ and let Ψ be a function on $\mathrm{M}_{(\mathcal{R} \sqcup \mathcal{A}_j^- \sqcup \mathcal{A}_j^+) \times \mathcal{B}^{\mathrm{c}}}(\mathcal{O}_E/\mathfrak{M}^2)$, and put*

$$\theta_\Psi(x,g) = \theta^{S^{\mathrm{c}},\tilde{W}_j}_{\upsilon^m,\nu_j^\flat \upsilon^m;\mathfrak{M}^2}[\Psi, P_j](x,(g,1)) \qquad (x \in \mathrm{G}(s^{\mathrm{c}}),\ g \in \mathrm{G}(s_j)).$$

Then for every $g \in \mathrm{G}(s_j)$ the function $h \mapsto \varphi^\circ(g_1(h,1))\,\theta_\Psi(g_1(h,1),g)\,m^{\mathrm{hol}}_{s^{\mathrm{c}},u_1}(h)$ is integrable on $G_\infty(s^{\mathrm{c}})$, and there are numbers $(c_i)_{i \in \mathrm{Mem}_j}$, independent of g, with

$$\sum_{i \in \mathrm{Mem}_j} c_i\,\mathsf{F}_i(g) = \int_{G_\infty(s^{\mathrm{c}})} \varphi^\circ(g_1(h,1))\,\theta_\Psi(g_1(h,1),g)\,m^{\mathrm{hol}}_{s^{\mathrm{c}},u_1}(h)\,dh \qquad (g \in \mathrm{G}(s_j)).$$

Proof. Let o be the order on $\mathcal{B}^{\mathrm{c}}$ with $o((1,a)) = -a$, $o((2,a)) = a-1$; it is injective and positive-first for s.

Step 0: the carrier, the model function and the kernel. The character χ_M is tame of weight 0 for $(\mathfrak{M},\chi)$, χ the character of $\mathcal{O}_E/\mathfrak{M}$ given by its tame part on units. So by (b) of Lemma 6.6.62, φ° is continuous, left invariant under $\Gamma_{S^{\mathrm{c}}}$, right invariant under $\mathbf{K}_0(\mathcal{B}^{\mathrm{c}})$, and $|\varphi^\circ|\,\mathrm{ht}^B_{S^{\mathrm{c}}}$ is bounded for every B. The weights u_1 and $u_3(\tau)$ of Lemma 6.6.66 are K-dominant for (s,o) with holomorphic gap $300n_*$ and $|u_3(\tau)(b')| \le 5b$ ((c) there), and $3m \le t$ ((AD2) of Definition 2.3.4); so Lemma 6.6.68 with $u = u_3$, $u' = u_1$ gives $\int |m_1(h)|\,|h|^{6n_*}dh < \infty$, and since $|h| \ge 1$ there, m_1 is integrable. By (a) of Lemma 6.6.66, $\beta = \nu_j^\flat \upsilon^m$ has trivial tame part, and υ^m has weight m and trivial tame part. Fix $g \in \mathrm{G}(s_j)$ and put $\psi_g(x) = \varphi^\circ(x)\,\theta_\Psi(x,g)$ on $\mathrm{G}(s)$. Then ψ_g is of level $\mathfrak{n}$: it is continuous ((2) of Lemma 4.1.1, the theta functional being bounded) and left invariant under $\Gamma_{S^{\mathrm{c}}}$ (Lemma 6.2.5); for $k \in \mathbf{K}_{\mathcal{B}^{\mathrm{c}}}(\mathfrak{n})$, (6) of Lemma 4.1.1 gives $\omega((1,k),1)\mathrm{mod}_{\mathfrak{n}}(\Psi, v_{P_j}) = \beta(\det k)^{-1}\mathrm{mod}_{\mathfrak{n}}(\Psi_{k,1}, v_{P_j}) = \mathrm{mod}_{\mathfrak{n}}(\Psi, v_{P_j})$, as $\bar{k} = 1$ and $\beta(\det k) = 1$ (Definition 3.1.1), so $\theta_\Psi(x(1,k),g) = \theta_\Psi(x,g)$, and φ° is invariant too; and by Lemma 6.2.28 there are Y, A with $|\theta_\Psi(x,g)| \le Y'\,\mathrm{ht}_{S^{\mathrm{c}}}(x)^A$, Y' depending on g, so $|\psi_g|\,\mathrm{ht}^M_{S^{\mathrm{c}}} \le Y'|\varphi^\circ|\,\mathrm{ht}^{M+A}_{S^{\mathrm{c}}}$ is bounded.

Step 1: the reproducing formula on $(S^{\mathrm{c}}, W^\flat)$. Fix an injective $e\colon \mathcal{B}^{\mathrm{c}} \to \mathcal{B}^\flat$ ($3m \le 6m+1$) and let $p = \{(1,a) : 1 \le a \le 2m\}$, q the second block, $i = e|_p$, $d = \Delta'$, and

$$Q_0 = \prod_{\tau \in \Phi} \det\big((X_{\tau,e(1,a),(1,a')})_{1 \le a,a' \le 2m}\big)^{\Delta'}.$$

By Lemma 6.6.65 (signs $+1$ on p, -1 on q, $n' = \mathcal{B}^\flat$), $Q_0 \ne 0$ and for every bounded linear functional ℓ on $\mathcal{F}_{S^{\mathrm{c}},W^\flat}$ the function $x \mapsto \ell(\Omega_{0,0}(x,1)v_{Q_0})$ is holomorphic of weight $u_{\Delta'} = (-\Delta';\,6m+1) = u_1$ for o; as $\Omega_{m,0}(x,1) = \Omega_{0,0}(x,1)$ (the weight m multiplies by $\det 1$), the hypothesis of Lemma 6.6.53 on Q_0 holds for $(\gamma',\gamma) = (m,0)$. The weights u_1 are constant on each block, hence K-dominant. $\mathfrak{n} = \pi_M^2\mathcal{O}_E$ is a modulus with a generator, $2\#\mathcal{B}^{\mathrm{c}} = 6m < 6m+1$, and $\mathbf{K}_{\mathfrak{n}}[\{1\}] = \mathbf{K}_{\mathcal{B}^\flat}(\mathfrak{n})$, so the classes of Lemma 6.6.53 are $\mathrm{Cl}^\flat$ with the representatives $x_{\mathfrak{c}}$.

Step 2: the span. Let $\mathfrak{c} \in \mathrm{Cl}^\flat$, $Y_{\mathfrak{c}} \in \mathrm{GL}_{\mathcal{B}^\flat}(E)$ the matrix of the short row family, and $y \in G_\infty(1_{\mathcal{B}^\flat})$. By the second formula of (5) of Lemma 4.1.1 (the point 1 of $G_\infty(s)$ and y are block diagonal, all signs of $W^\flat$ being $+1$), $\Omega_{m,0}(1,y)v_{Q_0}$ is a scalar times $v_{Q_0'}$, where Q_0' is Q_0 after the substitution $X_{\tau,a,b} \mapsto \sum_{c \in \mathcal{B}^\flat}(y_\tau)_{ca}X_{\tau,c,b}$ of the variables with b in the positive block (the only ones in Q_0). So the matrix $(X_{\tau,e(1,a),(1,a')})_{a,a'}$ becomes $N_\tau X_\tau$, where $X_\tau = (X_{\tau,c,(1,a')})_{c \in \mathcal{B}^\flat,a'}$

and N_τ is the $2m \times \mathcal{B}^\flat$ matrix whose row a is the row $e(1,a)$ of y_τ^{T}. As $\tau(Y_{\mathfrak{c}})$ is invertible, $N_\tau = M_\tau\,\tau(Y_{\mathfrak{c}})$ with $M_\tau = N_\tau\tau(Y_{\mathfrak{c}})^{-1}$, and multilinearity of the determinant in the rows gives

$$\det(N_\tau X_\tau) = \sum_{r\colon \{1,\dots,2m\}\to\mathcal{B}^\flat} \Big(\prod_a (M_\tau)_{a,r(a)}\Big)\det\big(\tau(Y_{\mathfrak{c}})[r]\,X_\tau\big),$$

$\tau(Y_{\mathfrak{c}})[r]$ the matrix whose row a is the row $r(a)$ of $\tau(Y_{\mathfrak{c}})$. Expanding the product over τ and the Δ' factors, Q_0' is a linear combination of the polynomials $Q_{\mathfrak{c},\lambda} = \prod_\tau \prod_{k=1}^{\Delta'} \det(\tau(Y_{\mathfrak{c}})[\lambda_\tau(k,\cdot)]X_\tau)$, $\lambda = (\lambda_\tau)$ running over the families of maps $\lambda_\tau\colon \{1,\dots,\Delta'\}\times\{1,\dots,2m\}\to\mathcal{B}^\flat$. These form a finite set ι, and $Q_{\mathfrak{c},\lambda}$ is the flat determinant product $P_i^\flat$ of every choice $i=(\Lambda,\mathfrak{c},\lambda)$, read with the rows $\mathcal{B}^\flat$. So the span hypothesis of Lemma 6.6.53 holds, and with Step 1 that lemma gives numbers $a_{\mathfrak{c},\lambda}$, depending only on g_1, such that for every g the function $h\mapsto \psi_g(g_1(h,1))m_1(h)$ is integrable and

$$\int_{G_\infty(s)} \psi_g(g_1(h,1))\,m_1(h)\,dh = \sum_{\mathfrak{c},\lambda} a_{\mathfrak{c},\lambda}\int_{[\mathrm{U}(S^{\mathrm{c}})]} \psi_g(x)\,\theta_2^{\mathfrak{c},\lambda}(x)\,dx,$$

the right-hand integrands being integrable.

Step 3: the product of the kernels and the Fourier expansion. By (a) of Lemma 6.3.80 (target spaces $\tilde W_j$ and $W^\flat$, rows $\mathcal{R}\sqcup\mathcal{A}_j^-\sqcup\mathcal{A}_j^+$ and $\mathcal{B}^\flat$) and (d) of Lemma 6.3.52 ($\alpha=\upsilon^m$, $\beta=\nu_j^\flat\upsilon^m$, $\beta'=1$), for $x\in\mathrm{G}(s)$

$$\theta_\Psi(x,g)\,\theta_2^{\mathfrak{c},\lambda}(x) = \theta^{S^{\mathrm{c}},W_j'}_{\upsilon^m,\nu_j^\flat\upsilon^m;\mathfrak{n}}\big[\Psi\otimes\delta_e,\ P_jQ_{\mathfrak{c},\lambda}\big](x,(g,1,x_{\mathfrak{c}})).$$

The function $\Psi\otimes\delta_e$ lives on the finite abelian group $G=\mathrm{M}_{\mathcal{B}_j\times\mathcal{B}^{\mathrm{c}}}(R)$. The map $\Lambda\mapsto\psi_\Lambda$ from $\mathrm{M}_{\mathcal{B}^{\mathrm{c}}\times\mathcal{B}_j}(R)$ to the characters of G is a homomorphism; it is injective: if $\psi_\Lambda=1$, then taking y with the single entry t at (b,a) gives $\psi_{\mathfrak{n}}(t\Lambda_{ab})=1$ for all $t\in R$, and $\psi_{\mathfrak{n}}$ is primitive: if $a'\in\widehat{\mathcal{O}}_E$ and $\psi_{E,f}(\jmath\alpha\bar\alpha\varpi_{\mathfrak{n}}^{-1}a't)=1$ for all $t\in\widehat{\mathcal{O}}_E$, then $\overline{\varpi_{\mathfrak{n}}^{-1}a'}$ lies in the annihilator of $\widehat{\mathcal{O}}_E$ under $(u,v)\mapsto\psi_{E,f}(\jmath\alpha\bar\alpha\bar uv)$, which is $\widehat{\mathcal{O}}_E$ (Lemma 4.1.3), so $a'\in\varpi_{\mathfrak{n}}\widehat{\mathcal{O}}_E$, i.e. $a'\equiv 0$ modulo $\mathfrak{n}$. Both groups have the same order, so every character of G is some ψ_Λ, and the characters of a finite abelian group span the functions on it: $\Psi\otimes\delta_e=\sum_\Lambda f(\Lambda)\psi_\Lambda$ for some numbers $f(\Lambda)$. As the datum $\mathrm{mod}_{\mathfrak{n}}(\Phi,v)$ and the theta kernel are linear in Φ,

$$\theta_\Psi(x,g)\,\theta_2^{\mathfrak{c},\lambda}(x) = \sum_\Lambda f(\Lambda)\,\theta^{S^{\mathrm{c}},W_j'}_{\upsilon^m,\nu_j^\flat\upsilon^m;\mathfrak{n}}\big[\psi_\Lambda,P_jP^\flat_{(\Lambda,\mathfrak{c},\lambda)}\big](x,(g,1,x_{\mathfrak{c}})).$$

Each term times φ° is integrable on $[\mathrm{U}(S^{\mathrm{c}})]$ (bounded by Step 0's argument and Lemma 6.2.28; finite volume, Theorem 4.2.18), and by (XE5) its integral is $\mathsf{F}_{(\Lambda,\mathfrak{c},\lambda)}(g)$. Hence $\int\psi_g\,\theta_2^{\mathfrak{c},\lambda} = \sum_\Lambda f(\Lambda)\,\mathsf{F}_{(\Lambda,\mathfrak{c},\lambda)}(g)$.

Step 4: conclusion. Since $\psi_g(g_1(h,1))m_1(h)=\varphi^\circ(g_1(h,1))\theta_\Psi(g_1(h,1),g)m^{\mathrm{hol}}_{s^{\mathrm{c}},u_1}(h)$, Steps 2 and 3 give the integrability and the identity with $c_i=a_{\mathfrak{c}_i,\lambda_i}f(\Lambda_i)$ for $i=(\Lambda_i,\mathfrak{c}_i,\lambda_i)\in\mathrm{Mem}_j$; these depend on g_1 and Ψ but not on g. □

Lemma 6.6.80. *Let D be admissible data and $n_*=2ft^3$. Let n be a nonempty finite index set with $\#n\le t$, s signs on n, o a positive-first order for s, and $u=(u_v)_{v\in\Phi}$, $u'=(u'_v)_{v\in\Phi}$ weights such that every u_v and every u'_v is K-dominant for (s,o) and has holomorphic gap $300n_*$ for s, and $|u_v(i)|\le 5b$ for all v, i. For $h\in G_\infty(s)$ let $|h|=\max_{v\in\Phi}\|h_v\|$ (operator norms). Let $F\colon\mathrm{G}(s)\to\mathbb{C}$ be continuous, bounded and adelically holomorphic of weight u for o, let z be a peak of F, let $K_0\colon G_\infty(s)\to\mathbb{C}$ with $|K_0|\le C$ for some $C\ge 0$, and let $\delta\ge 0$. Assume that for*

every peak x of F there is a function $K\colon G_\infty(s)\to\mathbb{C}$, holomorphic of weight $u-u'$ for o, with $|K(h)-K_0(h)|\le\delta|h|^{6n_*}$ for all h and

$$\int_{G_\infty(s)}\overline{F(x\,(h,1))}\,K(h)\,m^{\mathrm{hol}}_{s,u'}(h)\,dh=0.$$

Then $|K_0(1)|\le(4b^{n_*}+1)\,\delta$.

Proof. Write $N=\#n$, $m_u=m^{\mathrm{hol}}_{s,u}$, $m_{u'}=m^{\mathrm{hol}}_{s,u'}$, and $\chi(\theta)=\prod_{v,i}e^{2\pi\mathrm{i}\theta o(i)u_v(i)}$.

Step 1: the integrals of the model functions. By Lemma 6.6.68, every $h\in G_\infty(s)$ has $|h|\ge1$, and

$$I:=\int_{G_\infty(s)}|m_u|^2\in(0,\infty),\qquad J:=\int_{G_\infty(s)}|m_{u'}(h)|\,|h|^{6n_*}dh\le b^{n_*}I<\infty.\tag{6.160}$$

Step 2: rotated peaks. For $g\in\mathrm{G}(s)$ the function $\varphi_g(h)=F(g(h,1))$ is holomorphic of weight u for o. For a family e of diagonal matrices with entries of modulus 1 and $h\in G_\infty(s)$, (HV3) at he with the family 1 and at h with the family e gives $\varphi_g(he)=\tilde\varphi_g(he)=e^u\varphi_g(h)$. So $F(z(d(\theta),1))=\chi(\theta)F(z)$ and $|F(z(d(\theta),1))|=|F(z)|$: every $z_\theta=z(d(\theta),1)$ is a peak. Since $|F(z)|>\frac12\sup|F|\ge0$, $F(z)\ne0$. For each $\theta\in\mathbb{R}/\mathbb{Z}$ let K_θ be a function given by the hypothesis for $x=z_\theta$, and put

$$W_\theta(h)=\overline{F(z(d(\theta)h,1))\chi(\theta)^{-1}},\qquad f=K_0'\,m_{u'},\quad K_0':=K_{\theta=0}.$$

Then $|W_\theta(h)|\le\sup|F|<2|F(z)|$; $(\theta,h)\mapsto W_\theta(h)$ is continuous; and since $z_\theta(h,1)=z(d(\theta)h,1)$, the hypothesis for z_θ gives $\int W_\theta K_\theta m_{u'}=\overline{\chi(\theta)^{-1}}\cdot0=0$.

Step 3: integrability. The functions K_θ, $m_{u'}$ are continuous (restrictions of functions analytic at every point of $G_\infty(s)$). As $|h|\ge1$, $|K_0'(h)|\le C+\delta|h|^{6n_*}\le(C+\delta)|h|^{6n_*}$, so f is integrable by (6.160), and $(\theta,h)\mapsto f(h)W_\theta(h)$ is integrable on $\mathbb{R}/\mathbb{Z}\times G_\infty(s)$, being dominated by $2|F(z)||f(h)|$. The function f is holomorphic of weight u for o: if $\tilde K$, $\tilde m$ are extensions of K_0', $m_{u'}$ as in (HV3) ((i) of Lemma 6.6.70), $\tilde K\tilde m$ is analytic at hb with value $b^{u-u'}K_0'(h)\,b^{u'}m_{u'}(h)=b^uf(h)$.

Step 4: each rotation is small. For every θ, by Step 2, $\int fW_\theta=\int(K_0'-K_\theta)\,m_{u'}\,W_\theta$, and $|K_0'-K_\theta|\le|K_0'-K_0|+|K_0-K_\theta|\le2\delta|h|^{6n_*}$; so by (6.160)

$$\Big|\int_{G_\infty(s)}f\,W_\theta\Big|\le2\delta\cdot2|F(z)|\cdot J\le4\delta\,|F(z)|\,b^{n_*}I.$$

Step 5: the mean over the rotations. By Lemma 6.6.72 for φ_z (with $\varphi_z(1)=F(z)$), $\int_{\mathbb{R}/\mathbb{Z}}W_\theta(h)\,d\theta=\overline{m_u(h)F(z)}$. By Fubini (Step 3), $h\mapsto f(h)\overline{m_u(h)}\,\overline{F(z)}$ is integrable, hence so is $f\overline{m_u}$ ($F(z)\ne0$), and

$$\int_{\mathbb{R}/\mathbb{Z}}\int_{G_\infty(s)}f\,W_\theta\,dh\,d\theta=\overline{F(z)}\int_{G_\infty(s)}f\,\overline{m_u}=\overline{F(z)}\,f(1)\,I$$

by Lemma 6.6.71. With Step 4, $|F(z)|\,|f(1)|\,I\le4\delta|F(z)|b^{n_*}I$, and as $|F(z)|I>0$ and $f(1)=K_0'(1)m_{u'}(1)=K_0'(1)$ ((i) of Lemma 6.6.70), $|K_0'(1)|\le4\delta b^{n_*}$.

Step 6: conclusion. $|1|=1$, so $|K_0(1)|\le|K_0'(1)|+|K_0'(1)-K_0(1)|\le4\delta b^{n_*}+\delta$. □

The last group proves that the periods of false lifts vanish (Proposition 6.6.81): a form of a descent pair that is killed by the hyperplane sums at the chosen places, by its normalised descent and by the Hecke polynomial at the lift root at one split place has vanishing theta

periods over a complementary definite block. The proof climbs a Witt tower. At the bottom, the vanishing of the normalised descent spreads to all data that are the vacuum off the level places (Lemma 6.6.84). The tower step (Lemma 6.6.86) passes vanishing to the space plus a hyperbolic plane: the descents there are bounded (Lemma 6.6.87) and killed by a transferred Hecke polynomial with a small root, and a bounded spherical function killed by such a polynomial vanishes (Lemma 6.6.89); in our reading this is the trivial bound for Satake parameters [248, 109], used in the contrapositive. At the top, vanishing descents kill the periods (Lemma 6.6.90). In our reading the argument is a vanishing counterpart of the tower property (Proposition 6.2.2); for Rallis's towers and the injectivity of theta liftings see [236, 237], and for the convergence of theta integrals over definite groups [323].

Proposition 6.6.81. *Let O be a descent pair with ranks $1 \le r < n$ that fits a parameter, $W = W(O)$, $\Sigma = S_O \cup \bar{S}_O$ and $\xi_O = v^+_{l_O}$. Let $\Xi \in \mathrm{Form}(O)$, and suppose that $\psi_{O,w} \neq 1$ for every $w \in Q_O$; that $\sum_{s \in H(\lambda)} \psi_{O,w}(\det s)^{-1}[s]_w \Xi = 0$ for every $w \in M_O$ and every nonzero row $\lambda \in k_w^r$; that $\mathrm{Th}_O(\Xi) = 0$; and that $\overline{H_w(X_*(w))}\,\Xi = 0$ for some finite place w with $w \neq \bar{w}$ and $w, \bar{w} \notin S_O$. For $\zeta \in \mathbb{V}_O$ put $Y_\zeta(h) = \overline{\langle \zeta, \rho_{\kappa_O,e_O}(h_\infty)^{-1}\Xi(h_f)\rangle_{\kappa_O}}$ on $\mathrm{G}(1_n)$. Let $\{1,\dots,n\} = N \sqcup l$ with $2r+2\#N+3 < n$, $\mathrm{I}_n = \mathrm{I}_N \oplus \mathrm{I}_l$, and $\iota\colon \mathrm{G}(1_N)\times\mathrm{G}(1_l) \to \mathrm{G}(1_n)$ the block-diagonal embedding. Then*

$$\int_{[\mathrm{U}(\mathrm{I}_l)]} \theta^{W,\mathrm{I}_l}_{\chi_O,b}(\varphi)(g,h_2)\, Y_\zeta\big(\iota(h_1,h_2)\,(1,\gamma)\big)\, dh_2 = 0$$

for every $\zeta \in \mathbb{V}_O$, every automorphic character b of $\mathrm{U}(1)$, every datum $\varphi \in \mathcal{M}_{W,\mathrm{I}_l}$ that is the vacuum at the finite places outside Σ, every $g \in \mathrm{G}(\varepsilon_O)$, every $h_1 \in \mathrm{G}(1_N)$ and every $\gamma \in G_f(n)$ whose components outside Σ are 1.

Proof. Write $\rho = \rho_{\kappa_O,e_O}$ and $\langle\,,\rangle = \langle\,,\rangle_{\kappa_O}$. Every Y_ζ is continuous, left invariant under the rational points of $\mathrm{U}(\mathrm{I}_n)$ (by (DP3) and (DP5), $\Xi(\gamma_f h_f) = \rho(\gamma_\infty)\Xi(h_f)$), and right invariant under every $k \in \mathbf{K}_0(n)$ whose components at Σ are 1: such a k has reductions 1 at S_O, so $k \in \mathbf{K}(O)$ with $\tilde{\eta}_O(k) = 1$ and $\Xi(h_f k) = \Xi(h_f)$. In particular Y_ζ is right invariant under an open subgroup of $G_f(n)$. Since $\langle\,,\rangle$ is conjugate-linear in its first argument, $\zeta \mapsto Y_\zeta(h)$ is linear.

Vanishing descents to W. Put $Y_0 = Y_{\xi_O}$. By Lemma 6.6.82 (O has rank $r \ge 1$ and fits a parameter), the descent D_{χ_O} of Y_0 to W vanishes on $v_P \otimes \Upsilon(O)$ for every polynomial P. This is (VT3) of Lemma 6.6.84 for the linear form $L = \langle \xi_O, \cdot\rangle$, for which the function Y there is Y_0; (VT1) and (VT2) are hypotheses. By Lemma 6.6.84, $D_{\chi_O}(\varphi) = 0$ for every datum φ on $\mathrm{Hom}(W, \mathrm{I}_n)$ that is the vacuum at the finite places outside Σ: the descents of Y_0 to W twisted by χ_O vanish off Σ.

The Hecke equation of Y_0. Let w be the place of the hypothesis, $q = \mathrm{N}w$ and $X = \overline{X_*(w)}$. For $\gamma' \in \mathrm{GL}_n(\mathcal{O}_{E,w})$ the element $\iota_w(\gamma')$ of $G_f(n)$ (component γ' at w, the forced component at $\bar{w}$, 1 elsewhere) lies in $\mathbf{K}_0(n)$ and has components 1 at Σ, as $w, \bar{w} \notin S_O$; so Y_0 is right invariant under $K_w = \iota_w(\mathrm{GL}_n(\mathcal{O}_{E,w}))$. The operators $T_{w,i}$ of (DP10) are the sums of right translates of Lemma 6.6.89; taking the coefficient $\langle \xi_O, \rho(h_\infty)^{-1}\cdot\rangle$ (linear, and it commutes with right translation by finite elements) and then complex conjugates (the coefficients $q^{i(i-1)/2}$ are real) turns $H_w(X_*(w))\Xi = 0$ into

$$\sum_{i=0}^{n} (-X)^i q^{\binom{i}{2}}\, T_{w,i} Y_0 = 0, \qquad \text{i.e.} \quad H^{\mathrm{I}_n}_w(X)\, Y_0 = 0.$$

By (DP10), $X_*(w) = (\chi_{O,w}\{r\}\, q^{n-r-1})^{-1}$ with $\chi_{O,w}\{r\} = \chi_O[r](\varpi_w)\, q^{r/2}$, and $|\chi_O[r](\varpi_w)| = 1$, so $|X| = (q^{r/2}q^{n-r-1})^{-1}$. The base change of υ is unramified at $\bar{w}$: for a unit u of $E_{\bar{w}}$, read as an idele with components 1 elsewhere, $u/\bar{u}$ has unit finite components and infinite components 1,

so $\upsilon(u/\bar{u}) = 1$ by (WD3); hence the base change of $\chi_O = \upsilon^{r_-(O)}$ is unramified at $\bar{w}$ too. Finally $w \notin \Sigma$, $w \neq \bar{w}$, and $2r + 2\#N + 3 < n$.

Vanishing descents to $W \oplus \mathbb{H}^{\#N}$. By Lemma 6.6.86, applied to W (of rank r), $c = \#N$, $V = \mathrm{I}_n$, $\nu = \chi_O$, $\Sigma_0 = S_O$, the function Y_0, the place w and the number X (its hypotheses were checked in the two previous paragraphs), the descents of Y_0 to $U = W \oplus \mathbb{H}^{\#N}$ twisted by $a = \chi_O \upsilon^{\#N}$ vanish off Σ.

Translations preserve vanishing descents. Let Y' be continuous on $\mathrm{G}(1_n)$, left invariant under the rational points and right invariant under an open subgroup, with descents to U twisted by a vanishing off Σ, and let $x \in \mathrm{G}(1_n)$ have finite components 1 outside Σ. Then the descents of $Y'_x = Y'(\cdot\, x)$ to U twisted by a vanish off Σ. Indeed, by the right invariance of the measure on $[\mathrm{U}(\mathrm{I}_n)]$ and since $\omega_{a,1}$ is a representation ((2) of Lemma 4.1.1), the descent of Y'_x by φ is the descent of Y' by $\varphi' = \omega_{a,1}(1, x^{-1})\varphi$. If φ is the vacuum at the finite places outside Σ, so is φ': for an integral finite adelic z with $z_v = 0$ for $v \in \Sigma$, $\rho_f(z)$ is a scalar times $\rho(\mathrm{fv}(z))$, $\mathrm{fv}(z)$ has infinite component 0 and finite components 0 at Σ, so the element $(1, x^{-1})$ of the pair, acting on $\mathrm{Hom}(U, \mathrm{I}_n)$ by left multiplication, fixes $\mathrm{fv}(z)$; by (2) of Lemma 4.1.1, $\omega_{a,1}(1, x^{-1})$ commutes with $\rho(\mathrm{fv}(z))$, hence $\rho_f(z)\varphi' = \omega_{a,1}(1, x^{-1})\rho_f(z)\varphi = \varphi'$.

Vanishing descents of every Y_ζ and of its translates. For $x \in G_\infty(1_n) = \prod_\tau \mathrm{U}(n)$ let $\pi(x)$ be the adjoint of $\rho(x)^{-1}$ for the positive definite form $\langle\,,\rangle$; then π is a representation of $G_\infty(1_n)$ on $\mathbb{V}_O$, and $Y_\zeta(h(x,1)) = \overline{\langle \rho(x)^{-1\,*}\zeta, \rho(h_\infty)^{-1}\Xi(h_f)\rangle}$ with $\rho(x)^{-1\,*} = \pi(x)$, i.e. $Y_0(\cdot\,(x,1)) = Y_{\pi(x)\xi_O}$. By the previous paragraph and linearity in ζ, the descents of Y_ζ to U twisted by a vanish off Σ for every ζ in the span of $\{\pi(x)\xi_O\}$. This span is $\mathbb{V}_O$: it is a nonzero π-invariant subspace ($\xi_O = \mathrm{ev}_{\kappa_O}(y^{l_O}) \neq 0$: y^{l_O} has in row a at τ a single entry 1, in the column $c_\tau(a) = \#\{a' : l_{O,\tau}(a') > l_{O,\tau}(a)\}$, and c_τ is a bijection onto $\{0, \dots, n-1\}$ as $l_{O,\tau}$ is injective; for the column filling with $T(\tau, h, j) = \{a : c_\tau(a) < h\}$ each factor of Δ_T at y^{l_O} is the determinant of a permutation matrix, so $\Delta_T(y^{l_O}) = \pm 1$, and Δ_T is an integral combination of the standard basis (b_i), so some $b_i(y^{l_O}) \neq 0$), its orthogonal complement is ρ-invariant, and $\mathbb{V}_O$ is irreducible under $\rho(G_\infty(1_n))$ (the Weyl module of weight κ_O is an irreducible polynomial representation of $\prod_\tau \mathrm{GL}_n(\mathbb{C})$, the twist by $\det^{e_O}$ and complex conjugation of the matrices do not change invariant subspaces, and $\prod_\tau \mathrm{U}(n)$ is Zariski dense in $\prod_\tau \mathrm{GL}_n(\mathbb{C})$). Applying the previous paragraph once more with $x = (1, \gamma)$, the descents of $Y_\zeta(\cdot\,(1,\gamma))$ to U twisted by a vanish off Σ, for every ζ and every $\gamma \in G_f(n)$ with components 1 outside Σ.

The periods. Fix ζ, γ and let $Y = Y_\zeta(\cdot\,(1,\gamma))$. It is continuous and left invariant under the rational points, and right invariant under every $k \in \mathbf{K}_0(n)$ with components 1 at Σ, because such k commutes with $(1,\gamma)$ (their nontrivial finite components lie at disjoint sets of places) and fixes Y_ζ. The character $a' = \chi_O = \upsilon^{r_-(O)}$ satisfies $a'(\det k) = 1$ for every $k \in \mathbf{K}_0(n)$: $\det k$ has unit finite components and infinite components 1, so $\upsilon(\det k) = 1$ by (WD3). By Lemma 6.6.90, applied to W, $N \sqcup l$, a' (so that $a'\upsilon^{\#N} = a$), $A = S_O$ and this Y, the integral of the statement vanishes for every b, φ, g and h_1 as there. □

Lemma 6.6.82. *Let O be a descent pair with ranks $1 \leq r < n$ that fits a parameter, $W = W(O)$ and $\xi_O = v^+_{l_O}$. Let $\Xi \in \mathrm{Form}(O)$ with $\mathrm{Th}_O(\Xi) = 0$, and let*

$$Y_0(h) = \overline{\langle \xi_O,\ \rho_{\kappa_O, e_O}(h_\infty)^{-1}\, \Xi(h_f)\rangle_{\kappa_O}} \qquad (h = (h_\infty, h_f) \in \mathrm{U}(\mathrm{I}_n)(\mathbb{A}_F)).$$

Then the descent D_{χ_O} of Y_0 to W vanishes on $v_P \otimes \Upsilon(O)$ for every polynomial P in the Fock variables of $\mathrm{Hom}(W, \mathrm{I}_n)$.

Proof. Let l be a parameter that fits O, $\Phi = v_{\mathrm{Har}_O(\xi_O)}$, $\Upsilon = \Upsilon(O)$, and $\gamma(\tau) = r_-(O)$, $\gamma'(\tau) = \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)$ the weights of $\chi_O = \upsilon^{r_-(O)}$ and $\chi'_O = \mu_O \upsilon^{\lfloor n/2 \rfloor}$ (υ has weight 1).

The theta kernel. For $h \in \mathrm{G}(1_n)$ and $u \in \mathrm{G}(\varepsilon_O)$ put $\omega'(h,u) = \omega_O(u,h)$. Then (ω', θ_O) is a theta kernel for (I_n, W) on $\mathrm{Hom}(W, \mathrm{I}_n)$ in the sense of (ED3): ω' is a homomorphism with $(p,\varphi) \mapsto \omega'(p)\varphi$ continuous by (2) of Lemma 4.1.1, θ_O is a bounded linear functional, and $\theta_O(\omega'(\gamma h, \gamma' u)\varphi) = \theta_O(\omega'(h,u)\varphi)$ for rational points γ, γ' by Lemma 6.2.5. By (5) of Lemma 4.1.1 (with $\alpha = \chi_O$ of weight γ and $\beta = \chi'_O$ of weight γ'), $\omega'((y,1),(x,1))(v \otimes f) = (\Omega_{\gamma,\gamma'}(x,y)v) \otimes f$ for $y \in G_\infty(1_n)$, $x \in G_\infty(\varepsilon_O)$: the kernel has real part $(y,x) \mapsto \Omega_{\gamma,\gamma'}(x,y)$. Its lift (ED3) of a function F on $\mathrm{G}(1_n)$ with datum φ is

$$L_\varphi F(u) = \int_{[\mathrm{U}(\mathrm{I}_n)]} \theta^{W,\mathrm{I}_n}_{\chi_O,\chi'_O}(\varphi)(u,h)\, F(h)\, dh.$$

The harmonic descent vanishes. By (DP8), $\mathrm{J}_O(\xi_O) = \Phi \otimes \Upsilon$. By (2) of Lemma 6.6.83 (its function Y_Ξ is Y_0), $0 = \mathrm{Th}_O(\Xi)(u) = c_O \overline{L_{\Phi\otimes\Upsilon} Y_0(u)}$ with $c_O > 0$, so $L_{\Phi\otimes\Upsilon} Y_0 = 0$.

Polynomial data. By Lemma 6.4.15(e), Φ is an edge for the weights (γ,γ') from $e_{l/2}$ to $e_{l_O/2}$; by (ED4) it is in particular a half edge for $(y,x) \mapsto \Omega_{\gamma,\gamma'}(x,y)$ from $\overline{e_{l_O/2}}$ to $\overline{e_{l/2}}$. By (1) of Lemma 6.6.83, Y_0 is a form of type $\overline{e_{l_O/2}}$ on I_n. Let P be a polynomial in the Fock variables of $\mathrm{Hom}(W, \mathrm{I}_n)$. By (d) of (ED4), for the spaces $(\mathrm{I}_n, W, \mathrm{Hom}(W,\mathrm{I}_n))$, the theta kernel (ω', θ_O), the vector Υ and the form Y_0, there is a continuous compactly supported a on $G_\infty(\varepsilon_O)$ with

$$L_{v_P\otimes\Upsilon} Y_0 = R(a)\, L_{\Phi\otimes\Upsilon} Y_0 = 0.$$

The character. By (WR4) of Definition 3.2.18, $\omega_{\chi_O,1}(u,h) = \chi'_O(\det u)\, \omega_{\chi_O,\chi'_O}(u,h)$, so $\theta^{W,\mathrm{I}_n}_{\chi_O,1}(\varphi)(u,h) = \chi'_O(\det u)\, \theta^{W,\mathrm{I}_n}_{\chi_O,\chi'_O}(\varphi)(u,h)$ and

$$D_{\chi_O}(v_P \otimes \Upsilon)(u) = \chi'_O(\det u)\, L_{v_P\otimes\Upsilon} Y_0(u) = 0 \qquad (u \in \mathrm{G}(\varepsilon_O)).$$

□

Lemma 6.6.83. *Let O be a descent pair with ranks $r < n$, $W = W(O)$, $\xi_O = v^+_{l_O}$, $\rho = \rho_{\kappa_O, e_O}$ and $\langle\,,\rangle = \langle\,,\rangle_{\kappa_O}$. For $\Xi \in \mathrm{Form}(O)$ let*

$$Y_\Xi(h) = \overline{\langle \xi_O,\ \rho(h_\infty)^{-1}\, \Xi(h_f)\rangle} \qquad (h = (h_\infty, h_f) \in \mathrm{G}(1_n)).$$

Then:

(1) Y_Ξ is a form of type $\overline{e_{l_O/2}}$ on I_n;

(2) there is $c_O > 0$, depending only on O, such that for every $\Xi \in \mathrm{Form}(O)$ and every $u \in \mathrm{G}(\varepsilon_O)$

$$\mathrm{Th}_O(\Xi)(u) = c_O \cdot \overline{\int_{[\mathrm{U}(\mathrm{I}_n)]} \theta^{W,\mathrm{I}_n}_{\chi_O,\chi'_O}(\mathrm{J}_O(\xi_O))(u,h)\, Y_\Xi(h)\, dh}\,.$$

Proof. Put $\kappa = \kappa_O$, $e = e_O$, $d = \#I_\kappa = \dim \mathbb{V}_O$, $S_O = M_O \cup Q_O$, and let $\gamma(\tau) = r_-(O)$, $\gamma'(\tau) = \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau)$; by (ii) of Lemma 3.2.11, υ has weight 1 at every $\tau \in \Phi$, so $\chi_O = \upsilon^{r_-(O)}$ and $\chi'_O = \mu_O \upsilon^{\lfloor n/2 \rfloor}$ have the weights γ and γ'. The matrices $S_\kappa(z)$ have entries in $\mathbb{Z}[z]$ (the minors of zy are integral combinations of those of y by the Cauchy–Binet formula, and (b_j) is a $\mathbb{Z}$-basis of ∇_κ), so $S_\kappa(\bar z) = \overline{S_\kappa(z)}$, and the same holds for $\kappa^\pm_O$. The highest vector $\xi_O = \mathrm{ev}_\kappa(y^{l_O})$ has integer entries, as y^{l_O} has entries 0 and 1; so $\bar\xi_O = \xi_O$. By the sidecalc of (DP11), l_O is a doubled Harish-Chandra parameter of (κ, e) for the index set $\{1,\dots,n\}$.

Step 1: real points act on the theta datum through ρ. We show, for $y \in G_\infty(1_n)$ and $\xi \in \mathbb{V}_O$,

$$\omega_O(1,(y,1))\, \mathrm{J}_O(\bar\xi) = \mathrm{J}_O\big(\overline{\rho(y)\xi}\big). \tag{6.161}$$

This is (a) of Lemma 6.6.31.

Step 2: the level acts on the theta datum through $\tilde{\eta}_O$. We show $\omega_O(1,(1,k))\mathrm{J}_O(\xi) = \tilde{\eta}_O(k)\,\mathrm{J}_O(\xi)$ for $k \in \mathbf{K}(O)$, $\xi \in \mathbb{V}_O$. This is (b) of Lemma 6.6.31.

Step 3: the integrand. Fix $\Xi \in \mathrm{Form}(O)$ and $u \in \mathrm{G}(\varepsilon_O)$. By (DP5), $\Xi(\gamma_f g k) = \tilde{\eta}_O(k)\rho(\gamma_\infty)\Xi(g)$ for $\gamma \in \Gamma$, $g \in G_f(n)$, $k \in \mathbf{K}(O)$; the group K_1 of the $k \in \mathbf{K}_0(n)$ with $\mathrm{red}_w k = 1$ for $w \in S_O$ is open, lies in $\mathbf{K}(O)$ and has $\tilde{\eta}_O(K_1) = 1$. So Ξ is an algebraic form of weight (κ, e) on $\{1,\dots,n\}$, and by unitarity (Lemma 4.6.10(c)) $\overline{Y_\Xi(y,g)} = \langle \xi_O, \rho(y)^{-1}\Xi(g)\rangle = F_\Xi^{l_O}(y,g)$. Let

$$f(y,g) = \overline{\theta^{W,\mathrm{I}_n}_{\chi_O,\chi_O'}(\mathrm{J}_O(\xi_O))(u,(y,g))}\; F_\Xi^{l_O}(y,g) \qquad ((y,g) \in \mathrm{G}(1_n)).$$

As $(u,(y,g)) = (u,(1,g))\cdot(1,(y,1))$, ω_O is a representation ((2) of Lemma 4.1.1), $\xi_O = \bar{\xi}_O$, and by (6.161) and (DP9),

$$\overline{\theta_O(\omega_O(u,(y,g))\mathrm{J}_O(\xi_O))} = \overline{\theta_O(\omega_O(u,(1,g))\mathrm{J}_O(\overline{\rho(y)\xi_O}))} = \mathcal{C}_O(u,g)(\rho(y)\xi_O),$$

so $f(y,g) = \mathcal{C}_O(u,g)(\rho(y)\xi_O)\,F_\Xi^{l_O}(y,g)$. The function f is continuous (the kernel by (2) of Lemma 4.1.1, θ_O being a bounded linear functional; $F_\Xi^{l_O}$ by (b) of Lemma 6.4.72) and left Γ-invariant (Lemma 6.2.5; (b) of Lemma 6.4.72). It is right invariant under $\mathbf{K}(O)$ in g: by Step 2, $f(y,gk) = \overline{\tilde{\eta}_O(k)}\,\tilde{\eta}_O(k)\,f(y,g) = f(y,g)$, as $|\tilde{\eta}_O(k)| = 1$ (the $\psi_{O,w}$ are characters of finite groups).

Step 4: integration over the real points. The Cayley measure of $G_\infty(1_n)$ is a probability measure (Lemma 5.5.21, at every τ). The linear form $\mathcal{C}_O(u,g)$ on the finite-dimensional space $\mathbb{V}_O$ commutes with the integral of the continuous function $y \mapsto F_\Xi^{l_O}(y,g)\rho(y)\xi_O$, so by (WM7) and (b) of Lemma 6.4.72,

$$\begin{aligned} H(g) := \int_{G_\infty(1_n)} f(y,g)\,dy &= \mathcal{C}_O(u,g)\Big(\int F_\Xi^{l_O}(y,g)\,\rho(y)\xi_O\,dy\Big) \\ &= \mathcal{C}_O(u,g)(d^{-1}(F_\Xi^{l_O})^\flat_{l_O}(g)) = d^{-1}\,\mathcal{C}_O(u,g)(\Xi(g)). \end{aligned}$$

Step 5: unfolding. By (2) of Lemma 6.4.74 for $V = \mathbb{C}$, $\mathrm{K} = \mathrm{K}' = \mathbf{K}(O)$ and $c_n = 1$, and then (1) of Lemma 6.4.74 with $\mathrm{K} = \mathrm{K}' = \mathbf{K}(O)$ (where $\Gamma_h' = \Gamma_f \cap h\mathbf{K}(O)h^{-1}$ is the group Γ_h of the class sum),

$$\begin{aligned} \int_{[\mathrm{U}(\mathrm{I}_n)]} f &= \frac{1}{[\mathbf{K}_0(n):\mathbf{K}(O)]}\sum_{[h]}\frac{H(h)}{\#\Gamma_h} \\ &= \frac{1}{d\,[\mathbf{K}_0(n):\mathbf{K}(O)]}\sum_{[h]}\frac{1}{\#\Gamma_h}\,\mathcal{C}_O(u,h)(\Xi(h)) = \frac{\mathrm{Th}_O(\Xi)(u)}{d\,[\mathbf{K}_0(n):\mathbf{K}(O)]} \end{aligned}$$

by (DP9). Since $f = \overline{\theta^{W,\mathrm{I}_n}_{\chi_O,\chi_O'}(\mathrm{J}_O(\xi_O))(u,\cdot)\,Y_\Xi}$, this is (2) with $c_O = d\,[\mathbf{K}_0(n):\mathbf{K}(O)]$, a positive integer depending only on O.

Step 6: (1). By Step 3, $F_\Xi^{l_O}$ is the function of an algebraic form of weight (κ,e), so it is a form of type $e_{l_O/2}$ on I_n by Lemma 6.4.70. Complex conjugation preserves continuity, boundedness, left Γ-invariance and right invariance under an open subgroup, $\overline{e_{l_O/2}}$ is integrable exactly when $e_{l_O/2}$ is, and $R(\overline{e_{l_O/2}})\overline{F} = \overline{R(e_{l_O/2})F} = \overline{F}$ for $F = F_\Xi^{l_O}$ ((ED2) of Definition 3.2.29). So $Y_\Xi = \overline{F_\Xi^{l_O}}$ is a form of type $\overline{e_{l_O/2}}$ on I_n. □

Lemma 6.6.84. *Let O be a descent pair with ranks $r < n$, $W = W(O)$ and $\Sigma = S_O \cup \bar{S}_O$. Let $\Xi \in \mathrm{Form}(O)$, let L be a linear form on $\mathbb{V}_O$, and let Y be the function on $\mathrm{G}(1_n) = \mathrm{U}(\mathrm{I}_n)(\mathbb{A}_F)$ given by*

$$Y(h) = \overline{L(\rho_{\kappa_O,e_O}(h_\infty)^{-1}\,\Xi(h_f))} \qquad (h = (h_\infty,h_f)).$$

For a place $w \in M_O$ and a nonzero row $\lambda \in k_w^r$ let $H(\lambda)$ be the group of the invertible matrices $s \in \mathrm{GL}_r(k_w)$ of the form $s = 1 + v\lambda$ with a column $v \in k_w^r$, and for $s \in \mathrm{GL}_r(k_w)$ let $[s]_w$ be the action on $\mathrm{Form}(O)$ of the element of $\mathrm{GL}_r^{\mathrm{ch}}(O)$ with component s at w and 1 at the other chosen places. Suppose:

(VT1) *$\psi_{O,w} \neq 1$ for every $w \in Q_O$;*

(VT2) *$\sum_{s \in H(\lambda)} \psi_{O,w}(\det s)^{-1}\,[s]_w \Xi = 0$ for every $w \in M_O$ and every nonzero row $\lambda \in k_w^r$;*

(VT3) *the descent D_{χ_O} of Y to W vanishes on $v_P \otimes \Upsilon(O)$ for every polynomial P in the Fock variables of $\mathrm{Hom}(W, \mathrm{I}_n)$.*

Then $D_{\chi_O}(\varphi) = 0$ for every datum $\varphi \in \mathcal{M}_{W,\mathrm{I}_n}$ that is the vacuum at the finite places outside Σ.

Proof. Write $S = S_O$, ϖ_w for the fixed uniformisers ($w \in S$), $\mathrm{Box}_m = \prod_{w \in S} \varpi_w^m \mathrm{M}_{n\times r}(\mathcal{O}_{E,w})$, and $C_m(x) = x + \mathrm{Box}_m$ for an $n \times r$ matrix x over $\prod_{w\in S} E_w$. All functions Ψ of finite adelic $n \times r$ matrices below depend only on the components at S; for $g \in \prod_{w\in S} \mathrm{GL}_n(E_w)$ and $a \in \prod_{w\in S} \mathrm{GL}_r(E_w)$ put $(g \cdot \Psi)(X) = \Psi(g^{-1}X)$ and $(\Psi \cdot a)(X) = \Psi(Xa)$. For $k \in G_f(n)$ and a function Ξ' on $G_f(n)$, $R(k)\Xi'(h) = \Xi'(hk)$. An element $g \in \prod_{w\in S} \mathrm{GL}_n(\mathcal{O}_{E,w})$ has a *lift* $k_g \in \mathbf{K}_0(n)$: the element with component g_w at $w \in S$, the component at $\bar{w}$ forced by $k^*k = 1$ (the conjugate of the inverse transpose of g_w, integral and invertible), and 1 elsewhere; this is possible because no place of S is conjugate to itself or to another (DP1). In the same way an element of $\prod_{w\in S} \mathrm{GL}_r(E_w)$ is the family of components at S of an element of $G_f(r)$ supported on Σ.

The function Y satisfies the hypotheses of Lemma 6.6.85. Let $K_1 \subseteq \mathbf{K}_0(n)$ be the open subgroup of the elements whose reductions at the places of S are 1. It lies in $\mathbf{K}(O)$ and $\tilde{\eta}_O = 1$ on it, so Ξ is right invariant under K_1 by (DP5); hence Ξ is locally constant and Y is continuous and right invariant under K_1. For a rational point γ, $\Xi(\gamma_f h_f) = \rho_{\kappa_O,e_O}(\gamma_\infty)\Xi(h_f)$ by (DP3) and (DP5), so $Y(\gamma h) = Y(h)$. Thus Lemma 6.6.85 applies with $V = \mathrm{I}_n$, $a = \chi_O$ and $A = S$ ($S \cap \bar{S} = \emptyset$ by (DP1)).

Fix a polynomial P and $u \in \mathrm{G}(\varepsilon_O)$. For $\Psi \in \mathcal{T}_S$ and a function $\Xi'\colon G_f(n) \to \mathbb{V}_O$ with $\Xi'(\gamma_f h) = \rho_{\kappa_O,e_O}(\gamma_\infty)\Xi'(h)$ for rational γ and right invariant under an open subgroup, let $Y_{\Xi'}$ be formed from Ξ' as Y from Ξ and

$$B(\Psi, \Xi') = \int_{[\mathrm{U}(\mathrm{I}_n)]} \theta^{W,\mathrm{I}_n}_{\chi_O,1}(v_P \otimes (\Psi \otimes_S v^0))(u,h)\, Y_{\Xi'}(h)\, dh.$$

It is linear in Ψ (the map $\Psi \mapsto \Psi \otimes_S v^0$ is linear by (WR6)) and conjugate-linear in Ξ'. By (ii) of Lemma 6.6.85, applied for all P and u, it suffices to show: for every x there is m_0 with $B(\mathbf{1}_{x,m}, \Xi) = 0$ for all $m \geq m_0$.

Translation by the level. Let $k \in \mathbf{K}_0(n)$ and $\Psi \in \mathcal{T}_S$. Then $k \cdot \Psi \in \mathcal{T}_S$, where k acts through its components at S, and

$$B(k \cdot \Psi, R(k)\Xi) = B(\Psi, \Xi). \tag{6.162}$$

Let $k' \in \mathbf{K}_0(n)$ have the components of k at $S \cup \bar{S}$ and 1 elsewhere (it is unitary because the condition $k^*k = 1$ couples only the components at w and $\bar{w}$). By (4) of Lemma 4.1.1 with $g = 1$, $h = k'$, and by (5) there, $\omega_{\chi_O,1}((1,1),(1,k'))(v_P \otimes (\Psi \otimes_S v^0)) = c\, v_P \otimes ((k \cdot \Psi) \otimes_S v^0)$ with $k \cdot \Psi \in \mathcal{T}_S$ and $c = \chi_O[r](\det_S k')\,|\det_S k'|^{-r/2}$. The idele $\det_S k'$ has unit components at S and 1 elsewhere; so $|\det_S k'| = 1$, and υ, hence $\chi_O = \upsilon^{r-(O)}$ and $\chi_O[r] = \tilde{\chi}_O \upsilon^{-r}$, is 1 on it

and on its quotient by its conjugate, by (WD3): $c = 1$. Since $\omega_{\chi_O,1}$ is a representation ((2) of Lemma 4.1.1) and the quotient measure of $[\mathrm{U}(\mathrm{I}_n)]$ is right invariant,

$$B(k\cdot\Psi,\Xi') = \int \theta\big(\omega(u,h(1,k'))\,v_P\otimes(\Psi\otimes_S v^0)\big)Y_{\Xi'}(h)\,dh = B\big(\Psi,R(k'^{-1})\Xi'\big)$$

for every admissible Ξ'; with $\Xi' = R(k')\Xi$ this is $B(k\cdot\Psi,R(k')\Xi) = B(\Psi,\Xi)$. Finally $k'^{-1}k \in \mathbf{K}_0(n)$ has reductions 1 at S, so it lies in $\mathbf{K}(O)$ with $\tilde\eta_O(k'^{-1}k) = 1$, and $R(k)\Xi = R(k')R(k'^{-1}k)\Xi = R(k')\Xi$ by (DP5). This proves (6.162).

Translates of the block function pair to zero. Let Φ_O be the function of (DP7), so that $\Upsilon(O) = \Phi_O\otimes_S v^0$, and let $a\in G_f(r)$ be supported on Σ. Then $\Phi_O\cdot a\in\mathcal{T}_S$ and $B(\Phi_O\cdot a,\Xi) = 0$. Indeed, by (4) of Lemma 4.1.1 with $g = a$, $h = 1$ (the character of $\mathrm{U}(W)$ is trivial) and by (5), $\omega_{\chi_O,1}((1,a),(1,1))(v_P\otimes\Upsilon(O)) = c_a\,v_P\otimes((\Phi_O\cdot a)\otimes_S v^0)$ with $\Phi_O\cdot a\in\mathcal{T}_S$ and $c_a = |\det_S a|^{n/2}\upsilon(\det_S a)^n\neq 0$. Hence, $\omega_{\chi_O,1}$ being a representation,

$$c_a\,B(\Phi_O\cdot a,\Xi) = \int_{[\mathrm{U}(\mathrm{I}_n)]}\theta^{W,\mathrm{I}_n}_{\chi_O,1}\big(v_P\otimes\Upsilon(O)\big)(u(1,a),h)\,Y(h)\,dh = D_{\chi_O}\big(v_P\otimes\Upsilon(O)\big)(u(1,a)) = 0$$

by (VT3).

The level group and the block function. Let J be the group of $g\in\prod_{w\in S}\mathrm{GL}_n(\mathcal{O}_{E,w})$ whose family of reductions lies in $\mathbf{L}(O)$, and $\eta(g) = \eta_O((g_w \bmod w)_w)$, a character of J with values of modulus 1. The lift k_g of $g\in J$ lies in $\mathbf{K}(O)$ and $\tilde\eta_O(k_g) = \eta(g)$, so $R(k_g)\Xi = \eta(g)\Xi$, and $k_g\cdot\Psi = g\cdot\Psi$. We show: for a as above and $g\in J$,

$$g\cdot(\Phi_O\cdot a) = \eta(g)\,(\Phi_O\cdot a). \tag{6.163}$$

Put $Z = Xa$. As $g_w\in\mathrm{GL}_n(\mathcal{O}_{E,w})$, $g^{-1}Z$ is integral at S if and only if Z is; both sides vanish otherwise. Let Z be integral with reductions $\bar Z_w$, and $\bar g_w$ the reductions of g. At $w\in M_O$, $\bar g_w\mathrm{Inc}_O = \mathrm{Inc}_O$, so $\bar g_w^{-1}\bar Z_w = \mathrm{Inc}_O$ if and only if $\bar Z_w = \mathrm{Inc}_O$. At $w\in Q_O$, $\bar g_w$ maps the span of the basis vectors outside the image of inc_O into itself (its entries in the inc-rows and the other columns vanish), hence so does $\bar g_w^{-1}$; therefore the inc-rows of $\bar g_w^{-1}\bar Z_w$ are the inc-block of $\bar g_w^{-1}$ times the inc-rows of $\bar Z_w$, and the inc-block of $\bar g_w^{-1}$ is the inverse of that of $\bar g_w$. So $\psi_{O,w}(\det(\text{inc-rows of }\bar g_w^{-1}\bar Z_w)) = \psi_{O,w}(\det(\text{inc-block of }\bar g_w))^{-1}\psi_{O,w}(\det(\text{inc-rows of }\bar Z_w))$ (both sides 0 if the last determinant is 0). Taking the product over Q_O, $\Phi_O(g^{-1}Z) = \eta(g)\Phi_O(Z)$, which is (6.163).

Moreover J is transitive on the support of Φ_O: if Z is integral at S with $\Phi_O(Z)\neq 0$, then $Z = g\,\mathrm{Inc}_O$ for some $g\in J$. Take $g_w = 1 + (Z_w - \mathrm{Inc}_O)\mathrm{Inc}_O^{\mathsf{T}}$: it sends the basis vector of index $\mathrm{inc}_O(c)$ to the column c of Z_w and fixes the other basis vectors, so $g_w\mathrm{Inc}_O = Z_w$; its determinant is that of the inc-rows Z'_w of Z_w. At $w\in M_O$, $\bar Z_w = \mathrm{Inc}_O$, so $\bar g_w = 1$; at $w\in Q_O$, $\psi_{O,w}(\det\bar Z'_w)\neq 0$ forces $\det\bar Z'_w\neq 0$, and the entries of g_w in the inc-rows and the other columns vanish. In both cases $g_w\in\mathrm{GL}_n(\mathcal{O}_{E,w})$ and $(\bar g_w)_w\in\mathbf{L}(O)$.

Points with an integral block ratio at every place. Say that x has an *integral block ratio* at $w\in S$ if the inc-rows x'_w of x_w form an invertible matrix and the matrix $x''_wx'^{-1}_w$, x''_w the other rows, has entries in $\mathcal{O}_{E,w}$, and in $\varpi_w\mathcal{O}_{E,w}$ if $w\in M_O$. Suppose this holds at every $w\in S$. Let $a\in G_f(r)$ be supported on Σ with $a_w = x'^{-1}_w$ at $w\in S$, and $\Psi_0 = \Phi_O\cdot a$. The matrix x_wa_w has inc-rows 1_r and other rows $x''_wx'^{-1}_w$; so xa is integral at S, reduces to Inc_O at M_O and has inc-rows 1_r at Q_O, and $\Psi_0(x) = 1$. Since $\Phi_O(Z)$ depends only on Z modulo Box_1, and $\mathrm{Box}_ma\subseteq\mathrm{Box}_1$ for m at least some m_1 (the a_w are fixed matrices), Ψ_0 is constant on $C_m(x)$, equal to 1, for $m\geq m_1$.

Fix $m\geq m_1$ and choose $N\geq 1$ such that $(g-1)x\in\mathrm{Box}_m$ for every $g\in\prod_{w\in S}\mathrm{GL}_n(\mathcal{O}_{E,w})$ with $g\equiv 1$ modulo ϖ_w^N at every $w\in S$. Let J_N be the normal subgroup of these g in J; it has

finite index, $\eta = 1$ on it, and $g \cdot \mathbf{1}_{x,m} = \mathbf{1}_{x,m}$ for $g \in J_N$ (as $g^{-1}(x + \mathrm{Box}_m) = g^{-1}x + \mathrm{Box}_m$). Put

$$\Psi_1 = \frac{1}{[J : J_N]} \sum_{g \in J/J_N} \overline{\eta(g)}\, g \cdot \mathbf{1}_{x,m}.$$

Each $g \cdot \mathbf{1}_{x,m} = \mathbf{1}_{gx,m}$ lies in $\mathcal{T}_S$ by (i) of Lemma 6.6.85, and by (6.162) with $k = k_g$ and conjugate-linearity, $\overline{\eta(g)}\, B(g \cdot \mathbf{1}_{x,m}, \Xi) = B(g \cdot \mathbf{1}_{x,m}, \eta(g)\Xi) = B(\mathbf{1}_{x,m}, \Xi)$. Hence $B(\Psi_1, \Xi) = B(\mathbf{1}_{x,m}, \Xi)$. We show $\Psi_1 = \Psi_1(\mathrm{Inc}_O a^{-1})\, \Psi_0$. Re-indexing the sum gives $g \cdot \Psi_1 = \eta(g)\Psi_1$ for $g \in J$. If $\Psi_1(X) \neq 0$, then $g^{-1}X \in C_m(x)$ for some $g \in J$, so $1 = \Psi_0(g^{-1}X) = (g \cdot \Psi_0)(X) = \eta(g)\Psi_0(X)$ by (6.163), and $\Psi_0(X) \neq 0$. If $\Psi_0(X) \neq 0$, then $\Phi_O(Xa) \neq 0$, so $Xa = g\,\mathrm{Inc}_O$ with $g \in J$ by transitivity, i.e. $X = g(\mathrm{Inc}_O a^{-1})$ at S; then $\Psi_1(X) = \eta(g)^{-1}\Psi_1(\mathrm{Inc}_O a^{-1})$ and $\Psi_0(X) = \eta(g)^{-1}\Psi_0(\mathrm{Inc}_O a^{-1}) = \eta(g)^{-1}$, as $\Phi_O(\mathrm{Inc}_O) = 1$. So the two functions are proportional, and $B(\mathbf{1}_{x,m}, \Xi) = \Psi_1(\mathrm{Inc}_O a^{-1})\, B(\Psi_0, \Xi) = 0$ by the previous paragraph but one. So $m_0 = m_1$ works.

An annihilating row where the block ratio is not integral. Suppose x has no integral block ratio at some $w_0 \in S$, and write $\mathcal{O} = \mathcal{O}_{E,w_0}$. Then there is a row $l \in \mathcal{O}^n$ with $l x_{w_0} = 0$, whose inc-part $\tilde{\lambda} = (l_{\mathrm{inc}_O(c)})_c$ has nonzero reduction $\lambda \in k_{w_0}^r$, and whose other entries lie in $\varpi_{w_0}\mathcal{O}$ if $w_0 \in Q_O$. If x'_{w_0} is singular, take a nonzero row μ with $\mu x'_{w_0} = 0$, scaled so that its largest entry has absolute value 1, and let l be μ at the inc-indices and 0 elsewhere. Otherwise let z_j be the row of $x''_{w_0} x'^{-1}_{w_0}$ of a row index j outside the image of inc_O, and $l^{(j)}$ the row with entry 1 at j, $-z_j$ at the inc-indices and 0 elsewhere; then $l^{(j)} x_{w_0} = 0$, as the row j of x_{w_0} is $z_j x'_{w_0}$. If some z_j has an entry of absolute value > 1, let c be one of largest absolute value and $l = c^{-1} l^{(j)}$: its inc-part has largest entry of absolute value 1 and its entry at j lies in $\varpi_{w_0}\mathcal{O}$. If all z_j are integral, the failure forces $w_0 \in M_O$ and some $z_j \not\equiv 0$; take $l = l^{(j)}$.

Points without an integral block ratio pair to zero. Let w_0, l, λ be as just found and $H = H(\lambda)$, with $\psi = \psi_{O,w_0}$. For $s \in H$ write $s = 1 + v\lambda$, lift v to a column $\tilde{v} \in \mathcal{O}^r$ and put $h_s = 1_n + (\mathrm{Inc}_O \tilde{v})\, l$. Then $h_s x_{w_0} = x_{w_0}$; $\det h_s = 1 + l\,\mathrm{Inc}_O \tilde{v} = 1 + \tilde{\lambda}\tilde{v}$ reduces to $1 + \lambda v = \det s \neq 0$, so $h_s \in \mathrm{GL}_n(\mathcal{O})$; and in the decomposition of the indices into the image of inc_O and the rest the reduction of h_s is $\left(\begin{smallmatrix} s & v\bar{l}'' \\ 0 & 1 \end{smallmatrix}\right)$, $\bar{l}''$ the reduction of the other entries of l. Let g_s have component h_s at w_0 and 1 at the other places of S, and k_s its lift. As $g_s x = x$ and $g_s \mathrm{Box}_m = \mathrm{Box}_m$, $k_s \cdot \mathbf{1}_{x,m} = \mathbf{1}_{x,m}$, and (6.162) gives $B(\mathbf{1}_{x,m}, R(k_s)\Xi) = B(\mathbf{1}_{x,m}, \Xi)$ for every m. If $w_0 \in Q_O$, then $\bar{l}'' = 0$, the reduction of h_s is $\mathrm{diag}(s, 1)$, so $k_s \in \mathbf{K}(O)$ and $R(k_s)\Xi = \psi(\det s)^{-1}\Xi$. If $w_0 \in M_O$, let $d_s \in \mathbf{K}_0(n)$ have reduction $\mathrm{diag}(s, 1)$ at w_0 and 1 at the other places of S, so that $[s]_{w_0}\Xi = \psi(\det s) R(d_s)\Xi$ by (DP5); the reductions of $d_s^{-1}k_s$ are $\left(\begin{smallmatrix} 1 & s^{-1}v\bar{l}'' \\ 0 & 1 \end{smallmatrix}\right)$ at w_0, which fixes Inc_O, and 1 at the other places, so $d_s^{-1}k_s \in \mathbf{K}(O)$ with $\tilde{\eta}_O = 1$ (η_O reads only the places of Q_O), and $R(k_s)\Xi = R(d_s)\Xi = \psi(\det s)^{-1}[s]_{w_0}\Xi$. Summing over $s \in H$, by additivity in the second argument,

$$\#H \cdot B(\mathbf{1}_{x,m}, \Xi) = B\Big(\mathbf{1}_{x,m}, \sum_{s \in H} R(k_s)\Xi\Big).$$

For $w_0 \in M_O$ the sum is $\sum_s \psi(\det s)^{-1}[s]_{w_0}\Xi = 0$ by (VT2). For $w_0 \in Q_O$ it is $\big(\sum_{s \in H} \psi(\det s)^{-1}\big)\Xi$, and this coefficient is 0: as $\lambda \neq 0$, $v \mapsto 1 + v\lambda$ is a bijection from the columns with $1 + \lambda v \neq 0$ onto H (applying $1 + v\lambda$ to a vector e with $\lambda e = 1$ recovers v), $\det(1 + v\lambda) = 1 + \lambda v$, and $v \mapsto \lambda v$ is a surjective linear map $k_{w_0}^r \to k_{w_0}$ with fibres of $(\mathrm{N}w_0)^{r-1}$ elements, so the coefficient is $(\mathrm{N}w_0)^{r-1} \sum_{t \in k_{w_0}^\times} \psi(t)^{-1} = 0$ by (VT1). As $\#H \geq 1$, $B(\mathbf{1}_{x,m}, \Xi) = 0$ for every m, and $m_0 = 0$ works.

Every x falls under one of the two cases, which proves the claim and the lemma. □

Lemma 6.6.85. *Let W be a space with index set r_W, V a standard definite space with index set n, a an automorphic character of* U(1)*, A a finite set of finite places of E with $A \cap \bar{A} = \emptyset$, and*

$\Sigma = A \cup \bar{A}$. *Fix a uniformiser* ϖ_w *of* E_w *for* $w \in A$. *For an* $n \times r_W$ *matrix* $x = (x_w)_{w\in A}$ *over* $\prod_{w\in A} E_w$ *and an integer* $m \geq 0$ *let* $\mathbf{1}_{x,m}$ *be the function on finite adelic* $n \times r_W$ *matrices* X *equal to* 1 *if* $X_w - x_w \in \varpi_w^m \mathrm{M}_{n\times r_W}(\mathcal{O}_{E,w})$ *for every* $w \in A$, *and* 0 *otherwise. A datum* $\varphi \in \mathcal{M}_{W,V}$ *is* the vacuum at the finite places outside Σ *if* $\rho_f(z)\varphi = \varphi$ *for every integral finite adelic* $n \times r_W$ *matrix* z *with* $z_w = 0$ *for all* $w \in \Sigma$. *Let* $Y\colon \mathrm{U}(V)(\mathbb{A}_F) \to \mathbb{C}$ *be continuous, left invariant under* $\mathrm{U}(V)(F)$ *and right invariant under an open subgroup of* $\mathrm{U}(V)(\mathbb{A}_{F,f})$, *and for* $\varphi \in \mathcal{M}_{W,V}$ *let*

$$D_a(\varphi)(g) = \int_{[\mathrm{U}(V)]} \theta_{a,1}^{W,V}(\varphi)(g,h)\, Y(h)\, dh \qquad (g \in \mathrm{U}(W)(\mathbb{A}_F))$$

be the descent *of* Y *by* φ. *Then:*

(i) $\mathbf{1}_{x,m} \in \mathcal{T}_A$ *for every* x *and* m;

(ii) if for every polynomial P *in the Fock variables of* $\mathrm{Hom}(W,V)$ *and every* x *there is* m_0 *with* $D_a(v_P \otimes (\mathbf{1}_{x,m} \otimes_A v^0)) = 0$ *for all* $m \geq m_0$, *then* $D_a(\varphi) = 0$ *for every datum* φ *that is the vacuum at the finite places outside* Σ.

Proof. Put $\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$, $Q_A = \prod_{w\in A} \mathrm{M}_{n\times r_W}(E_w)$ and $L_A = \prod_{w\in A} \mathrm{M}_{n\times r_W}(\mathcal{O}_{E,w})$.

(i). By (1) of Lemma 4.1.3, $\mathcal{T}_A$ is the space of locally constant compactly supported functions on Q_A; $\mathbf{1}_{x,m}$, the indicator of the compact open set $x + \varpi^m L_A$, is one of them.

Two Heisenberg facts. (h1) By (WD7), for vectors u, u' of the theta model,

$$(\rho(u)\rho(u')f)(x) = \beta(u,x)\beta(u',x)\beta(u',u)^{-1} f(x-u-u'),$$

so $\rho(u)\rho(u') = \beta(u,u')\beta(u',u)^{-1}\rho(u')\rho(u)$; if u and u' have no common nonzero component, $\beta(u,u') = \beta(u',u) = 1$ and they commute, and then also $\rho_f(x_1+x_2) = \rho_f(x_1)\rho_f(x_2)$ for finite x_1, x_2 with disjoint supports (the phases of (WD8) multiply). (h2) For finite z, y, $\rho_f(z)\rho_f(y) = c(z,y)\rho_f(y)\rho_f(z)$ with

$$c(z,y) = \psi_0((\bar{y}^{\mathsf{T}} z - \bar{z}^{\mathsf{T}} y)/2) = \psi_0(\bar{y}^{\mathsf{T}} z).$$

By Lemma 4.1.3, $(\jmath\alpha\bar{\alpha})\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1}\widehat{\mathcal{O}}_E$, so at every finite place w the annihilator of $\mathcal{O}_{E,w}^N$ under $(u,v) \mapsto \psi_0(\bar{u}^{\mathsf{T}} v)$ (read at w) is $\mathcal{O}_{E,w}^N$.

(ii). Let Z be the compact group of the $z \in \widehat{\mathcal{O}}_E^N$ with $z_w = 0$ for $w \in \Sigma$, with its Haar probability measure dz, and $\mathcal{V} \subseteq \mathcal{M}_{W,V}$ the data that are the vacuum at the finite places outside Σ, i.e. fixed by every $\rho_f(z)$, $z \in Z$. Fix $g \in \mathrm{U}(W)(\mathbb{A}_F)$ and put $L(\varphi) = D_a(\varphi)(g)$.

L is a bounded linear form. By Theorem A.4.5, $[\mathrm{U}(V)]$ is compact of finite volume and Y is bounded; choose a compact K with $\mathrm{U}(V)(F)K = \mathrm{U}(V)(\mathbb{A}_F)$. By (2) of Lemma 4.1.1, $h \mapsto \omega_{a,1}(g,h)\varphi$ is continuous for each φ, so by the uniform boundedness principle $M_g = \sup_{h\in K} \|\omega_{a,1}(g,h)\| < \infty$; by Lemma 6.2.5, $|\theta_{a,1}^{W,V}(\varphi)(g,\gamma h)| = |\theta_{a,1}^{W,V}(\varphi)(g,h)|$ for rational γ, so $|\theta_{a,1}^{W,V}(\varphi)(g,h)| \leq \|\theta\| M_g \|\varphi\|$ for all h, θ the theta functional of $\mathrm{Hom}(W,V)$. Hence $|L(\varphi)| \leq \mathrm{vol}([\mathrm{U}(V)]) \sup|Y|\, \|\theta\| M_g \|\varphi\|$.

The averaged form. Each $\rho_f(z)$ is an isometry of $\mathcal{M}_{W,V}$ ($|\beta| = 1$ and μ is translation invariant), and $z \mapsto \rho_f(z)\varphi$ is continuous (continuity of translation in L^1, and dominated convergence for the factor β). So $L'(\varphi) = \int_Z L(\rho_f(z)\varphi)\, dz$ is a bounded linear form, and $L' = L$ on $\mathcal{V}$. It suffices to show $L' = 0$, and by Lemma 6.3.30 (for the theta model of $\mathrm{Hom}(W,V)$ and $Y = \mathbb{C}$) it suffices that $L'(v_P \otimes \rho_f(y)v^0) = 0$ for every polynomial P in the Fock variables and every $y \in \mathbb{A}_{E,f}^N$.

Evaluation on the test data. For $z \in Z$, $\mathrm{fv}(z)$ is finite and $\mathrm{rv}(\cdot)$ infinite, so by (h1) $\rho_f(z)$ commutes with every $\rho(\mathrm{rv}(\cdot))$ and hence with $v \otimes (\cdot)$ (a Bochner integral of these operators, (WR6)). By (h2) and (5) of Lemma 6.2.33 ($\rho_f(z)v^0 = v^0$),

$$L'(v_P \otimes \rho_f(y)v^0) = \Big(\int_Z c(z,y)\,dz\Big)\, L(v_P \otimes \rho_f(y)v^0).$$

Write $y = y_\Sigma + y'$ with y_Σ supported on Σ and y' off Σ. As $c(z,y) = \psi_0(\bar{y}'^{\mathsf{T}} z)$ for $z \in Z$, the integral of this character of Z is 0 unless it is trivial; it is trivial only if $y'_w \in \mathcal{O}_{E,w}^N$ for every finite $w \notin \Sigma$ (test with z supported at one place, and use the annihilator above). Then $y' \in \widehat{\mathcal{O}}_E^N$, and by (h1) and (5) of Lemma 6.2.33, $\rho_f(y)v^0 = \rho_f(y_\Sigma)\rho_f(y')v^0 = \rho_f(y_\Sigma)v^0$. By (WD8), (1) and (2) of Lemma 4.1.3 and (2) of Lemma 6.2.33 ($v^0 = P^A v^0 = \mathbf{1}_A \otimes_A v^0$),

$$\rho_f(y_\Sigma)v^0 = \mathrm{ph}\,\rho(\mathrm{fv}(y_\Sigma))(\mathbf{1}_A \otimes_A v^0) = \mathrm{ph}\,\big(R_A(y_\Sigma)\mathbf{1}_A\big) \otimes_A v^0, \qquad |\mathrm{ph}| = 1,$$

with $\Psi = R_A(y_\Sigma)\mathbf{1}_A \in \mathcal{T}_A$ (R_A the operator of Lemma 4.1.3). Being locally constant with compact support, Ψ is constant on the cosets of $\varpi^{m_1} L_A$ for some m_1 and vanishes off finitely many of them; for $m \geq m_1$ each such coset is the disjoint union of finitely many sets $x + \varpi^m L_A$, so $\Psi = \sum_k c_k \mathbf{1}_{x_k,m}$ with finitely many x_k independent of $m \geq m_1$. By linearity of $\Psi \mapsto v_P \otimes (\Psi \otimes_A v^0)$ and the hypothesis (taking $m \geq m_1$ and $m \geq m_0(P, x_k)$ for all k),

$$L(v_P \otimes \rho_f(y)v^0) = \mathrm{ph} \sum_k c_k\, D_a(v_P \otimes (\mathbf{1}_{x_k,m} \otimes_A v^0))(g) = 0.$$

Hence $L' = 0$, and $D_a(\varphi)(g) = L(\varphi) = L'(\varphi) = 0$ for every $\varphi \in \mathcal{V}$ and every g. □

Lemma 6.6.86. *For a space W and an integer $c \geq 0$ let $W \oplus \mathbb{H}^c$ be the iterated hyperbolic sum: $W \oplus \mathbb{H}^0 = W$ and $W \oplus \mathbb{H}^{i+1} = (W \oplus \mathbb{H}^i) \oplus \mathbb{H}$. For a finite set Σ of finite places, say that the descents of Y to a space U twisted by ν* vanish off Σ *if the descent of Y to $\mathrm{U}(U)$ twisted by ν is 0 for every datum on $\mathrm{Hom}(U,V)$ that is the vacuum at every finite place outside Σ.*

Let W be a space of rank r_W, $c \geq 0$ an integer, and V a standard definite space of rank n with $2r_W + 2c + 3 < n$. Let ν be an automorphic character of $\mathrm{U}(1)$, Σ_0 a finite set of finite places with $\Sigma_0 \cap \bar{\Sigma}_0 = \emptyset$, $\Sigma = \Sigma_0 \cup \bar{\Sigma}_0$, and Y a continuous function on $\mathrm{U}(V)(\mathbb{A}_F)$, left invariant under $\mathrm{U}(V)(F)$ and right invariant under an open subgroup of $\mathrm{U}(V)(\mathbb{A}_{F,f})$. Suppose that the descents of Y to W twisted by ν vanish off Σ. Suppose that there are a finite place $w \notin \Sigma$ with $w \neq \bar{w}$ and a number $X \in \mathbb{C}$ such that

- *the base changes $\tilde{\nu}$ and $\tilde{\upsilon}$ of ν and of υ are unramified at $\bar{w}$;*
- *Y is right invariant under K_w and $H_w^V(X)\,Y = 0$;*
- *$|X| = \big(q^{r_W/2}\, q^{n - r_W - 1}\big)^{-1}$, where $q = \mathrm{N}w$.*

Then the descents of Y to $W \oplus \mathbb{H}^c$ twisted by $\nu\upsilon^c$ vanish off Σ.

Proof. For $0 \leq i \leq c$ put $W_i = W \oplus \mathbb{H}^i$, a space of rank $r_W + 2i$, and $\nu_i = \nu\upsilon^i$. We prove by induction on $i \leq c$ the claim (V_i): the descents of Y to W_i twisted by ν_i vanish off Σ. Since $W_0 = W$ and $\nu_0 = \nu$, (V_0) is the hypothesis, and (V_c) is the assertion.

Let $0 \leq i < c$ and assume (V_i). Put $W' = W_{i+1} = W_i \oplus \mathbb{H}$, of rank $m' = r_W + 2i + 2$, and $k = n - m'$; then $k \geq 1$, because $m' \leq r_W + 2c < n$. By Lemma 6.6.85, applied to W', the character ν_{i+1} and the set Σ_0, it suffices to show that the descent

$$F(g) = \int_{[\mathrm{U}(V)]} \theta^{W',V}_{\nu_{i+1},1}(\varphi)(g,h)\, Y(h)\, dh$$

vanishes for every datum $\varphi = v_P \otimes \Phi \otimes \Phi^\circ$ with P a polynomial in the entries of $\mathrm{Hom}(W', V)$ at the infinite places and $\Phi = \mathbf{1}_{x,m}$ the indicator of a class at the places of Σ_0, as in (i) of Lemma 6.6.85; it lies in $\mathcal{T}_{\Sigma_0}$ and is a Schwartz–Bruhat function on $\mathrm{Hom}(W', V)\big(\prod_{u\in\Sigma_0} E_u\big)$. Fix such a φ.

F **is bounded.** Apply Lemma 6.6.87 to $U = W_i$ and the character ν_i: its hypothesis is (V_i), and $\nu_i \upsilon = \nu_{i+1}$; so F is a bounded function on $\mathrm{U}(W')(\mathbb{A}_F)$.

F **satisfies a Hecke equation.** As $w \notin \Sigma$ and Σ is stable under conjugation, neither w nor $\bar{w}$ lies in Σ_0, so φ is the vacuum at w and $\bar{w}$ in the sense of Lemma 6.6.11: for an integral finite adelic z with components 0 outside $\{w, \bar{w}\}$, $\rho_f(z)$ fixes $\Phi \otimes_{\Sigma_0} v^0$ by (5) of Lemma 6.2.33, and it commutes with the operators $\rho(\mathrm{rv}(y))$ of (WR6), whose vectors have finite components 0 (the bicharacter of (WD6) is 1 on a pair of vectors with disjoint supports), hence with the insertion $v_P \otimes (\cdot)$. Base change is multiplicative, $\tilde{\nu}_{i+1} = \tilde{\nu}\,\tilde{\upsilon}^{\,i+1}$, so $\tilde{\nu}_{i+1}$ is unramified at $\bar{w}$. Apply Lemma 6.6.11 with $S = W'$, $T = V$ (so $n_S = m'$, $n_T = n = m' + k$), $a = \nu_{i+1}$ and $y = Y$: the function $F = F_Y$ is right invariant under K_w, and there are α, β with $|\alpha| = q^{m'/2}$, $|\beta| = q^{n/2}$ such that, with $X' = q^k \alpha \beta^{-1} X$,

$$0 = F_{H_w^V(X)Y} = \prod_{j=0}^{k-1} (1 - q^j \alpha X) \cdot H_w^{W'}(X')\, F;$$

the left side vanishes because $H_w^V(X)Y = 0$ and the descent is linear in the function.

The sizes. By hypothesis $|X| = q^{-(2n - r_W - 2)/2}$. For $0 \le j \le k-1$,

$$|q^j \alpha X| \le q^{\,k-1+\frac{m'}{2}-\frac{2n-r_W-2}{2}} = q^{\frac{r_W-m'}{2}} = q^{-(i+1)} < 1,$$

so every factor $1 - q^j \alpha X$ is nonzero and $H_w^{W'}(X')F = 0$. Moreover $|X'| = q^{k+\frac{m'}{2}-\frac{n}{2}}|X| = q^{k/2}|X|$, and

$$q^{m'+1}|X'| = q^{\frac{m'+r_W+4-n}{2}} = q^{\frac{2r_W+2i+6-n}{2}} \le 1,$$

because $n \ge 2r_W + 2c + 4 \ge 2r_W + 2(i+1) + 4$, by the rank hypothesis and $i + 1 \le c$.

Conclusion of the step. The adelic unitary group of W' is $G_\infty \times G_f^{(m')}$ through its frames. The function F on it is bounded, right invariant under K_w, and killed by $H_w^{W'}(X') = \sum_{i'=0}^{m'} (-X')^{i'} q^{\binom{i'}{2}} T_{w,i'}$, with $q^{m'+1}|X'| \le 1$. By Lemma 6.6.89, $F = 0$. This proves (V_{i+1}) and the lemma. □

Lemma 6.6.87. *Let U be a space, $U \oplus \mathbb{H}$ its orthogonal sum with the hyperbolic plane, V a standard definite space, a an automorphic character of $\mathrm{U}(1)$, A a finite set of finite places with $A \cap \bar{A} = \emptyset$, $\Sigma = A \cup \bar{A}$, and Y a continuous function on $\mathrm{U}(V)(\mathbb{A}_F)$, left invariant under $\mathrm{U}(V)(F)$ and right invariant under an open subgroup of $\mathrm{U}(V)(\mathbb{A}_{F,f})$. Suppose that the descent of Y to $\mathrm{U}(U)$ twisted by a vanishes for every datum on $\mathrm{Hom}(U, V)$ that is the vacuum at every finite place outside Σ. Then for every polynomial P in the Fock variables of $\mathrm{Hom}(U \oplus \mathbb{H}, V)$ and every $\Phi \in \mathcal{T}_A$, the descent*

$$g \longmapsto \int_{[\mathrm{U}(V)]} \theta_{a\upsilon,1}^{U\oplus\mathbb{H},V}(v_P \otimes \Phi \otimes \Phi^\circ)(g,h)\, Y(h)\, dh$$

is a bounded function on $\mathrm{U}(U \oplus \mathbb{H})(\mathbb{A}_F)$.

Proof. Write $W' = U \oplus \mathbb{H}$, with signs s' on $n' = n_U \sqcup \{1, 2\}$ (the two indices of $\mathbb{H}$ last, of signs $+1$, -1), $V = \mathrm{I}_n$ (point 1, so $\mathrm{G}(1_n) = \{h : h^* h = 1\}$), $\varphi = v_P \otimes (\Phi \otimes_A v^0)$, $\omega = \omega_{a\upsilon,1}$ on $\mathcal{M}_{W',V}$

and $D(g)$ the descent of the statement. Call a datum *the vacuum off* Σ if it is the vacuum at the finite places outside Σ. By Theorem A.4.5, $[\mathrm{U}(V)]$ is compact of finite volume and Y is bounded.

Step 0: the splitting. Let $l = e_1$, $k = e_2$ (the basis vectors of $\mathbb{H}$) and i the inclusion of E^{n_U}. Then $k^*J_{W'}k = l^*J_{W'}l = 0$, $l^*J_{W'}k = 1$, $i^*J_{W'}i = J_U$, $i^*J_{W'}k = i^*J_{W'}l = 0$: (k, l, i) is a splitting of W' over U along a one-element set. It is base-adapted: $(l_{\mathbb{A}}\ k_{\mathbb{A}}\ i_{\mathbb{A}})$ is, up to the order of the columns, block diagonal with blocks $x_{\mathbb{H}}^{-1}$ and 1, $x_{\mathbb{H}}$ the point of $\mathbb{H}$ ((WR1) of Definition 3.2.18), whose real component is unitary and whose finite component is integral of unit determinant (the unit of (WD9) times $c_0 + \bar{c}_0 = 1$). Consider the induced splitting of $\mathrm{Hom}(W', V)$ over $\mathrm{Hom}(U, V)$ along $p_1 = n \times \{1\}$, with zero term Z and adelic frames $\kappa_{\mathbb{A}}, \lambda_{\mathbb{A}}, \iota_{\mathbb{A}}$ (the induced k, l, i). As $x_V = 1$, $J_V = 1$, they act on adelic matrices by $\kappa_{\mathbb{A}}b = b\,\mathrm{pr}_l$, $\lambda_{\mathbb{A}}b = b\,\mathrm{pr}_k$, $\iota_{\mathbb{A}}w = w\,\mathrm{pr}_i$. Hence the slices of the space $\mathrm{Hom}(W', V)$ along the induced splitting are the Hom slices, and the frame matrix of the induced splitting is $1_n \otimes M^{\mathsf{T}}$, where M has row blocks pr_k, pr_l, pr_i, i.e. $M = \mathrm{diag}(1, 1, \varepsilon_{s_U})(l_{\mathbb{A}}\ k_{\mathbb{A}}\ i_{\mathbb{A}})^*\varepsilon_{s'}$: unitary at the infinite places and integral at the finite places (where $\varepsilon = 1$); so the induced splitting is base-adapted, and at the finite places $\iota_{\mathbb{A}}$ is integral with $\iota_{\mathbb{A}}^*\iota_{\mathbb{A}} = 1$ (from $i^*J_{W'}i = J_U$).

Let $\theta' = \theta_{\mathrm{Hom}(W',V)}$, $\theta_0 = \theta_{\mathrm{Hom}(U,V)}$ and $\ell = \theta' - \theta_0 \circ Z$, a bounded linear form ((7) of Lemma 4.1.1). For $F \in \mathcal{M}_{W',V}$:

(z1) $\theta_0(Z(F)) = \int_{\mathbb{A}_E^{p_1}/E^{p_1}} \theta'(\rho(\kappa_{\mathbb{A}}b)F)\,db$, by (MX4) at $y = 0$;

(z2) $Z(\rho(\kappa_{\mathbb{A}}b)F) = Z(F)$ for adelic b, and $Z(\rho(\iota_{\mathbb{A}}w)F) = \rho(w)Z(F)$, by (1) of Lemma 6.2.44 at $x = 0$ for the induced splitting;

(z3) $\theta'(\rho(\kappa_{\mathbb{A}}b)F) = \theta'(F)$ for $b \in E^{p_1}$, since $\kappa_{\mathbb{A}}b$ is then a rational vector of $\mathrm{Hom}(W', V)$ read through its point (Lemma 4.1.17);

(z4) if F is the vacuum off Σ, so is $Z(F)$: for w integral with components 0 at Σ, $w' = \iota_{\mathbb{A},f}w$ is integral with components 0 at Σ and $\bar{w}'^{\mathsf{T}}w' = \bar{w}^{\mathsf{T}}w$, so $\rho_f(w)$ and $\rho_f(w')$ carry the same phase (WD8), and by (z2) $\rho_f(w)Z(F) = Z(\rho_f(w')F) = Z(F)$.

By (z2), (z3), $\ell(\rho(\kappa_{\mathbb{A}}b)F) = \ell(F)$ for $b \in E^{p_1}$, and by (z1), (z2), $\int_{\mathbb{A}_E^{p_1}/E^{p_1}} \ell(\rho(\kappa_{\mathbb{A}}b)F)\,db = 0$.

Step 1: the zero term contributes nothing. We show $\int_{[\mathrm{U}(V)]} \theta_0(Z(\omega(g, h)\varphi))Y(h)\,dh = 0$ for every $g \in \mathrm{G}(s')$. The integrand is continuous and left invariant under the rational points: by (2) of Lemma 6.2.45 with the element 1 of $\mathrm{G}(s')$, $Z(\omega(1, \gamma)F) = \omega_{a,1}(1, \gamma)Z(F)$, and θ_0 is invariant under rational points (Lemma 6.2.5). By Lemma 6.6.88, $g = q\kappa$ with $q = \mathrm{rad}(c_1, b_1)\,\mathrm{lev}(m, g_0)$, $g_0 \in \mathrm{G}(s_U)$, and κ unitary at the infinite places and in $\mathbf{K}_0(n')$ at the finite places. By (MX2), $qk_{\mathbb{A}} = k_{\mathbb{A}}m$ and $\mathrm{pr}_i q i_{\mathbb{A}} = g_0$ ($\mathrm{rad}(c_1, b_1)$ fixes $k_{\mathbb{A}}$, as $\mathrm{pr}_l k_{\mathbb{A}} = \mathrm{pr}_i k_{\mathbb{A}} = 0$, and $\mathrm{pr}_i\,\mathrm{rad}(c_1, b_1)\,i_{\mathbb{A}} = 1$, as $\mathrm{pr}_l k_{\mathbb{A}} = \mathrm{pr}_l i_{\mathbb{A}} = 0$, $\mathrm{pr}_i k_{\mathbb{A}} = 0$). Let $\varphi' = \omega(\kappa, 1)\varphi$. It is the vacuum off Σ: φ is ((5) of Lemma 6.2.33 for $\Phi \otimes_A v^0$, and $\rho_f(z)$ commutes with the real Heisenberg operators of (WR6), vectors with disjoint supports having bicharacter 1); by (2) of Lemma 4.1.1, $\rho(z)\omega(\kappa, 1) = \omega(\kappa, 1)\rho(z\kappa)$ for vectors z of $\mathrm{Hom}(W', V)$ ($X \mapsto X\kappa^{-1}$ being the pair map), and for z integral with components 0 at Σ so is $z\kappa_f$, with $\mathrm{tr}((z\kappa_f)^*z\kappa_f) = \mathrm{tr}(z^*z)$, so the phases of (WD8) agree and $\rho_f(z)\varphi' = \omega(\kappa, 1)\rho_f(z\kappa_f)\varphi = \varphi'$. By (2) of Lemma 6.2.45 at $x = 0$ (characters $(a, 1)$, so $av^{\#p} = av$),

$$Z(\omega(g, h)\varphi) = Z(\omega(q, h)\varphi') = c_m\,\omega_{a,1}(g_0, h)\,Z(\varphi'), \qquad c_m = v(\det m)^n|\det m|^{n/2}.$$

So the integral is $c_m\,D_a(Z(\varphi'))(g_0)$, the descent of Y to U by $Z(\varphi')$, which is the vacuum off Σ by (z4); it vanishes by hypothesis.

Step 2: a uniform bound for ℓ. Let $g \in \mathrm{G}(s')$ and $H = \mathrm{ht}_{W'}(g)$. If $H > 2$, Lemma 6.2.30 gives a nonzero u in the lattice of W'_g with $u^* J_{W'} u = 0$ and length at most $2/H$; as k is isotropic and nonzero, $Eu \to E^{n'}$, $u \mapsto k$, is an injective isometry, and Witt's theorem A.6.7 gives $\gamma \in \mathrm{U}(W')(F)$ with $\gamma u = k$. If $H \le 2$ put $\gamma = 1$. Let $\gamma_{\mathbb{A}} = x_{W'}^{-1} \gamma x_{W'}$ and $g' = \gamma_{\mathbb{A}} g$. As $x_{W'} g' = \gamma x_{W'} g$, $u \mapsto \gamma u$ carries the lattice of W'_g onto that of $W'_{g'}$ preserving lengths; so $\mathrm{ht}_{W'}(g') = H$, and if $H > 2$, k lies in the lattice of $W'_{g'}$ with length at most $2/H$.

The heights on V are 1: for $h \in \mathrm{G}(1_n)$ and a nonzero u in the lattice of V_h, $u^* u \in F$ is totally positive and, h being unitary, equal at the finite places to the integral value $(h^{-1}u)^*(h^{-1}u)$; so $u^*u \in \mathcal{O}_F \setminus \{0\}$ and the squared length is $\mathrm{Tr}_{F/\mathbb{Q}}(u^*u) \ge [F : \mathbb{Q}] \ge 1$. By (VP3) of Definition 3.2.2 V is a framed hermitian space (a framed space of rank $\#n$ in the sense of (FT1)) and $\lambda_V(h)$ is the first minimum of (VP4) for both readings; by what was just shown $\lambda_V(h) = 1$, and Lemma 5.8.21 at h gives $d \in \mathrm{GL}_n(E)$ whose columns lie in the lattice of V_h and have lengths at most $Z_0 = f\, n\, |d_E|$, $f = [F : \mathbb{Q}]$.

We apply (2) of Lemma 6.2.31 with $S = W'$, $T = V$, the characters $(av, 1)$, the set A, $\Psi = \Phi$ (so $f_0 = \Phi \otimes_A v^0$ and $v_P \otimes f_0 = \varphi$), the index sets n and $\{1\}$, $N = 0$, $r = 1$, the functional ℓ and the points (g', h). Its hypotheses: (a) by Lemma 6.2.28 for (W', V) there are Y_1, A_1 with $|\theta'(\omega(g', h)F)| \le Y_1 \|F\| H^{A_1}$ (as $\mathrm{ht}_V(h) = 1$); by (z1), $|\theta_0(Z\omega(g', h)F)| \le \sup_b |\theta'(\rho(\kappa_{\mathbb{A}} b)\omega(g', h)F)|$, and $\rho(\kappa_{\mathbb{A}} b)\omega(g', h) = \omega(g', h)\rho(z_b)$ for a vector z_b ((2) of Lemma 4.1.1), $\rho(z_b)$ an isometry; so $|\ell(\omega(g', h)F)| \le 2Y_1 \|F\| H^{A_1}$. (b) If $H > 2$, (d, k) is a short pair for $(g', h, \ell, Z_0, 2/H)$, and $Z_0 \cdot 2/H = 2Z_0 r/H$: (i) holds by the choices of d and γ; (ii) $k^* J_{W'} k = 0$; for (iii), (iv), the group $Z_{d,k}(R) = \{dYk^* J_{W'} x_{W'} : Y \in \mathrm{M}_{n\times 1}(R)\}$ is $\{\kappa_{\mathbb{A}}(dY)\}$, so (iii) is the invariance of ℓ under $\rho(\kappa_{\mathbb{A}} b)$, $b \in E^{p_1}$, and, $Y \mapsto dY$ inducing an automorphism of the compact group $\mathbb{A}_E^{p_1}/E^{p_1}$ that preserves its Haar probability measure, the mean in (iv) is $\int \ell(\rho(\kappa_{\mathbb{A}} b)F)\, db = 0$ (end of Step 0). As $\mathrm{mom}_{A_1}(v_P) < \infty$ (XD3), we obtain, with $K = C' \mathrm{mom}_{A_1}(v_P)$ independent of g and h,

$$\big|\ell(\omega(g', h)\varphi)\big| \le K.$$

Step 3: conclusion. By Lemma 6.2.5, $\theta^{W',V}_{av,1}(\varphi)(g', h) = \theta^{W',V}_{av,1}(\varphi)(g, h)$, so $D(g) = D(g')$. Writing $\theta' = \ell + \theta_0 \circ Z$ and using Step 1 at g',

$$D(g) = \int_{[\mathrm{U}(V)]} \ell(\omega(g', h)\varphi) Y(h)\, dh, \qquad |D(g)| \le K \sup|Y| \,\mathrm{vol}([\mathrm{U}(V)])$$

by Step 2, for every g. □

Lemma 6.6.88. *Let V, V_0 be spaces with points with signs s on n and s_0 on n_0, p a nonempty index set and (k, l, i) a base-adapted splitting of V over V_0 along p. Let $K_\infty(s)$ be the subgroup of $G_\infty(s)$ of the elements whose component at every infinite place is a unitary matrix. Then every $g \in \mathrm{G}(s)$ can be written*

$$g = \mathrm{rad}(c, b)\, \mathrm{lev}(m, g_0)\, \kappa$$

with an adelic $n_0 \times p$ matrix c, an adelic $p \times p$ matrix b, $m \in \mathrm{GL}_p(\mathbb{A}_E)$, $g_0 \in \mathrm{G}(s_0)$ and $\kappa \in K_\infty(s) \times \mathbf{K}_0(n)$.

Proof. Write $k_{\mathbb{A}}, l_{\mathbb{A}}, i_{\mathbb{A}}$ for the adelic frames and $M = (l_{\mathbb{A}} \mid k_{\mathbb{A}} \mid i_{\mathbb{A}})$. As $x_V^* J_V x_V = \varepsilon_s$, the relations of (MX1) give the Gram matrices

$$k_{\mathbb{A}}^* \varepsilon_s k_{\mathbb{A}} = 0, \quad l_{\mathbb{A}}^* \varepsilon_s l_{\mathbb{A}} = 0, \quad l_{\mathbb{A}}^* \varepsilon_s k_{\mathbb{A}} = 1, \quad i_{\mathbb{A}}^* \varepsilon_s i_{\mathbb{A}} = \varepsilon_{s_0}, \quad i_{\mathbb{A}}^* \varepsilon_s k_{\mathbb{A}} = 0, \quad i_{\mathbb{A}}^* \varepsilon_s l_{\mathbb{A}} = 0, \quad (6.164)$$

and every adelic vector is $x = l_{\mathbb{A}} \mathrm{pr}_l(x) + k_{\mathbb{A}} \mathrm{pr}_k(x) + i_{\mathbb{A}} \mathrm{pr}_i(x)$; so M is invertible, with inverse the matrix of rows $\mathrm{pr}_l = k_{\mathbb{A}}^* \varepsilon_s$, $\mathrm{pr}_k = l_{\mathbb{A}}^* \varepsilon_s$, $\mathrm{pr}_i = \varepsilon_{s_0} i_{\mathbb{A}}^* \varepsilon_s$, and pr_k is a left inverse of $k_{\mathbb{A}}$. Since

the splitting is base-adapted, M_f has entries in $\widehat{\mathcal{O}}_E$, and so has M_f^{-1}, whose rows are k_f^*, l_f^*, i_f^* ($\varepsilon_s = 1$ at the finite places); in particular k_f has the left inverse l_f^* over $\widehat{\mathcal{O}}_E$, and $k_f^* k_f = 0$.

Step 1: a compact element moves the frame into its own span. We show that there are $\kappa \in K_\infty(s) \times \mathbf{K}_0(n)$ and $m \in \mathrm{GL}_p(\mathbb{A}_E)$ with $g\kappa\, k_{\mathbb{A}} = k_{\mathbb{A}} m$.

Infinite places. Fix $v \in \Phi$, $D = \mathrm{diag}(s)$, $K_v = \mathrm{U}(D) \cap \mathrm{U}(n)$. For an $n \times p$ complex matrix X of rank p with $X^* D X = 0$ put $R_X = (X^*X)^{1/2} \in \mathrm{GL}_p(\mathbb{C})$ and $Q_X = X R_X^{-1}$; then $Q_X^* Q_X = 1$ and $Q_X^* D Q_X = 0$. Splitting the rows by sign, $Q_X^* Q_X = Q_{X,+}^* Q_{X,+} + Q_{X,-}^* Q_{X,-}$ and $Q_X^* D Q_X = Q_{X,+}^* Q_{X,+} - Q_{X,-}^* Q_{X,-}$, so $\sqrt{2}\, Q_{X,\pm}$ have orthonormal columns in $\mathbb{C}^{n_\pm}$, $n_\pm$ the number of indices of sign ± 1. Take $X_0 = (k_{\mathbb{A}})_v$ and $X = g_v^{-1}(k_{\mathbb{A}})_v$: both have rank p (pr_k is a left inverse of $k_{\mathbb{A}}$) and are isotropic for D by (6.164) and $g_v \in \mathrm{U}(D)$. Completing the orthonormal columns of $\sqrt{2}Q_{X_0,\pm}$ and $\sqrt{2}Q_{X,\pm}$ to orthonormal bases of $\mathbb{C}^{n_\pm}$ gives unitary $u_\pm$ with $u_\pm Q_{X_0,\pm} = Q_{X,\pm}$; the block diagonal matrix $\kappa_v = u_+ \oplus u_-$ (for the partition of the indices by sign) is unitary and commutes with D, so $\kappa_v \in K_v$, and $\kappa_v Q_{X_0} = Q_X$. Hence $g_v \kappa_v (k_{\mathbb{A}})_v = g_v \kappa_v Q_{X_0} R_{X_0} = g_v X R_X^{-1} R_{X_0} = (k_{\mathbb{A}})_v m_v$ with $m_v = R_X^{-1} R_{X_0} \in \mathrm{GL}_p(\mathbb{C})$.

Finite places. Let $Z = g_f^{-1} k_f$, an $n \times p$ matrix over $\mathbb{A}_{E,f}$ with the left inverse $\mathrm{pr}_{k,f}\, g_f$. For a finite place w of E, the ring $\mathcal{O}_{E,w}$ is a discrete valuation ring, so $L_w = Z_w E_w^p \cap \mathcal{O}_{E,w}^n$ is free of rank p and $\mathcal{O}_{E,w}^n / L_w$, being finitely generated and torsion free, is free; so L_w is a direct summand of $\mathcal{O}_{E,w}^n$ and $L_w = Z_w b'_w \mathcal{O}_{E,w}^p$ for some $b'_w \in \mathrm{GL}_p(E_w)$, whence $Z_w b'_w$ has a left inverse over $\mathcal{O}_{E,w}$. For all but finitely many w, $g_w \in \mathrm{GL}_n(\mathcal{O}_{E,w})$, so Z_w itself has a left inverse over $\mathcal{O}_{E,w}$ (as k_f has one) and we take $b'_w = 1$. Thus $b' = (b'_w)_w \in \mathrm{GL}_p(\mathbb{A}_{E,f})$ and $Y = Z b'$ has a left inverse over $\widehat{\mathcal{O}}_E$. As $g_f^* g_f = 1$, $Y^* Y = b'^* k_f^* k_f b' = 0 = k_f^* k_f$. By (2) of Lemma 3.2.6 for $X = k_f$ and Y there is $\kappa' \in \mathrm{GL}_n(\widehat{\mathcal{O}}_E)$ with $\kappa'^* \kappa' = 1$, i.e. $\kappa' \in \mathbf{K}_0(n)$, and $\kappa' k_f = g_f^{-1} k_f b'$; so $g_f \kappa' k_f = k_f b'$.

Put $\kappa = ((\kappa_v)_v, \kappa') \in K_\infty(s) \times \mathbf{K}_0(n)$ and $m = ((m_v)_v, b') \in \mathrm{GL}_p(\mathbb{A}_E)$; then $g\kappa k_{\mathbb{A}} = k_{\mathbb{A}} m$.

Step 2: an element stabilising the frame is a radical element times a Levi element. Let $h \in \mathrm{G}(s)$ and $m \in \mathrm{GL}_p(\mathbb{A}_E)$ with $h k_{\mathbb{A}} = k_{\mathbb{A}} m$. From $h^* \varepsilon_s = \varepsilon_s h^{-1}$ and $h^{-1} k_{\mathbb{A}} = k_{\mathbb{A}} m^{-1}$ we get $k_{\mathbb{A}}^* \varepsilon_s h = (m^{-1})^* k_{\mathbb{A}}^* \varepsilon_s$, so $\mathrm{pr}_l(h i_{\mathbb{A}}) = (m^{-1})^* k_{\mathbb{A}}^* \varepsilon_s i_{\mathbb{A}} = 0$ and

$$h i_{\mathbb{A}} = i_{\mathbb{A}} g_0 + k_{\mathbb{A}} B, \qquad g_0 = \mathrm{pr}_i(h i_{\mathbb{A}}),\ B = \mathrm{pr}_k(h i_{\mathbb{A}}).$$

By (6.164), $\varepsilon_{s_0} = (h i_{\mathbb{A}})^* \varepsilon_s (h i_{\mathbb{A}}) = g_0^* \varepsilon_{s_0} g_0$, so $g_0 \in \mathrm{G}(s_0)$. Write $h l_{\mathbb{A}} = l_{\mathbb{A}} X + i_{\mathbb{A}} Y + k_{\mathbb{A}} Z'$. Since h preserves the form, (6.164) gives

$$\begin{aligned} 1 &= (h l_{\mathbb{A}})^* \varepsilon_s (h k_{\mathbb{A}}) = X^* m, \\ 0 &= (h l_{\mathbb{A}})^* \varepsilon_s (h i_{\mathbb{A}}) = X^* B + Y^* \varepsilon_{s_0} g_0, \\ 0 &= (h l_{\mathbb{A}})^* \varepsilon_s (h l_{\mathbb{A}}) = X^* Z' + Z'^* X + Y^* \varepsilon_{s_0} Y. \end{aligned}$$

So $X = (m^{-1})^*$; with $c = \varepsilon_{s_0} (g_0^{-1})^* B^*$ (an adelic $n_0 \times p$ matrix, $\varepsilon_{s_0}^{-1} = \varepsilon_{s_0}$) the second relation reads $Y = -cX$; and with $Z = Z' X^{-1}$ the third reads $Z + Z^* = -c^* \varepsilon_{s_0} c$. Hence $Z_1 = Z + \frac{1}{2} c^* \varepsilon_{s_0} c$ satisfies $Z_1^* = -Z_1$. Let $b = -\jmath Z_1$; as $\bar{\jmath} = -\jmath$, $b + b^* = -2\jmath Z_1$, so $Z = -\frac{1}{2}(\jmath^{-1}(b + b^*) + c^* \varepsilon_{s_0} c)$. Now compare h with $q = \mathrm{rad}(c, b)\, \mathrm{lev}(m, g_0)$ on the columns of M, using pr_l, pr_k, pr_i on $l_{\mathbb{A}}, k_{\mathbb{A}}, i_{\mathbb{A}}$ ((MX1), (MX2)): $\mathrm{lev}(m, g_0)$ sends $k_{\mathbb{A}}, i_{\mathbb{A}}, l_{\mathbb{A}}$ to $k_{\mathbb{A}} m$, $i_{\mathbb{A}} g_0$, $l_{\mathbb{A}} X$, and $\mathrm{rad}(c, b)$ sends $k_{\mathbb{A}}$, $i_{\mathbb{A}}$, $l_{\mathbb{A}}$ to $k_{\mathbb{A}}$, $i_{\mathbb{A}} + k_{\mathbb{A}} c^* \varepsilon_{s_0}$, $l_{\mathbb{A}} - i_{\mathbb{A}} c + k_{\mathbb{A}} Z$. So

$$q k_{\mathbb{A}} = k_{\mathbb{A}} m, \qquad q i_{\mathbb{A}} = i_{\mathbb{A}} g_0 + k_{\mathbb{A}} c^* \varepsilon_{s_0} g_0 = i_{\mathbb{A}} g_0 + k_{\mathbb{A}} B, \qquad q l_{\mathbb{A}} = l_{\mathbb{A}} X - i_{\mathbb{A}} c X + k_{\mathbb{A}} Z X = h l_{\mathbb{A}},$$

as $c^* \varepsilon_{s_0} g_0 = B$. Since M is invertible, $h = q = \mathrm{rad}(c, b)\, \mathrm{lev}(m, g_0)$.

Conclusion. With κ, m of Step 1, Step 2 for $h = g\kappa$ gives $g\kappa = \mathrm{rad}(c, b)\, \mathrm{lev}(m, g_0)$, i.e. $g = \mathrm{rad}(c, b)\, \mathrm{lev}(m, g_0)\, \kappa^{-1}$ with $\kappa^{-1} \in K_\infty(s) \times \mathbf{K}_0(n)$. □

Lemma 6.6.89. *Let $n \geq 0$, let $G_f^{(n)} = \{g \in \mathrm{GL}_n(\mathbb{A}_{E,f}) : \bar{g}^{\mathsf{T}} g = 1\}$, let $s = (s_v)_{v\in\Phi}$ be signs $s_v \colon \{1,\dots,n\} \to \{\pm 1\}$, and let $G = G_\infty \times G_f^{(n)}$ with $G_\infty = \prod_{v\in\Phi} \mathrm{U}(\mathrm{diag}(s_v))$; the group $G_f^{(n)}$ acts on functions on G by right translation. Let w be a finite place of E with $w \neq \bar{w}$, $\mathcal{O}_w$ the integers of E_w and ϖ_w a uniformiser. For $\gamma \in \mathrm{GL}_n(E_w)$ let $\iota_w(\gamma) \in G_f^{(n)}$ be the element with component γ at w, $(\bar{\gamma}^{\mathsf{T}})^{-1}$ at $\bar{w}$ and 1 at the other finite places, and put $K_w = \iota_w(\mathrm{GL}_n(\mathcal{O}_w))$. For $0 \leq i \leq n$ let $R_i \subset \mathrm{GL}_n(E_w)$ be a set of representatives of the cosets $\mathrm{GL}_n(\mathcal{O}_w)\,\delta_i\,\mathrm{GL}_n(\mathcal{O}_w)/\mathrm{GL}_n(\mathcal{O}_w)$, where $\delta_i = \mathrm{diag}(\varpi_w 1_i, 1_{n-i})$, and for a function F on G that is right invariant under K_w let*

$$(T_{w,i}F)(g) = \sum_{\gamma\in R_i} F(g\,\iota_w(\gamma)) \qquad (g \in G).$$

Let $X \in \mathbb{C}$ with $(\mathrm{N}w)^{n+1}|X| \leq 1$, and let $F \colon G \to \mathbb{C}$ be a bounded function, right invariant under K_w, with

$$\sum_{i=0}^{n} (-X)^i\,(\mathrm{N}w)^{\binom{i}{2}}\,T_{w,i}F = 0.$$

Then $F = 0$.

Proof. Write $q = \mathrm{N}w$, so $q \geq 2$, and $K_0 = \mathrm{GL}_n(\mathcal{O}_w)$. The map ι_w is a homomorphism into $G_f^{(n)}$: complex conjugation exchanges the components at w and $\bar{w}$, and at w the product $\bar{h}^{\mathsf{T}} h$ for $h = \iota_w(\gamma)$ is $((\gamma^{\mathsf{T}})^{-1})^{\mathsf{T}}\gamma = 1$, and likewise at $\bar{w}$.

The operators and the number of their terms. If $\gamma \in \mathrm{GL}_n(E_w)$ and $k \in K_0$, then $F(g\,\iota_w(\gamma k)) = F(g\,\iota_w(\gamma)\,\iota_w(k)) = F(g\,\iota_w(\gamma))$ by the invariance of F, so $T_{w,i}F$ does not depend on the choice of R_i. Since $\delta_0 = 1$, we may take $R_0 = \{1\}$, so $T_{w,0}F = F$. We show that $\#R_i$ is the number $\binom{n}{i}_q$ of i-dimensional subspaces of k_w^n. The map $\gamma K_0 \mapsto L = \gamma\mathcal{O}_w^n$ is well defined and injective on cosets, since $\gamma\mathcal{O}_w^n = \gamma'\mathcal{O}_w^n$ exactly when $\gamma^{-1}\gamma' \in K_0$. If $\gamma = k\delta_i k'$ with $k, k' \in K_0$, then $L = k\delta_i\mathcal{O}_w^n$ satisfies $\varpi_w\mathcal{O}_w^n \subseteq L \subseteq \mathcal{O}_w^n$ with $\dim_{k_w} L/\varpi_w\mathcal{O}_w^n = n - i$, because $\delta_i\mathcal{O}_w^n$ does and k preserves $\mathcal{O}_w^n$ and $\varpi_w\mathcal{O}_w^n$. Conversely let L be an $\mathcal{O}_w$-module with these properties. It is free of rank n, being finitely generated and torsion free over the discrete valuation ring $\mathcal{O}_w$ and containing $\varpi_w\mathcal{O}_w^n$; so $L = \gamma\mathcal{O}_w^n$ for some $\gamma \in \mathrm{GL}_n(E_w)$. By the elementary divisor theorem over $\mathcal{O}_w$, $\gamma = k\,\mathrm{diag}(\varpi_w^{a_1},\dots,\varpi_w^{a_n})\,k'$ with $k, k' \in K_0$ and $a_l \in \mathbb{Z}$; then $L \subseteq \mathcal{O}_w^n$ gives $a_l \geq 0$, $\varpi_w\mathcal{O}_w^n \subseteq L$ gives $a_l \leq 1$, and $\dim L/\varpi_w\mathcal{O}_w^n = \#\{l : a_l = 0\} = n - i$ shows that exactly i of the a_l equal 1. Conjugating by a permutation matrix, which lies in K_0, puts γ in $K_0\delta_i K_0$. Hence the cosets of R_i correspond to the modules L, and these correspond, by $L \mapsto L/\varpi_w\mathcal{O}_w^n$, to the subspaces of dimension $n - i$ of $k_w^n = \mathcal{O}_w^n/\varpi_w\mathcal{O}_w^n$. Their number is $\binom{n}{n-i}_q = \binom{n}{i}_q$.

The total weight of the relation is below 2. By Gauss's q-binomial theorem,

$$\sum_{i=0}^{n} q^{\binom{i}{2}} \binom{n}{i}_q y^i = \prod_{j=0}^{n-1} (1 + q^j y)$$

for every y. At $y = |X|$, using $1 + u \leq e^u$,

$$\sum_{i=0}^{n} |X|^i q^{\binom{i}{2}} \#R_i = \prod_{j=0}^{n-1} (1 + q^j|X|) \leq \exp\Bigl(|X| \sum_{j=0}^{n-1} q^j\Bigr) \leq \exp(q^n|X|) \leq e^{1/2} < 2.$$

So $\theta = \sum_{i=1}^{n} |X|^i q^{\binom{i}{2}} \#R_i < 1$.

Conclusion. Let $S = \sup_{g\in G} |F(g)|$, finite since F is bounded. Evaluating the relation at $g \in G$ and using $T_{w,0}F = F$,

$$F(g) = -\sum_{i=1}^{n} (-X)^i q^{\binom{i}{2}} \sum_{\gamma\in R_i} F(g\,\iota_w(\gamma)), \qquad \text{so} \qquad |F(g)| \leq \theta S.$$

Taking the supremum over g gives $S \le \theta S$, hence $(1-\theta)S \le 0$ and $S = 0$, since $\theta < 1$ and $S \ge 0$. So $F = 0$. □

Lemma 6.6.90. *Let W be a space with signs ε_W, $n \ge 1$, and $\{1,\dots,n\} = N \sqcup l$ a decomposition of the index set of I_n, so that $\mathrm{I}_n = \mathrm{I}_N \oplus \mathrm{I}_l$ with the standard definite spaces on these index sets; let $\iota\colon \mathrm{G}(1_N) \times \mathrm{G}(1_l) \to \mathrm{G}(1_n)$ be the block-diagonal embedding. Let A be a finite set of finite places with $A \cap \bar{A} = \emptyset$, $\Sigma = A \cup \bar{A}$, and a an automorphic character of $\mathrm{U}(1)$ with $a(\det k) = 1$ for every $k \in \mathbf{K}_0(n)$ whose components at the places of Σ are 1. Let $Y\colon \mathrm{G}(1_n) \to \mathbb{C}$ be continuous, left invariant under the rational points, and right invariant under every $k \in \mathbf{K}_0(n)$ whose components at the places of Σ are 1. Suppose that the descents of Y to $W \oplus \mathbb{H}^{\#N}$ twisted by $a\upsilon^{\#N}$ vanish off Σ. Then for every automorphic character b of $\mathrm{U}(1)$, every datum $\varphi \in \mathcal{M}_{W,\mathrm{I}_l}$ that is the vacuum at the finite places outside Σ, every $g \in \mathrm{G}(\varepsilon_W)$ and every $h_1 \in \mathrm{G}(1_N)$,*

$$\int_{[\mathrm{U}(\mathrm{I}_l)]} \theta_{a,b}^{W,\mathrm{I}_l}(\varphi)(g,h_2)\, Y\big(\iota(h_1,h_2)\big)\, dh_2 = 0.$$

Proof. Write $c = \#N$, and call a datum *the vacuum off* Σ if it is the vacuum at the finite places outside Σ. By the associativity of orthogonal sums ((VP5) of Definition 3.2.2), $W \oplus \mathbb{H}^c$ is the orthogonal sum of W and $\mathbb{H}^c = 0 \oplus \mathbb{H}^c$, with the concatenated index sets. Fix b, φ, g and h_1 as in the statement. All integrals below are over the compact quotients $[\mathrm{U}(\mathrm{I}_m)]$ of continuous functions.

Step 1: a complementary datum. By (iii) of Lemma 6.6.91 (for the pair (W, I_N) and the characters a, b^{-1}) there is $\varphi'' \in \mathcal{M}_{W,\mathrm{I}_N}$, the vacuum off Σ, with $K'' = \theta_{a,b^{-1}}^{W,\mathrm{I}_N}(\varphi'')(g,h_1) \ne 0$. Let $\Phi = \varphi'' \boxtimes^{\mathrm{tg}} \varphi \in \mathcal{M}_{W,\mathrm{I}_n}$ (target product for $\mathrm{I}_n = \mathrm{I}_N \oplus \mathrm{I}_l$); by (ii) of Lemma 6.6.91 it is the vacuum off Σ.

Step 2: the function Y_2. Put $\vartheta(h) = \theta_{a,1}^{W,\mathrm{I}_n}(\Phi)(g,h)$ and $Y_2 = \vartheta\, Y$ on $\mathrm{G}(1_n)$. It is continuous ($\omega_{a,1}$ is continuous and θ bounded, (2) of Lemma 4.1.1). It is left invariant under Γ_{I_n}: ϑ by Lemma 6.2.5, Y by hypothesis. It is right invariant under every $k \in \mathbf{K}_0(n)$ with components 1 at Σ: by (i) of Lemma 6.6.91, $\vartheta(hk) = \theta(\omega_{a,1}(g,h)\,\omega_{a,1}((1,1),(1,k))\Phi) = a(\det k)\vartheta(h) = \vartheta(h)$ by the hypothesis on a, and $Y(hk) = Y(h)$. The descents of Y_2 to $\mathbb{H}^c$ twisted by υ^c vanish off Σ: let $\varphi' \in \mathcal{M}_{\mathbb{H}^c,\mathrm{I}_n}$ be the vacuum off Σ and $g' \in \mathrm{U}(\mathbb{H}^c)(\mathbb{A}_F)$. The source product $\Phi \boxtimes^{\mathrm{sc}} \varphi' \in \mathcal{M}_{W\oplus\mathbb{H}^c,\mathrm{I}_n}$ is the vacuum off Σ ((ii) of Lemma 6.6.91), and by the exact factorisation (d) of Lemma 6.3.52 for source sums (characters $a\upsilon^c$ on the target, 1 on the source)

$$\theta_{a\upsilon^c,1}^{W\oplus\mathbb{H}^c,\mathrm{I}_n}\big(\Phi \boxtimes^{\mathrm{sc}} \varphi'\big)\big(\mathrm{diag}(g,g'),h\big) = \vartheta(h)\,\theta_{\upsilon^c,1}^{\mathbb{H}^c,\mathrm{I}_n}(\varphi')(g',h).$$

Integrating against $Y(h)$, the descent of Y_2 to $\mathbb{H}^c$ by φ' at g' is the descent of Y to $W \oplus \mathbb{H}^c$ twisted by $a\upsilon^c$ by $\Phi \boxtimes^{\mathrm{sc}} \varphi'$ at $\mathrm{diag}(g,g')$, which is 0 by hypothesis.

Step 3: the period. By Lemma 6.6.92 for $n_1 = N$, $n_2 = l$ and Y_2, $\int_{[\mathrm{U}(\mathrm{I}_l)]} Y_2(\iota(h_1,h_2))\, dh_2 = 0$. By the exact factorisation (d) of Lemma 6.3.52 for target sums (character a on the target, $1 = b^{-1}b$ on the source),

$$\vartheta(\iota(h_1,h_2)) = \theta_{a,1}^{W,\mathrm{I}_N\oplus\mathrm{I}_l}(\varphi'' \boxtimes^{\mathrm{tg}} \varphi)\big(g,\mathrm{diag}(h_1,h_2)\big) = K''\,\theta_{a,b}^{W,\mathrm{I}_l}(\varphi)(g,h_2).$$

Hence $K'' \int \theta_{a,b}^{W,\mathrm{I}_l}(\varphi)(g,h_2)\, Y(\iota(h_1,h_2))\, dh_2 = 0$, and $K'' \ne 0$. □

Lemma 6.6.91. *Let A be a finite set of finite places of E with $A \cap \bar{A} = \emptyset$, $\Sigma = A \cup \bar{A}$, and call a datum* the vacuum off Σ *if it is the vacuum at the finite places outside Σ. Let S, T be spaces with points with signs on n and n', and α, β automorphic characters of $\mathrm{U}(1)$.*

(i) *If $\varphi \in \mathcal{M}_{S,T}$ is the vacuum off Σ, and $k \in \mathbf{K}_0(n)$, $k' \in \mathbf{K}_0(n')$ have components 1 at the places of Σ, then $\omega_{\alpha,\beta}((1,k),(1,k'))\varphi = \alpha(\det k')\,\beta(\det k)^{-1}\,\varphi$.*

(ii) *Source and target products of data that are the vacuum off Σ are the vacuum off Σ.*

(iii) *For every $(g,h) \in \mathrm{G}(s) \times \mathrm{G}(s')$ there is a datum $\varphi \in \mathcal{M}_{S,T}$ that is the vacuum off Σ with $\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h) \neq 0$.*

Proof. Let $\eta = s' \otimes s$ on $m = n' \times n$, $\mathcal{M} = \mathcal{M}_{S,T} = \mathcal{M}_\eta$, Θ the theta functional of $\mathrm{Hom}(S,T)$, and let R be the set of finite places of F not below a place of Σ; as $\Sigma = \bar{\Sigma}$, the finite places of E above R are those outside Σ, so a datum is the vacuum off Σ exactly when it is a vacuum above R in the sense of (7) of Lemma 6.2.33.

(i). By (WR4), $\omega_{\alpha,\beta}((1,k),(1,k')) = \alpha(\det k')\beta(\det k)^{-1}\omega_\eta(\kappa)$ with κ the image of the pair under the pair map, $X \mapsto k'Xk^{-1}$. Its infinite components are 1, its finite components are integral unitary matrices, and its components at Σ are 1. (a) $c = \langle v^0, \omega_\eta(\kappa)v^0\rangle = \overline{\Delta_\eta(\kappa)}^{-1} = 1$: the archimedean factor of $\Delta_\eta(\kappa)$ is a determinant of an identity block, and $\widehat{\mathcal{O}}_E^m \cap \kappa_f^{-1}\widehat{\mathcal{O}}_E^m = \widehat{\mathcal{O}}_E^m$, so $u = 1$ in (WR3). (b) By (7) of Lemma 6.2.33 for vacua above R and $\gamma = \kappa$ (its components are 1 at the infinite places and at the finite places not above R, which are the places of Σ), $\langle\varphi, \omega_\eta(\kappa)\varphi\rangle = c\langle\varphi,\varphi\rangle = \langle\varphi,\varphi\rangle$; as $\omega_\eta(\kappa)$ preserves the inner product,

$$\langle\omega_\eta(\kappa)\varphi - \varphi, \omega_\eta(\kappa)\varphi - \varphi\rangle = \langle\varphi,\varphi\rangle - 2\,\mathrm{Re}\langle\varphi,\omega_\eta(\kappa)\varphi\rangle + \langle\varphi,\varphi\rangle = (2 - 2\,\mathrm{Re}\,c)\langle\varphi,\varphi\rangle = 0.$$

A vector g of $\mathcal{M}$ with $\int |g|^2 d\mu = 0$ vanishes μ-almost everywhere, hence is 0 (its continuous representative is given by the reproducing identity). So $\omega_\eta(\kappa)\varphi = \varphi$, which is (i).

(ii). Let $f = f_1 \boxtimes_e f_2$ be a product (WR6) of vacua off Σ, for a bijection $e\colon m_1 \sqcup m_2 \to m$, and z integral with components 0 at Σ, with blocks z_1, z_2 through e. The bicharacter (WD6) and the phase of (WD8) are products over the coordinates, so $(\rho_f(z)f)(x) = (\rho_f(z_1)f_1)(x_1)\,(\rho_f(z_2)f_2)(x_2) = f_1(x_1)f_2(x_2) = f(x)$, with x_1, x_2 the blocks of x. Source and target products are products of this kind.

(iii). Let k be the image of (g,h) under the pair map and $u = \omega_{\alpha,\beta}(g,h)v^0$; $\langle u,u\rangle = \langle v^0, v^0\rangle = 1$. By orthogonality of theta functions for the theta functional Θ, $\int_{\mathbb{A}_E^m/L} |\Theta(\rho(x)u)|^2 dx = 1$, L the rational vectors of $\mathrm{Hom}(S,T)$; so the continuous L-periodic function $G(x) = |\Theta(\rho(x)u)|$ is not identically 0. Let W_Σ be the set of $x \in \mathbb{A}_E^m$ whose components at the finite places outside Σ are 0. By strong approximation for the additive group (E is dense in the adeles of E away from the infinite places), L is dense in the product of the components at the finite places outside Σ (the point of $\mathrm{Hom}(S,T)$ is a homeomorphism there), so $L + W_\Sigma$ is dense; as G is continuous and L-periodic, $G(x) \neq 0$ for some $x \in W_\Sigma$. Put $\varphi = \rho(k^{-1}x)v^0$. Since k^{-1} acts place by place, $k^{-1}x \in W_\Sigma$, so for z integral with components 0 at Σ the vectors $\mathrm{fv}(z)$ and $k^{-1}x$ have disjoint supports, $\rho_f(z)$ commutes with $\rho(k^{-1}x)$, and $\rho_f(z)\varphi = \rho(k^{-1}x)\rho_f(z)v^0 = \varphi$: φ is the vacuum off Σ. Finally $\omega_{\alpha,\beta}(g,h)\varphi = \rho(x)\omega_{\alpha,\beta}(g,h)v^0 = \rho(x)u$ (the scalar characters cancel in $\omega\rho(k^{-1}x)\omega^{-1} = \rho(x)$), so $|\theta^{S,T}_{\alpha,\beta}(\varphi)(g,h)| = |\Theta(\rho(x)u)| = G(x) \neq 0$. □

Lemma 6.6.92. *Let $n = n_1 \sqcup n_2$ be finite index sets, $\mathrm{I}_n = \mathrm{I}_{n_1} \oplus \mathrm{I}_{n_2}$, $\iota\colon \mathrm{G}(1_{n_1}) \times \mathrm{G}(1_{n_2}) \to \mathrm{G}(1_n)$ the block-diagonal embedding, and $\mathbb{H}^c = 0 \oplus \mathbb{H}^c$ with $c = \#n_1$, 0 the zero space. Let A be a finite set of finite places with $A \cap \bar{A} = \emptyset$ and $\Sigma = A \cup \bar{A}$. Let $Y\colon \mathrm{G}(1_n) \to \mathbb{C}$ be continuous, left invariant under Γ_{I_n}, and right invariant under every $k \in \mathbf{K}_0(n)$ whose components at the places of Σ are 1. If the descents of Y to $\mathbb{H}^c$ twisted by v^c vanish off Σ, then*

$$\int_{[\mathrm{U}(\mathrm{I}_{n_2})]} Y\big(\iota(h_1,h_2)\big)\,dh_2 = 0 \qquad \text{for every } h_1 \in \mathrm{G}(1_{n_1}).$$

Proof. Write $c = \#n_1$, $\Gamma = \Gamma_{\mathrm{I}_n}$, and let $x_0 \in \mathrm{M}_{n\times n_1}(E)$ have entries δ_{ij} ($i \in n$, $j \in n_1$). The point of I_m is 1, so $\Gamma_{\mathrm{I}_m} = \mathrm{U}(\mathrm{I}_m)(F)$ and $\mathrm{G}(1_m) = \{g \in \mathrm{GL}_m(\mathbb{A}_E) : g^*g = 1\} = G_\infty(1_m)\times G_f(m)$, with $G_\infty(1_m)$ compact. By Theorem A.4.5, $[\mathrm{U}(\mathrm{I}_m)]$ is compact of finite measure for $m \neq \emptyset$, and Y is bounded; for $m = \emptyset$ the group is trivial of mass 1. The groups $\mathrm{G}(1_m)$ are unimodular (a compact group times the finite adelic points of a reductive group), so their quotient measures are invariant under right translations. For $\gamma \in \Gamma_{\mathrm{I}_{n_2}}$, $\iota(1,\gamma) \in \Gamma$, so functions $a \mapsto Y(\iota(h_1,a)q)$ are read on $[\mathrm{U}(\mathrm{I}_{n_2})]$.

Step 0: $c = 0$. Then $n_2 = n$, ι is the identity, $\mathrm{Hom}(0,\mathrm{I}_n)$ has rank 0, its theta model is $\mathbb{C}$ with $v^0 = 1$, its theta functional and all its Weil and Heisenberg operators are the identity (the rank-0 conventions of Definition 3.2.12). The datum 1 is the vacuum off Σ, and its descent is the constant $\int_{[\mathrm{U}(\mathrm{I}_n)]} Y$, which therefore vanishes. From now on $c \geq 1$.

Step 1: the splitting. Index the c planes of $S = 0 \oplus \mathbb{H}^c$ by n_1 (a fixed bijection), so that S has index set $n_1 \times \{1,2\}$, signs $s(j,1) = 1$, $s(j,2) = -1$, and J_S and the point x_S are block diagonal with the blocks $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and the point of $\mathbb{H}$ on $\{(j,1),(j,2)\}$. Let l, k be the $(n_1 \times \{1,2\}) \times n_1$ matrices with columns $e_{(j,1)}$, $e_{(j,2)}$. Then $k^*J_Sk = l^*J_Sl = 0$ and $l^*J_Sk = 1$: $(k, l, \emptyset)$ is the tautological splitting of S over the zero space along n_1. It is base-adapted: $(l_{\mathbb{A}}\ k_{\mathbb{A}}) = x_S^{-1}(l\ k)$ with $(l\ k)$ a permutation matrix; the real component $2^{-1/2}\left(\begin{smallmatrix}1&-1\\1&1\end{smallmatrix}\right)$ of the point of $\mathbb{H}$ is unitary, and its finite component (WR1) has entries in $\widehat{\mathcal{O}}_E$ and determinant equal to the unit of (WD9) (as $c_0 + \bar{c}_0 = 1$ by (WD1)), so its inverse is integral. Consider the induced splitting of the space $\mathrm{Hom}(S,\mathrm{I}_n)$ over the zero space $\mathrm{Hom}(0,\mathrm{I}_n)$ along $n \times n_1$ (MX4). As $x_{\mathrm{I}_n} = 1$ and $J_{\mathrm{I}_n} = 1$, its adelic frames act on adelic $n \times n_1$ matrices by

$$k_{\mathbb{A}}^{\mathrm{Hom}} b = b\,\mathrm{pr}_l, \qquad l_{\mathbb{A}}^{\mathrm{Hom}} b = b\,\mathrm{pr}_k.$$

Hence (i) the slices of the space $\mathrm{Hom}(S,\mathrm{I}_n)$ along the induced splitting are the Hom slices $f[x]$ of (MX4); (ii) the induced splitting is base-adapted: its frame matrix is $1_n \otimes M^{\mathsf{T}}$ with M the square matrix with row blocks pr_k, pr_l, that is $M = (l_{\mathbb{A}}\ k_{\mathbb{A}})^*\varepsilon_s$, unitary at the infinite places and integral at the finite places, where $\varepsilon_s = 1$.

Step 2: test data. Let b be an adelic $n \times n_1$ matrix with $b_w = 0$ at every finite place $w \notin \Sigma$, and $f_b = \rho(y_b)v^0 \in \mathcal{M}_{S,\mathrm{I}_n}$ with $y_b = k_{\mathbb{A}}^{\mathrm{Hom}} b$; the components of y_b at the finite places outside Σ vanish. By (WD7), $\rho(u)\rho(y) = \beta(u,y)\beta(y,u)^{-1}\rho(y)\rho(u)$ for vectors u, y of the theta model (β its bicharacter). If z is an integral finite adelic vector of $\mathrm{Hom}(S,\mathrm{I}_n)$ with $z_w = 0$ for $w \in \Sigma$, then $\mathrm{fv}(z)$ and y_b have no common nonzero component, so $\beta(\mathrm{fv}(z), y_b) = \beta(y_b, \mathrm{fv}(z)) = 1$ and $\rho_f(z)f_b = \rho(y_b)\rho_f(z)v^0 = f_b$ by (5) of Lemma 6.2.33. So f_b is the vacuum off Σ, and by hypothesis

$$\Theta_b(g') = \int_{[\mathrm{U}(\mathrm{I}_n)]} \theta^{S,\mathrm{I}_n}_{\upsilon^c,1}(f_b)(g',h)\,Y(h)\,dh = 0 \qquad (g' \in \mathrm{G}(s)).$$

Step 3: the terms at a Levi element. Let $m = \alpha h_1 \in \mathrm{GL}_{n_1}(\mathbb{A}_E)$, α the Weil idele read with components 1 at the infinite places, and $g_0 = \mathrm{lev}(m,1) = k_{\mathbb{A}} m\,\mathrm{pr}_k + l_{\mathbb{A}}(m^{-1})^*\mathrm{pr}_l \in \mathrm{G}(s)$ (MX2). As $\mathrm{pr}_k k_{\mathbb{A}} = \mathrm{pr}_l l_{\mathbb{A}} = 1$ and $\mathrm{pr}_l k_{\mathbb{A}} = \mathrm{pr}_k l_{\mathbb{A}} = 0$, $g_0 k_{\mathbb{A}} = k_{\mathbb{A}} m$ and $g_0^{-1} l_{\mathbb{A}} = l_{\mathbb{A}} m^*$. Apply (2) of Lemma 6.2.45 to $(V, V_0, T) = (S, 0, \mathrm{I}_n)$ and the characters $(1,1)$ (so that $1 \cdot \upsilon^{\#n_1} = \upsilon^c$): here $B = \mathrm{pr}_k g_0^{-1} l_{\mathbb{A}} = 0$, W is empty, and $\omega_{1,1}(1,h)$ and $\rho(0)$ on the rank-0 model are the identity. The theta term (MX6) for the splitting, I_n and $(\upsilon^c, 1)$ is $\mathrm{term}_x(f)(g,h) = (\omega_{\upsilon^c,1}(g,h)f)[x]$, the theta functional of the zero space being the identity. Hence, for $x \in \mathrm{M}_{n\times n_1}(E)$ and $h \in \mathrm{G}(1_n)$,

$$\mathrm{term}_x(f)(g_0,h) = \upsilon(\det m)^{\#n}\,|\det m|^{\#n/2}\,f[h^{-1}xm], \tag{6.165}$$

and likewise, with $g = 1$ and $h = \gamma \in \Gamma$, $\mathrm{term}_x(f)(g_0,\gamma h) = \mathrm{term}_{\gamma^{-1}x}(f)(g_0,h)$, using $\omega(g_0,\gamma h) = \omega(1,\gamma)\omega(g_0,h)$ ((2) of Lemma 4.1.1).

Step 4: integrable terms. By Step 1 (i) and (2) of Lemma 6.2.44 for the induced splitting, $T(h) = \sum_x |\mathrm{term}_x(f_b)(g_0,h)\,Y(h)| \le M \sup|Y|\,\|\omega_{v^c,1}(g_0,h)f_b\|$, the norm of the rank-0 model being $|\cdot|$; the right side is continuous in h by (2) of Lemma 4.1.1. By Step 3, T is left Γ-invariant (γ^{-1} permutes $\mathrm{M}_{n\times n_1}(E)$), so its integral over any Borel set of representatives of $\Gamma\backslash\mathrm{G}(1_n)$ equals $\int_{\Gamma\backslash\mathrm{G}(1_n)} T$. As the quotient is compact and the quotient map open, there is a compact K with $\Gamma K = \mathrm{G}(1_n)$; cover K by finitely many Vg_i, V an open neighbourhood of 1 with $VV^{-1}\cap\Gamma = \{1\}$; then $\bigcup_i (Vg_i\cap K)\setminus\Gamma\big(\bigcup_{j<i} Vg_j\cap K\big)$ is a Borel set of representatives inside K, over which $\int T \le \mathrm{vol}(K)\,M\sup|Y|\sup_K\|\omega(g_0,\cdot)f_b\| < \infty$. So f_b has integrable terms at g_0 with weight Y.

Step 5: the Fourier coefficient. For $h\in\mathrm{G}(1_n)$ put $q(h) = \iota(h_1,1)^{-1}h$ and $\mathrm{fr}(h) = h^{-1}x_0h_1 = q(h)^{-1}x_0$ (as $\iota(h_1,1)x_0 = x_0h_1$); $\mathrm{fr}(h)^*\mathrm{fr}(h) = 1$ at every place. Let $\mathbf{1}_{\mathrm{int}}(h)$ be 1 if the finite components of $\mathrm{fr}(h)$ are integral and 0 otherwise; it is locally constant. By (6.165) at $x = x_0$, $h^{-1}x_0m = \alpha\,\mathrm{fr}(h)$. By Step 1 (i) and (1) of Lemma 6.2.44 (with $a=0$, $w=0$), $f_b[y] = \psi_E(\jmath\, b^*y)\,v^0[y]$, where $b^*y = \sum_{i,j}\bar b_{ij}y_{ij}$; by Step 1 (ii) and Lemma 6.2.11 (at $g=1$, in the basis 1), $v^0[y] = C_1|e_{n\times n_1}(y)|^2$ with $C_1\neq 0$ independent of b and y (the rank-0 vacuum vector is 1). By (WD5), as α is 1 at the infinite places, $\mathrm{Im}\,\jmath_v = 1/\sqrt{d}$ and $\mathrm{tr}(\mathrm{fr}(h)_v^*\mathrm{fr}(h)_v) = c$ for $v\in\Phi$, and $\alpha\,\mathrm{fr}(h)_f \in \alpha\widehat{\mathcal{O}}_E^{n\times n_1}$ exactly when $\mathrm{fr}(h)_f$ is integral, $|e_{n\times n_1}(\alpha\,\mathrm{fr}(h))|^2 = e^{-2\pi c\#\Phi/\sqrt{d}}\,\mathbf{1}_{\mathrm{int}}(h)$. Hence

$$\mathrm{term}_{x_0}(f_b)(g_0,h) = c_*\,\mathrm{ch}_b(\mathrm{fr}(h))\,\mathbf{1}_{\mathrm{int}}(h),$$
$$\mathrm{ch}_b(y) = \psi_E(\jmath\, b^*\alpha y),\quad c_* = C_1\upsilon(\det m)^{\#n}|\det m|^{\#n/2}e^{-2\pi c\#\Phi/\sqrt{d}} \neq 0.$$

The Gram matrix $x_0^*x_0 = 1$ is invertible, and $\Gamma_{x_0} = \{\gamma\in\Gamma : \gamma x_0 = x_0\} = \iota(1,\Gamma_{\mathrm{I}_{n_2}})$. By (2) of Lemma 6.3.25 (for $W=S$, $W_0=0$, $V=\mathrm{I}_n$, the characters $(\upsilon^c,1)$, the datum f_b, the weight Y and $g=g_0$; Step 4), the centre coefficient at $\jmath\cdot 1$ of $\Theta_b = 0$ at g_0, which is 0, is the integral of $\mathrm{term}_{x_0}(f_b)(g_0,h)Y(h)$ over $\Gamma_{x_0}\backslash\mathrm{G}(1_n)$. Dividing by c_*: for every b as in Step 2,

$$h\mapsto \mathrm{ch}_b(\mathrm{fr}(h))\,\mathbf{1}_{\mathrm{int}}(h)\,Y(h) \text{ is integrable on } \Gamma_{x_0}\backslash\mathrm{G}(1_n) \text{ with integral 0.} \tag{6.166}$$

Step 6: the period. For $q\in\mathrm{G}(1_n)$ let $\mathrm{per}(q) = \int_{[\mathrm{U}(\mathrm{I}_{n_2})]} Y(\iota(h_1,a)q)\,da$; the claim is $\mathrm{per}(1) = 0$.

(a) per is continuous, by dominated convergence (Y continuous and bounded, the quotient of finite measure).

(b) $\mathrm{per}(\iota(1,v)q) = \mathrm{per}(q)$ for $v\in\mathrm{G}(1_{n_2})$, since $\iota(h_1,a)\iota(1,v) = \iota(h_1,av)$ and the quotient measure is right invariant; and $\mathrm{per}(qk) = \mathrm{per}(q)$ for $k\in\mathbf{K}_0(n)$ with components 1 at Σ, by the invariance of Y.

(c) If $q^{-1}x_0 = q'^{-1}x_0$, then $q'q^{-1}$ fixes x_0, so $q'q^{-1} = \iota(1,v)$ (sidecalc of Step 5) and $\mathrm{per}(q') = \mathrm{per}(q)$ by (b).

(d) Let $q,q'\in G_\infty(1_n)\times\mathbf{K}_0(n)$ with $(q^{-1}x_0)_v = (q'^{-1}x_0)_v$ for $v\in\Phi\cup\Sigma$. Let $\kappa\in\mathbf{K}_0(n)$ have components $(q_f^{-1}q'_f)_w$ at the finite places $w\notin\Sigma$ and 1 at Σ, and $k = (1,\kappa)$. Then $(qk)^{-1}x_0$ has the components of $q'^{-1}x_0$ at every place, so $\mathrm{per}(q') = \mathrm{per}(qk) = \mathrm{per}(q)$ by (c) and (b).

(e) If $\mathbf{1}_{\mathrm{int}}(h) = 1$, there is $q'\in G_\infty(1_n)\times\mathbf{K}_0(n)$ with $q'^{-1}x_0 = \mathrm{fr}(h)$: x_0 and $\mathrm{fr}(h)_f$ are integral with left inverses x_0^*, $\mathrm{fr}(h)_f^*$ and equal Gram matrices 1, so (2) of Lemma 3.2.6 gives $d\in\mathrm{GL}_n(\widehat{\mathcal{O}}_E)$ with $d^*d = 1$ and $dx_0 = \mathrm{fr}(h)_f$; then $d\in\mathbf{K}_0(n)$ and $q' = (q(h)_\infty, d^{-1})$.

Step 7: Stone–Weierstrass. Let $\pi(q) = ((q^{-1}x_0)_v)_{v\in\Phi\cup\Sigma} \in \prod_{v\in\Phi\cup\Sigma} \mathrm{M}_{n\times n_1}(E_v)$ and $K_{\mathrm{fr}} = \pi(G_\infty(1_n)\times \mathbf{K}_0(n))$, a compact Hausdorff space. As π is a continuous surjection of a compact space onto K_{fr}, hence a quotient map, (d) and (a) give a continuous $J\colon K_{\mathrm{fr}}\to\mathbb{C}$ with $J\circ\pi = \overline{\mathrm{per}}$. For b as in Step 2, $\mathrm{ch}_b(y)$ depends only on $(y_v)_{v\in\Phi\cup\Sigma}$ and so defines a function on K_{fr}; $\mathrm{ch}_b\mathrm{ch}_{b'} = \mathrm{ch}_{b+b'}$, $\overline{\mathrm{ch}_b} = \mathrm{ch}_{-b}$, $\mathrm{ch}_0 = 1$. They separate points: ψ_E is nontrivial on each completion E_v (for $v\in\Phi$ by (WD4); for finite v because otherwise it would be trivial on the dense subgroup $E+E_v$ of $\mathbb{A}_E$, by strong approximation for the additive group, contradicting (a) of Theorem A.5.2); if $y, y'\in K_{\mathrm{fr}}$ differ in the entry (i,j) at v, the b with the single entry $t\in E_v$ there gives $\mathrm{ch}_b(y)/\mathrm{ch}_b(y') = \psi_E(\bar{t}\jmath\alpha_v\delta)$ with $\delta\neq 0$, which is $\neq 1$ for some t. By the Stone–Weierstrass theorem their span is uniformly dense in the continuous functions on K_{fr}.

Let $F(h) = \overline{\mathrm{per}(q(h))}\,\mathbf{1}_{\mathrm{int}}(h)\,Y(h)$. It is continuous, and left Γ_{x_0}-invariant: $q(\iota(1,\gamma)h) = \iota(1,\gamma)q(h)$, $\mathrm{fr}(\iota(1,\gamma)h) = \mathrm{fr}(h)$, and (b). If $\mathbf{1}_{\mathrm{int}}(h) = 1$, then by (e) and (c), $(\mathrm{fr}(h)_v)_{v\in\Phi\cup\Sigma} = \pi(q')\in K_{\mathrm{fr}}$ and $J(\pi(q')) = \overline{\mathrm{per}(q(h))}$. Given $\epsilon>0$ choose $Q = \sum_k c_k\mathrm{ch}_{b_k}$ with $|J-Q|\le\epsilon$ on K_{fr}; then $|F(h) - Q(\mathrm{fr}(h))\mathbf{1}_{\mathrm{int}}(h)Y(h)| \le \epsilon\,\mathbf{1}_{\mathrm{int}}(h)|Y(h)|$. By (6.166) ($b=0$ for integrability, $b=b_k$ for the integrals), F is integrable on $\Gamma_{x_0}\backslash\mathrm{G}(1_n)$ and $|\int F| \le \epsilon\int \mathbf{1}_{\mathrm{int}}|Y|$. Hence $\int_{\Gamma_{x_0}\backslash\mathrm{G}(1_n)} F = 0$.

Step 8: positivity. Let $\varsigma(h) = \mathbf{1}_{\mathrm{int}}(h)\,|\mathrm{per}(q(h))|^2 \ge 0$, continuous by (a). For every h, as $\mathrm{fr}(\iota(1,a)h) = \mathrm{fr}(h)$, $\mathrm{per}(q(\iota(1,a)h)) = \mathrm{per}(\iota(1,a)q(h)) = \mathrm{per}(q(h))$ and $\iota(1,a)h = \iota(h_1,a)q(h)$,

$$\int_{[\mathrm{U}(\mathrm{I}_{n_2})]} F(\iota(1,a)h)\,da = \overline{\mathrm{per}(q(h))}\,\mathbf{1}_{\mathrm{int}}(h)\int_{[\mathrm{U}(\mathrm{I}_{n_2})]} Y(\iota(h_1,a)q(h))\,da = \varsigma(h),$$

the integral converging absolutely. Suppose $\mathrm{per}(1)\neq 0$. At $h^\dagger = \iota(h_1,1)$, $q(h^\dagger) = 1$ and $\mathrm{fr}(h^\dagger) = x_0$ is integral, so $\varsigma(h^\dagger) = |\mathrm{per}(1)|^2 > 0$, and $\varsigma>0$ near $h^\dagger$. Apply Lemma 6.2.47 to $\mathrm{G}(1_n)$, the trivial group as first factor, $\mathrm{G}(1_{n_2})$ with the embedding $a\mapsto\iota(1,a)$ (a continuous injective homomorphism and a homeomorphism onto its closed image, the stabiliser of x_0), the discrete subgroup $\Gamma_{\mathrm{I}_{n_2}}$ (with image Γ_{x_0}, discrete), the function ς, the constant 1 and, at every h, the element 1: it gives $\int_{\Gamma_{x_0}\backslash\mathrm{G}(1_n)} F = R$ with $R\neq 0$, contradicting Step 7. So $\mathrm{per}(1) = \int_{[\mathrm{U}(\mathrm{I}_{n_2})]} Y(\iota(h_1,h_2))\,dh_2 = 0$. □

6.7 Hecke values and test forms

This section bounds the Hecke ratios ρ_j of the comparison along a regular family (Proposition 6.7.1), as the cost of one lift requires. The Hecke value product factors into ratios of main and of ramified critical pairs (Lemma 6.7.2). The ramified ratios are small along a regular family by definition, and the work is in the main ratios, whose inverses have height at most $6\beta_{\mathrm{den}} - h_{\mathrm{dir}}$.

The key step realises Hecke L-values as squared norms of theta lifts from U(1) (Proposition 6.7.13): for a critical pair $p = (\nu, z)$ with $z\ge 2$, the algebraic part of the lift of Definition 4.2.5 is a positive constant, independent of the twist, times the partial L-value, so that a main ratio is a ratio of squared norms. The proof writes the norm, by the Rallis formula for sharp data, as a series of vacuum coefficients, whose sum in rank one is the partial L-value times an Euler product. For theta series attached to Hecke characters see [128], and for theta functions with complex multiplication [272, 274]; the unramified computation is that of the doubling method [231, 187], and for theta lifts from U(1) see [121].

The inverse main ratio is then a product of four numbers whose heights add: the squared norm of the untwisted lift, bounded through its conjugates and denominators (Lemma 6.7.4); the pairing of two test forms projected to the line of the lift, bounded by the denominator condition of the regular family; the inverse square of a flat ramified ratio, which is small; and a term bounded at each prime ℓ by $\ell^{2F(\ell)}$ through the choice of the test forms. So the finite prime sums enter through the test forms and the archimedean sizes through the untwisted norm, and

their difference is the direction amount (Definition 2.4.2). For the algebraicity of critical Hecke L-values of CM fields see [274, 159]. The architecture follows [259].

Proposition 6.7.1. *Let θ be a regular family of m-fold twists and $1 \le j \le m$.*

(a) For every main pair p for j, the family $\iota \mapsto r_p(\theta(\iota)^j)^{-1}$ has height at most $6\beta_{\mathrm{den}} - h_{\mathrm{dir}}$.

(b) There are $e \in \mathbb{Z}$, main pairs $p_1, \dots, p_{k/2}$ for j, ramified pairs $q_1, \dots, q_{5m}$ and signs $s_1, \dots, s_{5m} \in \{\pm 1\}$, independent of the twist, with

$$\Gamma_j(\vartheta) = \Gamma_j(1)\,\mathrm{N}Q^e\Big(\prod_{a=1}^{k/2} r_{p_a}(\vartheta)\prod_{b=1}^{5m} r_{q_b}(\vartheta, s_b)\Big)^2 \qquad \text{for every twist } \vartheta.$$

(c) The family $\iota \mapsto \rho_j(\theta(\iota)^j)^{-1}$ has height at most $10\,m\,x_1 + k\,(6\beta_{\mathrm{den}} - h_{\mathrm{dir}})$.

Proof. Part (a). Let $p = (\nu, z)$ be a main pair for j; by (RH1), $z \ge 10$. For a prime identification $\iota \mid \ell$ put $\vartheta_\iota = \theta(\iota)^j$, a twist with the modulus Q_ι of $\theta(\iota)$, so $\ell \nmid \mathrm{N}Q_\iota$ by (SR1). Write $\varphi_\iota = \varphi_{p,\vartheta_\iota}$, the lift of p at ϑ_ι (the vector paired in Proposition 6.7.17), a scalar multiple of $\lambda_{p,\vartheta_\iota}$ by (OL4), and $\beta = \beta_{\mathrm{den}}$.

Step 1: a flat ramified pair. Let $\nu_{\mathrm{fl}} = \chi_M \upsilon^{-2}$. Weights and tame parts multiply, so ν_{fl} has weight -2 at every infinite place and tame part ω, which is trivial off the chosen places and, at a chosen place w, equal to $\eta_w^{q_2}$ of order $q_2 > 1$. As $1 \le 2 \le t$ and $2 \le 4b$, $p_{\mathrm{fl}} = (\nu_{\mathrm{fl}}, 2)$ is a critical pair; it is flat and ramified. For every tame character ϑ, $r_{p_{\mathrm{fl}}}(\vartheta) > 0$: numerator and denominator are absolute values of absolutely convergent Euler products at $z = 2$ with factors $(1 - \tilde{\mu}(x_w)\mathrm{N}w^{-2})^{-1} \ne 0$, $|\tilde{\mu}| = 1$, hence not 0. Write $r_\iota = r_{p_{\mathrm{fl}}}(\vartheta_\iota)$.

Step 2: test forms at ι and $\bar{\iota}$. Let $c = c_{\mathrm{tf}} \ne 0$ be the number of Proposition 6.7.17 for j, p and p_{fl}. Apply Proposition 6.7.17 to ϑ_ι with the embeddings ι and $\bar{\iota}$ (both over ℓ, and $\ell \nmid \mathrm{N}Q_\iota$): there are test forms Ξ_ι, Ξ'_ι for ℓ of O_{p,ϑ_ι} with $\|\overline{\iota(c\,r_\iota)}\|\ell^{-F(\ell)} \le \|\iota\langle \Xi_\iota, \varphi_\iota\rangle\|$ and $\|\bar{\iota}(c\,r_\iota)\|\ell^{-F(\ell)} \le \|\bar{\iota}\langle \Xi'_\iota, \varphi_\iota\rangle\|$. Since $\bar{\iota}(u) = \iota(\bar{u})$, r_ι is real and $\overline{\langle \Xi', \varphi\rangle} = \langle \varphi, \Xi'\rangle$, the product of the two inequalities is

$$\|\iota(c\bar{c}\,r_\iota^2)\|\,\ell^{-2F(\ell)} \le \|\iota(P_\iota)\|, \qquad P_\iota = \langle \Xi_\iota, \varphi_\iota\rangle\langle \varphi_\iota, \Xi'_\iota\rangle. \tag{6.167}$$

The left side is positive, so $P_\iota \ne 0$ and $\varphi_\iota \ne 0$; hence $c_s \ne 0$, and $\eta_P \ne 0$ or $j_p = 0$, so the scale factor $c_s^{2z}(\pi_M \eta_P)^{j_p}$ of (OL4) is not 0 (as $\pi_M \ne 0$), and Proposition 6.7.13 applies to p.

Step 3: the inverse ratio as a product of four factors. By Proposition 6.7.13, $r_p(\vartheta_\iota) = \mathrm{alg}_p(\vartheta_\iota)/\mathrm{alg}_p(1) > 0$; by (OL4), with $\mathrm{N}Q = 1$ for the trivial twist,

$$r_p(\vartheta_\iota)^{-1} = \mathrm{N}Q_\iota^{2z-1}\|\varphi_{p,1}\|^2/\|\varphi_\iota\|^2.$$

Let $\mathrm{pr} = \mathrm{pr}_{p,\vartheta_\iota}$, the orthogonal projection to $\mathbb{C}\lambda_{p,\vartheta_\iota} = \mathbb{C}\varphi_\iota$. For a nonzero vector φ of an inner product space and vectors Ξ, Ξ',

$$\langle \mathrm{pr}\,\Xi, \mathrm{pr}\,\Xi'\rangle = \langle \Xi, \varphi\rangle\langle \varphi, \Xi'\rangle/\|\varphi\|^2$$

(expand $\mathrm{pr}\,\Xi$ along $\varphi/\|\varphi\|$); we apply it to $\varphi = \varphi_\iota$. Hence, P_ι and r_ι being nonzero,

$$r_p(\vartheta_\iota)^{-1} = \mathrm{N}Q_\iota^{2z-1} \cdot y_\iota^{(1)}\,y_\iota^{(2)}\,y_\iota^{(3)}\,y_\iota^{(4)},$$
$$y_\iota^{(1)} = \|\varphi_{p,1}\|^2,\ y_\iota^{(2)} = \langle \mathrm{pr}\,\Xi_\iota, \mathrm{pr}\,\Xi'_\iota\rangle,\ y_\iota^{(3)} = r_\iota^{-2},\ y_\iota^{(4)} = \frac{r_\iota^2}{P_\iota}.$$

Step 4: the heights of the factors. (i) By Lemma 6.7.4, $y^{(1)}$ has height at most $\beta - \sum_\sigma s_\infty(\kappa(\sigma))$. (ii) By (RW) there is a denominator bound (γ_ℓ) for (θ, j); by (DB2) and (DL2), applied to the main pair p and the test forms Ξ_ι, Ξ'_ι for ℓ, $\|\iota(y_\iota^{(2)})\| \le e^{\gamma_\ell}$, and by (DB1) $\sum_{\ell\in T}\gamma_\ell \le \beta$ for every finite set T; so $y^{(2)}$ has height at most β by Lemma 3.1.8 (c). (iii) p_{fl} is a ramified critical pair, so by (SR2) with $s=1$ the family $\iota \mapsto r_\iota^{-1}$ has height at most x_1, and $y^{(3)}$ at most $2x_1$ by Lemma 3.1.8 (e). (iv) For $y^{(4)}$ take $H = c\bar c \ne 0$ in the definition of height: for every automorphism σ, $|\sigma(H)| = |\sigma(c)|\,|\sigma(\bar c)| \le e^\alpha$ ($\sigma(\bar c)$ is the value at c of the automorphism $u \mapsto \sigma(\bar u)$) with $\alpha = 2(fmb/k + 3x^{34})$ by Proposition 6.7.17; by (6.167), $\|\iota(Hy_\iota^{(4)})\| \le \ell^{2F(\ell)}$, with $2F(\ell) \ge 0$ (a sum of nonnegative terms); and for every finite set S of primes, by Lemma 6.7.12 and Lemma 6.7.11,

$$\alpha + \sum_{\ell\in S} 2F(\ell)\log\ell \le 2\beta + \sum_{\sigma\in\Phi} s_{\mathrm{fin}}(\kappa(\sigma),\sigma).$$

So $y^{(4)}$ has height at most $2\beta + \sum_\sigma s_{\mathrm{fin}}(\kappa(\sigma),\sigma)$.

Step 5: conclusion. By Lemma 3.1.8 (e) the product $y^{(1)}y^{(2)}y^{(3)}y^{(4)}$ has height at most $4\beta + 2x_1 - h_{\mathrm{dir}}$, and by Lemma 3.1.8 (b) (the positive integers $\mathrm{N}Q_\iota$ are prime to ℓ by (SR1), exponent $2z-1$) so does $\iota \mapsto r_p(\vartheta_\iota)^{-1}$. As $x_1 \le \beta$ by Lemma 6.7.11, Lemma 3.1.8 (a) gives the height bound $6\beta - h_{\mathrm{dir}}$.

Part (b). By Lemma 6.7.2 (c), for every twist ϑ, $\Gamma_j(\vartheta) = \Gamma_j(1)\,\mathrm{N}Q^e(\cdots)^2$ with $e = z_*^2 + z_+^2 + z_-^2$ and, inside the square, the ratios at ϑ of the $k/2$ lower points (ν_*, z), $7\lfloor m/2\rfloor < z \le z_*$, which are main pairs for j by Lemma 6.7.2 (a), and the ratios with signs of the $5m$ lower points listed in Lemma 6.7.2 (b), which are ramified pairs. Numbering the former $p_1,\dots,p_{k/2}$ and the latter $q_1,\dots,q_{5m}$ with their signs s_b gives (b); e, the pairs and the signs depend only on j.

Part (c) from (a), (b). By (b) with $\vartheta = \theta(\iota)^j$, the number $\rho_j(\vartheta)^{-1}$ is $\mathrm{N}Q^{-e}$ times the squares of $k/2$ inverse ratios of main pairs and $5m$ inverse ratios of ramified pairs. By (a) each squared main inverse ratio has height at most $2(6\beta_{\mathrm{den}} - h_{\mathrm{dir}})$; by the small-ratio condition (SR2) each ramified inverse ratio $r_{q_b}(\theta(\iota)^j, s_b)^{-1}$ has height at most x_1, so its square at most $2x_1$; and the factor $\mathrm{N}Q^{-e}$, prime to $\ell(\iota)$, costs nothing (height calculus, Lemma 3.1.8). Heights add over products, so the height is at most $k(6\beta_{\mathrm{den}} - h_{\mathrm{dir}}) + 5m\cdot 2x_1$. □

Lemma 6.7.2. *Let $1 \le j \le m$ and put $e = z_*^2 + z_+^2 + z_-^2$.*

(a) For $7\lfloor m/2\rfloor < z \le z_*$ the lower point (ν_*, z) of p_j^* is a main pair for j; there are $k/2$ such z.

(b) The $5m$ lower points (ν_*, z) with $1 \le z \le 7\lfloor m/2\rfloor$, (ν_+, z) with $1 \le z \le m$ and (ν_-, z) with $1 \le z \le \lfloor m/2\rfloor$ are ramified pairs; attach to them the signs s_*, s_+, s_- respectively.

(c) For every twist ϑ, $\Gamma_j(\vartheta) > 0$ and

$$\Gamma_j(\vartheta) = \Gamma_j(1)\,\mathrm{N}Q^e\Big(\prod_{z=7\lfloor m/2\rfloor+1}^{z_*} r_{(\nu_*,z)}(\vartheta)\prod_{z=1}^{7\lfloor m/2\rfloor} r_{(\nu_*,z)}(\vartheta,s_*)\prod_{z=1}^{m} r_{(\nu_+,z)}(\vartheta,s_+)\prod_{z=1}^{\lfloor m/2\rfloor} r_{(\nu_-,z)}(\vartheta,s_-)\Big)^2.$$

Proof. Numerical facts. By (AD2), $m = 2^{e_1+1}$ is even and $r = 3^{e'+1}$ is odd, so $t = 5r$ is odd, $k = r + 1 - m$ is even and $\lfloor m/2\rfloor = m/2$. By (AD1), $\log x \ge 100\log\log b \ge 100\cdot 10^{12} C_P$ with $C_P \ge q_0 \ge 1$, so $x \ge 4$; by (AD2), $m \ge \sqrt{x} \ge 2$ and in fact $m \ge 10$ (as $x \ge e^{10^{14}}$), and

$r \ge x^5 \ge 12\sqrt{x}+1 \ge 6m+1$. Hence $n_* = j + (t-11m-1)/2$ is an integer ($t-11m-1$ is even) with $n_* \ge 0$ (as $t = 5r > 11m+1$) and $n_* \le m + (5r-1)/2 \le 4r = t - r$ (as $2m - 1 \le 3r$).

(a). The character $\nu_* = \mu_j^0 \upsilon^{n_*}$ has weight $-\kappa_j + n_*$ (weights add under products; μ_j^0 has weight $-\kappa_j$ and υ weight 1) and tame part ω^j (υ has trivial tame part). So it is of weight $-c$ with $c = \kappa_j - n_*$, and every lower point (ν_*, z) is a critical pair. Let $7m/2 < z \le z_*$. (Mp1): $z > 7m/2 \ge m$ and $z > 7m/2 \ge 35 \ge 10$; $z \le z_* = 7m/2 + k/2 = (r+1+6m)/2 \le r$, since $6m+1 \le r$. (Mp2): the tame part is ω^j. (Mp3): $\kappa_j(v) + r - t \le \kappa_j(v) - n_* \le \kappa_j(v)$ is $0 \le n_* \le t - r$, shown above. The number of such z is $z_* - 7m/2 = k/2$.

(b). The tame parts of ν_*, ν_+, ν_- are ω^j, ω^j, ω^{-j} (Definition 6.6.1). By Lemma 6.7.3, for $1 \le |n| \le 8m$, $\omega_w^n \ne 1$ exactly at the chosen places w; as $1 \le j \le m$, the tame part $\omega^{\pm j}$ is nontrivial exactly at the chosen places, so each of these critical pairs is ramified. Their number is $7m/2 + m + m/2 = 5m$.

(c). Fix a twist ϑ with modulus Q. Its places are prime to MN (Definition 3.1.2), and the chosen places lie above $p_0 p_1 = M$, so $S_0 \cap Q = \emptyset$ and $\prod_{w \in S_0 \cup Q} \mathrm{N}w = \mathrm{N}S_0 \cdot \mathrm{N}Q$ with $\mathrm{N}S_0 = \prod_{w \in S_0} \mathrm{N}w$.

Nonvanishing. Let λ be one of the characters ν_i^{-1}, $(\nu_i \vartheta^{s_i})^{-1}$ and $T \in \{S_0, S_0 \cup Q\}$. It has weight c_i, where ν_i has weight $-c_i$ with $c_i(v) \ge z_i \ge 1$ (ϑ has weight 0). The base change $\tilde{\lambda}(x) = \lambda(x/\bar{x})$ is a unitary Hecke character of E whose restriction to the circle $\{x_v \in \mathbb{C} : |x_v| = 1\}$ at an infinite place v is $x_v \mapsto x_v^{2c_i(v)} \ne 1$; so $\tilde{\lambda}$ is not of the form $|\cdot|_{\mathbb{A}}^{i\tau}$, and by Theorem A.1.12 (3) its L-function is holomorphic and has no zeros in $\mathrm{Re}\, s \ge 1$. The partial L-function $L^T(s, \lambda)$ differs from it by finitely many factors $1 - \tilde{\lambda}(x_w)\mathrm{N}w^{-s}$ at unramified places, which do not vanish for $\mathrm{Re}\, s \ge 1$ since $|\tilde{\lambda}(x_w)\mathrm{N}w^{-s}| \le \mathrm{N}w^{-1} < 1$. Hence $L^T(z, \lambda) \ne 0$ for every integer $z \ge 1$. In particular every factor of $\Gamma_j(\vartheta)$ in (6.129) is a positive real number, so $\Gamma_j(\vartheta) > 0$, and the ratios below have nonzero denominators.

Factorisation. For $i \in \{*, +, -\}$ and $1 \le z \le z_i$, the definition of the ratio of the lower point (ν_i, z) gives

$$\left|L^{S_0 \cup Q}(z, (\nu_i \vartheta^{s_i})^{-1})\right|^2 = \left|L^{S_0}(z, \nu_i^{-1})\right|^2 r_{(\nu_i, z)}(\vartheta, s_i)^2,$$

as the denominator $|L^{S_0}(z, \nu_i^{-1})|$ is nonzero. Inserting this in (XD7) of Definition 3.2.22,

$$\begin{aligned} L_*(S_0 \cup Q, (\nu_i \vartheta^{s_i})^{-1}; z_i) &= \mathrm{N}Q^{z_i^2}\ \mathrm{N}S_0^{z_i^2} \prod_{z=1}^{z_i} \left|L^{S_0}(z, \nu_i^{-1})\right|^2 \prod_{z=1}^{z_i} r_{(\nu_i, z)}(\vartheta, s_i)^2 \\ &= \mathrm{N}Q^{z_i^2}\, L_*(S_0, \nu_i^{-1}; z_i) \prod_{z=1}^{z_i} r_{(\nu_i, z)}(\vartheta, s_i)^2. \end{aligned}$$

For the trivial twist ($Q = \emptyset$, $\vartheta^{s_i} = 1$) the left side is $L_*(S_0, \nu_i^{-1}; z_i)$, so $\Gamma_j(1) = \prod_i L_*(S_0, \nu_i^{-1}; z_i)$. Multiplying over $i \in \{*, +, -\}$ gives $\Gamma_j(\vartheta) = \Gamma_j(1)\, \mathrm{N}Q^e \big(\prod_i \prod_{z=1}^{z_i} r_{(\nu_i, z)}(\vartheta, s_i)\big)^2$. Splitting the range $1 \le z \le z_*$ at $7m/2$ and using $s_* = 1$, so that $r_{(\nu_*, z)}(\vartheta, s_*) = r_{(\nu_*, z)}(\vartheta)$, gives the displayed formula. □

Lemma 6.7.3. *Let D be admissible data. For every finite place w of E and every integer n with $1 \le |n| \le 8m$, $\omega_w^n \ne 1$ if and only if w is a chosen place.*

Proof. Put s for the exponent of the admissible datum. By (AD2) of Definition 2.3.4, there is an integer $e \ge 0$ with

$$m = 2^{e+1} \ge 2, \qquad s = 22(e+1), \qquad 2^s = m^{22}.$$

The parameter integers q_0, d are positive and $q_0 d \ge 2$. Consequently their defining formula gives

$$q_2 = (q_0 d)^{s+6} \ge 2^{s+6} = 64 m^{22} > 8m. \tag{1}$$

This comparison already suffices; it requires no divisibility assertion involving q_1. The related size inequality also holds directly:

$$q_1 d = 3q_0 d\, l_0(s)\, 2^s = 3q_0 d\, l_0(s) m^{22} \geq m^{22} > 8m,$$

since $l_0(s) \geq 1$ and $m^{21} \geq 2^{21} > 8$.

Fix w and n as in the statement. If w is not chosen, the definition in Definition 3.2.26 gives $\omega_w = 1$, so $\omega_w^n = 1$.

If w is chosen, the same definition fixes η_w of exact order q_2^2 and sets $\omega_w = \eta_w^{q_2}$. For every positive integer a,

$$\omega_w^a = 1 \quad \Longleftrightarrow \quad \eta_w^{q_2 a} = 1 \quad \Longleftrightarrow \quad q_2^2 \mid q_2 a \quad \Longleftrightarrow \quad q_2 \mid a.$$

Thus ω_w has exact order q_2. Take $a = |n|$. The assumptions and (1) give $0 < a \leq 8m < q_2$, so $q_2 \nmid a$ and $\omega_w^a \neq 1$. If $n > 0$ this is the desired assertion. If $n < 0$, then $\omega_w^n = (\omega_w^{|n|})^{-1}$, and an inverse equals the identity exactly when the original element does. Hence $\omega_w^n \neq 1$ also in this case. The two cases for w prove the equivalence. □

Lemma 6.7.4. *Let $1 \leq j \leq m$ and let p be a main pair for j. Then the constant family $\iota \mapsto \|\varphi_{p,1}\|^2$ has height at most $\beta_{\mathrm{den}} - \sum_{\sigma\in\Phi} s_\infty(\kappa(\sigma))$.*

Proof. Let $p = (\nu, z)$, ν of weight $-c$; by (RH1), $10 \leq z \leq r$.

Step 1: a flat pair of the same rank. The character υ^{-z} of $\mathrm{U}(1)$ has weight $-z$ at every infinite place and trivial tame part. As $1 \leq z \leq r \leq t$ and $z \leq r \leq 3x^5 \leq b \leq 4b$ (items 2 and 3 of Lemma 4.2.67), $q = (\upsilon^{-z}, z)$ is a critical pair with $c_q(\tau) = z$ for every τ, and $z \geq 2$. Its row lengths $c_q(\tau) - z$ vanish, so $j_q = 0$ and $n_q := n_{\upsilon^{-z}} = M^{4fz}$. Put $y_q = N_{\mathrm{ch}}\, n_q^2\, \mathrm{alg}_q(1) = N_{\mathrm{ch}} n_q^2 \|\varphi_{q,1}\|^2$.

Step 2: y_q is integral at every prime identification. Let O be the descent pair of $X(q,1)$, of rank $2z$. Its weight is 0, so $\mathbb{V}_O = \mathbb{C}$ with $\langle a, b\rangle_0 = \bar{a}b$, and by (DP5)

$$y_q = \sum_{[h]} \frac{N_{\mathrm{ch}}}{\#\Gamma_h}\, \overline{n_q \varphi_{q,1}(h)}\; n_q \varphi_{q,1}(h),$$

the sum over $\mathrm{U}(\mathrm{I}_{2z})(E)\backslash G_f(2z)/\mathbf{K}(O)$. Each $N_{\mathrm{ch}}/\#\Gamma_h$ is an integer by Lemma 6.7.9 ($2z \leq 2r \leq rt$, $\mathbf{K}(O) \subseteq \mathbf{K}_0(2z)$ open). Let $\iota \mid \ell$ be a prime identification and $\bar{\iota}$ its conjugate; both are embeddings $\mathbb{C} \to \overline{\mathbb{Q}}_\ell \subset \mathbb{C}_\ell$, and $\ell \nmid \mathrm{N}Q = 1$. By Lemma 6.7.7 for q and the trivial twist, the weight-0 form $n_q\varphi_{q,1}$ is integral at ι and at $\bar{\iota}$: $\|\iota(n_q\varphi_{q,1}(h))\| \leq 1$ and $\|\iota(\overline{n_q\varphi_{q,1}(h)})\| = \|\bar{\iota}(n_q\varphi_{q,1}(h))\| \leq 1$. By the ultrametric inequality $\|\iota(y_q)\| \leq 1$.

Step 3: the size of the integer factor. By Lemma 6.7.10, $N_{\mathrm{ch}} \geq 1$ and $\log N_{\mathrm{ch}} \leq x^{33}$. By items 2–5 of Lemma 4.2.67 ($f \leq x^{12}$, $z \leq r \leq 3x^5$, $\log M \leq 22 \log b$, $\log b \leq x^{1/100}$), $\log n_q^2 = 8fz \log M \leq 8 \cdot x^{12} \cdot 3x^5 \cdot 22\, x^{1/100} \leq x^{19}$ and $8f \leq 8x^{12}$. By Lemma 6.7.11, $3x^{34} \leq \frac{fmb}{k} + 3x^{34} \leq \beta_{\mathrm{den}}$. Hence

$$\log(N_{\mathrm{ch}} n_q^2) + 8f \leq x^{33} + x^{19} + 8x^{12} \leq 3x^{34} \leq \beta_{\mathrm{den}}. \tag{6.168}$$

Step 4: the jet sizes. For $\tau \in \Phi$, (RH3) and $z \leq r$ give $c(\tau) - z \geq \kappa_j(\tau) + r - t - r = \kappa(\tau) + (2j+1)r^8 - 5r \geq \kappa(\tau)$. The function s_∞ is nondecreasing on $\mathbb{Z}_{\geq 0}$: $s_\infty(i+1) - s_\infty(i) = \log(i+1) + \log(M/\Omega)$, and $M \geq \Omega$, because $\mathrm{rd}_E \geq 1$ gives $\Omega \leq 2\pi e^{|c_{\mathrm{CS}}(\overline{d})|} \leq e^{C_P+2} \leq e^{\log\log b} = \log b \leq b^{20} \leq M$ (by (AD1), $10^{12} C_P \leq \log\log b$, and $C_P \geq 64$ by item 1 of Lemma 4.2.67). Hence

$$\sum_{\tau\in\Phi} s_\infty(\kappa(\tau)) \leq \sum_{\tau\in\Phi} s_\infty(c(\tau) - z). \tag{6.169}$$

Step 5: the multiplier. By Lemma 6.7.5 (for this z and $\sigma = 1$), $\|\varphi_{p,1}\|^2 \neq 0$ and $\|\varphi_{q,1}\|^2 \neq 0$. Put $H = y_q/\|\varphi_{p,1}\|^2$, which is nonzero, $\alpha = \beta_{\mathrm{den}} - \sum_\tau s_\infty(\kappa(\tau))$ and $h_\ell = 0$ for every prime ℓ. (HG2): $H\,\|\varphi_{p,1}\|^2 = y_q$ is integral at every ι by Step 2. (HG3): $\alpha + 0 \leq \alpha$. (HG1): let σ be an automorphism of $\mathbb{C}$. It maps K to itself; if it does not fix K, then σ', $\sigma'(w) = \overline{\sigma(w)}$, fixes K and $|\sigma'(H)| = |\sigma(H)|$, so we may assume that σ fixes K. Let C be the constant of Lemma 6.7.5 for z and σ. Applied to q ($c_q(\tau) - z = 0$ and $s_\infty(0) = 0$) and to p, it gives $\log|\sigma(\|\varphi_{q,1}\|^2)| = C + r_q$ with $r_q \leq 4f$ and $\log|\sigma(\|\varphi_{p,1}\|^2)| = C + r_p + \sum_\tau s_\infty(c(\tau) - z)$ with $r_p \geq -4f$. As $N_{\mathrm{ch}} n_q^2$ is a positive integer, σ fixes it, and by (6.169) and (6.168)

$$\log|\sigma(H)| = \log(N_{\mathrm{ch}} n_q^2) + r_q - r_p - \sum_\tau s_\infty(c(\tau) - z) \leq \log(N_{\mathrm{ch}} n_q^2) + 8f - \sum_\tau s_\infty(\kappa(\tau)) \leq \alpha.$$

So the constant family $\iota \mapsto \|\varphi_{p,1}\|^2$ has height at most $\alpha = \beta_{\mathrm{den}} - \sum_{\sigma\in\Phi} s_\infty(\kappa(\sigma))$. □

Lemma 6.7.5. *Let $z \geq 2$ and let σ be an automorphism of $\mathbb{C}$ fixing K. There is $C \in \mathbb{R}$ such that for every critical pair $p = (\nu, z)$, with ν of weight $-c$, we have $\|\varphi_{p,1}\|^2 \neq 0$ and*

$$\log|\sigma(\|\varphi_{p,1}\|^2)| = C + r + \sum_{\tau\in\Phi} s_\infty(c(\tau) - z)$$

for some real $r \geq -4f$, with moreover $r \leq 4f$ if $c(\tau) = z$ for every $\tau \in \Phi$.

Proof. Fix $z \geq 2$ and σ, put $n = 2z$, and let $a > 0$ be the number of Lemma 6.7.6 and $B_z > 0$ the number of Proposition 6.7.13, both for this z (and σ). Put

$$C = \log|\sigma(|c_s|^{4z})| + \log a + \log B_z,$$

which is a real number ($c_s \neq 0$ by (OL4), so $\sigma(|c_s|^{4z}) \neq 0$) depending only on D, z and σ. Let $p = (\nu, z)$ be a critical pair, ν of weight $-c$, $X = X(p, 1)$ and $J = j_p = \sum_\tau(c(\tau) - z)$.

Step 1: the scale factor. By (OL4), $\varphi_{p,1} = c_s^{2z}(\pi_M \eta_P)^J \lambda_X$, so, the inner product being sesquilinear and $|\pi_M|^2 = M$,

$$\|\varphi_{p,1}\|^2 = |c_s|^{4z}\, M^J\, |\eta_P|^{2J} \|\lambda_X\|^2.$$

At the trivial twist ($Q = \emptyset$, $\mathrm{N}Q = 1$), $\mathrm{alg}_p(1) = \|\varphi_{p,1}\|^2$, which is positive by Proposition 6.7.13; in particular $\|\varphi_{p,1}\|^2 \neq 0$. As $M^J \in \mathbb{Q}$ is fixed by σ,

$$|\sigma(\|\varphi_{p,1}\|^2)| = |\sigma(|c_s|^{4z})|\, M^J\, |\sigma(|\eta_P|^{2J}\|\lambda_X\|^2)|. \tag{6.170}$$

Step 2: the conjugate pair. By Lemma 6.7.6 there is a critical pair $p' = (\nu', z)$, ν' of weight $-c'$ with $c'(\sigma\tau) = c(\tau)$ for all $\tau \in \Phi$, such that, with $X' = X(p', 1)$,

$$|\sigma(|\eta_P|^{2J}\|\lambda_X\|^2)| = a\,\Omega^{-J}\,\|\lambda_{X'}\|^2.$$

As $\tau \mapsto \sigma\tau$ is a permutation of Φ, $j_{p'} = \sum_\tau(c'(\sigma\tau) - z) = J$ and $\prod_\tau(c'(\tau) - z)! = \prod_\tau(c(\tau) - z)!$.

Step 3: the norm of the conjugate lift. By Step 1 for p' and by Proposition 6.7.13 for p' at the trivial twist,

$$\begin{aligned}|c_s^{2z}(\pi_M\eta_P)^J|^2\|\lambda_{X'}\|^2 &= \mathrm{alg}_{p'}(1)\\ &= |c_s^{2z}(\pi_M\eta_P)^J|^2 B_z \prod_\tau (c'(\tau) - z)! \cdot d' \cdot L^{S_M}(z, \nu'), \qquad d' = \dim \mathbb{V}_{O_{X'}},\end{aligned}$$

and the scale factor is nonzero ($c_s \neq 0$, $\pi_M \neq 0$, $\eta_P \neq 0$ by (OL4)). Hence, by Step 2,

$$\|\lambda_{X'}\|^2 = B_z \prod_\tau (c(\tau) - z)! \cdot d' \cdot L^{S_M}(z, \nu') > 0.$$

Step 4: the L-value at $z \geq 2$. The character ν' takes values in the unit circle (a continuous character of the compact group $\mathrm{U}(1)(E)\backslash\mathrm{U}(1)(\mathbb{A})$), hence so does $\tilde{\nu}'$, and $L^{S_M}(z, \nu') = \prod_w (1 - a_w q_w^{-z})^{-1}$ over the finite places w of E outside $S_M \cup \bar{S}_M$, $q_w = \mathrm{N}w$, with $a_w = \tilde{\nu}'(\varpi_w)$ of modulus 1 if $\tilde{\nu}'$ is unramified at w and $a_w = 0$ otherwise. For real $0 \leq x < 1$ and $|u| \leq x$, $1 - x \leq |1 - u| \leq 1 + x$, so $|\log|1 - u|| \leq -\log(1 - x)$. With $x = q_w^{-z} \leq q_w^{-2}$, and since above each rational prime ℓ there are at most $[E : \mathbb{Q}] = 2f$ places, each with $q_w \geq \ell$,

$$\left|\log |L^{S_M}(z, \nu')|\right| \leq \sum_w -\log(1 - q_w^{-2}) \leq 2f \sum_\ell -\log(1 - \ell^{-2}) = 2f \log \zeta(2) = 2f \log \frac{\pi^2}{6} < f.$$

Step 5: assembling. By (6.170) and Steps 2 and 3,

$$\log|\sigma(\|\varphi_{p,1}\|^2)| = C + J \log M - J \log \Omega + \sum_\tau \log(c(\tau) - z)! + r, \qquad r = \log d' + \log|L^{S_M}(z, \nu')|.$$

By the definition of the jet size, $\sum_\tau s_\infty(c(\tau) - z) = \sum_\tau \log(c(\tau) - z)! + J \log(M/\Omega)$, which is the claimed formula. As $\mathbb{V}_{O_{X'}} = \mathbb{C}^{I_\kappa}$ is nonzero, $d' \geq 1$, and Step 4 gives $r > -f \geq -4f$. If $c(\tau) = z$ for every τ, then $c'(\tau) = z$ for every τ, the weight $\kappa_{O_{X'}}$ is zero, its only column filling is the empty one with $\Delta_\emptyset = 1$, so I_κ has one element, $d' = 1$ and $r = \log|L^{S_M}(z, \nu')| < f \leq 4f$. □

Lemma 6.7.6. *For an automorphism σ of $\mathbb{C}$ fixing K and $\tau \in \Phi$, let $\sigma\tau \in \Phi$ be the place induced by $\sigma \circ \iota_\tau$, where ι_τ is the embedding of E over K inducing τ; so $\tau \mapsto \sigma\tau$ is a permutation of Φ. Let $z \geq 2$ be an integer and σ an automorphism of $\mathbb{C}$ fixing K. There is a real number $a > 0$ such that for every critical pair $p = (\nu, z)$, with ν of weight $-c$, there is a critical pair $p' = (\nu', z)$, with ν' of weight $-c'$, such that $c'(\sigma\tau) = c(\tau)$ for every $\tau \in \Phi$ and*

$$\left|\sigma\left(|\eta_P|^{2j_p} \|\lambda_{X(p,1)}\|^2\right)\right| = a\, \Omega_s^{-j_p} \|\lambda_{X(p',1)}\|^2.$$

Proof. **The common transport is fixed before the critical pair.** Fix $z \geq 2$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, put $n = 2z$, and let $\mathcal{A}$ be the chosen places. Apply Lemma 4.2.14 in the standard positive rank-n model, identified with $\mathrm{Hom}(\mathrm{I}_1, \mathrm{I}_n)$ in column coordinates. Both have Gram matrix and adelic point equal to the identity; the sign-preserving coordinate bijection identifies their canonical functionals exactly by Lemma 4.1.17, with the same fixed line factors. Write $L_\sigma, t_\sigma, u_\sigma, C_\sigma$ for its transport, scale, chosen-place norm-one idele and nonzero vacuum scalar. Its Taylor, unitary-action and chosen-place assertions apply in this exact model. They give $|t_\sigma|^2 = \Omega_s^{-1}$ and fix these witnesses independently of all row lengths and critical characters. Write $e(\tau) = \sigma\tau$ and $m_\sigma(g) = \sigma(\Delta_n(g))/\Delta_n(g)$ for a rank-n finite unitary g.

Use the same assertions for $\sigma^\dagger = \mathrm{conj} \circ \sigma \circ \mathrm{conj}$, with its own fixed witnesses. Every embedding of a CM field commutes with complex conjugation, so $\sigma^\dagger|_E = \sigma|_E$ and both induce e. The character-conjugation assertion of Lemma 4.2.14 gives, for every critical pair $p = (\nu, z)$, one automorphic character ν' obtained from either automorphism. It has finite character $\sigma \circ \nu_f$, archimedean weight $-c'$ with $c'(e\tau) = c(\tau)$, and the same triviality conditions at finite subgroups. Thus $z \leq c' \leq 4b$, its tame part is trivial off the same chosen places, and $1 \leq z \leq t$ is unchanged. Therefore $p' = (\nu', z)$ is critical. Put

$$X = X(p, 1), \quad Y = X(p', 1), \quad \ell_\tau = c(\tau) - z, \quad J = \sum_\tau \ell_\tau.$$

The row lengths of Y are $\ell'_{e\tau} = \ell_\tau$, so $j_p = j_{p'} = J$.

The untwisted finite datum has one transport scalar. There are no twist places in either X or Y. By (DP7) their position function is the same rational-valued indicator: it is 1 on integral position vectors whose reduction at every chosen place is the fixed incidence column, and 0 otherwise. Denote its Schrödinger vector by $B_{\mathcal{A}}$. The chosen-place assertion (CT5) of Lemma 4.2.14, applied to this indicator, gives

$$L_\sigma B_{\mathcal{A}} = C_\sigma \omega(u_\sigma) B_{\mathcal{A}}. \tag{1}$$

Here and below $\omega(u_\sigma)$ means the scalar unitary action on the rank-n column space. In particular C_σ does not depend on p.

The character and multiplier factors match the same pair action. Write $\chi_X = \nu \upsilon^z$ and $\chi_Y = \nu' \upsilon^z$. For finite points $g \in G_f(1)$ and $h \in G_f(n)$, let g' be scalar multiplication by g^{-1} on the rank-n column and let h' be multiplication by h. Then

$$\omega_X(g,h) = \chi_X(g)^{-1} \omega(g') \omega(h').$$

The representation and commuting-action identities here are those of Lemma 4.1.1(2).

We check the scalar multiplier without commuting a field automorphism with complex conjugation. The hypotheses of Lemma 4.1.13 hold for the fixed finite model: c_0 is integral by (WD1), its lattice is self-dual by Lemma 4.1.3, and (WD3) makes the finite splitting character unitary, trivial on finite integral units and inverse under conjugation. Its normalized lattice measure and multiplier are the ones in (CT6). For a rank-one finite unitary v, put $\delta = \Delta_1(v)$. That lemma gives

$$\bar{\delta} = \upsilon(v)\delta, \qquad \delta\bar{\delta} = I(v) \in \mathbb{Z}_{>0}.$$

Applying σ to their algebraic product gives

$$\left(\frac{\sigma\delta}{\delta}\right)^2 = \frac{\upsilon(v)}{\sigma(\upsilon(v))}.$$

The direct-sum identity of the same lemma, with $v = g^{-1}$ and $n = 2z$, consequently gives

$$m_\sigma(g') = \left(\frac{\sigma(\upsilon(g))}{\upsilon(g)}\right)^z, \qquad \sigma(\chi_X(g)) = \chi_Y(g) m_\sigma(g'). \tag{2}$$

Apply (CT6) of Lemma 4.2.14 successively to the two finite unitary factors, and then use (1) and (2). The scalar u_σ commutes with g' and h', so

$$L_\sigma\big(\omega_X(g,h) B_{\mathcal{A}}\big) = C_\sigma \chi_Y(u_\sigma^{-1}) m_\sigma(h')\, \omega_Y(g u_\sigma^{-1}, h) B_{\mathcal{A}}. \tag{3}$$

Indeed the argument of the last underlying unitary is $hZ(gu_\sigma^{-1})^{-1} = u_\sigma h Z g^{-1}$, and its pair character is $\chi_Y(g)^{-1}\chi_Y(u_\sigma)$. This gives exactly the inverse character factor in (3).

The kernel coordinates are ordinary derivatives with the stated Gram normalization. For a one-row weight the standard basis of (WM1) is indexed by multi-indices $\mathbf{r} = (r_{\tau,i})$ with $\sum_i r_{\tau,i} = \ell_\tau$ and consists of the distinct monomials $\prod_{\tau,i} y_{\tau,i,0}^{r_{\tau,i}}$. Every such monomial occurs among the products of height-one minors, and these monomials are linearly independent; hence any basis consisting of products of minors is this basis, up to its indexing. The filling matrix has a row containing 1 precisely in the coordinate of its monomial. There are $\ell_\tau! / \prod_i r_{\tau,i}!$ labelled fillings at each place with those multiplicities. Therefore the Gram matrix is diagonal, with

$$G_{\mathbf{r},\mathbf{r}} = \prod_\tau \frac{\ell_\tau!}{\prod_i r_{\tau,i}!}, \qquad \mathrm{Pol}_\kappa(\xi) = \sum_{\mathbf{r}} G_{\mathbf{r},\mathbf{r}} \xi_{\mathbf{r}} X^{\mathbf{r}}. \tag{4}$$

The fixed mixed Cartan matrix in (DP8) is the identity after the zero-weight factor is removed. To prove this for the actual fixed choice, (c) and (d) of Lemma 3.2.20 show that it commutes with every Schur matrix and fixes the highest evaluation vector. Its orbit spans: if a linear form vanishes on that orbit, its homogeneous polynomial vanishes on every tuple of nonzero first columns of invertible matrices. These tuples form a dense open set, so the polynomial is zero; independence of the monomials makes the linear form zero. Thus the matrix is the identity. The positive letter substitution sends $y_{\tau,i,0}$ to the unique column variable $X_{\tau,i}$, and the zero-weight negative factor is 1. Consequently $\mathrm{Har}_X(\xi) = \mathrm{Pol}_\kappa(\xi)$.

For $F_X(g,h) = \omega_X(g,h)B_{\mathcal{A}}$ let $\mathcal{E}_F$ be the entire germ of the common transport lemma. By Lemma 4.1.5, the finite pair operator commutes with the real datum operation. By Lemma 4.2.2, the theta value of $v_{X^{\mathbf{r}}} \otimes F$ is exactly $\partial^{\mathbf{r}}\mathcal{E}_F(0)$. Substitute (4) in (DP8) and (OL2), and compare the coefficient of each arbitrary $\xi_{\mathbf{r}}$. The same nonzero integer $G_{\mathbf{r},\mathbf{r}}$ occurs on both sides and cancels. Hence

$$k_X(g)(h)_{\mathbf{r}} = \partial^{\mathbf{r}}\mathcal{E}_{F_X(g,h)}(Z)\Big|_{Z=0}. \tag{5}$$

This includes zero row lengths, and inserts no factorial divisor.

Taylor transport passes through the rank-one class sum. All class sums used here are finite. Indeed apply Lemma 6.4.74(1) to the zero scalar function, with maximal integral level $\mathbf{K}_0$ and with $\mathbf{K}'$ equal to the rank-one chosen level or the target level (DP4). These are open residue-defined subgroups, so the lemma gives finite class sets and finite stabilizers in both ranks. For a multi-index $\mathbf{r}$ of the weight of X, put $\mathbf{r}^e_{e\tau,i} = r_{\tau,i}$. Define the coordinate pullback from the weight of Y to that of X by $(R_e w)_{\mathbf{r}} = w_{\mathbf{r}^e}$. It is a bijection of coordinate spaces. Every multi-index in (5) has total degree J. Apply (CT1)–(CT2) to $F_X(g,h)$ and then (3). The ordinary chain rule and the explicit permutation convention of (CT1) give

$$\sigma\big(\eta_P^J k_X(g)(h)\big) = t_\sigma^J C_\sigma \chi_Y(u_\sigma^{-1}) m_\sigma(h')\, R_e k_Y(g u_\sigma^{-1})(h). \tag{6}$$

Right multiplication by u_σ^{-1} permutes the double cosets of the rank-one class sum (OL3). This group is abelian, so its stabilizer $\mathrm{U}(1)(E) \cap g\mathbf{K}_X^{\mathrm{ch}} g^{-1}$ is unchanged under this translation. The two critical pairs have the same chosen places and the same rank-one level. Thus their class sums have identical rational weights $1/\#\Gamma_g$, fixed by σ. Sum (6) to obtain

$$\sigma\big(\eta_P^J \lambda_X(h)\big) = t_\sigma^J C_\sigma \chi_Y(u_\sigma^{-1}) m_\sigma(h')\, R_e \lambda_Y(h). \tag{7}$$

All witnesses preceding p remain independent of it.

Polarization uses the conjugate automorphism, rather than commuting it with conjugation. Write $A_X = \eta_P^J \lambda_X$ and $B_Y = \lambda_Y$. Apply (7) both to σ and to $\sigma^\dagger$; the critical pair p' and the permutation e are the same, as established above. Formula (4) shows that the two Gram matrices are carried to one another by R_e, since the row lengths are merely permuted. Both target levels are the same: in (DP4) the chosen-place level depends on the fixed incidence column, and there are no twist places or level characters. Their norm sums therefore have the same rational stabilizer weights.

For any complex number w one has $\sigma(\bar{w}) = \overline{\sigma^\dagger(w)}$ by definition. Expand the finite class sum $\langle A_X, A_X\rangle = \sum_{[h]} (\#\Gamma_h)^{-1} \overline{A_X(h)}^{\mathsf{T}} G_X A_X(h)$, apply this identity coordinatewise and use (7) for its two factors. By (CT7), the point-dependent multiplier cancels: $m_\sigma(h')\overline{m_{\sigma^\dagger}(h')} = 1$. The result is

$$\sigma\langle A_X, A_X\rangle = \overline{t_{\sigma^\dagger}^J C_{\sigma^\dagger} \chi_Y(u_{\sigma^\dagger}^{-1})}\, t_\sigma^J C_\sigma \chi_Y(u_\sigma^{-1}) \langle B_Y, B_Y\rangle. \tag{8}$$

Every character value displayed here has modulus one because χ_Y is unitary. Both scales have squared modulus Ω_s^{-1}. Set $a = |C_\sigma|\,|C_{\sigma^\dagger}| > 0$. It is fixed before p, because both common

transport scalars were fixed before its character and row lengths. Taking absolute values in (8) gives

$$\left|\sigma\left(|\eta_P|^{2j_p}\|\lambda_{X(p,1)}\|^2\right)\right| = a\,\Omega_s^{-j_p}\|\lambda_{X(p',1)}\|^2.$$

The identities also hold for a zero lift and for $J = 0$, without division by either lift. This proves the frozen assertion. □

Lemma 6.7.7. *Let $q = (\nu_q, z_q)$ be a flat critical pair (Definition 4.2.4) with $z_q \geq 2$, let ϑ be a tame character with modulus Q, ℓ a prime with $\ell \nmid \mathrm{N}Q$ and $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism. Then the form $n_{\nu_q}\varphi_{q,\vartheta}$ is integral at j.*

Proof. Let $X = X(q,\vartheta)$ and $N = 2z_q$. The row lengths $c_q(\tau) - z_q$ vanish, so $j_q = 0$, the weight of O_X is 0, $\varphi_{q,\vartheta} = c_s^N\lambda_X$ and $n_{\nu_q} = M^{4fz_q} = M^{2fN}$. For a weight-0 form ($e_O = 0$, $S_0 = 1$) integrality at j means $\|j(\Xi(h))\| \leq 1$ for every $h \in G_f(N)$. The group $\mathbf{K}_X^{\mathrm{ch}}$ is an open subgroup of the stabiliser of $\widehat{\mathcal{O}}_E$ in $G_f(1)$ whose elements reduce to 1 at every chosen place, and a chosen place exists ($M > 1$); so by (c) of Lemma 4.6.2, for the definite line I_1 with the lattice $\mathcal{O}_E$, every $\Gamma_g = \mathrm{U}(1)(E) \cap g\mathbf{K}_X^{\mathrm{ch}}g^{-1}$ is trivial. Hence

$$n_{\nu_q}\varphi_{q,\vartheta}(h) = \sum_{[g]} M^{2fN} c_s^N\, k_X(g)(h),$$

a finite sum (the class set is finite, [33]). By (a) of Lemma 6.7.8 ($\ell \nmid \mathrm{N}Q$) each term has $\|j(\cdot)\| \leq 1$, and by the ultrametric inequality so has the sum. So $n_{\nu_q}\varphi_{q,\vartheta}$ is integral at j. □

Lemma 6.7.8. *Let $q = (\nu_q, z_q)$ be a flat critical pair (Definition 4.2.4) with $z_q \geq 2$, and put $N = 2z_q$. Let $g \in G_f(1)$ and $h \in G_f(N)$.*

(a) If ϑ is a tame character with modulus Q, ℓ a prime with $\ell \nmid \mathrm{N}Q$ and $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism, then $\|j(M^{2fN}c_s^N\, k_{X(q,\vartheta)}(g)(h))\| \leq 1$.

(b) For every automorphism σ of $\mathbb{C}$,

$$|\sigma(c_s^N\, k_{X(q,1)}(g)(h))| \leq M^{fN}\exp\bigl(C_P\,(q_1N+1)^2 + fN\log 4\bigr).$$

Proof. Let $X = X(q,\vartheta)$ and $O = O_X$. By (OL4), X has rank N, row lengths $c(\tau) - z_q = 0$ (the weight of ν_q is $-z_q$ at every τ), character $\nu_q\vartheta$, chosen places M_O the chosen places of D and twist places Q_O the places of Q. By (OL1), O has ranks $1 < N$, $W(O)$ is the standard definite line, $\kappa_O = 0$, $e_O = 0$, $\chi_O = 1$ and $\chi'_O = \nu_q\vartheta\upsilon^{z_q}$; the tame part $(\psi_{O,w})_w$ of $\mu_O = \nu_q\vartheta$ is trivial off $M_O \cup Q_O$.

Step 1: the kernel form is a theta value of the block datum. The weight 0 has exactly one column filling, the empty one, whose product of minors is 1; so the standard basis of $\nabla_0 = \mathbb{Z}$ is (1), $\mathbb{V}_O = \mathbb{C}$, $F_0 = G_0 = (1)$, $\mathrm{Pol}_0(\xi) = \xi$ and $\mathrm{ev}_0(y) = (1)$. For $\kappa^+ = \kappa^- = 0$, (c) of Lemma 3.2.20 reads $C_O \cdot 1 = 1$, so $C_O = (1)$, and since the letter substitutions fix constants, $\mathrm{Har}_O(\xi)$ is the constant polynomial ξ. As v_P is linear in P and data are linear in the Fock vector, $\mathrm{J}_O(\xi) = \xi\, v_1 \otimes \Upsilon(O) = \xi\,\Upsilon(O)$, by (4) of Lemma 6.2.33 for $A = S_O$ and $\Psi = \Phi_O$. Taking $\xi = 1$ in (OL2), with $G_0 = (1)$,

$$k_X(g)(h) = \theta_O(\omega_O((1,g),(1,h))\,\Upsilon(O)). \tag{6.171}$$

By (VP5), $\mathrm{Hom}(W(O), \mathrm{I}_N)$ has index set $N \times \{1\}$, identified with $\{1,\dots,N\}$, all signs $+1$, matrix 1 and point 1: it is the standard space V_s of the signs $s = (+1,\dots,+1)$, that is I_N. Since $k_{V_s} = 1$ in (WC2), (WC3) gives $\theta_O = \theta_{\mathrm{I}_N}$, the canonical functional used in Lemma 6.4.83.

Step 2: the pair action is a root of unity times a finite unitary operator. The pair map of (VP5) sends $((1,g),(1,h))$ to $X \mapsto hXg^{-1}$, i.e. to the element of $\mathrm{G}(s)$ with infinite component 1 and finite component $k_f = g^{-1}h \in G_f(N)$. By (DP6) and (WR4), with $\chi_O = 1$,

$$\omega_O((1,g),(1,h)) = \chi'_O((1,g))^{-1}\,\omega_s((1,k_f)).$$

We show that $\chi'_O((1,g))$ is a root of unity. The weights of ν_q, ϑ, υ are $-z_q$, 0, 1, and each has a tame part ((OL1), Definition 3.1.2, (WD3)); multiplying the defining identities of Definition 3.1.1, χ'_O has weight 0 and a tame part ψ'. Let h_E be the class number of E. The ideal of the finite idele g to the power h_E is principal, so $g^{h_E} = \gamma_0 u_0$ with $\gamma_0 \in E^\times$ and $u_0 \in \widehat{\mathcal{O}}_E^\times$. As $\bar{g}g = 1$, conjugation gives $g^{-h_E} = \bar{\gamma}_0\bar{u}_0$, hence $g^{2h_E} = \gamma u$ with $\gamma = \gamma_0/\bar{\gamma}_0 \in \mathrm{U}(1)(E)$ and $u = u_0/\bar{u}_0 \in \widehat{\mathcal{O}}_E^\times$, $u\bar{u} = 1$. In $\mathrm{U}(1)(\mathbb{A})$ this reads $(1,g)^{2h_E} = \gamma\cdot(\gamma_\infty^{-1},u)$, γ_∞ the infinite components of γ. Since χ'_O is trivial on $\mathrm{U}(1)(E)$ and has weight 0, and (γ_∞^{-1},u) has integral finite components,

$$\chi'_O((1,g))^{2h_E} = \prod_w \psi'_w(u \bmod w),$$

a finite product of values of characters of the finite groups $k_w^\times$, hence a root of unity; so $\chi'_O((1,g))$ is a root of unity.

Step 3: finite Fourier expansion of the block datum. Let $A = S_O = M_O \cup Q_O$; by (DP1) it contains the conjugate of none of its members, so $A \cap \bar{A} = \emptyset$. Fix a uniformiser at each $w \in A$ and let $\varpi \in \mathbb{A}_{E,f}^\times$ have these components at A and 1 elsewhere. Let $L_A = \prod_{w\in A}\mathcal{O}_{E,w}^N$, $R_A = \prod_{w\in A} k_w$ and $d_A = \#R_A = \prod_{w\in A}\mathrm{N}w$; reduction identifies $L_A/\varpi L_A$ with R_A^N, and vectors supported at A are written as elements of $\prod_{w\in A} E_w^N$. Put $\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$; by Lemma 4.1.3, $\widehat{\mathcal{O}}_E^N$ is its own annihilator under $(u,v)\mapsto\psi_0(\bar{u}^{\mathsf{T}}v)$, and in particular $\psi_0(\widehat{\mathcal{O}}_E) = 1$. Let $Y_A = \varpi^{-1}L_A$. For $z \in Y_A$ the character $x \mapsto \psi_0(z^{\mathsf{T}}x)$ of L_A is trivial on ϖL_A and depends only on z mod L_A. If it is trivial on L_A, then $\psi_0(z^{\mathsf{T}}v) = 1$ for every $v \in \widehat{\mathcal{O}}_E^N$ (as z is supported at A), so $\bar{z}$, hence z, is integral, i.e. $z \in L_A$. Thus $z \mapsto \psi_0(z^{\mathsf{T}}\cdot)$ is an injective homomorphism from Y_A/L_A, of order d_A^N, into the character group of R_A^N, of the same order, hence a bijection. Orthogonality of characters of the finite group R_A^N gives, for x integral at A and $u \in R_A^N$ with a lift $\tilde{u} \in L_A$, that $d_A^{-N}\sum_{z\in Y_A/L_A}\psi_0(z^{\mathsf{T}}(x_A - \tilde{u}))$ is 1 if x_A mod $\varpi = u$ and 0 otherwise. Summing against Ψ_O, for every finite adelic vector x,

$$\Phi_O(x) = \sum_{z\in Y_A/L_A}\widehat{\Phi}(z)\,\psi_0(z^{\mathsf{T}}x_A)\,\mathbf{1}_A(x), \qquad \widehat{\Phi}(z) = d_A^{-N}\sum_{u\in R_A^N}\Psi_O(u)\,\psi_0(-z^{\mathsf{T}}\tilde{u}),$$

the sum over a fixed set of representatives z. For such z let $y(z)$ be the finite adelic vector supported at $\bar{A}$ with $\overline{y(z)} = z$. Then $y(z)_A = 0$, and (WR6) gives $\tau_{y(z)}(x) = \psi_0(z^{\mathsf{T}}x_A)\mathbf{1}_A(x)$; so $\Phi_O = \sum_z \widehat{\Phi}(z)\tau_{y(z)}$. For y supported at $A\cup\bar{A}$, (2) of Lemma 4.1.3 applied to $\mathbf{1}_A$ gives $\tau_y \otimes_A f = \rho(\mathrm{fv}(y))(\mathbf{1}_A\otimes_A f)$, and $\mathbf{1}_A\otimes_A v^0 = P^A v^0 = v^0$ by (1) of Lemma 4.1.3 and (2) of Lemma 6.2.33. By linearity of $\Psi\mapsto\Psi\otimes_A v^0$,

$$\Upsilon(O) = \sum_z \widehat{\Phi}(z)\,\rho(\mathrm{fv}(y(z)))v^0. \tag{6.172}$$

Step 4: the kernel value as a combination of moved-vacuum theta values. By (2) of Lemma 4.1.1, $\omega_s((1,k_f))\rho(\mathrm{fv}(y)) = \rho((1,k_f)\mathrm{fv}(y))\omega_s((1,k_f))$, and $(1,k_f)\mathrm{fv}(y) = (0,k_f\alpha y) = \mathrm{fv}(k_f y)$, α being a scalar. With $\mathcal{V}_N(x,k) = \theta_{\mathrm{I}_N}(\rho(\mathrm{fv}(x))\omega_s((1,k))(c_s^N v^0))$ as in Lemma 6.4.83, (6.171), Step 2 and (6.172) give

$$c_s^N k_X(g)(h) = \chi'_O((1,g))^{-1}\sum_z \widehat{\Phi}(z)\,\mathcal{V}_N(k_f y(z),k_f). \tag{6.173}$$

Step 5: (a). Every value of Ψ_O is 0 or a product of values of characters of the groups $k_w^\times$, and every value of ψ_0 is $\exp(2\pi i \operatorname{Tr}_{E/\mathbb{Q}}(a))$ for some $a \in E$; all are 0 or roots of unity, so $d_A^N \widehat{\Phi}(z)$ is integral over $\mathbb{Z}$. Every $w \in A$ contains $M\,\mathrm{N}Q$: a chosen place contains π_M, hence $M = \pi_M \bar{\pi}_M$, and a place w of Q contains the rational prime below it, which divides $\mathrm{N}w$ and so $\mathrm{N}Q$. By the Chinese remainder theorem $R_A = \mathcal{O}_E/\prod_{w\in A} w$ is a quotient of $\mathcal{O}_E/M\mathrm{N}Q\,\mathcal{O}_E$, of order $(M\mathrm{N}Q)^{2f}$ as $[E:\mathbb{Q}] = 2f$. So $t = (M\mathrm{N}Q)^{2fN}/d_A^N$ is an integer and $M^{2fN}\widehat{\Phi}(z) = t\,\mathrm{N}Q^{-2fN}\, d_A^N \widehat{\Phi}(z)$. As $\ell \nmid \mathrm{N}Q$, $\|j(M^{2fN}\widehat{\Phi}(z))\| \le 1$. Further $\|j(\chi'_O((1,g))^{-1})\| = 1$ by Step 2, and $\|j(\mathcal{V}_N(k_f y(z), k_f))\| \le 1$ by (1) of Lemma 6.4.83. Multiplying (6.173) by M^{2fN}, applying j and the ultrametric inequality to the finite sum gives (a).

Step 6: (b). Now $\vartheta = 1$, so $Q = \emptyset$, $A = M_O$, and the product over Q_O in (DP7) is empty: Ψ_O is the indicator of the single point $u_0 = (\mathrm{Inc}_O)_{w\in M_O}$ of R_A^N. Hence $\widehat{\Phi}(z) = d_A^{-N}\psi_0(-z^{\mathsf{T}}\tilde{u}_0)$, and $|\sigma(\widehat{\Phi}(z))| = d_A^{-N}$ for every automorphism σ of $\mathbb{C}$, since σ maps roots of unity to roots of unity. Also $|\sigma(\chi'_O((1,g))^{-1})| = 1$, and $|\sigma(\mathcal{V}_N(k_f y(z), k_f))| \le \exp(C_P(q_1 N+1)^2)$ by (2) of Lemma 6.4.83. Applying σ to (6.173), a sum of d_A^N terms,

$$|\sigma(c_s^N k_{X(q,1)}(g)(h))| \le d_A^N \cdot d_A^{-N} \exp(C_P(q_1N+1)^2) \le M^{fN} \exp\bigl(C_P(q_1N+1)^2 + fN\log 4\bigr),$$

the last step as $M \ge 1$ and $fN\log 4 \ge 0$. This is (b). □

Lemma 6.7.9. *Let D be admissible data, $1 \le n \le rt$, $K' \subseteq \mathbf{K}_0(n)$ an open subgroup and $h \in G_f(n)$. Then $\Gamma_h = \mathrm{U}(\mathrm{I}_n)(E) \cap hK'h^{-1}$ is a finite group whose order divides N_{ch}.*

Proof. The group $\mathrm{U}(\mathrm{I}_n)(E)$ is discrete in $G_\infty(1)\times G_f(n)$ and $G_\infty(1)\times hK'h^{-1}$ is compact, so Γ_h is finite. As $M = p_0p_1 > 1$, π_M is not a unit of $\mathcal{O}_E$, so there is a chosen place w. For $\gamma \in \Gamma_h$ the element $h^{-1}\gamma h$ lies in $K' \subseteq \mathbf{K}_0(n)$, so its component at w lies in $\mathrm{GL}_n(\mathcal{O}_{E,w})$, and $\gamma \mapsto \mathrm{red}_w(h^{-1}\gamma h)$ is a homomorphism $\Gamma_h \to \mathrm{GL}_n(k_w)$. If γ is in its kernel, then $(h^{-1}\gamma h)_w$ has finite order and is $\equiv 1 \pmod{w}$, so it is 1 by (b) of Lemma 4.6.2; hence $\gamma_w = 1$, and $\gamma = 1$, a rational point being determined by its component at w. So $\#\Gamma_h$ divides $\#\mathrm{GL}_n(k_w)$. As $n \le rt$, $\mathrm{GL}_n(k_w)$ is a subgroup of $\mathrm{GL}_{rt}(k_w)$ (block diagonal with the identity of rank $rt-n$), so $\#\mathrm{GL}_n(k_w)$ divides $\#\mathrm{GL}_{rt}(k_w)$, which divides N_{ch} by (CX1). □

Lemma 6.7.10. *For admissible data D, $1 \le N_{\mathrm{ch}}$ and $\log N_{\mathrm{ch}} \le x^{33}$.*

Proof. Each factor $\#\mathrm{GL}_{rt}(k_w)$ is a positive integer at most $(\mathrm{N}w)^{(rt)^2}$. By Lemma 4.2.66, item 3, and Lemma 4.2.67, items 3–5,

$$\log N_{\mathrm{ch}} \le (rt)^2 \sum_{w \text{ chosen}} \log \mathrm{N}w = (rt)^2 f \log M \le (45x^{10})^2 \cdot x^{12} \cdot 22 \log b \le x^{33},$$

using $rt = 5r^2 \le 45x^{10}$, $f \le x^{12}$ and $\log b \le x^{1/100}$. □

Lemma 6.7.11. *For admissible data D,*

$$x_1 \le \beta_{\mathrm{den}} \qquad \textit{and} \qquad \frac{f\,m\,b}{k} + 3x^{34} \le \beta_{\mathrm{den}}.$$

Proof. By Lemma 2.3.6, $\log x \ge 10^{14} C_P \ge 10^{14}$ (as $C_P \ge q_0 \ge 18480$), $\log x_1 \ge 10^6 \log x$ and $\log b \ge 10^6 \log x_1$; so $x \ge e^{10^{14}}$, $\log b \ge 10^{12}\log x$, that is $b \ge x^{10^{12}}$, and $\log b \ge 2$. By (AD2), $m \le 2\sqrt{x}$ and $x^5 \le r \le 3x^5$, so $k = r + 1 - m$ satisfies $1 \le x^5 + 1 - 2\sqrt{x} \le k \le r \le 3x^5$; and $f \ge 1$. Hence

$$\beta_{\mathrm{den}} = \frac{f\,x\,b\,(\log b)^3}{m\,k} \ge \frac{x\,b}{2\sqrt{x}\cdot 3x^5} = \frac{b}{6x^{9/2}} \ge \frac{b}{x^5} \ge x^{10^{12}-5},$$

using $6 \le x^{1/2}$.

First inequality. $\log x_1 \le 10^{-6}\log b$ gives $x_1 \le b^{10^{-6}}$, and $b^{1-10^{-6}} \ge x^{(1-10^{-6})10^{12}} \ge x^5$, so $x_1 \le b^{10^{-6}} \le b/x^5 \le \beta_{\mathrm{den}}$.

Second inequality. $\beta_{\mathrm{den}}/\frac{fmb}{k} = \frac{x(\log b)^3}{m^2} \ge \frac{x\cdot 8}{4x} = 2$, so $\frac{fmb}{k} \le \frac{1}{2}\beta_{\mathrm{den}}$; and $3x^{34} \le \frac{1}{2}x^{10^{12}-5} \le \frac{1}{2}\beta_{\mathrm{den}}$. Adding, $\frac{fmb}{k} + 3x^{34} \le \beta_{\mathrm{den}}$. □

Lemma 6.7.12. *For every finite set T of primes,*

$$\sum_{\ell\in T} 2F(\ell)\log\ell \;\le\; \sum_{\sigma\in\Phi} s_{\mathrm{fin}}(\kappa(\sigma),\sigma).$$

Proof. Fix $\sigma\in\Phi$ and put $y=\kappa(\sigma)$ and $g(\nu)=\Lambda_K(\nu)\big(y/\nu-\kappa(-\nu\sigma)\big)_+$ for integers $\nu\ge 1$. Each $g(\nu)\ge 0$, since $\Lambda\ge 0$ and $1-\chi_K(\nu)\in\{0,1,2\}$. If $g(\nu)\ne 0$ then $y/\nu>\kappa(-\nu\sigma)\ge 1$, so $\nu<y=\kappa(\sigma)\le a$ (both values b and a/x_1 of κ are at most $a=xb$). Hence $s_{\mathrm{fin}}(y,\sigma)=\sum_{\nu\ge1} g(\nu)$, a finite sum; as Λ_K vanishes off prime powers and distinct pairs (p,e) give distinct p^e,

$$s_{\mathrm{fin}}(y,\sigma)=\sum_p\sum_{e\ge1} g(p^e) \;\ge\; \sum_{p\in T}\sum_{e\ge1} g(p^e).$$

For a prime p and $e\ge1$, $\Lambda_K(p^e)=\log p\,(1-\chi_K(p)^e)$, which is 0 if $\chi_K(p)=1$; $2\log p$ for odd e and 0 for even e if $\chi_K(p)=-1$; and $\log p$ if $p=d$ ($\chi_K(d)=0$). In all cases $\Lambda_K(p^e)=2\log p\cdot v(\pi_p)\,\mathbf{1}_{I_p}(e)$, so $\sum_{e\ge1} g(p^e)=2\log p\cdot s_p(\kappa(\sigma),\sigma)$. Summing over $\sigma\in\Phi$,

$$\sum_{\sigma\in\Phi} s_{\mathrm{fin}}(\kappa(\sigma),\sigma) \;\ge\; \sum_{p\in T} 2\log p\sum_{\sigma\in\Phi} s_p(\kappa(\sigma),\sigma) = \sum_{p\in T} 2F(p)\log p.$$

□

Proposition 6.7.13. *Let $z\ge 2$. There is a real number $B_z>0$, depending only on D and z, with the following property. Let $p=(\nu,z)$ be a critical pair, ν of weight $-c$, and put*

$$A=\big|c_s^{2z}(\pi_M\eta_P)^{j_p}\big|^2\,B_z\prod_{\tau\in\Phi}(c(\tau)-z)!\cdot\dim\mathbb{V}_{O_{X(p,1)}}.$$

Then for every tame character ϑ with modulus Q

$$\mathrm{alg}_p(\vartheta)=A\cdot L^{S_M\cup Q}(z,\nu\vartheta)\qquad\text{and}\qquad \mathrm{alg}_p(\vartheta)>0.$$

In particular $A\ne0$, $L^{S_M\cup Q}(z,\nu\vartheta)\ne0$, and $r_p(\vartheta)=\mathrm{alg}_p(\vartheta)/\mathrm{alg}_p(1)$.

Proof. Fix p and a tame character ϑ with modulus Q. Put $n=2z$, $X=X(p,\vartheta)$, $O=O_X$, $S=S_M\cup Q$ (the places of the level system of X), and $q_w=\mathrm{N}w$. By (DP1) the places of S are distinct, and none is the conjugate of itself or of another one; the character of X is $\nu_X=\nu\vartheta$, with tame part trivial off S, and its row lengths $\ell_X(\tau)=c(\tau)-z$ depend only on p.

Step 1: two indices. Let w be a finite place with $w\ne\bar w$. The relation $g^*g=1$ determines the component of $g\in G_f(m)$ at $\bar w$ from the one at w, which is an arbitrary element of $\mathrm{GL}_m(E_w)$; integral points correspond to $\mathrm{GL}_m(\mathcal{O}_{E,w})$. As the places of S are pairwise distinct and not conjugate to one another, reduction $\mathbf{K}_0(m)\to\prod_{w\in S}\mathrm{GL}_m(k_w)$ is surjective (prescribe the components at $w\in S$, the forced ones at $\bar w$, and 1 elsewhere; $\mathrm{GL}_m(\mathcal{O}_{E,w})\to\mathrm{GL}_m(k_w)$ is onto). Hence $[\mathbf{K}_0(1):\mathbf{K}_X^{\mathrm{ch}}]=\prod_{w\in S_M}\#k_w^\times=\prod_{w\in S_M}(q_w-1)$, and $[\mathbf{K}_0(n):\mathbf{K}(O)]=[\prod_{w\in S}\mathrm{GL}_n(k_w):\mathbf{L}(O)]$, where $\mathbf{L}(O)$ is the product of its local conditions (DP4) for the one index $i_0=\mathrm{inc}_O(1)$: at

$w \in S_M$, j_w fixes the basis vector e_{i_0}; at $w \in Q$, row i_0 of j_w vanishes outside column i_0, i.e. j_w^{T} preserves the line $k_w e_{i_0}$. Since $\mathrm{GL}_n(k_w)$ acts transitively on the $q_w^n - 1$ nonzero vectors and (through $j \mapsto j^{\mathsf{T}}$) on the $(q_w^n - 1)/(q_w - 1)$ lines,

$$[\mathbf{K}_0(n) : \mathbf{K}(O)] = \prod_{w \in S_M} (q_w^n - 1) \prod_{w \in Q} \frac{q_w^n - 1}{q_w - 1}.$$

Step 2: the norm of λ_X as a Petersson norm. By Lemma 6.7.14 (a), (b),

$$\|\lambda_X\|^2 = [\mathbf{K}_0(1) : \mathbf{K}_X^{\mathrm{ch}}]^2 \, [\mathbf{K}_0(n) : \mathbf{K}(O)] \, \dim \mathbb{V}_O \, \langle f_X, f_X \rangle_{\mathrm{I}_n},$$

with f_X the lift of the constant function $\mathbf{1}$ of Lemma 6.7.14 (c).

Step 3: the inner product formula. We apply Lemma 6.5.12 with $(S, T) = (\mathrm{I}_1, \mathrm{I}_n)$, so $2 \operatorname{rk} S = 2 < n = \operatorname{rk} T$; the characters $a = \chi_O = v^0 = 1$ (all signs of I_1 are $+1$) and $b = \chi_O' = \nu_X v^z$, of weights $\gamma = 0$ and $\gamma' = -(\ell_X + z) + z = -\ell_X$; the Fock vector $v = v_{\mathrm{Har}_O(\xi_X)}$, an edge for the weights $(0, -\ell_X)$ from the type $\mathbf{1}$ to e_X' by Lemma 6.7.14 (c); S_0 the set of places of the maximal totally real subfield E^+ (the base field of Lemma 6.5.12) below the places of S (so its complement Σ consists of the finite places of E^+ below no place of S, and $k_v \in K_v$ for $v \in \Sigma$ means that k is integral at the places of E outside $S \cup \bar{S}$), $J = J_O = \mathbf{K}_X^{\mathrm{ch}}$ and $F = \Upsilon(O) = \Phi_O \otimes_S v^0$. The vector v is a Schwartz vector, being the Fock vector of a polynomial (XD3). The hypotheses on F: by Lemma 6.5.10 (a), $\langle F, F \rangle \neq 0$ and $\langle F, F \rangle / [\mathbf{K}_0(1) : J] = c_O$; by Lemma 6.2.33 (4), $v_1 \otimes F = F$; by Lemma 6.5.10 (b′), F is a vacuum above Σ; this is (SH1). By Lemma 6.5.10 (c), for $k \in G_f(1)$ integral off $S \cup \bar{S}$, $\langle F, \omega_{1,b}((1,k),1)F \rangle = \langle F, \omega_O((1,k),1)F \rangle$ is $\langle F, F \rangle$ for $k \in J$ and 0 otherwise (SH2); all signs of I_1 agree (SH3); and by Lemma 6.5.10 (e), with $\omega_{a,b} = \omega_O$, F is fixed by $\omega_{a,b}((1,k),(1,k'))$ for k, k' in open subgroups of $G_f(1)$, $G_f(n)$ (SH4). So F is sharp for J off S_0 with mass c_O, where, O having ranks $1 < n$,

$$c_O = \prod_{w \in S_M} \frac{1}{q_w^n (q_w - 1)} \prod_{w \in Q} \frac{(q_w - 1)\, q_w^{n-1}}{q_w^n}.$$

By Lemma 6.7.14 (c) the lift by $v \otimes F = \mathrm{J}_O(\xi_X)$ of the constant function $\mathbf{1}$, a J-invariant form of type $\mathbf{1}$, is f_X. With $y = y' = \mathbf{1}$, and $\langle \mathbf{1}, \rho(q)\mathbf{1} \rangle_{\mathrm{I}_1} = V_1$, the volume of $[\mathrm{U}(\mathrm{I}_1)]$ (finite and positive, the quotient being compact), Lemma 6.5.12 gives, with its constant $A_v \neq 0$ depending only on I_1, I_n and v,

$$\langle f_X, f_X \rangle_{\mathrm{I}_n} = A_v \, c_O \, V_1 \sum_q \mathrm{mc}(q),$$

the sum running over $G_f(1)$ modulo the points integral outside $S \cup \bar{S}$, represented by points with component 1 at $S \cup \bar{S}$; the matrix coefficient of the unramified vector for b and $\operatorname{rk} T = 2z$ is the normalised vacuum coefficient mc of Lemma 6.7.16 for ν_X and $k = z$.

Step 4: the vacuum sum. By Lemma 6.7.15, for ν_X, the set S, $D = S \cup \bar{S}$ and $k = z \geq 2$ (its hypothesis $\psi_w = 1$ off S holds), mc is constant on each coset at representatives with component 1 at D, and the sum is $L^S(z, \nu_X) \prod_v e_v(z)$ over the places v of F below no place of D. The places of F below D are those below the $w \in S$; each such v splits as $w\bar{w}$, and $w \mapsto v$ is injective on S. With $P_z = \prod_v e_v(z)$ over all finite places of F (absolutely convergent and positive by (c) of Lemma 6.7.15),

$$\sum_q \mathrm{mc}(q) = L^S(z, \nu\vartheta)\, P_z \prod_{w \in S} (1 - q_w^{-n})^{-1}.$$

Step 5: collecting the factors. By (OL4), $\mathrm{alg}_p(\vartheta) = \left| c_s^{2z} (\pi_M \eta_P)^{j_P} \right|^2 \|\lambda_X\|^2 / \mathrm{N}Q^{2z-1}$. Insert Steps 1–4. At $w \in S_M$ the place contributes $(q_w - 1)^2 \cdot (q_w^n - 1) \cdot \frac{1}{q_w^n (q_w - 1)} \cdot \frac{q_w^n}{q_w^n - 1} = q_w - 1$, and at

$w \in Q$ it contributes $\frac{q_w^n-1}{q_w-1}\cdot\frac{(q_w-1)q_w^{n-1}}{q_w^n}\cdot\frac{q_w^n}{q_w^n-1} = q_w^{n-1}$; the product of the latter is $\mathrm{N}Q^{2z-1}$. Hence

$$\mathrm{alg}_p(\vartheta) = \left|c_s^{2z}(\pi_M\eta_P)^{j_p}\right|^2 \prod_{w\in S_M}(q_w-1)\cdot\dim\mathbb{V}_O\cdot A_v\,V_1\,P_z\cdot L^{S_M\cup Q}(z,\nu\vartheta). \tag{6.174}$$

The constant A_v. The type on I_1 is $e=\mathbf{1}$, so $\overline{e(x)}/e(1)=1$ and, by its definition in Lemma 6.5.12, $A_v = \mathrm{vol}([\mathrm{U}(\mathrm{I}_n)])\,\lambda_{\mathrm{I}_1}\int_{G_\infty(+)}\langle v,\Omega_{0,-\ell_X}(x,1)v\rangle\,dx$, with $\lambda_{\mathrm{I}_1}>0$ the constant of Lemma 6.5.12 for $S=\mathrm{I}_1$ and $G_\infty(+)=\mathrm{U}(1)^\Phi$. By Lemma 6.7.14 (d), $\Omega_{0,-\ell_X}(x,1)v=v$, and $G_\infty(+)$ has mass 1 (Lemma 5.5.21 for $N=1$), so the integral is $\|v\|^2$. Again by Lemma 6.7.14 (d), $v=v_{P_0}$ with $P_0=\prod_\tau X_{\tau,1,1}^{\ell_X(\tau)}$. All signs of $\mathrm{Hom}(\mathrm{I}_1,\mathrm{I}_n)$ are $+1$, so by (WR5) $v_{P_0}(x)=P_0(\bar{x})\,e^{-x^*x/2}$ on $\mathbb{C}^{\Phi\times n\times 1}$, and by Fubini over the coordinates

$$\|v\|^2=\int|P_0(\bar{x})|^2e^{-x^*x}\,d\nu_\iota(x)=\prod_{\tau\in\Phi}\Big(\frac{1}{\pi}\int_{\mathbb{C}}|y|^{2\ell_X(\tau)}e^{-|y|^2}dy\Big)\cdot\Big(\frac{1}{\pi}\int_{\mathbb{C}}e^{-|y|^2}dy\Big)^{f(n-1)}=\prod_{\tau\in\Phi}\ell_X(\tau)!,$$

since $\frac{1}{\pi}\int_{\mathbb{C}}|y|^{2m}e^{-|y|^2}dy=\int_0^\infty u^me^{-u}\,du=m!$ (polar coordinates, $u=|y|^2$). Hence $A_v=\mathrm{vol}([\mathrm{U}(\mathrm{I}_n)])\,\lambda_{\mathrm{I}_1}\prod_\tau(c(\tau)-z)!$. *The constant A.* Put

$$B_z=\prod_{w\in S_M}(q_w-1)\cdot\mathrm{vol}([\mathrm{U}(\mathrm{I}_{2z})])\,\lambda_{\mathrm{I}_1}\,V_1\,P_z.$$

It depends only on D (through E and S_M) and z, and it is positive: $q_w\ge 2$, the quotients $[\mathrm{U}(\mathrm{I}_m)]$ ($m=1,2z$) are compact and carry the quotient of a Haar measure, so V_1 and $\mathrm{vol}([\mathrm{U}(\mathrm{I}_{2z})])$ are finite and positive, $\lambda_{\mathrm{I}_1}>0$ and $P_z>0$. As $X(p,\vartheta)$ and $X(p,1)$ have the same row lengths and rank, $O=O_{X(p,\vartheta)}$ and $O_{X(p,1)}$ have the same weight κ_O (one row of length $c(\tau)-z$ at τ), so $\dim\mathbb{V}_O=\dim\mathbb{V}_{O_{X(p,1)}}$. Now (6.174) reads $\mathrm{alg}_p(\vartheta)=A\,L^{S_M\cup Q}(z,\nu\vartheta)$ with A as in the statement, which does not depend on ϑ or Q. Finally $A>0$: $c_s\neq 0$ by (OL4), $\pi_M\neq 0$ as $|\pi_M|^2=M$, $\eta_P\neq 0$ as s_0, u_{CS} and Ω_{CM} are nonzero (OL4), $\dim\mathbb{V}_O\ge 1$, and $B_z>0$.

Step 6: positivity and the ratio. $L^S(z,\nu\vartheta)\neq 0$ by (c) of Lemma 6.7.15 (for $\nu_X=\nu\vartheta$, S and $k=z\ge 2$). So $\mathrm{alg}_p(\vartheta)\neq 0$; as a squared norm divided by $\mathrm{N}Q^{2z-1}>0$ it is ≥ 0, hence >0. For an automorphic character μ of $\mathrm{U}(1)$, $\widetilde{\mu^{-1}}(x)=\mu(\bar{x}/x)=\tilde{\mu}(\bar{x})$, and conjugation permutes the places off $S\cup\bar{S}$ preserving norms, so $L^S(z,\mu^{-1})=L^S(z,\mu)$. Therefore, by the definition of the ratio and the identity just proved for ϑ and for the trivial twist ($Q=\emptyset$),

$$r_p(\vartheta)=\frac{|L^{S_M\cup Q}(z,\nu\vartheta)|}{|L^{S_M}(z,\nu)|}=\frac{|\mathrm{alg}_p(\vartheta)/A|}{|\mathrm{alg}_p(1)/A|}=\frac{\mathrm{alg}_p(\vartheta)}{\mathrm{alg}_p(1)}.$$

□

Lemma 6.7.14. *Let X be L-value data of rank n, $O=O_X$, $\xi_X=\xi_O$, let*

$$f_X(h)=\int_{[\mathrm{U}(\mathrm{I}_1)]}\theta_O(\omega_O(u,h)\,\mathrm{J}_O(\xi_X))\,du\qquad(h\in\mathrm{U}(\mathrm{I}_n)(\mathbb{A})),$$

and let $\Theta_X\in\mathrm{Form}(O)$ be the algebraic form attached to f_X. Then:

(a) $\lambda_X=[\mathbf{K}_0(1):\mathbf{K}_X^{\mathrm{ch}}]\,\Theta_X$;

(b) $\langle\Theta_X,\Theta_X\rangle=[\mathbf{K}_0(n):\mathbf{K}(O)]\cdot\dim\mathbb{V}_O\cdot\langle f_X,f_X\rangle_{\mathrm{I}_n}$;

(c) the pair (ω_O, θ_O) is a theta kernel for $(\mathrm{I}_1, \mathrm{I}_n)$ on $\mathrm{Hom}(\mathrm{I}_1, \mathrm{I}_n)$ with real part $\Omega_{0,-\ell_X}$, and $f_X = L_{\mathrm{J}_O(\xi_X)}\mathbf{1}$ is its lift of the constant function $\mathbf{1}$ on $\mathrm{U}(\mathrm{I}_1)(\mathbb{A})$ with the datum $\mathrm{J}_O(\xi_X) = v_{\mathrm{Har}_O(\xi_X)} \otimes \Upsilon(O)$; the constant function $\mathbf{1}$ is a form of the type $\mathbf{1}$ (the constant function 1 on the real points of $\mathrm{U}(\mathrm{I}_1)$) on I_1; and there is a type e'_X on the spaces with the signs of I_n, depending only on n and ℓ_X, such that $v_{\mathrm{Har}_O(\xi_X)}$ is an edge for the weights $(0, -\ell_X)$ from $\mathbf{1}$ to e'_X;

(d) in the Fock variables $X_{\tau,i,1}$ of $\mathrm{Hom}(\mathrm{I}_1, \mathrm{I}_n)$ of (DP8), $\mathrm{Har}_O(\xi_X) = \prod_{\tau\in\Phi} X_{\tau,1,1}^{\ell_X(\tau)}$, and $\Omega_{0,-\ell_X}(x,1)\, v_{\mathrm{Har}_O(\xi_X)} = v_{\mathrm{Har}_O(\xi_X)}$ for every $x \in G_\infty(+) = \mathrm{U}(1)^\Phi$.

Proof. Throughout, $O = O_X$, $\kappa = \kappa_O$, $\ell = \ell_X$, $\xi = \xi_X$, $d = \dim \mathbb{V}_O = \#I_\kappa$, $G = G_\kappa$, $\langle\,,\rangle = \langle\,,\rangle_\kappa$ on $\mathbb{V}_O$, and $K = \mathbf{K}_X^{\mathrm{ch}}$. By (OL1), $\kappa_O^- = 0$, so $e_O = 0$, $\kappa(\tau,1) = \ell(\tau)$ and $\kappa(\tau,h) = 0$ for $h \geq 2$; we write $\rho = \rho_{\kappa,0}$ on $G_\infty(1) = \prod_{\tau\in\Phi} \mathrm{U}(n)$, so that $\rho_O(\gamma) = \rho(\gamma_\infty)$ for rational points γ of $\mathrm{U}(\mathrm{I}_n)$. Also $\chi_O = \upsilon^0 = 1$ has weight 0, and $\chi'_O = \nu_X \upsilon^{\lfloor n/2\rfloor}$ has weight $-(\ell + \lfloor n/2\rfloor) + \lfloor n/2\rfloor = -\ell$. For definite signs the measure of $G_\infty(s)$ is a probability measure (Lemma 5.5.21), invariant under left and right translations and hence under inversion (Lemma 5.2.41); an open subgroup $K' \subseteq \mathbf{K}_0(m)$ has measure $[\mathbf{K}_0(m) : K']^{-1}$.

Step 1: four facts on $\mathbb{V}_O$. By (a) of Lemma 4.6.10, S_κ is multiplicative and, for all families $z = (z_\tau)$ of complex $n \times n$ matrices,

$$S_\kappa(z)^{\mathsf{T}} G = G\, S_\kappa(z^{\mathsf{T}}). \tag{6.175}$$

(1a) *Unitarity.* By (b) and (c) of Lemma 4.6.10 with $e = 0$, ρ is a unitary representation of $G_\infty(1)$ for $\langle\,,\rangle$, $\rho(h^{-1}) = S_\kappa(h^{\mathsf{T}})$, and G is positive definite. In particular $w \in \mathbb{V}_O$ is determined by the numbers $\eta^{\mathsf{T}} G w$, $\eta \in \mathbb{V}_O$. (1b) *Substitution.* For $\eta \in \mathbb{V}_O$, $\mathrm{Pol}_\kappa(\eta)(zy) = \sum_{i,j} (G\eta)_i S_\kappa(z)_{ij} b_j(y) = \mathrm{Pol}_\kappa(S_\kappa(z^{\mathsf{T}})\eta)(y)$, by (6.175) and the symmetry of G. (1c) *Irreducibility.* As κ has one row of length $\ell(\tau)$ at τ, $\nabla_\kappa \otimes \mathbb{C}$ is the space of polynomials homogeneous of degree $\ell(\tau)$ in $y_{\tau,1,0}, \ldots, y_{\tau,n,0}$ for every τ, on which $\prod_\tau \mathrm{GL}_n(\mathbb{C})$ acts by $P \mapsto P(z\,\cdot)$ through the matrices $S_\kappa(z)^{\mathsf{T}}$; this is the tensor product over τ of symmetric powers of the dual of the standard representation, which is irreducible [92]. A subspace of $\mathbb{V}_O$ invariant under all $S_\kappa(z)$ has an invariant annihilator in $\nabla_\kappa \otimes \mathbb{C}$, so S_κ is irreducible; as $\prod_\tau \mathrm{U}(n)$ is Zariski dense in $\prod_\tau \mathrm{GL}_n(\mathbb{C})$ and $h \mapsto \bar{h}$ is an automorphism of it, ρ is irreducible. (1d) *The highest vector.* $\xi = \mathrm{ev}_\kappa(y_{\mathrm{id}})$ has integral entries, and $F_\kappa \xi = (\Delta_T(y_{\mathrm{id}}))_{T\in\mathcal{C}_\kappa}$. Here $\Delta_T(y) = \prod_{\tau,j} y_{\tau,a(\tau,j),0}$ with $T(\tau,1,j) = \{a(\tau,j)\}$, and in column 0 the array y_{id} has entry 1 in row 1 only; so $\Delta_T(y_{\mathrm{id}})$ is 1 for the filling with all rows equal to 1 and 0 otherwise. Hence $\|\xi\|^2 = |F_\kappa\xi|^2 = 1$ and $\mathrm{Pol}_\kappa(\xi)(y) = \xi^{\mathsf{T}} G\, \mathrm{ev}_\kappa(y) = (F_\kappa\xi)\cdot(F_\kappa \mathrm{ev}_\kappa(y)) = \prod_\tau y_{\tau,1,0}^{\ell(\tau)}$. We use Schur's orthogonality in the following form: if π is a continuous irreducible unitary representation of a compact group C with Haar probability measure on a Hilbert space of finite dimension d_π, then for all vectors a, b, c

$$\int_C \langle \pi(y)a, b\rangle\, \pi(y)c\, dy = d_\pi^{-1} \langle a, c\rangle\, b. \tag{6.176}$$

Indeed the operator $b \mapsto \int \langle \pi(y)a, b\rangle \pi(y)c\, dy$ commutes with every $\pi(z)$ (substitute $y \mapsto zy$), so it is a scalar by Schur's lemma, and its trace is $\int \langle \pi(y)a, \pi(y)c\rangle\, dy = \langle a, c\rangle$.

Step 2: the theta datum under the real points. By (d) of Lemma 3.2.20 for $(\kappa_O^+, \kappa_O^-) = (\kappa, 0)$ (so I_{κ^-} is a point, $S_{\kappa^-} = 1$, $G_{\kappa^-} = 1$, Pol_{κ^-} is the identity of $\mathbb{C}$ and $e^- = 0$), the matrix $C_O \in \mathrm{M}_{I_\kappa}(\mathbb{Z})$ commutes with every $S_\kappa(z)$, and by (DP8) $\mathrm{Har}_O(\eta) = \mathrm{Let}_1(\mathrm{Pol}_\kappa(C_O\eta))$, where Let_1 renames $y_{\tau,a,0}$ as $X_{\tau,a,1}$ (no variable of column $c \geq 1$ occurs). By (1b) with $z = y^{\mathsf{T}}$, for $y \in G_\infty(1)$,

$$\mathrm{Har}_O(\eta)((y_\tau^{\mathsf{T}} X_\tau)_\tau) = \mathrm{Har}_O(S_\kappa(y)\eta)(X), \qquad X_\tau = (X_{\tau,a,1})_a, \tag{6.177}$$

and every $\mathrm{Har}_O(\eta)$ is homogeneous of degree $\ell(\tau)$ in the variables at τ. By (c) of Lemma 3.2.20, $C_O\xi = \xi$, so $\mathrm{Har}_O(\xi) = P_0 := \prod_\tau X_{\tau,1,1}^{\ell(\tau)}$ by (1d). By Lemma 6.3.77, (ω_O, θ_O) is a theta kernel for $(\mathrm{I}_1, \mathrm{I}_n)$ on $\mathrm{Hom}(\mathrm{I}_1, \mathrm{I}_n)$ with real part $\Omega := \Omega_{0,-\ell}$. All signs are $+1$, so (5) of Lemma 4.1.1 applies to all $x \in G_\infty(+) = \mathrm{U}(1)^\Phi$ and $y \in G_\infty(1)$, with the factor $\prod_\tau x_\tau^{\ell(\tau)}$ and the substitution $X_{\tau,a,1} \mapsto x_\tau^{-1} \sum_i (y_\tau)_{ia} X_{\tau,i,1}$: for P homogeneous of degree $m(\tau)$ in the variables at τ,

$$\Omega(x,y)\, v_P = \prod_\tau x_\tau^{\ell(\tau)-m(\tau)}\, v_{P\circ y^{\mathsf{T}}}, \qquad (P\circ y^{\mathsf{T}})(X) = P((y_\tau^{\mathsf{T}} X_\tau)_\tau). \tag{6.178}$$

With (6.177), $\Omega(x,y) v_{\mathrm{Har}_O(\eta)} = v_{\mathrm{Har}_O(S_\kappa(y)\eta)}$, and the first formula of (5) of Lemma 4.1.1 gives $\omega_O((x,g),(y,h))\mathrm{J}_O(\eta) = \omega_O((1,g),(1,h))\mathrm{J}_O(S_\kappa(y)\eta)$. By (OL2), (6.175) and (1a), for $x \in G_\infty(+)$, $g \in G_f(1)$, $y \in G_\infty(1)$, $h \in G_f(n)$, $\eta \in \mathbb{V}_O$,

$$\theta_O(\omega_O((x,g),(y,h))\,\mathrm{J}_O(\eta)) = (S_\kappa(y)\eta)^{\mathsf{T}} G\, k_X(g)(h) = \eta^{\mathsf{T}} G\, \rho(y^{-1})\, k_X(g)(h); \tag{6.179}$$

it does not depend on x.

Step 3: the kernel is constant on the classes. The rational points of I_1 are $\mathrm{U}(1)(E)$, embedded diagonally; we identify γ with its finite component, as (OL3) does. For $\gamma \in \mathrm{U}(1)(E)$ the theta-kernel invariance (ED3) gives $\theta_O(\omega_O((\gamma_\infty,\gamma g),(1,h))\varphi) = \theta_O(\omega_O((1,g),(1,h))\varphi)$; by (6.179) at $x = \gamma_\infty$ and at $x = 1$ (both with $y = 1$), $\eta^{\mathsf{T}} G k_X(\gamma g)(h) = \eta^{\mathsf{T}} G k_X(g)(h)$ for all η, so $k_X(\gamma g) = k_X(g)$ by (1a). Let $k \in K$. By (2) of Lemma 4.1.1, $\omega_O((1,gk),(1,h)) = \omega_O((1,g),(1,h))\,\omega_O((1,k),(1,1))$. If $S_O \neq \emptyset$, then by (DP7) $\Upsilon(O) = \Phi_O \otimes_{S_O} v^0$ with $\Phi_O = \Psi_O^\natural$ for the modulus $\mathfrak{m} = \prod_{w\in S_O} w$, and (6) of Lemma 4.1.1 (with $\alpha = \chi_O = 1$, $\beta = \chi_O'$, $k' = 1$) gives

$$\omega_O((1,k),(1,1))(v\otimes\Upsilon(O)) = \chi_O'(k)^{-1}\, v \otimes (\Psi_O(\,\cdot\,\bar{k})^\natural \otimes_{S_O} v^0),$$

$\bar{k} = (k \bmod w)_{w\in S_O}$. As $\bar{k}_w = 1$ at the chosen places and $r = 1$, (DP7) gives $\Psi_O(X\bar{k}) = \prod_{w\in Q_X} \psi_{X,w}(\bar{k}_w)\,\Psi_O(X)$, ψ_X the tame part of $\mu_O = \nu_X$. The finite components of k are integral and its infinite components are 1, so by Definition 3.1.2, (OL1) and the trivial tame part of υ, $\chi_O'(k) = \nu_X(k)\upsilon(k)^{\lfloor n/2\rfloor} = \prod_{w\in M_X\cup Q_X} \psi_{X,w}(\bar{k}_w) = \prod_{w\in Q_X} \psi_{X,w}(\bar{k}_w)$. The two factors cancel, so $\omega_O((1,k),(1,1))\mathrm{J}_O(\eta) = \mathrm{J}_O(\eta)$. If $S_O = \emptyset$, then $\Upsilon(O) = v^0$, ν_X has trivial tame part, and the last formula of (6) gives the same. By (OL2), $k_X(gk) = k_X(g)$. So $g \mapsto k_X(g)$ is constant on the double cosets $\mathrm{U}(1)(E)\, g\, K$, and by (6.179), for fixed (y,h), the function $(x,g) \mapsto \theta_O(\omega_O((x,g),(y,h))\mathrm{J}_O(\xi))$ depends only on g and is left $\mathrm{U}(1)(E)$- and right K-invariant.

Step 4: unfolding, and (a). Let $m \in \{1,n\}$, $V = \mathrm{I}_m$ with rational points Γ_V, and $K' \subseteq \mathbf{K}_0(m)$ open. As V is definite (for $m = 1$, a hermitian line), $\mathrm{U}(V)$ is anisotropic over F, $[\mathrm{U}(V)]$ is compact [37] and $\Gamma_V\backslash G_f(m)/K'$ is finite [33]. For a representative h, the finite group $\Gamma_h = \Gamma_V \cap hK'h^{-1}$ (discrete and compact) acts freely on $G_\infty(1_m)\times hK'$ by left multiplication, the quotient maps bijectively onto the classes of $\Gamma_V hK'$ in $[\mathrm{U}(V)]$, and these sets partition $[\mathrm{U}(V)]$. Hence for Ψ bounded measurable on $\mathrm{G}(1_m)$, left Γ_V-invariant and with $\Psi(y,hk) = \Psi(y,h)$ for $k \in K'$,

$$\int_{[\mathrm{U}(V)]} \Psi = [\mathbf{K}_0(m) : K']^{-1} \sum_{[h]} \frac{1}{\#\Gamma_h} \int_{G_\infty(1_m)} \Psi(y,h)\, dy. \tag{6.180}$$

For $m = 1$, $K' = K$ ($G_f(1)$ is abelian, so Γ_g is that of (OL3)) and

$$\Psi(x,g) = \theta_O(\omega_O((x,g),(y,h))\mathrm{J}_O(\xi)),$$

Step 3, (6.179) and the mass 1 of $G_\infty(+)$ give, with (OL3), ξ real and (1a),

$$f_X(y,h) = [\mathbf{K}_0(1) : K]^{-1}\, \xi^{\mathsf{T}} G \rho(y^{-1}) \lambda_X(h) = [\mathbf{K}_0(1) : K]^{-1}\, \langle \rho(y)\xi, \lambda_X(h)\rangle. \tag{6.181}$$

By (LF5) and (6.176) for ρ (irreducible by (1c)) with $a = c = \xi$, $b = \lambda_X(h)$, and $\|\xi\| = 1$ (1d),

$$\Theta_X(h) = d\,[\mathbf{K}_0(1) : K]^{-1} \int_{G_\infty(1)} \langle \rho(y)\xi, \lambda_X(h)\rangle\, \rho(y)\xi\, dy = [\mathbf{K}_0(1) : K]^{-1} \lambda_X(h).$$

As $\lambda_X \in \mathrm{Form}(O)$ (OL2, OL3), $\Theta_X \in \mathrm{Form}(O)$, and (a) holds.

Step 5: (b). By (6.181) and (a), $f_X(y,h) = \langle \rho(y)\xi, \Theta_X(h)\rangle$. The function $\Psi = |f_X|^2$ on $\mathrm{G}(1_n)$ is left invariant under the rational points γ ($\Theta_X(\gamma_f h) = \rho(\gamma_\infty)\Theta_X(h)$ by (DP5), and (1a)) and right invariant under $\mathbf{K}(O)$ ($|\tilde{\eta}_O| = 1$, the $\psi_{O,w}$ being characters of finite groups). By (6.180) with $m = n$, $K' = \mathbf{K}(O)$, and by (6.176) paired with $w = \Theta_X(h)$, which gives $\int |\langle \rho(y)\xi, w\rangle|^2 dy = \langle w, d^{-1}\|\xi\|^2 w\rangle = d^{-1}\|w\|^2$,

$$\langle f_X, f_X\rangle_{\mathrm{I}_n} = [\mathbf{K}_0(n) : \mathbf{K}(O)]^{-1} \sum_{[h]} \frac{1}{\#\Gamma_h}\, d^{-1}\|\Theta_X(h)\|^2 = [\mathbf{K}_0(n) : \mathbf{K}(O)]^{-1} d^{-1} \langle \Theta_X, \Theta_X\rangle,$$

the last sum being the class sum of (DP5). This is (b).

Step 6: (c). The first assertion is Step 2. By (ED3),

$$L_{\mathrm{J}_O(\xi)}\mathbf{1}(h) = \int_{[\mathrm{U}(\mathrm{I}_1)]} \theta_O(\omega_O(u,h)\mathrm{J}_O(\xi))\, du = f_X(h).$$

The constant $\mathbf{1}$ is continuous, bounded, invariant on both sides, and $R(\mathbf{1})\mathbf{1} = \mathbf{1}$ (mass 1), so it is a form of type $\mathbf{1}$ on I_1. Let $\Phi = v_{P_0}$ ($P_0 \neq 0$, so $\Phi \neq 0$), let $H \subset \mathcal{F}_\varepsilon$ be the span of the v_P with P homogeneous of degree $\ell(\tau)$ in the variables at every τ, and $\pi(y) = \Omega(1,y)|_H$ for $y \in G_\infty(1)$. By (6.178), $\pi(y)v_P = v_{P\circ y^{\mathsf{T}}}$, so H is invariant and $\pi(yy') = \pi(y)\pi(y')$; π is unitary ((2) of Lemma 4.1.1: $\omega_\varepsilon^\infty$ preserves the inner product) and continuous (its matrix entries are polynomials in y), and it is irreducible, being the action $P \mapsto P\circ y^{\mathsf{T}}$ on polynomials homogeneous of degree $\ell(\tau)$ at τ, irreducible as in (1c). Put $d_H = \dim H$, $c_H = d_H/\|\Phi\|^2$ and

$$e'_X(y) = c_H\, \langle \pi(y)\Phi, \Phi\rangle \qquad (y \in G_\infty(1)),$$

a type on the spaces with the signs of I_n; it depends only on n and ℓ. By (6.176) for π: (i) $\int e'_X(y)\pi(y)\Phi\, dy = \Phi$; (ii) $w = c_H \int \langle \pi(y)\Phi, w\rangle \pi(y)\Phi\, dy$ for $w \in H$; (iii) $\int e'_X(y)\pi(y)w\, dy = \|\Phi\|^{-2}\langle \Phi, w\rangle\Phi$ for $w \in H$. Also $\Omega(x,1)\Phi = \Phi$ for $x \in G_\infty(+)$, by (6.178) with $m = \ell$, and $\Omega(1,z)v_P = \zeta^m v_P$ for P of degree m and central $z = (\zeta_\tau 1_n)_\tau$, $\zeta^m = \prod_\tau \zeta_\tau^{m(\tau)}$. *Half edge for Ω from $\mathbf{1}$ to e'_X.* (a) $\mathbf{1}$ is continuous and integrable, $\mathbf{1}(x^{-1}) = \overline{\mathbf{1}(x)}$, $\mathbf{1}(1) \neq 0$, $\int \mathbf{1}\, dz = 1 = \mathbf{1}(x^{-1})\mathbf{1}(y)/\mathbf{1}(1)$; on the compact group $G_\infty(+)$ all operator norms are 1, and $a = \mathbf{1}$ is continuous with compact support with $\int a(x)\mathbf{1}(x^{-1}y)dx = \mathbf{1}(y)$: a cusp kernel of order 0. (b) The zeta integral is $\int \langle \Phi, \Omega(x,1)\Phi\rangle\, dx = \|\Phi\|^2 \neq 0$. For (c) and (d) fix spaces with points S, T, V with the signs $+1$ on $\{1\}$, on n and on $n \times \{1\}$, a theta kernel (ω, θ) for (S,T) on V with real part Ω, $f \in \mathcal{M}_\eta$ and a form F of type $\mathbf{1}$ on S, and put $\Lambda_h(v) = L_{v\otimes f}F(h)$ for $v \in \mathcal{F}_\varepsilon$, $h \in \mathrm{G}(s')$. This is defined and linear in v: $[\mathrm{U}(S)]$ is compact (S is a hermitian line), $g \mapsto \theta(\omega(g,h)(v \otimes f))$ is continuous and left Γ_S-invariant, hence bounded, and F is bounded. Since ω is a homomorphism with real part Ω, $\Lambda_{h(y,1)}(v) = \Lambda_h(\Omega(1,y)v)$. Since $R(\mathbf{1})F = F$ and the measure of $G_\infty(s)$ is invariant, $F(g(x,1)) = F(g)$; substituting $g \mapsto g(x,1)$ in $[\mathrm{U}(S)]$ (the quotient measure is right invariant) gives $\Lambda_h(\Omega(x,1)v) = \Lambda_h(v)$. (d) Let $P = \sum_m P_m$ be a polynomial, P_m homogeneous of multidegree m. By (6.178) and the last identity, $\Lambda_h(v_P) = \sum_m \prod_\tau x_\tau^{\ell(\tau)-m(\tau)} \Lambda_h(v_{P_m})$ for every x; integrating over $G_\infty(+)$ gives $\Lambda_h(v_P) = \Lambda_h(v_{P_\ell})$. Put $a_P(y) = c_H\langle \pi(y)\Phi, v_{P_\ell}\rangle$, continuous on the compact group $G_\infty(1)$. By (ii), and since Λ_h is linear on the finite-dimensional space H, $\Lambda_h(v_{P_\ell}) = \int a_P(y)\Lambda_h(\pi(y)\Phi)\, dy = \int a_P(y) L_{\Phi\otimes f}F(h(y,1))\, dy = R(a_P)L_{\Phi\otimes f}F(h)$. (c) Let $L = L_{\Phi\otimes f}F$ have a level K''. It is left Γ_T-invariant by (ED3). It is continuous: on $C \times N$, C the

closure of a relatively compact Borel fundamental domain of Γ_S and N a compact neighbourhood of h_0, the function $(g,h)\mapsto\theta(\omega(g,h)(\Phi\otimes f))$ is uniformly continuous. It is bounded: T is definite, so $\Gamma_T\backslash G_f(n)/K''$ is finite and $G_\infty(s')$ is compact. Finally $R(e'_X)L(h)=\int e'_X(y)\Lambda_h(\pi(y)\Phi)\,dy=\Lambda_h(\Phi)=L(h)$ by (i). So L is a form of type e'_X. *Half edge for* $(y,x)\mapsto\Omega(x,y)$ *from* $\bar e'_X$ *to* $\bar{\mathbf{1}}=\mathbf{1}$. (a) $\bar e'_X(y)=c_H\langle\Phi,\pi(y)\Phi\rangle$ is continuous, $\bar e'_X(y^{-1})=\overline{\bar e'_X(y)}$ (unitarity), $\bar e'_X(1)=d_H\neq0$, and with (6.176) for $a=\pi(x)\Phi$, $b=\pi(y)\Phi$, $c=\Phi$, paired with Φ,

$$\int\bar e'_X(z)\,\bar e'_X((zx)^{-1}y)\,dz=c_H^2\int\langle\Phi,\pi(z)\Phi\rangle\langle\pi(z)\pi(x)\Phi,\pi(y)\Phi\rangle\,dz$$
$$=\frac{c_H^2}{d_H}\langle\pi(x)\Phi,\Phi\rangle\langle\Phi,\pi(y)\Phi\rangle=\frac{\bar e'_X(x^{-1})\,\bar e'_X(y)}{\bar e'_X(1)};$$

operator norms on $G_\infty(1)$ are 1, and $a=\bar e'_X$ reproduces $\bar e'_X$ (the identity at $x=1$): a cusp kernel of order 0. (b) The zeta integral $\int\langle\Phi,\pi(y)\Phi\rangle\,e'_X(y)\,d_H^{-1}dy=c_Hd_H^{-1}\int|\langle\Phi,\pi(y)\Phi\rangle|^2dy$ is positive (continuous, nonnegative, equal to $\|\Phi\|^4$ at $y=1$). For (c), (d) fix S', T', V' with the signs $+1$ on n, on $\{1\}$ and on $n\times\{1\}$, a theta kernel (ω,θ) for (S',T') on V' with $\omega((y,1),(x,1))(v\otimes f)=(\Omega(x,y)v)\otimes f$, f, and a form F of type $\bar e'_X$ on S'; put $\Lambda'_g(v)=L_{v\otimes f}F(g)$, defined and linear as before ($[\mathrm{U}(S')]$ is compact). As before $\Lambda'_{g(x,1)}(v)=\Lambda'_g(\Omega(x,1)v)$, and substituting $h\mapsto h(y^{-1},1)$ gives $\int\theta(\omega(h,g)(v\otimes f))F(h(y,1))\,dh=\Lambda'_g(\Omega(1,y^{-1})v)$. (c) If $L'=L_{\Phi\otimes f}F$ has a level, it is continuous, left $\Gamma_{T'}$-invariant and bounded as in (c) above (T' is a hermitian line), and $L'(g(x,1))=\Lambda'_g(\Omega(x,1)\Phi)=L'(g)$, so $R(\mathbf{1})L'=L'$: it is a form of type $\mathbf{1}$. (d) Let $P=\sum_m P_m$ as before. Inserting $F=R(\bar e'_X)F$, applying Fubini's theorem (bounded integrand, finite measures), the last identity and $y\mapsto y^{-1}$ (with $\bar e'_X(y^{-1})=e'_X(y)$), $\Lambda'_g(v_P)=\sum_m J_m$ with $J_m=\int e'_X(y)\Lambda'_g(\pi_m(y)v_{P_m})\,dy$, $\pi_m(y)v_{P_m}=\Omega(1,y)v_{P_m}=v_{P_m\circ y^{\mathsf T}}$. For $m\neq\ell$ choose a central z with $\zeta^{m-\ell}\neq1$; as $e'_X(zy)=\overline{\zeta^\ell}e'_X(y)$ and $\Omega(1,zy)v_{P_m}=\zeta^m\Omega(1,y)v_{P_m}$, the substitution $y\mapsto zy$ gives $J_m=\zeta^{m-\ell}J_m$, so $J_m=0$. For $m=\ell$, (iii) and the linearity of Λ'_g on H give $J_\ell=\|\Phi\|^{-2}\langle\Phi,v_{P_\ell}\rangle L'(g)$. With the constant $a=\|\Phi\|^{-2}\langle\Phi,v_{P_\ell}\rangle$ on the compact group $G_\infty(+)$ and $R(\mathbf{1})L'=L'$, $L_{v_P\otimes f}F=R(a)L'$. So $v_{\mathrm{Har}_O(\xi_X)}=\Phi$ is an edge for the weights $(0,-\ell_X)$ from $\mathbf{1}$ to e'_X.

Step 7: (d). Step 2 gives $\mathrm{Har}_O(\xi)=P_0=\prod_\tau X_{\tau,1,1}^{\ell(\tau)}$, and (6.178) with $y=1$ and $m=\ell$ gives $\Omega(x,1)v_{P_0}=v_{P_0}$ for $x\in G_\infty(+)$. □

Lemma 6.7.15. *Let ν be an automorphic character of* U(1) *with a weight and a tame part ψ, let S be a finite set of finite places of E with $\psi_w=1$ for every finite place $w\notin S$, put $D=S\cup\bar S$, let $k\geq2$ be an integer, T a space of rank $2k$, and* mc *the normalised vacuum coefficient for $b=\nu v^k$. Let $K^D\subseteq\mathrm{U}(1)(\mathbb{A}_f)$ be the subgroup of the points that are integral at every finite place not in D.*

(a) Every coset c of K^D in $\mathrm{U}(1)(\mathbb{A}_f)$ contains points whose component is 1 at every place of D, and mc *takes one value* $\mathrm{mc}(c)$ *at all of them.*

(b) The family $(\mathrm{mc}(c))_c$, indexed by the cosets $c\in\mathrm{U}(1)(\mathbb{A}_f)/K^D$, is absolutely summable, and

$$\sum_{c\in\mathrm{U}(1)(\mathbb{A}_f)/K^D}\mathrm{mc}(c)=L^S(k,\nu)\prod_v e_v(k),$$

where v runs over the finite places of the totally real subfield F of E that lie below no place of D, the product converges absolutely, and $e_v(k)=1-\mathrm{N}w^{-2k}$ if v splits in E as $w\bar w$ with $w\neq\bar w$, and $e_v(k)=1-\mathrm{N}w^{-k}$ if w is the only place of E above v.

(c) *Define $e_v(k)$ by the same formula for every finite place v of F. For every set V' of finite places of F the product $\prod_{v\in V'} e_v(k)$ converges absolutely and is a positive real number. The partial L-function is the absolutely convergent Euler product $L^S(k,\nu)=\prod_{u\notin D}\bigl(1-\tilde\nu(x_u)\mathrm{N}u^{-k}\bigr)^{-1}$ over the finite places u of E not in D, x_u an idele with a uniformiser at u and 1 elsewhere, with nonzero factors; in particular $L^S(k,\nu)\neq 0$.*

Proof. Write $\mathrm{U}(1)(F_v)=\{u\in(E\otimes_F F_v)^\times : u\bar u=1\}$ for the component of $\mathrm{U}(1)(\mathbb{A}_f)$ above a finite place v of F. Since $\bar D=D$, a place v of F lies below a place of D exactly when all places of E above v lie in D. For a finite place u of E let x_u be the idele with a uniformiser ϖ_u of E_u at u and 1 elsewhere, as in Lemma 6.7.16 (c).

Step 1: the local groups. If w is the only place of E above v, then $E\otimes_F F_v=E_w$ is a field on which conjugation is the nontrivial automorphism over F_v; it preserves $|\cdot|_w$, so $|u|_w^2=|u\bar u|_w=1$ for $u\in\mathrm{U}(1)(F_v)$, and every such u is a unit: every point of $\mathrm{U}(1)(\mathbb{A}_f)$ is integral at w. If v splits as $w\bar w$, then $E\otimes_F F_v=E_w\times E_{\bar w}$, conjugation exchanges the factors, and $u\bar u=1$ says $u_{\bar w}=\overline{u_w}^{-1}$. So $u\mapsto u_w$ is an isomorphism $\mathrm{U}(1)(F_v)\to E_w^\times$, and $|u_{\bar w}|_{\bar w}=|u_w|_w^{-1}$; hence u is integral at w and at $\bar w$ if and only if $\mathrm{ord}_w(u_w)=0$.

Step 2: $\tilde\nu$ off D. Let $u\notin D$ be a finite place, $t\in\mathcal{O}_{E,u}^\times$ and y the idele with component t at u and 1 elsewhere. Then $y/\bar y\in\mathrm{U}(1)(\mathbb{A})$ has archimedean components 1, unit components at u and $\bar u$ (namely t and $\bar t^{-1}$ if $\bar u\neq u$, and $t/\bar t$ if $\bar u=u$) and components 1 elsewhere. By the defining formula of weight and tame part, $\tilde\nu(y)=\nu(y/\bar y)$ is a product of values of ψ_u and $\psi_{\bar u}$, which are trivial because $u,\bar u\notin D\supseteq S$; so $\tilde\nu(y)=1$. Thus $\tilde\nu$ is unramified at every $u\notin D$; its value $\tilde\nu(x_u)$ does not depend on the uniformiser and has absolute value 1. Two consequences.

(i) If v splits as $w\bar w$ below no place of D, then $\bar x_w$ is an idele $x_{\bar w}$ (component $\bar\varpi_w$, a uniformiser of $E_{\bar w}$) and $\tilde\nu(\bar x_w)=\nu(\bar x_w/x_w)=\nu(x_w/\bar x_w)^{-1}=\tilde\nu(x_w)^{-1}$.

(ii) If $w\notin D$ is the only place above v, then $\tilde\nu(x_w)=1$: if w is unramified over v, take $\varpi_w\in F_v$, so $x_w=\bar x_w$; if w is ramified over v, then $x_w/\bar x_w$ has component $\varpi_w/\bar\varpi_w$ at w, a unit of norm one, and 1 elsewhere, so $\nu(x_w/\bar x_w)=\psi_w(\varpi_w/\bar\varpi_w \bmod w)=1$.

Hence $L^S(s,\nu)=\prod_{u\notin D}(1-\tilde\nu(x_u)\mathrm{N}u^{-s})^{-1}$, absolutely convergent for $\mathrm{Re}\,s>1$; grouping the factors by the places v of F below no place of D, $L^S(k,\nu)=\prod_v L_v(k)$, absolutely convergent, with

$$\begin{aligned} L_v(k)&=\bigl(1-\tilde\nu(x_w)\mathrm{N}w^{-k}\bigr)^{-1}\bigl(1-\tilde\nu(x_w)^{-1}\mathrm{N}w^{-k}\bigr)^{-1} && (v=w\bar w \text{ split}),\\ L_v(k)&=\bigl(1-\mathrm{N}w^{-k}\bigr)^{-1} && (\text{otherwise}). \end{aligned} \tag{6.182}$$

Step 3: the cosets; proof of (a). Let V be the set of places of F that split in E and lie below no place of D. For $v\in V$ fix a place w_v above v and a uniformiser of E_{w_v}, write $\mathrm{N}_v=\mathrm{N}w_v$ and $\gamma_{v,z}=\gamma_{w_v,z}$ ($z\in\mathbb{Z}$) for the points of Lemma 6.7.16 (c). Let $\mathbb{Z}^{(V)}$ be the group of finitely supported maps $V\to\mathbb{Z}$, and define $\mathrm{ord}\colon\mathrm{U}(1)(\mathbb{A}_f)\to\mathbb{Z}^{(V)}$ by $q\mapsto(\mathrm{ord}_{w_v}(q_{w_v}))_{v\in V}$; the image is finitely supported because a finite adelic point is integral at almost all places, and ord is a homomorphism. Its kernel is K^D: by Step 1 every point is integral at the places above a nonsplit v, and at the two places above $v\in V$ it is integral exactly when its coordinate at v vanishes; the places above the remaining v lie in D, where K^D imposes nothing. It is surjective: for $z\in\mathbb{Z}^{(V)}$ the point $\gamma_z=\prod_{v\in\mathrm{supp}\,z}\gamma_{v,z_v}$ has $\mathrm{ord}(\gamma_z)=z$. So the cosets of K^D correspond to $\mathbb{Z}^{(V)}$, and the coset of z contains γ_z, whose component is 1 at every place of D. If q,q' lie in one coset and both have component 1 at every place of D, then $\kappa=q^{-1}q'\in K^D$ has component 1 at the places of D, hence is integral at every finite place, and Lemma 6.7.16 (b) (with the left factor 1; its hypothesis on D holds because $\psi_w=1$ for $w\notin S$ and $D\supseteq S$) gives $\mathrm{mc}(q')=\mathrm{mc}(q\kappa)=\mathrm{mc}(q)$. This proves (a), and $\mathrm{mc}(c)=\mathrm{mc}(\gamma_z)$ for the coset c of z.

Step 4: the values. The factors γ_{v,z_v} of γ_z have pairwise disjoint supports ($\gamma_{v,z}$ has components different from 1 only at w_v, $\bar{w}_v$), and if two points q, q' have, at every finite place, one of their two components equal to 1, and u, u' are finite ideles as in Lemma 6.7.16 (a) for q and q', then uu' is one for qq' (the condition $uu'\widehat{\mathcal{O}}_E = \widehat{\mathcal{O}}_E \cap qq'\widehat{\mathcal{O}}_E$ is checked place by place), so the formula Lemma 6.7.16 (a) and the multiplicativity of b, υ and $|\cdot|$ give $\mathrm{mc}(qq') = \mathrm{mc}(q)\,\mathrm{mc}(q')$; induction on $\#\operatorname{supp} z$ gives $\mathrm{mc}(\gamma_z) = \prod_{v\in\operatorname{supp} z} \mathrm{mc}(\gamma_{v,z_v})$. Put $c_v = \tilde{\nu}(\bar{x}_{w_v})$, so $\tilde{\nu}(x_{w_v}) = c_v^{-1}$ by Step 2 (i) and $|c_v| = 1$. By Lemma 6.7.16 (c),

$$\mathrm{mc}(\gamma_{v,z}) = a_v(z) := c_v^z\,\mathrm{N}_v^{-k|z|} \qquad (z \in \mathbb{Z}),$$

since $(c_v^{-1}\mathrm{N}_v^{-k})^{-z} = c_v^z\mathrm{N}_v^{kz}$ for $z < 0$. In particular $a_v(0) = 1$ and $|a_v(z)| = \mathrm{N}_v^{-k|z|}$.

Step 5: absolute summability. Put $B_v = \sum_{z\in\mathbb{Z}} |a_v(z)| = 1 + \beta_v$ with $\beta_v = 2\mathrm{N}_v^{-k}/(1-\mathrm{N}_v^{-k}) \le 4\mathrm{N}_v^{-k}$. Above each rational prime p there are at most $[F:\mathbb{Q}]$ places of F, each of norm at least p, and for $v \in V$ the number N_v is the norm of v; so $\sum_{v\in V}\beta_v \le 4[F:\mathbb{Q}]\sum_p p^{-k} < \infty$ as $k \ge 2$. For a finite set $V' \subseteq V$, distributivity gives

$$\sum_{\operatorname{supp} z\subseteq V'}\ \prod_{v\in V'} |a_v(z_v)| = \prod_{v\in V'} B_v \le \exp\Bigl(\sum_{v\in V}\beta_v\Bigr), \qquad \sum_{\operatorname{supp} z\subseteq V'}\ \prod_{v\in V'} a_v(z_v) = \prod_{v\in V'} A_v,$$

where $A_v = \sum_{z\in\mathbb{Z}} a_v(z)$. Every finite subset of $\mathbb{Z}^{(V)}$ consists of maps supported in some finite V', so $(\mathrm{mc}(\gamma_z))_z$ is absolutely summable with sum of moduli $B = \prod_{v\in V} B_v$ (a convergent product of numbers ≥ 1). Enumerate $V = \{v_1, v_2, \dots\}$ and put $V_n = \{v_1, \dots, v_n\}$; then

$$\Bigl|\sum_{z\in\mathbb{Z}^{(V)}} \mathrm{mc}(\gamma_z) - \prod_{i\le n} A_{v_i}\Bigr| \le \sum_{\operatorname{supp} z\not\subseteq V_n} |\mathrm{mc}(\gamma_z)| = B - \prod_{i\le n} B_{v_i} \longrightarrow 0,$$

so the sum in (b) is $\lim_n \prod_{i\le n} A_{v_i}$ (if V is finite the limit is reached).

Step 6: the local sums. With $\xi = c_v\mathrm{N}_v^{-k}$ and $\xi' = c_v^{-1}\mathrm{N}_v^{-k}$, both of modulus $\mathrm{N}_v^{-k} < 1$, the two geometric series give

$$\begin{aligned} A_v &= \sum_{z\ge 0}\xi^z + \sum_{z\ge 1}\xi'^z = \frac{1}{1-\xi} + \frac{\xi'}{1-\xi'} = \frac{1-\xi\xi'}{(1-\xi)(1-\xi')} \\ &= \frac{1-\mathrm{N}_v^{-2k}}{(1-c_v\mathrm{N}_v^{-k})(1-c_v^{-1}\mathrm{N}_v^{-k})} = e_v(k)\,L_v(k), \end{aligned}$$

by (6.182) with $\tilde{\nu}(x_{w_v})^{\pm 1} = c_v^{\mp 1}$.

Step 7: conclusion. For a place v of F below no place of D that does not split, (6.182) gives $e_v(k)L_v(k) = 1$. The products $\prod_v e_v(k)$ and $\prod_v L_v(k) = L^S(k,\nu)$ over the places v of F below no place of D converge absolutely, as $|1-e_v(k)| \le \mathrm{N}w^{-k}$ and $|1-L_v(k)^{-1}| \le 3\mathrm{N}w^{-k}$ with $\sum_v \mathrm{N}w^{-k} < \infty$ as in Step 5; so their product is the absolutely convergent product $\prod_v e_v(k)L_v(k)$, whose limit does not depend on the order of the factors. Its factors are 1 off V and A_v on V, so it equals $\lim_n \prod_{i\le n} A_{v_i}$, which by Step 5 is the sum in (b). This proves (b).

Step 8: proof of (c). For every finite place v of F and a place w above it, $0 < e_v(k) < 1$ and $1 - e_v(k) \le \mathrm{N}w^{-k} \le \mathrm{N}v^{-k}$ (as $\mathrm{N}w \ge \mathrm{N}v$). Above each rational prime p there are at most $[F:\mathbb{Q}]$ places of F, each of norm at least p, so $\sum_v \mathrm{N}v^{-k} \le [F:\mathbb{Q}]\sum_p p^{-k} < \infty$ as $k \ge 2$. Since $-\log(1-t) \le 2t$ for $0 \le t \le \frac{1}{2}$ and $1 - e_v(k) \le 2^{-k} \le \frac{1}{4}$, the series $\sum_{v\in V'} \log e_v(k)$ converges absolutely, and $\prod_{v\in V'} e_v(k) = \exp\sum_{v\in V'}\log e_v(k)$ is an absolutely convergent product with value in $(0,1]$. For the Euler product: by Step 2 the factor at $u \notin D$ is $(1-\xi_u)^{-1}$ with $|\xi_u| = \mathrm{N}u^{-k} \le 2^{-k}$, so it is defined and nonzero, and $\sum_{u\notin D}|\xi_u| \le 2[F:\mathbb{Q}]\sum_p p^{-k} < \infty$ (at most two places of E above each place of F); the identity with $L^S(k,\nu)$ is Step 2. With the principal branch, $|\log(1-\xi_u)| \le 2|\xi_u|$, so $\prod_u(1-\xi_u)^{-1} = \exp(-\sum_u \log(1-\xi_u))$ converges absolutely and is not 0. □

Lemma 6.7.16. *Let $S = \mathrm{I}_1$, so that $\mathrm{G}(s) = \mathrm{U}(1)(\mathbb{A})$ and the finite points form $G_f(1) = \{q \in \mathbb{A}_{E,f}^{\times} : q\bar{q} = 1\}$. Let $k \geq 1$ be an integer, T a space of rank $2k$, ν an automorphic character of $\mathrm{U}(1)$ with a weight and a tame part $\psi = (\psi_w)_w$, and $b = \nu\upsilon^k$. For $q \in G_f(1)$ let*

$$\mathrm{mc}(q) = \frac{\langle v^0, \omega_{1,b}(q,1)\, v^0\rangle}{\langle v^0, v^0\rangle}$$

be the ***normalised vacuum coefficient****, formed in $\mathcal{M}_{S,T}$. A point $q \in G_f(1)$ is* integral at *a finite place w if $q_w \in \mathcal{O}_{E,w}$.*

(a) For $q \in G_f(1)$ and every finite idele u with $u\widehat{\mathcal{O}}_E = \widehat{\mathcal{O}}_E \cap q\widehat{\mathcal{O}}_E$,

$$\mathrm{mc}(q) = b(q)^{-1}\, \upsilon(u)^{2k}\, |u|^k.$$

(b) Let D be a set of finite places of E with $\psi_w = 1$ for every finite place $w \notin D$. Then $\mathrm{mc}(\kappa q \kappa') = \mathrm{mc}(q)$ for every $q \in G_f(1)$ and all $\kappa, \kappa' \in G_f(1)$ that are integral at every finite place and have component 1 at every place of D.

(c) Let w be a finite place with $\bar{w} \neq w$, ϖ_w a uniformiser of E_w, $x_w \in \mathbb{A}_{E,f}^{\times}$ the idele with component ϖ_w at w and 1 elsewhere, and $\bar{x}_w$ its conjugate (component $\bar{\varpi}_w$ at $\bar{w}$, 1 elsewhere). For $z \in \mathbb{Z}$ the point $\gamma_{w,z} = (x_w/\bar{x}_w)^z$ lies in $G_f(1)$ (component ϖ_w^z at w, $\bar{\varpi}_w^{-z}$ at $\bar{w}$, 1 elsewhere), and

$$\mathrm{mc}(\gamma_{w,z}) = \begin{cases} \big(\tilde{\nu}(\bar{x}_w)\, \mathrm{N}w^{-k}\big)^z, & z \geq 0, \\ \big(\tilde{\nu}(x_w)\, \mathrm{N}w^{-k}\big)^{-z}, & z < 0. \end{cases}$$

Proof. Throughout, $L_1 = \widehat{\mathcal{O}}_E$. Two facts from (WD3) are used: $\upsilon(\bar{x}) = \upsilon(x)^{-1}$ for every idele x, and $\upsilon(x) = 1$ for every idele x whose finite components are units and whose infinite components are 1. Conjugation maps E_w isometrically onto $E_{\bar{w}}$ and $\mathrm{N}\bar{w} = \mathrm{N}w$, so $|\bar{x}| = |x|$.

Step 1: the ideles u. Let $q \in G_f(1)$. At a finite place w, $\mathcal{O}_{E,w} \cap q_w\mathcal{O}_{E,w} = \varpi_w^{m_w}\mathcal{O}_{E,w}$ with $m_w = \max(0, \mathrm{ord}_w\, q_w)$, and $m_w = 0$ for almost all w because q_w is a unit for almost all w. Hence finite ideles u with $uL_1 = L_1 \cap qL_1$ exist, and two of them differ by a factor whose finite components are units; that factor has υ-value 1 and norm 1. So the right side of (a) does not depend on the choice of u. Moreover $q^{-1} = \bar{q}$, so $L_1 \cap q^{-1}L_1 = L_1 \cap \bar{q}L_1$ is the conjugate of $L_1 \cap qL_1$, that is $\bar{u}L_1$.

Step 2: proof of (a). By (WR4) with $\alpha = 1$, $\beta = b$, the point $q \in \mathrm{G}(s)$ and $h = 1 \in \mathrm{G}(s')$, s' the signs of T on its index set n' with $\#n' = 2k$,

$$\omega_{1,b}(q,1) = b(q)^{-1}\, \omega_{s'}(h_q), \qquad h_q = 1_{2k} \otimes (q^{-1})^{\mathsf{T}} = q^{-1} \cdot 1_{2k}.$$

The point h_q has infinite components 1. By Lemma 6.2.13(1),(2),

$$\langle v^0, v^0\rangle = 1, \qquad \langle v^0, \omega_{s'}(h_q)v^0\rangle = \overline{\Delta_{s'}(h_q)}^{-1}.$$

For the idele u of (a), the matrix $u' = u \cdot 1_{2k}$ satisfies

$$u'L_1^{2k} = L_1^{2k} \cap h_q^{-1}L_1^{2k} = (L_1 \cap qL_1)^{2k}.$$

Thus (WR3), whose infinite factor is 1 here, gives $\Delta_{s'}(h_q) = \upsilon(u)^{2k}|u|^{-k}$. Since $|\upsilon(u)| = 1$ and $|u| > 0$, its inverse conjugate is $\upsilon(u)^{2k}|u|^k$. Multiplying by $b(q)^{-1}$ and using the denominator 1 proves (a).

Step 3: proof of (b). Let κ be as in (b). At a finite place w with $\bar{w} \neq w$, $(\kappa\bar{\kappa})_w = \kappa_w \overline{\kappa_{\bar{w}}} = 1$ with both factors integral, so κ_w is a unit; at $w = \bar{w}$, $|\kappa_w|_w^2 = |\kappa_w \bar{\kappa}_w|_w = 1$. So all finite components of κ, and likewise of κ', are units, $\kappa q \kappa' L_1 = qL_1$, and the same u serves for q and $\kappa q \kappa'$ in (a). Further $b(\kappa) = \nu(\kappa)\upsilon(\kappa)^k$ with $\upsilon(\kappa) = 1$ by the second fact above, and, κ having integral finite components and infinite components 1, the defining formula of weight and tame part gives $\nu(\kappa) = \prod_w \psi_w(\kappa_w \bmod w) = 1$, because $\kappa_w = 1$ for $w \in D$ and $\psi_w = 1$ for $w \notin D$. Likewise $b(\kappa') = 1$. So $b(\kappa q \kappa') = b(q)$, and (a) gives $\mathrm{mc}(\kappa q \kappa') = \mathrm{mc}(q)$.

Step 4: proof of (c). Since $\bar{w} \neq w$, $\gamma = \gamma_{w,z}$ has the components stated and $\gamma\bar{\gamma} = 1$, so $\gamma \in G_f(1)$. Its value under b is

$$b(\gamma) = \nu\big((x_w/\bar{x}_w)^z\big)\,\big(\upsilon(x_w)\upsilon(\bar{x}_w)^{-1}\big)^{kz} = \tilde{\nu}(x_w)^z\,\upsilon(x_w)^{2kz}.$$

If $z \geq 0$, then $L_1 \cap \gamma L_1$ has component $\varpi_w^z \mathcal{O}_{E,w}$ at w and $\mathcal{O}_{E,w'}$ at every other finite place w' (at $\bar{w}$ the component $\bar{\varpi}_w^{-z}$ of γ has order $-z \leq 0$); so $u = x_w^z$, $|u| = \mathrm{N}w^{-z}$ and $\upsilon(u)^{2k} = \upsilon(x_w)^{2kz}$. By (a), $\mathrm{mc}(\gamma) = \tilde{\nu}(x_w)^{-z}\mathrm{N}w^{-kz}$, and $\tilde{\nu}(x_w)^{-1} = \nu(\bar{x}_w/x_w) = \tilde{\nu}(\bar{x}_w)$. If $z < 0$, the component of $L_1 \cap \gamma L_1$ is $\bar{\varpi}_w^{-z}\mathcal{O}_{E,\bar{w}}$ at $\bar{w}$ and integral elsewhere, so $u = \bar{x}_w^{-z}$, $|u| = \mathrm{N}w^z$ and $\upsilon(u)^{2k} = \upsilon(\bar{x}_w)^{-2kz} = \upsilon(x_w)^{2kz}$; by (a), $\mathrm{mc}(\gamma) = \tilde{\nu}(x_w)^{-z}\mathrm{N}w^{kz} = \big(\tilde{\nu}(x_w)\mathrm{N}w^{-k}\big)^{-z}$. □

The test forms (Proposition 6.7.17) are the finite half of the bound: for every prime $\ell \nmid \mathrm{N}Q$ they pair with the scaled lift of a main pair to a number divisible by at most $\ell^{F(\ell)}$, up to one number $c_{\mathrm{tf}} \neq 0$ with small conjugates, chosen before the prime. They are built on four blocks of the definite space from point forms, which are integral and orthogonal to the false lifts by the vanishing of their periods (Proposition 6.6.81), and their pairing with every kernel form is a theta kernel on twelve blocks times three values of flat pairs (Lemma 6.7.18). The twelve-block kernel is handled by the sharpness of the jet valuation bound (Proposition 6.7.24), and a multiplicative pigeonhole over the class characters of U(1) keeps the product large (Lemma 6.7.22). The jet pair and the vacuum scales are those fixed in Section 4. For the ℓ-adic integrality of Hecke L-values of CM fields, the background of the construction, see [159, 272].

Proposition 6.7.17. *Let $1 \leq j \leq m$, let $p = (\nu, z)$ be a main pair for j, and let $p_{\mathrm{fl}} = (\nu_{\mathrm{fl}}, 2)$ be a flat critical pair. There is $c_{\mathrm{tf}} \in \mathbb{C} \setminus \{0\}$ with $|\sigma(c_{\mathrm{tf}})| \leq \exp(fmb/k + 3x^{34})$ for every automorphism σ of $\mathbb{C}$, such that for every prime ℓ, every embedding $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ and every twist ϑ with $\ell \nmid \mathrm{N}Q$ there is a test form Ξ for ℓ of $O_{p,\vartheta}$ with*

$$\big\|\iota\big(c_{\mathrm{tf}}\, r_{p_{\mathrm{fl}}}(\vartheta)\big)\big\| \cdot \ell^{-F(\ell)} \;\leq\; \big\|\iota\langle \Xi, \varphi_{p,\vartheta}\rangle\big\|,$$

where $\varphi_{p,\vartheta}$ is the lift of p at ϑ.

Proof. Let $p = (\nu, z)$, ν of weight $-c$, and $l(\tau) = c(\tau) - z$, so $\sum_\tau l(\tau) = j_p$.

Step 1: the jet degrees, the vacuum scale and nonvanishing. By (RH1), (RH3) and $z \leq r$, for every $\tau \in \Phi$: $l(\tau) \geq \kappa_j(\tau) + r - t - r = \kappa(\tau) + (2j+1)r^8 - 5r \geq \kappa(\tau)$, and $l(\tau) \leq \kappa_j(\tau) - z \leq \kappa(\tau) + (2m+1)r^8 \leq \kappa(\tau) + x^{42}$, since $m \leq 2\sqrt{x}$, $r \leq 3x^5$ by (AD2) and $(4\sqrt{x}+1)3^8x^{40} \leq x^{42}$ as $x \geq e^{10^{14}}$ by Lemma 2.3.6. So Proposition 6.7.24 applies to l; let $t_{\mathrm{shp}} \neq 0$ be its number, $|\sigma(t_{\mathrm{shp}})| \leq \exp(fmb/k)$. Applied at any prime p' and any ring homomorphism $\mathbb{C} \to \mathbb{C}_{p'}$ (one exists: both fields are algebraically closed of characteristic 0 and of the cardinality of the continuum), its conclusion is a positive lower bound for the norm of a multiple of c_s^{12}, so $c_s \neq 0$. Hence for every flat critical pair q with $z_q \geq 2$ the scale factor $c_s^{2z_q}$ of $\varphi_{q,\vartheta'}$ is nonzero (row lengths 0), and Proposition 6.7.13 gives $\mathrm{alg}_q(\vartheta') > 0$ and $r_q(\vartheta') = \mathrm{alg}_q(\vartheta')/\mathrm{alg}_q(1)$; as $N_{\mathrm{ch}}, n_{\nu_q} \geq 1$,

$$I_q^{N_{\mathrm{ch}}}(\vartheta') > 0, \qquad I_q^{N_{\mathrm{ch}}}(\vartheta') = r_q(\vartheta')\, I_q^{N_{\mathrm{ch}}}(1). \tag{6.183}$$

Step 2: the group G and the flat pairs over it. Let G be the set of automorphic characters of $\mathrm{U}(1)$ of weight 0 whose tame part is trivial off the chosen places; it is a group under multiplication. Let $\mathbf{K}^{\mathrm{ch}} \subseteq G_f(1)$ be the points integral at every finite place and congruent to 1 modulo every chosen place, and $C = \mathrm{U}(1)(E)\backslash G_f(1)/\mathbf{K}^{\mathrm{ch}}$, a finite abelian group (Lemma 6.7.22 (a); $G_f(1)$ is commutative). Every $\chi \in G$ is trivial on $\mathrm{U}(1)(\mathbb{R})$ (weight 0), on $\mathrm{U}(1)(E)$ and on $\mathbf{K}^{\mathrm{ch}}$ (its tame part is trivial off the chosen places and the residues at the chosen places are 1), so $g \mapsto \chi(g)$ is a character of C that determines χ, as $\mathrm{U}(1)(\mathbb{A}) = \mathrm{U}(1)(\mathbb{R}) \times G_f(1)$; hence G is finite. By (AD2) and Lemma 2.3.6, $m \geq \sqrt{x} \geq 12$, so $z \geq m+1 \geq 12$ by (RH1). For $\chi \in G$ put

$$q_1(\chi) = (\chi \upsilon^{-(z-10)},\, z-10), \qquad q_3(\chi) = (\chi \upsilon^{-2},\, 2).$$

These are flat critical pairs: $\chi\upsilon^{-z'}$ has weight $-z'$ and the tame part of χ, and $1 \leq z' \leq z \leq t$, $z' \leq z \leq c(\tau) \leq 4b$ because p is critical. As $\mathbb{C}$ is algebraically closed, choose $V \in \mathbb{C}$ with

$$V^{\#G} = \prod_{\chi \in G} I^{N_{\mathrm{ch}}}_{q_3(\chi)}(1)\, I^{N_{\mathrm{ch}}}_{q_1(\chi)}(1);$$

$V \neq 0$ by (6.183). Put $c_{\mathrm{tf}} = t_{\mathrm{shp}}\, I^{N_{\mathrm{ch}}}_{p_{\mathrm{fl}}}(1)\, V \neq 0$.

Step 3: the conjugates of c_{tf}. Let σ be an automorphism of $\mathbb{C}$. By Lemma 6.7.22 (b) for $\iota = \sigma$, $a(\chi) = I^{N_{\mathrm{ch}}}_{q_3(\chi)}(1)$, $b(\chi) = I^{N_{\mathrm{ch}}}_{q_1(\chi)}(1)$ and $\psi = 1$, there are $\varphi', \chi' \in G$ with $|\sigma(V)| \leq |\sigma(I^{N_{\mathrm{ch}}}_{q_3(\varphi')}(1))|\,|\sigma(I^{N_{\mathrm{ch}}}_{q_1(\chi')}(1))| \leq \exp(x^{34})^2$ by Lemma 6.7.23. With $|\sigma(t_{\mathrm{shp}})| \leq \exp(fmb/k)$ and $|\sigma(I^{N_{\mathrm{ch}}}_{p_{\mathrm{fl}}}(1))| \leq \exp(x^{34})$ (Lemma 6.7.23), $|\sigma(c_{\mathrm{tf}})| \leq \exp(fmb/k + 3x^{34})$.

Step 4: the choices at (ℓ, ι, ϑ). Fix a prime ℓ, a ring homomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ and a twist ϑ with modulus Q, $\ell \nmid \mathrm{N}Q$, and let $X = X(p, \vartheta)$; then $\ell_X = l$ and $\mathbf{K}^{\mathrm{ch}}_X = \mathbf{K}^{\mathrm{ch}}$ (the chosen places of X are those of D). Let $\mathfrak{m} = \pi_M \mathcal{O}_E$ and $F_{\mathfrak{m},y_0}$ the indicator datum of Proposition 6.7.24. By Proposition 6.7.24 at (ℓ, ι) there are γ with $\sum_i \gamma(\tau, i) = l(\tau)$ and $x_0 \in R^{12}_{\mathfrak{m}}$ with

$$\ell^{-F(\ell)} \leq \Big\| \iota\Big(\frac{c_s^{12}}{t_{\mathrm{shp}}} \prod_\tau l(\tau)!\, [Z^\gamma] T(F_{\mathfrak{m},x_0}) \Big) \Big\|. \tag{6.184}$$

Let $\chi_0 = \nu \upsilon^{z-2} \nu_{\mathrm{fl}}^{-1}$ and, for $g \in G_f(1)$, $s_0(g) = \theta^{\mathrm{I}_1,\mathrm{I}_{12}}_{1,\chi_0}(v_{X^\gamma} \otimes F_{\mathfrak{m},x_0})((1,g),(1,1))$. The germ θ_F of Proposition 6.7.24 is $\mathscr{E}_{V_{12},F}$ of Lemma 4.2.2, defined for $F = F_{\mathfrak{m},x_0}$ in the finite span $\mathcal{M}_{\mathrm{fin},V_{12}}$ there: by (4) and (2) of Lemma 6.2.33 and (WR6), $F_{\mathfrak{m},x_0} = v_1 \otimes (\Psi_{\mathfrak{m},x_0} \otimes_{\Sigma(\mathfrak{m})} v^0) = \Psi_{\mathfrak{m},x_0} \otimes_{\Sigma(\mathfrak{m})} v^0 = \sum_y c_y(\Psi_{\mathfrak{m},x_0})\, \rho(\mathrm{fv}(y))\, v^0$, a finite combination of finite Heisenberg translates of the vacuum, and the datum $v_{X^\gamma} \otimes F_{\mathfrak{m},x_0}$ of (WR6) is $v_{X^\gamma} \otimes (\Psi_{\mathfrak{m},x_0} \otimes_{\Sigma(\mathfrak{m})} v^0)$. By Lemma 4.2.2 (with $Q = X^\gamma$, the constant $\pi_M \eta_P$, $F_0 = F_{\mathfrak{m},x_0}$, and v_P linear in P), the Taylor coefficient is $[Z^\gamma]T(F_{\mathfrak{m},x_0}) = (\pi_M \eta_P)^{j_p}\, \theta_{\mathrm{Hom}(\mathrm{I}_1,\mathrm{I}_{12})}(v_{X^\gamma} \otimes F_{\mathfrak{m},x_0})/\gamma!$, $\gamma! = \prod_{\tau,i} \gamma(\tau,i)!$. At $g = 1$ the pair map gives the identity of G, $\chi_0(1) = 1$, $\Delta(1) = 1$, and the Weil operator at 1 is the identity on the theta model by its defining reproducing identity (WD7); so $s_0(1) = \theta_{\mathrm{Hom}(\mathrm{I}_1,\mathrm{I}_{12})}(v_{X^\gamma} \otimes F_{\mathfrak{m},x_0})$. With $T_\gamma = \prod_\tau l(\tau)!/\gamma!$, (6.184) becomes

$$\|\iota(t_{\mathrm{shp}})\|\, \ell^{-F(\ell)} \leq \big\|\iota\big(c_s^{12}(\pi_M \eta_P)^{j_p} T_\gamma\big)\big\|\, \|\iota(s_0(1))\|. \tag{6.185}$$

By Lemma 6.7.22 (a), with the chosen places as S, $K = \mathbf{K}^{\mathrm{ch}}$, a set R of representatives of C and $s = s_0$, there are $z_0 \in G_f(1)$ and $\psi \in G$ with $\|\iota(s_0(1))\| \leq \|\iota(A)\|$, $A = \frac{1}{\#C} \sum_{g \in R} \psi(gz_0) s_0(gz_0)$. By Lemma 6.7.22 (b) for ι, a, b, V as in Step 3 and this ψ, there are $\psi_3, \psi_1 \in G$ with $\psi_3\psi_1 = \psi$ and $\|\iota(V)\| \leq \|\iota(W)\|$, $W = I^{N_{\mathrm{ch}}}_{q_3(\psi_3)}(1)\, I^{N_{\mathrm{ch}}}_{q_1(\psi_1)}(1)$.

Step 5: the test form and its pairing with the lift. Apply Lemma 6.7.18 to j, p, ℓ, ϑ, the flat pairs $q = q_1(\psi_1)$, $q_2 = p_{\mathrm{fl}}$, $q_3 = q_3(\psi_3)$, x_0, $\alpha = \gamma$ and $\chi = \chi_0\psi^{-1}$; the character condition

holds since $\chi\cdot\nu_{\mathrm{fl}}\vartheta\upsilon^2\cdot\psi_3\cdot\psi_1=\chi_0\nu_{\mathrm{fl}}\vartheta\upsilon^2=\nu\vartheta\upsilon^z$. It gives a test form Ξ for ℓ of $O_{p,\vartheta}$ with, for all $g\in G_f(1)$,

$$\langle\Xi,k_X(g)\rangle\,h_{\mathrm{U}}^3c_s^{2z-12}=T_\gamma\,W\,I_{p_{\mathrm{fl}}}^{N_{\mathrm{ch}}}(\vartheta)\,s_\chi(g),$$

where s_χ is s_0 with χ in place of χ_0. By (WR4), $\omega_{1,\chi}((1,g),(1,1))=\chi(g)^{-1}\omega_{1,1}((1,g),(1,1))$, so $s_\chi(g)=\psi(g)s_0(g)$. By Lemma 4.6.2 (c) (the definite line I_1 with its standard lattice, a chosen place w, and $\mathbf{K}^{\mathrm{ch}}$ acting trivially modulo w) all stabilisers Γ_g are trivial, so $h_{\mathrm{U}}=\#C$ and, by (OL3), $\lambda_{p,\vartheta}=\sum_{g\in R}k_X(g)$. As C is a group, $g\mapsto gz_0$ permutes the classes, and k_X is constant on classes, so $\lambda_{p,\vartheta}=\sum_{g\in R}k_X(gz_0)$. Pairing with Ξ (linear in the second argument), multiplying by $c_s^{12}(\pi_M\eta_P)^{j_p}$ and using (6.183) for p_{fl}, with $r=r_{p_{\mathrm{fl}}}(\vartheta)$:

$$(\#C)^2\,\langle\Xi,\varphi_{p,\vartheta}\rangle=r\,I_{p_{\mathrm{fl}}}^{N_{\mathrm{ch}}}(1)\,W\,c_s^{12}(\pi_M\eta_P)^{j_p}\,T_\gamma\,A. \tag{6.186}$$

Step 6: the inequality. By the choices of Step 4 and (6.185),

$$\begin{aligned}\|\iota(c_{\mathrm{tf}}\,r)\|\,\ell^{-F(\ell)}&=\|\iota(r\,I_{p_{\mathrm{fl}}}^{N_{\mathrm{ch}}}(1))\|\,\|\iota(V)\|\,\|\iota(t_{\mathrm{shp}})\|\,\ell^{-F(\ell)}\\&\le\|\iota(r\,I_{p_{\mathrm{fl}}}^{N_{\mathrm{ch}}}(1))\|\,\|\iota(W)\|\,\|\iota(c_s^{12}(\pi_M\eta_P)^{j_p}T_\gamma)\|\,\|\iota(A)\|,\end{aligned}$$

which is $\|\iota((\#C)^2\langle\Xi,\varphi_{p,\vartheta}\rangle)\|$ by (6.186). Since $\#C$ is a positive integer, $\|\iota(\#C)\|\le1$, and the claim $\|\iota(c_{\mathrm{tf}}r_{p_{\mathrm{fl}}}(\vartheta))\|\,\ell^{-F(\ell)}\le\|\iota\langle\Xi,\varphi_{p,\vartheta}\rangle\|$ follows. □

Lemma 6.7.18. *Let $1\le j\le m$, let $p=(\nu,z)$ be a main pair for j, let ℓ be a prime and ϑ a twist with modulus Q and $\ell\nmid\mathrm{N}Q$; let $X=X(p,\vartheta)$, so that $\ell_X(\tau)=c(\tau)-z$ for the weight $-c$ of ν. Let $q=(\nu_1,z-10)$, $q_2=(\nu_2,2)$ and $q_3=(\nu_3,2)$ be flat critical pairs. Let $\mathfrak{m}=\pi_M\mathcal{O}_E$, $x_0\in R_{\mathfrak{m}}^{12}$, let $\alpha\colon\Phi\times\{1,\dots,12\}\to\mathbb{Z}_{\ge0}$ satisfy $\sum_{i=1}^{12}\alpha(\tau,i)=\ell_X(\tau)$ for every $\tau\in\Phi$, and let χ be an automorphic character of* $\mathrm{U}(1)$ *with*

$$\chi\cdot\nu_2\vartheta\upsilon^2\cdot\nu_3\upsilon^2\cdot\nu_1\upsilon^{z-10}=\nu\vartheta\upsilon^z.$$

For $g\in G_f(1)$ put

$$s(g)=\theta_{1,\chi}^{\mathrm{I}_1,\mathrm{I}_{12}}\big(v_{X^\alpha}\otimes F_{\mathfrak{m},x_0}\big)\big((1,g),(1,1)\big),$$

$X^\alpha=\prod_{\tau,i}X_{\tau,i}^{\alpha(\tau,i)}$ in the Fock variables of $\mathrm{Hom}(\mathrm{I}_1,\mathrm{I}_{12})$*, and let $h_{\mathrm{U}}=\sum_{[g]}1/\#\Gamma_g$, the class sum of* 1 *over* $\mathrm{U}(1)(E)\backslash G_f(1)/\mathbf{K}_X^{\mathrm{ch}}$*. Then there is a test form Ξ for ℓ of $O_{p,\vartheta}$ such that for every $g\in G_f(1)$*

$$\langle\Xi,k_X(g)\rangle\,h_{\mathrm{U}}^3\,c_s^{2z-12}=\frac{\prod_{\tau\in\Phi}\ell_X(\tau)!}{\prod_{\tau,i}\alpha(\tau,i)!}\,I_q^{N_{\mathrm{ch}}}(1)\,I_{q_3}^{N_{\mathrm{ch}}}(1)\,I_{q_2}^{N_{\mathrm{ch}}}(\vartheta)\,s(g).$$

Proof. Let $O=O_{p,\vartheta}$, $X_1=X(q,1)$, $X_2=X(q_2,\vartheta)$, $X_3=X(q_3,1)$, and let $F(h_1,h_2,h_3)\in\mathrm{Form}(O)$ be the forms of Lemma 6.7.19 for these data. For a flat pair $q'\in\{q,q_2,q_3\}$ and the tame character ϑ' attached to it (1, ϑ, 1), let $\mathrm{xs}_{q',\vartheta'}$ be the map of Lemma 6.7.21, formed with a fixed set of representatives and with values in $\mathrm{Form}(O)$. Put

$$G(h_2,h_3)=\mathrm{xs}_{q,1}\big(h_1\mapsto F(h_1,h_2,h_3)\big),\qquad H(h_2)=\mathrm{xs}_{q_3,1}\big(h_3\mapsto G(h_2,h_3)\big),\qquad\Xi=\mathrm{xs}_{q_2,\vartheta}(H).$$

The number h_{U} of Lemma 6.7.21 is the same for X_1, X_2, X_3 and X, the group $\mathbf{K}^{\mathrm{ch}}$ depending only on the chosen places.

Step 1: Ξ is integral above ℓ. Let $j\colon\mathbb{C}\to\mathbb{C}_\ell$ be a ring homomorphism. Each $F(h_1,h_2,h_3)$ is integral at j by Lemma 6.7.19 (ii). By Lemma 6.7.21 (a), applied to q and q_3 at the trivial character (modulus 1, prime to ℓ) and to q_2 at ϑ ($\ell\nmid\mathrm{N}Q$), the forms $G(h_2,h_3)$, $H(h_2)$ and Ξ are integral at j.

Step 2: Ξ is orthogonal to the false lifts. The forms orthogonal to the false lifts for $(e_{X-1}, \ker \mathrm{Th}_O)$ form a linear subspace, an intersection of orthogonal complements. By definition $G(h_2,h_3)$ is $N_{\mathrm{ch}} n_{\nu_1}^2$ times the form of Lemma 6.7.19 (iii), hence lies in it; $H(h_2)$ and Ξ are linear combinations of such forms.

Step 3: the pairing. Fix $g \in G_f(1)$ and let $L = \langle \cdot\,, k_X(g)\rangle$, conjugate-linear on $\mathrm{Form}(O)$. By Lemma 6.7.19 (i), $L(F(h_1,h_2,h_3)) = B_1 \cdot 1 \cdot k_{X_1}(g)(h_1)$ with $B_1 = T_\alpha\, s(g)\, k_{X_2}(g)(h_2)\, k_{X_3}(g)(h_3)$, so Lemma 6.7.21 (b) for q ($N = 2z-20$) gives $L_1(G(h_2,h_3)) = T_\alpha\, s(g)\, k_{X_2}(g)(h_2)\, I_q^{N_{\mathrm{ch}}}(1) \cdot k_{X_3}(g)(h_3)$ for the conjugate-linear form $L_1 = h_{\mathrm{U}} c_s^{2z-20} L$. Applying Lemma 6.7.21 (b) to L_1 and q_3 ($N=4$), $L_2 = h_{\mathrm{U}} c_s^4 L_1$ satisfies $L_2(H(h_2)) = T_\alpha\, s(g)\, I_q^{N_{\mathrm{ch}}}(1)\, I_{q_3}^{N_{\mathrm{ch}}}(1) \cdot k_{X_2}(g)(h_2)$; applying it to L_2 and q_2 at ϑ,

$$h_{\mathrm{U}} c_s^4 L_2(\Xi) = h_{\mathrm{U}}^3 c_s^{2z-12} \langle \Xi, k_X(g)\rangle = T_\alpha\, I_q^{N_{\mathrm{ch}}}(1)\, I_{q_3}^{N_{\mathrm{ch}}}(1)\, I_{q_2}^{N_{\mathrm{ch}}}(\vartheta)\, s(g).$$

With Steps 1 and 2, Ξ is a test form for ℓ of $O_{p,\vartheta}$ with the asserted pairing. □

Lemma 6.7.19. *Let $1 \le j \le m$, let $p = (\nu, z)$ be a main pair for j, ϑ a twist, $X = X(p,\vartheta)$ and $O = O_{p,\vartheta}$. Let $q = (\nu_1, z-10)$, $q_2 = (\nu_2, 2)$, $q_3 = (\nu_3, 2)$ be flat critical pairs, and put $X_1 = X(q,1)$, $X_2 = X(q_2,\vartheta)$, $X_3 = X(q_3,1)$. Let $x_0 \in R_{\mathfrak{m}}^{12}$, let $\alpha\colon \Phi \times \{1,\dots,12\} \to \mathbb{Z}_{\ge 0}$ with $\sum_i \alpha(\tau,i) = \ell_X(\tau)$ for every τ, and let χ be an automorphic character of $\mathrm{U}(1)$ with $\chi \cdot \nu_2\vartheta\upsilon^2 \cdot \nu_3\upsilon^2 \cdot \nu_1\upsilon^{z-10} = \nu\vartheta\upsilon^z$. For $g \in G_f(1)$ let $s(g) = \theta_{1,\chi}^{\mathrm{I}_1,\mathrm{I}_{12}}(v_{X^\alpha} \otimes F_{\mathfrak{m},x_0})((1,g),(1,1))$ and $T_\alpha = \prod_\tau \ell_X(\tau)! / \prod_{\tau,i} \alpha(\tau,i)!$. Then there are forms $F(h_1,h_2,h_3) \in \mathrm{Form}(O)$, for $h_1 \in G_f(2z-20)$ and $h_2, h_3 \in G_f(4)$, such that:*

(i) for all $g \in G_f(1)$ and all h_1, h_2, h_3,

$$\langle F(h_1,h_2,h_3), k_X(g)\rangle = T_\alpha\, s(g)\, k_{X_2}(g)(h_2)\, k_{X_3}(g)(h_3)\, k_{X_1}(g)(h_1);$$

(ii) every $F(h_1,h_2,h_3)$ is integral at every ring homomorphism $\mathbb{C} \to \mathbb{C}_\ell$, for every prime ℓ;

(iii) for every set R_1 of representatives of $\mathrm{U}(\mathrm{I}_{2z-20})(E)\backslash G_f(2z-20)/\mathbf{K}(O_{X_1})$ and all h_2, h_3, the form $\sum_{h_1 \in R_1} \frac{1}{\#\Gamma_{h_1}} \varphi_{q,1}(h_1)\, F(h_1,h_2,h_3)$, with $\Gamma_{h_1} = \mathrm{U}(\mathrm{I}_{2z-20})(E) \cap h_1 \mathbf{K}(O_{X_1}) h_1^{-1}$, is orthogonal to the false lifts for $(e_{X-1}, \ker \mathrm{Th}_O)$.

Proof. Put $n = 2z$ and split $\{1,\dots,n\}$ into the consecutive blocks B_1, B_2, B_3, B_4 of sizes $12, 4, 4, 2z-20$ (the data $X_1 = X(q,1)$ require $z - 10 \ge 2$), so that $\mathrm{I}_n = \mathrm{I}_{12} \oplus \mathrm{I}_4 \oplus \mathrm{I}_4 \oplus \mathrm{I}_{2z-20}$ with B_2, B_3, B_4 carrying the index sets of X_2, X_3, X_1. Let M be the chosen places, Q the modulus of ϑ, $S_O = M \cup Q$, $\mathfrak{m}' = \prod_{w \in S_O} w$ and $\mathfrak{m} = \pi_M \mathcal{O}_E = \prod_{w \in M} w$ (the chosen places are unramified and distinct, Lemma 4.6.2 (a)), so that $R_{\mathfrak{m}'} = \prod_{w \in S_O} k_w$ and, by (DP7) and (XD5), $\mathrm{J}_O(\xi) = \mathrm{mod}_{\mathfrak{m}'}(\Psi_O, v_{\mathrm{Har}_O(\xi)})$. For $h_1 \in G_f(2z-20)$, $h_2, h_3 \in G_f(4)$ put $d = \mathrm{diag}(1, h_2, h_3, h_1) \in G_f(n)$.

Step 1: the vector. By (OL1), (OL4) the weight $\kappa = \kappa_O$ has one row of length $\ell = \ell_X$ at each τ, so the products of minors are the monomials $y^\beta = \prod_{\tau,a} y_{\tau,a,0}^{\beta(\tau,a)}$ with $\sum_a \beta(\tau,a) = \ell(\tau)$, and the standard basis consists of all of them. A filling $T \in \mathcal{C}_\kappa$ chooses one row for each of the $\ell(\tau)$ columns at every τ and Δ_T is the monomial of these rows; so F_κ has entries $0,1$ with one 1 in each row, G_κ is diagonal, and $(G_\kappa)_{\beta\beta} = \prod_\tau \ell(\tau)! / \prod_{\tau,a} \beta(\tau,a)!$, the number of fillings with monomial y^β. Extend α by 0 to $\Phi \times \{1,\dots,n\}$ and let $e_\alpha \in \mathbb{V}_O(\mathbb{Z})$ be the coordinate vector of y^α; then $\mathrm{Pol}_\kappa(e_\alpha) = T_\alpha y^\alpha$. By Lemma 3.2.20 for $(\kappa, 0)$, C_O is a square integral matrix (I_{κ^-} is a point) with an integral left inverse, hence in $\mathrm{GL}_{I_\kappa}(\mathbb{Z})$, commuting with every $S_\kappa(z)$ by (d). Put $c = C_O^{-1} e_\alpha \in \mathbb{V}_O(\mathbb{Z})$. By (DP8), $\mathrm{Har}_O(c) = \mathrm{Let}_1(\mathrm{Pol}_\kappa(e_\alpha)) = T_\alpha X^\alpha$, $X^\alpha = \prod_{\tau, i \in B_1} X_{\tau,i,1}^{\alpha(\tau,i)}$. If $z' = (z'_\tau)$ is block diagonal with the identity on B_1, then $S_\kappa(z') e_\alpha = e_\alpha$: the column α of $S_\kappa(z')$

lists the coefficients of y^α in the $b_i(z'y)$, and $b_i(z'y) = b_i(y)$ if b_i involves only rows of B_1, while otherwise every monomial of $b_i(z'y)$ contains a variable of a row outside B_1. Hence $S_\kappa(z')c = c$. For the flat pairs q_i the weight is 0: $\mathbb{V} = \mathbb{C}$, $G = 1$, and by (c) of Lemma 3.2.20 the 1×1 matrix C is 1, so the theta datum of 1 is $\mathrm{mod}(\Psi, v_1)$ for the block datum Ψ of the pair.

Step 2: the point. Let i_2, i_3, i_1 be the indices $\mathrm{inc}(1)$ of $O_{X_2}, O_{X_3}, O_{X_1}$, placed in B_2, B_3, B_4, and $\iota_0 = \mathrm{inc}_O(1)$. For $w \in M$ let $Y_w \in k_w^n$ be the column with $x_0 \bmod w$ on B_1 and the unit vectors at i_2, i_3, i_1 on B_2, B_3, B_4; as $Y_w \neq 0$, choose $\bar{A}_w \in \mathrm{GL}_n(k_w)$ whose column ι_0 is Y_w. For $w \in Q$ let $\bar{A}_w$ be the permutation matrix of the transposition of ι_0 and i_2 (the identity if they agree); its row i_2 is $e_{\iota_0}^\mathsf{T}$, so the row ι_0 of $\bar{A}_w^{-1}$ is $e_{i_2}^\mathsf{T}$. Every $w \in S_O$ satisfies $w \neq \bar{w}$, and no member of S_O is conjugate to another (DP1). Let $A \in G_f(n)$ have at each $w \in S_O$ a component $A_w \in \mathrm{GL}_n(\mathcal{O}_{E,w})$ reducing to $\bar{A}_w$, at $\bar{w}$ the component $(\overline{A_w}^\mathsf{T})^{-1}$ forced by $A^*A = 1$, and 1 elsewhere; then $A \in \mathbf{K}_0(n)$. For $X = (X_w)_{w \in S_O}$, $X_w \in k_w^n$, write $X^{(i)}$ for the rows in B_i. Then $\bar{A}_w^{-1} X_w = \mathrm{Inc}_O$ iff $X_w = Y_w$ ($w \in M$), and the inc-row of $\bar{A}_w^{-1} X_w$ is the entry i_2 of X_w ($w \in Q$). The tame part of $\mu_O = \nu\vartheta$ at $w \in Q$ is that of ϑ, which is also that of $\mu_{O_{X_2}} = \nu_2\vartheta$, the tame parts of ν, ν_2 being trivial off M. Hence, by (DP7) for the four pairs,

$$\Psi_O(\bar{A}^{-1}X) = \Psi_0(X^{(1)})\, \Psi_{O_{X_2}}(X^{(2)})\, \Psi_{O_{X_3}}(X^{(3)} \bmod \mathfrak{m})\, \Psi_{O_{X_1}}(X^{(4)} \bmod \mathfrak{m}), \tag{6.187}$$

where Ψ_0 is the indicator of x_0 on $\mathrm{M}_{12\times 1}(R_\mathfrak{m})$, read on $R_{\mathfrak{m}'}$ through the reduction, and $S_{O_{X_2}} = S_O$, $S_{O_{X_3}} = S_{O_{X_1}} = M$.

Step 3: point forms. For $y \in G_f(n)$ and $v \in \mathbb{V}_O$ let

$$\delta_y(v)(x) = \sum_\gamma \tilde{\eta}_O(y^{-1}\gamma x)\, \rho_O(\gamma)^{-1} v \qquad (x \in G_f(n)),$$

the sum over the rational points γ of $\mathrm{U}(\mathrm{I}_n)$ with $y^{-1}\gamma x \in \mathbf{K}(O)$, a finite set (discrete and compact). Substituting $\gamma\gamma'$ for γ shows $\delta_y(v)(\gamma' x k) = \tilde{\eta}_O(k)\rho_O(\gamma')\delta_y(v)(x)$ for rational γ' and $k \in \mathbf{K}(O)$, so $\delta_y(v) \in \mathrm{Form}(O)$, supported on $\mathrm{U}(\mathrm{I}_n)(E)\, y\, \mathbf{K}(O)$. For $\varphi \in \mathrm{Form}(O)$ only the class of y contributes to the class sum (DP5), and for $\gamma \in \Gamma_y = \mathrm{U}(\mathrm{I}_n)(E) \cap y\mathbf{K}(O)y^{-1}$, with $k_\gamma = y^{-1}\gamma y$, $\rho_O(\gamma)\varphi(y) = \varphi(\gamma y) = \varphi(y k_\gamma) = \tilde{\eta}_O(k_\gamma)\varphi(y)$; as ρ_O is unitary for $\langle\,,\rangle_\kappa$ ((c) of Lemma 4.6.10) and $|\tilde{\eta}_O| = 1$, each term of $\langle \delta_y(v)(y), \varphi(y)\rangle_\kappa$ is $\langle v, \varphi(y)\rangle_\kappa$, whence

$$\langle \delta_y(v), \varphi\rangle = \frac{1}{\#\Gamma_y}\, \#\Gamma_y\, \langle v, \varphi(y)\rangle_\kappa = \langle v, \varphi(y)\rangle_\kappa. \tag{6.188}$$

Define $F(h_1, h_2, h_3) = \delta_{dA}(c)$.

Step 4: proof of (i). By (6.188), c real, and (OL2) with $\xi = \bar{c} = c$, $\langle F(h_1, h_2, h_3), k_X(g)\rangle = c^\mathsf{T} G_\kappa k_X(g)(dA) = \theta_O(\omega_O((1, g), (1, dA))\mathrm{J}_O(c))$. By (2) of Lemma 4.1.1, $\omega_O((1, g), (1, dA)) = \omega_O((1, g), (1, d))\, \omega_O((1, 1), (1, A))$, and by (6) there (with $k = 1$, $k' = A$, $\chi_O = 1$) and (6.187),

$$\omega_O((1, 1), (1, A))\mathrm{J}_O(c) = \mathrm{mod}_{\mathfrak{m}'}(\Psi_0 \otimes \Psi_{O_{X_2}} \otimes \Psi'_{O_{X_3}} \otimes \Psi'_{O_{X_1}},\ v_{T_\alpha X^\alpha \cdot 1 \cdot 1 \cdot 1}),$$

Ψ' denoting the function read modulo $\mathfrak{m}$. By (a) of Lemma 6.3.80, applied three times, this is the target product $T_\alpha \,\mathrm{mod}_{\mathfrak{m}'}(\Psi_0, v_{X^\alpha}) \boxtimes^{\mathrm{tg}} \mathrm{mod}_{\mathfrak{m}'}(\Psi_{O_{X_2}}, v_1) \boxtimes^{\mathrm{tg}} \mathrm{mod}_{\mathfrak{m}'}(\Psi'_{O_{X_3}}, v_1) \boxtimes^{\mathrm{tg}} \mathrm{mod}_{\mathfrak{m}'}(\Psi'_{O_{X_1}}, v_1)$ for $\mathrm{I}_n = \mathrm{I}_{12} \oplus \mathrm{I}_4 \oplus \mathrm{I}_4 \oplus \mathrm{I}_{2z-20}$. By Lemma 6.4.66, $\mathrm{mod}_{\mathfrak{m}'}(\Psi_0, v_{X^\alpha}) = \mathrm{mod}_\mathfrak{m}(\mathbf{1}_{x_0}, v_{X^\alpha}) = v_{X^\alpha} \otimes (\Psi_{\mathfrak{m},x_0} \otimes_M v^0)$, which is the datum $v_{X^\alpha} \otimes F_{\mathfrak{m},x_0}$ of $s(g)$, because $F_{\mathfrak{m},x_0} = v_1 \otimes (\Psi_{\mathfrak{m},x_0} \otimes_M v^0) = \Psi_{\mathfrak{m},x_0} \otimes_M v^0$ by (4) of Lemma 6.2.33 ($\Sigma(\mathfrak{m}) = M$), and $\mathrm{mod}_{\mathfrak{m}'}(\Psi'_{O_{X_i}}, v_1) = \mathrm{J}_{O_{X_i}}(1)$ ($i = 3, 1$), while $\mathrm{mod}_{\mathfrak{m}'}(\Psi_{O_{X_2}}, v_1) = \mathrm{J}_{O_{X_2}}(1)$ (Step 1). The characters: $\omega_O = \omega_{1,\chi'_O}$ by (DP6), with $\chi'_O = \nu\vartheta v^z = \chi \cdot \nu_2\vartheta v^2 \cdot \nu_3 v^2 \cdot \nu_1 v^{z-10}$ by hypothesis, the last three being $\chi'_{O_{X_2}}, \chi'_{O_{X_3}}, \chi'_{O_{X_1}}$, and

$\chi_{O_{X_i}} = 1$. So the exact factorisation (d) of Lemma 6.3.52 (target sums), applied three times at the source point $(1, g)$ and the block diagonal target point $(1, d)$, gives

$$\langle F(h_1,h_2,h_3), k_X(g)\rangle = T_\alpha\, s(g)\, \theta_{O_{X_2}}(\omega_{O_{X_2}}((1,g),(1,h_2))\mathrm{J}_{O_{X_2}}(1)) \cdot \theta_{O_{X_3}}(\cdots h_3 \cdots)\, \theta_{O_{X_1}}(\cdots h_1 \cdots),$$

where the last three factors are $k_{X_2}(g)(h_2)$, $k_{X_3}(g)(h_3)$, $k_{X_1}(g)(h_1)$ by (OL2) in weight 0. This is (i).

Step 5: proof of (ii). Let $j\colon \mathbb{C} \to \mathbb{C}_\ell$ be a ring homomorphism and, for $x \in G_f(n)$, $x_\ell = (x_{\ell,\tau})_\tau$ its ℓ-adic gauge at j (Definition 6.1.3); $e_O = 0$, so we must show that $S_\kappa(x_\ell^{-1})\, j(F(x))$ has integral coordinates. For a rational point γ, $j(\rho_O(\gamma)v) = S_\kappa(j(\overline{\gamma_\infty}))\, j(v) = S_\kappa(\gamma_\ell)\, j(v)$, as S_κ has integral polynomial entries and $j(\tau(\bar\gamma))$ is the image of γ at the place defining the gauge. Put $y = dA$. On the support, $x = \gamma' y k$ with $k \in \mathbf{K}(O)$ and $F(x) = \tilde\eta_O(k)\rho_O(\gamma')F(y)$, so, S_κ being multiplicative ((a) of Lemma 4.6.10), $S_\kappa(x_\ell^{-1})j(F(x)) = j(\tilde\eta_O(k))\, S_\kappa(k_\ell^{-1})\, S_\kappa(y_\ell^{-1})\, j(F(y))$, and $F(y) = \sum_{\gamma\in\Gamma_y} \tilde\eta_O(k_\gamma)\rho_O(\gamma)^{-1}c$ with $\gamma y = y k_\gamma$. Hence

$$S_\kappa(y_\ell^{-1})\, j(F(y)) = \sum_{\gamma\in\Gamma_y} j(\tilde\eta_O(k_\gamma))\, S_\kappa((k_\gamma)_\ell^{-1})\, S_\kappa(A_\ell^{-1})\, S_\kappa(d_\ell^{-1})\, c,$$

and $S_\kappa(d_\ell^{-1})c = c$ by Step 1, d_ℓ being block diagonal with the identity on B_1. The values of $\tilde\eta_O$ are roots of unity, so $\|j(\tilde\eta_O(\cdot))\| = 1$; $k, k_\gamma, A \in \mathbf{K}_0(n)$ have integral invertible components, so the matrices $S_\kappa(k_\ell^{\pm1})$, $S_\kappa((k_\gamma)_\ell^{-1})$, $S_\kappa(A_\ell^{-1})$ have integral entries; and c is integral. By the ultrametric inequality all coordinates have absolute value at most 1; off the support $F = 0$. So $F(h_1,h_2,h_3)$ is integral at j.

Step 6: proof of (iii). Let $N = B_1 \cup B_2 \cup B_3$ ($\#N = 20$) and $l = B_4$, so $\mathrm{I}_n = \mathrm{I}_N \oplus \mathrm{I}_l$, and let $\iota\colon \mathrm{G}(1_N) \times \mathrm{G}(1_l) \to \mathrm{G}(1_n)$ be the block diagonal embedding. Put $K_1 = \mathbf{K}(O_{X_1})$, $d_0 = \mathrm{diag}(1, h_2, h_3) \in G_f(20)$, so that $(1, d(h_1)A) = \iota((1,d_0),(1,h_1))\,(1,A)$ with $d(h_1) = \mathrm{diag}(1,h_2,h_3,h_1)$, and for $\zeta \in \mathbb{V}_O$ let $Y_\zeta(h) = \overline{\langle \zeta, \rho_O(h_\infty)^{-1}\Xi(h_f)\rangle_\kappa}$ on $\mathrm{G}(1_n)$, as in Proposition 6.6.81 ($e_O = 0$). Fix a finite place w with $w \neq \bar w$ and $w, \bar w \notin S_O$, and $\Xi \in e_{X^{-1}}\mathrm{Form}(O) \cap \ker H_w(X_*(w)) \cap \ker \mathrm{Th}_O$. Let $y = \sum_{h_1\in R_1} \frac{1}{\#\Gamma_{h_1}} \varphi_{q,1}(h_1) F(h_1,h_2,h_3)$; we show $\langle \Xi, y\rangle = 0$.

(1) *The pairing as a sum of periods.* The inner product of (DP5) is hermitian (G_κ is real symmetric), so by (6.188) $\langle \Xi, F(h_1,h_2,h_3)\rangle = \overline{\langle c, \Xi(d(h_1)A)\rangle_\kappa} = Y_c(\iota((1,d_0),(1,h_1))(1,A))$. The pair q is flat ($j_q = 0$), so by (OL4), (OL3) $\varphi_{q,1} = c_s^{2(z-10)} \sum_{[g]} \frac{1}{\#\Gamma_g} k_{X_1}(g)$ over $\mathrm{U}(1)(E)\backslash G_f(1)/\mathbf{K}^{\mathrm{ch}}_{X_1}$, and in weight 0 (OL2) gives $k_{X_1}(g)(h) = \theta_{1,b}^{\mathrm{I}_1,\mathrm{I}_l}(\varphi_1)((1,g),(1,h))$ with $\varphi_1 = \mathrm{J}_{O_{X_1}}(1)$ and $b = \chi'_{O_{X_1}}$ ((DP6), (XD2); $\chi_{O_{X_1}} = 1$). The inner product being linear in the second argument,

$$\langle \Xi, y\rangle = c_s^{2(z-10)} \sum_{[g]} \frac{1}{\#\Gamma_g} \sum_{h_1\in R_1} \frac{1}{\#\Gamma_{h_1}} f_g(1,h_1),$$

$$f_g(h') = \theta_{1,b}^{\mathrm{I}_1,\mathrm{I}_l}(\varphi_1)((1,g),h')\, Y_c(\iota((1,d_0),h')(1,A))$$

for $h' \in \mathrm{G}(1_l)$.

(2) *Invariance of f_g.* (a) Real points: $\varphi_1 = v_1 \otimes \Upsilon(O_{X_1})$ by (DP8) (Har = 1 in weight 0), so by (5) of Lemma 4.1.1 (all signs +1, the source character 1 of weight 0, $P = 1$) $\omega_{1,b}((1,g),(y,k))\varphi_1 = \omega_{1,b}((1,g),(1,k))\varphi_1$ for $y \in G_\infty(1_l)$; and $\rho_O(\iota(1,y))c = S_\kappa(\overline{\iota(1,y)})c = c$ by Step 1 ($\iota(1,y)$ is block diagonal with the identity on B_1), so by the unitarity of ρ_O ((c) of Lemma 4.6.10) $Y_c(x\,\iota(1,(y,1))) = Y_c(x)$ whenever $x_\infty = 1$. Hence $f_g(y,k) = f_g(1,k)$. (b) Rational points: by Lemma 6.3.77 the pair $(\omega_{O_{X_1}}, \theta_{O_{X_1}})$ is a theta kernel for $(\mathrm{I}_1, \mathrm{I}_l)$, hence invariant under the rational points of $\mathrm{U}(\mathrm{I}_l)$ ((ED2), (ED3) of Definition 3.2.29); and for a rational point γ' of $\mathrm{U}(\mathrm{I}_l)$,

$\iota((1,d_0),\gamma'h') = \iota(1,\gamma')\iota((1,d_0),h')$ with $\iota(1,\gamma')$ a rational point of $\mathrm{U}(\mathrm{I}_n)$, and $Y_c(\gamma x) = Y_c(x)$ for rational γ by (DP5) and the multiplicativity of ρ_O. So $f_g(\gamma'h') = f_g(h')$. (c) The level K_1: for $k \in K_1$, $k_{X_1}(g)(h'k) = \tilde{\eta}_{O_{X_1}}(k)k_{X_1}(g)(h') = k_{X_1}(g)(h')$ ($Q_{O_{X_1}} = \emptyset$), and $\iota((1,d_0),(1,h'k))(1,A) = \iota((1,d_0),(1,h'))(1,A)(1,k'')$ with $k'' = A^{-1}d_kA \in \mathbf{K}_0(n)$, $d_k = \mathrm{diag}(1_{20},k)$. At $w' \in M$, $\bar{k}_{w'}$ fixes the unit vector e_{i_1} of B_4 ((DP4) for O_{X_1}), so $\bar{d}_{k,w'}Y_{w'} = Y_{w'}$ and $\bar{k}''_{w'}e_{\iota_0} = \bar{A}_{w'}^{-1}\bar{d}_{k,w'}Y_{w'} = e_{\iota_0}$. At $w' \in Q$, $\bar{A}_{w'}$ is the transposition matrix P of ι_0 and $i_2 \in B_2$, and row i_2 of $\bar{d}_{k,w'}$ is $e_{i_2}^{\mathsf{T}}$, so row ι_0 of $\bar{k}''_{w'} = P\bar{d}_{k,w'}P$ is $e_{\iota_0}^{\mathsf{T}}$: it vanishes outside column ι_0 and its inc-block is 1. So $k'' \in \mathbf{K}(O)$ with $\tilde{\eta}_O(k'') = 1$ (DP4), whence $\Xi(xk'') = \Xi(x)$ and $f_g(h'k) = f_g(h')$.

(3) *Unfolding.* The space I_l is definite, so $[\mathrm{U}(\mathrm{I}_l)]$ is compact [37] and $\Gamma_l\backslash G_f(l)/K_1$ is finite [33], Γ_l the rational points. For $h \in R_1$ the sets $\Gamma_l(G_\infty(1_l) \times hK_1)$ partition $\mathrm{G}(1_l)$, an element $\gamma \in \Gamma_l$ maps $G_\infty(1_l) \times hK_1$ to a set meeting it iff $\gamma \in \Gamma_h$, and Γ_h (finite) acts freely (its real components determine it). With the measure of (VP1) of Definition 3.2.2, in which $G_\infty(1_l)$ has mass 1 (Lemma 5.5.21) and hK_1 has mass $[\mathbf{K}_0(l) : K_1]^{-1}$, (2) gives

$$\int_{[\mathrm{U}(\mathrm{I}_l)]} f_g(h')\,dh' = [\mathbf{K}_0(l) : K_1]^{-1} \sum_{h_1 \in R_1} \frac{1}{\#\Gamma_{h_1}} f_g(1,h_1).$$

(4) *The periods vanish.* We apply Proposition 6.6.81 to $O = O_{p,\vartheta}$ (ranks $1 < n$; it fits a parameter by Lemma 6.7.20), Ξ, $\zeta = c$, the partition $\{1,\dots,n\} = N \sqcup l$, the character b, the datum φ_1, the points $(1,g) \in \mathrm{G}(\varepsilon_O)$, $(1,d_0) \in \mathrm{G}(1_N)$ and $\gamma = A$. Its hypotheses: $2 \cdot 1 + 2 \cdot 20 + 3 = 45 < 2z$, since $z > m \geq \sqrt{x} \geq 23$ by (RH1), (AD2) and Lemma 2.3.6. The tame part of $\mu_O = \nu\vartheta$ is $\omega^j\psi_\vartheta$ ((RH2); tame parts multiply, Definition 3.1.2); as $\mathrm{N}Q$ is prime to M, ω is trivial on Q and ψ_ϑ is trivial at the chosen places. So for $w' \in Q_O = Q$, $\psi_{O,w'} = \psi_{\vartheta,w'} \neq 1$ (ϑ a twist); for $w' \in M_O = M$ and a nonzero $\lambda \in k_{w'}$, $H(\lambda) = k_{w'}^\times$ ($r = 1$), Ξ is fixed by every $[s]_{w'}$ ($\Xi \in e_{X-1}\mathrm{Form}(O)$, Definition 6.1.3), and $\psi_{O,w'} = \omega_{w'}^j \neq 1$ by Lemma 6.7.3 ($1 \leq j \leq m$), so $\sum_{s \in H(\lambda)} \psi_{O,w'}(\det s)^{-1}[s]_{w'}\Xi = (\sum_{s \in k_{w'}^\times} \psi_{O,w'}(s)^{-1})\Xi = 0$; $\mathrm{Th}_O(\Xi) = 0$; and $H_w(X_*(w))\Xi = 0$ with $w \neq \bar{w}$, $w, \bar{w} \notin S_O$. The datum φ_1 is the vacuum at the finite places outside $\Sigma = S_O \cup \bar{S}_O$: by (4) and (5) of Lemma 6.2.33, $\varphi_1 = \Phi_{O_{X_1}} \otimes_M v^0$ is fixed by $\rho_f(y')$ for every integral y' vanishing at $M \cup \bar{M} \subseteq \Sigma$. The components of A outside Σ are 1 (Step 2). As $\chi_O = v^{r_-(O)} = 1$, the conclusion of Proposition 6.6.81 is $\int_{[\mathrm{U}(\mathrm{I}_l)]} f_g = 0$. By (3) and (1), $\langle \Xi, y\rangle = 0$. So y is orthogonal to $\mathcal{F}_w(e_{X-1}, \ker \mathrm{Th}_O)$ for every such w, which is (iii). □

Lemma 6.7.20. *Let X be L-value data of rank $n \geq 2$. Then O_X fits the parameter $l = (l_\tau)_{\tau\in\Phi}$ with $l_\tau(1) = 0$ for every $\tau \in \Phi$.*

Proof. Write $O = O_X$ and $\ell = \ell_X$. By (OL1), O has ranks $1 < n$, $\varepsilon_O(1) = +1$ (so $r_+(O) = 1$, $r_-(O) = 0$), $\mathrm{ord}_O(1) = 0$, $\kappa_O^+(\tau,1) = \ell(\tau)$, $\kappa_O^+(\tau,i) = 0$ for $i \geq 2$, $\kappa_O^- = 0$, and $\mu_O = \nu_X$ has weight $\mathrm{wt}_O(\tau) = -(\ell(\tau) + \lfloor n/2 \rfloor)$. Hence (DP3) gives $\kappa_O(\tau,1) = \ell(\tau)$, $\kappa_O(\tau,i) = 0$ for $2 \leq i < n$ and $e_O = 0$, so the highest weight is $\lambda_{O,\tau}(0) = \ell(\tau)$ and $\lambda_{O,\tau}(i) = 0$ for $1 \leq i < n$, and by (DP11)

$$l_{O,\tau}(1) = 2\ell(\tau) + n - 1, \qquad l_{O,\tau}(a) = n + 1 - 2a \quad (2 \leq a \leq n).$$

(i). The weights of (DP11)(i) are $\gamma(\tau) = r_-(O) = 0$ and $\gamma'(\tau) = \lfloor n/2 \rfloor + \mathrm{wt}_O(\tau) = -\ell(\tau)$, the signs are $s = \varepsilon_O$ on $\{1\}$ and $t = \mathbf{1}_n$, and $N = 1 \leq N' = n$. Fix τ; we check the block rule for $(l_\tau, l_{O,\tau})$. (a) l_τ is injective, $l_\tau(1) = 0 \equiv N - 1 \pmod 2$, and $\{1\}$ carries one sign, so l_τ has every gap for s. As $\ell(\tau) \geq 0$, $l_{O,\tau}(1) \geq n - 1 > n - 3 = l_{O,\tau}(2)$, so $l_{O,\tau}$ is strictly decreasing, hence injective; all its values are $\equiv n - 1 \pmod 2$; and t is constant. So both have gap $500(N + N')^3$. (b) The shifted entries are $\tilde{l}_\tau(1) = l_\tau(1) - 2\gamma'(\tau) + N' = 2\ell(\tau) + n$ and $\tilde{l}'_\tau(b) = l_{O,\tau}(b) - 2\gamma(\tau) + N = l_{O,\tau}(b) + 1$, that is $\tilde{l}'_\tau(1) = 2\ell(\tau) + n$ and $\tilde{l}'_\tau(b) = n + 2 - 2b$ for

$2 \le b \le n$. The latter values are $\{n-2-2\beta : 0 \le \beta < n-1\} = [n-1]$, so $\tilde{l}'_\tau(\{1,\dots,n\}) = \tilde{l}_\tau(\{1\}) \cup [N'-N]$. (c) All signs of s and t are $+1$, and the only value of $\tilde{l}_\tau$ is $2\ell(\tau)+n > 0$; so whenever $\tilde{l}_\tau(1) = \tilde{l}'_\tau(b)$ we have $s(1) = t(b)$, as required.

(ii). The only index of sign $+1$ is $c = 1$, so the right side of (DP11)(ii) is $\#\{c' \in \{1\} : l_\tau(1) < l_\tau(c')\} = 0 = \mathrm{ord}_O(1)$.

(iii). $\kappa^+_O(\tau, i) = 0$ for $r_+(O) = 1 < i < n$, and $\kappa^-_O = 0$. □

Lemma 6.7.21. *Let $q = (\nu_q, z_q)$ be a flat critical pair with $z_q \ge 2$; put $N = 2z_q$. Let ϑ' be a tame character with modulus Q', $X' = X(q,\vartheta')$, $O' = O_{X'}$, $\varphi = \varphi_{q,\vartheta'}$ and $n = n_{\nu_q}$. Fix a set R of representatives of $\mathrm{U}(\mathrm{I}_N)(E)\backslash G_f(N)/\mathbf{K}(O')$, and for $h \in R$ let $\Gamma_h = \mathrm{U}(\mathrm{I}_N)(E) \cap h\mathbf{K}(O')h^{-1}$. Let O be a descent pair, and for a map $F\colon G_f(N) \to \mathrm{Form}(O)$ put*

$$\mathrm{xs}(F) = N_{\mathrm{ch}} \sum_{h \in R} \frac{1}{\#\Gamma_h} \frac{n^2}{\mathrm{N}Q'^{N-1}} \varphi(h)\, F(h) \ \in \mathrm{Form}(O).$$

Then:

(a) if ℓ is a prime with $\ell \nmid \mathrm{N}Q'$, $j\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism and $F(h)$ is integral at j for every h, then $\mathrm{xs}(F)$ is integral at j;

(b) if $g \in G_f(1)$, $A, B \in \mathbb{C}$ and $L\colon \mathrm{Form}(O) \to \mathbb{C}$ is conjugate-linear with $L(F(h)) = B\,A\,k_{X'}(g)(h)$ for every h, then

$$h_{\mathrm{U}}\, c_s^N\, L(\mathrm{xs}(F)) = B\, I_q^{N_{\mathrm{ch}}}(\vartheta')\, A,$$

where $h_{\mathrm{U}} = \sum_{[u]} 1/\#\Gamma_u$ is the class sum of 1 over $\mathrm{U}(1)(E)\backslash G_f(1)/\mathbf{K}^{\mathrm{ch}}_{X'}$.

Proof. A class sum of a function constant on the double cosets may be computed at the representatives R; this applies below to the inner products of (DP5).

Step 1: the stabilisers divide N_{ch}. Let $h \in G_f(N)$. As $N = 2z_q \le 2t \le rt$ and $\mathbf{K}(O') \subseteq \mathbf{K}_0(N)$ is open, $\Gamma_h = \mathrm{U}(\mathrm{I}_N)(E) \cap h\mathbf{K}(O')h^{-1}$ is finite and $\#\Gamma_h$ divides N_{ch} by Lemma 6.7.9.

Step 2: proof of (a). The forms integral at j form a set closed under sums (ultrametric inequality) and under multiplication by scalars c with $\|j(c)\| \le 1$ (the coordinates in the definition are linear in the form). The scalar of the summand at h is $\frac{N_{\mathrm{ch}}}{\#\Gamma_h} \cdot \frac{n}{\mathrm{N}Q'^{N-1}} \cdot n\varphi(h)$: the first factor is an integer by Step 1; $\|j(\mathrm{N}Q')\| = 1$ as $\ell \nmid \mathrm{N}Q'$; and $\|j(n\varphi(h))\| \le 1$ because $n\varphi$ is integral at j by Lemma 6.7.7 (for weight 0 integrality at j is this inequality at every point). So each summand, hence $\mathrm{xs}(F)$, is integral at j.

Step 3: central translation of kernel forms. Let $z, g \in G_f(1)$ and $h \in G_f(N)$, and write $z^{-1}h$ for $(z^{-1}1_N)h \in G_f(N)$. In rank one the pair points of $((1,zg),(1,h))$ and $((1,g),(1,z^{-1}h))$ coincide, $hX(zg)^{-1} = (z^{-1}h)Xg^{-1}$; comparing the characters of (DP6), $\omega_{O'}((1,zg),(1,h)) = \chi'_{O'}(z)^{-1}\omega_{O'}((1,g),(1,z^{-1}h))$, so

$$k_{X'}(zg)(h) = \chi'_{O'}(z)^{-1}\, k_{X'}(g)(z^{-1}h).$$

As $G_f(1)$ is commutative, $u \mapsto z^{-1}u$ permutes $\mathrm{U}(1)(E)\backslash G_f(1)/\mathbf{K}^{\mathrm{ch}}_{X'}$ and $\Gamma_u = \mathrm{U}(1)(E) \cap \mathbf{K}^{\mathrm{ch}}_{X'}$ does not depend on u; with the display (applied to $z^{-1}u$) this gives $\lambda_{X'}(zh) = \chi'_{O'}(z)^{-1}\lambda_{X'}(h)$.

Step 4: $\langle \lambda_{X'}, k_{X'}(g)\rangle$ does not depend on g. The scalar matrix $z1_N$ is central in $G_f(N)$ and commutes with the rational points, so $h \mapsto zh$ permutes the double cosets for $\mathbf{K}(O')$ and

preserves $\#\Gamma_h$. Hence, by Step 3 and $|\chi'_{O'}(z)| = 1$ (an automorphic character of $\mathrm{U}(1)$ takes values in the unit circle, $\mathrm{U}(1)(E)\backslash\mathrm{U}(1)(\mathbb{A})$ being compact),

$$\begin{aligned}\langle \lambda_{X'}, k_{X'}(zg)\rangle &= \chi'_{O'}(z)^{-1}\sum_{[h]}\frac{\overline{\lambda_{X'}(zh)}\,k_{X'}(g)(h)}{\#\Gamma_h}\\ &= \chi'_{O'}(z)^{-1}\overline{\chi'_{O'}(z)^{-1}}\,\langle \lambda_{X'}, k_{X'}(g)\rangle = \langle \lambda_{X'}, k_{X'}(g)\rangle.\end{aligned}$$

By (OL3) and linearity in the second argument,

$$\langle \lambda_{X'}, \lambda_{X'}\rangle = \sum_{[u]}\langle \lambda_{X'}, k_{X'}(u)\rangle/\#\Gamma_u = h_{\mathrm{U}}\langle \lambda_{X'}, k_{X'}(g)\rangle$$

for every g. Multiplying by $|c_s^N|^2$,

$$h_{\mathrm{U}}\, c_s^N\,\langle \varphi, k_{X'}(g)\rangle = \|\varphi\|^2. \tag{6.189}$$

Step 5: proof of (b). L is additive and $L(cx) = \bar{c}\,L(x)$; the weights $N_{\mathrm{ch}}/\#\Gamma_h$ and $n^2/\mathrm{N}Q'^{N-1}$ are real. So

$$L(\mathrm{xs}(F)) = \frac{N_{\mathrm{ch}}n^2}{\mathrm{N}Q'^{N-1}}\sum_{h\in R}\frac{\overline{\varphi(h)}\,B\,A\,k_{X'}(g)(h)}{\#\Gamma_h} = \frac{N_{\mathrm{ch}}n^2}{\mathrm{N}Q'^{N-1}}\,B\,A\,\langle \varphi, k_{X'}(g)\rangle,$$

and by (6.189), $h_{\mathrm{U}}c_s^N L(\mathrm{xs}(F)) = N_{\mathrm{ch}}n^2\,\|\varphi\|^2\,\mathrm{N}Q'^{-(2z_q-1)}\,B\,A = B\,I_q^{N_{\mathrm{ch}}}(\vartheta')\,A$ by (OL4). □

Lemma 6.7.22. *(a) Let S be a finite set of finite places of E, and $K \subseteq \mathrm{U}(1)(\mathbb{A}_f)$ the subgroup of the points that are integral at every finite place and congruent to 1 modulo every place of S. The set $C = \mathrm{U}(1)(E)\backslash\mathrm{U}(1)(\mathbb{A}_f)/K$ is finite. Let $R \subseteq \mathrm{U}(1)(\mathbb{A}_f)$ be a set of representatives of C, ℓ a prime, $\iota\colon \mathbb{C}\to\mathbb{C}_\ell$ a ring homomorphism and $s\colon \mathrm{U}(1)(\mathbb{A}_f)\to\mathbb{C}$ a function. Then there are $z \in \mathrm{U}(1)(\mathbb{A}_f)$ and an automorphic character χ of $\mathrm{U}(1)$ of weight 0 with a tame part trivial at every finite place outside S such that*

$$\|\iota(s(1))\| \le \Big\|\iota\Big(\frac{1}{\#C}\sum_{g\in R}\chi(gz)\,s(gz)\Big)\Big\|.$$

(b) Let G be a finite group, $\iota\colon \mathbb{C}\to\mathbb{F}$ a ring homomorphism into a normed field, $a, b\colon G\to\mathbb{C}$ functions and $V \in \mathbb{C}$ with $V^{\#G} = \prod_{\chi\in G} a(\chi)\,b(\chi)$. Then for every $\psi \in G$ there are $\varphi, \chi \in G$ with $\varphi\chi = \psi$ and $\|\iota(V)\| \le \|\iota(a(\varphi)\,b(\chi))\|$.

Proof. Proof of (a). The group $\mathrm{U}(1)(\mathbb{A}_f)$ is commutative, so C is the quotient group $A = \mathrm{U}(1)(\mathbb{A}_f)/\mathrm{U}(1)(E)K$; it is finite by the finiteness of class numbers [33] (K is an open subgroup of the integral points). Write $[g]$ for the class of g and $\widehat{A}$ for the group of characters of A; $\#\widehat{A} = \#A = \#C$.

Step 1: every character of A is an automorphic character of weight 0 with tame part trivial off S. Let $\lambda \in \widehat{A}$ and define $\chi_\lambda(u) = \lambda([u_f])$ for $u \in \mathrm{U}(1)(\mathbb{A})$ with finite part u_f. It is a homomorphism, trivial on $\mathrm{U}(1)(E)$, with values roots of unity (so of modulus 1), and continuous, being constant on the cosets of the open subgroup $\mathrm{U}(1)(\mathbb{A}_\infty)\times K$. Let $\mathrm{red}\colon \mathrm{U}(1)(\widehat{\mathcal{O}}) \to H = \prod_{w\in S} k_w^\times$ be reduction modulo the places of S on the points integral at every finite place; its kernel is K. So λ restricted to the integral points is $\theta\circ\mathrm{red}$ for a character θ of the subgroup $\mathrm{red}(\mathrm{U}(1)(\widehat{\mathcal{O}}))$ of the finite abelian group H. A character of a subgroup of a finite abelian group extends to the whole group, and a character of H is a product $\prod_{w\in S}\psi_w$ of characters ψ_w of

$k_w^\times$; put $\psi_w = 1$ for $w \notin S$. Then for every $u \in \mathrm{U}(1)(\mathbb{A})$ with integral finite components, $\chi_\lambda(u) = \prod_w \psi_w(u \bmod w) = \prod_{v\in\Phi} u_v^0 \prod_w \psi_w(u \bmod w)$: χ_λ has weight 0 and tame part ψ, trivial outside S.

Step 2: orthogonality. Let $r_1 \in R$ be the representative of the class of 1 and $z = r_1^{-1}$. For $\lambda \in \widehat{A}$ put

$$t(\lambda) = \frac{1}{\#C} \sum_{g\in R} \chi_\lambda(gz)\, s(gz) = \frac{1}{\#C} \sum_{g\in R} \lambda([g]\,[z])\, s(gz).$$

For $g \in R$ the class $[g][z] = [g][r_1]^{-1}$ is trivial exactly when $g = r_1$, as R contains one point of each class. By the orthogonality relation $\sum_{\lambda\in\widehat{A}} \lambda(x) = \#A$ for $x = 1$ and 0 for $x \neq 1$,

$$\sum_{\lambda\in\widehat{A}} t(\lambda) = \frac{1}{\#C} \sum_{g\in R} s(gz) \sum_{\lambda\in\widehat{A}} \lambda([g][z]) = \frac{\#A}{\#C}\, s(r_1 z) = s(1).$$

Step 3: the ultrametric inequality. Since ι is a ring homomorphism and the norm of $\mathbb{C}_\ell$ is ultrametric, $\|\iota(s(1))\| = \|\sum_\lambda \iota(t(\lambda))\| \le \max_\lambda \|\iota(t(\lambda))\|$. Take λ attaining the maximum and $\chi = \chi_\lambda$; with z as in Step 2 this is (a).

Proof of (b). Fix ψ. For $\chi \in G$ put $N(\chi) = \|\iota(a(\psi\chi^{-1})\, b(\chi))\|$ and choose χ_0 with $N(\chi_0)$ maximal (G is finite and nonempty). The map $\chi \mapsto \psi\chi^{-1}$ is a bijection of G, so, ι and the norm being multiplicative,

$$\|\iota(V)\|^{\#G} = \prod_{\chi\in G} \|\iota(a(\chi))\|\, \|\iota(b(\chi))\| = \prod_{\chi\in G} \|\iota(a(\psi\chi^{-1}))\|\, \|\iota(b(\chi))\| = \prod_{\chi\in G} N(\chi) \le N(\chi_0)^{\#G}.$$

Taking $\#G$-th roots of these non-negative numbers gives $\|\iota(V)\| \le N(\chi_0)$, and $\varphi = \psi\chi_0^{-1}$, $\chi = \chi_0$ satisfy $\varphi\chi = \psi$. □

Lemma 6.7.23. *Let $q = (\nu_q, z_q)$ be a flat critical pair with $z_q \ge 2$. Then for every automorphism σ of $\mathbb{C}$*

$$|\sigma(I_q^{N_{\mathrm{ch}}}(1))| \le \exp(x^{34}).$$

Proof. Let $X = X(q,1)$, $O = O_X$, $N = 2z_q$, $\varphi = \varphi_{q,1} = c_s^N \lambda_X$ and $y = I_q^{N_{\mathrm{ch}}}(1) = N_{\mathrm{ch}} n_{\nu_q}^2 \mathrm{alg}_q(1)$ ((OL4)). Put $B = M^{fN} \exp(C_P(q_1 N + 1)^2 + fN \log 4)$.

Step 1: class counts. The principal congruence subgroup of level $\pi_M \pi_N$ in $\mathbf{K}_0(1)$ lies in $\mathbf{K}_X^{\mathrm{ch}}$, and that in $\mathbf{K}_0(N)$ lies in $\mathbf{K}(O)$ (its elements reduce to 1 at the chosen places, and $Q = \emptyset$, (DP4)). As $1 \le N \le 2t \le n_m$ ($m \ge 2$), Lemma 4.6.4 gives at most $\exp(x^{30})$ classes in $\mathrm{U}(1)(E)\backslash G_f(1)/\mathbf{K}_X^{\mathrm{ch}}$ and in $\mathrm{U}(\mathrm{I}_N)(E)\backslash G_f(N)/\mathbf{K}(O)$.

Step 2: the conjugates of a value. Let τ be an automorphism of $\mathbb{C}$ and $h \in G_f(N)$. By (OL3), $\varphi(h) = \sum_{[g]} \frac{1}{\#\Gamma_g} c_s^N k_X(g)(h)$; as τ fixes rational numbers and $\#\Gamma_g \ge 1$, (b) of Lemma 6.7.8 and Step 1 give $|\tau(\varphi(h))| \le \exp(x^{30})\, B$.

Step 3: the conjugates of y. The weight of O is 0, so by (DP5) $\|\varphi\|^2 = \sum_{[h]} \frac{1}{\#\Gamma_h} \overline{\varphi(h)}\varphi(h)$. Let σ be an automorphism of $\mathbb{C}$ and $\sigma'(w) = \overline{\sigma(\bar{w})}$, also an automorphism; then $|\sigma(\bar{w})| = |\sigma'(w)|$. As $N_{\mathrm{ch}} n_{\nu_q}^2$ is a positive integer, Steps 1 and 2 (for $\tau = \sigma, \sigma'$) give

$$|\sigma(y)| \le N_{\mathrm{ch}} n_{\nu_q}^2 \sum_{[h]} |\sigma'(\varphi(h))|\, |\sigma(\varphi(h))| \le N_{\mathrm{ch}}\, n_{\nu_q}^2 \exp(3x^{30})\, B^2.$$

Step 4: sizes. By Lemma 6.7.10, $\log N_{\mathrm{ch}} \le x^{33}$. By Lemma 4.2.67 (items 2–5: $f \le x^{12}$, $z_q \le t \le 15x^5$, $\log M \le 22 \log b$, $\log b \le x^{1/100}$), $\log n_{\nu_q}^2 = 8 f z_q \log M \le x^{19}$, and $2fN \log M \le$

x^{19}, $2fN\log 4 \le x^{19}$. For the remaining term, by (AD2) $2^s = m^{22} \le 2^{22}x^{11}$ and $s \le 32\log x$, so $l_0(s) \le 81\max(s,1) \le 2592\log x$, and $q_0 \le C_P \le \log x$ (item 1 of Lemma 4.2.67 and (AD1)); hence $q_1 \le 3\cdot 2592\cdot 2^{22}(\log x)^2 x^{11}$, $q_1N+1 \le 10^{13}(\log x)^2x^{16}$ (as $N \le 30x^5$), and $2C_P(q_1N+1)^2 \le 2\cdot 10^{26}(\log x)^5x^{32} \le x^{33}$, since $x \ge e^{6.4\cdot 10^{15}}$ (item 1). So

$$\log|\sigma(y)| \le x^{33} + x^{19} + 3x^{30} + 2\log B \le x^{33} + x^{19} + 3x^{30} + x^{19} + x^{33} + x^{19} \le x^{34}.$$

□

The test forms need pairings that, at each prime p, are divisible by no more than $p^{F(p)}$. Such a pairing is a Taylor coefficient of a theta function at a CM point, and its divisibility is governed by the derivatives of theta functions of CM elliptic curves; in our reading $F(p)$ is the valuation that these derivatives force, so the requirement is that this divisibility is attained and not exceeded. The last group proves it (Proposition 6.7.24): for every degree l in the relevant range there is one number $t_{\mathrm{shp}} \neq 0$, with all conjugates at most $\exp(fmb/k)$ and chosen before the prime, such that at every prime some suitably normalised Taylor coefficient of degree l of the theta germ at an indicator datum has p-adic norm at least $p^{-F(p)}$.

The method has its origins in the p-adic theory of theta functions and Eisenstein series on CM elliptic curves [159, 272, 219, 13]. The curves enter through potentially good reduction [266, 281], the integrality of singular moduli, the extension of homomorphisms of semiabelian schemes [77, 178] and formal logarithms [124]. The proof has three steps. An arithmetic tight frame with a common unit row realises products of theta functions along its rows as theta germs (Lemma 6.7.35). The valuation budget is attained at each prime separately, from the derivative rate of one curve through six and then twelve copies (Lemma 6.7.25). Finite theta averaging passes to one indicator datum (Lemma 6.7.36), and the losses are absorbed into t_{shp}.

Proposition 6.7.24. *Let $V_{12} = \mathrm{Hom}(\mathrm{I}_1, \mathrm{I}_{12})$ for the standard definite spaces I_1, I_{12} of ranks 1 and 12, with coordinates $Z = (Z_{\tau,j})_{\tau\in\Phi,\,1\le j\le 12}$. For a modulus $\mathfrak{m}$ put $A = \Sigma(\mathfrak{m})$. For $y_0 \in R_{\mathfrak{m}}^{12}$ let $\Psi_{\mathfrak{m},y_0}$ be the function on finite adelic vectors that is the indicator of the class y_0 among vectors integral at A, with no restriction away from A. The **indicator datum** is*

$$F_{\mathfrak{m},y_0} = v_1 \otimes (\Psi_{\mathfrak{m},y_0} \otimes_A v^0)$$

in the polarized datum construction (WR6), where v_1 is the positive real vacuum of (WR5). It is a vector of the theta model; its theta values are those of (WC3) of Definition 4.1.15. The class indicator is used as the polarized Schrödinger input in (WR6). Let $T(F) = \sum_\gamma [Z^\gamma]T(F)\,Z^\gamma$ be the Taylor series at 0 of $Z \mapsto \theta_F(\pi_M\eta_P Z)$. Let $l\colon \Phi \to \mathbb{Z}_{\ge 0}$ satisfy $\kappa(\tau) \le l(\tau) \le \kappa(\tau) + x^{42}$ for every $\tau \in \Phi$. Then there is $t_{\mathrm{shp}} \in \mathbb{C}\setminus\{0\}$ with $|\sigma(t_{\mathrm{shp}})| \le \exp(fmb/k)$ for every automorphism σ of $\mathbb{C}$ such that, for the modulus $\mathfrak{m} = \pi_M\mathcal{O}_E$, every prime p and every ring homomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_p$, there are a multi-index γ on $\Phi \times \{1,\dots,12\}$ with $\sum_{j=1}^{12}\gamma(\tau,j) = l(\tau)$ for every τ, and $y_0 \in R_{\mathfrak{m}}^{12}$, with

$$p^{-F(p)} \;\le\; \Big\|\iota\Big(\frac{c_s^{12}}{t_{\mathrm{shp}}}\prod_{\tau\in\Phi} l(\tau)!\;\,[Z^\gamma]\,T(F_{\mathfrak{m},y_0})\Big)\Big\|.$$

Proof. **A fixed scalar before all prime embeddings.** Write v^0 for the rank-twelve vacuum in the prescribed positive model. Its finite theta function is not identically zero. Indeed Lemma 4.2.2 makes its entire theta function nonzero, because the map is injective and the vacuum has norm one by Lemma 4.1.10. Rational vectors are dense in its real coordinate space. The

exact rational finite-translation formula therefore gives a rational finite translate with nonzero theta value. Choose one finite x_*, once for all primes and embeddings, such that

$$\theta_{x_*}(v^0) \neq 0, \qquad \theta_x(F) = \theta(\rho(\mathrm{fv}(x))F).$$

Put

$$c_* = c_s^{12}\theta_{x_*}(v^0), \qquad B_* = \prod_{\tau\in\Phi} B(l(\tau)), \qquad t_{\mathrm{shp}} = M^{12f}c_*B_*^2, \tag{1}$$

where B is the positive integer product of Lemma 6.7.28. The fixed tensor normalization (WC1) of Definition 4.1.15 and the rank-one bounds of Proposition 4.2.1, for the simultaneous choice supplied by Lemma 4.2.3 and fixed as the common jet-pair and vacuum-scale choice, with C_P and its vacuum-constant summand as defined in Definition 2.3.3, give

$$|\sigma(c_*)| \leq \exp(12C_Pq_1^2) \qquad (\sigma \in \mathrm{Aut}(\mathbb{C}/K)).$$

The same bound holds for every automorphism of $\mathbb{C}$. Its restriction to the imaginary quadratic field is either the identity or conjugation; composing with ordinary complex conjugation in the latter case gives an automorphism over K and preserves the absolute value in question. Thus Lemma 6.7.28(2) gives

$$t_{\mathrm{shp}} \neq 0, \qquad |\sigma(t_{\mathrm{shp}})| \leq \exp(fmb/k) \quad (\sigma \in \mathrm{Aut}\,\mathbb{C}). \tag{2}$$

Everything in (1) has been chosen before p, ι.

An attained twelve-copy product. Choose the single frame (I, w_i, c_i) of Lemma 6.7.35. It too is fixed before every prime identification. Now fix p, ι, put

$$u_i = w_i/\bar{\pi}_M, \qquad v_{\mathrm{loc}} = \iota(\pi_M\beta), \qquad \beta = s_0/u_{\mathrm{CS}}.$$

The same lemma gives the full two-index Gram resolution for u_i, the simultaneous unit row, and, at nonsplit primes, integrality of every u_i at every place above p and $|v_{\mathrm{loc}}|_p = 1$. Moreover its normalized forms are exactly

$$L_i(Z) = \sum_\tau \iota(\beta\overline{\tau(w_i)})Z_\tau = \sum_\tau v_{\mathrm{loc}}\iota(\overline{\tau(u_i)})Z_\tau.$$

Consequently every hypothesis of Lemma 6.7.25 holds. That lemma supplies twelve multi-indices α_c, and nonnegative integer exponents $e_{i,c}$, with

$$\sum_{c=1}^{12}\alpha_c = l, \qquad |e_c| \leq |l|,$$

whose product coefficient norm is at least p^{-R_p}, where

$$R_p = F(p) + 2v_p(B_*) - \sum_\tau v_p(l(\tau)!). \tag{3}$$

The scale bounds Lemma 2.3.6 give $\kappa(\tau) \leq b$ and $x^{42} \leq b$, so $|l| \leq 2fb$. Since $c_i \geq 6fb$, one has $3e_{i,c} \leq 3|l| \leq c_i$.

Realization in the prescribed model. Apply Lemma 4.2.36 with its finite coordinate set $\{1, \ldots, 12\}$, $\delta = \bar{\pi}_M$, and these integer exponents. It gives an actual finite-part vector F_0, with no additional scalar, whose Taylor series at $\pi_M\eta_PZ$ is the complex monomial theta product with forms $\sum_\tau \beta\overline{\tau(w_i)}Z_{\tau,c}$. After applying ι, its coefficients are precisely those used in (3). No continuity of ι is needed: each coefficient of a composition with a zero-constant linear form

is a finite algebraic sum of coefficients. Define $\gamma(\tau,c)=(\alpha_c)_\tau$. Distinct copies have distinct variables, so coefficient extraction factors and gives

$$\sum_{c=1}^{12}\gamma(\tau,c)=l(\tau),\qquad |\iota[Z^\gamma]T(F_0)|_p\ge p^{-R_p}. \tag{4}$$

The realization lemma also proves

$$|\iota\theta_x(F_0)|_p\le 1\quad\text{for every finite } x,\qquad \rho_f(\alpha^{-1}\pi_M\lambda_f)F_0=F_0\quad(\lambda\in\mathfrak{a}^{12}). \tag{5}$$

Here the period is $\pi_M\mathfrak{a}^{12}$, because $\bar\delta=\pi_M$; the dual translation denominator is $\bar\pi_M$. Thus (5) is exactly the period hypothesis of Lemma 6.7.36.

Indicator selection and cancellation. The map $\mathscr{L}(F)=[Z^\gamma]T(F)$ is complex-linear on the actual finite part. Apply Lemma 6.7.36 to F_0, this functional and x_*. It produces $y_0\in R_{\mathfrak{m}}^{12}$ satisfying

$$|\iota[Z^\gamma]T(F_{\mathfrak{m},y_0})|_p\ge\left|\iota\Big(M^{12f}[Z^\gamma]T(F_0)\theta_{x_*}(v^0)\Big)\right|_p.$$

Multiply this inequality by $\left|\iota(c_s^{12}\prod_\tau l(\tau)!/t_{\mathrm{shp}})\right|_p$. The fixed nonzero vacuum value and the level factor cancel by (1), giving, with (3)–(4),

$$\left|\iota\left(\frac{c_s^{12}}{t_{\mathrm{shp}}}\prod_\tau l(\tau)!\,[Z^\gamma]T(F_{\mathfrak{m},y_0})\right)\right|_p\ge p^{-R_p-\sum_\tau v_p(l(\tau)!)+2v_p(B_*)}=p^{-F(p)}.$$

This is the required estimate for the scalar of (1), whose conjugate bound is (2). The local choices γ,y_0,F_0 may depend on p,ι; the scalar does not. □

Lemma 6.7.25. *Fix a prime embedding $\iota:\mathbb{C}\hookrightarrow\mathbb{C}_p$. Let I be finite, $u_i\in E$, and $c_i\ge 0$ integers, such that for all $\tau,\sigma\in\Phi$,*

$$\sum_i c_i\,\tau(u_i)\overline{\sigma(u_i)}=\mathbf{1}_{\tau=\sigma}.$$

Take $v_{\mathrm{loc}}\in\mathbb{C}_p$, put

$$L_i(Z)=\sum_{\tau\in\Phi}v_{\mathrm{loc}}\,\iota(\overline{\tau(u_i)})Z_\tau,\quad f_i=y(L_i),\quad \Theta=\prod_i\theta(L_i)^{c_i},$$

and suppose some i_0 has $\left|v_{\mathrm{loc}}\iota(\overline{\tau(u_{i_0})})\right|_p=1$ for every τ. If p is nonsplit, also suppose that every u_i is integral at every place above p and $|v_{\mathrm{loc}}|_p=1$. For every $\kappa(\tau)\le l_\tau\le\kappa(\tau)+x^{42}$, there are twelve multi-indices $\alpha_c\in\mathbb{N}^\Phi$ and $e_c\in\mathbb{N}^I$ satisfying

$$\sum_{c=1}^{12}\alpha_c=l,\qquad |e_c|\le|l|,$$

and

$$\prod_{c=1}^{12}\left|[Z^{\alpha_c}]\left(\prod_i f_i^{e_{c,i}}\Theta\right)\right|_p\ge p^{-\left(F(p)+2\sum_\tau v_p(B(l_\tau))-\sum_\tau v_p(l_\tau!)\right)}.$$

Proof. **The supplied resolution has the exact inert normal form.** First suppose p is inert. It is odd by Lemma 4.2.45. Write $L_i^0=\sum_\tau\iota(\overline{\tau(u_i)})Z_\tau$, so $L_i=v_{\mathrm{loc}}L_i^0$. All their coefficients are integral. For every positive odd e, Lemma 4.2.45 gives

$$\iota(\overline{\tau(u_i)})^{p^e}\equiv\iota((u(-p^e)\tau)(u_i))\pmod{pO}.$$

The quotient O/pO has characteristic p, so the power of a sum distributes in this quotient. Multiply by the coefficients of L_i^0, sum with multiplicities c_i, and apply the stated resolution identity, including its two embedding indices. The result is

$$\sum_i c_i (L_i^0)^{p^e} L_i^0 \equiv \sum_\tau Z_\tau^{p^e} Z_{u(-p^e)\tau} = \Gamma_e \pmod{pO[[Z]]}.$$

No one-embedding norm identity would suffice here.

Use the one-curve factorization Lemma 4.2.39 and its scale ϖ, of valuation $\rho = 1/(p^2-1)$. Since $v_p(p\varpi^{p+1}/p) = 1/(p-1)$, Lemma 4.2.46 applies to the displayed congruences with multiplier one. It gives

$$\prod_i \theta(\varpi L_i^0)^{c_i} = W_0(Z) \exp\left(\frac{\varpi^{p+1}}{p} \sum_{e \text{ odd}, h \geq 0} \lambda_{eh} \Gamma_e^{p^h} \right), \qquad W_0 \in O[[Z]].$$

The constant coefficient is one, so $W_0(0) = 1$ and W_0 is a unit. The coefficients λ_{e0} are units by the inert assertion of Lemma 4.2.39. Substitute $v_{\mathrm{loc}} Z$. It preserves integrality and changes λ_{eh} to $\lambda_{eh} v_{\mathrm{loc}}^{(p^e+1)p^h}$, still integral and still unit for $h = 0$. Thus Θ has exactly the normal form and all hypotheses required in Lemma 6.7.33 for its partner maps.

At every nonsplit prime the resolution also supplies an integral span for the forms. Indeed put $a_{\tau i} = c_i \iota(\tau(u_i)) \in O$. Then

$$\sum_i a_{\tau i} v_{\mathrm{loc}} \iota(\overline{\sigma(u_i)}) = v_{\mathrm{loc}} \mathbf{1}_{\tau=\sigma}.$$

The last assertion of Lemma 6.7.29 therefore supplies, in any residual degree, a six-copy monomial witness of product norm at least one.

Split primes. Here $F(p) = 0$ and $\gamma_p = 0$. The unit form and Lemma 6.7.29 give six copies in degree l, with product norm at least

$$|\prod_\tau l_\tau!|_p^{-1} = p^{\sum_\tau v_p(l_\tau!)}.$$

Append six zero-degree copies with monomial factor one. Their coefficients are $\Theta(0) = 1$. Since the loss valuations are nonnegative, this proves the required, possibly weaker, bound. No integrality of the other forms is used in this case.

Small nonsplit primes. Suppose $p \leq k$. Apply Lemma 6.7.26, obtaining $S, e, j, \nu', \nu'', \eta, J$ and $l = \nu' + \nu'' + \eta$. If S is empty, take six zero-degree copies as the coupling witness. If S is nonempty, the packing says p is inert; the normal form already proved and Lemma 6.7.33 give a six-copy monomial witness in degree $\nu' + \nu''$, with valuation budget

$$\gamma_p |\nu' + \nu''| - J - V_p(\nu') - V_p(\nu'').$$

This same displayed budget is zero in the empty case. Here $V_p(\omega) = \sum_\tau v_p(\omega_\tau!)$.

The unit form and Lemma 6.7.29 supply another six copies in degree η, with budget $\gamma_p|\eta| - V_p(\eta)$. Concatenate the witnesses. Their aggregate degree is l and their product norm is at least p^{-R}, where

$$R = \gamma_p |l| - J - V_p(\nu') - V_p(\nu'') - V_p(\eta).$$

The factorial-splitting inequality and the small-prime budget (1) in Lemma 6.7.26 imply

$$R \leq \gamma_p |l| - J + 3f(\log b + 2) - V_p(l) \leq F(p) + \sum_\tau v_p(B(l_\tau)) - V_p(l).$$

Adding the nonnegative extra loss sum gives the asserted bound. This covers the ramified prime: its coupling set is empty, and its residual rate is exactly the ramified one-curve rate.

Large nonsplit primes. Now $p > k$. The packing lemma says it is inert, and chooses all exponents equal to one. The negative-budget assertion of Lemma 6.7.33 supplies six copies of degree $\nu' + \nu''$ with product norm at least p^J. The integral-span assertion established above supplies six copies in degree η with norm at least one. Their concatenation has budget $-J$. By the large-prime budget (2) of Lemma 6.7.26,

$$-J \le F(p) + 2\sum_{\tau} v_p(B(l_\tau)) - V_p(l),$$

as required.

Each use of the six-copy lemmas bounds every monomial-factor degree by the degree of that witness. Its aggregate degree is coordinatewise at most l, so every such degree is at most $|l|$. The appended zero-degree copies have this property as well. This proves all conclusions. □

Lemma 6.7.26. *Let p be nonsplit in K and let $\kappa(\tau) \le l_\tau \le \kappa(\tau) + x^{42}$ be integers. Put $r_e(\tau) = u(-p^e)\tau$ and*

$$\gamma_p = \begin{cases} p/(p^2-1), & p \text{ inert}, \\ 1/(2(p-1)), & p = d, \end{cases} \qquad V_p(\nu) = \sum_{\tau\in\Phi} v_p(\nu_\tau!).$$

There exist $S \subseteq \Phi$, odd positive integers e_τ and integers $j_\tau \ge 0$ for $\tau \in S$, and multi-indices

$$\nu' = \sum_{\tau\in S} p^{e_\tau} j_\tau \mathbf{1}_\tau, \quad \nu'' = \sum_{\tau\in S} j_\tau \mathbf{1}_{r_{e_\tau}(\tau)}, \quad \eta \ge 0, \qquad l = \nu' + \nu'' + \eta,$$

with $J = \sum_{\tau\in S} j_\tau$, such that the unique-lowest coupling condition of Lemma 6.7.34 holds for the maps r_e and this S, e. If $S \ne \varnothing$, p is inert. In addition,

$$V_p(l) \le V_p(\nu') + V_p(\nu'') + V_p(\eta) + 3f(\log b + 2).$$

If $p \le k$, the choices satisfy

$$\gamma_p|l| - J + 3f(\log b + 2) \le F(p) + \sum_{\tau} v_p(B(l_\tau)). \tag{1}$$

If $p > k$, then p is inert, $S = \Phi$, $e_\tau = 1$, and the choices satisfy

$$V_p(l) - J \le F(p) + 2\sum_{\tau} v_p(B(l_\tau)). \tag{2}$$

Proof. **Elementary preliminary bounds and partners.** The scale bounds Lemma 2.3.6 imply $x \ge 10^6$. As $r \ge x^5$, $m \le 2\sqrt{x}$, one has $k = r+1-m \ge x$, while $d \le C_P \le 10^{-14}\log x < x$. Thus $d \le k$. Also $k \le 4x^5$, whereas $2^s = m^{22} \ge x^{11}$, so $k \le 2^s$. Put $H = l_0(s)$ and $L = \log b$.

The primes $2, 3, 5, 7, 11$ split in K. For two use $d \equiv 7 \pmod 8$. Each odd prime in this list divides q_0 and hence $d+1$; the discriminant $-d$ is then a nonzero square modulo that prime, so it splits. Every prime dividing $q_1 = 3q_0H2^s$ similarly splits, since H is a power of three. Thus nonsplit $p \ge 13$ and $p \nmid q_1$. In particular each r_e is a permutation, and for odd e it equals g^e, with g the permutation $\tau \mapsto u(-p)\tau$.

There are no pairs of b-places $\tau, r_e\tau$ at $p = d$, for any $e \ge 1$. They would have the same class modulo q_0, hence $-d^e \equiv 1 \pmod{q_0}$. But $d \equiv q_0/2 - 1$, q_0 is divisible by four, and therefore $d^2 \equiv 1 \pmod{q_0}$. Its powers are 1 or d, neither congruent to -1, a contradiction.

Small primes: choose all short pairs. Suppose $p \le k$. Call (τ, e) short when $e \in I_p$, $2e < H$, and $\tau, r_e\tau \in \Phi_b$. Ramified short pairs do not exist by the preceding paragraph; otherwise the prime is inert and e is positive odd. The sparsity in the datum implies the forbidden-triple hypothesis of Lemma 6.7.27 for Φ_b, g, H: a triple at times $0 < i < j < H$ would have ratios $(-p)^i, (-p)^j$, which are of the permitted signs times p^i, p^j modulo the prime-to-p part of q_1. The prime is at most $k \le kt$, so sparsity applies.

Let S be all short-pair starts, choose their unique exponents e_τ, and put $j_\tau = \lfloor l_\tau / p^{e_\tau} \rfloor$. At ramified p, take S empty. We have $l_\tau \le 2b$, by Lemma 6.7.28; on Φ_b, $l_\tau \ge b$. The matching and capacity assertions of Lemma 6.7.27 give the unique-lowest condition and $\nu' + \nu'' \le l$. Put $\eta = l - \nu' - \nu''$.

We prove its quantitative budget. Extend $j_\tau = 0$ off S, and define

$$R_\tau = \sum_{e \in I_p} \min(\kappa(\tau)/p^e, \kappa(r_e\tau)).$$

All these positive series converge by comparison with $b \sum_{e \ge 1} p^{-e}$. The elementary identity $v - (v - w)_+ = \min(v, w)$, and the geometric sums over I_p, give

$$\gamma_p \sum_\tau \kappa(\tau) - F(p) = v_p(\pi_p) \sum_\tau R_\tau \le \sum_\tau R_\tau. \tag{3}$$

Here the geometric sum is $p/(p^2 - 1)$ in the inert case, and $1/(p - 1)$ in the ramified case, multiplied by $v_p(\pi_p) = 1/2$ in the latter.

Put $n_0 = \lceil H/2 \rceil$. For $e < n_0$, any nonshort term of R_τ is at most a/x_1: either its start or its end is not a b-place and has weight a/x_1. If a short term exists, it is unique and bounded by

$$\kappa(\tau)/p^{e_\tau} \le l_\tau / p^{e_\tau} < j_\tau + 1.$$

For the tail, $H > 27s$ implies $n_0 - 1 \ge s$, hence $p^{n_0 - 1} \ge 2^s \ge k$. Therefore

$$\sum_{e \ge n_0} b/p^e = \frac{bp^{-n_0}}{1 - p^{-1}} \le \frac{2b}{kp}.$$

Since $n_0 \le H$, both cases, including absence of a short term, give

$$R_\tau \le j_\tau + 1 + H\, a/x_1 + 2b/(kp).$$

Also $0 \le \gamma_p \le 1$, so $\gamma_p |l| \le \gamma_p \sum_\tau \kappa(\tau) + f x^{42}$. Combine this inequality with (3), subtract J, and add $3f(L + 2)$. Apply the small-prime bound of Lemma 6.7.28 at each l_τ. This proves (1).

Large primes: couple every place once in exponent. Suppose $p > k$. Since $d \le k$, this nonsplit prime is inert. Take $S = \Phi$, $e_\tau = 1$, and

$$i_\tau = \left\lfloor \frac{\max(l_\tau - n_*, 0)}{p} \right\rfloor, \qquad j_\tau = \min(i_\tau, l_{r_1\tau}).$$

As $l_\tau \le 2b$ and $p > k$, $i_\tau \le 2b/k \le n_*$. The unique-lowest condition follows because the only positive odd exponent at most one is one and r_1 is a permutation. At a point $\sigma = r_1\tau$, the contributions are $pj_\sigma + j_\tau$. They are bounded by

$$\max(l_\sigma - n_*, 0) + \min(n_*, l_\sigma) = l_\sigma.$$

Thus the capacity and residual η again exist.

For a single τ, the consecutive-product divisibilities of Lemma 6.7.28 and one step of Legendre's formula give

$$v_p(l_\tau!) \le i_\tau + v_p(i_\tau!) + v_p(B(l_\tau)) \le i_\tau + 2v_p(B(l_\tau)).$$

The second step uses $i_\tau \le n_*$, so $i_\tau! \mid n_*! \mid B(l_\tau)$. Since $n_* \ge x^{42}$, the truncated numerator defining i_τ is at most $\kappa(\tau)$. Hence

$$i_\tau - j_\tau \le (\kappa(\tau)/p - \kappa(r_1\tau))_+ \le s_p(\kappa(\tau), \tau).$$

The last term is the first nonnegative summand of the inert local prime sum. Sum over τ to obtain (2).

Bound the factorial cost of splitting three degrees. For $0 \le v \le 2b$ and $p \ge 3$, choose $h_0 = \lceil L \rceil + 1$. Then $v \le 2e^L < e^{L+1} \le p^{h_0}$. Writing $\mathrm{dig}_p(v)$ for the sum of the base-p digits, Legendre's formula gives

$$v_p(v!) = (v - \mathrm{dig}_p(v))/(p-1), \qquad 0 \le \mathrm{dig}_p(v)/(p-1) \le h_0 \le L + 2.$$

Thus $v/(p-1) - (L+2) \le v_p(v!) \le v/(p-1)$. At each τ, apply the upper bound to $l_\tau!$ and the lower bound to the three factorials of $\nu'_\tau, \nu''_\tau, \eta_\tau$. Their linear terms cancel, giving error at most $3(L+2)$. All arguments are at most $l_\tau \le 2b$. Summing over the f places proves the remaining assertion. □

Lemma 6.7.27. *Let Q be a finite set, $g : Q \to Q$ a permutation, $B \subseteq Q$, and $H \ge 3$ an integer. Suppose there do not exist $\tau \in B$ and integers $0 < i < j < H$ for which $g^i\tau, g^j\tau \in B$. Consider all pairs (τ, e) with $\tau \in B$, e positive odd, $2e < H$, and $g^e\tau \in B$. Then each start belongs to at most one such pair, each end belongs to at most one such pair, and no end is a start.*

Let S be any subset of their starts and e_τ the corresponding exponent. For $\tau, \sigma \in S$ and positive odd $a \le e_\tau$,

$$g^a\sigma = g^{e_\tau}\tau \quad \Longrightarrow \quad (\sigma, a) = (\tau, e_\tau).$$

Furthermore, let $p \ge 2$, let $b_0 > 0$, and let l_τ be nonnegative integers for $\tau \in Q$, with $l_\tau \le 2b_0$ for all τ and $l_\tau \ge b_0$ for $\tau \in B$. Put

$$j_\tau = \lfloor l_\tau / p^{e_\tau} \rfloor, \qquad \nu' = \sum_{\tau \in S} p^{e_\tau} j_\tau \mathbf{1}_\tau, \quad \nu'' = \sum_{\tau \in S} j_\tau \mathbf{1}_{g^{e_\tau}\tau}.$$

Then $\nu' + \nu'' \le l$ coordinatewise.

Proof. Suppose two pairs have the same start, with exponents $e < e'$. The three points at times $0, e, e'$, all in B, contradict the hypothesis, since $0 < e < e' < H$. Thus the exponent at a start is unique.

Suppose two pairs $(\tau, e), (\sigma, e')$ have the same end. If $e = e'$, injectivity of g^e gives $\tau = \sigma$. If, say, $e < e'$, then $\tau = g^{e'-e}\sigma$. The three points starting at σ, at times $0, e' - e, e'$, lie in B; these times satisfy $0 < e' - e < e' < H$. This again contradicts sparsity. Thus ends also determine their pairs uniquely.

If an end $g^e\tau$ were a start σ of a pair of exponent e', then the times $0, e, e + e'$ from τ would all lie in B. They are strictly increasing, and $e + e' < H$ follows by adding $2e < H$ and $2e' < H$. This is impossible.

For the claimed unique-lowest condition, the pair (σ, a) itself belongs to the considered collection: its start lies in B, its end equals the end of (τ, e_τ) and lies in B, it is positive odd, and $2a \le 2e_\tau < H$. Uniqueness at an end therefore identifies the two pairs.

Finally a start cannot receive an end contribution, so at a start τ the sum $\nu' + \nu''$ equals $p^{e_\tau}\lfloor l_\tau / p^{e_\tau} \rfloor \le l_\tau$. At an end it receives exactly one contribution, namely j_τ from its unique pair. Since $e_\tau \ge 1$ and $p \ge 2$,

$$j_\tau \le l_\tau / p^{e_\tau} \le 2b_0/2 = b_0.$$

The degree at that end is therefore at most its prescribed l, because the end lies in B. At a point which is neither a start nor an end the degree is zero. These cases prove the coordinatewise bound. □

Lemma 6.7.28. *Put $L = \log b$, $H = l_0(s)$, $n_* = \lceil 6b/k \rceil$, and, for an integer $v \ge 0$, put*

$$v^- = \max(v - n_*, 0), \qquad B(v) = \prod_{i=1}^{n_*} (v^- + i).$$

Then $n_* \ge x^{42}$, $B(v) > 0$, and, for every prime p,

$$v_p(n_*!) \le v_p(B(v)), \qquad v_p(v!) \le v_p(v^-!) + v_p(B(v)).$$

For $p \le k$, one has

$$3(L+2) + x^{42} + 1 + H\, a/x_1 + 2b/(kp) \le v_p(B(v)). \tag{1}$$

If $\kappa(\tau) \le l_\tau \le \kappa(\tau) + x^{42}$ for all $\tau \in \Phi$, and a nonzero scalar $c_0 \in \mathbb{C}$ satisfies

$$|\sigma(c_0)| \le \exp(12 C_P q_1^2) \quad (\sigma \in \operatorname{Aut} \mathbb{C}),$$

then

$$t = M^{12f} c_0 \prod_{\tau \in \Phi} B(l_\tau)^2 \ne 0, \qquad |\sigma(t)| \le \exp(fmb/k) \quad (\sigma \in \operatorname{Aut} \mathbb{C}). \tag{2}$$

Proof. **Record quantitative scale consequences.** Write $w = \log x$. Those bounds and the defining scale inequalities give

$$x \ge 10^6, \quad L \ge 100, \quad \log x_1 \ge 10^6 w, \quad L \ge 10^6 \log x_1 \ge 10^{12} w, \quad L \le x^{1/100}. \tag{3}$$

The last inequality is the exponential of $100 \log L \le w$. In particular $b \ge x^{10^{12}}$ and $x_1 \le b^{10^{-6}}$. The rank formulas give $k = r + 1 - m \le 3x^5 + 1 \le 4x^5 \le x^6$, and $m \ge \sqrt{x}$. Since $s \ge 1$, $H \le 81 s \le 2592 w$. Thus $H + 1 \le x^2$ for $x \ge 10^6$: use $w \le \sqrt{x}$, followed by $2592\sqrt{x} + 1 \le x^2$.

For later use the explicit tower formula gives

$$q_1 = 3 q_0 H 2^s \le 7776\, 2^{22} q_0 w x^{11} \le w^2 x^{11} \le x^{13}. \tag{4}$$

Here $2^s = m^{22} \le 2^{22} x^{11}$, $q_0 \le C_P \le 10^{-14} w$, and $7776\, 2^{22} < 10^{14}$. Also $C_P \le x$, $f \ge 1$, $2 \le k \le b$, and $\log M \le 22L$.

Put $\varepsilon = a/x_1 = xb/x_1$. The inequalities just proved imply

$$3(L+2) + x^{42} + 1 \le \varepsilon, \qquad (H+1)\varepsilon \le b/k^2, \qquad \varepsilon \ge 1. \tag{5}$$

Here are direct estimates for both nontrivial assertions. First $\varepsilon \ge b^{1-10^{-6}}$, $x^{42} \le b^{1/4}$, and $3(L+2) + 1 \le b^{1/4}$. For the latter, $e^{L/4} \ge L^3/384 \ge 3L + 7$ for $L \ge 100$. Since $2b^{1/4} \le b^{1/2} \le b^{1-10^{-6}}$, the first assertion follows. For the second,

$$(H+1) x k^2 \le x^2 x x^{12} = x^{15} \le x_1,$$

by $\log x_1 \ge 10^6 \log x$; cancel x_1 and multiply by b.

Consecutive products supply the loss. The product $B(v)$ consists of n_* positive consecutive integers, so it is positive and divisible by $n_*!$, since $B(v) = n_*!\binom{v^- + n_*}{n_*}$. Furthermore $v \le v^- + n_*$, whence $v! \mid (v^- + n_*)! = v^-!B(v)$. Taking valuations gives the first two assertions. From (3) and $k \le x^6$ we have $6b/k \ge x^{42}$, so $n_* \ge x^{42}$.

Legendre's formula gives $v_p(n_*!) \ge \lfloor n_*/p \rfloor \ge n_*/p - 1$. For $p \le k$, put $D_p = b/(kp)$. By (5), $D_p \ge b/k^2 \ge (H+1)\varepsilon \ge 1$. Also $n_*/p \ge 6D_p$. Consequently

$$v_p(B(v)) \ge n_*/p - 1 \ge 3D_p.$$

The expression on the left of (1) is at most

$$\varepsilon + H\varepsilon + 2D_p \le D_p + 2D_p = 3D_p$$

by (5). This proves (1), uniformly for all v.

The same loss has a small archimedean cost. Since $x_1 \ge x$, every $\kappa(\tau) \le b$. Thus $l_\tau \le b + x^{42} \le 2b$. Also

$$n_* \le 6b/k + 1 \le 7b/k, \qquad 2b + n_* \le 6b, \qquad \log(2b + n_*) \le 2L.$$

The middle estimate uses $k \ge 2$, and the last uses $b \ge 6$. Each factor of $B(l_\tau)$ is at most $2b + n_*$, so

$$\log B(l_\tau) \le 14bL/k.$$

For any automorphism σ, it fixes the positive integer factors of t, and the assumed bound on c_0 therefore yields

$$\log|\sigma(t)| \le 264fL + 28fbL/k + 12C_P q_1^2.$$

If a logarithm is not taken, the same inequality follows by multiplying the corresponding positive exponential bounds. Divide the displayed upper bound by $fb/k > 0$. Its three terms are bounded as follows, using (3)–(4):

$$264Lk/b \le 264x^7/x^{10^{12}} \le 1, \quad 28L \le 28x^{1/100} \le \sqrt{x}/2, \quad 12C_P q_1^2 k/(fb) \le 12x^{33}/x^{10^{12}} \le 1.$$

The middle inequality holds for $x \ge 10^6$, since $56 \le x^{49/100}$. The sum is at most $2 + \sqrt{x}/2 \le \sqrt{x} \le m$. This proves (2). Every factor in its defining product is nonzero, so $t \ne 0$, completing the proof. □

Lemma 6.7.29. *Use the exact normalized theta function θ, translated curve parameter y, prime embedding and ring $O = \mathcal{O}_{\mathbb{C}_p}$ of Lemma 4.2.37, and the number γ_p of Lemma 6.7.30. Let X, I be finite sets, let $L_i(Z) = \sum_{x \in X} \mu_{ix} Z_x$ with $\mu_{ix} \in \mathbb{C}_p$, and let $m_i \ge 0$. Put*

$$f_i = y(L_i(Z)), \quad R_\mu = \{A((f_i)_i) : A \in O[[U_i : i \in I]]\}, \quad \Theta(Z) = \prod_i \theta(L_i(Z))^{m_i}.$$

There exist $\delta_0, \dots, \delta_5 \in \mathbb{Z}_p$, with $\delta_0 = 1$ and $\sum_c \delta_c^2 = 0$, such that

$$U(Z) = \prod_{c=0}^{5} \Theta(\delta_c Z) \in R_\mu, \qquad U^{-1} \in R_\mu, \qquad U(0) = 1.$$

If some $i_0 \in I$ satisfies $|\mu_{i_0 x}|_p = 1$ for every $x \in X$, then for every $\nu \in \mathbb{N}^X$ there are $\alpha_c \in \mathbb{N}^X$ and $e_c \in \mathbb{N}^I$, $0 \le c \le 5$, with

$$\sum_c \alpha_c = \nu, \quad |e_c| \le |\nu|, \qquad \prod_{c=0}^{5} \left| [Z^{\alpha_c}] \left(\prod_i f_i^{e_{c,i}} \Theta \right) \right|_p \ge p^{-\gamma_p |\nu|} \left| \prod_x \nu_x! \right|_p^{-1}. \tag{1}$$

If instead there are $a_{xi} \in O$ and $u \in O^\times$ with

$$\sum_i a_{xi}\mu_{ix'} = u\,\mathbf{1}_{x=x'},$$

then the same existence assertion holds with the right side of (1) replaced by 1.

Proof. **Choose integral scalars whose squares cancel.** For odd p, the sets of squares and of minus one minus a square in $\mathbb{F}_p$ both have cardinality $(p+1)/2$, and therefore intersect. This gives $a^2+b^2=-1$ modulo p, with at least one coordinate nonzero. Holding the other coordinate fixed, successive lifting with the unit derivative $2a$ or $2b$ gives a solution in $\mathbb{Z}_p$. Thus minus one is a sum of four squares in $\mathbb{Z}_p$, adding two zeros if needed. For $p=2$, minus seven has a square root in $\mathbb{Z}_2$. Explicitly start with an odd solution modulo eight. If $v^2 \equiv -7 \pmod{2^h}$, $h \geq 3$, adding either zero or 2^{h-1} changes its square by either zero or 2^h modulo 2^{h+1}, since v is odd. Choose the correction giving a solution modulo 2^{h+1}. These compatible corrections converge to the required root. Then

$$-1 = 1^2 + 1^2 + 2^2 + (\sqrt{-7})^2.$$

In either case choose $a_1, \ldots, a_4 \in \mathbb{Z}_p$ with sum of squares minus one and take $(\delta_0, \ldots, \delta_5) = (1, a_1, a_2, a_3, a_4, 0)$.

Cancel the quadratic exponential in the same parameter. Put $g = -\log\theta$, so $\theta = \exp(-g)$ as formal series and $g(0)=0$. The last assertion of Lemma 4.2.37 gives $\exp(\delta_c^2 g(z) - g(\delta_c z)) \in O[[y]]$. Their product, because $\sum_c \delta_c^2 = 0$, equals $\prod_c \theta(\delta_c z)$. It has constant coefficient one. Thus it and its reciprocal belong to $O[[y]]$: in an outer representation $A(y)$, the constant coefficient of A is one, so its formal reciprocal lies in $O[[U]]$. Substitute $L_i(Z)$, raise to m_i, and multiply over i. This gives $U, U^{-1} \in R_\mu$ and $U(0)=1$, with exactly the stated six scalars. No integrality of the μ_{ix} is needed for these formal substitutions.

Use the attained one-variable derivative. Let $d_0 = |\nu|$. By Lemma 6.7.30, there is a power $A(z) = y(z)^j \in O[[y]]$ for which

$$|d_0![z^{d_0}]A|_p \geq p^{-\gamma_p d_0}.$$

Put $B(Z) = A(L_{i_0}(Z)) \in R_\mu$. The multinomial formula gives

$$[Z^\nu]B = \frac{d_0!}{\prod_x \nu_x!}[z^{d_0}]A \prod_x \mu_{i_0 x}^{\nu_x}.$$

All factors in the last product are units in norm. Hence this coefficient has norm at least the right side of (1). Take $H_0 = BU^{-1} \in R_\mu$, and $H_c = 1$ for $c > 0$. Then, since $\delta_0 = 1$,

$$\prod_{c=0}^{5} (H_c\Theta)(\delta_c Z) = B(Z).$$

Apply Lemma 6.7.32 with one new copy of X, coefficient matrix (δ_c), and target ν. It yields the required aggregate exponents and monomial factors. Its coefficient inequality transfers the displayed lower bound and proves (1).

An integral span supplies a unit coefficient directly. Under the last hypothesis, put

$$B_x(Z) = \sum_i a_{xi} y(L_i(Z)) \in R_\mu.$$

Since $y(z) = z+$ higher-degree terms, $B_x = uZ_x+$ terms of total degree at least two. Thus $\prod_x B_x^{\nu_x}$ has unique lowest total-degree term $u^{|\nu|}Z^\nu$. Multiplication by Θ, whose constant coefficient is one, leaves this coefficient unchanged. It has norm one. Apply Lemma 6.7.31 to this

integral formal factor and multiplier Θ at degree ν. It gives one monomial factor of degree at most $|\nu|$ with coefficient norm at least one. Use that factor and degree ν in the first copy, and use factor one and degree zero in the remaining five copies. Their coefficients are $\Theta(0) = 1$, proving the final assertion. □

Lemma 6.7.30. *Let $d \equiv 7 \pmod 8$ be prime, let $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$, and let $M \subset \mathbb{C}$ be an $\mathcal{O}_K$-stable lattice with marked point t satisfying the hypotheses of Lemma 4.2.6. Use its normalized equation $Y^2 + aXY = X^3 + X$ and invariant differential dz. Fix an embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, and let $y(z) \in \mathbb{C}_p[[z]]$ be the coefficientwise image of*

$$\frac{\wp_M'(z+t) - a(\wp_M(z+t) - \wp_M(t))}{2}.$$

Put $O = \mathcal{O}_{\mathbb{C}_p}$ and $R = \{A(y(z)) : A \in O[[U]]\}$. Set

$$\gamma_p = \begin{cases} 0, & p \text{ split in } K, \\ p/(p^2-1), & p \text{ inert in } K, \\ 1/(2(p-1)), & p = d. \end{cases}$$

For every integer $n \geq 0$, there is an integer j, $0 \leq j \leq n$, such that

$$p^{-n\gamma_p} \leq |n![z^n]y(z)^j|_p = \max_{A \in O[[U]]} |n![z^n]A(y(z))|_p \leq 1.$$

Proof. **The normalized local ring has integral differentiation and addition.** The all-prime good model is Lemma 4.2.6. At its marked section $Q = (0,0)$, the derivative with respect to X of $Y^2 + aXY - X^3 - X$ is -1. Formal coefficient comparison therefore solves uniquely $X = X_0(Y) \in Y^2 O[[Y]]$. The analytic differential normalization identifies the completion at Q with R, and gives $y(0) = 0$, $y'(0) = 1$. Differentiating the equation in its invariant coordinate gives

$$x' = 2y + ax, \qquad y' = 1 + 3x^2 - ay, \qquad x = X_0(y).$$

Thus $y' \in O[[y]]$, and the formal chain rule shows that R is stable under differentiation. This calculation involves no division by two and holds also at $p = 2$.

Translate the elliptic group law so that Q is its identity. Translation by an integral section is defined over O; hence the completed group law has coefficients in O. The invariant differential is still dz, so in the analytic coordinate its addition is $z + w$. Consequently

$$y(z+w) \in O[[y(z), y(w)]].$$

The normalized formal logarithm of this law is the compositional inverse of $y(z)$.

Construct a scaled parameter at split primes. For odd nonsplit primes, Lemma 4.2.41 supplies a generator T of R, numbers q, ρ, and a scalar ϖ with

$$T(\varpi z) = \varpi Y_0(z), \quad v_p(\varpi) = \rho, \quad Y_0 \in O[[z]], \quad Y_0 \equiv \sum_{h \geq 0} z^{q^h} \pmod{\pi O[[z]]}, \tag{1}$$

where $(q, \rho) = (p^2, 1/(p^2-1))$ in the inert case, and $(q, \rho) = (p, 1/(2(p-1)))$ if $p = d$. We establish the corresponding assertion with $q = p, \rho = 1/(p-1), \pi = p$ at split primes, including two.

Over a finite local extension the full CM action is defined and extends to the same good elliptic scheme by Theorem A.2.19: the base is a noetherian normal discrete valuation ring, its generic point is open dense, and the scheme is abelian. Uniqueness preserves the ring action. The two idempotents in $\mathcal{O}_K \otimes \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p$ split its p-divisible group into factors of height one. Their heights are one on the generic Tate module and remain one since the corresponding finite flat kernels have constant rank. Their dimensions sum to one, so over the algebraically closed residue field exactly one is connected of dimension one and the other is etale. The formal group is the connected factor. Its multiplication-by-p kernel has rank p and is supported at its identity in the special fibre. Therefore multiplication by p on the one-dimensional special-fibre formal group has order exactly p: its kernel algebra is the quotient by its defining series, whose length is that order.

For $u \in \mathbb{Z}_p$, the formal multiplication series $[u]$ is integral. Indeed for integers $u_m \to u$, the coefficients of $y(u_m y^{-1}(U))$ lie in O; in each degree they are polynomials in u_m over $\mathbb{C}_p$, so converge to those of $y(uy^{-1}(U))$, and O is closed. In particular $y(\zeta z) \in R$ for every $(p-1)$st root of unity $\zeta \in \mathbb{Z}_p$. Put

$$T(z) = \frac{1}{p-1} \sum_{\zeta^{p-1}=1} \zeta^{-1} y(\zeta z).$$

It belongs to R, has linear term z, and has only degrees congruent to one modulo $p-1$. Its inverse change from the parameter y is integral, by recursive coefficient comparison; thus $O[[T]] = R$. The multiplication series in this parameter is

$$T(pz) = g(T(z)), \qquad g(U) = pU + b_1 U^p + \sum_{i \geq 2} b_i U^{1+i(p-1)}, \qquad b_i \in O.$$

The support follows by commuting with all the displayed roots of unity. The special-fibre order just proved makes b_1 a unit. These assertions also apply for $p = 2$, when the averaging group is trivial.

Choose $\varpi^{p-1} = -p/b_1$, so $v_p(\varpi) = 1/(p-1)$. Direct scaling gives

$$g(\varpi U) = p\varpi(U - U^p + pE(U)), \qquad E \in U^2 O[[U]].$$

For $Y_0 = T(\varpi z)/\varpi$, the resulting equation is

$$Y_0(pz) = p(Y_0 - Y_0^p + pE(Y_0)).$$

For $n \geq 2$, the coefficient multiplying $[z^n]Y_0$ is $p^{n-1} - 1$, a unit, while the remaining coefficient uses only lower degrees. Induction makes Y_0 integral. Reduction modulo pO gives $Y_0 - Y_0^p \equiv z$, hence (1) with $q = p$, by iteration in each fixed degree. This proves the required split-prime assertion without changing the differential scale.

Derivative maxima are attained and submultiplicative. For every n, apply Lemma 6.7.31 to the one series y, the multiplier one, and degree n. It gives $0 \leq j \leq n$ maximizing the norm of $[z^n]A(y)$ over all integral A. Denote the attained maximum after multiplication by $n!$ by c_n. Differentiation stability and integral constant terms give

$$0 \leq c_n \leq 1, \qquad c_0 = 1.$$

We claim $c_{m+n} \leq c_m c_n$. Choose a maximizing power y^j in degree $m+n$. Its value at $z+w$ is an integral series in $y(z), y(w)$. Apply Lemma 6.7.31 to these two zero-constant series and the coefficient at $z^m w^n$. Some product $y(z)^u y(w)^v$ bounds that coefficient. Multiplying by $m!n!$, and using

$$m!n![z^m w^n] y(z+w)^j = (m+n)![z^{m+n}] y(z)^j,$$

bounds it by $c_m c_n$, as required.

Prime-power coefficients force the rate in every degree. In (1), if $N = q^h$ and $r = N - 1$, the coefficient $[z^N]Y_0$ is congruent to one modulo πO, and therefore has norm one. Thus

$$|[z^N]T|_p = p^{r\rho}.$$

Legendre's formula gives $v_p(N!) \leq (N-1)/(p-1)$. In all three cases, $\gamma_p = 1/(p-1) - \rho \geq 0$, so, since $T \in R$,

$$c_N \geq |N![z^N]T|_p \geq p^{-r\gamma_p} \geq p^{-N\gamma_p}. \tag{2}$$

These N are arbitrarily large.

For completeness, put $\eta = p^{-\gamma_p} \in (0,1]$. Suppose that for some $n > 0$ one had $c_n < \eta^n$. Write $N = hn + r$, $0 \leq r < n$. Submultiplicativity and $c_r \leq 1$ imply $c_N \leq c_n^h$. At the arbitrarily large N in (2) this gives

$$\eta^n(\eta^n)^h \leq \eta^N \leq c_n^h, \qquad \eta^n \leq (c_n/\eta^n)^h.$$

The last right-hand side tends to zero, a contradiction. Hence $c_n \geq \eta^n$ for every positive n; for zero it is equality. Combining this with the attained maximum proves the assertion. □

Lemma 6.7.31. *Let L be a field with a nonarchimedean absolute value and valuation ring $O = \{a \in L : |a| \leq 1\}$. Let I, X be finite sets, let $f_i \in L[[Z_x : x \in X]]$ have zero constant term for every $i \in I$, and let $s \in L[[Z_x : x \in X]]$. For every $\alpha \in \mathbb{N}^X$, there exists $m \in \mathbb{N}^I$, with $|m| \leq |\alpha|$, such that simultaneously for all $G \in O[[U_i : i \in I]]$,*

$$|[Z^\alpha](G((f_i)_i)s)| \leq \left|[Z^\alpha]\left(\prod_{i\in I} f_i^{m_i} s\right)\right|.$$

Proof. Put $d = |\alpha|$. For every multi-index $b \in \mathbb{N}^I$ with $|b| \leq d$, put

$$a_b = [Z^\alpha]\left(\prod_i f_i^{b_i} s\right).$$

The set of these b is finite and nonempty, since it contains zero. Choose m on this set maximizing $|a_b|$. Each f_i has zero constant term, so every monomial of $\prod_i f_i^{b_i}$ has total degree at least $|b|$; this remains true if a factor is zero. Multiplication by s, which has no negative degrees, cannot lower that degree. Consequently terms of G of degree greater than d make no contribution to the coefficient at α. Formal substitution therefore gives the finite identity

$$[Z^\alpha](G((f_i)_i)s) = \sum_{|b|\leq d} [U^b]G\, a_b.$$

All coefficients $[U^b]G$ have absolute value at most one. The ultrametric inequality bounds the absolute value of this finite sum by $\max_{|b|\leq d}|a_b| = |a_m|$. The chosen m depends only on the f_i, s, α, so the bound holds for every G at once. □

Lemma 6.7.32. *Let L be a nonarchimedean valued field with valuation ring O. Let X, I, K be finite sets and let $n \geq 1$. Let $f_i \in L[[Z_x : x \in X]]$ have zero constant term, let $\Theta \in L[[Z_x : x \in X]]$, and let $H_c = G_c((f_i)_i)$ with $G_c \in O[[U_i : i \in I]]$, for $1 \leq c \leq n$. Take $A_{ck} \in O$ and set*

$$P(W) = \prod_{c=1}^{n} (H_c\Theta)\left(\left(\sum_{k\in K} A_{ck}W_{xk}\right)_{x\in X}\right).$$

For every $\beta \in \mathbb{N}^{X\times K}$, put $\nu_x = \sum_k \beta_{xk}$. There exist $\alpha_c \in \mathbb{N}^X$ and $e_c \in \mathbb{N}^I$ such that

$$\sum_c \alpha_c = \nu, \qquad |e_c| \le |\alpha_c| \le |\nu|,$$

and

$$|[W^\beta]P| \le \prod_{c=1}^{n} \left| [Z^{\alpha_c}] \left(\prod_i f_i^{e_{c,i}} \Theta \right) \right|.$$

Proof. Write $B_c = H_c\Theta$. The coefficient of the finite product defining P is the finite sum over all tuples $(\gamma_c)_{c=1}^n$ of nonnegative multi-indices on $X \times K$ with $\sum_c \gamma_c = \beta$, of

$$\prod_c [W^{\gamma_c}] B_c \left(\left(\sum_k A_{ck} W_{xk} \right)_x \right).$$

The indexing set is nonempty, since one may place β in the first copy and zero in the others. Choose a tuple whose summand has largest absolute value. The ultrametric inequality bounds $|[W^\beta]P|$ by that summand's absolute value. Put $\alpha_{c,x} = \sum_k \gamma_{c,xk}$. Then $\sum_c \alpha_c = \nu$.

Linear substitution gives the exact formula

$$[W^{\gamma_c}] B_c \left(\left(\sum_k A_{ck} W_{xk} \right)_x \right) = [Z^{\alpha_c}] B_c \prod_{x\in X} \left(\binom{\alpha_{c,x}}{(\gamma_{c,xk})_{k\in K}} \prod_{k\in K} A_{ck}^{\gamma_{c,xk}} \right).$$

Every multinomial coefficient is an integer and has absolute value at most one in a nonarchimedean valued field, by repeated use of the triangle inequality on sums of ones. The entries of A also have absolute value at most one. Hence

$$|[W^\beta]P| \le \prod_c |[Z^{\alpha_c}](H_c\Theta)|.$$

For each c, apply Lemma 6.7.31 with $s = \Theta$ and target α_c. It supplies e_c of degree at most $|\alpha_c|$ for which the corresponding monomial coefficient dominates that of $G_c((f_i)_i)\Theta$. Multiply these finitely many inequalities. Finally $\alpha_c \le \sum_d \alpha_d = \nu$ coordinatewise, so $|\alpha_c| \le |\nu|$. The argument also covers an empty variable set or an empty set K, since all coefficient expansions above remain finite and the zero multi-index is available. □

Lemma 6.7.33. *Use the normalized CM theta setting and the series $f_i = y(L_i(Z))$, R_μ, $\Theta = \prod_i \theta(L_i)^{m_i}$ of Lemma 6.7.29, with variable set Q. Let $p > 2$ be inert in K, put $\rho = 1/(p^2-1)$, $\gamma = p/(p^2-1)$, and assume $\mu_{i\tau} \in O$. Take exactly the maps r_a, coefficients λ_{ab}, subset S, odd exponents e_τ, multiplicities j_τ, series Ξ and degrees ν', ν'' satisfying the coupling hypotheses of Lemma 6.7.34; put $\nu = \nu' + \nu''$ and $J = \sum_{\tau\in S} j_\tau$. Suppose*

$$v_p(\varpi) = \rho, \qquad \Theta(\varpi Z) = W(Z) \exp\left(\frac{\varpi^{p+1}}{p} \Xi(Z) \right), \qquad W \in O[[Z]]^\times.$$

Then there are six multi-indices $\alpha_c \in \mathbb{N}^Q$ and six $e_c \in \mathbb{N}^I$, with $\sum_c \alpha_c = \nu$ and $|e_c| \le |\nu|$, such that

$$\prod_{c=0}^{5} \left| [Z^{\alpha_c}] \left(\prod_i f_i^{e_{c,i}} \Theta \right) \right|_p \ge p^{-r_0}, \qquad r_0 = \frac{J}{p-1} - J - \rho|\nu| - v_p(\nu''!).$$

Here $\nu''! = \prod_\tau \nu_\tau''!$. Moreover

$$r_0 \le -J, \qquad r_0 \le \gamma|\nu| - J - v_p(\nu'!) - v_p(\nu''!).$$

Proof. Choose the six scalars δ_c and the cancelling product

$$U = \prod_c \Theta(\delta_c Z)$$

supplied by Lemma 6.7.29. Its coefficients belong to R_μ, and $U(0) = 1$. We first check integrality after scaling. The parameter assertion Lemma 4.2.41 implies $y(\varpi z) \in O[[z]]$, with zero constant coefficient, since $v_p(\varpi) = \rho$. This assertion holds at any scalar of valuation at least ρ, not just the particular root chosen in that lemma. As the coefficients of each L_i are integral, $f_i(\varpi Z) = y(\varpi L_i(Z))$ is integral. An integral formal series in these zero-constant series remains integral. Therefore

$$U(\varpi Z) \in O[[Z]], \qquad U(\varpi Z)(0) = 1.$$

Put $F(Z) = \Theta(\varpi Z)$ and $\kappa = \varpi^{p+1}/p$. Then

$$|\kappa|_p = p^{1-(p+1)\rho} = p^{1-1/(p-1)} > 1,$$

because $p > 2$. All hypotheses of Lemma 6.7.34 hold for F, W, and the product $\prod_c F(\delta_c Z) = U(\varpi Z)$. Consequently, with

$$G(Z', Z'') = \Theta(Z' + Z'') \prod_{c=1}^{5} \Theta(\delta_c Z'),$$

that result gives

$$|\nu''!|_p \, |\varpi|_p^{|\nu|} |[(Z')^{\nu'}(Z'')^{\nu''}]G|_p = |\kappa|_p^J.$$

The scaling factor is exactly $\varpi^{|\nu'|+|\nu''|}$, since substitution of $\varpi Z', \varpi Z''$ multiplies each homogeneous coefficient by that power. Rearranging the displayed norm identity yields

$$|[(Z')^{\nu'}(Z'')^{\nu''}]G|_p = p^{(1-1/(p-1))J + \rho|\nu| + v_p(\nu''!)} = p^{-r_0}.$$

Apply Lemma 6.7.32 to six copies with all integral formal multipliers equal to one, and to the integral two-copy coefficient matrix

$$A_{c,0} = \delta_c, \qquad A_{c,1} = \mathbf{1}_{c=0}.$$

Its substituted product is precisely G, because $\delta_0 = 1$. The target is (ν', ν''), whose aggregate is ν. This gives the stated monomial witnesses and norm bound.

For the two final inequalities, each $e_\tau \geq 1$, hence

$$|\nu'| = \sum_{\tau \in S} p^{e_\tau} j_\tau \geq pJ, \qquad |\nu''| = J.$$

Since $\rho(p+1) = 1/(p-1)$, one has $\rho|\nu| \geq J/(p-1)$. Also $v_p(\nu''!) \geq 0$. Substitution in r_0 gives $r_0 \leq -J$. Finally Legendre's formula gives $v_p(\nu'!) \leq |\nu'|/(p-1)$. Using $\gamma = 1/(p-1) - \rho$ and $|\nu| = |\nu'| + J$, subtract the proposed second upper bound from r_0. The difference is exactly

$$v_p(\nu'!) - |\nu'|/(p-1) \leq 0.$$

This proves both inequalities, including the empty-coupling case. □

Lemma 6.7.34. *Let p be prime, $O = \mathcal{O}_{\mathbb{C}_p}$, and let Q be a finite set. For each odd positive integer a, let $r_a : Q \to Q$ be a map, and take $\lambda_{ab} \in O$ for $b \geq 0$, with $\lambda_{a0} \in O^\times$. Put*

$$\Gamma_a(Z) = \sum_{\tau \in Q} Z_\tau^{p^a} Z_{r_a(\tau)}, \qquad \Xi(Z) = \sum_{\substack{a \geq 1 \text{ odd} \\ b \geq 0}} \lambda_{ab} \Gamma_a(Z)^{p^b}.$$

Let $S \subseteq Q$, and choose odd positive integers e_τ for $\tau \in S$, such that

$$\tau, \sigma \in S, \quad a \leq e_\tau, \quad a \text{ odd}, \quad r_a(\sigma) = r_{e_\tau}(\tau) \implies (\sigma, a) = (\tau, e_\tau).$$

Let $j_\tau \geq 0$ for $\tau \in S$, and put

$$J = \sum_{\tau \in S} j_\tau, \quad \nu' = \sum_{\tau \in S} p^{e_\tau} j_\tau \, \mathbf{1}_\tau, \quad \nu'' = \sum_{\tau \in S} j_\tau \, \mathbf{1}_{r_{e_\tau}(\tau)}.$$

Suppose $W \in O[[Z]]^\times$, $\kappa \in \mathbb{C}_p$ with $|\kappa|_p > 1$, and $F = W \exp(\kappa \Xi)$. Let $n \geq 1$, let $\delta_0, \ldots, \delta_{n-1} \in \mathbb{C}_p$ with $\delta_0 = 1$, and suppose

$$U(Z) = \prod_{c=0}^{n-1} F(\delta_c Z) \in O[[Z]], \qquad U(0) = 1.$$

Then, for

$$G(Z', Z'') = F(Z' + Z'') \prod_{c=1}^{n-1} F(\delta_c Z'),$$

one has the exact attained coefficient norm

$$\left| \prod_{\tau \in Q} \nu''_\tau! \right|_p \left| [(Z')^{\nu'} (Z'')^{\nu''}] G \right|_p = |\kappa|_p^J.$$

Proof. **Repeated derivatives retain their leading reduction.** The sum defining Ξ is well-defined coefficientwise: a term with indices a, b has degree $(p^a + 1)p^b$, and only finitely many pairs have bounded degree. Put $V_\tau = \partial \Xi / \partial Z_{r_{e_\tau}(\tau)}$. Since W is an integral unit, $B_x = W^{-1} \partial_x W$ is integral for every x. Formal differentiation gives

$$\partial_x F = (\kappa \partial_x \Xi + B_x) F.$$

Only the first factor of G involves Z'', so its logarithmic derivative in Z''_x is $\kappa(\partial_x \Xi)(Z' + Z'') + B_x(Z' + Z'')$. All series in this expression except κ have integral coefficients. If $q \in O[[Z', Z'']]$, then

$$\partial_{Z''_x}(qG) = \kappa q_{\text{new}} G, \qquad q_{\text{new}} = q(\partial_x \Xi)(Z' + Z'') + \kappa^{-1}\big(\partial_{Z''_x} q + q B_x(Z' + Z'')\big).$$

The series q_{new} is integral, and its reduction modulo the maximal ideal is the product of the reductions of q and $(\partial_x \Xi)(Z' + Z'')$, because $|\kappa^{-1}|_p < 1$. Starting with $q = 1$, differentiate j_τ times in $Z''_{r_{e_\tau}(\tau)}$, for every $\tau \in S$. This gives an integral q such that

$$\partial_{Z''}^{\nu''} G = \kappa^J q G, \qquad \overline{q(Z', 0)} = \prod_{\tau \in S} \overline{V_\tau(Z')}^{j_\tau}.$$

The selected coefficient is a unit. In the residue field of characteristic p, differentiation annihilates the terms of Ξ with $b > 0$, and also annihilates the derivative of $Z_\sigma^{p^a}$. Hence

$$\overline{V_\tau} = \sum_{\substack{a \geq 1 \text{ odd} \\ \sigma \in Q \\ r_a(\sigma) = r_{e_\tau}(\tau)}} \overline{\lambda_{a0}} Z_\sigma^{p^a}.$$

Set $Z_\sigma = 0$ for $\sigma \notin S$. The uniqueness hypothesis says that among all terms of this series of total degree at most p^{e_τ}, exactly one remains: $\overline{\lambda_{e_\tau,0}} Z_\tau^{p^{e_\tau}}$. Its coefficient is nonzero. Products of series each having a unique nonzero term of smallest total degree have the product of those terms as their unique term of smallest total degree: expanding the product, any use of a higher-degree term raises the total degree. The same assertion holds for powers. Since $\overline{U}(0) = 1$, it follows that the coefficient at ν' of $q(Z', 0)U(Z')$ reduces to

$$\prod_{\tau \in S} \overline{\lambda_{e_\tau,0}}^{j_\tau} \neq 0.$$

Setting the variables outside S to zero does not change this coefficient, because ν' is supported on S. Its coefficient before reduction is integral and has norm one.

Pass from derivatives to the coefficient. By the product hypothesis, $G(Z', 0) = U(Z')$. Extract the coefficient at $(Z')^{\nu'}$ after setting $Z'' = 0$ in the derivative identity. The elementary coefficient formula for ordinary derivatives gives

$$\left(\prod_{\tau \in Q} \nu_\tau''! \right) [(Z')^{\nu'} (Z'')^{\nu''}] G = \kappa^J [Z^{\nu'}](q(Z, 0)U(Z)).$$

Taking norms and using the unit coefficient just proved yields the asserted equality. This argument includes $S = \varnothing$ or $J = 0$; the empty product of leading coefficients is one. □

Lemma 6.7.35. *Put $\omega = (1 + \sqrt{-d})/2$, let ζ be the fixed primitive q_1-th root, and put $D_{\mathrm{rel}} = \mathrm{cyc}'_{q_1}(\zeta)$, where cyc_{q_1} is the cyclotomic polynomial. Use the prescribed finite idele*

$$\alpha_w = \begin{cases} D_{\mathrm{rel}}^{-1}, & \omega \in w, \\ 1, & \omega \notin w, \end{cases} \qquad \mathfrak{a} = E \cap \alpha \widehat{\mathcal{O}}_E.$$

Let $\chi_{\mathrm{fin}} = \upsilon_f |\cdot|^{-1/2}$ be the fixed algebraic finite splitting character: its exponents are zero on the chosen CM type and minus one on the conjugate type. Use the same fixed vacuum root and period factor as the arithmetic theta normalization:

$$s_0^f = \chi_{\mathrm{fin}}(\alpha)^{-1}, \qquad \beta = s_0 / u_{\mathrm{CS}}, \qquad |\iota u_{\mathrm{CS}}|_p = 1 \quad \text{for every } p, \iota.$$

For $\phi_\tau(z) = \iota(\overline{\tau(z)})$, let α_{ϕ_τ} denote the image of the component of α at the prime induced by ϕ_τ. Then, for every prime p, every prime identification $\iota : \mathbb{C} \to \mathbb{C}_p$, and every $\tau \in \Phi$,

$$|\iota\beta|_p = |\alpha_{\phi_\tau}|_p^{-1}, \qquad |\iota(\beta\overline{\tau(w)})|_p \leq 1 \quad (w \in \mathfrak{a}).$$

In particular β is a unit at every nonsplit prime of K.

There exist a finite set I, vectors $w_i \in \mathfrak{a}$, and integers $c_i \geq 6fb$, chosen before all p, ι, such that

$$\sum_{i \in I} c_i \, \tau(w_i) \overline{\sigma(w_i)} = M \mathbf{1}_{\tau = \sigma} \qquad (\tau, \sigma \in \Phi).$$

For every p, ι there exists $i_0 \in I$, allowed to depend on p, ι, such that

$$|\iota(\beta\overline{\tau(w_{i_0})})|_p = 1 \qquad \textit{simultaneously for every } \tau \in \Phi.$$

Put $u_i = w_i/\bar{\pi}_M$ and $v_{\mathrm{loc}} = \iota(\pi_M\beta)$. Then

$$\sum_i c_i\tau(u_i)\overline{\sigma(u_i)} = \mathbf{1}_{\tau=\sigma}, \qquad v_{\mathrm{loc}}\iota(\overline{\tau(u_i)}) = \iota(\beta\overline{\tau(w_i)}).$$

At every nonsplit prime p, every u_i is integral at every place above p, and $|v_{\mathrm{loc}}|_p = 1$. No new vacuum scalar or theta functional is chosen.

Proof. **1. The exact finite-character valuation and period factor.** The infinity type of χ_{fin} means that on principal finite ideles $z \in E^\times$ its value is $\prod_{\tau\in\Phi}\overline{\tau(z)}$. On finite idele units its image is finite. Indeed the unit group is compact and profinite, and a continuous character into the circle has finite image, since the circle has no small subgroups. For any finite idele x, finiteness of the ideal class group gives a positive power whose ideal is principal, say the ideal of z. Raising to a further power kills the character of the remaining finite unit. Consequently for some integer $n > 0$,

$$\chi_{\mathrm{fin}}(x)^n = \prod_{\tau\in\Phi}\overline{\tau(z)}, \qquad x^n \text{ and } z \text{ have the same finite valuations.}$$

Applying ι, taking absolute values and then the positive n-th root gives

$$|\iota\chi_{\mathrm{fin}}(x)|_p = \prod_{\tau\in\Phi}|x_{\phi_\tau}|_p. \tag{1}$$

The absolute value of α_{ϕ_τ} is independent of τ. The condition $\omega \in w_{\phi_\tau}$ reads $|\iota\bar{\omega}|_p < 1$, independent of τ, because these embeddings fix K. Also all conjugates of D_{rel} have equal valuations at any fixed finite prime. This can be seen without a different comparison: write the derivative as the product of $\zeta - \zeta^a$ over the other primitive conjugates. For v prime to q_1, each quotient $(1-\zeta^{v(a-1)})/(1-\zeta^{a-1})$ is an algebraic integer, by a finite geometric sum; its reciprocal is also an algebraic integer by choosing the inverse of v modulo q_1. It is a unit. The remaining root-of-unity factors are units as well. Thus a Galois conjugate of D_{rel} differs from it by a unit.

Write the common absolute value as A_p. Equation (1) yields $|\iota\chi_{\mathrm{fin}}(\alpha)|_p = A_p^f$; the specified f-th-root equation gives $|\iota s_0|_p = A_p^{-1}$. For the specified common jet-pair and vacuum-scale choice, the all-prime unit assertion of Lemma 4.2.3 gives $|\iota u_{\mathrm{CS}}|_p = 1$. Hence

$$|\iota\beta|_p = A_p^{-1}, \qquad |\iota(\beta\overline{\tau(w)})|_p = |(\alpha^{-1}w)_{\phi_\tau}|_p \le 1 \quad (w \in \mathfrak{a}). \tag{2}$$

This proves the period assertions apart from the nonsplit specialization below.

2. A fixed cyclotomic fractional frame. Abbreviate $q = q_1$ and $U_0 = (d+1)/4$. Every prime dividing q divides U_0. For odd primes this follows from $q_0 \mid 2(d+1)$ and the tower primes $2, 3$; at 2, $d \equiv 7 \pmod 8$ makes U_0 even. Choose z_0, $0 \le z_0 < q$, such that

$$U_0 z_0^2 + z_0 + 1 = qe_0 \tag{3}$$

for an integer $e_0 > 0$. Such a residue exists because $z \mapsto U_0 z^2 + z$ is a permutation modulo q: a difference factors as $(z - z')(1 + U_0(z + z'))$, whose second factor is a unit modulo q. Put $t_0 = \bar{\omega} + U_0 z_0$. Then

$$t_0 + \bar{t}_0 = 1 + 2U_0 z_0, \quad t_0\bar{t}_0 = U_0 q e_0, \quad (\omega + U_0 z_0)t_0 = U_0 q e_0. \tag{4}$$

For each rational prime $\ell \mid q$, write $q = \ell^{a_\ell+1}Q_\ell$, with $\ell \nmid Q_\ell$, and put $r_\ell = q/\ell$, $a_\ell^* = \zeta^{r_\ell}$. There is $V_\ell \in \mathcal{O}_E$, dividing Q_ℓ, such that

$$D_{\mathrm{rel}}(a_\ell^* - 1)Q_\ell = qV_\ell. \tag{5}$$

Here is an elementary verification. The cyclotomic identity $\mathrm{cyc}_{Q_\ell}(T^{\ell^{a_\ell+1}}) = \mathrm{cyc}_q(T)\mathrm{cyc}_{Q_\ell}(T^{\ell^{a_\ell}})$, differentiated at ζ, shows that $\ell^{a_\ell+1}$ divides $D_{\mathrm{rel}}\mathrm{cyc}_{Q_\ell}(\zeta^{\ell^{a_\ell}})$. Since $\mathrm{cyc}_{Q_\ell}(T) \mid T^{Q_\ell} - 1$, this latter cyclotomic value divides $a_\ell^* - 1$. Thus q divides the left side of (5). On the other hand $T^q - 1 = (T^{r_\ell} - 1)\mathrm{cyc}_q(T)R(T)$ for an integral polynomial R. Differentiating and multiplying by ζ gives $q = D_{\mathrm{rel}}(a_\ell^* - 1)R(\zeta)\zeta$. Together with (5), cancellation of q gives $Q_\ell = V_\ell R(\zeta)\zeta$, proving the divisibility assertion.

Put $\gamma_\ell = t_0 Q_\ell/q$, and form the finite family

$$v_{\ell,h,k,i} = \zeta^h \left(\mathbf{1}_{k=0}(a_\ell^*)^i + \gamma_\ell((a_\ell^*)^i - (a_\ell^*)^k)\right), \quad 0 \le h < r_\ell, \quad 0 \le k, i < \ell. \tag{6}$$

Every vector in (6) lies in $\mathfrak{a}$. If $\omega \in w$, (5) shows that $D_{\mathrm{rel}}\gamma_\ell(a_\ell^* - 1) = t_0 V_\ell$ is integral, so the required $D_{\mathrm{rel}}v$ is integral at w. If $\omega \notin w$, and $w \nmid q$, integrality is immediate. If $w \mid q$ while ω is a unit, then $\bar\omega \in w$, because every prime divisor of q divides $U_0 = \omega\bar\omega$. Hence $\omega + U_0 z_0$ is a unit at w, and the last equation in (4) shows that t_0/q is integral there. Thus γ_ℓ, and hence v, is integral at w. This verifies membership at every finite place.

The Gram sum of the ℓ-family is $C_\ell \mathbf{1}_{\tau=\sigma}$, where

$$C_\ell = q + Q_\ell(1 + 2U_0(z_0 + \ell Q_\ell e_0)). \tag{7}$$

For clarity, expand the two bracketed factors in (6). The sums of the powers of either primitive ℓ-th root are zero. If $G = \sum_{i<\ell} \tau(a_\ell^*)^i\overline{\sigma(a_\ell^*)^i}$, the sum over k, i is $G(1 + \gamma_\ell + \bar\gamma_\ell + 2\ell\gamma_\ell\bar\gamma_\ell)$. Multiplication by the sum over $h < r_\ell$ turns G into the complete root-of-unity sum $q\mathbf{1}_{\tau=\sigma}$. Substituting (4) in the diagonal expression gives exactly (7).

Let $C = \sum_{\ell\mid q} C_\ell$. This positive integer is prime to q. Modulo a prime $\ell_0 \mid q$, every summand with $\ell \ne \ell_0$ vanishes, since Q_ℓ is divisible by ℓ_0. The remaining summand is $Q_{\ell_0} \not\equiv 0$, since q and U_0 vanish modulo ℓ_0.

We retain a crude explicit bound. From (3), $e_0 \le 2U_0 q$, while $z_0 < q$, $Q_\ell \le q$, and $\ell Q_\ell \le q$. Equation (7) gives $C_\ell \le 2q + 2U_0 q^2 + 4U_0^2 q^3 \le 8U_0^2 q^3$. There are at most q prime factors, so in particular

$$C \le 8U_0^2 q^5. \tag{8}$$

All this data was chosen without any prime identification.

3. The common unit row. The powers ζ^i, $0 \le i < q$, form a frame of length q, by root-of-unity orthogonality. Their union with (6) contains a vector that has a unit normalized coefficient at all CM embeddings simultaneously, for every prime identification. If $|\iota\beta|_p = 1$, use $1 = \zeta^0$.

Otherwise (2) and the definition of α imply that $p \mid q$, and that ω belongs to every prime induced by ϕ_τ. Use the one vector

$$v_{p,0,1,0} = \gamma_p(1 - a_p^*).$$

Equation (5) says $D_{\mathrm{rel}}v_{p,0,1,0} = -t_0 V_p$. At each of those primes, $t_0 \equiv 1$: ω and U_0 vanish there. Also V_p is a unit there, since it divides Q_p, which is prime to p. Formula (2) therefore gives

$$|\iota(\beta\overline{\tau(v_{p,0,1,0})})|_p = |\phi_\tau(t_0 V_p)|_p = 1 \quad \text{for every } \tau. \tag{9}$$

The selected index depends on p, ι, but the finite family does not.

4. The exact length M and large multiplicities. Put $R = 6fb$, an integer. The admissible-data inequalities give $d \le b$, $q \le b$, $f \le q \le b$, and $b \ge 7$. Indeed $q \mid b$; $f = \varphi(q) \le q$; and $\log\log b \ge 10^{12} C_P \ge 2d$ implies $b \ge e^d$. Thus

$$(dq)^6(R+1) \le 7b^{14} \le b^{20} \le M. \tag{10}$$

As $U_0 \le d$ and $d \ge 7$, $9U_0^2 \le d^6$. Using (8),

$$R(C+q) + qC \le 9U_0^2 q^6 (R+1) \le (dq)^6(R+1) \le M. \tag{11}$$

Because $\gcd(q, C) = 1$, every integer at least $(q-1)(C-1)$ is a nonnegative combination of q and C. One short proof is to choose $B \in \{0, \dots, q-1\}$ with BC congruent to the given integer modulo q; its difference from BC is a multiple of q greater than $-q$, hence nonnegative. Apply this to $M - R(C+q)$, which is at least qC by (11). Obtain $A, B \ge 0$ with $Aq + BC = M - R(C+q)$. Give each root-of-unity vector multiplicity $A + R$, and every vector of (6) multiplicity $B + R$. Their full two-index Gram sum is

$$((A+R)q + (B+R)C)\mathbf{1}_{\tau=\sigma} = M\mathbf{1}_{\tau=\sigma}.$$

Every multiplicity is at least R, and the unit row of Section 3 is retained with positive multiplicity. This is the required single fixed family.

5. Nonsplit primes and the resolution at scale $\bar\pi_M$. Every rational prime dividing q splits in K. For odd primes, $-d \equiv 1 \pmod p$, so the defining quadratic has two distinct roots. At 2, the polynomial of ω reduces to $T(T-1)$, again with distinct roots. Thus a nonsplit prime does not divide q. Since $D_{\mathrm{rel}} \mid q$ (differentiate $T^q - 1 = \mathrm{cyc}_q(T)R(T)$ at ζ), the idele α is a unit at all its primes. Equation (2) proves that β is a unit, and membership in $\mathfrak{a}$ proves that all w_i are integral there.

The primes p_0, p_1 are also split in K: they are congruent to one modulo d, so $\chi_K(p_i) = 1$; neither equals d. Therefore π_M and $\bar\pi_M$ are units at every nonsplit prime. Dividing the Gram identity by $\tau(\bar\pi_M)\overline{\sigma(\bar\pi_M)} = \bar\pi_M \pi_M = M$ gives the claimed resolution. Finally $\pi_M \beta\, \overline{\tau(w_i/\bar\pi_M)} = \beta\overline{\tau(w_i)}$, which proves the exact linear-form identity and the remaining unit assertion. □

Lemma 6.7.36. *Write $\theta_x(F) = \theta(\rho(\mathrm{fv}(x))F)$. Let F belong to the finite part and satisfy*

$$\rho_f(\alpha^{-1}\pi_M \lambda_f)F = F \qquad (\lambda \in \mathfrak{a}^{12}).$$

Fix a prime embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, and suppose $|\iota\theta_x(F)|_p \le 1$ for every finite x. Then for every complex-linear functional $\mathscr{L}$ on the finite part and every finite x, there exists $a_0 \in (\mathcal{O}_E/\mathfrak{m})^{12}$ with

$$\Big|\iota\Big(M^{12f}\mathscr{L}(F)\theta_x(v^0)\Big)\Big|_p \le |\iota\mathscr{L}(F_{\mathfrak{m},a_0})|_p\,.$$

No restriction on $p \mid M$ is imposed.

Proof. **The exact lattice and finite pairing.** Put $A_1 = \sqrt{2\pi/\sqrt{d}}$ and $\Gamma = \{A_1(\tau\lambda_c)_{\tau,c} : \lambda \in \mathfrak{a}^{12}\}$. This lattice is self-dual for $E_1(z,w) = \pi^{-1}\operatorname{Im}\sum z_{\tau,c}\bar w_{\tau,c}$. Indeed (WD2) gives $\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E/K}^{-1}$. The different tower formula gives $\jmath\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{D}_{E/\mathbb{Q}}^{-1}$. The trace-dual ideal of $\mathfrak{a}$, for the alternating trace form $\mathrm{Tr}_{E/\mathbb{Q}}(\jmath\lambda\bar\mu)$, is therefore $\jmath^{-1}\mathfrak{D}_{E/\mathbb{Q}}^{-1}\bar{\mathfrak{a}}^{-1} = \mathfrak{a}$. This trace form is $-E_1$ in the coordinates above, which has the same integral dual. The different and annihilator normalizations are also checked explicitly in Lemma 4.1.10 and Lemma 4.1.3.

Write

$$G = \Gamma/\pi_M\Gamma, \qquad Y = \bar{\pi}_M^{-1}\Gamma/\Gamma, \qquad D_1 = M^{12f}.$$

Both finite groups have cardinal D_1: multiplication by π_M on the underlying rational space has determinant of absolute value $\mathrm{Norm}_{E/\mathbb{Q}}(\pi_M)^{12} = M^{12f}$. The pairing

$$\chi_y(\lambda) = \exp\left(2i\,\mathrm{Im}\sum_{\tau,c}\lambda_{\tau,c}\bar{y}_{\tau,c}\right) \tag{1}$$

is well-defined on $G \times Y$ and perfect. Well-definedness uses the adjoint of multiplication by π_M, which is multiplication by $\bar{\pi}_M$. If one argument pairs trivially with every other, self-duality puts it in the indicated denominator lattice; cardinal equality then proves perfection. Choose rational representatives for both quotients, identifying their coordinates with vectors in $\mathfrak{a}^{12}$ and $\bar{\pi}_M^{-1}\mathfrak{a}^{12}$, respectively.

Averaging with no denominator. For a rational vector u, let $T_u = \rho(0, u_f)$, and for $\lambda \in \mathfrak{a}^{12}$ let $S_\lambda = \rho_f(\alpha^{-1}\lambda_f)$. The signed operators S_λ are a representation of the additive integral lattice: expand (WD8), use $c_0 + \bar{c}_0 = 1$, and use the integral annihilator for $\psi_{E,f}(\jmath\alpha\bar{\alpha}\,\cdot)$. They fix the vacuum. The exact raw translation formula of Lemma 4.2.2 identifies their signed real lattice action. Thus the fixed space of all S_λ in the finite part has dimension one: its entire image belongs to the principal theta line for Γ, of dimension one by Theorem A.2.16, with Hermitian form $\pi^{-1}\sum z\bar{w}$. The entire map is injective and the vacuum is a nonzero vector in that line, by Lemma 4.2.2 and Lemma 4.1.10. Hence that fixed space is exactly $\mathbb{C}v^0$.

For $y \in Y$, the vector $T_{-y}F$ still has the specified $\pi_M\Gamma$ signed periods: its commutator with each such period is one by (1). Consequently the finite average

$$\mathcal{A}T_{-y}F := \sum_{\lambda\in G} S_\lambda T_{-y}F = b_y v^0 \tag{2}$$

is defined independently of the representatives and is fixed by every S_λ. No division by $|G|$ occurs. The raw projective law and rational invariance, with their signs as in Lemma 4.2.2, give

$$T_y S_\lambda T_{-y} = \chi_y(\lambda) S_\lambda.$$

Character orthogonality on the perfect pairing (1) now gives the exact vector identity

$$\sum_{y\in Y} b_y T_y v^0 = \sum_{y\in Y} T_y \mathcal{A} T_{-y} F = D_1 F. \tag{3}$$

Indeed the coefficient of $S_\lambda F$ is zero for nonzero $\lambda \in G$ and D_1 at zero.

Applying a finite translate and the theta functional to (2) gives

$$|\iota(b_y\theta_x(v^0))|_p \le 1. \tag{4}$$

Every summand is a finite theta value of F times a root of unity: this follows by composing the raw and signed operators (WD7)–(WD8). The assumed integrality and the ultrametric triangle inequality prove (4), also when $p \mid M$.

Each representative y has $\bar{\pi}_M y \in \mathfrak{a}^{12}$. Thus Lemma 6.7.37 gives $T_y v^0 = \sum_{a_0} \xi_{y,a_0} F_{\mathfrak{m},a_0}$ with roots of unity ξ_{y,a_0}. Apply $\mathscr{L}$ to (3), multiply by $\theta_x(v^0)$, and substitute these expansions:

$$D_1 \mathscr{L}(F)\theta_x(v^0) = \sum_{y,a_0} \xi_{y,a_0}(b_y\theta_x(v^0))\mathscr{L}(F_{\mathfrak{m},a_0}).$$

All coefficients on the right have prime-adic norm at most one by (4). The norm of this finite sum is at most the largest norm of an indicator value. Choose a residue class attaining that finite maximum. This is the required inequality. □

Lemma 6.7.37. *Put $\mathfrak{a} = E \cap \alpha\widehat{\mathcal{O}}_E$, $\mathfrak{m} = (\pi_M)$ and $A = \Sigma(\mathfrak{m})$. For $a_0 \in (\mathcal{O}_E/\mathfrak{m})^{12}$, let Ψ_{a_0} be its class indicator in the integral position coordinates at A, and put*

$$F_{\mathfrak{m},a_0} = v_1 \otimes (\Psi_{a_0} \otimes_A v^0).$$

For every $u \in E^{12}$ satisfying $\bar{\pi}_M u \in \mathfrak{a}^{12}$, there are roots of unity $\xi_{a_0} \in \mathbb{C}$ such that, as vectors of the prescribed model,

$$\rho(0, u_f)v^0 = \sum_{a_0 \in (\mathcal{O}_E/\mathfrak{m})^{12}} \xi_{a_0} F_{\mathfrak{m},a_0}.$$

Consequently these coefficients have norm one under every prime embedding.

Proof. Set $y = \alpha^{-1}u_f$. The hypothesis is precisely $\bar{\pi}_M y \in \widehat{\mathcal{O}}_E^{12}$. At a place in A, $\bar{\pi}_M$ is a unit, so y_A is integral. Outside $A \cup \bar{A}$, both π_M and $\bar{\pi}_M$ are units, so y is integral. At the places of A, the conjugate vector satisfies

$$\pi_M \bar{y} \in \mathcal{O}_A^{12}. \tag{1}$$

Here the selected places and their conjugates are disjoint because the principal level primes are split and (π_M) selects exactly one member of each pair.

First remove the coordinates outside $A \cup \bar{A}$. A raw integral finite translation acts on their vacuum by a root of unity. Indeed its vacuum coefficient has modulus one and is a finite additive-character value, by (WD5); the evaluation identity and invariant norm in Lemma 4.1.10(1), or the equality case of Cauchy–Schwarz, show that the translated unit vacuum is a scalar multiple of itself. That scalar is the conjugate of its vacuum coefficient and hence a root of unity. The vacuum and Heisenberg operators factor over the selected and remaining coordinates. Disjoint finite supports have trivial cross bicharacter. Thus it remains to expand the selected translate, with one overall root of unity retained.

Use $\psi_0(t) = \psi_{E,f}(\jmath\alpha\bar{\alpha}t)$ and the exact raw operator of Lemma 4.1.3(2). Its action on the position box $B = \mathbf{1}_{\mathcal{O}_A^{12}}$ is

$$(R_A(y)B)(x) = \psi_0\Big(\bar{y}^{\mathsf{T}}(2x + y)_A/2\Big)\, B(x + y_A) = \psi_0\Big(\bar{y}^{\mathsf{T}} y_A/2\Big)\, \psi_0(\bar{y}^{\mathsf{T}} x_A) B(x). \tag{2}$$

The second equality uses the already proved integrality of y_A. The arithmetic normalization in Lemma 4.1.3 says that the integral lattice is its own annihilator for $\psi_0(\bar{z}^{\mathsf{T}}x)$. Equation (1) therefore implies that the last character in (2) is trivial on $\pi_M\mathcal{O}_A^{12}$. It is constant on every class modulo $\mathfrak{m}$.

Choose any integral representative z_0 of a_0. The scalar

$$\xi'_{a_0} = \psi_0\Big(\bar{y}^{\mathsf{T}} y_A/2 + \bar{y}^{\mathsf{T}}(z_0)_A\Big)$$

is independent of the representative by (1), and is a root of unity because a finite adelic additive character has finite order at each point. Thus the equality of position functions is exactly

$$R_A(y)B = \sum_{a_0} \xi'_{a_0} \Psi_{a_0}. \tag{3}$$

This is an expansion of a character on residue classes, not an average over characters, and has no denominator.

By Lemma 4.1.3(2)–(4), the fixed map $\Psi \mapsto \Psi \otimes_A v^0$ takes (3) to the equality of the actual selected coherent vectors. Its image of B is the same vacuum, and its raw intertwining has exactly the phase used in (2). Inserting v_1 leaves the positive real vacuum fixed, again by clause (4), and leaves all finite factors unchanged. Restoring the overall root of unity from the remaining finite coordinates gives the asserted coefficients ξ_{a_0}. Every root of unity has prime-adic norm one under every embedding, which proves the last assertion. □

7 Regular twist families: congruences and local loss

This section proves that every datum has a regular family of twists (Theorem 7.3.1), the arithmetic input of Theorem C. A regular family (Definition 6.1.3) consists of one m-fold twist θ_ι for each prime identification $\iota \mid \ell$, with modulus prime to ℓ, such that the inverse ratios of Hecke L-values at the ramified critical pairs form a family of height at most x_1 and the denominators of the projections to the spaces of lifts are bounded by numbers whose sum over all primes is at most β_{den}.

The twist is chosen one prime at a time, and nothing ties the choices at different primes together. The price of the choice at ℓ consists of multiples of the ℓ-adic valuations of a few integers fixed by the datum and a local loss at the places above ℓ. Weighted with $\log \ell$, the valuations of a fixed integer sum over all primes to at most its logarithm, and this is why the prices add up to an amount that the denominator budget absorbs (Proposition 7.3.3). At one prime three inputs are needed: a twist that makes the Selmer groups of finitely many Galois representations small and the ratios of L-values ℓ-adically close to units, for all its powers τ^j, $1 \le j \le m$, at once, which is the twisting theorem (Theorem 7.4.3); the Galois representations to which it is applied, the main summands attached to the Hecke eigensystems of the good forms (Proposition 7.5.1); and a passage from Selmer groups to denominators. A denominator at ℓ measures congruences between lifts and the forms orthogonal to them, and such congruences produce Galois cohomology classes, after Ribet and Mazur–Wiles [241, 197, 327] and, for unitary groups, Bellaïche–Chenevier [20, 21, 19]. Here the classes are controlled through a common Ribet element, whose leading coefficient is bounded by a Selmer length, a ramification term, the local loss above ℓ and the odd classes of the Tate twists of a CM field (Theorem 7.6.6).

Section 7.1 comes first: it fixes the conventions of the section on class field theory, p-adic Hodge theory, Galois cohomology and ℓ-adic coefficients, and derives from published theorems what is used about the Galois representations attached to eigenforms on definite unitary groups. Section 7.2 sets up the local and global vocabulary and proves the bound for the odd classes (Theorem 7.2.8). Section 7.3 proves the existence of regular families from a twist with few congruences at one prime (Proposition 7.3.2) and the sum of the prices. The three inputs at one prime follow: the twisting theorem (Section 7.4), the main summands (Section 7.5) and the denominators, with Ribet's element and prime triangles of auxiliary places (Section 7.6). Section 7.7 bounds the local loss above ℓ. From earlier sections it uses the ramification and the sizes of the datum, the integral theta jets of Section 4, the exchange identity and the norms of lifts of Section 6, and the equality of traces of Section 5.

7.1 Galois representations of eigenforms on definite unitary groups

The section opens with four conventions used throughout Section 7 and in the Galois-theoretic quotations of the Appendix: class field theory, normalised so that uniformisers correspond to arithmetic Frobenius elements [9, 224, 260, 297]; p-adic Hodge theory, with Fontaine's period rings and Hodge–Tate weights normalised so that the cyclotomic character has weight 1 [85, 86, 44]; continuous Galois cohomology, with local and Poitou–Tate duality and the Greenberg–Wiles formula [265, 295, 201, 225, 63]; and ℓ-adic coefficients, automorphic representations and Satake parameters, in the normalisations of [58, 49, 109, 122, 261, 60, 15, 68].

The congruences of Section 7 are congruences between Hecke eigensystems of algebraic modular forms on definite unitary groups [110], and they become Galois cohomology through the Galois representations attached to these eigensystems. The rest of the section derives what is used about these representations, with explicit normalisations, from published theorems quoted in the Appendix: base change to GL_n [177, 185], the Galois representations of polarised regular

algebraic cuspidal representations [15, 276, 56], and the local Langlands correspondence with supercuspidal supports [122, 130, 254, 318]. For the construction of these representations see also [58], for the endoscopic classification [209, 156], for local–global compatibility [302, 47, 48, 16], for complex conjugation [300], for automorphy lifting [305] and for families and Selmer groups [21]. Weil–Deligne representations and conductors rest on Grothendieck's monodromy theorem and Serre's Artin conductor [267, 263].

The main statement, Theorem 7.1.4, attaches to an automorphic representation of a definite unitary group a conjugate self-dual Galois representation with the Hecke–Frobenius relation at the unramified places. Its local–global compatibility is given at the split places not above ℓ (Theorem 7.1.8), at the places above ℓ (Theorem 7.1.9) and for depth-zero parahoric types (Theorem 7.1.11). The proofs pass through the discrete spectrum of GL_n [207, 292, 149, 269, 230], through Zelevinsky's segments [332, 24], through the Weil–Deligne description of crystalline representations [85, 86, 22] (for modular forms the p-adic Hodge theory at ℓ goes back to [76, 247]), and, for depth-zero types, through twisted spherical blocks (Proposition 7.1.12), proved from Roche's equivalence and the spherical Hecke algebra [243, 248, 50, 23, 80]; for types of depth zero see [217, 218, 46].

The section closes with three further inputs of Section 7: a Higman operator that exhibits algebraic modular forms as direct summands of induced modules over a group ring (Proposition 7.1.14; see [62]), the form of freeness behind the method of Taylor–Wiles [301, 60]; the ℓ-adic avatars of algebraic Hecke characters (Theorem 7.1.15), with the Bloch–Kato Selmer groups (Definition 7.1.16; [28]); and the multiplicities of depth-zero types of compact type, specialised from Shotton [278] (Theorem 7.1.18).

Class field theory is normalised as in [224] and [9]: uniformisers correspond to arithmetic Frobenius elements. For a non-archimedean local field F_v, $\mathrm{Art}_{F_v}\colon F_v^\times \to G_{F_v}^{ab}$ is the local Artin map and F_v^{ur} the maximal unramified extension; for a number field F, $\mathrm{Art}_F\colon C_F \to G_F^{ab}$ is the global Artin map and F^{ab} the maximal abelian extension. The **norm group** of a finite abelian extension E/F (respectively E_w/F_v) is $\mathrm{N}_{E/F}C_E \subseteq C_F$ (respectively $\mathrm{N}_{E_w/F_v}E_w^\times \subseteq F_v^\times$).

Notation for p-adic Hodge theory is that of [85] and [44]. Let K be a finite extension of $\mathbb{Q}_p$, G_K its absolute Galois group and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$. For a perfectoid field L, $L^\flat$ denotes its tilt. Fix a compatible system $\underline{\zeta} = (\zeta_{p^n})_{n\geq 0}$ of p-power roots of unity. The period rings are the following:

- $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p^\flat})$, with $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$;
- B_{dR}^+, the $\ker\theta$-adic completion of $A_{\mathrm{inf}}[1/p]$, and $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$, where $t = \log[\underline{\zeta}]$, with $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$;
- $A_{\mathrm{cris}} \subset B_{\mathrm{cris}}^+$ and $B_{\mathrm{cris}} = B_{\mathrm{cris}}^+[1/t]$, with Frobenius φ;
- B_{st};
- A_{max}, B_{max}^+, B_{max} (Colmez): with $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{\mathbb{C}_p^\flat}$ given by $\underline{\zeta}$, $\mu = [\varepsilon] - 1$ and $\xi = \sum_{i=0}^{p-1}[\varepsilon^{1/p}]^i \in A_{\mathrm{inf}}$ (a generator of $\ker\theta$, with $\xi\,([\varepsilon^{1/p}] - 1) = \mu$), A_{max} is the p-adic completion of $A_{\mathrm{inf}}[X]/(pX - \xi)$, $B_{\mathrm{max}}^+ = A_{\mathrm{max}}[1/p]$ and $B_{\mathrm{max}} = B_{\mathrm{max}}^+[1/\mu]$, with the Frobenius φ and the action of G_K.

We use, as part of this convention, the standard comparisons $B_{\mathrm{cris}}^+ \subseteq B_{\mathrm{max}}^+ \subseteq B_{\mathrm{dR}}^+$, $\varphi(B_{\mathrm{max}}^+) \subseteq B_{\mathrm{cris}}^+$, $B_{\mathrm{cris}} \subseteq B_{\mathrm{max}} \subseteq B_{\mathrm{dR}}$, and that V is crystalline if and only if $(B_{\mathrm{max}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ has dimension $\dim_{\mathbb{Q}_p} V$ over $B_{\mathrm{max}}^{G_K}$ [85]. A finite-dimensional $\mathbb{Q}_p$-representation V of G_K is **de Rham**, **semistable** or **crystalline** if $(B \otimes_{\mathbb{Q}_p} V)^{G_K}$ has dimension $\dim_{\mathbb{Q}_p} V$ over B^{G_K} for

$B = B_{\mathrm{dR}}, B_{\mathrm{st}}, B_{\mathrm{cris}}$ respectively. Hodge–Tate weights are normalised so that the cyclotomic character χ_{cyc} has Hodge–Tate weight 1.

Galois cohomology is continuous cohomology of profinite groups, as in [265] and [201]. For a discrete module M (or, for $\mathbb{Z}_\ell$-coefficients, a continuous ℓ-adic module M), we write $H^i(F, M)$ for $H^i(G_F, M)$ and $H^i(F_v, M)$ for $H^i(G_{F_v}, M)$. Tate cohomology of a finite group is denoted $\hat{H}^i$. For a prime ℓ, χ_{cyc} is the ℓ-adic cyclotomic character, $\mathbb{Z}_\ell(k)$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(k)$ are the twists by χ_{cyc}^k, and μ_{ℓ^r} is the group of ℓ^r-th roots of unity. For a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module M, the Cartier dual is $M^\vee = \mathrm{Hom}(M, \mu_{\ell^r})$. The unramified classes are

$$H^1_{\mathrm{ur}}(F_v, M) = \ker\bigl(H^1(F_v, M) \to H^1(I_v, M)\bigr),$$

and inv_v is the invariant map of the local Brauer group. For a finite set S of places containing those above ℓ and ∞,

$$\mathrm{Sha}^i_S(F, M) = \ker\Bigl(H^i(G_{F,S}, M) \to \prod_{v \in S} H^i(F_v, M)\Bigr).$$

For a number field E, a prime ℓ, the set S_ℓ of places of E above ℓ, the Galois group G_{E,S_ℓ} of the maximal extension of E unramified outside S_ℓ, and $k \in \mathbb{Z}$, put

$$\mathrm{Sha}^1_\ell(E, k) = \ker\Bigl(H^1(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)) \to \prod_{v \in S_\ell} H^1(E_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))\Bigr),$$

with continuous cohomology, and let $\mathrm{len}_\ell(E, k) \in \mathbb{N} \cup \{\infty\}$ be its length as a $\mathbb{Z}_\ell$-module.

Local Tate duality, the local and global Euler characteristic formulas and Poitou–Tate duality are used in the form of [201], Ch. I, and the Greenberg–Wiles formula for Selmer groups in the form of [63].

We fix the notation for ℓ-adic coefficients, Galois representations and automorphic representations used in this paper, following [60] and [15].

Coefficients. Let ℓ be a prime, let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$ with absolute value $|\cdot|_\ell$ and valuation v_ℓ normalised by $v_\ell(\ell) = 1$, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism of fields. A **coefficient ring** is a $\mathbb{Z}_\ell$-subalgebra $\mathcal{O} \subset \overline{\mathbb{Q}}_\ell$ that is a discrete valuation ring finite over $\mathbb{Z}_\ell$. Its ramification index is $e_{\mathcal{O}} = \mathrm{length}_{\mathcal{O}}(\mathcal{O}/\ell\mathcal{O})$. For an $\mathcal{O}$-module M of finite length, the **normalised length** of M is $\mathrm{length}_{\mathcal{O}}(M)/e_{\mathcal{O}}$, so that $\ell^{[\mathcal{O}:\mathbb{Z}_\ell]\cdot \mathrm{length}_{\mathcal{O}}(M)/e_{\mathcal{O}}} = \#M$. For a character θ of a group with values in $\mathcal{O}^\times$, we write $\mathcal{O}^\vee(\theta)$ for $\mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ with the discrete topology and with the group acting through θ. Dirichlet characters with values in $\mathcal{O}^\times$ are identified with characters of $G_{\mathbb{Q}}$ by class field theory.

Galois representations. An ℓ-adic representation of G_E is a continuous homomorphism $r\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. We write $r^\vee$ for its contragredient. For a CM field E with complex conjugation c, we set $r^c(g) = r(cgc)$. Two representations are **isomorphic** if they are conjugate in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

Automorphic side. For a number field E, the notion of a **regular algebraic** cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is that of [58]. For E CM, π is **conjugate self-dual** if $\pi^\vee \cong \pi^c$. An ℓ-adic representation r of G_E is **attached to** π, written $r \leftrightarrow_\iota \pi$, if the following holds for all but finitely many finite places w of E at which π is unramified: r is unramified at w, and ι carries the characteristic polynomial of $r(\mathrm{Frob}_w^{-1})$ (a geometric Frobenius) to $\prod_\alpha (X - \mathrm{N}w^{(n-1)/2}\alpha)$, where α runs over the Satake parameters of π_w.

The **Satake parameters** of π_w are defined as follows. Let $K_w = \mathrm{GL}_n(\mathcal{O}_{E_w})$. For a uniformiser ϖ_w and $0 \le j \le n$, let $d_j = \mathrm{diag}(\varpi_w, \dots, \varpi_w, 1, \dots, 1)$ with j entries ϖ_w, and let T_j be the Hecke operator of the double coset $K_w d_j K_w$, acting on $\pi_w^{K_w}$ by $x \mapsto \sum_{gK_w \subseteq K_w d_j K_w} \pi_w(g)x$.

The Satake parameters of π_w form the multiset $\{\alpha_1,\dots,\alpha_n\}\subset\mathbb{C}^\times$ such that, for every uniformiser ϖ_w and every such j, the operator T_j is the scalar

$$T_j = \mathrm{N}w^{j(n-j)/2}\, e_j(\alpha_1,\dots,\alpha_n),$$

where e_j is the j-th elementary symmetric polynomial. They are the values $\chi_i(\varpi_w)$ of unramified characters $\chi_1,\dots,\chi_n$ of $E_w^\times$ such that π_w is a subquotient of the normalised induction of $\chi_1\otimes\cdots\otimes\chi_n$ [49], [109]. Thus the relation $r\leftrightarrow_\iota\pi$ is the normalisation $\mathrm{rec}(\pi_w\otimes|\det|^{(1-n)/2})$ of [122]. For E CM, r is **automorphic of conjugate self-dual type** if $r\leftrightarrow_\iota\pi$ for some regular algebraic conjugate self-dual cuspidal π. A normalised newform f of weight k, level $\Gamma_1(M)$ and character χ has the Galois representation $\rho_{f,\iota}\colon G_{\mathbb{Q}}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ constructed in [68].

Algebraic Hecke characters. Let E be a number field and let $n=(n_\varphi)_\varphi$ be integers indexed by the field embeddings $\varphi\colon E\to\mathbb{C}$. A (not necessarily unitary) Hecke character χ of E is **algebraic of infinity type** n if

$$\chi_\infty(k)=\prod_{\varphi\colon E\to\mathbb{C}}\varphi(k)^{n_\varphi}\qquad\text{for every } k\in E^\times \text{ positive at every real place,}$$

where $\chi_\infty=\prod_{v\mid\infty}\chi_v$ and k is embedded diagonally in $\prod_{v\mid\infty}E_v^\times$. For these characters the global Artin map is used in the **geometric normalisation** $\mathrm{Art}_E^{\mathrm{geo}}(x)=\mathrm{Art}_E(x)^{-1}$, which sends uniformisers to geometric Frobenius elements. The **ℓ-adic character attached to** (χ,ι) is the continuous character $\chi_\iota\colon G_E\to\overline{\mathbb{Q}}_\ell^\times$ such that

$$\chi_\iota(\sigma)=\iota^{-1}\Big(\chi(x)\prod_{v\mid\infty}\prod_i i(x_v)^{-n_i}\Big)\cdot\prod_{w\mid\ell}\prod_j j(x_w)^{n_{\iota\circ j}}$$

for every $x\in\mathbb{A}_E^\times$ and every $\sigma\in G_E$ whose image in G_E^{ab} is $\mathrm{Art}_E^{\mathrm{geo}}(x)$. Here i runs over the continuous embeddings $E_v\to\mathbb{C}$, j runs over the continuous embeddings $E_w\to\overline{\mathbb{Q}}_\ell$, and n_i, $n_{\iota\circ j}$ are the exponents of their restrictions to E. This character exists and is unique [261].

Sign of the Hodge jumps. With these normalisations, let $w\nmid\ell$ be a finite place at which χ is unramified. Then χ_ι is unramified at w and $\chi_\iota(\mathrm{Frob}_w^{-1})=\iota^{-1}\chi_w(\varpi_w)$ for an arithmetic Frobenius Frob_w and a uniformiser ϖ_w. At a place $w\mid\ell$, the restriction $\chi_\iota|_{G_{E_w}}$ is de Rham and, for every embedding $\tau\colon E\to\overline{\mathbb{Q}}_\ell$ inducing w, its Hodge–Tate weight at τ is $n_{\iota\circ\tau}$ when χ_{cyc} has weight 1; equivalently, its only Hodge jump at τ (a jump of the Hodge filtration, the negative of a Hodge–Tate weight) is $-n_{\iota\circ\tau}$ [261], [85]. For $E=\mathbb{Q}$ and $\chi=|\cdot|_{\mathbb{A}}$, of infinity type 1, we have $\chi_\iota=\chi_{\mathrm{cyc}}$, with Hodge jump -1. The character defined by the same formula with Art_E in place of $\mathrm{Art}_E^{\mathrm{geo}}$ is χ_ι^{-1}, with Hodge jump $n_{\iota\circ\tau}$ at τ.

Definition 7.1.1. Number fields are subfields of a fixed algebraic closure $\overline{\mathbb{Q}}$; ℓ is a prime and $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ an isomorphism. For a number field F and a place $\bar v$ of $\overline{\mathbb{Q}}$ above a finite place v of F, $D_{\bar v}\supseteq I_{\bar v}$ are the decomposition and inertia subgroups of G_F at $\bar v$, and an **arithmetic Frobenius at** $\bar v$ is an element of $D_{\bar v}$ acting on the residue field of $\bar v$ as $x\mapsto x^{q_v}$. We write χ_{cyc} also for its restriction to G_F, with values in $\mathbb{Z}_\ell^\times\subset\overline{\mathbb{Q}}_\ell^\times$. For $M\geq1$, a character of G_F is **of level** M if it is trivial on $G_{F(\mu_M)}$. For a subgroup $D\subseteq G_F$ and characters $\kappa_1,\dots,\kappa_n\colon D\to\overline{\mathbb{Q}}_\ell^\times$, a representation $r\colon G_F\to\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is **triangular on** D **with diagonal** $(\kappa_1,\dots,\kappa_n)$ if there is $g\in\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that, for every $s\in D$, $g\,r(s)\,g^{-1}$ is upper triangular with diagonal entries $\kappa_1(s),\dots,\kappa_n(s)$.

Level and weights. Let $n,M\geq1$ and $m=(m_1,\dots,m_n)\in\mathbb{Z}^n$. A continuous $r\colon G_F\to\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ has **level** M **and weights** m if

(Lw1) for every finite place $v \nmid \ell$ of F there are a place $\bar{v} \mid v$ and characters $\chi_1, \dots, \chi_n$ of G_F of level M such that $r|_{I_{\bar{v}}}$ is conjugate to $s \mapsto \mathrm{diag}(\chi_1(s), \dots, \chi_n(s))$;

(Lw2) for every place $v \mid \ell$ of F there are a place $\bar{v} \mid v$, characters $\chi_1, \dots, \chi_n$ of G_F of level M and characters $\kappa_1, \dots, \kappa_n$ of $D_{\bar{v}}$ such that r is triangular on $D_{\bar{v}}$ with diagonal $(\kappa_1, \dots, \kappa_n)$, $\kappa_i(s) = \chi_{\mathrm{cyc}}(s)^{m_i}\chi_i(s)$ for all i and $s \in I_{\bar{v}}$, and either $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}}$ for all i, j, or $r|_{D_{\bar{v}}}$ is potentially crystalline.

It is **generic ordinary of level** M **and weights** m if for every place $v \mid \ell$ of F there are $\bar{v}$, χ_i and κ_i as in (Lw2) with $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}}$ for all i, j.

Witness conditions. Let F be a CM field and $r\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ a representation. For a place $v \mid \ell$ of F, r is **generic at** v if for every place $\bar{v} \mid v$ it is triangular on $D_{\bar{v}}$ with some diagonal (κ_i) such that $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}}$ for all i, j; it is **potentially generic at** v if for every $\bar{v} \mid v$ it is triangular on $D_{\bar{v}}$ with some diagonal (κ_i) such that $\kappa_i^a \neq (\kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}})^a$ for all i, j and all integers $a \geq 1$. The representation r satisfies the **witness conditions** if

(Wc1) r is potentially generic at every place $v \mid \ell$ at which it is generic;

(Wc2) for every regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that r is semisimple and $r \leftrightarrow_\iota \pi$, there is a real number τ_π such that at every finite place v of F with $v \nmid \ell$, or with $v \mid \ell$ and r potentially generic at v, the local component π_v has a cuspidal support consisting of characters of $F_v^\times$, and every character ξ occurring in a cuspidal support of π_v satisfies $q_v^{\tau_\pi - 1/2} < |\xi(\varpi_v)| < q_v^{\tau_\pi + 1/2}$.

(The number $|\xi(\varpi_v)|$ does not depend on the uniformiser, since ξ has finite image on $\mathcal{O}_{F_v}^\times$.) A representation W of $G_\mathbb{Q}$ satisfies the **witness conditions over imaginary quadratic fields** if $W|_{G_{K'}}$ satisfies the witness conditions for every imaginary quadratic field K' such that $W|_{G_{K'}}$ is irreducible.

Admissible representations. Let K be an imaginary quadratic field with discriminant d_K and let $e, N \geq 1$ be integers. A representation $\rho\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ is **admissible of stable type** for (ι, e, N) if

(Sb1) ρ is irreducible;

(Sb2) $\rho \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type and satisfies the witness conditions;

(Sb3) ρ has level N and weights $(e, 0, -e)$;

(Sb4) ρ is generic ordinary of level N and weights $(e, 0, -e)$.

For $M \geq 1$, ρ is **admissible of endoscopic type** for (ι, e, M) if there is a representation $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that

(Eo1) W is irreducible and satisfies the witness conditions over imaginary quadratic fields;

(Eo2) $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is semisimple, and there is a normalised newform $f = \sum_{n \geq 1} a_n q^n$ of weight $2e + 1$, some level and Nebentypus ε_f such that for all but finitely many primes p and every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p, $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is trivial on $I_{\bar{p}}$ and $\iota(\det(X - (W \otimes \chi_{\mathrm{cyc}}^{-e})(\sigma^{-1}))) = X^2 - a_p X + \varepsilon_f(p)p^{2e}$ for every arithmetic Frobenius σ at $\bar{p}$;

(Eo3) W has level M and weights $(e, -e)$, and $\det W$ is a character of level M' for some $M' \geq 1$;

(Eo4) ρ is conjugate to $s \mapsto \begin{pmatrix} a(s) & 0 & b(s) \\ 0 & 1 & 0 \\ c(s) & 0 & d(s) \end{pmatrix}$, where $W(s) = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}$ for $s \in G_K \subseteq G_{\mathbb{Q}}$.

A representation $\rho\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ is **admissible of weight** e **and level** N (for K and ι) if it is admissible of stable type for (ι, e, N) or admissible of endoscopic type for $(\iota, e, N|d_K|)$.

Definition 7.1.2. Let v and ℓ be primes and $I_v \subset G_{\mathbb{Q}_v}$ the inertia subgroup. For w in the Weil group $W_{\mathbb{Q}_v} \subseteq G_{\mathbb{Q}_v}$ let $n(w) \in \mathbb{Z}$ be the integer with $w(x) = x^{v^{n(w)}}$ on the residue field of $\overline{\mathbb{Q}}_v$.

A **Weil–Deligne representation** of $W_{\mathbb{Q}_v}$ over $\overline{\mathbb{Q}}_\ell$ is a triple $D = (V, \rho, N)$ of a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space V, a homomorphism $\rho\colon W_{\mathbb{Q}_v} \to \mathrm{GL}(V)$ trivial on an open subgroup of I_v, and a nilpotent endomorphism N of V (the **monodromy**) with

$$\rho(w)\, N\, \rho(w)^{-1} = v^{n(w)} N \qquad (w \in W_{\mathbb{Q}_v}). \tag{7.1}$$

Its **conductor exponent** is

$$\mathrm{a}(D) = \mathrm{a}\big(\rho|_{I_v}\big) + \dim V^{I_v} - \dim\big(\ker N\big)^{I_v},$$

where, for a finite-dimensional representation θ of I_v trivial on an open subgroup, $\mathrm{a}(\theta)$ is its Artin conductor ([263], Chapter VI, §2: the sum of the codimension of the invariants and the Swan conductor, computed on any finite quotient of I_v through which θ factors).

For a continuous representation $r\colon G_{\mathbb{Q}_v} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, the **Weil–Deligne representation** $\mathrm{WD}(r)$ is defined when $v \neq \ell$, or when $v = \ell$ and r is potentially semistable:

(Wd1) if $v \neq \ell$: by Grothendieck's monodromy theorem there are a unique nilpotent N and an open subgroup $J \subseteq I_v$ with $r(\sigma) = \exp(t_\ell(\sigma)N)$ for $\sigma \in J$, where $t_\ell\colon I_v \to \mathbb{Z}_\ell$ is a nonzero homomorphism (a choice of tame character); $\mathrm{WD}(r) = (\overline{\mathbb{Q}}_\ell^{\,n}, \rho, N)$ with $\rho(\Phi^m\sigma) = r(\Phi^m\sigma)\exp(-t_\ell(\sigma)N)$ for a fixed $\Phi \in W_{\mathbb{Q}_v}$ with $n(\Phi) = -1$, $m \in \mathbb{Z}$ and $\sigma \in I_v$;

(Wd2) if $v = \ell$ and r is potentially semistable: $\mathrm{WD}(r)$ is the Weil–Deligne representation attached by Fontaine to the filtered $(\varphi, N, G_{\mathbb{Q}_\ell})$-module $D_{\mathrm{pst}}(r)$ ([86]).

Both are well defined up to isomorphism (independent of the choices), and the **conductor exponent of** r is $\mathrm{a}(r) = \mathrm{a}(\mathrm{WD}(r))$.

Frobenius-semisimplification. Let $D = (V, \rho, N)$ be a Weil–Deligne representation and $\Phi \in W_{\mathbb{Q}_v}$ an element with $n(\Phi) = -1$ (a lift of geometric Frobenius). Write $\rho(\Phi) = \rho(\Phi)_s\rho(\Phi)_u$ for the Jordan decomposition into commuting semisimple and unipotent parts. The **Frobenius-semisimplification** of D is $D^{F\text{-}ss} = (V, \rho_s, N)$, where $\rho_s(\Phi^m\sigma) = \rho(\Phi)_s^m\rho(\sigma)$ for $m \in \mathbb{Z}$ and $\sigma \in I_v$. It is a Weil–Deligne representation, independent of Φ up to isomorphism, and D is **Frobenius-semisimple** if $D \cong D^{F\text{-}ss}$, i.e. if $\rho(\Phi)$ is semisimple.

All of the above applies without change to a finite extension K of $\mathbb{Q}_v$ in place of $\mathbb{Q}_v$, with $W_K \supseteq I_K$, and the cardinality q_K of the residue field of K in place of v in the definition of $n(\cdot)$ and in (7.1).

Definition 7.1.3. Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field with a fixed embedding $\tau\colon K \to \mathbb{C}$, $\bar\tau = \overline{\tau(\cdot)}$ the other one, and $n \geq 1$. Put $G_\infty = \mathrm{GL}_n(K \otimes_{\mathbb{Q}} \mathbb{R})$, identified with $\mathrm{GL}_n(\mathbb{C})$ through τ; $K_\infty = \mathrm{U}(n)\cdot\mathbb{R}_{>0} \subset G_\infty$ ($\mathbb{R}_{>0}$ as scalar matrices); $\mathfrak{g} = \mathrm{Lie}(G_\infty)\otimes_{\mathbb{R}}\mathbb{C} \cong \mathfrak{gl}_n(\mathbb{C})\times\mathfrak{gl}_n(\mathbb{C})$, the two factors corresponding to τ and $\bar\tau$. Put $\widehat{\mathcal{O}}_K = \prod_{v\nmid\infty} \mathcal{O}_{K_v} \subset \mathbb{A}_{K,f}$. For $N \geq 1$, $\mathrm{K}_n(N) \subset \mathrm{GL}_n(\widehat{\mathcal{O}}_K)$ is the kernel of reduction modulo $N\widehat{\mathcal{O}}_K$.

Infinitesimal characters. For a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$, the **infinitesimal character of π at τ** (resp. at $\bar\tau$) is the multiset of n complex numbers given by the τ- (resp. $\bar\tau$-) factor of the infinitesimal character of π_∞, in the normalisation of [58], in which the trivial representation has infinitesimal character $\{\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2}\}$ at both embeddings.

Coefficient systems. For $\mu \in \mathbb{Z}^n$ with $\mu_1 \geq \cdots \geq \mu_n$ let (ρ_μ, V_μ) be the irreducible algebraic representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight μ; its infinitesimal character is $\{\mu_i + \frac{n+1}{2} - i : 1 \leq i \leq n\}$ and that of its dual is $\{-\mu_i - \frac{n+1}{2} + i\}$. For two such μ, μ', $V_{\mu,\mu'} = V_\mu \otimes_{\mathbb{C}} V_{\mu'}$ with $g \in G_\infty$ acting by $\rho_\mu(g) \otimes \rho_{\mu'}(\bar g)$; $\mathrm{GL}_n(K) \subset G_\infty$ acts by restriction. For a cuspidal π, $H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_{\mu,\mu'})$ is the relative Lie algebra cohomology of the $(\mathfrak{g}, K_\infty)$-module of K_∞-finite vectors of π_∞ tensored with $V_{\mu,\mu'}$; π is **cohomological for** $V_{\mu,\mu'}$ if it is nonzero in some degree.

Arithmetic groups of a level. For an open compact subgroup $U \subseteq \mathrm{GL}_n(\mathbb{A}_{K,f})$ the double coset space $\mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_{K,f})/U$ is finite ([33]); for representatives $g_1, \dots, g_h$ put $\Gamma_j(U) = \mathrm{GL}_n(K) \cap g_j U g_j^{-1}$, acting on $V_{\mu,\mu'}$ through $\mathrm{GL}_n(K) \subset G_\infty$, and

$$h^*(U, V_{\mu,\mu'}) = \sum_{j=1}^{h} \sum_{q \geq 0} \dim_{\mathbb{C}} H^q(\Gamma_j(U), V_{\mu,\mu'}) \in \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

which does not depend on the representatives (replacing g_j by $\gamma g_j u$ with $\gamma \in \mathrm{GL}_n(K)$, $u \in U$ conjugates $\Gamma_j(U)$ by γ, compatibly with the action on $V_{\mu,\mu'}$). It is the total dimension of the cohomology of the locally symmetric space of level U with coefficients in $V_{\mu,\mu'}$.

Theorem 7.1.4. *Let F be a totally real field with $[F : \mathbb{Q}] \geq 2$, let E/F be a totally imaginary quadratic extension with non-trivial automorphism c, let $n \geq 1$, and let U be an inner form of the quasi-split unitary group in n variables for E/F such that $U(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact. Let σ be an automorphic representation of $U(\mathbb{A}_F)$, ℓ a prime and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Let $\pi = \boxplus_{i=1}^{s} \pi_i$ be a weak base change of σ in the sense of [177], Corollaire 5.3: each π_i is a discrete automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ with $\pi_i^c \cong \pi_i^\vee$, $\sum_i n_i = n$, and π_∞ is cohomological with infinitesimal character the base change of that of σ_∞. Write $\pi = \Pi_1 \boxplus \cdots \boxplus \Pi_k$ as an isobaric sum of cuspidal automorphic representations Π_j of $\mathrm{GL}_{m_j}(\mathbb{A}_E)$, $\sum_j m_j = n$. Then:*

(Ua0) *each $\Pi'_j := \Pi_j \otimes |\det|^{(m_j - n)/2}$ is a regular algebraic polarizable cuspidal automorphic representation of $\mathrm{GL}_{m_j}(\mathbb{A}_E)$, and π_w is unramified at every finite place w of E above a finite place v of F at which E/F and U are unramified and σ_v is spherical;*

and the continuous semisimple representation

$$r = r_{\ell,\iota}(\sigma) := \bigoplus_{j=1}^{k} r_{\ell,\iota}(\Pi'_j)\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

satisfies:

(Ua1) $r^c \cong r^\vee \otimes \chi_{\mathrm{cyc}}^{1-n}$;

(Ua2) *for every finite place $w \nmid \ell$ of E at which π_w is unramified, every place $\bar w$ of $\overline{\mathbb{Q}}$ above w and every arithmetic Frobenius element* Frob *at $\bar w$, r is trivial on $I_{\bar w}$ and*

$$\iota\Big(\det\big(X - r(\mathrm{Frob}^{-1})\big)\Big) = \prod_{\alpha} (X - q_w^{(n-1)/2}\alpha),$$

α running over the Satake parameters of π_w with multiplicity; in particular the polynomial on the right, transported by ι^{-1}, kills $r(\mathrm{Frob}^{-1})$.

Up to isomorphism, r is the unique continuous semisimple representation satisfying (Ua2) at all but finitely many w.

Proof. Write each discrete summand as $\pi_i = \mathrm{Sp}(\tau_i, a_i)$, with τ_i unitary cuspidal, by the discrete-spectrum classification of [207]. Its cuspidal isobaric constituents are

$$\tau_i |\det|^{s_{i,b}}, \qquad s_{i,b} = (a_i - 1)/2 - b, \quad 0 \le b < a_i.$$

The uniqueness in this classification, applied to $\mathrm{Sp}(\tau_i^c, a_i) \cong \pi_i^c \cong \pi_i^\vee \cong \mathrm{Sp}(\tau_i^\vee, a_i)$, gives $\tau_i^c \cong \tau_i^\vee$. Here duality reverses the list of exponents, which is symmetric about zero. The local parameters of the isobaric sum, including monodromy, are the direct sums of the parameters of its cuspidal constituents.

We check regular algebraicity, including the rank-dependent shift. At each complex embedding, the infinitesimal-character entries of the cohomological representation π are n distinct numbers in $(n-1)/2 + \mathbb{Z}$. This follows also directly from its algebraic coefficient system: for a dominant integral highest weight λ, the entries are $\lambda_i + (n + 1 - 2i)/2$, and consecutive entries differ by at least one. Infinitesimal characters are preserved under taking subquotients of normalized parabolic induction, and the infinitesimal character of the induced representation is the union of the inducing infinitesimal characters. Thus the entries of a rank-m_j constituent Π_j form a subset of this distinct list. Twisting by $|\det|^{(m_j-n)/2}$ adds $(m_j - n)/2$ to each entry, putting them in $(m_j - 1)/2 + \mathbb{Z}$. If these entries, in decreasing order, are $z_1, \dots, z_{m_j}$, then

$$\lambda_i' = z_i - (m_j + 1 - 2i)/2$$

are integers with $\lambda_i' \ge \lambda_{i+1}'$. They are therefore the highest weight of an irreducible algebraic representation. The same computation applies at the conjugate embedding. This proves that Π_j' is regular algebraic.

If $\Pi_j = \tau_i |\det|^s$, put $d = s + (m_j - n)/2$. Then

$$(\Pi_j')^c \cong \tau_i^\vee |\det|^d \cong (\Pi_j')^\vee \otimes (|\cdot|_{\mathbb{A}_F}^{2d} \circ \mathrm{N}_{E/F} \circ \det).$$

The character $|\cdot|_{\mathbb{A}_F}^{2d}$ is trivial on $F^\times$, and its value at -1 is 1 at every real place. Thus it is a polarization character of the required kind. Cuspidality is preserved by twisting. This checks all the hypotheses of the polarized regular-algebraic cuspidal theorem [15, Theorem 2.1.1] for every Π_j'. Its continuous semisimple representations have the direct sum r in the statement.

At a place where π_w is unramified, its local parameter is unramified with zero monodromy. The direct-sum formula for the local parameter shows that each constituent parameter has both properties, since its inertia action and monodromy are direct summands. By the unramified local Langlands correspondence, every $\Pi_{j,w}$ is spherical, as is its unramified twist $\Pi_{j,w}'$. If α is a Satake parameter of $\Pi_{j,w}$, the corresponding parameter of $\Pi_{j,w}'$ is $q_w^{(n-m_j)/2}\alpha$. The cuspidal Galois theorem therefore gives, at geometric Frobenius, the eigenvalue

$$q_w^{(m_j-1)/2} q_w^{(n-m_j)/2} \alpha = q_w^{(n-1)/2}\alpha.$$

The union over j is exactly the Satake multiset of π_w, proving (Ua2), including triviality on inertia. Cayley–Hamilton gives its annihilating-polynomial assertion. The spherical clause of (Ua0) is the unramified base-change assertion of [177, Corollaire 5.3]; its hypotheses here are precisely the stated unramifiedness of the extension and group and the spherical hypothesis on σ_v.

It remains to prove duality and uniqueness. Since each π_i is conjugate self-dual, so is π. Away from finitely many places, the Satake multiset at w^c is the inverse of that at w. At a

geometric Frobenius at w, the eigenvalues of r^c are consequently $q_w^{(n-1)/2}\alpha^{-1}$. Those of $r^\vee \otimes \chi_{\mathrm{cyc}}^{1-n}$ are the same, because χ_{cyc} takes the value q_w^{-1} at geometric Frobenius. Their characteristic polynomials agree at all such places. Chebotarev density and continuity extend equality of their traces to every element of G_E: in every finite quotient, each conjugacy class occurs at infinitely many unramified places outside any fixed finite exceptional set. The characteristic-zero Brauer–Nesbitt theorem now identifies these semisimple representations, proving (Ua1). The identical argument applied to any other continuous semisimple representation satisfying (Ua2) proves uniqueness. □

Theorem 7.1.5. *Let F be a number field and $\mathbb{A}_F$ its adeles.*

(Mw1) *Let $m, k \geq 1$ and let τ be a unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$. The normalised parabolic induction to $\mathrm{GL}_{km}(\mathbb{A}_F)$, from the standard parabolic subgroup of type $(m, \dots, m)$, of $\tau|\det|^{(k-1)/2} \otimes \tau|\det|^{(k-3)/2} \otimes \cdots \otimes \tau|\det|^{(1-k)/2}$ has a unique irreducible quotient $\mathrm{Sp}(\tau, k)$, and $\mathrm{Sp}(\tau, k)$ occurs in the discrete spectrum of $\mathrm{GL}_{km}(\mathbb{A}_F)$. Conversely, every irreducible automorphic representation of $\mathrm{GL}_N(\mathbb{A}_F)$ in the discrete spectrum is isomorphic to $\mathrm{Sp}(\tau, k)$ for a unique pair (τ, k) with $km = N$.*

(Mw2) *Let $r \geq 1$, and for $1 \leq i \leq r$ let τ_i be a unitary cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ and $k_i \geq 1$; let $\Pi = \mathrm{Sp}(\tau_1, k_1) \boxplus \cdots \boxplus \mathrm{Sp}(\tau_r, k_r)$ be the isobaric sum, an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, $n = \sum_i k_i m_i$. Then for every finite place w of F, with $K = F_w$ and $|\cdot|$ the normalised absolute value of K,*

$$\mathrm{rec}_K(\Pi_w) \cong \bigoplus_{i=1}^{r} \bigoplus_{a=0}^{k_i-1} \mathrm{rec}_K(\tau_{i,w}) \otimes \mathrm{rec}_K\big(|\cdot|^{(k_i-1)/2-a}\big) \tag{7.2}$$

as Weil–Deligne representations (monodromy operators included).

Proof. The main theorem of [207], with fixed unitary central character and square-integrability modulo the centre, proves (Mw1). Its quotient $J(\tau, k)$ is precisely $\mathrm{Sp}(\tau, k)$: both use normalised induction from the upper standard parabolic and the descending exponents $((k-1)/2, \dots, (1-k)/2)$. This also gives uniqueness of the pair, including the cuspidal case $k = 1$.

Fix a finite place w, put $K = F_w$, and write $\nu(g) = |\det g|_K$. We first check the hypotheses needed for the local calculation. The local component τ_w is unitary: fix nonzero vectors in the other factors of the restricted tensor product for τ and restrict its positive invariant Hermitian form to the resulting copy of τ_w. It is also generic. Indeed the cuspidal Whittaker expansion of [269, 230] gives a nonzero Whittaker functional Λ on τ for a nontrivial character of $F\backslash\mathbb{A}_F$. Some pure tensor has nonzero value under Λ, since pure tensors span the smooth automorphic representation. Fix its factors away from w. Evaluation of Λ on the remaining factor is a nonzero functional on τ_w transforming under $U_m(K)$ by the local nondegenerate Whittaker character. Each local component of a nontrivial adelic character trivial on F is nontrivial. This proves the required local genericity. When $m = 1$, the unipotent group is trivial and the same assertion is immediate.

The generic unitary classification, as explicitly stated in [207], I.10–I.11, now gives

$$\tau_w \cong \delta_1\nu^{e_1} \times \cdots \times \delta_b\nu^{e_b}, \qquad \tfrac{1}{2} > e_1 \geq \cdots \geq e_b > -\tfrac{1}{2}, \tag{1}$$

where each δ_j is irreducible, unitary and square-integrable, and the displayed induction is irreducible. Thus τ_w is the Langlands quotient for the multiset

$$c = \{\delta_j\nu^{e_j} : 1 \leq j \leq b\}.$$

Here and below multisets retain every multiplicity.

Put $s_a = (k-1)/2 - a$ for $0 \le a < k$. By transitivity of normalised induction the local inducing representation

$$I_w = \tau_w \nu^{s_0} \times \cdots \times \tau_w \nu^{s_{k-1}}$$

is the induction of the square-integrable twists $\delta_j \nu^{e_j + s_a}$, ordered first by increasing a and then by increasing j. This order is the Langlands order. Within a fixed a, the exponents are nonincreasing by (1). If $a < a'$, then

$$(e_j + s_a) - (e_{j'} + s_{a'}) = (a' - a) + (e_j - e_{j'}) > (a' - a) - 1 \ge 0. \tag{2}$$

Consequently every exponent in one level is strictly larger than every exponent in a later level. For two linked Zelevinsky segments, precedence requires both endpoints, and hence their real centre, to be strictly smaller. Therefore this nonincreasing-centre order also has the nonprecedence condition in (Ln6). The standard-module form of the Langlands classification, [207], I.2(1), identifies the unique irreducible quotient of I_w with

$$L(c_k), \qquad c_k = \{\delta_j \nu^{e_j + s_a} : 1 \le j \le b,\ 0 \le a < k\}. \tag{3}$$

We check that (3) is the actual local component of the global quotient. The global inducing representation is the restricted tensor product of the local inducing representations, and its nonzero quotient map to $\mathrm{Sp}(\tau, k)$ gives a nonzero K-local map $I_w \to \mathrm{Sp}(\tau, k)_w$. More explicitly, choose an inducing pure tensor with nonzero image and a linear coefficient functional on the other output factors that does not kill this image. Fix the other inducing factors and apply that coefficient functional. The resulting map intertwines $\mathrm{GL}_{km}(K)$, because this group acts only on the factor at w. Its target is irreducible, so it is surjective. The uniqueness just proved therefore gives

$$\mathrm{Sp}(\tau, k)_w \cong L(c_k). \tag{4}$$

This argument applies to the smooth restricted tensor products; any vector used in extracting the coefficient is a finite sum of pure tensors.

Apply (Ln6) and (Ln2) to (1), (3) and (4). They give, as full Weil–Deligne representations,

$$\begin{aligned} \mathrm{rec}_K(\tau_w) &\cong \bigoplus_{j=1}^{b} \mathrm{rec}_K(\delta_j) \otimes \mathrm{rec}_K(|\cdot|^{e_j}), \\ \mathrm{rec}_K(\mathrm{Sp}(\tau, k)_w) &\cong \bigoplus_{a=0}^{k-1} \bigoplus_{j=1}^{b} \mathrm{rec}_K(\delta_j) \otimes \mathrm{rec}_K(|\cdot|^{e_j + s_a}) \\ &\cong \bigoplus_{a=0}^{k-1} \mathrm{rec}_K(\tau_w) \otimes \mathrm{rec}_K(|\cdot|^{s_a}). \end{aligned} \tag{5}$$

The character identity used in the last equality follows from (Ln1), so it uses the same geometric Artin map on both sides.

It remains to combine the summands in (Mw2). For each i let c_{i,k_i} be the multiset in (3) for τ_i, k_i. The representation $L(c_{i,k_i}) = \mathrm{Sp}(\tau_i, k_i)_w$ is unitary, since its global representation lies in the discrete L^2 spectrum; restricting the invariant positive form to a local factor proves this as above. Hence Tadić's unitary Langlands-data formula, [292], Appendix Corollary A.7, applies successively and gives

$$\mathrm{Sp}(\tau_1, k_1)_w \times \cdots \times \mathrm{Sp}(\tau_r, k_r)_w \cong L\left(\sum_{i=1}^{r} c_{i,k_i}\right). \tag{6}$$

In particular this entire normalised induction is irreducible. The local isobaric construction takes the irreducible constituent associated with the union of the inducing Langlands data. Equivalently, it is the Langlands quotient of that union; (6) shows it is the entire local induced representation here. This is the local component Π_w of the isobaric sum in the hypothesis (for isobaric sums see [149]). No existence assertion for a new global isobaric sum is needed: the automorphic representation Π is given in (Mw2). Applying (Ln6) to the union in (6), and then (5) for each i, proves (7.2).

Finally, (Ln6) is an isomorphism of Weil–Deligne representations, not just of their Weil semisimplifications. On a summand in (5) its monodromy is the monodromy of $\mathrm{rec}_K(\delta_j)$ tensored with the identity of a one-dimensional character, whose own monodromy is zero. Both regrouping in (5) and the direct sum in (6) preserve these operators. Thus the asserted identity includes the full monodromy operators, with no temperedness assumption on τ_w and no replacement by a semisimplified parameter. □

Theorem 7.1.6. *Let F be a CM field with maximal totally real subfield F^+ and complex conjugation c, $n \geq 1$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and π a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is **polarizable**: there is a continuous character $\chi\colon \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$, with $\chi_v(-1)$ independent of the infinite place v of F^+, such that $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathrm{N}_{F/F^+} \circ \det)$. Then there is a continuous semisimple representation $r_{\ell,\iota}(\pi)\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that, for every finite place v of F and every place $\bar{v}$ of $\overline{\mathbb{Q}}$ above v, writing $r_{\bar{v}}$ for the restriction of $r_{\ell,\iota}(\pi)$ to $D_{\bar{v}}$ (identified with G_{F_v}), $W_{\bar{v}} \subseteq D_{\bar{v}}$ for its Weil group and $\mathcal{O}_v = \mathcal{O}_{F_v}$:*

(Bg1) *if $v \nmid \ell$ and π_v is unramified, then $r_{\ell,\iota}(\pi)$ is trivial on $I_{\bar{v}}$ and $\iota(\det(X - r_{\ell,\iota}(\pi)(\sigma^{-1}))) = \prod_\alpha (X - q_v^{(n-1)/2}\alpha)$ for every arithmetic Frobenius σ at $\bar{v}$, α running over the Satake parameters of π_v with multiplicity; in particular $r_{\ell,\iota}(\pi) \leftrightarrow_\iota \pi$;*

(Bg2) *if $v \mid \ell$, then $r_{\bar{v}}$ is de Rham, and for every field embedding $\tau\colon F_v \to \overline{\mathbb{Q}}_\ell$ over $\mathbb{Q}_\ell$ the τ-Hodge–Tate weights of $r_{\bar{v}}$ are the n numbers $z - \frac{n-1}{2}$, z running over the infinitesimal character of π at the complex embedding $\iota \circ \tau|_F$ of F;*

(Bg3) *if $U \subseteq \mathcal{O}_v^\times$ is a subgroup and there are characters $\Psi_1, \dots, \Psi_n$ of $W_{\bar{v}}$, each trivial on the image of $\mathrm{Art}_{F_v}(U)$ in $W_{\bar{v}}^{\mathrm{ab}}$, such that the semisimplification of the restriction to $I_{\bar{v}}$ of the representation ρ of $\mathrm{WD}(r_{\bar{v}})$ is isomorphic to $\bigoplus_i \Psi_i|_{I_{\bar{v}}}$ (for $v \mid \ell$, $\mathrm{WD}(r_{\bar{v}})$ is defined by (Bg2), since de Rham representations are potentially semistable), then π_v is isomorphic to an irreducible subquotient of the normalised parabolic induction from the upper triangular Borel subgroup of $\mathrm{GL}_n(F_v)$ of $\xi_1 \otimes \cdots \otimes \xi_n$, for some characters $\xi_1, \dots, \xi_n$ of $F_v^\times$ trivial on U;*

(Bg4) *if $v \mid \ell$ and π_v is unramified, then $r_{\bar{v}}$ is crystalline, $\mathrm{WD}(r_{\bar{v}}) = (V, \rho, N)$ has $N = 0$ and ρ trivial on $I_{\bar{v}}$, and $\iota(\det(X - \rho(\Phi))) = \prod_\alpha (X - q_v^{(n-1)/2}\alpha)$ for every $\Phi \in W_{\bar{v}}$ with $n(\Phi) = -1$ (a geometric Frobenius), α as in (Bg1);*

(Bg5) *if π_v is isomorphic to an irreducible subquotient of the normalised parabolic induction from the upper triangular Borel subgroup of $\mathrm{GL}_n(F_v)$ of $\xi_1 \otimes \cdots \otimes \xi_n$, for characters $\xi_1, \dots, \xi_n$ of $F_v^\times$, then the semisimplification of the representation ρ of $\mathrm{WD}(r_{\bar{v}})$ is isomorphic to $\lambda_1 \oplus \cdots \oplus \lambda_n$, where λ_i is the character of $W_{\bar{v}}$ with $\iota(\lambda_i(\mathrm{Art}_{F_v}(x)^{-1})) = \xi_i(x)\,|x|_v^{(1-n)/2}$ for $x \in F_v^\times$ (characters of $W_{\bar{v}}$ factor through $W_{\bar{v}}^{\mathrm{ab}}$, identified with $F_v^\times$ by Art_{F_v});*

(Bg7) *if $v \nmid \ell$, then the Frobenius-semisimplification of* $\mathrm{WD}(r_{\bar v})$*, transported to* $\mathbb{C}$ *by* ι*, is isomorphic to* $\mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})$*, where* rec_{F_v} *is the local Langlands correspondence of Harris–Taylor (uniformisers corresponding to geometric Frobenius elements); in particular the monodromy operators of* $\mathrm{WD}(r_{\bar v})$ *and of* $\mathrm{rec}_{F_v}(\pi_v)$ *are conjugate.*

Proof. First normalize the polarizing character. Its common sign at infinity is either $(-1)^n$ or its negative. In the second case replace χ by $\chi\delta_{F/F^+}$, where δ_{F/F^+} is the quadratic character of F^+ associated to F/F^+. At every real place $\delta_{F/F^+}(-1) = -1$, whereas $\delta_{F/F^+} \circ \mathrm{N}_{F/F^+} = 1$ by class field theory. Thus the relation for π^c is unchanged and the normalized pair satisfies the precise hypotheses of Theorem A.4.11. Take its representation r.

For $v \nmid \ell$, conclusion (1) of that theorem gives (Bg7). For $v \mid \ell$, Theorem 7.1.7 gives, on forgetting monodromy and semisimplifying,

$$\iota\, \rho_v^{\mathrm{ss}} \simeq \mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})^{\mathrm{ss}}. \tag{*}$$

Here and below the superscript on the right means semisimplification of the underlying Weil representation. Frobenius semisimplification does not change that representation's semisimplification. Hence (*) holds at every finite place. The de Rham conclusion of Theorem A.4.11 and Theorem 7.1.10(1) ensure that this Weil–Deligne representation exists above ℓ.

To obtain (Bg2), write $a_i = a_{\iota\tau,i}$. In the convention of the quoted existence theorem the weights are $h_i = a_i + n - i$. The infinitesimal character of the dual algebraic representation is the unordered tuple

$$z_i = -a_i - \frac{n+1}{2} + i = \frac{n-1}{2} - h_i.$$

The project convention assigns the cyclotomic character weight 1, so its Hodge–Tate weights are $-h_i = z_i - (n-1)/2$, as asserted.

Suppose π_v is unramified, with Satake parameters α_i. The unramified local correspondence is the direct sum of the unramified characters with geometric-Frobenius eigenvalues α_i. Twisting by $|\det|_v^{(1-n)/2}$ therefore multiplies each eigenvalue by

$$|\varpi_v|_v^{(1-n)/2} = q_v^{(n-1)/2}.$$

It leaves inertia trivial and monodromy zero. For $v \nmid \ell$, (Bg7) gives these assertions for $\mathrm{WD}(r_v)$. The construction (Wd1) then gives $r_v|_{I_v} = 1$, since its monodromy vanishes. An inverse arithmetic Frobenius is geometric Frobenius, proving (Bg1), including the indicated characteristic polynomial. For $v \mid \ell$, an unramified π_v has an Iwahori fixed vector, so conclusion (3) of Theorem A.4.11 gives crystallinity and the full Weil–Deligne comparison. The identical eigenvalue computation proves all of (Bg4). In (Bg7) the extra twist is unramified and tensoring a Weil–Deligne representation with a character changes its Weil action but leaves its monodromy operator N unchanged, which proves the final monodromy assertion.

We prove (Bg3). Choose the supercuspidal support $(\sigma_j)_j$ of π_v, with σ_j on $\mathrm{GL}_{n_j}(F_v)$. By Theorem A.4.10 and the twisting law, the right hand side of (*) is the sum of the irreducible Weil representations

$$R_j = \mathrm{rec}_{F_v}(\sigma_j) \otimes \mathrm{rec}_{F_v}(|\cdot|_v^{(1-n)/2}).$$

Their irreducibility follows from (Ln5), and the second factor is unramified. Restricting (*) to inertia is compatible with semisimplification: the inertia image of a Weil–Deligne representation is finite, and finite-group representations in characteristic zero are semisimple. Thus $R_j|_{I_v}$ contains some $\iota\Psi_i|_{I_v}$. The nonzero subspace

$$\{x : R_j(t)x = \iota\Psi_i(t)x \text{ for every } t \in I_v\}$$

is W_v-stable: normality of I_v and the fact that Ψ_i is a character of W_v give the same eigenvalue equation after applying $R_j(w)$. Irreducibility implies this subspace is all of R_j. Consequently $R_j \otimes (\iota\Psi_i)^{-1}$ factors through $W_v/I_v \simeq \mathbb{Z}$. An irreducible finite-dimensional complex representation of this group has dimension one: its generator has an eigenvector and that eigenline is invariant. Therefore $n_j = 1$ for every j. The defining support induction is now the required normalized Borel induction of characters $\xi_1, \dots, \xi_n$.

The multiset of their inertia characters equals that of the $\iota\Psi_i$, because the twist in (*) is unramified. Local reciprocity maps units into inertia, and for $x \in U$ each Ψ_i is trivial on $\mathrm{Art}_{F_v}(x)$. Reversing arithmetic to geometric reciprocity replaces x by x^{-1}, which does not affect triviality. The norm character is also trivial on units. Hence every ξ_j is trivial on U, proving (Bg3).

Finally suppose the principal-series hypothesis of (Bg5) holds. Its inducing characters are supercuspidal representations of $\mathrm{GL}_1(F_v)$. Uniqueness of supercuspidal support in Theorem A.4.10 makes their unordered tuple the support of π_v, even when the induction is reducible. Applying that theorem to (*) gives the sum of the characters

$$\mathrm{rec}_{F_v}(\xi_i|\cdot|_v^{(1-n)/2}).$$

By (Ln1), geometric reciprocity sends x to the value $\xi_i(x)|x|_v^{(1-n)/2}$. The map Art_{F_v} in the statement is arithmetic reciprocity, so its inverse element is geometric reciprocity at x. Pulling coefficients back by ι gives precisely the displayed formula for λ_i in (Bg5). □

Theorem 7.1.7. *Let F be an imaginary CM field, $n \geq 1$, ℓ a prime, and $\iota : \overline{\mathbb{Q}}_\ell \longrightarrow \mathbb{C}$ an isomorphism. Let (π, χ) be regular algebraic, essentially conjugate self-dual and cuspidal: π is cuspidal on $\mathrm{GL}_n(\mathbb{A}_F)$ and regular algebraic, $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \longrightarrow \mathbb{C}^\times$ is continuous with $\chi_v(-1)$ independent of $v \mid \infty$, and*

$$\pi^c \simeq \pi^\vee \otimes (\chi \circ \mathrm{N}_{F/F^+} \circ \det).$$

For a place $v \mid \ell$, put

$$D = \iota\,\mathrm{WD}\big(r_{\ell,\iota}(\pi)|_{G_{F_v}}\big)^{F\text{-}ss}, \qquad D' = \mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2}).$$

Then $D \prec D'$ in the following precise sense. Their semisimplified Weil representations are isomorphic. Partition the irreducible Weil representations into equivalence classes modulo unramified character twists. On each such class, decompose the corresponding Weil–Deligne summand as a direct sum of special representations $\mathrm{Sp}_{m_j}(s_j)$, order their positive block lengths $m_1 \geq m_2 \geq \cdots$, and pad the list with zeros. For every class and every $k \geq 1$,

$$\sum_{j=1}^{k} m_j(D) \leq \sum_{j=1}^{k} m_j(D').$$

Moreover $D \simeq D'$ if π has Shin-regular weight: either n is odd, or, writing π_∞ with the infinitesimal character of the dual of an algebraic representation of highest weights $a_{\tau,1} \geq \cdots \geq a_{\tau,n}$, there are an embedding τ and an odd $j < n$ with $a_{\tau,j} > a_{\tau,j+1}$.

Proof. We first justify the associated representation in the opening convention under precisely the displayed hypotheses. Write $s = \chi_v(-1) \in \{1, -1\}$, independent of the real place v of F^+. Global reciprocity Theorem A.2.8, applied to the quadratic CM extension F/F^+, gives a quadratic character κ of C_{F^+} with kernel $\mathrm{N}_{F/F^+}C_F$. Thus $\kappa \circ \mathrm{N}_{F/F^+} = 1$. At every real place the extension is $\mathbb{C}/\mathbb{R}$, so the real-place clause of that theorem gives $\kappa_v(-1) = -1$. Choose $e \in \{0, 1\}$ with $s(-1)^e = (-1)^n$ and put $\chi_0 = \chi\kappa^e$. The character χ_0 is continuous, has the

real signs required by Theorem A.4.11, and has the same norm pullback as χ. Consequently the displayed polarization of π holds with χ_0 as well. Regular algebraicity and cuspidality concern the unchanged π. Applying Theorem A.4.11 to (π, χ_0) therefore gives a continuous semisimple representation r_0 and its stated unramified compatibility.

Apply the published comparison Theorem A.4.15 to the original pair (π, χ), whose common-sign hypothesis is exactly the one in the present statement. Denote its associated continuous semisimple representation by R. We check that this is r_0, so that the auxiliary sign adjustment does not change the representation named here. At every finite place outside the union of the two finite exceptional sets and the places above ℓ, both representations have the characteristic polynomial prescribed by the same unramified local Langlands parameter of π, with the same twist $|\det|^{(1-n)/2}$ and the same geometric-Frobenius convention. Their traces agree at those Frobenius elements.

For completeness, Chebotarev Theorem A.2.6 makes these conjugacy classes dense: in any finite Galois quotient every conjugacy class occurs at a place outside the given finite exceptional set, because its Frobenius set has positive Dirichlet density. Conjugation invariance and continuity of the two trace functions then give equality of their traces on all of G_F. Equivalently, for any open normal subgroup and any conjugacy class in its quotient there is a Frobenius representative in that class; after conjugation this supplies an element in the prescribed coset, which proves the required density. The representations are finite dimensional and semisimple over the algebraically closed characteristic-zero field $\overline{\mathbb{Q}}_\ell$. Item 2 of Lemma 7.7.9 therefore gives $R \simeq r_0$. This also proves independence from the auxiliary character χ_0 and identifies the canonical $r_{\ell,\iota}(\pi)$ in the statement with the representation supplied in the convention Theorem A.4.11.

Now Theorem A.4.15 gives, at the specified $v \mid \ell$, precisely $D \prec D'$ with the underlying Weil semisimplification and the classwise block-length partial sums printed in the statement. Isomorphism $R \simeq r_0$ remains an isomorphism after restriction and after the Weil–Deligne and Frobenius-semisimplification operations. The same quoted theorem gives $D \simeq D'$ under exactly the displayed Shin-regular condition. The normalization agrees on both sides by construction; Frobenius-semisimplification retains monodromy. Thus this step uses the full published monodromy comparison, rather than deducing monodromy from equality of semisimplified Weil representations. □

Theorem 7.1.8. *Let F be a totally real field with $[F : \mathbb{Q}] \geq 2$, E/F a totally imaginary quadratic extension, $n \geq 1$, U an inner form of the quasi-split unitary group in n variables for E/F with $U(F \otimes_{\mathbb{Q}} \mathbb{R})$ compact, σ an automorphic representation of $U(\mathbb{A}_F)$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, π a weak base change of σ to $\mathrm{GL}_n(\mathbb{A}_E)$ in the sense of [177], and $r\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ the continuous semisimple representation such that, for all but finitely many finite places w of E at which π_w is unramified, r is unramified at w and ι carries the characteristic polynomial of a geometric Frobenius at w to $\prod_\alpha (X - q_w^{(n-1)/2}\alpha)$, α over the Satake parameters of π_w. Let $v \nmid \ell$ be a finite place of F that splits in E, $w \mid v$ a place of E, $\iota_w\colon U(F_v) \to \mathrm{GL}_n(E_w)$ the isomorphism determined by w (well defined up to conjugacy), $\bar{w}$ a place of $\overline{\mathbb{Q}}$ above w, and $(V, \rho, N) = \mathrm{WD}(r|_{D_{\bar{w}}})$. Then:*

(Us1) $\pi_w \cong \sigma_v \circ \iota_w^{-1}$;

(Us2) $(V, \rho, N)^{F\text{-}ss} \cong \iota^{-1}\mathrm{rec}_{E_w}(\pi_w \otimes |\det|_w^{(1-n)/2})$;

(Us3) *if the monodromy operator of $\mathrm{rec}_{E_w}(\pi_w)$ is zero, then $N = 0$, $r(\gamma)$ is semisimple for every $\gamma \in I_{\bar{w}}$, and $\iota(\det(X - r(\gamma)))$ is the characteristic polynomial of γ in $\mathrm{rec}_{E_w}(\pi_w)$, for every $\gamma \in I_{\bar{w}}$; this applies in particular when π_w is supercuspidal (for instance of depth zero), since then $\mathrm{rec}_{E_w}(\pi_w)$ is irreducible with zero monodromy;*

(Us4) *let $B \subseteq \mathrm{GL}_n(E_w)$ be the upper triangular Borel subgroup with unipotent radical N_B and modulus character δ_B, and let $r_B(\pi_w) = \delta_B^{-1/2} \otimes (\pi_w)_{N_B}$ be the normalised Jacquet module; if a character $\chi = \chi_1 \otimes \cdots \otimes \chi_n$ of the diagonal torus is a Jordan–Hölder constituent of $r_B(\pi_w)$ and, for a uniformiser ϖ_w,*

$$|\chi_1(\varpi_w)| \;\geq\; |\chi_2(\varpi_w)| \;\geq\; \dots \;\geq\; |\chi_n(\varpi_w)|,$$

then the monodromy operator of $\mathrm{rec}_{E_w}(\pi_w)$ is zero (so (Us3) applies), and $\rho^{ss} \cong \lambda_1 \oplus \cdots \oplus \lambda_n$ with $\iota(\lambda_i(\mathrm{Art}_{E_w}(x)^{-1})) = \chi_i(x)\,|x|_w^{(1-n)/2}$ for $x \in E_w^\times$;

(Us5) *(Steinberg exclusion) for $n \geq 2$ and every character μ of $E_w^\times$, $r_B(\mathrm{St}_n \otimes (\mu \circ \det))$ is the single character $\mu\delta_B^{1/2}$, whose i-th component $\mu|\cdot|_w^{(n+1-2i)/2}$ has $|\mu(\varpi_w)|\,q_w^{-(n+1-2i)/2}$ strictly increasing in i, so the hypothesis of (Us4) fails for it; for the character $\mu \circ \det$ of $\mathrm{GL}_n(E_w)$, r_B is the single character $\mu\delta_B^{-1/2}$, which satisfies it.*

Proof. By Theorem 7.1.4 and its uniqueness assertion, the representation in the hypothesis is the direct sum of the polarized regular-algebraic cuspidal representations attached to $\Pi'_j = \Pi_j|\det|^{(m_j-n)/2}$.

At a split place v, the identification $E \otimes_F F_v = E_w \times E_{w^c}$ identifies the unitary group with $\mathrm{GL}_n(E_w)$: the second matrix is determined by the inverse adjoint of the first. The corresponding split base-change map has first component the identity and second component its conjugate contragredient. Thus its first local representation is $\sigma_v \circ \iota_w^{-1}$. The split-place assertion of [177, Corollaire 5.3] gives (Us1).

For each Π'_j, apply the away-from-ℓ local–global compatibility theorem [15, Theorem 2.1.1]. All its cuspidality, regular algebraicity and polarization hypotheses were checked in Theorem 7.1.4. Its local parameter is

$$\iota\,\mathrm{WD}(r_{\ell,\iota}(\Pi'_j)|_{D_{\bar{w}}})^{F\text{-}ss} \cong \mathrm{rec}_{E_w}(\Pi_{j,w}|\det|^{(1-n)/2}),$$

since $(m_j - n)/2 + (1 - m_j)/2 = (1 - n)/2$. Formation of Weil–Deligne representations and Frobenius semisimplification commutes with direct sums. The parameter of the isobaric sum is the direct sum of the constituent parameters, including monodromy, by the discrete-spectrum parameter formula and local Langlands. Taking their sum proves (Us2).

Frobenius semisimplification leaves monodromy and inertia unchanged. An unramified twist likewise leaves inertia unchanged and preserves the vanishing of monodromy. Consequently zero monodromy of $\mathrm{rec}_{E_w}(\pi_w)$ implies $N = 0$. The Weil–Deligne comparison then gives $r(\gamma) = \rho(\gamma)$ for $\gamma \in I_{\bar{w}}$. The inertia image of ρ is finite, so every such matrix has finite order and is semisimple in characteristic zero. Its characteristic polynomial is that furnished by (Us2). This proves (Us3); the supercuspidal case is (Ln5) of Theorem A.4.9.

We prove the local assertion in (Us4) with the induction direction explicit. The finite-length representation $r_B(\pi_w)$ of the abelian torus is finite dimensional and decomposes into simultaneous generalized character spaces. If χ occurs, its generalized character space has a quotient χ; projection onto that space therefore gives a nonzero quotient $r_B(\pi_w) \to \chi$. Normalized Frobenius reciprocity gives a nonzero map

$$\pi_w \longrightarrow I(\chi_1, \dots, \chi_n).$$

It is injective because π_w is irreducible. Write $\chi_i = \mu_i|\cdot|^{s_i}$ with μ_i unitary. The absolute-value inequality is exactly $s_1 \leq \cdots \leq s_n$. Smooth contragredient duality is exact on admissible representations and carries this normalized induction to $I(\chi_1^{-1}, \dots, \chi_n^{-1})$. Hence $\pi_w^\vee$ is an irreducible

quotient of the latter induction. Its exponents $-s_i$ are nonincreasing, so Theorem A.4.23 identifies it with the singleton-segment Langlands quotient. By (Ln6) and (Ln1) of Theorem A.4.9,

$$\mathrm{rec}_{E_w}(\pi_w^\vee) \cong \bigoplus_i \mathrm{rec}_{E_w}(\chi_i^{-1}).$$

Duality in (Ln3) now gives $\mathrm{rec}_{E_w}(\pi_w) \cong \bigoplus_i \mathrm{rec}_{E_w}(\chi_i)$, so its monodromy vanishes. Combining this equality with (Us2) and the unramified normalization $|\det|^{(1-n)/2}$ gives precisely the characters λ_i of (Us4). Normalized Frobenius reciprocity and admissible smooth duality here are the standard induction–Jacquet adjunction of [24]; no comparison with an Iwahori Hecke operator is being assumed.

For (Us5), we compute the actual normalised Jacquet module by the segment formula of [332, Proposition 9.5]. Put $K = E_w$, let $\nu(g) = |\det g|_w$, and set

$$\xi_0 = |\cdot|_w^{(1-n)/2}$$

as a representation of $\mathrm{GL}_1(K)$. This is a smooth irreducible supercuspidal representation of $\mathrm{GL}_1(K)$: that group has no proper parabolic subgroup. The segment has length n, and its essentially square-integrable representation is

$$\delta([\xi_0, \nu^{n-1}\xi_0]) = \mathrm{St}_n.$$

The equality is the centered-character example in [332, Example 9.2]. In particular it is the ordinary Steinberg representation, not a determinant twist of it. Its segment orientation is the unique irreducible subrepresentation of $\nu^{(n-1)/2} \times \cdots \times \nu^{(1-n)/2}$, with all inductions normalised and all parabolics upper block-triangular.

For a character ξ of $K^\times$ and a segment of length $h > 1$, the published formula with $m = 1$, $i = p = 1$ reads

$$r_{h-1,1}\big(\delta([\xi, \nu^{h-1}\xi])\big) \cong \delta([\nu\xi, \nu^{h-1}\xi]) \boxtimes \xi.$$

The formula applies at each succeeding stage, since $\nu^a\xi$ is still an irreducible character of $\mathrm{GL}_1(K)$. Iterating it in the first block, and using the length-one identity $\delta([\xi,\xi]) = \xi$, gives

$$r_{B_h}\big(\delta([\xi, \nu^{h-1}\xi])\big) \cong \nu^{h-1}\xi \boxtimes \nu^{h-2}\xi \boxtimes \cdots \boxtimes \xi.$$

Here normalised Jacquet transitivity introduces no further twist. Indeed N_{B_h} is the semidirect product of the unipotent radical of $P_{h-1,1}$ and the upper unipotent subgroup in its first Levi block, so taking their coinvariants successively takes precisely N_{B_h}-coinvariants. For $t = \mathrm{diag}(t_1, \dots, t_h)$ the normalising factors multiply correctly because

$$\delta_{P_{h-1,1}}(t)\, \delta_{B_{h-1}}(\mathrm{diag}(t_1, \dots, t_{h-1})) = \left(\prod_{i<h} |t_i/t_h|_w\right)\left(\prod_{i<j<h} |t_i/t_j|_w\right) = \delta_{B_h}(t).$$

These are isomorphisms of modules at every step. The final module is one-dimensional, so there is neither an omitted extension nor a passage to its semisimplification.

Taking $h = n$ and $\xi = \xi_0$, the i-th character is

$$\nu^{n-i}\xi_0 = |\cdot|_w^{(n+1-2i)/2}.$$

On the other hand the determinant of the conjugation action of t on the upper unipotent radical gives

$$\delta_B(t) = \prod_{i<j} |t_i/t_j|_w = \prod_{i=1}^{n} |t_i|_w^{n+1-2i}.$$

Thus the preceding one-dimensional module is exactly $\delta_B^{1/2}$.

Twisting a representation by $\mu\circ\det$ leaves its unipotent coinvariant quotient space unchanged, since $\det u = 1$ for $u \in N_B$. Its torus action is multiplied by $\prod_i \mu(t_i)$. This proves

$$r_B\big(\mathrm{St}_n \otimes (\mu\circ\det)\big) = \boxtimes_{i=1}^{n} \mu|\cdot|_w^{(n+1-2i)/2}.$$

At a uniformiser its i-th absolute value is $|\mu(\varpi_w)|q_w^{-(n+1-2i)/2}$. The ratio of the value at $i+1$ to that at i is $q_w > 1$, so for $n \geq 2$ the descending-absolute-value hypothesis in (Us4) fails. For the character representation $\mu\circ\det$, the unipotent action is already trivial, and the definition of the normalised Jacquet functor directly gives

$$r_B(\mu\circ\det) = \boxtimes_{i=1}^{n} \mu|\cdot|_w^{-(n+1-2i)/2} = \mu\delta_B^{-1/2}.$$

Its successive absolute-value ratios at ϖ_w are $q_w^{-1} < 1$, so it satisfies the hypothesis. This proves (Us5). □

Theorem 7.1.9. *Let F be a totally real field with $[F:\mathbb{Q}] \geq 2$, E/F a totally imaginary quadratic extension, $n \geq 1$, U an inner form of the quasi-split unitary group in n variables for E/F with $U(F\otimes_{\mathbb{Q}}\mathbb{R})$ compact, σ an automorphic representation of $U(\mathbb{A}_F)$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, π a weak base change of σ to $\mathrm{GL}_n(\mathbb{A}_E)$ in the sense of [177], and $r\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ the continuous semisimple representation such that, for all but finitely many finite places w of E at which π_w is unramified, r is unramified at w and ι carries the characteristic polynomial of a geometric Frobenius at w to $\prod_\alpha(X - q_w^{(n-1)/2}\alpha)$, α over the Satake parameters of π_w. Let $w \mid \ell$ be a place of E, $\bar{w}$ a place of $\overline{\mathbb{Q}}$ above it, and $r_{\bar{w}} = r|_{D_{\bar{w}}}$, viewed as a representation of G_{E_w}. Then:*

(Ul1) *$r_{\bar{w}}$ is de Rham, and for every field embedding $\tau\colon E_w \to \overline{\mathbb{Q}}_\ell$ over $\mathbb{Q}_\ell$ its τ-Hodge–Tate weights are the n numbers $z - \frac{n-1}{2}$, z running over the infinitesimal character of π_∞ at the complex embedding $\iota\circ\tau|_E$; this infinitesimal character is the base change of that of σ_∞;*

(Ul2) *writing $\mathrm{WD}(r_{\bar{w}}) = (V,\rho,N)$, the semisimplification of ρ is isomorphic to that of the representation of $W_{\bar{w}}$ in $\iota^{-1}\mathrm{rec}_{E_w}(\pi_w \otimes |\det|_w^{(1-n)/2})$, and if the monodromy operator of $\mathrm{rec}_{E_w}(\pi_w)$ is zero then $N = 0$;*

(Ul3) *if the monodromy operator of $\mathrm{rec}_{E_w}(\pi_w)$ is zero (in particular if π_w is supercuspidal, or satisfies the Jacquet-module condition (Us4) of Theorem 7.1.8 at w), then $r_{\bar{w}}$ is potentially crystalline, $\mathrm{WD}(r_{\bar{w}})^{F\text{-}ss} \cong \iota^{-1}\mathrm{rec}_{E_w}(\pi_w \otimes |\det|_w^{(1-n)/2})$, and $r_{\bar{w}}|_{G_K}$ is crystalline for every finite extension K/E_w inside $\overline{\mathbb{Q}}_\ell$ such that ρ is trivial on the inertia subgroup of W_K;*

(Ul4) *if π_w is unramified, then $r_{\bar{w}}$ is crystalline, $N = 0$, ρ is trivial on the inertia subgroup of $W_{\bar{w}}$, and $\iota(\det(X - \rho(\Phi))) = \prod_\alpha(X - q_w^{(n-1)/2}\alpha)$ for every $\Phi \in W_{\bar{w}}$ with $n(\Phi) = -1$, α running over the Satake parameters of π_w with multiplicity; this holds in particular when w lies above a place v of F at which E/F and U are unramified and σ_v is spherical.*

Proof. By uniqueness in Theorem 7.1.4, the given r is $\bigoplus_j r_{\ell,\iota}(\Pi'_j)$, where $\Pi'_j = \Pi_j|\det|^{(m_j-n)/2}$ is regular algebraic, polarizable and cuspidal. All these hypotheses were proved there. For each summand the cuspidal Galois theorem gives de Rhamness and the Hodge–Tate weights of (Bg2). If z is an infinitesimal-character entry of Π_j, the corresponding entry for Π'_j is $z + (m_j - n)/2$, and its Hodge–Tate weight is

$$z + (m_j - n)/2 - (m_j - 1)/2 = z - (n-1)/2.$$

The infinitesimal character of π is the union of those of the Π_j, so taking the multiset union proves (Ul1). The assertion about the coefficient system of σ_∞ is the infinitesimal-character clause of [177, Corollaire 5.3].

Put

$$D_j = \mathrm{WD}(r_{\ell,\iota}(\Pi'_j)|_{D_{\bar{w}}}), \qquad D'_j = \iota^{-1}\mathrm{rec}_{E_w}(\Pi_{j,w}|\det|^{(1-n)/2}).$$

For $m_j \geq 2$, the monodromy-domination theorem [16, Theorem 1.1] gives $D_j^{F\text{-}ss} \prec D'_j$. In particular their semisimplified Weil representations agree. If D'_j has zero monodromy, each of its special-representation blocks has length one. The partial-sum inequality for its largest block then forces each block of D_j to have length at most one; hence D_j has zero monodromy. For $m_j = 1$, monodromy is automatically zero, and the equality of the Weil characters is (Bg5), applied to the rank-one local representation.

The parameter of $\pi_w|\det|^{(1-n)/2}$ is the direct sum of the D'_j, including monodromy. Indeed the local discrete-spectrum formula gives the constituent direct sum, and

$$(m_j - n)/2 + (1 - m_j)/2 = (1 - n)/2.$$

Weil–Deligne formation commutes with direct sums. Thus summing the semisimplified comparisons proves the first assertion of (Ul2). If the total automorphic monodromy is zero, each summand monodromy is zero; the preceding domination argument then gives zero monodromy for each D_j and hence for their sum. This proves all of (Ul2).

By (Ul1), $r_{\bar{w}}$ is de Rham and hence potentially semistable. When its monodromy is zero, the crystalline descent criterion of Theorem 7.1.10 says that it is potentially crystalline, and that its restriction to any finite extension K/E_w is crystalline whenever its Weil representation is trivial on inertia at K. Both parameters in (Ul2) now have zero monodromy. A Frobenius-semisimple Weil representation with finite inertia image is semisimple, so their equal semisimplifications are the asserted isomorphic Frobenius semisimplifications. This proves (Ul3). The supercuspidal and sorted-Jacquet cases have zero automorphic monodromy by the purely local assertions (Ln5) and (Us4), respectively; the latter argument concerns only $\mathrm{GL}_n(E_w)$ and is independent of whether w divides ℓ.

If π_w is spherical, the direct-sum local parameter formula makes each $\Pi_{j,w}$ spherical: its parameter is a direct summand of an unramified parameter with zero monodromy. Its unramified twist $\Pi'_{j,w}$ is therefore spherical too. For each such summand apply (Bg4) of Theorem 7.1.6. Their sum is crystalline, has zero monodromy and trivial Weil inertia, and its geometric-Frobenius eigenvalues are

$$q_w^{(m_j-1)/2} q_w^{(n-m_j)/2}\alpha = q_w^{(n-1)/2}\alpha.$$

The multiset union over j is the Satake multiset of π_w. Trivial inertia means this characteristic polynomial is the same for every Weil element of degree -1, proving (Ul4). Its last clause is the spherical base-change assertion (Ua0) of Theorem 7.1.4. □

Theorem 7.1.10. *Let ℓ be a prime, K a finite extension of $\mathbb{Q}_\ell$, and $r\colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ a continuous homomorphism. Then $r(G_K) \subseteq \mathrm{GL}_n(E)$ for some finite extension E of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$; r is called **de Rham**, **semistable** or **crystalline** if the $\mathbb{Q}_\ell$-linear representation E^n of G_K is, and **potentially semistable** (**potentially crystalline**) if $r|_{G_{K'}}$ is semistable (crystalline) for some finite extension K' of K inside $\overline{\mathbb{Q}}_\ell$; these notions do not depend on E.*

1. *Crystalline implies semistable, semistable implies potentially semistable, and r is de Rham if and only if it is potentially semistable. In particular $\mathrm{WD}(r)$ is defined by (Wd2) for every de Rham r.*

2. *Let r be potentially semistable, $\mathrm{WD}(r) = (V, \rho, N)$, and K' a finite extension of K inside $\overline{\mathbb{Q}}_\ell$. Then $r|_{G_{K'}}$ is semistable if and only if ρ is trivial on $I_{K'}$, and crystalline if and only if ρ is trivial on $I_{K'}$ and $N = 0$. In particular r is potentially crystalline if and only if $N = 0$.*

3. *Consequently r is crystalline if and only if r is de Rham and $\mathrm{WD}(r)$ is unramified, that is, ρ is trivial on I_K and $N = 0$.*

Proof. **The continuous representation has a finite coefficient field.** For each positive integer d, let E_d be the compositum in $\overline{\mathbb{Q}}_\ell$ of all extensions of $\mathbb{Q}_\ell$ of degree at most d. There are finitely many such extensions by Krasner's finiteness theorem for local fields, so E_d is finite, complete, and closed. Every algebraic element belongs to some E_d. Set

$$H_d = r^{-1}\big(\mathrm{GL}_n(E_d)\big).$$

This is a closed subgroup of G_K: $\mathrm{GL}_n(E_d)$ is the intersection of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with the closed matrix space $\mathrm{M}_n(E_d)$. The compact Hausdorff space G_K is the countable union of the H_d. Baire's theorem gives an H_d with nonempty interior. Translating an interior neighbourhood by an element of this subgroup shows that H_d contains a neighbourhood of 1, so it is open. Its index is finite by compactness. Choose coset representatives $g_1, \dots, g_s$. Adjoin the finitely many entries of $r(g_i)$ to E_d, obtaining a finite extension E. For $g = g_i h$ with $h \in H_d$, the matrix $r(g) = r(g_i)r(h)$ has entries in E and is invertible over E. This proves the field-of-values assertion. This is also the argument of Lemma 7.1.17(1).

The admissibility notions do not depend on coefficients. For a finite extension E_1/E, choose an E-basis of E_1. It identifies the underlying $\mathbb{Q}_\ell[G_K]$-module $E_1 \otimes_E E^n$ with the direct sum of $[E_1 : E]$ copies of E^n. Period invariants commute with finite direct sums. Their dimensions are bounded above by the representation dimensions for each of the three period rings, and equality for a finite direct sum is therefore equivalent to equality for every summand. Hence de Rham, semistable and crystalline admissibility are unchanged by extending E. Two possible coefficient fields are compared inside their finite compositum. The same argument after each finite extension of K proves independence for the potential notions.

We recall precisely the period facts used here: the comparison maps are injective, $B_{\mathrm{cris}}^{G_K} = B_{\mathrm{st}}^{G_K} = K_0$, $B_{\mathrm{dR}}^{G_K} = K$, and

$$B_{\mathrm{cris}} \subset B_{\mathrm{st}}, \qquad K \otimes_{K_0} B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}.$$

These are the regular-period-ring comparison statements of [44], Theorem 5.2.1, Theorem 9.2.10 and Proposition 9.2.11. They give the preceding dimension bounds, and show directly that crystalline implies semistable and semistable implies de Rham.

De Rham admissibility descends along finite extensions of K. To see this, it suffices to use a finite Galois extension L/K. The space $D_{\mathrm{dR},L}(U) = (B_{\mathrm{dR}} \otimes U)^{G_L}$ has a semilinear action of $\mathrm{Gal}(L/K)$ and its invariants are $D_{\mathrm{dR},K}(U)$. Finite Galois descent gives $L \otimes_K D_{\mathrm{dR},K}(U) \simeq D_{\mathrm{dR},L}(U)$. Thus if the latter has full dimension, so does the former. Passing to a Galois closure proves the assertion for every finite extension. Conversely, a de Rham comparison isomorphism remains an isomorphism after restriction, which proves preservation under restriction. Berger's monodromy theorem [22] now supplies de Rham implies potentially semistable. A potentially semistable representation is de Rham over an extension and hence over K by the descent just proved. This proves item 1.

Restriction preserves the completed module and its inertia and monodromy. Write $U = E^n$ as a $\mathbb{Q}_\ell$-representation, and form the completed module A in Fontaine's theorem

[86], Theorem 2.3.2 and Remark 2.3.3. For a finite extension K'/K, $I_{K'}$ is an open subgroup of I_K. The open subgroups contained in $I_{K'}$ are cofinal among those of I_K: replace any open H by $H \cap I_{K'}$. The defining union for A is therefore unchanged by restriction. The residue algebraic closure, its Witt fraction field P_0, and the operator N are unchanged as well. Fontaine's semistability and crystalline criteria applied over K' consequently say

$$U|_{G_{K'}} \text{ semistable} \iff I_{K'} \text{ acts trivially on } A,$$

$$U|_{G_{K'}} \text{ crystalline} \iff I_{K'} \text{ acts trivially on } A \text{ and } N = 0.$$

These criteria apply to the E-linear representation r with the stated Weil–Deligne coefficients. Indeed the E-action commutes with every period action. Fontaine's Weil functor commutes with scalar extension and direct sums. Extending the coefficient field to $\overline{\mathbb{Q}}_\ell$ decomposes the underlying $\mathbb{Q}_\ell$-representation into the components indexed by embeddings $E \hookrightarrow \overline{\mathbb{Q}}_\ell$; the chosen inclusion gives $\mathrm{WD}(r)$. The other components are its coefficient-Galois conjugates. Triviality of an inertia matrix and vanishing of a monodromy matrix are preserved and reflected by a field embedding, so they hold on the chosen component exactly when they hold on every component. Faithfulness of field extension then reflects them on the original module. In the Weil functor, the Frobenius correction is the identity on inertia; hence the inertia matrices are exactly $\rho|_{I_{K'}}$. Its monodromy is the same operator after the faithful scalar changes. No Frobenius eigenvalue or arithmetic/geometric Frobenius choice enters either criterion. This proves the two equivalences of item 2.

The inertia action has finite image. To kill it after a finite extension, let J be its open kernel in I_K. The subspace topology on $I_K \subset G_K$ gives an open neighbourhood U_0 of 1 in G_K with $U_0 \cap I_K \subset J$. Choose an open normal subgroup $H \subset U_0$ and let K' be its finite fixed field. Then $I_{K'} = H \cap I_K \subset J$. If $N = 0$, the crystalline criterion over this K' proves potential crystallinity. Conversely, crystallinity over any K' forces the same operator N to vanish, by the second equivalence. This proves the final assertion of item 2.

Finally, a crystalline r is de Rham by item 1 and has $\rho|_{I_K} = 1$, $N = 0$ by item 2. If r is de Rham and its Weil–Deligne representation is unramified in the stated sense, item 1 makes it potentially semistable and item 2 with $K' = K$ makes it crystalline. This proves item 3. □

Theorem 7.1.11. *Let F be a totally real field with $[F : \mathbb{Q}] \geq 2$, E/F a totally imaginary quadratic extension, $n \geq 1$, U an inner form of the quasi-split unitary group in n variables for E/F with $U(F \otimes_{\mathbb{Q}} \mathbb{R})$ compact, σ an automorphic representation of $U(\mathbb{A}_F)$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, π a weak base change of σ to $\mathrm{GL}_n(\mathbb{A}_E)$ in the sense of [177], and $r\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ the continuous semisimple representation such that, for all but finitely many finite places w of E at which π_w is unramified, r is unramified at w and ι carries the characteristic polynomial of a geometric Frobenius at w to $\prod_\alpha (X - q_w^{(n-1)/2}\alpha)$, α over the Satake parameters of π_w. Let $v \nmid \ell$ be a finite place of F split in E, $w \mid v$ a place of E and $\bar{w}$ a place of $\overline{\mathbb{Q}}$ above w; recall $\pi_w \cong \sigma_v \circ \iota_w^{-1}$ for the isomorphism $\iota_w\colon U(F_v) \to \mathrm{GL}_n(E_w)$ determined by w, so the hypothesis below may equally be read on σ_v. Let $n = n_1 + \cdots + n_k$ with $n_a \geq 1$, let $P \subseteq \mathrm{GL}_n$ be the standard (block upper triangular) parabolic subgroup with Levi factor $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$, let $J \subseteq \mathrm{GL}_n(\mathcal{O}_{E_w})$ be the preimage of $P(k_w)$ under reduction modulo ϖ_w, and let $\chi_1, \ldots, \chi_k\colon k_w^\times \to \mathbb{C}^\times$ be **pairwise distinct** characters. Suppose that π_w has a non-zero vector y with*

$$\pi_w(g)\, y = \prod_{a=1}^{k} \chi_a\big(\det \overline{g_{aa}}\big)\, y \qquad (g \in J),$$

where g_{aa} is the a-th diagonal block of g and the bar is reduction modulo ϖ_w. Then:

(Up1) $\pi_w \cong \mathrm{Ind}_P^{\mathrm{GL}_n}(\rho_1 \otimes \cdots \otimes \rho_k)$ *(normalised induction, irreducible), where* $\rho_a = \rho_a^\circ \otimes (\tilde{\chi}_a \circ \det)$ *with* ρ_a° *an irreducible unramified (spherical) representation of* $\mathrm{GL}_{n_a}(E_w)$ *and* $\tilde{\chi}_a$ *a character of* $E_w^\times$ *whose restriction to* $\mathcal{O}_{E_w}^\times$ *is* χ_a *composed with reduction;*

(Up2) *the monodromy operators of* $\mathrm{rec}_{E_w}(\pi_w)$ *and of* $\mathrm{WD}(r|_{D_{\bar{w}}})$ *are zero;*

(Up3) *for every* $\gamma \in I_{\bar{w}}$, $r(\gamma)$ *is semisimple and*

$$\det(X - r(\gamma)) = \prod_{a=1}^{k} (X - \iota^{-1}\chi_a(\bar{u}))^{n_a},$$

where $u \in \mathcal{O}_{E_w}^\times$ *is such that the image of* γ *in the abelianised Weil group of* E_w *is* $\mathrm{Art}_{E_w}(u)^{-1}$*; in particular* $\prod_a (X - \iota^{-1}\chi_a(\bar{u}))$ *kills* $r(\gamma)$.

This covers in particular: $k = 1$ *(*$J = \mathrm{GL}_n(\mathcal{O}_{E_w})$ *acting through* $\chi_1 \circ \det$*, any* χ_1*); and* $k = 2$*,* $(n_1, n_2) = (1, n-1)$*,* $\chi_2 = 1 \neq \chi_1$ *(*J *the stabiliser of a line modulo* ϖ_w*, acting through* $\chi_1(\overline{g_{11}})$*). It does not cover types with* $\chi_a = \chi_b$ *for some* $a \neq b$ *(for instance the Iwahori subgroup with trivial character, where* St_n *occurs); those are governed by the Jacquet-module criterion with antitone exponents and the Steinberg exclusion stated for split places not above* ℓ.

Proof. Apply Proposition 7.1.12 with local field $K = E_w$, the given block partition, the pairwise distinct characters χ_a and the nonzero vector y. Its hypotheses are exactly the parahoric transformation law in the statement. It yields the irreducible normalized induction and the twisted spherical representations ρ_a in (Up1).

Each spherical block ρ_a° is the Langlands quotient of a normalized induction of unramified characters. Twisting it by $\tilde{\chi}_a \circ \det$ gives the corresponding quotient of the twisted characters. Local Langlands therefore gives a parameter with zero monodromy whose inertia is the scalar character $\mathrm{rec}_{E_w}(\tilde{\chi}_a)|_I$ repeated n_a times. The parameter of the irreducible normalized induction of the ρ_a is their direct sum: concatenate their Langlands multisegments and use (Ln6); irreducibility identifies this induction with its Langlands quotient. Consequently

$$\mathrm{rec}_{E_w}(\pi_w)|_I \cong \bigoplus_a \mathrm{rec}_{E_w}(\tilde{\chi}_a)|_I^{\oplus n_a}, \qquad N_{\mathrm{rec}(\pi_w)} = 0.$$

Here twisting a spherical parameter by a character uses (Ln2), and unramified parameters have zero monodromy.

The global hypotheses and the split place $v \nmid \ell$ meet those of Theorem 7.1.8. Its (Us2) identifies the Frobenius-semisimplified Galois parameter with $\iota^{-1}\mathrm{rec}_{E_w}(\pi_w |\det|^{(1-n)/2})$; the unramified norm twist changes neither inertia nor zero monodromy. By (Us3), the Galois monodromy vanishes and every $r(\gamma)$ for $\gamma \in I_{\bar{w}}$ is semisimple. This proves (Up2).

For u as in (Up3), the arithmetic Artin convention is the inverse of the geometric Artin convention in local Langlands. Write $\lambda_a = \iota^{-1}\mathrm{rec}_{E_w}(\tilde{\chi}_a)$ for the transported one-dimensional parameter. Then

$$\iota(\lambda_a(\mathrm{Art}_{E_w}(u)^{-1})) = \tilde{\chi}_a(u) = \chi_a(\bar{u}).$$

The factor $|u|^{(1-n)/2}$ is 1. The direct sum above consequently gives the characteristic polynomial in (Up3), with multiplicities n_a. Since $r(\gamma)$ is semisimple, its minimal polynomial is a product of distinct linear factors drawn from those eigenvalues. It therefore divides $\prod_a (X - \iota^{-1}\chi_a(\bar{u}))$, even if some of these evaluated character values coincide. This proves the smaller annihilator too.

For one block the stated type is precisely a determinant twist of the maximal compact type. For the partition $(1, n-1)$, the inverse image of the upper block parabolic is the stabilizer

of a line modulo the uniformizer; its determinant character on the first block is $\chi_1(\bar{g}_{11})$. The hypothesis $\chi_1 \neq 1 = \chi_2$ is exactly the distinctness needed by the local block theorem. If two block characters are equal, that hypothesis fails; no conclusion is deduced by this argument. The separate sorted-Jacquet criterion is (Us4) of Theorem 7.1.8. □

Proposition 7.1.12. *Let K be a nonarchimedean local field of characteristic zero, with integer ring $\mathcal{O}_K$, residue field k_K and a uniformizer. Let $n = n_1 + \cdots + n_d$ with $n_a \geq 1$, let P be the standard upper block triangular parabolic of GL_n, and let J be the inverse image of $P(k_K)$ in $\mathrm{GL}_n(\mathcal{O}_K)$. Let $\chi_a : k_K^\times \to \mathbb{C}^\times$ be pairwise distinct characters. Suppose an irreducible smooth representation π of $\mathrm{GL}_n(K)$ contains a nonzero vector v satisfying*

$$\pi(g)v = \prod_{a=1}^{d} \chi_a(\det \overline{g_{aa}})\, v \qquad (g \in J).$$

Then

$$\pi \cong \mathrm{Ind}_{P(K)}^{\mathrm{GL}_n(K)}(\rho_1 \otimes \cdots \otimes \rho_d)$$

with normalized induction, which is irreducible, where $\rho_a = \rho_a^\circ \otimes (\widetilde{\chi}_a \circ \det)$, ρ_a° is irreducible spherical on $\mathrm{GL}_{n_a}(K)$, and $\widetilde{\chi}_a : K^\times \to \mathbb{C}^\times$ restricts on $\mathcal{O}_K^\times$ to χ_a composed with reduction.

Proof. Write $G = \mathrm{GL}_n(K)$ and $M = \prod_a \mathrm{GL}_{n_a}(K)$. Let $I \subseteq J$ be the upper Iwahori subgroup and put $I_M = I \cap M$ and $K_M = \prod_a \mathrm{GL}_{n_a}(\mathcal{O}_K)$. Inflate the character

$$\eta(\mathrm{diag}(t_i)) = \prod_a \chi_a\left(\prod_{i \text{ in block } a} \overline{t_i} \right)$$

from the diagonal residue torus to a character λ of I. The character prescribed in the hypothesis on J will be denoted θ. Its restriction to I is λ, so v belongs to the nonzero (I, λ)-isotypic space of π. Since the block characters are pairwise distinct, the roots on which η is trivial are exactly the roots within individual blocks. Hence the equal-character group in Roche's depth-zero type theorem is precisely M, with no additional permutation of blocks. The theorem applies to general linear groups for every residue characteristic; see [243, Remark4.14 and Theorems7.7,8.2,9.4].

We identify the effect of the larger compact group J under this equivalence. Choose each $\widetilde{\chi}_a$ to have value 1 at the uniformizer. Within a block, untwisting by $\widetilde{\chi}_a \circ \det$ turns λ into the trivial Iwahori character. Let $q = |k_K|$ and $W_M = \prod_a S_{n_a}$. Give I volume 1, and denote by f_w the twisted Iwahori Hecke basis function for $w \in W_M$, normalized by $f_w(w) = \theta(w)^{-1}$. Finite Bruhat decomposition of each diagonal block gives

$$J = \coprod_{w \in W_M} IwI, \qquad \mathrm{vol}(J) = \sum_{w \in W_M} q^{\ell(w)}.$$

The operator projecting onto the θ-isotypic space for J is therefore the Hecke idempotent

$$e_{J,\theta} = \frac{\sum_{w \in W_M} f_w}{\sum_{w \in W_M} q^{\ell(w)}}.$$

Indeed it is integration of $\theta(j)^{-1}\pi(j)$ over J, divided by its volume; integration over any left I-coset has weight 1. On every within-block finite Hecke algebra the type isomorphism, after the determinant untwist just made, sends f_w to the ordinary basis element T_w for (M, I_M). Thus it sends $e_{J,\theta}$ to

$$\frac{\sum_{w \in W_M} T_w}{\sum_{w \in W_M} q^{\ell(w)}} = e_{K_M},$$

the projector onto K_M-fixed vectors. This calculation also checks the scalar normalization; a support correspondence alone would not suffice.

Let $\rho^\circ = \mathcal{F}(\pi)$ under the exact type equivalence. Irreducibility is preserved, and $e_{J,\theta}v = v \neq 0$, so the projector calculation shows that ρ° is spherical on M. By finite-product decomposition and the spherical principal-series parametrization,

$$\rho^\circ \simeq \bigotimes_a \rho_a^\circ,$$

where every ρ_a° is irreducible spherical on $\mathrm{GL}_{n_a}(K)$. Here one uses the classical admissibility of irreducible smooth general-linear representations, so the finite-product decomposition applies. See [23] and [80] for this standard product and admissibility setting, and [318] for the spherical parametrization.

For each a, choose unramified characters $\nu_{a,1}, \dots, \nu_{a,n_a}$ in a Langlands order such that ρ_a° is the unique irreducible quotient of their normalized upper-Borel induction. Let ν be their concatenation on $T(K)$. Then ρ° is the unique irreducible quotient of $i_{B\cap M}^M \nu$. To see uniqueness directly, any irreducible quotient on the product group is a tensor product; restriction of its quotient map to each factor forces that factor to be the unique irreducible quotient of the corresponding ordered principal series. Normalized torus-induction compatibility of $\mathcal{F}$ consequently shows that π is the unique irreducible quotient of

$$i_B^G(\widetilde{\eta}\nu),$$

where $\widetilde{\eta}$ is the concatenation of the extended block characters. No order comparison between two different blocks is required here.

Put $\rho_a = \rho_a^\circ \otimes (\widetilde{\chi}_a \circ \det)$. Its cuspidal support consists of unramified twists of the character $\widetilde{\chi}_a$ of $\mathrm{GL}_1(K)$. For $a \neq b$, these cuspidal lines are different: an unramified twist does not change the restriction to $\mathcal{O}_K^\times$, whereas $\chi_a \neq \chi_b$. Thus

$$\Sigma = i_{P(K)}^G \left(\bigotimes_a \rho_a \right)$$

is irreducible by [332, Proposition8.5]; the hypothesis of that result is satisfied because no segment on one of these lines can be linked to a segment on another. Exactness and transitivity of normalized parabolic induction, applied to the block quotient maps chosen above, give a surjection $i_B^G(\widetilde{\eta}\nu) \twoheadrightarrow \Sigma$. Its unique irreducible quotient is π, so $\Sigma \simeq \pi$. This proves every assertion, with the required unit restrictions on the characters $\widetilde{\chi}_a$. □

Theorem 7.1.13. *Let F be a nonarchimedean local field, $G = \mathrm{GL}_n(F)$, T its diagonal torus, B its upper Borel and I its standard Iwahori subgroup. Let $\eta : T(\mathcal{O}_F) \to \mathbb{C}^\times$ be trivial on $T(1+\mathfrak{p}_F)$, with equal coordinate characters occurring in contiguous blocks. Let M be the corresponding block Levi and $I_M = I \cap M$. Inflate η to a character λ of I. Extend each block character to $F^\times$ with value 1 at a fixed uniformizer, obtaining $\widetilde{\eta}$ on $T(F)$.*

There is an exact equivalence between the category generated by the (I, λ)-isotypic spaces in G and the category generated by the I_M-fixed spaces in M. It comes from a support-preserving Hecke algebra isomorphism and satisfies

$$\mathcal{F}\big(i_B^G(\widetilde{\eta}\nu)\big) \simeq i_{B\cap M}^M(\nu)$$

for every unramified character ν of $T(F)$, with normalized induction. On each finite Weyl-group Hecke subalgebra inside a block, the isomorphism is the usual Iwahori Hecke isomorphism after untwisting by the extended block determinant character.

Proof. Write $\eta(t_1,\dots,t_n)=\prod_i \eta_i(t_i)$ and let q be the residue cardinality. Each ratio $\eta_i\eta_j^{-1}$ is trivial on $1+\mathfrak{p}_F$. Its conductor in the convention of Theorem A.5.32 is therefore one, including when the ratio is trivial. The positive root depth is $\lfloor 1/2\rfloor = 0$, and the negative root depth is $\lceil 1/2\rceil = 1$. Thus the subgroup in that theorem is generated by diagonal units, upper elementary matrices with integral entries and lower elementary matrices with entries in $\mathfrak{p}_F$. These generate exactly I: elimination of the reduction modulo $\mathfrak{p}_F$, followed by integral elementary elimination, gives its Iwahori decomposition. Its character is the prescribed inflation λ, since it is η on diagonal units and trivial on both unipotent factors.

The character on the coroot $u\mapsto \operatorname{diag}(1,\dots,u,\dots,u^{-1},\dots,1)$ is $\eta_i(u)\eta_j(u)^{-1}$. It is trivial exactly when the two coordinate characters agree. Consequently the equal-character root system consists precisely of the roots within the stated contiguous blocks. The associated group is M, with the inherited upper Borel $B\cap M$ and Iwahori I_M. Its Weyl group is the product of the symmetric groups on those blocks. A permutation stabilizes η precisely when it preserves each equal-character class, so this Weyl group is the full stabilizer; there is no additional component group.

Extend a block character by

$$\widetilde{\eta}_a(\varpi^m u)=\eta_a(u)\qquad (m\in\mathbb{Z},\ u\in\mathcal{O}_F^\times).$$

This is a smooth character, has value one at the fixed uniformizer, and extends the prescribed unit character. Its product on the diagonal torus is exactly the extension used in Theorem A.5.32. The type equivalences and support-preserving algebra isomorphism of that theorem therefore have exactly the two categories in the statement. The isotypic functors are exact: averaging a smooth vector over a compact subgroup with its inverse character is an idempotent projector, and applying that projector to a lift in a surjection gives a lift in the isotypic subspace. Restriction of scalars through the Hecke algebra isomorphism is exact as well. Composing with the inverse equivalence gives the asserted exact functor $\mathcal{F}$.

Its torus-induction compatibility is the normalized one in Theorem A.5.32. Explicitly, our induction consists of functions transforming under the upper Borel by $\delta_B^{1/2}\widetilde{\eta}\nu$, and on M by $\delta_{B\cap M}^{1/2}\nu$. These are the same upper Borels and positive roots used in that theorem. For a diagonal element t,

$$\delta_B(t)=\prod_{i<j}|t_i/t_j|_F,\qquad \delta_{B\cap M}(t)=\prod_{\substack{i<j\\ i,j\text{ in one block}}}|t_i/t_j|_F.$$

The remaining factors are the modulus of the block parabolic: on $m=\operatorname{diag}(m_a)$ it is

$$\delta_P(m)=\prod_{a<b}|\det m_a|_F^{n_b}|\det m_b|_F^{-n_a}.$$

Thus the normalized torus embeddings have the prescribed half-moduli, rather than an unnormalized induction with a discarded cross-block factor. The published normalized compatibility yields $\mathcal{F}(i_B^G(\widetilde{\eta}\nu))\simeq i_{B\cap M}^M(\nu)$ with precisely these conventions.

It remains to determine the finite-block scalar of the same support-preserving isomorphism. Give I and I_M mass one, and use the Hecke convention

$$f(i_1gi_2)=\lambda(i_1)^{-1}f(g)\lambda(i_2)^{-1},\qquad f\cdot v=\int_G f(g)\pi(g)v\,dg.$$

For one block define $\theta_a=\widetilde{\eta}_a\circ\det$. It extends λ on that block's Iwahori. Multiplication of a scalar Hecke function by θ_a^{-1} identifies the untwisted convolution algebra on this block with the

twisted one: in the convolution integral the two factors of the character multiply to $\theta_a(g)^{-1}$. In particular it sends the unit to the unit.

Let s interchange two adjacent coordinates within the block, choose its permutation matrix $\dot{s}$, and let f_s be the twisted function supported on $I\dot{s}I$ with value $\theta_a(\dot{s})^{-1}$ there. The character extends to the rank-one parahoric generated by I and $\dot{s}$, through the block determinant. After untwisting, reduction to its two-dimensional residue quotient gives the ordinary double coset of a Borel in $\mathrm{GL}_2(k_F)$. It has q right Borel cosets. For the convolution square, the coefficient at the identity counts these q cosets; the coefficient at $\dot{s}$ counts the $q-1$ nonzero residue parameters. There is no other support, since this rank-one quotient has exactly the two Bruhat cells. Hence, with e_λ the unit,

$$f_s * f_s = q e_\lambda + (q-1) f_s.$$

The identical count for M gives $T_s^2 = q + (q-1)T_s$ for its ordinary Iwahori generator.

Choose the support-preserving isomorphism supplied by Theorem A.5.32, in the direction from the M algebra to the G algebra. Its image of T_s has support exactly $I\dot{s}I$, whose twisted function space is one dimensional. It is therefore $c_s f_s$ for some nonzero scalar c_s. Applying the isomorphism to the quadratic relation and comparing the disjoint identity and reflection supports gives

$$c_s^2 = 1, \qquad c_s^2(q-1) = c_s(q-1).$$

Since $q > 1$, these equalities force $c_s = 1$. Thus the normalized-induction-compatible isomorphism itself, without changing its torus action, has the required simple-generator normalization. For a reduced word in the finite block Weyl group, multiplication of the corresponding root subgroups gives the usual length-additive double-coset product; its residue parameters are independent, so the leading coefficient is one. Multiplying the simple generators therefore gives the same basis element for every finite Weyl-group element. This proves the last assertion on each finite block algebra, with the exact determinant untwist and no undetermined sign or scalar. □

Proposition 7.1.14. *Let F^+ be a totally real field, G a connected reductive group over F^+ such that $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, $U_0 \supseteq U \supseteq V$ compact open subgroups of $G(\mathbb{A}_{F^+}^\infty)$ with V normal in U, and $\Delta = U/V$. Let A be a commutative ring, W an A-module with an A-linear action of U, and*

$$S(V,W) = \{ f\colon G(\mathbb{A}_{F^+}^\infty) \to W \ :\ f(\gamma g v) = v^{-1} f(g) \text{ for all } \gamma \in G(F^+),\ g \in G(\mathbb{A}_{F^+}^\infty),\ v \in V \},$$

with Δ acting by $(uf)(g) = u\, f(gu)$. Then:

(HM1) *for every $t \in G(\mathbb{A}_{F^+}^\infty)$ the group $\Gamma_t = t^{-1}G(F^+)t \cap U_0$ is finite, and only finitely many orders $\#\Gamma_t$ occur; let c be their least common multiple;*

(HM2) *there is an A-linear endomorphism Ψ of $S(V,W)$ such that*

$$\sum_{u \in \Delta} u^{-1}\, \Psi\, u = c \cdot \mathrm{id}_{S(V,W)};$$

(HM3) *in particular, if c is invertible in A, then $S(V,W)$ is a direct summand of an induced $A[\Delta]$-module, hence projective over $A[\Delta]$ when $S(V,W)$ is projective over A (for instance when W is projective over A and $\Gamma_t \cap V = \{1\}$ for every t); and if $\Gamma_t \cap U = \{1\}$ for every t, one may take $c = 1$.*

Proof. Put $S = S(V, W)$. The action is well defined: for $a \in U$, $v \in V$,

$$(af)(\gamma g v) = af(ga(a^{-1}va)) = a(a^{-1}va)^{-1} f(ga) = v^{-1}(af)(g).$$

Also $(vf)(g) = vf(gv) = f(g)$, so it factors through the finite group Δ.

We first prove (HM1). Choose a faithful algebraic representation of G over F^+. The matrix entries of the compact finite adelic set tU_0t^{-1} lie in a fixed fractional ideal at all finite places. The archimedean matrix entries of $G(F^+)$ lie in a bounded set, because $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact. A fractional ideal is a lattice in the product of archimedean completions, so only finitely many rational matrices meet both conditions. Thus Γ_t is finite. Borel's finiteness of class numbers gives finitely many representatives for $G(F^+)\backslash G(\mathbb{A}_{F^+}^{\infty})/U_0$, by [33]. If $t' = \gamma t a$ with $\gamma \in G(F^+)$ and $a \in U_0$, then $\Gamma_{t'} = a^{-1}\Gamma_t a$. Consequently only finitely many orders occur, and their least common multiple c is a positive integer.

Let $X = G(F^+)\backslash G(\mathbb{A}_{F^+}^{\infty})/V$. This is finite by the same finiteness theorem, applied to V. It has a right Δ action. For $x = G(F^+)tV$ its stabilizer is

$$H_x = (t^{-1}G(F^+)t \cap U)V/V.$$

Indeed $G(F^+)tuV = G(F^+)tV$ if and only if $u \in (t^{-1}G(F^+)t \cap U)V$. Thus H_x is an image of a subgroup of Γ_t, and $\#H_x$ divides c.

Choose one point from each Δ orbit and let $R \subseteq X$ be their set. Define an integer-valued function on X by

$$b(x) = \begin{cases} c/\#H_x, & x \in R, \\ 0, & x \notin R. \end{cases}$$

Define $(\Psi f)(g) = b([g])f(g)$. The scalar function is left $G(F^+)$-invariant and right V-invariant. Hence Ψ is an A-linear endomorphism of S. For $u \in \Delta$, computed using any lift to U,

$$(u^{-1}\Psi u f)(g) = b([g]u^{-1})f(g).$$

For each $x \in X$ there are exactly $\#H_x$ elements $u \in \Delta$ for which xu^{-1} is the selected representative of its orbit. All stabilizers on an orbit have the same order. Therefore

$$\sum_{u \in \Delta} b(xu^{-1}) = c, \qquad \sum_{u \in \Delta} u^{-1}\Psi u = c\,\mathrm{id}_S,$$

which proves (HM2).

If c is invertible, put $\Psi' = c^{-1}\Psi$. The multiplication map $p : A[\Delta] \otimes_A S \to S$, $p(u \otimes s) = us$, has the section

$$j(s) = \sum_{u \in \Delta} u^{-1} \otimes \Psi'(us).$$

The displayed averaging identity gives $pj = \mathrm{id}_S$. For $a \in \Delta$, substituting $v = ua$ shows $j(as) = aj(s)$, so the section is $A[\Delta]$-linear. Thus S is a direct summand of an induced module. If S is A-projective, $A[\Delta] \otimes_A S$ is $A[\Delta]$-projective, and so is S. If $\Gamma_t \cap V = \{1\}$ for every t, evaluation at representatives of X identifies S with a finite direct sum of copies of W: the unique right V expression makes arbitrary values in W permissible. This proves the stated sufficient condition for A-projectivity.

Finally, if $\Gamma_t \cap U = \{1\}$ for all t, all the groups H_x are trivial. Taking b to be the indicator of R gives the same identity with multiplier 1. This is the last assertion of (HM3). □

Theorem 7.1.15. *Let E be a number field, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, and χ an algebraic Hecke character of E of infinity type $n = (n_\tau)_{\tau\colon E\to\mathbb{C}}$. Let $\mu\colon \mathrm{Gal}(\overline{E}/E) \to \overline{\mathbb{Q}}_\ell^\times$ be the continuous ℓ-adic character attached to (χ,ι) through the Artin map. Then for every place w of E above ℓ, the restriction of μ to a decomposition group at w is de Rham and potentially crystalline, it is crystalline if χ is unramified at w, and for every embedding $\tau\colon E \to \overline{\mathbb{Q}}_\ell$ inducing w its Hodge–Tate weight at τ is $-n_{\iota\circ\tau}$.*

Proof. Fix $w \mid \ell$, put $K = E_w$, and write $\eta = \mu|_{G_K}$. By Lemma 7.1.17(1), the values of η lie in a finite extension $L/\mathbb{Q}_\ell$. Enlarge L to contain all images of K in $\overline{\mathbb{Q}}_\ell$ and the finitely many roots of unity occurring below. Such enlargement does not change any admissibility property: as an underlying $\mathbb{Q}_\ell$-representation it replaces a representation by finitely many copies, and period invariants commute with finite direct sums. Set $V = L(\eta)$ and $d = [L : \mathbb{Q}_\ell]$. Its image lies in $\mathcal{O}_L^\times$, since the image of the compact group G_K under the valuation homomorphism to $\mathbb{Z}$ is a finite subgroup and hence zero.

The actual reciprocity formula determines the weights and the finite unit factor. For $\tau : K \hookrightarrow L$, let $a_\tau = n_{\iota\circ(\tau|_E)}$ and put

$$A(x) = \prod_{\tau:K\hookrightarrow L} \tau(x)^{a_\tau} \qquad (x \in K^\times).$$

This is the map on $\mathbb{Q}_\ell$-points of a morphism $\mathrm{Res}_{K/\mathbb{Q}_\ell}\mathbb{G}_m \to \mathrm{Res}_{L/\mathbb{Q}_\ell}\mathbb{G}_m$. Apply the defining avatar formula to the idele with component $u \in \mathcal{O}_K^\times$ at w and component 1 elsewhere. The use of arithmetic reciprocity for $\mu = \chi_\iota^{-1}$ gives

$$\eta(\mathrm{Art}_K(u)) = \iota^{-1}\chi_w(u)\,A(u). \tag{1}$$

The local Hecke character is smooth on $\mathcal{O}_K^\times$, so $\epsilon_0(u) = \iota^{-1}\chi_w(u)$ is a finite-image character there. On its open kernel, (1) is exactly the local algebraicity formula. Tate's character theorem [261], Chapter III, Appendix A.5–A.6, therefore gives Hodge–Tate weight $-a_\tau$ at τ. In particular the arithmetic Artin map has not introduced an additional inversion: on a unit its algebraic exponent is a_τ, whereas the weight is $-a_\tau$ when the cyclotomic character has weight 1.

We next prove potential crystallinity using the whole unit group. Choose a uniformiser ϖ_K and extend ϵ_0 to the finite-image continuous character

$$\epsilon_K : K^\times = \varpi_K^{\mathbb{Z}}\mathcal{O}_K^\times \longrightarrow L^\times, \qquad \epsilon_K(\varpi_K^m u) = \epsilon_0(u).$$

Local class field theory gives a finite-image character $\epsilon : G_K^{\mathrm{ab}} \to L^\times$ with $\epsilon \circ \mathrm{Art}_K = \epsilon_K$. The character $\eta_0 = \eta\epsilon^{-1}$ now satisfies

$$\eta_0(\mathrm{Art}_K(u)) = A(u) \quad \text{for every } u \in \mathcal{O}_K^\times.$$

The crystalline unit criterion of Fontaine–Serre [264], Section 2.3, makes $L(\eta_0)$ crystalline over K.

Let F/K be the finite extension cut out by $\ker\epsilon$. On G_F the characters η and η_0 coincide. Crystalline admissibility is preserved under this restriction, as follows directly from crystalline comparison. If $M_0 = D_{\mathrm{cris},K}(L(\eta_0))$, comparison identifies $B_{\mathrm{cris}} \otimes_{K_0} M_0$ with $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L(\eta_0)$. Thus a K_0-basis of M_0 remains linearly independent over F_0 inside the periods, where F_0 is the maximal unramified subfield of F. The subspace $F_0 \otimes_{K_0} M_0$ of $D_{\mathrm{cris},F}(L(\eta_0))$ has dimension d over F_0; the comparison dimension bound is d, so equality holds. Hence $V|_{G_F}$ is crystalline. This proves potential crystallinity of η. By Theorem 7.1.10(1), crystalline implies semistable and potential semistability implies de Rham over K. Therefore η is de Rham. If χ is unramified

at w, then $\epsilon_0 = 1$ and (1) agrees with A on all of $\mathcal{O}_K^\times$ already; the same crystalline unit criterion proves crystallinity over K itself.

For completeness, the coefficient rank of the semistable period union can be proved rather than inferred from its $\mathbb{Q}_\ell^{\mathrm{ur}}$-dimension. This also supplies the rank-one period assertion with the actual coefficient action. Retain the extension F over which V is crystalline, and set

$$M = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)^{G_F}, \qquad C = L \otimes_{\mathbb{Q}_\ell} F_0, \qquad R = L \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}.$$

The L-action on V commutes with G_F, so M is a C-module. Crystalline comparison is L-linear and gives an R-module isomorphism

$$R \otimes_C M = B_{\mathrm{cris}} \otimes_{F_0} M \;\simeq\; B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L = R. \tag{2}$$

The algebra C is finite étale over F_0, hence a finite product of fields. The map $C \to R = C \otimes_{F_0} B_{\mathrm{cris}}$ is faithfully flat, since B_{cris} is a nonzero F_0-vector space. Write $C = \prod_i C_i$ and $M_i = e_i M$ for its field factors and idempotents. Each M_i is a finite-dimensional C_i-space, of some dimension r_i. Faithful flatness gives $R_i = e_i R \neq 0$, and (2) gives $R_i^{r_i} \simeq R_i$. Reducing this isomorphism modulo a maximal ideal of the nonzero commutative ring R_i gives an isomorphism of vector spaces of dimensions r_i and 1. Thus every $r_i = 1$. Choosing a basis in each factor proves

$$M \simeq C$$

as C-modules. In particular freeness of rank one is a consequence of comparison and faithful descent, not of the dimension count alone.

Put $Q = \mathbb{Q}_\ell^{\mathrm{ur}}$ inside the canonical period rings and

$$D = \bigcup_{K'/K \text{ finite}} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_\ell} V)^{G_{K'}}.$$

There is a natural $L \otimes_{\mathbb{Q}_\ell} Q$-linear injection

$$Q \otimes_{F_0} M \;\longrightarrow D. \tag{3}$$

Indeed crystalline comparison makes an F_0-basis of M linearly independent over B_{cris}, and therefore over its subfield Q, proving injectivity. Every element of the left side is a finite sum of products of periods fixed by G_F and elements of finite unramified extensions of $\mathbb{Q}_\ell$. One finite extension of K containing F and all these unramified elements fixes that sum, proving that (3) has image in D. The representation is de Rham, as proved above; Berger's dimension theorem [22] gives $\dim_Q D = d$. The source of (3) also has dimension d over Q. Therefore (3) is an isomorphism, and

$$D \simeq Q \otimes_{F_0} C \simeq L \otimes_{\mathbb{Q}_\ell} Q$$

as modules over $L \otimes_{\mathbb{Q}_\ell} Q$. This proves the asserted coefficient freeness explicitly. Moreover this description represents every element of D by crystalline periods; the canonical monodromy vanishes on B_{cris} and on Q [85], Section 3.2, so it vanishes on D.

Finally, embeddings $\tau : E \to \overline{\mathbb{Q}}_\ell$ inducing w are exactly the restrictions of the embeddings of K used in (1). The weight already computed is therefore $-n_{\iota\circ\tau}$ for every such τ, which proves all the claims. □

Definition 7.1.16. Let $F \subset \overline{\mathbb{Q}}$ be a number field and ℓ a prime. Let L be a field with $\mathbb{Q}_\ell \subseteq L \subseteq \overline{\mathbb{Q}}_\ell$ which is finite over $\mathbb{Q}_\ell$ or equal to $\overline{\mathbb{Q}}_\ell$, with the topology of $|\cdot|_\ell$, and let V be a finite-dimensional L-vector space with a continuous L-linear action of G_F. For a finite place

v of F, a **decomposition group at** v is the image $D_j \subseteq G_F$ of $\mathrm{Gal}(\overline{F_v}/F_v)$ under restriction along an embedding $j\colon \overline{\mathbb{Q}} \to \overline{F_v}$ over F; $I_j \subseteq D_j$ is the image of the inertia group. For a closed subgroup $D \subseteq G_F$, $H^1(D,V)$ is computed with continuous cocycles.

Local conditions. Let v be a place of F and D a decomposition group at v (for v archimedean: the image of $\mathrm{Gal}(\mathbb{C}/F_v)$ under an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$ over F). The subspace $H^1_f(D,V) \subseteq H^1(D,V)$ consists of the classes $[c]$ such that:

- if v is finite and $v \nmid \ell$, $D = D_j$: there is $x \in V$ with $c(g) = gx - x$ for all $g \in I_j$;
- if $v \mid \ell$, $D = D_j$: there is $y \in V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $c(g) \otimes 1 = gy - y$ for all $g \in D_j$, where $D_j \cong \mathrm{Gal}(\overline{F_v}/F_v)$ acts on V and on B_{cris} (formed for $\overline{F_v} = \overline{\mathbb{Q}}_\ell$) diagonally;
- if v is archimedean: $H^1_f(D,V) = 0$.

Rational coefficients. A continuous 1-cocycle $c\colon G_F \to V$ is a **Bloch–Kato cocycle** if for every place v of F and every decomposition group D at v the class of $c|_D$ lies in $H^1_f(D,V)$. The **Bloch–Kato Selmer group**

$$H^1_f(F,V) \subseteq H^1(F,V)$$

is the space of classes of Bloch–Kato cocycles.

Divisible coefficients [158], 14.1. Let L be finite over $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$, and let $T \subset V$ be a G_F-stable $\mathcal{O}_L$-lattice; $V/T \cong T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is a discrete G_F-module. The **Selmer group** of T over F is

$$\mathrm{Sel}(F,T) = \{x \in H^1(F,V/T) \ :\ x|_D \in \mathrm{Im}(H^1_f(D,V) \to H^1(D,V/T)) \\ \text{for every place } v \text{ and every decomposition group } D \text{ at } v\}.$$

Lemma 7.1.17. *Let ℓ be a prime.*

1. *Let Γ be a profinite group and $\rho\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ a continuous homomorphism. There is a finite extension $E \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ with $\rho(\Gamma) \subseteq \mathrm{GL}_n(E)$.*
2. *Let $F \subset \overline{\mathbb{Q}}$ be a number field, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and V a finite-dimensional E-vector space with a continuous E-linear action of G_F. If $H^1_f(F,V) = 0$, then $H^1_f(F, V \otimes_E \overline{\mathbb{Q}}_\ell) = 0$.*
3. *Let Γ be a profinite group, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and V a finite-dimensional E-vector space with a continuous E-linear action of Γ. Every continuous 1-cocycle $c\colon \Gamma \to V \otimes_E \overline{\mathbb{Q}}_\ell$ takes values in $V \otimes_E E'$ for some finite extension $E' \subset \overline{\mathbb{Q}}_\ell$ of E.*

Proof. For $d \geq 1$ let $E_d \subset \overline{\mathbb{Q}}_\ell$ be the compositum of the finitely many extensions of $\mathbb{Q}_\ell$ of degree at most d (Krasner); E_d is finite over $\mathbb{Q}_\ell$, hence complete and closed in $\overline{\mathbb{Q}}_\ell$, and $\overline{\mathbb{Q}}_\ell = \bigcup_d E_d$. The same holds with $\mathbb{Q}_\ell$ replaced by any finite extension E (write E_d for the compositum of E with the extensions of E of degree at most d).

Item 1. Each $\mathrm{GL}_n(E_d) = \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \cap \mathrm{M}_n(E_d)$ is a closed subgroup, so $\Gamma = \bigcup_d \rho^{-1}(\mathrm{GL}_n(E_d))$ is a countable union of closed subgroups. Γ is compact Hausdorff, so by Baire's theorem one of them, U say, has non-empty interior; a subgroup with non-empty interior is open. Let $g_1, \ldots, g_s$

represent Γ/U and let $E \supseteq E_d$ be a finite extension containing the entries of $\rho(g_1), \dots, \rho(g_s)$. Then $\rho(g_i u) = \rho(g_i)\rho(u) \in \mathrm{GL}_n(E)$ for all i and $u \in U$.

Item 3. Put $V' = V \otimes_E \overline{\mathbb{Q}}_\ell$. Each $V \otimes_E E_d$ is a finite-dimensional space over the complete field E_d, hence closed in V', and is Γ-stable. So $\Gamma = \bigcup_d c^{-1}(V \otimes_E E_d)$, and by Baire's theorem one of these closed sets, say for d, contains a coset gH of an open normal subgroup H (open normal subgroups form a basis of neighbourhoods of 1 in a profinite group). For $h \in H$, $c(h) = g^{-1}\big(c(gh) - c(g)\big) \in V \otimes_E E_d$. Let $g_1, \dots, g_s$ represent Γ/H and $E' \supseteq E_d$ finite containing the coordinates of $c(g_1), \dots, c(g_s)$ in an E-basis of V; then $c(g_i h) = c(g_i) + g_i c(h) \in V \otimes_E E'$.

Item 2. Put $V' = V \otimes_E \overline{\mathbb{Q}}_\ell$ and let c be a Bloch–Kato cocycle of V'.

(a) By item 3 (with $\Gamma = G_F$), c takes values in $V \otimes_E E'$ for a finite extension E'/E.

(b) *c is a Bloch–Kato cocycle of $V \otimes_E E'$.* Choose an E'-linear $\varpi\colon \overline{\mathbb{Q}}_\ell \to E'$ with $\varpi|_{E'} = \mathrm{id}$ and put $\Pi = \mathrm{id}_V \otimes \varpi\colon V' \to V \otimes_E E'$, which is G_F-equivariant and the identity on $V \otimes_E E'$. If $c(g) = gx - x$ on an inertia group (or on an archimedean decomposition group), then $c(g) = \Pi(c(g)) = g\Pi(x) - \Pi(x)$ there; at a place above ℓ, the $\mathbb{Q}_\ell$-linear equivariant map $\Pi \otimes \mathrm{id}$ into $(V \otimes_E E') \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ fixes $c(g) \otimes 1$ and carries y with $c(g) \otimes 1 = gy - y$ to such an element. The topology of $V \otimes_E E'$ is the subspace topology, so c is continuous into it.

(c) *Conclusion.* An E-basis $b_1, \dots, b_m$ of E' gives a G_F-equivariant E-linear homeomorphism $V \otimes_E E' \cong V^m$. The conditions of Definition 7.1.16 depend only on the underlying topological $\mathbb{Q}_\ell[G_F]$-module and are compatible with the inclusions and projections of a finite direct sum; hence $H^1_f(F, V \otimes_E E') \cong H^1_f(F, V)^m = 0$, and $c(g) = gz - z$ for some $z \in V \otimes_E E' \subseteq V'$. So $H^1_f(F, V') = 0$. □

Theorem 7.1.18. *Let $F/\mathbb{Q}_p$ be finite with residue field k, let $K = \mathrm{GL}_n(\mathcal{O}_F)$ and $K^1 = \ker(K \to \mathrm{GL}_n(k))$. Fix finitely many distinct irreducible cuspidal representations ρ of finite general linear groups over k, of ranks d_ρ, with positive multiplicities m_ρ and $\sum_\rho m_\rho d_\rho = n$. For each ρ choose a depth-zero supercuspidal representation π_ρ of $\mathrm{GL}_{d_\rho}(F)$ containing its inflation. Suppose their inertial cuspidal lines are pairwise distinct. Put $\tau_\rho = \mathrm{rec}_F(\pi_\rho)|_{I_F}$; local Langlands and reciprocity are geometrically normalised.*

For every function P assigning to ρ a partition of m_ρ, let $\bar\sigma_P$ be the finite Green representation with these cuspidal partition data, with the convention that (a) on ρ gives the generalised finite Steinberg representation $\mathrm{St}(\rho, a)$ and that $(1, \dots, 1)$ on the trivial rank-one cuspidal representation gives the trivial representation of $\mathrm{GL}_a(k)$. For another such function P' put

$$\bar\Pi_{P'} = \mathrm{HC}\text{-}\mathrm{Ind} \underset{\rho,j}{\boxtimes} \mathrm{St}(\rho, P'(\rho)_j).$$

There are irreducible finite representations $\bar\nu_P$ of $\mathrm{GL}_n(k)$, in this same finite cuspidal series, satisfying:

1. *for every P',*
$$\dim \mathrm{Hom}_{\mathrm{GL}_n(k)}(\bar\nu_P, \bar\Pi_{P'}) = \dim \mathrm{Hom}_{\mathrm{GL}_n(k)}(\bar\sigma_P, \bar\Pi_{P'});$$
2. *if an irreducible admissible smooth complex representation π of $\mathrm{GL}_n(F)$ contains $\mathrm{Inf}_K \bar\nu_P$, its local Langlands parameter has semisimple inertia $\bigoplus_\rho \tau_\rho^{\oplus m_\rho}$, and its monodromy partition $P_\pi(\rho)$ on each inertial cuspidal line is dominated by $P(\rho)$. Here a Weil–Deligne summand $\mathrm{Sp}(\sigma, a)$ contributes one part a on the line of σ, and domination means*
$$\sum_{j=1}^{b} P_\pi(\rho)_j \le \sum_{j=1}^{b} P(\rho)_j \quad (b \ge 1), \qquad |P_\pi(\rho)| = |P(\rho)| = m_\rho.$$

The square matrix $\left(\dim \mathrm{Hom}(\bar{\sigma}_P, \bar{\Pi}_{P'})\right)_{P,P'}$ *is unitriangular in an order refining dominance.*

Proof. Choose an auxiliary coefficient prime $\ell \neq p$ and transport the characteristic-zero representation theory to an algebraic closure of an ℓ-adic field. A field isomorphism back to $\mathbb{C}$ preserves irreducibility, invariant spaces and the dimensions of the Hom spaces here. No reduction modulo ℓ is taken.

We first identify the data of the compact construction. For each ρ, its given depth-zero supercuspidal line has the degenerate maximal simple type with order $\mathrm{M}_{d_\rho}(\mathcal{O}_F)$, zero simple character on the principal congruence subgroup, trivial extension $\kappa_\rho = 1$, and finite cuspidal representation ρ. In the zero-stratum notation the stratum field is F itself, not a nontrivial extension. These assertions are the degenerate data in the construction of [278, Sections 6.4 and 6.9]: the stabilizer of the standard lattice is $\mathrm{GL}_{d_\rho}(\mathcal{O}_F)$, its congruence kernel is $1+\varpi_F \mathrm{M}_{d_\rho}(\mathcal{O}_F)$, and their quotient is $\mathrm{GL}_{d_\rho}(k)$; tensoring the trivial extension with ρ is precisely its inflation. The hypothesis that π_ρ contains that inflation matches the finite cuspidal label to the specified supercuspidal line.

All these zero simple characters have the same degenerate endo-class, even though the supercuspidal inertial lines are distinct. Consequently the construction on their total vector space is the homogeneous construction, with one maximal order $\mathrm{M}_n(\mathcal{O}_F)$. In its notation

$$J = K, \qquad J^1 = K^1, \qquad J/J^1 = \mathrm{GL}_n(k), \qquad \kappa = 1.$$

The choices are compatible on every Levi: restriction of the trivial extension remains trivial. The finite cuspidal label matched to the basic inertial type τ_ρ is the original ρ. This also shows why different depth-zero cuspidal lines do not require a nontrivial cover between different endo-classes.

The pairwise-distinct-line hypothesis makes the assignment

$$\mathcal{P}(\tau_\rho) = P(\rho), \qquad \mathcal{P}(\tau_0) = \varnothing \quad (\tau_0 \notin \{\tau_\rho\})$$

unambiguous. Its weighted degree is n. Here the basic type is the restriction of an irreducible Weil parameter: the supercuspidal assertion follows from (Ln5) of Theorem A.4.9. Take the compact representation $\sigma(\mathfrak{S}_{\mathcal{P}})$ in the published construction. Its finite factor is irreducible by the definition of an SZ datum. Since $J = K$ and $\kappa = 1$, induction from J to K is the identity and it is therefore the inflation of an irreducible finite representation; denote that finite representation by $\bar{\nu}_P$. This conclusion uses the degenerate construction and does not identify its finite factor with $\bar{\sigma}_P$.

Fix P'. Choose a discrete pair (M', π') attached to $\mathcal{P}'$ by taking one essentially discrete-series segment of length $P'(\rho)_j$ on the line of π_ρ for every nonzero part. Arbitrary unramified twists of those factors do not affect their compact restrictions. Choose any parabolic Q' with Levi M'. The homogeneous invariant-functor formula of [278, Corollary 6.19], with the just-identified $J^1 = K^1$ and $\kappa = 1$, gives

$$\left(\mathrm{Ind}_{Q'}^{G}\, \pi'\right)^{K^1} \simeq \mathrm{HC}\text{-}\mathrm{Ind} \boxtimes_{\rho,j} \mathrm{St}(\rho, P'(\rho)_j) = \bar{\Pi}_{P'}. \tag{1}$$

Every input of that formula has the same zero simple character; all the required extensions are trivial and compatible; M^0 is the product of the original supercuspidal blocks and is contained in M'. Thus its homogeneous hypothesis is satisfied. The normalization in (1) introduces no finite modulus factor: restriction to K identifies the inducing functions through $G = Q'K$, and the parabolic modulus is 1 on $Q' \cap K$. Taking K^1-invariants therefore reads the finite parabolic and its ordinary Harish–Chandra induction.

A K-map from $\operatorname{Inf}\bar{\nu}_P$ has image in the K^1-fixed space, and reduction modulo K^1 identifies such maps in both directions. Hence (1) and [278, Theorem 6.20] give

$$\begin{aligned}\dim \operatorname{Hom}_{\mathrm{GL}_n(k)}(\bar{\nu}_P, \bar{\Pi}_{P'}) &= \dim \operatorname{Hom}_K\left(\sigma(\mathfrak{S}_{\mathcal{P}}), \operatorname{Ind}_{Q'}^{G}\pi'\right)\\ &= \prod_{\rho} m(P(\rho), P'(\rho)).\end{aligned} \tag{2}$$

The finite formula of [278, Corollary 6.10] identifies the last expression with $\dim \operatorname{Hom}(\bar{\sigma}_P, \bar{\Pi}_{P'})$. Its convention is exactly the convention in the statement: the symmetric-group labeling is twisted by sign, the single part (a) corresponds to $\mathrm{St}(\rho, a)$, and the all-one partition on the trivial rank-one line corresponds to the trivial representation. No transposition of partitions is made. This proves item 1.

Set $P' = P$ in (2). Every diagonal Kostka factor is 1, so $\bar{\nu}_P$ occurs in $\bar{\Pi}_P$. The latter is in the prescribed finite cuspidal series: its generalized Steinberg factors have the prescribed repeated cuspidal supports and Harish–Chandra induction concatenates those supports. Equivalently, by transitivity it is a constituent of induction of exactly the original cuspidal multiset. Thus $\bar{\nu}_P$ is in that same finite series, as asserted.

Now suppose that π is irreducible admissible and contains $\operatorname{Inf}\bar{\nu}_P = \sigma(\mathfrak{S}_{\mathcal{P}})$. Write its Langlands discrete-pair data as (M'', π'') and its corresponding partition function as $\mathcal{P}_\pi$. The parameter compatibility with the Langlands classification in (Ln6) of Theorem A.4.9 identifies a length-a segment with a Weil–Deligne block $\mathrm{Sp}(\sigma, a)$; thus $\mathcal{P}_\pi$ is exactly the list of the monodromy-block sizes on each basic inertial line. Apply [278, Corollary 6.21(1)] to this quotient and the same constructed compact representation. It gives

$$\mathcal{P}_\pi(\tau_0) \preceq \mathcal{P}(\tau_0) \quad \text{for every basic inertial type } \tau_0.$$

By the definition of this order, the sizes agree on every line and all initial partial sums satisfy the displayed dominance inequalities. On every line outside the chosen list, equality of sizes with the empty partition forces the partition to be empty. On the line τ_ρ its size is m_ρ. The semisimple inertia is therefore $\bigoplus_\rho \tau_\rho^{\oplus m_\rho}$, and the inequalities are precisely item 2.

Shotton writes this parameter as $r_\ell(\pi) = \mathrm{rec}_F(\pi \otimes |\det|^{(1-n)/2})$. By (Ln2), the difference from $\mathrm{rec}_F(\pi)$ is tensoring with an unramified one-dimensional parameter. Its restriction to inertia is 1 and its monodromy is 0; tensoring leaves the original monodromy endomorphism and all its Jordan-block sizes unchanged. Thus both the inertia statement and the partition inequalities just proved hold for the geometric normalization in the statement.

Finally the same finite formula gives the matrix entries as the products in (2). They vanish unless $P(\rho)$ dominates $P'(\rho)$ on every line, and all diagonal entries are 1. There are finitely many partition functions with the prescribed sizes. Ordering them by a linear extension of this componentwise dominance order makes the matrix triangular with diagonal 1. This proves the final assertion and completes the specialization. □

7.2 Crystalline classes, auxiliary twists and the odd Sha bound

This section sets up the vocabulary of Section 7 and proves its one global bound on class groups. It opens with the local and global definitions: crystalline classes of local lattices, auxiliary twists, Tate twists with their odd and even classes, Taylor–Wiles places and prime triangles, and the objects of Kolyvagin's method. It closes with the main summands of ℓ-adic representations and the denominators of projections (Definition 7.2.30, Definition 7.2.31).

The theorem of the section, Theorem 7.2.8, bounds the odd classes $\mathrm{Sha}^-(e,\nu)$ of the Tate twists over a CM field $E \subset \mathbb{Q}(\mu_N)$ at all primes at once: there is an integer X with $1 \le X \le 2^{4(N+t+3)^{27}} h_E$ such that $\mathrm{len}_{\mathcal{O}}\mathrm{Sha}^-(e,\nu) \le v_{\mathcal{O}}(X)$ for every prime ℓ, every coefficient ring, every $0 \le e < t$ and every ν. Such bounds belong to the theory of cyclotomic fields, in which the odd parts of class groups and of Selmer groups are controlled by Bernoulli numbers and Stickelberger elements and their orders are described by the main conjecture [317, 197, 327]. Only an integer uniform in ℓ is needed here, and the proof uses Stickelberger's theorem [290] and the Euler system of Gauss sums with its Kolyvagin derivatives [304, 169, 244, 107] (for the Galois cohomology of Tate twists see [287]). It has four steps. An integer X_e satisfies the twisted Stickelberger congruences at every prime, by the theorem of von Staudt and Clausen and the generalised Bernoulli numbers (Lemma 7.2.9). These congruences annihilate the odd cocycles (Proposition 7.2.10). A bounded number of evaluations detects the odd cocycles (Proposition 7.2.27): at large primes by Poitou–Tate duality [295] and a reciprocity law derived from the Brauer group of $\mathbb{Q}$ [297, 202], and at small primes by a detecting set from Minkowski's bound. Finally, for $e = 0$ the odd classes are unramified homomorphisms, which gives the factor h_E (Lemma 7.2.29).

At the end of the section a convention fixes the odd-class integer X_{odd} of the clearing bound (Definition 6.6.15), for which only a size bound was assumed, to be this integer X for $N = q_1 d$, the given t and the field E. From then on $\mathrm{len}_{\mathcal{O}}\mathrm{Sha}^-(e,\nu) \le v_{\mathcal{O}}(X_{\mathrm{odd}})$. The size bound alone makes the sum of the clearing bounds over all primes negligible (Proposition 7.3.3), and the specified value enters the bound for the denominators (Theorem 7.6.6).

Definition 7.2.1. Let ℓ be a prime. Throughout, Γ is a topological group with a homomorphism $j\colon \Gamma \to G_\ell$ which is an open embedding; Γ acts on $B_{\max}$, $\mathbb{C}_\ell$ and $\overline{\mathbb{Q}}_\ell$ through j, and the filter *near* 1 on Γ is the pull-back along j of the neighbourhood filter of 1 in G_ℓ. An element $s \in \Gamma$ is *inertial* if $j(s)$ lies in the inertia group of G_ℓ.

Coefficient rings and lattices. A **coefficient ring at** ℓ is a coefficient ring $\mathcal{O}$ whose fraction field is Galois over $\mathbb{Q}_\ell$ (this Galois condition is a hypothesis of the notions below and of the results that use them, except for the crystalline cocycles, the crystalline relations and the crystalline condition with module coefficients, whose coefficient domain is stated with them); coefficient rings at ℓ are ordered by inclusion, and a property holds **for all large coefficient rings** if there is $\mathcal{O}_0$ such that it holds for every $\mathcal{O} \supseteq \mathcal{O}_0$. The normalised length of a finite-length $\mathcal{O}$-module X is $\mathrm{nl}(X) = \mathrm{length}_{\mathcal{O}}(X) \cdot v(\varpi)$, with ϖ a uniformiser and $v(\ell) = 1$ (so $\mathrm{nl}(\mathcal{O}/\ell) = 1$), and $\mathrm{nl} = \infty$ for infinite length. A **Galois lattice** L over $\mathcal{O}$ is a finite free $\mathcal{O}$-module with an $\mathcal{O}$-linear action ρ_L of Γ. Its **Tate dual** is $L^\vee(1) = \mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O}(1))$, with Γ acting on $\mathcal{O}(1)$ by $\chi_{\mathrm{cyc}} \circ j$; $Y \otimes L$ is $Y \otimes_{\mathcal{O}} L$ with the diagonal action.

Periods. For a filter F on Γ, L is **admissible along** F if the elements $y \in B_{\max} \otimes_{\mathbb{Z}_\ell} L$ with $j(g) \otimes \rho_L(g)\, y = y$ for F-almost all g span $B_{\max} \otimes_{\mathbb{Z}_\ell} L$ over $B_{\max}$; L is **potentially crystalline** if it is admissible along the filter near 1. For the next three constructions only (crystalline cocycles, crystalline relations and the crystalline condition with module coefficients), $\mathcal{O}$ may be the ring of integers of any finite extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, and L any finite free $\mathcal{O}$-module with an $\mathcal{O}$-linear action ρ_L of Γ; their definitions use no Galois condition. The **crystalline cocycles** of L are

$$Z_f(L) = \{c\colon \Gamma \to L \ :\ \exists\, y \in B_{\max} \otimes_{\mathbb{Z}_\ell} L \ \ \forall g \in \Gamma,\ \ 1 \otimes c(g) = (j(g) \otimes \rho_L(g))\, y - y\},$$

an $\mathcal{O}$-submodule of the functions $\Gamma \to L$. The **crystalline relations** of L form the $\mathcal{O}$-module $\mathcal{R}_f(L)$ of finitely supported functions $\theta\colon \Gamma \to \mathrm{End}_{\mathcal{O}}(L)$ with $\sum_{g\in\Gamma} \theta(g)\, c(g) = 0$ for every $c \in Z_f(L)$. For an $\mathcal{O}$-module B, a function $z\colon \Gamma \to B \otimes_{\mathcal{O}} L$ satisfies the **crystalline condition of**

L **with coefficients in** B if

$$\sum_{g\in\Gamma}(\mathrm{id}_B\otimes\theta(g))\,z(g)=0 \ \text{ in } B\otimes_{\mathcal{O}}L \qquad \text{for every } \theta\in\mathcal{R}_f(L)$$

(no cocycle identity or continuity is imposed). For $L=\mathcal{O}^d$, $\mathrm{End}_{\mathcal{O}}(L)=M_d(\mathcal{O})$ and $B\otimes_{\mathcal{O}}L=B^d$. For a $\mathbb{Z}_\ell$-algebra map $\gamma\colon\mathcal{O}\to\mathbb{C}_\ell$ and $u\in\mathbb{Z}$, the **jump multiplicity** $\mathrm{jump}(L,\gamma,u)$ is the $\overline{\mathbb{Q}}_\ell$-dimension (zero if infinite) of the space of γ-semilinear maps $\lambda\colon L\to\mathbb{C}_\ell$ with $\lambda(\rho_L(g)x)=\chi_{\mathrm{cyc}}(j(g))^{-u}\,j(g)\lambda(x)$ for all g near 1. A multiset J of integers **lists the jumps of** L **at** γ if $\mathrm{jump}(L,\gamma,u)$ is the multiplicity of u in J for every u. For potentially crystalline L, an inertial s and $Q\in\mathcal{O}[X]$, Q **kills** s **on** L if Q, evaluated at the $B_{\max}$-linear endomorphism of $B_{\max}\otimes L$ that agrees with $j(s)\otimes\rho_L(s)$ on the potentially crystalline periods, is zero.

Classes. Let F be a filter on Γ, K a relation between elements $s\in\Gamma$ and polynomials $Q\in\mathcal{O}[X]$, and U a map from the $\mathbb{Z}_\ell$-algebra maps $\gamma\colon\mathcal{O}\to\mathbb{C}_\ell$ to finite subsets of $\mathbb{Z}$. A lattice Y lies in the **crystalline class** $\mathrm{Cr}(F,K,U)$ if

(CY1) Y is admissible along the supremum of F and the filter near 1;

(CY2) $\mathrm{jump}(Y,\gamma,u)\neq 0$ implies $u\in U(\gamma)$;

(CY3) if $K(s,Q)$ holds and s is inertial, then Q kills s on Y.

For $K=\bot$ (the empty relation) (CY3) is void.

Pairings and cup-orthogonality. Fix the embedding of $\mathbb{Z}/\ell^m$ into $\overline{\mathbb{Q}}_\ell^{\times}$ given by a compatible system of ℓ-power roots of unity. A **pairing** P of lattices Y and L modulo ℓ^m consists of $m\geq 0$, an element $\psi\in(Y\otimes L)^\vee(1)$, a $\mathbb{Z}_\ell$-linear map $\phi\colon\mathcal{O}\to\mathbb{Z}_\ell$ and a crystalline cocycle $c_P\in Z_f((Y\otimes L)^\vee(1))$ with $\ell^m c_P(g)=g\psi-\psi$ for all $g\in\Gamma$; it gives the $\mathbb{Z}_\ell$-bilinear map $b_P(y,x)=\phi(\psi(y\otimes x))\bmod\ell^m$. For functions $a\colon\Gamma\to Y$ and $c\colon\Gamma\to L$, the **cup product** of a and c under P is the 2-cochain $(g,h)\mapsto b_P(a(g),\rho_L(g)c(h))\in\mathbb{Z}/\ell^m\subset\overline{\mathbb{Q}}_\ell^{\times}$; it is a **cup coboundary** if it is the coboundary of a continuous 1-cochain $\Gamma\to\overline{\mathbb{Q}}_\ell^{\times}$ (Γ acting through j). For a class $\mathcal{Q}$ of lattices, the **cup-orthogonal module** $L^{\perp\mathcal{Q}}$ is the $\mathcal{O}$-module of functions $c\colon\Gamma\to L$ such that for every $Y\in\mathcal{Q}$, every pairing P of Y and L and every $a\in Z_f(Y)$, the cup product of a and c under P is a cup coboundary.

Dual defect and cup exponents. The **dual defect** of L for $\mathcal{Q}$ is

$$\mathrm{dd}(\mathcal{Q},L)=\mathrm{length}_{\mathcal{O}}(Z_f(L)/(Z_f(L)\cap L^{\perp\mathcal{Q}}))\in\mathbb{N}\cup\{\infty\},$$

and $\mathrm{nl}\,\mathrm{dd}(\mathcal{Q},L)$ is its normalised value. An integer $e\geq 0$ is a **cup exponent** of L for $\mathcal{Q}$ if $\ell^e Z_f(L)\subseteq L^{\perp\mathcal{Q}}$.

Definition 7.2.2. Fix a prime ℓ and a prime identification $\iota\colon\mathbb{C}\to\overline{\mathbb{Q}}_\ell$. Let $G_E=\mathrm{Gal}(\overline{\mathbb{Q}}/E)$, let v_ℓ be the valuation of the convention (with $v_\ell(0)=\infty$), and for a finite place w of E prime to ℓ let a **tame generator** at w be an element of an inertia group above w that maps to a generator of its tame quotient. Coefficient rings $\mathcal{O}$ (the rings of integers of the finite extensions of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$) are ordered by inclusion, and a property holds **for all large** $\mathcal{O}$ if there is $\mathcal{O}_0$ such that it holds for every $\mathcal{O}\supseteq\mathcal{O}_0$. For a finite $\mathcal{O}$-module X, $\mathrm{nl}(X)=\mathrm{len}_{\mathcal{O}}(X)/\mathrm{len}_{\mathcal{O}}(\mathcal{O}/\ell)$ is its normalised length, and $\mathrm{nl}=\infty$ for modules of infinite length. For a place A of $\overline{\mathbb{Q}}$ write $G_A\supseteq I_A$ for its decomposition and inertia groups in G_E. An **ℓ-adic chart** at a place A above ℓ is an embedding $e\colon\overline{\mathbb{Q}}\to\overline{\mathbb{Q}}_\ell$ inducing A together with the open embedding $j\colon G_A\to G_\ell$ it defines; for every place $v\mid\ell$ of E fix a place A_v of $\overline{\mathbb{Q}}$ over v and a chart (e_v,j_v) at A_v. For a homomorphism $u\colon G_E\to M_n(\mathcal{O})$ the **local lattice** $L_v(u)$ is $\mathcal{O}^n$ with G_{A_v} acting through u, a Galois lattice for $\Gamma=G_{A_v}$ via j_v.

(AX1) Bad places. The **bad places** S_ℓ^{bad} are the finite places of E dividing ℓMN; they contain S_ℓ.

(AX2) Auxiliary sets. A finite set Q of finite places of E is **auxiliary** for a set S of places if every $w \in Q$ has prime norm $\mathrm{N}w$, $w \notin S$, $\bar{w} \notin S$ and $\bar{w} \notin Q$.

(AX3) Characters from the norm-one torus. A continuous character $\chi\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ **comes from** $\mathrm{U}(1)$ **with modulus** Q if $\chi(\mathrm{Art}(x)) = \eta(x/\bar{x})$ for all ideles x of E, for some idele class character η of finite order and modulus supported on Q that is trivial on the local components at the places above ℓ and their conjugates; here Art is the Artin map.

(AX4) Avatar. The **inverse avatar** of a tame character ϑ is the character $\psi_\vartheta\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ with $\psi_\vartheta(\mathrm{Art}(x)) = \iota(\vartheta(x/\bar{x})^{-1})$ for all ideles x of E, where ϑ is read as a character of $\mathrm{U}(1)(\mathbb{A})/\mathrm{U}(1)(E)$ (it has weight 0, hence finite order).

(AX5) Auxiliary twists. For a set S of finite places, a finite set Q, a character $\chi\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ and $c \in [0,\infty]$, the triple (S,Q,χ) is an **auxiliary twist with ramification at most** c if Q is auxiliary for S, no place of Q lies above ℓ, χ comes from $\mathrm{U}(1)$ with modulus Q, and for some tame generators σ_w at the places $w \in Q$

$$\sum_{w\in Q} v_\ell(\chi(\sigma_w) - 1) \le c.$$

(AX6) Crystalline Selmer length. Let $\mathcal{O}$ be a coefficient ring, $\rho\colon G_E \to \mathrm{GL}_n(\mathcal{O})$ continuous, Q a finite set of finite places and $\gamma \in G_E$. The **Selmer group** $\mathrm{Sel}(\rho,Q,\gamma)$ is the $\mathcal{O}$-module of continuous cocycles $c\colon G_E \to (\mathrm{Frac}\,\mathcal{O}/\mathcal{O})^n$ for ρ that are unramified at every finite place not above ℓ, p_0 or p_1 and not in $Q \cup \bar{Q}$, have, for every place $v \mid \ell$ of E, restriction to G_{A_v} satisfying the crystalline condition of the local lattice $L_v(\rho)$ with coefficients in $\mathrm{Frac}\,\mathcal{O}/\mathcal{O}$, and vanish at γ. Its **length** is $\mathrm{nl}\,\mathrm{Sel}(\rho,Q,\gamma) \in [0,\infty]$.

(AX7) Short twisted Selmer groups. Let $R\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be continuous, $\chi\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ a character, Q a finite set of finite places, $\gamma \in G_E$, $\zeta \in \overline{\mathbb{Z}}_\ell$ and $\mathrm{sel} \in [0,\infty]$. The twist χR has **Selmer length at most** sel **at** (Q,γ,ζ) if γ lies in an inertia group above a place not dividing ℓ, and for all large $\mathcal{O}$ there are a continuous $u\colon G_E \to \mathrm{GL}_n(\mathcal{O})$ conjugate to R over $\overline{\mathbb{Q}}_\ell$ and a character $\chi'\colon G_E \to \mathcal{O}^\times$ equal to χ, with $\zeta \in \mathcal{O}$, such that

$$(\chi' u)(\gamma) = \zeta \cdot 1, \qquad \zeta - 1 \in \mathcal{O}^\times, \qquad \mathrm{nl}\,\mathrm{Sel}(\chi' u, Q, \gamma) \le \mathrm{sel}.$$

Definition 7.2.3. Let ℓ be a prime and let $E \subset \bar{\mathbb{Q}}$ be a CM field, with absolute Galois group $G_E = \mathrm{Gal}(\bar{\mathbb{Q}}/E)$. Fix a complex conjugation $c \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and write $g^c = cgc$ for $g \in G_E$. Let $\chi_{\mathrm{cyc}}\colon G_E \to \mathbb{Z}_\ell^\times$ be the ℓ-adic cyclotomic character. For a coefficient ring $\mathcal{O}$, $\mathrm{len}_\mathcal{O}$ is length over $\mathcal{O}$ and $v_\mathcal{O}$ its normalised valuation. For a submodule S and a submodule T of one $\mathcal{O}$-module, $\mathrm{len}(S \bmod T) = \mathrm{len}_\mathcal{O}(S/(S \cap T))$. Let S_ℓ be the set of places of E above ℓ; for a set S of finite places, $\bar{S}$ is the set of complex conjugate places.

Twists and Selmer modules. For integers $e, \nu \ge 0$, $\mathcal{O}/\ell^\nu(-e)$ is $\mathcal{O}/\ell^\nu$ with G_E acting through χ_{cyc}^{-e}. For a set S of finite places of E, $\mathrm{Sel}_S(e,\nu)$ is the $\mathcal{O}$-module of locally constant maps $z\colon G_E \to \mathcal{O}/\ell^\nu$ with $z(gh) = z(g) + \chi_{\mathrm{cyc}}(g)^{-e} z(h)$ (cocycles, not taken modulo coboundaries) that vanish on the inertia groups of all places of $\bar{\mathbb{Q}}$ above finite places of E outside S.

The local set. Λ_0 is the union of the inertia groups at the places of $\bar{\mathbb{Q}}$ above ℓ; for $e \neq 0$, $\Lambda_e = \Lambda_1$ is the set of elements g of the decomposition groups at these places with $\chi_{\mathrm{cyc}}(g) = 1$.

Odd and even classes.

(OE1) The **odd classes** $\mathrm{Sha}^-(e,\nu)$ are the $z \in \mathrm{Sel}_{S_\ell}(e,\nu)$ that vanish on Λ_e and, if $e \neq 0$, satisfy $z(g^c) = -z(g)$ for all g. For $e = 0$ these are the everywhere unramified continuous homomorphisms $G_E \to \mathcal{O}/\ell^\nu$.

(OE2) For a set S of finite places, the **even classes** $\mathrm{Ev}_S(e,\nu)$ are the $z \in \mathrm{Sel}_{S_\ell \cup S \cup \bar{S}}(e,\nu)$ with $z(g^c) = z(g)$ for all g and $z|_{\Lambda_1} = 0$.

Tame generators. Fix for every $m \geq 1$ a primitive m-th root of unity $\zeta_m \in \bar{\mathbb{Q}}$. Let A be a place of $\bar{\mathbb{Q}}$ above a finite place v of E, or of the maximal real subfield F of E, of residue characteristic p, with inertia group I_A in G_E, respectively G_F. For m prime to p, an element $\gamma \in I_A$ is a **tame generator of order** m at A if $\gamma(y)/y = \zeta_m$ for a uniformiser ϖ of the completion at v and an m-th root $y \in \bar{\mathbb{Q}}$ of ϖ (the quotient $\gamma(y)/y$ does not depend on the choice of ϖ and y, since inertia fixes the m-th roots of units and of unity). At each place A above a place of E one element $\tau_A \in I_A$ is fixed such that $\tau_A(y)/y$ is a primitive m-th root of unity for every m prime to p and every m-th root y of a uniformiser; it is **the chosen tame generator** at A.

Auxiliary places. A **Frobenius datum** is a finite place w of E of prime norm q_w, a place A of $\bar{\mathbb{Q}}$ above it, a geometric Frobenius φ at A and the chosen tame generator $\tau = \tau_A$. It is **ν-split** if $w \nmid \ell$ and φ fixes μ_{ℓ^ν} (then $\ell^\nu \mid q_w - 1$). For $M \geq 1$ and a finite index set J, **Taylor–Wiles places of level** M are Frobenius data $(w_j, \varphi_j, \tau_j)_{j \in J}$ with no two of the w_j equal or complex conjugate and $M \mid q_{w_j} - 1$ for all j.

Most Frobenii. A set X of Frobenius data contains **most Frobenii** if every nonempty open $V \subseteq G_E$ contains a nonempty open U such that for all but finitely many finite places w of E of prime norm, every Frobenius datum at w whose Frobenius lies in U belongs to X. For $g \in G_E$, a set W contains **most Frobenii near** g if there are such an X and an open neighbourhood U_g of g such that every datum of X with Frobenius in U_g belongs to W.

Kummer maps. With F the maximal real subfield of E as above, let $\chi_{\mathrm{cyc}}^+ \colon G_F \to \mathbb{Z}_\ell^\times$ the character that extends χ_{cyc} and is 1 at c. In this paragraph χ_{cyc} also denotes the ordinary ℓ-adic cyclotomic character of G_F, which restricts to that of G_E and takes the value -1 at c. Let $G_{F,\nu} = G_E \cap \ker(\chi_{\mathrm{cyc}} \bmod \ell^\nu)$, a normal subgroup of G_F that does not contain c. For $e \geq 0$ let M_e be $\mathbb{Z}/\ell^\nu$ with G_F acting through $(\chi_{\mathrm{cyc}}^+)^{-e}$. Let $x = (x_i)_{i \in I}$ be a finite family of ν-split Frobenius data, S_0 the set of primes of F above ℓ or below some x_i, and $T \supseteq S_0$ a finite set of primes of F. A **relation off** T is a locally constant homomorphism $y \colon G_{F,\nu} \to \mathrm{Hom}(M_e, \mathbb{Q}/\mathbb{Z})$, extended by 0 to G_F, with $y(gug^{-1}) = \chi_{\mathrm{cyc}}(g) \cdot y(u) \circ g^{-1}$ for $g \in G_F$ and $u \in G_{F,\nu}$ (ordinary cyclotomic character of G_F), zero on the inertia groups at the primes outside T, and such that $\sum_p y(\varphi_p)(z(\gamma_p)) = 0$ for every finite family of distinct primes $p \notin T$ of F with, for each p, a place A_p of $\bar{\mathbb{Q}}$ above p, an arithmetic Frobenius $\varphi_p \in G_{F,\nu}$ at A_p and a tame generator $\gamma_p \in G_F$ of order ℓ^ν at A_p, and every cocycle $z \colon G_F \to M_e$ that vanishes at the complex conjugations, is unramified outside S_0 and the p, vanishes on the decomposition groups at the primes of S_0 not above ℓ, and vanishes on the elements of the decomposition groups above ℓ with $\chi_{\mathrm{cyc}}^+ = 1$. A **Kummer map of exponent** e **off** T (for x) is a map $\psi \colon G_E \to \mathbb{Z}/\ell^\nu$ with $y(g)(a) = a\psi(g)/\ell^\nu$ for all $g \in G_E$, $a \in M_e$, for some relation y off T.

Prime triangles. Let $\nu, t, h, C \geq 0$ be integers, J a finite linearly ordered set, $\mathrm{ty} \colon J \to \{1, \ldots, t-1\}$ monotone (the **exponent**), $a \in G_E$, and for $j \in J$ let w_j be finite places, $\tau_j, \varphi_j \in G_E$. These form a **prime triangle for** a **with bounds** (h, C) if:

(PT1) $w_j \nmid \ell$, τ_j is the chosen tame generator at a place of $\bar{\mathbb{Q}}$ over w_j, and φ_j fixes μ_{ℓ^ν};

(PT2) there is a finite set $s \subset \ker \chi_{\mathrm{cyc}}$ of **test points** with $(\#J + \#s + 1)h \leq C$ such that for all $\mathcal{O}$ and e, every $z \in \mathrm{Sel}_{S_\ell}(e,\nu)$ with $z(a) = 0$ and $z|_s = 0$ is zero, and for every

$e \in \{1, \dots, t-1\}$ and $g \in s$ there is j with $\mathrm{ty}(j) = e$ and $z(\varphi_j) = 2z(g)$ for all $\mathcal{O}$ and all $z \in \mathrm{Ev}_{\emptyset}(e, \nu)$;

(PT3) for $0 \le e < t$, with $u = \chi_{\mathrm{cyc}}(a^{-1})$, $\prod_{0 \le k < t,\, k \ne e}(u^e - u^k)$ divides ℓ^h in $\mathbb{Z}_\ell$;

(PT4) for all $\mathcal{O}$, all j and all $z \in \mathrm{Ev}_{\{w_i\}}(\mathrm{ty}(j), \nu)$: if $z(\tau_i) = 0$ for all $i < j$ and $z(\varphi_i) = 0$ for all $i < j$ with $\mathrm{ty}(i) = \mathrm{ty}(j)$, then $\ell^h z(\tau_j) = 0$;

(PT5) for $0 < e < t$ and all $\mathcal{O}$, $4(\chi_{\mathrm{cyc}}(a)^e - 1)^2$ divides ℓ^h in $\mathcal{O}$.

Triangle cost. Let $\mathcal{O}$ be a discrete valuation ring, G a set, B an $\mathcal{O}$-module, Z a submodule of the maps $G \to B$, J a finite linearly ordered set, $\tau, \tau', \varphi, \varphi' \colon J \to G$, $I \subseteq J$, $C \in \mathcal{O}$ and $b \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. Then Z has **triangle cost** b **along** I **up to** C if there is $L \subseteq G$ such that, with $Z_0 = \{z \in Z : z|_L = 0\}$:

(TK1) for $j \in I$ and $z \in Z_0$ with $z(\tau_i) = z(\tau'_i)$ for all $i < j$ and $z(\varphi_i) = z(\varphi'_i)$ for all $i < j$ in I, one has $Cz(\tau_j) = Cz(\tau'_j)$;

(TK2) for every submodule $S \subseteq Z_0$ whose members vanish at all τ_j, τ'_j and satisfy $z(\varphi_j) = z(\varphi'_j)$ for $j \in I$,

$$\mathrm{len}_{\mathcal{O}}(Z/Z_0) + \mathrm{len}\big(S \bmod S[C]\big) \le b,$$

where $S[C]$ is the C-torsion of the maps $G \to B$.

In the application $\tau'_j = (\tau_j^c)^{-1}$ and $\varphi'_j = (\varphi_j^c)^{-1}$, so that $z(\tau_j) - z(\tau'_j)$ is the value at τ_j of the even part of z (up to the cocycle relation).

Definition 7.2.4. Let ℓ be an odd prime and $\mathcal{O}$ a coefficient ring, with maximal ideal $\mathfrak{m}$ and normalised valuation $v_{\mathcal{O}}$ ($v_{\mathcal{O}}$ of a uniformiser is 1), extended to the fraction field of $\mathcal{O}$. Write $\mathcal{O}^\vee = \mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$, a discrete $\mathcal{O}$-module. Fix an algebraic closure $\overline{\mathbb{Q}}$, so $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; for $N \ge 1$ let $\mu_N \subset \overline{\mathbb{Q}}$ be the N-th roots of unity and $U_N = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N))$. A **decomposition group** at a prime p is the stabiliser $D \subseteq G_{\mathbb{Q}}$ of a place of $\overline{\mathbb{Q}}$ above p; $I(D) \subseteq D$ is its inertia group, and a **Frobenius** of D is an element of D inducing $x \mapsto x^p$ on the residue field of that place. Let $\theta \colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ be a continuous character, and let $\check{\theta} = \chi_{\mathrm{cyc}}\theta^{-1}$ be its **dual character** (χ_{cyc} takes values in $\mathbb{Z}_\ell^\times \subseteq \mathcal{O}^\times$).

Kolyvagin levels. A **Kolyvagin level** of θ is a pair (N_0, m) of integers with $m \ge 1$, $N_0 \ge 1$ and $\ell \nmid N_0$ such that, with $N = N_0\ell^m$ (the **modulus** of the level),

(Kv1) $\theta(u) - 1 \in \ell^m\mathcal{O}$ for every $u \in U_N$;

(Kv2) $\theta(g) + 1 \in \ell^m\mathcal{O}$ for every $g \in G_{\mathbb{Q}}$ with $g(\zeta) = \zeta^{-1}$ for all $\zeta \in \mu_N$;

(Kv3) there is $g_1 \in G_{\mathbb{Q}}$ with $\theta(g_1) - \chi_{\mathrm{cyc}}(g_1) \in \mathcal{O}^\times$.

A **normaliser** for θ is an element $\delta \in G_{\mathbb{Q}}$ with $\theta(\delta) - 1 \in \mathcal{O}^\times$.

Normalised strict cocycles. Let δ be a normaliser for θ, Σ a finite set of primes, $a \in \mathcal{O}\backslash\{0\}$ and $n \ge 1$ an integer. Then $\mathrm{Z}_{\theta,\delta}(\Sigma, a; n)$ is the set of continuous maps $z \colon G_{\mathbb{Q}} \to \mathcal{O}^\vee$ such that

(Kd1) $z(gh) = z(g) + \theta(g)z(h)$ for all $g, h \in G_{\mathbb{Q}}$, and $z(\delta) = 0$;

(Kd2) $a\,z(g) = 0$ for all $g \in G_{\mathbb{Q}}$;

(Kd3) $z(g) = 0$ for every g in an inertia group at a prime $p \notin \Sigma$;

(Kd4) for every prime $p \in \Sigma$ and every decomposition group D at p there is $y \in \mathcal{O}^\vee$ with $ay = 0$ and $z(g) = \theta(g)y - y$ for all $g \in D$;

(Kd5) $z(g) = 0$ for every g in a decomposition group at a prime dividing n.

It is an $\mathcal{O}$-submodule of the module of all maps $G_{\mathbb{Q}} \to \mathcal{O}^\vee$; we put $\mathrm{Z}_{\theta,\delta}(\Sigma, a) = \mathrm{Z}_{\theta,\delta}(\Sigma, a; 1)$, and lengths are $\mathcal{O}$-lengths in $\mathbb{N} \cup \{\infty\}$.

Twisted Stickelberger values. Let $N \geq 2$ and b be integers. For $t \in (\mathbb{Z}/N\mathbb{Z})^\times$ let $\tilde{t} \in \{1, \dots, N-1\}$ be its representative and

$$h_N(b,t) = \Big\lfloor \frac{\tilde{t}\, b}{N} \Big\rfloor - \Big\lfloor \frac{(N-\tilde{t})\, b}{N} \Big\rfloor \in \mathbb{Z}.$$

A **family of lifts** modulo N is a family $\sigma = (\sigma_t)_{t \in (\mathbb{Z}/N\mathbb{Z})^\times}$ of elements of $G_{\mathbb{Q}}$ with $\sigma_t(\zeta) = \zeta^{\tilde{t}}$ for all $\zeta \in \mu_N$. The **twisted Stickelberger value** is

$$\mathrm{st}_{N,b}(\theta, \sigma) = \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} h_N(b,t)\, \theta(\sigma_t)^{-1} \in \mathcal{O}. \tag{7.3}$$

Definition 7.2.5. Let $\theta\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ be a continuous character with a Kolyvagin level (N_0, m) of modulus $N = N_0 \ell^m$, and let r be a prime with $r \equiv 1 \pmod{N}$.

A **Kolyvagin prime** for (N, r) is a prime p with $p \equiv 1 \pmod{N}$, $p \neq r$, and $\ell \nmid \mathrm{ord}_p(r)$, where $\mathrm{ord}_p(r)$ is the order of r in $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $\mathcal{L}(N, r)$ be the set of finite sequences $L = (p_1, \dots, p_k)$, $k \geq 0$, of pairwise distinct Kolyvagin primes for (N, r); put $|L| = p_1 \cdots p_k$ ($|L| = 1$ for the empty sequence $\emptyset$), and for a Kolyvagin prime p not occurring in L put $(p, L) = (p, p_1, \dots, p_k) \in \mathcal{L}(N, r)$.

Let D_0 be a decomposition group at r and $w \in \mathcal{O}^\vee$. A **Kolyvagin system** for $(\theta, N_0, m, r, D_0, w)$ is a family $(\kappa_L)_{L \in \mathcal{L}(N,r)}$ of continuous maps $\kappa_L\colon U_N \to \mathcal{O}^\vee$ such that for every $L \in \mathcal{L}(N, r)$:

(Ks1) $\kappa_L(uv) = \kappa_L(u) + \kappa_L(v)$ and $\ell^m \kappa_L(u) = 0$ for all $u, v \in U_N$;

(Ks2) $\kappa_L(gug^{-1}) = \check{\theta}(g)\, \kappa_L(u)$ for all $g \in G_{\mathbb{Q}}$ and $u \in U_N$;

(Ks3) $\kappa_L(u) = 0$ for every $u \in U_N$ lying in an inertia group at a prime not dividing $Nr|L|$;

(Ks4) for every Kolyvagin prime p for (N, r) not occurring in L, every decomposition group D at p and every Frobenius φ of D, there are $\tau \in I(D)$ and $u \in \mathcal{O}^\times$ with $\kappa_{(p,L)}(\tau) = u\, \kappa_L(\varphi)$;

(Ks5) there are $\tau_0 \in I(D_0)$ and $u_0 \in \mathcal{O}^\times$ with $\kappa_\emptyset(\tau_0) = u_0\, w$.

In (Ks4) and (Ks5) the elements φ, τ, τ_0 lie in U_N: a prime $p \equiv 1 \pmod{N}$ is unramified in $\mathbb{Q}(\mu_N)$ with residue degree 1, so D acts trivially on μ_N; the same holds for r and D_0.

Definition 7.2.6. Let $m \geq 2$ be an integer and $K_m = \mathbb{Q}(\mu_m) \subset \overline{\mathbb{Q}}$. For $t \in (\mathbb{Z}/m\mathbb{Z})^\times$, $\sigma_{m,t} \in \mathrm{Gal}(K_m/\mathbb{Q})$ is the automorphism with $\sigma_{m,t}(\zeta) = \zeta^{\tilde{t}}$ for $\zeta \in \mu_m$; $t \mapsto \sigma_{m,t}$ is an isomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \to \mathrm{Gal}(K_m/\mathbb{Q})$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_{K_m}$ not dividing m, p its residue characteristic and $k_\mathfrak{p} = \mathcal{O}_{K_m}/\mathfrak{p}$, a field with $|k_\mathfrak{p}| \equiv 1 \pmod{m}$. The **$m$-th power residue character** of $\mathfrak{p}$ is $\chi_\mathfrak{p}\colon k_\mathfrak{p}^\times \to \mu_m$, where $\chi_\mathfrak{p}(x \bmod \mathfrak{p})$ is the element of μ_m congruent to $x^{(|k_\mathfrak{p}|-1)/m}$ modulo $\mathfrak{p}$ ($x \in \mathcal{O}_{K_m} \setminus \mathfrak{p}$). For a primitive p-th root of unity $\zeta_p \in \overline{\mathbb{Q}}$ and $j \in \mathbb{Z}$, the **Gauss sums** of $\mathfrak{p}$ are

$$\mathrm{g}_\mathfrak{p}(j) = \sum_{c \in k_\mathfrak{p}^\times} \chi_\mathfrak{p}(c)^{-j}\, \zeta_p^{\mathrm{Tr}_{k_\mathfrak{p}/\mathbb{F}_p}(c)} \in \mathbb{Q}(\mu_{mp}); \tag{7.4}$$

for $m \nmid j$ they are nonzero, since $\mathrm{g}_{\mathfrak{p}}(j)\mathrm{g}_{\mathfrak{p}}(-j) = \chi_{\mathfrak{p}}(-1)^j |k_{\mathfrak{p}}|$ [317]. For an integer b with $1 \le b < m$, the **Stickelberger quotient** and the **Gauss-sum element** of $(m, b, \mathfrak{p})$ are

$$\mathrm{J}_{m,b}(\mathfrak{p}) = \frac{\mathrm{g}_{\mathfrak{p}}(1)^b}{\mathrm{g}_{\mathfrak{p}}(b)}, \qquad \mathrm{R}_{m,b}(\mathfrak{p}) = \frac{\mathrm{g}_{\mathfrak{p}}(1)^b\, \mathrm{g}_{\mathfrak{p}}(-b)}{\mathrm{g}_{\mathfrak{p}}(b)\, \mathrm{g}_{\mathfrak{p}}(-1)^b}, \tag{7.5}$$

formed with the same ζ_p.

Definition 7.2.7. Let ℓ be an odd prime, $N_0 \ge 1$ and $m \ge 1$ integers with $\ell \nmid N_0$, $N = N_0 \ell^m$, and r a prime with $r \equiv 1 \pmod N$. A **Kolyvagin integer** for (N, r) is an integer $|L|$ with $L \in \mathcal{L}(N, r)$, i.e. 1 or a product of distinct Kolyvagin primes for (N, r). Put $\mathbb{Q}(\mu_\infty) = \bigcup_{n \ge 1} K_n$. For a place w of $\overline{\mathbb{Q}}$ above a prime p, v_w is its valuation with $v_w(p) = 1$; z is a w-unit if $v_w(z) = 0$, and for w-units z, z' we write $z \equiv z' \pmod w$ if $v_w(z - z') > 0$.

Euler systems. An **Euler system for** (N, r) is a family (α_d), indexed by the Kolyvagin integers d for (N, r), with $\alpha_d \in K_{Nd}^{\times}$, together with integers $e_{d,p}$, indexed by the pairs of a Kolyvagin integer d and a Kolyvagin prime $p \nmid d$, such that for all such d and p:

(Eu1) α_d is a w-unit for every place w of $\overline{\mathbb{Q}}$ not above r;

(Eu2) $\ell \nmid e_{d,p}$, and for every $\tau \in G_{\mathbb{Q}}$ that fixes μ_{Nd} pointwise and acts on μ_p by $\zeta \mapsto \zeta^g$ with g a generator of $(\mathbb{Z}/p\mathbb{Z})^{\times}$, and every $\varphi \in G_{\mathbb{Q}}$ with $\varphi(\zeta) = \zeta^p$ for $\zeta \in \mu_{Nd}$,

$$\prod_{i=0}^{p-2} \tau^i(\alpha_{pd}) = \Big(\frac{\alpha_d}{\varphi^{-1}(\alpha_d)}\Big)^{e_{d,p}};$$

(Eu3) for every φ as in (Eu2) and every place w of $\overline{\mathbb{Q}}$ above p, $\alpha_{pd} \equiv \varphi^{-1}(\alpha_d)^{e_{d,p}} \pmod w$.

Kolyvagin derivatives. A **choice of generators** for (N, r) is a family (τ_p), indexed by the Kolyvagin primes p, of elements $\tau_p \in G_{\mathbb{Q}}$ that fix every root of unity of order prime to p and act on μ_p by $\zeta \mapsto \zeta^{g_p}$ with g_p a generator of $(\mathbb{Z}/p\mathbb{Z})^{\times}$. The **Kolyvagin derivatives** are the maps $\overline{\mathbb{Q}}^{\times} \to \overline{\mathbb{Q}}^{\times}$

$$\mathrm{Der}_p(z) = \prod_{i=0}^{p-2} \tau_p^i(z)^i, \qquad \mathrm{Der}_L = \mathrm{Der}_{p_1} \circ \cdots \circ \mathrm{Der}_{p_k} \quad (L = (p_1, \ldots, p_k) \in \mathcal{L}(N, r)), \tag{7.6}$$

with $\mathrm{Der}_{\emptyset}$ the identity.

Descended Kummer characters. Let $\delta \in G_{\mathbb{Q}}$ and $a \in \mathbb{Z}$ with $\delta(\zeta) = \zeta^a$ for $\zeta \in \mu_{\ell^m}$, and put $\mathrm{P}_{\delta,a}(z) = \delta(z)\, z^{-a}$ for $z \in \overline{\mathbb{Q}}^{\times}$. Fix a primitive ℓ^m-th root of unity ζ_{ℓ^m}. An element $y \in \mathbb{Q}(\mu_\infty)^{\times}$ is **ℓ^m-invariant over** K_N if for every $g \in U_N$ the element $g(y)/y$ is an ℓ^m-th power in $\mathbb{Q}(\mu_\infty)^{\times}$. For such y and $g \in U_N$, let $Q_y(g)$ be the set of the quotients

$$\frac{g(\mathrm{P}_{\delta,a}(Y))}{\mathrm{P}_{\delta,a}(Y)\, \mathrm{P}_{\delta,a}(z)}, \qquad Y \in \overline{\mathbb{Q}},\ Y^{\ell^m} = y, \qquad z \in \mathbb{Q}(\mu_\infty),\ z^{\ell^m} = g(y)/y.$$

If $Q_y(g) = \{\zeta_{\ell^m}^k\}$ for a class $k \in \mathbb{Z}/\ell^m\mathbb{Z}$, the **descended Kummer character** of y takes the value $\mathrm{k}_y(g) = k$ at g; it depends on $(\delta, a, \zeta_{\ell^m})$, which are fixed wherever it is used.

Theorem 7.2.8. *Let $N \ge 1$ and $t \ge 0$ be integers and let $E \subset \mathbb{Q}(\mu_N)$ be a CM field with class number h_E. There is an integer X with*

$$1 \le X \le 2^{4(N+t+3)^{27}}\, h_E$$

such that for every prime ℓ, every coefficient ring $\mathcal{O}$ over $\mathbb{Z}_\ell$ containing the roots of unity of order the exponent of $(\mathbb{Z}/N)^\times$, every $0 \le e < t$ and every $\nu \ge 0$,

$$\operatorname{len}_{\mathcal{O}} \operatorname{Sha}^-(e,\nu) \le v_{\mathcal{O}}(X).$$

Proof. Put $y = N + t + 3$. For each integer e with $0 < e < t$, choose an integer X_e from Lemma 7.2.9; these choices depend only on N and e. Thus

$$1 \le X_e \le 2^{(N+e+3)^6} \le 2^{y^6}, \qquad Y_e = 16\varphi(N)X_e^2 \ge 1.$$

Define the positive integer

$$X = h_E \left(\prod_{0<e<t} Y_e \right)^{y^{20}}.$$

An empty product is 1. This choice precedes every prime, coefficient ring and truncation.

We first check its size. Since $y \ge 4$, $\varphi(N) \le N \le y$ and $y \le 2^y$ give

$$16\varphi(N) \le 16y \le 2^{y+4} \le 2^{2y^6}.$$

Consequently $Y_e \le 2^{4y^6}$. The number of indices $0 < e < t$ is at most $t \le y$, whence

$$\left(\prod_{0<e<t} Y_e \right)^{y^{20}} \le \left(2^{4y^6 \cdot y} \right)^{y^{20}} = 2^{4y^{27}}.$$

This proves the stated upper bound and positivity of X.

Fix a prime ℓ, an allowed coefficient ring $\mathcal{O}$, an integer $0 \le e < t$ and a truncation $\nu \ge 0$. If $e = 0$, (OE1) identifies the module with the everywhere unramified homomorphisms to $\mathcal{O}/\ell^\nu$. The injective coefficient map

$$\mathcal{O}/\ell^\nu \longrightarrow \mathcal{O}[\ell^{-1}]/\mathcal{O}, \qquad a \bmod \ell^\nu \longmapsto \ell^{-\nu} a \bmod \mathcal{O}$$

embeds this module in the unramified homomorphism module of Lemma 7.2.29. The CM field E has no real place, so that lemma applies also when $\ell = 2$. Its normalized length is $v_\ell(h_E)$; equivalently its ordinary $\mathcal{O}$-length is $v_{\mathcal{O}}(\ell)v_\ell(h_E) = v_{\mathcal{O}}(h_E)$. Since $h_E \mid X$, this proves the required inequality for $e = 0$.

Suppose now $0 < e < t$. The two congruence properties of X_e are exactly the hypotheses of Proposition 7.2.10; hence Y_e kills every odd cocycle. Also $N + e + 3 \le y$ and $X_e \le 2^{y^6}$, so Proposition 7.2.27 supplies a finite set $\mathcal{S} \subseteq G_E$ of at most y^{20} elements detecting these cocycles. Evaluation therefore gives an injective $\mathcal{O}$-linear map

$$\operatorname{Sha}^-(e,\nu) \longrightarrow \prod_{g \in \mathcal{S}} (\mathcal{O}/\ell^\nu)[Y_e].$$

For a discrete valuation ring, the Y_e-torsion in $\mathcal{O}/\ell^\nu$ has length $\min\{v_{\mathcal{O}}(Y_e), \nu v_{\mathcal{O}}(\ell)\}$: writing Y_e as a unit times a uniformizer power reduces the kernel of multiplication to the final that many steps of the uniformizer filtration. Thus

$$\operatorname{len}_{\mathcal{O}} \operatorname{Sha}^-(e,\nu) \le \#\mathcal{S}\, v_{\mathcal{O}}(Y_e) \le v_{\mathcal{O}}(Y_e^{y^{20}}).$$

Since Y_e is a factor in the defining product, $Y_e^{y^{20}} \mid X$. Monotonicity of the valuation on positive integer divisibility gives the claimed bound by $v_{\mathcal{O}}(X)$. Every choice and estimate was uniform in the prime, coefficient ring and truncation, as required. □

Lemma 7.2.9. *Let $N, e \geq 1$ be integers. There is an integer X with*

$$1 \leq X \leq 2^{(N+e+3)^6}$$

satisfying both conditions below.

1. *For every prime ℓ there is an integer $c \geq 2$ with $c \equiv 1 \pmod{N}$, $(c, \ell) = 1$, and $(c^e - 1)(c^{e+1} - 1) \mid X$.*
2. *For every integer $c \geq 0$ with $c \equiv 1 \pmod{N}$, every characteristic-zero commutative domain $\mathcal{O}$, and every Dirichlet character ψ modulo N with values in $\mathcal{O}$ such that $\psi(-1) = -(-1)^e$, there exists $w \in \mathcal{O}$ such that for every integer $m \geq 1$ prime to c,*

$$m \mid (c^{e+1} - 1)X - 2w \sum_{a=0}^{Nm-1} \left\lfloor \frac{ca}{Nm} \right\rfloor a^e \psi(a) \quad \text{in } \mathcal{O}.$$

Proof. Put $h = (e+2)!$, and for every integer a define

$$\beta(a) = hN^{e+1} B_{e+1}(a/N).$$

These numbers are integers. Indeed $N^{e+1} B_{e+1}(a/N) = \sum_{j=0}^{e+1} \binom{e+1}{j} B_j a^{e+1-j} N^j$, and the denominator theorem of von Staudt–Clausen makes the denominator of B_j a divisor of $(e+2)!$ for $j \leq e+1$.

We first establish the polynomial congruence needed below. Let R be any characteristic-zero domain, let ψ be an R-valued Dirichlet character modulo N, and let $c \equiv 1 \pmod{N}$, $(c, m) = 1$, $m \geq 1$. Write

$$A_\psi = \sum_{a=0}^{N-1} \psi(a)\beta(a), \quad P = \sum_{a=0}^{Nm-1} \psi(a) a^{e+1}, \quad S = \sum_{a=0}^{Nm-1} \left\lfloor \frac{ca}{Nm} \right\rfloor a^e \psi(a).$$

The Bernoulli generating function gives

$$\sum_{i=0}^{m-1} (a + Ni)^{e+1} = \frac{N^{e+1}}{e+2} \big(B_{e+2}(a/N + m) - B_{e+2}(a/N) \big).$$

After multiplying by h, its coefficient of m is $\beta(a)$ and all higher coefficients are integers. For powers m^k, $k \geq 2$, this follows by expanding the Bernoulli polynomial: the coefficient is $(e+1)!\binom{e+2}{k} N^{e+1} B_{e+2-k}(a/N)$, whose Bernoulli denominators divide $(e+1)!$ because $e + 2 - k \leq e$. Consequently

$$hP - mA_\psi \in m^2 R,$$

by writing every integer in $[0, Nm)$ uniquely as $a + Ni$ and using $\psi(a + Ni) = \psi(a)$.

For $0 \leq a < Nm$, put $q_a = \lfloor ca/(Nm) \rfloor$ and $r_a = ca - Nmq_a$. Multiplication by c permutes the residues modulo Nm, since c is coprime to both N and m; moreover $\psi(r_a) = \psi(a)$. The binomial expansion of $(ca - Nmq_a)^{e+1}$, modulo $(Nm)^2$, therefore gives

$$(c^{e+1} - 1)P - (e+1)c^e NmS \in (Nm)^2 R.$$

Multiply by h and insert the preceding congruence for hP. The result is

$$m\big((c^{e+1} - 1)A_\psi - h(e+1)Nc^e S\big) \in m^2 R.$$

As R has characteristic zero and is a domain, multiplication by m is injective. Dividing this divisibility by m proves

$$h(e+1)Nc^eS-(c^{e+1}-1)A_\psi\in mR. \tag{*}$$

We next construct a nonzero integer determinant. On the finite abelian group $H=(\mathbb{Z}/N)^\times$, let $\langle u\rangle$ denote the representative in $[0,N)$ and put

$$g(u)=2\beta(\langle u\rangle)+(-1)^e\mathbf{1}_{u=1}+\mathbf{1}_{u=-1},\qquad T_{s,t}=g(ts^{-1})\quad(s,t\in H).$$

For every complex character ψ of H, the vector $(\psi(t))_{t\in H}$ is an eigenvector of T, with eigenvalue

$$\lambda_\psi=2A_\psi+(-1)^e+\psi(-1).$$

All these eigenvalues are nonzero. To check this with imprimitive characters included, the Bernoulli evaluation and the distribution identity give

$$A_\psi=-h(e+1)NL(-e,\psi).$$

For a primitive character this is [317, Chapter 4, Theorem 4.2]. For the character induced from a primitive ψ_* of conductor $f\mid N$, both sides acquire the factor $\prod_{p\mid N}(1-\psi_*(p)p^e)$. For the Bernoulli side, insert $\mathbf{1}_{(a,N)=1}=\sum_{d\mid(a,\prod_{p\mid N}p)}\mu(d)$, put $a=db$, and use $\sum_{i=0}^{q-1}B_j((x+i)/q)=q^{1-j}B_j(x)$. Terms with $(d,f)>1$ vanish; in the others $f\mid N/d$, so this distribution identity reduces the sum to conductor f and leaves the factor d^e. Summing $\mu(d)\psi_*(d)d^e$ gives the stated Euler product. This also covers $N=1$: $B_{e+1}(0)=B_{e+1}(1)$ since $e+1\geq 2$.

The primitive functional equation, the nonvanishing Euler product at $1+e>1$, and the gamma factor show that $L(-e,\psi_*)$ vanishes exactly when $\psi_*(-1)=(-1)^e$. The conductor-one case is the zeta functional equation. None of the imprimitive Euler factors vanishes, because $|\psi_*(p)p^e|$ is either 0 or $p^e>1$. Thus the same parity criterion holds for ψ. If $\psi(-1)=-(-1)^e$, then $A_\psi\neq 0$ and $\lambda_\psi=2A_\psi\neq 0$; otherwise $A_\psi=0$ and $\lambda_\psi=2(-1)^e\neq 0$. Characters form a basis of the functions on the finite abelian group H, so $D=\det T$ is a nonzero integer.

Set

$$X=\big((N+1)^e-1\big)\big((N+1)^{e+1}-1\big)\big((2N+1)^e-1\big)\big((2N+1)^{e+1}-1\big)|D|.$$

Every factor is positive. For any prime ℓ, at least one of $N+1$ and $2N+1$ is prime to ℓ: a prime dividing both would divide their difference N and then 1. Taking that value for c proves the first required property.

We verify the second property over an arbitrary R, not only over a field of complex numbers. The same eigenvector identity $Tv=\lambda_\psi v$ holds over R, with $v_t=\psi(t)$ and $v_1=1$. Multiplying by the adjugate matrix and taking coordinate 1 proves that λ_ψ divides D in R. It therefore divides X in R. Under the required parity, $\lambda_\psi=2A_\psi$, so choose $u\in R$ with $X=2uA_\psi$ and put

$$w=uh(e+1)Nc^e.$$

This w does not depend on m. Multiplying (*) by $-2u$ gives exactly the required divisibility for every m prime to c.

Finally we check the explicit size. Let $M=hN^{e+1}(e+1)!$ and $z=N+e+3$. The Bernoulli recurrence gives $|B_j|\leq j!$ by induction: after using the induction hypotheses, the normalized bound is $\sum_{r=2}^{j+1}1/r!<1$. Hence, for $0\leq x\leq 1$, $|B_{e+1}(x)|\leq(e+1)!\sum_{r=0}^{e+1}1/r!<3(e+1)!$. Thus $|\beta(a)|\leq 3M$ for $0\leq a<N$, and every entry of T has absolute value at most $6M+2\leq 64M$. If $r=\varphi(N)\leq N$, the determinant expansion yields

$$|D|\leq r!(64M)^r\leq N!(64M)^N.$$

The four remaining factors are at most $(2N+1)^{4e+2}$. Use $2N+1 \le 2^{2N}$, $j! \le 2^{j^2}$ and $N \le 2^N$. These bounds give $X \le 2^H$, where

$$H = 8Ne + 4N + N^2 + N\big(6 + (e+2)^2 + N(e+1) + (e+1)^2\big) = N\big(2e^2 + 14e + 15 + N(e+2)\big).$$

Since $z \ge e+4$, $2e^2 + 14e + 15 \le 2z^2$ and $N(e+2) \le z^2$. Therefore $H \le 3z^3 \le z^6$, proving $X \le 2^{(N+e+3)^6}$. All the asserted properties now hold for this one positive integer. □

Proposition 7.2.10. *Let $N, e \ge 1$, let $E \subseteq \mathbb{Q}(\mu_N)$ be a CM field. Suppose $X \ge 1$ is an integer with the following two properties: for each prime p, some integer $c \ge 2$ satisfies $c \equiv 1 \pmod N$, $(c,p) = 1$ and $(c^e - 1)(c^{e+1} - 1) \mid X$; and for every $c \ge 0$ with $c \equiv 1 \pmod N$, every characteristic-zero commutative domain R and every R-valued Dirichlet character ψ modulo N with $\psi(-1) = -(-1)^e$, some $w \in R$ satisfies*

$$m \mid (c^{e+1} - 1)X - 2w \sum_{a=0}^{Nm-1} \left\lfloor \frac{ca}{Nm} \right\rfloor a^e \psi(a) \quad \text{in } R$$

for every $m \ge 1$ prime to c. Then for every prime ℓ, every coefficient ring $\mathcal{O}$ over $\mathbb{Z}_\ell$ containing roots of unity of order the exponent of $(\mathbb{Z}/N)^\times$, and every $\nu \ge 0$,

$$16\varphi(N)X^2 \operatorname{Sha}^-(e,\nu) = 0.$$

Proof. Fix $\ell, \mathcal{O}, \nu$ and $x \in \operatorname{Sha}^-(e,\nu)$. Apply Lemma 7.2.11, with representatives chosen once. For every character ψ with $\psi(-1) = -(-1)^e$, the projected cocycle Z_ψ satisfies exactly the local conditions in Proposition 7.2.12. Part (OC1) therefore gives

$$4X^2 Z_\psi = 0.$$

Projection is $\mathcal{O}$-linear in x. Consequently all odd projected components of $4X^2 x$ vanish. The comparison conclusion of Lemma 7.2.11, applied to $4X^2 x$, gives

$$4\varphi(N)\,(4X^2 x) = 0.$$

This is the asserted annihilator $16\varphi(N)X^2$. The argument does not invert 2, $\varphi(N)$, or any factor of X, so it applies to every coefficient prime. □

Lemma 7.2.11. *Let $N, e \ge 1$, let $E \subseteq \mathbb{Q}(\mu_N)$ be a CM field, fix a complex conjugation c, and put $A = (\mathbb{Z}/N)^\times$. Choose representatives $s_u \in G_\mathbb{Q}$ for its quotient A with $s_1 = 1$. For $u \in A$ and $g \in G_\mathbb{Q}$ put*

$$b(u,g) = s_u(cgc^{-1}g^{-1})s_u^{-1} \in G_E.$$

For every prime ℓ, every coefficient ring $\mathcal{O}$ containing the roots of unity of order the exponent of A, every $\nu \ge 0$, every $x \in \operatorname{Sha}^-(e,\nu)$ and every character $\psi : A \to \mathcal{O}^\times$, put $\alpha_\psi = \chi_{\mathrm{cyc}}^e \psi$ and

$$Z_\psi(g) = \sum_{u \in A} \alpha_\psi(s_u) x(b(u,g)).$$

Then Z_ψ is locally constant, satisfies

$$Z_\psi(gh) = Z_\psi(g) + \alpha_\psi(g)^{-1} Z_\psi(h), \qquad Z_\psi(c) = 0,$$

and vanishes on every $g \in G_{\mathbb{Q}(\mu_N)}$ which lies in inertia at a prime different from ℓ, or lies in a decomposition group above ℓ with $\chi_{\mathrm{cyc}}(g) = 1$. Moreover, if $Z_\psi = 0$ for every ψ with $\psi(-1) = -(-1)^e$, then $4\varphi(N)x = 0$. The representatives and the maps b are independent of $\ell, \mathcal{O}, e, \nu$ and x.

Proof. The extension $\mathbb{Q}(\mu_N)/\mathbb{Q}$ is abelian. Therefore every commutator $cgc^{-1}g^{-1}$ fixes μ_N, and so do its conjugates. This proves that $b(u,g) \in G_E$. Representatives with $s_1 = 1$ exist by the surjectivity of the action on μ_N.

Fix the coefficient data and x. Write $\delta(g) = cgc^{-1}g^{-1}$. The cyclotomic character is trivial on every commutator, hence on every conjugate of $\delta(g)$. Consequently the cocycle identity for x is additive on products of such elements. It also gives, for $h, t \in G_E$ with $\chi_{\mathrm{cyc}}(t) = 1$,

$$x(hth^{-1}) = \chi_{\mathrm{cyc}}(h)^{-e} x(t).$$

Indeed, expand the three factors and use $x(h^{-1}) = -\chi_{\mathrm{cyc}}(h)^e x(h)$.

For $v, g \in G_{\mathbb{Q}}$ put $T_\psi(v,g) = \alpha_\psi(v) x(v\delta(g)v^{-1})$. The argument of x belongs to G_E. If $v' = hv$ with $h \in G_{\mathbb{Q}(\mu_N)}$, the preceding conjugation identity and $\psi(h) = 1$ give $T_\psi(v', g) = T_\psi(v, g)$. Thus $T_\psi(v,g)$ depends only on the image of v in A. The identity

$$\delta(gh) = \delta(g)\, g\delta(h) g^{-1}$$

then yields

$$T_\psi(v, gh) = T_\psi(v,g) + \alpha_\psi(g)^{-1} T_\psi(vg, h).$$

Right multiplication by the image of g permutes A. Summing over the representatives proves the asserted cocycle identity for Z_ψ. Since $\delta(c) = 1$, it also proves $Z_\psi(c) = 0$. All maps involved are continuous and x is locally constant with finite target, so Z_ψ is locally constant.

To verify the local conditions, let g be one of the specified elements in $G_{\mathbb{Q}(\mu_N)}$. For every $v \in G_{\mathbb{Q}}$, both vgv^{-1} and $vcgc^{-1}v^{-1}$ belong to G_E. In the inertia case they belong to inertia at primes of E not dividing ℓ. In the decomposition case they belong to decomposition groups above ℓ, and their cyclotomic character is 1. The defining local conditions of x therefore make both their values zero. The cocycle identity makes $x(v\delta(g)v^{-1}) = 0$, proving $Z_\psi(g) = 0$.

It remains to prove the comparison factor. Orthogonality in the characteristic-zero ring $\mathcal{O}$, followed by reduction modulo ℓ^ν, gives

$$\sum_\psi \psi(u) = \begin{cases} |A| & u = 1, \\ 0 & u \neq 1. \end{cases}$$

There are exactly $|A|$ characters because $\mathcal{O}$ contains all roots of unity of order the exponent of A. This identity uses no division by $|A|$, even when $\ell \mid |A|$. Since $s_1 = 1$, for $h \in G_E$ it gives

$$\sum_\psi Z_\psi(h) = |A|\, x(\delta(h)) = |A|\big(x(chc^{-1}) - x(h)\big).$$

The last equality follows by expanding $chc^{-1}h^{-1}$ and using equality of the cyclotomic characters at h and chc^{-1}.

The scalar $\alpha_\psi(c)$ is either 1 or -1. If it is 1, the cocycle identity and $Z_\psi(c) = 0$ imply $Z_\psi(chc^{-1}) = Z_\psi(h)$. If it is -1, the parity condition on ψ is exactly $\psi(-1) = -(-1)^e$, and the hypothesis makes Z_ψ zero. Hence the sum over all ψ is invariant under $h \mapsto chc^{-1}$. Comparing the displayed identity at h and at chc^{-1} gives

$$2|A|\big(x(chc^{-1}) - x(h)\big) = 0.$$

Because $e > 0$, oddness means $x(chc^{-1}) = -x(h)$. Thus $4|A|x(h) = 0$ for every h, as required. All the initial choices were made using only N, E, c, proving the final uniformity assertion. □

Proposition 7.2.12. *Let $N, e \geq 1$. Suppose $X \geq 1$ is an integer with the following two properties: for each prime p, some integer $c \geq 2$ satisfies $c \equiv 1 \pmod{N}$, $(c,p) = 1$ and $(c^e - 1)(c^{e+1} - 1) \mid X$; and for every $c \geq 0$ with $c \equiv 1 \pmod{N}$, every characteristic-zero commutative domain R and every R-valued Dirichlet character ψ modulo N with $\psi(-1) = -(-1)^e$, some $w \in R$ satisfies*

$$m \mid (c^{e+1} - 1)X - 2w \sum_{a=0}^{Nm-1} \left\lfloor \frac{ca}{Nm} \right\rfloor a^e \psi(a) \quad \text{in } R$$

for every $m \geq 1$ prime to c. Fix a prime ℓ, a ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$, a Dirichlet character $\psi : (\mathbb{Z}/N)^\times \to \mathcal{O}^\times$ with $\psi(-1) = -(-1)^e$, and an integer $\nu \geq 0$. Inflate ψ to $G_\mathbb{Q}$ by its action on μ_N, and put $\alpha = \chi_{\mathrm{cyc}}^e \psi$. Fix a complex conjugation c and put $K_N = G_{\mathbb{Q}(\mu_N)}$. Let $\mathcal{C}$ be the $\mathcal{O}$-module of locally constant maps $z : G_\mathbb{Q} \to \mathcal{O}/\ell^\nu$ satisfying

$$z(gh) = z(g) + \alpha(g)^{-1} z(h), \qquad z(c) = 0,$$

and the following local conditions: $z(g) = 0$ whenever $g \in K_N$ belongs to an inertia group at a prime different from ℓ, or belongs to a decomposition group above ℓ and satisfies $\chi_{\mathrm{cyc}}(g) = 1$. Then:

(OC1) $4X^2\mathcal{C} = 0$.

(OC2) *If $\ell > N + e + 3$, there are $v_\ell(X) + 1$ elements $\gamma_i \in G_\mathbb{Q}$ such that $z \in \mathcal{C}$ and $z(\gamma_i) = 0$ for every i imply $z = 0$.*

Proof. Part (OC1) is Proposition 7.2.13, with exactly the hypotheses displayed here. We prove (OC2). If $\nu = 0$, the module is zero and any list of the required length detects it. Suppose $\nu \geq 1$, and put

$$m = v_\ell(X) + 1, \qquad R_m = \mathcal{O}/\ell^m, \qquad \beta = \chi_{\mathrm{cyc}}^{e+1}\psi, \qquad L = N\ell^m.$$

The field of coefficients and the truncation are fixed throughout this argument.

Choose $1 \leq a < N\ell$ with $a \equiv 1 \pmod{N}$ and with primitive residue in $\mathbb{F}_\ell^\times$. This is possible since $\ell > N$. Both $a^e - 1$ and $a^{e+1} - 1$ are units of $\mathcal{O}$, because $0 < e < e + 1 < \ell - 1$. In particular $a \neq 1$ and $a < L$. Choose γ in inertia above ℓ with $\chi_{\mathrm{cyc}}(\gamma) = a$; inertia fixes μ_N, so $\psi(\gamma) = 1$ and $\beta(\gamma) - 1 = a^{e+1} - 1$ is a unit.

Assume for a contradiction that no m elements detect $\mathcal{C}$. Thus every list of m elements has a nonzero member of $\mathcal{C}$ vanishing at all of them. We will use the following consequence: for every set T of at most m primes congruent to 1 modulo $N\ell$, there exists a nonzero $z \in \mathcal{C}$ killed by ℓ which is zero on every decomposition group above the primes of T. Choose a Frobenius at each prime of T, pad this list to length m, and choose a nonzero cocycle vanishing on the list. Multiply it by its last nonzero power of ℓ. The resulting z is killed by ℓ and still vanishes at the chosen Frobenii. At these primes the coefficient action on its values is trivial, and its inertia values are zero by the component local condition. Its restriction is consequently an unramified homomorphism zero at a generator of the unramified quotient, hence zero on the decomposition group. The cocycle conjugation formula transports this zero to every conjugate decomposition group. This proves the consequence.

We also need a finite simultaneous set of Frobenius tests. Let $\mathcal{Y}$ be the set of locally constant β-cocycles into R_m that vanish on inertia over $\mathbb{Q}(\mu_N)$ away from ℓ. Both $\mathcal{Y}$ and $\mathcal{C}$ are finite. Indeed, outside the primes dividing $N\ell$ the full inertia group fixes μ_N, so these cocycles are

unramified there; their coefficient actions are also unramified there. They therefore belong to the continuous cocycle groups for $G_{\mathbb{Q},S}$ with S consisting of those primes and infinity. The cohomology groups H^1 are finite by global finiteness, and the groups of coboundaries have at most the number of coefficient elements; hence the cocycle groups are finite. This is the finiteness assertion in [201, Chapter I, Theorem4.10].

For any cyclotomic subgroup $G_{\mathbb{Q}(\mu_f)}$, a nonzero $z \in \mathcal{C}$ does not vanish identically on that subgroup. Otherwise every commutator with c would be in its zero subgroup, so the cocycle identity and $z(c) = 0$ would imply $(\alpha(c)^{-1} - 1)z = -2z = 0$, impossible since ℓ is odd. Similarly, a nonzero $y \in \mathcal{Y}$ with $y(\gamma) = 0$ is nonzero on every such cyclotomic subgroup: the same commutator identity would give $(\beta(\gamma) - 1)y = 0$.

Put $f = aL$. For each pair consisting of $z \in \mathcal{C} \setminus \{0\}$ and $y \in \mathcal{Y}$ whose restriction to $G_{\mathbb{Q}(\mu_f)}$ is nonzero, choose a prime $q \equiv 1 \pmod f$, a place above it and a Frobenius φ such that $z(\varphi) \neq 0$ and $y(\varphi) \neq 0$. Such a choice follows from Chebotarev: the zero sets of the two cocycles are open subgroups, their intersections with the cyclotomic subgroup are proper, and a group cannot be the union of two proper subgroups. Thus there is a nonempty open subset of that subgroup on which both values are nonzero. Passing to a finite Galois quotient and choosing the place above a prime in the resulting conjugacy class gives the desired Frobenius representative. Since both cocycle sets are finite, the set Q of primes chosen in this way is finite. Every $q \in Q$ satisfies $q \equiv 1 \pmod{aL}$, and it has the following property:

$$0 \neq z \in \mathcal{C}, \quad y \in \mathcal{Y}, \quad y|_{G_{\mathbb{Q}(\mu_f)}} \neq 0$$
$$\implies \text{some } q \in Q \text{ has a Frobenius } \varphi \text{ with } z(\varphi)y(\varphi) \text{ separately nonzero.}$$

Only the two separate nonzero values are asserted; no product in a coefficient ring with zero divisors is used.

Choose a nonzero ℓ-torsion $z_0 \in \mathcal{C}$ by the preceding consequence with $T = \varnothing$. Odd visibility and Chebotarev give a prime $p \equiv 1 \pmod{L \prod_{q \in Q} q}$ and a Frobenius at p on which z_0 is nonzero. In particular z_0 is not zero on its decomposition group. The prime p is not in Q. Apply Proposition 7.2.17 to a, γ, Q, p, obtaining the actual dual cocycles κ_S and the bottom identity (DS4).

Call a finite set T of primes tight in this proof if all its primes are 1 modulo $N\ell$ and every ℓ-torsion β-cocycle into R_m unramified over $\mathbb{Q}(\mu_N)$ away from ℓ and T is unramified over that field away from ℓ. This is a property of the displayed cocycles, not an additional hypothesis. The empty set is tight. Moreover, if T is tight and an ℓ-torsion $z \in \mathcal{C}$ is zero above every prime in T but not above a prime $r \notin T$ congruent to 1 modulo $N\ell$, then $T \cup \{r\}$ is tight. To prove this, apply Lemma 7.2.15 to remove the possible inertia at r from any such dual cocycle, and then apply tightness of T. The initial choice of z_0, p shows in particular that $\{p\}$ is tight.

We prove by downward induction on $i = m, m-1, \ldots, 0$ that

$$S \subseteq Q, \quad \#S \leq i, \quad \{p\} \cup S \text{ tight}$$
$$\implies \ell^i \kappa_S \text{ is unramified over } \mathbb{Q}(\mu_N) \text{ away from } \ell.$$

For $i = m$ the cocycle is zero. Assume the assertion for $i+1$, and let S satisfy the conditions at $i < m$. By induction, $y = \ell^{i+1}\kappa_S$ belongs to $\mathcal{Y}$ and is zero at γ. If $y \neq 0$, it is visible on $G_{\mathbb{Q}(\mu_f)}$. The set $\{p\} \cup S$ has at most $i + 1 \leq m$ elements, so the consequence of blindness supplies a nonzero ℓ-torsion $z \in \mathcal{C}$ zero above it. The defining property of Q supplies $q \in Q$ and a Frobenius φ with $y(\varphi) \neq 0$ and $z(\varphi) \neq 0$. Hence $q \notin S \cup \{p\}$, and z is not zero above q. The tightness-extension assertion makes $\{p\} \cup S \cup \{q\}$ tight. Apply the induction hypothesis to $S \cup \{q\}$: the cocycle $\ell^{i+1}\kappa_{S \cup \{q\}}$ is unramified away from ℓ over $\mathbb{Q}(\mu_N)$. By (DS3), some inertia

element τ at q satisfies

$$y(\varphi) = \ell^{i+1}\kappa_{S\cup\{q\}}(\tau) = 0.$$

Here the full inertia group at q fixes μ_N, because $q \nmid N$. This contradicts its choice. We have proved $\ell^{i+1}\kappa_S = 0$. Thus $\ell^i\kappa_S$ is killed by ℓ. It is unramified away from $\ell, \{p\} \cup S$ by (DS2), and tightness removes the latter primes. This completes the induction.

Take $i = 0$ and $S = \varnothing$. Since $\{p\}$ is tight, $\kappa_\varnothing$ is unramified away from ℓ over $\mathbb{Q}(\mu_N)$, and hence zero on inertia at p. Its bottom value (DS4), whose scalar is a unit, gives

$$V_m = 0 \quad \text{in } R_m,$$

where V_m is the displayed floor-difference sum at level $N\ell^m$.

Put $A_0 = (a^{e+1} - 1)X$. Apply Lemma 7.2.26 with $n = m$. Here $N, e, m \geq 1$, the chosen a satisfies $a \equiv 1 \pmod N$ and $\ell \nmid a$, and the character has exactly the stated odd parity. The single multiplier w in the hypothesis supplies its two congruences at levels ℓ^m and ℓ^{m-1}, including the case $m = 1$ where the lower congruence is modulo 1. Its floor-difference sum is the present V_m. Since $V_m = 0$ in R_m, the lemma gives $\ell^m \mid A_0$ in $\mathcal{O}$. The factor $a^{e+1} - 1$ is a unit by our primitive-residue choice, so $\ell^m \mid X$. This is impossible:

$$v_\mathcal{O}(X) = v_\mathcal{O}(\ell)v_\ell(X) < v_\mathcal{O}(\ell)(v_\ell(X) + 1) = v_\mathcal{O}(\ell^m).$$

The contradiction proves that $m = v_\ell(X) + 1$ detecting elements exist, establishing (OC2). □

Proposition 7.2.13. *Let $N, e \geq 1$. Suppose $X \geq 1$ is an integer with the following two properties: for each prime p, some integer $c \geq 2$ satisfies $c \equiv 1 \pmod N$, $(c, p) = 1$ and $(c^e - 1)(c^{e+1} - 1) \mid X$; and for every $c \geq 0$ with $c \equiv 1 \pmod N$, every characteristic-zero commutative domain R and every R-valued Dirichlet character ψ modulo N with $\psi(-1) = -(-1)^e$, some $w \in R$ satisfies*

$$m \mid (c^{e+1} - 1)X - 2w \sum_{a=0}^{Nm-1} \left\lfloor \frac{ca}{Nm} \right\rfloor a^e\psi(a) \quad \text{in } R$$

for every $m \geq 1$ prime to c. Fix a prime ℓ, a ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$, a Dirichlet character $\psi : (\mathbb{Z}/N)^\times \to \mathcal{O}^\times$ with $\psi(-1) = -(-1)^e$, and an integer $\nu \geq 0$. Inflate ψ to $G_\mathbb{Q}$ by its action on μ_N, and put $\alpha = \chi_{\mathrm{cyc}}^e\psi$. Fix a complex conjugation c and put $K_N = G_{\mathbb{Q}(\mu_N)}$. Let $\mathcal{C}$ be the $\mathcal{O}$-module of locally constant maps $z : G_\mathbb{Q} \to \mathcal{O}/\ell^\nu$ satisfying

$$z(gh) = z(g) + \alpha(g)^{-1}z(h), \qquad z(c) = 0,$$

and the following local conditions: $z(g) = 0$ whenever $g \in K_N$ belongs to an inertia group at a prime different from ℓ, or belongs to a decomposition group above ℓ and satisfies $\chi_{\mathrm{cyc}}(g) = 1$. Then $4X^2\mathcal{C} = 0$.

Proof. Write $R = \mathcal{O}/\ell^\nu$ and fix $z \in \mathcal{C}$. If $\nu = 0$, then $R = 0$ and there is nothing to prove. Choose the integer $a \geq 2$ in the first hypothesis for the prime ℓ, and write

$$X = (a^e - 1)(a^{e+1} - 1)b, \qquad b \in \mathbb{Z}_{\geq 1}.$$

Put $r = \nu + 2$, $L = N\ell^r$, $K = \mathbb{Q}(\mu_L)$, and $d = 2(a^e - 1)$. Thus $L \geq 4$, so K has no real embeddings. The action through α^{-1} is trivial on G_K with values in R. Hence $z|_{G_K}$, and therefore $h = d\,z|_{G_K}$, are continuous homomorphisms to the finite additive group of R.

We first prove that h is everywhere unramified. Away from ℓ this follows directly from the local condition, since $G_K \subseteq K_N$. For each decomposition group D above ℓ, apply Lemma 7.2.14 to the given a and the cocycle z, with coefficient exponent ν and $s = r = \nu + 2$. Its local zero condition is exactly the displayed defining condition of $\mathcal{C}$, and its congruence and coprimality hypotheses are those in the choice of a. Thus $(a^e - 1)z$ vanishes on $D \cap G_K$. The multiple $h = 2(a^e - 1)z|_{G_K}$ therefore vanishes on the entire decomposition group of K above ℓ, proving the required assertion.

The finite-image homomorphism h cuts out a finite abelian extension of K. Global reciprocity, with arithmetic Frobenius, identifies its unramified ideal map with a homomorphism from the ideal class group: local units map to inertia, principal ideles map to 1, and there are no real-place conditions. In particular the value on every principal fractional ideal is zero. This is precisely the finite-extension reciprocity theorem of [202, Chapter V, Proposition5.2 and Theorem5.3].

We next construct the particular principal ideal to which we apply this fact. Let $p \equiv 1 \pmod{L}$ be a rational prime, let $\mathfrak{p}$ be a prime of K above p, and let $\varphi \in G_K$ be an arithmetic Frobenius at a chosen place above $\mathfrak{p}$. For $u \in (\mathbb{Z}/L)^\times$, let $\widetilde{u} \in \{1, \dots, L-1\}$ be its representative and let σ_u act on μ_L by u. The residue field at $\mathfrak{p}$ has order p. Let $G(j)$ be the Gauss sum for the inverse jth power of its Lth power-residue character, using a primitive pth root of unity.

The classical Gauss-sum identities and ideal factorization give $G(j) \neq 0$ when $L \nmid j$ and

$$(G(1)^L) = \prod_u \sigma_u^{-1}(\mathfrak{p})^{\widetilde{u}};$$

see [317, Chapter6]. Define $J_0 = G(1)^a / G(a)$. An automorphism fixing K and sending the chosen pth root to its tth power multiplies $G(j)$ by the jth power of the residue-character value at t. The factors cancel in J_0, so $J_0 \in K^\times$. Extending σ_a to fix that pth root gives $G(a)^L = \sigma_a(G(1)^L)$. Consequently

$$(J_0^L) = \prod_u \sigma_u^{-1}(\mathfrak{p})^{a\widetilde{u} - \widetilde{au}} = \prod_u \sigma_u^{-1}(\mathfrak{p})^{L\lfloor a\widetilde{u}/L \rfloor}.$$

The group of fractional ideals is free abelian; cancelling the common exponent L proves the corresponding formula for (J_0). Thus $J = J_0 / c(J_0)$ lies in $K^\times$, is a unit away from p, and satisfies

$$(J) = \prod_u \sigma_u^{-1}(\mathfrak{p})^{k(u)}, \qquad k(u) = \left\lfloor \frac{a\widetilde{u}}{L} \right\rfloor - \left\lfloor \frac{a(L - \widetilde{u})}{L} \right\rfloor.$$

Choose lifts $s_u \in G_{\mathbb{Q}}$ of the σ_u. Frobenius at $\sigma_u^{-1}(\mathfrak{p})$ is represented by $s_u^{-1} \varphi s_u$. The cocycle identity, together with trivial action of φ on R, gives

$$z(s_u^{-1} \varphi s_u) = \alpha(s_u) z(\varphi) = \widetilde{u}^{\,e} \psi(\widetilde{u}) z(\varphi) \quad \text{in } R.$$

Apply the ideal homomorphism defined by h to the principal ideal (J). We obtain

$$V_r \, 2(a^e - 1) z(\varphi) = 0, \qquad V_r = \sum_{u \in (\mathbb{Z}/L)^\times} k(u) \widetilde{u}^{\,e} \psi(\widetilde{u}) \in \mathcal{O}.$$

Put $A = (a^{e+1} - 1)X$. Apply Lemma 7.2.26 with $n = r$ and the present $N, e, \ell, \mathcal{O}, \psi, a$. Its parity and positive-exponent hypotheses are part of our assumptions. The single multiplier w supplied by the second hypothesis for $a, \mathcal{O}, \psi$ works at both levels ℓ^r and ℓ^{r-1}, because these integers are prime to a. Its floor-difference sum is precisely V_r above. Hence

$$A \equiv q V_r \pmod{\ell^r \mathcal{O}}$$

for some $q \in \mathcal{O}$. This application includes $\ell \mid N$, $\ell = 2$, and ramified coefficient rings without altering the ideal of congruence.

Reducing to R and multiplying the reciprocity identity by bq now gives

$$2X^2 z(\varphi) = 2(a^e - 1)b\, A\, z(\varphi) = 0.$$

This holds at every such split prime and every choice of its Frobenius.

To conclude, put $w_0 = 2X^2 z$ and suppose that $2w_0 \neq 0$. Its restriction to G_K cannot be zero. Otherwise its cocycle would factor through the abelian quotient $\mathrm{Gal}(K/\mathbb{Q})$. Commutativity of this quotient and $w_0(c) = 0$ would give

$$(\alpha(c)^{-1} - 1)w_0(g) = (\alpha(g)^{-1} - 1)w_0(c) = 0$$

for every g. Since $\alpha(c) = -1$, this would imply $2w_0 = 0$, a contradiction.

Choose $g \in G_K$ with $w_0(g) \neq 0$. Local constancy and the finite coefficient action allow passage to a finite Galois quotient in which membership in G_K and this nonzero value are determined. Chebotarev applied to the conjugacy class of the image of g supplies a prime split completely in K and a place above it whose Frobenius has that image. Conjugating the chosen place permits the representative of that image to be exactly the chosen one. This contradicts the preceding vanishing of $2X^2 z(\varphi)$. Thus $2w_0 = 4X^2 z = 0$, as asserted. □

Lemma 7.2.14. *Let $N, e, \nu \geq 1$, let ℓ be a prime, let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, and let $\psi : (\mathbb{Z}/N)^\times \to \mathcal{O}^\times$ be a Dirichlet character inflated to $G_\mathbb{Q}$. Put $\alpha = \chi_{\mathrm{cyc}}^e \psi$ and $K_N = G_{\mathbb{Q}(\mu_N)}$. Fix a decomposition group D above ℓ. Suppose $z : G_\mathbb{Q} \to \mathcal{O}/\ell^\nu\mathcal{O}$ satisfies*

$$z(gh) = z(g) + \alpha(g)^{-1} z(h), \qquad z(h) = 0 \quad (h \in D \cap K_N,\ \chi_{\mathrm{cyc}}(h) = 1).$$

Let $a \geq 1$ satisfy $a \equiv 1 \pmod N$ and $\ell \nmid a$. Then, for every $s \geq \nu$,

$$(a^e - 1)z(g) = 0 \qquad (g \in D \cap G_{\mathbb{Q}(\mu_{N\ell^s})}).$$

If $a^e - 1$ is a unit of $\mathcal{O}$, the restriction of z to this intersection is zero. Neither a parity condition on ψ nor unramifiedness away from ℓ is needed.

Proof. We use the cyclotomic inertia description in Theorem A.2.10. It gives an element i of inertia in D with $\chi_{\mathrm{cyc}}(i) = a$. Here is the passage from the finite description to that assertion. For each $t \geq 1$, the inertia image on μ_{ℓ^t} is all of $(\mathbb{Z}/\ell^t)^\times$, so the closed subset of inertia having cyclotomic value congruent to a modulo ℓ^t is nonempty. These subsets are nested. Inertia is a closed subgroup of a profinite group and is compact, so their intersection is nonempty. An element in the intersection has the stated exact cyclotomic value. Inertia fixes roots of unity of order prime to ℓ, and $a \equiv 1 \pmod N$ makes its action trivial on the ℓ-primary roots in μ_N as well. Consequently $i \in K_N$, $\psi(i) = 1$ and $\alpha(i) = a^e$.

Fix $g \in D \cap G_{\mathbb{Q}(\mu_{N\ell^s})}$. Its action through ψ is trivial, and $\chi_{\mathrm{cyc}}(g) \equiv 1 \pmod{\ell^s}$. Since $s \geq \nu$, $\alpha(g)$ and its inverse act as 1 on $\mathcal{O}/\ell^\nu\mathcal{O}$. The commutator

$$\delta = igi^{-1}g^{-1}$$

belongs to D, fixes μ_N, and has cyclotomic character 1: the actions on roots of unity and the cyclotomic character are abelian characters. Thus $z(\delta) = 0$ by hypothesis and $\alpha(\delta) = 1$. Since $ig = \delta gi$, the cocycle identity gives $z(ig) = z(gi)$. Expanding both sides and using the trivial coefficient action of g yields

$$z(i) + a^{-e} z(g) = z(g) + z(i).$$

It follows that $(a^{-e}-1)z(g)=0$. Multiplication by the unit $-a^e$ proves the claimed annihilation. When a^e-1 is a unit, this forces $z(g)=0$. All computations take place in the literal quotient $\mathcal{O}/\ell^\nu\mathcal{O}$, so no reduction to an unramified coefficient ring or residue field is involved. □

Lemma 7.2.15. *Let $N, e \geq 1$, let $\ell > N+e+3$ be a prime, let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, and let $\psi : (\mathbb{Z}/N)^\times \to \mathcal{O}^\times$ satisfy $\psi(-1) = -(-1)^e$. Inflate ψ to $G_\mathbb{Q}$ and put $\alpha = \chi_{\mathrm{cyc}}^e\psi$, $\beta = \chi_{\mathrm{cyc}}^{e+1}\psi$ and $K_N = G_{\mathbb{Q}(\mu_N)}$. Let T be a finite set of primes congruent to 1 modulo $N\ell$, and let $r \notin T$ be another such prime. Suppose $\nu, m \geq 1$ and:*

(CP1) *$z : G_\mathbb{Q} \to \mathcal{O}/\ell^\nu$ is a locally constant α^{-1}-cocycle, killed by ℓ, zero at a fixed complex conjugation c, and zero on every element of K_N which lies in inertia away from ℓ or lies in a decomposition group above ℓ with cyclotomic character 1.*

(CP2) *z is zero on every decomposition group above every prime in T, but is not zero on a decomposition group above r.*

(CP3) *$y : G_\mathbb{Q} \to \mathcal{O}/\ell^m$ is a locally constant β-cocycle, killed by ℓ, and is zero on the intersection with K_N of every inertia group at a prime outside $\{\ell, r\} \cup T$.*

Then y vanishes on every inertia group above r.

Proof. Put $A = \mathcal{O}/\ell\mathcal{O}$. Multiplication by $\ell^{\nu-1}$ identifies A with $(\mathcal{O}/\ell^\nu)[\ell]$, and multiplication by ℓ^{m-1} gives the analogous identification for the target of y. We henceforth regard both cocycles as A-valued. These are equivariant identifications, so every zero and every local condition is preserved.

There is a coefficient field $k \subseteq A$, isomorphic to the residue field of A. Indeed, if its order is q, the solutions of $X^q - X = 0$ in the finite local ring A lift its residue-field elements uniquely: the derivative is -1 and the maximal ideal is nilpotent. In characteristic ℓ these solutions are closed under addition and multiplication, so form a field. Every root of unity of order prime to ℓ in A belongs to this field: its reduction has the same order, and uniqueness of such a root's lift follows from its derivative being a unit. Since the order of ψ divides $\varphi(N) < \ell$, the reduction of every $\psi(g)$ belongs to $k^\times$; the reduction of $\chi_{\mathrm{cyc}}(g)$ belongs to $\mathbb{F}_\ell^\times \subseteq k^\times$. Thus A is a finite-dimensional k-vector space on which the two Galois actions are scalar actions through k. Every k-linear map from A to k commutes with the appropriate action.

Let $K_0 = \mathbb{Q}(\mu_{N\ell})$. Its degree is $\varphi(N)(\ell-1)$, which is prime to ℓ, and both coefficient characters are trivial on G_{K_0}. At every place above ℓ, the restriction of z to the decomposition group in G_{K_0} is zero. Indeed choose a positive integer $a \equiv 1 \pmod N$ with primitive residue modulo ℓ by the Chinese remainder theorem, since $\ell > N$. As $0 < e < \ell - 1$, $a^e - 1$ is a unit of $\mathcal{O}$. Apply Lemma 7.2.14 to the present $A = \mathcal{O}/\ell\mathcal{O}$-valued cocycle with coefficient exponent 1 and $s = 1$. Condition (CP1) is exactly its required local vanishing on $D \cap K_N \cap \ker \chi_{\mathrm{cyc}}$, and all its other hypotheses follow from the choice of a. Its unit conclusion gives the asserted zero restriction. This argument uses the quotient by $\ell\mathcal{O}$ itself, and allows its nilpotents when $\mathcal{O}$ is ramified.

Fix a decomposition group D_r at which z is nonzero. Its inertia group lies in K_N and is away from ℓ, so z vanishes on it. The action on A is trivial on D_r, because $r \equiv 1 \pmod{N\ell}$. Thus $z|_{D_r}$ is an unramified homomorphism. If φ_r is an arithmetic Frobenius, then $z(\varphi_r) \neq 0$; otherwise its closed zero subgroup would contain inertia and the topological generator of the unramified quotient, and hence all of D_r. Choose a k-linear map $f : A \to k$ with $f(z(\varphi_r)) \neq 0$. Put $Z = f \circ z$.

Suppose, towards a contradiction, that y is nonzero at an inertia element above r. Conjugation of inertia and the cocycle formula show that its restriction to inertia is nonzero at the

chosen group D_r as well: for an inertia element t above r, its coefficient action is trivial, and $y(gtg^{-1}) = \beta(g)y(t)$. Choose such a $t \in I_r$ and a k-linear map $g : A \to k$ with $g(y(t)) \neq 0$, and put $Y = g \circ y$. Then Z and Y are cocycles for $k(\alpha^{-1})$ and $k(\beta)$ respectively. Multiplication of their values is a Galois-equivariant pairing into $k(1)$, since $\alpha^{-1}\beta = \chi_{\mathrm{cyc}}$. Their cup product gives a global class

$$\xi = [Z] \cup [Y] \in H^2(\mathbb{Q}, k(1)).$$

We show that this class is locally zero outside r. At a prime in T, z is identically zero on the decomposition group, so the cup product is zero. At a prime outside $T \cup \{\ell, r\}$, restrict to any completion of K_0. Both cocycles are unramified there, by (CP1) and (CP3), since the local inertia subgroup of G_{K_0} is contained in K_N. The coefficient modules are trivial over K_0, so their unramified classes are orthogonal by local Tate duality; see [201, Chapter I, Theorem2.6]. Thus the restricted cup product is zero. This includes primes dividing N and requires no local-coboundary hypothesis there. At ℓ, the restriction of z was proved zero, so again the restricted cup product is zero. Each of the local extension degrees divides $[K_0 : \mathbb{Q}]$ and is prime to ℓ. Restriction on the ℓ-torsion group $H^2(-, k(1))$ is injective, because corestriction after restriction is multiplication by that degree. Therefore these local cup products were already zero over $\mathbb{Q}$. Finally $H^2(\mathbb{R}, k(1)) = 0$, since ℓ is odd.

Apply global Brauer reciprocity coordinatewise on the finite-dimensional $\mathbb{F}_\ell$-space k. The class ξ is zero at all places other than r, so it is zero at r as well; see [224] and [201]. On D_r, both coefficient characters are trivial, and the unramified k-cohomology is one-dimensional over k. Its nonzero element $[Z]$ therefore spans it. The vanishing of the k-valued cup product implies that $[Y]$ is orthogonal to every multiple of $[Z]$. Using the perfect trace pairing $k \times k \to \mathbb{F}_\ell$ identifies $k(\beta)$ with the Cartier dual of $k(\alpha^{-1})$, so local Tate orthogonality makes $[Y]$ unramified. Since the local action is trivial, every coboundary is zero; hence the cocycle Y itself vanishes on inertia. This contradicts $Y(t) \neq 0$. Thus y is zero on every inertia group above r. □

Theorem 7.2.16. *Let ℓ be an odd prime and $R \geq 1$. For a place v of $\mathbb{Q}$ write $\mathrm{res}_v \colon H^2(G_{\mathbb{Q}}, \mu_{\ell^R}) \to H^2(G_{\mathbb{Q}_v}, \mu_{\ell^R})$ for the restriction to a decomposition group at v. Then:*

1. *$H^2(G_{\mathbb{R}}, \mu_{\ell^R}) = 0$, and for every prime p the map $\mathrm{inv}_p \colon H^2(G_{\mathbb{Q}_p}, \mu_{\ell^R}) \to \mathbb{Q}/\mathbb{Z}$, the composite of the map to the Brauer group $\mathrm{Br}(\mathbb{Q}_p)$ induced by $\mu_{\ell^R} \subset \overline{\mathbb{Q}}_p^{\times}$ with the Hasse invariant, is injective;*
2. *for every $\xi \in H^2(G_{\mathbb{Q}}, \mu_{\ell^R})$, $\mathrm{res}_p(\xi) = 0$ for all but finitely many primes p, and*

$$\sum_p \mathrm{inv}_p(\mathrm{res}_p\, \xi) = 0.$$

In particular, if $\mathrm{res}_v(\xi) = 0$ for every place $v \neq v_0$, then $\mathrm{res}_{v_0}(\xi) = 0$.

Proof. Put $n = \ell^R$. For $K = \mathbb{Q}$, $\mathbb{Q}_p$ or $\mathbb{R}$, choose a separable closure K^{sep}. All these fields have characteristic zero. Thus the map $x \mapsto x^n$ on $(K^{\mathrm{sep}})^{\times}$ is surjective and gives the exact sequence of discrete continuous G_K-modules

$$1 \longrightarrow \mu_n \longrightarrow (K^{\mathrm{sep}})^{\times} \xrightarrow{x \mapsto x^n} (K^{\mathrm{sep}})^{\times} \longrightarrow 1.$$

Use the canonical cohomological description $\mathrm{Br}(K) = H^2(G_K, (K^{\mathrm{sep}})^{\times})$ ([202], Chapter IV, Corollary 3.16). Hilbert's Theorem 90 gives $H^1(G_K, (K^{\mathrm{sep}})^{\times}) = 0$ ([265]). The relevant part of the long cohomology sequence is consequently

$$0 \longrightarrow H^2(G_K, \mu_n) \xrightarrow{j_K} \mathrm{Br}(K) \xrightarrow{n} \mathrm{Br}(K).$$

Here j_K is precisely the map induced by inclusion of μ_n in the multiplicative group. In particular it is injective, and its image is $\mathrm{Br}(K)[n]$. These assertions use exactness at the two displayed groups, not any assertion about surjectivity onto the whole Brauer group. Functoriality of the cohomology sequence gives

$$j_{\mathbb{Q}_v} \circ \mathrm{res}_v = \mathrm{res}_v^{\mathrm{Br}} \circ j_{\mathbb{Q}} \tag{Br1}$$

for each chosen decomposition group.

The local maps have the stated kernels. The local invariant theorem [202], Chapter IV, Section 4, identifies $\mathrm{Br}(\mathbb{Q}_p)$ with $\mathbb{Q}/\mathbb{Z}$ and $\mathrm{Br}(\mathbb{R})$ with $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Since n is odd, the latter group has no n-torsion: if $2b = nb = 0$, integers u, v with $2u + nv = 1$ give $b = 0$. The injective map $j_{\mathbb{R}}$ therefore has zero image, proving $H^2(G_{\mathbb{R}}, \mu_n) = 0$. For each prime p, the map called inv_p in the statement is $\mathrm{inv}_p^{\mathrm{Br}} \circ j_{\mathbb{Q}_p}$. The first factor is an isomorphism and the second is injective, so this composite is injective. This proves item 1, including the specified normalization of its map.

Global reciprocity passes through the Kummer injection. Let $\xi \in H^2(G_{\mathbb{Q}}, \mu_n)$ and put $b = j_{\mathbb{Q}}(\xi)$. Apply the Albert–Brauer–Hasse–Noether theorem with the number field $K = \mathbb{Q}$ ([202], Chapter VIII, Theorem 4.2). The localization of b belongs to the direct sum of the local Brauer groups, so its finite local restrictions vanish outside a finite set. By (Br1) and the injectivity of $j_{\mathbb{Q}_p}$, the same is true of $\mathrm{res}_p(\xi)$. The invariant sum of the localizations of b is zero. The infinite localization is zero, either by (Br1) and item 1 or because it is n-torsion in $\mathrm{Br}(\mathbb{R})$. Hence

$$0 = \sum_v \mathrm{inv}_v^{\mathrm{Br}}(\mathrm{res}_v^{\mathrm{Br}}\, b) = \sum_p \mathrm{inv}_p^{\mathrm{Br}}(j_{\mathbb{Q}_p}\, \mathrm{res}_p\, \xi) = \sum_p \mathrm{inv}_p(\mathrm{res}_p\, \xi).$$

Every sum here has finite support. This proves item 2.

Finally suppose all restrictions except possibly the one at v_0 vanish. If v_0 is the real place, item 1 gives the result. If $v_0 = p_0$ is finite, the sum in item 2 reduces to $\mathrm{inv}_{p_0}(\mathrm{res}_{p_0}\, \xi) = 0$. Injectivity from item 1 forces $\mathrm{res}_{p_0}\, \xi = 0$, as asserted. □

Proposition 7.2.17. *Let $N, e \geq 1$, let $\ell > N + e + 3$ be prime, let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, and let $\psi : (\mathbb{Z}/N)^\times \to \mathcal{O}^\times$ satisfy $\psi(-1) = -(-1)^e$. Put*

$$\alpha = \chi_{\mathrm{cyc}}^e \psi, \qquad \theta = \alpha^{-1}, \qquad \beta = \chi_{\mathrm{cyc}} \theta^{-1}.$$

Fix $m \geq 1$, put $L = N\ell^m$, and choose $1 \leq a < L$ with $a \equiv 1 \pmod N$, $(a, \ell) = 1$, and $a^{e+1} - 1 \in \mathcal{O}^\times$. Suppose $\gamma \in G_{\mathbb{Q}(\mu_N)}$ has $\chi_{\mathrm{cyc}}(\gamma) = a$. Let Q be a finite set of primes congruent to 1 modulo aL, and let p be a prime congruent to 1 modulo $L \prod_{q \in Q} q$. Fix a place above p. There are maps $\kappa_S : G_{\mathbb{Q}} \to \mathcal{O}/\ell^m$, for every $S \subseteq Q$, such that:

(DS1) *Each κ_S is a locally constant β-cocycle and $\kappa_S(\gamma) = 0$.*

(DS2) *κ_S vanishes on the intersection with $G_{\mathbb{Q}(\mu_N)}$ of every inertia group at a prime outside $\{\ell, p\} \cup S$.*

(DS3) *For $q \in Q \setminus S$, every place above q and every arithmetic Frobenius φ there, some inertia element τ at that place satisfies $\kappa_{S \cup \{q\}}(\tau) = \kappa_S(\varphi)$.*

(DS4) *At the fixed place above p there is an inertia element τ_0 with*

$$\kappa_\varnothing(\tau_0) = 2N(a^{-e} - a)^2 \sum_{u \in (\mathbb{Z}/L)^\times} \left(\left\lfloor \frac{a\widetilde{u}}{L} \right\rfloor - \left\lfloor \frac{a(L - \widetilde{u})}{L} \right\rfloor \right) \widetilde{u}^{\,e} \psi(\widetilde{u}) \quad \text{in } \mathcal{O}/\ell^m,$$

where $1 \leq \widetilde{u} < L$ represents u. The scalar $2N(a^{-e} - a)^2$ is a unit of $\mathcal{O}$.

Proof. Put $U = G_{\mathbb{Q}(\mu_L)}$, $M = \ell^m$ and $R = \mathcal{O}/M$. Since $\ell > N$, the integer N is prime to ℓ. The character θ has Kolyvagin level (N, m): on U its value is 1 modulo M, on elements inverting μ_L its value is -1 modulo M by the parity of ψ, and

$$\theta(\gamma) - \chi_{\mathrm{cyc}}(\gamma) = a^{-e} - a = a^{-e}(1 - a^{e+1}) \in \mathcal{O}^\times.$$

Also $\beta(\gamma) = a^{e+1}$, so $1 - \beta(\gamma)$ is a unit.

We use the explicit construction in Proposition 7.2.18, taking its auxiliary witness to be $\delta = \gamma$ and its auxiliary integer to be a, and taking the element $w_0 = M^{-1} \bmod \mathcal{O}$ in $\mathcal{O}[1/\ell]/\mathcal{O}$. We recall the construction and its properties to specify which system is used. Let $(\alpha_d, e_{d,q})$ be the Euler system provided by Proposition 7.2.19 for base modulus L, auxiliary prime p and exponent a. Every $q \in Q$ is one of its Kolyvagin primes: $q \equiv 1 \pmod{L}$, $q \neq p$, and $p \equiv 1 \pmod{q}$ implies $\mathrm{ord}_q(p) = 1$. Choose its inertia generators and a primitive Mth root of unity. For $S \subseteq Q$, let

$$y_S = \mathrm{Der}_S(\alpha_{\prod_{q \in S} q}).$$

The derivative is independent of the order of the primes, lies in $\mathbb{Q}(\mu_L \prod S)^\times$, is a unit at every place not above p, and is M-invariant over $\mathbb{Q}(\mu_L)$, by Lemma 7.2.23. Its descended Kummer map $k_S : U \to \mathbb{Z}/M$ is therefore a continuous homomorphism by Lemma 7.2.24. Choose lifts s_t of the units modulo L and put

$$\Lambda_S(h) = \sum_{t \in (\mathbb{Z}/L)^\times} k_S(s_t^{-1} h s_t)\, \beta(s_t) \quad (h \in U),$$

with values in R. This is the construction of Proposition 7.2.18 after identifying the M-torsion generated by w_0 with R.

For clarity, its equivariance is $\Lambda_S(ghg^{-1}) = \beta(g)\Lambda_S(h)$. To verify it, right multiplication of the representatives by g permutes their cosets modulo U; the discrepancy in each representative is an element of U. The homomorphism k_S is unchanged by inner conjugation by such an element, and β is 1 modulo M on U. Reindexing the sum gives the displayed identity.

At $q \in Q \setminus S$, the finite–singular identity Lemma 7.2.25, applied to each conjugate in this sum, gives an inertia element τ with

$$\Lambda_{S \cup \{q\}}(\tau) = -e_{\prod S, q} \Lambda_S(\varphi).$$

The multiplier is an integer prime to ℓ. Replacing τ by an integer power whose exponent is inverse to $-e_{\prod S,q}$ modulo M makes this identity exact without the multiplier; on this inertia group the maps are homomorphisms, since the local coefficient action is trivial. This replacement uses an integer unit, not an arbitrary $\mathcal{O}$-unit.

The same construction has the required stronger conductor-prime vanishing. Let $r \notin \{\ell, p\} \cup S$ and let h belong to inertia at r and to $G_{\mathbb{Q}(\mu_N)}$. Since $r \neq \ell$, inertia fixes μ_M, and hence $h \in U$. Each prime in S is different from r, so inertia also fixes their roots of unity. Thus h fixes $\mathbb{Q}(\mu_L \prod S)$ and y_S. The same holds for every conjugate $s_t^{-1} h s_t$ at its conjugate place. Since y_S is a unit at all places above r, item3 of Lemma 7.2.24 gives $k_S(s_t^{-1} h s_t) = 0$ and therefore $\Lambda_S(h) = 0$. This argument includes $r \mid N$; excluding such primes from the ramification assertion would not suffice.

We record the bottom-value calculation as well. Write

$$H(t) = \left\lfloor \frac{a\widetilde{t}}{L} \right\rfloor - \left\lfloor \frac{a(L - \widetilde{t})}{L} \right\rfloor,$$

and let t_γ be the action of γ on μ_L. The Euler-system valuation in Proposition 7.2.19 is $2N(H(tt_\gamma) - aH(t))$ at the chosen prime. The descended Kummer map applies the auxiliary operator once more. Item4 of Lemma 7.2.24 consequently gives an inertia element τ_0 with

$$\Lambda_\varnothing(\tau_0) = 2N \sum_t \bigl(H(tt_\gamma^2) - 2aH(tt_\gamma) + a^2 H(t)\bigr)\theta(s_t)^{-1}.$$

Reindex $t \mapsto tt_\gamma^{-j}$ in the term with exponent j. Since two corresponding representatives differ by U and $\theta|_U = 1$ modulo M, that term is $\theta(\gamma)^j \sum_t H(t)\theta(s_t)^{-1}$. Thus

$$\Lambda_\varnothing(\tau_0) = 2N(\theta(\gamma) - a)^2 \sum_t H(t)\theta(s_t)^{-1}.$$

Here $\theta(\gamma) = a^{-e}$ and $\theta(s_t)^{-1} \equiv \widetilde{t}^{\,e}\psi(\widetilde{t}) \pmod M$. This proves exactly the formula in (DS4). Its scalar is a unit because $\ell \nmid 2N$ and $a^{-e} - a$ is a unit. In particular the two auxiliary operators produce a unit square, not a further denominator.

It remains to extend the homomorphisms from U to the full absolute Galois group. Its quotient by U is abelian, so $g\gamma g^{-1}\gamma^{-1} \in U$ for every g. Define

$$\kappa_S(g) = (1 - \beta(\gamma))^{-1}\Lambda_S(g\gamma g^{-1}\gamma^{-1}).$$

The identity

$$gh\gamma(gh)^{-1}\gamma^{-1} = g(h\gamma h^{-1}\gamma^{-1})g^{-1}(g\gamma g^{-1}\gamma^{-1})$$

and equivariance of Λ_S prove the β-cocycle identity. The formula is locally constant, and it is zero at γ. For $h \in U$,

$$\Lambda_S(h\gamma h^{-1}\gamma^{-1}) = (1 - \beta(\gamma))\Lambda_S(h),$$

so $\kappa_S|_U = \Lambda_S$.

The inertia elements in the strengthened vanishing assertion belong to U, as already checked. The full decomposition groups at $q \in Q$ and at p also belong to U, because these primes split completely in $\mathbb{Q}(\mu_L)$. Hence the extension preserves all the inertia zeros, the finite–singular identities and the bottom value. These are (DS1)–(DS4). □

Proposition 7.2.18. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring and $\theta\colon G_\mathbb{Q} \to \mathcal{O}^\times$ a continuous character with a Kolyvagin level (N_0, m) of modulus $N = N_0\ell^m$. Let r be a prime with $r \equiv 1 \pmod N$, D_0 a decomposition group at r, b an integer with $0 \le b < N$ and $\gcd(b, N) = 1$, σ a family of lifts modulo N, and $w \in \mathcal{O}^\vee$ with $\ell^m w = 0$. Then there is a Kolyvagin system for $(\theta, N_0, m, r, D_0, \mathrm{st}_{N,b}(\theta, \sigma)\, w)$.*

Proof. Write $M = \ell^m$, $\mathrm{st} = \mathrm{st}_{N,b}(\theta, \sigma)$, and let $\bar{\mathfrak{r}}$ be the place stabilised by D_0. As $N \ge 3$ and $\gcd(b, N) = 1$, $1 \le b < N$. For $j \in \mathbb{Z}/M\mathbb{Z}$ and $v \in \mathcal{O}^\vee$ with $Mv = 0$, $j\,v$ means $\tilde{j}v$ for any integer $\tilde{j}$ in the class j. For $h \in U_N$, $\chi_{\mathrm{cyc}}(h) \equiv 1$ and, by (Kv1), $\theta(h) \equiv 1$ modulo $M\mathcal{O}$; hence $\check{\theta}(h) \equiv 1$, and $\check{\theta}(gh)v = \check{\theta}(g)v$, $\theta(gh)^{-1}v = \theta(g)^{-1}v$ for $g \in G_\mathbb{Q}$ and $Mv = 0$.

Step 1: construction. By (Kv3) choose $\delta \in G_\mathbb{Q}$ with $\theta(\delta) - \chi_{\mathrm{cyc}}(\delta) \in \mathcal{O}^\times$, an integer a with $\delta(\zeta) = \zeta^a$ for $\zeta \in \mu_M$ (so $\chi_{\mathrm{cyc}}(\delta) \equiv a \pmod M$), and let $t_\delta \in (\mathbb{Z}/N\mathbb{Z})^\times$ be the class of the action of δ on μ_N. For each Kolyvagin prime p let τ_p be an element of an inertia group at p acting on μ_p by a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$; it exists because p is totally ramified in $\mathbb{Q}(\mu_p)$, and it fixes every root of unity ζ' of order prime to p (as $\tau_p(\zeta') \equiv \zeta'$ modulo the place, and such roots of unity are pairwise incongruent there). So (τ_p) is a choice of generators for (N, r). Fix a primitive M-th root of unity ζ_{ℓ^m}, so that k_y is formed with $(\delta, a, \zeta_{\ell^m})$, and $\rho \in \overline{\mathbb{Q}}$ with $\rho^M = r$.

By Proposition 7.2.19, applied to $(\ell, N_0, m, r, \bar{\mathfrak{r}}, b, \delta, a)$, there is an Euler system $((\alpha_d), (e_{d,p}))$ for (N, r) with

$$v_{\bar{\mathfrak{r}}}(g(\alpha_1)) = 2N_0\big(h_N(b, t\,t_\delta) - a\,h_N(b,t)\big) \quad \text{if } g \in G_{\mathbb{Q}} \text{ acts on } \mu_N \text{ by } t. \tag{7.7}$$

For $L \in \mathcal{L}(N,r)$ let $y_L = \mathrm{Der}_L(\alpha_{|L|})$; by Lemma 7.2.23 it lies in $K^\times_{N|L|}$, is a w'-unit for every place w' not above r, and is ℓ^m-invariant over K_N, so by item 1 of Lemma 7.2.24 $\mathrm{k}_L = \mathrm{k}_{y_L} : U_N \to \mathbb{Z}/M\mathbb{Z}$ is a continuous homomorphism. Put

$$\kappa_L(u) = \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathrm{k}_L(\sigma_t^{-1} u \sigma_t)\, \check{\theta}(\sigma_t)\, w \qquad (u \in U_N),$$

which makes sense as U_N is normal in $G_{\mathbb{Q}}$. We check (Ks1)–(Ks5) for $(\kappa_L)_{L \in \mathcal{L}(N,r)}$.

Step 2: (Ks1). Each $u \mapsto \mathrm{k}_L(\sigma_t^{-1} u \sigma_t)$ is a continuous homomorphism into $\mathbb{Z}/M\mathbb{Z}$, so κ_L is continuous, additive, and killed by M.

Step 3: (Ks2). Let $g \in G_{\mathbb{Q}}$. For $t \in (\mathbb{Z}/N\mathbb{Z})^\times$ let t' be the class of the action of $g^{-1}\sigma_t$ on μ_N and $h_t = \sigma_{t'}^{-1} g^{-1} \sigma_t \in U_N$; $t \mapsto t'$ is a bijection. Then $\sigma_t^{-1}(gug^{-1})\sigma_t = h_t^{-1}(\sigma_{t'}^{-1} u \sigma_{t'}) h_t$, whose image under the homomorphism k_L into an abelian group is $\mathrm{k}_L(\sigma_{t'}^{-1} u \sigma_{t'})$, and $\check{\theta}(\sigma_t)w = \check{\theta}(g)\check{\theta}(\sigma_{t'})\check{\theta}(h_t)w = \check{\theta}(g)\check{\theta}(\sigma_{t'})w$. Summing, $\kappa_L(gug^{-1}) = \check{\theta}(g)\kappa_L(u)$.

Step 4: (Ks3). Let $u \in U_N$ lie in an inertia group I at a prime $p' \nmid Nr|L|$. For each t, $h = \sigma_t^{-1} u \sigma_t$ lies in the inertia group $\sigma_t^{-1} I \sigma_t$ at p'; it fixes the roots of unity of order prime to p' (as in Step 1), hence $K_{N|L|}$ and y_L. As $p' \neq \ell$ ($\ell \mid N$) and y_L is a unit at the places above $p' \neq r$, item 3 of Lemma 7.2.24 gives $\mathrm{k}_L(h) = 0$. So $\kappa_L(u) = 0$.

Step 5: (Ks4). Let $L \in \mathcal{L}(N,r)$, p a Kolyvagin prime not in L, D a decomposition group at p and φ a Frobenius of D. As p is totally ramified in $\mathbb{Q}(\mu_p)$, $I(D)$ maps onto $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$; choose $\tau \in I(D)$ agreeing with τ_p on μ_p. For $t \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\sigma_t^{-1} D \sigma_t$ is a decomposition group at p, $\sigma_t^{-1}\tau\sigma_t$ lies in its inertia group and agrees with τ_p on μ_p ($\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ is abelian), and $\sigma_t^{-1}\varphi\sigma_t$ is a Frobenius of it. Since $y_{(p,L)} = \mathrm{Der}_p(\mathrm{Der}_L(\alpha_{p|L|}))$, Lemma 7.2.25 gives $\mathrm{k}_{(p,L)}(\sigma_t^{-1}\tau\sigma_t) = -e\,\mathrm{k}_L(\sigma_t^{-1}\varphi\sigma_t)$ with $e = e_{|L|,p}$. Summing over t, $\kappa_{(p,L)}(\tau) = (-e)\,\kappa_L(\varphi)$, and $-e \in \mathcal{O}^\times$ as $\ell \nmid e$ by (Eu2).

Step 6: (Ks5). By item 4 of Lemma 7.2.24 there is $\tau_0 \in I(D_0)$ with $\tau_0(\rho) = \zeta_{\ell^m}\rho$. Let $t \in (\mathbb{Z}/N\mathbb{Z})^\times$; σ_t acts on $\mu_M \subseteq \mu_N$ by $\tilde{t}$, and $y_\emptyset = \alpha_1 \in K_N^\times$. Item 4 of Lemma 7.2.24 and (7.7), applied to $\sigma_t\delta$ (acting on μ_N by $t\,t_\delta$) and to σ_t, give in $\mathbb{Z}/M\mathbb{Z}$

$$\tilde{t}\,\mathrm{k}_\emptyset(\sigma_t^{-1}\tau_0\sigma_t) = v_{\bar{\mathfrak{r}}}(\sigma_t\delta(\alpha_1)) - a\,v_{\bar{\mathfrak{r}}}(\sigma_t(\alpha_1)) = H(t) := 2N_0\big(h_N(b, t\,t_\delta^2) - 2a\,h_N(b, t\,t_\delta) + a^2 h_N(b,t)\big).$$

As $\chi_{\mathrm{cyc}}(\sigma_t) \equiv \tilde{t} \pmod M$, $\check{\theta}(\sigma_t)w = \tilde{t}\,\theta(\sigma_t)^{-1}w$, so $\kappa_\emptyset(\tau_0) = \sum_t H(t)\,\theta(\sigma_t)^{-1}w$. For $j \geq 0$, substituting $t = t''t_\delta^{-j}$ and using that $\sigma_{t''t_\delta^{-j}}$ and $\sigma_{t''}\delta^{-j}$ agree on μ_N, hence differ by an element of U_N,

$$\sum_t h_N(b, t\,t_\delta^j)\,\theta(\sigma_t)^{-1}w = \sum_{t''} h_N(b,t'')\,\theta(\sigma_{t''})^{-1}\theta(\delta)^j w = \theta(\delta)^j\,\mathrm{st}\,w$$

by (7.3). Hence $\kappa_\emptyset(\tau_0) = u_0\,\mathrm{st}\,w$ with $u_0 = 2N_0(\theta(\delta)^2 - 2a\theta(\delta) + a^2) = 2N_0(\theta(\delta) - a)^2$. Here $\ell \nmid 2N_0$, and $\theta(\delta) - a = (\theta(\delta) - \chi_{\mathrm{cyc}}(\delta)) + (\chi_{\mathrm{cyc}}(\delta) - a)$ is a unit plus an element of $M\mathcal{O} \subseteq \mathfrak{m}$ ($m \geq 1$), hence a unit; so $u_0 \in \mathcal{O}^\times$.

Hence $(\kappa_L)_{L \in \mathcal{L}(N,r)}$ is a Kolyvagin system for $(\theta, N_0, m, r, D_0, \mathrm{st}\,w)$. □

Proposition 7.2.19. *Let ℓ be an odd prime, $N_0 \geq 1$ and $m \geq 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, r a prime with $r \equiv 1 \pmod N$, $\bar{\mathfrak{r}}$ a place of $\overline{\mathbb{Q}}$ above r, and b an integer with $1 \leq b < N$ and $\gcd(b, N) = 1$. Let $\delta \in G_{\mathbb{Q}}$ and $a \in \mathbb{Z}$ with $\delta(\zeta) = \zeta^a$ for $\zeta \in \mu_{\ell^m}$, and let $t_\delta \in (\mathbb{Z}/N\mathbb{Z})^\times$ be the*

class with $\delta(\zeta) = \zeta^{\tilde{t}_\delta}$ *for* $\zeta \in \mu_N$. *Then there is an Euler system* $((\alpha_d), (e_{d,p}))$ *for* (N, r) *such that, for every* $t \in (\mathbb{Z}/N\mathbb{Z})^\times$ *and every* $\sigma \in G_{\mathbb{Q}}$ *with* $\sigma(\zeta) = \zeta^{\tilde{t}}$ *for* $\zeta \in \mu_N$,

$$v_{\bar{\mathfrak{r}}}(\sigma(\alpha_1)) = 2N_0\big(h_N(b, t\,t_\delta) - a\,h_N(b,t)\big).$$

Proof. We use two classical facts on cyclotomic fields: (C1) for $n \geq 1$ and a prime $p \nmid n$, p is unramified in K_n with residue degree the order of p in $(\mathbb{Z}/n\mathbb{Z})^\times$; (C2) every root of unity in K_n lies in μ_{2n}. Write $\mathrm{P} = \mathrm{P}_{\delta,a}$. It is an endomorphism of the group $\overline{\mathbb{Q}}^\times$, it is trivial on μ_{ℓ^m}, and $g(\mathrm{P}(z)) = \mathrm{P}(g(z))$ for $g \in G_{\mathbb{Q}}$ and $z \in \mathbb{Q}(\mu_\infty)$, since $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ is abelian.

Construction. Let d be a Kolyvagin integer. Its prime factors are $\equiv 1 \pmod N$, hence $> N > b$, so $\gcd(b, Nd) = 1$; also $r \nmid Nd$, as $r \equiv 1 \pmod N$ and Kolyvagin primes differ from r. Let $\mathfrak{r}_d = \{z \in \mathcal{O}_{K_{Nd}} : v_{\bar{\mathfrak{r}}}(z) > 0\}$, a prime of K_{Nd} above r, and put

$$\mathrm{R}_d = \mathrm{R}_{Nd,b}(\mathfrak{r}_d) \in K_{Nd}^\times, \qquad \alpha_d = \mathrm{P}(\mathrm{R}_d)^{2N_0 d} = \big(\delta(\mathrm{R}_d)\,\mathrm{R}_d^{-a}\big)^{2N_0 d} \in K_{Nd}^\times,$$

using item 1 of Lemma 7.2.20 ($Nd \geq N \geq 3$) and that $K_{Nd}/\mathbb{Q}$ is Galois. For a Kolyvagin prime $p \nmid d$ let $s_{d,p} = [k_{\mathfrak{r}_{pd}} : k_{\mathfrak{r}_d}]$ and $e_{d,p} = p\,s_{d,p}$.

Step 1: $\ell \nmid e_{d,p}$. $p \neq \ell$, as $\ell \mid N$ and $p \equiv 1 \pmod N$. By (C1), $s_{d,p} = f_{pd}/f_d$ with f_n the order of r modulo Nn. As $(\mathbb{Z}/Npd\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/Nd\mathbb{Z})^\times$, $f_{pd} = \mathrm{lcm}(\mathrm{ord}_p(r), f_d)$, so $s_{d,p}$ divides $\mathrm{ord}_p(r)$, which is prime to ℓ because p is a Kolyvagin prime.

Step 2: (Eu1). By item 2 of Lemma 7.2.20, R_d is a unit at every prime of K_{Nd} not above r. Let w be a place of $\overline{\mathbb{Q}}$ not above r; then $v_w(\mathrm{R}_d) = 0$, and $v_w(\delta(\mathrm{R}_d)) = v_{w'}(\mathrm{R}_d) = 0$ for the place $w' = w \circ \delta$, again not above r. So α_d is a w-unit.

Step 3: (Eu2). Let τ and φ be as in (Eu2). As $p \nmid Nd$, $\mathrm{Gal}(K_{Npd}/K_{Nd})$ is cyclic of order $p-1$ and is generated by the restriction of τ; so $\prod_{i=0}^{p-2} \tau^i(z) = \mathrm{N}_{K_{Npd}/K_{Nd}}(z)$ for $z \in K_{Npd}$. Lemma 7.2.21 (with $m := Nd$, $q := p$, $\mathfrak{p}' := \mathfrak{r}_{pd}$, $\mathfrak{p} = \mathfrak{r}_d$; $b < p$, $\gcd(b, Nd) = 1$, $\mathfrak{r}_{pd} \nmid pNd$) gives a root of unity $\eta \in K_{Nd}$ with $\mathrm{N}_{K_{Npd}/K_{Nd}}(\mathrm{R}_{pd}) = \eta\,(\mathrm{R}_d/\varphi^{-1}(\mathrm{R}_d))^{s_{d,p}}$. As P commutes with the norm and with φ^{-1} on K_{Npd},

$$\begin{aligned}\prod_{i=0}^{p-2} \tau^i(\alpha_{pd}) &= \mathrm{N}_{K_{Npd}/K_{Nd}}\big(\mathrm{P}(\mathrm{R}_{pd})\big)^{2N_0 pd} = \mathrm{P}\big(\eta^{2N_0 pd}\big)\Big(\frac{\mathrm{P}(\mathrm{R}_d)}{\varphi^{-1}(\mathrm{P}(\mathrm{R}_d))}\Big)^{2N_0 pd\,s_{d,p}} \\ &= \mathrm{P}\big(\eta^{2N_0 pd}\big)\Big(\frac{\alpha_d}{\varphi^{-1}(\alpha_d)}\Big)^{e_{d,p}}.\end{aligned}$$

By (C2), $\eta^{2Nd} = 1$, so $\eta^{2N_0 d} \in \mu_{\ell^m}$ and $\mathrm{P}(\eta^{2N_0 pd}) = \mathrm{P}(\eta^{2N_0 d})^p = 1$.

Step 4: (Eu3). Let φ be as in (Eu2) and w a place above p; put $\mathfrak{Q} = \{z \in \mathcal{O}_{K_{Npd}} : v_w(z) > 0\}$. By Lemma 7.2.22 (same data as in Step 3), R_{pd} and $\varphi^{-1}(\mathrm{R}_d)$ are w-units and $\mathrm{R}_{pd} \equiv \varphi^{-1}(\mathrm{R}_d)^{s_{d,p}} \pmod w$. The same holds at the place $w \circ \delta$, also above p; since $v_w(\delta(z) - \delta(z')) = v_{w\circ\delta}(z - z')$ and $\delta\varphi^{-1} = \varphi^{-1}\delta$ on K_{Nd}, this gives $\delta(\mathrm{R}_{pd}) \equiv \varphi^{-1}(\delta(\mathrm{R}_d))^{s_{d,p}} \pmod w$. Congruences between w-units are compatible with products and inverses, so $\mathrm{P}(\mathrm{R}_{pd}) \equiv \varphi^{-1}(\mathrm{P}(\mathrm{R}_d))^{s_{d,p}} \pmod w$, and raising to the power $2N_0 pd$ gives $\alpha_{pd} \equiv \varphi^{-1}(\alpha_d)^{e_{d,p}} \pmod w$.

Step 5: the valuations of α_1. By (C1), $r \equiv 1 \pmod N$ splits completely in K_N: the primes $\sigma_{N,t}^{-1}(\mathfrak{r}_1)$, $t \in (\mathbb{Z}/N\mathbb{Z})^\times$, are pairwise distinct, and $v_{\bar{\mathfrak{r}}}$ restricts to the $\mathfrak{r}_1$-adic valuation $\mathrm{ord}_{\mathfrak{r}_1}$ on K_N. By item 2 of Lemma 7.2.20, $\mathrm{ord}_{\sigma_{N,t}^{-1}(\mathfrak{r}_1)}(\mathrm{R}_1) = h_N(b,t)$. If σ acts on μ_N by t, then $\sigma|_{K_N} = \sigma_{N,t}$ and $v_{\bar{\mathfrak{r}}}(\sigma(\mathrm{R}_1)) = \mathrm{ord}_{\mathfrak{r}_1}(\sigma_{N,t}(\mathrm{R}_1)) = \mathrm{ord}_{\sigma_{N,t}^{-1}(\mathfrak{r}_1)}(\mathrm{R}_1) = h_N(b,t)$. As $\sigma\delta$ acts on μ_N by $t\,t_\delta$,

$$v_{\bar{\mathfrak{r}}}(\sigma(\alpha_1)) = 2N_0\big(v_{\bar{\mathfrak{r}}}(\sigma\delta(\mathrm{R}_1)) - a\,v_{\bar{\mathfrak{r}}}(\sigma(\mathrm{R}_1))\big) = 2N_0\big(h_N(b, t\,t_\delta) - a\,h_N(b,t)\big).$$

□

Lemma 7.2.20. *Let $m \geq 2$, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_{K_m}$ not dividing m, with residue characteristic p, and let b be an integer with $1 \leq b < m$ and $\gcd(b, m) = 1$. Then:*

1. *$\mathrm{J}_{m,b}(\mathfrak{p})$ and $\mathrm{R}_{m,b}(\mathfrak{p})$ lie in $K_m^\times$ and do not depend on the choice of ζ_p;*

$$\mathrm{R}_{m,b}(\mathfrak{p})\,\sigma_{m,-1}(\mathrm{R}_{m,b}(\mathfrak{p})) = 1,$$

   and $|\iota(\mathrm{R}_{m,b}(\mathfrak{p}))| = 1$ for every field embedding $\iota\colon K_m \to \mathbb{C}$;

2. *as fractional ideals of K_m,*

$$\mathrm{R}_{m,b}(\mathfrak{p})\,\mathcal{O}_{K_m} = \prod_{t \in (\mathbb{Z}/m\mathbb{Z})^\times} \sigma_{m,t}^{-1}(\mathfrak{p})^{h_m(b,t)};$$

   in particular $\mathrm{R}_{m,b}(\mathfrak{p})$ is a unit at every prime of K_m not above p.

Proof. Write $\mathrm{g}(j) = \mathrm{g}_\mathfrak{p}(j)$, $\chi = \chi_\mathfrak{p}$, $k = k_\mathfrak{p}$, $\mathrm{J} = \mathrm{J}_{m,b}(\mathfrak{p})$, $\mathrm{R} = \mathrm{R}_{m,b}(\mathfrak{p})$ and $L = K_m(\zeta_p) = \mathbb{Q}(\mu_{mp})$. As $p \nmid m$, restriction gives $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{Gal}(K_m/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. For $c \in \mathbb{F}_p^\times$ let $\rho_c \in \mathrm{Gal}(L/K_m)$ be the element with $\rho_c(\zeta_p) = \zeta_p^c$ (these are all the elements of $\mathrm{Gal}(L/K_m)$), and for $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ let $\tilde{\sigma}_t \in \mathrm{Gal}(L/\mathbb{Q}(\zeta_p))$ be the element restricting to $\sigma_{m,t}$. The Gauss sum $\mathrm{g}(j)$ depends only on j modulo m, as χ takes values in μ_m.

Step 1: Galois action on Gauss sums. For $c \in \mathbb{F}_p^\times \subseteq k^\times$ we have $c\,\mathrm{Tr}_{k/\mathbb{F}_p}(x) = \mathrm{Tr}_{k/\mathbb{F}_p}(cx)$, so substituting $x \mapsto c^{-1}x$,

$$\rho_c(\mathrm{g}(j)) = \sum_{x \in k^\times} \chi(x)^{-j} \zeta_p^{\mathrm{Tr}(cx)} = \chi(c)^j\,\mathrm{g}(j). \tag{7.8}$$

As $\tilde{\sigma}_t$ fixes ζ_p and raises elements of μ_m to the power $\tilde{t}$, $\tilde{\sigma}_t(\mathrm{g}(j)) = \mathrm{g}(j\tilde{t})$. By item 1 of Theorem A.2.13, $\mathrm{g}(j) \neq 0$ for $m \nmid j$; this applies to $j = \pm 1, \pm b$, since $1 \leq b < m$.

Step 2: item 1. By (7.8), $\rho_c(\mathrm{J}) = \chi(c)^b \mathrm{g}(1)^b/(\chi(c)^b \mathrm{g}(b)) = \mathrm{J}$ for all c, so $\mathrm{J} \in L^{\mathrm{Gal}(L/K_m)} = K_m$, and $\mathrm{J} \neq 0$. By Step 1, $\tilde{\sigma}_{-1}(\mathrm{J}) = \mathrm{g}(-1)^b/\mathrm{g}(-b)$, so $\mathrm{R} = \mathrm{J}/\tilde{\sigma}_{-1}(\mathrm{J}) = \mathrm{J}/\sigma_{m,-1}(\mathrm{J}) \in K_m^\times$. Replacing ζ_p by ζ_p^c replaces each $\mathrm{g}(j)$ by $\rho_c(\mathrm{g}(j))$ (ρ_c fixes the values of χ), hence J by $\rho_c(\mathrm{J}) = \mathrm{J}$ and R by R. Since $\sigma_{m,-1}^2 = 1$, $\mathrm{R}\,\sigma_{m,-1}(\mathrm{R}) = \frac{\mathrm{J}}{\sigma_{m,-1}(\mathrm{J})} \cdot \frac{\sigma_{m,-1}(\mathrm{J})}{\mathrm{J}} = 1$. For an embedding $\iota\colon K_m \to \mathbb{C}$, $\iota \circ \sigma_{m,-1}$ and $\bar{\iota}$ agree on μ_m (both send ζ to $\iota(\zeta)^{-1} = \overline{\iota(\zeta)}$), hence on $K_m = \mathbb{Q}(\mu_m)$; so $|\iota(\mathrm{R})|^2 = \iota(\mathrm{R})\,\overline{\iota(\mathrm{R})} = \iota(\mathrm{R}\,\sigma_{m,-1}(\mathrm{R})) = 1$.

Step 3: item 2. By item 2 of Theorem A.2.13, $\mathrm{g}(1)^m \in K_m$ generates $\prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\tilde{t}}$. As $\gcd(b, m) = 1$, $\mathrm{g}(b)^m = \tilde{\sigma}_b(\mathrm{g}(1)^m) = \sigma_{m,b}(\mathrm{g}(1)^m)$ generates

$$\prod_t \sigma_{m,b}\sigma_{m,t}^{-1}(\mathfrak{p})^{\tilde{t}} = \prod_{t'} \sigma_{m,t'}^{-1}(\mathfrak{p})^{\widetilde{bt'}} \qquad (t = bt').$$

Hence $\mathrm{J}^m = \mathrm{g}(1)^{bm}/\mathrm{g}(b)^m$ generates $\prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{b\tilde{t} - \widetilde{b\tilde{t}}}$, and $b\tilde{t} - \widetilde{b\tilde{t}} = m\lfloor b\tilde{t}/m \rfloor$, because $b\tilde{t} \equiv \widetilde{b\tilde{t}} \pmod m$ and $0 < \widetilde{b\tilde{t}} < m$. The group of fractional ideals of K_m is free abelian, so

$$\mathrm{J}\,\mathcal{O}_{K_m} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\lfloor b\tilde{t}/m \rfloor}, \qquad \sigma_{m,-1}(\mathrm{J})\,\mathcal{O}_{K_m} = \prod_t \sigma_{m,-t}^{-1}(\mathfrak{p})^{\lfloor b\tilde{t}/m \rfloor} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\lfloor b(m-\tilde{t})/m \rfloor},$$

and $\mathrm{R}\,\mathcal{O}_{K_m} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\lfloor \tilde{t}b/m \rfloor - \lfloor (m-\tilde{t})b/m \rfloor} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{h_m(b,t)}$. All the primes $\sigma_{m,t}^{-1}(\mathfrak{p})$ lie above p. □

Lemma 7.2.21. *Let $m \geq 2$, let q be a prime with $q \nmid m$, and let b be an integer with $1 \leq b < m$, $b < q$ and $\gcd(b, m) = 1$. Let $\mathfrak{p}'$ be a prime ideal of $\mathcal{O}_{K_{qm}}$ not dividing qm, $\mathfrak{p} = \mathfrak{p}' \cap \mathcal{O}_{K_m}$, and $s = [k_{\mathfrak{p}'} : k_{\mathfrak{p}}]$. Let $\varphi \in G_{\mathbb{Q}}$ satisfy $\varphi(\zeta) = \zeta^q$ for $\zeta \in \mu_m$. Then there is a root of unity $\eta \in K_m$ with*

$$\mathrm{N}_{K_{qm}/K_m}(\mathrm{R}_{qm,b}(\mathfrak{p}')) = \eta \Big(\frac{\mathrm{R}_{m,b}(\mathfrak{p})}{\varphi^{-1}(\mathrm{R}_{m,b}(\mathfrak{p}))}\Big)^s.$$

Proof. Put $n = qm$, $\mathrm{R}' = \mathrm{R}_{n,b}(\mathfrak{p}')$, $\mathrm{R} = \mathrm{R}_{m,b}(\mathfrak{p})$, and let $u \in (\mathbb{Z}/m\mathbb{Z})^\times$ be the class of q, so that $\varphi|_{K_m} = \sigma_{m,u}$. For $t' \in (\mathbb{Z}/n\mathbb{Z})^\times$ with image $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ (written $t' \mapsto t$), $\sigma_{n,t'}|_{K_m} = \sigma_{m,t}$. As $1 \leq b < q$, $\gcd(b, n) = 1$, so Lemma 7.2.20 applies at both levels.

Step 1: a distribution identity. We show, for every $t \in (\mathbb{Z}/m\mathbb{Z})^\times$,

$$\sum_{t' \mapsto t} h_n(b, t') = h_m(b, t) - h_m(b, u^{-1}t). \tag{7.9}$$

The integers in $[1, n-1]$ congruent to $\tilde{t}$ modulo m are $\tilde{t} + mj$, $0 \leq j \leq q-1$; they are prime to m, and since m is invertible modulo q exactly one of them, for $j = j_0$, is divisible by q. So the representatives $\tilde{t}'$ of the classes $t' \mapsto t$ are the $\tilde{t} + mj$ with $j \neq j_0$. Write $\tilde{t} + mj_0 = qt_1$; then $1 \leq t_1 \leq m-1$ and $qt_1 \equiv \tilde{t} \pmod m$, so $t_1 = \widetilde{u^{-1}}t$. Write $bj = qa_j + i_j$ with $0 \leq i_j < q$; as $\gcd(b, q) = 1$, $j \mapsto i_j$ permutes $\{0, \dots, q-1\}$. With $z = b\tilde{t}/m$ and Hermite's identity $\sum_{i=0}^{q-1} \lfloor (z+i)/q \rfloor = \lfloor z \rfloor$,

$$\sum_{j=0}^{q-1} \Big\lfloor \frac{b(\tilde{t} + mj)}{n} \Big\rfloor = \sum_{j=0}^{q-1} \Big(a_j + \Big\lfloor \frac{z + i_j}{q} \Big\rfloor \Big) = C_b + \Big\lfloor \frac{b\tilde{t}}{m} \Big\rfloor, \qquad C_b = \sum_{j=0}^{q-1} \Big\lfloor \frac{bj}{q} \Big\rfloor,$$

and the term $j = j_0$ is $\lfloor bqt_1/n \rfloor = \lfloor bt_1/m \rfloor$. Hence $A(t) := \sum_{t' \mapsto t} \lfloor b\tilde{t}'/n \rfloor = \lfloor b\tilde{t}/m \rfloor + C_b - \lfloor b\,\widetilde{u^{-1}}t/m \rfloor$. Since $n - \tilde{t}' = \widetilde{-t'}$ and $t' \mapsto t$ if and only if $-t' \mapsto -t$, $\sum_{t' \mapsto t} \lfloor b(n - \tilde{t}')/n \rfloor = A(-t) = \lfloor b(m - \tilde{t})/m \rfloor + C_b - \lfloor b(m - \widetilde{u^{-1}}t)/m \rfloor$. Subtracting gives (7.9).

Step 2: ideals. The ideal norm $\mathrm{N} = \mathrm{N}_{K_n/K_m}$ on fractional ideals is multiplicative, satisfies $\mathrm{N}(\mathfrak{p}') = \mathfrak{p}^s$, $\mathrm{N}(x\mathcal{O}_{K_n}) = \mathrm{N}_{K_n/K_m}(x)\mathcal{O}_{K_m}$, and $\mathrm{N}(\sigma(\mathfrak{a})) = \sigma|_{K_m}(\mathrm{N}(\mathfrak{a}))$ for $\sigma \in \mathrm{Gal}(K_n/\mathbb{Q})$ ($K_n/\mathbb{Q}$ is abelian). By item 2 of Lemma 7.2.20 at level n and (7.9),

$$\mathrm{N}_{K_n/K_m}(\mathrm{R}')\mathcal{O}_{K_m} = \prod_{t'} \sigma_{m,t}^{-1}(\mathfrak{p})^{s\,h_n(b,t')} = \prod_{t} \sigma_{m,t}^{-1}(\mathfrak{p})^{s\,(h_m(b,t) - h_m(b,u^{-1}t))}.$$

Substituting $t = ut_1$, $\prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{h_m(b,u^{-1}t)} = \sigma_{m,u}^{-1}\big(\prod_{t_1} \sigma_{m,t_1}^{-1}(\mathfrak{p})^{h_m(b,t_1)}\big) = \varphi^{-1}(\mathrm{R})\mathcal{O}_{K_m}$ by item 2 of Lemma 7.2.20 at level m. Hence $\mathrm{N}_{K_n/K_m}(\mathrm{R}')$ and $(\mathrm{R}/\varphi^{-1}(\mathrm{R}))^s$ generate the same fractional ideal.

Step 3: the unit is a root of unity. So $\beta = \mathrm{N}_{K_n/K_m}(\mathrm{R}')\,(\varphi^{-1}(\mathrm{R})/\mathrm{R})^s$ is a unit of $\mathcal{O}_{K_m}$. Let $\iota\colon K_m \to \mathbb{C}$ be an embedding and ι' an extension to K_n. Then $\iota(\mathrm{N}_{K_n/K_m}(\mathrm{R}')) = \prod_{\rho \in \mathrm{Gal}(K_n/K_m)} (\iota' \circ \rho)(\mathrm{R}')$ has absolute value 1, and so do $\iota(\mathrm{R})$ and $\iota(\varphi^{-1}(\mathrm{R})) = (\iota \circ \varphi^{-1}|_{K_m})(\mathrm{R})$, by item 1 of Lemma 7.2.20 applied to the embeddings $\iota' \circ \rho$, ι, $\iota \circ \varphi^{-1}|_{K_m}$. So every conjugate of the algebraic integer β has absolute value 1, and β is a root of unity by Kronecker's theorem [317]. Put $\eta = \beta^{-1}$. □

Lemma 7.2.22. *Let $m \geq 2$, let q be a prime with $q \nmid m$, and let b be an integer with $1 \leq b < m$, $b < q$ and $\gcd(b, m) = 1$. Let $\mathfrak{p}'$ be a prime ideal of $\mathcal{O}_{K_{qm}}$ not dividing qm, $\mathfrak{p} = \mathfrak{p}' \cap \mathcal{O}_{K_m}$, $s = [k_{\mathfrak{p}'} : k_{\mathfrak{p}}]$, and let $\varphi \in G_{\mathbb{Q}}$ satisfy $\varphi(\zeta) = \zeta^q$ for $\zeta \in \mu_m$. Then for every prime ideal $\mathfrak{Q}$ of $\mathcal{O}_{K_{qm}}$ above q, the elements $\mathrm{R}_{qm,b}(\mathfrak{p}')$ and $\varphi^{-1}(\mathrm{R}_{m,b}(\mathfrak{p}))$ are $\mathfrak{Q}$-adic units and*

$$\mathrm{R}_{qm,b}(\mathfrak{p}') \equiv \varphi^{-1}(\mathrm{R}_{m,b}(\mathfrak{p}))^s \pmod{\mathfrak{Q}}.$$

Proof. Put $n = qm$, $\mathrm{R}' = \mathrm{R}_{n,b}(\mathfrak{p}')$, $\mathrm{R} = \mathrm{R}_{m,b}(\mathfrak{p})$; let p be the residue characteristic of $\mathfrak{p}'$, so $p \neq q$; $k = k_{\mathfrak{p}} \subseteq k' = k_{\mathfrak{p}'}$ (via $\mathcal{O}_{K_m}/\mathfrak{p} \to \mathcal{O}_{K_n}/\mathfrak{p}'$), $\chi = \chi_{\mathfrak{p}}$, $\chi' = \chi_{\mathfrak{p}'}$, and $\mathrm{g}(j) = \mathrm{g}_{\mathfrak{p}}(j)$, $\mathrm{g}'(j) = \mathrm{g}_{\mathfrak{p}'}(j)$, formed with the same ζ_p. As $1 \leq b < q$, $\gcd(b, n) = 1$. Let $L = \mathbb{Q}(\mu_{np})$.

Step 1: the residue characters. For $c \in k'^{\times}$, $\chi'(c)^q = \chi(\mathrm{N}_{k'/k}(c))$. Indeed, let $x \in \mathcal{O}_{K_n}$ lift c and $y \in \mathcal{O}_{K_m}$ lift $\mathrm{N}_{k'/k}(c) = c^{(|k'|-1)/(|k|-1)}$. Modulo $\mathfrak{p}'$, $\chi'(c)^q \equiv x^{(|k'|-1)/m}$ and $\chi(\mathrm{N}_{k'/k}(c)) \equiv y^{(|k|-1)/m} \equiv x^{(|k'|-1)/m}$; both sides lie in μ_n, whose elements are pairwise incongruent modulo $\mathfrak{p}'$ as $p \nmid n$.

Step 2: q-th powers of Gauss sums. Let $j \in \{\pm 1, \pm b\}$; then $m \nmid j$ and χ^{-j} is a nontrivial character of $k^{\times}$. In a commutative ring $(a_1 + \cdots + a_r)^q \equiv a_1^q + \cdots + a_r^q$ modulo q. Hence, in $\mathcal{O}_L$ modulo q, using $q \operatorname{Tr}_{k'/\mathbb{F}_p}(c) = \operatorname{Tr}_{k'/\mathbb{F}_p}(qc)$, the substitution $c \mapsto q^{-1}c$ in $k'^{\times}$ ($q \in \mathbb{F}_p^{\times}$), Step 1, and Theorem A.2.14 for $\alpha = \chi^{-j}$:

$$\begin{aligned} \mathrm{g}'(j)^q &\equiv \sum_{c \in k'^{\times}} \chi'(c)^{-jq} \zeta_p^{\operatorname{Tr}(qc)} = \chi'(q)^{jq} \sum_{c \in k'^{\times}} \chi\big(\mathrm{N}_{k'/k}(c)\big)^{-j} \zeta_p^{\operatorname{Tr}(c)} \\ &= \chi'(q)^{jq}\, \mathrm{g}_{k'}(\chi^{-j} \circ \mathrm{N}_{k'/k}) = \chi'(q)^{jq} (-1)^{s-1} \mathrm{g}(j)^s. \end{aligned} \tag{7.10}$$

Step 3: $\mathrm{R}'^q \equiv \mathrm{R}^s$ modulo $\mathfrak{Q}$. By item 1 of Theorem A.2.13, $\mathrm{g}(j)\mathrm{g}(-j) = \pm|k|$ and $\mathrm{g}'(j)\mathrm{g}'(-j) = \pm|k'|$ are powers of p up to sign, so these Gauss sums are units at every prime of L above q. Let $\widetilde{\mathfrak{Q}}$ be a prime of $\mathcal{O}_L$ above $\mathfrak{Q}$. Dividing the congruences (7.10) according to (7.5) in the local ring at $\widetilde{\mathfrak{Q}}$, the factors $\chi'(q)^{\pm q}$, $\chi'(q)^{\pm bq}$ and the signs cancel:

$$\mathrm{R}'^q = \frac{\mathrm{g}'(1)^{qb}\, \mathrm{g}'(-b)^q}{\mathrm{g}'(b)^q\, \mathrm{g}'(-1)^{qb}} \equiv \frac{\chi'(q)^{qb}\chi'(q)^{-bq}}{\chi'(q)^{bq}\chi'(q)^{-qb}} \cdot \frac{(-1)^{(s-1)(b+1)}}{(-1)^{(s-1)(1+b)}} \cdot \mathrm{R}^s = \mathrm{R}^s.$$

Both sides lie in K_n, so $\mathrm{R}'^q \equiv \mathrm{R}^s$ modulo $\mathfrak{Q} = \widetilde{\mathfrak{Q}} \cap \mathcal{O}_{K_n}$.

Step 4: Frobenius. Since $q \nmid m$, q is unramified in K_m and the Frobenius of every prime $\mathfrak{q}$ of K_m above q is $\sigma_{m,u} = \varphi|_{K_m}$, u the class of q; so $\varphi(z) \equiv z^q$ modulo $\mathfrak{q}$ for z in the local ring at $\mathfrak{q}$. By item 2 of Lemma 7.2.20, R, $\varphi^{-1}(\mathrm{R})$ and R' are units at every prime not above p, in particular at $\mathfrak{q} = \mathfrak{Q} \cap \mathcal{O}_{K_m}$ and at $\mathfrak{Q}$. Applied to $z = \varphi^{-1}(\mathrm{R})$ this gives $\mathrm{R} \equiv \varphi^{-1}(\mathrm{R})^q$ modulo $\mathfrak{q}$, hence $\mathrm{R}'^q \equiv \big(\varphi^{-1}(\mathrm{R})^s\big)^q$ modulo $\mathfrak{Q}$.

Step 5: conclusion. In the residue field $\mathcal{O}_{K_n}/\mathfrak{Q}$, of characteristic q, the images of R' and $\varphi^{-1}(\mathrm{R})^s$ have the same q-th power; as $x \mapsto x^q$ is injective on a field of characteristic q, they are equal. □

Lemma 7.2.23. *Let ℓ be an odd prime, $N_0 \geq 1$ and $m \geq 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, r a prime with $r \equiv 1 \pmod{N}$, $((\alpha_d), (e_{d,p}))$ an Euler system for (N, r) and (τ_p) a choice of generators for (N, r). For $L \in \mathcal{L}(N, r)$ put $y_L = \mathrm{Der}_L(\alpha_{|L|})$. Then $y_L \in K_{N|L|}^{\times}$, y_L does not change when the primes of L are permuted, y_L is a w-unit for every place w of $\overline{\mathbb{Q}}$ not above r, and for every $g \in U_N$ there is $z \in K_{N|L|}^{\times}$ with*

$$z^{\ell^m} = g(y_L)/y_L.$$

In particular y_L is ℓ^m-invariant over K_N.

Proof. Write $M = \ell^m$ in this proof. For a Kolyvagin prime p put $\mathrm{N}_p(z) = \prod_{i=0}^{p-2} \tau_p^i(z)$.

Step 1: elementary properties. Let n be a Kolyvagin integer. Every element of $G_{\mathbb{Q}}$ maps K_{Nn} onto itself, and elements of $G_{\mathbb{Q}}$ commute on K_{Nn} ($K_{Nn}/\mathbb{Q}$ is abelian). Hence each Der_p

is an endomorphism of the group $K_{Nn}^\times$, $\mathrm{Der}_p \circ \mathrm{Der}_{p'} = \mathrm{Der}_{p'} \circ \mathrm{Der}_p$ on $K_{Nn}^\times$, and $g(\mathrm{Der}_p(z)) = \mathrm{Der}_p(g(z))$, $g(\mathrm{N}_p(z)) = \mathrm{N}_p(g(z))$ for $g \in G_\mathbb{Q}$ and $z \in K_{Nn}^\times$. So $y_L \in K_{N|L|}^\times$ and y_L depends only on the set of primes of L. If z is a w-unit for every place w not above r, so is $g(z)$, as $v_w(g(z)) = v_{w\circ g}(z)$; with (Eu1) this shows that y_L is such a unit.

Step 2: telescoping. Let $p \mid n$. Then τ_p^{p-1} fixes K_{Nn}: it fixes the roots of unity of order prime to p and acts on μ_p by $g_p^{p-1} = 1$. For $z \in K_{Nn}^\times$,

$$\frac{\tau_p(\mathrm{Der}_p(z))}{\mathrm{Der}_p(z)} = \frac{z^{p-1}}{\mathrm{N}_p(z)}. \tag{7.11}$$

Step 3: induction on the length of L. If $L = \emptyset$, $y_L = \alpha_1 \in K_N$ is fixed by U_N and $z = 1$ works. Let L have length $k \geq 1$ and assume the claim for sequences of length $k-1$. Put $n = |L|$ and let S be the set of $g \in U_N$ for which some $z \in K_{Nn}^\times$ has $z^M = g(y_L)/y_L$.

(a) $U_{Nn} \subseteq S$, with $z = 1$, since $y_L \in K_{Nn}$.

(b) S is closed under products: if z_g, z_h serve for g, h, then $g(z_h)z_g \in K_{Nn}^\times$ and $(g(z_h)z_g)^M = g(h(y_L)/y_L) \cdot g(y_L)/y_L = gh(y_L)/y_L$.

(c) $\tau_p \in S$ for every prime p of L. First $\tau_p \in U_N$, as $p \nmid N$. By Step 1 we may write $L = (p, L_0)$ with $L_0 \in \mathcal{L}(N, r)$ of length $k-1$; put $n_0 = |L_0|$, so $n = pn_0$, and $w_0 = \mathrm{Der}_{L_0}(\alpha_n)$, so $y_L = \mathrm{Der}_p(w_0)$. Choose $\varphi \in G_\mathbb{Q}$ with $\varphi(\zeta) = \zeta^p$ for $\zeta \in \mu_{Nn_0}$ (possible as $p \nmid Nn_0$); as $p \equiv 1 \pmod N$, $\varphi \in U_N$. Since τ_p fixes μ_{Nn_0} and acts on μ_p by a generator, (Eu2) for $d = n_0$ gives $\mathrm{N}_p(\alpha_n) = (\alpha_{n_0}/\varphi^{-1}(\alpha_{n_0}))^e$ with $e = e_{n_0,p}$. Applying Der_{L_0}, which commutes with N_p on $K_{Nn}^\times$ and with φ^{-1} on $K_{Nn_0}^\times$ (Step 1), $\mathrm{N}_p(w_0) = (y_{L_0}/\varphi^{-1}(y_{L_0}))^e$. By the induction hypothesis, applied to L_0 and $\varphi^{-1} \in U_N$, there is $z_0 \in K_{Nn_0}^\times$ with $z_0^M = \varphi^{-1}(y_{L_0})/y_{L_0}$. By (7.11), and as $M \mid N \mid p-1$,

$$\frac{\tau_p(y_L)}{y_L} = \frac{w_0^{p-1}}{\mathrm{N}_p(w_0)} = w_0^{p-1}\Big(\frac{\varphi^{-1}(y_{L_0})}{y_{L_0}}\Big)^e = (w_0^{(p-1)/M} z_0^e)^M,$$

with $w_0^{(p-1)/M} z_0^e \in K_{Nn}^\times$.

(d) $S = U_N$. Restriction maps U_N onto $\mathrm{Gal}(K_{Nn}/K_N)$ with kernel U_{Nn}, and $\mathrm{Gal}(K_{Nn}/K_N) \cong (\mathbb{Z}/n\mathbb{Z})^\times \cong \prod_{p|n}(\mathbb{Z}/p\mathbb{Z})^\times$ through the action on μ_n; the image of τ_p is a generator of the factor at p and trivial at the other factors. So every element of this finite group is a product of nonnegative powers of the images of the τ_p, every $g \in U_N$ is of the form $\tau_{p_1}^{c_1} \cdots \tau_{p_k}^{c_k} h$ with $c_i \geq 0$ and $h \in U_{Nn}$, and $g \in S$ by (a)–(c).

Finally, $K_{N|L|} \subseteq \mathbb{Q}(\mu_\infty)$, so y_L is ℓ^m-invariant over K_N. □

Lemma 7.2.24. *Let ℓ be an odd prime, $N_0 \geq 1$ and $m \geq 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, $\delta \in G_\mathbb{Q}$ and $a \in \mathbb{Z}$ with $\delta(\zeta) = \zeta^a$ for $\zeta \in \mu_{\ell^m}$, ζ_{ℓ^m} a primitive ℓ^m-th root of unity, and $y \in \mathbb{Q}(\mu_\infty)^\times$ ℓ^m-invariant over K_N. Write $\mathrm{P} = \mathrm{P}_{\delta,a}$.*

1. *For every $g \in U_N$, $Q_y(g)$ consists of one ℓ^m-th root of unity; the map $\mathrm{k}_y \colon U_N \to \mathbb{Z}/\ell^m\mathbb{Z}$ so defined is a continuous homomorphism.*

2. *If $g \in U_N$ fixes y, then $\zeta_{\ell^m}^{\mathrm{k}_y(g)} = g(\mathrm{P}(Y))/\mathrm{P}(Y)$ for every $Y \in \overline{\mathbb{Q}}$ with $Y^{\ell^m} = y$.*

3. *Let $p' \neq \ell$ be a prime such that y is a w-unit for every place w of $\overline{\mathbb{Q}}$ above p', and let $g \in U_N$ fix y and lie in an inertia group $I(D)$ at p'. Then $\mathrm{k}_y(g) = 0$.*

4. *Let r be a prime with $r \equiv 1 \pmod N$, D_0 a decomposition group at r, the stabiliser of the place $\bar{\mathfrak{r}}$, and $\rho \in \overline{\mathbb{Q}}$ with $\rho^{\ell^m} = r$. There is $t_0 \in I(D_0)$ with $t_0(\rho) = \zeta_{\ell^m}\rho$. If moreover $y \in K_N^\times$, then for every $\sigma \in G_\mathbb{Q}$, with $c \in \mathbb{Z}$ such that $\sigma(\zeta) = \zeta^c$ for $\zeta \in \mu_{\ell^m}$, one has $\sigma^{-1}t_0\sigma \in U_N$ and*

$$c \cdot \mathrm{k}_y(\sigma^{-1}t_0\sigma) = v_{\bar{\mathfrak{r}}}(\sigma(\mathrm{P}(y))) \quad \text{in } \mathbb{Z}/\ell^m\mathbb{Z}.$$

Proof. Write $M = \ell^m$ and $\zeta = \zeta_{\ell^m}$. The map P is an endomorphism of $\overline{\mathbb{Q}}^\times$, trivial on μ_M, and $g(\mathrm{P}(x)) = \mathrm{P}(g(x))$ for $g \in G_\mathbb{Q}$ and $x \in \mathbb{Q}(\mu_\infty)$, as $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ is abelian. Elements of U_N fix $\mu_M \subseteq \mu_N$.

Item 1. Let $g \in U_N$, $Y^M = y$, and $z \in \mathbb{Q}(\mu_\infty)$ with $z^M = g(y)/y$ (it exists by invariance). Then $\mathrm{P}(Y)^M = \mathrm{P}(y)$, $g(\mathrm{P}(Y))^M = g(\mathrm{P}(y)) = \mathrm{P}(g(y))$ and $\mathrm{P}(z)^M = \mathrm{P}(g(y))/\mathrm{P}(y)$, so the quotient $g(\mathrm{P}(Y))/(\mathrm{P}(Y)\mathrm{P}(z))$ lies in μ_M. Any other choices are ξY and $\xi' z$ with $\xi, \xi' \in \mu_M$, and $\mathrm{P}(\xi Y) = \mathrm{P}(Y)$, $\mathrm{P}(\xi' z) = \mathrm{P}(z)$; so $Q_y(g)$ is one element $\zeta^{\mathrm{k}_y(g)}$. For $g, h \in U_N$ with z_g, z_h as above, $z = g(z_h)z_g \in \mathbb{Q}(\mu_\infty)$ satisfies $z^M = gh(y)/y$ and $\mathrm{P}(z) = g(\mathrm{P}(z_h))\mathrm{P}(z_g)$, so

$$\frac{gh(\mathrm{P}(Y))}{\mathrm{P}(Y)\mathrm{P}(z)} = g\Big(\frac{h(\mathrm{P}(Y))}{\mathrm{P}(Y)\mathrm{P}(z_h)}\Big) \cdot \frac{g(\mathrm{P}(Y))}{\mathrm{P}(Y)\mathrm{P}(z_g)} = g\big(\zeta^{\mathrm{k}_y(h)}\big)\zeta^{\mathrm{k}_y(g)} = \zeta^{\mathrm{k}_y(h)+\mathrm{k}_y(g)},$$

and k_y is a homomorphism. The subgroup U' of the $h \in U_N$ fixing y and $\mathrm{P}(Y)$ is open; for $h \in U'$ one may take $z = 1$, so $\zeta^{\mathrm{k}_y(h)} = h(\mathrm{P}(Y))/\mathrm{P}(Y) = 1$. A homomorphism into a finite group that vanishes on an open subgroup is continuous.

Item 2. Take $z = 1$ in the definition.

Item 3. Let w be the place of which D is the stabiliser, and $V = \mathrm{P}(Y)$, so $V^M = \mathrm{P}(y) = \delta(y)y^{-a}$. As $v_w(\delta(y)) = v_{w\circ\delta}(y) = 0$, $\mathrm{P}(y)$ and hence V are w-units. Since $g \in I(D)$ acts trivially on the residue field of w, $g(V)/V \equiv 1 \pmod w$; it lies in μ_M, whose elements are pairwise incongruent modulo w as $p' \nmid M$. So $g(V)/V = 1$, and $\mathrm{k}_y(g) = 0$ by item 2.

Item 4: the element t_0. Let $E = K_N(\rho)$; it contains μ_M and all the roots $\xi\rho$ of $X^M - r$, so $E/\mathbb{Q}$ is Galois. As $r \equiv 1 \pmod N$, r splits completely in K_N, so the completion of K_N at $\bar{\mathfrak{r}}$ is $\mathbb{Q}_r$; there $X^M - r$ is an Eisenstein polynomial, so the completion of E is totally ramified of degree M over $\mathbb{Q}_r$. Hence $[E : K_N] = M$, $X^M - r$ is irreducible over K_N, and the inertia group of $E/\mathbb{Q}$ at $\bar{\mathfrak{r}}$, which lies in $\mathrm{Gal}(E/K_N)$ (K_N is unramified at r) and has order M, is all of $\mathrm{Gal}(E/K_N)$. It contains the automorphism $\rho \mapsto \zeta\rho$. Since $I(D_0)$ maps onto the inertia group of E at $\bar{\mathfrak{r}}$, some $t_0 \in I(D_0)$ satisfies $t_0(\rho) = \zeta\rho$.

Item 4: the value. As $r \nmid N$, inertia groups at r fix K_N, so $t_0 \in U_N$ and $h = \sigma^{-1}t_0\sigma \in U_N$; h fixes $y \in K_N$. Let $Y^M = y$, $V = \sigma(\mathrm{P}(Y))$ and $u = \sigma(\mathrm{P}(y)) = V^M \in K_N^\times$. By item 2, $\zeta^{\mathrm{k}_y(h)} = h(\mathrm{P}(Y))/\mathrm{P}(Y) = \sigma^{-1}(t_0(V)/V)$. Let $v = v_{\bar{\mathfrak{r}}}(u) \in \mathbb{Z}$ (K_N is unramified at r) and $V_0 = V\rho^{-v}$; then $V_0^M = ur^{-v}$ is a $\bar{\mathfrak{r}}$-unit, hence so is V_0, and as in item 3 ($r \nmid M$) $t_0(V_0) = V_0$. So $t_0(V)/V = (t_0(\rho)/\rho)^v = \zeta^v$, and $\zeta^{\mathrm{k}_y(h)} = \sigma^{-1}(\zeta^v) = \zeta^{vc'}$ with $cc' \equiv 1 \pmod M$. Hence $c\,\mathrm{k}_y(h) = v$ in $\mathbb{Z}/M\mathbb{Z}$. □

Lemma 7.2.25. *Let ℓ be an odd prime, $N_0 \geq 1$ and $m \geq 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, r a prime with $r \equiv 1 \pmod N$, $((\alpha_d), (e_{d,p}))$ an Euler system for (N, r), (τ_p) a choice of generators, and δ, a, ζ_{ℓ^m} as in the definition of k_y. For $L \in \mathcal{L}(N, r)$ put $y_L = \mathrm{Der}_L(\alpha_{|L|})$. Let $L \in \mathcal{L}(N, r)$, p a Kolyvagin prime for (N, r) not occurring in L, D a decomposition group at p, φ a Frobenius of D, and $t \in I(D)$ with $t(\zeta) = \tau_p(\zeta)$ for $\zeta \in \mu_p$. Then $\varphi, t \in U_N$ and*

$$\mathrm{k}_{y_{(p,L)}}(t) = -\,e_{|L|,p}\,\mathrm{k}_{y_L}(\varphi) \quad \text{in } \mathbb{Z}/\ell^m\mathbb{Z}.$$

Proof. Write $M = \ell^m$, $\zeta = \zeta_{\ell^m}$, $\mathrm{P} = \mathrm{P}_{\delta,a}$, $n = |L|$, $e = e_{n,p}$, $k' = (p-1)/M \in \mathbb{Z}$ ($M \mid N \mid p-1$), $\mathrm{N}_p(z) = \prod_{i=0}^{p-2} \tau_p^i(z)$, and let w be the place stabilised by D. Put $y = y_L \in K_{Nn}^\times$, $w_0 = \mathrm{Der}_L(\alpha_{pn}) \in K_{Npn}^\times$ and $y' = y_{(p,L)} = \mathrm{Der}_p(w_0)$. Elements of $G_\mathbb{Q}$ commute on the abelian field K_{Npn}, so Der_L, Der_p, N_p and P commute with them there.

Step 1: t and φ. For $\zeta' \in \mu_{Nn}$ we have $t(\zeta') \equiv \zeta'$ and $\varphi(\zeta') \equiv \zeta'^p$ modulo w; both sides lie in μ_{Nn}, whose elements are pairwise incongruent modulo w as $p \nmid Nn$; so t fixes μ_{Nn} and $\varphi(\zeta') = \zeta'^p$. As $p \equiv 1 \pmod N$, $t, \varphi \in U_N$. Since τ_p also fixes μ_{Nn} and agrees with t on μ_p, t and τ_p agree on K_{Npn}.

Step 2: a root of $t(y')/y'$. As τ_p^{p-1} fixes K_{Npn}, $\tau_p(\mathrm{Der}_p(w_0))/\mathrm{Der}_p(w_0) = w_0^{p-1}/\mathrm{N}_p(w_0)$.

By (Eu2) for $d = n$, with τ_p (which fixes μ_{Nn} and acts on μ_p by a generator) and φ, and as Der_L is multiplicative and commutes with N_p and φ^{-1}, $\mathrm{N}_p(w_0) = \mathrm{Der}_L(\mathrm{N}_p(\alpha_{pn})) = (y/\varphi^{-1}(y))^e$. By Lemma 7.2.23 applied to $\varphi^{-1} \in U_N$, there is $z' \in K_{Nn}^\times$ with $z'^M = \varphi^{-1}(y)/y$. Then $z = w_0^{k'} z'^e \in K_{Npn}^\times$ satisfies

$$z^M = w_0^{p-1}\Big(\frac{\varphi^{-1}(y)}{y}\Big)^e = \frac{\tau_p(y')}{y'} = \frac{t(y')}{y'}.$$

Step 3: an identity. Choose $Y, Y' \in \overline{\mathbb{Q}}$ with $Y^M = y'$ and $Y'^M = y$, and put $c = w_0/\varphi^{-1}(y)^e \in K_{Npn}^\times$. By the definition of k, $\zeta^{\mathrm{k}_{y'}(t)} = t(\mathrm{P}(Y))/(\mathrm{P}(Y)\mathrm{P}(z))$ and $\zeta^{\mathrm{k}_y(\varphi^{-1})} = \varphi^{-1}(\mathrm{P}(Y'))/(\mathrm{P}(Y')\mathrm{P}(z'))$. Hence

$$\zeta^{\mathrm{k}_{y'}(t) - e\,\mathrm{k}_y(\varphi^{-1})} = \frac{t(\mathrm{P}(Y))}{\mathrm{P}(Y)} \cdot \Big(\frac{\mathrm{P}(Y')}{\varphi^{-1}(\mathrm{P}(Y'))^p}\Big)^e \cdot \mathrm{P}(c)^{-k'}. \tag{7.12}$$

Step 4: each factor is $\equiv 1 \pmod w$. The elements y, y', w_0 are products of integer powers of Galois conjugates of α_n, α_{pn}, hence w-units by (Eu1) ($p \neq r$); so are $\mathrm{P}(y)$, $\mathrm{P}(y')$ ($v_w \circ \delta = v_{w\circ\delta}$), hence $\mathrm{P}(Y)$, $\mathrm{P}(Y')$, and $x = \varphi^{-1}(\mathrm{P}(Y'))$ (φ stabilises w). As $t \in I(D)$ acts trivially on the residue field of w, $t(\mathrm{P}(Y)) \equiv \mathrm{P}(Y)$. As φ is a Frobenius of D, $\mathrm{P}(Y') = \varphi(x) \equiv x^p$. Finally, by (Eu3) for $d = n$, $\alpha_{pn}/\varphi^{-1}(\alpha_n)^e \equiv 1$ modulo every place above p; $c = \mathrm{Der}_L(\alpha_{pn}/\varphi^{-1}(\alpha_n)^e)$ is a product of integer powers of its conjugates $g(\cdot)$, $g \in G_\mathbb{Q}$, and $v_{w'}(g(x') - 1) = v_{w'\circ g}(x' - 1)$; so $c \equiv 1$ modulo every place above p, hence $\delta(c) \equiv 1$, $c^{-a} \equiv 1$ and $\mathrm{P}(c) \equiv 1 \pmod w$.

Step 5: conclusion. By (7.12) and Step 4, the M-th root of unity $\zeta^{\mathrm{k}_{y'}(t) - e\mathrm{k}_y(\varphi^{-1})}$ is $\equiv 1 \pmod w$; as $p \nmid M$ it equals 1. So $\mathrm{k}_{y'}(t) = e\,\mathrm{k}_y(\varphi^{-1}) = -e\,\mathrm{k}_y(\varphi)$, k_y being a homomorphism. □

Lemma 7.2.26. *Let $N, e, n \geq 1$, let ℓ be a prime, and let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$. Let $\psi : (\mathbb{Z}/N)^\times \to \mathcal{O}^\times$ be a Dirichlet character, extended to all integers by zero on nonunits, and assume $\psi(-1) = -(-1)^e$. Let $a \geq 1$ satisfy $a \equiv 1 \pmod N$ and $\ell \nmid a$. For $j \geq 0$ put*

$$T_j = \sum_{x=0}^{N\ell^j - 1} \Big\lfloor \frac{ax}{N\ell^j} \Big\rfloor x^e \psi(x).$$

Put $L = N\ell^n$ and

$$V_n = \sum_{\substack{1 \leq u < L \\ (u,L)=1}} \Big(\Big\lfloor \frac{au}{L} \Big\rfloor - \Big\lfloor \frac{a(L-u)}{L} \Big\rfloor \Big) u^e \psi(u).$$

Suppose $A, w \in \mathcal{O}$ satisfy

$$A \equiv 2wT_n \pmod{\ell^n\mathcal{O}}, \qquad A \equiv 2wT_{n-1} \pmod{\ell^{n-1}\mathcal{O}}.$$

Then

$$(1 - \ell^e\psi(\ell))A \equiv wV_n \pmod{\ell^n\mathcal{O}}.$$

The factor $1-\ell^e\psi(\ell)$ is a unit in $\mathcal{O}$. In particular $A \equiv qV_n \pmod{\ell^n\mathcal{O}}$ for some $q \in \mathcal{O}$, and $V_n \equiv 0 \pmod{\ell^n\mathcal{O}}$ implies $A \equiv 0 \pmod{\ell^n\mathcal{O}}$. No assumption $\ell \nmid N$, $\ell > 2$, or unramifiedness of $\mathcal{O}$ is required.

Proof. Write

$$U_n = \sum_{\substack{1 \le u < N\ell^n \\ (u, N\ell^n)=1}} \left\lfloor \frac{au}{N\ell^n} \right\rfloor u^e\psi(u), \qquad H = \ell^e\psi(\ell).$$

Partition the defining sum for T_n according as ℓ divides x. A term with $\ell \nmid x$ and $(x, N) > 1$ is zero, so the terms in the first part sum to U_n. If $\ell \nmid N$, write $x = \ell y$ in the second part, with $0 \le y < N\ell^{n-1}$. Its floor is $\lfloor ay/(N\ell^{n-1}) \rfloor$ and $x^e\psi(x) = Hy^e\psi(y)$, including the cases where the character is zero. If $\ell \mid N$, every term in the second part is zero and $H = 0$. Thus in all cases

$$T_n = U_n + HT_{n-1}.$$

The second assumed congruence may be multiplied by H modulo ℓ^n: indeed $H \in \ell\mathcal{O}$ because $e \ge 1$. Subtracting it from the first gives

$$(1-H)A \equiv 2wU_n \pmod{\ell^n\mathcal{O}}. \tag{1}$$

The map $u \mapsto L - u$ permutes the unit representatives $1 \le u < L$. In the second floor sum defining V_n, make this substitution. As $N \mid L$, one has $\psi(L-u) = \psi(-1)\psi(u)$ exactly, and

$$(L-u)^e \equiv (-u)^e \pmod{\ell^n\mathcal{O}}.$$

Their product is congruent to $-u^e\psi(u)$ by the parity hypothesis. Multiplying by the integer floor and summing therefore shows that this second sum is congruent to $-U_n$. Hence $V_n \equiv 2U_n \pmod{\ell^n\mathcal{O}}$. Together with (1) this proves the asserted congruence, without cancelling 2.

Finally H lies in the maximal ideal of $\mathcal{O}$, since ℓ does. Therefore $1-H$ is a unit. Multiplication by its inverse gives the last two conclusions, with $q = (1-H)^{-1}w$. This uses the actual ideal $\ell^n\mathcal{O}$ throughout and so is unchanged when $\mathcal{O}$ is ramified. □

Proposition 7.2.27. *Let $N, e \ge 1$, let $E \subseteq \mathbb{Q}(\mu_N)$ be a CM field, and let y be an integer with $y \ge N + e + 3$. Suppose $X \ge 1$ is an integer with the following two properties: for each prime p, some integer $c \ge 2$ satisfies $c \equiv 1 \pmod N$, $(c, p) = 1$ and $(c^e - 1)(c^{e+1} - 1) \mid X$; and for every $c \ge 0$ with $c \equiv 1 \pmod N$, every characteristic-zero commutative domain R and every R-valued Dirichlet character ψ modulo N with $\psi(-1) = -(-1)^e$, some $w \in R$ satisfies*

$$m \mid (c^{e+1} - 1)X - 2w \sum_{a=0}^{Nm-1} \left\lfloor \frac{ca}{Nm} \right\rfloor a^e\psi(a) \quad \text{in } R$$

for every $m \ge 1$ prime to c. Assume also $X \le 2^{y^6}$. For every prime ℓ, every coefficient ring $\mathcal{O}$ over $\mathbb{Z}_\ell$ containing roots of unity of order the exponent of $(\mathbb{Z}/N)^\times$, and every $\nu \ge 0$, there is a finite set $\mathcal{S} \subseteq G_E$ with $\#\mathcal{S} \le y^{20}$ such that

$$z \in \mathrm{Sha}^-(e, \nu), \qquad z(g) = 0 \ \ (g \in \mathcal{S}) \quad \Longrightarrow \quad z = 0.$$

Proof. Fix $\ell, \mathcal{O}, \nu$. If $\ell \le y^2$, apply Lemma 7.2.28. Since $N \le y$, its fixed set has at most y^{20} elements and detects every unramified-outside-ℓ Tate-twist cocycle, hence also the submodule $\mathrm{Sha}^{-}(e,\nu)$.

Suppose now that $\ell > y^2$. In particular $\ell > N + e + 3$. Choose the representatives and the elements $b(u,g)$ of Lemma 7.2.11. For every character ψ with $\psi(-1) = -(-1)^e$, choose the $r+1$ detecting elements $\gamma_{\psi,i}$ supplied by (OC2) of Proposition 7.2.12, where $r = v_\ell(X)$. Let

$$\mathcal{S} = \{b(u, \gamma_{\psi,i}) : u \in (\mathbb{Z}/N)^\times,\ \psi(-1) = -(-1)^e,\ 0 \le i \le r\}.$$

There are at most $\varphi(N)$ characters, and $\varphi(N) \le N \le y$. Also

$$2^r \le \ell^r \le X \le 2^{y^6},$$

so $r \le y^6$. As $y \ge N + e + 3 \ge 5$,

$$\#\mathcal{S} \le \varphi(N)^2(r+1) \le y^2(y^6+1) \le 2y^8 \le y^{20}.$$

If $x \in \mathrm{Sha}^{-}(e,\nu)$ vanishes on $\mathcal{S}$, the projection formula of Lemma 7.2.11 makes every odd Z_ψ vanish at every $\gamma_{\psi,i}$. Its local conditions are precisely those in Proposition 7.2.12, so (OC2) makes every odd Z_ψ zero. The comparison conclusion gives $4\varphi(N)x = 0$. Finally

$$0 < 4\varphi(N) \le 4N < (N+e+3)^2 \le y^2 < \ell.$$

Thus $4\varphi(N)$ is a unit in $\mathcal{O}$, and $x = 0$. This proves detection in the large-prime case as well. □

Lemma 7.2.28. *Let $N \ge 1$ be an integer, let $E \subseteq \mathbb{Q}(\mu_N)$ be a number field, and let ℓ be a prime. There is a finite set $D \subseteq G_E$ such that*

$$\#D \le y^{20} \qquad (N \le y,\ \ell \le y^2,\ y \in \mathbb{Z}_{\ge 0}).$$

The same set D has the following property for every ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$ and every $e, \nu \ge 0$: if a locally constant map $z : G_E \to \mathcal{O}/\ell^\nu$ satisfies

$$z(gh) = z(g) + \chi_{\mathrm{cyc}}(g)^{-e} z(h),$$

vanishes on every inertia group above a finite place of E not dividing ℓ, and vanishes on D, then $z = 0$.

Moreover, for every $a \in G_E$ such that $\chi_{\mathrm{cyc}}(a)$ topologically generates $\chi_{\mathrm{cyc}}(G_E)$, there is a finite set $D_a \subseteq \ker \chi_{\mathrm{cyc}}$, chosen independently of $\mathcal{O}, e, \nu$, with the same bound $\#D_a \le y^{20}$, such that any such cocycle vanishing at a and on D_a is zero.

Proof. Put $M = N\ell^2$, $K = \mathbb{Q}(\mu_M)$ and $d = [K : \mathbb{Q}] = \varphi(M)$. Thus $M \ge 4$, K is totally complex and $E \subseteq K$. We first bound the number of generators needed to detect homomorphisms on G_K.

Let A be the maximal abelian profinite quotient of G_K which has exponent dividing M and in which all inertia groups away from ℓ vanish. For every finite quotient B of A, global reciprocity for its finite abelian extension gives a surjection from the idele class group of K onto B, with local units mapping onto inertia; see [202, Chapter V, Proposition 5.2 and Theorem 5.3]. Let U be the image in B of the product of the local unit groups at the places above ℓ. The quotient B/U is an everywhere unramified quotient. Its reciprocity map factors through the ideal class group: the ideal map on finite ideles has kernel the product of all local unit groups,

and principal ideles map to principal ideals. The infinite local groups are connected, so have trivial image in B. Consequently

$$|B| \le h_K \prod_{v|\ell} [\mathcal{O}_{K_v}^\times : (\mathcal{O}_{K_v}^\times)^M].$$

The local unit structure theorem gives, for a finite extension $F/\mathbb{Q}_\ell$, a topological decomposition $\mathcal{O}_F^\times \simeq \mu(F) \times \mathbb{Z}_\ell^{[F:\mathbb{Q}_\ell]}$, where $\mu(F)$ is finite cyclic. In particular

$$[\mathcal{O}_F^\times : (\mathcal{O}_F^\times)^M] = \gcd(M, |\mu(F)|)\ell^{[F:\mathbb{Q}_\ell]v_\ell(M)} \le M^{[F:\mathbb{Q}_\ell]+1}.$$

Here the decomposition includes the case $\ell = 2$; no assertion that the entire principal-unit group is torsion free is needed. Summing the local degrees and bounding the number of places by d shows that the product is at most M^{2d}. These local unit and reciprocity facts are the standard local and global class field theorems of [224].

We next bound h_K explicitly. The cyclotomic discriminant formula gives

$$|\operatorname{disc}(K)| = \frac{M^{\varphi(M)}}{\prod_{p|M} p^{\varphi(M)/(p-1)}} \le M^d.$$

Write $d = 2r$. In Minkowski's ideal-class bound the constant is at most $2^{-r}(4/\pi)^r = (2/\pi)^r \le 1$: the first r factors of $(2r)!/(2r)^{2r}$ are at most $1/2$, and the rest are at most 1. Thus every ideal class has an integral ideal of norm at most $X = |\operatorname{disc}(K)|^{1/2}$. At a rational prime, the Euler factor counting ideals is a product of at most d factors $(1 - T^f)^{-1}$ with $f \ge 1$. Its coefficients are bounded by those of $(1-T)^{-d}$. Hence the number of ideals of norm n is at most the number of ordered d-tuples of positive integers with product n. The number with product at most X is at most $\lfloor X \rfloor^d \le X^d$, since every entry is at most X. It follows that

$$h_K \le |\operatorname{disc}(K)|^{d/2} \le M^{d^2/2}.$$

The discriminant formula and Minkowski bound used here are classical; see [224]. Combining the estimates, and using $d \le M$ and $\log_2 M \le M$, gives

$$|B| \le M^{d^2/2+2d} \le 2^{M^3/2+2M^2} \le 2^P, \qquad P = M^3 + 5M^2 + M.$$

This bound holds for every finite quotient of A, so A is finite. Indeed, an infinite profinite group has finite quotients of arbitrarily large order: otherwise a finite quotient of maximal order would have kernel contained in every open normal subgroup and hence trivial kernel. A finite group of order at most 2^P has a generating set of at most P elements, by adjoining an element outside the subgroup already generated and observing that the order at least doubles at every strict enlargement. Choose lifts $b_1, \dots, b_q \in G_K$ of such generators, with $q \le P$. Every continuous homomorphism from G_K to a finite abelian group killed by M, unramified outside ℓ, vanishes if it vanishes at the b_i.

Choose coset representatives C for G_K in G_E and put $D = C \cup \{b_1, \dots, b_q\}$, viewing G_K as a subgroup of G_E. These choices depend only on E, N, ℓ. Since $[K : E] \le \varphi(M) \le M$,

$$\#D \le M^3 + 5M^2 + 2M \le 8M^3.$$

If $N \le y$ and $\ell \le y^2$, then $y \ge 2$, $M \le y^5$, and $8M^3 \le 8y^{15} \le y^{20}$.

We prove detection by induction on an integer k such that $\ell^k z = 0$, allowing all the coefficient rings and twists in the statement throughout the induction. The case $k = 0$ is immediate. If

$\ell^{k+1}z = 0$, then $\ell^k Mz = 0$, since $\ell \mid M$. The cocycle Mz has the same unramifiedness and vanishes on D, so the induction hypothesis gives $Mz = 0$.

For $x \in G_K$, the cyclotomic character satisfies $\chi_{\mathrm{cyc}}(x) \equiv 1 \pmod{\ell^{v_\ell(M)}}$. Consequently $\chi_{\mathrm{cyc}}(x)^{-e} - 1 \in M\mathbb{Z}_\ell$, because the factor of M prime to ℓ is a unit in $\mathbb{Z}_\ell$. As every value of z is killed by M, the cocycle identity therefore becomes

$$z(xg) = z(x) + z(g) \qquad (x \in G_K,\ g \in G_E).$$

In particular $z|_{G_K}$ is an M-torsion continuous homomorphism. It is unramified outside ℓ: inertia in G_K at such a place is contained in inertia at the place below it in E, which also does not divide ℓ. It vanishes on all b_i, so is zero by their construction. Writing $g = xc$ with $x \in G_K$ and $c \in C$ now gives $z(g) = z(x) + z(c) = 0$. This completes the induction and proves the first assertion.

Finally fix a as in the second assertion and let H be the closure of the cyclic subgroup generated by a. Compactness and continuity imply $\chi_{\mathrm{cyc}}(H) = \chi_{\mathrm{cyc}}(G_E)$. For each $p \in D$, choose $q_p \in H$ with $\chi_{\mathrm{cyc}}(q_p) = \chi_{\mathrm{cyc}}(p)$ and put $h_p = pq_p^{-1} \in \ker \chi_{\mathrm{cyc}}$. Take $D_a = \{h_p : p \in D\}$, so its cardinality does not increase. The zero set of a continuous cocycle is a closed subgroup: the cocycle identity proves closure under products, and its value at gg^{-1} proves closure under inverses. If $z(a) = 0$, this subgroup contains H. If also $z(h_p) = 0$ for all p, then $z(p) = z(h_p q_p) = 0$. The first assertion yields $z = 0$. □

Lemma 7.2.29. *Let E be a number field, ℓ a prime and $\mathcal{O}$ a coefficient ring for ℓ. Let*

$$\mathrm{Hom}_{\mathrm{ur}}(G_E, \mathcal{O}[\ell^{-1}]/\mathcal{O})$$

be the $\mathcal{O}$-module of continuous homomorphisms $G_E \to \mathcal{O}[\ell^{-1}]/\mathcal{O}$ (discrete topology) that vanish on the inertia group I_w of every finite place w of E. If ℓ is odd or E has no real place, this module has finite length and its normalised length is $v_\ell(\#\mathrm{Cl}(E))$.

Proof. Write $K = \mathcal{O}[\ell^{-1}]$, $U = \mathrm{Hom}_{\mathrm{ur}}(G_E, K/\mathcal{O})$, and let H be the Hilbert class field of E, the maximal abelian extension of E unramified at all places, finite and infinite; by class field theory the Artin map induces $\mathrm{Gal}(H/E) \cong \mathrm{Cl}(E)$ [224] (Ch. VI, the Hilbert class field).

Every $\varphi \in U$ factors through $\mathrm{Gal}(H/E)$. The kernel of φ is open (continuity into a discrete group) and contains the commutator subgroup (the target is abelian), so its fixed field L is a finite abelian extension of E and φ induces an injection $\mathrm{Gal}(L/E) \hookrightarrow K/\mathcal{O}$. For a finite place w the inertia group of L/E at w is the image of I_w, on which φ vanishes, so L/E is unramified at w. For a real place, the decomposition group of L/E is generated by the image σ of a complex conjugation and has order at most 2, so $2\varphi(\sigma) = 0$; if ℓ is odd, multiplication by 2 is injective on $K/\mathcal{O}$ and $\varphi(\sigma) = 0$, and if E has no real place there is nothing to check. Hence L/E is unramified at all places, $L \subseteq H$, and φ factors through $\mathrm{Gal}(H/E)$. Conversely, every homomorphism $\mathrm{Gal}(H/E) \to K/\mathcal{O}$ composed with the restriction $G_E \to \mathrm{Gal}(H/E)$ is continuous and vanishes on every I_w, because H/E is unramified at w. So restriction induces an isomorphism of $\mathcal{O}$-modules $U \cong \mathrm{Hom}(\mathrm{Gal}(H/E), K/\mathcal{O}) \cong \mathrm{Hom}(\mathrm{Cl}(E), K/\mathcal{O})$.

The length. Write $\mathrm{Cl}(E) \cong \bigoplus_i \mathbb{Z}/n_i\mathbb{Z}$, a finite sum, and $n_i = \ell^{a_i}u_i$ with $\ell \nmid u_i$. Then $\mathrm{Hom}(\mathbb{Z}/n_i\mathbb{Z}, K/\mathcal{O})$ is the n_i-torsion of $K/\mathcal{O}$; as u_i is a unit of $\mathcal{O}$ this is the ℓ^{a_i}-torsion $\ell^{-a_i}\mathcal{O}/\mathcal{O} \cong \mathcal{O}/\ell^{a_i}\mathcal{O}$, of length $a_i e_{\mathcal{O}}$ (the quotients $\ell^j\mathcal{O}/\ell^{j+1}\mathcal{O} \cong \mathcal{O}/\ell\mathcal{O}$ have length $e_{\mathcal{O}}$). Hence U has length $e_{\mathcal{O}} \sum_i a_i = e_{\mathcal{O}}\, v_\ell(\#\mathrm{Cl}(E))$, and its normalised length is $v_\ell(\#\mathrm{Cl}(E))$. □

From now on, for all admissible data D, the odd-class integer X_{odd} of (CX3) of Definition 6.6.15 is the integer X of Theorem 7.2.8 for the modulus $N = q_1 d$, the given t and the field

E, chosen once for each D. The theorem applies: $E = K(\zeta_{q_1})$ with $K = \mathbb{Q}(\sqrt{-d}) \subseteq \mathbb{Q}(\zeta_d)$, as $d \equiv 3 \pmod 4$, so E is a CM field contained in $\mathbb{Q}(\mu_{q_1 d})$. Its bound $1 \le X \le 2^{4(q_1 d+t+3)^{27}}\,\#\mathrm{Cl}(E)$ is the property that (CX3) assumes, and in addition

$$\operatorname{len}_{\mathcal{O}} \mathrm{Sha}^{-}(e,\nu) \le v_{\mathcal{O}}(X_{\mathrm{odd}})$$

for every prime ℓ, every coefficient ring $\mathcal{O}$ over $\mathbb{Z}_\ell$ containing the roots of unity of order the exponent of $(\mathbb{Z}/q_1 d)^\times$, every $0 \le e < t$ and every $\nu \ge 0$. The denominator errors e_ℓ and the clearing bounds β_ℓ are taken with this value of X_{odd}.

Definition 7.2.30. Fix a prime ℓ and a prime identification $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. An **$\ell$-adic representation** of rank n is a continuous homomorphism $R\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Let $c \in G_{\mathbb{Q}}$ be a complex conjugation, $g^c = cgc^{-1}$, and let χ_{cyc} be the ℓ-adic cyclotomic character. A **lattice in** R over a coefficient ring $\mathcal{O}$ is a continuous $u\colon G_E \to \mathrm{GL}_n(\mathcal{O})$ conjugate to R over $\overline{\mathbb{Q}}_\ell$. Let $t \ge 0$ be an integer and J a function from the embeddings $E \to \mathbb{C}$ to finite multisets of integers.

(MS1) Conjugate self-duality. R is **conjugate self-dual of twist** t if the trace form on the $\overline{\mathbb{Q}}_\ell$-span of $R(G_E)$ is nondegenerate and $\operatorname{tr} R(g^c) = \chi_{\mathrm{cyc}}(g)^{1-\max(t,1)} \operatorname{tr} R(g^{-1})$ for all $g \in G_E$.

(MS2) Level $\emptyset$. It is moreover **of level $\emptyset$ with jumps** J if at every finite place v of E: R is of finite type at v, has Hodge–Tate jumps J at v (read through ι), and is good at v (unramified, and crystalline if $v \mid \ell$) when $v \nmid p_0 p_1$.

(MS3) Main summand like. R of rank n is **main-summand like with string length t and jumps** J if (MS1) and (MS2) hold, J is a main-summand weight system of rank n and string length t, and:

1. at every place A of $\overline{\mathbb{Q}}$ over a place $v \mid p_0 p_1$ of E with $v \nmid \ell$ there are γ in the inertia group of A, equal to the tame generator of A if v is chosen, and an integer N_v prime to v, such that R is trivial on the N_v-th powers of inertia at A and
$$v_\ell \det\bigl(R(\gamma) - 1\bigr) \le n\, v_\ell(N_{\mathrm{ch}}), \qquad v_\ell \det\bigl(R((\gamma^c)^{-1}) - 1\bigr) \le n\, v_\ell(N_{\mathrm{ch}});$$
2. for all large $\mathcal{O}$ and every lattice u in R over $\mathcal{O}$, the $\mathcal{O}$-span of $u(G_E)$ is $M_n(\mathcal{O})$, and some $g \in G_E$ with $\chi_{\mathrm{cyc}}(g) \in \{p_0^n, p_1^n\}$ has $u(g)$ scalar;
3. some γ in an inertia group above a place $v \mid p_0 p_1$, $v \nmid \ell$, has $R(\gamma) = y \cdot 1$ with $\|y^{\ell^a j} - 1\| = 1$ for all $a \ge 0$ and $1 \le j \le 4n+2$.

(MS4) Main summand. Let O be a descent pair with rank difference t_O and let $z = (z_{w,i})$ be a system of Hecke eigenvalues. R is a **main summand for** (O, z) if R is main-summand like with string length t_O for some jumps, and: for every tame character ϑ and every descent pair O' with the same source rank and the same target rank as O and with $\mu_{O'} = \mu_O \vartheta$ for the characters of (DP2) of Definition 3.2.23, every Hecke eigenform y for O' whose lift Hecke polynomials are those of z, and every representation S attached to y, the normalised representation S' of S for O' (MS6) satisfies

$$\operatorname{tr} S'(g) = \psi_\vartheta(g) \operatorname{tr} R(g) + \sum_{e=0}^{t_{O'}-1} \chi_{\mathrm{cyc}}(g)^{-e} \qquad (g \in G_E).$$

(MS5) Lift settings. A descent pair O of source rank n with labels $c = (c_i)$ at its chosen places is a **lift setting** for a twist ϑ if its places are the level places of ϑ, its tame part is $\omega^j\psi$ for some $1 \le j \le m$ (with ω the fixed residue characters of the chosen places and ψ the tame part of ϑ), n and t_O are odd with $n \le r_D$ and $t_O \le t_D$, every c_i is a label of degree n whose scaled type polynomial does not vanish at 1, and over each of p_0, p_1 some c_i is primitive with $n = r_D$ or equals $X - 1$ with $n = 1$.

(MS6) Normalising character. Let O be a descent pair whose space $W(O)$ has rank r_O (the source rank) and whose character is χ_O, a character of $\mathrm{U}(1)(\mathbb{A})$. The **normalising character** $\nu_O \colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ is the ℓ-adic character attached through ι to the algebraic Hecke character

$$x \longmapsto \chi_O(x/\bar{x})^{-1}\, \upsilon(x)^{r_O}\, |x|_{\mathbb{A}_E}^{r_O/2} \qquad (x \in \mathbb{A}_E^\times),$$

so that $\nu_O(\varphi_w) = \iota\big(\chi_O(\varpi_w/\bar{\varpi}_w)^{-1}\upsilon(\varpi_w)^{r_O} q_w^{-r_O/2}\big)$ for a geometric Frobenius φ_w at every finite place $w \nmid \ell$ where this character is unramified, ϖ_w the idele with a uniformiser at w and 1 elsewhere. The **normalised representation** of an ℓ-adic representation S for O is $S' = \nu_O \otimes S$.

Definition 7.2.31. Let $\iota \mid \ell$ be a prime identification, O a descent pair with space of forms $\mathcal{A}(V)$ and source rank $n = \operatorname{rk} W$, e an idempotent of the group algebra of $\prod_{w \in S_M} \mathrm{GL}_n(k_w)$ acting on $\mathcal{A}(V)$, $Z \subseteq \mathcal{A}(V)$ a subspace with orthogonal projection pr_Z, and $\beta \in \mathbb{R}$.

(DR1) Denominator of a projection. The projection to Z has **denominator at most** β **at** ι **for** e if there is $B \in \mathrm{End}_{\mathbb{C}}(Z)$ with

$$\|\iota(\det B)\| \ge \exp\big(-(\dim Z)\, n\, \beta\big)$$

such that for every $y \in \mathcal{A}(V)$ that is integral at ι and orthogonal to the false lifts for $(e, Z^\perp)$, the form $B(\mathrm{pr}_Z\, y)$ is integral at ι and $\|\iota\langle y', B(\mathrm{pr}_Z\, y)\rangle\| \le 1$ for every $y' \in \mathcal{A}(V)$ integral above ℓ.

(DR2) Lift generators. Let ϑ be a twist and $\mathrm{sel} \in [0, \infty]$. A form $y \in Z$ is a **lift generator of** Z **for** $(\vartheta, e, \mathrm{sel})$ if (i) y lies in the range of the type projection of a tuple of labels c such that (O, c) is a lift setting for ϑ and that range lies in $e\,\mathcal{A}(V)$; (ii) there are a system of Hecke eigenvalues z whose lift Hecke polynomials are those of y, a descent pair O' whose character times ϑ is the character of O, and a main summand R for (O', z) such that $\psi_\vartheta R$ has Selmer length at most sel at (Q, γ, ζ) for some $\gamma \in G_E$ and $\zeta \in \overline{\mathbb{Z}}_\ell$, where Q is the modulus of ϑ.

(DR3) Negligible numbers. For an integer $C \ge 0$, a family (e_ℓ) of reals indexed by the primes is **negligible with exponent** C if $e_\ell \ge 0$ for all ℓ and $\sum_{\ell \in T} e_\ell \log \ell \le x^C$ for every finite set T of primes.

(DR4) Class number at the chosen level. The **chosen level** $\mathrm{K}_{\mathrm{ch}} \subseteq \mathrm{U}(1)(\mathbb{A}_f)$ is the group of u with $u_w \in \mathcal{O}_{E_w}^\times$ for every finite place w and $u_w \equiv 1$ modulo w for every chosen place w. The **class number at the chosen level** is $h_{\mathrm{ch}} = \#\big(\mathrm{U}(1)(E)\backslash \mathrm{U}(1)(\mathbb{A}_f)/\mathrm{K}_{\mathrm{ch}}\big)$.

(DR5) Weak lifts. A system x of Hecke eigenvalues of forms for O, with attached ℓ-adic representation r_x, is a **weak lift** if $\det\big(1 - r'_x(g)\big) = 0$ for every $g \in G_E$, where $r'_x = \nu_O \otimes r_x$ is the normalised representation of r_x for O.

7.3 Existence of regular families

This section proves Theorem 7.3.1. The family is chosen one prime identification at a time: for $\iota \mid \ell$, Proposition 7.3.2 gives an m-fold twist with modulus prime to ℓ whose ratios of L-values and denominators are bounded at ι by the twist error e'_ℓ and by $\gamma_\ell = (\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log \ell$, with β_ℓ the clearing bound of Definition 6.6.15. The twist is assembled from the next three sections: the main summands (Section 7.5), the twisting theorem applied to them (Section 7.4), and the passage from Selmer bounds to denominators (Lemma 7.6.2, Section 7.6). In our reading, the freedom to take a different twist at each prime identification is what makes this possible: only the sum of the bounds over the primes is controlled.

The family is regular because these bounds add up (Proposition 7.3.3). Apart from the local defect bounds, every term of β_ℓ and of e'_ℓ is a multiple of the ℓ-adic valuation of the chosen group order N_{ch} or of the odd-class integer X_{odd}, or the summable term $2^{-\ell}$, and for $1 \le a \le n$ the sum $\sum_\ell v_\ell(a)\log \ell$ is at most $\log n$ (Lemma 7.3.4). It therefore suffices to bound these integers by the sizes of the datum (Lemma 7.3.5, Lemma 7.3.6); the local defect bounds are summed directly.

Theorem 7.3.1. *For all admissible data D there is a regular family $\theta = (\theta_\iota)$ of m-fold twists, one for each prime identification ι.*

Proof. **Choice of the family.** For each prime identification $\iota \mid \ell$, Proposition 7.3.2 gives an m-fold twist θ_ι with $\ell \nmid \mathrm{N}Q(\theta_\iota)$, with

$$\left\|\iota(r_p(\theta_\iota^j, s)^{-1})\right\| \le \ell^{e'_\ell}$$

for every $1 \le j \le m$, every ramified critical pair p and every $s \in \{\pm 1\}$, and such that for every $1 \le j \le m$ the twist θ_ι^j has denominators at most $\gamma_\ell = (\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log \ell$ at ι for j. Choose one such twist for each ι. Then (SR1) holds.

Small ratios (SR2). Fix j, p and s, and let $y_\iota = r_p(\theta_\iota^j, s)^{-1}$. Take $H = 1$, $\alpha = 0$ and $h_\ell = e'_\ell \ge 0$ in the definition of height. (HG1) holds as $|\sigma(1)| = 1$; (HG2) is the displayed bound of the choice; for (HG3), every finite set S of primes has

$$\sum_{\ell \in S} e'_\ell \log \ell \le x^{999} \le x_1$$

by Proposition 7.3.3(c) and $x^{10^6} \le x_1$ (Proposition 2.3.5, with $x \ge 1$). So the family (y_ι) has height at most x_1.

Denominators (RW). For each $1 \le j \le m$ the family (γ_ℓ) is a denominator bound for (θ, j): (DB2) holds by the choice of θ_ι, and (DB1) is Proposition 7.3.3(b), which gives $\sum_{\ell \in T} \gamma_\ell \le \beta_{\mathrm{den}}$ for every finite set T of primes. Hence θ is regular. □

Proposition 7.3.2. *Let D be admissible data, ℓ a prime and $\iota \mid \ell$ a prime identification. There is an m-fold twist τ such that:*

1. *$\ell \nmid \mathrm{N}Q(\tau)$;*
2. *for every $1 \le j \le m$, every ramified critical pair p and every $s \in \{\pm 1\}$, $\left\|\iota(r_p(\tau^j, s)^{-1})\right\| \le \ell^{e'_\ell}$;*
3. *for every $1 \le j \le m$, τ^j has denominators at most $(\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log \ell$ at ι for j.*

Proof. **Step 1: finitely many main summands.** The systems z of Hecke eigenvalues occurring in the good forms on W_m are finitely many, as these forms span a finite-dimensional space. For each $1 \le j \le m$ and each such z, Proposition 7.5.1(1) gives a main summand $R_{j,z}$ of rank r for $((W_j, V^{\mathrm{def}})$ at $1, z)$. The main pairs are ramified (by (RH2), as $\omega_w^j \neq 1$ exactly at the chosen places) and finitely many; for each main pair p for j, Proposition 7.5.1(2) gives a rank-one main summand $R_{p,j}$ for $(O_{p,1}, 1)$. All are main-summand like.

Step 2: the twist. Theorem 7.4.3, applied to the finite family of all $R_{j,z}$ and $R_{p,j}$, gives an m-fold twist τ such that every $\xi = \tau^j$ $(1 \le j \le m)$ satisfies its items 1–3.

Item 1. The modulus Q of τ is that of τ^1, auxiliary for S_ℓ: each $w \in Q$ has prime norm and does not lie above ℓ, so $\mathrm{N}w \neq \ell$ and $\ell \nmid \mathrm{N}Q$.

Item 2. By item 3 of Theorem 7.4.3, $v_\ell(\iota(r_p(\tau^j, s))) \le e'_\ell < \infty$; so $r_p(\tau^j, s) \neq 0$ and $\|\iota(r_p(\tau^j, s)^{-1})\| = \ell^{\,v_\ell(\iota(r_p(\tau^j,s)))} \le \ell^{\,e'_\ell}$.

Item 3. Fix j and put $\beta' = (\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log\ell \ge \beta_\ell \log\ell$. By items 1–2 of Theorem 7.4.3, $(S_\ell, Q, \psi_{\tau^j})$ is an auxiliary twist with ramification at most e'_ℓ and each $\psi_{\tau^j} R_{j,z}$, $\psi_{\tau^j} R_{p,j}$ has Selmer length at most e'_ℓ at (Q, γ, ζ) for some γ, ζ. Then Lemma 7.6.2(2) gives (DL1) for τ^j at ι for j with β', and Lemma 7.6.2(3), for every main pair p for j, gives (DL2) with $e^{\beta'}$. □

Proposition 7.3.3. *For admissible data D and every finite set T of primes:*

(a) $\displaystyle\sum_{\ell\in T} \mathrm{loc}_\ell \log\ell \le 4\cdot 10^6 e^{180}\, f\,\frac{m}{k}\, b\log b;$

(b) $\displaystyle\sum_{\ell\in T} (\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log\ell \le \beta_{\mathrm{den}} = \frac{f\,a\,(\log b)^3}{m\,k};$

(c) the families (e'_ℓ), $(v_\ell(X_{\mathrm{odd}}))$, (e_ℓ) and $(v_\ell(h_{\mathrm{ch}}))$ are negligible with exponents 999, 2000, 4000 *and* 30 *respectively.*

Proof. All sums below run over $\ell \in T$; every summand is non-negative. Sizes are from Lemma 4.2.67: $f \le x^{12}$, $q_1 d \le x^{13}$, $r \le 3x^5$, $t \le 15x^5$, $h \le 3^8 x^{40}$, $m^2 \le 4x$, $\log b \le x^{1/100}$, $x^{10^{12}} \le b$, $\log b \ge e^{6.4\cdot 10^{13}}$. Weighted valuation sums are bounded by Lemma 7.3.4.

Step 1: item (c). For the twist error, by Lemma 6.7.10,

$$\sum e'_\ell \log\ell = (4f+3)r\sum v_\ell(N_{\mathrm{ch}})\log\ell + \sum 2^{-\ell}\log\ell \le 7x^{12}\cdot 3x^5\cdot x^{33} + 2 \le x^{999}.$$

For the odd-class integer, $1 \le X_{\mathrm{odd}} \le 2^{4(q_1 d+t+3)^{27}} h_E$ by Theorem 7.2.8, so with Lemma 7.3.5

$$\sum v_\ell(X_{\mathrm{odd}})\log\ell \le \log X_{\mathrm{odd}} \le 4(2x^{13})^{27} + x^{29} \le x^{2000}.$$

For the denominator error, put $c = 8ft + 2 + (4t+1)h^{22} + h \le x^{900}$; then

$$\sum e_\ell \log\ell = t\sum v_\ell(X_{\mathrm{odd}})\log\ell + c\sum v_\ell(N_{\mathrm{ch}})\log\ell \le 15x^{2005} + x^{933} \le x^{4000}.$$

Finally $\sum v_\ell(h_{\mathrm{ch}})\log\ell \le \log h_{\mathrm{ch}} \le x^{30}$ by Lemma 7.3.6.

Step 2: item (a). Let $Y = (2b)!\,\Pi^{h^3}$ with $\Pi = \prod_{\max(b-30h^3,0)<a\le b+30h^3} a \le (b+30h^3)^{60h^3}$, so $\lambda_\ell = v_\ell(Y)$. Then

$$\sum \mathrm{loc}_\ell \log\ell = 4f\cdot 10^6 \frac{m}{k}\Big(b\sum v_\ell(M)\log\ell + \sum v_\ell(Y)\log\ell\Big) \le 4f\cdot 10^6\frac{m}{k}(b\log M + \log Y).$$

Here $\log M \le 22\log b$, $\log(2b)! \le 2b\log(2b) \le 4b\log b$, and $\log \Pi^{h^3} \le 60h^6\log(b+30h^3) \le 120h^6\log b \le b\log b$, as $120h^6 \le 120\cdot 3^{48}x^{240} \le b$ and $b+30h^3 \le b^2$. So $b\log M + \log Y \le 27\,b\log b \le e^{180}b\log b$.

Step 3: item (b). By (CX5), $\beta_\ell + v_\ell(h_{\mathrm{ch}}) = \mathrm{loc}_\ell + (1+2t)e'_\ell + e_\ell + v_\ell(N_{\mathrm{ch}}) + v_\ell(h_{\mathrm{ch}})$. By Steps 1–2 (with the sharper constant 27) and Lemma 6.7.10,

$$\sum(\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log\ell \le 1.08\cdot 10^8 f\frac{m}{k}b\log b + (1+30x^5)x^{999} + x^{4000} + x^{33} + x^{30}.$$

The first term is at most $\beta_{\mathrm{den}}/2 = fxb(\log b)^3/(2mk)$, since this amounts to $2.16\cdot 10^8 m^2 \le x(\log b)^2$, and $m^2 \le 4x$. The remaining terms are at most $2x^{4000}$, and

$$\frac{\beta_{\mathrm{den}}}{2} \ge \frac{xb}{2mk} \ge \frac{xb}{2\cdot 2\sqrt{x}\cdot 3x^5} \ge \frac{x^{10^{12}}}{12x^{4.5}} \ge 2x^{4000},$$

using $f \ge 1$, $\log b \ge 1$, $k \le r$. Adding gives (b). □

Lemma 7.3.4. *Let T be a finite set of primes. For all integers $1 \le a \le n$,*

$$\sum_{\ell\in T} v_\ell(a)\log\ell \le \log n, \qquad \text{and} \qquad \sum_{\ell\in T} 2^{-\ell}\log\ell \le 2.$$

Proof. By unique factorisation, $\sum_{\ell\in T} v_\ell(a)\log\ell \le \sum_{\ell \text{ prime}} v_\ell(a)\log\ell = \log a \le \log n$, all terms being non-negative. For the second sum, $\log\ell \le \ell$, so it is at most $\sum_{n\ge1} n2^{-n} = \frac{1/2}{(1-1/2)^2} = 2$. □

Lemma 7.3.5. *For admissible data D, $\log h_E \le 26x^{24}\log x \le x^{29}$.*

Proof. **Discriminant.** As $d \equiv 3 \pmod 4$, $K \subseteq \mathbb{Q}(\zeta_d)$, so $E \subseteq L = \mathbb{Q}(\zeta_{q_1 d})$. The root discriminant does not decrease in towers ($|d_L| = |d_E|^{[L:E]}\,\mathrm{N}(\mathfrak{d}_{L/E})$), so $\mathrm{rd}_E \le \mathrm{rd}_L \le q_1 d$ (for $L = \mathbb{Q}(\zeta_n)$, $|d_L|^{1/[L:\mathbb{Q}]} = n/\prod_{p\mid n} p^{1/(p-1)} \le n$). With $[E:\mathbb{Q}] = 2f$ this gives $\log|d_E| \le 2f\log(q_1 d)$.

Class number. Every ideal class contains an integral ideal of norm at most the Minkowski bound $B_E \le |d_E|^{1/2}$. The number of integral ideals of norm j is at most the coefficient $\tau_{2f}(j)$ of $\zeta(s)^{2f}$ (compare the Euler products of ζ_E and $\zeta^{[E:\mathbb{Q}]}$), and $\tau_{2f}(j) \le \tau(j)^{2f-1} \le j^{2f-1}$. So $h_E \le \sum_{j\le B_E} j^{2f-1} \le B_E^{2f}$ and

$$\log h_E \le 2f\cdot\tfrac{1}{2}\log|d_E| \le 2f^2\log(q_1 d) \le 2x^{24}\cdot 13\log x$$

by Lemma 4.2.67, item 4. Finally $26\log x \le x^5$ as $x \ge e^{6.4\cdot 10^{15}}$. □

Lemma 7.3.6. *For admissible data D, $1 \le h_{\mathrm{ch}}$ and $\log h_{\mathrm{ch}} \le x^{30}$.*

Proof. **Maximal level.** Let $K_{\max} = \prod_w \mathrm{U}(1)(\mathcal{O}_w)$, where $\mathrm{U}(1)(\mathcal{O}_w)$ is the group of u_w with $u_w\bar{u}_w = 1$ and u_w a unit at the places of E above the place v of E^+ under w. At a place v of E^+ inert or ramified in E, $\mathrm{U}(1)(E_v^+)$ is compact, so it equals $\mathrm{U}(1)(\mathcal{O}_v)$. At a split $v = w\bar{w}$, $u \mapsto u_w$ identifies $\mathrm{U}(1)(E_v^+)$ with $E_w^\times$, and $\mathrm{U}(1)(E_v^+)/\mathrm{U}(1)(\mathcal{O}_v) \cong \mathbb{Z}$ by the valuation at w. Thus $u \mapsto \prod_{v \text{ split}} w^{\mathrm{ord}_w(u_w)}$ identifies $\mathrm{U}(1)(\mathbb{A}_f)/K_{\max}$ with the group I^1 of fractional ideals $\mathfrak{a}$ of E with $\mathfrak{a}\bar{\mathfrak{a}} = \mathcal{O}_E$. Each $\mathfrak{a} \in I^1$ is $\mathfrak{b}/\bar{\mathfrak{b}}$ for an ideal $\mathfrak{b}$ supported on one place above each split v, and $z \in \mathrm{U}(1)(E)$ maps to (z), where by Hilbert 90 $z = y/\bar{y}$ with $y \in E^\times$. Hence $[\mathfrak{b}] \mapsto \mathfrak{b}/\bar{\mathfrak{b}}$ is a well-defined surjection from the class group of E onto $\mathrm{U}(1)(E)\backslash\mathrm{U}(1)(\mathbb{A}_f)/K_{\max}$, and the latter has at most h_E elements.

Chosen level. Each double coset for $K_{\max}$ is the union of at most $[K_{\max} : K_{\mathrm{ch}}]$ double cosets for K_{ch}. At a chosen place w (split over E^+, as $\bar{w}$ contains $\bar{\pi}_M$, which is prime to w), the condition $u_w \equiv 1$ modulo w has index $\mathrm{N}w - 1$ in $\mathcal{O}_w^\times$. By Lemma 4.2.66, item 3, $[K_{\max} : K_{\mathrm{ch}}] \le \prod_{w \text{ chosen}} \mathrm{N}w = M^f$. So

$$\log h_{\mathrm{ch}} \le \log h_E + f \log M \le x^{29} + 22x^{12} \log b \le x^{30}$$

by Lemma 7.3.5 and Lemma 4.2.67, items 2, 4 and 5. □

7.4 The twisting theorem

The twisting theorem (Theorem 7.4.3) chooses the twist at one prime. For finitely many main-summand-like representations it gives an m-fold twist τ, anticyclotomic of ℓ-power order and ramified only at auxiliary places, such that every power τ^j, $1 \le j \le m$, at once has ramification at most e'_ℓ, twists each representation to Selmer length at most e'_ℓ, and keeps the ℓ-adic valuations of the ratios of L-values at most e'_ℓ.

The theorem is reduced in two steps to a general form for Selmer data and Stickelberger systems (Definition 7.4.1). Main-summand-like representations give Selmer data whose local conditions above ℓ are the crystalline cocycles (Proposition 7.4.4; see [28, 85]). The ratios of L-values are governed by Stickelberger systems (Proposition 7.4.9), built from a class sum of theta values at the critical integer one (Lemma 7.4.10) through the exchange identity and the expression of Hecke L-values as squared norms of lifts. In our reading they are a finite, integral analogue of Stickelberger elements and of the ℓ-adic interpolation of Hecke L-values [290, 317, 159]; for the valuations of L-values in families of twists see [78, 283].

The general form (Theorem 7.4.11) is proved in three steps. Ties at auxiliary places, found by class field theory and Chebotarev's theorem [54], cut the Selmer modules down until their length is bounded by the global and local terms (Proposition 7.4.13), through the Greenberg–Wiles formula and Poitou–Tate duality [328, 63, 225, 295, 201]. The twisted Selmer lengths and the Stickelberger values then become valuations of polynomials in the variables $\zeta_{\mathfrak{q}} - 1$ (Proposition 7.4.12; with Mazur's finiteness condition [196]). Finally the roots of unity are chosen so that every polynomial loses little at all m powers at once (Lemma 7.4.22). Auxiliary primes that control Selmer groups go back to Kolyvagin [169, 244] and to Taylor and Wiles [301]; in our reading the dependence on the variables $\zeta - 1$ is of the kind studied in Iwasawa theory. For bounds of Selmer groups of automorphic Galois representations of unitary type see also [306, 226].

Definition 7.4.1. Fix a prime ℓ, a finite set S_0 of finite places of E containing the places above ℓ, and $g_0 \in G_E$. For every finite place v of E fix a place A_v of $\bar{\mathbb{Q}}$ above v, for $v \mid \ell$ the place A_v fixed in Definition 7.2.2, and let $D_v \supseteq I_v$ be its decomposition and inertia groups in G_E. Let $\mathcal{O}$ be a coefficient ring with uniformiser ϖ and normalised valuation v_ϖ ($v_\varpi(0) = \infty$), and for $0 \ne a \in \mathcal{O}$ and a continuous $\rho\colon G_E \to \mathrm{GL}_n(\mathcal{O})$ let $W_a = (\mathcal{O}/a)^n$ with the action ρ. Then $\mathrm{Hom}_{G_E}(W_a, W_a^{*c})$ is the $\mathcal{O}$-module of $\mathcal{O}$-linear maps $\eta\colon W_a \to \mathrm{Hom}_{\mathbb{Z}}(W_a, \mathbb{Q}/\mathbb{Z})$ with

$$\eta(\rho(g)u)(m) = \eta(u)\big(\chi_{\mathrm{cyc}}(g^c)\,\rho(g^c)^{-1}m\big) \qquad (g \in G_E,\ u, m \in W_a).$$

For an $\mathcal{O}$-module M and a finite set Σ of finite places, a cocycle $c\colon G_E \to M^n$ (for ρ or a twist of ρ) is **unramified outside** Σ if it vanishes on the inertia groups of all places of $\bar{\mathbb{Q}}$ above the finite places not in Σ; cocycles are not taken modulo coboundaries.

(SD) Selmer datum. A **Selmer datum** for (S_0, g_0) over $\mathcal{O}$ of rank $n \ge 1$ is

$$T = (\rho, S, (Z_w)_{w\mid\ell}, \gamma_*, \zeta_*) :$$

a continuous $\rho\colon G_E \to \mathrm{GL}_n(\mathcal{O})$ with $\mathcal{O}$-span of $\rho(G_E)$ equal to $M_n(\mathcal{O})$; a finite set S with $\{w \mid \ell\} \subseteq S \subseteq S_0$ outside which ρ is unramified; for each $w \mid \ell$ a finitely generated $\mathcal{O}$-module Z_w of cocycles $D_w \to \mathcal{O}^n$ for $\rho|_{D_w}$, containing the coboundaries $g \mapsto \rho(g)b - b$ ($b \in \mathcal{O}^n$), with $\sum_{w|\ell} \operatorname{rk} Z_w \le (\#\{w \mid \ell\} + f_E)\, n$ (f_E the number of infinite places); an element γ_* of an inertia group above a place of S prime to ℓ and $\zeta_* \in \mathcal{O}$ with $\rho(\gamma_*) = \rho((\gamma_*^c)^{-1}) = \zeta_* \cdot 1$ and $\zeta_*^2 - 1 \in \mathcal{O}^\times$.

(SF) Terms. The **global term** of T is $\sup_{a \neq 0} \operatorname{len}_{\mathcal{O}} \operatorname{Hom}_{G_E}(W_a, W_a^{*c})$; the **local term** at $v \in S$, $v \nmid \ell$, is

$$\sup_{a \neq 0} \big(\operatorname{len}_{\mathcal{O}} C_v(a) \mathbin{\dot{-}} \operatorname{len}_{\mathcal{O}} W_a\big),$$

where $C_v(a)$ is the $\mathcal{O}$-module of the restrictions to D_v of all continuous cocycles $G_E \to W_a$ for ρ (no local condition) and $x \mathbin{\dot{-}} y = \max(x - y, 0)$ on $[0, \infty]$ ($\infty \mathbin{\dot{-}} y = \infty$); the **first-prime term** is $v_\varpi \det(\rho(g_0) - 1) + v_\varpi \det(\rho((g_0^c)^{-1}) - 1)$, both summands counted also when $g_0 = (g_0^c)^{-1}$. The datum requires all three finite, and β_T is their sum (local terms summed over $v \in S$, $v \nmid \ell$).

(SW) Twisted Selmer groups. A **relation** of Z_w ($w \mid \ell$) is a finitely supported family $(A_g)_{g \in D_w}$ of matrices in $M_n(\mathcal{O})$ with $\sum_g A_g z(g) = 0$ for every $z \in Z_w$. For an $\mathcal{O}$-algebra $\mathcal{O}'$ and an $\mathcal{O}'$-module M, a function $c\colon G_E \to M^n$ **obeys the local conditions of** T if $\sum_g A_g c(g) = 0$ in M^n for every $w \mid \ell$ and every relation (A_g) of Z_w (the matrices acting through $\mathcal{O} \to \mathcal{O}'$). For a finite set Q and a character $\psi\colon G_E \to \mathcal{O}'^\times$ with $\mathcal{O} \subseteq \mathcal{O}'$ coefficient rings, $\mathrm{Sel}_Q(T\psi)$ is the $\mathcal{O}'$-module of continuous cocycles $c\colon G_E \to (\operatorname{Frac}\mathcal{O}'/\mathcal{O}')^n$ for $g \mapsto \psi(g)\rho(g)$ that are unramified outside $S \cup Q \cup \bar{Q}$, obey the local conditions of T, and satisfy $c(\gamma_*) = 0$. The twist has **length at most** B if for all large $\mathcal{O}' \supseteq \mathcal{O}$ the character ψ takes values in $\mathcal{O}'$ and $\operatorname{nl} \mathrm{Sel}_Q(T\psi) \le B$.

(SK) Stickelberger system. For a finite set Q, Cl_Q is the class group $\mathrm{U}(1)(E)\backslash\mathrm{U}(1)(\mathbb{A}_f)/\mathrm{K}_Q$, with K_Q the integral units congruent to 1 at the places of Q; a character ψ from U(1) with modulus Q factors through Cl_Q and extends linearly to $\overline{\mathbb{Z}}_\ell[\mathrm{Cl}_Q]$. A **Stickelberger system** for (S_0, g_0) is $Y = (\vartheta, (e_1, \dots, e_u), (\Theta_Q)_Q)$: a continuous character $\vartheta\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ unramified outside S_0 with $\vartheta(g_0) \neq 1$, integers $e_1, \dots, e_u$, and $\Theta_Q \in \overline{\mathbb{Z}}_\ell[\mathrm{Cl}_Q]$ for every finite Q, with $1(\Theta_\emptyset) \neq 0$ and, whenever $Q \cup \{P\}$ is auxiliary for S_0, $P \notin Q$, ψ comes from U(1) with modulus Q and φ is a Frobenius at P,

$$\psi(\Theta_{Q\cup\{P\}}) = \psi(\Theta_Q) \prod_{i=1}^{u} \big(1 - \alpha\, \mathrm{N}P^{-e_i}\big)\big(1 - \alpha^{-1}\mathrm{N}P^{-e_i}\big), \qquad \alpha = \vartheta(\varphi)\psi(\varphi).$$

Its number is $\beta_Y = v_\ell(1(\Theta_\emptyset)) + 2u\, v_\ell(\vartheta(g_0) - 1)$.

(SM) Small twists. For Selmer data T_i over $\mathcal{O}$, Stickelberger systems Y_k, $\varepsilon > 0$, a finite set Q and $\psi\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$: ψ is an **ε-small twist** at Q if ψ comes from U(1) with modulus Q, has ℓ-power order, is ramified at every place of Q, the twist of each T_i by ψ has length at most $\operatorname{nl}(\beta_{T_i}) + \varepsilon$, $v_\ell(\psi(\Theta_Q^{(k)})) \le \beta_{Y_k} + \varepsilon$ for each k, and $\sum_{w \in Q \cup \bar{Q}} v_\ell(\psi(\gamma_w) - 1) \le \varepsilon$ for all tame generators γ_w.

(SQ) Torsion level. An integer M_0 is a **torsion level** for the data T_i if $\operatorname{nl}(\beta_{T_i}) + 1 < M_0$ and $\#(\mathcal{O}/\varpi) > 2n_i \operatorname{len}_{\mathcal{O}}(\mathcal{O}/\ell^{M_0}) + 1$ for each i (n_i the rank of T_i).

Definition 7.4.2. Let p be a prime, K a finite extension of $\mathbb{Q}_p$, L a finite extension of $\mathbb{Q}_p$, and V a finite-dimensional L-vector space with a continuous L-linear action of G_K. Put

$$D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \qquad \mathrm{Fil}^j D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \cap (\mathrm{Fil}^j B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \quad (j \in \mathbb{Z}).$$

This is a module over $K \otimes_{\mathbb{Q}_p} L$, where $K \subset B_{\mathrm{dR}}$ acts on the first factor and L on V, with a decreasing filtration by $K \otimes_{\mathbb{Q}_p} L$-submodules. For a $\mathbb{Q}_p$-embedding $\tau\colon K \to L$ let

$$D_{\mathrm{dR},\tau}(V) = L \otimes_{K \otimes_{\mathbb{Q}_p} L,\, \tau \otimes \mathrm{id}} D_{\mathrm{dR}}(V),$$

with the filtration $\mathrm{Fil}^j D_{\mathrm{dR},\tau}(V)$ given by the image of $L \otimes \mathrm{Fil}^j D_{\mathrm{dR}}(V)$. An integer a is a **Hodge jump of V at τ** if $\mathrm{Fil}^{a+1} D_{\mathrm{dR},\tau}(V) \neq \mathrm{Fil}^a D_{\mathrm{dR},\tau}(V)$.

If $F \subseteq K$ is a subfield and $\tau_0\colon F \to L$ is a $\mathbb{Q}_p$-embedding, the **embeddings of K over τ_0** are the $\mathbb{Q}_p$-embeddings $\tau\colon K \to L$ with $\tau|_F = \tau_0$.

Theorem 7.4.3. *Let D be admissible data, ℓ a prime, $\iota \mid \ell$ a prime identification, and $(R_i)_{i \in I}$ finitely many ℓ-adic representations, each main-summand like for some string length and jumps. Then there is an m-fold twist τ such that for every $1 \le j \le m$ the twist $\xi = \tau^j$, with modulus Q, satisfies:*

1. *$(S_\ell^{\mathrm{bad}}, Q, \psi_\xi)$ is an auxiliary twist with ramification at most e'_ℓ;*
2. *for every i there are $\gamma_i \in G_E$ and $\zeta_i \in \overline{\mathbb{Z}}_\ell$ such that $\psi_\xi R_i$ has Selmer length at most e'_ℓ at (Q, γ_i, ζ_i);*
3. *for every ramified critical pair p and every $s \in \{\pm 1\}$, $v_\ell\big(\iota(r_p(\xi, s))\big) \le e'_\ell$.*

Proof. Put $c_\ell = (4f+3)r\, v_\ell(N_{\mathrm{ch}})$ and $\varepsilon = 2^{-\ell}$, so $e'_\ell = c_\ell + \varepsilon$.

Step 1: data and systems. By Proposition 7.4.4 there are a chosen place $w_0 \nmid \ell$ of norm p_{i_0}, a tame generator g_0 at w_0 (inertial above a place of S_ℓ^{bad} not above ℓ, as $p_{i_0} \mid M$), a coefficient ring $\mathcal{O}$ and Selmer data T_i for $(S_\ell^{\mathrm{bad}}, g_0)$ with representation a lattice u_i in R_i, places those above $\ell p_0 p_1$, local modules the crystalline cocycles, $\mathrm{nl}(\beta_{T_i}) \le c_\ell$, and torsion level $c_\ell + 2$. There are finitely many ramified critical pairs. Their weights and integers $z \le t$ lie in finite sets. For each fixed weight, the ratio of any two such characters has weight zero and factors through the finite tame class group at the chosen places, which has finitely many characters. Thus only finitely many characters occur for each weight; for each such p and each s, Proposition 7.4.9(1) gives a Stickelberger system $Y_{p,s}$ for $(S_\ell^{\mathrm{bad}}, g_0)$.

Step 2: the twist. Theorem 7.4.11, applied with $S_0 = S_\ell^{\mathrm{bad}}$, m and ε, gives Q auxiliary for S_ℓ^{bad} and ψ such that ψ^j is an ε-small twist at Q for (T_i), $(Y_{p,s})$, for $1 \le j \le m$. As ψ comes from $\mathrm{U}(1)$ with modulus Q, Proposition 7.4.9(2) gives a tame character τ with modulus Q and $\psi = \psi_\tau$; then $\psi^j = \psi_{\tau^j}$. Since ψ^j is ramified at every $w \in Q$ and, by local class field theory, its restriction to inertia at w is the tame part of τ^j at w transported by ι, the tame part of τ^j is nontrivial at every $w \in Q$ for $1 \le j \le m$: τ is an m-fold twist.

Step 3: the three conclusions for $\xi = \tau^j$. Item 1: Q is auxiliary for S_ℓ^{bad}, so no place of Q is above ℓ; ψ_ξ comes from $\mathrm{U}(1)$ with modulus Q; and $\sum_{w \in Q} v_\ell(\psi_\xi(\sigma_w) - 1) \le \varepsilon \le e'_\ell$ for tame generators σ_w. Item 2: take $\gamma_i = \gamma_*$ and $\zeta_i = \zeta_*$ of T_i. The element γ_* is inertial at a place of S_ℓ^{bad} not above ℓ, where ψ_ξ is unramified, so $(\psi_\xi u_i)(\gamma_*) = \zeta_* \cdot 1$; $\zeta_* - 1$ is a unit since $\zeta_*^2 - 1$ is. The actual crystalline Selmer group over every sufficiently large $\mathcal{O}' \supseteq \mathcal{O}$ is contained in the group cut out by the mapped relations of T_i. We verify the local inclusion. At a place above ℓ, ψ_ξ is trivial, so the integral local lattice is $L' = L \otimes_{\mathcal{O}} \mathcal{O}'$ with the original action. For

every $\mathcal{O}$-linear functional $f : \mathcal{O}' \to \mathcal{O}$, coordinatewise application to a crystalline cocycle z' for L' gives a crystalline cocycle for L: apply $1 \otimes f$ to its period witness to obtain

$$1 \otimes f(z'(g)) = g(1 \otimes f)y - (1 \otimes f)y.$$

The map is equivariant since the matrices of L have entries in $\mathcal{O}$. If $\sum_g A(g)z(g) = 0$ is any fixed matrix relation holding on $Z_f(L)$, it follows that $f(\sum_g A(g)z'(g)) = 0$ for every f. The dual basis of the finite free $\mathcal{O}$-module $\mathcal{O}'$ separates its elements, hence this relation holds on $Z_f(L')$. Thus every mapped relation of T_i is a relation of the extended integral crystalline module. A torsion cocycle satisfying all the latter relations satisfies the former ones. The ramification and γ_* conditions are identical, giving the claimed inclusion of Selmer groups. Length is monotone under inclusion, so (AX6) has normalized length at most $\mathrm{nl}(\beta_{T_i})+\varepsilon \le e'_\ell$ over every sufficiently large coefficient ring. Item 3: $v_\ell(\psi_\xi(\Theta_Q)) \le \beta_{Y_{p,s}}+\varepsilon$, so by Proposition 7.4.9(1), $v_\ell(\iota(r_p(\xi,s))) \le 4v_\ell(p_{i_0}-1)+\varepsilon$. As $p_{i_0}-1 = \#\mathrm{GL}_1(k_{w_0})$ divides $\#\mathrm{GL}_{rt}(k_{w_0})$, which divides N_{ch}, $4v_\ell(p_{i_0}-1) \le 4v_\ell(N_{\mathrm{ch}}) \le c_\ell$. □

Proposition 7.4.4. *Let D be admissible data, ℓ a prime, $\iota \mid \ell$ a prime identification, and $c_\ell = (4f+3)\,r\,v_\ell(N_{\mathrm{ch}})$.*

1. *There is a chosen place w_0 not above ℓ; it has norm p_{i_0} for some $i_0 \in \{0,1\}$. Let g_0 be a tame generator at w_0.*
2. *Let $(R_i)_{i\in I}$ be finitely many ℓ-adic representations, each main-summand like for some string length and jumps. Then there are a coefficient ring $\mathcal{O}$ and Selmer data T_i for $(S_\ell^{\mathrm{bad}}, g_0)$ over $\mathcal{O}$ such that, for every i: the representation of T_i is a lattice in R_i; the set of places of T_i is the set of places above ℓ, p_0 or p_1; its local modules Z_w ($w \mid \ell$) are the crystalline cocycles Z_f; and $\mathrm{nl}(\beta_{T_i}) \le c_\ell$. Moreover $c_\ell + 2$ is a torsion level for the T_i.*

Proof. By Lemma 4.2.66, each of the distinct primes p_0, p_1 has f chosen places, of norm p_0, p_1 respectively. At most one equals ℓ. Choose w_0 above the other and fix the indicated tame generator g_0.

We first fix a coefficient ring and lattices for the finite family. A continuous representation of the profinite group G_E in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is defined over a finite extension of $\mathbb{Q}_\ell$. Indeed, there are countably many finite extensions inside $\overline{\mathbb{Q}}_\ell$, and their matrix groups are closed. Their inverse images cover G_E. The Baire category theorem gives one inverse image with nonempty interior; being a subgroup, it is open. Adjoining the entries of the images of finitely many coset representatives gives a finite extension containing the whole image. The image is compact, hence bounded, and the $\mathcal{O}$-span of the translates of $\mathcal{O}^n$ is contained in a fixed fractional lattice. This span is a finite free stable lattice of rank n. For the finitely many representations take a common finite Galois coefficient extension, sufficiently large for the full matrix-span assertion in (MS3)(ii). Further extensions preserve lattices and that assertion.

Fix one representation, of rank n and string length t', and a lattice ρ. Put $q = \max(t',1)-1$ and $V_\ell = v_\ell(N_{\mathrm{ch}})$. We have $1 \le n \le r$, $t' \le t$, and $q+1 \le t$ because $t = 5r \ge 1$. We prove the required bounds uniformly after any further coefficient extension.

Conjugating a scalar. If $\rho(g) = uI$, then u is a unit and

$$\rho(g^{-c}) = \chi_{\mathrm{cyc}}(g)^q u I, \qquad g^{-c} = (g^c)^{-1}.$$

Here is a direct derivation that uses only (MS1). For every $x \in G_E$, apply its trace identity at $x^c g^{-1}$ and at x^c. Since $\chi_{\mathrm{cyc}}(x^c) = \chi_{\mathrm{cyc}}(x)$, the two identities give

$$\mathrm{tr}(\rho(x)\rho(g^{-c})) = \chi_{\mathrm{cyc}}(g)^q u \,\mathrm{tr}\,\rho(x).$$

The matrices $\rho(x)$ span $M_n(\mathcal{O})$, so the identity holds with every matrix in place of $\rho(x)$. Testing against the matrix units proves the asserted matrix equality. The scalar u is a unit because uI is invertible.

The global term. By (MS3)(ii) choose g acting by uI and with $\chi_{\mathrm{cyc}}(g) = p_i^n$ for some $i \in \{0, 1\}$. This equality also implies $p_i \neq \ell$, as a cyclotomic value is an ℓ-adic unit. On the conjugate Cartier dual of $W = (\mathcal{O}/\eta)^n$, the element g acts by $\chi_{\mathrm{cyc}}(g^c)\chi_{\mathrm{cyc}}(g)^q u$. Consequently every equivariant map $F : W \to W^{*c}$ is killed by

$$\delta = u(1 - p_i^d), \qquad d = n(q+1), \qquad 1 \leq d \leq rt.$$

Evaluation at the first basis vector injects these maps into $W^{*c}[\delta]$. To check injectivity, a map vanishing there vanishes on all its G_E-translates by equivariance; these translates span W, since the matrices $\rho(G_E)$ span all matrix units. The δ-torsion of the character module of a finite module has the same length as its quotient by δ, hence as its δ-torsion: this follows by taking the dual of the quotient and comparing the kernel and cokernel of multiplication by δ on a finite group. Thus

$$\operatorname{len}_{\mathcal{O}} \operatorname{Hom}_{G_E}(W, W^{*c}) \leq \operatorname{len}_{\mathcal{O}} W[\delta] \leq n v_{\varpi}(\delta).$$

The last inequality follows coordinatewise from $\operatorname{len}(\mathcal{O}/\eta)[\delta] = \min(v_{\varpi}(\eta), v_{\varpi}(\delta))$. Since $p_i^d - 1$ is a factor in

$$\#\mathrm{GL}_{rt}(\mathbb{F}_{p_i}) = p_i^{rt(rt-1)/2} \prod_{\nu=1}^{rt} (p_i^{\nu} - 1),$$

and a chosen place above p_i occurs in N_{ch}, we obtain $v_{\varpi}(\delta)/e_{\mathcal{O}} \leq V_\ell$. Taking the supremum over $\eta \neq 0$ bounds the normalized global term by nV_ℓ.

The terms away from ℓ. Let $v \mid p_0 p_1$, $v \nmid \ell$, and take an inertia element γ_v with the determinant bound of (MS3)(i). For any finite quotient $W = (\mathcal{O}/\eta)^n$, restriction of global cocycles has length at most the length of all local cocycles. The coboundaries have length $\operatorname{len} W - \operatorname{len} H^0(E_v, W)$. Local Tate duality and the local Euler characteristic formula, [201, Chapter I, Corollary 2.3 and Theorem 2.8], therefore give

$$\operatorname{len} Z^1(E_v, W) = \operatorname{len} W + \operatorname{len} H^2(E_v, W) = \operatorname{len} W + \operatorname{len} H^0(E_v, W^\vee).$$

The inertia element γ_v has cyclotomic value 1. Hence the last fixed module embeds into the kernel of the dual of $\rho(\gamma_v) - I$. That kernel has the same length as the cokernel of $\rho(\gamma_v) - I$ on W, hence as its kernel on W. Smith normal form over $\mathcal{O}$ bounds this length by $v_{\varpi} \det(\rho(\gamma_v) - I)$. Taking the nonnegative excess over $\operatorname{len} W$ and the supremum proves that the normalized local term at v is at most nV_ℓ. There are at most $4f$ places above $p_0 p_1$, by complete splitting in Lemma 4.2.66. The total normalized local term is consequently at most $4fnV_\ell$. At the chosen w_0, (MS3)(i) applies to the specified g_0 and to g_0^{-c}, so the two-index first-prime term is at most $2nV_\ell$. All these terms are finite and

$$\operatorname{nl}(\beta_T) \leq (4f + 3)nV_\ell \leq c_\ell.$$

The local modules and distinguished inertia element. Take S to be the places above ℓ, p_0, p_1 and $Z_w = Z_f(L_w(\rho))$ via the fixed chart j_w at A_w for $w \mid \ell$. The local finite-type condition of (MS2) is potentially crystalline at these places. The modules are finite free by Lemma 7.4.5, and contain coboundaries: the coboundary of v has period witness $1 \otimes v$. The same lemma gives

$$\sum_{w \mid \ell} \operatorname{rank}_{\mathcal{O}} Z_w \leq \#\{w \mid \ell\}\, n + \sum_{\sigma : E \hookrightarrow \mathbb{C}} \#\{u \in J(\sigma) : u < 0\} \leq (\#\{w \mid \ell\} + f)n.$$

For the last inequality group embeddings into the f conjugate pairs and apply (DW3) of Definition 6.6.14. The representation is unramified outside S by (MS2).

Choose γ_* and its scalar ζ_* by (MS3)(iii). Its place is prime to ℓ, so $\chi_{\mathrm{cyc}}(\gamma_*) = 1$; the scalar-conjugation calculation proves $\rho(\gamma_*^{-c}) = \zeta_* I$ as well. Taking $a = 0, j = 2$ in (MS3)(iii) shows that $\zeta_*^2 - 1$ is a unit. We have now checked every requirement on the Selmer datum, including its distinguished inertia element.

Finally put $M_0 = c_\ell + 2$, an integer. Choose a further unramified coefficient extension whose residue field has more than $2n_i e_{\mathcal{O}} M_0 + 1$ elements for every member of the finite family. This is possible because unramified extensions have arbitrarily large residue fields and leave $e_{\mathcal{O}}$ unchanged. The constructions and bounds above apply over that extension. They give $\mathrm{nl}(\beta_{T_i}) + 1 \le c_\ell + 1 < M_0$ and precisely the residue-field inequalities in the torsion-level definition. Thus M_0 is a common torsion level. If the family is empty, only the initial choice of w_0 is needed. □

Lemma 7.4.5. *Let E be a number field, ℓ a prime, $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ an isomorphism, $e\colon E \to \overline{\mathbb{Q}}_\ell$ an embedding, and $j\colon \Gamma \to G_\ell$ an open embedding onto the subgroup of elements fixing $e(E)$ pointwise. Let $\mathcal{O}$ be a coefficient ring, L a potentially crystalline Galois lattice over $\mathcal{O}$, and J a family of finite multisets of integers indexed by the complex embeddings of E such that, for every $g \in G_\ell$, $J(\sigma_{e,g})$ lists the jumps of L at the embedding $\gamma_g\colon a \mapsto g(a)$ of $\mathcal{O}$ into $\mathbb{C}_\ell$. Then $Z_f(L)$ is a finitely generated free $\mathcal{O}$-module and*

$$\mathrm{rank}_{\mathcal{O}} Z_f(L) = \mathrm{rank}_{\mathcal{O}} L + \sum_{\sigma \in \{\sigma_{e,g}\colon g \in G_\ell\}} \#\{u \in J(\sigma) : u < 0\}.$$

Proof. Put $K_\ell = \overline{e(E)} \subset \overline{\mathbb{Q}}_\ell$, $n_v = [K_\ell : \mathbb{Q}_\ell]$, $F_{\mathrm{c}} = \mathrm{Frac}\,\mathcal{O}$, $n_{\mathrm{c}} = [F_{\mathrm{c}} : \mathbb{Q}_\ell]$, and $V = L \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Thus $j(\Gamma) = G_{K_\ell}$. By Lemma 7.4.6, the action and crystalline cocycles are continuous, V is de Rham, $Z_f(L)$ is finite free, and

$$Z_f(L) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = Z_f(V), \tag{1}$$

$$\dim_{\mathbb{Q}_\ell} Z_f(V) = \dim_{\mathbb{Q}_\ell} V + \dim_{\mathbb{Q}_\ell} (D_{\mathrm{dR}}(V)/\mathrm{Fil}^0). \tag{2}$$

The lemma applies to this actual completion, with exactly the potentially crystalline period condition assumed here.

We check the filtration sign before counting. In the jump definition a functional of jump u satisfies $\lambda(\rho(g)x) = \chi_{\mathrm{cyc}}(g)^{-u} g\lambda(x)$. It is thus an equivariant functional to $\mathbb{C}_\ell(-u)$. The comparison of Theorem A.3.11 identifies its multiplicity with the multiplicity of filtration degree u. In particular for $L = \mathbb{Z}_\ell(1)$ the identity functional has $u = -1$, agreeing with its nonzero gr^{-1}. The jumps here are the negatives of the cyclotomic-positive weights in the Hodge–Tate convention; there is no index shift. Passing to the open subgroup used for potential crystallinity does not change these multiplicities: the Hodge–Tate comparison is preserved under finite extension and the dimensions of the filtered de Rham pieces are preserved by scalar extension.

For each embedding $\gamma : F_{\mathrm{c}} \to \mathbb{C}_\ell$, let $\mathcal{J}_\gamma$ be the jump multiset of its coefficient component, and put $\mathcal{J}_{\mathrm{tot}} = \sum_\gamma \mathcal{J}_\gamma$. The Hodge–Tate decomposition of the underlying $\mathbb{Q}_\ell$-representation is obtained by summing these components. Therefore Theorem A.3.11, or equivalently the filtered count in Lemma 7.4.7(3) applied after extending coefficients, gives

$$\dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 = n_v \,\#\{u \in \mathcal{J}_{\mathrm{tot}} : u < 0\}. \tag{3}$$

Here the factor n_v converts the dimension over K_ℓ into dimension over $\mathbb{Q}_\ell$.

It remains to identify the multiplicity in (3) with that in the statement; this is a finite double count, so it does not require any choice of compatible coset representatives. Choose a finite Galois extension $F_*/\mathbb{Q}_\ell$ containing K_ℓ and F_{c}. The maps

$$g \longmapsto g|_{F_{\mathrm{c}}}, \qquad g \longmapsto \iota^{-1} g^{-1} e$$

factor through $G_* = \mathrm{Gal}(F_*/\mathbb{Q}_\ell)$. The first has n_{c} values, each with $\#G_*/n_{\mathrm{c}}$ preimages. The second has n_v values, each with $\#G_*/n_v$ preimages: embeddings of E inducing this place extend uniquely to embeddings of its completion, and there are n_v of those. By the hypothesis on J, for each g the two resulting multisets are equal. Summing over G_* and cancelling its order gives

$$n_v \, \mathcal{J}_{\mathrm{tot}} = n_{\mathrm{c}} \sum_{\sigma \in \{\sigma_{e,g}\}} J(\sigma). \tag{4}$$

Taking negative multiplicities in (4), then using (1)–(3) and $\dim_{\mathbb{Q}_\ell} V = n_{\mathrm{c}} \operatorname{rank}_{\mathcal{O}} L$, yields

$$n_{\mathrm{c}} \operatorname{rank}_{\mathcal{O}} Z_f(L) = n_{\mathrm{c}} \operatorname{rank}_{\mathcal{O}} L + n_{\mathrm{c}} \sum_{\sigma} \#\{u \in J(\sigma) : u < 0\}.$$

Division by $n_{\mathrm{c}} > 0$ proves the formula. □

Lemma 7.4.6. *Let $K_\ell/\mathbb{Q}_\ell$ be finite, let $\mathcal{O}$ be a coefficient ring, and let L be a finite free $\mathcal{O}$-module with an $\mathcal{O}$-linear action of G_{K_ℓ} that is potentially crystalline in the $B_{\max}$-period sense. Put $V = L[1/\ell]$, $n_v = [K_\ell : \mathbb{Q}_\ell]$, and $n_{\mathrm{c}} = [\mathrm{Frac}\,\mathcal{O} : \mathbb{Q}_\ell]$. Then:*

1. *The action on L, and every cocycle in $Z_f(L)$, are continuous. The representation V is potentially crystalline in the usual B_{cris}-sense and is de Rham.*
2. *For any continuous finite-dimensional $\mathbb{Q}_\ell$-representation W and any cocycle $c : G_{K_\ell} \to W$, the existence of a $B_{\max}$-coboundary witness is equivalent to the existence of a B_{cris}-coboundary witness.*
3. *$Z_f(L)$ is finite free over $\mathcal{O}$, its rationalization is the space of crystalline cocycles of V, and*

$$n_{\mathrm{c}} \operatorname{rank}_{\mathcal{O}} Z_f(L) = \dim_{\mathbb{Q}_\ell} V + \dim_{\mathbb{Q}_\ell} (D_{\mathrm{dR}}(V)/\mathrm{Fil}^0).$$

In particular

$$\operatorname{rank}_{\mathcal{O}} Z_f(L) \le (1 + n_v) \operatorname{rank}_{\mathcal{O}} L \le 2 n_v \operatorname{rank}_{\mathcal{O}} L.$$

Proof. We first record the integral period-ring calculation, including the prime 2. Choose a compatible system of ℓ-power roots of ℓ, and put $\xi_0 = [\ell^\flat] - \ell$, a generator of $\ker\theta$. The ring $A_{\max}$ is the ℓ-adic completion of $A_{\mathrm{inf}}[\xi_0/\ell]$, and A_{cris} is the ℓ-adic completion of its divided-power envelope with divided powers compatible with those of (ℓ). Both embed in B_{dR}^+. The divided power $\xi_0^{[j]} = \xi_0^j/j!$ belongs to $A_{\max}$, since

$$\xi_0^{[j]} = (\xi_0/\ell)^j \, \frac{\ell^j}{j!}, \qquad v_\ell(j!) \le j.$$

The resulting map of the divided-power envelope extends by ℓ-adic completion, giving $A_{\mathrm{cris}} \subset A_{\max}$. Conversely Frobenius sends the generator ξ_0/ℓ into A_{cris}:

$$\frac{\varphi(\xi_0)}{\ell} = \frac{(\xi_0 + \ell)^\ell - \ell}{\ell} = (\ell - 1)!\, \xi_0^{[\ell]} + \sum_{j=1}^{\ell-1} \binom{\ell}{j} \ell^{\ell-j-1} \xi_0^j + \ell^{\ell-1} - 1.$$

All coefficients are integral. Frobenius is continuous and preserves A_{inf}, so it extends from $A_{\max}$ into A_{cris}. Inverting ℓ and the Tate period, whose Frobenius is ℓ times itself, yields

$$\varphi(B_{\max}) \subset B_{\mathrm{cris}} \subset B_{\max}. \tag{1}$$

These maps commute with Galois and Frobenius fixes $\mathbb{Q}_\ell$.

For (2), if $1 \otimes c(g) = gy - y$ in $B_{\max} \otimes W$, applying $\varphi \otimes 1$ gives the same identity with witness $(\varphi \otimes 1)y$ in $B_{\mathrm{cris}} \otimes W$. The reverse implication is the second inclusion of (1).

To prove continuity in (1) of the statement, choose finitely many potentially crystalline periods spanning $B_{\max} \otimes L$, and intersect the finitely many open subgroups on which they are fixed. For a rational basis e_i of V, write $e_i = \sum_j b_{ij} v_j$ in those periods. On the resulting open subgroup, $ge_i = \sum_j g(b_{ij}) v_j$. The coefficients of the latter expression in the rational basis are continuous functions into $B_{\max}$, and belong to $\mathbb{Q}_\ell$. The usual topology of $B_{\max}$ induces the ℓ-adic topology on $\mathbb{Q}_\ell$; hence the representation matrices are continuous. The finitely many cosets give continuity on the full group. A crystalline cocycle has a period witness, and the same calculation, now using the continuous action just proved, shows that $gy - y$ and thus $c(g)$ are continuous. No basis of $B_{\max} \otimes V$ needs to be extracted from the chosen generators.

Applying φ to all coefficients and periods in the spanning expressions shows that the $\varphi(v_j)$ span $B_{\mathrm{cris}} \otimes V$ and are fixed on that open subgroup. Thus V is crystalline over its finite fixed field. It is de Rham there because $B_{\mathrm{cris}} \subset B_{\mathrm{dR}}$. It is de Rham over K_ℓ as well: enlarge that field extension to a finite Galois extension, take its finite-dimensional de Rham period space, and descend its semilinear action by ordinary Galois descent. The invariants have the same dimension over K_ℓ as the original space over the extension. This proves (1).

For (3), local Tate finiteness Theorem A.3.1 supplies the finite-cohomology hypotheses of Theorem A.3.2, so $H^1(G_{K_\ell}, L)$ is a finitely generated $\mathcal{O}$-module. Continuous coboundaries form a quotient of L; therefore the continuous cocycle module, and its submodule $Z_f(L)$, are finitely generated. The latter is torsion free as a module of functions into L, hence free over the discrete valuation ring $\mathcal{O}$.

Every continuous rational cocycle has bounded image, by compactness of G_{K_ℓ}. Scaling it and its period witness by a power of ℓ puts it in $Z_f(L)$. This proves the rationalization assertion. Its quotient by the rational coboundaries is $H^1_f(K_\ell, V)$, by (2). The rational coboundaries have dimension $\dim V - \dim H^0(K_\ell, V)$. The formula of Theorem A.3.6 therefore gives

$$\dim_{\mathbb{Q}_\ell} Z_f(V) = \dim_{\mathbb{Q}_\ell} V + \dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(V)/\mathrm{Fil}^0.$$

This is the asserted rank formula. Finally $D_{\mathrm{dR}}(V)$ has dimension $\dim_{\mathbb{Q}_\ell} V$ over K_ℓ; its quotient has $\mathbb{Q}_\ell$-dimension at most $n_v \dim_{\mathbb{Q}_\ell} V$. Divide by n_{c} and use $n_v \geq 1$. □

Lemma 7.4.7. *Let F be a number field, ℓ a prime, v a place of F above ℓ, $D \subseteq G_F$ a decomposition group at v, and $n_v = [F_v : \mathbb{Q}_\ell]$. Let K be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$ and uniformiser ϖ, W a K-vector space of finite dimension d with a continuous K-linear action of D which is de Rham as a $\mathbb{Q}_\ell$-representation, $\Lambda \subset W$ a D-stable $\mathcal{O}$-lattice,* $\mathrm{Cs} = W/\Lambda$, $H = H^1(D, \mathrm{Cs})$ *and* $F' = H^1_f(D, \mathrm{Cs})$, *the image of $H^1_f(D, W)$ under $H^1(D, W) \to H$. Put*

$$d^0(W) = \dim_K \mathrm{Fil}^0 D_{\mathrm{dR}}(W), \qquad D_{\mathrm{dR}}(W) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} W)^D,$$

K acting through W. Let $W' = \mathrm{Hom}_{\mathbb{Q}_\ell}(W, \mathbb{Q}_\ell(1))$ and $\Lambda' = \mathrm{Hom}_{\mathbb{Z}_\ell}(\Lambda, \mathbb{Z}_\ell(1))$, with D acting by $(gf)(w) = g\, f(g^{-1}w)$ and $\mathcal{O}$ by $(\alpha f)(w) = f(\alpha w)$, and $A' = W'/\Lambda'$. For an integer $a \geq 0$ let $Y_a = \{z \in H : \varpi^a z \in F'\}$. Then:

1. $\mathrm{nl}(Y_a/F') = \dfrac{a}{e_{\mathcal{O}}}\, d^0(W) + \mathrm{nl}\big(H^2(D, \Lambda/\varpi^a\Lambda)\big)$;

2. $\mathrm{nl}\big(H^2(D,\Lambda/\varpi^a\Lambda)\big) = \mathrm{nl}\big(H^0(D,A')[\varpi^a]\big) \le \mathrm{nl}\big(H^0(D,A')\big)$, *and* $H^0(D,A')$ *is finite if* $H^0(D,W')=0$;

3. $d^0(W) \le n_v\, d$; *and if* K *contains* $\tau(F_v)$ *for every* $\mathbb{Q}_\ell$*-embedding* τ *of* F_v *into* $\overline{\mathbb{Q}}_\ell$, *then* $d^0(W) = \sum_\tau \sum_{a\ge 0} \dim_K\big(\mathrm{Fil}^a D_{\mathrm{dR},\tau}(W)/\mathrm{Fil}^{a+1} D_{\mathrm{dR},\tau}(W)\big)$, *the number of Hodge jumps* $a\ge 0$ *of* W *counted with multiplicity over all* τ.

Proof. **Step 0: two facts on torsion modules.** (i) A torsion $\mathcal{O}$-module X with $X[\varpi]$ finite is isomorphic to $(K/\mathcal{O})^c \oplus X_0$ with X_0 finite (dual form of the structure theorem for finitely generated $\mathcal{O}$-modules); c is the *corank* of X, and it is the corank of any divisible submodule X_1 with X/X_1 finite (the image of X_1 in $X/(K/\mathcal{O})^c \cong X_0$ is divisible and finite, hence 0, so $X_1 \subseteq (K/\mathcal{O})^c$ with finite divisible quotient, hence $X_1 = (K/\mathcal{O})^c$). Every $\mathcal{O}$-submodule of $\mathrm{Cs} \cong (K/\mathcal{O})^d$ satisfies the hypothesis. (ii) For such X and $a \ge 0$,

$$\operatorname{length} X[\varpi^a] - \operatorname{length} X/\varpi^a X = a\, c,$$

because on $(K/\mathcal{O})^c$ multiplication by ϖ^a is onto with kernel of length ac, and on the finite X_0 its kernel and cokernel have equal length.

Step 1: $F' \cong (K/\mathcal{O})^{c_f}$ **with** $c_f = \dim_K H^1_f(D,W)$**.** The long exact sequence of $0\to\Lambda\to W\to \mathrm{Cs}\to 0$ (a set-theoretic section of $W\to\mathrm{Cs}$ is continuous, Cs being discrete) shows that the kernel of $H^1_f(D,W)\to H$ is $L = \iota(H^1(D,\Lambda))\cap H^1_f(D,W)$, $\iota\colon H^1(D,\Lambda)\to H^1(D,W)$. By Theorem A.3.2, $H^1(D,\Lambda)$ is finitely generated, so L is finitely generated, and torsion-free as a submodule of a K-vector space, hence free. L spans $H^1_f(D,W)$: if $x=[c]\in H^1_f(D,W)$, the image $c(D)$ of the compact group D is compact, hence contained in $\varpi^{-s}\Lambda$ for some s (the open sets $\varpi^{-s}\Lambda$ increase and cover W), so $\varpi^s c$ is a continuous cocycle with values in Λ and $\varpi^s x\in L$ ($H^1_f(D,W)$ is a K-subspace). Thus L is a lattice in $H^1_f(D,W)$ and $F'\cong H^1_f(D,W)/L \cong (K/\mathcal{O})^{c_f}$. In particular F' is divisible and $\operatorname{length} F'[\varpi^a] = a\, c_f$.

Step 2: $X = H^0(D,\mathrm{Cs})$ **has corank** $h^0 = \dim_K H^0(D,W)$**.** In $0\to H^0(D,\Lambda)\to H^0(D,W)\to X \xrightarrow{\partial} H^1(D,\Lambda)$, $H^0(D,\Lambda)=\Lambda\cap H^0(D,W)$ is a lattice in $H^0(D,W)$ (finitely generated as a submodule of Λ; it spans, since $\varpi^s w\in\Lambda$ for $w\in W$ and large s), so the image of $H^0(D,W)$ in X is $\cong (K/\mathcal{O})^{h^0}$. The image of ∂ consists of torsion elements of the finitely generated module $H^1(D,\Lambda)$ (Theorem A.3.2), so it is finite. By Step 0(i), X has corank h^0.

Step 3: the length of Y_a/F'**.** Since F' is divisible, $Y_a = H[\varpi^a]+F'$: if $\varpi^a z=\varpi^a f'$ with $f'\in F'$, then $z-f'\in H[\varpi^a]$. Hence $Y_a/F' \cong H[\varpi^a]/(H[\varpi^a]\cap F') = H[\varpi^a]/F'[\varpi^a]$, and by Step 1

$$\operatorname{length}(Y_a/F') = \operatorname{length} H[\varpi^a] - a\, c_f.$$

Let $M = \mathrm{Cs}[\varpi^a] = \varpi^{-a}\Lambda/\Lambda \cong \Lambda/\varpi^a\Lambda$, a finite $\mathcal{O}$-module of length ad killed by ℓ^r, $r=\lceil a/e_{\mathcal{O}}\rceil$. The sequence $0\to M\to \mathrm{Cs}\xrightarrow{\varpi^a}\mathrm{Cs}\to 0$ is exact (Cs is divisible) and gives the exact sequence

$$0\to X/\varpi^a X\to H^1(D,M)\to H[\varpi^a]\to 0,$$

and $H^0(D,M)=X[\varpi^a]$. By Theorem A.3.1(4),

$$\operatorname{length} H^1(D,M) = \operatorname{length} H^0(D,M) + \operatorname{length} H^2(D,M) + n_v\, a\, d.$$

With Step 0(ii) and Step 2,

$$\begin{aligned}\operatorname{length} H[\varpi^a] &= \big(\operatorname{length} X[\varpi^a] - \operatorname{length} X/\varpi^a X\big) + \operatorname{length} H^2(D,M) + n_v ad\\ &= a\, h^0 + \operatorname{length} H^2(D,M) + n_v ad,\end{aligned}$$

so $\mathrm{length}(Y_a/F') = a\,(h^0 + n_v d - c_f) + \mathrm{length}\, H^2(D, M)$.

Step 4: the dimension count. W is de Rham as a $\mathbb{Q}_\ell$-representation, so by Theorem 7.4.8

$$\dim_{\mathbb{Q}_\ell} H^1_f(D, W) = \dim_{\mathbb{Q}_\ell} H^0(D, W) + \dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(W) - \dim_{\mathbb{Q}_\ell} \mathrm{Fil}^0 D_{\mathrm{dR}}(W).$$

All four spaces are K-vector spaces (K acts through W, commuting with D and preserving $\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes W$, and Definition 7.1.16 defines H^1_f by a K-linear condition), so we may divide by $[K : \mathbb{Q}_\ell]$. Being de Rham, $\dim_{F_v} D_{\mathrm{dR}}(W) = \dim_{\mathbb{Q}_\ell} W = d[K : \mathbb{Q}_\ell]$ ($B^D_{\mathrm{dR}} = F_v$), so $\dim_K D_{\mathrm{dR}}(W) = n_v d$. Hence $c_f = h^0 + n_v d - d^0(W)$, and Step 3 gives $\mathrm{length}(Y_a/F') = a\, d^0(W) + \mathrm{length}\, H^2(D, M)$. Dividing by $e_{\mathcal{O}}$ gives item 1.

Item 2. By Theorem A.3.1(2) with $i = 2$, $\#H^2(D, M) = \#H^0(D, M^\vee)$. Since Λ is free over $\mathbb{Z}_\ell$, $\mathrm{Hom}_{\mathbb{Z}_\ell}(\Lambda, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) = W'/\Lambda' = A'$, and $M^\vee = \mathrm{Hom}(\Lambda/\varpi^a\Lambda, \mu_{\ell^r})$ is the submodule of homomorphisms vanishing on $\varpi^a\Lambda$, i.e. $A'[\varpi^a]$ for the $\mathcal{O}$-structure of the statement; the identification is D-equivariant. So $H^0(D, M^\vee) = H^0(D, A')[\varpi^a]$; both $H^2(D, M)$ and this group are finite $\mathcal{O}$-modules ($M^\vee$ is an $\mathcal{O}$-module through M), and equal orders give equal lengths. If $H^0(D, W') = 0$, the sequence $0 = H^0(D, W') \to H^0(D, A') \xrightarrow{\partial} H^1(D, \Lambda')$ of $0 \to \Lambda' \to W' \to A' \to 0$ embeds the torsion module $H^0(D, A')$ into the torsion of $H^1(D, \Lambda')$, which is finite by Theorem A.3.2 (Λ' is a finitely generated $\mathcal{O}$-module with continuous action).

Item 3. $d^0(W) \le \dim_K D_{\mathrm{dR}}(W) = n_v d$ (Step 4). If K contains every $\tau(F_v)$, then $F_v \otimes_{\mathbb{Q}_\ell} K \cong \prod_\tau K$ ($F_v/\mathbb{Q}_\ell$ is separable), with idempotents e_τ; $D_{\mathrm{dR}}(W)$ and each $\mathrm{Fil}^a D_{\mathrm{dR}}(W)$ are $F_v \otimes K$-modules, so $\mathrm{Fil}^a D_{\mathrm{dR}}(W) = \bigoplus_\tau e_\tau \mathrm{Fil}^a D_{\mathrm{dR}}(W)$ and $e_\tau \mathrm{Fil}^a D_{\mathrm{dR}}(W) = \mathrm{Fil}^a D_{\mathrm{dR},\tau}(W)$. The filtration of the finite-dimensional space $D_{\mathrm{dR},\tau}(W)$ is decreasing, with $\mathrm{Fil}^a = 0$ for large a (as $\bigcap_a t^a B^+_{\mathrm{dR}} = 0$), so $\dim_K \mathrm{Fil}^0 D_{\mathrm{dR},\tau}(W) = \sum_{a \ge 0} \dim_K(\mathrm{Fil}^a/\mathrm{Fil}^{a+1})$, and summing over τ gives the formula. □

Theorem 7.4.8. *Let p be a prime, K a finite extension of $\mathbb{Q}_p$ and V a finite-dimensional continuous $\mathbb{Q}_p$-linear representation of G_K which is de Rham. Put $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and $\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \cap (\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$. Then*

$$\dim_{\mathbb{Q}_p} H^1_f(G_K, V) = \dim_{\mathbb{Q}_p} H^0(G_K, V) + \dim_{\mathbb{Q}_p}(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)).$$

Proof. Apply Theorem A.3.6 with $\ell = p$, $M = K$ and the given V. Its hypotheses hold: $K/\mathbb{Q}_p$ is finite and V is finite-dimensional, continuous and de Rham.

The two descriptions of the filtration agree inside $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$. Indeed $\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ is a G_K-stable subspace, and for every stable subspace U of a representation W one has $U^{G_K} = U \cap W^{G_K}$: membership in either side says precisely that the vector lies in U and is fixed by every element of G_K. Thus

$$(\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \cap (\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = \mathrm{Fil}^0 D_{\mathrm{dR}}(V).$$

The local finite condition is the same kernel in both statements: a cocycle class maps to zero after tensoring with B_{cris} exactly when it has a coboundary witness there, as in Definition 7.1.16. The invariant space $H^0(G_K, V)$ is unchanged. Consequently the dimension formula of Theorem A.3.6, with all dimensions over $\mathbb{Q}_p$, is the displayed formula after interchanging its two summands.

□

Proposition 7.4.9. *Let D be admissible data, ℓ a prime, $\iota \mid \ell$ a prime identification, w_0 a chosen place not above ℓ and g_0 a tame generator at w_0.*

1. *For every ramified critical pair p and every $s \in \{\pm 1\}$ there is a Stickelberger system Y for $(S_\ell^{\mathrm{bad}}, g_0)$ such that for every twist ξ whose modulus Q is auxiliary for S_ℓ^{bad} and whose inverse avatar ψ_ξ comes from* $\mathrm{U}(1)$ *with modulus Q, and every $b \in [0,\infty]$:*

$$v_\ell(\psi_\xi(\Theta_Q)) \le \beta_Y + b \implies v_\ell(\iota(r_p(\xi,s))) \le 4\, v_\ell(\mathrm{N}w_0 - 1) + b.$$

2. *Every character $\psi\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ that comes from* $\mathrm{U}(1)$ *with a modulus Q auxiliary for S_ℓ^{bad} is the inverse avatar of a tame character with modulus Q.*

Proof. We construct the required normalized measures. At critical integer $z = 1$ use the literal rank-two measure of Lemma 7.4.10; its Euler exponents are $\{0,1\}$, and its ratio exponent is two.

Suppose $z \ge 2$, put $n = 2z$, $j = \sum_\tau (c_\tau - z)$ and

$$n_\nu = M^{4fz}(Md)^j, \qquad C = N_{\mathrm{ch}}^3 \det(G_\kappa) n_\nu^2, \qquad F(\xi) = C\,\mathrm{alg}_p(\xi). \tag{A}$$

The positive integer C depends on p and D, not on ξ or Q. Here κ is the fixed one-row weight of these L-value data. The matrix $G_\kappa = F_\kappa^{\mathsf{T}} F_\kappa$ is a positive-definite integer matrix, since the filling matrix contains the standard-basis rows. Thus its determinant is a positive integer. We prove directly that (A) is an integral finite-character measure, retaining every stabilizer weight.

For a fixed declared modulus Q, expand the source class sum and its finite residue data. Choose source-line classes indexed by a and target classes indexed by t, at their actual levels, and include the finitely many twist-residue indices in a. They are independent of ξ among characters with this declared modulus. In the kernel, replace ξ by the weight-zero trivial character with this same modulus; write $K_a(t)$ for the resulting OL4-normalized point kernel, and u_a for its source norm-one point, including its residue translate. The determinant factor in (WR4) and the residue-character expansion isolate all the character dependence:

$$\varphi_{p,\xi}(t) = \sum_a \gamma_a^{-1} \xi(u_a)^{-1} K_a(t), \qquad \gamma_a = |\Gamma_a|. \tag{B}$$

This is an expansion of the original datum, not an integration over a refined source level: write its multiplicative residue character as the sum of its values times the residue indicators; a compact lift moves each indicator to the base indicator. Covariance then moves that lift to the point and contributes exactly the character value in (B). Every sum is finite, and no inverse cardinality is introduced by this indicator expansion.

Let γ_t be the original target-class stabilizer. Expanding the Petersson norm in (OL4) gives

$$\mathrm{alg}_p(\xi) = \sum_{a,b,t} \frac{\langle K_b(t), K_a(t)\rangle_\kappa}{(\mathrm{N}Q)^{2z-1}\gamma_a\gamma_b\gamma_t}\, \xi(u_b u_a^{-1}). \tag{C}$$

The coefficients and points in (C) precede ξ. The maximal-level class sets are finite by Lemma 4.6.4; the actual levels are finite-index residue refinements, so their class sets remain finite. Every one of the three stabilizer orders divides N_{ch}, by Lemma 6.7.9. Indeed apply that lemma to the same class representative and its actual open level subgroup $K' \subset K_0(m)$, with $m = 1$ on the source and $m = 2z$ on the target. These are standard definite spaces, so the E-matrix points are precisely the rational points of the indicated unitary group over E^+. A critical pair has $z \le t_D$, and admissibility gives $r_D \ge 3$, hence $1 \le m \le 2t_D \le r_D t_D$. The residue refinements only shrink K' and do not replace the class representative. Thus the lemma applies separately to γ_a, γ_b, γ_t. Its auxiliary odd reduction prime is independent of the coefficient prime ℓ; in particular this argument includes $\ell = 2$.

We establish the needed point-kernel integrality with the exact OL4 scale. Let $\ell \nmid \mathrm{N}Q$ and fix any identification ι. Move each target representative and each source-line representative rationally to be integral above ℓ. Indeed apply Lemma 4.2.64 separately to the actual standard target space I_n and source line I_1, with the finite set of all places of E above ℓ. Its auxiliary finite factor is 1 at these places, so the rational unitary components are dense there. The local inverses of the representatives lie in the nonempty open sets of movers making the translated representatives integral with integral inverse. Density gives the required rational movers. The DP covariance is an exact unitary change of coordinates on the Weyl module, so it preserves the Gram pairing in (C). On the source line the form being lifted has weight zero, so its rational move introduces no extra coefficient. Movers can depend on ℓ, ι, but the original points and the Gram coefficient in (C) are unchanged.

We now check preservation of the precise finite span to which the theta estimate applies. Put $\mathcal{O}_\iota = \{c : |\iota(c)|_\ell \le 1\}$ and let

$$\mathcal{V}_{M,\iota} = \mathrm{span}_{\mathcal{O}_\iota}\{\rho(f_v(x))c_s^n v_0 : \ Mx_w \in \mathcal{O}_{E_w}^n \text{ for every } w \mid \ell\}. \tag{E}$$

There is no restriction on x away from ℓ. This is the actual finite Heisenberg span of Proposition 4.2.35, in the scaled coordinates of Lemma 4.1.3; it is not a lattice selected from a complex isomorphism.

First the finite datum in (OL4), multiplied by $M^{2fn} = M^{4fz}$, belongs to (E). To see this, use the self-dual integral polarization and the actual coefficient transform of Lemma 4.1.3(1)–(4). An indicator of a residue class modulo $\mathfrak{m}$ is

$$\mathbf{1}_{b+\mathfrak{m}}(v) = \frac{1}{|\mathcal{O}_E/\mathfrak{m}|} \sum_{\chi \in (\mathcal{O}_E/\mathfrak{m})^\vee} \chi(v-b).$$

Here $v \in \mathcal{O}_E$; both sides are extended by zero off the integral position lattice. Apply this in each of the n coordinates, with the unchanged vacuum off the finitely many indicated places. Self-duality identifies its additive characters with frequencies in $\mathfrak{m}^{-1}$ times the dual integral lattice. Here $\mathfrak{m}$ divides $(M\mathrm{N}Q)$ and $\ell \nmid \mathrm{N}Q$, so at every $w \mid \ell$ these frequencies are M-integral. The ℓ-part of the Fourier denominator divides the ℓ-part of $|\mathcal{O}_E/(M)|^n = M^{2fn}$. All character coefficients are roots of unity. Thus this multiplier clears every denominator in $\mathcal{O}_\iota$. In the polarized realization these characters times the integral indicator are exactly the Heisenberg translates in Lemma 4.1.3(2); a pure dual-polarization translate has no extra quadratic term. Any phase from another permitted realization is a root of unity and hence a unit. The multiplicative residue-character expansion used in (B) adds only root-of-unity coefficients and integral residue lifts. This proves the assertion for the actual finite datum, with its specified factor c_s^n.

Next let (g, h) be one of the integral-at-ℓ pair points obtained above, including a residue lift. On the Hom coordinates its linear transformation is the actual map induced by g, h; denote it by G. Both G and G^{-1} preserve the standard local lattice at every $w \mid \ell$. The scaled coordinates multiply by a scalar and therefore have the same property. The product-coordinate identification in Proposition 4.1.14 identifies this finite pair action with the normalized operator of Lemma 4.1.13, with its specified determinant-character factors. This is the same realization as Lemma 4.1.3, not an abstract equivalent representation. Write L for its standard finite lattice,

$$J = L \cap G^{-1}L, \qquad I = [L : J].$$

At $w \mid \ell$ one has $J_w = L_w$, so I is prime to ℓ. Representatives of L/J may be chosen with component zero at every such w. The finite-vacuum formula and covariance in Lemma 4.1.13 give, for a generator in (E),

$$\Omega(G)\rho(f_v(x))c_s^n v_0 = \frac{\Delta(G)}{I} \sum_{y \in L/J} \epsilon_y \, \rho(f_v(Gx + Gy))c_s^n v_0, \tag{F}$$

where the ϵ_y include precisely the Heisenberg multiplication phases. They are roots of unity. Every $Gx + Gy$ in this formula remains M-integral above ℓ.

We check the scalar in (F), including the pair-character factors. In the definition of $\Delta(G)$ choose the lattice matrix carrying L onto J to be integral with integral inverse above ℓ; this is possible since $J_w = L_w$ there. Its determinant is a unit above ℓ. The finite absolute-value factors of this determinant involve only rational primes different from ℓ, and are ℓ-adic units, also for their square roots. The algebraic Hecke-character factors at an idele integral and invertible above ℓ are units as well. Here is the elementary reason. Raise its fractional ideal to the order of its ideal class and write the resulting idele as a principal idele times a finite integral unit idele. The principal generator is a unit at every place above ℓ. Its algebraic infinity-type value, at any identification above ℓ, is a product of integral powers of the images of this generator and its conjugates, hence is a unit. The value on the compact finite unit group is of finite order, since the character is smooth there, hence is a root of unity. Taking the positive power back preserves valuation zero. If the normalization has half-integral weights, first square it to clear those weights and use the same argument. This proves the assertion for the character in $\Delta(G)$ and for both prescribed determinant twists of the pair action. Since I is also a unit, every coefficient in (F) lies in $\mathcal{O}_\iota^\times$. Consequently the actual finite pair operator preserves (E).

For completeness, the residue substitution in this application is the one in Lemma 4.1.16: its integral pair action sends the residue function to the function obtained by the indicated left and right changes of variables and multiplies it by the two determinant characters. These are exactly the integral residue lifts and unit factors just checked, so no projection average or extra denominator is hidden in (F). The same lemma and the product compatibility of Proposition 4.1.14 leave the real Fock vector unchanged under this finite action. We have therefore proved preservation of the actual finite span with the actual real polynomial, in Hom rank n, rather than inferring it from unitarity. At $\ell = 2$ no division by 2 has been made in the coefficient ring: the phases in the exact Heisenberg formula, including its quadratic phases, are roots of unity and are still units. The Fourier denominator is the only denominator cleared by M^{4fz}.

Apply Proposition 4.2.35 with $\zeta_{\mathrm{sc}} = \pi_M$ to the integral polynomial tested in (OL2), namely the integer polynomial obtained from $\mathrm{Pol}_\kappa(e_i)$ under the integral theta substitution. Its degree is exactly j. The germ identity Lemma 4.2.2 gives the same factor $c_s^{2z}(\pi_M\eta_P)^j$ as (OL4), and the estimate is

$$\left|\iota\big(M^{4fz}\, e_i^{\mathsf{T}} G_\kappa K_a(t)\big)\right|_\ell \le |\iota(\pi_M/(Md))|_\ell^j .$$

The element π_M is an algebraic integer. Multiplying by $(Md)^j$ proves that every coordinate of

$$A_a = n_\nu G_\kappa K_a(t)$$

is integral at this identification. The assertion holds at every identification, including $\iota \circ \mathrm{conj}$. It follows that

$$\det(G_\kappa) n_\nu^2 \langle K_b(t), K_a(t)\rangle_\kappa = \overline{A_b}^{\mathsf{T}} \operatorname{adj}(G_\kappa) A_a$$

is integral. The remaining denominator $(\mathrm{N}Q)^{2z-1}$ is an ℓ-adic unit, and $N_{\mathrm{ch}}^3/(\gamma_a\gamma_b\gamma_t)$ is an integer. Multiplying (C) by C therefore proves the simultaneous all-prime integral finite expansion for (A). No assertion about integrality of arbitrary coherent coefficients, or of the scalar c_s itself, was used.

The scale in (OL4) is nonzero by Proposition 4.2.1 and Lemma 4.2.3. Hence Proposition 6.7.13 applies at every tame character, and gives $F(1) > 0$ and

$$F(\xi)/F(1) = r_p(\xi). \tag{D}$$

The same proposition identifies $\mathrm{alg}_p(\xi)$ with one fixed nonzero multiple of the partial L-value. On adding an unramified split place of norm q, deleting the two local Euler factors therefore multiplies it by

$$(1 - yq^{-z})(1 - y^{-1}q^{-z}), \qquad y = (\nu\xi)(\varpi_v/\bar{\varpi}_v).$$

The second coefficient is y^{-1} because the base-change character is anti-conjugate. The finite Euler deletion is valid in the absolutely convergent region $z \geq 2$. It applies even when the character has smaller conductor than its declared modulus. This proves all the required measure properties in the higher-integer case.

We now pass from either of these actual measures to the system. For the resulting measure put $K = \{0, 1\}$ and $d_{\mathrm{rat}} = 2$ if $z = 1$, and $K = \{z\}$ and $d_{\mathrm{rat}} = 1$ otherwise. Fix the sign s. For an auxiliary modulus Q, choose the finite expansion of F for all tame characters with declared modulus Q, including characters with smaller conductor. A weight-zero character is trivial on the archimedean norm-one group, so $t(u_a)$ depends only on the finite class of u_a in Cl_Q. Define

$$\Theta_Q = \sum_a \iota(b_a)[u_a]_Q^{-s}.$$

Since $\ell \nmid \mathrm{N}Q$, all its coefficients are integral at ℓ. The inverse-avatar convention gives

$$\psi_\xi(\Theta_Q) = \iota F(\xi^s). \tag{1}$$

This identity holds for every character of declared modulus Q, whether or not its conductor is all of Q. At nonauxiliary moduli set $\Theta_Q = 0$; every transition required in (SK) has both its moduli auxiliary.

Let ϑ be the geometric avatar of the base change of ν^{-s}. The local avatar formula shows that it is unramified off S_ℓ^{bad}. On g_0 its value is the ramified residue-character value, or its inverse. It is therefore different from 1 and its order divides $L = \mathrm{N}w_0 - 1$. At an auxiliary arithmetic Frobenius φ the same formula yields

$$\vartheta(\varphi)\psi_\xi(\varphi) = \iota\big((\nu\xi^s)(\varpi_P/\bar{\varpi}_P)^s\big).$$

The product over the two signs is unchanged by raising this value to s. Apply the Euler relation for F to (1), using the character ξ with its enlarged declared modulus. This proves exactly the required norm relation, with exponent list K. Moreover

$$1(\Theta_\emptyset) = \iota F(1) \neq 0.$$

Thus these data form the required Stickelberger system.

Write $\zeta = \vartheta(g_0)$. Since $\zeta \neq 1$ and $\zeta^L = 1$, the equality $\sum_{i=0}^{L-1} \zeta^i = 0$ shows that $L = \sum_{i=0}^{L-1}(1 - \zeta^i)$ is divisible by $1 - \zeta$ in the ring of integers of the coefficient field. Hence

$$v_\ell(\zeta - 1) \leq v_\ell(L), \qquad \beta_Y \leq v_\ell(\iota F(1)) + 4v_\ell(L). \tag{2}$$

For a twist ξ, its power ξ^s is again a twist. The asserted value of F and the fact that $\mathrm{N}Q$ is an ℓ-adic unit give

$$|\psi_\xi(\Theta_Q)|_\ell = |\iota F(1)|_\ell \, |\iota r_p(\xi, s)|_\ell^{d_{\mathrm{rat}}}. \tag{3}$$

If $|\iota r_p(\xi, s)|_\ell \geq 1$, its nonnegative truncated valuation is zero. Otherwise $\iota F(1)$ is integral, so (3) gives additivity for those nonnegative valuations:

$$v_\ell(\psi_\xi(\Theta_Q)) = v_\ell(\iota F(1)) + d_{\mathrm{rat}}\, v_\ell(\iota r_p(\xi, s)).$$

As $d_{\mathrm{rat}} \geq 1$, (2) and the hypothesis imply the asserted bound. This argument also covers an infinite value of the ratio valuation; for finite b it would contradict the finite bound, and for $b = \infty$ the conclusion is immediate.

Finally let ψ be a character from $\mathrm{U}(1)$ of modulus Q. By its defining descent it is a character of Cl_Q. Invert it according to the inverse-avatar reciprocity convention, pull it back to the finite adelic norm-one group, and extend trivially on its archimedean factor. Rational invariance follows from the quotient by $\mathrm{U}(1)(E)$. The resulting automorphic character has weight zero; its local characters are trivial outside Q and on the principal units at Q. It is therefore a tame character of modulus Q, and its inverse avatar is ψ. This proves the second assertion. □

Lemma 7.4.10. *Let $p = (\nu, 1)$ be a ramified critical pair, of weight $-c_\tau$. Put $B = 2\sum_{\tau\in\Phi}(c_\tau - 1)$ and*

$$N = M^{16f} c_s^4 (M^2 d\eta_P)^B \neq 0.$$

For a tame character ξ with declared modulus Q, put $\alpha_\xi = \nu\xi\upsilon$, let $\mathfrak{m}_Q$ have exponent two at the chosen places and exponent one at Q, and let χ_ξ be the character modulo $\mathfrak{m}_Q$ formed from the tame part of $\nu\xi$. Let $P(Z) = \prod_\tau \det(Z_\tau)^{c_\tau - 1}$, and let

$$\Lambda_\xi = \sum_{h \bmod \mathbf{K}_2}^{\mathrm{cl}} \theta^{I_2, I_2}_{\alpha_\xi, \alpha_\xi; \mathfrak{m}_Q}[\chi_\xi \circ \det, P](1, h), \qquad F(\xi) = |N_{\mathrm{ch}} N \Lambda_\xi|^2.$$

Then:

1. *$F(1) \neq 0$. For every declared modulus Q there are finitely many norm-one finite adelic points u_a and coefficients $b_a \in \mathbb{C}$, all chosen before ξ, such that*

$$F(\xi) = \sum_a b_a \xi(u_a)$$

   for every tame character with declared modulus Q. Every b_a is integral under every prime identification at every rational prime not dividing $\mathrm{N}Q$.

2. *If ξ' is the same automorphic character and tame part as ξ, with declared modulus $Q \cup \{v\}$, where $v \notin Q$ and this enlarged modulus is conjugate-disjoint from the chosen places and their conjugates, then, writing $q = \mathrm{N}v$ and $y = (\nu\xi)(\varpi_v/\bar{\varpi}_v)$,*

$$F(\xi') = F(\xi) \prod_{\epsilon = \pm 1} (1 - y^\epsilon)(1 - y^\epsilon/q).$$

3. *For every genuine twist ξ,*

$$F(\xi) = F(1)\, \mathrm{N}Q\, r_p(\xi)^2.$$

The last assertion is not imposed on tame characters whose actual conductor is smaller than their declared modulus.

Proof. Put $\delta = M^2$. We use the same theta functional and finite coordinates throughout.

We first prove the uniform clearing assertion needed for the finite class expansion. The underlying positive Hom space has rank four. Finite Fourier inversion modulo $\mathfrak{m}$ writes its residue datum as a sum of finite Heisenberg translates of the vacuum, with coefficients equal to residue values times roots of unity divided by $\mathrm{Norm}(\mathfrak{m})^4$. This is the ordinary character orthogonality formula on the additive residue group; the self-duality and phase convention are precisely those of Lemma 4.1.3. If $q = \mathrm{N}Q$, then $(q\delta) \subseteq \mathfrak{m}_Q$, so

$$\mathrm{Norm}(\mathfrak{m}_Q)^4 \mid (q\delta)^{8f}.$$

At a rational prime $\ell \nmid q$, multiplication by δ^{8f} therefore clears the Fourier denominator. The resulting vectors belong to the integral span of

$$\rho(\mathrm{fv}(x))c_s^4 v^0, \qquad \delta x_w \in \mathcal{O}_{E,w}^4 \quad (w \mid \ell). \tag{1}$$

No integrality of c_s itself is asserted.

Finite pair operators integral above ℓ preserve this module. Indeed the literal finite vacuum formula in Lemma 4.1.13 is

$$\Omega(g)v^0 = \frac{\Delta(g)}{I(g)} \sum_{y \in L/(L \cap g^{-1}L)} H(gy)v^0.$$

When g is integral above ℓ, this finite quotient has no ℓ-part, its representatives may be chosen with zero component above ℓ, and $I(g)$ and $\Delta(g)$ are ℓ-adic units. For $\Delta(g)$ the latter assertion follows from its defining unramified determinant character and norm factor: both local valuation and norm exponent above ℓ vanish. Covariance sends a vector x in (1) to gx, still satisfying (1). The determinant-character factors of the pair operator are also units. To check the algebraic-character valuation, take a finite ideal-class power of the finite idele to make its ideal principal, generated by $a \in E^\times$. Since the original idele is a unit at all places above ℓ, so is a. The principal-idele weight formula expresses that power of the character value as a product of conjugates of a and a finite-unit character value. Every conjugate of a is an ℓ-adic unit at every identification, and finite-unit values are roots of unity. The original character value consequently has valuation zero. These are the literal pair operators by Proposition 4.1.14, so no separately normalized action is being substituted.

Apply Proposition 4.2.35 with $\zeta_{\mathrm{sc}} = \delta d$ to this module and the integral homogeneous polynomial P. Its ordinary degree-B derivative is integral at every prime identification. By Lemma 4.2.2 it is exactly

$$\delta^{8f} c_s^4 (\delta d \eta_P)^B \theta_{\mathrm{Hom}(I_2, I_2)}(v_P \otimes \text{the translated finite datum}).$$

Thus the scalar clearing the actual standard kernel is precisely the stated N. All its factors are nonzero by Proposition 4.2.1 and Lemma 4.2.3. The construction is independent of Q, the point, the prime and the identification.

We explain why this applies to the actual class representatives. The determinant polynomial transforms at infinity by $\prod_\tau \det^{c_\tau - 1}$, which cancels the archimedean factor of α_ξ, of weight $-(c_\tau - 1)$. At the finite standard level its residue character is canceled by $\chi_\xi \circ \det$. The kernel defining the period is therefore rationally invariant and has the original standard right level. For a fixed rational prime ℓ, apply Lemma 4.2.64 to the standard space I_2, with the finite set of all places of E above ℓ. Its auxiliary finite factor is 1 at these places, so the rational unitary components are dense there. For each finite class representative h, the conditions that $\gamma_w h_w$ and its inverse are integral above ℓ form a nonempty open set of possible rational movers, containing the local inverse of h. Choose such a γ. The mover may depend on the prime and identification; rational invariance leaves the original kernel value unchanged. Hence the clearing assertion holds at the fixed representatives.

Apply the stated common supplier Lemma 6.7.9 to $n = 2$, $K' = \mathbf{K}_2 = \mathbf{K}_0(2)$ and each SAME finite class representative h. The rank bound holds because the admissible target rank is a positive integer, $t = 5r$, and thus $1 \le 2 \le rt$. The standard level is an open subgroup of itself. It follows that

$$\Gamma_h = \mathrm{U}(I_2)(F) \cap h\mathbf{K}_2 h^{-1} \quad \text{is finite}, \qquad |\Gamma_h| \mid N_{\mathrm{ch}}.$$

Here the rational unitary matrices are those called $\mathrm{U}(I_2)(E)$ in that supplier; their entries lie in E and their unitary group is the group over F. No change of the class representative or of its original maximal-level stabilizer has been made. The class set is finite by Lemma 4.6.4, and $N_{\mathrm{ch}}/|\Gamma_h|$ is an integer for every one of its original weights.

Fix Q. Expand the residue determinant character as a sum of indicators of invertible residue matrices. Move each indicator to the identity indicator by a lift in the standard compact group. The representation law and the determinant-character factor show that all dependence on ξ is a value $\xi(u_a)$; the remaining kernel has the fixed base character $\nu\upsilon$ and the identity residue datum of this same declared modulus. Thus, over the finite set of original lattice classes and residue matrices,

$$N_{\mathrm{ch}} N\Lambda_\xi = \sum_a d_a \xi(u_a), \qquad d_a = \frac{N_{\mathrm{ch}}}{|\Gamma_a|}\, N\theta_Q(g_a). \tag{2}$$

The representatives, points and coefficients are chosen before ξ. No refinement of the standard class weights is used. The preceding two paragraphs prove that each d_a is integral at every identification away from q. Since ξ is unitary,

$$F(\xi) = \sum_{a,b} d_a \overline{d_b}\, \xi(u_a u_b^{-1}). \tag{3}$$

For a fixed identification ι, apply the same integrality assertion also to $\iota \circ \mathrm{conj}$. This proves integrality of the conjugated coefficient in (3). It proves the finite expansion simultaneously at all identifications.

We prove the Euler relation without requiring full conductor. At a new place v, the old finite datum is the vacuum. For a subspace $U \subseteq k_v^2$ of dimension i, take its transposed basic Hecke representative, and put $X = (qy)^{-1}$. Write g_U for the basic representative, so its reduction has kernel U, and the action uses $h_U = g_U^{\mathsf{T}}$. Its precise position coefficient is

$$(qy)^i\, \mathbf{1}_{Z \text{ integral}}\, \mathbf{1}_{U \subseteq \ker(\bar{Z}^{\mathsf{T}})}.$$

Indeed the split-place formula of Lemma 4.1.16 gives the scalar $q^i y^i$ and the position condition $h_U^{-1} Z$ integral. To check its support, Smith form gives

$$g_U^{-1}\mathcal{O}_v^2 = \{\varpi_v^{-1} u : u \in \mathcal{O}_v^2,\ \bar{u} \in U\}.$$

Consequently $Z^{\mathsf{T}} g_U^{-1}$ is integral if and only if Z is integral and $\bar{Z}^{\mathsf{T}}$ kills U. This is equivalent to the asserted condition on $h_U^{-1} Z$; each individual term is zero outside integral positions. Multiplication by $(-X)^i q^{i(i-1)/2}$ cancels the scalar. Summing over U therefore inserts

$$\sum_{U \subseteq \ker(\bar{Z}^{\mathsf{T}})} (-1)^{\dim U} q^{\binom{\dim U}{2}} = \prod_{j=0}^{\dim \ker(\bar{Z}^{\mathsf{T}})-1} (1 - q^j) = \mathbf{1}_{\det \bar{Z} \neq 0}. \tag{4}$$

For dimension at most two the product formula follows directly: the respective sums are 1, $1-1$, and $1-(q+1)+q$. This is precisely the new residue datum, since its new local character is trivial. Apply the theta functional to this finite vector identity and integrate over the same compact quotient. The total weighted kernel has standard right level, so its integral is its standard class sum. We integrate each translated term before using Haar invariance; an individual translated function is not asserted to have that right level or to have its bare fixed-class sum equal to its integral. Each translated integral equals the original period. There are $1, q+1, 1$ subspaces in the three dimensions, so

$$\Lambda_{\xi'} = (1 - (q+1)X + qX^2)\Lambda_\xi = (1 - X)(1 - qX)\Lambda_\xi.$$

Taking squared complex absolute values and using $|y| = 1$ gives assertion 2.

It remains to prove nonvanishing and the ratio for genuine twists. We use the stated class-sum identity of Proposition 6.3.67; in particular no kernel-calculus assertion is borrowed only from a proof. Here is its complete specialization. Put

$$n = m = a = \{1, 2\}, \qquad l = b = p = \emptyset, \qquad W = I_2, \quad V' = 0, \quad d_{\mathrm{ex}} = 1, \quad |\iota_\tau| = c_\tau - 1.$$

Use the natural bijections $m \sqcup \emptyset \to n$, $\emptyset \sqcup \emptyset \to \emptyset$, $\emptyset \sqcup a \to m$ and $\emptyset \sqcup a \to n$. Then $|a| = |b| + 2d_{\mathrm{ex}} = 2$ and $2|l| + 2 = |n| = 2$, so both numerical hypotheses of that proposition hold. Each $\eta_{\tau k}$ is the unique 0×2 matrix. The defining stacked matrix consists of its two variable rows; hence $\Delta^j(e; \eta) = P$. Both matrices defining the matched polynomial have zero rows and columns, so their determinants, and therefore $Q_{f,j}(\eta)$, are 1.

The fixed index sets and ι give ONE real constant C, chosen before every modulus and automorphic character in Proposition 6.3.67. Fix a genuine twist ξ and specialize that proposition to

$$\mathfrak{m} = \mathfrak{m}_Q, \qquad \alpha = \beta = \alpha_\xi = \nu\xi\upsilon, \qquad \chi = \chi_\xi, \qquad \Phi(X) = \chi_\xi(\det X), \qquad \Psi = 1.$$

Its character weight is $-(c_\tau - 1) = |l| - |\iota_\tau|$, since υ has weight 1. Its tame part is the tame part of $\nu\xi$, since υ has trivial tame part. The chosen places and Q are disjoint, ν is ramified exactly at the chosen places, and a genuine twist has nontrivial tame component at every place of Q. Thus α is tame for the WHOLE declared modulus, as that proposition requires. The functions are matched for (σ, χ_ξ^{-1}): the only matrices in the support of Φ are the invertible 2×2 matrices, every such matrix is k^{-1} for some invertible k, and

$$\Phi(k^{-1}) = \chi_\xi(\det k)^{-1} = \chi_\xi^{-1}(\det k)\Psi(k_b).$$

Here k_b is the unique matrix with zero columns. These statements also verify the zero-extended values off the required supports.

Consequently Proposition 6.3.67 gives γ_ξ with

$$L_\xi \neq 0, \qquad L_\xi|\gamma_\xi|^2 = C, \qquad \gamma_\xi\Lambda_\xi = \int_{[\mathrm{U}(0)]} \theta^{0,I_2}_{1,\alpha_\xi;\mathfrak{m}_Q}[1,1](x,1)\,dx, \quad L_\xi = L_*(\mathfrak{m}_Q, \alpha_\xi^{-1}\upsilon; 1). \quad (5)$$

The right side is literally 1. Indeed $\mathrm{Hom}(0, I_2)$ is zero, its Fock and finite data are the empty tensor 1, and its normalized theta functional has value 1. In (XD2) the only possible character factor is $(\alpha_\xi/\beta)(\det 1) = 1$; the source determinant is the empty determinant 1. The group $\mathrm{U}(0)$ and its rational quotient have one element and measure one, from the empty-product quotient measure of (XD1). There is therefore no unspecified Haar mass or character-dependent normalization on this side.

It follows from (5) that $\gamma_\xi\Lambda_\xi = 1$, so both factors are nonzero. Moreover $\alpha_\xi^{-1}\upsilon = (\nu\xi)^{-1}$ and the set of places of $\mathfrak{m}_Q$ is $S_M \cup Q$. Thus the definition (XD7) gives

$$L_\xi = \Big(\prod_{w \in S_M \cup Q} \mathrm{N}w\Big)|L^{S_M \cup Q}(1, (\nu\xi)^{-1})|^2 > 0.$$

Since $|\gamma_\xi|^2|\Lambda_\xi|^2 = 1$, equation (5) yields

$$C|\Lambda_\xi|^2 = L_\xi. \quad (6)$$

The trivial character with empty Q is a genuine twist, so (5) at $\xi = 1$ also proves $\Lambda_1 \neq 0$ and $C > 0$. Divide (6) by its value at $\xi = 1$. The fixed chosen-place norm product cancels, and the ratio in Definition 4.2.4 is defined with the now nonzero denominator. Therefore

$$\frac{|\Lambda_\xi|^2}{|\Lambda_1|^2} = \mathrm{N}Q\,\frac{|L^{S_M \cup Q}(1, (\nu\xi)^{-1})|^2}{|L^{S_M}(1, \nu^{-1})|^2} = \mathrm{N}Q\, r_p(\xi)^2.$$

Multiplication by the SAME nonzero $|N_{\mathrm{ch}}N|^2$ proves assertion 3 and $F(1) \neq 0$. The full-conductor check above applies only to genuine twists; no such nonvanishing assertion is made for a tame character with an enlarged declared modulus. □

Theorem 7.4.11. *Let ℓ be a prime, S_0 a finite set of finite places of E containing those above ℓ, $g_0 \in G_E$ in an inertia group above a place of S_0 not above ℓ, $\mathcal{O}$ a coefficient ring, $(T_i)_{i\in I}$ finitely many Selmer data for (S_0, g_0) over $\mathcal{O}$ with a torsion level M_0, and $(Y_k)_{k\in\kappa}$ finitely many Stickelberger systems for (S_0, g_0). Let $m \geq 1$ and $\varepsilon > 0$. Then there are a set Q of places, auxiliary for S_0, and a character $\psi\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ such that ψ^j is an ε-small twist at Q for (T_i), (Y_k) for every $1 \leq j \leq m$.*

Proof. Write $b_i = \mathrm{nl}(\beta_{T_i})$ and $d_k = \beta_{Y_k}$. These are finite by the defining finiteness conditions. Replacing the requested error by $\varepsilon_0 = \min(\varepsilon, 1/2) > 0$ is sufficient, since every inequality defining a small twist is preserved on increasing the allowance.

Apply Proposition 7.4.12 to obtain its integer $s_0 \geq 1$. Choose a rational number $r > 0$ such that

$$s_0 r \leq \varepsilon_0/2.$$

In particular $r < 1$ and $s_0 r < 1$. Put

$$B = \sum_{i\in I}(b_i + 1) + \sum_{k\in\kappa}(d_k + 1).$$

Apply Lemma 7.4.22 with this B, the variable bound s_0, the given m, radius r, and error $\varepsilon_0/2$. Let H be its uniform order bound. Only now choose $N \geq H$ sufficiently large for Proposition 7.4.12, and choose its auxiliary set Q, characters $\lambda_{\mathfrak{q}}$, polynomials P_i, P_k, and points x_i, x_k.

The number of variables is $\#Q \leq s_0$. Their point coordinates have valuation at least r, and

$$v_\ell(P_i(x_i)) \leq b_i + s_0 r \leq b_i + 1, \qquad v_\ell(P_k(x_k)) \leq d_k + s_0 r \leq d_k + 1.$$

Thus the sum of their evaluation valuations is at most B, as required by the torsion specialization lemma. It gives roots $\zeta_{\mathfrak{q}}$ of order dividing ℓ^H, hence also ℓ^N, such that for $1 \leq j \leq m$, on writing $y_j = (\zeta_{\mathfrak{q}}^j - 1)_{\mathfrak{q}\in Q}$,

$$0 < v_\ell(\zeta_{\mathfrak{q}}^j - 1) \leq r, \qquad v_\ell(P_i(y_j)) \leq b_i + \varepsilon_0, \qquad v_\ell(P_k(y_j)) \leq d_k + \varepsilon_0. \tag{1}$$

Define

$$\psi(g) = \prod_{\mathfrak{q}\in Q} \zeta_{\mathfrak{q}}^{\lambda_{\mathfrak{q}}(g)} \quad (g \in G_E).$$

The exponent is read modulo ℓ^N; the definition is independent of its representative because each root has order dividing ℓ^N. This is a continuous character of ℓ-power order. Each factor comes from the norm-one torus with modulus $\{\mathfrak{q}\}$, so their product comes from that torus with modulus Q, is anticyclotomic, and is trivial on the decomposition groups above ℓ. The same holds for every power of ψ.

Fix $1 \leq j \leq m$. The evaluation identity of Proposition 7.4.12, applied to $(\zeta_{\mathfrak{q}}^j)_{\mathfrak{q}}$, gives

$$v_\ell(\psi^j(\Theta_Q^{(k)})) = v_\ell(P_k(y_j)) \leq d_k + \varepsilon_0.$$

Since $\varepsilon_0 < 1$, (1) also gives $v_\ell(P_i(y_j)) < b_i + 1$. The Selmer-length conclusion of the same proposition therefore applies and bounds the twisted Selmer length of T_i by

$$v_\ell(P_i(y_j)) \leq b_i + \varepsilon_0.$$

It remains to check ramification and the sum at tame generators. Let $\gamma_{\mathfrak{q}}$ be any tame generator at $\mathfrak{q}$. Every other $\lambda_{\mathfrak{p}}$ is unramified there, since Q is auxiliary and disjoint from $\bar{Q}$. Moreover the residue characteristic at $\mathfrak{q}$ differs from ℓ. Its wild inertia has no nontrivial image in the finite ℓ-group, and a tame generator maps to a generator of the cyclic image of inertia. Because $\lambda_{\mathfrak{q}}$ is onto on inertia, $\lambda_{\mathfrak{q}}(\gamma_{\mathfrak{q}})$ is a unit modulo ℓ. Raising an ℓ-power root of unity to such a unit preserves its order and the valuation of its difference from one. It follows from (1) that

$$0 < v_\ell(\psi^j(\gamma_{\mathfrak{q}}) - 1) = v_\ell(\zeta_{\mathfrak{q}}^j - 1) \le r.$$

In particular ψ^j is ramified at $\mathfrak{q}$. Conjugate inertia has the inverse character by anticyclotomicity, and inversion preserves this valuation. The same bound consequently holds for any tame generator at $\bar{\mathfrak{q}}$, including generators in any conjugate choice of decomposition subgroup. Hence

$$\sum_{w \in Q \cup \bar{Q}} v_\ell(\psi^j(\gamma_w) - 1) \le 2\#Q\, r \le s_0 r \le \varepsilon_0.$$

All conditions for an ε_0-small twist are now proved simultaneously for every j. Increasing the allowance to ε proves the assertion. If Q or either indexing family is empty, the products and sums used here have their usual empty values, and the same argument applies. □

Proposition 7.4.12. *Let ℓ, S_0, g_0, $\mathcal{O}$, the Selmer data $(T_i)_{i\in I}$ and the torsion level M_0 be as in Proposition 7.4.13, and let $(Y_k)_{k\in\kappa}$ be finitely many Stickelberger systems for (S_0, g_0). Then there is an integer $s_0 \ge 1$ such that for every rational $r > 0$ with $s_0 r < 1$ and every sufficiently large N there are:*

- *a set Q of places, auxiliary for S_0, with $2\#Q \le s_0$;*
- *for each $\mathfrak{q} \in Q$ a continuous homomorphism $\lambda_{\mathfrak{q}} \colon G_E \to \mathbb{Z}/\ell^N$ from $\mathrm{U}(1)$ with modulus $\{\mathfrak{q}\}$ that is onto on inertia at $\mathfrak{q}$;*
- *polynomials $P_i, P_k \in \overline{\mathbb{Z}}_\ell[X_{\mathfrak{q}} : \mathfrak{q} \in Q]$ and points $x_i, x_k \in \overline{\mathbb{Z}}_\ell^{\,Q}$ whose coordinates have valuation at least r, with*

$$v_\ell(P_i(x_i)) \le \mathrm{nl}(\beta_{T_i}) + s_0 r, \qquad v_\ell(P_k(x_k)) \le \beta_{Y_k} + s_0 r;$$

such that for all ℓ^N-th roots of unity $(\zeta_{\mathfrak{q}})_{\mathfrak{q}\in Q}$, with $\psi = \prod_{\mathfrak{q}} \zeta_{\mathfrak{q}}^{\lambda_{\mathfrak{q}}}$ and $y = (\zeta_{\mathfrak{q}} - 1)_{\mathfrak{q}}$: $\psi(\Theta_Q^{(k)}) = P_k(y)$ for every k, and for every i with $v_\ell(P_i(y)) < \mathrm{nl}(\beta_{T_i}) + 1$ the twist of T_i by ψ has length at most $v_\ell(P_i(y))$ at Q.

Proof. Write $b_i = \mathrm{nl}(\beta_{T_i})$, and let n_i be the rank of T_i. For the system Y_k write its character as ϑ_k, its number of exponents as u_k, and put $c_k = v_\ell(\vartheta_k(g_0) - 1)$. These numbers are finite. Every value of ϑ_k is an ℓ-adic unit: its image is compact, and its absolute values form a compact subgroup of $\mathbb{R}_{>0}$, which must be $\{1\}$.

Choose a neighborhood W of g_0 so that for every $g \in W$,

$$\rho_i(g^b) \equiv \rho_i(g_0^b) \pmod{\ell^{M_0}} \quad (i \in I,\ b \in \{+,-\}), \qquad v_\ell(\vartheta_k(g) - \vartheta_k(g_0)) > c_k \quad (k \in \kappa).$$

Here $g^+ = g$ and $g^- = (g^c)^{-1}$; if I is empty its congruences are omitted. Continuity gives such a neighborhood. Let J_W be the bound in the strengthened conclusion of Proposition 7.4.13. Choose an integer $C \ge 1$ with $n_i, u_k \le C$ for every index, and set

$$s_0 = 2C(J_W + 1).$$

This choice precedes r and N.

Fix a rational $r > 0$ with $s_0 r < 1$. Choose an integer $K \geq 0$ with $\ell^K r \geq 1$, and a positive integer H larger than every $\beta_{Y_k} + 1$. Take N sufficiently large that

$$N > M_0, \qquad N > c_k \ \ (k \in \kappa), \qquad N \geq K + H, \qquad N \geq K + M_0$$

(the last condition is omitted if I is empty). The strengthened descent statement gives Q, its first place v_0, characters λ_v, elements γ_v, φ_v and slopes $s_{v,i}$. In particular $\#Q \leq J_W$, so $2\#Q \leq s_0$. Each λ_v is onto inertia since $\lambda_v(\gamma_v) = 1$, is trivial on the decomposition groups above ℓ, and comes from the norm-one torus with modulus $\{v\}$. It is anticyclotomic, so $\lambda_v(g^c) = -\lambda_v(g)$. Characters at other auxiliary places are unramified, giving $\lambda_w(\gamma_v) = \delta_{wv}$. All hypotheses about these characters used below are therefore supplied.

We use the following elementary fact about a finite family of nonzero polynomials $F(T)$ with integral algebraic ℓ-adic coefficients. Put

$$G_r(F) = \min_d \{v_\ell(F_d) + dr\}.$$

There is a point t in a finite coefficient extension with $v_\ell(t) = r$ and $v_\ell(F(t)) = G_r(F)$ for every polynomial in the family. To prove this, choose an algebraic element π_r with valuation r. For each F, divide $F(\pi_r Z)$ by a coefficient of least valuation. The resulting polynomial is integral and has nonzero reduction. Enlarge the residue field so that one nonzero residue avoids the finitely many roots of all these reductions, and lift it to a unit z. Then $t = \pi_r z$ works. This argument also proves all the asserted points lie in $\overline{\mathbb{Z}}_\ell$.

We first construct the Selmer polynomials. Fix i, and let $\mathscr{L}_i$ be the fixed relations of T_i, including vanishing at γ_*. In all finite-coefficient modules take ramification set $S_i \cup Q \cup \bar{Q}$ and coefficient module $(\mathcal{O}'/\ell^{M_0})^{n_i}$. Let $\mathcal{A}_i(x)$ denote the module of Lemma 7.4.20 for these relations and characters, and let $\mathcal{B}_i(x)$ have in addition the ties at $Q \setminus \{v_0\}$. At $x = 0$ these are the original untwisted cocycle equations.

The all-ties module at zero has unnormalised length at most $\beta^\circ_{T_i}$ by descent. Apply Lemma 7.4.21 at $y = 0$ (using $r = 1, K = 0$ there) to remove just the ties at v_0, retaining all the other relations, with $w_{v_0,b} = g_0^b$. The required matrix congruences hold since $\varphi_{v_0} \in W$. The cost is the first-prime term. Consequently

$$\operatorname{nl} \mathcal{B}_i(0) \leq b_i.$$

Here $M_0 \geq 1$ whenever I is nonempty, because $b_i + 1 < M_0$.

By Lemma 7.4.20, choose a polynomial $G_i(X)$ sharp for $\mathcal{B}_i$ at zero and bounding its normalized length at every point over every coefficient extension. Restrict it to the line

$$x_{v_0} = T, \qquad x_v = s_{v,i} T \quad (v \neq v_0)$$

and call the resulting polynomial $g_i(T)$. Its constant term is nonzero and has valuation at most b_i, so $G_r(g_i) \leq b_i$. For every $v \neq v_0$ and $b \in \{+, -\}$ the polynomial

$$D_{v,b}(T) = \det\big((1 + T)\rho_i(\varphi_v^b) - I\big)$$

has degree n_i and unit leading coefficient. Thus $G_r(D_{v,b}) \leq n_i r$. Choose t_i attaining these finitely many Gauss valuations as above, and let x_i be its point on the line. All its coordinates have valuation at least r, and

$$\operatorname{nl} \mathcal{B}_i(x_i) \leq v_\ell(g_i(t_i)) \leq b_i.$$

Remove the ties at $Q\setminus\{v_0\}$ by Lemma 7.4.21, with $y = t_i$ and $w_{v,b} = \varphi_v^b$. The choice $N \geq K+M_0$ supplies its geometric-sum condition. It follows that

$$\operatorname{nl} \mathcal{A}_i(x_i) \leq b_i + 2n_i(\#Q-1)r \leq b_i + s_0 r.$$

Apply Lemma 7.4.20 once more, now sharply for $\mathcal{A}_i$ at x_i. Its polynomial P_i has the required small value there and bounds $\mathcal{A}_i(x)$ at every point.

At a tuple of ℓ^N-th roots of unity ζ, the polynomial twist is the actual character $\psi = \prod_v \zeta_v^{\lambda_v}$. Its local action above ℓ is unchanged, and it is unramified at the place of γ_*, since that place belongs to S_0 and not to $Q \cup \bar{Q}$. Thus $\mathcal{A}_i(\zeta - 1)$ is exactly the ℓ^{M_0}-torsion of the divisible-coefficient twisted Selmer module defined by the same fixed relations, under the map $a \mapsto a/\ell^{M_0}$. This assertion holds over every sufficiently large coefficient ring.

For completeness the strict threshold in the statement passes from this torsion to the whole module. A continuous cocycle into the discrete module $(\operatorname{Frac}\mathcal{O}'/\mathcal{O}')^{n_i}$ has finite image, so is killed by a power of ℓ. If an ℓ-primary torsion $\mathcal{O}'$-module A has

$$\operatorname{nl} A[\ell^{M_0}] < M_0,$$

then $A = A[\ell^{M_0}]$: an element not killed by ℓ^{M_0} would generate a cyclic submodule whose ℓ^{M_0}-torsion has length at least $M_0 v_{\varpi'}(\ell)$, a contradiction. Since $b_i + 1 < M_0$, the hypothesis $v_\ell(P_i(\zeta-1)) < b_i + 1$ gives precisely this strict bound. It follows that the full twisted module has normalized length at most $v_\ell(P_i(\zeta-1))$, as required.

We next construct the system polynomials, using only the defining norm relations. Fix k, and suppress its index. A character λ_v from the norm-one torus gives a cyclic character of Cl_Q. Fix a primitive ℓ^N-th root to identify its values with $\mathbb{Z}/\ell^N$. For each class d let $a_v(d)$ be this additive value. If $\Theta_Q = \sum_d A_d[d]$, put

$$P(X) = \sum_d A_d \prod_{v\in Q} (1+X_v)^{[a_v(d)]_{\ell^N}}.$$

It has integral coefficients and, for every root tuple, satisfies $P(\zeta-1) = \psi(\Theta_Q)$.

Let $F_A(T)$ be the corresponding one-variable polynomial obtained from Θ_A using just the character λ_{v_0}, for $v_0 \in A \subseteq Q$. In particular

$$F_Q(T) = P(X_{v_0} = T,\ X_v = 0 \text{ for } v \neq v_0).$$

The first-prime norm relation at the trivial character gives

$$v_\ell(F_{\{v_0\}}(0)) = \beta_Y.$$

Indeed $\varphi_{v_0} \in W$ makes $\vartheta(\varphi_{v_0}) - \vartheta(g_0)$ have valuation greater than $c = v_\ell(\vartheta(g_0)-1)$; and $\mathrm{N}v_0 \equiv 1 \pmod{\ell^N}$ with $N > c$ makes $(\mathrm{N}v_0)^{-e} - 1$ have valuation greater than c for every exponent e. Each of the two Euler factors for that exponent therefore has valuation exactly c. Their product contributes $2uc$, added to $v_\ell(1(\Theta_\emptyset))$.

Write $q = \ell^N$ and $f_N(T) = (1+T)^q - 1$. For $v \in Q \setminus \{v_0\}$ put

$$d_{v,e} = \vartheta(\varphi_v)(\mathrm{N}v)^{-e}, \qquad d'_{v,e} = \vartheta(\varphi_v)^{-1}(\mathrm{N}v)^{-e}.$$

Both are units. Applied to every character $\zeta^{\lambda_{v_0}}$, $\zeta^q = 1$, the system norm relation and $\lambda_{v_0}(\varphi_v) = 1$ imply

$$(1+T)^{u(\#Q-1)} F_Q(T) \equiv F_{\{v_0\}}(T) \prod_{v\in Q\setminus\{v_0\}} \prod_e [1 - d_{v,e}(1+T)]\,[(1+T) - d'_{v,e}] \pmod{f_N(T)}.$$

To justify the polynomial congruence, first do one insertion. After multiplying by $(1+T)^u$ its difference vanishes at all q distinct values $\zeta-1$, and hence is divisible by the monic polynomial f_N. Monic division preserves integrality. Iterate the insertions to obtain the displayed congruence.

Each linear factor in the product has unit leading coefficient, so has Gauss valuation at most r; and $G_r(F_{\{v_0\}}) \le \beta_Y$. Choose t_k of valuation r attaining the Gauss valuations of that polynomial and all these factors. The right side at t_k has valuation at most

$$\beta_Y + 2u(\#Q-1)r \le \beta_Y + s_0 r < \beta_Y + 1 < H.$$

The binomial inequality $v_\ell((1+t)^\ell - 1) \ge \min\{\ell v_\ell(t), v_\ell(t)+1\}$ shows that $v_\ell((1+t_k)^{\ell^K}-1) \ge 1$. Each further ℓ-th power raises this valuation by at least 1. Consequently $v_\ell(f_N(t_k)) \ge 1 + N - K > H$, since $N \ge K+H$. The congruence error is therefore of valuation at least H. Also $1+t_k$ is a unit. The ultrametric inequality consequently makes $v_\ell(F_Q(t_k))$ equal to the valuation of the right side, so it is at most $\beta_Y + s_0 r$. Set the first coordinate of x_k equal to t_k and all other coordinates equal to zero. Their valuations are at least r, and $P(x_k) = F_Q(t_k)$. This is the required polynomial and point for Y_k.

The constructions apply to every sufficiently large N after r is fixed. Empty families simply omit their corresponding steps. All polynomials and chosen points involve only finitely many algebraic coefficients and so lie in the specified algebraic closure. This proves every conclusion. □

Proposition 7.4.13. *Let ℓ be a prime, S_0 a finite set of finite places of E containing those above ℓ, $g_0 \in G_E$ in an inertia group above a place of S_0 not above ℓ, $\mathcal{O}$ a coefficient ring, $(T_i)_{i\in I}$ finitely many Selmer data for (S_0, g_0) over $\mathcal{O}$ with representations ρ_i and places S_i, and $M_0 \ge 0$ a torsion level for them. Then there is an integer J_0 such that for every $N \ge 1$ there are a set J of at most J_0 places, auxiliary for S_0, and for each $v \in J$ a tame generator γ_v and a Frobenius element φ_v at v, a character $h_v\colon G_E \to \mathbb{Z}/\ell^N$ from $\mathrm{U}(1)$ with modulus $\{v\}$ and $h_v(\gamma_v) = 1$, and slopes $s_{v,i} \in \mathcal{O}$ $(i \in I)$, such that for every i the $\mathcal{O}$-module of continuous cocycles $z\colon G_E \to (\mathcal{O}/\ell^{M_0})^{n_i}$ for ρ_i that are unramified outside $S_i \cup J \cup \bar{J}$, satisfy the local relations of T_i above ℓ, vanish at γ_*, and satisfy the ties*

$$z(\gamma_v) = s_{v,i}\, z(\varphi_v) \quad \text{and} \quad z(\bar{\gamma}_v) = s_{v,i}\, z(\bar{\varphi}_v) \qquad (v \in J)$$

(with $\bar{\gamma}_v, \bar{\varphi}_v$ the conjugate elements at $\bar{v}$) has $\mathcal{O}$-length at most β_{T_i}.

More precisely, let W be any neighborhood of g_0 in G_E. There is a bound J_W such that, for every integer $N > M_0$, there are data of the same kind with $\#J \le J_W$ and a distinguished $v_0 \in J$ satisfying all of the following:

1. *Each $\mathrm{N}v$ is a rational prime congruent to 1 modulo ℓ^N. The elements γ_v, φ_v lie in one chosen decomposition group at v, with γ_v a tame generator and φ_v arithmetic Frobenius. Write $\lambda_v = h_v$ additively. The characters λ_v are trivial on the decomposition groups at the places above ℓ.*
2. *The first slopes vanish: $s_{v_0,i} = 0$ for all i, and $\varphi_{v_0} \in W$.*
3. *For $v \in J \setminus \{v_0\}$ and $w \in J$,*

$$\lambda_w(\varphi_v) = \begin{cases} 1, & w = v_0, \\ 0, & w \ne v_0. \end{cases}$$

4. *No place in S_0 and no member of $J \setminus \{v_0\}$ lies above the rational prime $\mathrm{N}v_0$.*

5. *If $s_{v,i} \neq 0$, then*
$$\rho_i(\varphi_v) \equiv \rho_i((\varphi_v^c)^{-1}) \equiv I \pmod{\ell^{M_0}}.$$

6. *Let $\beta^{\circ}_{T_i}$ be the global term of T_i plus its local terms at the places of S_i prime to ℓ, with no first-prime term. The module with all the preceding cocycle, ramification and local relations and with the ties written as*
$$z(\gamma_v) = s_{v,i} z(\varphi_v), \qquad z((\gamma_v^c)^{-1}) = s_{v,i} z((\varphi_v^c)^{-1}) \quad (v \in J)$$
has unnormalised $\mathcal{O}$-length at most $\beta^{\circ}_{T_i}$.

For these choices the conjugate-inverse ties in the last clause are equivalent to the conjugate ties in the first conclusion. Indeed ρ_i is unramified at the auxiliary places, and for a nonzero slope both Frobenius matrices are the identity on the coefficient module; the cocycle identity $z(g^{-1}) = -\rho_i(g^{-1})z(g)$ then gives the equivalence. For a zero slope both versions simply require vanishing of the inertial value.

Proof. We use the cocycle local terms in (SF): at $v \nmid \ell$ they are the supremum of the nonnegative excess of the length of the restrictions of global cocycles over the coefficient-module length. Thus $\beta^{\circ}_{T_i}$ is exactly the sum occurring in Lemma 7.4.16. At $w \mid \ell$ the local relations are all finitely supported matrix-evaluation relations satisfied by Z_w.

First suppose the datum family is nonempty. The torsion-level inequality implies $M_0 \geq 1$. Fix a uniformizer ϖ and put $e = \mathrm{len}_{\mathcal{O}}(\mathcal{O}/\ell^{M_0})$. For $1 \leq k \leq e$, use coefficients $V_{i,k} = (\mathcal{O}/\varpi^k)^{n_i}$. All modules of continuous cocycles considered below are finite: their ramification is restricted to a finite set, the coefficient module is finite, and finiteness of global H^1 follows from Proposition 7.4.17 with unrestricted local conditions; the coboundaries also form a finite module.

Call a configuration any choice of J, v_0, the prime data, characters and slopes satisfying clauses 1–5 of the enriched conclusion, except for the length assertion, with the characters at every later Frobenius having the displayed first-prime pattern. Write $C_{i,k}(J)$ for its normalized tied cocycle module over $V_{i,k}$, using conjugate-inverse ties. Let $C_{i,k}(\emptyset)$ be the module with no auxiliary primes or ties, retaining the old ramification, local relations and vanishing at γ_*, and set
$$L = \sum_{i \in I} \sum_{k=1}^{e} \mathrm{len}_{\mathcal{O}} C_{i,k}(\emptyset), \qquad B = 1 + L.$$
These numbers depend only on the data and M_0.

Fix $N > M_0$ and a neighborhood W of g_0. A singleton configuration exists by Lemma 7.4.14. Apply it near g_0 with no prescribed auxiliary decomposition groups, and exclude all places above the rational primes below S_0. The normal-inertia hypothesis holds because g_0 is inertial at a place of S_0 prime to ℓ. Take its supplied prime as v_0 and set every first slope to zero. This gives all the required character and Frobenius clauses; the later-prime conditions are empty. No old place lies above $\mathrm{N}v_0$ by the avoidance just imposed. For every i, k, the singleton tied module equals $C_{i,k}(\emptyset)$. Indeed the zero ties at $v_0, \bar{v}_0$ kill their tame inertial values, hence all their inertia, and so restore the old unramified condition. Thus
$$\#J + \sum_{i,k} \mathrm{len}\, C_{i,k}(J) = B$$
for this singleton.

Among configurations satisfying

$$\#J + \sum_{i,k} \operatorname{len} C_{i,k}(J) \le B, \tag{1}$$

choose one minimizing the integer $\sum_{i,k} \operatorname{len} C_{i,k}(J)$. This is possible by well-ordering. We show that at this configuration every new-prime Frobenius pair of a member of $C_{i,k}(J)$ lies in the tame-pair image of the module relaxed at the new prime.

Fix i, an admissible new tuple $(P, \gamma, \phi, \lambda)$ as in Lemma 7.4.16, with $M = M_0$, and a level k. Let $Y_{i,k}$ be the module with ramification also allowed at $P, \bar{P}$ but retaining the old ties. On it define

$$\Phi_0(u) = \big(u(\gamma), u((\gamma^c)^{-1})\big), \qquad \Phi_1(u) = \big(u(\phi), u((\phi^c)^{-1})\big).$$

Then $C_{i,k}(J) = Y_{i,k} \cap \ker \Phi_0$: a zero tame pair is equivalent to unramifiedness at the two new places, because the representation is unramified there and the coefficient module is an ℓ-group.

Apply Lemma 7.4.15 at each of the e levels for this datum. At level k it excludes at most $2n_i$ residues, and for every remaining residue represented by $t \in \mathcal{O}$ gives

$$\operatorname{len}\big(Y_{i,k} \cap \ker(t\Phi_0 - \Phi_1)\big) \le \operatorname{len}\big(C_{i,k}(J) \cap \Phi_1^{-1}(\Phi_0 Y_{i,k})\big). \tag{2}$$

The torsion-level residue-field bound is $\#(\mathcal{O}/\varpi) > 2n_i e + 1$. We may therefore choose one t avoiding all these exceptional residues and also zero. It is a unit. Add P to J, give datum i the slope t^{-1} at P, and give every other datum slope zero. The new module for datum i is the kernel on the left of (2), at every level. The modules for the other data are unchanged, since their new zero ties simply enforce unramifiedness at $P, \bar{P}$.

This is again a configuration. Residual triviality for the selected datum is part of admissibility; the other new slopes are zero. The old first-prime Frobenius pattern is preserved because the new character is trivial on the decomposition groups at $J \setminus \{v_0\}$. The pattern at P and its own zero character value are part of admissibility. The avoidance clauses and the first prime are preserved as well. If some member of some $C_{i,k}(J)$ had its Frobenius pair outside $\Phi_0 Y_{i,k}$, the right side of (2) would be a proper submodule of the finite module $C_{i,k}(J)$ and its length would drop by at least one. All other lengths would not increase. The new prime increases $\#J$ by one, so (1) would still hold while its minimized sum strictly decreased, a contradiction. Consequently for every i, k and every admissible tuple,

$$\Phi_1(C_{i,k}(J)) \subseteq \Phi_0(Y_{i,k}). \tag{3}$$

We now bound all the levels for this one configuration. Fix i and induct on k from 0 to e, with the level-zero module zero. If $C_{i,k}(J)$ has a cocycle with a unit coordinate, equation (3) supplies exactly its new-prime evaluation hypothesis in Lemma 7.4.16. That lemma gives

$$\operatorname{len} C_{i,k}(J) \le \beta^{\circ}_{T_i}. \tag{4}$$

Otherwise every value of every cocycle lies in $\varpi(\mathcal{O}/\varpi^k)^{n_i}$. The map $\mathcal{O}/\varpi^{k-1} \to \varpi\mathcal{O}/\varpi^k$, $x \mapsto \varpi x$, is an isomorphism onto that submodule. Dividing all values through this map identifies $C_{i,k}(J)$ with $C_{i,k-1}(J)$. The cocycle equations, all matrix-evaluation relations, the ties, the vanishing condition and unramifiedness all pass through this identification because they are homogeneous integral linear equations. Continuity is preserved since the coefficient modules are finite and discrete. Thus (4) follows from induction in this case too.

At $k = e$, the ideal (ϖ^e) is (ℓ^{M_0}), so (4) is the required strengthened length bound. Equation (1) gives $\#J \le B$. The same bound works for every $N > M_0$ and every chosen W, and hence supplies the stated J_W.

If the datum family is empty, the singleton auxiliary-character construction alone proves the enriched conclusion, with bound 1; there are no length or residue-field conditions to check. Here the explicit hypothesis $M_0 \geq 0$ ensures that $N > M_0$ is a positive exponent.

Finally fix any $N \geq 1$. Perform the enriched construction at an integer $N' > \max\{M_0, N\}$, with $W = G_E$, and reduce its characters modulo ℓ^N. They are still one-prime characters with value 1 on the chosen tame generators; the norm congruence and all required auxiliary clauses persist. The primes, slopes and cocycle modules are unchanged. Their lengths are at most $\beta^{\circ}_{T_i} \leq \beta_{T_i}$, since the first-prime term is nonnegative. Thus one uniform bound J_0 gives the first conclusion for every N.

To compare the two displayed kinds of ties, inertia acts trivially on the coefficient module at each auxiliary prime. If the slope is nonzero, the relevant Frobenius also acts trivially. The cocycle identity $u(g^{-1}) = -\rho(g^{-1})u(g)$ therefore changes both sides of the conjugate tie by the same minus sign. For zero slope both versions say that the inertial value is zero. This proves the final equivalence and completes the descent. □

Lemma 7.4.14. *Let E be a CM field, c complex conjugation on G_E, ℓ a prime, $N \geq 1$, and S, R finite sets of finite places of E. Write $\bar{S}$ for the conjugate set and Σ_ℓ for the places above ℓ. Suppose $z \in G_E$ belongs to the closed normal subgroup generated by the inertia groups above the places of $R \setminus (S \cup \bar{S} \cup \Sigma_\ell)$, and let W be a neighborhood of z. Then there are a place P outside $S \cup \bar{S} \cup \Sigma_\ell \cup R \cup \bar{R}$, of rational-prime norm Q, a decomposition group at P, a tame generator γ in that group, an arithmetic Frobenius ϕ in the same group, and a continuous homomorphism $\lambda : G_E \to \mathbb{Z}/\ell^N$ such that*

$$P \neq \bar{P}, \qquad Q \equiv 1 \pmod{\ell^N}, \qquad \phi \in W, \qquad \lambda(\gamma) = 1, \qquad \lambda(\phi) = 0.$$

The character λ is unramified outside $\{P, \bar{P}\}$ and trivial on every decomposition group at the places of $S \cup \bar{S} \cup \Sigma_\ell$. More precisely there is a continuous homomorphism $\eta : \mathbb{A}_{E,f}^{\times}/E^{\times} \to \mathbb{Z}/\ell^N$ of conductor dividing P, trivial on each local multiplicative group at the places of $S \cup \bar{S} \cup \Sigma_\ell$, for which

$$\lambda(\mathrm{Art}_E(x)) = \eta(x/\bar{x}) \qquad (x \in \mathbb{A}_{E,f}^{\times}).$$

In particular $\lambda(c(g)) = -\lambda(g)$, and λ has exact order ℓ^N.

Proof. Put $q = \ell^N$ and $A = S \cup \bar{S} \cup \Sigma_\ell$. We first choose a finite conjugation-stable set T of places containing A, disjoint from $(R \cup \bar{R}) \setminus A$, such that the ideal class group is generated by the classes of primes in T. The class group is finite by Minkowski's theorem. For each of its finitely many classes choose a fractional ideal and multiply by a principal fractional ideal to make its valuations zero at $(R \cup \bar{R}) \setminus A$. Such a multiplier exists by weak approximation at these finitely many places. The prime factors of these representatives, their conjugates, and A give the required T.

Let $U_T = \mathcal{O}_{E,T}^{\times}$. Its valuation map into $\mathbb{Z}^T$ has kernel $\mathcal{O}_E^{\times}$. Dirichlet's theorem and finite generation of subgroups of $\mathbb{Z}^T$ imply that U_T has finitely many generators $u_1, \ldots, u_d$. Choose qth roots of these generators and let L/E be the splitting field of their binomials together with μ_q. This is a finite Galois extension unramified outside T: at a place outside T, the residue characteristic is prime to q and each u_j is a unit. Over the maximal unramified extension its residue has a qth root and Hensel's lemma lifts that root, since q is a unit. The roots of unity are unramified there for the same reason. Consequently the inertia groups used to generate the normal subgroup in the hypothesis act trivially on L. Their closed normal closure does too, and z fixes L.

Choose an open normal subgroup $K \subseteq G_E$ with $zK \subseteq W$, and replace it by $K \cap G_L$. Thus zK is a nonempty open coset contained in $W \cap G_L$. Apply Chebotarev to the conjugacy class of the image of z in the finite quotient G_E/K. After choosing a place above the resulting prime, its Frobenius has exactly that image, rather than just a conjugate image. We may exclude any prescribed finite set of primes, in particular $T \cup R \cup \bar{R}$, the primes ramified in the field fixed by K, and the primes above rational primes ramified in $E/\mathbb{Q}$. We may also require the norm to be a rational prime: primes of residue degree at least two over $\mathbb{Q}$ have Dirichlet density zero, since their contribution is bounded by $[E : \mathbb{Q}] \sum_p p^{-2s}$ as $s \downarrow 1$, whereas the Chebotarev class has positive density. We obtain P with prime norm Q, and an arithmetic Frobenius σ such that every Frobenius in the chosen decomposition group lies in zK. Indeed their quotients lie in inertia, which maps trivially to G_E/K. The image of σ on μ_q is trivial, so $Q \equiv 1 \pmod{q}$.

We have $P \neq \bar{P}$. Otherwise the rational prime below P, which is unramified in E, would be inert in the quadratic CM extension E/E^+ at that place; its residue degree in $E/\mathbb{Q}$ would be even, contrary to $\mathrm{N}P = Q$. Since T is conjugation-stable, both P and $\bar{P}$ lie outside T.

Choose a tame generator γ at the chosen place above P. By local reciprocity its image in the tame abelian inertia quotient corresponds to a generator $\bar{u}$ of the cyclic group $\mathbb{F}_Q^\times$. More explicitly tame inertia is procyclic, Frobenius acts on it by the Qth power, and its maximal abelian quotient inside the local Galois group is its quotient by $Q - 1$, identified by reciprocity with $\mathbb{F}_Q^\times$. Fix a uniformizer π of E_P and let

$$f : E_P^\times \longrightarrow \mathbb{Z}/q$$

be the homomorphism which takes the residue of $t\pi^{-\operatorname{ord}_P(t)}$ to its discrete logarithm to base $\bar{u}$, reduced modulo q. It is continuous, kills π and $1 + \mathfrak{m}_P$, If $u \in \mathcal{O}_{E_P}^\times$ lifts $\bar{u}$, then the local Artin image of u and γ have the same image in tame abelian inertia. Thus this discrete logarithm is normalized to take the value 1 on that image.

Each u_j is a qth power modulo P. In fact σ fixes its chosen qth root in L; reduction of that root at the chosen place above P is therefore fixed by the Qth power map, hence belongs to $\mathbb{F}_Q^\times$. Thus $f(u_j) = 0$, and f kills every T-unit.

Let I_T be the subgroup of finite ideles that are units outside T. Define $F : I_T \to \mathbb{Z}/q$ by $F(x) = f(x_P)$. Since the classes of the primes in T generate the class group, every finite idele belongs to $E^\times I_T$. The intersection $E^\times \cap I_T$ is exactly U_T, which F kills. Therefore the formula $\eta(ax) = F(x)$ for $a \in E^\times$, $x \in I_T$ is well-defined on all finite ideles, kills $E^\times$, and is a homomorphism. It is continuous because I_T is open and F is continuous. At a place in T it kills the entire local multiplicative group. At every place other than P it kills local units, and at P it kills $1 + \mathfrak{m}_P$. Thus its conductor divides P.

For completeness this idelic character is a character of the usual ray class group modulo P: the quotient of finite ideles by $E^\times$, all local units away from P, and $1 + \mathfrak{m}_P$ at P, is the ideal ray class group. To see the identification, use weak approximation to make the P-component congruent to 1, send the other local uniformizers to their fractional ideal, and observe that changing the multiplier changes that ideal by a principal ideal generated by an element congruent to 1 at P. This also proves that the kernel is exactly the indicated ray subgroup. The ray class field theorem therefore supplies a continuous Galois character $\mu : G_E \to \mathbb{Z}/q$ with $\mu(\mathrm{Art}_E(x)) = \eta(x)$.

Set $\lambda(g) = \mu(g) - \mu(c(g))$. Naturality of the Artin map under conjugation gives the displayed formula $\lambda(\mathrm{Art}_E(x)) = \eta(x/\bar{x})$. That naturality follows already on prime ideals from conjugation of arithmetic Frobenius, and hence on the finite ray class quotients by their Artin surjectivity. Because η is trivial on the local components in A, and $A = \bar{A}$, local reciprocity shows that λ is trivial on their decomposition groups. The same local compatibility shows that it is unramified

outside $P, \bar{P}$. At γ, the first term $\mu(\gamma)$ is 1, by the normalization of f, and $\mu(c(\gamma)) = 0$, since μ is unramified at $\bar{P}$. Thus $\lambda(\gamma) = 1$.

Finally choose an integer a representing $-\lambda(\sigma) \in \mathbb{Z}/q$ and put $\phi = \sigma\gamma^a$. This is still arithmetic Frobenius in the same decomposition group. Every such Frobenius lies in $zK \subseteq W$, and $\lambda(\phi) = \lambda(\sigma) + a = 0$. Surjectivity, exact order q, and $\lambda(c(g)) = -\lambda(g)$ now follow directly from the construction. □

Lemma 7.4.15. *Let $\mathcal{O}$ be a discrete valuation ring with residue field k, let $0 \neq a \in \mathcal{O}$, let $m \geq 0$, and let B be an $\mathcal{O}$-module. Given $\mathcal{O}$-linear maps*

$$\Phi_0, \Phi_1 : B \longrightarrow (\mathcal{O}/a)^m$$

and a submodule $Y \subseteq B$, there is a set $\mathcal{E} \subseteq k$ with $\#\mathcal{E} \leq m$ such that, for every $s \in \mathcal{O}$ whose residue is outside $\mathcal{E}$,

$$\mathrm{len}_{\mathcal{O}}\big(Y \cap \ker(s\Phi_0 - \Phi_1)\big) \leq \mathrm{len}_{\mathcal{O}}\big(Y \cap \ker \Phi_0 \cap \Phi_1^{-1}(\Phi_0(Y))\big).$$

Lengths are allowed to be infinite.

Proof. Write $N = (\mathcal{O}/a)^m$, and form the submodule

$$Z = \sum_{s \in \mathcal{O}} \big(Y \cap \ker(s\Phi_0 - \Phi_1)\big) \subseteq Y.$$

On the summand indexed by s one has $\Phi_1(z) = \Phi_0(sz)$. By additivity, therefore,

$$\Phi_1(Z) \subseteq \Phi_0(Z).$$

Let $\pi : \mathcal{O}^m \to N$ be reduction, and let $\mathcal{M} = \pi^{-1}(\Phi_0(Z))$. It is a submodule of a finite free module over the principal ideal domain $\mathcal{O}$, hence finite free of some rank $h \leq m$. Reduction gives a surjection $p : \mathcal{M} \to \Phi_0(Z)$.

Choose a basis of $\mathcal{M}$ and lift the images of its basis vectors through the surjection $\Phi_0 : Z \to \Phi_0(Z)$. This gives an $\mathcal{O}$-linear map $\sigma : \mathcal{M} \to Z$ with $\Phi_0\sigma = p$. Since $\Phi_1\sigma$ takes values in $\Phi_0(Z)$, lift its values on the same basis through p to obtain an endomorphism U of $\mathcal{M}$ satisfying

$$pU = \Phi_1\sigma.$$

Let $\mathcal{E}$ be the roots in k of the reduction of the characteristic polynomial of U. That polynomial is monic of degree h, so $\#\mathcal{E} \leq h \leq m$; if $h = 0$ it is 1 and the set is empty.

If the residue of s avoids $\mathcal{E}$, the determinant of $sI - U$ is a unit. Thus $sI - U$ is an automorphism of $\mathcal{M}$, by the adjugate formula, and

$$(s\Phi_0 - \Phi_1)\sigma = p(sI - U).$$

Consequently the image of $s\Phi_0 - \Phi_1$ on Z contains $p(\mathcal{M}) = \Phi_0(Z)$. The reverse inclusion follows from $\Phi_1(Z) \subseteq \Phi_0(Z)$, so the two images are equal.

Both images have finite length, as submodules of N. The two exact sequences on Z with maps Φ_0 and $s\Phi_0 - \Phi_1$ therefore show that their kernels have the same length. If Z has finite length this follows by ordinary length additivity and subtraction of the common finite image length. If Z has infinite length, both kernels have infinite length, since a module with a finite-length submodule and finite-length quotient has finite length. This proves the assertion about kernel lengths also in that case.

By definition of Z, the whole module $Y \cap \ker(s\Phi_0 - \Phi_1)$ lies in Z. Hence it equals $Z \cap \ker(s\Phi_0 - \Phi_1)$, whose length equals that of $Z \cap \ker \Phi_0$. Finally

$$Z \cap \ker \Phi_0 \subseteq Y \cap \ker \Phi_0 \cap \Phi_1^{-1}(\Phi_0(Y)),$$

because $Z \subseteq Y$ and $\Phi_1(Z) \subseteq \Phi_0(Z) \subseteq \Phi_0(Y)$. Monotonicity of length proves the claimed inequality. □

Lemma 7.4.16. *Let $T = (\rho, S, (Z_w), \gamma_*, \zeta_*)$ be a Selmer datum over $\mathcal{O}$ of rank n, with $S \subseteq S_0$. Let $1 \leq M < N$, let $0 \neq a \in \mathcal{O}$ divide ℓ^M, and put $V = (\mathcal{O}/a)^n$. In this statement B_{gl} and B_v denote the global term and the local terms of T already defined in (SF) of Definition 7.4.1; its restriction modules are denoted there by $C_v(b)$. These terms are finite by the Selmer-datum requirement.*

Let J be auxiliary for S_0, with $v_0 \in J$. For each $v \in J$, choose in one decomposition group a tame generator γ_v, an arithmetic Frobenius ϕ_v, a slope $s_v \in \mathcal{O}$, and an additive one-prime character $\lambda_v : G_E \to \mathbb{Z}/\ell^N$ with $\lambda_v(\gamma_v) = 1$, trivial on the decomposition groups above ℓ. Suppose $\mathrm{N}v$ is a rational prime congruent to 1 modulo ℓ^N, $s_{v_0} = 0$, and

$$s_v \neq 0 \implies \rho(\phi_v) \equiv \rho((\phi_v^c)^{-1}) \equiv I \pmod{\ell^M}.$$

At a place above ℓ, impose every finitely supported matrix-evaluation relation $\sum_g A_g u(g) = 0$ that holds for every $u \in Z_w$. For this statement only, let C be the module of continuous global cocycles into V, unramified outside $S \cup J \cup \bar{J}$, satisfying these local relations, vanishing at γ_*, and satisfying

$$u(\gamma_v) = s_v u(\phi_v), \qquad u((\gamma_v^c)^{-1}) = s_v u((\phi_v^c)^{-1}) \qquad (v \in J).$$

For a new place P, let C_P, also local to this statement, be the same module with ramification also allowed at $P, \bar{P}$, retaining only the ties at J.

Suppose there is $z \in C$ with a unit coordinate at some element of G_E, having the following property. For every tuple $(P, \gamma, \phi, \lambda)$ satisfying all of these conditions:

- *P has rational-prime norm congruent to 1 modulo ℓ^N, lies outside $S_0 \cup \bar{S}_0 \cup J \cup \bar{J}$, and does not lie above $\mathrm{N}v_0$;*
- *γ, ϕ are a tame generator and arithmetic Frobenius in one decomposition group at P;*
- *$\lambda : G_E \to \mathbb{Z}/\ell^N$ is a one-prime character of modulus $\{P\}$, trivial on the decomposition groups above ℓ and at $J \setminus \{v_0\}$, with $\lambda(\gamma) = 1$ and $\lambda(\phi) = 0$;*
- *for every $w \in J$, $\lambda_w(\phi) = 1$ if $w = v_0$ and 0 otherwise, and $\rho(\phi) \equiv \rho((\phi^c)^{-1}) \equiv I \pmod{\ell^M}$;*

there is $z_P \in C_P$ with

$$z_P(\gamma) = z(\phi), \qquad z_P((\gamma^c)^{-1}) = z((\phi^c)^{-1}).$$

Then

$$\mathrm{len}_{\mathcal{O}} C \leq B_{\mathrm{gl}} + \sum_{\substack{v \in S \\ v \nmid \ell}} B_v.$$

Proof. We first check the finite-level local conditions. For a finitely generated rank-d module Z of integral vector-valued functions, regard its elements as scalar functions on the set I of pairs consisting of an argument and a vector coordinate. Put

$$L = (\mathrm{Frac}(\mathcal{O})Z) \cap \mathcal{O}^I.$$

Some d coordinates are injective on the fraction-field span of Z. Their nonzero determinant bounds the denominators of every element of L in a basis of Z. Thus $L \subseteq \varpi^{-b}Z$ for some b, and L is finite free of rank d. A basis of L stays linearly independent modulo ϖ as a family of functions. Otherwise a primitive linear combination of that basis would be divisible coordinatewise by ϖ, and its quotient by ϖ would belong to L, a contradiction. Consequently some d coordinates give a unit determinant on this basis. Projection on them identifies L with $\mathcal{O}^d$; every other coordinate is an integral linear combination of these selected coordinates.

Every finite-coordinate linear relation on Z also holds on L, since they have the same fraction-field span. Conversely the graph equations just obtained are such relations. Over $\mathcal{O}/a$, therefore, the solution module of all the relations is the image of L, isomorphic to $(\mathcal{O}/a)^d$. Scalar coordinate relations are equivalent to matrix-evaluation relations: place any scalar relation in a fixed row of the output vector. It follows that the local continuous cocycles satisfying the prescribed relations at $w \mid \ell$ have length at most

$$(\mathrm{rk}_{\mathcal{O}} Z_w)\,\mathrm{len}_{\mathcal{O}}(\mathcal{O}/a). \tag{1}$$

They contain all local coboundaries, because each finite-level vector lifts to an integral vector and its integral coboundary lies in Z_w. This proves the needed bound without identifying this solution module with the reduction of Z_w.

The inputs from (SD) used below are the following: ρ is unramified outside S; its image spans $M_n(\mathcal{O})$; each Z_w is finitely generated and contains integral coboundaries; its ranks have the stated total bound; and γ_* is inertial at a place of S prime to ℓ, with $\rho(\gamma_*) = \zeta_* I$, $\rho(\gamma_*^c) = \zeta_*^{-1} I$ and $\zeta_*^2 - 1$ a unit. In particular both $\zeta_* - 1$ and $\zeta_* + 1$ are units, and these properties survive reduction modulo a. Auxiliary places and their conjugates are outside S_0, so inertia acts trivially on V there. We now apply global duality with local conditions containing coboundaries. At a place $v \in S$ prime to ℓ, take $C_v(a)$. At $w \mid \ell$, take the continuous local cocycles satisfying the relations just considered. At a place in $J \cup \bar{J}$, take all local cocycles satisfying its displayed tie, using γ_v, ϕ_v at v and $(\gamma_v^c)^{-1}, (\phi_v^c)^{-1}$ at $\bar{v}$. These local conditions contain coboundaries. For a zero slope they are unramified conditions. For a nonzero slope, inertia acts trivially and the relevant Frobenius acts as I on V, so both evaluations of a coboundary are zero.

At an auxiliary place a tied local cocycle is determined by its Frobenius value. Indeed a zero Frobenius value forces a zero value at the tame generator. Wild inertia has no nonzero additive homomorphism to the finite ℓ-group V, and the tame generator detects every such homomorphism on tame inertia. The cocycle then vanishes on inertia and on Frobenius, hence everywhere by continuity. Thus each auxiliary local condition has length at most $\mathrm{len}_{\mathcal{O}} V$. At the first prime, where the slope is zero, it is exactly the unramified condition.

Let $\mathcal{L}_v \subseteq H^1(E_v, V)$ be the images of these local cocycle conditions, and let $\mathcal{D}$ be the ordinary dual Selmer group in $H^1(G_{E,S\cup J\cup\bar{J}}, V^\vee)$ for their local Tate annihilators, where $V^\vee = \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Q}/\mathbb{Z})(1)$. The module C is isomorphic to $H^1_{\mathcal{L}}(V)$. To see this, every cohomology class has a unique representative vanishing at γ_*, since $\rho(\gamma_*) - I = (\zeta_* - 1)I$ is invertible on V. Adding a coboundary preserves every local condition above. Conversely their membership says exactly that the normalized representative satisfies the local relations and ties defining C. The conditions at old primes away from ℓ impose no further restriction on a global cocycle, by the definition of $C_v(a)$.

Both global invariant groups $H^0(V)$ and $H^0(V^\vee)$ are zero. The inertial element γ_* acts by ζ_* on the first and by ζ_*^{-1} on the second, since the cyclotomic character is trivial on inertia away from ℓ. Both scalars differ from 1 by units. If Z_v denotes a local cocycle condition, then

$$\operatorname{len}\mathcal{L}_v - \operatorname{len}H^0(E_v,V) = \operatorname{len}Z_v - \operatorname{len}V, \tag{2}$$

because the local coboundaries have length $\operatorname{len}V - \operatorname{len}H^0(E_v,V)$. The formula of Proposition 7.4.17, converted from orders to $\mathcal{O}$-lengths, therefore gives

$$\operatorname{len}C = \operatorname{len}\mathcal{D} + \sum_{v\in S\cup J\cup\bar{J}} (\operatorname{len}Z_v - \operatorname{len}V) - f_E \operatorname{len}V. \tag{3}$$

Here every complex place contributes $\operatorname{len}V$. The formula applies at every ℓ, including 2, because E has no real places. Using (1), the rank bound in (SD), the auxiliary-place bound, and the definition of B_v, equation (3) implies

$$\operatorname{len}C \le \operatorname{len}\mathcal{D} + \sum_{\substack{v\in S\\ v\nmid\ell}} B_v. \tag{4}$$

All the groups used here are finite by the same global and local duality theorems.

It remains to bound $\mathcal{D}$. Fix a compatible system of ℓ-power roots of unity to write the underlying additive dual as $D = \operatorname{Hom}_{\mathbb{Z}}(V,\mathbb{Q}/\mathbb{Z})$; its action is

$$(gd)(v) = d(\chi_{\mathrm{cyc}}(g)\rho(g)^{-1}v).$$

Normalize each dual cocycle by requiring its value at γ_* to be zero; this representative is unique because $\zeta_*^{-1}-1$ is a unit. Let H be the subgroup generated by $[g,\gamma_*]$ and $[g,\gamma_*^c]$. By Lemma 7.4.18, it is normal, conjugation-stable, killed by ρ and every abelian character, and $z(H)$ spans V. Restrict the normalized dual cocycles to H and call the image Y. These are additive homomorphisms with

$$y(ghg^{-1})(v) = y(h)\big(\chi_{\mathrm{cyc}}(g)\rho(g)^{-1}v\big).$$

Restriction is injective. Indeed, if the normalized dual cocycle is zero on H, expanding it at $[g,\gamma_*]$ gives $(1-\zeta_*^{-1})y(g)=0$ for every g, so it is zero. Hence $\operatorname{len}Y = \operatorname{len}\mathcal{D}$.

We prove diagonal mirror isotropy on H. Fix $y\in Y$, its normalized global representative, $h\in H$, and $r\in\mathcal{O}$. Put $\tau_0=\gamma_{v_0}$. The dual class is unramified at $v_0,\bar{v}_0$: its local condition is the annihilator of the unramified condition, which is again unramified by local Tate duality [201, Chapter I, Theorem 2.6]. Inertia acts trivially there on both modules, so this means

$$y(\tau_0) = y(\tau_0^c) = 0.$$

Likewise $z(\tau_0)=z(\tau_0^c)=0$ by the zero first slope. Therefore the diagonal expression

$$y(g)(rz(g)) - y(g^c)(rz(g^c)) \tag{5}$$

has the same value at $g=h$ and $g=\tau_0 h$.

Apply Lemma 7.4.14 with its forbidden set $S_{\mathrm{aux}} = J\setminus\{v_0\}$. Choose its set R to contain $S_0\cup\bar{S}_0\cup J\cup\bar{J}$ and all places above $\mathrm{N}v_0$. The forbidden decomposition groups are therefore those above $S_{\mathrm{aux}}\cup\bar{S}_{\mathrm{aux}}\cup\Sigma_\ell$. The normal-inertia hypothesis is satisfied by $\tau_0 h$: the old place containing γ_* and its conjugate belong to $S\cup\bar{S}$, are prime to ℓ, and lie outside $J\cup\bar{J}$ by (AX2). Each generating commutator of H lies in the normal subgroup generated by those two inertia

groups. Also v_0 belongs to R and lies outside $S_{\mathrm{aux}} \cup \bar{S}_{\mathrm{aux}} \cup \Sigma_\ell$, so τ_0 belongs to the permitted inertia. Thus $\tau_0 h$ belongs to the required closed normal subgroup generated by inertia above $R \setminus (S_{\mathrm{aux}} \cup \bar{S}_{\mathrm{aux}} \cup \Sigma_\ell)$. Enlarging R only enlarges this subgroup. The supplied prime is outside $R \cup \bar{R}$, so it meets all the required exclusions; the supplied character has modulus $\{P\}$ and is trivial on the required decomposition groups.

Choose the neighborhood of $\tau_0 h$ so that (5) is constant, the two representation matrices are I modulo ℓ^M, and the old characters take the required values. These conditions hold at $\tau_0 h$: ρ kills both τ_0 and H; conjugation preserves H and ρ also kills τ_0^c; every λ_w kills H; and $\lambda_w(\tau_0) = 1$ for $w = v_0$ and 0 otherwise. The last assertion follows from the normalization at v_0 and unramifiedness at the other disjoint auxiliary primes. The finite-valued cocycles and characters are locally constant, so this is indeed a neighborhood. The supplied tuple $(P, \gamma, \phi, \lambda)$ is admissible in the statement. In particular (5) at ϕ equals (5) at h.

Choose z_P using the hypothesis. Since $\rho(\gamma^c) = I$ and $\rho(\phi^c) = I$ on V, the cocycle identity for inverses converts the second equality into $z_P(\gamma^c) = z(\phi^c)$. Global cup reciprocity for rz_P and y says that the sum of their local cup invariants is zero [201, Chapter I, Theorem 4.10]. At the old places it is zero by the dual local conditions. Away from the old places and $P, \bar{P}$, both classes are unramified, so local unramified orthogonality makes it zero. Only the two new places remain.

We spell out their common normalization. Put $q = \ell^M$. At P the local action on V and its Cartier dual is trivial, and μ_q is contained in E_P. Scalar homomorphisms $G_{E_P} \to \mathbb{Z}/q$ have independent coordinates given by evaluation on γ, ϕ: wild inertia contributes nothing and the tame relation is $\phi\gamma\phi^{-1} = \gamma^{\mathrm{N}P}$, with $\mathrm{N}P \equiv 1 \pmod q$. Thus this scalar cohomology is $(\mathbb{Z}/q)^2$. Local Tate perfectness and unramified orthogonality [201, Chapter I, Corollary 2.3 and Theorem 2.6] show that pairing a tame-coordinate generator with the unramified line is multiplication by a unit $u_P \in (\mathbb{Z}/q)^\times$. Perfectness is essential here: it makes this multiplication invertible, rather than merely nonzero.

For an arbitrary finite $\mathbb{Z}/q$-module V, naturality under the map $\mathbb{Z}/q \to V$ taking 1 to v gives the same formula with $u_P d(v)$, where d is the dual Frobenius value. Indeed a cocycle with tame value v differs from the image of the scalar tame-coordinate generator by an unramified cocycle, whose pairing with an unramified dual class is zero. This argument does not require V to be free over $\mathbb{Z}/q$. At $\bar{P}$ use the transported generators γ^c, ϕ^c. To compare coefficients, transport the scalar tame generator and scalar unramified character from P to $\bar{P}$ by c. Their evaluation coordinates remain $(1, 0)$ and $(0, 1)$ on γ^c, ϕ^c. The invariant map on the Brauer group is preserved by this field isomorphism. However the identification of the Cartier dual scalar module with $\mathbb{Z}/q$ uses a fixed primitive root ξ_q, and its transported identification uses $c(\xi_q) = \xi_q^{-1}$. Expressing that transported dual generator in the original fixed-root identification multiplies it by -1. By bilinearity and naturality, the cup coefficient at $\bar{P}$ in that original identification is therefore exactly $-u_P$, with no change of the invariant map itself. The two new local contributions consequently give

$$u_P\big(y(\phi)(rz_P(\gamma)) - y(\phi^c)(rz_P(\gamma^c))\big) = 0.$$

Cancel the unit and substitute the two prescribed values. Expression (5) at ϕ, and therefore at h, is zero. This proves the diagonal isotropy for every h, r, y; the argument includes $\ell = 2$, when the displayed negative may equal the positive at exponent 2.

All the hypotheses of Lemma 7.4.19 now hold for Y, H, z and the scalar inertial element. It gives an injection

$$Y \hookrightarrow \mathrm{Hom}_{\mathcal{O}, G_E}(V, V^{*c}).$$

Thus $\operatorname{len} \mathcal{D} = \operatorname{len} Y \le B_{\mathrm{gl}}$. Combining this with (4) proves the assertion. □

Proposition 7.4.17. *Let F be a number field and ℓ a prime; assume that ℓ is odd or that F has no real place. Let S be a finite set of places of F containing the places above ℓ and the archimedean places, $S_f \subseteq S$ its finite places, $r \geq 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with a continuous action of $G_{F,S}$. For $v \in S_f$ let*

$$\langle\ ,\ \rangle_v \colon H^1(F_v, M) \times H^1(F_v, M^\vee) \to H^2(F_v, \mu_{\ell^r}) \xrightarrow{\mathrm{inv}_v} \mathbb{Q}/\mathbb{Z}$$

*be the cup product pairing. A **family of local conditions** for M is a family $\mathcal{L} = (L_v)_{v \in S_f}$ of subgroups $L_v \subseteq H^1(F_v, M)$; its **dual family** is $\mathcal{L}^\perp = (L_v^\perp)_{v \in S_f}$, $L_v^\perp \subseteq H^1(F_v, M^\vee)$ the annihilator of L_v under $\langle\ ,\ \rangle_v$; the **Selmer groups** are*

$$H^1_{\mathcal{L}}(M) = \{x \in H^1(G_{F,S}, M) :\ \mathrm{res}_v(x) \in L_v \text{ for all } v \in S_f\},$$
$$H^1_{\mathcal{L}^\perp}(M^\vee) = \{y \in H^1(G_{F,S}, M^\vee) :\ \mathrm{res}_v(y) \in L_v^\perp \text{ for all } v \in S_f\}.$$

Then $H^1_{\mathcal{L}}(M)$ and $H^1_{\mathcal{L}^\perp}(M^\vee)$ are finite and

$$\#H^1_{\mathcal{L}}(M) \cdot \#H^0(G_{F,S}, M^\vee) \cdot \prod_{v \in S} \#H^0(F_v, M) = \#H^1_{\mathcal{L}^\perp}(M^\vee) \cdot \#H^0(G_{F,S}, M) \cdot \prod_{v \in S_f} \#L_v,$$

where the product on the left runs over all places of S, the archimedean ones included ($H^0(F_v, M)$ is the subgroup fixed by a complex conjugation at a real place v, and M at a complex place).

Proof. We extend the prescribed conditions to all places, identify the two Selmer groups, and then specialize the classical quotient formula.

First, inflate M from $G_{F,S}$ to G_F. Since S contains every place above ℓ, the roots of unity μ_{ℓ^r} are unramified outside S: locally at residue characteristic different from ℓ, roots of unity of ℓ-power order lie in an unramified extension, by lifting their simple residue-field roots. Thus $M^\vee$ also factors through $G_{F,S}$, and it is the Cartier dual in the classical theorem with $n = \ell^r$. In particular its global invariants, and those of M, agree for G_F and $G_{F,S}$.

At finite $v \in S$ retain the given L_v. At finite $v \notin S$ set

$$\widetilde{L}_v = H^1_{\mathrm{ur}}(F_v, M) = H^1(G_{F_v}/I_v, M).$$

Here inertia acts trivially on both modules. The residue characteristic is different from ℓ, so Theorem A.3.1(3) identifies $\widetilde{L}_v^\perp$ with the unramified subgroup for $M^\vee$. At each infinite place set $\widetilde{L}_v = 0$. Under the hypothesis in the statement, $H^1(F_v, M)$ and $H^1(F_v, M^\vee)$ are both zero: at a complex place the Galois group is trivial, and at a real place its order is two and is invertible on these odd-primary modules. Averaging over that finite group contracts positive-degree cohomology. Thus zero is the whole local group for each module at infinity, and the two infinite local conditions are exact annihilators. In particular this argument includes $\ell = 2$ when F has no real place, without imposing an odd-prime hypothesis there.

The all-place Selmer group for $\widetilde{L}$ is exactly the inflation of the displayed group $H^1_{\mathcal{L}}(M)$. Here is the descent check. Put $N = \ker(G_F \to G_{F,S})$; it acts trivially on M. The restriction to an inertia subgroup outside S of a global cocycle representing an unramified local class is zero as a function: coboundaries on that inertia subgroup vanish because its action is trivial. The cocycle is consequently zero on every conjugate of such an inertia subgroup, since its action factors through $G_{F,S}$. These inertia subgroups topologically normally generate N, by the definition of the maximal extension unramified outside S. The cocycle restricted to N is a continuous homomorphism to the finite group M, and therefore vanishes on N. It descends to a continuous cocycle of $G_{F,S}$. Inflation is injective on cohomology: if an inflated cocycle is a coboundary

on G_F, the same coboundary formula already holds on its quotient. Conversely every quotient cocycle is unramified outside S. This proves the asserted equality of groups, with the local conditions inside S unchanged. The identical argument applies to the dual module and its annihilator conditions.

For $v \notin S$ the unramified cohomology is the cokernel of $\mathrm{Fr}_v - 1$ on the finite group M, while $H^0(F_v, M)$ is its kernel. Indeed continuous cocycles of the procyclic unramified quotient are determined by their value on Frobenius, modulo these coboundaries; every value gives a continuous cocycle, by taking a multiple of the order of the action and of the exponent of M. An endomorphism of a finite group has equally sized kernel and cokernel. Hence

$$\#\widetilde{L}_v = \#H^0(F_v, M) \qquad (v \notin S).$$

All corresponding factors in the classical product are one. At infinity $\#\widetilde{L}_v = 1$, whereas the denominator is the ordinary invariant group, not a Tate quotient: it is M^{c_v} at a real place and M at a complex place.

The classical Greenberg–Wiles theorem, [63], Theorem 2.19, now gives finiteness of both identified Selmer groups and

$$\frac{\#H^1_{\mathcal{L}}(M)}{\#H^1_{\mathcal{L}^\perp}(M^\vee)} = \frac{\#H^0(G_{F,S}, M)}{\#H^0(G_{F,S}, M^\vee)} \frac{\prod_{v \in S_f} \#L_v}{\prod_{v \in S} \#H^0(F_v, M)}.$$

Every denominator is the positive order of a finite group, so multiplying by them gives exactly the asserted identity. □

Lemma 7.4.18. *Let R be a commutative ring, $n \geq 1$, G a group with an involutive automorphism c, and $\rho : G \to \mathrm{GL}_n(R)$ a representation whose image spans $\mathrm{Mat}_n(R)$ over R. Suppose $\gamma \in G$ and $\zeta \in R^\times$ satisfy*

$$\rho(\gamma) = \zeta I, \qquad \rho(c(\gamma)) = \zeta^{-1} I, \qquad \zeta - 1 \in R^\times.$$

Let H be the subgroup generated by $[g, \gamma]$ and $[g, c(\gamma)]$ for $g \in G$, where $[u, v] = uvu^{-1}v^{-1}$. Then H is normal, $c(H) = H$, and $\rho(H) = \{I\}$. Every homomorphism from G to an abelian group is trivial on H. If $z : G \to R^n$ is a cocycle for ρ with $z(\gamma) = 0$ and some coordinate of some $z(g)$ is a unit, then

$$\mathrm{span}_R\, z(H) = R^n.$$

Proof. For $\delta \in \{\gamma, c(\gamma)\}$ and $a, g \in G$, direct multiplication gives

$$a[g, \delta]a^{-1} = [ag, \delta][a, \delta]^{-1}.$$

Thus conjugation by every a preserves the subgroup generated by these commutators; applying the same assertion to a^{-1} proves normality. Applying c interchanges the two families of generators, so $c(H) = H$. Both $\rho(\gamma)$ and $\rho(c(\gamma))$ are scalar matrices, hence the images of all the generating commutators are I. Any homomorphism to an abelian group also kills each generating commutator.

Put $X = \mathrm{span}_R\, z(H)$. Since $\rho(H) = \{I\}$, the cocycle identity and $z(g^{-1}) = -\rho(g)^{-1} z(g)$ give

$$z(aha^{-1}) = \rho(a) z(h) \qquad (a \in G,\ h \in H).$$

Normality therefore makes X stable under $\rho(G)$, and hence under every matrix in $\mathrm{Mat}_n(R)$ by the spanning hypothesis. Also $z(\gamma) = z(\gamma^{-1}) = 0$, and expanding the cocycle at the commutator yields

$$z([g,\gamma]) = z(g) + \rho(g)\rho(\gamma)z(g^{-1}) = (1-\zeta)z(g).$$

Choose g and a coordinate j for which $z(g)_j$ is a unit. Since $1-\zeta$ is a unit, the vector $u = z([g,\gamma]) \in X$ has a unit jth coordinate. For every coordinate i, stability under the matrix unit E_{ij} gives $u_j e_i \in X$. Multiplication by u_j^{-1} gives $e_i \in X$. All standard basis vectors belong to X, which proves the asserted equality. $\square$

Lemma 7.4.19. *Let $\mathcal{O}$ be a commutative ring, V an $\mathcal{O}$-module, G a group, c an involutive automorphism of G, and H a normal subgroup with $c(H) = H$. Let $\rho : G \to \mathrm{Aut}_{\mathcal{O}}(V)$ and $\chi : G \to \mathcal{O}^{\times}$ be homomorphisms, both trivial on H, and let $z : G \to V$ be a cocycle for ρ. Suppose that $z(H)$ generates V over $\mathcal{O}$. Put $D = \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Q}/\mathbb{Z})$, with $(r\lambda)(v) = \lambda(rv)$. Let Y be an $\mathcal{O}$-submodule of the additive homomorphisms $H \to D$ such that, for every $y \in Y$, $g \in G$, $h \in H$, $v \in V$,*

$$y(ghg^{-1})(v) = y(h)\big(\chi(g)\rho(g)^{-1}v\big).$$

Assume that there are $\gamma \in G$ and $\zeta \in \mathcal{O}^{\times}$ with

$$\rho(\gamma) = \zeta I, \qquad \rho(c(\gamma)) = \zeta^{-1} I, \qquad \chi(\gamma) = \chi(c(\gamma)) = 1, \qquad \zeta^2 - 1 \in \mathcal{O}^{\times},$$

and that for every $y \in Y$, $h \in H$, $r \in \mathcal{O}$,

$$y(h)(rz(h)) = y(c(h))(rz(c(h))).$$

Then for each $y \in Y$ there is a unique $\mathcal{O}$-linear map $A_y : V \to D$ with

$$A_y(z(h)) = y(c(h)) \qquad (h \in H).$$

The assignment $y \mapsto A_y$ is an injective $\mathcal{O}$-linear map into the module of $\mathcal{O}$-linear maps $A : V \to D$ satisfying

$$A(\rho(g)v)(w) = A(v)\big(\chi(c(g))\rho(c(g))^{-1}w\big) \qquad (g \in G,\ v, w \in V).$$

In particular, if $\mathcal{O}$ is a discrete valuation ring, the length of Y is at most the length of this latter module.

Proof. Fix $y \in Y$. For $r \in \mathcal{O}$ and $h, k \in H$ write

$$F_r(h,k) = y(k)(rz(h)) - y(c(h))(rz(c(k))) \in \mathbb{Q}/\mathbb{Z}.$$

Because ρ is trivial on H, the restriction of z to H is additive. Thus F_r is additive in each group argument, and additive in r. The isotropy hypothesis is $F_r(h,h) = 0$. Expanding at hk gives

$$F_r(h,k) + F_r(k,h) = 0. \tag{1}$$

For $h \in H$ the cocycle identity gives $z(ghg^{-1}) = \rho(g)z(h)$: in its expansion the terms involving $z(g)$ cancel because $\rho(h) = I$. Using this identity and the covariance of y, the scalar hypotheses give

$$F_r(\gamma h \gamma^{-1}, k) = F_{\zeta r}(h,k), \qquad F_{\zeta r}(k, \gamma h \gamma^{-1}) = F_r(k,h).$$

Indeed in the first equality both $z(h)$ and the functional $y(c(h))$ acquire a factor ζ; in the second, both occurrences involving h acquire a factor ζ^{-1}, which cancels the displayed factor ζ. Apply

(1) with parameter ζr and arguments $\gamma h \gamma^{-1}, k$. Subtract (1) with parameter r and arguments h, k. It follows that

$$F_{(\zeta^2-1)r}(h,k) = 0.$$

Since $\zeta^2 - 1$ is a unit, we have

$$y(k)(rz(h)) = y(c(h))(rz(c(k))) \qquad (r \in \mathcal{O},\ h,k \in H). \tag{2}$$

We next show that the prescribed values of A_y respect all linear relations among the $z(h)$. Suppose $\sum_{j=1}^{q} r_j z(h_j) = 0$. Put $\lambda = \sum_j r_j y(c(h_j)) \in D$. For $k \in H$ and $r \in \mathcal{O}$, equation (2) gives

$$\lambda(rz(c(k))) = \sum_j y(c(h_j))(rr_j z(c(k))) = y(k)\left(r \sum_j r_j z(h_j)\right) = 0.$$

The vectors $z(c(k))$ generate V, since $c(H) = H$. Every vector of V is therefore a finite sum of the tested vectors $rz(c(k))$. The additive functional λ is zero. Consequently

$$A_y\left(\sum_j r_j z(h_j)\right) = \sum_j r_j y(c(h_j))$$

is a well-defined $\mathcal{O}$-linear map $V \to D$. Generation also proves uniqueness.

For $g \in G$, $h \in H$ and $w \in V$, normality of H and the cocycle identity yield

$$\begin{aligned} A_y(\rho(g)z(h))(w) &= A_y(z(ghg^{-1}))(w) \\ &= y(c(g)c(h)c(g)^{-1})(w) \\ &= y(c(h))(\chi(c(g))\rho(c(g))^{-1}w) \\ &= A_y(z(h))(\chi(c(g))\rho(c(g))^{-1}w). \end{aligned}$$

Extend this equality $\mathcal{O}$-linearly from the generating set $z(H)$ to obtain the required equivariance. The formula on this generating set shows that $y \mapsto A_y$ preserves addition and scalar multiplication. If $A_y = 0$, then $y(c(h)) = 0$ for all $h \in H$, hence $y = 0$ because $c(H) = H$. This proves injectivity. The final assertion follows from monotonicity of module length under an injection. □

Lemma 7.4.20. *Let E be a number field, ℓ a prime, $\mathcal{O}$ the integer ring of a finite extension of $\mathbb{Q}_\ell$, $M, q \geq 1$ integers, $n \geq 1$, J a finite set, and*

$$\rho : G_E \longrightarrow \mathrm{GL}_n(\mathcal{O}), \qquad \lambda_j : G_E \longrightarrow \mathbb{Z}/q\mathbb{Z} \quad (j \in J)$$

continuous homomorphisms. Let S be a finite set of finite places and $\mathscr{L}$ any family of finitely supported functions $G_E \to M_n(\mathcal{O})$. For $g \in G_E$ let $[\lambda_j(g)]_q \in \{0, \ldots, q-1\}$ be its least nonnegative representative. For any coefficient extension $\mathcal{O}' \supseteq \mathcal{O}$ and $x \in (\mathcal{O}')^J$, put

$$\rho_x(g) = \left(\prod_{j \in J} (1 + x_j)^{[\lambda_j(g)]_q}\right) \rho(g).$$

Let $\mathcal{S}_{\mathcal{O}'}(x)$ be the $\mathcal{O}'$-module of continuous functions $z : G_E \to (\mathcal{O}'/\ell^M)^n$ satisfying

$$z(gh) = z(g) + \rho_x(g)z(h) \quad (g,h \in G_E), \qquad z(\gamma) = 0 \quad (\gamma \in I_v,\ v \notin S),$$

where all choices of inertia groups above v are included, and

$$\sum_{g\in G_E} A(g)z(g) = 0 \quad (A \in \mathscr{L}).$$

The coefficients in these formulas are reduced modulo ℓ^M. No multiplicativity of the function ρ_x is required.

For every coefficient extension $\mathcal{O}_0 \supseteq \mathcal{O}$ and every $x_0 \in (\mathcal{O}_0)^J$, there is a nonzero polynomial $P \in \mathcal{O}[X_j : j \in J]$ such that

$$v_\ell(P(x_0)) = \mathrm{nl}_{\mathcal{O}_0}\mathcal{S}_{\mathcal{O}_0}(x_0), \qquad \mathrm{nl}_{\mathcal{O}'}\mathcal{S}_{\mathcal{O}'}(x) \le v_\ell(P(x))$$

for every coefficient extension $\mathcal{O}' \supseteq \mathcal{O}$ and every $x \in (\mathcal{O}')^J$. All the displayed modules have finite length.

Proof. We first produce a finite quotient that works for every point and coefficient extension. Let

$$H = \ker(\rho \bmod \ell^M) \cap \bigcap_{j\in J} \ker \lambda_j.$$

This is an open normal subgroup of G_E. Write F for its fixed field, so $H = G_F$, and let S_F be the places of F above S. The set of continuous characters

$$H \longrightarrow \mathbb{Z}/\ell^M\mathbb{Z}$$

unramified outside S_F is finite. Indeed the cited finiteness condition gives this for $M = 1$. Reduction modulo ℓ maps the set for exponent ℓ^a into the finite set for exponent ℓ; the difference of two elements in one fiber takes values in $\ell(\mathbb{Z}/\ell^a) \simeq \mathbb{Z}/\ell^{a-1}$, so induction proves finiteness for every a.

Intersect their kernels, obtaining an open subgroup of H, and take its normal core in G_E. Denote the resulting open normal subgroup by $K \subseteq H$. Thus every such character of H vanishes on K. We claim that every function z in every module of the statement vanishes on K. On H the matrices $\rho_x(g)$ act identically modulo ℓ^M for every x, because each $\lambda_j(g)$ is zero and $\rho(g) \equiv I$. The cocycle equation therefore makes $z|_H$ additive. For any additive character $\nu : (\mathcal{O}'/\ell^M)^n \to \mathbb{Z}/\ell^M$, its composite with $z|_H$ is a continuous character unramified outside S_F: inertia over F at a place outside S_F is contained in the corresponding inertia over E, where z vanishes. Hence $\nu z(k) = 0$ for $k \in K$. These additive characters separate points. Explicitly, a finite abelian group killed by ℓ^M is a direct sum of cyclic groups of orders ℓ^a with $a \le M$, and the map $\mathbb{Z}/\ell^a \to \mathbb{Z}/\ell^M$ given by multiplication by ℓ^{M-a} detects every nonzero element in that summand. Thus $z(K) = 0$. The cocycle equation now gives

$$z(gk) = z(g) + \rho_x(g)z(k) = z(g) \qquad (g \in G_E,\ k \in K).$$

All the functions in question factor through the same finite set $\Delta = G_E/K$.

We next write their equations over one polynomial ring. Put $\Lambda = \mathcal{O}[X_j : j \in J]$, and for $g \in G_E$ put

$$U_g(X) = \prod_{j\in J}(1 + X_j)^{[\lambda_j(g)]_q}, \qquad R_g(X) = U_g(X)\rho(g).$$

There are $c = n\#\Delta$ scalar coordinates, indexed by $(d, i) \in \Delta \times \{1, \dots, n\}$. Writing w_d for their n-component column, form the rows of all the linear equations

$$w_{[gh]} - w_{[g]} - R_g(X)w_{[h]} = 0, \qquad w_{[\gamma]} = 0 \quad (\gamma \in I_v,\ v \notin S), \qquad \sum_g A(g)w_{[g]} = 0 \quad (A \in \mathscr{L}).$$

Each component is a row in Λ^c. Include also the rows $\ell^M e_i$, $1 \le i \le c$, and let $\mathscr{R}$ be the submodule generated by all these rows. Since $\mathcal{O}$ is Noetherian and J is finite, Λ is Noetherian. Consequently $\mathscr{R}$ is generated by finitely many of the displayed rows: start with a finite set of module generators and expand them as finite linear combinations of the displayed rows. Let $A(X)$ be the resulting finite matrix, with c columns.

For every $\mathcal{O}'$ and x, pullback along $G_E \to \Delta$ identifies $\mathcal{S}_{\mathcal{O}'}(x)$ with

$$\ker\bigl(A(x) : (\mathcal{O}'/\ell^M)^c \longrightarrow (\mathcal{O}'/\ell^M)^{\#\mathrm{rows}(A)}\bigr).$$

One direction follows from the factorization just proved. Conversely a vector satisfying the specialized rows gives a function on Δ, hence a continuous function on G_E, and the rows impose exactly every equation in the statement. Finite row generation is preserved under specialization, so there are no additional equations to check.

We finally compute these lengths by maximal minors. Fix $\mathcal{O}'$ and x, write ϖ' for a uniformizer and $e' = v_{\varpi'}(\ell)$. The row module of $A(x)$ contains $\ell^M(\mathcal{O}')^c$, since this containment already held over Λ. Smith normal form therefore has exactly c nonzero diagonal invariant factors $(\varpi')^{d_1}, \dots, (\varpi')^{d_c}$, with $0 \le d_i \le Me'$. On $\mathcal{O}'/\ell^M$, the kernel of multiplication by $(\varpi')^{d_i}$ has length d_i. Thus the module above has length $\sum_i d_i$. The ideal generated by the c by c minors of $A(x)$ is $(\varpi')^{\sum_i d_i}$. To see this last assertion directly, invertible row and column operations preserve the ideal of maximal minors, by the determinant expansion and their inverse operations; for the Smith matrix the only nonzero maximal minors generate the product of the diagonal factors. It follows that

$$\mathrm{nl}_{\mathcal{O}'}\mathcal{S}_{\mathcal{O}'}(x) = \min_D v_\ell(D(x)),$$

where D ranges over the finitely many maximal-minor polynomials of $A(X)$. There is a nonzero evaluated minor, because the row module contains $\ell^M(\mathcal{O}')^c$.

At $(\mathcal{O}_0, x_0)$ choose a minor attaining this minimum and call its polynomial P. Its evaluation at x_0 is nonzero, so P itself is nonzero. The same minimum formula at every other point gives the required upper bound by $v_\ell(P(x))$, and at x_0 it gives equality. The matrix kernel lies in a finite module for each coefficient ring, proving finite length as well. □

Lemma 7.4.21. *Let E be a CM field with complex conjugation c, let ℓ be a prime, $\mathcal{O}$ a coefficient ring, $M, N, n \ge 1$, and $\rho : G_E \to \mathrm{GL}_n(\mathcal{O})$ continuous. Let J be a finite set of places with a distinguished $v_0 \in J$, disjoint from its conjugate and from the places above ℓ. Assume that ρ is unramified at every place of $J \cup \bar{J}$. For each $v \in J$ fix a tame generator γ_v and an arithmetic Frobenius φ_v in the same decomposition group, and require $\mathrm{N}v \equiv 1 \pmod{\ell^N}$. Put $g^+ = g$ and $g^- = (g^c)^{-1}$.*

Suppose that continuous additive characters $\lambda_w : G_E \to \mathbb{Z}/\ell^N$ satisfy

$$\lambda_w(g^c) = -\lambda_w(g), \qquad \lambda_w(\gamma_v) = \delta_{wv}, \qquad \lambda_w(\varphi_v) = \delta_{wv_0} \quad (v \ne v_0),$$

and are unramified outside $\{w, \bar{w}\}$. Let $s_v \in \mathcal{O}$, with $s_{v_0} = 0$, and assume

$$s_v \ne 0 \implies \rho(\varphi_v^+) \equiv \rho(\varphi_v^-) \equiv I \pmod{\ell^M}.$$

Let $\mathcal{O}' \supseteq \mathcal{O}$ be a coefficient extension, $r > 0$, $K \ge 0$ an integer with $\ell^K r \ge 1$, $N \ge K + M$, and $y \in \mathcal{O}'$ with $v_\ell(y) \ge r$. Put $u = 1 + y$ and

$$a_{v_0} = u, \qquad a_v = 1 + s_v y \quad (v \ne v_0), \qquad \eta_y(g) = \prod_{v \in J} a_v^{[\lambda_v(g)]_{\ell^N}},$$

where brackets denote the least nonnegative representative. Modulo ℓ^M, this defines a character with unit values, hence $\sigma_y = \eta_y\rho$ defines a representation on $W = (\mathcal{O}'/\ell^M)^n$.

Let $I \subseteq J$, with $y = 0$ if $v_0 \in I$, and let V be any $\mathcal{O}'$-submodule of the continuous σ_y-cocycles $G_E \to W$. Let V_I be cut out by

$$z(\gamma_v^b) = s_v z(\varphi_v^b) \qquad (v \in I,\ b \in \{+,-\}).$$

For each such (v,b) choose $w_{v,b} \in G_E$ with $\rho(w_{v,b}) \equiv \rho(\varphi_v^b) \pmod{\ell^M}$. Then V/V_I has finite length and

$$\mathrm{nl}_{\mathcal{O}'}(V/V_I) \le \sum_{v\in I}\sum_{b\in\{+,-\}} v_\ell \det(u\rho(w_{v,b}) - I).$$

Proof. We first prove the geometric-sum estimate that makes the affine local calculation exact. Let $a = 1 + t \in \mathcal{O}'$ with $v_\ell(t) \ge r$, allowing $t = 0$. The binomial theorem gives

$$v_\ell(a^\ell - 1) \ge \min\{\ell v_\ell(a-1),\, v_\ell(a-1) + 1\}.$$

Induction implies $v_\ell(a^{\ell^j} - 1) \ge \min\{\ell^j r, 1\}$ for $j \le K$, and hence this valuation is at least 1 for all $j \ge K$. For $j \ge K$ the integral polynomial

$$1 + a^{\ell^j} + \cdots + (a^{\ell^j})^{\ell-1}$$

has valuation at least 1: modulo an element of valuation at least 1 all its ℓ summands are 1, and its constant sum is ℓ. The factorization

$$\sum_{h=0}^{\ell^N-1} a^h = \prod_{j=0}^{N-1}\left(\sum_{b=0}^{\ell-1}(a^{\ell^j})^b\right)$$

therefore has valuation at least $N - K \ge M$. Multiplication by $a - 1$ also gives $a^{\ell^N} \equiv 1 \pmod{\ell^M}$. These assertions hold for every a_v, since s_v is integral, and for u. Each a_v is a unit because $v_\ell(a_v - 1) > 0$. The cyclic character $g \mapsto a_v^{\lambda_v(g)}$ is thus well defined modulo ℓ^M, proving the assertion about η_y and σ_y.

Fix a cocycle $z \in V$. It defines a continuous affine action on the finite set W by

$$T_g(m) = \sigma_y(g)m + z(g), \qquad T_gT_h = T_{gh}.$$

At a place $v \in J$ the linear action of wild inertia is trivial. The representation ρ is unramified there, and each λ_w kills wild inertia, which is pro-residue-characteristic, whereas its target is an ℓ-group. Restriction of z to wild inertia is then a continuous homomorphism to the additive ℓ-group W, so it vanishes. Thus the affine action kills wild inertia.

We recall explicitly the tame relation used here. In the maximal tame quotient of a local Galois group of residue cardinality Q, arithmetic Frobenius acts on inertia by the Q-power map. Indeed over the maximal unramified extension, every finite tame extension of ramification degree d prime to the residue characteristic is obtained by adjoining a d-th root of a uniformizer: units have d-th roots by Hensel's lemma, and the residue field is algebraically closed. Inertia multiplies this root by a d-th root of unity; arithmetic Frobenius raises those roots of unity to their Q-th powers. Passing over all d gives the relation. Changing the Frobenius lift by an inertia element does not change it, since tame inertia is abelian.

At γ_v, the linear part of T_{γ_v} is $a_v I$, by the character-coordinate identities and unramifiedness of ρ. Its ℓ^N-th iterate has linear part $a_v^{\ell^N} I = I$ on W, and translation part

$$\left(\sum_{h=0}^{\ell^N-1} a_v^h\right) z(\gamma_v) = 0.$$

Hence $T_{\gamma_v}^{\ell^N} = 1$. Since $\mathrm{N}v \equiv 1 \pmod{\ell^N}$, the tame Frobenius relation gives

$$T_{\varphi_v} T_{\gamma_v} = T_{\gamma_v} T_{\varphi_v}.$$

At the conjugate place, the linear part at γ_v^c is $a_v^{-1} I$. Since $v_\ell(a_v^{-1} - 1) = v_\ell(a_v - 1) \geq r$, the same geometric-sum argument proves $T_{\gamma_v^c}^{\ell^N} = 1$. The transported arithmetic Frobenius φ_v^c therefore commutes with it. Inverting this commuting pair proves the same assertion for γ_v^-, φ_v^-.

For clarity, the character values at both signs agree: $\lambda_w(g^-) = \lambda_w(g)$. At $v \neq v_0$ we consequently have

$$\sigma_y(\gamma_v^b) = a_v I, \qquad \sigma_y(\varphi_v^b) = u\rho(\varphi_v^b).$$

The equality of the translation parts of the two commuting affine transformations yields, with $B_{v,b} = u\rho(\varphi_v^b) - I$,

$$B_{v,b} z(\gamma_v^b) = (a_v - 1) z(\varphi_v^b).$$

It follows that

$$B_{v,b}\big(z(\gamma_v^b) - s_v z(\varphi_v^b)\big) = s_v u\big(I - \rho(\varphi_v^b)\big) z(\varphi_v^b) = 0.$$

If $s_v = 0$ this is immediate; otherwise the residual Frobenius hypothesis makes the matrix on the right zero on W. If $v = v_0 \in I$, the imposed condition $y = 0$ makes every $a_w = 1$ and $\eta_y = 1$; also $s_{v_0} = 0$. The same commuting identity then says $(\rho(\varphi_{v_0}^b) - I) z(\gamma_{v_0}^b) = 0$. Thus the displayed kernel assertion is valid for every $(v, b) \in I \times \{+, -\}$.

Send $z \in V$ to its tuple of tie errors

$$\big(z(\gamma_v^b) - s_v z(\varphi_v^b)\big)_{v,b}.$$

Its kernel is exactly V_I, and its image is contained in the product of the kernels of $B_{v,b}$ on W. The same kernels are obtained on replacing φ_v^b by $w_{v,b}$, by the assumed congruence. This product is a finite module, proving finite length of V/V_I.

Finally, for any matrix $B \in M_n(\mathcal{O}')$, Smith normal form shows

$$\mathrm{len}_{\mathcal{O}'} \ker(B : W \to W) \leq v_{\varpi'}(\det B).$$

If $\det B = 0$, the right side is infinite and the assertion is immediate. Otherwise the diagonal invariant factors of B give kernel length $\sum_j \min\{v_{\varpi'}(d_j), M v_{\varpi'}(\ell)\}$, at most the valuation of their product. Sum this inequality over (v, b) and divide by $v_{\varpi'}(\ell)$ to obtain the required normalized bound. □

Lemma 7.4.22. *Let ℓ be prime, let v be the valuation on $\overline{\mathbb{Q}}_\ell$ normalized by $v(\ell) = 1$ and $v(0) = +\infty$, and put $\overline{\mathbb{Z}}_\ell = \{a : v(a) \geq 0\}$. Let $s_0 \geq 0$ and $m \geq 1$ be integers, let $0 < \rho < 1$ be rational, and let $B \geq 0$ and $\varepsilon > 0$ be real. There is an integer $H \geq 1$, depending only on $\ell, s_0, m, \rho, B, \varepsilon$, with the following property.*

For any finite set U with $\#U \leq s_0$, any finite set I, any polynomials $P_i \in \overline{\mathbb{Z}}_\ell[X_u : u \in U]$, and any points $x_i \in \overline{\mathbb{Z}}_\ell^U$ satisfying

$$v(x_{i,u}) \geq \rho, \qquad \sum_{i \in I} v(P_i(x_i)) \leq B,$$

there are ℓ^H-th roots of unity $(\zeta_u)_{u\in U}$ such that, for every $u \in U$ and $1 \le j \le m$,

$$0 < v(\zeta_u^j - 1) \le \rho,$$

and, simultaneously for every $i \in I$ and $1 \le j \le m$,

$$v\Big(P_i((\zeta_u^j - 1)_{u\in U})\Big) \le v(P_i(x_i)) + \varepsilon.$$

In particular H is chosen before the sets, polynomials and points.

Proof. **Gauss valuations reduce the one-variable problem to roots at one radius.** For a nonzero integral polynomial $g(Y) = \sum a_d Y^d$ and a rational $t \ge 0$, write

$$G_t(g) = \min_d (v(a_d) + dt).$$

These valuations are additive on products. Indeed choose an algebraic number of valuation t, scale the variable by it, and divide each polynomial by a coefficient of minimum valuation. The normalized polynomials have integral coefficients and nonzero reduction; their product has nonzero reduction because the residue ring is a field. This proves multiplicativity. Factoring g over the algebraic closure therefore gives

$$G_t(g) = C + \sum_{\alpha:\, v(\alpha)\ge 0} \min(t, v(\alpha)), \qquad C = G_0(g) \ge 0, \tag{1}$$

with roots counted with multiplicity. The factors belonging to roots of negative valuation are absorbed in C. For an integral y with $v(y) = t > 0$, only roots of valuation t can contribute an excess over (1). If R_t is their multiset, then

$$v(g(y)) = G_t(g) + \sum_{\alpha\in R_t} (v(y - \alpha) - t). \tag{2}$$

Both formulas also cover a nonzero constant polynomial.

For $h \ge 1$ put $r_h = 1/\varphi(\ell^h)$. Every primitive ℓ^h-th root ζ has $v(\zeta - 1) = r_h$. One elementary proof uses the Eisenstein polynomial $\Phi_{\ell^h}(1 + Y)$: its constant term is ℓ, its reduction is $Y^{\varphi(\ell^h)}$, and its roots are conjugate under the local Galois group, which preserves the valuation. Their valuations are consequently all $1/\varphi(\ell^h)$. The r_h are distinct and decrease to zero.

Fix positive rational t_0, a finite bound B_0, and $0 < \delta < 1$. Choose $H_0 \ge 1$ such that

$$B_0 < \delta \sum_{\substack{1\le h\le H_0\\ r_h\le t_0}} \frac{1}{h+1}. \tag{3}$$

Such an integer exists because the condition $r_h \le t_0$ holds eventually and the harmonic series diverges. Formula (1), and distinctness of the radii, show that every integral polynomial $g \ne 0$ with $G_{t_0}(g) \le B_0$ satisfies

$$\sum_{\substack{1\le h\le H_0\\ r_h\le t_0}} \#R_{r_h}\, r_h \le G_{t_0}(g) \le B_0.$$

Thus some h in this sum has

$$\#R_{r_h}\, r_h \le \frac{\delta}{h+1}. \tag{4}$$

Otherwise summing the strict reverse inequalities contradicts (3).

We next choose a primitive root at this radius. If $v(\alpha) > 0$, then $1+\alpha$ is a unit. For $f(Y) = (Y+1+\alpha)^{\ell^h} - 1$, all its roots are integral and its linear coefficient has valuation h. Factoring this monic polynomial and applying its Gauss valuation at radius one gives

$$\sum_{\zeta^{\ell^h}=1} \min(v(\zeta - 1 - \alpha), 1) = G_1(f) \le h + 1. \tag{5}$$

Restricting to primitive roots preserves the inequality. There are $\varphi(\ell^h) = 1/r_h$ such roots. Summing (5) over $\alpha \in R_{r_h}$, then averaging, gives a primitive ζ with

$$\sum_{\alpha \in R_{r_h}} \min(v(\zeta - 1 - \alpha), 1) \le \#R_{r_h}\, r_h(h+1) \le \delta.$$

Because $\delta < 1$, each summand is strictly less than one. Its truncation can therefore be removed. Formula (2), with $y = \zeta - 1$, proves

$$v(g(\zeta - 1)) \le G_{r_h}(g) + \delta. \tag{6}$$

In particular this evaluation is nonzero. The integer H_0 in (3) depends on t_0, B_0, δ, ℓ, not on the degree or coefficients of g.

One coordinate can be replaced simultaneously for all polynomials. First consider the assertion with $m = 1$, an arbitrary rational radius $t_0 > 0$, and error $e > 0$; no upper bound on t_0 is needed here. Choose

$$0 < \delta < \min(1, e/(s_0 + 1)),$$

and choose H_0 by (3) with $B_0 = B + e$. Order U. Suppose some coordinates have already been replaced by common values of the form $\zeta_u - 1$, with $0 < v(\zeta_u - 1) \le t_0$, while the remaining coordinates of the i-th point retain their original values. Let b_i be the current evaluation valuation. Initially $\sum_i b_i \le B$.

For the next coordinate, specialize all other coordinates in P_i at its current point and call the resulting integral polynomial $g_i(Y)$. Its evaluation at the unchanged coordinate is nonzero, since the current b_i are finite; hence $g_i \neq 0$. The unchanged coordinate has valuation at least t_0, so $G_{t_0}(g_i) \le b_i$. Apply (6) to $g = \prod_i g_i$; when I is empty this product is one. At every stage $\sum_i b_i \le B + \#U\,\delta < B + e$, so the same integer H_0 applies.

For the chosen root ζ, write $a_i = G_{r_h}(g_i)$ and $b_i' = v(g_i(\zeta - 1))$. Each $b_i' \ge a_i$, and (6) with multiplicativity gives

$$\sum_i b_i' \le \sum_i a_i + \delta.$$

Consequently $b_i' \le a_i + \delta \le b_i + \delta$ for each i, and also $\sum_i b_i' \le \sum_i b_i + \delta$. These two inequalities prove the induction assertions for both the individual valuations and their sum. After at most s_0 replacements, each valuation has increased by less than e. Every chosen root has order dividing ℓ^{H_0}. This proves the one-power assertion uniformly in the number of polynomials. If U is empty, no replacement is needed and the same conclusion is immediate.

Polynomial substitution makes the first m powers simultaneous. Return to the hypotheses of the statement. For each i, each $1 \le j \le m$, and each coordinate u, choose $x'_{i,j,u}$ with

$$(1 + x'_{i,j,u})^j = 1 + x_{i,u}, \qquad v(x'_{i,j,u}) \ge \rho/m. \tag{7}$$

To justify this choice, if $x_{i,u} = 0$ take $x' = 0$. Otherwise the j roots y of $y^j = 1 + x_{i,u}$ are units. The product of $1 - y$ over these roots is $-x_{i,u}$. All factors have nonnegative valuation, so one has valuation at least $v(x_{i,u})/j \ge \rho/m$. Take $x' = y - 1$.

Set

$$Q_{i,j}(X) = P_i(((1 + X_u)^j - 1)_{u \in U}).$$

These have integral coefficients, and (7) gives $Q_{i,j}(x'_{i,j}) = P_i(x_i)$. Thus their total evaluation valuation is at most mB. Apply the one-power assertion, with radius $t_0 = \rho/m$, bound mB, and error $e = \varepsilon$, to this finite family indexed by (i, j). It supplies roots ζ_u of order dividing a uniformly fixed ℓ^H, with

$$0 < v(\zeta_u - 1) \le \rho/m, \qquad v(Q_{i,j}((\zeta_u - 1)_u)) \le v(P_i(x_i)) + \varepsilon.$$

The latter is the required polynomial inequality.

It remains to check the stated distances for powers. A selected ζ_u is primitive of order ℓ^h, for some $h \ge 1$, and has distance valuation $r_h \le \rho/m < 1/m$. If $b = v_\ell(j)$, then $\ell^b \le j \le m$. Necessarily $h > b$: otherwise $r_h \ge 1/\ell^h \ge 1/j \ge 1/m$, a contradiction. Hence ζ_u^j is primitive of order ℓ^{h-b}, and

$$v(\zeta_u^j - 1) = r_{h-b} = \ell^b r_h \le j\rho/m \le \rho.$$

It is positive and finite. All choices of H were made before the polynomials and points, which proves the full quantifier assertion. □

7.5 Main summands from eigensystems

This section supplies the Galois representations to which the twisting theorem is applied (Proposition 7.5.1): for each Hecke eigensystem of the good forms, and for each main pair, a main summand (Definition 7.2.30). This is an ℓ-adic representation R that describes at once the Galois representations of all eigenforms with the same lift Hecke polynomials, with normalised traces $\psi_\vartheta \operatorname{tr} R + \sum_{e<t} \chi_{\mathrm{cyc}}^{-e}$. This reducible shape is what Ribet's method turns into cohomology classes (Theorem 7.6.6). In our reading it is the Galois shadow of the lift from rank r to rank $r + t$, of the kind met for non-tempered forms on unitary groups by Bellaïche and Chenevier [20, 21, 19]. The representation R is also main-summand like: conjugate self-dual, crystalline where required, with controlled tame inertia and large image. These are the hypotheses under which it gives a Selmer datum.

The representations come from those attached to eigenforms on definite unitary groups through base change [177], as derived in Section 7.1 (see [58, 122, 276, 56, 15, 302, 47, 48, 110]). The rank-one summands are the ℓ-adic characters of algebraic Hecke characters, and Chebotarev's theorem [54] enters the proof. The one local statement proved here recognises the inertia type at the places above $p_0 p_1$ (Lemma 7.5.2), from the local Langlands correspondence [122, 130] and the depth-zero types of Shotton [278]; for Green's cuspidal representations and for types see [106, 46, 253]. The section closes with the representations of a type and the Ribet cocycles (Definition 7.5.3), read by the next two sections, and with one calibration of local reciprocity (Lemma 7.5.4).

Proposition 7.5.1. *Let D be admissible data, ℓ a prime and $\iota \mid \ell$ a prime identification.*

1. *For every $1 \le j \le m$ and every system $z = (z_{w,i})$ of Hecke eigenvalues that occurs in the good forms on W_m, there is an ℓ-adic representation R of rank r that is a main summand for the descent pair (W_j, V^{def}) at the trivial twist and z.*

2. *For every $1 \le j \le m$ and every main pair p for j there is an ℓ-adic representation of rank one that is a main summand for $O_{p,1}$ and the constant system 1.*

Proof. We first construct the initial representation. In the first assertion choose a nonzero form with eigensystem z. Since the good idempotent is the sum of the mutually orthogonal products of its Green idempotents, one good family of labels has a nonzero projection of this form. Hecke operators away from the level commute with these idempotents, so the projection retains z. Decomposing its automorphic span into irreducible constituents gives a constituent with this eigensystem and these finite types. The group at infinity is compact because the space is W_m. The discrete-series parameter is the specified regular integral parameter ς. Definite-unitary base change [177] and attachment Theorem 7.1.4 therefore give a semisimple representation R_0 of rank r_D, with

$$R_0^c \simeq R_0^\vee \chi_{\mathrm{cyc}}^{1-r_D}, \qquad J_{0,\sigma} = \{(r_D-1)/2 - \epsilon\varsigma_q : 1 \le q \le r_D\}, \quad \epsilon = \mathrm{sgn}(\sigma). \tag{1}$$

For the tame normalization use Lemma 7.5.4 with the local chosen field, the fixed faithful character of the Green labels, and $f = n!$. Every irreducible-factor degree d satisfies $1 \le d \le n$ and $d \mid n!$. The lemma identifies its regular Green character on an inertia element of exponent a with $T(x)^a$, not its inverse, for every root and its Frobenius conjugates. This is the same finite inertia representation on the potentially crystalline Weil–Deligne module when the place divides ℓ. The inverse-character clause of the lemma gives exactly the scaled roots after the inverse avatar of μ_O is applied. The local statements in Theorem 7.1.8 and Theorem 7.1.9, together with the case $t = 0$ of Lemma 7.5.2, identify its chosen tame types with the label roots and give zero monodromy. Away from the chosen places it is unramified, and crystalline at places over ℓ; at the chosen places over ℓ it is potentially crystalline with the specified finite inertia type. Apply the squarefree-label clause of Theorem A.5.29 to the distinct irreducible factors of the label. It gives precisely the irreducible finite Harish–Chandra product of their distinct regular cuspidal representations, so the hypothesis of Lemma 7.5.2 with $t = 0$ is the type selected by the idempotent. The harmless sign of the Green character cancels in that idempotent.

In the second assertion put $R_0 = 1$, of rank $n = 1$ with jump zero and labels $X - 1$. In the first assertion put $n = r_D$. In either case let O be the descent pair in question, let $t = t_O$, and put $A = (n - t)/2 \in \mathbb{Z}$. Define

$$\eta_O = \mathrm{avatar}^{\mathrm{geo}}\big(\mu_O^{-1} |\cdot|_E^A\big), \qquad R = \eta_O R_0. \tag{2}$$

Here μ_O is the character in (DP2). Its base change is understood in (2). Since this character is anti-conjugate apart from its norm power,

$$\eta_O^c \eta_O = \chi_{\mathrm{cyc}}^{2A}, \qquad R^c \simeq R^\vee \chi_{\mathrm{cyc}}^{1-t}. \tag{3}$$

The algebraic-character theorem [261], [85], in the avatar convention stated above, gives the jump shift

$$J_\sigma = J_{0,\sigma} - (A - \epsilon\,\mathrm{weight}_O(\sigma)). \tag{4}$$

It also preserves the required potential crystallinity and gives crystallinity away from the chosen places. Its finite character is trivial away from those places, because the descent pair here is at the trivial twist. Thus R has level $\emptyset$. The trace form on the image algebra is nondegenerate: for a semisimple representation in characteristic zero that algebra is a product of full matrix algebras, and the representation trace on each factor is a positive integral multiplicity times the matrix trace. Twisting by a character changes neither this conclusion nor semisimplicity. This proves (MS1)–(MS2).

We check the weight conditions explicitly. Write $h = h_D$, $\Delta = 2h$ and $\kappa_j = \kappa + j\Delta + h$. For the first assertion, (1)–(4) give

$$J_\sigma = \{(t_D - 1)/2 - \epsilon(\varsigma_q + \kappa_j) : 1 \le q \le r_D\}. \tag{5}$$

For the first $k-1$ entries, $\varsigma_q = k/2 - q$, hence $|\epsilon u + \kappa_j| \le (t_D-1)/2 + k/2 - 1 \le t_D$. For the remaining entries $\varsigma_q = -\kappa - i\Delta$, $1 \le i \le m$. Their jumps are

$$u = (t_D-1)/2 - \epsilon((j-i)\Delta + h),$$

and therefore $\Delta/4 \le |u| \le 2m\Delta$. These m entries are inside $2h^2$, because $2m \le h$; the other entries are outside it because $\kappa_j - t_D > 2h^2$. For completeness, the scale inequalities and the numerical bounds of Lemma 4.2.67 imply

$$\Delta^2 = 4r_D^{16} \le x^{81} \le x_1, \qquad \kappa \ge b/x_1 \ge x_1,$$

which give the last strict inequality. Thus k times the number of small jumps is $km \le r_D m$. Conjugate jumps pair to $t_D - 1 \ge 0$, so at most r_D are negative in each conjugate pair of embeddings.

For the second assertion, write the main pair as (ν, z) and its weight as $-c$. Its string length is $t = 2z-1$ and (4) gives $u = z - 1 - \epsilon c$. Since $\kappa_j + r_D - t_D \le c \le \kappa_j$ and $z \le r_D$,

$$|\epsilon u + \kappa_j| \le t_D, \qquad |u| > 2h_D^2.$$

There are no small jumps, and the conjugate jumps sum to $2z - 2 \ge 0$. These are all the weight conditions for rank one.

We next verify the three arithmetic conditions in (MS3). By Lemma 4.2.66, the chosen completions above p_i are $\mathbb{Q}_{p_i}$ and their residue fields are $\mathbb{F}_{p_i}$. At a chosen place over p_i, (2) scales the original label roots by the inverse of the residue character of μ_O. This multiplier lies in $\mathbb{F}_{p_i}^\times$ and has order dividing q_2. The allowed-label condition implies that the scaled polynomial does not vanish at 1. At a primitive place its degree is $n = r_D$; in the rank-one case it is the scaled root of $X - 1$.

Fix any $p \in \{p_0, p_1\}$ different from ℓ. Every scaled inertia root belongs to $\mathbb{F}_{p^a}^\times$ for some $a \le n$, so its order divides $p^{n!} - 1$. Inertia is semisimple of this tame type, and hence is killed by that integer, which is prime to p. At a chosen tame generator every eigenvalue λ is different from 1. Since $\lambda^{p^a-1} = 1$,

$$v_\ell(\lambda - 1) \le v_\ell(p^a - 1) \le v_\ell(N_{\mathrm{ch}}).$$

The last inequality follows because $a \le n \le r_D t_D$ and $p^a - 1$ is a factor of the order of $\mathrm{GL}_{r_D t_D}(\mathbb{F}_p)$, one of the factors defining N_{ch}. Summing over the eigenvalues gives the determinant bound. Conjugate duality (3) on inertia gives the same characteristic polynomial at the inverse conjugate element, hence the second determinant bound. At an unchosen conjugate place choose the inverse conjugate of the chosen generator. This proves (MS3)(1).

At a primitive place scaling preserves primitivity. Indeed $q_2^2 \mid p - 1$, and multiplication of a generator of order $H = p^n - 1$ by a q_2-th root changes its exponent by a multiple of H/q_2. Every prime dividing H still divides H/q_2, so the new exponent remains coprime to H. The roots $\zeta, \zeta^p, \dots, \zeta^{p^{n-1}}$ have pairwise unit differences at every ℓ. To see this, write $n = 3^d$. If $\ell \ne 3$, the 3-primary part separates them, since

$$v_3(p^n - 1) = v_3(p-1) + d, \quad v_3(p^a - p^b) = v_3(p-1) + v_3(a-b) < v_3(p^n - 1)$$

for distinct $0 \le a, b < n$. If $\ell = 3$, the integer $\Phi_{3^d}(p) = p^{2n/3} + p^{n/3} + 1$ has 3-valuation one and is greater than 3. A prime divisor $L \ne 3$ of it has $\mathrm{ord}_L(p) = n$: the order divides n and cannot divide $n/3$, since the common divisor of this cyclotomic value with $p^{n/3} - 1$ divides 3. The surviving L-primary part separates the roots. Prime-to-ℓ roots of unity reduce injectively, proving the claim.

Take the coefficient field large enough to contain these roots. For every lattice, the integral Lagrange projectors

$$E_a = \prod_{b \neq a} \frac{u(\gamma) - \zeta_b}{\zeta_a - \zeta_b}$$

belong to the algebra spanned by the image and split the lattice into rank-one free direct summands. Arithmetic Frobenius cycles them by invertible maps. Products $E_b u(F)^v E_a$ give unit multiples of all matrix units. The image span is therefore the full matrix algebra on every lattice. Moreover $u(F)^n$ is diagonal and commutes with $u(F)$, so its diagonal entries are equal. It is scalar, and $\chi_{\mathrm{cyc}}(F^n) = p^n$. The rank-one case is immediate. This proves (MS3)(2).

The common power $e = (p^n - 1)/(p - 1)$ of primitive inertia gives a scalar Y; its residue value has order $p - 1$. In rank one take $e = 1$; its residue value has order $q_2/\gcd(q_2, j)$, where $1 \leq j \leq m$. In both cases $y^v = 1$ implies $q_2 \mid jv$. If $Y^{\ell^a v} - 1$ were nonunit, then $Y^{\ell^a v}$ would have ℓ-power order, so $q_2 \mid j\ell^{a+b}v$ for some $b \geq 0$. Choose $p^* \in \{2, 5\}$ different from ℓ. This forces $(p^*)^s \mid jv$, whereas

$$0 < jv \leq m(4r_D + 2) < 2^s = m^{22} \leq (p^*)^s$$

for $1 \leq v \leq 4n + 2$. The strict inequality follows from $m \leq 2\sqrt{x}$, $r_D \leq 3x^5$, $m^{22} \geq x^{11}$ and the lower bound on x. This proves (MS3)(3).

Finally we verify the trace identity. Let O' have the same source rank n and target rank $n+t$ as O, and let $\mu_{O'} = \mu_O \vartheta$. Fix any eigenform y and any attached representation S quantified in (MS4). Throughout this comparison write $S' = \nu_{O'} \otimes S$, the normalized representation of this very S for this very O' in (MS6). At an unramified split place put $q = \mathrm{N}w$, $\alpha = \chi_{O',w}\{n\}$ and $\beta = \chi'_{O',w}\{n + t\}$. The coefficient of $-X$ in the defining lift polynomial is

$$s_{w,1} = q^t \alpha \beta^{-1} z_{w,1} + \alpha \sum_{e=0}^{t-1} q^e.$$

The geometric Frobenius value of the normalizing character is α^{-1}. Since the target rank is even, (DP6) gives

$$q^t/\beta = \vartheta(\varpi_w/\bar{\varpi}_w)^{-1} \mu_O(\varpi_w/\bar{\varpi}_w)^{-1} q^{(t-n)/2}.$$

Attachment and (2) now give

$$\operatorname{tr} S'(g) = \psi_\vartheta(g) \operatorname{tr} R(g) + \sum_{e=0}^{t-1} \chi_{\mathrm{cyc}}(g)^{-e}$$

on those geometric Frobenii. They are dense for continuous class functions: Chebotarev applies outside every prescribed finite set, and places of residue degree greater than one have density zero. The same equality therefore holds for every $g \in G_E$. □

Lemma 7.5.2. *Let F be a finite extension of $\mathbb{Q}_p$, with residue field k of order q, let $K_n = \mathrm{GL}_n(\mathcal{O}_F)$, and use the local Langlands correspondence with geometric Artin reciprocity. Let $d_1, \ldots, d_s \geq 1$ and $t \geq 0$, with $n = d_1 + \cdots + d_s + t \geq 1$. For each j let k_j/k be the extension of degree d_j, and let $\theta_j : k_j^\times \to \mathbb{C}^\times$ be a regular character, meaning that its Frobenius orbit has size d_j. Let ρ_j be the irreducible cuspidal Green representation of $\mathrm{GL}_{d_j}(k)$ associated to θ_j. Suppose that the pairs $(d_j, [\theta_j])$ are distinct, where brackets denote Frobenius orbit. If $t > 0$, let $\xi : k^\times \to \mathbb{C}^\times$ be a character, and suppose that no $d_j = 1$ has $\theta_j = \xi$.*

Let λ be the inflation to K_n of the finite Harish–Chandra induction

$$\operatorname{Ind}_{\bar{P}}^{\mathrm{GL}_n(k)} \left(\rho_1 \boxtimes \cdots \boxtimes \rho_s \boxtimes (\xi \circ \det_{\mathrm{GL}_t(k)}) \right),$$

where $\bar{P}$ has the displayed blocks, and the last factor is omitted when $t = 0$. If an irreducible admissible smooth complex representation π of $\mathrm{GL}_n(F)$ contains λ on restriction to K_n, then its Weil–Deligne representation (r_π, N_π) satisfies $N_\pi = 0$ and

$$r_\pi|_{I_F} \simeq \bigoplus_{j=1}^{s} \bigoplus_{\sigma \in \mathrm{Gal}(k_j/k)} (\theta_j^\sigma \circ \mathrm{Art}_{F_j}^{-1})|_{I_F} \ \oplus \ (\xi \circ \mathrm{Art}_F^{-1})|_{I_F}^{\oplus t}.$$

Here F_j/F is the unramified extension of degree d_j, the characters are inflated through residue units, and the last summand is omitted if $t = 0$.

Proof. We first identify the compact representation to which the depth-zero occurrence theorem applies. For every finite cuspidal representation ρ occurring here choose a depth-zero supercuspidal representation π_ρ containing it. Such a representation is obtained by extending ρ to $F^\times \mathrm{GL}_{d_\rho}(\mathcal{O}_F)$ with any compatible unitary central character and compactly inducing. The intertwining test is particularly simple: in the Cartan decomposition, unequal valuations in a diagonal representative induce a proper parabolic on the finite block; its unipotent radical has no invariants on the cuspidal ρ. Thus the intertwining is exactly $F^\times \mathrm{GL}_{d_\rho}(\mathcal{O}_F)$, and the compact-induction irreducibility criterion applies. For a rank-one character choose its extension to $F^\times$ directly.

For the Green factor ρ_j, the regular depth-zero local Langlands theorem [74, Proposition 2.4.1(i)] identifies

$$\tau_j := \mathrm{rec}_F(\pi_{\rho_j})|_{I_F} = \bigoplus_{\sigma \in \mathrm{Gal}(k_j/k)} (\theta_j^\sigma \circ \mathrm{Art}_{F_j}^{-1})|_{I_F}, \qquad N_{\pi_{\rho_j}} = 0.$$

Regularity gives the full orbit; distinct pairs $(d_j, [\theta_j])$ give distinct inertial cuspidal lines. If $t > 0$, the line of the extension of ξ is different from all these lines: a basic tame orbit can equal a rank-one orbit only if its degree is one, and the latter equality is excluded by hypothesis. Local Langlands for a character identifies its inertia with $\xi \circ \mathrm{Art}_F^{-1}$.

Let S be the finite cuspidal support consisting of one copy of every ρ_j and, when $t > 0$, t copies of the rank-one cuspidal character ξ. Let P_0 put the partition (1) on each ρ_j and $(1, \dots, 1)$, with t parts, on ξ. The finite Green representation $\bar{\sigma}_{P_0}$ is precisely

$$\mathrm{HC}{-}\mathrm{Ind}(\rho_1 \boxtimes \cdots \boxtimes \rho_s \boxtimes (\xi \circ \det_{\mathrm{GL}_t(k)})). \tag{*}$$

We justify the particular spherical choice in this equality. For the trivial rank-one cuspidal character, the partition $(1, \dots, 1)$ is the trivial representation of $\mathrm{GL}_t(k)$, while (t) is Steinberg. Twisting by $\xi \circ \det$ gives the stated character block for ξ. If $s = 0$, (*) is just the character $\xi \circ \det$ by the partition convention in Theorem 7.1.18; no distinct-cuspidal theorem is needed. For $s \geq 1$, for the product of the distinct cuspidal factors ρ_j, use Theorem A.5.29: they satisfy its distinctness hypothesis by the distinct Frobenius orbits, so their finite Harish–Chandra product is the irreducible Green representation on those factors. When $t = 0$ this is already (*). When $t > 0$ it remains to check the additional, possibly noncuspidal, character block; this is the following finite comparison. In this case its irreducibility can also be read directly from Mackey's formula. A double coset contributing to its endomorphism algebra must carry every cuspidal block whole to a block of the same rank and the same cuspidal representation; splitting one induces proper-parabolic unipotent invariants and gives zero. Distinctness rules out whole-block exchanges, and the ξ-block cannot exchange a constituent with a ρ_j by the disjointness just checked. The remaining double coset is the identity and has one-dimensional intertwining. Thus the induced representation in (*) has endomorphism algebra $\mathbb{C}$, so it is irreducible by

finite-group semisimplicity. Its Green data are the union of the displayed factor data, proving (*).

There is a literal finite averaging check of this spherical block. Refine the last block into t one-dimensional blocks and twist its character to 1. The finite Iwahori Hecke operators T_w, $w \in \mathfrak{S}_t$, are the double-coset sums with value 1 at the chosen Weyl representative. They satisfy

$$T_s^2 = (q-1)T_s + q, \qquad e_t = \frac{\sum_{w\in\mathfrak{S}_t} T_w}{\sum_{w\in\mathfrak{S}_t} q^{\ell(w)}}.$$

Pairing w with sw in the sum gives $T_s e_t = e_t T_s = q e_t$, and the square of the numerator is the numerator times the displayed denominator. Hence e_t is an idempotent. It is exactly averaging over $\mathrm{GL}_t(k)$ on the trivial block, since its Bruhat cell has $q^{\ell(w)}$ cosets. Therefore its image in the refined parahoric multiplicity space is the multiplicity space for the coarser parahoric with its trivial $\mathrm{GL}_t(k)$ block. Frobenius reciprocity identifies the latter with the compact type in (*). For $t = 0, 1$ this is the identity. For $t = 2$ the idempotent is $(1+T)/(1+q)$: it fixes the spherical eigenvalue $T = q$ and kills the Steinberg eigenvalue $T = -1$. This calculation fixes both the normalization and the choice of partition.

We now compare this actual finite representation with the compact representations in the published occurrence theorem, without using an abstract uniqueness assertion about inertial types. Apply [278, Theorem 6.20 and Corollary 6.21(1)] in the depth-zero form of Theorem 7.1.18. Write $\bar{\nu}_{P_0}$ for the finite representation there. For every partition function P' in the fixed cuspidal series S, that theorem gives

$$\langle \bar{\nu}_{P_0}, \bar{\Pi}_{P'} \rangle = \langle \bar{\sigma}_{P_0}, \bar{\Pi}_{P'} \rangle.$$

Here the brackets are ordinary finite character inner products. The finite irreducible characters $\bar{\sigma}_Q$, indexed by the partition functions Q in S, are an orthonormal basis for its representation group. The test characters $\bar{\Pi}_{P'}$ are another basis after tensoring with $\mathbb{Q}$, since their multiplicity matrix is unitriangular. Write $[\bar{\nu}_{P_0}] - [\bar{\sigma}_{P_0}] = \sum_Q b_Q [\bar{\sigma}_Q]$. The displayed equalities multiply the vector (b_Q) by that invertible matrix and give zero. Hence every $b_Q = 0$. Finite complex representations are semisimple, so equality of their classes is an actual isomorphism:

$$\bar{\nu}_{P_0} \simeq \bar{\sigma}_{P_0}.$$

In the depth-zero construction the compact group is K_n itself, the simple character is trivial and its extension is trivial; thus these finite representations are inflated to exactly the same K_n. This comparison uses the published multiplicity characterization and the explicit finite averaging above.

By hypothesis $\pi|_{K_n}$ contains the representation in (*), and hence contains $\mathrm{Inf}_{K_n} \bar{\nu}_{P_0}$. The occurrence theorem now says that the inertial cuspidal support of its local Langlands parameter is S, with the basic parameters τ_j and the character of ξ identified above, and that its monodromy partition on each line is dominated by P_0. A partition of an integer m dominated by $(1, \dots, 1)$ can have no part greater than 1: the dominance inequality with one part already gives this. As its total is m, all its parts are 1. Consequently every Weil–Deligne segment has length one and $N_\pi = 0$. The same occurrence assertion gives its full semisimple inertia. Finite inertia is semisimple in characteristic zero, so this is the actual inertia representation and is the direct sum displayed in the statement.

All arguments take place over $\mathbb{C}$. The unramified rank-dependent twist in the published local Langlands normalization is trivial on inertia and preserves zero monodromy, so it does not alter either conclusion. The coefficient-prime application therefore requires no exclusion of a residue prime at this local step. □

Definition 7.5.3. Fix a prime ℓ and an isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. The decomposition and inertia groups $G_A \supseteq I_A$, the ℓ-adic charts, the fixed places A_v and charts (e_v, j_v) for $v \mid \ell$, and the local lattices $L_v(u)$ are those of Definition 7.2.2. For $g \in G_\ell$ let $\gamma_g\colon \mathcal{O} \to \mathbb{C}_\ell$, $a \mapsto g(a)$.

Let $R\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be continuous.

(RT1) Killing. Let A be a place of $\overline{\mathbb{Q}}$, $s \in I_A$ and $Q \in \overline{\mathbb{Q}}_\ell[X]$. Q **kills** s **on** R **at** A if $Q(R(s)) = 0$ when $A \nmid \ell$, and, when $A \mid \ell$, for every chart (e, j) at A and every lattice u of R over a coefficient ring containing the coefficients of Q, Q kills s on $\mathcal{O}^n$ with G_A acting through u (via j).

(RT2) Tame exponents. Let w be a finite place of E with residue field $\mathbb{F}_w$, A a place of $\overline{\mathbb{Q}}$ over w, $T\colon \overline{\mathbb{F}}_w^\times \to \mathbb{C}^\times$ an injective character (one is fixed once and for all, the **faithful character** of $\mathbb{F}_w$) and $f \geq 1$. An element $s \in I_A$ has **tame exponent** $a \in \mathbb{N}$ (for T, f) if there are a uniformiser π of E at w and $z \in \overline{\mathbb{Q}}$ with $z^{\mathrm{N}(w)^f - 1} = \pi$ such that $s(z)/z$ is integral at A and T sends its residue at A (in $\overline{\mathbb{F}}_w$, through a residue map extending $\mathcal{O}_E \to \mathbb{F}_w$) to $\exp(2\pi i\, a/(\mathrm{N}(w)^f - 1))$.

(RT3) Labels. For $c \in \mathbb{F}_w[X]$, R is **labelled by** c **at** A (for T, f) if for every $s \in I_A$ of tame exponent a the polynomial

$$\prod_x (X - \iota(T(x)^a)), \qquad x \text{ over the roots of } c \text{ in } \overline{\mathbb{F}}_w \text{ with multiplicity},$$

kills s on R at A. For a character χ of $\mathbb{F}_w^\times$, $x_\chi \in \mathbb{F}_w^\times$ is the element that **names** χ: $T^a(x_\chi) = \chi(\bar{u})$ whenever s has tame exponent a and is the local Artin symbol of a unit u with residue $\bar{u}$; $\chi \mapsto x_\chi$ is a group isomorphism.

(RT4) Level S**.** Let $k \geq 0$, J a function from the embeddings $E \to \mathbb{C}$ to finite multisets of integers, and S a set of finite places of E. R is **conjugate self-dual of twist** k**, jumps** J **and level** S if it is conjugate self-dual of twist k (MS1) and at every finite place v with $v \notin S$ or $v \mid \ell p_0 p_1$, and every A over v, R is of finite type at A, has the Hodge–Tate jumps J at A, and is good at A if $v \nmid p_0 p_1$. Level $\emptyset$ is (MS2).

(RT5) Galois data of a descent pair. Of a descent pair O the Galois side uses only: its index set I_O of chosen places w_i ($i \in I_O$); its source rank r_O (the rank of its space W) and rank n_O (that of V); the rank difference $t_O = n_O - r_O$; the tame part $\psi_O = (\psi_{O,w})_w$ of its character of $\mathrm{U}(1)$.

(RT6) Representations of a type. Let $c = (c_i)_{i \in I_O}$ with $c_i \in \mathbb{F}_{w_i}[X]$. R is a **representation of type** (O, k, J, c, S) if (RT4) holds and, for every $i \in I_O$ and every place A of $\overline{\mathbb{Q}}$, R is labelled by c_i at A for the faithful character of $\mathbb{F}_{w_i}$ and $f = r_O!$ (the condition is empty unless A lies over w_i).

(RT7) Scaled type polynomial and Ribet type. The **scaled type polynomial** $\mathrm{tl}(c, i) \in \mathbb{F}_{w_i}[X]$ is the polynomial of degree $\deg c_i$ whose roots are those of c_i multiplied by the element $x_{\psi_{O,w_i}^{-1}}$ naming the inverse of the tame part at w_i; this is the scaled type polynomial of (MS5). For $t \geq 0$ the **Ribet type** of (O, c) is $c^{\sharp,t} = \big(\mathrm{tl}(c, i)\,(X-1)^t\big)_i$, and R is **of Ribet type** $(O, c; t, J, S)$ if it is a representation of type $(O, t, J, c^{\sharp,t}, S)$.

Block lattices. Let $r' \geq 1$, $t \geq 0$ and $T\colon G_E \to M_{r'}(\mathcal{O})$. The **summands** of T with string t are $\mathrm{blk}_T(*) = T$ and $\mathrm{blk}_T(e) = \mathcal{O}(-e)$, $g \mapsto \chi_{\mathrm{cyc}}(g)^{-e}$, for $0 \leq e < t$ (χ_{cyc} the ℓ-adic cyclotomic character). For a bijection ϵ from $\{*\} \times \{1, \dots, r'\} \sqcup \{0, \dots, t-1\}$ to an index set n,

the **block lattice** $u = \epsilon_*(T \oplus \bigoplus_{e<t} \mathcal{O}(-e)) \colon G_E \to M_n(\mathcal{O})$ is the block-diagonal homomorphism with these blocks, placed by ϵ.

Ribet cocycles and crystalline colength. Let $w = (w_x)_{x \in Y}$ be homomorphisms $G_E \to M_\sigma(\mathcal{O})$, $\rho \colon G_E \to M_d(\mathcal{O})$ and $a \in \mathcal{O}$. A **relation** is a finitely supported $\theta \colon G_E \to M_d(\mathcal{O})$, and $c \colon G_E \to B^d$ **obeys** θ if $\sum_g \theta(g)c(g) = 0$. The **Ribet cocycles**

$$\mathcal{Z}_w(\rho; a) = \Big\{ c \colon G_E \to (\mathcal{O}/a)^d \text{ a cocycle for } \rho : \sum_g \vartheta(g)\, c(g) = 0$$
$$\text{for every finitely supported } \vartheta \colon G_E \to \mathcal{O} \text{ with } \sum_g \vartheta(g)(w_x(g) - 1) = 0 \ \forall x \Big\}$$

form an $\mathcal{O}$-module (no continuity is imposed). The **crystalline relations** $\mathcal{R}_{\mathrm{cris}}(\rho)$ are the elements $\theta \colon G_{A_v} \to M_d(\mathcal{O})$ of $\mathcal{R}_f(L_v(\rho))$ (the relations obeyed by every element of $Z_f(L_v(\rho))$, Definition 7.2.1), for some $v \mid \ell$, extended by zero to G_E. The **crystalline colength** is

$$\mathrm{cc}(\rho; w, M) = \mathrm{length}_{\mathcal{O}}\Big(\mathcal{Z}_w(\rho; \ell^M) / \big(\mathcal{Z}_w(\rho; \ell^M) \cap \{c \text{ obeying every } \theta \in \mathcal{R}_{\mathrm{cris}}(\rho)\} \big) \Big)$$
$$\in \mathbb{N} \cup \{\infty\} \qquad (M \geq 0).$$

Lemma 7.5.4. *Let $F/\mathbb{Q}_p$ be finite with residue field k of cardinality q, and fix an injective character $T : \overline{k}^\times \to \mathbb{C}^\times$. Let $f \geq 1$, let $s \in I_F$ have tame exponent a at level f, and let $d \mid f$, $d \geq 1$. For $x \in k_d^\times$, where k_d/k has degree d, define its cyclic-pairing character θ_x by*

$$\theta_x(u) = T(x)^b \quad \text{if} \quad T(u) = \exp(2\pi i b/(q^d - 1)).$$

Let F_d/F be the unramified extension of degree d. Then

$$(\theta_x \circ \mathrm{Art}^{-1}_{F_d,\mathrm{geo}})(s) = T(x)^a.$$

The left side uses inflation from residue units, so it is independent of the chosen unit representative under reciprocity. This identity holds for every Frobenius conjugate of x. If $\chi : k^\times \to \mathbb{C}^\times$ is named by x_χ as in (RT3), multiplying this inertial character by $\chi \circ \mathrm{Art}^{-1}_{F,\mathrm{geo}}$ replaces its parameter x by $x_\chi x$. In particular an inverse character replaces it by $x_\chi^{-1} x$.

Proof. Put $m = q^d - 1$ and $N = q^f - 1$. Choose the uniformiser π and z in the tame-exponent definition, so $z^N = \pi$ and

$$T(\overline{s(z)/z}) = \exp(2\pi i a/N).$$

Set $z_d = z^{N/m}$. Inertia fixes all prime-to-p roots of unity and F_d, and therefore

$$z_d^m = \pi, \qquad T(\overline{s(z_d)/z_d}) = \exp(2\pi i a/m). \tag{1}$$

The quotient in (1) is an mth root of unity; reduction on such roots is injective.

Here is the reciprocity sign calculation. In any local field K with residue cardinality Q and containing μ_m, $m \mid Q - 1$, the Hilbert symbol with arithmetic reciprocity satisfies

$$\mathrm{Art}_{K,\mathrm{ar}}(b)(a^{1/m})/a^{1/m} = (a, b)_m, \qquad (a, b)_m = (b, a)_m^{-1}. \tag{2}$$

These are [202, Chapter III, Theorem 4.4(b), Remark 4.5]. To evaluate them for a unit u and uniformiser π, choose $v^m = u$. Such a root lies in an unramified extension: choose a residue root

and lift it by Hensel's lemma, since m is prime to p. Arithmetic Frobenius sends its residue to its Qth power. Hence

$$\overline{(u,\pi)_m} = \bar{u}^{(Q-1)/m}, \qquad \overline{\mathrm{Art}_{K,\mathrm{geo}}(u)(\pi^{1/m})/\pi^{1/m}} = \bar{u}^{(Q-1)/m}. \tag{3}$$

The second equality follows by first using skew symmetry in (2), then inverting the arithmetic Artin map. This accounts for both inversions; it is not the arithmetic formula with its sign unchanged.

Apply (3) to $K = F_d$, whose residue cardinality is $Q = q^d$, with the present $m = Q - 1$. The image of inertia in the abelianized local group is the image of the unit group, by local reciprocity Theorem A.2.7. Thus take a unit u representing the abelianized image of s. Equation (3) and (1) give

$$T(\bar{u}) = \exp(2\pi i a/m).$$

Changing the representative changes neither its residue nor the value of any residue character: its tame abelian image is detected faithfully by its action on z_d. By the defining cyclic pairing, $\theta_x(\bar{u}) = T(x)^a$, as required. Replacing x by any q-power conjugate proves the conjugate assertions.

For the final assertion apply the already proved identity with $d = 1$ to $\chi = \theta_{x_\chi}$. It gives

$$(\chi \circ \mathrm{Art}^{-1}_{F,\mathrm{geo}})(s) = T(x_\chi)^a.$$

Multiplying this equality by the identity for x gives $T(x_\chi x)^a$, which is the identity for parameter $x_\chi x$. This holds for every inertia element: its quotient $s(z)/z$ belongs to μ_N, and injectivity of T therefore assigns it some exponent a modulo N. The inverse-character version follows by inversion. No regularity assumption, factorial bound or coefficient-prime restriction is used. □

7.6 Denominators: Ribet elements and prime triangles

This section bounds the denominators of the projections to the spaces of lifts. For the twist at one prime the result is Lemma 7.6.2: if the twisted main summands have Selmer length at most e'_ℓ, the projection to the lifts has denominator at most $\beta_\ell \log \ell$ at ι. It is the case of the lifts of a general bound (Theorem 7.6.3).

A denominator measures congruences between lifts and the forms orthogonal to them. The systems of the lifts are weak: their normalised Galois representations contain the trivial character, so the functions $\delta_g(x) = \det(1 - N_x(g))$, which lie in the integral Hecke algebra (Lemma 7.6.13), vanish on them. Ribet's idea [241], in the form of Mazur and Wiles [197, 327], is that the congruences between such systems and the others are measured by Galois cohomology classes of the reducible representation. Here it takes the form of a common Ribet element of the functions δ_g (Theorem 7.6.6), whose top coefficient is bounded by the Selmer length, the ramification of the twist, the local loss above ℓ, and the odd classes with the cost of a prime triangle; under the hypotheses this is at most $r'\beta_\ell$. Auxiliary operators turn the Ribet element into an integral operator that clears the projection (Proposition 7.6.11).

Ribet's element rests on Ribet's lemma to first order in tie variables over a discrete valuation ring (Lemma 7.6.7), through Fitting ideals [79, 227], Amitsur's formula for the determinant of a sum of matrices [6] and a Cayley–Hamilton argument; working with the traces and determinants of families goes back to the pseudo-representations of Wiles [326] (see [55, 21, 19]). The cocycles of the block are bounded by the Selmer length and the local loss of Section 7.7. The cocycles of the Tate twists are controlled by prime triangles, and no bound for their even classes is used: along a triangle their Selmer modules are bounded by the odd classes (Proposition 7.6.10), which Theorem 7.2.8 bounds at all primes at once.

The auxiliary places come from two existence results. Prime triangles exist at every level (Theorem 7.6.15); they are built from Kummer maps and Chebotarev's theorem, in the manner of Kolyvagin [169, 244], with a reciprocity law from Poitou–Tate duality [295, 201]. Taylor–Wiles places give chains of Hecke roots (Proposition 7.6.12), from a local statement on spherical and pro-p Iwahori Hecke operators (Lemma 7.6.5; see [248, 35, 50, 114]); places of this kind go back to Taylor and Wiles [301, 328] (see [63, 60, 305]). In the dictionary at an auxiliary place (Lemma 7.6.13) tame generators act as diamond operators, Frobenius acts through U_{ϖ}, and the pro-p Iwahori eigencharacters are read as Jacquet exponents [51, 310].

Definition 7.6.1. Let E be a CM field, $n \geq 1$, V the standard definite hermitian space of rank n over E with unitary group $\mathrm{U}(V)$, κ a weight, ℓ a prime and $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ an isomorphism; $\bar{\mathbb{Z}}_\ell$ is the ring of integers of $\bar{\mathbb{Q}}_\ell$. For a finite place w of E, k_w is its residue field, $q_w = \#k_w$.

(AL1) Base data. Base data B consist of a finite set B_{pl} of finite places of E, a space of forms of weight κ on V of a fixed type at B_{pl}, and a common multiple c_B of the orders of the finite groups $\mathrm{U}(V)(E^+) \cap g\mathrm{K}_0 g^{-1}$, where K_0 is the standard maximal level.

(AL2) Level L and systems. For a finite family $L = (w_j)$ of places split over E^+ and outside $B_{\mathrm{pl}} \cup \bar{B}_{\mathrm{pl}}$, the **forms of level** L have the type of B at B_{pl}, principal level at the w_j (the kernel of reduction to $\mathrm{GL}_n(k_{w_j})$) and maximal level elsewhere; $L = \emptyset$ is the **base level**. A form is **integral** if its values are ι-integral. A **system of level** L is a space of forms with given Hecke eigenvalues at almost all places that contains a nonzero form of level L; it has a semisimple Galois representation $r_x\colon G_E \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, and at a split place $w \nmid \ell$ where it is unramified a **Hecke polynomial** $P_{x,w} = \prod_\alpha (X - q_w^{(n-1)/2}\alpha)$, α over its Satake parameters. A place w is **off level** L if it lies outside $B_{\mathrm{pl}} \cup \bar{B}_{\mathrm{pl}}$, does not divide ℓ, and the forms of level L of every system are Hecke eigenforms at w.

(AL3) Integral Hecke algebra. $\mathbb{T}_L$ is the ring of functions t from the systems of level L to $\bar{\mathbb{Q}}_\ell$ such that the operator acting on each system x by the scalar $\iota^{-1}(t(x))$ preserves the integral forms of level L.

(AL4) Traces and the tied ideal. Let $a\colon G_E \to \bar{\mathbb{Z}}_\ell^\times$ be a character and $N_x = a \otimes r_x$. Two systems x, x' are **congruent** if for all $g \in G_E$ and $i \geq 0$ the i-th coefficients $c_i(x,g)$, $c_i(x',g)$ of the characteristic polynomials of $N_x(g)$ and $N_{x'}(g)$ satisfy $|c_i(x,g) - c_i(x',g)|_\ell < 1$. For $P \in \bar{\mathbb{Z}}_\ell[X]$ and $g, h \in G_E$ write $\mathrm{tr}[P(h)g]$ for $x \mapsto \mathrm{tr}\big(P(N_x(h))N_x(g)\big)$. For Taylor–Wiles places $Q = (w_j, \varphi_j, \tau_j)_{j\in J}$ of level ℓ^ν, polynomials $H_j \in \bar{\mathbb{Z}}_\ell[X]$ and variables α_j, the **tied ideal** $\mathfrak{a} \subseteq \mathbb{T}_Q[\alpha_j]_{j\in J}$ is generated by the $\ell^\nu \mathrm{tr}[g]$ and the traces $\mathrm{tr}[r_j g]$ of the **tied relations** $r_j = \tau_j - 1 - \alpha_j(\varphi_j - 1)H_j(\varphi_j)$, $g \in G_E$ (when these traces lie in $\mathbb{T}_Q[\alpha_j]$).

(AL5) Root chains. A system x of base level has a **root chain** at w_j for (a, t, H_j) if there is $S \in \mathbb{C}[X]$ dividing $\iota^{-1}(H_j)$ (the polynomial obtained by applying ι^{-1} to the coefficients of H_j) with

$$P_{x,w_j}\big(\iota^{-1}(a(\varphi_j))^{-1}X\big) = \prod_{0\leq i<t} (X - q_{w_j}^i) \cdot S(X).$$

(AL6) Trace ideals and Ribet elements. Let $\mathcal{O}$ be a discrete valuation ring with fraction field K, G a group, Y a set, $u_x\colon G \to M_m(K)$ for $x \in Y$, $p \in \mathcal{O}$, $H_j \in \mathcal{O}[X]$ for $j \in J$, and a finite set T of relations with $\tau_t, \varphi_t \in G$ and an index $j_t \in J$ for each $t \in T$ (several relations may share one variable); let S be an $\mathcal{O}$-algebra of families $Y \to K[X_j]_{j\in J}$ containing the variables and the coefficients of the characteristic polynomials of the $u_x(g)$, and $\mathcal{J}$ an S-submodule. Then $\mathcal{J}$ is a **trace ideal** if it contains $x \mapsto p\,\mathrm{tr}\,u_x(g)$ and $x \mapsto \mathrm{tr}\big(u_x(\rho_t)u_x(g)\big)$

for the tied relations $\rho_t = \tau_t - 1 - X_{j_t}(\varphi_t - 1)H_{j_t}(\varphi_t)$, $t \in T$, and all $g \in G$ (with u_x extended linearly). For functions $\beta_i \colon Y \to K$ that vanish on a subset $\mathcal{X} \subseteq Y$, a point $\pi \in \mathcal{X}$ and $d \in K$, a **common Ribet element** with coefficient d at π is $f \in S$ such that the constant families β_i lie in S, $f\beta_i \in \mathcal{J}$ for all i, for $x \in \mathcal{X}$ the polynomial $f(x)$ has no monomial properly dividing $\prod_j X_j$, and the coefficient of $\prod_j X_j$ in $f(\pi)$ is d.

(AL7) Iwahori operator. Let w be a finite place of E and ϖ a uniformiser of E_w. The **prounipotent Iwahori subgroup** $\mathrm{Iw}_1(w) \subseteq \mathrm{GL}_n(\mathcal{O}_{E_w})$ is the group of matrices whose reduction modulo ϖ is upper triangular with all diagonal entries 1; let $t_\varpi = \mathrm{diag}(\varpi, 1, \dots, 1)$. For a smooth representation π of $\mathrm{GL}_n(E_w)$ on a complex vector space W, the **Iwahori operator** U_ϖ on the $\mathrm{Iw}_1(w)$-fixed vectors is

$$U_\varpi y = \sum_{h \in \mathrm{Iw}_1(w) t_\varpi \mathrm{Iw}_1(w)/\mathrm{Iw}_1(w)} \pi(h)\, y,$$

the sum over representatives h of the finitely many cosets $h\,\mathrm{Iw}_1(w)$ contained in

$$\mathrm{Iw}_1(w) t_\varpi \mathrm{Iw}_1(w);$$

it does not depend on the representatives and maps $\mathrm{Iw}_1(w)$-fixed vectors to $\mathrm{Iw}_1(w)$-fixed vectors.

Lemma 7.6.2. *Let D be admissible data, ℓ a prime, $\iota \mid \ell$ a prime identification, ϑ a twist such that $(S_\ell, Q_\vartheta, \psi_\vartheta)$ is an auxiliary twist with ramification at most e'_ℓ, and $1 \le j \le m$.*

1. *The space $\mathcal{L}_{j,\vartheta}$ is spanned by forms $y \in \mathcal{L}_{j,\vartheta}$ that lie in the range of the type projection of a lift setting for ϑ contained in $e_0\mathcal{A}(V^{\mathrm{def}})$, and whose lift Hecke polynomials are those of a system z of Hecke eigenvalues occurring in the good forms on W_m.*
2. *Suppose every such system z has a main summand R_z for $((W_j, V^{\mathrm{def}})$ at $1, z)$ with $\psi_\vartheta R_z$ of Selmer length at most e'_ℓ at $(Q_\vartheta, \gamma, \zeta)$ for some γ, ζ. Then the projection to $\mathcal{L}_{j,\vartheta}$ has denominator at most $\beta_\ell \log \ell$ at ι for e_0, and ϑ satisfies (DL1) at ι for j with every $\beta' \ge \beta_\ell \log \ell$.*
3. *Let p be a main pair for j and R a main summand for $(O_{p,1}, 1)$ with $\psi_\vartheta R$ of Selmer length at most e'_ℓ at some $(Q_\vartheta, \gamma, \zeta)$. Then the projection to $\mathbb{C}\lambda_{p,\vartheta}$ has denominator at most $\beta_\ell \log \ell$ at ι for e_{X-1}, and for all test forms Ξ, Ξ' for ℓ of $O_{p,\vartheta}$*

$$\left\|\iota\langle \mathrm{pr}_{p,\vartheta}\Xi, \mathrm{pr}_{p,\vartheta}\Xi'\rangle\right\| \le \exp((\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log \ell).$$

Proof. Put $\beta = \beta_\ell \log \ell$. First the auxiliary-twist hypothesis also holds with S_ℓ^{bad} in place of S_ℓ. Indeed a twist has $\mathrm{N}Q_\vartheta$ prime to MN, so neither a place of Q_ϑ nor its conjugate divides MN. The assumed auxiliary condition excludes the places above ℓ, their conjugates, and conjugate pairs in Q_ϑ. Thus Q_ϑ is auxiliary for the places dividing ℓMN. The character and ramification conditions in (AX5) are unchanged. This verifies exactly the auxiliary hypothesis whenever we use Theorem 7.6.3.

Spanning forms and their actual types. Write $O = O_{j,\vartheta}$, $\mathcal{A}_j = \mathcal{A}_M(W_j)$, and $E_c = \bigotimes_w e_{c_w}$ for a good family c. The finiteness and spectral decomposition in Proposition 5.8.4 give

$$e_0\mathcal{A}_j = \bigoplus_z (e_0\mathcal{A}_j \cap \mathcal{A}_j[z]),$$

with only finitely many nonzero summands. Chosen-place translations commute with the spherical operators off the chosen places: the adelic translations have disjoint local supports. Consequently every E_c preserves every eigenpart. Since $e_0 = \sum_{c\ \mathrm{good}} E_c$, the good forms are spanned by vectors $a = E_c a_z$, where $a_z \in \mathcal{A}_j[z]$. Their images

$$y = \Theta_{j,\vartheta}(a)$$

span $\mathcal{L}_{j,\vartheta}$. By Lemma 6.6.10(i), extended linearly to the group algebra,

$$\Theta_{j,\vartheta}(E_c a_z) = p_c \Theta_{j,\vartheta}(a_z). \tag{1}$$

Thus y lies in the range of this very type projector. Moreover $e_0 E_c = E_c$, so the whole range of p_c, not just the vector y, lies in $[e_0]\mathrm{Form}(O)$.

We check that (O, c) is a lift setting. Its source rank is r, its rank difference is $t = 5r$, and both are odd by (AD2). Its places are the chosen places and Q_ϑ. By (LF1)–(LF4) its tame part is $\omega^j \psi_\vartheta$, with $1 \le j \le m$. Every c_w is a label of degree r, and over each of p_0, p_1 one is primitive, by goodness. At a chosen place the twist has trivial tame part, because its modulus is prime to M. Let $x_w \in k_w^\times$ be the element naming the character ω_w^j under the fixed faithful character. Then $x_w^{q_2} = 1$. The scaled type polynomial is

$$x_w^{-r} c_w(x_w X).$$

It vanishes at 1 precisely when $c_w(x_w) = 0$, which the allowed-for-q_2 condition excludes. This checks every condition of (MS5). The scaling in this check concerns the type polynomial in the lift setting; the tuple in (1) is the same c, since the action (DP5) already incorporates its determinant character.

The full Hecke polynomials of these images. We give the polynomial identity, rather than infer it from vanishing at the lift root. Let $w \ne \bar{w}$ be off the places of O, put $q = \mathrm{N}w$, $n = r + t$, and

$$\alpha = \chi_{O,w}\{r\}, \qquad \alpha' = \chi'_{O,w}\{n\}, \qquad c_w^* = q^t \alpha/\alpha', \qquad P_w(X) = \prod_{b=0}^{t-1} (1 - q^b \alpha X).$$

In this paragraph c_w^* is just a scalar, distinct from the label c_w. For $a \in \mathcal{A}_j[z]$ we claim

$$H_w(X)\Theta_{j,\vartheta}(a) = P_w(X)\, H^{(r)}_{z(w)}(c_w^* X)\, \Theta_{j,\vartheta}(a) \qquad (X \in \mathbb{C}). \tag{2}$$

Here are the hypotheses and the transposition giving (2). By Lemma 6.5.11 and Lemma 6.4.15, the theta datum in (LF5) is an edge from the prescribed source type to the target type. Therefore the lift L before algebraic attachment and its transpose L^{t} satisfy Lemma 6.3.76(1)–(2). The datum is the vacuum at $w, \bar{w}$: these places are outside the finite block datum, and Lemma 6.2.33(5) fixes its finite vacuum under integral Heisenberg translations. These translations commute with the real creation operators, as their supports are disjoint. Its invariance under the target spherical subgroup follows from Lemma 4.1.1(6). The base change of $\chi_O = \upsilon^{r_-(O)}$ is unramified there. Thus Lemma 6.6.11 applies.

For clarity, this application uses the bilinear transpose, so no conjugation of X is involved. If F is a spherical form of the conjugate target type, transposition of the finite double cosets and Lemma 6.3.76(2) give

$$\int F\, H_w^{(n)}(X) L a = \int a\, L^{\mathsf{t}}(H_{\bar{w}}^{(n)}(X) F). \tag{3}$$

The inverse double coset at w is the positive double coset at $\bar{w}$, under the unitary identification; inversion preserves Haar measure. More explicitly the inverse of the coset of $\mathrm{diag}(\varpi_w 1_i, 1)$ at w acts at $\bar{w}$ by $\overline{(\,\cdot\,)^{-1}}^{\mathsf{T}}$, which is the positive coset of $\mathrm{diag}(\bar{\varpi}_w 1_i, 1)$. This proves the transposition coefficient by coefficient.

The actual kernel of L^{t} is $\chi'_O(\det g)^{-1}$ times the kernel with characters $(\chi_O, 1)$. Apply Lemma 6.6.11 at $\bar{w}$ to the latter kernel. Its first scalar is α; before this determinant twist its second scalar is $\upsilon(\varpi_w)^{-n} q^{n/2}$. Moving the factor $\chi'_O(\det g)^{-1}$ through the source double coset of number i multiplies its coefficient by $\chi'_O(\varpi_{\bar{w}}/\bar{\varpi}_{\bar{w}})^i = \widetilde{\chi'_O}(\varpi_w)^{-i}$. Consequently the second scalar becomes exactly $\alpha' = \widetilde{\chi'_O}(\varpi_w)\upsilon(\varpi_w)^{-n} q^{n/2}$, and the transfer reads

$$L^{\mathsf{t}}(H^{(n)}_{\bar{w}}(X)F) = P_w(X)\, H^{(r)}_{\bar{w}}(c_w^* X) L^{\mathsf{t}} F. \tag{4}$$

Transposing the source double cosets back in (3), and using $H^{(r)}_w(c_w^* X)a = H^{(r)}_{z(w)}(c_w^* X)a$, proves that the difference in (2), before algebraic attachment, pairs to zero with every such F. All the integrals and their finite translates are justified by Lemma 6.3.76(2). The difference G is itself a spherical form of the target type, so take $F = \bar{G}$. Positivity and full support of the quotient measure give $G = 0$. Algebraic attachment in (LF5) commutes with finite right translations, which proves (2) on $\mathrm{Form}(O)$. This is precisely (DP10), at every required place and every X.

For a nonzero spanning image $y = \Theta_{j,\vartheta}(a)$, its source vector a is nonzero. Hence z occurs in the good forms on W_j. By Proposition 5.8.4(ii)–(iii), the good z-parts on W_j and W_m have the same dimension, so z also occurs on W_m. Together with (1), (2), and the lift-setting check, this proves item 1. Zero images can be omitted from the spanning family.

Apply the denominator theorem to the space of lifts. Assume the hypothesis of item 2 and take one of the nonzero spanning forms just constructed. In (DR2) choose the tuple c from (1), the system z, the supplied main summand R_z, and $O' = O_{j,1}$. These two descent pairs have the same source and target ranks, and their characters in (DP2) satisfy the exact identity

$$\mu_{O_{j,\vartheta}} = \mu_{O_{j,1}}\, \vartheta. \tag{5}$$

This follows from $\mu_j^0 \vartheta$ in (LF4); it is an identity of the characters μ, not of the splitting factor χ_O. The hypothesis supplies the required γ, ζ and Selmer bound. Thus every chosen spanning form satisfies every clause of (DR2), with $\mathrm{sel} = e'_\ell$. Apply Theorem 7.6.3 to $Z = \mathcal{L}_{j,\vartheta}$ and $e = e_0$, using the auxiliary check at the beginning. It gives (DR1) with $\beta = \beta_\ell \log \ell$. The source rank here is exactly r. The integrality part of (DR1) is the clearing assertion (DL1), and for $\beta' \geq \beta$ the same endomorphism has

$$\|\iota(\det B)\| \geq e^{-(\dim Z) r\beta} \geq e^{-(\dim Z) r\beta'}.$$

This proves all of item 2, including $Z = 0$, where the determinant on the zero space is 1.

The actual rank-one lift generator. For item 3 write $p = (\nu, z_p)$, $X = X(p, \vartheta)$, $O = O_{p,\vartheta}$, and $\lambda = \lambda_{p,\vartheta} = \lambda_X$. Its source has rank 1, its target rank is $2z_p$, and its rank difference is $2z_p - 1$, an odd integer at most $2r - 1 \leq t$. Its character is $\nu\vartheta$, whose tame part is $\omega^j \psi_\vartheta$. Choose every label equal to $X - 1$. These are labels of degree one and satisfy the rank-one alternative over each level prime. Their scaled polynomials do not vanish at 1: the tame character of ν at a chosen place is $\omega_w^j \neq 1$. Indeed ω_w has order q_2, $1 \leq j \leq m < q_2$; the last inequality follows from $2^s = m^{22}$ and $q_2 = (q_0 d)^{s+6} > 2^s$. The twist is trivial at the chosen places. Thus $(O, (X-1)_w)$ is a lift setting for ϑ.

We verify the type projection on the actual form. By Lemma 6.7.14(a),(c), λ is the specified finite index times the algebraic theta lift of the constant function on U(1). For residue units

$s = (s_w)$, choose an integral source point a_s with these reductions at the chosen places and reduction 1 at Q_ϑ; choose a target point k_s acting by s_w on the inc-line and as the identity on its complement. The finite function in (DP7) satisfies $\Psi_O(k_s^{-1}(\,\cdot\,)a_s) = \Psi_O$: the chosen-place condition $X_w = \mathrm{Inc}_O$ is preserved by $k_s\mathrm{Inc}_O = \mathrm{Inc}_O a_s$, and the twist-place factors are unchanged. Since $\chi_O = 1$, Lemma 4.1.1(6) gives

$$\omega_O(a_s, k_s)\mathrm{J}_O(\xi) = \Big(\prod_w \psi_{O,w}(s_w)\Big)^{-1}\mathrm{J}_O(\xi).$$

The constant source function is unchanged by a_s; Lemma 6.3.76(3) therefore shows that target translation by k_s multiplies its lift by the reciprocal displayed character. The action (DP5) includes the character itself, so every $[s]$ fixes λ. For GL_1, the Green character of the label $X-1$ is the trivial character. Its tensor type projector is consequently the literal finite mean of the operators $[s]$, and fixes λ.

The same full polynomial calculation (3)–(4) applies here: Lemma 6.7.14(c) supplies the edge and the actual lift of $\mathbf{1}$, while each source Hecke operator on this constant function has eigenvalue 1. Explicitly, with $N = 2z_p$, $q = \mathrm{N}w$, $\alpha = \chi_{O,w}\{1\}$ and $\alpha' = \chi'_{O,w}\{N\}$, the source identity is $H_w^{(1)}(Y)\mathbf{1} = (1-Y)\mathbf{1}$, and hence

$$H_w^{(N)}(X)\lambda = \prod_{b=0}^{N-2}(1-q^b\alpha X)\,(1-q^{N-1}\alpha(\alpha')^{-1}X)\lambda.$$

This is exactly (DP10) for the system 1, for all X. Also

$$\mu_{O_{p,\vartheta}} = \mu_{O_{p,1}}\,\vartheta.$$

The given main summand and Selmer bound now verify (DR2) for λ in its line, with the actual projector e_{X-1}. Scalar multiples remain lift generators, and this line is spanned by λ. If $\lambda = 0$, its space is zero and the spanning assertion is vacuous. Applying Theorem 7.6.3 proves the denominator assertion of item 3 in either case.

Central averaging changes the test subspace. It remains to deduce the stated pairing bound; test forms are orthogonal to false lifts in $\ker \mathrm{Th}_O$, which need not equal $(\mathbb{C}\lambda)^\perp$. We make this change explicitly.

Let $A_f = \mathrm{U}(1)(\mathbb{A}_f)$ and define the normalised central action

$$T(a)f = \chi'_O(a)\,R(\mathrm{scalar}(a))f \qquad (a \in A_f). \tag{6}$$

Characters in (6) are evaluated at the finite point with infinite part 1. These operators are unitary, $T(a)^* = T(a^{-1})$, and commute with the chosen-place action and all Hecke polynomial operators, since scalar matrices commute with every right translation. The scalar-pair formula in (WR4), with $\chi_O = 1$, says that the pair $(a, \mathrm{scalar}(a))$ acts on the theta datum by $\chi'_O(a)^{-1}$. Applying (OL2) therefore gives the exact identities

$$T(a)^{-1}k_X(g) = k_X(ga), \qquad T(a)\lambda = \lambda. \tag{7}$$

The second follows by translating the class sum (OL3); multiplication by a^{-1} permutes the classes with their stabiliser weights. These identities also check the normalisation of (6).

Let $C_{\mathrm{ch}} = \mathrm{U}(1)(E)\backslash A_f/\mathrm{K}_{\mathrm{ch}}$ and $h = \#C_{\mathrm{ch}} = h_{\mathrm{ch}}$. It is finite by Lemma 6.7.22(a) and is nonempty because it contains the identity class. By Lemma 4.6.2(c), all its rational stabilisers are trivial. Choose representatives a_c integral at every place above ℓ. Apply Lemma 4.2.64

to the standard rank-one space I_1, with the finite set of all places of E above ℓ. Its auxiliary finite factor has component 1 there, so rational norm-one elements are dense in the product of these local groups. For an initial representative, the local inverse lies in the nonempty open set of multipliers making its product integral with integral inverse above ℓ. Choose a rational norm-one multiplier in that set. Left multiplication by it changes no double coset, giving the required representatives.

For these representatives $T(a_c)^{-1}$ preserves forms integral at every embedding above ℓ. In the gauge of Definition 6.1.3, translation by their scalar matrices and its inverse preserve the integral coordinate lattice, since all their ℓ-components are units. The scalar $\chi'_O(a_c)^{-1}$ is an ℓ-adic unit at every embedding. To check the latter from its algebraic weight, take a positive power killing the class of a_c in the finite abelian group C_{ch}, and write $a_c^d = \gamma u$, with $\gamma \in \mathrm{U}(1)(E)$ and $u \in \mathrm{K}_{\mathrm{ch}}$. The value on u is a product of tame character values, hence roots of unity. The value on γ_f is the inverse of its archimedean algebraic monomial. Every conjugate of γ above ℓ is a unit because both a_c and u are, so this monomial is an ℓ-adic unit. Taking the d-th power proves the assertion.

For any $v \perp \lambda$, (DP9), (OL2), and (7) show

$$\sum_{c \in C_{\mathrm{ch}}} T(a_c) v \in \ker \mathrm{Th}_O. \tag{8}$$

Indeed evaluation of the descent at a finite point g is pairing with $k_X(g)$; the infinite factor is scalar in rank one. By adjointness of T, its value on the sum is the pairing of v with $\sum_c T(a_c)^{-1} k_X(g) = \sum_c k_X(g a_c) = \lambda$. Here the last equality is the unweighted class sum, since the stabilisers are trivial; multiplying all classes by g permutes them.

Let Ξ, Ξ' be the test forms in the statement and put

$$\Psi = \sum_{c \in C_{\mathrm{ch}}} T(a_c)^{-1} \Xi'. \tag{9}$$

It is integral at ι. We claim that it is orthogonal to the false lifts for $(e_{X-1}, (\mathbb{C}\lambda)^{\perp})$. Fix an allowed split place w and a vector v in that false-lift subspace. Each $T(a_c)v$ remains in the range of e_{X-1} and in $\ker H_w(X_*(w))$, by the commutation just proved. By (8) their sum is in $\ker \mathrm{Th}_O$. The test condition on Ξ' therefore gives

$$\langle v, \Psi \rangle = \left\langle \sum_c T(a_c) v, \Xi' \right\rangle = 0.$$

This proves the claim at every allowed place. Since every $T(a_c)$ fixes λ and is unitary,

$$P_{\mathbb{C}\lambda} \Psi = h P_{\mathbb{C}\lambda} \Xi'. \tag{10}$$

If $\lambda = 0$, the desired pairing is zero. Otherwise its line has dimension one, so the endomorphism B furnished by (DR1) is multiplication by a scalar b, with $\|\iota(b)\| \geq e^{-\beta}$, because the source rank is one. Apply the pairing part of (DR1) to the integral, line-tested vector Ψ and the form Ξ, which is integral above ℓ. Using (10) gives

$$1 \geq \|\iota \langle \Xi, b P_{\mathbb{C}\lambda} \Psi \rangle\| = \|\iota(b)\| \, \|\iota(h \langle \Xi, P_{\mathbb{C}\lambda} \Xi' \rangle)\|.$$

Hence

$$\|\iota(h \langle \Xi, P_{\mathbb{C}\lambda} \Xi' \rangle)\| \leq e^{\beta}.$$

Orthogonal projection is self-adjoint, so the pairing here equals $\langle P_{\mathbb{C}\lambda}\Xi, P_{\mathbb{C}\lambda}\Xi'\rangle$. Finally $\|\iota(h)\| = \ell^{-v_\ell(h)}$, with $h \geq 1$. Dividing by this norm proves

$$\|\iota\langle \mathrm{pr}_{p,\vartheta}\Xi, \mathrm{pr}_{p,\vartheta}\Xi'\rangle\| \leq \exp\big((\beta_\ell + v_\ell(h_{\mathrm{ch}}))\log \ell\big).$$

This is exactly the remaining conclusion. □

Theorem 7.6.3. *Let D be admissible data, ℓ a prime, $\iota \mid \ell$ a prime identification, and ϑ a twist such that $(S_\ell^{\mathrm{bad}}, Q_\vartheta, \psi_\vartheta)$ is an auxiliary twist with ramification at most e'_ℓ. Let O be a descent pair, e an idempotent of the group algebra of the product of the groups $\mathrm{GL}_{\mathrm{rk}W}(k_w)$ over the chosen places, and $Z \subseteq \mathcal{A}(V)$ a subspace spanned by lift generators of Z for (ϑ, e, e'_ℓ). Then the projection to Z has denominator at most $\beta_\ell \log \ell$ at ι for e.*

Proof. Put $r' = \mathrm{rk}\, W(O)$, $t = t_O$, $n = r' + t$, $C = N_{\mathrm{ch}}$, and

$$L = e'_\ell + 2te'_\ell + r'\mathrm{loc}_\ell + e_\ell.$$

All these bounds are finite. If $Z = 0$, take its identity operator: its determinant is the empty product 1, and every projected form is zero. Hence suppose $Z \neq 0$. A nonzero lift generator supplies a lift setting, so $1 \leq r' \leq r_D$ and $1 \leq t \leq t_D$. We first construct an operator for each nonzero lift generator v_0 and then make their eigenvalue bounds simultaneous.

Admissibility gives $E = E_s$ and $5 \mid q_0 \mid q_1(s)$, so $\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) \subseteq E_s^+$. The real number $\zeta_5 + \zeta_5^{-1}$ satisfies $X^2 + X - 1 = 0$, whose discriminant 5 is not a rational square; hence $[E^+ : \mathbb{Q}] \geq 2$, as required by the dictionary and auxiliary-operator inputs.

Choose the tuple c in (DR2) for v_0, and write p_c for its orthogonal type projection. The actual Green character projectors are mutually orthogonal and commute with the spherical Hecke algebra. Every lift generator belongs to the range of one of them; therefore p_c either fixes it or kills it. It follows that Z and $Z^\perp$ are stable under p_c. They are also stable under the Hecke polynomial operators and their adjoints: each generator has the Hecke polynomials of a lift, so its line is stable, and the spherical Hecke algebra is closed under adjoints. Thus all these operators commute with pr_Z.

Apply Lemma 7.6.4 to (O, c) and ϑ. It supplies actual base data B with $c_B = C$, containing $p_c\mathrm{Form}(O)$ at base level; Cp_c preserves integral forms at ι; and every deeper-level normalised representation

$$N_x = \nu_O \otimes r_x$$

has the Ribet type and the quadratic inertia relation required by Theorem 7.6.6. In particular their characteristic coefficients are integral. One may also see this directly from continuity: the eigenvalues of a compact group image have valuation zero. Let x_0 be the system of v_0, and let $\mathcal{X}$ be the systems containing nonzero lift generators of Z, allowing arbitrary Selmer-length bounds. At every deeper level we use the inverse image of this same set of systems.

For $x \in \mathcal{X}$, (DR2) and (MS4) give a main-summand-like representation R_x with

$$\mathrm{tr}\, N_x(g) = \psi_\vartheta(g)\,\mathrm{tr}\, R_x(g) + \sum_{a<t} \chi_{\mathrm{cyc}}(g)^{-a}.$$

They also give the scalar lattice element with scalar minus 1 a unit needed in the Ribet theorem. For the system x_0 choose these data from v_0, so that its twisted Selmer length is at most e'_ℓ. If several lift generators have the same system, this choice is made once for that system; their normalized representations and their traces agree. Every $x \in \mathcal{X}$ is weak, since the term $a = 0$ in this semisimple trace decomposition contributes a trivial character. Indeed semisimple

characteristic-zero representations with equal traces are isomorphic, so the decomposition has the indicated trivial direct summand.

Write

$$\delta_g(x) = \det(1 - N_x(g)).$$

These functions vanish on $\mathcal{X}$. Their characteristic-coefficient expressions belong to the integral operator algebra by Lemma 7.6.13; the same holds at every deeper level. There are only finitely many systems at base level. For each that is not weak choose g_k with $\lambda_k = \delta_{g_k}(x_k) \neq 0$, and put

$$\delta_0 = \prod_k (\lambda_k - \delta_{g_k}), \qquad n_0 = \prod_k \lambda_k.$$

The factors are integral, $n_0 \neq 0$, $\delta_0 = n_0$ on $\mathcal{X}$, and δ_0 vanishes on every nonweak base system. Empty products are 1.

We choose the cyclotomic element with its prescribed cost. The image of G_E under χ_{cyc} is procyclic: for odd ℓ it is a closed subgroup of $\mathbb{Z}_\ell^\times$, and for $\ell = 2$ it lies in $1 + 16\mathbb{Z}_2$ because $\mu_{16} \subset E$. Let u be a topological generator. If $\ell > h_D^2$, then ℓ is unramified in E, by Lemma 4.2.66 and the size bounds in Lemma 4.2.67. The extension $E \cap \mathbb{Q}(\mu_{\ell^\infty})$ is unramified at ℓ and totally ramified at ℓ, hence is $\mathbb{Q}$. The cyclotomic image is therefore all of $\mathbb{Z}_\ell^\times$. Choose u with primitive residue; since $0 < a < t_D < \ell - 1$, every $u^a - 1$ is a unit. Set $h = 0$ in this case.

For $\ell \leq h_D^2$ set $h = v_\ell(C)$. Choose $i \in \{0, 1\}$ with $p_i \neq \ell$; the cyclotomic image contains p_i, the value at an arithmetic Frobenius above this split rational prime. For $0 < a < t_D$, the integer $p_i^a - 1$ divides $\#\mathrm{GL}_{r_D t_D}(\mathbb{F}_{p_i})$, since its order formula contains that factor, and hence divides C. Since u topologically generates the image, p_i is a limit of integral powers of u. Each $u^{ab} - 1$ is divisible in $\mathbb{Z}_\ell$ by $u^a - 1$, also for negative integers b. Passing to the limit gives $u^a - 1 \mid p_i^a - 1$, and consequently $u^a - 1 \mid \ell^h$. Choose g_1 with cyclotomic value u. In both cases $h \leq v_\ell(C)$ and the hypotheses of Proposition 7.6.12 hold.

That proposition supplies one finite index set J, independent of the truncation, and for each $\nu > 0$ auxiliary Taylor–Wiles places forming a prime triangle for g_1^{-1} with the required bounds. At these places every weak base system congruent to x_0 has the root chain with integral complementary polynomials H_j satisfying $H_j(1) = 1$. Choose ν large enough that

$$2n!(n_0 C^2)^{\#J+1} \mid \ell^\nu.$$

This is possible since the scalar on the left is nonzero and J was fixed before ν. The places avoid all bad places, and ν_O is trivial on their inertia. The supplied Ribet types and quadratic inertia relations hold simultaneously at this deeper level.

For every trace ideal of the deeper-level N_x, apply Theorem 7.6.6, with $\mathcal{X}$ as above, the chosen main summands, $\mathrm{ram} = \mathrm{sel} = e'_\ell$, and the prime triangle just obtained. Every hypothesis has now been supplied: the local types and quadratic relation come from the actual base-level construction, the trace decompositions and scalar lattices from the lift generators, and the bounded Selmer length at x_0 from v_0. It gives a common Ribet element whose coefficient d satisfies

$$v_\ell(C^2 d) \leq L.$$

Apply item 2 of Proposition 7.6.11 with $m = C^2$, $c_B = C$, the functions δ_g, and the displayed δ_0, n_0. A base system with $\delta_0 \neq 0$ is weak, so the root-chain hypothesis follows from the preceding construction. We obtain a function s on the deeper-level systems with

$$v_\ell(C^2 s(x_0)) \leq L.$$

If l, z are base-level forms, Cl is integral, z is supported on $\mathcal{X}$, and $z = l$ on every system where $\delta_0 \neq 0$, the same conclusion gives that $C^2 sz$ is integral.

We apply this assertion to

$$l = p_c y, \qquad z = p_c \mathrm{pr}_Z y$$

for an integral form y orthogonal to the false lifts. The first required integrality follows from the base-level projector estimate. The form z is supported on $\mathcal{X}$ because Z is spanned by lift generators and p_c either fixes or kills each generator. We justify their agreement at the weak base systems; it is this step that uses false-lift orthogonality.

Choose a split place w off the base level, not dividing ℓ, at which all base-level systems have their Hecke polynomials; there are infinitely many such places. For a weak system its normalised Frobenius matrix has eigenvalue 1. The local chain theorem Lemma 7.6.5, with the normalization of (MS6) and (DP10), continues this root to the full string of length t. Its last root is $X_*(w)^{-1}$, so the reversed Hecke polynomial operator $H_w(X_*(w))$ kills its base-level vectors. This holds for all weak base systems at this same good place.

Put $q = p_c(y - \mathrm{pr}_Z y)$. It lies in $p_c \mathrm{Form}(O) \cap Z^{\perp}$. This space is stable under the commuting normal Hecke operators, so it decomposes orthogonally into their simultaneous systems. Its weak-system component q_{wk} is killed by $H_w(X_*(w))$, lies in the range of p_c, and lies in $Z^{\perp}$. By (DR2), the range of p_c is contained in $e\mathrm{Form}(O)$. Thus q_{wk} is a false lift. Self-adjointness of p_c and $p_c q_{\mathrm{wk}} = q_{\mathrm{wk}}$ give

$$\langle q_{\mathrm{wk}}, q \rangle = \langle q_{\mathrm{wk}}, y - \mathrm{pr}_Z y \rangle = 0.$$

The last equality uses the assumed orthogonality of y to false lifts and the orthogonality of q_{wk} to Z. As this is the orthogonal weak component, its squared norm is zero; hence it vanishes. Therefore l and z agree at every weak base system, in particular wherever $\delta_0 \neq 0$. The auxiliary-operator conclusion proves that

$$G_{v_0} \mathrm{pr}_Z y := C^2 \, \mathrm{diag}(s) \, p_c \mathrm{pr}_Z y$$

is integral. On every lift generator v, this operator is scalar: p_c kills v or fixes it, and in the latter case v belongs to one system. Its scalar at v_0 has valuation at most L.

Choose a basis $b_1, \ldots, b_a$ of Z among the nonzero lift generators. For each b_i, the preceding construction gives an endomorphism G_i diagonal on this basis, preserving the required integrality after projection, with

$$G_i b_j = a_{ij} b_j, \qquad v_\ell(\iota(a_{ii})) \leq L.$$

The image of G_i on Z lies in Z because it is diagonal on the generators, so these are indeed endomorphisms. Choose integral coefficients d_i such that

$$\left| \sum_i d_i \iota(a_{ij}) \right|_\ell = \max_i |\iota(a_{ij})|_\ell \quad \text{for every } j.$$

Here is a simultaneous choice, with the coefficient fields specified. The coefficient field of Z and its endomorphisms in (DR1) is $\mathbb{C}$; the CM field E is not a required field of definition for the operator. By Definition 3.1.3, $\iota \colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ is a field isomorphism on all of $\mathbb{C}$. The finite set of coefficients $\iota(a_{ij})$ lies in a finite extension $K_\ell / \mathbb{Q}_\ell$ inside the fixed algebraic closure. In each nonzero column divide by an entry of maximal absolute value; its reduction is a nonzero linear form in the variables $\bar{d}_i$. Enlarge the residue field until its cardinality exceeds the number of columns. The union of the resulting proper hyperplanes then has cardinality at most a times a codimension-one hyperplane and cannot fill that field's a-dimensional vector space. Choose

a point outside this union and lift its coordinates integrally. This gives the equality in every column.

Take the residue-field enlargement inside a finite unramified extension K'_ℓ/K_ℓ in the same fixed algebraic closure, and take each $d_i \in \mathcal{O}_{K'_\ell}$. Define $c_i = \iota^{-1}(d_i) \in \mathbb{C}$ and $\Lambda = \sum_i c_i G_i \in \mathrm{End}_{\mathbb{C}}(Z)$. These are permitted complex scalars in (DR1), not a change of the datum E or of any representation's field of definition. Since $|\iota(c_i)|_\ell = |d_i|_\ell \le 1$, the ring $\mathcal{O}_\iota$ of integral complex scalars acts on integral forms: in every gauge, multiplication of every coordinate by c_i preserves integrality. Thus Λ still sends all the projected test forms to integral forms. Its eigenvalue on b_j is $\mu_j = \sum_i c_i a_{ij}$, so $\iota(\mu_j) = \sum_i d_i \iota(a_{ij})$. The extended absolute value restricts to the original one on K_ℓ; the simultaneous equality therefore introduces no valuation loss. Its eigenvalue μ_j on b_j satisfies

$$v_\ell(\iota(\mu_j)) \le L$$

by the diagonal bound for G_j. Put $B = C\Lambda$. By (CX5), $t \le t_D$ and $r' \ge 1$,

$$\begin{aligned} L + v_\ell(C) &= r'\mathrm{loc}_\ell + (1+2t)e'_\ell + e_\ell + v_\ell(C) \\ &\le r'\big(\mathrm{loc}_\ell + (1+2t_D)e'_\ell + e_\ell + v_\ell(C)\big) = r'\beta_\ell. \end{aligned}$$

Multiplying the eigenvalues therefore gives

$$|\iota(\det B)|_\ell \ge \exp\big(-(\dim Z) r' \beta_\ell \log \ell\big).$$

Also $B\mathrm{pr}_Z y$ is integral, since $\Lambda \mathrm{pr}_Z y$ is integral and C is an integer.

It remains to prove the pairing clause, for which self-adjointness of the type projector alone would not suffice. Let f be integral above ℓ and $h = \Lambda \mathrm{pr}_Z y$ integral at ι. At an adelic representative g, let A_τ be its matrices in the gauge at ι. Unitarity identifies the matrices in the gauge at $\iota \circ$ complex conjugation with $(A_\tau^{-1})^{\mathsf{T}}$. Thus the gauged coordinate vectors of f and h are integral and their contractions equal $\iota\langle f(g), h(g)\rangle_{\kappa_O}$. To check this equality, write G_{κ_O} for the integer Gram matrix of the Schur module. The tensor/exterior-power construction of (WM1)–(WM2) gives

$$\rho_{\kappa_O, e_O}(A)^{\mathsf{T}} G_{\kappa_O} = G_{\kappa_O} \rho_{\kappa_O, e_O}(A^{\mathsf{T}}).$$

On tensor powers this is the ordinary transpose identity; restricting to exterior powers and then to the Schur image preserves it. The determinant twist satisfies the same identity. The two gauge factors are inverse transposes, so they cancel in this contraction. Since the Gram matrix has integer entries, each local contracted value is integral.

The global inner product is the class sum of these values divided by the finite rational stabiliser orders. Every such order divides C, as proved in the base-level construction; this also holds at the smaller level of the forms because its stabiliser is a subgroup of the maximal-level stabiliser. Hence

$$|\iota(C\langle f, h\rangle)|_\ell \le 1.$$

Since $B = C\Lambda$, this is exactly the pairing condition of (DR1). Together with the determinant and integrality assertions it proves the claimed denominator bound. □

Lemma 7.6.4. *Let (O, c) be a lift setting for a twist ϑ, and suppose that Q_ϑ is auxiliary for S_ℓ^{bad}. Write $r' = r_O$, $n = n_O$, $t = n - r'$, $S_O = M_O \cup Q_O$, and N_{ch} for the chosen group order. There are a multiset of jumps J at every embedding of E and base data B of the weight of O, with $B_{\mathrm{pl}} = S_O$ and $c_B = N_{\mathrm{ch}}$, such that:*

1. *every $p_c y$, $y \in \mathrm{Form}(O)$, is a form of base level for B;*

2. *if y is integral at ι (respectively above ℓ), then $N_{\mathrm{ch}}p_c y$ is integral at ι (respectively above ℓ);*

3. *for every finite family U of auxiliary level places outside S_ℓ^{bad} and every system x of level U for B, its normalised representation $N_x = \nu_O \otimes r_x$ is of Ribet type*

$$(O, c; t, J, Q_\vartheta \cup \bar{Q}_\vartheta \cup U \cup \bar{U}),$$

and for every element g of inertia above $Q_\vartheta \cup \bar{Q}_\vartheta$,

$$(N_x(g) - \psi_\vartheta(g)1)(N_x(g) - 1) = 0.$$

In particular, when a chosen place divides ℓ, its label polynomial kills the inertia operator on the potentially crystalline Weil–Deligne period module, in the sense of (RT1), for every chart and lattice; it is not asserted to kill the raw Galois matrix there.

Proof. Write $\mathscr{F}$ for the smooth automorphic forms of the weight of O on the definite standard unitary group. At the finite set S_O, let π_O act by right translation at these places and their conjugates, and by the identity at the other places. Let $\mathscr{V}$ be its subspace with the level and character prescribed in (DP4)–(DP5). On $\mathscr{V}$ the chosen finite groups act by (DP5). Put

$$\rho_c = \prod_{i \in I_O} e_{c_i}, \qquad \mathscr{Y} = \{v \in \mathscr{V} : \rho_c v = v\}.$$

The idempotents in this formula are the actual Green character idempotents. Define B to have type $\mathscr{Y}$ at S_O, maximal level away from S_O, and the weight of O. If an operator T on $\mathscr{F}$ commutes with $\pi_O(k)$ for every k in the maximal compact group, then it preserves $\mathscr{V}$: applying T to the equation $\pi_O(k)v = \eta_O(k)v$ gives the same equation for Tv. It also commutes with the action of every element of the chosen finite group algebra, since each such action is a linear combination of these translations. Thus $T\mathscr{Y} \subseteq \mathscr{Y}$. In particular all translations at places outside S_O and their conjugates preserve this type. The level defining $\mathscr{V}$ is open. These assertions give the form and level conditions for base data.

We check their prescribed stabiliser multiplier. By the lift-setting bounds,

$$n = r' + t \le r_D + t_D \le r_D t_D;$$

here $r_D \ge 3$ and $t_D = 5r_D$. Apply Lemma 6.7.9 to the same maximal-level representative, with the standard definite space of rank n and $K' = \mathrm{K}_0(n)$. The preceding bound checks $1 \le n \le r_D t_D$; hence its stabiliser is finite and its order divides N_{ch}. The group written using E-matrix points there is the same rational unitary group over E^+ here. This argument is independent of the coefficient prime, including when it is 2. Thus $c_B = N_{\mathrm{ch}}$ has precisely the common-multiple property required in (AL1) of Definition 7.6.1. Finiteness of the class set and its stabilisers follows from definiteness and the usual compact adelic quotient. The definition of p_c is the action of ρ_c on the forms of O; its image lies in $\mathscr{Y}$, and those forms have maximal level away from S_O. This proves item 1.

For item 2 let $G_c = \prod_i \mathrm{GL}_{r'}(k_{w_i})$. The central projector onto an irreducible finite representation λ has coefficients

$$\frac{\dim \lambda}{\#G_c} \operatorname{tr} \lambda(g^{-1}).$$

Every trace in this expression is an algebraic integer, because the eigenvalues of every finite-order matrix are roots of unity. A sum of distinct such projectors has the same integrality property

after multiplication by $\#G_c$. The Green projector ρ_c is such a sum (in the squarefree case it is a single irreducible projector at each place). The chosen group order is divisible by $\#G_c$: each $\mathrm{GL}_{r'}(k_w)$ embeds block diagonally into $\mathrm{GL}_{r_D t_D}(k_w)$. Consequently $N_{\mathrm{ch}}\rho_c$ has algebraic-integral coefficients. The additional factors in (DP5) are roots of unity. Each translating element belongs to the maximal level, so it preserves integral values in every Schur gauge, also when its residue characteristic is ℓ. Taking the finite sum proves the asserted integrality at every identification above ℓ.

Let now x be a system of level U. Choose a nonzero simultaneous eigenform f in its type space, so $\rho_c f = f$. Its associated automorphic representation has a weak base change and the semisimple attached representation r_x by Theorem 7.1.4. Multiplicity spaces cause no ambiguity: all constituents with this system have isomorphic attached representations, by the uniqueness there. At a place split over E^+, the local representation on the unitary side identifies with the corresponding general-linear representation in the base change. This is the split local assertion of [177, Corollaire 5.3], with no restriction on the coefficient prime.

At a chosen place w_i, the first block of the parahoric has rank r' and carries the Green representation labelled by the scaled polynomial $\mathrm{tl}(c,i)$; the complementary block of rank t is trivial. To see the scaling directly, (DP5) makes the action on the first block equal to its ordinary translation multiplied by $\psi_{O,w_i}(\det)$. Thus ordinary translation carries its inverse determinant twist, whose Green parameters are multiplied by $x_{\psi_{O,w_i}^{-1}}$. Since $\rho_c f = f \neq 0$, its local factor cannot annihilate f. Finite-group Frobenius reciprocity gives the actual finite Harish–Chandra product of this first-block Green representation and the trivial representation of $\mathrm{GL}_t(k_{w_i})$ in the compact restriction. Factor $\mathrm{tl}(c,i)$ into distinct irreducible polynomials. The squarefree-label clause of Theorem A.5.29, applied to these distinct monic irreducible factors, identifies its signed Green character with the irreducible finite Harish–Chandra product of their regular cuspidal representations. The sign cancels in the central idempotent, since both the character degree and its conjugate character acquire that sign. No factor is the trivial rank-one character, since $\mathrm{tl}(c,i)(1) \neq 0$ in a lift setting. Hence the hypotheses of Lemma 7.5.2 hold.

For the tame normalization apply Lemma 7.5.4 with $F = E_{w_i}$, $f = r'!$ and d the degree of the chosen irreducible factor. One has $1 \leq d \leq r'$, hence $d \mid r'!$. The faithful character is exactly the fixed character used in the Green label and in (RT2). That lemma gives $T(x)^a$ for each root x, including every Frobenius conjugate, with geometric Artin reciprocity. Its inverse-character clause also gives the multiplier $x_{\psi_{O,w_i}^{-1}}$ already found from (DP5). The complementary trivial block has value 1. Thus Lemma 7.5.2 gives zero monodromy and exactly the roots of $\mathrm{tl}(c,i)(X-1)^t$, with the required tame values.

Away from ℓ, apply Theorem 7.1.8, (Us2)–(Us3). The unramified normalising factor in its local Langlands comparison changes neither inertia nor monodromy. Inertia acts through a finite group and is semisimple in characteristic zero, so the displayed label polynomial kills the actual matrix. At a chosen place above ℓ, apply Theorem 7.1.9, (Ul2)–(Ul3). It gives potential crystallinity and the same finite inertia action on the Weil–Deligne module. This action is semisimple, so the same polynomial kills it. The potentially crystalline period comparison transports this equality to the period operator of (RT1): on the potentially crystalline periods inertia acts as the Weil operator, and extension of scalars is faithful. Identifying the crystalline invariants computed using $B_{\max}$ and B_{cris} is legitimate: $\varphi(B_{\max}) \subset B_{\mathrm{cris}} \subset B_{\max}$ and the finite-dimensional invariant space is stable under bijective Frobenius, so their invariant spaces coincide. This argument applies to every lattice after tensoring with its fraction field. Changing a chart conjugates the decomposition group and transports both the period ring and its operator, preserving the polynomial identity. Thus all charts and lattices in (RT1) are covered.

Off $S_O \cup \bar{S}_O \cup U \cup \bar{U}$, the eigenform is spherical. The extension E/E^+ is unramified at every

finite place: if the residue characteristic is not 3, inertia cannot act by complex conjugation on the roots $\mu_3 \subset E$; at residue characteristic 3 use $\mu_4 \subset E$ instead. Roots of unity of order prime to the residue characteristic lie in the maximal unramified extension, and complex conjugation acts nontrivially on either indicated group. The standard hermitian lattice is self-dual, so its maximal compact subgroup is hyperspecial over this unramified extension. The spherical local parameter is unramified with monodromy zero. The unramified local assertions in Theorem 7.1.4 and Theorem 7.1.9, (Ul4), now give good reduction at all the required places. At the chosen places the preceding argument supplies finite type, including potential crystallinity at ℓ. The de Rham property and jumps at all embeddings follow from (Ul1). This proves the required local level assertions for r_x.

The normalising character ν_O is unramified at every finite place away from ℓ, crystalline at the places above ℓ, and has an unramified Weil–Deligne character there: $\chi_O = \upsilon^{r_-(O)}$ has trivial finite tame part and υ is trivial on finite units. Algebraic Hecke-character reciprocity therefore gives these assertions, and its Hodge jump is minus its infinity exponent. Let J be the jumps of the weight of O shifted by these Hodge jumps. Tensoring preserves the local finite-type and good-reduction assertions and leaves every label unchanged. Moreover

$$\nu_O\, \nu_O^c = \chi_{\mathrm{cyc}}^{r'}, \qquad r_x^c \simeq r_x^\vee \chi_{\mathrm{cyc}}^{1-n},$$

the first equality following by multiplying the defining Hecke characters and the second from (Ua1). Thus

$$N_x^c \simeq N_x^\vee \chi_{\mathrm{cyc}}^{1-t}.$$

Its matrix span is a semisimple algebra, since r_x is semisimple, and the trace pairing on each simple factor is a positive integral multiple of its usual nondegenerate matrix trace pairing. Hence it is nondegenerate. Since $t \geq 1$, this is exactly (MS1) with twist t. Together with the local conclusions it proves the Ribet-type assertion.

At a twist place $w \in Q_\vartheta$, (DP4) says that a parahoric block of rank r' acts on f by $\psi_{O,w}^{-1} \circ \det$, and its rank-$t$ complement acts trivially. The lift-setting identity gives $\psi_{O,w} = \vartheta_w$, since the fixed chosen-place character is trivial here. These two characters are distinct because w belongs to the modulus of the tame twist. The scalar-block parahoric theorem 7.1.11 gives zero monodromy, finite semisimple inertia and its two characters. Geometric reciprocity and the inverse-avatar definition identify their values at g as 1 and $\psi_\vartheta(g)$. A semisimple matrix with these eigenvalues is killed by $(X - \psi_\vartheta(g))(X - 1)$. The normalising character is trivial on this inertia, as $w \nmid \ell$; hence the same polynomial kills $N_x(g)$. At a conjugate place apply conjugate self-duality to $(g^c)^{-1}$. Cyclotomic twists are trivial on inertia away from ℓ, and the torus character satisfies $\psi_\vartheta((g^c)^{-1}) = \psi_\vartheta(g)$. Transporting the polynomial through the duality therefore gives the identical quadratic relation. All the assertions in item 3 follow. □

Lemma 7.6.5. *Let F be a nonarchimedean local field, $\mathcal{O}_F$ its valuation ring, ϖ a uniformiser and q the cardinality of its residue field. Let $N \geq 1$, $K = \mathrm{GL}_N(\mathcal{O}_F)$, and let $I_1 \subset K$ be the inverse image of the upper unitriangular subgroup modulo ϖ. Let Π be a smooth complex representation of $\mathrm{GL}_N(F)$ with an invariant positive definite Hermitian form B, conjugate linear in its first argument. For $0 \leq j \leq N$, let T_j be the sum-normalised spherical Hecke operator of $\mathrm{diag}(\varpi 1_j, 1_{N-j})$. Suppose $0 \neq x \in \Pi^K$ and $T_j x = \lambda_j x$, and put*

$$P(X) = \sum_{j=0}^{N} (-1)^j q^{j(j-1)/2} \lambda_j X^{N-j}.$$

Let U be the sum of the right-coset operators in $I_1 \,\mathrm{diag}(\varpi, 1, \ldots, 1) I_1 / I_1$, acting on Π^{I_1}. Then:

1. *If $P(c) = 0$ and $P(qc) \neq 0$, then $|c|^2 \geq q^{N-2}$. If $k \geq 0$, $P(c) = 0$ and $|c|^2 q^k = q^N$, then $P(cq^i) = 0$ for every $0 \leq i < k$.*

2. *Let $d \neq 0$, $k \geq 0$, and $R \in \mathbb{C}[X]$. If*

$$P(d^{-1}X) = \prod_{i=0}^{k} (X - q^i)\, S(X), \qquad S \mid R,$$

then for every $y \in \Pi^{I_1}$,

$$B\big(x,\, (dU - q^k)R(dU)y\big) = 0.$$

Proof. **The lattice count.** For $0 \leq a \leq N$ and $j \geq 0$, define $A_{a,j}x$ to be the sum of $\Pi(g)x$ over the right K-cosets gK whose lattices $g\mathcal{O}_F^N$ satisfy

$$\varpi\mathcal{O}_F^N \subseteq g\mathcal{O}_F^N \subseteq \mathcal{O}_F^N, \quad \dim_{\mathbb{F}_q}(\mathcal{O}_F^N / g\mathcal{O}_F^N) = j, \quad e_1, \ldots, e_a \in g\mathcal{O}_F^N.$$

These sums are finite and independent of representatives. Write $v_j = A_{1,j}x$, and put $n = N - 1$. Then $v_0 = x$, $v_{n+1} = 0$, and $A_{0,j}x = \lambda_j x$. Each v_j is fixed by I_1, since its defining lattice condition is stable under upper triangular matrices.

Let M be the mean over I_1, and on Π put

$$V = q^n M\Pi(\mathrm{diag}(\varpi, 1, \ldots, 1))M, \qquad V' = q^n M\Pi(\mathrm{diag}(\varpi^{-1}, 1, \ldots, 1))M.$$

The mean is an orthogonal projection: smoothness makes each mean a finite average, and invariance of B gives self-adjointness. Thus V' is the adjoint of V. On Π^{I_1}, V is exactly U. Indeed the right cosets of the indicated double coset are represented by $u\ \mathrm{diag}(\varpi, 1, \ldots, 1)$, where u is upper unitriangular with its first row to the right of the diagonal running independently through $\mathbb{F}_q$. There are q^n such cosets, and their sum is $q^n M\Pi(\mathrm{diag}(\varpi, 1, \ldots, 1))$ on an I_1-fixed vector.

We give the count that will be used twice. With coordinates numbered $0, \ldots, N-1$, let b_a have ϖ in position a and 1 elsewhere. If a linear map L is unchanged by upper unitriangular row operations supported in row a, then

$$q^{N-1-a} L\Pi(b_a) A_{a+1,j}x = q^j L(A_{a,j+1}x - A_{a+1,j+1}x). \tag{1}$$

To verify (1), sum the row vectors $(0, \ldots, 0, 1, u_{a+1}, \ldots, u_{N-1})$ over all q^{N-1-a} choices and use the corresponding upper unitriangular matrices. For a lattice counted by $A_{a,j+1}$, a row contributes precisely when it annihilates the reduction of that lattice. There are no such rows if the lattice contains e_a. Otherwise restriction of the coordinate at a to its annihilator is a nonzero linear functional on a vector space of dimension $j+1$. Its fibre over 1 has q^j elements. These are all the contributions, giving exactly the difference on the right of (1).

Taking $a = 0$ and $L = M$ in (1) gives the first recurrence below. Apply (1) also to the representation $g \mapsto \Pi((g^{\mathsf{t}})^{-1})$. Duality of lattices, $L \mapsto \varpi L^\vee$, interchanges colength j with $N - a - j$ among lattices containing the first a basis vectors. For $a = 0$ its diagonal correction is $\varpi^{-1}1_N$; for $a = 1$ it is $\mathrm{diag}(1, \varpi^{-1}, \ldots, \varpi^{-1})$. Substituting these two corrections in (1), and applying M, gives

$$\begin{aligned} Vv_j &= q^j(\lambda_{j+1}x - v_{j+1}) \quad (0 \leq j \leq n), \\ V'v_{i+1} &= q^{n-i-1}\big((\lambda_{i+1}/q^n)V'x - v_i\big) \quad (0 \leq i < n), \\ V'x &= q^n\lambda_N^{-1}v_n, \qquad Vv_n = q^n\lambda_N x. \end{aligned} \tag{2}$$

For clarity, the dual count before substituting $A_{0,i+1}x = \lambda_{i+1}x$ is

$$q^n M\Pi(b_0^{-1})A_{1,i+1}x = q^{n-i-1}(M\Pi(b_0^{-1})A_{0,i+1}x - A_{1,i}x).$$

At top degree $A_{0,N}x = \Pi(\varpi 1_N)x = \lambda_N x$, and multiplication by b_0^{-1} gives $A_{1,n}x$. This proves the third formula of (2); $\lambda_N \neq 0$ because the central translation is invertible. The fourth formula follows by reversing that translation. Thus all the recurrences, including their endpoint normalisations, follow from the single finite count (1).

The adjoint calculation. Let

$$P_m(X) = \sum_{j=0}^{m} (-1)^j q^{j(j-1)/2} \lambda_j X^{m-j}.$$

Put $a_m = (-1)^m q^{m(m-1)/2}$. Thus $a_0 = 1$ and $a_{m+1} = -q^m a_m$. Since $\lambda_0 = 1$, $P_0(V)x = x = a_0 v_0$. For $0 \leq m \leq n$, the polynomial recurrence and the first line of (2) give

$$\begin{aligned} P_{m+1}(V)x &= VP_m(V)x + a_{m+1}\lambda_{m+1}x \\ &= a_m q^m(\lambda_{m+1}x - v_{m+1}) + a_{m+1}\lambda_{m+1}x \\ &= a_{m+1}v_{m+1}. \end{aligned}$$

This proves by induction, including $m = N = n+1$ and $v_N = 0$, that

$$P_m(V)x = a_m v_m \quad (0 \leq m \leq N), \qquad P(V)x = 0. \tag{3}$$

The span of $v_0, \ldots, v_n$ is preserved by V' by (2).

For $c \in \mathbb{C}$, define the polynomial

$$D_{m,c}(X) = \sum_{j=0}^{m-1} a_j \lambda_j \sum_{\ell=0}^{m-j-1} X^{m-j-1-\ell} c^\ell \quad (m \geq 1), \qquad D_{0,c} = 0.$$

The identity $(X-c)\sum_{\ell=0}^{h-1} X^{h-1-\ell}c^\ell = X^h - c^h$ shows, without division by any scalar, that

$$(X-c)D_{m,c}(X) = P_m(X) - P_m(c).$$

It also gives $D_{m+1,c} = P_m + cD_{m,c}$. Iterating this recurrence, starting with $D_{0,c} = 0$, yields the universal polynomial identity

$$D_{m,c}(X) = \sum_{h=0}^{m-1} c^{m-1-h} P_h(X).$$

Indeed multiplication by c gives the terms with $h < m$, and addition of P_m gives the last term for $m+1$. Thus, with $D_c = D_{N,c}$ and $z_c = D_c(V)x$, (3) gives

$$z_c = \sum_{m=0}^{n} a_m c^{n-m} v_m.$$

Here and below $c^0 = 1$, including at $c = 0$, and an empty sum is zero.

For $1 \leq m \leq n$, the second recurrence of (2), with $i = m-1$, reads

$$V' v_m = q^{n-m}\big((\lambda_m/q^n)V'x - v_{m-1}\big).$$

Apply this formula term by term to the preceding expression for z_c, keeping its $m = 0$ term separate. It gives

$$cV'z_c = \left(c^{n+1} + \sum_{m=1}^{n} a_m \lambda_m q^{-m} c^{n+1-m} \right) V'x \\ - \sum_{m=1}^{n} a_m q^{n-m} c^{n+1-m} v_{m-1}.$$

Since $-a_m q^{n-m} = q^{n-1} a_{m-1}$, the last sum, with its displayed minus sign, equals

$$q^{n-1} \sum_{h=0}^{n-1} a_h c^{n-h} v_h = q^{n-1} z_c - q^{n-1} a_n v_n.$$

This remains valid when $n = 0$: the sum is empty and its right side is $q^{-1}(z_c - v_0) = 0$. The endpoint recurrence in (2) gives $v_n = \lambda_N q^{-n} V'x$, so the correction term is $-a_n \lambda_N q^{-1} V'x = a_N \lambda_N q^{-N} V'x$. Together with $\lambda_0 = a_0 = 1$, the complete coefficient of $V'x$ is therefore

$$\sum_{m=0}^{N} a_m \lambda_m q^{-m} c^{N-m} = q^{-N} P(qc).$$

We have proved, for every $c \in \mathbb{C}$, the exact identity

$$cV'z_c = \frac{P(qc)}{q^N} V'x + q^{N-2} z_c. \tag{4}$$

All divisions in this coefficient calculation are by powers of q, which is nonzero. The endpoint $\lambda_N \neq 0$ and the endpoint recurrence used above were proved in (2). In particular no root hypothesis or division by c was used. When $N = 1$ the same computation has $n = 0$, $z_c = x$ and $V'x = \lambda_1^{-1} x$; its two sides are both $c\lambda_1^{-1} x$. When $c = 0$, (4) follows from the same finite polynomial calculation, with its constant powers interpreted as above. If $P(c) = 0$, (3) also gives $Vz_c = cz_c$. Pair (4) with $V'x$, using $VV'x = q^{2n}x$, which follows from the last two formulas of (2). The result is

$$c(q^N - 1) B(x, z_c) = P(qc) B(x, x). \tag{5}$$

If $P(qc) \neq 0$, (5) implies $z_c \neq 0$. Pairing (4) with z_c and using (5) and adjointness gives

$$(|c|^2 - q^{N-2}) B(z_c, z_c) = \frac{|P(qc)|^2 B(x, x)}{q^N (q^N - 1)} \geq 0. \tag{6}$$

Positive definiteness proves the first assertion of item 1.

We record also the case of two consecutive roots. Suppose $c \neq 0$, $P(c) = P(qc) = 0$ and $P = (X - c)F$. Then $F = D_c$. By (4), $V'z_c = c^{-1} q^{N-2} z_c$, and by (5), $B(x, z_c) = 0$. Adjointness and induction on degree now give $B(G(V)x, z_c) = 0$ for every polynomial G: the constant case is (5), and multiplication by V moves to the scalar action of V' on z_c. Taking $G = D_c$ and using positivity gives

$$F(V)x = 0. \tag{7}$$

Continuation of roots. The roots of P, with multiplicity, are stable under $c \mapsto q^{N-1}/\bar{c}$. Here is a coefficient check. The adjoint of T_j is the central translation by $\varpi^{-1} 1_N$ followed by T_{N-j}. Pairing this identity with x gives $\bar{\lambda}_j = \bar{\lambda}_N \lambda_{N-j}$. Insert these equations in the displayed formula for P, reverse the indices j and $N - j$, and use $\binom{N}{2} + \binom{j}{2} + j = \binom{N-j}{2} + Nj$. The

conjugate reciprocal polynomial has the same roots as P, with precisely the factor q^{N-1}. Its constant coefficient is nonzero, so its roots are all nonzero.

Now suppose $|c|^2 q^k = q^N$. The assertion is empty for $k = 0$. The reflection just proved sends cq^i to cq^{k-1-i}. If $i \geq 1$ is the first missing root before the midpoint, then cq^{i-1} is a root and cq^i is not. Equation (6) gives

$$q^{N-k+2i-2} = |cq^{i-1}|^2 \geq q^{N-2}, \quad \text{hence } 2i \geq k.$$

This contradicts $i < k/2$. Consequently every root in the lower half of the proposed chain occurs, and reflection supplies the upper half. In the middle case $k = 2i$, the desired root is the reflection of the already present root at $i - 1$. This proves item 1 in all cases.

Cancellation and orthogonality. We first give the cancellation step without a multiplicity assumption. Suppose

$$P(X) = \prod_{i=0}^{k-1} (dX - q^i)\, G(X), \qquad P(q^k/d) = 0, \quad d \neq 0.$$

Then

$$G(V)x = 0. \tag{8}$$

Induct on k. For $k = 0$ this is (3). At the next step, the induction hypothesis says $(dV - q^k)G(V)x = 0$. Thus $w = G(V)x$ is a V-eigenvector of eigenvalue q^k/d. Apply (7) to the root q^k/d and its successor. The factorisation of P shows that (7) says $Q(V)w = 0$, where $Q(X) = \prod_{i<k}(dX - q^i)$, up to a nonzero scalar. Since $Q(q^k/d) \neq 0$ for $q > 1$, this forces $w = 0$. This proves (8).

In item 2 put

$$G(X) = (dX - q^k)S(dX).$$

The assumed factorisation gives $P(X) = \prod_{i<k}(dX - q^i)G(X)$, and q^k/d is a root. Hence (8) gives $G(V)x = 0$. By (3), every v_j is a nonzero scalar multiple of a polynomial in V applied to x; consequently $G(V)$ kills every v_j. Let W be their span. It is V'-stable by (2), so $\bar{G}(V')x \in W$ and

$$B(\bar{G}(V')x, \bar{G}(V')x) = B(x, G(V)\bar{G}(V')x) = 0.$$

Thus $\bar{G}(V')x = 0$. Adjointness gives $B(x, G(V)y) = 0$ for every y. Write $R = SF$. The operator in item 2 is $F(dV)G(V)$ on I_1-fixed vectors; moving the additional polynomial across the pairing, or applying the same argument to $F(dX)G(X)$, gives the required zero. The operator V preserves Π^{I_1} and equals U there, so this is exactly the stated sum-normalised identity. □

Theorem 7.6.6. *Let D be admissible data, ℓ a prime, $\iota\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$, ϑ a twist, (O, c) a lift setting for ϑ with source rank r' and rank difference t', $n = r' + t'$, and (S_ℓ, Q, ψ) an auxiliary twist with ramification at most* ram. *Let $(w_j, \tau_j, \varphi_j)_{j\in J}$ with exponent* ty$\colon J \to \{1, \dots, t' - 1\}$ *be a prime triangle for some g_0 with bounds $(h_D h,\ h_D^{22} h)$, where $h \leq v_\ell(N_{\mathrm{ch}})$; put $W = \{w_j\}$. Let u_x $(x \in Y$, Y finite$)$ be ℓ-adic representations of rank n such that:*

(a) each u_x is of Ribet type $(O, c; t', J', Q \cup \bar{Q} \cup W \cup \bar{W})$ (RT7), for one function J' of jumps;

(b) on inertia above $Q \cup \bar{Q}$, $(u_x(g) - \psi(g))(u_x(g) - 1) = 0$.

Let $\mathcal{X} \subseteq Y$, and for $x \in \mathcal{X}$ let R_x be main-summand like of rank r' and string length t' with

$$\operatorname{tr} u_x(g) = \psi(g) \operatorname{tr} R_x(g) + \sum_{e<t'} \mathrm{cyc}(g)^{-e} \qquad (g \in G_E),$$

and with $\gamma_x \in G_E$ in an inertia group off ℓ and $\zeta_x \in \bar{\mathbb{Z}}_\ell$, $\zeta_x - 1$ a unit, such that for all large $\mathcal{O}$ some lattice of ψR_x has γ_x acting by ζ_x. Let S, $\mathcal{J}$ be a trace ideal for the u_x, $p = \ell^\nu$ and the tied relations at (τ_j, φ_j) with $H_j \in \bar{\mathbb{Z}}_\ell[X]$, $H_j(1) = 1$. Let $\pi \in \mathcal{X}$ be such that ψR_π has Selmer length at most $\mathrm{sel} < \infty$ *at $(Q, \gamma_\pi, \zeta_\pi)$. Then the functions $x \mapsto \det(1 - u_x(g))$, $g \in G_E$, have a common Ribet element at π whose coefficient d of $\prod_j X_j$ satisfies*

$$v_\ell(d) + 2\, v_\ell(N_{\mathrm{ch}}) \le \mathrm{sel} + 2t'\, \mathrm{ram} + r'\, \mathrm{loc}_\ell + e_\ell.$$

Proof. We verify the hypotheses of item 2 of Lemma 7.6.7 and keep track of its constants. Valuation inequalities are understood in $[0, \infty]$, so a zero value of an inertia polynomial causes no exceptional case.

Integral models and blocks. The character ψ has finite order, is unramified outside $Q \cup \bar{Q}$, is trivial on the decomposition groups above ℓ, and satisfies

$$\psi(g^c) = \psi(g)^{-1}, \qquad \psi((g^c)^{-1}) = \psi(g).$$

Indeed these identities hold on Artin symbols by (AX3), since conjugation exchanges $x/\bar{x}$ with its inverse; local compatibility and density in Theorem A.2.7 give the assertions on Galois groups.

All the finitely many representations and characters can be defined over one finite extension of $\mathbb{Q}_\ell$. Here is the usual descent argument for this assertion. There are countably many finite subextensions of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$. The compact image of a continuous representation is the union of its closed intersections with the matrix groups over these fields. The Baire theorem gives one such intersection with nonempty relative interior. Its inverse image is a subgroup with nonempty interior, hence an open subgroup. Adjoining the matrix entries of representatives for its finitely many cosets gives a finite coefficient field for the whole representation. A compact subgroup over this field preserves a lattice: the span of its orbit of the standard lattice is bounded, contains the standard lattice, and is a finite module over the valuation ring.

All these representations are finite dimensional over the algebraically closed characteristic-zero field $\overline{\mathbb{Q}}_\ell$. By (RT7), (RT6), (RT4) and (MS1), the trace pairing on the matrix span of each u_x is nondegenerate. By (MS3) and (MS1), the same holds for each R_x. Item1 of Lemma 7.7.9 therefore makes them semisimple. Multiplication by the nonzero scalar character $\psi(g)$ leaves invariant subspaces and invariant complements unchanged, so $T_x = \psi R_x$ is semisimple. The one-dimensional characters χ^{-e} are simple, and their direct sum with T_x is semisimple. The trace equality in the hypothesis and item2 of Lemma 7.7.9 now give

$$u_x \simeq T_x \oplus \bigoplus_{0 \le e < t} \chi^{-e}, \qquad T_x = \psi R_x, \quad x \in \mathcal{X}. \tag{1}$$

Choose a coefficient ring $\mathcal{O}$, large for all these models, all coefficients H_j, all scalars ζ_x, the assertions (MS3), the Selmer bound at π, and Corollary 7.7.4. Replace the representations by conjugate integral models. For $x \in \mathcal{X}$ choose (1) as the integral model, using at π the lattice supplied by (AX7). Thus $T_\pi(\gamma_\pi) = \zeta_\pi$, its Selmer module has normalised length at most sel, and $\zeta_\pi - 1$ is a unit. Since ψ is trivial above ℓ, T_π agrees there with a lattice of R_π. The span of $T_\pi(G_E)$ is $M_r(\mathcal{O})$: multiplication of the individual matrices of R_π by the units $\psi(g)$ leaves their span unchanged. Also $\chi(\gamma_x) = 1$, because γ_x is inertial away from ℓ. These prove (R1) and (R2). Conjugating a model leaves all trace and determinant functions and their trace ideal unchanged.

The doubled relations. Put $h^- = (h^c)^{-1}$ for any $h \in G_E$. Conjugate self-duality gives, for $i \ge 0$,

$$\mathrm{tr}\, u_x((h^-)^i g) = \chi((h^-)^i g)^{1-t}\, \mathrm{tr}\, u_x(h^i g^-). \tag{2}$$

Here cyclic invariance of the trace moves h^i past g^-. On the two inertia generators τ_j, τ_j^- the cyclotomic character is 1; on φ_j, φ_j^- it is congruent to 1 modulo $p = \ell^\nu$. Expand $(X-1)H_j(X)$ in monomials and use (2). The trace of the negative tied relation, multiplied on the right by $u_x(g)$, is $\chi(g)^{1-t}$ times the positive tied relation at g^-, plus X_j times an integral linear combination of the functions $p \operatorname{tr} u_x(k)$. All these functions belong to $\mathcal{J}$. This proves the trace-ideal assertion for both signs. Moreover

$$H_j(\chi(\varphi_j^\pm)^{-e}) \equiv H_j(1) = 1 \pmod{p} \quad (0 \le e < t).$$

This is (R3). Take $a = g_0$ and $C = \ell^{Hh}$. Condition (PT3), with $u = \chi(a^{-1})$, is precisely (R4), since $\chi(a)^{-e} = u^e$.

Continuity and the block count. We spell out why the scalar-relation condition in (R5)–(R7) supplies continuity, although continuity was not imposed in the definition of Ribet cocycles. Let

$$L_0 = \operatorname{span}_{\mathcal{O}}\{(u_x(g) - 1)_{x \in Y} : g \in G_E\} \subseteq \prod_{x \in Y} M_n(\mathcal{O}).$$

This is a finite free $\mathcal{O}$-module. The relation condition on a map z says exactly that $z(g)$ is the value on $(u_x(g)-1)_x$ of an $\mathcal{O}$-linear map on L_0. Such a map with values in $\mathcal{O}/\ell^M$, or in $(K/\mathcal{O})^r$, has finite image: choose a finite basis of L_0 and one common annihilator for its images. It is continuous. Indeed the topology induced on L_0 by the ambient finite free module is its valuation topology, by the elementary-divisor theorem. The original representations are continuous, so z is locally constant. If $u_x(g) = 1$ for all x, the same relation condition gives $z(g) = 0$.

In particular these cocycles are unramified outside the allowed ramification of the u_x. If, at a place away from ℓ where the coefficient representation is unramified, a cocycle vanishes at a tame generator, it vanishes on inertia: its restriction is a continuous homomorphism, the wild pro-residue-characteristic subgroup has zero image in an ℓ-primary group, and the pro-ℓ part of tame inertia is topologically generated by that generator. This argument applies at the places of $W \cup \bar{W}$ outside the already allowed set above $p_0 p_1$ and $Q \cup \bar{Q}$.

Let Z_M be the module in (R5) with values in $(\mathcal{O}/\ell^M)^r$, vanishing at all $\tau_j^\pm$ and at γ_π. The preceding argument makes its members continuous and unramified away from the places above $\ell p_0 p_1$ and $Q \cup \bar{Q}$. Let $Z_{M,f}$ be its submodule obeying the crystalline relations. Multiplication by ℓ^{-M} embeds $Z_{M,f}$ into $\operatorname{Sel}(T_\pi, Q, \gamma_\pi)$; both the crystalline relations and the vanishing conditions are preserved by this injection. Consequently

$$\operatorname{length} Z_M \le \operatorname{cc}(T_\pi; u, M) + \operatorname{length} \operatorname{Sel}(T_\pi, Q, \gamma_\pi) \le \operatorname{loss}_T + \operatorname{length} \operatorname{Sel}(T_\pi, Q, \gamma_\pi). \tag{3}$$

The second inequality is Corollary 7.7.4, item 1. Every $K/\mathcal{O}$-valued relation-obeying cocycle has bounded denominator by the finite generation of L_0 just proved. Thus the module in (R5) is the increasing union of these finite-level modules. A uniform finite bound on their lengths bounds the length of the union: any finite strictly increasing chain is already contained in one finite level. We may therefore take

$$b_0 = \operatorname{loss}_T + \operatorname{length} \operatorname{Sel}(T_\pi, Q, \gamma_\pi). \tag{4}$$

Old inertia relations. First take the tame generators δ_w in (AX5), for $w \in Q$, and also δ_w^-. The polynomial

$$o_w(X) = X - \psi(\delta_w)$$

has $o_w(X)(X-1)$ killing every $u_x(\delta_w)$ by hypothesis (b). The same holds for the negative generator since it lies above $\bar{w}$ and $\psi(\delta_w^-) = \psi(\delta_w)$. Their combined cost is at most $2\,\mathrm{ram}$.

Next, at each place above p_0p_1 away from ℓ, take a tame generator and its negative conjugate. One can use at most $8f$ elements, since $[E:\mathbb{Q}] = 2f$ and there are two rational level primes. Their images under every u_x are killed by $X^{N_{\mathrm{ch}}}-1$. To check this assertion, at a chosen place the roots of the labelled polynomial lie in extensions of the residue field of degrees at most r', together with the root 1 of the string. Their multiplicative orders therefore divide numbers $\#k_w^a - 1$, $1 \le a \le r'$, each of which divides $\#\mathrm{GL}_{r_D t_D}(k_w)$ and hence N_{ch}. Inertia has finite image by the finite-type condition, so its matrices are semisimple in characteristic zero; a repeated root 1 in the label does not produce a nontrivial Jordan block. At the conjugate place the same assertion follows from (MS1): the semisimple inertia representation is the inverse conjugate of the one just considered, and inversion preserves these orders. These places cover the level places. The resulting polynomial is valid on every lattice, as it is already zero over the coefficient field.

For all these level generators use

$$o_w(X) = 1 + X + \cdots + X^{N_{\mathrm{ch}}-1}.$$

Its value at 1 is N_{ch}, including when the label has a multiple root 1. All the old generators are inertial away from ℓ, so $\chi^{-e}(\delta_w) = 1$. We have proved (R6) with

$$\sum_w v_\ell(o_w(1)) \le 2\,\mathrm{ram} + 8fh_N. \tag{5}$$

The character counts. For $0 \le e < t$ write

$$c_e = \mathrm{cc}(\mathcal{O}(-e); u, \nu), \qquad b_e = c_e + \mathrm{length}_{\mathcal{O}} \mathrm{Sha}^-(e,\nu). \tag{6}$$

Consider any module Z occurring in (R7). Its members are continuous by the L_0 argument. Vanishing at the old inertia generators makes them unramified at $Q \cup \bar{Q}$ and at the level primes away from ℓ, since their coefficient character is unramified there. Away from these and $W \cup \bar{W}$ all the u_x are unramified. Hence

$$Z \subseteq \mathrm{Sel}_{S_\ell \cup W \cup \bar{W}}(e,\nu).$$

If a member is unramified outside ℓ, it vanishes at γ_π, since that element is inertial away from ℓ. The normalising relation in (R7), evaluated at $g = a$, then gives

$$(\chi(a)^{-e} - 1) z(a) = 0.$$

The crystalline colength of Z is at most c_e, since intersecting a submodule with the crystalline-relation module injects its quotient into the defining quotient for c_e. Item 3 of Proposition 7.6.10 now proves triangle cost b_e along $\mathrm{ty}^{-1}(e)$ up to C.

Let s_0 be the test set of (PT2) and put

$$s' = s_0 \cup \{a\} \cup \{\tau_j, \tau_j^- : j \in J\}.$$

If a member of Z vanishes on s', it is unramified at the triangle places by the tame-inertia argument, hence lies in $\mathrm{Sel}_{S_\ell}(e,\nu)$. It vanishes at a and on s_0, so (PT2) makes it zero. This proves every part of (R7). Put $K_0 = \#J + \#s_0 + 1$. Then

$$\#s' \le 2K_0, \qquad K_0 H h \le H^{22} h. \tag{7}$$

By item 2 of Corollary 7.7.4, (4), (6), (AX7), and Theorem 7.2.8,

$$\frac{b_0 + \sum_{e<t} b_e}{v_{\mathcal{O}}(\ell)} \le \mathrm{sel} + r'\mathrm{loc}_\ell + t\, v_\ell(X_{\mathrm{odd}}). \tag{8}$$

The coefficient ring was enlarged to contain the finitely many roots of unity required by the odd-class bound.

Application and constants. Apply Lemma 7.6.7, item 2, over $\mathcal{O}$, with the blocks (1), $p = \ell^{\nu}$, the tied relations indexed by $J \times \{+,-\}$, $C = \ell^{Hh}$, the old generators just constructed, and the test set s'. Equations (3)–(8) verify all its hypotheses. Divide its length-valued inequality by $v_{\mathcal{O}}(\ell)$, so that its valuation becomes v_ℓ. It supplies a common Ribet element with top coefficient d satisfying

$$\begin{aligned} v_\ell(d) &\le \mathrm{sel} + r'\mathrm{loc}_\ell + t\, v_\ell(X_{\mathrm{odd}}) + 2t\,\mathrm{ram} + 8fth_N \\ &\quad + (2t\,\#s' + \#J + 1)Hh \\ &\le \mathrm{sel} + r'\mathrm{loc}_\ell + 2t\,\mathrm{ram} + t_D v_\ell(X_{\mathrm{odd}}) + \big(8ft_D + (4t_D+1)H^{22}\big)h_N. \end{aligned} \tag{9}$$

For the second inequality, use $t \le t_D$, $h \le h_N$, $\#s' \le 2K_0$, and $(2t\,\#s' + \#J + 1)Hh \le (4t+1)K_0Hh \le (4t_D+1)H^{22}h$. After adding $2h_N$ to the left, (CX4) bounds the right by $\mathrm{sel} + 2t\,\mathrm{ram} + r'\mathrm{loc}_\ell + e_\ell$; its additional term Hh_N is nonnegative. Finally $\det(1 - u_x(g))$ is among the determinant combinations in that lemma, by taking the coefficient -1 at g and zero elsewhere. This is exactly the required common Ribet element. □

Lemma 7.6.7. *Let $\mathcal{O}$ be a discrete valuation ring with valuation v and fraction field K, G a group, J a finite linearly ordered set, $R = \mathcal{O}[X_j]_{j\in J}$, $C \in \mathcal{O}$ and $0 \ne p \in \mathcal{O}$.*

1. *(Fitting block.) Let s be a finite set, $\Phi\colon \mathcal{O}^n \to R^s$ $\mathcal{O}$-linear, $z\colon G \to \mathcal{O}^n$, $y(g) = \Phi(z(g))$, $\mathcal{N} \subseteq R^s$ a submodule with $\Phi(p\,\mathcal{O}^n) \subseteq \mathcal{N}$, and $\tau, \tau', \varphi, \varphi'\colon J \to G$, $I \subseteq J$. Suppose that modulo $\mathcal{N}$ the basis vector of R^s at $g \in s$ is $y(g)$, $y(\tau_j) \equiv X_j y(\varphi_j)$ and $y(\tau'_j) \equiv X_j y(\varphi'_j)$ for all j, and every i below a member of I lies in I or has $X_i R^s \subseteq \mathcal{N}$. Let Z be the module of the maps $\Psi \circ z\colon G \to \mathcal{O}/p$ for $\mathcal{O}$-linear $\Psi\colon \mathcal{O}^n \to \mathcal{O}/p$ vanishing on $\Phi^{-1}(\mathcal{N})$, and suppose Z has triangle cost b along I up to C. Then the Fitting ideal of $R^s/\mathcal{N}$ contains $F\prod_{j\in I} X_j$ for some $F \in R$ whose constant coefficient has valuation at most $b + (\#I + \#s)\,v(C)$.*

2. *(Ribet's lemma.) Let $u_x\colon G \to \mathrm{GL}_m(\mathcal{O})$, $x \in Y$ (Y finite), be representations, L an $\mathcal{O}$-algebra, S and $\mathcal{J}$ as in (AL6) with values in $L[X_j]$, and $\mathcal{X} \subseteq Y$, $\pi \in \mathcal{X}$. Assume:*

   (R1) *for $x \in \mathcal{X}$, $u_x = \rho_x \oplus \bigoplus_{e<t} \xi_e$ (a block of rank r and characters $\xi_e\colon G \to \mathcal{O}^\times$, $\xi_0 = 1$, $t \ge 1$), and $\rho_\pi(G)$ spans $M_r(\mathcal{O})$;*

   (R2) *$\rho_x(\gamma_x) = y_x \cdot 1$ with $y_x - 1 \in \mathcal{O}^\times$ and $\xi_e(\gamma_x) = 1$;*

   (R3) *$\mathcal{J}$ is a trace ideal for (u,p) and the relations at τ_k, φ_k, $k = (j,\pm) \in J \times \{\pm\}$, with polynomials ε_j such that $p \mid \varepsilon_j(\xi_e(\varphi_k)) - 1$;*

   (R4) *for some $a \in G$ and every e, $\prod_{k\ne e}(\xi_e(a) - \xi_k(a))$ divides C; $\mathrm{ty}\colon J \to \{0,\dots,t-1\}$ is monotone;*

   (R5) *the module of cocycles $c\colon G \to (K/\mathcal{O})^r$ of ρ_π with*

$$\sum_g \theta_g c(g) = 0 \quad \text{whenever} \quad \sum_g \theta_g (u_x(g) - 1) = 0 \text{ for all } x,$$

   vanishing at all τ_k and at γ_π, has length at most b_0;

   (R6) *finitely many $\delta_w \in G$ are trivial on the characters and $o_w(X)(X-1)$ kills every $u_x(\delta_w)$;*

   (R7) *for each e, every module of character cocycles $G \to \mathcal{O}/p$ for ξ_e (cocycles with the same relation condition, vanishing at the δ_w, and with $(\xi_e(g) - 1)c(a) = 0$ for all g when $c(\gamma_\pi) = 0$) has triangle cost b_e along $\mathrm{ty}^{-1}(e)$ up to C for $\tau'_j = \tau_{(j,-)}$, $\varphi'_j = \varphi_{(j,-)}$, and its members vanishing on a fixed finite set $s \subseteq G$ are 0.*

Then the determinants of all combinations $\sum_g \omega_g(u_x(g)-1)$ lie in S and vanish on $\mathcal{X}$, and they have a common Ribet element at π whose coefficient of $\prod_j X_j$ is the image of some $d \in \mathcal{O}$ with

$$v(d) \le b_0 + \sum_{e<t} b_e + t\Big(\sum_w v(o_w(1)) + 2\,\#s\,v(C)\Big) + (\#J+1)\,v(C).$$

Proof. Item 1. Write $d_s = \#s$ and $I_0 = (X_j : j \in J) \subset R$. If the asserted upper bound is infinite, take $F = 0$. If s is empty, the module presented has rank zero, its Fitting ideal is R, and $F = 1$ suffices. We may therefore assume that s is nonempty and the bound is finite. In particular $C \neq 0$ and $b < \infty$. The basis congruences and $\Phi(p\mathcal{O}^n) \subseteq \mathcal{N}$ imply $pR^s \subseteq \mathcal{N}$: multiply each congruence $\Phi(z(g)) \equiv e_g$ by p.

We use two elementary facts about modules of rows. If $U \subset R^s$, $a \in R^s$, and $da \in U$ for $d \in R$, then

$$d\ \mathrm{Fitt}(R^s/(U+Ra)) \subseteq \mathrm{Fitt}(R^s/U).$$

Indeed expand any determinant with rows in $U + Ra$ by multilinearity. Terms with two rows proportional to a are zero. After multiplication by d, a term with one such row has all its rows in U, as does the term with none. The claim follows for the ideal generated by those determinants.

Here is the corresponding length version. For any R-submodule $U \supseteq \mathcal{N}$ put

$$H_U = \mathcal{O}^n/\Phi^{-1}(U), \qquad \mathcal{Z}(U) = \{\Psi \circ z : \Psi \in \mathrm{Hom}_{\mathcal{O}}(H_U, \mathcal{O}/p)\}.$$

The module H_U has finite length and is killed by p. The functor $D(H) = \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O}/p)$ is an exact length-preserving duality on the finite-length modules killed by p, as is seen by writing them as sums of $\mathcal{O}/\varpi^a$ with $a \le v(p)$.

Let L_0 be a set supplied by triangle cost and let Z_0 be the submodule of $Z = \mathcal{Z}(\mathcal{N})$ vanishing on L_0. Pull Z_0 back under the surjection $D(H_{\mathcal{N}}) \to Z$, obtaining $B \subseteq D(H_{\mathcal{N}})$. Let $A \subseteq H_{\mathcal{N}}$ be the annihilator of B. By finite-length duality,

$$\mathrm{length}(A) = \mathrm{length}(D(H_{\mathcal{N}})/B) = \mathrm{length}(Z/Z_0) =: \delta.$$

Choose generators of a direct-sum decomposition of A into cyclic modules, and lift them to vectors $a_i \in \mathcal{O}^n$. If the corresponding annihilators are $d_i\mathcal{O}$, then $d_i\Phi(a_i) \in \mathcal{N}$. Set

$$K = \mathcal{N} + \sum_i R\Phi(a_i), \qquad e_0 = \prod_i d_i.$$

The determinant fact just proved, applied successively, gives

$$e_0\,\mathrm{Fitt}(R^s/K) \subseteq \mathrm{Fitt}(R^s/\mathcal{N}), \qquad v(e_0) = \delta.$$

A form factoring through H_K annihilates A and therefore belongs to B by double annihilation. Consequently $\mathcal{Z}(U) \subseteq Z_0$ for every $U \supseteq K$.

Next put $a_j = y(\varphi_j) - y(\varphi'_j)$. Starting with K, adjoin the rows a_j for $j \in I$ in increasing order. We claim that at the stage before adjoining a_j one has $CX_ja_j \in U$, where U is the current module of rows. For $i < j$, the hypothesis on the earlier indices gives $X_ia_i \in U$: if $i \in I$ its row was adjoined already, and otherwise $X_iR^s \subseteq \mathcal{N}$. The tie congruences now give

$$\Phi(z(\tau_i) - z(\tau'_i)) \in U \quad (i < j).$$

Every $\Psi \in D(H_U)$ thus gives a function in Z_0 agreeing at the earlier tame pairs. It also agrees at the earlier Frobenius pairs with indices in I, since their rows a_i lie in U. Condition (TK1) implies

$$\Psi\big(C(z(\tau_j) - z(\tau_j'))\big) = 0.$$

Duality separates points of the finite module H_U, so $\Phi(C(z(\tau_j) - z(\tau_j'))) \in U$. Subtracting C times the difference of the two tie congruences yields $CX_j a_j \in U$, as claimed. The determinant fact therefore gives, at the end of these ordered additions, a module $T \supseteq K$ with all $a_j \in T$ for $j \in I$ and

$$\left(\prod_{j\in I} CX_j\right) \operatorname{Fitt}(R^s/T) \subseteq \operatorname{Fitt}(R^s/K).$$

It remains to find an element of $\operatorname{Fitt}(R^s/T)$ with controlled constant coefficient. Put $T_0 = T + I_0 R^s$. A form defining a member of $\mathcal{Z}(T_0)$ kills every $z(\tau_j)$ and $z(\tau_j')$: their images under Φ are, modulo $\mathcal{N}$, multiples of X_j. It also kills $z(\varphi_j) - z(\varphi_j')$ for $j \in I$, since the corresponding rows were added to T. Moreover $\mathcal{Z}(T_0) \subseteq Z_0$. Hence (TK2), with $S = \mathcal{Z}(T_0)$, gives

$$\delta + \operatorname{length}\big(\mathcal{Z}(T_0)/\mathcal{Z}(T_0)[C]\big) \le b.$$

Specialise all variables to zero and let $M = \mathcal{O}^s/\operatorname{ct}(T)$, where ct is the coordinatewise constant coefficient. This module is killed by p. The map induced by $\operatorname{ct}\circ\Phi$ surjects onto M, since it sends each $z(g)$, $g \in s$, to the corresponding basis class. It induces an isomorphism $H_{T_0} \simeq M$. Evaluation on all the $z(g)$ is injective on $D(M)$, because those basis classes generate M. Thus

$$\mathcal{Z}(T_0) \simeq D(M), \qquad \operatorname{length}\big(\mathcal{Z}(T_0)/\mathcal{Z}(T_0)[C]\big) = \operatorname{length}(CM).$$

Write the elementary divisors of M as $\mathcal{O}/\varpi^{a_i}$, with at most d_s summands. Then

$$\operatorname{length}(M) = \sum_i a_i \le \sum_i \max(a_i - v(C), 0) + d_s v(C) = \operatorname{length}(CM) + d_s v(C).$$

Formation of the Fitting ideal commutes with this specialisation, directly from its determinant generators. Over the discrete valuation ring, $\operatorname{Fitt}_{\mathcal{O}}(M)$ is generated by an element of valuation $\operatorname{length}(M)$. Lift a linear combination of specialised determinant generators giving such a generator. This gives $D_0 \in \operatorname{Fitt}(R^s/T)$ with

$$v(\operatorname{ct}(D_0)) \le \operatorname{length}(CM) + d_s v(C).$$

Now take $F = e_0 C^{\#I} D_0$. The two determinant inclusions show $F \prod_{j\in I} X_j \in \operatorname{Fitt}(R^s/\mathcal{N})$, while

$$\begin{aligned} v(\operatorname{ct}(F)) &\le \delta + \#I\, v(C) + \operatorname{length}(CM) + d_s v(C) \\ &\le b + (\#I + \#s) v(C). \end{aligned}$$

This proves item 1.

Item 2. First, every determinant in the conclusion belongs to S, by Lemma 7.6.9, applied to the family u_x with coefficients in S. On $\mathcal{X}$ it vanishes because the summand $\xi_0 = 1$ kills every $u_x(g) - 1$. If the asserted valuation bound is infinite, the zero multiplier and $d = 0$ give the conclusion. Assume henceforth that the bound is finite. Then $C \neq 0$, every b_e is finite and every $o_w(1)$ is nonzero.

We construct the group-algebra rows to which Lemma 7.6.8 will apply. In the remainder of this proof the letter s denotes the finite test set in (R7). Put $A = R[G]$, and extend the representations and the characters R-linearly to A. Let $\mathfrak{k}$ be the common kernel of all these

extended u_x, and let $I_{\mathrm{aug}} \subset A$ be the augmentation ideal. Choose a finite ordered list of generators consisting of the δ_w, a further finite list indexed by s_0, and the test points indexed by s, such that the families $(u_x(g_i)-1)_x$ span the families $(u_x(g)-1)_x$ over $\mathcal{O}$. Such a list exists because Y is finite and a submodule of the finite free $\mathcal{O}$-module $\prod_x \mathrm{M}_m(\mathcal{O})$ is finitely generated. Write its index set as

$$\mathcal{I} = W \amalg s_0 \amalg s, \qquad u_x(g) - 1 = \sum_{i \in \mathcal{I}} z_i(g)(u_x(g_i) - 1), \quad z_i(g) \in \mathcal{O}.$$

Coordinates may be chosen independently for each g; uniqueness is unnecessary. Set $v_i = g_i - 1$, and let $I = \sum_i A v_i$. The coordinate equation shows $I_{\mathrm{aug}} \subseteq I + \mathfrak{k}$.

Put $\gamma = \gamma_\pi$ and choose $c_1 \in \mathcal{O}$ with $c_1(y_\pi - 1) = 1$. Define

$$n_0 = c_1(\gamma - 1), \qquad \theta = n_0 + (1 - n_0)(a - 1).$$

The character ξ_0 kills θ, while $\rho_\pi(n_0) = 1$. Let $N \subseteq A^{\mathcal{I}}$ be the module of rows r for which $\sum_i r_i v_i$ lies in

$$\mathfrak{k} + pI_{\mathrm{aug}} + \sum_{j,\pm} A\big(\tau_{(j,\pm)} - 1 - X_j \varepsilon_j(\varphi_{(j,\pm)})(\varphi_{(j,\pm)} - 1)\big) + I_{\mathrm{aug}}\theta.$$

All kernels and ideals in this display are taken inside A; the last three terms are left submodules. Let $\mathfrak{a}$ be the two-sided trace ideal after extending coefficients from R to S, as in Lemma 7.6.8. The trace hypotheses (R3) put the first three displayed terms in $\mathfrak{a}$. For pI_{aug} use the differences $p\,\mathrm{tr}(u_x(gh)) - p\,\mathrm{tr}(u_x(h))$; for a tie use its prescribed trace against every group element, then S-linearity and cyclicity of trace. Since $I_{\mathrm{aug}} \subseteq I + \mathfrak{k}$, every row of N has

$$\sum_i r_i v_i \in \mathfrak{a} + I\theta.$$

Thus every square matrix whose rows are in N satisfies the row hypothesis of that lemma.

Here are the row identities needed to construct those matrices. Regard $z(g)$ as a row with central coefficients. The coordinate equation gives rows in N of the following forms:

$$z(gh) - z(g) - gz(h), \qquad z(g_i) - e_i, \qquad \sum_g \omega_g z(g) \quad \text{if } \sum_g \omega_g(u_x(g) - 1) = 0 \text{ for every } x.$$

Indeed their row values $\sum_i r_i v_i$ are killed by every u_x. Also $pA^{\mathcal{I}} \subseteq N$. For each auxiliary tie there is the row

$$z(\tau_{(j,\pm)}) - X_j \varepsilon_j(\varphi_{(j,\pm)}) z(\varphi_{(j,\pm)}) \in N.$$

Finally set $r_0 = c_1 z(\gamma) + (1 - n_0) z(a)$. Its row value is θ modulo $\mathfrak{k}$, so $(g-1)r_0 \in N$ for every g.

We next construct one matrix B_0 of rows in N with

$$D_x = \det\big(\rho_x((B_0)_{ij})\big)_{i,j} \in R, \qquad v(\mathrm{ct}(D_\pi)) \le b_0.$$

Read rows at $X = 0$ through ρ_π. Their image is a left $\mathrm{M}_r(\mathcal{O})$-submodule of the rows of $r \times r$ blocks, because $\rho_\pi(G)$ spans that matrix ring. For $r > 0$, multiplication by the elementary matrices shows that this image consists exactly of the $r \times (r\#\mathcal{I})$ matrices all of whose scalar rows lie in one lattice $M \subseteq \mathcal{O}^{r\#\mathcal{I}}$. It contains $p\mathcal{O}^{r\#\mathcal{I}}$.

A vector $(w_i)_i \in ((K/\mathcal{O})^r)^{\mathcal{I}}$ annihilating these rows gives $c(g) = \sum_i z_i(g) w_i$. The first and third row identities make c a cocycle satisfying the relation condition in (R5); the second gives $c(g_i) = w_i$. The tie rows at $X = 0$ give $c(\tau_{(j,\pm)}) = 0$. The row $(\gamma - 1)r_0$, read through ρ_π, is

precisely $z(\gamma)$: its entries are $(y_\pi - 1)c_1 z_i(\gamma)$, because $\rho_\pi(n_0) = 1$. Thus $c(\gamma) = 0$. The map from the annihilator to the module in (R5) is injective, since $c(g_i) = w_i$. Duality with $K/\mathcal{O}$ therefore gives

$$\operatorname{length}(\mathcal{O}^{r\#\mathcal{I}}/M) \le b_0.$$

Choose a basis of M, group its rows into blocks of r rows, and lift each block to a row in N. This gives B_0 with the asserted determinant bound by elementary divisors. If $r = 0$, take $B_0 = 0$; its represented determinant is the empty determinant 1 and the same bound holds.

We now construct matrices controlling all characters simultaneously. Under a character ξ_e, forget the W-coordinates of rows. If an R-linear map of these remaining coordinates sends the image of N into a module U, every function $\Psi \circ z$ obtained from a form modulo p annihilating its inverse image is a character cocycle of the kind in (R7). To check this, apply the first three row identities to get respectively the cocycle law, the values at generators, and the relation condition. The δ_w-generator row gives value zero at δ_w after its coordinate is forgotten. Since $\xi_e(\gamma) = 1$, the normaliser row gives

$$(\xi_e(g) - 1)\big(c_1 c(\gamma) + c(a)\big) = 0.$$

If $c(\gamma) = 0$ this is exactly the last condition of (R7). These verifications also hold after enlarging the image module U.

For each fixed e, project these character rows to the s_0-coordinates alone. Every corresponding dual function vanishes at the test points s, by $z(g_i) - e_i \in N$, and hence is zero by (R7). The finite $\mathcal{O}/p$ duality used in item 1 now shows that every unit vector on s_0 is in this projected row module: it is the image of $z(g_i)$ for the corresponding generator, and all dual forms kill it. Choose rows in N whose s_0-part under ξ_e is the identity matrix, and call their s_0-entry matrix E_e before evaluating the character. If s_0 is empty, there are no such rows and all subsequent elimination statements use the empty identity matrix.

These choices can be made simultaneous without a division. Expand

$$\mathbf{1} - \prod_{e<t} (\mathbf{1} - E_e)$$

by telescoping, using any fixed order of e. Each term is a left matrix multiple of one E_e, so its rows lift to rows of N. At ξ_h the product contains the zero matrix $\mathbf{1} - \xi_h(E_h)$; hence the displayed matrix evaluates to the identity at every ξ_h. We have therefore obtained rows $U_i \in N$, $i \in s_0$, whose s_0-part is the identity under every character.

For each e, use the rows U_i to eliminate the s_0-coordinates from the character image of N, and let $K_e \subseteq R^s$ be the remaining image. Explicitly, a row with character coordinates $(b_i)_{i\in s_0}$ and $(c_j)_{j\in s}$ maps to

$$(c_j - \sum_{i\in s_0} b_i\, \xi_e((U_i)_j))_{j\in s}.$$

Put

$$M_e = \prod_{\mathrm{ty}(j)=e} X_j, \qquad M_{<e} = \prod_{\mathrm{ty}(j)<e} X_j, \qquad \mathfrak{q} = (X_j^2 : j \in J), \qquad \mathfrak{q}_e = (\mathfrak{q} : M_{<e}).$$

Apply item 1 to $K_e + \mathfrak{q}_e R^s$. Its map Φ is obtained from the constant coordinate row $z(g)$ by the character evaluation and elimination just described. The s-generator congruences give its basis vectors; p times its image lies in K_e. The tie congruences hold with factor 1: under ξ_e, $\varepsilon_j(\varphi_{(j,\pm)}) = 1 + pc$ by (R3), and adding the row $pX_j c\, z(\varphi_{(j,\pm)}) \in N$ removes the extra term. Its module of forms has the character-cocycle properties checked above, so has the required triangle

cost b_e by (R7). Finally, if $i < j$ and $\mathrm{ty}(j) = e$, monotonicity gives either $\mathrm{ty}(i) = e$ or $\mathrm{ty}(i) < e$. In the latter case $X_i \in \mathfrak{q}_e$, because $X_i M_{<e}$ is divisible by X_i^2. These check every hypothesis of item 1.

We obtain polynomials G_e with

$$v(\mathrm{ct}(G_e)) \le b_e + (\#\{j : \mathrm{ty}(j) = e\} + \#s)v(C), \qquad G_e M_e \in \mathrm{Fitt}(R^s/(K_e + \mathfrak{q}_e R^s)).$$

Expanding determinant generators of the enlarged module gives

$$d_e + \eta_e = G_e M_e, \qquad d_e \in \mathrm{Fitt}(R^s/K_e), \quad \eta_e \in \mathfrak{q}_e.$$

Indeed every determinant term using a row from $\mathfrak{q}_e R^s$ lies in $\mathfrak{q}_e$. Express d_e as a finite sum

$$d_e = \sum_i w_{e,i} \det B_{e,i}, \qquad w_{e,i} \in R,$$

where the rows of $B_{e,i}$ lie in K_e. Lift these rows to rows $B'_{e,i,j} \in N$ whose s_0-part under ξ_e is zero and whose s-part is $B_{e,i,j}$. Such lifts follow from the definition of K_e by subtracting the corresponding multiples of the rows U_i.

Condition (R4) gives elements $q_e \in \mathcal{O}[G]$ with

$$\xi_h(q_e) = \begin{cases} C & h = e, \\ 0 & h \ne e. \end{cases}$$

Namely multiply $\prod_{h \ne e}(a - \xi_h(a))$ by $C/\prod_{h \ne e}(\xi_e(a) - \xi_h(a)) \in \mathcal{O}$. For each choice of one i_e for every e, form a square matrix $M_{\mathbf{i}}$ of rows in N as follows. Its W-rows have the single entry $o_w(\delta_w)$ in coordinate w; they lie in N by (R6). Its s_0-rows are U_i. Its s-rows are $\sum_e q_e B'_{e,i_e,j}$. Under ξ_h this is block upper triangular, with diagonal blocks $(o_w(1))_w$, the identity on s_0, and CB_{h,i_h} on s. Consequently, with

$$c_0 = \left(\prod_w o_w(1)\right) C^{\#s},$$

one has

$$\det \xi_e(M_{\mathbf{i}}) = c_0 \det B_{e,i_e}, \qquad \sum_{\mathbf{i}} \left(\prod_e w_{e,i_e}\right) \prod_e \det \xi_e(M_{\mathbf{i}}) = c_0^t \prod_e d_e.$$

If one determinant sum is empty, the same formulas hold with the empty index set.

It remains to combine these character matrices with B_0. There is an element $k_0 \in \mathcal{O}[G]$ which is 1 on every character and zero on every block ρ_x, $x \in \mathcal{X}$. Take an ordered product, over $x \in \mathcal{X}$, of $(y_x - \gamma_x)/(y_x - 1)$: every character reads each factor as 1, and at any block the corresponding factor is zero. For each $\mathbf{i}$ put

$$\widetilde{B}_{\mathbf{i}} = (1 - k_0)B_0 + k_0 M_{\mathbf{i}}.$$

Its rows lie in N, it reads as B_0 on each block and as $M_{\mathbf{i}}$ on each character. Apply Lemma 7.6.8 after extending scalars to S, with this matrix and with the testing element $k_0 q_0$. This element is zero on the blocks and on all characters except ξ_0, where its value is C; moreover its product with θ is zero on every u_x, $x \in \mathcal{X}$, since $\xi_0(\theta) = 0$. The resulting trace multiplier $f_{\mathbf{i}} \in S$ therefore has

$$f_{\mathbf{i}}(x) = CD_x \prod_{e<t} \det \xi_e(M_{\mathbf{i}}) \quad (x \in \mathcal{X}),$$

and multiplies the determinant of every represented element of I into $\mathcal{J}$. The same holds for every combination of the $g-1$, since it agrees in all the representations with an element of I. Taking the weighted sum gives $f \in S$ with the common multiplier property and

$$f(x) = Cc_0^t D_x \prod_{e<t} d_e \quad (x \in \mathcal{X}).$$

To read its required coefficients, telescope the product using $d_e + \eta_e = G_e M_e$ in increasing order of e. Every error contains $\eta_e M_{<e}$ and hence lies in $\mathfrak{q}$. Thus

$$\prod_e d_e \equiv \left(\prod_e G_e\right) \prod_{j\in J} X_j \pmod{\mathfrak{q}}.$$

No monomial dividing $\prod_j X_j$ belongs to $\mathfrak{q}$. This congruence proves both the vanishing of all properly dividing monomials of $f(x)$ and the assertion that its full squarefree coefficient at π is the image of

$$d = Cc_0^t \operatorname{ct}(D_\pi) \prod_e \operatorname{ct}(G_e) \in \mathcal{O}.$$

Finally,

$$\begin{aligned} v(d) &\le v(C) + t\left(\sum_w v(o_w(1)) + \#s\, v(C)\right) + b_0 + \sum_e \Big(b_e + (\#\mathrm{ty}^{-1}(e) + \#s)v(C)\Big) \\ &= b_0 + \sum_e b_e + t\left(\sum_w v(o_w(1)) + 2\#s\, v(C)\right) + (\#J+1)v(C). \end{aligned}$$

This proves item 2 and the lemma. □

Lemma 7.6.8. *Let L be a commutative ring, Y a set, $n \ge 1$, G a group, and $u_y : \mathrm{G} \to \mathrm{GL}_n(L)$ representations for $y \in Y$. Let $S \subseteq L^Y$ be a subring containing the functions given by every coefficient of the characteristic polynomial of $u_y(g)$, for every $g \in \mathrm{G}$. Put $A = S[\mathrm{G}]$ and let*

$$\rho : A \longrightarrow \mathrm{M}_n(L^Y)$$

be the induced S-algebra homomorphism. Let $\mathfrak{J} \subseteq L^Y$ be an S-submodule and put

$$\mathfrak{a} = \{a \in A : \operatorname{tr}(\rho(ab)) \in \mathfrak{J} \text{ for every } b \in A\}.$$

Let $v_1, \dots, v_k \in A$, let $I = \sum_i Av_i$ be their left ideal, and let $\theta \in A$. Suppose $B \in \mathrm{M}_k(A)$ satisfies

$$\sum_j B_{ij} v_j \in \mathfrak{a} + I\theta \qquad (1 \le i \le k).$$

Then there is $P(T) \in S[T]$ such that

$$P(0) = \det\big(\rho(B_{ij})\big)_{i,j}, \qquad IP(\theta) \subseteq \mathfrak{a},$$

where the first determinant is taken after flattening the $n \times n$ blocks and is viewed in L^Y. Moreover, for every $w \in I$ and $a \in A$,

$$D_w := \det \rho(w) \in S, \qquad f_a := \operatorname{tr}(\rho(aP(\theta))) \in S, \qquad D_w f_a \in \mathfrak{J}.$$

Proof. The module $\mathfrak{a}$ is a two-sided ideal of A. Right multiplication preserves its defining condition, and for left multiplication use $\operatorname{tr}(\rho(cab)) = \operatorname{tr}(\rho(abc))$. It contains $\ker \rho$.

We first record the coefficient-ring consequence of the determinant lemma. Every entry of $\rho(a)$ is a finite S-combination of the matrices $u_y(g)$. Thus Lemma 7.6.9 applies to every square block matrix with entries from $\rho(A)$ and puts its determinant in S. It applies also with extra commuting polynomial variables: embed $S[T_1, \dots, T_d]$ coefficientwise in $(L[T_1, \dots, T_d])^Y$, and use the same representations, regarded as constant matrices. In particular all characteristic coefficients of such block matrices lie in the corresponding polynomial ring over S, by applying the determinant assertion to $Z\mathbf{1} - M$ and reading its coefficients in Z. All hypotheses of that lemma hold here: L is commutative, the representations are multiplicative, and their characteristic coefficients are in S by assumption.

If $k = 0$, then $I = 0$, and $P = 1$ has all the required properties, including $D_0 = 0$ since $n \geq 1$. Hence assume $k \geq 1$. Choose $a_i \in \mathfrak{a}$ and coefficients $C_{ij} \in A$ such that

$$\sum_j B_{ij} v_j = a_i + \sum_j C_{ij} v_j \theta.$$

Put $\mathcal{B}(T) = B - TC \in \mathrm{M}_k(A[T])$ and flatten its image under ρ to a matrix $\widetilde{\mathcal{B}}(T)$ of size kn over $(L^Y)[T]$. The preceding coefficient statement shows

$$P(T) := \det \widetilde{\mathcal{B}}(T) \in S[T].$$

The adjugate identity, expressed using the Cayley–Hamilton polynomial, gives a matrix $Q(T) \in \mathrm{M}_k(A[T])$ for which

$$\rho\big(Q(T)\mathcal{B}(T)\big) = P(T)\mathbf{1}.$$

Indeed the adjugate of the $kn \times kn$ represented matrix is a polynomial of degree at most $kn - 1$ in that matrix, with coefficients among its characteristic coefficients. Those coefficients lie in $S[T]$. Apply exactly that polynomial to $\mathcal{B}(T)$ before representing and flattening. This proves the displayed identity without requiring surjectivity of ρ. Evaluation at $T = 0$ gives the stated formula for $P(0)$.

To use this identity without commuting θ with the coefficients, act on $A/\mathfrak{a}$ by left multiplication of A and let T act by right multiplication by θ. These actions commute. Our row relation says $\mathcal{B}(T)\bar{v} = 0$ for the column $\bar{v} = (v_i \bmod \mathfrak{a})_i$. Every coefficient of $Q(T)\mathcal{B}(T) - P(T)\mathbf{1}$ lies in $\ker \rho \subseteq \mathfrak{a}$, so its action on this quotient is zero. It follows that

$$P(T)\bar{v} = 0, \qquad v_i P(\theta) \in \mathfrak{a} \quad (1 \leq i \leq k).$$

Since $\mathfrak{a}$ is a left ideal, $IP(\theta) \subseteq \mathfrak{a}$.

Finally the coefficient statement shows $D_w \in S$ and $f_a \in S$, the latter because the trace is minus the next-to-leading characteristic coefficient. Cayley–Hamilton for $\rho(w)$ gives an element $w' \in A$, polynomial in w with coefficients in S, such that

$$\rho(w'w) = D_w \mathbf{1}.$$

As $w \in I$, the element $w'w$ belongs to I, and therefore $a(w'w)P(\theta) \in \mathfrak{a}$. Its trace belongs to $\mathfrak{J}$ by the definition of $\mathfrak{a}$, taking the testing element to be 1. Its represented matrix is $D_w \rho(aP(\theta))$, whose trace is $D_w f_a$. This proves the final assertion. □

Lemma 7.6.9. *For a square matrix Z over a commutative ring and an integer $j \geq 0$, let $\sigma_j(Z)$ be the sum of the principal $j \times j$ minors of Z, so that $\sigma_0(Z) = 1$. A $g \times g$ matrix $\mathbf{X} = (\mathbf{X}_{pq})$*

whose entries are $n \times n$ matrices is identified with the $gn \times gn$ matrix having $\mathbf{X}_{pq}$ as its block in position (p,q), and $\det \mathbf{X}$ is the determinant of the latter.

Let F be a commutative ring, Y a set, $n \geq 1$, G a monoid, and for every $y \in Y$ let $\varrho_y \colon G \to \mathrm{M}_n(F)$ be a homomorphism of monoids. Let A be a subring of the ring F^Y of functions $Y \to F$ (pointwise operations) such that the functions $y \mapsto \sigma_j(\varrho_y(h))$ lie in A for every $h \in G$ and $1 \leq j \leq n$. Call a function $f \colon Y \to \mathrm{M}_n(F)$ an A-combination *if $f(y) = \sum_{h \in H} a_h(y)\, \varrho_y(h)$ for a finite $H \subseteq G$ and $a_h \in A$. Then for every $g \geq 1$ and every $g \times g$ matrix $\mathbf{X} = (\mathbf{X}_{pq})$ of A-combinations, the function $y \mapsto \det \mathbf{X}(y)$ lies in A, where $\mathbf{X}(y) = (\mathbf{X}_{pq}(y))_{p,q}$.*

Proof. **Step 1: words.** Choose a finite $H \subseteq G$ and $a_{pqh} \in A$ with $\mathbf{X}_{pq}(y) = \sum_{h \in H} a_{pqh}(y) \varrho_y(h)$ for all p, q (take the union of the finite sets of the entries and coefficients 0 elsewhere). For $Z \in \mathrm{M}_n(F)$ let $e_{pq} \otimes Z \in \mathrm{M}_{gn}(F)$ be the matrix with Z in the block (p,q) and 0 elsewhere; $(e_{pq} \otimes Z)(e_{p'q'} \otimes Z')$ is $e_{pq'} \otimes ZZ'$ if $q = p'$ and 0 otherwise. Enumerate the triples $i = (p,q,h)$ $(1 \leq p, q \leq g,\ h \in H)$ as $1, \ldots, k$, and for $y \in Y$ put $A_i(y) = a_{pqh}(y)\, e_{pq} \otimes \varrho_y(h)$, so that $\mathbf{X}(y) = \sum_i A_i(y)$, of size $N = gn$. For a word $w = i_1 \cdots i_m$, $i_\nu = (p_\nu, q_\nu, h_\nu)$, put $a_w = \prod_\nu a_{p_\nu q_\nu h_\nu} \in A$ and $h_w = h_1 \cdots h_m \in G$. As ϱ_y is multiplicative, $A_w(y) = A_{i_1}(y) \cdots A_{i_m}(y)$ equals $a_w(y)\, e_{p_1 q_m} \otimes \varrho_y(h_w)$ if $q_\nu = p_{\nu+1}$ for all $\nu < m$, and 0 otherwise. For $j \geq 1$, $\sigma_j(e_{pq} \otimes Z) = 0$ if $p \neq q$ (a principal minor of positive size of this matrix has a zero row or column) and $\sigma_j(e_{pp} \otimes Z) = \sigma_j(Z)$ (the principal minors that do not lie in the block (p,p) vanish). With $\sigma_j(bZ) = b^j \sigma_j(Z)$ for $b \in F$, for every word w and $j \geq 1$

$$\sigma_j\big(A_w(y)\big) = \epsilon_w\, a_w(y)^j\, \sigma_j\big(\varrho_y(h_w)\big), \qquad \epsilon_w \in \{0, 1\},$$

where $\epsilon_w = 1$ exactly when $q_\nu = p_{\nu+1}$ for all $\nu < m$ and $p_1 = q_m$; ϵ_w, a_w, h_w do not depend on y.

Step 2: Amitsur's formula. Fix y. By Theorem A.6.6 over $C = F$ with the matrices $A_i(y)$,

$$\det\Big(1 - \sum_i t_i A_i(y)\Big) = \prod_{w \text{ Lyndon}} \det(1 - t_w A_w(y)) = \prod_w \sum_{j=0}^{N} (-1)^j \sigma_j\big(A_w(y)\big)\, t_w^j$$

in $F[[t_1, \ldots, t_k]]$ (for an $N \times N$ matrix Z, $\det(1 - XZ) = \sum_j (-1)^j \sigma_j(Z) X^j$). The left side is $\sum_{j=0}^{N} (-1)^j \sigma_j(\sum_i t_i A_i(y))$, and $\sigma_j(\sum_i t_i A_i(y))$ is homogeneous of degree j in t; so its homogeneous part of degree N is $(-1)^N \det(\sum_i t_i A_i(y))$. On the right, t_w has degree the length $|w|$ of w, so only the finitely many Lyndon words of length at most N contribute in degree N, and the homogeneous part of degree N is the finite sum, over the families (j_w) of non-negative integers indexed by these words with $\sum_w j_w |w| = N$, of $\prod_w (-1)^{j_w} \sigma_{j_w}(A_w(y))\, t_w^{j_w}$. Equating the two homogeneous polynomials of degree N and evaluating at $t_1 = \cdots = t_k = 1$, with Step 1,

$$\det \mathbf{X}(y) = (-1)^N \sum_{(j_w)} \prod_w (-1)^{j_w} \big(\epsilon_w a_w(y)^{j_w}\big)^{[j_w \geq 1]}\, \sigma_{j_w}\big(\varrho_y(h_w)\big),$$

where the index set, the signs, ϵ_w, a_w and h_w do not depend on y, and $\sigma_0 = 1$.

Step 3: conclusion. Each a_w is in A, each function $y \mapsto \sigma_j(\varrho_y(h_w))$ with $j \geq 1$ is in A by hypothesis, and the constants $0, \pm 1$ are in A; A is closed under sums and products. So the right side of the last display, as a function of y, lies in A. □

Proposition 7.6.10. *Let ℓ be a prime, E a CM field containing μ_{5040}, $\mathcal{O}$ a coefficient ring and $\nu \geq 0$.*

1. *Let $e \neq 0$, let S be a set of finite places of E, and let $z \in \mathrm{Sel}_{\Sigma_\ell \cup S \cup \bar{S}}(e,\nu)$ vanish on Λ_e. Then $z + z \circ (g \mapsto g^c) \in \mathrm{Ev}_S(e,\nu)$.*

2. *Let $(w_j, \tau_j, \varphi_j)_{j \in J}$ with exponent* ty *form a prime triangle for g_0 with bounds (h, C), and let $0 < e < t$. Let S be a submodule of $\mathrm{Sel}_{\Sigma_\ell}(e,\nu)$ whose members z vanish on Λ_e, satisfy $(\mathrm{cyc}(g_0)^{-e} - 1)z(g_0) = 0$, and satisfy $(z + z \circ (g \mapsto g^c))(\varphi_j) = 0$ for all j with $\mathrm{ty}(j) = e$. Then*
$$\mathrm{len}(S \bmod S[\ell^h]) \leq \mathrm{len}_{\mathcal{O}} \mathrm{Sha}^-(e,\nu).$$

3. *With the triangle of item 2, let $0 \leq e < t$, $W = \{w_j\}$, and let $Z \subseteq \mathrm{Sel}_{\Sigma_\ell \cup W \cup \bar{W}}(e,\nu)$ be a submodule such that $(\mathrm{cyc}(g_0)^{-e} - 1)z(g_0) = 0$ for every $z \in Z \cap \mathrm{Sel}_{\Sigma_\ell}(e,\nu)$. Let Z_f be the module of maps $z\colon G_E \to \mathcal{O}/\ell^\nu$ such that, for every $v \mid \ell$, the restriction $z|_{G_{A_v}}$ satisfies the crystalline condition of the local lattice $L_v = \mathcal{O}(-e)|_{G_{A_v}}$, with action $g \mapsto \mathrm{cyc}(g)^{-e}$, with coefficients in $\mathcal{O}/\ell^\nu$, for $\Gamma = G_{A_v}$ via the fixed chart $j_v\colon G_{A_v} \to G_\ell$ of Definition 7.2.2, and let $b \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with*
$$\mathrm{len}(Z \bmod Z_f) + \mathrm{len}_{\mathcal{O}} \mathrm{Sha}^-(e,\nu) \leq b.$$
Then Z has triangle cost b along $I = \{j : \mathrm{ty}(j) = e\}$ up to ℓ^h, for $\tau'_j = (\tau_j^c)^{-1}$ and $\varphi'_j = (\varphi_j^c)^{-1}$.

Proof. Preliminaries. As E is CM, c normalises G_E, and $g \mapsto g^c$ is a continuous involutive automorphism of G_E with $\mathrm{cyc}(g^c) = \mathrm{cyc}(g)$. Conjugation by c maps the inertia (decomposition) group of a place A of $\bar{\mathbb{Q}}$ onto that of cA, which lies above the complex conjugate place; so it preserves Λ_e, and it maps inertia above places outside a conjugation-stable set Σ to inertia above places outside Σ. Hence for $z \in \mathrm{Sel}_\Sigma(e,\nu)$ with Σ conjugation-stable, $z^c \in \mathrm{Sel}_\Sigma(e,\nu)$: it is locally constant, it vanishes on the required inertia groups, and
$$z^c(gh) = z(g^c) + \mathrm{cyc}(g^c)^{-e} z(h^c) = z^c(g) + \mathrm{cyc}(g)^{-e} z^c(h).$$
If z vanishes on Λ_e, so does z^c.

Item 1. $\Sigma_\ell \cup S \cup \bar{S}$ is conjugation-stable, so $w = z + z^c \in \mathrm{Sel}_{\Sigma_\ell \cup S \cup \bar{S}}(e,\nu)$; $w(g^c) = z^c(g) + z(g) = w(g)$ since $(g^c)^c = g$; and w vanishes on $\Lambda_1 = \Lambda_e$ ($e \neq 0$). So $w \in \mathrm{Ev}_S(e,\nu)$.

Item 2. Let $\theta\colon S \to \mathrm{Sel}_{\Sigma_\ell}(e,\nu)$, $\theta(z) = z^c - z$. By the preliminaries $\theta(z)$ lies in $\mathrm{Sel}_{\Sigma_\ell}(e,\nu)$ and vanishes on Λ_e, and $\theta(z)(g^c) = z(g) - z^c(g) = -\theta(z)(g)$; so $\theta(S) \subseteq \mathrm{Sha}^-(e,\nu)$. Let $z \in \ker\theta$, so $z^c = z$. By item 1 with $S = \emptyset$, $2z = z + z^c \in \mathrm{Ev}_\emptyset(e,\nu)$. Let $g \in s$, the test points of (PT2); there is j with $\mathrm{ty}(j) = e$ and $(2z)(\varphi_j) = 2 \cdot (2z)(g)$; by hypothesis $(2z)(\varphi_j) = (z + z^c)(\varphi_j) = 0$, so $4z(g) = 0$. Put $d = \mathrm{cyc}(g_0)^{-e} - 1$. Then $4d\,z \in \mathrm{Sel}_{\Sigma_\ell}(e,\nu)$ vanishes on s and at g_0 (as $d\,z(g_0) = 0$), so $4d\,z = 0$ by the detection clause of (PT2). By (PT5), $4(\mathrm{cyc}(g_0)^e - 1)^2$ divides ℓ^h; as $d = -\mathrm{cyc}(g_0)^{-e}(\mathrm{cyc}(g_0)^e - 1)$ with $\mathrm{cyc}(g_0)^{-e}$ a unit, $\ell^h = 4d^2\alpha$ for some $\alpha \in \mathcal{O}$, and $\ell^h z = \alpha d\,(4d\,z) = 0$. Hence $\ker\theta \subseteq S \cap S[\ell^h]$, and
$$\mathrm{len}(S \bmod S[\ell^h]) \leq \mathrm{len}_{\mathcal{O}}(S/\ker\theta) = \mathrm{len}_{\mathcal{O}}\theta(S) \leq \mathrm{len}_{\mathcal{O}}\mathrm{Sha}^-(e,\nu).$$

Item 3. We first establish the local vanishing used below. Put $K = E_v$ for a place $v \mid \ell$, $L = \mathcal{O}(-e)$ and $V = L \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. A crystalline integral cocycle of L determines a class in the usual $H^1_f(G_K, V)$. The rational representation V is finite dimensional and continuous: it is a direct sum over $\mathbb{Q}_\ell$ of cyclotomic Tate twists, since the coefficient action is trivial. Thus Lemma 7.4.6(2), with $K_\ell = K$ and $W = V$, identifies the $B_{\max}$-coboundary condition with the usual B_{cris}-condition. Its statement covers every coefficient prime, including $\ell = 2$, and uses the

same Frobenius, fixing $\mathbb{Q}_\ell$ and satisfying $\varphi(t) = \ell t$; hence no Tate-twist scalar is introduced in the witness comparison. Continuity of an integral cocycle follows from its period witness and the natural continuous Galois action on the period ring.

Suppose first that $e > 0$. As a $\mathbb{Q}_\ell$-representation, V is a direct sum of copies of $\mathbb{Q}_\ell(-e)$. Its filtered de Rham module has $\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V)$, and $H^0(G_K, V) = 0$, since the cyclotomic image of G_K is open and its eth power is nontrivial. The dimension formula of Bloch–Kato, [28, Corollary 3.8.4], gives $H^1_f(G_K, V) = 0$. Consequently every integral crystalline cocycle c has the form $c(g) = (\mathrm{cyc}(g)^{-e} - 1)x$ for an $x \in V$. It vanishes whenever $\mathrm{cyc}(g) = 1$. If $e = 0$, the action is trivial, so cocycles are homomorphisms and coboundaries are zero. By [28, Example 3.9, the case $r = 0$], the finite local classes of the trivial representation are unramified. Thus every integral crystalline cocycle vanishes on inertia. These arguments apply after restriction of scalars from $\mathcal{O}$ and at every finite extension $K/\mathbb{Q}_\ell$, including $\ell = 2$.

The crystalline condition on a map modulo ℓ^ν is expressed by the linear evaluation relations of the integral module $Z_f(L)$. Explicitly, if finitely many $a_i \in \mathcal{O}$ and $s_i \in G_K$ satisfy $\sum_i a_i c(s_i) = 0$ for every $c \in Z_f(L)$, the map is required to satisfy the same relation modulo ℓ^ν. In particular, the one-term relation at any element of the local set just considered holds for every such map. If charts at one chosen extension of each v to $\bar{\mathbb{Q}}$ are used, this also gives vanishing on the other charts. Conjugate an element g of such a chart to sgs^{-1} in the chosen chart. In both cases in question $\mathrm{cyc}(g)^{-e} = 1$, and the cocycle identity gives

$$z(sgs^{-1}) = \mathrm{cyc}(s)^{-e} z(g).$$

The scalar is a unit, so vanishing transports back to g. We have therefore proved

$$z \in Z \cap Z_f \quad \Longrightarrow \quad z|_{\Lambda_e} = 0.$$

Take $L_0 = \Lambda_e$ as the set in the definition of triangle cost, and put $Z_0 = \{z \in Z : z|_{\Lambda_e} = 0\}$. The inclusion $Z \cap Z_f \subseteq Z_0$ yields the surjection $Z/(Z \cap Z_f) \twoheadrightarrow Z/Z_0$ and hence

$$\mathrm{len}_{\mathcal{O}}(Z/Z_0) \le \mathrm{len}(Z \bmod Z_f).$$

At each τ_j the cyclotomic character is 1, and at each φ_j it is 1 modulo ℓ^ν, by (PT1). The cocycle identity at an inverse therefore gives, for every $z \in Z$,

$$z(\tau_j') = -z(\tau_j^c), \qquad z(\varphi_j') = -z(\varphi_j^c).$$

Thus agreement at either of these pairs is exactly the vanishing of $z + z^c$ at the corresponding unprimed element.

For (TK1), take $j \in I$ and $z \in Z_0$ with the required earlier agreements. Then $\mathrm{ty}(j) = e > 0$, since every triangle exponent is positive. Item 1 shows that $z + z^c$ is in $\mathrm{Ev}_W(e, \nu)$. The earlier agreements give all the vanishing hypotheses in (PT4), which implies

$$\ell^h(z + z^c)(\tau_j) = 0 \quad \Longleftrightarrow \quad \ell^h z(\tau_j) = \ell^h z(\tau_j').$$

If $e = 0$, I is empty and (TK1) has no instances.

For (TK2), let $S \subseteq Z_0$ be a submodule as in that condition. Each $z \in S$ vanishes on every τ_j and τ_j', so it is unramified at all the w_j and their conjugates. To check this last assertion, the cyclotomic action on inertia away from ℓ is trivial; the restriction of z is therefore a continuous homomorphism to an ℓ-primary group. Wild inertia, being pro-residue-characteristic, maps to zero, and on tame inertia only the pro-ℓ factor can have nonzero image. Its chosen generator has

value zero by hypothesis. The same argument at conjugate places proves the assertion there. All other inertia groups away from ℓ were already killed by membership in Z. Hence $S \subseteq \mathrm{Sel}_{\Sigma_\ell}(e,\nu)$.

If $e = 0$, each member of S also vanishes on Λ_0, so $S \subseteq \mathrm{Sha}^-(0,\nu)$ by (OE1), without a sign condition. Therefore $\mathrm{len}(S \bmod S[\ell^h]) \le \mathrm{len}_{\mathcal{O}}\mathrm{Sha}^-(0,\nu)$. If $e > 0$, the agreement at φ_j, φ_j' for $j \in I$ says exactly $(z + z^c)(\varphi_j) = 0$. The hypothesis concerning g_0 holds on S, since its members are unramified outside ℓ. The local vanishing holds since $S \subseteq Z_0$. Item 2 consequently gives $\mathrm{len}(S \bmod S[\ell^h]) \le \mathrm{len}_{\mathcal{O}}\mathrm{Sha}^-(e,\nu)$. Adding this inequality to the bound for Z/Z_0 proves (TK2) in both cases, and completes the proof. □

Proposition 7.6.11. *Assume $[E^+ : \mathbb{Q}] \ge 2$. Let B be base data of rank n, $Q = (w_j, \varphi_j, \tau_j)_{j\in J}$ Taylor–Wiles places of level ℓ^ν off the base level, $a : G_E \to \overline{\mathbb{Z}}_\ell^\times$ a character with $a(\tau_j) = 1$, $N_x = a \otimes r_x$, $H_j \in \overline{\mathbb{Z}}_\ell[X]$ and $t \ge 1$. In this statement a root chain at w_j for (a, t, H_j) means the identity*

$$P_{x,w_j}(a(\varphi_j)^{-1}X) = \prod_{i<t}(X - q_{w_j}^i)S(X), \qquad S \mid H_j$$

over $\mathbb{C}$, with coefficients transported through ι^{-1}. Write $\mathrm{Nm} = \sum_{g\in\prod_j \mathrm{GL}_n(k_{w_j})} g$, and $\equiv$ for equality after any ring map on the integral operator ring that kills each T for which $2n!\,T$ is divisible by ℓ^ν.

1. *Let $\delta = c_B^2\delta'$ with $\delta' \in \mathbb{T}_Q$. Suppose every system of base level with $\delta'(x) \ne 0$ has a root chain at every w_j. Then there are integral operators $\mathcal{N}_S$ ($S \subseteq J$) on the forms of level Q, commuting with $\mathbb{T}_Q$, with $\mathcal{N}_\emptyset = \mathrm{Nm}$, such that, for $\mathcal{N} = \sum_S \mathcal{N}_S\alpha_S$ and $\alpha_S = \prod_{j\in S}\alpha_j$, the coefficient of $\prod_j \alpha_j$ in $\mathcal{N}F$ is $\equiv 0$ for every $F \in \mathfrak{a}$, and*

$$\delta^{\#J}\mathcal{N} \equiv \mathcal{Y}\,\mathrm{Nm}$$

   for an integral operator polynomial $\mathcal{Y} = \sum_S \mathcal{Y}_S\alpha_S$ with $\mathcal{Y}_\emptyset = \delta^{\#J}$.

2. *Let $\mathcal{X}$ be a subset of the systems of level Q, let $\pi \in \mathcal{X}$ be of base level, and let δ_i be functions on the systems vanishing on $\mathcal{X}$. Let $\mathrm{sel}(k)$ be finitely many indices, let $\lambda_k \in \overline{\mathbb{Z}}_\ell$, and put*

$$\delta_0 = \prod_k(\lambda_k - \delta_{\mathrm{sel}(k)}), \qquad n_0 = \prod_k \lambda_k.$$

   Two systems are called congruent if all characteristic coefficients of $N_x(g)$ and $N_\pi(g)$ are equal modulo the maximal ideal of $\overline{\mathbb{Z}}_\ell$, for every $g \in G_E$. Suppose the N_x have integral characteristic polynomials; every system of base level congruent to π with $\delta_0(x) \ne 0$ has a root chain at every w_j; $2n!(n_0c_B^2)^{\#J+1}$ divides ℓ^ν; and, for a fixed scalar m and a bound β, every trace ideal of the N_x has a common Ribet element for the δ_i relative to $\mathcal{X}$, at π, whose coefficient d satisfies $v_\ell(md) \le \beta$. Then there is a function s on the systems with $v_\ell(m\,s(\pi)) \le \beta$ such that $d'c_B\,s\,y$ is integral for every integer $d' \ge 1$ and every pair of forms l, y of base level for which $d'l$ is integral, y is supported on $\mathcal{X}$, and y agrees with l at every system where $\delta_0 \ne 0$.

Proof. **Integral averaging with the prescribed multiplier.** Every subgroup of Γ acts through the standard maximal level, and the stabiliser of a point of its finite double-coset set is an image of a subgroup of $\mathrm{U}(V)(E^+) \cap gK_0g^{-1}$. Its order divides c by the common-multiple hypothesis (AL1) of Definition 7.6.1 and Lagrange's theorem: pass first to the subgroup of that finite rational stabiliser, then to its image. Here B is arbitrary base data; no admissible-datum rank restriction or replacement of c_B by N_{ch} is being imposed. Therefore Proposition 7.1.14, or

its explicit orbit weights $c/\#H$, gives an integral endomorphism Ψ_Δ for every subgroup $\Delta \subseteq \Gamma$ such that

$$\sum_{g\in\Delta} g^{-1}\Psi_\Delta g = c. \tag{1}$$

The averaging operator may be chosen to preserve the prescribed base type: form its orbit weights after quotienting by the maximal compact factors at the fixed base places, and pull those weights back. The resulting scalar multiplication commutes with translation by those factors; the base-type space is stable under every such operator. The stabilisers on this quotient are still images of subgroups of the same finite rational stabilisers, so the same integer c works. There is no need for trivial stabilisers. In particular, for $\Delta = \Gamma$ and any base-level vector z,

$$\operatorname{Nm}\Psi_\Gamma z = cz. \tag{2}$$

Indeed every g fixes z, and one can replace g^{-1} by g in the sum.

The local derivative. Fix j, write $q = q_{w_j}$, $\tau = \tau_j$, $\varphi = \varphi_j$, and $G = \mathrm{GL}_n(k_{w_j})$. Let $H \subseteq G$ stabilise the first coordinate line. Choose $\bar z$ and $\mathrm{ind} : k_{w_j}^\times \to \mathbb{Z}/m_\nu$ as in Lemma 7.6.13, with $\mathrm{ind}(\bar z) = 1$, and put

$$b(h) = \mathrm{ind}(h_{11}) \quad (h \in H), \qquad b_G(g) = \mathrm{ind}(\det g) \quad (g \in G).$$

The transfer of b is b_G, by item 4 of that lemma. All actions in the formulas below are the integral actions on forms.

For an integer-valued function F on $\mathbb{Z}/m_\nu$ let

$$W_H(F) = \sum_{h\in H} F(b(h))h, \qquad W_G(F) = \sum_{g\in G} F(b_G(g))g.$$

Let λ be any integer lift of the identity on $\mathbb{Z}/m_\nu$ and put

$$p_j(X) = (X - q^{t-1})H_j(X), \qquad \widetilde U = a(\varphi)U_\varpi, \qquad B(F) = p_j(\widetilde U)W_H(F).$$

These are integral operators supported at w_j. To check the domain, $W_H(F)$ maps into the vectors fixed by I_1: the character b is trivial there. The sum-normalised U_ϖ maps I_1-fixed forms into themselves and is an integral sum of local translates. Thus the displayed polynomial defines the required operator on all deep-level forms. It also preserves a parahoric character on the first entry. This follows directly from its q^{n-1} representatives $u\,\mathrm{diag}(\varpi, 1, \dots, 1)$: the parahoric permutes these representatives modulo the same parahoric, retaining their first-entry character. In particular $HB(1) = B(1)$. Right multiplication gives the weight-shift identity

$$B(F)h = B(F(\,\cdot - b(h))). \tag{3}$$

Choose left-coset representatives s for G/H, set $T_+ = \sum_s s$, $T_- = \sum_s s^{-1}$, and define

$$\partial_j = -T_+ B(\lambda) T_-. \tag{4}$$

Let $\sigma : \mathcal{E} \to A$ be any reduction killing each operator T for which $\kappa_0 T \in m_\nu\mathcal{E}$. We prove the two local identities

$$\mathrm{Nm}_j \operatorname{tr}[(\tau - 1)g] = 0, \qquad \partial_j \operatorname{tr}[(\tau - 1)g] \equiv \mathrm{Nm}_j \operatorname{tr}[(\varphi - 1)H_j(\varphi)g]. \tag{5}$$

Here and below congruence means equality after σ. Differences of integer weights divisible by m_ν disappear after σ.

Write $t_g = \mathrm{tr}[(\tau-1)g]$ and $t_{p,g} = \mathrm{tr}[(\tau-1)g\,p_j(\varphi)]$. By item 3 of Lemma 7.6.13,

$$B(\lambda)t_g \equiv W_H(\lambda)t_{p,g}. \tag{6}$$

We include the factor in this deduction. Set $E(F) = B(F)t_g - W_H(F)t_{p,g}$. Fourier decomposition for the finite cyclic group shows $E(F(\cdot - n!) - F) = 0$: on a character ψ with $\psi(n!) \neq 1$ this is exactly the dictionary, and on the other characters the difference is zero. For $F(u) = \lambda(u)(\lambda(u) + n!)$ one has

$$F(u - n!) - F(u) \equiv -2n!\lambda(u) \pmod{m_\nu}.$$

Applying the additive map E shows that $2n!E(\lambda)$ is a multiple of m_ν by an integral operator. The definition of σ proves (6), including at $\ell = 2$.

Two finite-group calculations finish (5). First,

$$T_+ W_H(\lambda)T_- \equiv W_G(\lambda), \tag{7}$$

because the coefficient of a fixed g on the left is the sum of the lifts of b on the coset components of gs; modulo m_ν that sum is the transfer $b_G(g)$. Second, for $\gamma = \mathrm{diag}(\bar{z}, 1, \ldots, 1)$,

$$(\gamma - 1)W_G(\lambda) \equiv -\mathrm{Nm}_j. \tag{8}$$

This is the change of variable $g \mapsto \gamma g$ and $\lambda(u-1) - \lambda(u) \equiv -1$. Item 2 of Lemma 7.6.13, applied to the determinant-character Fourier components, says

$$W_G(\lambda)t_{p,g} = (\gamma - 1)W_G(\lambda)\mathrm{tr}[p_j(\varphi)g].$$

The same formula with constant weight gives $\mathrm{Nm}_j t_g = 0$. Since all Hecke operators commute with local translations, (4), (6)–(8) give $\partial_j t_g \equiv \mathrm{Nm}_j \mathrm{tr}[p_j(\varphi)g]$. Finally $q \equiv 1 \pmod{m_\nu}$ replaces p_j by $(X-1)H_j$ and proves (5).

We also need the exact form of the failure of ∂_j to commute with G: for each $g \in G$ there is $Z_g \in \mathcal{E}$ such that

$$c\,\partial_j(g-1) \equiv \mathrm{Nm}_j Z_g, \qquad \delta' \,\mathrm{Nm}\, Z_g = 0. \tag{9}$$

Here δ' is any diagonal integral operator with the root-chain hypothesis in item 1. That hypothesis and Lemma 7.6.5, item 2, give

$$\delta' \,\mathrm{Nm}\, B(1) = 0. \tag{10}$$

To see the passage from pairings to (10), $\mathrm{Nm}\, B(1)$ has base level. A base-level spherical eigenvector in a system with $\delta' \neq 0$ is orthogonal to every value of $B(1)$ by the local chain lemma, and hence to every value of its norm since that eigenvector is fixed by Γ. The local operators preserve each Hecke system and commute with δ'. Expanding a base-level form in its orthogonal spherical Hecke eigenspaces and using positive definiteness proves (10).

For completeness, the finite calculation proving (9) is as follows. Right multiplication by g permutes the terms s^{-1} of T_- up to left factors in H. Equation (3) and the linear congruence for λ therefore give an integral operator Q_g with

$$B(\lambda)T_-(g-1) \equiv B(1)Q_g.$$

Choose Ψ_H as in (1). Since H fixes $B(1)$ from the left,

$$cB(1) = \Big(\sum_{h \in H} h\Big)\Psi_H B(1).$$

Multiplying by T_+ turns the sum over H into Nm_j, so (9) holds with $Z_g = -\Psi_H B(1) Q_g$. To check its vanishing assertion, put $L = \delta' \mathrm{Nm}$. Then $Lh = L$ for $h \in H$ and $LB(1) = 0$ by (10). Consequently

$$\#H\, L\Psi_H B(1) = L\Big(\sum_{h\in H} h\Big)\Psi_H B(1) = cLB(1) = 0.$$

The operator ring acts on a complex vector space and has no nonzero integer torsion. Thus $L\Psi_H B(1) = 0$, proving (9).

The family and its factorisation. Local operators at distinct auxiliary places commute: their supporting adelic groups commute. Put

$$\mathcal{N}_S = \prod_{j\in S} \partial_j \prod_{j\notin S} \mathrm{Nm}_j, \qquad \mathcal{N} = \prod_j (\mathrm{Nm}_j + \alpha_j \partial_j). \tag{11}$$

These are integral, commute with $\mathbb{T}_Q$, and $\mathcal{N}_\emptyset = \mathrm{Nm}$. For a generator $t_g - \alpha_j r_g$ of the tied ideal, pair in the top coefficient of $\mathcal{N}F(t_g - \alpha_j r_g)$ the subsets not containing j with those containing it. Formula (5) makes the two terms cancel; the remaining terms contain $\mathrm{Nm}_j t_g = 0$. Generators divisible by m_ν disappear after σ. Additivity and multiplication by arbitrary polynomials then give the first assertion of item 1.

We prove the second assertion without dividing by a stabiliser order. Put $D = c^2\delta'$. For every subset S, induction on $\#S$ gives an integral Y'_S with

$$D^{\#S}\mathcal{N}_S \equiv Y'_S \mathrm{Nm}, \qquad Y'_\emptyset = 1. \tag{12}$$

For the induction step let $\#S = e+1$ and set $A_0 = c\delta' D^e \mathcal{N}_S$. For a group element in a factor $j \notin S$, the right factor Nm_j makes $A_0(g-1) = 0$. For $j \in S$, write $\mathcal{N}_S = N'\partial_j$ and $\mathcal{N}_{S\setminus\{j\}} = N'\mathrm{Nm}_j$. By (9),

$$A_0(g-1) \equiv \delta' D^e \mathcal{N}_{S\setminus\{j\}} Z_g = 0.$$

Indeed (12) for the smaller subset factors $D^e\mathcal{N}_{S\setminus\{j\}}$ through Nm; move δ' to its right before using that factorisation, and then use $\delta'\mathrm{Nm}Z_g = 0$. Thus A_0 is right Γ-invariant after reduction. Applying (1) for Γ yields $cA_0 \equiv A_0\Psi_\Gamma \mathrm{Nm}$, which is (12) since $cA_0 = D^{e+1}\mathcal{N}_S$. Multiply (12) by the remaining power $D^{\#J-\#S}$ and take $\mathcal{Y}_S = D^{\#J-\#S}Y'_S$. This gives $D^{\#J}\mathcal{N} \equiv \mathcal{Y}\mathrm{Nm}$ with $\mathcal{Y}_\emptyset = D^{\#J}$, as required.

The integral congruence projector. We now prove item 2. Congruence means equality modulo the maximal ideal of all characteristic coefficients of $N_x(g)$ and $N_\pi(g)$, for every g and every coefficient. There is an integral idempotent $A \in \mathbb{T}_Q$ whose value is 1 on the systems congruent to π and 0 on the others. Here is an explicit construction. For each of the finitely many noncongruent systems x, choose one characteristic-coefficient difference b_x which is a unit at x and is zero at π. All these functions lie in $\mathbb{T}_Q$ by item 1 of Lemma 7.6.13. Their values lie in one finite coefficient extension. If its residue cardinality is Q_0, then $b_x^{(Q_0-1)Q_0^a}$ tends, as $a \to \infty$, to 1 at its unit values and to 0 at its nonunits. Thus the product of $1 - b_x^{(Q_0-1)Q_0^a}$ tends to the desired indicator. The Hecke algebra is closed: a convergent family of diagonal operators preserves the integral lattice in the limit, as that lattice is closed and the space is finite dimensional. Hence A belongs to it. This construction also proves $A(\pi) = 1$.

Choose a common finite coefficient field for the finitely many representation models, the coefficients of H_j, and the scalars used here; the representations descend by the compact-image and Baire argument, and their integral models are obtained from a bounded orbit of a lattice. Intersect $\mathbb{T}_Q$ with the families valued in this field. Apply the common-Ribet hypothesis to the polynomial ring over this integral Hecke algebra, embedded in the ring of polynomial families on the finite set of systems, and to the image of the tied ideal. This is a trace ideal by Lemma 7.6.13,

item 1. The embedding is injective, as comparison of every coefficient at every system shows. The resulting common element therefore gives a polynomial $f \in \mathbb{T}_Q[\alpha_j]$ such that

$$f\delta_i \in \mathfrak{a}, \quad f_S(x) = 0 \ (x \in \mathcal{X},\ S \subsetneq J), \quad v_\ell(m f_J(\pi)) \le \beta. \tag{13}$$

It also shows that each δ_i belongs to $\mathbb{T}_Q$, by taking the constant coefficient of a polynomial representing its constant family. Set

$$d_0 = \prod_k (\lambda_k - \delta_{\mathrm{sel}(k)}), \quad n_* = n_0 c^2, \quad i_0 = \#J + 1, \quad D = c^2 A d_0.$$

The annihilator of f modulo $\mathfrak{a}$ is an ideal containing every δ_i. Reducing the displayed product modulo that ideal gives

$$(d_0 - n_0) f \in \mathfrak{a}. \tag{14}$$

The root-chain hypothesis applies to Ad_0, since its nonzero values at base level are precisely among the congruent systems with $d_0 \neq 0$. Item 1 consequently applies with $\delta' = Ad_0$.

Extraction of the top coefficient. Let $k_0 = \#J$. Put

$$G_0 = \sum_{S \subseteq J} \mathcal{N}_{J \setminus S} f_S, \qquad G_1 = \sum_{S \subseteq J} \mathcal{Y}_{J \setminus S} \mathrm{Nm}\, D f_S.$$

Here the $\mathcal{Y}$ are those of item 1 for D, so $\mathcal{Y}_\emptyset = D^{k_0}$. There is an integral operator W with

$$n_*^{i_0} A^{i_0} G_0 - G_1 = n_*^{i_0} b W \tag{15}$$

for some integral scalar b. To obtain (15), take the quotient of $\mathcal{E}$ by the scalar $\ell^\nu/(2n!)$. This is allowed by the divisibility hypothesis, and its kernel is that scalar times $\mathcal{E}$; the quotient has the reduction property used in item 1 since $\mathcal{E}$ has no integer torsion. Also $\ell^\nu/(2n!) = n_*^{i_0} b$ with b integral. Modulo this quotient, (14) and the top-coefficient annihilation give

$$n_*^{i_0} A^{i_0} G_0 \equiv D^{i_0} G_0,$$

because $D^{i_0} - n_*^{i_0} A^{i_0}$ is a multiple of $d_0 - n_0$. The factorisation of item 1, applied term by term, gives $D^{i_0} G_0 \equiv G_1$. This proves (15).

Let l, y, d' be as in item 2. Since all δ_i vanish on $\mathcal{X}$ and y is supported on $\mathcal{X}$,

$$d_0 y = n_0 y, \qquad f_S y = 0 \quad (S \subsetneq J).$$

Since y agrees with l on the systems where $d_0 \neq 0$, one also has $Ad_0 l = n_0 A y$. The vector $u = \Psi_\Gamma(d' l)$ is integral, and (2) gives $\mathrm{Nm}\, u = d' c\, l$. Move the diagonal Hecke operators past the norm in the expression for $G_1 u$. The preceding identities eliminate every proper subset S, while the last term uses $\mathcal{Y}_\emptyset = D^{k_0}$. The result is

$$G_1 u = n_*^{i_0} d' c\, A^{i_0} f_J y. \tag{16}$$

Indeed the norm first contributes $d'c$, the factor $D = c^2 A d_0$ contributes $n_* A$ on y, and D^{k_0} contributes $n_*^{k_0} A^{k_0}$.

Apply (15) to u, insert (16), and cancel the nonzero scalar $n_*^{i_0}$. It is nonzero because it divides $\ell^\nu/(2n!)$. We obtain

$$d' c\, A^{i_0} f_J y = (A^{i_0} G_0 - bW) u.$$

The right side is an integral operator applied to an integral vector. Thus it is integral. Taking $s = A^{i_0} f_J$, one has $s(\pi) = f_J(\pi)$, so (13) gives the required valuation bound and the claimed clearing for every d'. □

Proposition 7.6.12. *Let ℓ be prime, $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, and let ϑ have modulus Q_ϑ with $(S_\ell, Q_\vartheta, \psi_\vartheta)$ an auxiliary twist. Let (O, c) be a lift setting for ϑ, with source rank r', rank difference t_O, and $n = r' + t_O$. Put $\widetilde{\chi}_O(z) = \chi_O(z/\bar{z})$. The normalising character ν_O of (MS6) of Definition 7.2.30 is the geometric ℓ-adic avatar, read through ι, of*

$$\widetilde{\chi}_O^{-1} \upsilon^{r'} |\cdot|_E^{r'/2},$$

where υ is the splitting character of (WD3) of Definition 3.2.12. Let B be base data of rank n, and for every system x set $u_x = \nu_O \otimes r_x$. Suppose these representations have integral characteristic polynomials. Let x_0 be a system of base level, and let R be main-summand like of rank r' and string length t_O, with

$$\operatorname{tr} u_{x_0}(g) = \psi_\vartheta(g) \operatorname{tr} R(g) + \sum_{e<t_O} \chi_{\mathrm{cyc}}(g)^{-e}.$$

Let $g_1 \in G_E$ have cyclotomic image a topological generator of the cyclotomic image of G_E, let $h \geq 0$ satisfy $\chi_{\mathrm{cyc}}(g_1)^a - 1 \mid \ell^h$ for $0 < a < t_D$, and suppose $\ell \leq h_D^2$ or $h = 0$. Then there are a finite linearly ordered J and monotone $\mathrm{ty} : J \to \{1, \dots, t_O - 1\}$, independent of ν, such that for every $\nu > 0$ there are Taylor–Wiles places (w_j, φ_j, τ_j) of level ℓ^ν with:

1. *they form a prime triangle for g_1^{-1} with bounds $(h_D h, h_D^{22} h)$;*
2. *the places avoid S_ℓ^{bad}, are off the base level, and ν_O is unramified there;*
3. *there are $e_j \in \overline{\mathbb{Z}}_\ell[X]$ with $e_j(1) = 1$ such that every weak base system x congruent to x_0 satisfies*
$$P_{x,w_j}(\nu_O(\varphi_j)^{-1} X) = \prod_{i<t_O} (X - q_{w_j}^i)\, S_{x,j}(X), \qquad S_{x,j} \mid e_j,$$
with coefficients transported through ι^{-1}. Here weak means $\det(1 - u_x(g)) = 0$ for all $g \in G_E$, and congruence means equality modulo the maximal ideal of every characteristic coefficient at every g.

Proof. **Normalisation and weak lifts.** At a split place w away from the ramification of a, write

$$X_0(w) = \widetilde{\chi}_O(\varpi_w) \upsilon(\varpi_w)^{-r'} q_w^{r'/2}.$$

Here $\widetilde{\chi}_O(z) = \chi_O(z/\bar{z})$. The Hecke character defining a is algebraic: at an infinite place where χ_O has weight w, its infinity factor is $z^{r'-w}\bar{z}^w$. Thus its geometric avatar exists by the algebraic-character convention. The characters in the formula for X_0 are unitary, so $|X_0(w)|^2 = q_w^{r'}$. Evaluation of the defining Hecke character $\widetilde{\chi}_O^{-1}\upsilon^{r'}|\cdot|_E^{r'/2}$ at a uniformiser, using $|\varpi_w|_E = q_w^{-1}$ and geometric reciprocity, gives

$$a(\varphi_w) = \iota(X_0(w)^{-1}), \qquad \chi(\varphi_w) = q_w^{-1}. \tag{1}$$

Let x be a weak lift of base level. At a split unramified place where the Hecke–Frobenius equality of Theorem 7.1.4 holds, the root 1 of $C_x(\varphi_w)$ makes $X_0(w)$ a root of $P_{x,w}$. A nonzero base-level eigenform is a spherical vector in the unitary local representation on automorphic forms. Its invariant positive definite form is the class-sum inner product. Since $|X_0(w)|^2 q_w^t = q_w^n$, item 1 of Lemma 7.6.5 gives the distinct roots $X_0(w) q_w^e$, $0 \leq e < t$. Therefore

$$P_{x,w}(a(\varphi_w)^{-1} X) = \prod_{e<t} (X - q_w^e)\, A_{x,w}(X). \tag{2}$$

Coefficients in (2) are transported through ι^{-1}.

The integral cofactor. If B is monic integral and AB is integral, then A is integral, by division by a monic integral polynomial. If also $AB \equiv A'B$ coefficientwise modulo the maximal ideal, then $A \equiv A'$ coefficientwise. Indeed the Gauss norm of a monic integral polynomial is 1, and multiplicativity gives $\|A - A'\|_{\mathrm{G}} = \|(A - A')B\|_{\mathrm{G}} < 1$. In particular, if $|A'(1)|_\ell = 1$, then $|A(1)|_\ell = 1$, and $A/A(1)$ is integral with value 1 at 1.

Put $b = a(\varphi_w)$ and transport (2) to $\overline{\mathbb{Q}}_\ell$. Exact unramified compatibility and twisting give

$$C_x(\varphi_w; X) = b^n P_{x,w}(b^{-1}X) = \big(b^n A_{x,w}(X)\big) B_{\varphi_w}(X). \tag{3}$$

The character a has unit values by compactness, and B_{φ_w} is monic integral. Newton identities applied to the trace hypothesis at all powers g^j give

$$C_{x_0}(g; X) = \det(X - \psi_\vartheta(g)R(g))\, B_g(X). \tag{4}$$

Suppose

$$|\det(1 - \psi_\vartheta(\varphi_w)R(\varphi_w))|_\ell = 1. \tag{5}$$

If x is congruent to x_0, the monic-factor argument applied to (3), (4) makes $b^n A_{x,w}$ integral with unit value at 1. Since b is a unit, the same holds for $A_{x,w}$. Dividing by that value gives an integral polynomial $e_{x,w}$ with $e_{x,w}(1) = 1$ such that the cofactor in (2) divides $e_{x,w}$.

Frobenius data near suitable translates. Put $T(g) = \psi_\vartheta(g)R(g)$. Let $\mathcal{W}$ consist of the Frobenius data whose places avoid the bad places, are off the base level and the ramification of a, satisfy the Hecke–Frobenius equalities for the finitely many base systems, and satisfy $|\det(1 - T(\varphi))|_\ell = 1$. All conditions except the last exclude only finitely many places, including the exceptions to (1).

For every $v \geq 0$ and $g \in G_E$, we show that some $\delta \in \ker \chi$ makes $\mathcal{W}$ contain most Frobenii near $\delta^{\ell^v} g$. By (MS3), there is an inertial γ above $p_0 p_1$, away from ℓ, with

$$R(\gamma) = z\,1, \qquad |z^{\ell^a j} - 1|_\ell = 1 \quad (a \geq 0,\ 1 \leq j \leq 4r' + 2). \tag{6}$$

The modulus of a twist is prime to MN, by Definition 3.1.2; hence $\psi_\vartheta(\gamma) = 1$. Also $\chi(\gamma) = 1$. Thus

$$T((\gamma^i)^{\ell^v} g) = (z^{\ell^v})^i T(g). \tag{7}$$

For each eigenvalue μ of $T(g)$, among $0 \leq i \leq r'$ at most one can have $|1 - (z^{\ell^v})^i \mu|_\ell < 1$. Two such integers would give $|(z^{\ell^v})^{i-i'} - 1|_\ell < 1$, contrary to (6). There are r' eigenvalues and $r' + 1$ integers, so some i avoids all these inequalities. Compactness supplies an integral lattice for T, so its eigenvalues are units; consequently all r' factors have norm exactly 1. Equation (7) proves $|\det(1 - T((\gamma^i)^{\ell^v} g))|_\ell = 1$. The unit condition is open by continuity. Take $\delta = \gamma^i$ and shrink a neighbourhood of $\delta^{\ell^v} g$. Its Frobenius data satisfy (5), and all but finitely many satisfy the other conditions defining $\mathcal{W}$. This is precisely the required most-Frobenii property. Nonemptiness of the Frobenius filters and removal of the finite exceptional sets follow from Theorem A.2.6.

The triangle and the common polynomial. Apply Theorem 7.6.15 with conductor $q_1 d$, length t and $y = H$. By Lemma 4.2.67 and Lemma 4.2.66,

$$q_1 d + t \leq x^{13} + 15x^5 < x^{40} \leq H, \qquad E \subseteq \mathbb{Q}(\mu_{q_1 d}).$$

The tower formula gives $5040 \mid q_1$, so E contains μ_{5040} and has at least four infinite places. The generator, separation and coefficient-prime assumptions are those of the statement, restricted to $t \leq t_D$. The near-Frobenius hypothesis was just proved. The resulting finite ordered J and

monotone exponent are independent of ν; for each $\nu > 0$ the theorem gives the required triangle, all its Frobenius data in $\mathcal{W}$.

There are finitely many weak base systems congruent to x_0. For each resulting place use their polynomials constructed from (2)–(5) and put

$$e_j = \prod_{\substack{x \text{ weak, base} \\ x \equiv x_0}} e_{x,w_j}.$$

The empty product is 1. Every factor is integral and has value 1 at 1, so e_j does too; every cofactor in (2) divides it. This proves all three conclusions simultaneously. □

Lemma 7.6.13. *Assume $[E^+ : \mathbb{Q}] \geq 2$. Let B be base data, L a level, $a\colon G_E \to \bar{\mathbb{Z}}_\ell^\times$ a character and $N_x = a \otimes r_x$.*

1. *For every $g \in G_E$ and $i \geq 0$, the function sending a system x of level L to the i-th coefficient of the characteristic polynomial of $N_x(g)$ lies in $\mathbb{T}_L$; hence $\operatorname{tr}[P(h)g] \in \mathbb{T}_L$ for $P \in \bar{\mathbb{Z}}_\ell[X]$, $g, h \in G_E$.*

Let now $Q = (w_j, \varphi_j, \tau_j)_{j \in J}$ be Taylor–Wiles places of level ℓ^ν, all off the base level, let $a(\tau_j) = 1$ for all j, fix j, $w = w_j$, $\varphi = \varphi_j$, $\tau = \tau_j$, and identify the decomposition group at the place of $\bar{\mathbb{Q}}$ of the datum with G_{E_w}. Let $\varpi \in E_w$ be a uniformiser and $u \in \mathcal{O}_{E_w}^\times$ a unit such that $\operatorname{Art}_{E_w}(\varpi)^{-1}$ and $\operatorname{Art}_{E_w}(u)^{-1}$ are the images of φ and of τ in $G_{E_w}^{\mathrm{ab}}$, let $\bar{z} \in k_w^\times$ be the reduction of u (a generator of $k_w^\times$, as τ generates tame inertia), and fix a homomorphism $\operatorname{ind}\colon k_w^\times \to \mathbb{Z}/\ell^\nu$ with $\operatorname{ind}(\bar{z}) = 1$, and let $\psi\colon \mathbb{Z}/\ell^\nu \to \bar{\mathbb{Z}}_\ell^\times$ be a character. Let y be a form of level Q in a system x.

2. *(Tame generator.) If $\mathrm{GL}_n(k_w)$ acts on y through $\psi \circ \operatorname{ind} \circ \det$, then for all $g \in G_E$ and $p \in \bar{\mathbb{Z}}_\ell[X]$,*

$$\operatorname{tr}\big[(\tau - 1)\, g\, p(\varphi)\big]\, y = (\langle \bar{z} \rangle - 1)\operatorname{tr}\big[g\, p(\varphi)\big]\, y, \qquad \langle \bar{z} \rangle = \operatorname{diag}(\bar{z}, 1, \ldots, 1).$$

3. *(Frobenius.) If the stabiliser in $\mathrm{GL}_n(k_w)$ of the line spanned by the first basis vector acts on y through $g \mapsto \psi(\operatorname{ind}(g_{11}))$ and $\psi(n!) \neq 1$, then for all g and p,*

$$\operatorname{tr}\big[(\tau - 1)\, g\, p(\varphi)\big]\, y = p(\tilde{U}) \operatorname{tr}\big[(\tau - 1)\, g\big]\, y, \qquad \tilde{U} = a(\varphi)\, U_\varpi,$$

where, with $I_1 \subseteq \mathrm{GL}_n(\mathcal{O}_{E_w})$ the pro-p Iwahori subgroup (the matrices whose reduction is upper unitriangular) and $t_\varpi = \operatorname{diag}(\varpi, 1, \ldots, 1)$, $U_\varpi y = \sum_{h \in I_1 t_\varpi I_1 / I_1} h\, y$ is the double-coset operator of t_ϖ on I_1-fixed forms, for the uniformiser ϖ fixed above (under the hypothesis of this item I_1 fixes y, since its elements reduce into the stabiliser of the line with $g_{11} = 1$, and U_ϖ maps I_1-fixed forms to I_1-fixed forms).

4. *(Transfer.) Over a finite field k, the transfer from $\mathrm{GL}_n(k)$ to the stabiliser of a line, composed with the first diagonal entry, is the determinant.*

Proof. Put $F = E^+$; the stated degree hypothesis gives $[F : \mathbb{Q}] \geq 2$. The space of forms of a fixed weight and finite level is finite-dimensional: by Theorem A.4.5, the adelic quotient is compact, and its quotient by the compact group at infinity and an open compact finite level has finitely many points. Evaluation at representatives embeds the space into a finite direct sum of copies of the weight space. Its integral forms are the intersection with the corresponding integral value modules; in particular that set is closed for the ℓ-adic topology transported by ι.

We will also use the irreducible automorphic decomposition on these spaces. There is no proper F-parabolic subgroup of $\mathrm{U}(V)$: such a subgroup would give a nonzero isotropic vector over E, contrary to definiteness at a real place. The cuspidality condition is therefore vacuous. Apply Theorem 7.6.14(a) to the compact quotient, take the prescribed infinity type and finite-level fixed vectors, and use their finite dimension. This gives a finite sum of irreducible automorphic components. Their projections commute with the local group action, the away-from-level Hecke operators and the finite type projectors. The factorisation Theorem A.4.25, with the finite adelic group separated into its factor at w and the remaining factors, supplies the irreducible local components used below. The compact infinity factor consists of discrete-series representations, and property $(*)$ in Theorem A.4.16 is automatic; that theorem therefore supplies weak base change for every such component. Every component in a system x has the same attached semisimple Galois representation r_x: its unramified characteristic polynomials are those of x, and the uniqueness assertion of Theorem 7.1.4 identifies the representations.

Integral characteristic coefficients. For a split place w off level and $0 \le k \le n$, let $T_{w,k}$ denote the double-coset sum for $\mathrm{diag}(\varpi_w 1_k, 1_{n-k})$. Each summand preserves the integral value condition, so $T_{w,k}$ preserves integral forms. By Theorem A.4.24, its scalar on x is

$$q_w^{k(n-k)/2} e_k(\alpha_{x,w,1}, \dots, \alpha_{x,w,n}).$$

Consequently the coefficient of X^{n-k} in the Hecke polynomial is the scalar of

$$(-1)^k q_w^{k(k-1)/2} T_{w,k}.$$

The multiplier is an integer. All coefficients of these Hecke polynomials therefore lie in $\mathbb{T}_L$.

The algebra $\mathbb{T}_L$ is closed in the finite product of the coefficient fields indexed by the systems occurring at this level. Indeed the diagonal-scalar map from that product to the endomorphisms of the finite-dimensional form space is continuous; preservation of integral forms is a closed condition, since for each integral form its image must belong to the closed integral value module. This argument does not require that $\bar{\mathbb{Z}}_\ell$ be a discrete valuation ring or that an integral basis over that ring be chosen.

Fix $g \in G_E$. Chebotarev, Theorem A.2.6, applied to simultaneous finite quotients for the finitely many representations at this level, approximates g, up to conjugacy, by geometric Frobenius elements at places off level. These places can be required to split over F: take the finite Galois extensions stable under $\mathrm{Gal}(E/F)$ and apply Chebotarev over F to an element restricting to the identity on E, choosing the place above it which gives the required conjugacy class over E. Finitely many forbidden places may be omitted. Continuity of characteristic coefficients and (Ua2) of Theorem 7.1.4 now imply that every characteristic coefficient of $r_x(g)$, simultaneously as a function of x, belongs to $\mathbb{T}_L$.

For $0 \le i \le n$, the coefficient of X^i in $\det(X - a(g) r_x(g))$ is $a(g)^{n-i}$ times the corresponding coefficient for $r_x(g)$. Since $a(g)$ is integral, this coefficient belongs to $\mathbb{T}_L$; coefficients with $i > n$ are zero. Finally, writing $P(X) = \sum_j P_j X^j$ gives directly

$$\mathrm{tr}[P(h)g] = \sum_j P_j \ \mathrm{tr}[h^j g].$$

A trace is minus the coefficient of X^{n-1}, so this expression lies in $\mathbb{T}_L$. No division by a factorial or by an integer is used.

The tame scalar. Write $\zeta = \psi(1)$. Apply (Up3) of Theorem 7.1.11 with one block and the character $\iota^{-1}\psi \circ \mathrm{ind}$. The prescribed relation $\mathrm{Art}_{E_w}(u)^{-1} = \tau$ in the abelianisation gives

$$r_x(\tau) = \zeta I.$$

Here semisimplicity is essential and is part of (Up3). Since $a(\tau)=1$, also $N_x(\tau)=\zeta I$. The diamond operator in the statement acts on y by ζ. Thus, for every g and p,

$$\operatorname{tr}\big((N_x(\tau)-I)N_x(g)p(N_x(\varphi))\big)=(\zeta-1)\operatorname{tr}\big(N_x(g)p(N_x(\varphi))\big),$$

which proves item 2. No commutation of $N_x(g)$ with a local operator is being asserted.

The line type and its double-coset scalar. For item 3 suppose first that $n\geq 2$. Set $K=\mathrm{GL}_n(\mathcal{O}_{E_w})$, let J be the inverse image in K of the line stabiliser, and put $\chi=\iota^{-1}\psi\circ\det$. The hypothesis $\psi(n!)\neq 1$ implies $\chi\neq 1$. On each nonzero irreducible automorphic component of y, (Up1) of Theorem 7.1.11, with the two blocks $(1,n-1)$, identifies the local representation with the irreducible normalised induction

$$\pi_w=\operatorname{Ind}_P^{\mathrm{GL}_n(E_w)}(\rho_1\otimes\rho_2),$$

where ρ_1 is a character restricting to χ on units and ρ_2 is irreducible spherical. Write a vector in this induction as f, with right translation action and $f(pg)=\delta_P(p)^{1/2}(\rho_1\otimes\rho_2)(p)f(g)$.

We verify explicitly the part of this induced model seen by the original line type. The double quotient $(P\cap K)\backslash K/J$ has two elements, represented by 1 and the transposition $w_0=(12)$. Indeed the finite parabolic stabilising the first line has two orbits on lines: that line and its complement. Reduction gives these two cells since its kernel is contained in J. A J-eigenvector of character $j\mapsto\chi(\bar{j}_{11})$ is determined on a cell by its value at its representative.

Choose a unit b with $\chi(\bar{b})\neq 1$, and put $h=\operatorname{diag}(1,bI_{n-1})$. It belongs to $P\cap w_0Jw_0^{-1}$. Its inducing action is trivial. To see this on the entire space of ρ_2, not just on a spherical line, observe that the central unit bI_{n-1} fixes a nonzero K_{n-1}-fixed vector, commutes with $\mathrm{GL}_{n-1}(E_w)$, and that vector generates the irreducible representation ρ_2. On the other hand the first diagonal entry of $w_0^{-1}hw_0$ is b. Comparing left and right equivariance at w_0 therefore gives

$$f(w_0)=\chi(\bar{b})f(w_0),\qquad f(w_0)=0.$$

Evaluation $f\mapsto f(1)$ is consequently injective on the J-eigenspace. No assertion that all compact elements act trivially on a spherical representation has been used.

Put $t=\operatorname{diag}(\varpi,1,\dots,1)$ and $n(v)=\left(\begin{smallmatrix}1 & v\\ 0 & I_{n-1}\end{smallmatrix}\right)$. For v running through $\mathcal{O}_{E_w}^{n-1}/\varpi\mathcal{O}_{E_w}^{n-1}$, the matrices $n(v)t$ represent both JtJ/J and I_1tI_1/I_1. This follows by reducing the first upper row: conjugation by t multiplies those entries by ϖ, while all other quotient coordinates are unchanged. In particular there are exactly q_w^{n-1} representatives.

This sum preserves the actual J-eigenspace. For $j\in J$, write $jn(v)t=n(v')th_v$ with $h_v\in J$. The map $v\mapsto v'$ is a permutation of the quotient. Comparing first columns shows $(h_v)_{i1}\in\varpi^2\mathcal{O}_{E_w}$ for $i>1$ and $(h_v)_{11}\equiv j_{11}\pmod{\varpi}$. Hence $\chi(\overline{(h_v)_{11}})=\chi(\bar{j}_{11})$, and summing proves the required equivariance. Thus injective evaluation may be applied to $U_\varpi f$. Since all $n(v)t$ belong to P,

$$(U_\varpi f)(1)=q_w^{n-1}\delta_P(t)^{1/2}\rho_1(\varpi)f(1)=q_w^{(n-1)/2}\rho_1(\varpi)f(1).$$

It follows that

$$U_\varpi f=\alpha f,\qquad \alpha=q_w^{(n-1)/2}\rho_1(\varpi).\tag{$*$}$$

This is the double-coset sum, not the mean.

Identification with the Frobenius on the ramified line. By (Up2)–(Up3) of Theorem 7.1.11, inertia is semisimple, the monodromy is zero, and its nontrivial character $\psi\circ\det$ occurs once, the remaining inertia characters being trivial. Thus

$$\mathcal{L}=\operatorname{im}(r_x(\tau)-I)$$

is that one-dimensional character space. It is stable under Frobenius: its inertia character is obtained from local reciprocity and hence is invariant under conjugation by the local Weil group. By (Us2) of Theorem 7.1.8, the parameter is the direct sum of the parameters of $\rho_1|\cdot|^{(1-n)/2}$ and $\rho_2|\det|^{(1-n)/2}$. Here the compatibility with this irreducible induction follows from (Ln6) of Theorem A.4.9; equivalently, order the unramified segments of the spherical factor in a Langlands order. Its parameter is unramified by the spherical description in Theorem A.4.24 and (Ln1), (Ln6). Therefore the unique nontrivial inertia line is the first summand.

Our Artin map takes uniformisers to arithmetic Frobenius, whereas (Ln1) uses geometric Frobenius. Its value at the prescribed $\varphi = \mathrm{Art}_{E_w}(\varpi)^{-1}$ is consequently $\rho_1(\varpi)|\varpi|^{(1-n)/2} = \alpha$. Frobenius semisimplification cannot change a scalar on a one-dimensional invariant space. We have proved the exact identity

$$r_x(\varphi)(r_x(\tau) - I) = \iota(\alpha)(r_x(\tau) - I). \tag{**}$$

In particular the scalar $\iota(\alpha)$ is determined by r_x, independently of the irreducible automorphic component. By (*) it follows that U_ϖ acts on the entire original vector y by $\iota^{-1}(\lambda)$, where λ is this scalar; decomposing y into its finitely many nonzero automorphic components justifies this even when y is not initially specified to belong to a single irreducible representation.

For $n = 1$ the same conclusion is immediate without a two-block assertion: the local irreducible representation is a character ρ_1, there is one double-coset representative, $U_\varpi = \rho_1(\varpi)$, and (Ln1) gives (**). In all ranks $a(\tau) = 1$ and multiplication by $a(\varphi)$ therefore give, with $D = N_x(\tau) - I$, $F_x = N_x(\varphi)$ and $\tilde\lambda = a(\varphi)\lambda$,

$$F_x D = \tilde\lambda D, \qquad p(F_x)D = p(\tilde\lambda)D, \qquad p(\tilde U)y = \iota^{-1}(p(\tilde\lambda))y.$$

Finally cyclicity of trace, valid for arbitrary $N_x(g)$, yields

$$\operatorname{tr}(DN_x(g)p(F_x)) = \operatorname{tr}(p(F_x)DN_x(g)) = p(\tilde\lambda)\operatorname{tr}(DN_x(g)).$$

This is item 3 with exactly the fixed uniformiser, twist and double-coset convention of the statement.

Transfer. Let $\mathscr{L}$ be the set of lines of k^n, and choose $v_L \neq 0$ on every line, with $v_{ke_1} = e_1$. Define $c(g, L) \in k^\times$ by $gv_L = c(g, L)v_{gL}$. The coset formula for transfer, followed by the first entry on the stabiliser, is

$$T(g) = \prod_{L \in \mathscr{L}} c(g, L).$$

Changing any v_L cancels between the factors indexed by L and $g^{-1}L$, and $c(gh, L) = c(h, L)c(g, hL)$ proves that T is a homomorphism. A transvection has order the characteristic of k, whereas $k^\times$ has order $\#k - 1$, so its image under T is 1.

For $d = \mathrm{diag}(b, 1, \dots, 1)$, the line ke_1 contributes b and every line in $H = \langle e_2, \dots, e_n\rangle$ contributes 1. The remaining lines are uniquely $k(e_1 + w)$ with $0 \neq w \in H$. Their permutation is $w \mapsto b^{-1}w$. If b has order m, each orbit has length m and its product of scalar factors is $b^m = 1$. Hence $T(d) = b = \det d$. Transvections together with these diagonal matrices generate $\mathrm{GL}_n(k)$ by Gaussian elimination, proving $T = \det$. For $n = 1$ the transfer is the identity; if $k = \mathbb{F}_2$, the target group is trivial. This completes all four assertions. □

Theorem 7.6.14. *Let E/F be a CM extension, V a hermitian space over E of rank $r \geq 1$, $G = \mathrm{U}(V)$, $G_\infty = \prod_\rho G(F_\rho)$ the product over the real places of F, μ a Haar measure on $G(\mathbb{A}_F)$, R the right regular representation of $G(\mathbb{A}_F)$ on $L^2 = L^2(G(F)\backslash G(\mathbb{A}_F))$ (quotient of μ by counting measure), and $L^2_{\mathrm{cusp}} \subseteq L^2$ the cuspidal subspace: for a nonzero totally isotropic*

E-subspace $C \subseteq V$ let N_C be the unipotent radical of the parabolic subgroup of G stabilising C; then L^2_{cusp} is the set of classes that have a left $G(F)$-invariant Borel representative ϕ such that for every such C and almost every $x \in G(\mathbb{A}_F)$ the integral $\int_{N_C(F)\backslash N_C(\mathbb{A}_F)} \phi(nx)\,dn$ converges absolutely and vanishes, dn being the quotient of a Haar measure of $N_C(\mathbb{A}_F)$ by the counting measure on $N_C(F)$. Let $C_c^\infty(G(\mathbb{A}_F))$ be the span of the functions $g \mapsto f_\infty(g_\infty)f_{\mathrm{f}}(g_{\mathrm{f}})$ with f_∞ smooth with compact support on G_∞ and f_{f} locally constant with compact support on $G(\mathbb{A}_{F,\mathrm{f}})$, and $R(f) = \int f(g)R(g)\,d\mu(g)$.

(a) L^2_{cusp} is a closed $G(\mathbb{A}_F)$-invariant subspace of L^2. For every irreducible unitary representation π of $G(\mathbb{A}_F)$ on a Hilbert space H_π, the space $\mathrm{Hom}_{G(\mathbb{A}_F)}(H_\pi, L^2_{\mathrm{cusp}})$ of bounded linear maps intertwining π with R has finite dimension $m(\pi)$. Let $L^2_{\mathrm{cusp}}[\pi]$ be the closure of the span of the images of these maps. Then $L^2_{\mathrm{cusp}}[\pi]$ is unitarily equivalent, as a representation of $G(\mathbb{A}_F)$, to the orthogonal direct sum of $m(\pi)$ copies of π; $L^2_{\mathrm{cusp}}[\pi]$ and $L^2_{\mathrm{cusp}}[\pi']$ are orthogonal for inequivalent π, π'; and L^2_{cusp} is the Hilbert direct sum of the $L^2_{\mathrm{cusp}}[\pi]$, π running over the equivalence classes of irreducible unitary representations of $G(\mathbb{A}_F)$.

(b) For every $f \in C_c^\infty(G(\mathbb{A}_F))$ the operator $R(f)$ maps L^2_{cusp} into itself, and its restriction to L^2_{cusp} is of trace class.

Proof. The published inputs are the cuspidal decomposition and Hilbert–Schmidt theorem in [7], Section 12, Theorem 12.1 and its proof, printed p.64, and the finite convolution factorisation theorem [72], Theorem 3.1. We give the normalization and trace-class deduction.

Apply the former to the connected reductive $\mathbb{Q}$-group $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{U}(V)$. Its rational points and adelic points identify with $G(F)$ and $G(\mathbb{A}_F)$, respectively. There is no nontrivial rational character: over F every character factors through the determinant torus $\mathrm{U}(1)$, which is anisotropic, and the same assertion for the restriction of scalars follows by base change and Galois invariance. Consequently Arthur's subgroup of adelic points on which all rational-character absolute values equal 1 is the entire adelic group. Thus no central split direction or fixed-central-character quotient is missing here. Haar measures differ only by positive scalars; scaling the Hilbert norm and convolution operator by constants preserves the stated decomposition and operator ideals.

The cuspidality conditions agree. A proper parabolic of a unitary group is a stabilizer of a nonempty isotropic flag. If C is its smallest member, its unipotent radical N contains N_C as a normal subgroup. Integration of its constant term first over $N_C(F)\backslash N_C(\mathbb{A}_F)$ and then over the quotient gives zero under the stated hypothesis. These unipotent adelic quotients are compact; Fubini applies for almost every translate to a locally integrable representative, and the L^2 representative is locally integrable by Cauchy–Schwarz. Conversely the parabolic stabilizing each nonzero isotropic C is one of the proper parabolics. Hence vanishing of all proper-parabolic constant terms is equivalent to the displayed condition.

Arthur's theorem gives a closed invariant Hilbert subspace and an orthogonal discrete decomposition into irreducible unitary representations with finite multiplicities. If π is one of these irreducibles, bounded Hilbert-space Schur's lemma identifies its intertwining maps into this direct sum with its finite-dimensional multiplicity space: maps to an inequivalent irreducible vanish, and maps to an equivalent copy are scalar multiples of a unitary equivalence. Thus the isotypic closures, the dimensions $m(\pi)$, and their orthogonality are precisely those of part (a), including zero multiplicity for absent π.

The proof of Arthur's theorem states that $R_{\mathrm{cusp}}(h)$ is Hilbert–Schmidt for every smooth compactly supported test function h. We derive part (b), rather than identify trace class with that assertion. Factor the chosen Haar measure as a product of archimedean and finite Haar

measures. It suffices by the definition of the test space to take $f = f_\infty \otimes f_{\mathrm{f}}$. By the Dixmier–Malliavin theorem there are finitely many $a_i, b_i \in C_c^\infty(G_\infty)$ with

$$f_\infty = \sum_i a_i * b_i.$$

Choose a compact open subgroup U of the finite adelic group fixing f_{f} under left translation, and put $e_U = \mathbf{1}_U/\mu_{\mathrm{f}}(U)$. Local constancy and compact support permit such a U; averaging gives $e_U * f_{\mathrm{f}} = f_{\mathrm{f}}$. Consequently

$$f = \sum_i (a_i \otimes e_U) * (b_i \otimes f_{\mathrm{f}}), \qquad R_{\mathrm{cusp}}(f) = \sum_i R_{\mathrm{cusp}}(a_i \otimes e_U) R_{\mathrm{cusp}}(b_i \otimes f_{\mathrm{f}}).$$

Every factor is Hilbert–Schmidt by the cited cuspidal theorem. A product of two Hilbert–Schmidt operators is trace class, and a finite sum of trace-class operators is trace class. Invariance of the cusp subspace, already proved in part (a), justifies each restriction and composition. This proves part (b). □

Theorem 7.6.15. *Let ℓ be a prime, $N \geq 1$, and let $E \subseteq \mathbb{Q}(\mu_N)$ be a CM field containing μ_{5040} with at least four infinite places. Let $t, y, h \geq 0$ be integers with $N + t < y$, and let $\gamma \in G_E$ be such that $\operatorname{cyc}(\gamma)$ generates the closure of $\operatorname{cyc}(G_E)$ topologically, $\operatorname{cyc}(\gamma)^a - 1$ divides ℓ^h in $\mathbb{Z}_\ell$ for $0 < a < t$, and $\ell \leq y^2$ or $h = 0$. Let W be a set of Frobenius data such that for every $\nu \geq 0$ and $g \in G_E$ there is $\delta \in \ker \operatorname{cyc}$ for which W contains most Frobenii near $\delta^{\ell^\nu} g$.*

Then there are a finite linearly ordered set J and a monotone map $\operatorname{ty}\colon J \to \{1, \dots, t-1\}$ such that for every $\nu > 0$ there are Taylor–Wiles places $(w_j, \varphi_j, \tau_j)_{j \in J}$ of level ℓ^ν, all in W, that form a prime triangle for γ^{-1} with bounds $(yh,\, y^{22}h)$.

Proof. Write $\chi = \chi_{\mathrm{cyc}}$, $u = \chi(\gamma)$ and $a = \gamma^{-1}$. By Lemma 7.2.28, applied to a, choose a finite set $s \subseteq \ker \chi$, independently of the coefficient ring, exponent and truncation, which detects every cocycle unramified outside ℓ after its value at a is prescribed to be zero. If $\ell \leq y^2$, it satisfies $\#s \leq y^{20}$. The existence of a finite such set holds also without this inequality. Order s and set

$$J = \{1, \dots, t-1\} \times s$$

with lexicographic order and $\operatorname{ty}(e, g) = e$. When $t \leq 1$ the first factor and hence J are empty. This choice is independent of ν.

We record the elementary divisibility estimates used in the construction. The hypothesis on u implies, for distinct b, k in $\{0, \dots, t-1\}$,

$$u^b - u^k \mid \ell^h, \qquad D_b := \prod_{\substack{0 \leq k < t \\ k \neq b}} (u^b - u^k) \mid \ell^{(t-1)h}.$$

Indeed one factors out a unit and uses $0 < |b - k| < t$. Since E contains μ_4, $u^b - 1$ is divisible by 4 in $\mathbb{Z}_\ell$. If $0 < b < t$, it also divides ℓ^h, and consequently $4 \mid \ell^h$. Thus, whenever $J \neq \emptyset$,

$$4D_b \mid \ell^{(t+2)h} \mid \ell^{yh}, \qquad 4(u^b - 1)^2 \mid \ell^{3h} \mid \ell^{yh} \quad (0 < b < t).$$

Here $t + 2 \leq y$ follows from $N + t < y$ and $N \geq 1$. These statements include $\ell = 2$: the factor 4 is retained rather than cancelled.

Fix $\nu > 0$ and put $R_\nu = \mathbb{Z}/\ell^\nu$ and $G_\nu = \ker(\chi \bmod \ell^\nu)$. We construct, in the order of J, a datum (w_j, φ_j, τ_j), a Kummer map ψ_j of exponent $\operatorname{ty}(j)$, and a finite exceptional set T_j of primes of the maximal real field F. The set T_j contains the primes above ℓ and those below

the earlier w_i of the same exponent. The reductions of the maps of any fixed exponent will be linearly independent as functions $G_\nu \to \mathbb{F}_\ell$. We require the prime below w_j to avoid T_i for all $i \le j$ and all previously chosen primes. At later stages we require

$$\psi_i(\varphi_j) = 0 \quad (i < j), \qquad 2\psi_j(\varphi_j) \mid \ell^{yh} \quad \text{in } R_\nu,$$

and, if $j = (e, g)$,

$$z(\varphi_j) = 2z(g) \quad \text{for every coefficient ring } \mathcal{O} \text{ and } z \in \mathrm{Ev}_\emptyset(e, \nu).$$

Here is the induction step, including the simultaneous choice of the values. Let $j = (e, g)$ and let x be the family of earlier data whose exponent is e. Item 3 of Proposition 7.6.16 gives a finite T_j and a Kummer map ψ_j whose reduction does not lie in the span of the reductions of the earlier maps of exponent e. To see this directly from that item, if every available reduction belonged to that span, their whole span would have dimension at most $\#x$, contrary to its conclusion. Items 1 and 3 therefore preserve linear independence at the new stage.

The homomorphisms on G_ν consisting of ψ_j and these earlier maps jointly map onto $R_\nu^{1+\#x}$. In fact their image is an R_ν-submodule, and independence of the component linear forms modulo ℓ says that its reduction modulo ℓ is the whole target. If its quotient were nonzero, multiplication by ℓ would be surjective on that finite quotient and hence, on iterating ν times, would force it to be zero, a contradiction. Choose $v_0 \in G_\nu$ with $\psi_j(v_0) = 1$ and every earlier map of exponent e zero at v_0.

Conjugation separates the other exponents without imposing independence between different exponents. If ψ has exponent d, item 1 of Proposition 7.6.16 gives

$$\psi(\gamma v \gamma^{-1}) = u^{d+1}\psi(v) \qquad (v \in G_\nu).$$

Choose integers representing u^{-1} and $-u^k$ modulo ℓ^ν. Replacing v by the product of the corresponding powers of $\gamma v \gamma^{-1}$ and v multiplies every such $\psi(v)$ by $u^d - u^k$. Perform this operation once for every $0 \le k < t$ other than e. The resulting $v \in G_\nu$ satisfies

$$\psi_i(v) = 0 \quad (i < j), \qquad \psi_j(v) = D_e \bmod \ell^\nu.$$

Maps of exponent e remain zero from their initial value; every map of another exponent acquires its zero factor in the product. The construction takes place in G_ν because that subgroup is normal.

Apply the hypothesis on W to

$$g\, g^c\, v (v^c)^{-1}.$$

It supplies $\delta \in \ker \chi$ such that W contains most Frobenii near

$$\sigma = \delta^{\ell^\nu}\, g\, g^c\, v (v^c)^{-1}.$$

All factors lie in G_ν. Every Kummer map under consideration is a homomorphism on this subgroup and is odd under c, so

$$\psi_i(\sigma) = 2\psi_i(v) = 0 \quad (i < j), \qquad \psi_j(\sigma) = 2D_e.$$

The term involving δ is zero because its multiple is ℓ^ν. Every even cocycle z of exponent e unramified outside ℓ is a homomorphism on G_ν and is invariant under c. Consequently

$$z(\sigma) = 2z(g).$$

These equalities hold on one open neighbourhood of σ, simultaneously for all the Kummer maps already chosen and all coefficient rings and even cocycles in the last assertion. For the latter uniformity it suffices to use scalar cocycles with values in R_ν: every additive map $\mathcal{O}/\ell^\nu \to R_\nu$ is R_ν-linear, carries a cocycle to a scalar cocycle, and these maps separate points. There are only finitely many scalar cocycles unramified outside ℓ. For completeness, finiteness follows from the finiteness condition in Theorem A.3.19, applied to the open subgroup killing the finite action: its homomorphisms to $\mathbb{F}_\ell$ are finite in number; successive exact sequences for the target $\mathbb{Z}/\ell^m$ give finiteness for R_ν, and a cocycle on the whole group is determined by its restriction and finitely many coset values. Intersect their finitely many neighbourhoods of constancy. Also intersect with the open subgroup G_ν.

The definition of most Frobenii, followed by Theorem A.2.6, now gives a prime-norm datum in this neighbourhood and in W, outside the finitely many primes above ℓ, the T_i for $i \le j$, and the previously used primes of F. More explicitly, inside the neighbourhood and the neighbourhood witnessing most Frobenii near σ, that definition supplies a nonempty open subset on which all sufficiently large prime-norm data are allowed. Chebotarev supplies infinitely many such data there, so the finite exclusions can be imposed. Take this as (w_j, φ_j, τ_j), using the fixed chosen tame generator. The divisibility estimate above gives

$$2\psi_j(\varphi_j) = 4D_e \mid \ell^{yh} \quad \text{in } R_\nu.$$

This proves every induction requirement.

The primes below the w_j are pairwise distinct. A prime-norm place of E cannot be fixed by complex conjugation here: E/F is unramified at finite places, while an inert place has residue degree 2. To check the unramified assertion without imposing an additional hypothesis, use the cyclotomic subfield $\mathbb{Q}(\mu_{5040}) \subseteq E$. At a prime of residue characteristic p, complex conjugation acts nontrivially on a root of unity of order prime to p in this subfield (use μ_3 if $p \neq 3$ and μ_4 if $p = 3$); inertia acts trivially on these prime-to-p roots. Thus inertia in E/F cannot be its nontrivial element. It follows that the w_j are pairwise distinct and not conjugate. Since $\varphi_j \in G_\nu$ and $w_j \nmid \ell$, the cyclotomic Frobenius formula gives $\ell^\nu \mid q_{w_j} - 1$. The family is therefore a Taylor–Wiles family and satisfies (PT1).

The detecting set s gives the detection clause of (PT2) for $a = \gamma^{-1}$, and the index (e, g) supplies its asserted equality. If $\ell \le y^2$, then

$$\#J = (t-1)\#s, \qquad \#J + \#s + 1 = t\#s + 1 \le (t+2)y^{20} \le y^{21}.$$

Multiplying by yh proves its cost bound. If $h = 0$, the same cost bound is automatic, without a bound for $\#s$. The initial product estimate proves (PT3), since $\chi(a^{-1}) = u$. The second estimate, and the fact that $u^{-e} - 1$ differs from $u^e - 1$ by a unit, prove (PT5).

Finally fix j and an even cocycle z as in (PT4). Vanishing at the earlier chosen tame generators makes z unramified at their places and their conjugates. Indeed inertia acts trivially on the Tate twist away from ℓ, its wild pro-residue-characteristic subgroup has no nonzero image in an ℓ-primary group, and its pro-ℓ tame quotient is generated topologically by any chosen tame generator; evenness gives the assertion at the conjugate place. Thus z is an even class supported at the places with indices $i \ge j$. Apply item 2 of Proposition 7.6.16 to ψ_j, with its initial family the earlier places of exponent $\mathrm{ty}(j)$ and its final family the places $i \ge j$. The prescribed Frobenius zeros are exactly the hypothesis of (PT4); all final primes avoid T_j. Every term of its reciprocity sum except the jth term is zero, because $\psi_j(\varphi_i) = 0$ for $i > j$. Hence

$$2\psi_j(\varphi_j) z(\widetilde{\tau}_j) = 0$$

for a tame generator normalised over F. The value of z at the fixed chosen tame generator τ_j is an integer multiple of its value at $\widetilde{\tau}_j$: both maps factor through the same finite cyclic pro-ℓ tame quotient. The divisibility $2\psi_j(\varphi_j) \mid \ell^{yh}$ therefore gives $\ell^{yh} z(\tau_j) = 0$. This is (PT4) and completes the proof. □

Proposition 7.6.16. *Let ℓ be a prime, $\nu \geq 0$, E a CM field containing μ_{5040}, x a finite family of ν-split Frobenius data and $e \geq 0$. Write $G_\nu = G_E \cap G_{F,\nu}$.*

1. *Every Kummer map ψ of exponent e off a finite set T is locally constant, and on G_ν it is a homomorphism with $\psi(h^c) = -\psi(h)$ and $\psi(ghg^{-1}) = \chi_{\mathrm{cyc}}(g)^{e+1}\psi(h)$ for $g \in G_E$, $h \in G_\nu$.*
2. *(Reciprocity.) Let T contain the primes of F above ℓ or below x, let $(w_j, \varphi_j, \tau_j)_j$ be a finite family of ν-split Frobenius data above distinct primes of F outside T, with tame generators τ_j normalised over F, and let ψ be a Kummer map of exponent e off T. Then for every $z \in \mathrm{Ev}_{\{w_j\}}(e, \nu)$ that vanishes at the Frobenius elements of x,*

$$\sum_j \psi(\varphi_j) \cdot 2z(\tau_j) = 0.$$

3. *If $\nu > 0$, E has at least four infinite places and the x_i lie above distinct primes of F, there is a finite set T of primes of F containing those above ℓ or below the x_i such that the reductions modulo ℓ of the Kummer maps of exponent e off T, as functions $G_\nu \to \mathbb{F}_\ell$, span a space of dimension greater than the number of the x_i.*

Proof. First note that E/F is unramified at every finite place. If its inertia at a place of residue characteristic p were nontrivial, it would act on E as complex conjugation. This is impossible on the prime-to-p roots of unity in E: conjugation acts nontrivially on μ_3 when $p \neq 3$ and on μ_4 when $p = 3$, whereas inertia fixes them. It follows that κ is unramified outside ℓ, since χ is. At every real place κ is trivial. Also

$$\kappa(G_F) = \chi(G_E).$$

For $\ell = 2$ this image lies in $1 + 16\mathbb{Z}_2$, since E contains μ_{16}. For odd ℓ every closed subgroup of $\mathbb{Z}_\ell^\times$ is procyclic; for $\ell = 2$ the same is true for closed subgroups of $1 + 4\mathbb{Z}_2$. The image of κ on a decomposition group above ℓ is infinite: its restriction to the index-at-most-two subgroup belonging to E is a local cyclotomic character and has open image in $\mathbb{Z}_\ell^\times$.

For item 1, let y be the defining relation. Its restriction to $G_{F,\nu} \cap G_E = G_\nu$ is a homomorphism. Evaluate its character at $1 \in M_e$ and identify $(1/\ell^\nu)\mathbb{Z}/\mathbb{Z}$ with R_ν. This gives the homomorphism assertion for ψ and its local constancy. Equivariance of the relation is

$$y(gvg^{-1})(a) = \chi(g)\, y(v)(\kappa(g)^e a) \qquad (v \in G_\nu,\ g \in G_F).$$

At $g = c$ the scalar is -1, and at $g \in G_E$ it is $\chi(g)^{e+1}$. Evaluation at $a = 1$ proves both asserted transformation laws. When $\nu = 0$ all target groups are zero.

For item 2, it is enough to test the asserted equality after every additive map $f : \mathcal{O}/\ell^\nu \to R_\nu$. These maps are R_ν-linear and separate points: a finite abelian group of exponent dividing ℓ^ν is separated by its characters with values in R_ν. Replace z temporarily by $f \circ z$. Its evenness permits an extension

$$z^+(g) = z(g), \qquad z^+(gc) = z(g) \quad (g \in G_E).$$

The cocycle rule on $G_F = G_E \rtimes \langle c \rangle$ follows directly from the rule on G_E, the identity $z(g^c) = z(g)$, and $\kappa(c) = 1$. We use $2z^+$, rather than assume that z^+ itself vanishes at every complex

conjugation. If c' is any complex conjugation, then $(c')^2 = 1$ and $\kappa(c') = 1$, so the cocycle identity gives $2z^+(c') = 0$.

This extension satisfies all the finite local conditions required in the defining reciprocity relation. Away from ℓ inertia in G_F lies in G_E by the unramified assertion above, so it has the stated unramifiedness. At a decomposition group above ℓ, if $\kappa(g) = 1$ then $g^2 \in G_E$, $\chi(g^2) = 1$, and g^2 belongs to the corresponding decomposition group over E. Thus

$$2z^+(g) = z^+(g^2) = 0$$

by the defining local condition of an even class. At a prime below an initial x_i, the class is unramified, since this prime is outside the final family and outside ℓ. Its value at the specified Frobenius is zero. Inertia and that Frobenius topologically generate the decomposition group, so the cocycle vanishes on the whole group; the same holds for $2z^+$. Therefore the defining relation may be applied to $2z^+$ and to the primes below the final w_j.

Its Frobenius convention is arithmetic, whereas φ_j is geometric. Item 1 gives $\psi(\varphi_j^{-1}) = -\psi(\varphi_j)$, because $\varphi_j \in G_\nu$. The relation, with this common minus sign removed, is exactly

$$\sum_j \psi(\varphi_j)\, 2f(z(\tau_j)) = 0.$$

Separation by the maps f proves the asserted equality for $\mathcal{O}/\ell^\nu$.

We prove item 3 by a finite-level duality count. Let $d = [F : \mathbb{Q}]$, let k be the number of the x_i, and let S_f be the primes of F above ℓ or below the x_i. A prime-norm place of E splits over F: the extension is unramified and an inert place would have residue degree two. Thus these latter primes are distinct, are not above ℓ, and their decomposition groups act trivially on κ mod ℓ^ν.

For every $0 \le m \le \nu$, set

$$A_m = R_m(\kappa^{-e}), \qquad A_m^\vee = \mathrm{Hom}(A_m, \mu_{\ell^m}).$$

After compatible choices of bases, the latter is R_m with character $\chi\kappa^e$. At $m = 0$ all groups below have order one. Let $\mathcal{Z}_m$ be the literal group of cocycles in A_m which are unramified outside S_f, vanish at every complex conjugation, are zero on the whole decomposition groups below x, and vanish on the kernel of κ in decomposition groups above ℓ. It is essential here that these are cocycles, before quotienting by coboundaries.

Let $L_{v,m} \subseteq H^1(F_v, A_m)$ be the cohomological local conditions just specified. At a real place it is zero. Let $\mathcal{Y}_m$ be the dual Selmer group in $H^1(G_{F,S_f\cup S_\infty}, A_m^\vee)$: at each finite $v \in S_f$ impose $L_{v,m}^\perp$ for local Tate duality, and at every real place impose the whole group $H^1(F_v, A_m^\vee)$. All these groups are finite by finite-module Galois duality. We shall prove the exact identity

$$|\mathcal{Y}_m| = |\mathcal{Z}_m|\,|H^0(F, A_m^\vee)|\ell^{m(d+k-1)}. \tag{1}$$

Here are the real-place terms in this identity. Use Theorem A.3.3, with Tate cohomology at the real places, together with the global Euler characteristic of Theorem A.3.5 and the finite local Euler characteristic of Theorem A.3.1. For clarity the order calculation is given explicitly. Write $H^i = H^i(G_{F,S_f\cup S_\infty}, A_m)$, $P^i = \prod_{v\in S_f\cup S_\infty} H^i(F_v, A_m)$ for $i = 1, 2$, and let $\Lambda = \prod_v L_{v,m}$. Let $\mathcal{S}_m$ be the cohomological primal Selmer group. The annihilator identity for the two global localisation images gives

$$\frac{|\mathcal{S}_m|}{|\mathcal{Y}_m|} = \frac{|H^1|\,|\Lambda|}{|\mathrm{Sha}^1(F, A_m^\vee)|\,|P^1|}.$$

Indeed if B is the image of primal localisation, its dual image is $B^\perp$, and $|B^\perp \cap \Lambda^\perp| = |P^1|\,|B \cap \Lambda|/(|B|\,|\Lambda|)$. The perfect pairing between $\mathrm{Sha}^1(F, A_m^\vee)$ and $\mathrm{Sha}^2(F, A_m)$, and the end of the Poitou–Tate sequence, give

$$|\mathrm{Sha}^1(F, A_m^\vee)| = \frac{|H^2|\,|H^0(F, A_m^\vee)|}{|P^2|}.$$

These are the all-place assertions, not assertions with the real summands omitted.

The module A_m is fixed by every real conjugation and F is totally real. Its truncated global Euler characteristic is therefore one: $|H^0|\,|H^2| = |H^1|$. At a real place

$$H^1(F_v, A_m) = A_m[2], \qquad H^2(F_v, A_m) = A_m/2A_m,$$

which have equal order, also when $\ell = 2$. At the finite places local Euler characteristic gives

$$\frac{|P^1|}{|P^2|} = |A_m|^d \prod_{v \in S_f} |H^0(F_v, A_m)|.$$

Consequently

$$\frac{|\mathcal{S}_m|}{|\mathcal{Y}_m|} = \frac{|H^0(F, A_m)|}{|H^0(F, A_m^\vee)|} |A_m|^{-d} \prod_{v \in S_f} \frac{|L_{v,m}|}{|H^0(F_v, A_m)|}. \tag{2}$$

This calculation displays the cancellation of the real Tate groups; none is presumed to vanish at $\ell = 2$.

At a place above ℓ, allowed local cocycles are precisely the inflations from the infinite procyclic group $\kappa(G_{F_v})$. Such a cocycle is determined by its value at a topological generator, and every value in A_m occurs. To verify the latter assertion, write its action as multiplication by b. If $b-1$ is a unit, any desired value is the coboundary of its quotient by $b-1$. Otherwise the finite subgroup of order prime to ℓ acts trivially, and on the pro-ℓ cyclic factor the formula on integer powers is $(1 + b + \cdots + b^{n-1})a$; it extends continuously to that factor for every $a \in A_m$. The assertion follows, and these cocycles lift through every reduction $A_m \to A_{m-1}$. Their group has order $|A_m|$. Since the local coboundary group has order $|A_m|/|H^0(F_v, A_m)|$, we obtain

$$|L_{v,m}| = |H^0(F_v, A_m)| \quad (v \mid \ell).$$

At each prime below x, $L_{v,m} = 0$ and A_m is fixed, so the factor in (2) is $|A_m|^{-1}$.

All global coboundaries belong to $\mathcal{Z}_m$. They vanish at real conjugations and factor through κ above ℓ, and at a prime below x the action on A_m is trivial, so their restriction is zero. They number $|A_m|/|H^0(F, A_m)|$. Substitute $|\mathcal{S}_m| = |\mathcal{Z}_m|\,|H^0(F, A_m)|/|A_m|$ and the just-computed local factors into (2). This proves (1).

We next compare consecutive levels without assuming that arbitrary dual classes lift. Reduction of the primal local condition from m to $m-1$ is surjective on cohomology: above ℓ the preceding one-generator description lifts every allowed cocycle; below x and at the real places the condition is zero. Compatibility of the local cup products therefore implies the following precise assertion. If a class $b \in H^1(G_{F,S_f \cup S_\infty}, A_{m-1}^\vee)$ has image in $\mathcal{Y}_m$ under the injection $A_{m-1}^\vee \hookrightarrow A_m^\vee$, then $b \in \mathcal{Y}_{m-1}$. Indeed every element of $L_{v,m-1}$ lifts to $L_{v,m}$, so its pairing with b is the pairing of that lift with the image of b, and is zero.

Apply the long exact sequence for

$$0 \longrightarrow A_{m-1}^\vee \longrightarrow A_m^\vee \longrightarrow A_1^\vee \longrightarrow 0.$$

Its exactness and the preceding assertion give

$$|\ker(\mathcal{Y}_m \longrightarrow H^1(F, A_1^\vee))| \le |\mathcal{Y}_{m-1}|.$$

No injectivity on H^1 of the first coefficient injection is needed. The coefficient injection $A_{m-1} \hookrightarrow A_m$ does inject the literal cocycle groups $\mathcal{Z}_{m-1} \hookrightarrow \mathcal{Z}_m$, and $H^0(F, A_{m-1}^\vee)$ also injects into $H^0(F, A_m^\vee)$. Dividing the two identities (1) consequently shows that the image of $\mathcal{Y}_m$ in $H^1(F, A_1^\vee)$ has at least

$$\ell^{\,d+k-1}$$

elements. It is an $\mathbb{F}_\ell$-subspace, hence has dimension at least $d + k - 1$.

Take $m = \nu$ and put $U = G_E \cap \ker(\chi \bmod \ell^\nu)$. This subgroup fixes $A_1^\vee$. The quotient G_F/U is generated by two elements: the map

$$G_F/U \longrightarrow \{1, -1\} \times \mathrm{im}(\kappa \bmod \ell^\nu)$$

given by (η, κ) is an isomorphism. Its kernel is U, and it is surjective because c gives $(-1, 1)$ and a generator of the cyclic second factor has a lift in G_E. The second factor is cyclic, including for $\ell = 2$ by the containment in $1 + 16\mathbb{Z}_2$. A cocycle on a two-generated group with values in the one-dimensional space $A_1^\vee$ is determined by two values, so

$$\dim_{\mathbb{F}_\ell} H^1(G_F/U, A_1^\vee) \le 2.$$

Inflation–restriction thus shows that the restrictions to U of the primitive reductions from $\mathcal{Y}_\nu$ have dimension at least $d + k - 3 \ge k + 1$.

It remains to check that these restricted classes really give the relations in the statement. Choose a cocycle representative of a class in $\mathcal{Y}_\nu$. On U the dual action is trivial, so its restriction is a homomorphism, independent of the representative. View it as

$$y : U \longrightarrow \mathrm{Hom}(A_\nu, \mathbb{Q}/\mathbb{Z})$$

and extend it by zero off U. It is locally constant, is unramified outside S_f, and has the equivariance with the ordinary cyclotomic prefactor: this follows either from the cocycle identity or from the dual action $\chi\kappa^e$. To verify reciprocity, take any further finite family of split primes outside S_f and any primal cocycle satisfying the required conditions and unramified outside this family and S_f. Apply the annihilator assertion of the same all-prime global duality, equivalently global cup-product reciprocity as in Theorem A.2.8, to their two global cohomology classes. The contributions at S_f vanish by the dual local conditions, and those at real places vanish because the primal class is zero there. Outside the new family both local classes are unramified, so their pairing is zero by unramified orthogonality. At a new split prime the dual class is unramified; the unramified–tame local pairing is

$$y(\varphi_{\mathrm{arith}})(z(\tau)).$$

Here τ multiplies an ℓ^ν-th root of a uniformiser by the chosen root of unity. This normalisation follows by pairing an unramified character taking the arithmetic Frobenius to $1/\ell^\nu$ with the Kummer class of that uniformiser; bilinearity gives the displayed formula. Reversing the order of the cup product changes every term by the same sign, which does not affect the zero sum. The sum of these terms is therefore zero. This proves all the requirements of a relation off $T = S_f$.

Identify its restriction to G_E with the corresponding Kummer map ψ. Reducing this map modulo ℓ and restricting it to $U = G_\nu$ is precisely the restriction in the preceding dimension estimate. Hence the reductions of the Kummer maps off this finite T span a space of dimension at least $k + 1 > k$. Since T contains the primes above ℓ and below x, this proves item 3 and the proposition. □

7.7 Local loss at the places above ℓ

Ribet's method produces an extension class from a congruence [241, 197, 327, 284, 20, 19]. At the places above ℓ, however, the class is only known to be crystalline up to a loss: the lattice that carries the congruence is crystalline, but the cocycle cut out of it need not satisfy the Bloch–Kato condition [28] on the nose. This section bounds the loss (Corollary 7.7.4): the Ribet cocycles of a lift setting lose at most $r'\,\mathrm{loc}_\ell$, the local term of Ribet's element. The organisation follows [259].

The loss is measured by dual defects of crystalline classes of lattices (Definition 7.2.1), with the crystalline cocycles defined through Colmez's ring $B_{\max}$ [85, 87, 61, 44]. Local Tate duality [295, 201, 225] turns a bound for the dual defect into a bound for the Ribet cocycles (Lemma 7.7.6), and the defect is bounded by a cup exponent: an explicit power of ℓ that makes a cup product of crystalline cocycles a coboundary (Proposition 7.7.13). The idea of the section is to produce cup exponents from Breuil–Kisin modules, in the form of Kisin and of Bhatt, Morrow and Scholze [160, 25]. After its definitions the section has two groups: the first assembles the bound from one place to all places above ℓ and all summands (Proposition 7.7.7, Proposition 7.7.11), and the second proves the bound at one place (Theorem 7.7.12) with the cup exponents behind it.

Definition 7.7.1. Let ℓ be a prime and Γ, j as above.

Kummer charts. A **Kummer chart** for Γ consists of an integer $f \geq 1$, with $\Lambda = W(\mathbb{F}_{\ell^f})$ (where $\mathbb{F}_{\ell^f} \subset \mathcal{O}^\flat_{\mathbb{C}_\ell}$ is the set of solutions of $x^{\ell^f} = x$), an element $\varpi^\flat \in \mathcal{O}^\flat_{\mathbb{C}_\ell}$ with $z = [\varpi^\flat]$ not a unit of A_{inf}, an integer $e \geq 1$, and a Λ-algebra map $\Lambda[[X]] \to A_{\mathrm{inf}}$, $X \mapsto z$, sending $X^e A + \ell$ to ξ times a unit, for a unit power series A; the image of $X^e A + \ell$ is written E. A Kummer chart also carries a subgroup $\Gamma' \subseteq \Gamma$ of index prime to ℓ, containing every g near 1, with $j(\Gamma')$ fixing Λ and $\theta(z)$ and containing every element of G_ℓ that fixes them, and such that for every $l \geq 0$ some $g \in \Gamma'$ has $j(g)z = [\epsilon]^l z$ (ϵ a compatible system of ℓ-power roots of unity in $\mathcal{O}^\flat_{\mathbb{C}_\ell}$). It is a chart **for** Γ if $j(\Gamma)$ fixes Λ pointwise and $\theta(z)$, $ef = [G_\ell : j(\Gamma)]$, the chart is unramified when $j(\Gamma)$ contains the inertia group, and $j(\Gamma)$ contains an element acting as $x \mapsto x^{\ell^f}$ on the residue field of $\overline{\mathbb{Z}}_\ell$. The numbers e and f are the **ramification** and the **residue degree**; the chart is **unramified** if $e = 1$ and $E = z + \ell$; its **different bound** is $d = 0$ if $e = 1$ and $d = v_\ell(e) + 1$ otherwise.

Letters. The **letters** of a chart are the elements of $\mathbb{F}_{\ell^f}^\times$; the **letter Frobenius** is $i \mapsto i^\ell$. A **letter root** is $\zeta \in \mathcal{O}$ with $\zeta^{\ell^f - 1} = 1$; an embedding $\gamma\colon \mathcal{O} \to \mathbb{C}_\ell$ **lies over** the letter i if $\gamma(\zeta) = \theta([i])$. For jump sets U (finite sets $U(\gamma) \subset \mathbb{Z}$, one for each γ), the **letter jump set** is $U_\zeta(i) = \{0\} \cup \bigcup_{\gamma \text{ over } i} U(\gamma)$.

Windows and gaps. For $\eta, b, W \in \mathbb{N}$ and a sign $\varepsilon \in \{\pm 1\}$, a finite set $V \subset \mathbb{Z}$ **fits the windows** $(\eta, \varepsilon, b, W)$ if every $u \in V$ satisfies $|u| \leq \eta$ or $b \leq \varepsilon u \leq b + W$. For a prime q, $n \in \mathbb{N}$ and $b \in \mathbb{Z}$, the **gap condition** $\mathrm{Ng}(q, n, b)$ says that q divides no nonzero integer y with $|y| \leq n$, $|y - b| \leq n$ or $|y + b| \leq n$.

Tame types. For $r, N \in \mathbb{N}$, a set $\mathcal{K}$ of polynomials over $\mathcal{O}$ is of **tame type** (ℓ, r, N) if $X^{e'} - 1 \in \mathcal{K}$ with $e' = \mathrm{lcm}\{\ell^i - 1 : 1 \leq i \leq r\}$, and $\mathcal{K}$ contains the image of a monic $q \in \mathbb{Z}_\ell[X]$ of degree at most N.

Size conditions. Natural numbers $(Q, m, h, G, b, \delta, \eta, W)$ satisfy the **crystalline size conditions** if

(KZ1) $Q > 0$ and $99\,(Q^2 + (W+1)Q + Q + W + m) \leq h$;

(KZ2) $99\,h^3 \leq G$ and $99\,(Gh^3 + G) \leq b$;

(KZ3) $2\log(4b) < m$;

(KZ4) $\delta < \eta$ and $99\,((\eta + W)Q + \eta + h) \le h^3$.

For a lattice L, $h, \mu, \eta \in \mathbb{N}$ and jump sets U, U', L has **few low jumps** (h, μ, η, U, U') if at every γ each negative $u \in U(\gamma) \cup U'(\gamma)$ satisfies $2u \le -h$, and $\sum_{|u| \le \eta} \mathrm{jump}(L, \gamma, u) \le \mu$.

Letters reading thickness. Let k be a Kummer chart for Γ, an index set I with values $\mathrm{lt}(g) \in \mathbb{C}_\ell$ $(g \in I)$, a predicate th (*thick*) and signs $\varepsilon(g)$ on I, $Q, h \in \mathbb{N}$, and functions T (a predicate) and s (a sign) on $\mathbb{C}_\ell$. The letters **read thickness** if: $ef = [G_\ell : j(\Gamma)]$ and $e \le h$; for every g with $\mathrm{th}(g)$ or $Q < \ell$, $\mathrm{th}(g) \Leftrightarrow T(\mathrm{lt}(g))$ and $\varepsilon(g) = s(\mathrm{lt}(g))$; either k is unramified and $\ell \ge 7$, or $\ell \le Q$ and s is constant; and, if $\ell \le Q$, there are no $z \in \mathbb{C}_\ell$ and $a, a' \in [1, \lfloor \log_\ell h^3 \rfloor]$ with $T(z)$, $T(z^{\ell^a})$ and $T(z^{\ell^{a+a'}})$.

Pairing sides. Over a Kummer chart with letter root $\zeta \in \mathcal{O}$, a **pairing side** S consists of a lattice Y that is crystalline over Γ' (admissible along the filter of subsets of Γ containing Γ'), finite jump sets $S(i) \ni 0$ at the letters i containing the jumps of Y at the embeddings over i, a Galois lattice C (the **parameter lattice**) and an $\mathcal{O}$-linear map $\alpha\colon C \to Z_f(Y)$. For a pairing P of the lattices of two sides S, S', $e \in \mathbb{N}$, $c \in C$ and $c' \in C'$, P is a **scaled cup coboundary** at (e, c, c') if the cup product of $\alpha(c)$ and $\ell^e \alpha'(c')$ under P is a cup coboundary. The **Tate dual side** of P has lattice $(Y \otimes Y')^\vee(1)$, parameter lattice $\mathcal{O}$ with $1 \mapsto c_P$, and jump sets $\{0\} \cup \{-u - u' - 1 : u \in S(i), u' \in S'(i)\}$.

Definition 7.7.2. Fix a prime ℓ, Γ and j, and a Kummer chart for Γ with data $f, \Lambda, z, e, E, \Gamma'$ as in Definition 7.7.1; G_ℓ acts on B_{dR}, and $g \in \Gamma'$ acts through $j(g)$. Let $C^\flat$ be the fraction field of $\mathcal{O}^\flat_{\mathbb{C}_\ell}$ and $W(C^\flat)$ its ring of Witt vectors, with Frobenius φ and the action of G_ℓ; $A_{\mathrm{inf}} \subseteq W(C^\flat)$. Put $\tau = [\varepsilon^{1/\ell}] - 1 \in A_{\mathrm{inf}}$, and for $n \in \mathbb{Z}$ write $\mathrm{Fil}^n = t^n B^+_{\mathrm{dR}}$.

(BF1) Lattices with letter jump sets. A **lattice with letter jump sets** over a coefficient ring $\mathcal{O}$ with a letter root $\zeta \in \mathcal{O}$ is a Galois lattice L for Γ over $\mathcal{O}$ (via j) that is crystalline over Γ' with jumps at each embedding $\gamma\colon \mathcal{O} \to \mathbb{C}_\ell$ in a finite set $U(\gamma) \subset \mathbb{Z}$ (that is, of class Cr for the filter of subsets of Γ containing Γ' and the jump sets U), together with finite sets $U(i) \subset \mathbb{Z}$, one for each letter i, such that $U(\gamma) \subseteq U(i)$ for every γ over i. The **multiplicity** of u at i is $\sum_{\gamma \text{ over } i} \mathrm{jump}(L, \gamma, u)$.

(BF2) Ambient module, letter parts. The **ambient module** of L is $W(C^\flat) \otimes_{\mathbb{Z}_\ell} L$, with φ acting as $\varphi \otimes 1$ and $g \in \Gamma$ as $j(g) \otimes g$; these actions are semilinear over A_{inf}. The element ζ acts through L. For a letter i, the **letter-i part** is the A_{inf}-submodule $\{x : \zeta x = [i]x\}$, and the **letter projector** is

$$\mathrm{pr}_i = \frac{1}{\ell^f - 1} \sum_{n=0}^{\ell^f - 2} \left([i]^{-1}\zeta\right)^n,$$

an idempotent onto the letter-i part; the pr_i sum to the identity.

(BF3) Forms. A **form** of L is an A_{inf}-linear map $\lambda\colon W(C^\flat) \otimes L \to B_{\mathrm{dR}}$ with $\lambda(1 \otimes gy) = j(g)\,\lambda(1 \otimes y)$ for all $g \in \Gamma'$ and $y \in L$. It is **crystalline** if there is a $\mathbb{Z}_\ell$-linear map $m\colon L \to B_{\mathrm{max}}$ with $\lambda(1 \otimes y) = m(y)$ in B_{dR} for all y. It is **of type** $u \in \mathbb{Z}$ **at the letter** i if moreover $\lambda(\mathrm{pr}_i(1 \otimes y)) \in \mathrm{Fil}^{-u}$ for all $y \in L$.

(BF4) Chart rings and value fields. The **chart ring** $\mathcal{S}_{\mathrm{ch}} \subseteq A_{\mathrm{inf}}$ is the image of the chart map $\Lambda[[X]] \to A_{\mathrm{inf}}$, $X \mapsto z$. The ring $\mathcal{S}_B$ is the set of $y \in B^+_{\mathrm{max}}$ such that multiplication by y commutes with every $\mathcal{S}_{\mathrm{ch}}$-linear endomorphism of B^+_{max} that maps A_{max} into A_{max}.

For $a \geq 0$ the **value field** $\mathcal{K}_a \subseteq B_{\mathrm{dR}}$ is the subfield generated over the fixed field in $\overline{\mathbb{Q}}_\ell$ of $j(\Gamma')$ by the products $b \cdot \varphi^{-a}(s)$ with $b \in \mathcal{S}_B$ and $s \in \mathcal{S}_{\mathrm{ch}}$ (images in B_{dR}).

(BF5) The modules. Let

$$M(L) = \{x : \tau^n x \in A_{\mathrm{inf}} \otimes L \text{ for some } n \geq 0, \\ \text{and } \lambda(\varphi^{n+1}x) \in B_{\mathrm{dR}}^+ \text{ for all } n \geq 0 \text{ and all forms } \lambda\},$$

an A_{inf}-submodule of the ambient module; for $l \geq 0$, $\varphi^l M(L) = \{x : \varphi^{-l}x \in M(L)\}$. The **Breuil–Kisin module** of L is the $\mathcal{S}_{\mathrm{ch}}$-module

$$N(L) = \{x \in M(L) : \ \lambda(x) \in \mathcal{K}_0 \text{ for every form } \lambda\},$$

and its **localisation** is

$$N_{\mathrm{loc}}(L) = \{x : Px \in M(L) \text{ for some finite product } P \text{ of elements } \varphi^a(E),\ a \geq 0, \\ \text{and } \lambda(x) \in \mathcal{K}_0 \text{ for every form } \lambda\}.$$

For a letter i, the **letter-i part** of an element x is $\mathrm{pr}_i(x)$.

Definition 7.7.3. Fix a Kummer chart for Γ. The constructions (BF2)–(BF5) of Definition 7.7.2 are used for any Galois lattice L with finite sets $U(i)$ of integers at the letters i. For a letter i and $a \geq 0$, $i^{(a)}$ is the letter whose ℓ^a-th power is i (so that φ^a maps the letter-$i^{(a)}$ part to the letter-i part).

(DQ1) Orders. An **exponent function** is a map $o\colon \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$. Let $M_{\mathrm{loc}}(L)$ be the A_{inf}-module of the x in the ambient module with $Px \in M(L)$ for some finite product P of elements $\varphi^a(E)$, $a \geq 0$. The module of elements **with orders o at the letter i** is

$$\mathfrak{X}_L(i;o) = \Big\{x \in M_{\mathrm{loc}}(L) \text{ in the letter-}i \text{ part} : \ \lambda(\varphi^{-a}x) \in \mathrm{Fil}^{o(a,u)} \\ \text{for all } a \geq 0,\ u \in U(i^{(a)}) \text{ and forms } \lambda \text{ of type } u \text{ at } i^{(a)}\Big\},$$

an A_{inf}-submodule.

(DQ2) Standard orders. For $l \geq 0$ the **standard orders of level l** are $\mathrm{std}_l(a,u) = 0$ for $a < l$ and $\mathrm{std}_l(a,u) = -u$ for $a \geq l$. The **standard part** of level l at i is $\mathfrak{N}_{l,i}(L) = N_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i;\mathrm{std}_l)$.

(DQ3) Extension of a side and constants. For a pairing side $S = (Y, (S(i)), C, \alpha)$, the **extension** $\tilde{Y}_S$ is the Galois lattice $Y \times C$ with $g(y,c) = (gy + \alpha(c)(g),\ c)$ and the sets $S(i)$ at the letters; the **trivial lattice** is C with the trivial action and the set $\{0\}$ at every letter; π_S is the extension to ambient modules of the projection $\tilde{Y}_S \to C$. For $c \in C$ the **constant** of c at i is $\mathrm{cst}_{S,i}(c) = \mathrm{pr}_i(1 \otimes c)$ in the ambient module of C.

(DQ4) Lift ideals and cokernel killers. For $l, r \geq 0$ the **lift ideal** $J_l(i;r) \subseteq A_{\mathrm{inf}}$ of S is the ideal generated by r and by the $s \in A_{\mathrm{inf}}$ with

$$s \cdot \mathrm{pr}_i(A_{\mathrm{inf}} \otimes C) \subseteq \pi_S\big(\mathfrak{X}_{\tilde{Y}_S}(i;\mathrm{std}_l)\big),$$

where $A_{\mathrm{inf}} \otimes C$ is the A_{inf}-span of the $1 \otimes c$. An element $s \in A_{\mathrm{inf}}$ **kills the cokernel** of S at i if $s \in J_0(i;0)$; for $\kappa, \rho \geq 0$ write $\mathrm{Kill}_S(i;\kappa,\rho)$ if $\ell^\kappa E^\rho$ kills the cokernel of S at i.

Corollary 7.7.4. *Let D, ℓ, ι, ϑ, (O,c), r', t, $n=r'+t$, R, J, J', S, Y, π, $(\rho_x)_{x\in Y}$, ϵ and the classes $\mathcal{Q}_v$ be as in Proposition 7.7.7. Then for all large coefficient rings $\mathcal{O}$: whenever $T,T_0\colon G_E\to M_{r'}(\mathcal{O})$ and $u=(u_x)_{x\in Y}$, $u_x\colon G_E\to M_n(\mathcal{O})$, satisfy the hypotheses there (u_x a lattice in ρ_x, u_π the block lattice of T placed by ϵ, T_0 a lattice in R agreeing with T on G_{A_v} for all $v\mid\ell$), the **loss** $\operatorname{loss}_T=\sum_{v\mid\ell}\operatorname{dd}(\mathcal{Q}_v,L_v(T)^\vee(1))$ satisfies*

1. $\operatorname{cc}(T;u,M)\le\operatorname{loss}_T$ *for every* $M\ge 0$;
2. $\operatorname{nl}(\operatorname{loss}_T)+\sum_{e<t}\operatorname{nl}\operatorname{cc}(\mathcal{O}(-e);u,\nu)\le r'\cdot\operatorname{loc}_\ell$ *for every* $\nu\ge 0$;

in particular $\sum_{i\in\{*,0,\dots,t-1\}}\operatorname{nl}\operatorname{cc}(\operatorname{blk}_T(i);u,M)\le r'\operatorname{loc}_\ell$ *for every* M. *Here* $\operatorname{nl}=v(\varpi)\cdot\operatorname{length}$ *for a uniformiser* ϖ *of* $\mathcal{O}$, $v(\ell)=1$.

Proof. Take $\mathcal{O}$ large for Proposition 7.7.7, and T,T_0,u as in the statement. By Proposition 7.7.7, for every $v\mid\ell$ the class $\mathcal{Q}_v$ contains $L_v(u_x)$ for all $x\in Y$ and $L_v(\operatorname{blk}_T(i))$ for all i, and

$$\sum_i\sum_{v\mid\ell}\operatorname{nl}\operatorname{dd}(\mathcal{Q}_v,L_v(\operatorname{blk}_T(i))^\vee(1))\le r'\operatorname{loc}_\ell. \tag{$*$}$$

For every i and every $M\ge 0$, Lemma 7.7.5 with $\rho=\operatorname{blk}_T(i)$ and the classes $\mathcal{Q}_v$ gives

$$\operatorname{cc}\big(\operatorname{blk}_T(i);u,M\big)\le\sum_{v\mid\ell}\operatorname{dd}\big(\mathcal{Q}_v,L_v(\operatorname{blk}_T(i))^\vee(1)\big).$$

For $i=*$ the right side is loss_T, which is item 1. For $i=e<t$ apply nl (monotone and additive) with $M=\nu$, add $\operatorname{nl}(\operatorname{loss}_T)$, and use $(*)$: this is item 2. The last claim follows from items 1 and 2 with $\nu=M$. □

Lemma 7.7.5. *Let E be a CM field as in Definition 2.3.4, ℓ a prime, $\mathcal{O}$ a coefficient ring, Y a finite set, $u_x\colon G_E\to M_n(\mathcal{O})$ ($x\in Y$) and $\rho\colon G_E\to M_d(\mathcal{O})$ homomorphisms. For every place $v\mid\ell$ of E let $\mathcal{Q}_v=\operatorname{Cr}(F_v,K_v,U_v)$ be a class of Galois lattices for G_{A_v} (via j_v) containing $L_v(u_x)$ for every x and containing $L_v(\rho)$. Then for every $M\ge 0$*

$$\operatorname{cc}(\rho;u,M)\le\sum_{v\mid\ell}\operatorname{dd}\big(\mathcal{Q}_v,L_v(\rho)^\vee(1)\big).$$

Proof. Write $\mathcal{Z}=\mathcal{Z}_u(\rho;\ell^M)$ and, for $v\mid\ell$, let $\mathcal{C}_v$ be the module of $c\colon G_E\to(\mathcal{O}/\ell^M)^d$ whose restriction to G_{A_v} obeys every relation obeyed by all of $Z_f(L_v(\rho))$.

The crystalline relations are the local ones. A relation θ on G_{A_v} extended by zero to $\tilde\theta$ on G_E satisfies $\sum_{g\in G_E}\tilde\theta(g)c(g)=\sum_{g\in G_{A_v}}\theta(g)c(g)$. Hence c obeys every element of $\mathcal{R}_{\mathrm{cris}}(\rho)$ if and only if $c\in\bigcap_{v\mid\ell}\mathcal{C}_v$, and

$$\operatorname{cc}(\rho;u,M)=\operatorname{length}_{\mathcal{O}}\Big(\mathcal{Z}/\mathcal{Z}\cap\textstyle\bigcap_v\mathcal{C}_v\Big).$$

Splitting over the places. The map $\mathcal{Z}/(\mathcal{Z}\cap\bigcap_v\mathcal{C}_v)\to\bigoplus_{v\mid\ell}\mathcal{Z}/(\mathcal{Z}\cap\mathcal{C}_v)$ induced by the quotient maps is injective (its kernel is $\bigcap_v(\mathcal{Z}\cap\mathcal{C}_v)$ modulo itself), and there are finitely many $v\mid\ell$. Length is monotone under injections and additive on finite direct sums (in $\mathbb{N}\cup\{\infty\}$), so

$$\operatorname{cc}(\rho;u,M)\le\sum_{v\mid\ell}\operatorname{length}_{\mathcal{O}}(\mathcal{Z}/(\mathcal{Z}\cap\mathcal{C}_v)).$$

Each place. For $v\mid\ell$ apply Lemma 7.7.6 with $\Gamma=G_{A_v}$, $j=j_v$, $G=G_E$, η the inclusion, $m=M$ and $\mathcal{Q}=\mathcal{Q}_v$: its hypotheses are that $\mathcal{Q}_v$ contains $L_v(u_x)$ and $L_v(\rho)$, and its module $\mathcal{C}$ is $\mathcal{C}_v$. It gives $\operatorname{length}_{\mathcal{O}}(\mathcal{Z}/(\mathcal{Z}\cap\mathcal{C}_v))\le\operatorname{dd}(\mathcal{Q}_v,L_v(\rho)^\vee(1))$. Summing gives the claim. □

Lemma 7.7.6. *Let ℓ be a prime, Γ a topological group with an open embedding $j\colon \Gamma \to G_\ell$, G a group, $\eta\colon \Gamma \to G$ a homomorphism, $\mathcal{O}$ a coefficient ring, Y a finite set, $u_x\colon G \to M_n(\mathcal{O})$ $(x \in Y)$ and $\rho\colon G \to M_d(\mathcal{O})$ homomorphisms, and $m \geq 0$. Let $\mathcal{Q} = \mathrm{InClass}(F, K, U)$ be a class of Galois lattices for Γ containing $\mathcal{O}^n$ with Γ acting by $u_x \circ \eta$ for every x, and containing $L = \mathcal{O}^d$ with Γ acting by $\rho \circ \eta$. Let $\mathcal{C}$ be the module of $c\colon G \to (\mathcal{O}/\ell^m)^d$ such that $c \circ \eta$ obeys every relation $\theta\colon \Gamma \to M_d(\mathcal{O})$ obeyed by all elements of $Z_f(L)$. Then*

$$\mathrm{length}_{\mathcal{O}}\Big(\mathcal{Z}_u(\rho;\ell^m)/\big(\mathcal{Z}_u(\rho;\ell^m)\cap\mathcal{C}\big)\Big) \leq \mathrm{dd}\big(\mathcal{Q}, L^\vee(1)\big).$$

Proof. Put $K_\ell = \overline{\mathbb{Q}}_\ell^{\,j(\Gamma)}$, so that Γ is identified topologically with G_{K_ℓ} for a finite extension $K_\ell/\mathbb{Q}_\ell$. All cohomology in the proof is continuous cohomology for this group. If $m = 0$, the quotient in the statement is zero. Assume $m > 0$ and put $a = \ell^m$ and $L = \mathcal{O}^d$ with its stated action.

By Lemma 7.4.6, the actions and crystalline cocycles of all potentially crystalline lattices here are continuous; their rational representations are de Rham, and the $B_{\max}$ and B_{cris} cocycle conditions agree. The lemma also supplies finite freeness and the rational crystalline-cocycle dimension formula used below.

We construct an actual lift for each Ribet cocycle. Let

$$A = \prod_{x\in Y} M_n(\mathcal{O}), \qquad D(g) = (u_x(g) - 1)_{x\in Y}, \qquad N = \mathrm{span}_{\mathcal{O}}\{D(g) : g \in G\}.$$

Let $w \in \mathcal{Z}_u(\rho; a)$. Its defining relations make the rule

$$f\Big(\sum_g b_g D(g)\Big) = \sum_g b_g w(g)$$

a well-defined $\mathcal{O}$-linear map $N \to L/aL$. The identity $u(h)D(g) = D(hg) - D(h)$ and the cocycle identity for w show that N is stable under left multiplication by $u(h)$ and $f(u(h)v) = \rho(h)f(v)$.

Choose k such that $N \cap a^k A \subseteq aN$. Such a k exists by elementary divisors: in a suitable basis of the finite free module A, write $N = \bigoplus_{i=1}^{s} \varpi^{b_i}\mathcal{O}e_i$, with $b_i \geq 0$ and ϖ a uniformiser; it suffices that $k\, v_\varpi(a) \geq v_\varpi(a) + \max_i b_i$. If $N = 0$, any $k \geq 1$ suffices. Therefore f extends to a map

$$t : Y_w = N + a^k A \longrightarrow L/aL, \qquad t(n + a^k v) = f(n).$$

It is well defined since $f(aN) = 0$, and is equivariant for the left action of G. The module Y_w is finite free and contains $a^k A$. Hence it is a full lattice in $A \otimes \mathbb{Q}_\ell$. Restricted along η, its rational representation is the direct sum of the representations $u_x \circ \eta$, each repeated n times. It lies in $\mathcal{Q}$: direct sums preserve (CY1), (CY2) and (CY3), and replacing a lattice by another full lattice changes neither the period representation, its jump multiplicities, nor its period operators.

The map $b(g) = D(\eta(g)) \in Y_w$ is a cocycle and $t(b(g)) = w(\eta(g))$. The vector $a^k 1 \in a^k A \subseteq Y_w$ satisfies

$$g(a^k 1) - a^k 1 = a^k b(g).$$

Since a is invertible in $B_{\max}$, this exhibits b itself as a coboundary after tensoring with $B_{\max}$. Thus $b \in Z_f(Y_w)$. We have proved that every restriction of a Ribet cocycle is an equivariant image of a crystalline cocycle of a lattice of $\mathcal{Q}$. This construction does not assert that it is the reduction of a cocycle of L.

We next give the finite duality comparison needed for such images. Choose an integral linear functional $f_0 : \mathcal{O} \to \mathbb{Z}_\ell$ for which $(x, y) \mapsto f_0(xy)$ is perfect over $\mathbb{Z}_\ell$. It exists because the inverse

different of the finite local extension $\mathcal{O}[1/\ell]/\mathbb{Q}_\ell$ is a free rank-one $\mathcal{O}$-module: if δ generates it, take $f_0(x) = \mathrm{Tr}(\delta x)$. Evaluation followed by f_0 identifies $L' = L^\vee(1)$ with the $\mathbb{Z}_\ell$-Tate dual of the underlying $\mathbb{Z}_\ell$-lattice of L. Put

$$V = L/aL, \quad V' = L'/aL', \quad A_f = \mathrm{red}\, Z_f(L), \quad A'_f = \mathrm{red}\, Z_f(L'),$$

where the last two modules are modules of continuous cocycles, before quotienting by coboundaries.

These reductions are exact annihilators under the local cup pairing on cocycles. Here are the details, including their sizes. Write $d_0 = \mathrm{rank}_{\mathbb{Z}_\ell} L$ and $n_0 = [K_\ell : \mathbb{Q}_\ell]$. By local cohomological finiteness, Theorem A.3.2 and Theorem A.3.1, $Z^1(K_\ell, L)$ is finitely generated: its quotient by coboundaries is $H^1(K_\ell, L)$, and the coboundaries are an image of L. The module $Z_f(L)$ is a submodule, hence finitely generated and free; it is saturated in the functions $\Gamma \to L$, since division by a nonzero scalar is possible for its period witness. Its reduction therefore has kernel $aZ_f(L)$, and the same assertions hold for L'.

A rational continuous cocycle has bounded image because Γ is compact, so a power of ℓ multiplies it into the integral lattice; its period witness scales by the same power. After rationalising, the quotient of $Z_f(L)$ by coboundaries is $H^1_f(K_\ell, L \otimes \mathbb{Q}_\ell)$, by the period comparison already proved. Bloch–Kato's dimension formula [28], Corollary 3.8.4, gives

$$\mathrm{rank}_{\mathbb{Z}_\ell} Z_f(L) = d_0 + \dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(L \otimes \mathbb{Q}_\ell)/\mathrm{Fil}^0.$$

The formula for L' is analogous. For the two Tate-dual de Rham representations the filtration jumps are paired as u and $-u-1$; exactly one of these is negative. Their two de Rham quotient dimensions consequently sum to $n_0 d_0$. Thus

$$|A_f|\,|A'_f| = \ell^{m(2+n_0)d_0} = |V|^{2+n_0}.$$

Rational Bloch–Kato orthogonality [28], Proposition 3.8, makes the cup product of the integral crystalline cocycles zero: its class in $H^2(K_\ell, \mathbb{Z}_\ell(1))$ maps to zero rationally, and this integral group is $\mathbb{Z}_\ell$, hence torsion free. Their reductions are therefore orthogonal.

For clarity, finite Tate duality gives the corresponding size of annihilators on cocycles as well as on classes. Write $B^1(V)$ for finite coboundaries. For any module B of cocycles of V' containing $B^1(V')$, perfect duality on H^1 gives

$$|\mathrm{Ann}(B)|\,|B| = |B^1(V)|\,|B^1(V')|\,|H^1(K_\ell, V)| = |V|^{2+n_0};$$

the last equality uses $|B^1(V)| = |V|/|H^0(K_\ell, V)|$, the analogous identity for V', and the local Euler characteristic formula. Both A_f, A'_f contain all finite coboundaries, by lifting their defining vectors integrally. Their orthogonality and the displayed size identity imply $A_f = \mathrm{Ann}(A'_f)$. This argument is valid at $\ell = 2$ as well. The local duality statements apply to the finite local field K_ℓ; equivalently one may realise K_ℓ as the completion of a number field by approximating a defining polynomial and applying Krasner's lemma.

Let $H = Z_f(L') \cap (L')^{\perp_{\mathcal{Q}}}$ and let $\overline{H}$ be its reduction in A'_f. If a continuous cocycle $s : \Gamma \to V$ has the form $t \circ b$, with $b \in Z_f(Y_0)$, $Y_0 \in \mathcal{Q}$ and $t : Y_0 \to V$ equivariant, then s annihilates $\overline{H}$. Indeed, lift t to an $\mathcal{O}$-linear map $q : Y_0 \to L$, possible since Y_0 is free. Its defect of equivariance is divisible by a. The evaluation form

$$\psi(y \otimes z) = z(q(y)) \quad \text{on } Y_0 \otimes L'$$

therefore satisfies $g\psi - \psi = ac_P(g)$ for an integral cocycle c_P. This cocycle is crystalline, with period witness ψ/a. Together with f_0 these are exactly the data of a pairing in the definition

of cup-orthogonality. Its cup product with b and any element of H is a coboundary by that definition, and is the ordinary evaluation cup product of s with the reduction of that element. This proves the assertion. The same applies to every element of A_f, using $Y_0 = L$ and the reduction map.

Let S be the module of restrictions along η of $\mathcal{Z}_u(\rho; a)$. Its elements have the preceding form by our construction, so $A_f + S \subseteq \mathrm{Ann}(\overline{H})$. Finite duality and $A_f = \mathrm{Ann}(A_f')$ give

$$|S/(S \cap A_f)| \leq |\mathrm{Ann}(\overline{H})/A_f| = |A_f'/\overline{H}| \leq |Z_f(L')/H|.$$

The last inequality follows from the surjective reduction map on these quotients; if the last quotient is infinite the asserted length bound is immediate. All finite modules here are $\mathcal{O}$-modules, so comparison of cardinalities is comparison of $\mathcal{O}$-lengths, dividing logarithms by the logarithm of the residue-field cardinality.

Finally let B be the module of functions $\Gamma \to V$ obeying every relation of $Z_f(L)$. Reductions of its elements obey those relations, so $A_f \subseteq B$. By definition $\mathcal{C}$ is the inverse image of B under restriction along η. The first isomorphism theorem and this inclusion yield

$$\begin{aligned} \mathrm{length}_{\mathcal{O}}(\mathcal{Z}_u(\rho; a)/(\mathcal{Z}_u(\rho; a) \cap \mathcal{C})) &= \mathrm{length}_{\mathcal{O}}(S/(S \cap B)) \\ &\leq \mathrm{length}_{\mathcal{O}}(S/(S \cap A_f)) \\ &\leq \mathrm{length}_{\mathcal{O}}(Z_f(L')/H) = \mathrm{dd}(\mathcal{Q}, L^{\vee}(1)). \end{aligned}$$

This is the required inequality. □

Proposition 7.7.7. *Let D be admissible data, ℓ a prime, $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, ϑ a twist, and (O, c) a lift setting for ϑ with source rank r' and rank difference t; put $n = r' + t$. Let R be an ℓ-adic representation of rank r', main-summand like with string length t and jumps J; let J' be a function from the embeddings $E \to \mathbb{C}$ to finite multisets of integers and S a set of finite places of E; let Y be a finite set, $\pi \in Y$, and ρ_x $(x \in Y)$ ℓ-adic representations of rank n of Ribet type $(O, c; t, J', S)$; let ϵ be a bijection placing the blocks. For $v \mid \ell$ define the class $\mathcal{Q}_v = \mathrm{Cr}(F_v, K_v, U_v)$ for G_{A_v} by: F_v the trivial filter $\perp$ if $v \mid p_0 p_1$ and $\top$ otherwise; $K_v(s, Q)$ if and only if $v \mid p_0 p_1$ and Q kills s at A_v on every representation of rank n of Ribet type $(O, c; t, J', S)$; $U_v(\gamma_g)$ the set of members of $J'(\sigma_{e_v|_E, g})$.*

Then for all large coefficient rings $\mathcal{O}$ the following holds. Let $T, T_0\colon G_E \to M_{r'}(\mathcal{O})$ and $u_x\colon G_E \to M_n(\mathcal{O})$ $(x \in Y)$ be homomorphisms such that u_x is a lattice in ρ_x, u_π is the block lattice of T with string t placed by ϵ, T_0 is a lattice in R, and $T = T_0$ on G_{A_v} for every $v \mid \ell$. Then for every $v \mid \ell$ the class $\mathcal{Q}_v$ contains $L_v(u_x)$ for every $x \in Y$ and $L_v(\mathrm{blk}_T(i))$ for every $i \in \{*, 0, \ldots, t-1\}$, and

$$\sum_{i \in \{*, 0, \ldots, t-1\}} \sum_{v \mid \ell} \mathrm{nl}\, \mathrm{dd}\Big(\mathcal{Q}_v,\, L_v\big(\mathrm{blk}_T(i)\big)^{\vee}(1)\Big) \leq r' \cdot \mathrm{loc}_\ell.$$

Proof. U_v is well defined. Every $\mathbb{Z}_\ell$-algebra map $\mathcal{O} \to \mathbb{C}_\ell$ is γ_g for some $g \in G_\ell$. By (RT4), ρ_π has the jumps J' at A_v, so for every lattice of ρ_π, $J'(\sigma_{e_v|_E, g})$ lists the jumps of its local lattice at γ_g; two g with the same γ_g therefore give the same multiset.

Large rings. Let $\mathcal{O}_0$ be a coefficient ring above which the conclusion of Proposition 7.7.11 holds for D, ℓ, ι, the charts $(e_v|_E, j_v)$ and the weight system J; J is a main-summand weight system of rank r' and length t by (MS3). Fix $\mathcal{O} \supseteq \mathcal{O}_0$ and T, T_0, u_x as in the statement, and $v \mid \ell$.

Step 1: the lattices $L_v(u_x)$ lie in $\mathcal{Q}_v$. By (RT4) at $v \mid \ell$, ρ_x is of finite type at A_v, so $L_v(u_x)$ is potentially crystalline, which is (CY1) when $F_v = \bot$ (the supremum of $\bot$ and the filter near 1 is the filter near 1); if $v \nmid p_0p_1$, ρ_x is good at A_v, so $L_v(u_x)$ is crystalline, which is (CY1) for $F_v = \top$. (CY2): the jumps of $L_v(u_x)$ at γ_g are listed by $J'(\sigma_{e_v|_E,g})$, whose members form $U_v(\gamma_g)$. (CY3): if $K_v(s,Q)$ holds then Q kills s on every representation of the Ribet type, in particular on ρ_x, and by (RT1) on its lattice u_x.

Step 2: the summands lie in $\mathcal{Q}_v$. Via ϵ, $L_v(u_\pi) = \bigoplus_i L_v(\mathrm{blk}_T(i))$ as Galois lattices for G_{A_v}, and the inclusions and projections are equivariant. (CY1): $B_{\max} \otimes L_v(u_\pi) = \bigoplus_i B_{\max} \otimes L_v(\mathrm{blk}_T(i))$ equivariantly, a period fixed along a filter has fixed components, and the projection of a spanning set spans each summand. (CY2): a γ-semilinear map on a summand extends by zero to the sum, so $\mathrm{jump}(L_v(\mathrm{blk}_T(i)),\gamma,u) \le \mathrm{jump}(L_v(u_\pi),\gamma,u)$, and a jump of a summand lies in $U_v(\gamma)$ by Step 1. (CY3): the $B_{\max}$-linear operator of s on $B_{\max} \otimes L_v(u_\pi)$ (determined by its values on the spanning periods) is the direct sum of those of the summands, so Q kills it only if it kills each. Jump multiplicities are additive over the direct sum for the same reason:

$$\mathrm{jump}(L_v(u_\pi),\gamma,u) = \sum_i \mathrm{jump}(L_v(\mathrm{blk}_T(i)),\gamma,u).$$

Step 3: the hypotheses (i)–(iv) of Theorem 7.7.12 at v, for $L_{\mathrm{main}} = L_v(T)$ and $L_e = L_v(\mathcal{O}(-e))$, $e < t$, with $(F,K,U) = (F_v,K_v,U_v)$ and $(e,j) = (e_v|_E, j_v)$. (i) If $v \mid p_0p_1$, then $t \le t_D$ by (MS5), and Lemma 7.7.8 gives $\ell \in \{p_0,p_1\}$ and that $K_v(s,\cdot)$ is of tame type $(\ell, r_D, (r_D+t_D)r_D)$ for every inertial s. If $v \nmid p_0p_1$, then $F_v = \top$ and K_v holds for no pair. (ii) If $u \in U_v(\gamma_g)$, then u is a member of $J'(\sigma_{e_v|_E,g})$, so $\mathrm{jump}(L_v(u_\pi),\gamma_g,u) \ne 0$, and by the additivity of Step 2 some summand has the jump u at γ_g. (iii) is Lemma 7.7.10, applied to R, T_0 and T. (iv) $\mathrm{rank}_{\mathcal{O}} L_v(T) = r'$; moreover $\mathrm{rank}_{\mathcal{O}} L_v(\mathcal{O}(-e)) = 1$.

Step 4: the sum. By Steps 2 and 3 the families $(L_v(\mathrm{blk}_T(i)))_i$, $v \mid \ell$, with the classes $\mathcal{Q}_v$ satisfy every hypothesis of Proposition 7.7.11 over $\mathcal{O} \supseteq \mathcal{O}_0$, whose conclusion is the displayed bound. Steps 1 and 2 are the membership claims. □

Lemma 7.7.8. *Let D be admissible data, ℓ a prime, $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, ϑ a twist, (O,c) a lift setting for ϑ with source rank r', $0 \le t \le t_D$, J' a function from the embeddings $E \to \mathbb{C}$ to finite multisets of integers, S a set of finite places of E and $n \ge 1$. Let v be a place of E above ℓ with $v \mid p_0p_1$. Then $\ell \in \{p_0,p_1\}$, and for every coefficient ring $\mathcal{O}$ and every $s \in G_{A_v}$ with $j_v(s)$ inertial, the property $K_v(s,\cdot)$ of $Q \in \mathcal{O}[X]$,*

$$K_v(s,Q) \iff Q \text{ kills } s \text{ at } A_v \text{ on every } R\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \text{ of Ribet type } (O,c;t,J',S),$$

is of tame type $(\ell,\, r_D,\, (r_D+t_D)\, r_D)$.

Proof. The residue characteristic of v is both ℓ and one of p_0,p_1; hence $\ell = p_i$ for some i. By Lemma 4.2.66, this prime splits completely in E. In particular its residue fields are $\mathbb{F}_\ell$, and exactly one place of each complex-conjugate pair above ℓ is chosen. Indeed the factors of (π_M) in the quadratic imaginary subfield choose one prime above each p_i, whereas $(\bar\pi_M)$ chooses the other.

We first construct a multiset $\mathcal{U}$, depending on the fixed local element and the lift setting but not on the representation, such that

$$\#\mathcal{U} = r' + t, \qquad \forall y \in \mathcal{U}\ \exists\, 1 \le d_y \le r_D : y^{\ell^{d_y}-1} = 1, \tag{1}$$

and its root polynomial kills that element on every representation in the assertion.

Suppose first that v is chosen. Write $P_v = \mathrm{tl}(c, v)$, the scaled label polynomial of Definition 7.5.3, (RT7). It is a monic label of degree r' over $\mathbb{F}_\ell$, with nonzero constant term; scaling its roots preserves these properties. Every root x therefore belongs to $\mathbb{F}_{\ell^{d_x}}^\times$ for some $1 \le d_x \le r'$. Here d_x is the degree of its irreducible factor, which is at most the degree of P_v. Include t further roots equal to 1, obtaining the roots with multiplicity of $P_v(X)(X-1)^t$.

Put $d_* = r'!$. All these roots lie in $\mathbb{F}_{\ell^{d_*}}^\times$. Fix the faithful residue-field character T_v used in the definition of the local type. There is an integer a_{tm} giving the tame exponent of $j_v(s)$ on this finite field: choose a uniformiser π_v, a root $z_v^{\ell^{d_*}-1} = \pi_v$, and write the character value of the residue of $j_v(s)(z_v)/z_v$ as $\exp(2\pi i a_{\mathrm{tm}}/(\ell^{d_*}-1))$. This is possible since the exponent is prime to the residue characteristic and inertia acts on these roots by the corresponding roots of unity. Changing a_{tm} by a multiple of $\ell^{d_*}-1$ has no effect on any root under consideration. Set

$$\mathcal{U} = \big[\iota(T_v(x)^{a_{\mathrm{tm}}}) : P_v(x)(x-1)^t = 0\big],$$

with algebraic multiplicities. The labelled condition (RT3), applied to the Ribet polynomial (RT7), says precisely that

$$\prod_{y \in \mathcal{U}} (X - y) \tag{2}$$

kills s at A_v. The condition holds for every representation of that type, so this is a common polynomial. Its degree is $r'+t$. The order of the image of a root of P_v divides $\ell^{d_x}-1$, because T_v is multiplicative; the added root 1 has this property with $d_x = 1$. This proves (1), since $r' \le r_D$.

For completeness, transport to the unchosen place uses the actual conjugate self-duality, rather than an identification of chosen and unchosen labels. If v is unchosen, apply the preceding construction at $\bar{v}$ to the inertial element $c\, j_v(s)^{-1} c^{-1}$, where c is complex conjugation. The resulting multiset still has (1). The conjugate self-duality (MS1) identifies the representation at the conjugate place with its dual, up to a power of the cyclotomic character. To justify that identification from the trace formulation, work over the algebraically closed characteristic-zero field $\overline{\mathbb{Q}}_\ell$. The Ribet type (RT7), through (RT6) and (RT4), supplies (MS1), including nondegeneracy of the trace pairing on the finite-dimensional matrix span. Item1 of Lemma 7.7.9 therefore makes R semisimple. Conjugation $g \mapsto g^c$ is an automorphism of G_E, so $g \mapsto R(g^c)$ is semisimple. The dual of a simple finite-dimensional representation is simple: the annihilator of a proper nonzero invariant subspace in the dual would be a proper nonzero invariant subspace in the original. Thus duality preserves semisimplicity, as does twisting by a nonzero scalar character. In particular

$$g \longmapsto \chi_{\mathrm{cyc}}(g)^{1-\max(t,1)} \big(R(g^{-1})\big)^{\mathsf{T}}$$

is semisimple. The trace identity of (MS1) is exactly equality of the characters of these last two representations. Item2 of Lemma 7.7.9 identifies them, giving the asserted conjugate duality with its actual cyclotomic twist.

All the representations under consideration are potentially crystalline at both places by (RT4), since these are level primes. Their inertia operators on potentially crystalline periods have finite image. Duality of crystalline periods carries an inertia operator to its inverse transpose. The cyclotomic twist contributes no inertia on these periods: its crystalline period is generated by the usual Tate period and its associated Weil–Deligne inertia is trivial. Consequently the operator of $c\, j_v(s)^{-1} c^{-1}$ at $\bar{v}$ is conjugate to the transpose of the operator of $j_v(s)$ at v. A polynomial kills one if and only if it kills the other. Thus (2) kills at v as well. This reasoning works on every local realization in (RT1); the scalar polynomial and its roots are independent of that realization.

Let

$$e_{\mathrm{tm}} = \operatorname*{lcm}_{1 \le d \le r_D} (\ell^d - 1).$$

By (1), $y^{e_{\mathrm{tm}}} = 1$ for every $y \in \mathcal{U}$. A root of unity of order dividing $\ell^{d_y} - 1$ lies in the unramified extension of $\mathbb{Q}_\ell$ of degree d_y: its reduction is in $\mathbb{F}_{\ell^{d_y}}^{\times}$, and Hensel's lemma lifts that reduction uniquely, since the derivative of $X^{\ell^{d_y}-1} - 1$ is a unit at a root. Hence its minimal polynomial m_y over $\mathbb{Q}_\ell$ belongs to $\mathbb{Z}_\ell[X]$, is monic, and has degree at most d_y. Set

$$Q_{\mathrm{tm}} = \prod_{y \in \mathcal{U}} m_y.$$

It is monic, with degree at most $(r' + t)r_D \le (r_D + t_D)r_D$, and vanishes on $\mathcal{U}$.

Finally, every inertia operator on potentially crystalline periods has finite order: over a finite extension where the representation is crystalline, the open inertia subgroup acts trivially on its period module. In characteristic zero a finite-order operator is semisimple, since $X^N - 1$ has no repeated roots. Because (2) kills it, its eigenvalues belong to the set underlying $\mathcal{U}$. Every polynomial vanishing on that set therefore kills it. Apply this to $X^{e_{\mathrm{tm}}} - 1$ and to the image of Q_{tm} in $\mathcal{O}[X]$. These are exactly the two requirements for tame type $(\ell, r_D, (r_D + t_D)r_D)$. No equality between n and $r' + t$ was used: the common annihilator is supplied by the prescribed type, regardless of the representation's rank. □

Lemma 7.7.9. *Let k be an algebraically closed field of characteristic zero and G a group. All representations below are finite dimensional over k.*

1. *If $\rho : G \to \mathrm{GL}(V)$ has nondegenerate trace pairing $(a, b) \mapsto \mathrm{tr}_V(ab)$ on $A = \mathrm{span}_k \rho(G)$, then V is a semisimple G-module.*
2. *If $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are semisimple and $\mathrm{tr}\,\rho(g) = \mathrm{tr}\,\sigma(g)$ for every $g \in G$, then they are isomorphic as G-representations.*

Proof. For every representation, its matrix span is a finite-dimensional algebra with identity: products of group matrices are group matrices. A subspace is stable under this algebra if and only if it is stable under the group.

The trace pairing kills the radical. Let J be the Jacobson radical of A. It is nilpotent by the classical radical theorem and Artin–Wedderburn structure theorem (Jacobson, Basic Algebra II, second edition, 1989). For $a \in J$ and $b \in A$, the element ab belongs to J, so some power of ab is zero. As an endomorphism of V it is nilpotent and has trace zero. Nondegeneracy therefore forces $a = 0$. Hence $J = 0$ and the same classical structure theorem writes A as a finite product of full matrix algebras over k.

Here is the module decomposition explicitly. Let e_i be the central identity of the i-th factor, and $e_{ab}^{(i)}$ its matrix units, with size d_i. Any A-module X decomposes as $\bigoplus_i e_i X$. In $e_i X$ put $X_i = e_{11}^{(i)} X$. Every vector has the unique expression

$$x = \sum_{a=1}^{d_i} e_{a1}^{(i)} x_a, \qquad x_a = e_{1a}^{(i)} x \in X_i.$$

Existence follows from $\sum_a e_{aa}^{(i)} = e_i$ and uniqueness follows by applying each $e_{1a}^{(i)}$. In a basis of X_i this identifies $e_i X$ with $\dim_k X_i$ copies of the standard simple module k^{d_i}. Applying this to V proves item1. For $V = 0$ the conclusion is immediate and no structure theorem for the zero algebra is needed.

Equal traces recover the multiplicities in characteristic zero. For item2, if $V = W = 0$ there is nothing to prove. Otherwise let

$$B = \mathrm{span}_k\{(\rho(g), \sigma(g)) : g \in G\} \subseteq \mathrm{End}_k(V) \times \mathrm{End}_k(W).$$

The B-module $V \oplus W$ is semisimple by hypothesis and the observation in the first paragraph. Its action is faithful by construction. Let J_B be its nilpotent radical. On a simple summand S the subspace $J_B S$ is a submodule. If it were nonzero it would be S, and iteration would give $J_B^a S = S$ for every a, contradicting nilpotence. Thus J_B kills every simple summand and hence kills the faithful module $V \oplus W$. It follows that $J_B = 0$.

Write $B = \prod_i \mathrm{M}_{d_i}(k)$ using the same classical theorem. The explicit matrix-unit decomposition above gives multiplicities m_i and n_i for V and W. The trace identity on group elements extends k-linearly to all of B, since those pairs span B. At the central identity e_i the two traces are $m_i d_i$ and $n_i d_i$. Their equality in a characteristic-zero field implies the equality of integers $m_i = n_i$, since $d_i > 0$. Choosing an isomorphism between the multiplicity spaces in every factor yields a B-module isomorphism $V \simeq W$ and therefore a G-module isomorphism. No assertion of this integer recovery in positive characteristic is used. □

Lemma 7.7.10. *Let D be admissible data, ℓ a prime, $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, R an ℓ-adic representation of rank r' that is main-summand like with string length t and jumps J, $\mathcal{O}$ a coefficient ring, $T_0\colon G_E \to M_{r'}(\mathcal{O})$ a lattice in R, v a place of E above ℓ, and $T\colon G_E \to M_{r'}(\mathcal{O})$ a homomorphism with $T(g) = T_0(g)$ for all $g \in G_{A_v}$. Then for every $g \in G_\ell$:*

1. *$J(\sigma_{e_v|_E, g})$ lists the jumps of $L_v(T) = L_v(\mathrm{blk}_T(*))$ at γ_g;*
2. *for $0 \le e < t$, $\{e\}$ lists the jumps of $L_v(\mathcal{O}(-e)) = L_v(\mathrm{blk}_T(e))$ at γ_g.*

Proof. Item 1. As T and T_0 agree on G_{A_v}, $L_v(T) = L_v(T_0)$ as Galois lattices for G_{A_v}. By (MS2), part of (MS3), R has the Hodge–Tate jumps J at A_v: along the chart (e_v, j_v), the jumps of the local lattice of a lattice in R at the coefficient embedding γ_g are listed by J at the complex embedding $\iota^{-1} \circ g^{-1} \circ e_v|_E = \sigma_{e_v|_E, g}$. Applied to T_0 this is item 1.

Item 2. Let $L = \mathcal{O}$ with $\Gamma = G_{A_v}$ acting by χ^{-e}, χ the cyclotomic character (through j_v). A γ_g-semilinear $\lambda\colon \mathcal{O} \to \mathbb{C}_\ell$ is $\lambda(x) = \gamma_g(x)\, y$ with $y = \lambda(1)$. As χ takes values in $\mathbb{Z}_\ell^\times$ and γ_g is $\mathbb{Z}_\ell$-linear, the condition $\lambda(\chi(h)^{-e} x) = \chi(h)^{-u}\, h\lambda(x)$ reads $\chi(h)^{-e}\gamma_g(x) y = \chi(h)^{-u} h(\gamma_g(x))\, h(y)$. For h near 1, h fixes the finite extension $\gamma_g(\mathcal{O})[1/\ell]$, so the condition becomes

$$h(y) = \chi(h)^{\,u-e}\, y \qquad \text{for all } h \text{ in an open subgroup } H.$$

Shrink H if necessary so that its image in G_ℓ is an open subgroup fixing the finite coefficient field $\gamma_g(\mathcal{O})[1/\ell]$. Every such subgroup is G_F for a finite extension $F/\mathbb{Q}_\ell$. If $u \neq e$, the displayed equation is

$$\chi(h)^{e-u} h(y) = y \qquad (h \in G_F).$$

Apply Theorem A.3.17(2) with $p = \ell$ and the nonzero integer $j = e - u$: it gives $y = 0$. Thus $\mathrm{jump}(L, \gamma_g, u) = 0$. If $u = e$, apply Theorem A.3.17(1) to G_F: its invariants in $\mathbb{C}_\ell$ are F, since a finite extension is complete. Taking the union over open subgroups contained in the chart and fixing the coefficient field gives $\overline{\mathbb{Q}}_\ell$. Indeed every algebraic element is fixed by such a subgroup after intersection with its open stabilizer. Therefore the solution space is $\overline{\mathbb{Q}}_\ell \cdot \gamma_g$, of dimension 1. Both clauses hold for every prime, including $\ell = 2$, and the exponent $j = e - u$ agrees with the jump convention χ^{-u} in the definition. So $\{e\}$ lists the jumps. □

Proposition 7.7.11. *Let D be admissible data, ℓ a prime and $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ an isomorphism. For every place v of E above ℓ fix an embedding $e_v\colon E \to \overline{\mathbb{Q}}_\ell$ inducing v and an open embedding $j_v\colon \Gamma_v \to G_\ell$ onto the subgroup of elements fixing $e_v(E)$ pointwise. Let J be a main-summand weight system of rank r' and length t', with $1 \le r' \le r$. Then for all large coefficient rings $\mathcal{O}$ the following holds. Suppose that for every $v \mid \ell$ there are Galois lattices $(L_{v,c})_{c \in \{\mathrm{main}\} \sqcup \{0,\dots,t'-1\}}$*

over $\mathcal{O}$ (with $\Gamma = \Gamma_v$) and (F_v, K_v, U_v) satisfying (i)–(iv) of Theorem 7.7.12 for $(e, j) = (e_v, j_v)$, that $\operatorname{rank}_{\mathcal{O}} L_{v,c} = 1$ *for $c < t'$, and that every $L_{v,c}$ lies in* $\mathrm{Cr}(F_v, K_v, U_v)$. *Then*

$$\sum_c \sum_{v \mid \ell} \operatorname{nl} \operatorname{dd}\big(\mathrm{Cr}(F_v, K_v, U_v), L_{v,c}^\vee(1)\big) \le r' \cdot \mathrm{loc}_\ell.$$

Proof. Large rings. For each of the finitely many $v \mid \ell$, Theorem 7.7.12 gives a coefficient ring $\mathcal{O}_v$ above which its conclusion holds at v. The compositum of the finitely many finite Galois extensions of $\mathbb{Q}_\ell$ with rings of integers $\mathcal{O}_v$ is finite Galois; its ring of integers $\mathcal{O}_0$ contains every $\mathcal{O}_v$. Fix $\mathcal{O} \supseteq \mathcal{O}_0$.

Each place. Put $n_v = [G_\ell : j_v(\Gamma_v)]$. Since $j_v(\Gamma_v)$ is the absolute Galois group of the closure E_v of $e_v(E)$, $n_v = [E_v : \mathbb{Q}_\ell]$, and $\sum_{v \mid \ell} n_v = [E : \mathbb{Q}] = 2f$ (E is a CM field with f infinite places). By Theorem 7.7.12 at v, with (iv) and $\operatorname{rank} L_{v,c} = 1$ for $c < t'$,

$$\sum_c \operatorname{nl} \operatorname{dd}\big(\mathrm{Cr}(F_v, K_v, U_v), L_{v,c}^\vee(1)\big) \le n_v \Big(r' \operatorname{db}_\ell + t' \frac{\operatorname{db}_\ell}{t} \Big) \le n_v \, (r' + 1) \operatorname{db}_\ell \le 2 n_v \, r' \operatorname{db}_\ell,$$

using $t' \le t$ and $r' \ge 1$.

Sum over places. Summing over $v \mid \ell$ gives at most $2 \cdot 2f \cdot r' \operatorname{db}_\ell = r' \cdot 4f \operatorname{db}_\ell = r' \operatorname{loc}_\ell$. □

The bound at one place (Theorem 7.7.12) has two cases. At $\ell \in \{p_0, p_1\}$ inertia acts through a tame type, and a cup exponent explicit in the type exists (Proposition 7.7.14). At the other primes the class is purely crystalline, and a Kummer chart whose letters record which embeddings are b-places separates the thick letters, where the jumps are of the size b, from the thin ones (Lemma 7.7.17); the main summand has few low jumps. In both cases the lattices enter through their Tate duals, whose jumps are given by the Hodge–Tate comparison [298, 87]. The cup exponents come from writing a crystalline lattice as a Breuil–Kisin module (Proposition 7.7.16). This is the integral p-adic Hodge theory of [160, 25] (see also [161, 84, 312, 26, 89], and [88] for the Fontaine–Laffaille range), made explicit enough to keep track of every power of ℓ.

Theorem 7.7.12. *Let D be admissible data, ℓ a prime, $\iota \colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ an isomorphism, $e \colon E \to \overline{\mathbb{Q}}_\ell$ an embedding, and $j \colon \Gamma \to G_\ell$ an open embedding onto the subgroup of elements fixing $e(E)$ pointwise. Let J be a main-summand weight system of rank r' and length t'. Then for all large coefficient rings $\mathcal{O}$ the following holds. Let (L_c) be Galois lattices over $\mathcal{O}$ indexed by $c \in \{\mathrm{main}\} \sqcup \{0, 1, \dots, t' - 1\}$, and let F, K, U be a filter, a relation and jump sets such that:*

- *(i) either $\ell \in \{p_0, p_1\}$ and, for every inertial s, the set $\{Q : K(s, Q)\}$ is of tame type $(\ell, r, (r + t)r)$; or $F = \top$ and $K = \bot$;*
- *(ii) every $u \in U(\gamma)$ is a jump of some L_c at γ;*
- *(iii) for every $g \in G_\ell$, $J(\sigma_{e,g})$ lists the jumps of L_{main} at $\gamma_g \colon a \mapsto g(a)$, and $\{c\}$ lists the jumps of L_c at γ_g for $c < t'$;*
- *(iv) $\operatorname{rank}_{\mathcal{O}} L_{\mathrm{main}} = r'$.*

Then for every c with $L_c \in \mathrm{Cr}(F, K, U)$,

$$\operatorname{nl} \operatorname{dd}\big(\mathrm{Cr}(F, K, U), L_c^\vee(1)\big) \le [G_\ell : j(\Gamma)] \cdot \operatorname{rank}_{\mathcal{O}} L_c \cdot \frac{\operatorname{db}_\ell}{x_c}, \qquad x_{\mathrm{main}} = 1, \ x_c = t \ (c < t').$$

Proof. Write $n_\Gamma = [G_\ell : j(\Gamma)]$. Fix the index $j_0 \le m$ supplied by (DW1) of Definition 6.6.14. Put

$$\eta = h^2 + 1, \quad W = 4t, \quad \delta_0 = (2j_0 + 1)h - 2t, \quad S(\sigma) = \kappa(\sigma) + \delta_0 = S_{j_0}(\sigma).$$

All these quantities are the actual data quantities. By Lemma 7.7.31, $\delta_0 \ge 0$, the required crystalline size conditions hold, and the tame height and numerical comparisons below hold. By Lemma 7.7.32, choose a coefficient-ring threshold above which a chart, primitive letter root and thickness functions exist for this completion. This choice depends on the fixed data and completion, not on the subsequent lattices. Fix any coefficient ring above that threshold and lattices and class satisfying the statement.

Every coefficient embedding $\gamma : \mathcal{O} \to \mathbb{C}_\ell$ extends to some $g_\gamma \in G_\ell$. Put $\sigma_\gamma = \sigma_{e,g_\gamma}$ and $\varepsilon_\gamma = \mathrm{sg}(\sigma_\gamma)$. The hypotheses (ii)–(iii) imply

$$u \in U(\gamma) \quad \Longrightarrow \quad u \in J(\sigma_\gamma) \text{ or } u = c \text{ for some } 0 \le c < t'. \tag{18}$$

Indeed the jump of a summand is read from its listed multiset. This does not require the representative g_γ to be unique.

We record explicitly the windows and duality used in both cases. A low main jump has $|u| \le 4mh \le h^2$, by Lemma 4.2.67; a character jump is between 0 and $t-1$, also in $[-\eta, \eta]$. A high jump satisfies

$$\kappa_{j_0}(\sigma_\gamma) - t \le -\varepsilon_\gamma u \le \kappa_{j_0}(\sigma_\gamma) + t.$$

This interval is contained in $[S(\sigma_\gamma), S(\sigma_\gamma) + W]$. Thus $U(\gamma)$ fits $(\eta, -\varepsilon_\gamma, S(\sigma_\gamma), W)$. The set $-U(\gamma)-1$ fits the opposite windows $(\eta, \varepsilon_\gamma, S(\sigma_\gamma), W)$. For a low jump this follows from $|u+1| \le h^2 + 1$; for a character it follows from $c + 1 \le t \le \eta$; for a high jump it follows by widening the preceding interval by 1 at each end, which stays inside the displayed far interval because $t \ge 1$.

Fix c for which L_c belongs to the class, and put

$$V_c = L_c^\vee(1), \quad \rho_c = \mathrm{rank}_{\mathcal{O}} L_c, \quad x_c = 1 \ (c = \mathrm{main}), \quad x_c = t \ (c < t').$$

The lattice V_c is in $\mathrm{Cr}(F, K^-, -U - 1)$, with $K^-(s, Q) = K(s^{-1}, Q)$. Here is the admissibility check for the possibly nontrivial filter. Choose a finite spanning family of periods of L_c, each fixed along F and near 1, and intersect their stabilizers. The intersection is an open subgroup, large for both filters. The direct period bridge Lemma 7.4.6 makes the restricted rational representation crystalline in the usual sense. Its dual is crystalline by Theorem A.3.9; a crystalline period basis of the dual is fixed by that same subgroup and spans, so it is admissible along the required filter. The twist is crystalline. The comparison Theorem A.3.11 gives

$$\mathrm{jump}(V_c, \gamma, u) = \mathrm{jump}(L_c, \gamma, -u - 1), \tag{19}$$

because dualizing a $\mathbb{C}_\ell(-v)$-component and twisting by the cyclotomic character produces $\mathbb{C}_\ell(v+1)$, of jump $-v - 1$. Finally the period operator of an inertial element on the twisted dual is the inverse transpose of the operator on L_c; the cyclotomic crystalline period has trivial inertia. Thus a polynomial killing the inverse element on L_c kills the element on V_c. This checks all three class conditions, and the rank of V_c is ρ_c.

Tame case. Let $\ell = p_i$. By Lemma 4.2.66, the chosen completion is $\mathbb{Q}_\ell$; hence j is onto and all $\sigma_{e,g}$ coincide. The sign and the far start in the windows are consequently independent of γ. Apply Proposition 7.7.14(d) to V_c, with

$$(r, N, \eta, W, b) = (r, (r + t)r, h^2 + 1, 4t, S(\sigma_{e,1})).$$

The class and polynomial hypotheses are the duality check above and (i). The strict height inequality is Lemma 7.7.31(2). It gives the cup exponent

$$E_* = 3r^2((r+t)r+1)^2 + 150(\eta + W + 1).$$

By Proposition 7.7.13(a) and Lemma 7.7.31(2), since $1 \le x_c \le t$,

$$\operatorname{nl}\operatorname{dd}(\mathrm{Cr}(F,K,U),V_c) \le n_\Gamma\, 2E_*\,\rho_c \le n_\Gamma\,\rho_c\,\mathrm{db}_\ell/x_c.$$

Crystalline case. Now $F = \top$ and $K = \bot$. For each γ, replace the allowed dual set by its actual jump support:

$$U_c'(\gamma) = \{u \in -U(\gamma) - 1 : \mathrm{jump}(V_c,\gamma,u) \neq 0\}.$$

By (19) it still contains every jump of V_c, and is contained in the opposite windows already checked. Thus $V_c \in \mathrm{Cr}(\top,\bot,U_c')$. This restriction of the dual sets is important: character jumps from other summands are not imposed on the main summand's dual.

The chart from Lemma 7.7.32, restricted to the representatives g_γ, reads precisely the thickness $[\sigma_\gamma] \in \Phi_b$ and signs ε_γ. Its letter is $g_\gamma(\zeta) = \gamma(\zeta)$. Every clause of letters reading thickness therefore survives that restriction. Its thick far start is $b + \delta_0$, and its thin far start is $a/x_1 + \delta_0$, exactly the substitution in Lemma 7.7.31(1).

We check the alternative about low jumps of Proposition 7.7.13(b). If $c = \mathrm{main}$, each main jump has

$$|u| \ge h/2. \tag{20}$$

For low jumps this is $\Delta \le 4|u|$. For high jumps it follows from $|u| \ge \kappa_{j_0}(\sigma) - t \ge h - t \ge h/2$. By (18), a negative member of $U(\gamma)$ must be a main jump and hence satisfies $2u \le -h$. A negative $u \in U_c'(\gamma)$ corresponds by (19) to a nonnegative main jump $-u-1$; (20) gives $2u \le -h$ again. The multiplicity of dual jumps in $[-\eta,\eta]$ is at most the number of original main jumps with $|v| \le \eta + 1 = h^2 + 2 \le 2h^2$. By (DW2) that number is at most $\lfloor r'm/k \rfloor = \lfloor \rho_c m/(kx_c) \rfloor$. Thus the few-low-jump alternative holds with this value of μ.

If $c < t'$, (19) gives the unique dual jump $-c-1$, which lies in $[-\eta,0]$, because $c < t' \le t \le \eta$. Hence the other alternative holds; we may still choose $\mu = \lfloor \rho_c m/(kx_c) \rfloor$, since this alternative imposes no low-jump multiplicity bound.

All hypotheses of Proposition 7.7.13(b) are now established with

$$(Q,m,h,G,b,\delta,\eta,W) = (kt,m,h,a/x_1,b,\delta_0,h^2+1,4t).$$

It follows that the defect is at most

$$n_\Gamma 10^5 \left(\rho_c \left\lfloor \frac{2\lambda_\ell}{kt} \right\rfloor m + \left\lfloor \frac{\rho_c m}{kx_c} \right\rfloor \lambda_\ell\right).$$

Apply Lemma 7.7.31(3) to obtain $n_\Gamma \rho_c\,\mathrm{db}_\ell/x_c$, as required. Both cases use the same originally supplied lattice and class. □

Proposition 7.7.13. *Let ℓ be a prime, $j\colon \Gamma \to G_\ell$ an open embedding, $\mathcal{O}$ a coefficient ring and L a Galois lattice over $\mathcal{O}$; put $n_\Gamma = [G_\ell : j(\Gamma)]$.*

(a) If L is potentially crystalline and e is a cup exponent of L for a class $\mathcal{Q}$, then $\operatorname{nl}\operatorname{dd}(\mathcal{Q},L) \le n_\Gamma \cdot 2e \cdot \mathrm{rank}_{\mathcal{O}} L$.

(b) Let $(Q, m, h, G, b, \delta, \eta, W)$ satisfy (KZ1)–(KZ4), and for h, b put

$$\lambda = v_\ell\big((2b)!\,(\prod_{\max(b-30h^3,0)<a\le b+30h^3} a)^{h^3}\big).$$

Let each embedding $\gamma\colon \mathcal{O} \to \mathbb{C}_\ell$ be thick or thin, with a sign $\varepsilon(\gamma)$; let $b_\gamma = b$ if γ is thick and $b_\gamma = G$ otherwise, and let $U(\gamma)$ fit $(\eta, -\varepsilon(\gamma), b_\gamma + \delta, W)$ and $U'(\gamma)$ fit $(\eta, \varepsilon(\gamma), b_\gamma + \delta, W)$. Let $L \in \mathrm{Cr}(\top, \bot, U')$; let L have few low jumps (h, μ, η, U, U'), or let $U'(\gamma) \subseteq [-\eta, 0]$ for all γ; and let k be a Kummer chart for Γ with a letter root ζ such that the letters read thickness for $(\gamma \mapsto \gamma(\zeta), \mathit{thick}, \varepsilon, Q, h, T, s)$, for some T and s. Then

$$\mathrm{nl}\,\mathrm{dd}\big(\mathrm{Cr}(\top, \bot, U), L\big) \le n_\Gamma \cdot 10^5\Big(\mathrm{rank}_{\mathcal{O}} L \cdot \Big\lfloor \frac{2\lambda}{Q} \Big\rfloor \cdot m + \mu\lambda\Big).$$

Proof. (a). Let K_ℓ be the finite field extension fixed by $j(\Gamma)$, so that Γ is topologically identified with G_{K_ℓ} and $[K_\ell : \mathbb{Q}_\ell] = n_\Gamma$. Apply Lemma 7.4.6 to the actual lattice L. It gives a finite free module $Z_f(L)$ of rank at most

$$(1 + n_\Gamma)\,\mathrm{rank}_{\mathcal{O}} L \le 2n_\Gamma\,\mathrm{rank}_{\mathcal{O}} L.$$

By the cup-exponent hypothesis, $\ell^e Z_f(L) \subseteq Z_f(L) \cap L^{\perp_{\mathcal{Q}}}$. Consequently $Z_f(L)/(Z_f(L) \cap L^{\perp_{\mathcal{Q}}})$ is a quotient of $Z_f(L)/\ell^e Z_f(L)$. The latter is isomorphic to $(\mathcal{O}/\ell^e)^{\mathrm{rank}_{\mathcal{O}} Z_f(L)}$ and has normalised length $e\,\mathrm{rank}_{\mathcal{O}} Z_f(L)$, since $\mathrm{nl}(\mathcal{O}/\ell) = 1$. Monotonicity of length under quotients gives the claimed bound, including $e = 0$ and rank zero.

(b). Write $\rho = \mathrm{rank}_{\mathcal{O}} L$,

$$v_0 = \lambda, \qquad K_0 = \Big\lfloor \frac{2v_0}{Q} \Big\rfloor, \qquad J_0 = \lfloor \log_\ell h^3 \rfloor.$$

We use Proposition 7.7.14; its hypotheses are checked below. Two elementary facts about the actual integer defining v_0 are useful:

$$2b < \ell(v_0 + 1), \qquad 2v_0 < (K_0 + 1)Q. \tag{5}$$

The first follows from $v_\ell((2b)!) \ge \lfloor 2b/\ell \rfloor$, the second from division with remainder. Also, if a positive integer in $(b-30h^3, b+30h^3]$ is divisible by ℓ, then $v_0 \ge h^3$, because that product occurs to the power h^3.

Here and below the four size conditions imply

$$h \ge 99Q^2, \quad m \le h/99, \quad \eta + W \le h^3/(99Q), \quad \delta < \eta, \quad G \ge 99h^3, \quad b \ge 99G(h^3 + 1). \tag{6}$$

In particular $h \ge 99$, $\eta \ge 1$, $m \ge 1$, $\eta \le G$, and $\delta + W \le G$. For each integer $1 \le H \le 4b$,

$$\lfloor \log_\ell H \rfloor \le \frac{\log(4b)}{\log \ell} \le 2\log(4b) < m. \tag{7}$$

We used $\log \ell \ge \log 2 > 1/2$.

We transfer the jump sets from embeddings to letters without assuming thickness is read at a thin embedding when $\ell \le Q$. If a letter receives a jump from a thick embedding, the definition of letters reading thickness determines its thick label and sign. If $\ell > Q$, it determines both for every embedding over that letter. When $\ell \le Q$, a jump from a thin embedding, irrespective of its sign, belongs to $[-2G, 2G]$: its low part has radius $\eta \le G$, and its far part has absolute value

at most $G+\delta+W \le 2G$. Thus the two letter jump sets fit the required opposite windows with low radius $2G$ for $\ell \le Q$, and with the original radius η for $\ell > Q$. The extra root 0 in a letter jump set always belongs to its low window. Put $\beta_i = b+\delta$ at a thick letter and $G+\delta$ otherwise.

Small primes, $\ell \le Q$. In this case $K_0 \ge 1$ and

$$2(b+\delta) < h^2 K_0. \tag{8}$$

Here is a direct check. If $K_0 = 0$, then (5) gives $2b < Q(Q/2+1) \le 3Q^2/2$, contrary to (6). For $K_0 \ge 1$, (5) gives $\lambda < K_0 Q$, hence $2b < Q(K_0 Q+1) \le 2Q^2 K_0$. Thus $b < Q^2 K_0$. Also $\delta < b/2$ by (6), so $2(b+\delta) < 3b < 3Q^2 K_0 \le h^2 K_0$, proving (8).

Apply the sparse-letter assertion Proposition 7.7.14(c) with low radius $2G$, far width W, thick and thin starts $b+\delta, G+\delta$, gap length J_0, approximation parameter $2K_0$, and final auxiliary parameter K_0. We verify its numerical hypotheses. Since $\ell^{J_0} \le h^3 < \ell^{J_0+1}$, $J_0 \ge 1$, and (6) gives

$$10\ell^{J_0}(2G) \le 20Gh^3 \le b+\delta, \qquad 8e \le 8h \le h^3 < \ell^{J_0+1}.$$

The first is used in the unramified alternative, and the second in the constant-sign alternative; precisely one of those alternatives is supplied by letters reading thickness. Its same definition supplies the absence of three thick letters with both gaps between 1 and J_0. The depth inequality follows from (8):

$$2e(b+\delta+10G) \le 4h(b+\delta) < 2h^3 K_0 < \ell^{J_0+1}(2K_0).$$

For the remaining cost inequality, the chart's different bound d_{ch} is at most $\lfloor \log_\ell e \rfloor + 1$, or is zero. By (7), $J_0 < m$ and $d_{\mathrm{ch}} \le m$; hence $J_0 + 1 + d_{\mathrm{ch}} \le 2m \le h$. The integer $2(b+\delta+10G)$ is at most $4b$ by (6), so its logarithm in (7) is at most $m \le hG$. Therefore

$$20(J_0+1+d_{\mathrm{ch}})(2G) + \lfloor \log_\ell 2(b+\delta+10G) \rfloor \le 41Gh < K_0.$$

The last inequality follows from $h^2 K_0 > 2b \ge 198Gh^3$. Also $G+\delta+W \le 2G$, as required for the thin windows. The resulting cup exponent is at most

$$10^3\big(2K_0 + (J_0+1)K_0\big) \le 10^4 K_0(J_0+1) \le 10^4 K_0 m. \tag{9}$$

Large primes with large loss, $\ell > Q$ and $v_0 \ge h^3$. The chart is unramified and $\ell \ge 7$, by letters reading thickness. Apply Proposition 7.7.14(b) with thick and thin starts $b+\delta, G+\delta$, width W, low radius η, polynomial parameter K_0, and $W' = W + 4\eta$. For clarity the required inequalities follow as follows. We have $K_0 \ge 1$, $v_0 < K_0 Q$, and

$$K_0 > 2h^3/Q - 1, \qquad W + 4\eta \le 4h^3/(99Q) \le K_0.$$

By (5) and $Q+1 \le \ell$, $b < \ell(K_0 Q+1)/2 \le K_0 \ell^2/2$. Further $\delta \le \eta \le h^3/(99Q) \le v_0/(99Q) < K_0/99$, so $b+\delta < K_0\ell^2$. Similarly

$$2G \le \frac{2b}{99h^3} < \frac{\ell(v_0+1)}{99h^3} \le K_0\ell/99, \qquad 2\delta < 2K_0/99,$$

where $Q+1 \le h^3$ was used in the middle bound. Hence $2(G+\delta) < K_0\ell$. Positivity and ordering of the two starts are immediate from (6). Finally $2(b+\delta+W+4\eta) \le 4b$, again by (6). Its logarithm is at most $m-1$ by (7), so the cup exponent from (b) is bounded by

$$10^4\big(K_0 + \lfloor \log_\ell 2(b+\delta+W+4\eta) \rfloor\big) \le 10^4(K_0+m-1) \le 10^4 K_0 m. \tag{10}$$

Here $(K_0-1)(m-1) \ge 0$.

In either case (9) or (10), part (a) gives

$$\operatorname{nl}\operatorname{dd} \le 2n_\Gamma \rho\, 10^4 K_0 m \le n_\Gamma 10^5(\rho K_0 m + \mu v_0),$$

which is the desired inequality.

Large primes with small loss, $\ell > Q$ and $v_0 < h^3$. Set $a_0 = 2(v_0+1)$ and $R_0 = 6(a_0+\eta+W)$. No positive multiple of ℓ lies in $(b-30h^3, b+30h^3]$, by the observation following (5). Moreover

$$\ell > 2b/h^3 \ge 198G, \quad R_0 < 13h^3, \quad \delta + R_0 < 30h^3. \tag{11}$$

Indeed $v_0 + 1 \le h^3$ and (6) apply. Thus ℓ divides no nonzero integer in the low window of radius R_0, or within R_0 of either $\pm(G+\delta)$, since all their absolute values are $< \ell$. A multiple within R_0 of $\pm(b+\delta)$ would have positive absolute value in the prohibited product interval, by $\delta + R_0 < 30h^3$. This proves the gap condition at every letter.

The height and separation requirements in Proposition 7.7.14(a) also hold:

$$\beta_i + W + \eta \le (\ell-3)a_0, \qquad R_0 < \beta_i.$$

For the first, (5) gives $b < \ell a_0/2$, whereas $\delta + W + \eta + 3a_0 < 7h^3 < G < \ell a_0/2$, by (6), (11) and $a_0 \ge 2$. The second follows from $R_0 < 13h^3 < G$. The chart is unramified with $\ell \ge 7$, and $\eta \ge 1$, so all hypotheses of that assertion have now been checked.

If $U'(\gamma) \subseteq [-\eta, 0]$ for all γ, its vanishing conclusion gives the result immediately. Otherwise use the few-low-jump alternative. If $v_0 < \lfloor h/5 \rfloor$, then $2a_0 = 4(v_0+1) < h$; no negative allowed jump can lie in $[-a_0, 0)$, since such a jump would satisfy both $2u \le -h$ and $2u \ge -2a_0 > -h$. Again the defect vanishes.

It remains that $v_0 \ge \lfloor h/5 \rfloor$ and the few-low-jump condition holds. In particular $v_0 \ge 1$ and $a_0 + 1 = 2v_0 + 3 \le 6v_0$. From $h \ge 99Q(W+2)$ in (KZ1) we have $v_0 \ge 19Q(W+2)$, whence $W + 1 \le K_0$. The length conclusion of Proposition 7.7.14(a) and the assumed low-jump count now give

$$\operatorname{dd} \le 11 f_{\mathrm{ch}}\big(\mu(a_0+1) + \rho(W+1)\big)\operatorname{length}_{\mathcal{O}}(\mathcal{O}/\ell).$$

Here $f_{\mathrm{ch}} = n_\Gamma$, since the chart is unramified and its degree is n_Γ. Taking normalised length, using $\operatorname{nl}(\mathcal{O}/\ell) = 1$, $m \ge 1$, and the two preceding inequalities yields

$$\operatorname{nl}\operatorname{dd} \le 11 n_\Gamma(6\mu v_0 + \rho K_0) \le 10^5 n_\Gamma(\mu v_0 + \rho K_0 m).$$

This completes all three branches and proves (b). □

Proposition 7.7.14. *Let ℓ be a prime, Γ a topological group with an open embedding $j\colon \Gamma \to G_\ell$, $\mathcal{O}$ a coefficient ring, L a Galois lattice over $\mathcal{O}$, and U, U' jump sets. In (a)–(c), k is a Kummer chart for Γ with letter root $\zeta \in \mathcal{O}$, and $L \in \mathrm{Cr}(\top, \bot, U')$; write $\mathcal{Q} = \mathrm{Cr}(\top, \bot, U)$.*

(a) (Unramified, few low jumps.) Let k be unramified, $\ell \ge 5$, $\eta \ge 1$, $a, W \in \mathbb{N}$, and at each letter i let $b_i \in \mathbb{N}$ and $\varepsilon_i = \pm 1$ with $b_i + W + \eta \le (\ell-3)a$, $6(a+\eta+W) < b_i$, $\mathrm{Ng}(\ell, 6(a+\eta+W), b_i)$, $U_\zeta(i)$ fitting $(\eta, -\varepsilon_i, b_i, W)$ and $U'_\zeta(i)$ fitting $(\eta, \varepsilon_i, b_i, W)$. If no $u \in U(\gamma) \cup U'(\gamma)$ lies in $[-a, 0)$ for any γ, or $U'(\gamma) \subseteq [-\eta, 0]$ for all γ, then $\operatorname{dd}(\mathcal{Q}, L) = 0$. If $\sum_{|u| \le \eta} \mathrm{jump}(L, \gamma, u) \le \mu$ for all γ, then

$$\operatorname{dd}(\mathcal{Q}, L) \le 11 f\big(\mu(a+1) + \operatorname{rank}_{\mathcal{O}} L \cdot (W+1)\big) \cdot \operatorname{length}_{\mathcal{O}}(\mathcal{O}/\ell).$$

(b) (Any pattern.) Let k be unramified, $\ell \ge 7$, $\eta \ge 1$, and let T be a predicate on letters, $0 < G \le b$, $V + 4\eta \le W' \le K$, $b < K\ell^2$, $2G < K\ell$. If at each letter i, with $b_i = b$ for $T(i)$ and $b_i = G$ otherwise, $U_\zeta(i)$ fits $(\eta, -\varepsilon_i, b_i, V)$ and $U'_\zeta(i)$ fits $(\eta, \varepsilon_i, b_i, V)$, then $10^4(K + \lfloor \log_\ell 2(b+W') \rfloor)$ is a cup exponent of L for $\mathcal{Q}$.

(c) *(Sparse thick letters, any ramification.) Let $\eta \geq 1$, T, b_i, ε_i and the windows be as in (b), with $G + V \leq \eta$ and $J \geq 1$; let no three letters with T follow one another along the letter Frobenius with both gaps in $[1, J]$; let either k be unramified, $\ell \geq 7$ and $10\ell^J \eta \leq b$, or ε be constant and $8e \leq \ell^{J+1}$; and let $2e(b+5\eta) \leq \ell^{J+1}\delta$ and $20(J+1+d)\eta + \lfloor \log_\ell 2(b+5\eta) \rfloor \leq X$. Then $10^3(\delta + (J+1)X)$ is a cup exponent of L for $\mathcal{Q}$.*

(d) *(Tame types.) Let j be surjective, F a filter on Γ, K a relation with $K(s, \cdot)$ of tame type (ℓ, r, N) for every inertial s, and $L \in \mathrm{Cr}(F, K^-, U')$, where $K^-(s, Q) = K(s^{-1}, Q)$. Let $\varepsilon = \pm 1$ and $\eta, W, b \in \mathbb{N}$ with $U(\gamma)$ fitting $(\eta, -\varepsilon, b, W)$ and $U'(\gamma)$ fitting $(\eta, \varepsilon, b, W)$ for all γ, and $\ell > b + 6(W + \eta) + r^2(N+1)^2 + 2$. Then $3r^2(N+1)^2 + 150(\eta + W + 1)$ is a cup exponent of L for $\mathrm{Cr}(F, K, U)$.*

Proof. Write $p = \ell$.

From sides to the stated orthogonal module. By Lemma 7.4.6, $Z_f(L)$ is finite free and every crystalline cocycle is continuous. Regard L as a side with parameter lattice $Z_f(L)$, trivial action on that parameter, and cocycle map the identity. Its letter jump sets are $U'_\zeta(i)$. For every Y in the comparison class use the analogous universal side of Y. A pairing of these sides is precisely a pairing of Y, L in Definition 7.2.1; the additional third side is its stated Tate dual side. Consequently the assertion that every such pairing and every pair of parameters has a scaled cup coboundary at exponent h says exactly

$$p^h Z_f(L) \subseteq L^{\perp_{\mathcal{Q}}}. \tag{1}$$

The topology here is the given one; the continuous cochains supplied by the side results apply to every topology making j an open embedding. Thus no passage from algebraic to continuous cohomology is implicit in (1).

In (a), all jumps of the two sides and of the third side are at most $b_i + W + \eta \leq (p-3)a$. For the third side use $-u-v-1$: opposite far windows cannot both contribute a large negative jump, and direct separation into low–low, low–far and far–far gives this bound, since $b_i > 6(a + \eta + W) \geq \eta$. Hence Proposition 7.7.15(3) supplies E^a for all three cokernels. The small-negative-jump or nonpositive condition passes to letter jump sets; the added jump zero causes no problem. Apply Proposition 7.7.29(1) with $\kappa = 0$ and $d = 0$. It gives cup exponent zero, and (1) proves that the dual-defect quotient is zero.

For the length assertion let d_i be the number of embeddings of $\mathcal{O}$ over letter i, let $r_L = \mathrm{rank}_{\mathcal{O}} L$, and apply Proposition 7.7.29(2) to the universal side of L with

$$n_i = f d_i \mu, \qquad r_i = f d_i r_L.$$

These are valid bounds because actual letter multiplicities sum the multiplicities over precisely those embeddings, and the full jump multiset at each embedding has cardinality r_L. It gives one $\mathbb{Z}_p$-submodule $V \subseteq Z_f(L)$, chosen independently of all opposite sides and pairings, of colength at most

$$11 f\big(\mu(a+1) + r_L(W+1)\big) \sum_i d_i.$$

Uniformity gives $V \subseteq Z_f(L) \cap L^{\perp_{\mathcal{Q}}}$ even though V need not be $\mathcal{O}$-linear. Distinct letters have distinct $\theta([i])$, since their Teichmüller difference is a unit. Each embedding belongs to at most one letter, whence $\sum_i d_i \leq \mathrm{rank}_{\mathbb{Z}_p} \mathcal{O}$. If $f_{\mathcal{O}}$ is the residue degree of $\mathcal{O}$, a composition series gives $\mathrm{length}_{\mathbb{Z}_p} M = f_{\mathcal{O}} \,\mathrm{length}_{\mathcal{O}} M$ for any finite-length $\mathcal{O}$-module M, and $\mathrm{rank}_{\mathbb{Z}_p} \mathcal{O} = f_{\mathcal{O}} \,\mathrm{length}_{\mathcal{O}}(\mathcal{O}/p)$. Divide the resulting bound by $f_{\mathcal{O}} > 0$. This proves exactly the length bound in (a), including finiteness.

For (b) and (c), choose these universal sides and apply respectively the arbitrary-pattern and sparse-thick assertions of Proposition 7.7.27. Every hypothesis is precisely the corresponding hypothesis on the letter jump sets in the present statement. Formula (1) then gives the indicated cup exponents with the unchanged constants 10^4 and 10^3.

The tame chart and crystallinity over its subgroup. We give the additional argument for (d); no unramified-chart reduction is used. Put

$$e_0 = \operatorname{lcm}_{1\le i\le r}(p^i-1), \qquad m = r^2, \qquad \kappa = r^2(N+1)^2, \qquad n = b + 6(W+\eta). \tag{2}$$

The empty least common multiple is one. We have $r \le \kappa < p$, with the first inequality also valid at $r = 0$. Thus $f_0 = r!$ is positive and prime to p, and $e_0 \mid p^{f_0} - 1$. Moreover

$$e_0 \le \prod_{i=1}^{r}(p^i - 1) \le p^{r^2} = p^m, \tag{3}$$

including $r = 0$.

Identify Γ with $G_{\mathbb{Q}_p}$ by the surjective open embedding j. Choose $\varpi^{e_0} = p$ and the unramified extension of degree f_0, which contains μ_{e_0}. The subgroup fixing its constants and ϖ is open and normal of index prime to p: the associated splitting field of $X^{e_0} - p$ has degree dividing $e_0 f_0$, and both factors are prime to p. Use a compatible tower of p-power roots of ϖ to obtain $z = [\varpi^\flat]$. Put $E = p - z^{e_0}$. It lies in $\ker\theta = \xi A_{\mathrm{inf}}$. In the tilt valuation normalized by $v(\bar{\xi}) = 1$, one has $v(\bar{z}) = 1/e_0$, so the quotient $E/\xi \in A_{\mathrm{inf}}$ has reduction of valuation zero and is a unit. Thus this very E is associated to ξ. The map $\Lambda[[X]] \to A_{\mathrm{inf}}$, $X \mapsto z$, converges in the (p,z)-adic topology; it is injective by reduction modulo p, where the first nonzero series term has strictly smaller valuation than all later terms. It gives the required actual chart with $E = p + z^{e_0}(-1)$. Its stabilizer subgroup just chosen gives the auxiliary Γ' of the chart. The required automorphisms $z \mapsto [\epsilon]^h z$ exist in that subgroup: at each finite level the polynomials $X^{e_0 p^a} - p$ are Eisenstein over the maximal unramified extension, so inertia acts transitively on their roots; the compatible nonempty finite fibers give an automorphism on the inverse limit. The same argument with a primitive e_0-th root in the unramified constants gives an inertial s satisfying

$$sz = \xi z, \qquad \xi \in \Lambda^\times, \quad \bar{\xi} \text{ primitive of order } e_0, \qquad s^h z = \xi^h z. \tag{4}$$

Inertia fixes Λ. Normality holds because the splitting field already contains all e_0-th roots of unity. This is the required tame chart; its different bound is $d = 0$ if $e_0 = 1$ and $d = 1$ otherwise, since $p \nmid e_0$. We may take the letter root in $\mathcal{O}$ to be 1; no enlargement of the coefficient ring is required.

Every lattice in either class is potentially crystalline. The polynomial $X^{e_0} - 1$ allowed at every inertial element says that its inertial period operators have exponent dividing e_0. These operators form a representation of inertia with open kernel: choose a finite extension over which the representation is crystalline, and use its finite-dimensional crystalline periods. Wild inertia is pro-p, so its finite image is trivial, because its exponent divides the prime-to-p integer e_0. Tame inertia is procyclic, and an exponent-e_0 image factors through its cyclic quotient of order e_0. The inertia stabilizer of $\varpi = p^{1/e_0}$ is exactly the kernel of that quotient: over the maximal unramified field the extension $X^{e_0} - p$ is totally tamely ramified of degree e_0, and its automorphisms multiply ϖ by μ_{e_0}. Hence the period inertia is trivial on the inertia of Γ'.

To identify this with the classical crystalline criterion, choose a finite extension over which the representation is crystalline. The canonical $B_{\max}$ comparison with B_{cris} from Lemma 7.4.6 identifies these period inertia operators with the inertia action on D_{pst}, after scalar extension: both act on the same spanning crystalline periods, so they agree. Potential crystallinity gives

monodromy zero. Therefore Theorem 7.1.10(2) gives crystallinity over the finite fixed field of Γ'. The original jump sets are unchanged. This applies to K and K^-, since inversion preserves inertia. We may consequently use universal sides on this same tame chart for every $Y \in \mathrm{Cr}(F, K, U)$ and for L.

A tame polynomial supplies a pure-power killer. We prove the needed quantitative fact rather than assuming it from the tame type. Let a side have all jumps at most n, and let a monic $q \in \mathbb{Z}_p[X]$ kill the inertial period operator β_s on its lattice, with $\deg q = h$. Suppose

$$e_0 \le p^m, \qquad m(h+1) \le \kappa, \qquad n + \kappa + 2 < p. \tag{5}$$

Then p^κ kills its cokernel at every letter.

First $Q = q(X)(X-1)$ kills the period operator on the cocycle extension. Indeed the operator on the trivial parameter quotient is one, so $\beta_s - 1$ maps into the original lattice, on which $q(\beta_s) = 0$. The exactness used here is exactness of tensoring the split underlying lattice sequence, with its triangular action.

We next prove the divisibility that makes most lowering steps free. For x in the actual canonical N of that extension,

$$Q(s)x \in zM. \tag{6}$$

Normality of Γ' makes the conjugate $\mu_h(y) = s^{-h}\mu(s^h y)$ of each crystalline form another crystalline form. Its values on the lattice come from the conjugate $B_{\max}$-linear functional. On a potentially crystalline period vector the identity $\ell_1 \beta_s^h = \ell_{s^h}$ holds for these functionals: the usual semilinear action and the linear period action differ only by the action on a scalar fixed near one; such a scalar lies in a finite unramified field and is fixed by inertia. Check on the spanning period vectors and extend $B_{\max}$-linearly. Thus $Q(\beta_s) = 0$ gives $\sum_h Q_h \mu_h = 0$ on the lattice and then on its rational ambient span, which contains M. In particular

$$\mu(Q(s)x) = \sum_h Q_h\big(s^h \mu_h(x) - \mu_h(x)\big). \tag{7}$$

The uniform canonical N criterion of Proposition 7.7.16, item3(b), supplies one exponent b_0 clearing the crystalline-form values into $\mathcal{S}_B$. Use the actual scalar Galois law of Lemma 7.7.22, item7: every inertial s^h fixes Λ, so

$$z \mid s^h v - v \quad (v \in \mathcal{S}_B) \qquad \text{in } B^+_{\max}.$$

No stability of $\mathcal{S}_B$ under s is required. Clearing the two E^{b_0} denominators in a difference gives

$$E^{2b_0}\big(s^h \mu_h(x) - \mu_h(x)\big) \in zB^+_{\max}.$$

Here $s^h E/E$ is a unit of A_{inf}; its difference from one is divisible by z: power-series subtraction first gives $z \mid s^h E - E = E(s^h E/E - 1)$, and Lemma 7.7.30, with $m = 0, n = 1, h = 1$, cancels the factor E. This verifies the clearing assertion by subtraction of the two fractions. Equation(7) gives the same assertion for every crystalline form on $Q(s)x$. The scalar cancellation premise of the canonical divisibility criterion is precisely Lemma 7.7.30, with powers $m = 0, n = 1$. Normality of Γ' gives $Q(s)x \in M$ by canonical stability. Apply Proposition 7.7.16, item2(d), with divisor z, to obtain (6). Both divisibility and cancellation take place in the actual $B^+_{\max}$, not merely in its de Rham embedding.

Let J_i be the lift ideal with modulus $r_0 = p^{\kappa+1}$. If $p^t z^{k+1} \in J_i$, lift $(p^t z^{k+1} - r_0 u)\mathrm{cst}_i(c)$ to a canonical vector x, for all parameters c, with one common $u \in \mathfrak{S}$, by Lemma 7.7.21, item4.

Write $u = w + zv$, $w \in \Lambda$. Apply $Q(s)$ to the lift and divide by z using (6). Since $Q(1) = 0$, the terms in w cancel. Using (4) on the other terms gives

$$Q(\xi^{k+1})\, p^t z^k \in J_i. \tag{8}$$

The quotient of the lift lies in M and projecting to the letter preserves that property, so (8) is an actual lift-ideal inclusion. More explicitly the modulus correction after division is $r_0 \sum_h Q_h \xi^h s^h v$, a single chart scalar independent of c; thus the universal quantifier in the lift ideal is preserved. Cancellation of z in the ambient Witt vector tensor is valid because it is a unit in $W(\mathbb{C}_p^\flat)$.

The initial bound of Proposition 7.7.15(2) gives $z^\nu \in J_i$ with $(p-1)\nu \le e_0(n+\kappa)$. Condition (5) implies $\nu < e_0$. For $1 \le k < e_0$, the residues $\bar{\xi}^k$ are distinct. Since Q is monic, its reduction has at most $\deg Q = h + 1$ roots. Thus at most $h + 1$ descent steps have $Q(\xi^k)$ nonunit. At all other steps (8) lowers the power of z freely. At an exceptional step Lemma 7.7.23, item(e), with adjacent exponents, lowers the power at cost $1 + v_p(k) \le m$, since $k < e_0 \le p^m$. If $e_0 = 1$, no step is present. Consequently the total cost of lowering ν to zero is at most $m(h + 1) \le \kappa$, and $p^\kappa \in J_i$. The modulus is removed by the actual statement Lemma 7.7.23, item(f), at level zero. This proves the claimed pure-power killer. At $m = 0$, $e_0 = 1$ and $\nu = 0$, so no descent step is required.

The three polynomials and the final exponent. Choose monic polynomials q, q' of degree at most N allowed by $K(s, \cdot)$ and $K(s^{-1}, \cdot)$ respectively. On Y these kill β_s and $\beta_{s^{-1}}$, while on L, by the definition of K^-, they kill $\beta_{s^{-1}}$ and β_s. The inverse orientation is essential here.

A monic polynomial of degree at most N^2 kills the generator on $(Y \otimes L)^\vee(1)$. To construct one, take the characteristic polynomial of multiplication by $X \otimes X$ on the finite free algebra

$$\mathbb{Z}_p[X]/(q') \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[X]/(q).$$

Its rank is $\deg q' \deg q \le N^2$. Cayley–Hamilton gives a polynomial identity on that algebra, hence on the commuting period operators $\beta_{s^{-1}}$ of Y and L. Dualizing this identity gives the desired identity on the inverse dual operator, which is β_s on the Tate dual tensor; the Tate twist has inertial period operator one. Tensor and dual comparison are the canonical crystalline ones, compatible with the split underlying lattice maps, so the polynomial acts on the actual period operator. If one polynomial has degree zero, its value is one and its representation space is zero; the same construction is the zero-rank algebra with characteristic polynomial one, and the assertion remains valid.

The jumps of all three sides are at most n of (2). For the two original sides this follows immediately from the windows. For the third side, at most one of the original jumps is below $-\eta$, so $-u - v - 1 \le b + W + 2\eta \le n$; if $\eta = W = 0$ its upper bound is $b - 1 \le n$. Zero is also at most n. Conditions (5) hold for each of the three polynomials: their degrees are at most N, N, N^2, and $m(h+1) \le r^2(N+1)^2 = \kappa$. The assumed prime inequality is exactly $n+\kappa+2 < p$. We have therefore produced p^κ as a killer for all three cokernels at every letter.

Finally $p > b + 6(W + \eta)$ implies the required gap condition: a nonzero integer in any of those neighborhoods has absolute value at most $b + 6(W + \eta) < p$, so cannot be divisible by p. Apply the generic gap-branch argument of Proposition 7.7.29(1) with $a = 0$. The forbidden interval is empty, and the chart may be ramified. It yields exponent

$$3(\kappa + (48(\eta + W) + 4)d) \le 3\kappa + 150(\eta + W + 1), \tag{9}$$

since $d \le 1$. If $\eta = 0$, this application means the identical generic branch proof, which we spell out at this endpoint: killer points have orders $-u$; deepen level zero to $-u + \eta = -u$ at zero

cost, or to that depth with the same harmless allowance $(48W+4)d$. Opposite windows give one down side with $u \le 0$ and one up side with $u \ge 0$. The down point is a stop of orders $\max(0,-u)$, while the up-side central move has orders $\min(-u,0)$ at zero and $-u$ above, giving residual $\min(u,0)=0$. The all-letter walk of Proposition 7.7.25, which permits $\eta = 0$, gives the same bound (9). No strengthening of the statement of Proposition 7.7.29 is being assumed.

For every lattice Y of the specified class, every pairing, and every pair of crystalline cocycles, (9) is thus a continuous cup coboundary. Formula (1), now for $\mathrm{Cr}(F,K,U)$, proves (d) and completes the proof. □

Proposition 7.7.15. *Let k be a Kummer chart for Γ with ramification e, let S, S' be pairing sides over a coefficient ring $\mathcal{O}$ with letter root ζ, and let P be a pairing of their lattices Y, Y' modulo ℓ^m. Write $\widetilde{X} = W(\mathbb{C}_\ell^\flat) \otimes (Y \times C)$ and $\widetilde{X}'$ likewise, with Γ and φ acting, and $\langle\,,\rangle$ for the $W(\mathbb{C}_\ell^\flat)$-bilinear extension of b_P to the Y- and Y'-components. For $e_3 \ge 0$, a pair $(w,y) \in \widetilde{X} \times \widetilde{X}'$ is* safe *if $\ell^{e_3}\langle w, y\rangle$ lies in the image of $W(\mathfrak{m}^\flat)$; a move of the first component by y with the second at x' is safe if $(y, gx' - x')$ is safe for all $g \in \Gamma'$, and symmetrically.*

1. *(Pairing lemma.) Let x, w lie over $\ell^{e_1}c$ and x', w' over $\ell^{e_2}c'$ ($c \in C$, $c' \in C'$). Assume that the moves by $hx' - x'$ ($h \in \Gamma'$) are safe with x at rest and that four safe moves lead from $(\varphi x, \varphi x')$ through w and w' to (x, x'). Then P is a scaled cup coboundary at (e, c, c') for every $e \ge e_1 + e_2 + e_3$.*

2. *If every jump of S at every letter is at most ω, then for every $\kappa \ge 0$ and every letter i there is a with $(\ell-1)a \le e(\omega+\kappa)$ such that z^a lies in the lift ideal $J_0(i)$ of S with modulus $\ell^{\kappa+1}$.*

3. *If k is unramified, $\ell \ge 5$ and every jump of S at every letter is at most $(\ell-3)(s+t)$, then $\ell^s E^t$ kills the cokernel of S at every letter.*

4. *(Forward-letter bound.) Let $\omega, \omega', J, \kappa \in \mathbb{N}$ and i be a letter. Suppose all jumps of S at all letters are at most ω, and all jumps at $\sigma^t i$ are at most ω' for $1 \le t \le J$, where σ is the letter Frobenius. If*

$$8e \le \ell^{J+1}, \qquad 2Jd\omega' + \frac{(2J+4)\omega'}{\ell-1} + \frac{8e\omega}{\ell^{J+1}} \le \kappa,$$

where d is the chart's different bound, then ℓ^κ kills the cokernel of S at i.

Proof. **The pairing assertion.** Put $B = W(\mathbb{C}_\ell^\flat)$, and $D_m = B \otimes_{\mathbb{Z}_\ell} (\mathbb{Z}/\ell^m)(1)$. Write I_m for the image of $W(\mathfrak{m}^\flat)$ in D_m. We use additive notation in this Tate twist.

First we record the exact Witt facts needed below. The map $\varphi - 1$ is surjective on B/ℓ^m: solve $X^\ell - X = a$ in the algebraically closed field $\mathbb{C}_\ell^\flat$, subtract the chosen Teichmüller lift, divide the remaining error by ℓ, and repeat m times. Its kernel consists of the constant $\mathbb{Z}/\ell^m$. More generally,

$$\varphi v - v \in I_m \quad \Longrightarrow \quad v - 1 \otimes t \in I_m \quad \text{for a unique } t \in (\mathbb{Z}/\ell^m)(1). \tag{1}$$

Indeed, in the first Witt coordinate a negative valuation of v_0 would give valuation $\ell v(v_0) < v(v_0)$ for $v_0^\ell - v_0$, contradicting positive valuation. Thus v_0 is integral, and its residue is in $\mathbb{F}_\ell$. Subtract that constant, divide by ℓ, and repeat. At each step $W(\mathfrak{m}^\flat) \cap \ell B = \ell W(\mathfrak{m}^\flat)$, as is immediate from Witt coordinates over the perfect ideal $\mathfrak{m}^\flat$. This also proves uniqueness, or equivalently

$$(1 \otimes \mathbb{Z}/\ell^m)(1) \cap I_m = 0. \tag{2}$$

The subgroup I_m is preserved by Galois, Frobenius and multiplication by A_{inf}. No assertion of topological nilpotence of Frobenius is being used.

For later continuity, for each fixed $v \in D_m$ one has $gv - v \in I_m$ near 1. Choose a generator of the Tate twist. The cyclotomic character is 1 modulo ℓ^m near 1, and continuity on the finitely many Witt coordinates makes their differences have positive valuation near 1. The same argument applies to a fixed vector in $B \otimes L$, followed by any fixed B-linear map to D_m, provided the action on the finite free lattice L is continuous: write the vector in a lattice basis, make the finitely many lattice-action differences zero modulo ℓ^m, and make each scalar difference sufficiently small to offset the finitely many coefficients of the map. Continuity of the lattice actions and of the crystalline cocycles follows from Lemma 7.4.6. In particular every integral crystalline cocycle is divisible by ℓ^m near 1.

Let π, π' denote the projections to Y, Y'. Define

$$B_0(v,t) = \ell^{e_3} b_P(\pi v, t).$$

It is linear in the second variable; the notation $B_0(v, \pi' v')$ includes projection in both variables. Choose $y \in D_m$ with

$$\varphi y - y = B_0(w, \pi'(\varphi x' - w')) + B_0(x, \pi'(w' - x')), \tag{3}$$

using the surjectivity just proved. For $n \in \mathbb{Z} \times \Gamma$ define

$$F(n) = B_0(nx, n\pi' x') - n(B_0(x, \pi' x') - y) - B_0(x, \pi'(nx' - x')) - y. \tag{4}$$

Here $n\pi' x'$ means the action on the Y'-module, whereas $\pi'(nx')$ also includes the extension cocycle.

For $g \in \Gamma'$, bilinearity, commutation of g with φ, and (3) give

$$\begin{aligned}(\varphi - 1)F(g) = {} & B_0(gw - w, \pi' g(\varphi x' - w')) && (7.13)\\ & - B_0(\varphi x - w, \pi'(g\varphi x' - \varphi x')) && (7.14)\\ & - B_0(w - x, \pi'(gw' - w')) && (7.15)\\ & + B_0(gx - x, \pi' g(w' - x')). && (5)\end{aligned}$$

For completeness, the extension identity used in this expansion is $\pi(gv) = g\pi(v) + \alpha(\pi_C v)(g)$, with the analogous primed identity; φ fixes the constant quotient. Thus all extra extension terms cancel because x, w have the same constant quotient and x', w' have the same constant quotient. The four summands of (5) are exactly the four assumed safe moves. In the second-side convention a safe move by v' with v at rest means safety of $(gv - v, gv')$ for every $g \in \Gamma'$. Consequently (1) supplies a unique function $\kappa : \Gamma' \to (\mathbb{Z}/\ell^m)(1)$ such that

$$F(g) - 1 \otimes \kappa(g) \in I_m. \tag{6}$$

A second expansion of (4), using the same extension identity, gives

$$\begin{aligned}1 \otimes \ell^{e_1+e_2+e_3} b_P(\alpha(c)(g), g\alpha'(c')(h)) = {} & B_0(gx - x, \pi' g(hx' - x')) && (7.16)\\ & + gF(h) - F(gh) + F(g). && (7)\end{aligned}$$

The first term on the right is small by the additional cocycle-difference safety hypothesis. Substitute (6) in (7); Galois stability of I_m and (2) then give the exact equality

$$\ell^{e_1+e_2+e_3} b_P(\alpha(c)(g), g\alpha'(c')(h)) = g\kappa(h) - \kappa(gh) + \kappa(g). \tag{8}$$

We check continuity rather than treating κ as an arbitrary cochain. Near 1 the two cocycles vanish modulo ℓ^m, and (4) reduces to

$$F(g) = gy - y - B_0(x, g\pi' x' - \pi' x').$$

Both terms are small near 1 by the finite-coordinate observation above; hence (2) and (6) show that κ vanishes near 1. Equation (8) and the vanishing of $\alpha'(c')(h)$ modulo ℓ^m for h near 1 give $\kappa(gh) = \kappa(g)$ on one fixed identity neighbourhood, for every g. Thus κ is locally constant. The cup cocycle on Γ is also continuous, by continuity of the lattice actions and the cocycles. Its cohomology class in $H^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$ restricts to zero on the open subgroup Γ'. Restriction followed by corestriction multiplies this class by $[\Gamma : \Gamma']$, whereas its values show that it is killed by ℓ^m. These integers are coprime by the chart hypothesis, so the class is zero. This proves item 1 at exponent $e_1 + e_2 + e_3$; multiplication of the cochain by $\ell^{e-e_1-e_2-e_3}$ proves every larger exponent, including $m = 0$.

The actual lift-ideal suppliers. Write $A = A_{\mathrm{inf}}$, $S = \mathcal{S}_{\mathrm{ch}}$, and $J_l(r, i) = J_l(i; r)$. The actual standard-order and common chart-ring lifts are Lemma 7.7.21; the scalar operations, with arbitrary natural modulus r, are Lemma 7.7.23. In particular the following are literal projections of its clauses (b)–(e):

$$\varphi^l(s) \in J_l(r, \sigma^l i) \Longleftrightarrow s \in J_0(r, i), \qquad fE^\omega \in J_1(r, i) \quad (f \in J_0(r, i),\ u \le \omega). \tag{19}$$

$$\ell^c z^n \in J_1(r, \sigma i), \quad \ell a \le n, \quad c + v_\ell(n!) \le b + v_\ell((\ell a)!) \Longrightarrow \ell^b z^a \in J_0(r, i). \tag{20}$$

$$\ell^c z^a E^n \in J_1(r, i) \Longrightarrow \ell^{c+dn+v_\ell(n!)} z^{a+n} \in J_1(r, i). \tag{21}$$

$$\ell^c z^n \in J_0(r, i), \quad m \le n, \quad c + n + v_\ell(n!) \le b + m + v_\ell(m!) \Longrightarrow \ell^b z^m \in J_0(r, i). \tag{22}$$

The forward letter in (20) is σi; the formulas include every prime, including two, and zero exponents.

We now prove item 2. By the τ^h lattice comparison, each $J_0(\ell^{\kappa+1}, i)$ contains τ^h for a common h, as well as $\ell^{\kappa+1}$. Modulo ℓ the tilt valuation ring shows that z is in the radical of (ℓ, τ); hence some common power z^a lies in all these finitely many ideals. Choose the least such a. Since $z^e \in (E, \ell)\mathfrak{S}$, (19) and a binomial expansion give

$$z^{a+e(\omega+\kappa)} \in J_1(\ell^{\kappa+1}, \sigma i) \quad \text{for every } i.$$

Every term either contains $z^a E^\omega$ or contains $\ell^{\kappa+1}$. Put $b = \lfloor (a + e(\omega + \kappa))/\ell \rfloor$. Equation (20), with both valuation exponents equal to zero, gives z^b in every level-zero ideal: there is no multiple of ℓ strictly between ℓb and $a + e(\omega + \kappa)$, so their factorial valuations are equal. Minimality implies $a \le b$, hence $(\ell - 1)a \le e(\omega + \kappa)$, as required.

For item 3 put $q = s + t$, $r = \ell^{q+1}$, and choose a from item 2 with $(\ell - 1)a \le (\ell - 2)q$. Legendre's inequality $(\ell - 1)v_\ell(a!) \le a$ implies

$$(\ell - 2)(a + v_\ell(a!)) \le (\ell - 1)a \le (\ell - 2)q,$$

so $a + v_\ell(a!) \le q$. For every $0 \le m \le q$, multiply z^a up to $z^{\max(a,m)}$ and apply (22); this proves $\ell^{q-m} z^m \in J_0(r, i)$. In particular $\ell^q \in J_0(\ell^{q+1}, i)$. Write $I = J_0(0, i)$; the last membership says $\ell^q \in I + (\ell^{q+1})$, so $(1 - \ell u)\ell^q \in I$ for some $u \in A$. The factor is a unit by ℓ-adic completeness, and therefore the modulus can be removed. Finally

$$\ell^s E^t = \ell^s (z + \ell)^t = \sum_{m=0}^{t} \binom{t}{m} \ell^{q-m} z^m \in I.$$

This is exactly the asserted cokernel killer. The argument includes $s = t = 0$.

The forward-letter bound. We also prove the additional output in item 4, used by Proposition 7.7.27. Suppose the jumps at $\sigma^t i$ are at most ω' for $1 \le t \le J$. Equations (19)–(21) and (20) give the one-step implication

$$\ell^c z^n \in J_0(r, \sigma j) \implies \ell^{c+d\omega'+v_\ell(\omega'!)} z^{\lfloor (n+\omega')/\ell \rfloor} \in J_0(r, j). \tag{23}$$

For (23), multiply by $E^{\omega'}$ to reach level one, apply (21), then (20) with $a = \lfloor (n+\omega')/\ell \rfloor$. The factorial valuations of $n + \omega'$ and ℓa are equal because there is no intervening multiple of ℓ. Starting with the common z^a of item 2 modulo $r = \ell^{\kappa+1}$ and iterating back from $\sigma^J i$ to i, this yields $\ell^c z^n \in J_0(r, i)$ with

$$c \le J(d\omega' + v_\ell(\omega'!)), \qquad n \le a/\ell^J + \omega'/(\ell - 1). \tag{24}$$

The finite geometric sum proves the second inequality; the hypotheses concern the *forward* letters $\sigma^t i$, not the inverse letters.

Legendre's inequality and $(\ell - 1)a \le e(\omega + \kappa)$ imply

$$c \le J(d\omega' + \omega'/(\ell - 1)), \qquad \frac{a}{\ell^J} \le \frac{2e\omega}{\ell^{J+1}} + \frac{\kappa}{4},$$

where the second inequality uses $\ell \le 2(\ell - 1)$ and $8e \le \ell^{J+1}$. Since $v_\ell(n!) \le n$, (24) gives

$$c + n + v_\ell(n!) \le \frac{1}{2}\left(2Jd\omega' + \frac{(2J+4)\omega'}{\ell - 1} + \frac{8e\omega}{\ell^{J+1}}\right) + \frac{\kappa}{2} \le \kappa.$$

Equation (22) removes the remaining z^n, giving $\ell^\kappa \in J_0(\ell^{\kappa+1}, i)$. The unit $1 - \ell u$ argument above removes the modulus. Thus $\ell^\kappa \in J_0(0, i)$, exactly the asserted killer. Zero ω' and $J = 0$ are permitted whenever the displayed hypotheses hold. □

Proposition 7.7.16. *Let ℓ be a prime and Γ a group with an open embedding $j\colon \Gamma \to G_\ell$.*

1. *There is a Kummer chart for Γ.*

2. *Let k be a Kummer chart for Γ, with ring $\mathfrak{S} = \Lambda[[z]] \subset A_{\mathrm{inf}}$ and subgroup Γ'; let ϵ be a compatible system of ℓ-power roots of unity, $\mu = [\epsilon] - 1$, $\tau = \varphi^{-1}(\mu)$, and $B^+_{\mathrm{max}} = A_{\mathrm{max}}[1/\ell] \subset B_{\mathrm{max}}$, which contains A_{inf} and maps to B_{dR}. Let T be a lattice over a coefficient ring $\mathcal{O}$ (viewed over $\mathbb{Z}_\ell$) with an action of Γ that is crystalline over Γ', and let $h \ge 0$ be an integer such that every Hodge–Tate jump of T, as a $\mathbb{Z}_\ell$-representation of Γ', lies in $[-h, h]$. Call an A_{inf}-linear map $\lambda\colon W(\mathbb{C}^\flat_\ell) \otimes_{\mathbb{Z}_\ell} T \to B_{\mathrm{dR}}$ a* crystalline form *if $\lambda(1 \otimes gt) = j(g)\lambda(1 \otimes t)$ for all $g \in \Gamma'$, $t \in T$, and $t \mapsto \lambda(1 \otimes t)$ is the composite of a $\mathbb{Z}_\ell$-linear map $T \to B_{\mathrm{max}}$ with $B_{\mathrm{max}} \to B_{\mathrm{dR}}$. Then there are a finitely generated free $\mathfrak{S}$-submodule $N \subset A_{\mathrm{inf}}[1/\tau] \otimes_{\mathbb{Z}_\ell} T$ and its A_{inf}-span $M = A_{\mathrm{inf}}N$ such that:*

   (a) *$E^h \varphi(N) \subseteq N$;*

   (b) *$\tau^h M \subseteq A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} T$ and $\tau^h(A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} T) \subseteq M$;*

   (c) *$gM \subseteq M$ for every $g \in \Gamma$ with $g\Gamma' g^{-1} \subseteq \Gamma'$;*

   (d) *let $d \in A_{\mathrm{inf}}$ be such that, for all $m \ge 0$ and $r \in A_{\mathrm{inf}}$, if d divides $E^m r$ in B^+_{max} then d divides r in A_{inf}; if $x \in M$ and there is $n \ge 0$ such that $E^n \lambda(x)$ lies in the image of $d\,B^+_{\mathrm{max}}$ in B_{dR} for every crystalline form λ, then $x \in dM$.*

3. *Use the forms, the rings $S = \mathcal{S}_{\mathrm{ch}}, S_B = \mathcal{S}_B$, the value field $\mathcal{K}_0$, and the formulas for $M(T), N(T)$ in Definition 7.7.2, (BF2)–(BF5); these formulas do not require a choice of a letter root or jump sets. The modules in item 2 can be chosen to be these very modules $N(T)$ and $M(T)$. For $l \geq 0$ put*

$$V_l = \{x \in \varphi^l M(T) : \lambda(x) \in \mathcal{K}_0 \text{ for every form } \lambda\}.$$

Then:

(a) *For every S-linear map $f : A_{\mathrm{inf}} \to S$, every finite family x_i of ambient vectors with $\lambda(x_i) \in \mathcal{K}_0$ for every form, each satisfying $a_i x_i \in A_{\mathrm{inf}} \otimes T$ for some nonzero $a_i \in A_{\mathrm{inf}}$, and every $c_i \in A_{\mathrm{inf}}$,*

$$\sum_i c_i x_i = 0 \quad \Longrightarrow \quad \sum_i f(c_i) x_i = 0.$$

(b) *For every $l \geq 0$, V_l is the S-span of $\varphi^l N(T)$. For every crystalline form λ, $E^h \lambda(N(T)) \subset S_B$.*

(c) *If x is an ambient vector with $\tau^a x \in A_{\mathrm{inf}} \otimes T$ for some $a \geq 0$, then*

$$x \in M(T) \iff \exists n \geq 0 \ \forall \lambda \text{ crystalline} : E^n \lambda(x) \in B^+_{\max},$$
$$x \in N(T) \iff \exists n \geq 0 \ \forall \lambda \text{ crystalline} : E^n \lambda(x) \in S_B.$$

(d) *Let $l, A, \delta \geq 0$ be integers with $3e \leq \ell^l$ and $eA \leq \ell^l \delta$. For every $x \in V_l$,*

$$\ell^{A\delta} x \in \mathrm{span}_S\{y \in V_l : \ j(g)\lambda(y) - \lambda(y) \in \mathrm{Fil}^A \text{ for every form } \lambda \text{ and every } g \in \Gamma'\}.$$

Proof. Put $p = \ell$, $A = A_{\mathrm{inf}}$, $S = \mathfrak{S}$ and $B^+ = B^+_{\max}$. Item 1 is Lemma 7.7.17. For the remaining assertions fix a chart, T and h as in item 2, and write $r = \mathrm{rank}_{\mathbb{Z}_p} T$. We first establish the canonical A-lattice, its exact τ bounds and its normaliser stability.

Let K be the fixed field of $j(\Gamma')$, and put $K_0 = \mathrm{Frac}\,\Lambda$. The chart conditions identify $j(\Gamma')$ with the fixer of Λ and $\theta(z)$. Thus $K = \mathrm{Frac}(\Lambda)(\theta(z))$. Weierstrass preparation applied to the chart series $X^e a(X) + p$ replaces it by an associate which is an Eisenstein polynomial of degree e. It follows that $\theta(z)$ is a uniformiser of K, that $\Lambda = W(k_K)$ and that S is the usual Breuil–Kisin ring of K. This argument uses the chart map itself, not a new choice of compatible roots. Forgetting the coefficient ring makes T finite free over $\mathbb{Z}_p$. Continuity follows from Lemma 7.4.6(1). Crystallinity is over this same K: the $B_{\max}$-invariant basis and the standard period-ring comparison $\varphi(B_{\max}) \subset B_{\mathrm{cris}}$ give a B_{cris}-invariant basis after applying Frobenius. Extending that comparison to B_{dR} and using Theorem A.3.16 shows that $T[1/p]$ is de Rham and that $D_{\mathrm{dR}} = K \otimes_{K_0} D_{\mathrm{cris}}$.

Let $\mathfrak{M}$ be the module of Theorem A.3.12, and let $i_0 : S \to A$ be the chart map. Its embedding is $i_{\mathrm{B}} = \varphi_A i_0$: both the coefficients and the variable are carried through Frobenius. Put

$$\mathcal{M}_{\mathrm{B}} = A \otimes_{S, i_{\mathrm{B}}} \mathfrak{M}, \qquad \mathcal{M}_0 = (\varphi_A^{-1} \otimes 1)\mathcal{M}_{\mathrm{B}} \subset A[1/\tau] \otimes_{\mathbb{Z}_p} T. \tag{1}$$

Here the common ambient identification is the one in Theorem A.3.12(2); inverse Frobenius is applied to scalar coordinates in the fixed $\mathbb{Z}_p$-module T. In particular $\mathcal{M}_0$ is finite free over A and becomes the full ambient lattice after inverting τ. Since $i_{\mathrm{B}}(E)$ is associated to $\varphi(\xi) = \widetilde{\xi}$, the Breuil–Kisin Frobenius becomes an isomorphism on $\mathcal{M}_{\mathrm{B}}$ after inverting $\widetilde{\xi}$. Thus $\mathcal{M}_{\mathrm{B}}$ is a

finite free Breuil–Kisin–Fargues module in exactly the convention of Theorem A.3.14. Its étale lattice is the given T, by Theorem A.3.12(1). Write

$$\Xi = D_{\mathrm{dR}}(T[1/p]) \otimes_K B_{\mathrm{dR}}^+ \subset T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

The crystalline comparison Theorem A.3.12(3), followed by extension to B_{dR}^+, identifies the pair of $\mathcal{M}_{\mathrm{B}}$ under Theorem A.3.14 with (T, Ξ).

The Hodge jumps in the hypothesis lie in $[-h, h]$. The de Rham/Hodge–Tate comparison Theorem A.3.11 therefore gives

$$\xi^h \Xi \subset T \otimes B_{\mathrm{dR}}^+ \subset \xi^{-h} \Xi. \tag{2}$$

Choose a filtration-adapted basis d_i with jumps j_i. The graded comparison says that the reductions of $\xi^{-j_i} d_i$ form a basis of the residue space of $T \otimes B_{\mathrm{dR}}^+$. Hence these vectors form a basis over the discrete valuation ring B_{dR}^+, by their determinant. Since $|j_i| \leq h$, both inclusions in (2) follow. The symmetry of the interval makes a convention change of sign immaterial here.

Multiplication of $\mathcal{M}_{\mathrm{B}}$ by μ^b, for an integer b, gives a finite free Breuil–Kisin–Fargues module in the same ambient space. Indeed $\varphi(\mu)/\mu = \widetilde{\xi}$, so its Frobenius becomes an isomorphism after inverting $\widetilde{\xi}$. Its étale lattice is still T, because μ is a unit in $W(\mathbb{C}_p^\flat)$. Its de Rham lattice is $\xi^b \Xi$, since $\mu/\xi = \tau$ is a unit at θ. By (2), the identity on T is a morphism of pairs from $\mu^h \mathcal{M}_{\mathrm{B}}$ to $A \otimes T$, and from $A \otimes T$ to $\mu^{-h} \mathcal{M}_{\mathrm{B}}$. The full-faithfulness assertion of Theorem A.3.14 makes these actual inclusions in the common ambient module. Applying inverse scalar Frobenius gives

$$\tau^h \mathcal{M}_0 \subset A \otimes T \subset \tau^{-h} \mathcal{M}_0. \tag{3}$$

This proves exactly both τ bounds, without an unspecified exponent depending on the lattice.

We supply the scalar intersection calculation needed to identify the canonical module. For every $b \in A[1/\mu]$,

$$b \in A \quad \Longleftrightarrow \quad \varphi^n(b) \in B_{\mathrm{dR}}^+ \quad (n \geq 0). \tag{4}$$

First,

$$\bigcap_{n \geq 0} \ker(\theta \varphi^n : A \to \mathcal{O}_{\mathbb{C}_p}) = \mu A. \tag{5}$$

For the nontrivial inclusion, write an element a of the intersection as $a = \xi c$, $c \in A$, using $\ker \theta = \xi A$. For $n \geq 1$, the geometric-sum formula for the canonical $\xi = \mu/\tau$ gives $\theta(\varphi^n \xi) = p \neq 0$. Thus c satisfies every positive-residue equation. By Lemma 7.7.18, $c \in \tau B^+$; Lemma 7.7.30, with $m = 1$ and $n = h = 0$, gives $c \in \tau A$. Hence $a \in \xi \tau A = \mu A$. Conversely every $\varphi^n \mu$ has residue zero, proving (5).

For (4), write $b = a/\mu^q$. The right-hand tests put every $\varphi^n a$ in Fil^q. If $q > 0$, (5) first gives $a = \mu a_1$, $a_1 \in A$. Each $\varphi^n \mu$ has de Rham order one, because $\log(1 + \varphi^n \mu) = p^n t$; cancellation therefore leaves $\varphi^n a_1 \in \mathrm{Fil}^{q-1}$ for all n. Repeating gives $a \in \mu^q A$, as required. For $q = 0$ nothing is needed. The converse follows from $\varphi(A) = A$.

Next, for every $n \geq 0$,

$$(\varphi^n \mathcal{M}_{\mathrm{B}}) \otimes_A B_{\mathrm{dR}}^+ = \Xi. \tag{6}$$

The Frobenius comparison matrix of $\mathcal{M}_{\mathrm{B}}$ and its inverse have denominators only powers of $\widetilde{\xi}$. For its further twists the possible denominators are $\varphi^j \widetilde{\xi}$, $j \geq 0$. Each is a unit at θ, its θ-value being a nonzero power of p up to a unit. Thus successive Frobenius twists span the same de Rham lattice, which proves (6) from the case $n = 0$.

We also identify precisely what the forms test. Choose a basis of crystalline periods and let P be its matrix in $T \otimes B_{\mathrm{cris}}$. Its inverse rows give crystalline forms in the precise sense of

(BF3). Here the extension to the stated ambient module can be made as follows. The injection $A \subset B_{\mathrm{dR}}$ extends to its fraction field. The ambient module is A-torsion free, since T is finite free and $W(\mathbb{C}_p^\flat)$ is a domain. Its localization is therefore a vector space over $\mathrm{Frac}(A)$ containing $\mathrm{Frac}(A) \otimes T$. Extend each inverse row linearly from this finite-dimensional subspace to the whole localized space by choosing a vector-space complement, and restrict back to the ambient module. Its values on T lie in B_{cris} and have the required equivariance, since the columns of P are invariant. Those are exactly the additional conditions in (BF3); no continuity or equivariance on an unspecified complement is required there. In matrix expressions we extend only the restriction of a form to T, B_{dR}-linearly. If λ is any form, the row λP is fixed by G_K, hence has entries in K by Theorem A.3.16. Conversely these rows span all restrictions of forms to the fractional ambient span of T. Therefore

$$\lambda(y) \in B_{\mathrm{dR}}^+ \text{ for every form } \lambda \quad \Longleftrightarrow \quad y \in \Xi. \tag{7}$$

Only restrictions to that fractional span are used; no uniqueness of an arbitrary form on the larger ambient module is asserted.

An element of $\mathcal{M}_0$ satisfies the conditions of (BF5), by (3), (6) and (7). Conversely let x satisfy (BF5). Since a power of τ puts x in $A \otimes T$, (3) puts it in $\mathcal{M}_0[1/\tau]$. In an A-basis m_i of $\mathcal{M}_0$, write $x = \sum_i a_i m_i$, $a_i \in A[1/\tau]$. Equations (6)–(7) imply

$$\varphi^{m+1}(a_i) \in B_{\mathrm{dR}}^+ \quad (i,\ m \geq 0).$$

Apply (4) to $\varphi(a_i) \in A[1/\mu]$. It gives $\varphi(a_i) \in A$, and hence $a_i \in A$. Thus

$$M(T) = \mathcal{M}_0. \tag{8}$$

In particular the canonical $M(T)$ is finite free over A and satisfies item 2(b).

Normaliser stability follows directly from this canonical description. If $g\Gamma' g^{-1} \subset \Gamma'$ and λ is a form, then $x \mapsto j(g)^{-1}\lambda(gx)$ is again a form: for $h \in \Gamma'$, use $ghg^{-1} \in \Gamma'$ in its equivariance identity. The actions commute with Frobenius and preserve de Rham filtration. Also $g(\tau)/\tau$ is a unit of A, by the cyclotomic character and its binomial power series with unit constant term. Consequently g preserves the τ-saturated ambient span and all the defining filtration tests. This proves $gM(T) \subset M(T)$, namely item 2(c).

We next record two consequences of the scalar chart assertions of Lemma 7.7.22. We use the chart-linear retraction of item 4, $q_0 : A \to S$, $q_0|_S = \mathrm{id}_S$, and its S_B-linear extension. First,

$$B^+ \cap \mathrm{Frac}(S_B) = S_B. \tag{9}$$

Indeed, suppose $c, cd \in S_B$, $c \neq 0$, and $d \in B^+$. For every S-linear endomorphism u of B^+ preserving $A_{\max}$, and every $b \in B^+$, commutation with multiplication by c and cd gives $c\,u(db) = u(cdb) = cd\,u(b)$. Cancellation in the domain B^+ shows that multiplication by d commutes with u. This is exactly the definition of $d \in S_B$, proving (9). Also the extension of q_0 fixes S_B, since it is S_B-linear and takes 1 to 1.

Second, the positive-Frobenius integrality test extends from (4) to

$$b \in B_{\max}, \qquad b \in B^+ \quad \Longleftrightarrow \quad \varphi^n b \in B_{\mathrm{dR}}^+ \quad (n \geq 0). \tag{10}$$

For the nontrivial direction, first let $b \in B^+$ satisfy $\theta\varphi^n b = 0$ for every $n \geq 0$. By divisibility reflection in Lemma 7.7.22(2), $b/\xi \in B^+$. For $n \geq 1$, $\theta(\varphi^n \xi) = p \neq 0$, so b/ξ satisfies all the positive-residue equations. The positive-residue detector Lemma 7.7.18 therefore gives $b/\xi \in$

τB^+, hence $b \in \xi\tau B^+ = \mu B^+$. Each $\varphi^n\mu$ has de Rham order exactly one: $\log(1+\varphi^n\mu) = p^n t$, and the linear coefficient is nonzero. Consequently induction gives

$$\varphi^n b \in \mathrm{Fil}^m \ (n \geq 0) \quad \Longrightarrow \quad b \in \mu^m B^+ \qquad (b \in B^+,\ m \geq 0). \tag{11}$$

For $b = a/\mu^m \in B_{\max}$, the right side of (10) puts every $\varphi^n a$ in Fil^m; (11) cancels the denominator. The converse of (10) follows from $\varphi(B^+) \subset B^+$.

We make explicit the finite-dimensional Frobenius action on crystalline forms; inverse Frobenius on $B_{\max}$ is not required. Let $\lambda_1, \ldots, \lambda_r$ be the inverse rows of P. As invariant vectors for the dual representation they are independent over B_{cris}; the dimension bound of Theorem A.3.16, applied to that continuous dual representation, says that they form a K_0-basis of its crystalline invariant space. Frobenius commutes with Galois and preserves B_{cris}, so for $L = P^{-1}$ one has $\varphi(L) = FL$, $F \in M_r(K_0)$. Applying φ to $LL^{-1} = I$ shows that $\varphi(L)$ is nonsingular. Hence $\det F \neq 0$, and $F \in \mathrm{GL}_r(K_0)$. Since Frobenius on K_0 is an automorphism, this is a bijective semilinear action. The same conclusion holds for every positive iterate.

Every crystalline form is a K_0-linear combination of these rows. To check the assertion also for the $B_{\max}$-valued definition, apply Frobenius to its values: $\varphi(B_{\max}) \subset B_{\mathrm{cris}}$. This last inclusion follows from $\varphi(B^+) \subset B^+_{\mathrm{cris}}$ and $1/\varphi(\mu) \in B_{\mathrm{cris}}$. To see the latter without assuming an inverse in B^+_{cris}, put $c = \varphi(\mu)$. The series $R_c = \sum_{m\geq 1}(-1)^{m+1}c^{m-1}/m$ converges in B^+_{cris}: divided powers of $c \in \ker\theta$ make its mth term a divided power times a scalar of valuation tending to infinity. It satisfies $cR_c = \log(1+c) = pt$, so $c^{-1} = R_c/(pt) \in B_{\mathrm{cris}}$. The resulting crystalline row is a K_0-linear combination of the Frobenius-twisted basis rows. Apply the inverse automorphism of K_0 to those finitely many coefficients and use injectivity of Frobenius to recover the original row. Together with (7), we have proved that all forms are K-linear combinations of the λ_i, whereas all crystalline forms are K_0-linear combinations.

For $x \in M(T)$ and every crystalline form λ, one has the uniform bound

$$E^h\lambda(x) \in B^+. \tag{12}$$

Indeed $x \in T \otimes B^+_{\mathrm{dR}}$, since (3) bounds its denominator by a power of the de Rham unit τ. The inclusion $T \otimes B^+_{\mathrm{dR}} \subset \xi^{-h}\Xi$ in (2), and the fact that every λ_i is integral on Ξ, show $\lambda(x) \in \mathrm{Fil}^{-h}$. This proves the $n = 0$ test for $E^h\lambda(x)$. For $n \geq 1$, write $\lambda^{(n)}$ for the crystalline form whose values on T are φ^n of those of λ. Then

$$\varphi^n(E^h\lambda(x)) = \varphi^n(E)^h\,\lambda^{(n)}(\varphi^n x) \in B^+_{\mathrm{dR}}.$$

The defining tests of $M(T)$ apply to the last factor, and $\varphi^n(E)$ is a de Rham unit. The argument is legitimate in $B_{\max}$: $x \in A[1/\tau] \otimes T$, and $1/\tau = \xi/\mu \in B_{\max}$. Equation (10) proves (12).

Conversely, if x is in the τ-saturated ambient span and, for one $m \geq 0$, all crystalline forms satisfy $E^m\lambda(x) \in B^+$, apply each positive Frobenius power to these inclusions. The factors $\varphi^n(E)$, $n \geq 1$, are de Rham units. Bijectivity of Frobenius on the crystalline-form space therefore gives all the tests $\lambda_i(\varphi^n x) \in B^+_{\mathrm{dR}}$. Their K-linear combinations give the same assertion for every form. Hence $x \in M(T)$. With (12), this proves the first equivalence in item 3(c).

Let N_0 be the S-span of the inverse-Frobenius images of a basis of $\mathfrak{M}(T)$ in (1). The transport of its Breuil–Kisin Frobenius gives

$$N_0 \text{ finite free over } S, \qquad AN_0 = M(T), \qquad \varphi(N_0) \subset N_0[1/E], \tag{13}$$

and the Frobenius matrix belongs to $\mathrm{GL}_r(S[1/E])$. All statements concern the fixed ambient embedding: applying inverse scalar Frobenius to the matrix equality for $\mathfrak{M}(T)$ changes i_{B} to i_0

and its Eisenstein element to the actual E, up to a unit. By Lemma 7.7.19, every crystalline form takes N_0 into $\mathrm{Frac}(S_B)$. Thus every form takes N_0 into $\mathcal{K}_0$.

We first prove item 3(a), independently of the identification of $N(T)$. Apply a crystalline basis form to a relation $\sum_i c_i x_i = 0$ there. Its values at all x_i lie in $\mathcal{K}_0$; item 3 of Lemma 7.7.22, with $a = 0$, says that the exact scalar relation remains exact on replacing c_i by $f(c_i)$. Since the crystalline basis forms separate the fractional ambient span, $\sum_i f(c_i)x_i = 0$. The denominator hypothesis in item 3(a) places every x_i in this fractional span, so no assertion about arbitrary vectors outside that span is used.

It follows that

$$N(T) = N_0. \tag{14}$$

The inclusion $N_0 \subset N(T)$ follows from (8), (13) and the value-field conclusion. For the reverse inclusion choose an S-basis $n_1, \dots, n_r$ of N_0 and write $x \in N(T)$ as $x = \sum_i a_i n_i$, $a_i \in A$. Apply the just-proved relation law to $x - \sum_i a_i n_i = 0$ and the retraction q_0. It gives $x = \sum_i q_0(a_i) n_i$, which is in N_0. This proves finite freeness and the asserted canonical choice of both modules.

Combining (12) with the analytic value-field statement and (9) gives

$$E^h \lambda(N(T)) \subset S_B \tag{15}$$

for every crystalline form. We next prove the second equivalence of item 3(c). Its forward direction is (15). Conversely suppose the indicated τ-saturated x satisfies $E^m \lambda(x) \in S_B$ for all crystalline λ. The already-proved first equivalence gives $x \in M(T)$. In the basis n_i, write $x = \sum_i a_i n_i$, $a_i \in A$, and put

$$C = (\lambda_i(n_j))_{i,j}.$$

This matrix is nonsingular over $\mathrm{Frac}(S_B)$, since it is the change-of-basis matrix between two bases of the period comparison space. The column of all $\lambda_i(x)$ lies in $\mathrm{Frac}(S_B)^r$, so $a = C^{-1}(\lambda_i(x))_i$ does as well. Equation (9) puts every a_i in S_B. The extension of q_0 fixes S_B, and agrees on A with $q_0 : A \to S$; hence $a_i = q_0(a_i) \in S$. Thus $x \in N(T)$.

For $l \geq 0$, repeated use of (13) puts $\varphi^l N(T)$ inside $N(T)$ with denominators finite products of $\varphi^j(E)$. Those products belong to S, so every form takes $\varphi^l N(T)$ into $\mathcal{K}_0$. These vectors form an A-basis of $\varphi^l M(T)$. If $x \in V_l$, write it in that basis and apply the same q_0-relation argument. This proves the first assertion of item 3(b); its second assertion is (15).

For $x \in N(T)$, the bound (3) gives

$$E^h \varphi(x) \in \tau^{-h}(A \otimes T),$$

because μ is a unit times $E\tau$. Moreover, for $m \geq 0$ and every form λ,

$$\lambda(\varphi^{m+1}(E^h \varphi x)) = \varphi^{m+1}(E)^h \lambda(\varphi^{m+2} x) \in B^+_{\mathrm{dR}}.$$

Thus $E^h \varphi x \in M(T)$. By (13) its form values belong to $\mathcal{K}_0$. It is therefore in $N(T)$, proving item 2(a) with the exact exponent h.

The argument so far applies to $T^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$ as well. It is crystalline with the same symmetric Hodge bound, by Theorem A.3.9; the tensor functor in Theorem A.3.12 identifies its embedded lattice with the dual lattice of $M(T)$, and inverse scalar Frobenius preserves this evaluation pairing. This also identifies the dual basis $n_j^\vee$. More explicitly, let Q be the coordinate matrix of n_j in a $\mathbb{Z}_p$-basis of T, so $C = P^{-1}Q$. In the dual basis of $T^\vee$, the vectors $n_j^\vee$ have coordinate matrix $Q^{-\mathsf{t}}$, and the crystalline forms dual to the invariant period basis have row matrix P^{t}. Their value matrix is therefore $P^{\mathsf{t}} Q^{-\mathsf{t}} = C^{-\mathsf{t}}$. Applying (12) to these dual vectors proves

$$E^h C^{-1} \in M_r(B^+). \tag{16}$$

This argument also checks the dual lattice intrinsically: the evaluation pairing of the two published tensor-dual lattices has values in A, and is perfect before and after inverse scalar Frobenius.

We can now prove item 2(d). For $x = \sum_j a_j n_j \in M(T)$, $a_j \in A$, its crystalline-value column is Ca. The assumption in that item gives $E^m Ca \in d(B^+)^r$ for one $m \geq 0$. Multiplication by (16) gives $E^{m+h}a \in d(B^+)^r$. The stipulated cancellation property of d, applied separately to each a_j, gives $a_j \in dA$. Consequently $x \in dM(T)$. This argument includes $d = 0$: the matrix C is nonsingular and then $x = 0$.

It remains to prove the congruence span, keeping track of its uniformity for every form. In Lemma 7.7.20 take

$$Z = E^h C, \qquad Z' = E^h C^{-1}. \tag{17}$$

Equations (15)–(16) give $Z \in M_r(S_B)$, $Z' \in M_r(B^+)$, and $Z'Z = E^{2h} I_r$. Choose the chart element t_0 with $t_0 z = [\varepsilon]z$. It belongs to $j(\Gamma')$ and fixes Λ pointwise. Canonical stability applied to t_0 and t_0^{-1} gives

$$t_0(n_1, \ldots, n_r) = (n_1, \ldots, n_r)H, \qquad H \in \mathrm{GL}_r(A).$$

The equivariance of the forms gives $t_0 C = CH$. The ratio $t_0 E/E$ is an A-unit, since both generate $\ker\theta$; hence

$$t_0 Z = ZG, \qquad G = (t_0 E/E)^h H.$$

Thus all the hypotheses of Lemma 7.7.20 hold with its exponent H equal to $2h$. Apply it with $b = h$. Its factorization becomes

$$\varphi^l C = C_0 V, \tag{18}$$

after cancellation of $\varphi^l(E)^h$, where $C_0 \in \mathrm{GL}_r(B^+)$ is fixed by every g fixing Λ, and V is the evaluated invertible formal matrix. Let $F_l \in \mathrm{GL}_r(K_0)$ be determined by $(\lambda_i^{(l)})_i = F_l(\lambda_i)_i$. Therefore

$$(\lambda_i(\varphi^l n_j))_{i,j} = F_l^{-1} C_0 V. \tag{19}$$

For a coefficient column $y \in S^r$ belonging to the set $\mathcal{Y}_{A,V}$ of the radius lemma, put $x_y = \sum_j y_j \varphi^l n_j$. Then $x_y \in V_l$ by item 3(b), and $Vy - n_y \in E^A(B^+)^r$ for a natural-number column n_y. Equation (19) shows that each $\lambda_i(x_y)$ differs modulo Fil^A from the corresponding entry of $F_l^{-1} C_0 n_y$. These entries are fixed by $j(\Gamma')$: that group fixes Λ, K_0, and the natural numbers. Multiplication by $F_l^{-1} C_0$ preserves Fil^A, since its entries belong to B_{dR}^+. Every form is a K-linear combination of the λ_i; these coefficients are fixed by $j(\Gamma')$ and have de Rham order zero. Since the Galois action also preserves Fil^A, we obtain

$$j(g)\lambda(x_y) - \lambda(x_y) \in \mathrm{Fil}^A \quad (g \in \Gamma',\ \lambda \text{ any form}).$$

The final assertion of Lemma 7.7.20, with precisely the inequalities $3e \leq p^l$ and $eA \leq p^l \delta$, says that these columns y span a submodule containing $p^{4\delta} S^r$. The S-linear map $S^r \to V_l$, $y \mapsto x_y$, is an isomorphism by item 3(b). Its image gives exactly item 3(d). All arguments for rank zero use empty matrices and give the zero modules, so they also prove that case.

□

Lemma 7.7.17. *Let ℓ be prime and $j : \Gamma \to G_\ell$ an open embedding. There is a Kummer chart for Γ which fits Γ. It can be chosen with $\Gamma' = \Gamma$, with coefficient field the maximal unramified subfield of the finite extension $K/\mathbb{Q}_\ell$ fixed by $j(\Gamma)$, and with $\theta(z)$ a uniformiser of K.*

Proof. Write $p = \ell$. The open subgroup $j(\Gamma)$ is G_K for a finite extension $K/\mathbb{Q}_p$. Let K_0 be its maximal unramified subfield, let its residue field be $\mathbb{F}_{p^f}$, and let e be its ramification index.

Thus $ef = [K : \mathbb{Q}_p] = [G_p : j(\Gamma)]$. The canonical Teichmüller copy $\Lambda = W(\mathbb{F}_{p^f})$ in A_{inf} is carried by θ isomorphically onto $\mathcal{O}_{K_0}$: the prime-to-p roots of unity lift uniquely from the residue field. In particular G_K fixes this copy pointwise.

Choose a uniformiser π of K. If $e = 1$, choose $\pi = -p$. Choose compatible roots $\pi_n^p = \pi_{n-1}$, $\pi_0 = \pi$, in $\overline{\mathbb{Q}}_p$, and let $\varpi^\flat$ be the resulting tilt element. Put $z = [\varpi^\flat]$. Then $\theta(z) = \pi$, and z is not a unit. Substitution $X \mapsto z$ defines a Λ-algebra homomorphism

$$\Lambda[[X]] \longrightarrow A_{\mathrm{inf}}.$$

Here convergence uses the usual weak topology of A_{inf}, not convergence in its p-adic topology alone. More explicitly, $z \bmod p$ has positive valuation in $\mathcal{O}_{\mathbb{C}_p}^\flat$, so powers of $z \bmod p$ tend to zero; successively modulo p^n the Witt components give convergent sums in the complete valuation ring of the tilt. Their compatible limits define substitution, and continuity of addition and multiplication verifies the homomorphism identities.

We construct the required unit series without identifying a formal power series with a polynomial. Every element of $\mathcal{O}_K$ is a convergent series $\sum_{n\geq 0} a_n \pi^n$, with a_n Teichmüller representatives in $\mathcal{O}_{K_0}$. Indeed, successively subtract the representative of the residue and divide by π; completeness supplies the limit. Since π^e and $-p$ have the same valuation, there is $v \in \mathcal{O}_K^\times$ with $\pi^e v = -p$. Lift v by this expansion to $A(X) \in \Lambda[[X]]$. Its constant term has nonzero residue, so A is a unit. In the unramified case $v = 1$, and we take $A = 1$. Set

$$E = z^e A(z) + p.$$

Its image under θ is zero.

We check the stronger required assertion that E generates $\ker\theta$. Let ξ be any generator of that kernel. A generator may be chosen as $[p^\flat] - p$, where $p^\flat$ is a compatible system of roots of p; its reduction modulo p has valuation 1, with valuation normalised by $v(p) = 1$ on untilts. Since $E \in (\xi)$, write $E = \xi b$. The reduction of E is $(\varpi^\flat)^e \overline{A}(\varpi^\flat)$; its valuation is $e/e = 1$, because $A(z)$ is a unit. Consequently $\bar{b}$ has valuation zero and is a unit of the tilt valuation ring. An element of A_{inf} whose reduction modulo p is a unit is itself a unit, by p-adic completeness. Thus b is a unit, as required.

Take $\Gamma' = \Gamma$. Its index is one and it contains every element near one. Its image fixes Λ and $\theta(z) = \pi$. Conversely an element of G_p fixing these fixes $K_0(\pi) = K$: the equality follows either from the uniformiser expansion above, or from the Eisenstein minimal polynomial of π, of degree e. Thus the common fixer is exactly $G_K = j(\Gamma')$.

It remains to verify the Kummer translations, including their simultaneous compatibility at every level. Fix $a \geq 0$ and the compatible system ϵ_n of p-power roots of unity used in the chart. For each n, the polynomial $X^{p^n} - \pi$ is Eisenstein over K, hence irreducible. There is therefore a K-embedding carrying π_n to $\epsilon_n^a \pi_n$, and it extends to an element of G_K. The sets

$$C_n = \{g \in G_K : g\pi_n = \epsilon_n^a \pi_n\}$$

are nonempty closed subsets of the compact group G_K, and $C_{n+1} \subseteq C_n$, by compatibility of the roots. Compactness gives $g \in \bigcap_n C_n$. It satisfies $g\varpi^\flat = \epsilon^a \varpi^\flat$, and hence $gz = [\epsilon]^a z$. Pull it back through j. No assertion that this element fixes the roots of unity is needed.

Finally the residue quotient of G_K is the absolute Galois group of $\mathbb{F}_{p^f}$, so it contains a lift of arithmetic Frobenius, acting on the algebraic residue field by $x \mapsto x^{p^f}$. If $j(\Gamma)$ contains inertia, $K/\mathbb{Q}_p$ is unramified, so $e = 1$ and our choice gives $E = z + p$. All the chart conditions have now been checked. The whole-group fixer, the degree identity, the specified unramified case and the Frobenius lift also show that this chart fits Γ. □

Lemma 7.7.18. *For every prime p, including 2, and $y \in B$, if $\theta(\varphi^{j+1}y) = 0$ for every integer $j \geq 0$, then $y = \tau z$ for some $z \in B$.*

Proof. Write $R = \mathcal{O}_{\mathbb{C}_p^\flat}$ and $A = W(R)$. The ring R is a rank-one valuation domain of characteristic p; write ν for its real valuation, with $\nu(0) = +\infty$. Reduction modulo p identifies A/pA with R, the Frobenius induces $r \mapsto r^p$, and A is p-torsion-free, p-adically separated and complete. A bar below means reduction modulo p.

The integral detector. Let

$$I = \bigcap_{j \geq 0} \ker(\theta\varphi^j) \subseteq A.$$

This is an ideal, contains μ, and is p-saturated: if $px \in I$, then $p\theta(\varphi^j x) = 0$ in the characteristic-zero domain $\mathcal{O}_{\mathbb{C}_p}$, hence $x \in I$. The canonical identities are

$$\mu = \xi\tau, \qquad \varphi(\tau) = \mu, \qquad \theta(\varphi^k\xi) = p \quad (k \geq 1).$$

The last follows by applying $\theta\varphi^k$ to $\xi = \sum_{i=0}^{p-1}[\varepsilon^{1/p}]^i$: each summand becomes 1. Modulo p, put $t_0 = \bar{\tau}$. The primitive compatible root system gives $0 < \nu(t_0) < \infty$, and characteristic p gives

$$\bar{\mu} = t_0^p, \qquad \bar{\xi} = t_0^{p-1}.$$

In particular $\nu(\bar{\mu}) = p\nu(t_0)$ and $\nu(\bar{\xi}) = (p-1)\nu(t_0)$.

For $x \in I$, write $x = \xi b$, using $\ker\theta = (\xi)$. Then $\varphi b \in I$: for every $j \geq 0$,

$$0 = \theta(\varphi^{j+1}x) = p\,\theta(\varphi^{j+1}b),$$

so the second factor is zero. Starting with $x_0 = x$, repeat this construction, writing $x_i = \xi b_i$ and $x_{i+1} = \varphi b_i$. For every positive integer r,

$$\nu(\bar{x}_0) = \nu(\bar{\xi})\sum_{i=0}^{r-1} p^{-i} + p^{-r}\nu(\bar{x}_r) \geq \nu(\bar{\xi})\sum_{i=0}^{r-1} p^{-i}.$$

If a reduction is zero the desired inequality is immediate; otherwise this equality follows by multiplicativity of the valuation and $\overline{\varphi b_i} = \bar{b}_i^p$. Taking the limit proves $\nu(\bar{x}) \geq p\nu(\bar{\xi})/(p-1) = \nu(\bar{\mu})$. Thus $\bar{\mu}$ divides $\bar{x}$ in the valuation ring R.

Lift a quotient to $d_0 \in A$. Then $x - \mu d_0 = px_1'$ for some $x_1' \in A$; since both x and μd_0 lie in I, p-saturation implies $x_1' \in I$. Repeat. For each r,

$$x = \mu\sum_{i=0}^{r-1} p^i d_i + p^r x_r', \qquad x_r' \in I.$$

Completeness and separatedness give $x = \mu\sum_{i\geq 0} p^i d_i$. Consequently every $x \in I$ is divisible by μ in A.

If $a \in A$ satisfies $\theta(\varphi^{j+1}a) = 0$ for all $j \geq 0$, then $\varphi a \in I$, so $\varphi a = \mu b = \varphi(\tau)b$. Frobenius is bijective on A because R is perfect. Applying its inverse gives

$$a = \tau\varphi^{-1}(b). \tag{1}$$

Reduction of the actual completed ring modulo τ. Put $T = \xi/p \in A_{\max}$. Every $b \in A_{\max}$ has a restricted-series presentation

$$b = F(T), \qquad F(U) = \sum_{n\geq 0} a_n U^n, \quad a_n \in A, \quad a_n \longrightarrow 0 \text{ } p\text{-adically}.$$

Indeed $A_{\max}$ is the p-adic completion of $A[U]/(pU-\xi)$. Represent successive p-adic differences of an element of that completion by p^r times polynomials in $A[U]$; their sum is a restricted series. This gives the asserted presentation without assuming uniqueness of it.

The sum $a = F(1) = \sum_n a_n$ converges in A. Define

$$G(U) = \sum_{j\geq 0} \left(\sum_{n>j} a_n \right) U^j.$$

Each inner sum converges, and its values tend to zero p-adically as $j \to \infty$. Hence G is restricted and coefficient subtraction gives $F(U) - F(1) = (U-1)G(U)$. Furthermore $\xi - p$ is divisible by τ in A, since

$$\xi - p = \sum_{i=0}^{p-1} ((1+\tau)^i - 1).$$

Write $\xi - p = \tau c$. Thus in $B = A_{\max}[1/p]$,

$$b = a + \tau p^{-1} c G(T).$$

Every $y \in B$ can therefore be written

$$y = p^{-m} a + \tau w, \qquad m \geq 0,\ a \in A,\ w \in B. \tag{2}$$

For every $j \geq 0$, $\theta(\varphi^{j+1}\tau) = \theta(\varphi^j \mu) = 0$. Apply $\theta\varphi^{j+1}$ to (2). Its left side is zero by hypothesis and its τw term is zero; hence $\theta(\varphi^{j+1}a) = 0$. Equation (1) gives $a = \tau a'$, with $a' \in A$. Then $y = \tau(p^{-m}a' + w)$, proving the claim.

All products and sums above occur in the specified actual rings. There is no cancellation by τ in the completed ring, no assumption that Frobenius on B is onto, and no parity restriction: the same geometric sum and valuation argument hold at $p = 2$. □

Lemma 7.7.19. *Put $p = \ell$, $S = \mathcal{S}_{\mathrm{ch}} = \Lambda[[z]]$, and let T be a crystalline lattice over the local field of the chart. Realise the module of Theorem A.3.12 by its specified étale comparison in the actual ambient module, and denote its image by N_{B}. Here its coefficient map is Witt Frobenius and its variable maps to z^p. Put*

$$N_0 = (\varphi_{A_{\mathrm{inf}}}^{-1} \otimes 1) N_{\mathrm{B}}.$$

Thus N_0 is an S-module in the same ambient module, with the direct variable z. For every crystalline form λ of (BF3),

$$\lambda(N_0) \subseteq \mathrm{Frac}(\mathcal{S}_B).$$

No uniform denominator bound is asserted.

Proof. Write $K_0 = \mathrm{Frac}(\Lambda)$, and use the actual embeddings in the statement throughout.

Analytic scalars lie in the bicommutant. Let $\mathcal{O}^{\mathrm{an}} \subset K_0[[z]]$ be the ring in Theorem A.3.13. The chart equation is $E = z^e A(z) + p$, where $A(z) \in S^\times$, and E is a unit times a generator of $\ker\theta$. Consequently

$$z^e/p = (E/p - 1) A(z)^{-1} \in A_{\max}.$$

For $F = \sum_{n\geq 0} c_n z^n \in \mathcal{O}^{\mathrm{an}}$, the defining convergence condition at radius $p^{-1/e}$ gives $v_p(c_n) + n/e \longrightarrow +\infty$. Writing $n = eq + r$, $0 \leq r < e$, shows

$$c_n z^n = c_n p^q z^r (z^e/p)^q \in p^{\,v_p(c_n) + \lfloor n/e \rfloor} A_{\max}.$$

The sum therefore converges in $B^+_{\max}$, with its usual topology having $p^a A_{\max}$ as a neighbourhood basis. This is the natural analytic embedding of Theorem A.3.13.

Let Q be any S-linear endomorphism of $B^+_{\max}$ preserving $A_{\max}$. It is K_0-linear: it commutes with Λ, and commutation with multiplication by p implies commutation with its inverse. Moreover $Q(p^a A_{\max}) \subseteq p^a A_{\max}$, so Q is continuous. For the polynomial partial sums F_d, one has $Q(F_d x) = F_d Q(x)$ for every $x \in B^+_{\max}$. Both sides converge: multiplication by any fixed element of $B^+_{\max}$ is continuous, since that element belongs to $p^{-b} A_{\max}$ for some b. Passing to the limit gives $Q(Fx) = FQ(x)$. By the exact definition (BF4), $F \in \mathcal{S}_B$. Thus

$$\mathcal{O}^{\mathrm{an}} \subseteq \mathcal{S}_B, \qquad \mathcal{O}^{\mathrm{an}}[1/\varphi(\mathscr{L})] \subseteq \mathrm{Frac}(\mathcal{S}_B).$$

Here the Euler product $\varphi(\mathscr{L})$ is nonzero by its constant term 1; no assertion that its inverse is integral in $\mathcal{S}_B$ is needed. Also $S[1/E] \subseteq \mathrm{Frac}(\mathcal{S}_B)$, since polynomial multiplication and then continuity show $S \subseteq \mathcal{S}_B$, and $E \neq 0$.

The effective case on the actual embedded module. First suppose that the weights of T, in the convention of Theorem A.3.13, lie in $[0, h]$. Let $\mathfrak{M}$ denote the direct-embedding module of that theorem. The direct map i_0 and the map i_{B} of Theorem A.3.12 satisfy $i_{\mathrm{B}} = \varphi_{A_{\mathrm{inf}}} \circ i_0$, including on coefficients. The equivariant étale identifications in those two theorems identify

$$N_0 = \mathfrak{M} \quad \text{and} \quad S \cdot \varphi(N_0) = \mathfrak{M}^{\varphi}$$

inside the same $T \otimes W(\mathbb{C}_p^{\flat})$. More explicitly, evaluating an abstract basis through i_{B} gives the coefficientwise Frobenius of its evaluation through i_0; applying $\varphi^{-1}_{A_{\mathrm{inf}}} \otimes 1$ recovers precisely that latter evaluation. The étale identifications agree with the identity on T. This uses their specified comparisons, rather than choosing an abstract module isomorphism. The image of the linearised Frobenius after the direct evaluation is exactly the S-span of these Frobenius vectors.

Choose an S-basis $n_1, \ldots, n_d$ of N_0. Transporting the Breuil–Kisin Frobenius of Theorem A.3.12 through this direct evaluation gives

$$(\varphi n_1, \ldots, \varphi n_d) = (n_1, \ldots, n_d)A, \qquad A \in \mathrm{GL}_d(S[1/E]).$$

Let $d_1, \ldots, d_d$ be a K_0-basis of $D_{\mathrm{cris}}(T[1/p])$. The isomorphism after localisation in Theorem A.3.13, and its agreement with the actual crystalline comparison, express each φn_j as a linear combination of the d_i with coefficients in $\mathcal{O}^{\mathrm{an}}[1/\varphi(\mathscr{L})]$.

The crystalline form λ extends on $T \otimes B_{\max}$ to the $B_{\max}$-linear map defined by its values on T. For the vectors used here it agrees with the given form: item 2 of Theorem A.3.12, followed by inverse Frobenius, puts N_0 in $A_{\mathrm{inf}}[1/\tau] \otimes T$, since $\varphi^{-1}(\mu) = \tau$. Both maps agree on this localisation, by multiplying a vector by a sufficiently large power of the nonzero scalar τ. The same observation applies to $\varphi(N_0)$. Its equivariance implies $\lambda(d_i) \in B^{G_K}_{\max} = K_0$. Indeed the crystalline comparison identifies the fixed vectors with D_{cris}, and the scalar fixed field is K_0. Hence $\lambda(\varphi n_j) \in \mathrm{Frac}(\mathcal{S}_B)$. Multiplying the row of these values by A^{-1}, whose entries belong to $S[1/E]$, proves $\lambda(n_j) \in \mathrm{Frac}(\mathcal{S}_B)$ for every j. This proves the effective case without applying inverse Frobenius to an analytic power series.

Arbitrary signed weights. Choose a rank-one Tate lattice L whose weight is sufficiently large and nonnegative in the convention of Theorem A.3.13 that both L and $T \otimes L$ have all weights nonnegative. The tensor property of Theorem A.3.12, with its specified étale comparison, and coefficientwise inverse Frobenius give literal tensor identifications of the embedded modules

$$N_0(T \otimes L) = N_0(T) \otimes_S N_0(L).$$

Take an S-basis t_L of the rank-one module $N_0(L)$, and a nonzero crystalline form ρ of L. The actual rank-one crystalline comparison implies $\rho(t_L) \neq 0$. The effective case gives $\rho(t_L) \in \mathrm{Frac}(\mathcal{S}_B)$. For the original arbitrary crystalline form λ, its tensor product with ρ is a crystalline form of $T \otimes L$. Applying the effective case again gives, for each $n \in N_0(T)$,

$$\lambda(n)\rho(t_L) = (\lambda \otimes \rho)(n \otimes t_L) \in \mathrm{Frac}(\mathcal{S}_B).$$

Division by the nonzero scalar $\rho(t_L)$ proves the assertion. All comparisons and tensor products used here preserve the original embedded module; no different lattice or auxiliary frame is substituted. □

Lemma 7.7.20. *Let $r \geq 0$, $H \geq 0$, and let $t \in G_{\mathbb{Q}_p}$ fix Λ pointwise and $\theta(z)$, with $t(z) = [\varepsilon]z$. Suppose*

$$Z \in M_r(S_B), \qquad Z' \in M_r(B^+), \qquad Z'Z = E^H I_r, \qquad t(Z) = ZG, \quad G \in M_r(A_{\mathrm{inf}}).$$

For every $l, b \geq 0$ with $3e \leq p^l$, there are an evaluation homomorphism

$$\mathrm{ev}_l : S[[X]] \longrightarrow A_{\mathrm{max}}, \qquad \mathrm{ev}_l(X) = z^{p^l}/p^2,$$

a matrix $\mathcal{V} \in \mathrm{GL}_r(S[[X]])$, and a matrix $C \in \mathrm{GL}_r(B^+)$, such that, putting $V = \mathrm{ev}_l(\mathcal{V})$,

$$\varphi^l(Z) = \varphi^l(E)^b C V.$$

Every $g \in G_{\mathbb{Q}_p}$ fixing Λ pointwise fixes every entry of C. For any $A, \delta \geq 0$ with $eA \leq p^l\delta$, define

$$\mathcal{Y}_{A,V} = \{y \in S^r : \ Vy - n \in E^A(B^+)^r \text{ for some } n \in \mathbb{Z}^r_{\geq 0}\}.$$

Then

$$p^{4\delta} S^r \ \subseteq \ \mathrm{span}_S(\mathcal{Y}_{A,V}).$$

The rank-zero assertions are interpreted using empty matrices and the zero module.

Proof. **The actual scalar chart results apply to these matrices.** By Lemma 7.7.22(4)–(7), choose its single S_B-linear retraction $P : B^+ \to S_B$. It fixes S_B, maps A_{inf} into S and preserves A_{max}. For $\nu = E, z$, $a \geq 0$, and $x \in A_{\mathrm{max}}$, its stated approximation is

$$p^a P(x) = s + \nu^a w, \qquad s \in S, \quad w \in S_B. \tag{R0}$$

We also use the stated Frobenius stability of S_B, separation $\bigcap_n z^n B^+ = 0$, and Galois difference. The hypothesis on t, together with the actual Kummer chart condition, gives a preimage of t in the chart subgroup Γ'; hence that difference assertion gives $\tau z \mid t(y) - y$ for every $y \in S_B$. The separate result Lemma 7.7.30 supplies integral cancellation for every nonnegative m, n, h. These are statements about the actual scalar rings, with no canonical module as an input.

The chart identifies $S = \Lambda[[z]]$; its Frobenius preserves S, acting by Witt Frobenius on coefficients and sending z to z^p. The chart equation $E = p + z^e U(z)$, with $U(z)$ a unit, gives $\theta(z)^e \theta(U(z)) = -p$; thus $z \neq 0$. The compatible primitive cyclotomic system also gives $\tau \neq 0$.

Write $A_0 = A_{\mathrm{inf}}$ and $Q = [\varepsilon]$, to distinguish the coefficient ring from the exponent A in the conclusion. All displayed matrix identities are over the commutative domain B^+; its embeddings of A_0, S, A_{max} are suppressed. The retraction is applied entrywise. Because it fixes S_B and is S_B-linear, $P(TZ) = P(T)Z$ whenever Z has entries in S_B.

A constant matrix normalizes the constant term of Z. Apply P to $p^{-H}Z'Z = p^{-H}E^H I_r$. It gives

$$TZ = p^{-H}E^H I_r, \qquad T = P(p^{-H}Z') \in M_r(S_B). \tag{R1}$$

Every $y \in S_B$ has a decomposition

$$y = c + zw, \qquad c \in \Lambda[1/p], \quad w \in S_B. \tag{R2}$$

Indeed choose k with $p^k y \in A_{\max}$. The scalar approximation for $\nu = z, a = 1$, and $P(y) = y$, gives $p^{k+1}y = s + zw_0$; write the power series representing $s \in S$ as its constant in Λ plus z times another series, then divide by p^{k+1}. Apply (R2) to each entry of T, writing $T = B + zT_1$. Since $E - p \in zS$, also $E^H - p^H \in zS$, including $H = 0$. Equation (R1) becomes

$$BZ = I_r + zW, \qquad W \in M_r(S_B), \quad B \in M_r(\Lambda[1/p]). \tag{R3}$$

Every Galois element fixing Λ fixes B, and so does t. Put $Y = BZ$. Then $t(Y) = YG$.

The scalar difference law gives $\tau z \mid t(Z) - Z$ entrywise. Multiplication by Z' and the hypotheses imply

$$Z'(t(Z) - Z) = E^H(G - I_r).$$

Each entry of $G - I_r$ is integral in A_0. Integral cancellation therefore gives

$$\tau z \mid (G - I_r)_{ij} \quad \text{in } A_0. \tag{R4}$$

The needed lifting-the-exponent identity is an integral identity. For $n \geq 1$, put $v = v_p(n)$. There are $\alpha_n, \beta_n \in A_0$ with

$$\alpha_n(Q^n - 1) = \tau E(p^v + \beta_n z). \tag{R5}$$

Here is a proof valid also for $p = 2$. The period convention gives $Q - 1 = \tau\xi$, and E is associated to ξ. Applying Frobenius repeatedly, with $\varphi(\tau) = Q - 1$, gives

$$Q^{p^v} - 1 = u_v \tau \prod_{a=0}^{v} \varphi^a(E), \qquad u_v \in A_0^\times.$$

For $n = p^v n_0$, the quotient of $Q^n - 1$ by $Q^{p^v} - 1$ is the geometric sum with n_0 terms. It is a unit: $A_0 = W(\mathcal{O}^\flat_{\mathbb{C}_p})$ is local, Q has residue one, and this sum has nonzero residue $n_0 \bmod p$. Each $\varphi^a(E)$, for $a \geq 1$, is congruent to p modulo zA_0, by $E = p + z^e U(z)$ and $\varphi(z) = z^p$. Hence their product equals $p^v + \beta_n z$. Absorb the product of units into α_n. If $v = 0$, use the empty product one and $\beta_n = 0$. This proves (R5).

One row step keeps every cancellation integral. Fix a row index i. Suppose $q \geq 0$, $n \geq 0$, $W_n \in (S_B)^r$ and $A_n \in A_0^r$ satisfy

$$A_n = qY_i - z^{n+1}W_n. \tag{R6}$$

Put $v = v_p(n+1)$, and choose α, β from (R5) for $n + 1$. Write $E = p + zr_0$, $r_0 \in S$. By (R4) and the scalar difference assertion, there is $D \in (B^+)^r$ with

$$W_n(G - I_r) - Q^{n+1}(t(W_n) - W_n) = \tau z D.$$

Expanding $A_nG - t(A_n)$, using $t(Y) = YG$ and $t(z) = Qz$, gives exactly

$$A_nG - t(A_n) = z^{n+1}((Q^{n+1} - 1)W_n - \tau z D).$$

Consequently, with $W'' = \alpha D - (p^v r_0 + \beta E)W_n$, one has

$$\alpha(A_n G - t(A_n)) = \tau z^{n+1}(p^{v+1}W_n - zW''). \tag{R7}$$

The left side lies in A_0^r. Integral cancellation with $h = 0$ proves that the parenthesized vector is the image of a vector $x \in A_0^r$, not merely a fraction in B^+. Apply P to that equality. Since $P(x) \in S^r$ and $P(W_n) = W_n$, we obtain $s_n \in S^r$ and $W_{n+1} \in (S_B)^r$ with

$$p^{v_p(n+1)+1}W_n = s_n + zW_{n+1}. \tag{R8}$$

Set

$$q_{n+1} = p^{v_p(n+1)+1}q, \qquad A_{n+1} = p^{v_p(n+1)+1}A_n + z^{n+1}s_n.$$

Then $A_{n+1} = q_{n+1}Y_i - z^{n+2}W_{n+1}$. Thus the same integral row hypothesis (R6) is available for the next step. Initially take $q = 1$, the integral row $(I_r)_i$, and $W_0 = W_i$ from (R3).

The row recursion produces a unit power series after Frobenius. Put $N = p^l$ and $T = z^N/p^2$. Since E is a multiple of p in $A_{\max}$, the identity $E = p + z^e U(z)$ implies $z^e \in pA_{\max}$, because $U(z)$ is a unit. The inequality $N \geq 3e$ therefore gives $z^N \in p^3 A_{\max}$. Thus $T \in pA_{\max}$ and p-adic completeness defines the evaluation homomorphism ev_l on all of $S[[X]]$. This map sends every unit power series, and every invertible formal matrix, to a unit or an invertible matrix. Also $T \in zB^+$.

For a fixed entry of a row satisfying (R6), put

$$a_n = n - v_p(n!) + 2, \qquad g_n = p^{a_n}\varphi^l(W_{n,j}).$$

The elementary factorial identity gives

$$a_n = \big(n + 1 - v_p((n+1)!)\big) + v_p(n+1) + 1.$$

The first parenthesis is nonnegative: $v_p(m!) \leq m$ for every $m \geq 0$, from the sum $\sum_{k\geq 1}\lfloor m/p^k \rfloor$. Applying φ^l to (R8) consequently gives an exact recursion

$$g_n = d_n + Tg_{n+1}, \qquad d_n = p^{\,n+1-v_p((n+1)!)}\varphi^l(s_{n,j}) \in S. \tag{R9}$$

The exponent on g_{n+1} is correct because multiplication of T by p^2 is $\varphi^l(z)$. For $f(X) = \sum_{n\geq 0} d_n X^n \in S[[X]]$, iteration of (R9), together with the exact decomposition of f into its degree-$< m$ truncation and X^m times its shifted series, shows, for every m, that $g_0 - \mathrm{ev}_l(f) \in T^m B^+ \subseteq z^m B^+$. Separation by powers of z proves equality. At $n = 0$, $a_0 = 2$, so entrywise this constructs $F \in M_r(S[[X]])$ with

$$\mathrm{ev}_l(F) = p^2\varphi^l(W), \qquad \varphi^l(B)\varphi^l(Z) = \mathrm{ev}_l(I_r + XF). \tag{R10}$$

The formal matrix $\mathcal{U} = I_r + XF$ is invertible, with its inverse given coefficientwise by its geometric series. Thus the right side of (R10) is invertible. In a commutative ring, a square product is invertible only if both factors are: take determinants and use the adjugate formula. Hence $\varphi^l(B)$ is invertible in B^+.

Since $E = p + zr_0$,

$$\varphi^l(E) = p\ \mathrm{ev}_l(\gamma), \qquad \gamma = 1 + p\varphi^l(r_0)X \in S[[X]]^\times.$$

Define

$$\mathcal{V} = \gamma^{-b}\mathcal{U}, \qquad C = (p^b\varphi^l(B))^{-1}.$$

Substitution in (R10) now yields $\varphi^{l}(Z) = \varphi^{l}(E)^{b} C \operatorname{ev}_{l}(\mathcal{V})$. Every entry of B lies in $\Lambda[1/p]$; Frobenius preserves that field and commutes with the Galois action. Therefore every element fixing Λ fixes C. This proves the asserted factorization.

Denominator approximation gives the stated exponent 4δ. For every $f \in S[[X]]$ and integer $D \geq 0$, truncation at D, evaluated at $T = z^{N}/p^{2}$, gives

$$p^{2D} \operatorname{ev}_{l}(f) = s + z^{ND} \operatorname{ev}_{l}(f_{D}), \qquad s \in S, \quad f_{D} \in S[[X]]. \tag{R11}$$

Indeed for $f = \sum f_{n} X^{n}$, take $s = \sum_{n<D} p^{2(D-n)} f_{n} z^{Nn}$ and $f_{D} = \sum_{n \geq D} f_{n} X^{n-D}$. The retraction fixes each $\operatorname{ev}_{l}(f)$. To prove this without assuming continuity of P in the z-adic topology, apply P to (R11). Since $s, z \in S_{B}$, the difference $P(\operatorname{ev}_{l}(f)) - \operatorname{ev}_{l}(f)$ is divisible by z^{ND} in B^{+} for every D, after cancellation of the unit p^{2D}. Separation and $N > 0$ give the assertion.

Suppose $eA \leq N\delta$. The relation $z^{e} \in pA_{\max}$ gives $z^{N\delta} = p^{A} y$ for some $y \in A_{\max}$. Equation (R11), with $D = \delta$, and its image under P, therefore give

$$p^{2\delta} \operatorname{ev}_{l}(f) = s + p^{A} P(y \operatorname{ev}_{l}(f_{\delta})).$$

The approximation property for $\nu = E$ and exponent A writes the final term as $s' + E^{A} w$, $s' \in S, w \in S_{B}$. We have proved

$$\text{for every } f \in S[[X]] \text{ there is } s(f) \in S \text{ with } s(f) \equiv p^{2\delta} \operatorname{ev}_{l}(f) \pmod{E^{A} B^{+}}. \tag{R12}$$

No linearity of the choice $s(f)$ is needed.

Apply (R12) entrywise to $\mathcal{V}^{-1}$ and $\mathcal{V}$, obtaining matrices $M, M' \in M_{r}(S)$. Put $V = \operatorname{ev}_{l}(\mathcal{V})$. Modulo E^{A}, they satisfy

$$VM \equiv p^{2\delta} I_{r}, \qquad V(p^{4\delta} I_{r} - MM') \equiv 0.$$

Every column of M is therefore in $\mathcal{Y}_{A,V}$, with natural vector $p^{2\delta} e_{j}$; every column of $p^{4\delta} I_{r} - MM'$ is in that set with natural vector zero. Finally for every $x \in S^{r}$,

$$p^{4\delta} x = (p^{4\delta} I_{r} - MM')x + M(M'x).$$

Both summands lie in the S-span of those columns, proving the asserted inclusion. The argument includes $A = 0$ and $\delta = 0$. In rank zero all matrices are empty and the span inclusion is the equality of zero modules. □

Lemma 7.7.21. *For a fixed chart put $A = A_{\mathrm{inf}}$ and $S = \mathcal{S}_{\mathrm{ch}}$. There is an S-linear map $q_{0} : A \to S$, identity on S, such that the following assertions hold for every lattice L with letter jump sets.*

1. *The formula*

$$R_{L}\left(\sum_{\nu=1}^{r} a_{\nu} v_{\nu}\right) = \sum_{\nu=1}^{r} q_{0}(a_{\nu}) v_{\nu}, \qquad a_{\nu} \in A, \quad v_{\nu} \in N_{\mathrm{loc}}(L),$$

   defines an S-linear retraction from $AN_{\mathrm{loc}}(L)$ onto $N_{\mathrm{loc}}(L)$. It commutes with equivariant lattice maps, in particular the letter projectors and the inclusion and quotient maps of a side extension.

2. *For every $a \geq 0$, $n \in \mathbb{Z}$, form λ, $v \in \mathcal{K}_{a}$, and $y \in AN_{\mathrm{loc}}(L)$,*

$$\lambda(\varphi^{-a} y) - v \in \mathrm{Fil}^{n} \implies \lambda(\varphi^{-a} R_{L} y) - v \in \mathrm{Fil}^{n}.$$

   In particular R_{L} preserves every prescribed order module and every such congruence to a fixed-field constant.

3. *For every $l \geq 0$ and letter i,*

$$\mathfrak{X}_L(i;\mathrm{std}_l) = (\varphi^l M(L))_i, \qquad N_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i;\mathrm{std}_l) = \big(S \operatorname{span} \varphi^l N(L)\big)_i.$$

4. *For a side S_1, a natural modulus r, $f \in S \cap J_l(i;r)$, there is one $t \in S$ such that every $y \in N_{\mathrm{loc}}(C) \cap \mathfrak{X}_C(i;\mathrm{std}_l)$ has a lift $x \in N_{\mathrm{loc}}(\widetilde{Y}_{S_1}) \cap \mathfrak{X}_{\widetilde{Y}_{S_1}}(i;\mathrm{std}_l)$ with $\pi_{S_1} x = (f - rt)y$. The same t works for all y, hence for all constants.*

Proof. Let $S = \mathcal{S}_{\mathrm{ch}}$, $A = A_{\mathrm{inf}}$. We first construct the coefficient retraction used below. The regular sequence (p, z) in A_{inf} gives

$$\mathrm{gr}_{(p,z)} A_{\mathrm{inf}} = (A_{\mathrm{inf}}/(p,z))[P, Z].$$

This follows degree by degree by reducing a homogeneous relation modulo p, canceling the nonzero $\bar{z}$, and then dividing by p; iteration shows that every coefficient of a homogeneous relation lies in (p, z). The Witt-coordinate description makes A_{inf} complete and separated for (p, z): modulo p^m, the z-adic topology is the complete valuation topology on the finitely many tilt coordinates. Choose a basis $(\bar{b}_j)$ of $A_{\mathrm{inf}}/(p, z)$ over Λ/p, containing 1, and lifts b_j, with the lift of 1 equal to 1. The graded identity, followed by induction on the adic order and completeness, gives

$$A_{\mathrm{inf}} \simeq \varprojlim_n \bigoplus_j S/(p,z)^n S.$$

At each finite order only finitely many coefficients occur. The coordinate at 1 is an S-linear map $q_0 : A_{\mathrm{inf}} \to S$, identity on S. For a finite relation $\sum a_j v_j = 0$ with $v_j \in N_{\mathrm{loc}}(L)$, each vector is denominator-clearable in the ambient lattice and its form values belong to $\mathcal{K}_0$. The exact relation assertion of Proposition 7.7.16 therefore gives $\sum q_0(a_j) v_j = 0$. The coefficient map on $A \otimes_S N_{\mathrm{loc}}$ consequently factors through its actual image AN_{loc}, defining R_L. Its image lies in N_{loc}, and $q_0(1) = 1$ makes it a retraction. An equivariant lattice map takes canonical modules and their localizations to the corresponding canonical modules: integral denominators and Frobenius tests are preserved by pulling back forms, and value-field tests are preserved in the same way. On a finite sum $\sum a_j v_j$, its two composites with R both equal $\sum q_0(a_j) f(v_j)$. This proves naturality, including letter projectors and side maps.

For completeness, values at level a of $v \in N_{\mathrm{loc}}$ belong to $\mathcal{K}_a$. For $x \in N_{\mathrm{loc}}$, choose a finite product P of the E_b with $Px \in M$. Since $P \in S \subset \mathcal{K}_0$, all form values of Px are still in $\mathcal{K}_0$, so $Px \in N$. Thus N_{loc} is exactly this localization of N. Now $E^h \varphi N \subseteq N$ and $\varphi(E_b) = E_{b+1}$ imply $\varphi N_{\mathrm{loc}} \subseteq N_{\mathrm{loc}}$; and $M = AN$ gives $M_{\mathrm{loc}} = AN_{\mathrm{loc}}$ by clearing the finite localization product. Clear a nonzero ambient denominator of $\varphi^{-a} v$, multiply by a power of τ, and express the resulting vector using $M = AN$ and localization. Apply φ^a and then a chart coefficient coordinate which is nonzero on the coefficient of v. The exact relation law preserves this relation. Solving the evaluated relation expresses $\lambda(\varphi^{-a} v)$ as a quotient of finite sums of $\varphi^{-a}(S)$-multiples of $\mathcal{K}_0$-values. These belong to $\mathcal{K}_a$. Thus, for $y = \sum a_j v_j$,

$$\lambda(\varphi^{-a} y) - v = \sum_j \varphi^{-a}(a_j) \lambda(\varphi^{-a} v_j) + \varphi^{-a}(1)(-v).$$

Apply Lemma 7.7.22 to this finite sum, including its last term. It replaces a_j by $q_0(a_j)$ and fixes 1, proving item2 at every integer filtration. Naturality preserves the letter part, so all order modules are preserved.

The finite Hodge basis used in the standard tests. By Lemma 7.4.6 the representation is continuous and de Rham. By Theorem A.3.16, its invariant de Rham space has the full

dimension and gives the comparison isomorphism. Choose a basis d_m of this space adapted to its filtration, separately in each letter part; the projectors are fixed by the subgroup and preserve this filtration. Let u_m be the corresponding jumps. The natural graded comparison of Theorem A.3.11 says that the reductions of $e_m = t^{-u_m} d_m$ form a basis of the integral comparison lattice modulo t. Their comparison matrix has integral entries and unit determinant, hence they form a B_{dR}^+-basis of $B_{\mathrm{dR}}^+ \otimes L$ in that part. The invariant dual forms λ_m, with $\lambda_m(d_n) = \delta_{mn}$, have type u_m, since their values on e_m are t^{-u_m}. Their values extend to actual ambient forms by choosing a complement after localizing A_{inf} at all nonzero elements and extending by zero. All vectors evaluated below have a nonzero integral denominator, so their values do not depend on this auxiliary extension. Every u_m is an actual jump and lies in the prescribed jump set.

Actual standard orders and Frobenius pullback. Use (BF2)–(BF5), (DQ1)–(DQ2), and the invariant Hodge bases just constructed. Write $A = A_{\mathrm{inf}}$, $E_a = \varphi^a(E)$, and $T_l = \varphi^l(\tau)$. A subscript i means intersection with the actual letter-i part.

The following scalar facts are consequences of the fixed Fontaine elements and the chart $E = \xi$ times a unit:

$$T_l = u_l\, \tau \prod_{0 \le a < l} E_a, \qquad u_l \in A^\times, \tag{S1}$$

$$\theta(\varphi^{-a} c) = 0 \ (c \in A) \iff c \in E_a A, \qquad \varphi^{-a}(\tau) \in (B_{\mathrm{dR}}^+)^\times, \tag{S2}$$

and

$$\varphi^r(E_a) \in (B_{\mathrm{dR}}^+)^\times \quad (r \ge 1,\ a \ge 0). \tag{S3}$$

Here and below the scalar in a period-ring assertion is its prescribed image in B_{dR}. For (S1), use $\varphi(\tau) = \mu = \xi\tau$ and iterate, replacing ξ by its unit multiple E. For (S2), the kernel of θ is $\xi A = EA$, and Frobenius is an automorphism of A. Also $\theta(\varphi^{-a}\tau) = \zeta_{\ell^{a+1}} - 1 \neq 0$; this is a unit in the t-adic valuation ring B_{dR}^+. For (S3), $\theta(\varphi^b \xi) = \ell \neq 0$ for $b \ge 1$, directly from $\xi = \sum_{j=0}^{\ell-1} [\epsilon^{1/\ell}]^j$. Multiplication by a unit and its Frobenius images does not change nonvanishing. These facts include $\ell = 2$ and $l = 0$.

Positive Frobenius tests hold on the localization. If $x \in M_{\mathrm{loc}}(L)$, choose a finite product P of the E_a with $Px \in M(L)$. For every form λ and $r \ge 1$,

$$\lambda(\varphi^r(Px)) = \varphi^r(P)\lambda(\varphi^r x) \in B_{\mathrm{dR}}^+.$$

The scalar $\varphi^r(P)$ is a unit by (S3), so

$$\lambda(\varphi^r x) \in B_{\mathrm{dR}}^+ \qquad (r \ge 1). \tag{S4}$$

Each evaluation is on a vector whose denominator can be cleared by an element of A, so the displayed identity follows from A-linearity by clearing that denominator. No evaluation of an arbitrary ambient functional away from its fractional lattice span is involved.

A standard order permits cancellation of a high localization factor. Let $x \in \mathfrak{X}_L(i; \mathrm{std}_l)$, $a \ge l$, and $r \in A$. We prove

$$rE_a x \in A \otimes_{\mathbb{Z}_\ell} L \quad \Longrightarrow \quad rx \in A \otimes_{\mathbb{Z}_\ell} L. \tag{S5}$$

Choose a $\mathbb{Z}_\ell$-basis of L, with integral coordinate functionals κ_s. Let $j = i^{(a)}$. In this letter part choose invariant Hodge vectors $d_m = t^{u_m} e_m$, where the e_m are a B_{dR}^+-basis of the integral comparison lattice, and let λ_m be their invariant dual forms. The integers u_m are actual jumps and λ_m has type u_m. For any denominator-clearable y in that letter part,

$$\kappa_s(y) = \sum_m \kappa_s(d_m)\lambda_m(y), \qquad \kappa_s(d_m) \in \mathrm{Fil}^{u_m}. \tag{S6}$$

The coefficient assertion holds because $d_m = t^{u_m} e_m$ and the integral coordinate functional is B_{dR}^+-valued on the integral comparison lattice. Apply (S6) to $y = \varphi^{-a}x$. Its standard order at level $a \geq l$ gives $\lambda_m(y) \in \mathrm{Fil}^{-u_m}$, so

$$\kappa_s(\varphi^{-a}x) \in \mathrm{Fil}^0 \quad \text{for every } s. \tag{S7}$$

Write the integral vector $rE_a x$ in the chosen lattice basis, with coordinates $c_s \in A$. Then

$$\varphi^{-a}(c_s) = \varphi^{-a}(rE_a)\,\kappa_s(\varphi^{-a}x) \in \mathrm{Fil}^1.$$

By (S2), $c_s = E_a b_s$ for some $b_s \in A$. The vector with coordinates b_s is therefore integral and, after multiplication by E_a, equals $rE_a x$. The ambient module has no A-torsion; cancelling E_a proves (S5).

All localization denominators are absorbed by a power of T_l. Again let $x \in \mathfrak{X}_L(i;\mathrm{std}_l)$. Choose $P = \prod_{\nu=1}^m E_{a_\nu}$ with $Px \in M(L)$, so $\tau^n Px$ is integral for some n. By (S1), $T_l^n Px$ is integral. Remove the factors of P one at a time. At an intermediate stage let y be the product of the remaining factors times x, and suppose $T_l^b E_a y$ is integral. The vector y still has the same standard orders, because the order module is an A-submodule. If $a \geq l$, (S5) cancels E_a. If $a < l$, (S1) gives $T_l/E_a \in A$, so multiplication by this scalar makes $T_l^{b+1}y$ integral. The finite induction proves

$$T_l^b x \in A \otimes L \quad \text{for some } b \geq 0. \tag{S8}$$

Put $y = \varphi^{-l}x$. Applying inverse Frobenius to (S8) gives $\tau^b y \in A \otimes L$, the first condition defining $M(L)$. To verify the other condition, let λ be any form and $r \geq 1$. If $r > l$, (S4) gives $\lambda(\varphi^r y) = \lambda(\varphi^{r-l}x) \in B_{\mathrm{dR}}^+$. If $1 \leq r \leq l$, set $a = l - r < l$. The standard tests for x at level a put every typed invariant Hodge-basis form value on $\varphi^{-a}x = \varphi^r y$ in Fil^0. Any invariant form on that letter part is a finite linear combination of these dual Hodge forms with coefficients in the finite fixed field, hence coefficients in Fil^0. Thus its value too is integral. This proves all positive-Frobenius tests for y, and hence $y \in M(L)$. Frobenius permutes letter projectors, so x lies in the letter-i part of $\varphi^l M(L)$.

The reverse implication has the exact standard orders. Let $y \in M(L)$ in the letter-$i^{(l)}$ part and put $x = \varphi^l y$. Choose n with $\tau^n y \in A \otimes L$. Formula (S1) shows that the product $P = \prod_{a<l} E_a^n$ makes Px τ-saturated integral. Its positive-Frobenius form values are integral: they are those of $\varphi^{r+l}y$, multiplied by $\varphi^r(P) \in A$, for $r \geq 1$. Thus $Px \in M(L)$, so $x \in M_{\mathrm{loc}}(L)$.

For $a \geq 0$, the vector $\varphi^{-a}(\tau^n y)$ belongs to the integral A-lattice. A form of type u on its letter part takes it into Fil^{-u}. The scalar $\varphi^{-a}(\tau)^n$ is a unit by (S2), so y has orders $-u$ at every level. For $x = \varphi^l y$, a test at level $a \geq l$ is exactly the test on y at level $a - l$, giving order $-u$. At $a < l$ it is a positive-Frobenius value on y, giving order 0. This proves

$$\boxed{\mathfrak{X}_L(i;\mathrm{std}_l) = (\varphi^l M(L))_i.} \tag{S9}$$

The argument covers $l = 0$: there are no low factors and no low-level case.

The value-field intersection is precisely the existing descent clause. By (BF5), $N_{\mathrm{loc}}(L)$ consists of localized vectors all of whose form values belong to $\mathcal{K}_0$. The first equality (S9) therefore gives, without any descent assumption,

$$N_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i;\mathrm{std}_l) = \{x \in \varphi^l M(L) : \lambda(x) \in \mathcal{K}_0 \text{ for every form } \lambda\}_i. \tag{S10}$$

Item 3(b) of Proposition 7.7.16 states exactly that the unprojected set on the right is the S-span of $\varphi^l N(L)$. Applying that assertion to (S10) yields

$$\boxed{N_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i;\mathrm{std}_l) = (S\,\mathrm{span}\,\varphi^l N(L))_i.} \tag{S11}$$

Equations (S9) and (S11) prove item3.

To prove the common lift assertion, proceed as follows. The definition of $J_l(i;r)$ supplies ONE $t_0 \in A$ and, for each parameter c, a standard-order lift x_c with

$$\pi x_c = (f - rt_0)\mathrm{cst}_i(c).$$

Those standard modules lie in AN_{loc}, since they are $\varphi^l M$ and $M = AN$. Apply R. Its order preservation retains the standard orders, and its image belongs to N_{loc}. Naturality and S-linearity on the constant lattice give

$$\pi R(x_c) = (f - rq_0(t_0))\mathrm{cst}_i(c).$$

Here $f \in S$ and $q_0(1) = 1$. For the trivial parameter lattice, height zero gives $M = A \otimes C$; its canonical value-field part is $S \otimes C$. To see the latter equality directly, apply the exact coefficient relation law to $a(1 \otimes c) - 1(a \otimes c) = 0$ for a vector with value-field coordinates, using q_0; it forces each coordinate $a = q_0(a) \in S$. Frobenius and the letter projectors preserve the constants. Consequently the entire localized standard part of the trivial lattice is the S-span of these constants, at every level. Extend the constructed lifts by S-linearity to arbitrary y in that part. The single scalar $t = q_0(t_0) \in S$ is independent of c and y, as required. □

Lemma 7.7.22. *Put $S = \mathcal{S}_{\mathrm{ch}}$, $A = A_{\mathrm{inf}}$, and let K' be the algebraic field fixed by the chart subgroup. The following assertions hold.*

1. *Every S-linear $q : A \to S$ extends to an $\mathcal{S}_B$-linear map $\widetilde{q} : B^+ \to \mathcal{S}_B$. This extension agrees with q on A.*

2. *For every $c \in K'$ and $n \geq 0$, there is $t \in \mathcal{S}_B$ such that $t - c \in \xi^n B^+_{\mathrm{dR}}$. Moreover, for $b \in B^+$,*

$$E^n \mid b \text{ in } B^+ \quad \Longleftrightarrow \quad b \in \xi^n B^+_{\mathrm{dR}}.$$

   Here the displayed membership uses the fixed period-ring embedding.

3. *For every such q, $a \geq 0$, integer n, and finite families $r_j \in A$, $v_j \in \mathcal{K}_a$,*

$$\sum_j \varphi^{-a}(r_j)v_j \in \mathrm{Fil}^n \quad \Longrightarrow \quad \sum_j \varphi^{-a}(q(r_j))v_j \in \mathrm{Fil}^n.$$

   In particular an exact relation remains exact after this coefficient operation.

4. *Write $p = \ell$. There is an S-linear retraction $q_0 : A \to S$, $q_0|_S = \mathrm{id}$, whose extension $P : B^+ \to \mathcal{S}_B$ is $\mathcal{S}_B$-linear, fixes $\mathcal{S}_B$ pointwise, and satisfies $P(A) = S$ and $P(A_{\max}) \subseteq A_{\max}$. It is continuous for the p-adic topology defined by $A_{\max}$, and*

$$\mathcal{S}_B = \widehat{S[E/p]}^{\,p}[1/p]$$

   inside B^+; the integral completion here is its closure in $A_{\max}$.

5. *For this same P, for $\nu \in \{E, z\}$, $a \geq 0$ and $b \in A_{\max}$, there are $s \in S$ and $w \in \mathcal{S}_B$ with*

$$p^a P(b) = s + \nu^a w.$$

   For every $y \in \mathcal{S}_B$ and all sufficiently large integers k, there are $c \in \Lambda$ and $w \in \mathcal{S}_B$ such that $p^k y = c + zw$.

6. *Frobenius maps $\mathcal{S}_B$ into itself, and $\bigcap_{n\geq 0} z^n B^+ = 0$.*

7. *Put $\tau = \varphi^{-1}([\varepsilon] - 1)$ for the compatible roots of unity of the period-ring convention. For every $\gamma \in G_\ell$ fixing Λ pointwise and every $y \in \mathcal{S}_B$, one has $z \mid \gamma y - y$ in B^+. In addition, for every $g \in \Gamma'$ and $y \in \mathcal{S}_B$,*

$$z\tau \mid gy - y \quad \text{in } B^+.$$

Proof. Write $p = \ell$, $A = A_{\mathrm{inf}}$, and $B^+ = B^+_{\mathrm{max}}$. We first identify the scalar ring in (BF4); the identification is a conclusion, not an additional hypothesis. Put

$$R_A = A[E/p] \subset A[1/p], \qquad R_S = S[E/p], \qquad C_0 = \widehat{R_S}^{\,p}.$$

The completions below are separated completions. Since E and ξ differ by a unit of A, $R_A = A[\xi/p]$ and its p-adic completion is A_{max}. Indeed $A[X]/(pX - \xi)$ embeds in $A[1/p]$: reduction modulo p and cancellation of the nonzero element $\bar{\xi}$ show successively that its defining ideal is p-saturated. Its localization at p is $A[1/p]$, proving the asserted embedding.

A chart-linear retraction. The chart map identifies S with $\Lambda[[z]]$. For clarity, it is injective: a nonzero series, after removing its common power of p, has nonzero reduction in $k[[X]]$; its first nonzero term has strictly smaller valuation at $\bar{z}$ than its subsequent terms. Here $k = \Lambda/p$ and $0 < v(\bar{z}) < \infty$. Thus its value cannot vanish.

The ring A is complete and separated for $I = (p, z)$. One way to check this explicitly is to reduce modulo p^m. The Witt-coordinate description of A/p^m identifies its z-adic topology with the product valuation topology on its finitely many tilt coordinates (multiplication by $z = [\varpi^\flat]$ multiplies these by fixed positive powers of $\varpi^\flat$). Those coordinates are complete. Taking the inverse limit in m proves completeness for the ideals (p^m, z^n), which are cofinal with the powers of I. Also p, z is a regular sequence: A is p-torsion free and $A/p = \mathcal{O}_{\mathbb{C}_p^\flat}$ is a domain in which $\bar{z} \neq 0$. The sequence can be reversed. In fact, if $pa = zb$, reduction modulo p implies $b = pb_1$, so cancellation gives $a = zb_1$.

The elementary regular-sequence calculation consequently gives

$$\mathrm{gr}_I A = (A/IA)[P, Z]. \tag{4}$$

The variables on the right are the initial forms of p, z. This calculation applies without a noetherian hypothesis on A: in each degree the relations are generated by the relation between p and z; reducing successively modulo p and canceling $\bar{z}$ proves that there is no further homogeneous relation. Equivalently, the usual induction for a regular sequence identifies I^n/I^{n+1} with the direct sum of $n + 1$ copies of A/I.

Choose a k-basis $(\bar{b}_j)_{j\in J}$ of A/IA, containing $\bar{b}_0 = 1$, and lifts $b_j \in A$ with $b_0 = 1$. The map $\bigoplus_J S \to A$ sending the jth basis vector to b_j induces an isomorphism modulo every power of (p, z). This follows first modulo (p, z) by the choice of basis and then inductively from (4), since $\mathrm{gr}_{(p,z)} S = k[P, Z]$. Hence

$$A \simeq \varprojlim_n \bigoplus_J S/(p, z)^n S \tag{5}$$

as complete S-modules. Projection to coordinate 0 in (5) defines an S-linear $q_0 : A \to S$ with $q_0(s) = s$ for $s \in S$. Infinite bases cause no infinite summation ambiguity: at every finite adic order the expression in (5) has finite support.

Extensions and the bicommutant. Let $f : A \to A$ be any S-linear map, not necessarily a retraction. Extend it to $A[1/p]$ by $f(a/p^m) = f(a)/p^m$. It commutes with multiplication by E/p, so preserves R_A, and it preserves $p^n R_A$ for every n. It therefore extends uniquely to a

continuous map on $A_{\max}$, and then to a map $\widehat{f} : B^+ \to B^+$. This extension is $S[1/p]$-linear, preserves $A_{\max}$, and commutes with multiplication by E/p. All these assertions follow first on R_A and then by completion.

For $q : A \to S$, this construction maps R_A into R_S. The latter ring has the induced p-adic topology: apply the extension of q_0 on R_A to see that

$$R_S \cap p^n R_A = p^n R_S.$$

Thus C_0 embeds in $A_{\max}$. The extension $\widehat{q}_0$ is an idempotent on $A_{\max}$, and its image is exactly C_0: its values are limits of elements of R_S, and it fixes R_S and hence its closure. In particular this image is closed.

Let $C = C_0[1/p] \subset B^+$. Every S-linear endomorphism F of B^+ preserving $A_{\max}$ commutes with every element of C. Indeed it is automatically $\mathbb{Q}_p$-linear, commutes with E/p, and is p-adically continuous on $A_{\max}$ because it preserves $p^n A_{\max}$. It therefore commutes first with R_S, then with C_0, and then with C. This proves $C \subseteq \mathcal{S}_B$ by the actual bicommutant definition (BF4). Conversely multiplication by $b \in \mathcal{S}_B$ commutes with $\widehat{q}_0$, which is one of those endomorphisms. As $\widehat{q}_0(1) = 1$, this gives

$$b = \widehat{q}_0(b) \in C.$$

We have proved the exact equality

$$\mathcal{S}_B = \widehat{S[E/p]}^{\,p}[1/p]. \tag{6}$$

The same argument shows that every $\widehat{f}$ is $\mathcal{S}_B$-linear, directly from the bicommutant definition. If $f = q$ takes values in S, its extension takes values in $C = \mathcal{S}_B$. This proves item 1, with the claimed agreement on A. It also proves the auxiliary extension assertion for arbitrary S-linear endomorphisms of A that we shall use below.

Divisibility is reflected by the de Rham embedding. Every element of $A_{\max}$ is represented by a restricted series $\sum_{j\geq 0} a_j (E/p)^j$, where $a_j \in A$ tend to zero p-adically. To see surjectivity of this presentation, lift successive polynomial approximations modulo p^n and sum their p-adically convergent corrections. The map to $\mathcal{O}_{\mathbb{C}_p}$ sends this series to $\theta(a_0)$ and is the residue map induced by the fixed embedding into B^+_{dR}.

Let $b \in B^+$ have residue zero. Multiply by a power of p to make it integral and choose the preceding presentation. Then $\theta(a_0) = 0$, so $a_0 = E a_0'$ in A, since E generates $\ker\theta$. Consequently

$$\frac{1}{E}\sum_{j\geq 0} a_j (E/p)^j = a_0' + \frac{1}{p}\sum_{j\geq 1} a_j (E/p)^{j-1} \in B^+.$$

Thus $b/E \in B^+$. Repeating proves that $b \in \xi^n B^+_{\mathrm{dR}}$ implies $b \in E^n B^+$ for every $n \geq 0$: after each division the de Rham order decreases by one, since E is a uniformizer times a de Rham unit. The converse follows from the embedding. This proves the divisibility assertion in item 2, including $n = 0$.

The initial lift of every fixed algebraic constant. Put $\pi = \theta(z)$ and $K_0 = \operatorname{Frac}\Lambda$. The defining chart series is $X^e U(X) + p$, with U a unit. Weierstrass preparation over the complete discrete valuation ring Λ writes it as a unit times a monic Eisenstein polynomial of degree e: its lower coefficients are divisible by p, and its constant coefficient is p times a unit. Evaluation at π is zero. In particular π is algebraic, $K_1 = K_0(\pi)$ is totally ramified of degree e, and $\theta(S) = \mathcal{O}_{K_1}$. Here evaluation of convergent series gives the latter equality: $\Lambda[\pi]$ is already the complete integer ring, and polynomial representatives modulo the monic Eisenstein polynomial give all its elements.

The chart subgroup has image exactly the subgroup of G_p fixing Λ pointwise and fixing π, by both inclusions in the chart condition. The fixed field K' in the statement is therefore K_1, by ordinary finite Galois theory. In the presentation (6), $\theta(C_0) \subseteq \mathcal{O}_{K_1}$: the values of polynomial approximations lie there and converge p-adically. Conversely $S \subset C_0$ already maps onto $\mathcal{O}_{K_1}$. Thus

$$\theta(\mathcal{S}_B) = K'. \tag{7}$$

This proves the required initial lift; it is not inferred merely from a fixed-field name.

For $c \in K'$, choose $s \in \mathcal{S}_B$ with $\theta(s) = c$. Then $s - c \in \xi B^+_{\mathrm{dR}}$. Let P be the minimal polynomial of c over K_0. It is separable. Bezout's identity supplies $a, b \in K_0[X]$ with $aP + bP' = 1$. The polynomial $H(X) = X - b(X)P(X)$ satisfies $H(c) = c$ and $H'(c) = 0$. Thus $H(X) - c$ is divisible by $(X - c)^2$ over $K_0(c)$. Its coefficients are algebraic constants and hence belong to B^+_{dR}. Iterating H starting at s stays in $\mathcal{S}_B$ and doubles the order of its difference from c. This gives any prescribed order n; for $n = 0$ any such initial lift suffices. No inverse of $P'(s)$ in $\mathcal{S}_B$ has been assumed. Item 2 is proved.

Transfer for finite sums with fixed constants. We record the precise consequence of the preceding steps. For any S-linear $f : A \to A$, any finite $c_j \in K'$ and $y_j \in B^+$, put $U = \sum_j c_j y_j$ and $V = \sum_j c_j \widehat{f}(y_j)$, read in B^+_{dR}. For every $m \geq 0$, approximate all c_j by $t_j \in \mathcal{S}_B$ modulo ξ^m. If $U \in \xi^m B^+_{\mathrm{dR}}$, then $\sum_j t_j y_j$ has that same property. Divisibility reflection makes it a multiple of E^m in B^+. Apply the $\mathcal{S}_B$-linear $\widehat{f}$, and then replace t_j back by c_j modulo ξ^m. It follows that $V \in \xi^m B^+_{\mathrm{dR}}$. In the discrete valuation ring B^+_{dR} this proves

$$V = uU \quad \text{for some } u \in B^+_{\mathrm{dR}}. \tag{8}$$

If $U = 0$, apply all m and separatedness to get $V = 0$, and take $u = 0$; otherwise the order inequality gives (8).

All Frobenius levels and value-field denominators. Fix $a \geq 0$ and $q : A \to S$ as in item 3. Regard $f = \varphi^{-a} \circ q \circ \varphi^a$ as an endomorphism of A. It is S-linear, since $\varphi^a(S) \subset S$ and q is S-linear. Thus its extension in (8) is available; no assertion that $f(A) \subset S$ is required.

Let D_a be the K'-subalgebra of B_{dR} generated by the products $b\varphi^{-a}(s)$ with $b \in \mathcal{S}_B$ and $s \in S$. These products form a multiplicative monoid containing 1, so every element of D_a is a finite K'-linear sum of such products. By (BF4), $\mathcal{K}_a = \mathrm{Frac}(D_a)$. Clear a common nonzero denominator in the finitely many v_j, writing $v_j = dw_j$ with $d \in B^\times_{\mathrm{dR}}$ and $w_j \in D_a$. Expand all w_j into finite sums $c_{jk} b_{jk} \varphi^{-a}(s_{jk})$ of this form. Apply (8) to these constants and the elements

$$y_{jk} = b_{jk}\varphi^{-a}(s_{jk} r_j) \in B^+.$$

They belong to B^+ because $\varphi^{-a}(A) = A$ and $\mathcal{S}_B \subset B^+$. The extension is $\mathcal{S}_B$-linear and agrees with f on A, hence

$$\widehat{f}(y_{jk}) = b_{jk}\varphi^{-a}(s_{jk} q(r_j)).$$

Regrouping (8), then multiplying by the common d, proves

$$\sum_j \varphi^{-a}(q(r_j))v_j = u \sum_j \varphi^{-a}(r_j)v_j, \qquad u \in B^+_{\mathrm{dR}}.$$

Every Fil^n for integer n is a B^+_{dR}-module. The desired implication and the exact-relation assertion follow. This covers empty families and zero sums, and completes all three items without any canonical lattice module.

Projection and completed scalar ring. Use the retraction q_0 constructed above by choosing a residue basis containing 1 in the (p, z)-adic coordinate description of A. Its extension

P preserves $A_{\max}$, has image the integral completion C_0 of $S[E/p]$, and fixes C_0. The proved bicommutant equality is $\mathcal{S}_B = C_0[1/p]$, and every such extension is $\mathcal{S}_B$-linear. Thus P fixes $\mathcal{S}_B$, $P(A) = S$, and $P(A_{\max}) \subseteq C_0 \subseteq A_{\max}$. Preservation of $p^n A_{\max}$ proves continuity. This supplies the retraction and projection in clause 4.

Exact approximation, with its printed power of p. Put $X = \xi/p \in A_{\max}$. Suppose $\nu \in S$ and $\xi = pc + \nu d$ with $c, d \in A$. Every restricted series $F(X) = \sum_{j\geq 0} a_j X^j$ admits division

$$F(X) = F(c) + (X - c)G(X), \qquad F(c) \in A,\ G(X) \in A_{\max}.$$

Indeed $F(c) = \sum a_j c^j$ converges p-adically. The coefficient of X^i in the quotient is $\sum_{j>i} a_j c^{j-1-i}$; these sums converge, and their valuations tend to infinity with i, so the quotient is restricted. Since $p(X - c) = \nu d$, this gives $pb = a_1 + \nu b_1$ for every $b \in A_{\max}$, with $a_1 \in A$ and $b_1 \in A_{\max}$. Inductively,

$$p^a b = a_a + \nu^a b_a, \qquad a_a \in A,\ b_a \in A_{\max}.$$

For the induction multiply the previous identity by p, divide b_a once more, and absorb $\nu^a a_1$ into the new element of A. The case $a = 0$ uses $a_0 = 0, b_0 = b$. Applying P gives the asserted $p^a P(b) = q_0(a_a) + \nu^a P(b_a)$.

Both required ν satisfy the hypothesis: for $\nu = E$ use the association $\xi = Eu$; for $\nu = z$ use $E = z^e U + p$ and $\xi = Eu$, giving $\xi = pu + z(z^{e-1}Uu)$. Here u, U are units in the indicated actual rings. For $y \in \mathcal{S}_B$ choose k_0 with $p^{k_0} y \in A_{\max}$. For every $k \geq k_0 + 1$, apply the case $\nu = z, a = 1$ to $p^{k-1}y$, which P fixes. Split the resulting $s \in S = \Lambda[[z]]$ as its constant coefficient plus zs_1. This proves the eventual constant-plus-z assertion with exactly the same P.

Frobenius stability. Write $E = z^e U + p$, $U \in S^\times$. Frobenius preserves S, sends z to z^p, and fixes p. Hence

$$\frac{\varphi(E)}{p} = \frac{(E-p)^p}{p}\,\frac{\varphi(U)}{U^p} + 1 \in S[E/p].$$

The expansion is integral: its term E^p/p is $p^{p-1}(E/p)^p$, and the other coefficients of $(E-p)^p/p$ are integral powers of p times powers of E; this includes $p = 2$. Thus Frobenius preserves $S[E/p]$. It is p-adically continuous and preserves its completion, proving $\varphi(\mathcal{S}_B) \subseteq \mathcal{S}_B$.

Separation by powers of z. Reduction of the defining presentation gives

$$A_{\max}/pA_{\max} \simeq (A/(p, \xi))[X]. \tag{S1}$$

The p-adic completion does not change this reduction. A monic polynomial in X is a non-zero-divisor in this polynomial ring, even though its coefficient ring need not be a domain. Consequently, for a monic $F \in A[X]$, multiplication by $F(X)$ on $A_{\max}$ reflects divisibility by p^r for every r: reduce modulo p, cancel the monic polynomial, and iterate. In particular if $F(X)$ divides $b \in A_{\max}$ in B^+, its quotient is already in $A_{\max}$. Write that quotient as $p^{-r}c$, and cancel p^r from $F(X)c = p^r b$ using the preceding assertion.

We claim $\bigcap_{m\geq 0}(\xi - pu)^m B^+ = 0$ for every $u \in A$. If a nonzero element lay in this intersection, clear a power of p, then remove its largest integral power of p to obtain $b \in A_{\max}$ not divisible by p. Such a largest power exists by p-adic separatedness. Since $\xi - pu = p(X - u)$ and p is a unit in B^+, every $(X - u)^m$ divides b in B^+, and hence in $A_{\max}$ by monic cancellation. Reduction in (S1) gives a nonzero polynomial divisible by the monic polynomial $(X - \bar{u})^m$ for every m. Degree makes this impossible over any coefficient ring. This proves the claim.

Finally take $\xi = Eu$ and $E = z^e U + p$. Then $\xi - pu = z^e Uu$, with $Uu \in A^\times$. An element divisible by every power of z is therefore divisible by every power of $\xi - pu$, and is zero. No closedness of a z-ideal after inverting p was assumed.

The Galois differences. The only additional input for this paragraph is the exact scalar result of Lemma 7.7.18: for $b \in B^+$, the conditions $\theta\varphi^{i+1}(b) = 0$ for all $i \geq 0$ imply $\tau \mid b$ in B^+. We apply this scalar detector to the quotient below.

First let $\gamma \in G_\ell$ fix Λ pointwise, with no assumption on $\theta(z)$, and let $y \in \mathcal{S}_B$. The eventual constant-plus-z assertion gives $p^k y = c + zw$ with $c \in \Lambda$. The element γ fixes c, and $\gamma z/z$ is a unit of A: z is a Teichmuller lift and the Galois action preserves the valuation of its nonzero tilt element, so their quotient is a unit of the tilt valuation ring and its Teichmuller lift is a unit of A. Consequently the exact identity

$$\gamma y - y = p^{-k} z\big((\gamma z/z)\gamma w - w\big)$$

proves $z \mid \gamma y - y$ in B^+. Galois acts on B^+, and no membership of γw in $\mathcal{S}_B$ is required. This proves the z-divisibility assertion, including every inertial γ fixing Λ. For the remaining assertion specialize to $g \in \Gamma'$ and write $gy - y = zb$ with $b \in B^+$.

For each $i \geq 0$, Frobenius stability gives $\varphi^{i+1} y \in \mathcal{S}_B$. Clear a power of p to make it integral in $A_{\max}$, then apply the proved approximation with $\nu = E, a = 1$. It yields

$$p^k \varphi^{i+1} y = s + Ew, \qquad s \in S,\ w \in \mathcal{S}_B.$$

The chart group fixes $\theta(S) = K'$ pointwise and $\theta(E) = 0$. Applying $\theta(g-1)$ proves $\theta\varphi^{i+1}(gy - y) = 0$; Frobenius commutes with Galois and $p^k \neq 0$. In $gy - y = zb$, the other factor has $\theta\varphi^{i+1}(z) = \theta(z)^{p^{i+1}} \neq 0$. Thus $\theta\varphi^{i+1}(b) = 0$ for every i. The separately supplied detector gives $\tau \mid b$, and hence $z\tau \mid gy - y$. No invariance of $\mathcal{S}_B$ under g, and no topological closedness of the ideal $(z\tau)$ in B^+, is needed. □

Lemma 7.7.23. *Put $p = \ell$, $\delta = z\tau$, and $\sigma(i) = i^p$, and let d be the chart's different bound. The following assertions hold for every prime p, including 2, every letter i, and all natural numbers appearing as levels, moduli or exponents.*

(a) Suppose $f \in \mathcal{S}_{\mathrm{ch}} \cap J_l(i;r)$, $g \in \Gamma'$ and $\alpha, \beta, s \in A_{\mathrm{inf}}$. If

$$\delta(s + \beta f) = \alpha(gf - f),$$

then $s \in J_l(i;r)$. At level $l = 1$ the same conclusion holds with δE in place of δ.

(b) For every $s \in A_{\mathrm{inf}}$,

$$\varphi^l(s) \in J_l(\sigma^l i; r) \quad \Longleftrightarrow \quad s \in J_0(i;r).$$

If the jumps of S at i are at most ω, then $f \in J_0(i;r)$ implies $fE^\omega \in J_1(i;r)$.

(c) If $p^c z^n \in J_1(\sigma i; r)$, $pa \leq n$, and

$$c + v_p(n!) \leq b + v_p((pa)!),$$

then $p^b z^a \in J_0(i;r)$.

(d) If $p^c z^a E^n \in J_1(i;r)$, then

$$p^{c+dn+v_p(n!)} z^{a+n} \in J_1(i;r).$$

(e) If $p^c z^n \in J_0(i;r)$, $m \leq n$, and

$$c + n + v_p(n!) \leq b + m + v_p(m!),$$

then $p^b z^m \in J_0(i;r)$.

(f) If $p^c \in J_l(i; p^{c+1})$, then $p^c \in J_l(i; 0)$.

Proof. Write $A = A_{\mathrm{inf}}$, $\mathcal{S} = \mathcal{S}_{\mathrm{ch}}$, and $X_{l,i} = \mathfrak{X}_{\tilde{Y}_S}(i; \mathrm{std}_l)$. Let $\pi = \pi_S$, and abbreviate $\mathrm{cst}_{S,i}(c)$ by c_i. By (DQ4), membership in a lift ideal means exactly

$$s \in J_l(i; r) \quad \Longleftrightarrow \quad \exists t \in A \;\; \forall c \in C \;\; \exists x_c \in X_{l,i} : \pi(x_c) = (s - rt)c_i. \tag{1}$$

Indeed the multipliers taking the entire constant module into $\pi(X_{l,i})$ form an ideal, and $J_l(i; r)$ is its sum with (r). The constants c_i, for $c \in C$, generate the constant module over A, so testing them is sufficient. The constants c_i lie in the trivial lattice's localised standard part at every level: their form values are in the fixed field, so have filtration at least zero, and Frobenius fixes the underlying $1 \otimes c$. The letter projector has coefficients in the chart's unramified ring and preserves these properties. If $s \in \mathcal{S}$, Lemma 7.7.21 permits the same t to be chosen in $\mathcal{S}$, with all x_c in $N_{\mathrm{loc}} \cap X_{l,i}$.

Divided differences and lift ideals. We first prove the following general observation. Let D be either δ, at any level, or δE, at level one. For $f \in \mathcal{S} \cap J_l(i; r)$, choose t, x_c as in (1) with $s = f$. The chart-ring difference $gt - t$ is divisible by D; write $gt - t = Dv$. The module $X_{l,i}$ is stable under Γ': the subgroup fixes the letter coefficients, the forms are equivariant, and it preserves every de Rham filtration. Hence $gx_c - x_c \in X_{l,i}$. By Lemma 7.7.24, for each c there is $y_c \in X_{l,i}$ with $Dy_c = gx_c - x_c$. Since g fixes the constants,

$$D\pi(y_c) = (gf - f - r(gt - t))c_i.$$

If $Ds_0 = \alpha(gf - f)$, this implies

$$D\pi(\alpha y_c) = D(s_0 - r\alpha v)c_i.$$

Multiplication by D is injective on the ambient module of the trivial lattice: both $z\tau$ and E are units in $W(\mathbb{C}_p^\flat)$. For E, its reduction modulo p is nonzero; the same holds for $z\tau$. Cancellation and (1) give $s_0 \in J_l(i; r)$. Apply this with $s_0 = s + \beta f$ and subtract βf, which already belongs to the ideal. This proves (a).

Frobenius and raising the level. The standard-order assertion of Lemma 7.7.21 says

$$\varphi^l(X_{0,i}) = X_{l,\sigma^l i}.$$

Projection commutes with Frobenius, $\varphi^l(c_i) = c_{\sigma^l i}$, $\varphi(r) = r$, and Frobenius is an automorphism of A. Apply φ^l, or its inverse, to every lift and the common scalar witness in (1). This proves the equivalence in (b), with the forward letter $\sigma^l i$.

For its second assertion, multiply every lift in (1) by E^ω. At inverse-Frobenius level zero, this raises the filtration of a form of type u from $-u$ to $\omega - u \geq 0$. At positive inverse-Frobenius levels E is a unit for the de Rham filtration, so no order changes there. One way to see this unit assertion is that $\theta(\varphi^{-a}E) \neq 0$ for $a > 0$: otherwise E would be associated to $\varphi^a E$, whereas $\theta(\varphi^a E) \neq 0$, as follows by evaluating its Eisenstein series at z^{p^a}, whose positive-degree terms have valuation at least $p^a > 1$, while the constant term has valuation one. Thus $E^\omega x_c \in X_{1,i}$; its projection is $(fE^\omega - rtE^\omega)c_i$. Equation (1) proves the assertion.

Two scalar identities. Put $q = [\varepsilon]$. Since E generates $\ker\theta$, the identity $q - 1 = \tau\xi$ gives

$$q - 1 = uE\tau \qquad (u \in A^\times). \tag{2}$$

For each integer $n \geq 1$, with $v = v_p(n)$, there exist $\alpha, \beta \in A$ such that

$$\tau E(p^v + \beta z) = \alpha(q^n - 1). \tag{3}$$

Here is the factorisation proving this, including at $p = 2$. Frobenius sends τ to $q - 1$. Applying its successive powers to (2) and multiplying the resulting associated factors shows that

$$q^{p^v} - 1 \text{ is associated to } \tau \prod_{a=0}^{v} \varphi^a(E).$$

Write $n = p^v n_0$, $p \nmid n_0$. The quotient of $q^n - 1$ by $q^{p^v} - 1$ is the geometric sum of n_0 powers of q^{p^v}. It is a unit: in the residue field of the local ring A, q is one and the sum is the nonzero element n_0 mod p. Each $\varphi^a(E)$ is congruent to p modulo z, since the chart expression is $E = z^e A_0(z) + p$. Therefore $\prod_{a=1}^{v} \varphi^a(E) = p^v + \beta z$ for some $\beta \in A$, proving (3) after absorbing a unit. An empty product when $v = 0$ is one.

For each $n \geq 1$ there are $g \in \Gamma'$, $\omega \in A^\times$ and $X, \upsilon, b_1 \in A$ such that

$$gE = \omega E, \qquad \delta X = \omega^n - 1, \qquad \upsilon X = p^{d+v_p(n)} + E b_1. \tag{4}$$

We give the precise different estimate. Choose g with $gz = qz$; the chart supplies it, and g fixes Λ and θz. Formal power-series subtraction, using (2), gives

$$\delta E \mid gs - s \quad (s \in \mathcal{S}).$$

The generators gE, E of $\ker \theta$ are associated; after cancellation in the last divisibility, $\delta \mid \omega - 1$. Thus $X = (\omega^n - 1)/\delta$ belongs to A.

By Weierstrass preparation for the chart series $z^e A_0(z) + p$, choose a monic Eisenstein polynomial F of degree e and $G \in \mathcal{S}$ with $F(z) = EG$. If $e = 1$, $F' = 1$. If $e > 1$, write $F(T) = \sum_{j=0}^{e} a_j T^j$, with $a_e = 1$. Since $v_p(\theta z) = 1/e$, the nonzero terms $j a_j (\theta z)^{j-1}$ of $F'(\theta z)$ have different fractional valuations $(j-1)/e$. Hence their minimum is unique, and

$$v_p(F'(\theta z)) \leq v_p(e) + (e-1)/e < d.$$

In the case $e = 1$, the corresponding bound is $v_p(F'(\theta z)) = 0 = d$.

Write $gz = z + Ew$, where $w = u\tau z$ by (2). Taylor's polynomial identity gives $F(z + Ew) = F(z) + F'(z)Ew + E^2 w^2 R$. On the other hand this is $g(EG) = \omega E\, gG$. Cancel E, apply θ, and use $\theta(gG) = \theta G$, because g fixes Λ and θz. We obtain

$$\theta(\omega - 1)\theta G = F'(\theta z)\theta w.$$

Thus, in the valuation ring $\mathcal{O}_{\mathbb{C}_p}$, $\theta(\omega - 1)$ divides $p^d \theta\delta$. Moreover $\delta \mid \omega - 1$, and

$$v_p(\theta\delta) = 1/e + 1/(p-1) > 1/(p-1).$$

Here the normalisation of τ follows already from the period convention: $\xi = \sum_{j=0}^{p-1} [\varepsilon^{1/p}]^j$ generates $\ker \theta$, so $\theta[\varepsilon^{1/p}]$ is a nontrivial p-th root of unity, and $v_p(\theta\tau) = 1/(p-1)$. For any nonzero a with $v_p(a) > 1/(p-1)$, binomial expansion shows

$$v_p((1+a)^n - 1) = v_p(a) + v_p(n).$$

Indeed a prime-to-p power has its unique least-valued term na. A p-power step has its unique least-valued term pa, because for $2 \leq j < p$ the term $\binom{p}{j} a^j$ has larger valuation, and $p v_p(a) > 1 + v_p(a)$. Iterating proves the formula. This strict inequality includes $p = 2$. Applying it to $\theta(\omega - 1)$ shows

$$v_p(\theta X) \leq d + v_p(n).$$

The Taylor identity also shows $\theta(\omega-1)\neq 0$, so this cancellation is legitimate. Surjectivity of $\theta : A \to \mathcal{O}_{\mathbb{C}_p}$ now lifts the integral quotient $p^{d+v_p(n)}/\theta X$ to υ, giving (4).

Lowering powers of z at level one. For $n \geq 1$, choose g with $gz = qz$, and apply (3). Multiplying (3) by $p^c z^n$ gives the exact identity

$$\delta E(p^{c+v_p(n)} z^{n-1} + \beta\, p^c z^n) = \alpha(g(p^c z^n) - p^c z^n).$$

Part (a) at level one therefore lowers z^n to z^{n-1}, at price $v_p(n)$, with no change of letter. Starting from the hypothesis of (c), lower from n to pa. The accumulated price is $v_p(n!) - v_p((pa)!)$. The assumed inequality permits multiplication by the remaining nonnegative power of p, giving $p^b z^{pa} \in J_1(\sigma i; r)$. Since $\varphi(p^b z^a) = p^b z^{pa}$, (b) gives $p^b z^a \in J_0(i; r)$. If $n = pa$, no lowering is performed; if $a = 0$, the last descent is to $z^0 = 1$. Thus the proof includes both endpoints.

Trading E for z. Suppose $n \geq 1$, and write $F_0 = p^c z^a E^n \in J_1(i; r)$. Choose (4), and write $gz = z(1+\tau E\upsilon)$, which is possible by the chart-ring divisibility already proved. There is $b_2 \in A$ with $(1+\tau E\upsilon)^a = 1 + \tau E\upsilon b_2$, including $a = 0$. Set $k_n = d + v_p(n)$ and

$$\beta_0 = zb_1 + \upsilon v b_2 \omega^n.$$

Direct multiplication, using $\delta X = \omega^n - 1$ and $\upsilon X = p^{k_n} + Eb_1$, gives

$$\delta E(p^{c+k_n} z^{a+1} E^{n-1} + \beta_0 F_0) = z\upsilon(gF_0 - F_0).$$

By (a), $p^{c+k_n} z^{a+1} E^{n-1} \in J_1(i; r)$. Repeat for $n, n-1, \ldots, 1$. The total price is $\sum_{j=1}^{n}(d + v_p(j)) = dn + v_p(n!)$, proving (d). When $n = 0$, (d) is its hypothesis.

Lowering at level zero. Since $E - p = ze_0$ for some $e_0 \in \mathcal{S}$, equation (3) implies

$$\tau(p^{v_p(n)+1} + z(p^{v_p(n)} e_0 + \beta E)) = \alpha(q^n - 1) \qquad (n \geq 1).$$

Multiply by $p^c z^n$ and apply (a), now with divisor δ. It follows that

$$p^c z^n \in J_0(i; r) \quad \Longrightarrow \quad p^{c+v_p(n)+1} z^{n-1} \in J_0(i; r).$$

Iterating from n down to m costs

$$(n - m) + v_p(n!) - v_p(m!).$$

The inequality in (e) is exactly the assertion that this total cost, added to c, is at most b. Multiplication by the remaining power of p proves (e). Empty iteration covers $m = n$.

Removing the modulus. By definition $J_l(i; r) = J_l(i; 0) + (r)$. If $p^c \in J_l(i; p^{c+1})$, there exist $u \in A$ and $v \in J_l(i; 0)$ with $p^c = v + p^{c+1}u$. Hence $(1 - pu)p^c = v$. The element $1 - pu$ is a unit, with inverse $\sum_{a\geq 0}(pu)^a$ converging in the p-adically complete ring A. Therefore $p^c \in J_l(i; 0)$, proving (f), also when $c = 0$. No part of this proof restricts the natural modulus r or excludes the prime 2. □

Lemma 7.7.24. *Put $\delta = z\tau$. For every lattice L with letter jump sets, $g \in \Gamma'$ and $x \in N_{\mathrm{loc}}(L)$,*

$$D_g x := \frac{gx - x}{\delta} \in A_{\mathrm{inf}} N_{\mathrm{loc}}(L).$$

For every letter i and exponent function o,

$$gx - x \in \mathfrak{X}_L(i; o) \quad \Longrightarrow \quad D_g x \in \mathfrak{X}_L(i; o).$$

Moreover, if $x \in N_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i;\mathrm{std}_1)$*, then*

$$\frac{gx - x}{\delta E} \in \mathfrak{X}_L(i;\mathrm{std}_1).$$

The quotients are taken in the existing ambient Witt-vector module. For every chart scalar $s \in \mathcal{S}_{\mathrm{ch}}$ *one also has*

$$\delta E \mid gs - s \qquad \text{in } A_{\mathrm{inf}}.$$

Proof. Put $A = A_{\mathrm{inf}}$, $S = \mathcal{S}_{\mathrm{ch}}$, $B^+ = B^+_{\mathrm{max}}$, $E_a = \varphi^a E$, and $\delta = z\tau$. First we record the scalar differences used below. If $g \in \Gamma'$, then g fixes the unramified coefficients and θz. The compatible roots of gz/z therefore form a compatible system of p-power roots of unity, so $gz = q^c z$ for $q = [\varepsilon]$ and $c \in \mathbb{Z}_p$. Since

$$q - 1 = \tau\xi, \qquad \xi = uE, \quad u \in A^\times,$$

the convergent binomial identity for $q^c - 1$ gives

$$\delta E \mid gz - z.$$

Formal power-series subtraction, or its finite truncations followed by completeness, now gives

$$\delta E \mid gs - s \quad (s \in S). \tag{1}$$

This includes $s = E_a$ for every a.

Division on the actual canonical module. Suppose first that $x \in N(L)$. By Proposition 7.7.16, for every crystalline form λ,

$$s = E^h \lambda(x) \in \mathcal{S}_B$$

for a common h. Item7 of Lemma 7.7.22 gives $gs - s \in \delta B^+$. Also $gE - E \in \delta A$ by (1). Consequently

$$E^{2h}(g\lambda(x) - \lambda(x)) \in \delta B^+. \tag{2}$$

Indeed put $u = gE/E \in A^\times$. Equation (1) gives $u - 1 \in \delta A$; therefore

$$E^h(g\lambda(x) - \lambda(x)) = u^{-h} gs - s \in \delta B^+,$$

which is stronger than (2). The vector $gx - x$ lies in $M(L)$, by canonical Galois stability. The cancellation theorem Lemma 7.7.30 with powers $1, 1$ gives precisely the scalar hypothesis of Proposition 7.7.16, item2(d), for $d = \delta$. Applying that criterion to (2), and using equivariance of λ, gives

$$gx - x \in \delta M(L). \tag{3}$$

Every $x \in N_{\mathrm{loc}}(L)$ is killed into $N(L)$ by a finite product of the E_a: clear its localization denominator into $M(L)$; multiplication by this product preserves its $\mathcal{K}_0$-valued tests, so the result is in the actual $N(L)$. It therefore suffices to pass (3) backwards through one denominator. Suppose $E_a x$ already has the required property. Then

$$g(E_a x) - E_a x = E_a(gx - x) + (gE_a - E_a)gx.$$

The second term divided by δ belongs to $AM_{\mathrm{loc}}(L) = AN_{\mathrm{loc}}(L)$, by (1), canonical Galois stability, and localization. The left side divided by δ belongs there by induction. Hence $E_a(gx - x)/\delta$

belongs there. This module is saturated by the E_a, because it is the A-span of the localization, so (3) follows for x. This proves the first assertion.

Orders are unchanged by division by δ. For every inverse-Frobenius level a,

$$\theta(\varphi^{-a}z) \neq 0, \qquad \theta(\varphi^{-a}\tau) = \zeta_{p^{a+1}} - 1 \neq 0.$$

Thus $\varphi^{-a}\delta$ is a unit of B^+_{dR}. The quotient in the first assertion belongs to the actual localized ambient module. For every typed form,

$$\lambda(\varphi^{-a}((gx - x)/\delta)) = \varphi^{-a}(\delta)^{-1}\lambda(\varphi^{-a}(gx - x)).$$

Multiplication by this de Rham unit preserves every integer filtration. The letter part is unchanged, since g fixes the letter coefficients and δ is a scalar. This proves the prescribed-order assertion.

The additional level-one factor. Let $x \in N_{\mathrm{loc}}(L) \cap \mathfrak{X}_L(i; \mathrm{std}_1)$. Its value $v = \lambda(x)$ at level zero lies in $\mathcal{K}_0 \cap B^+_{\mathrm{dR}}$, for every form in a tested type. Choose $b \geq 0$ and $r \in A$ with $\theta(r) = p^b\theta(v)$. Then $r - p^b v \in \mathrm{Fil}^1$. Applying Lemma 7.7.22 with the coefficient retraction of Lemma 7.7.21 gives

$$v - p^{-b}q_0(r) \in \mathrm{Fil}^1.$$

Since g fixes S modulo E, equation (1) implies $gv - v \in \mathrm{Fil}^1$. Equivariance identifies this difference with $\lambda(gx - x)$. Dividing by δE consequently leaves it in Fil^0, the required level-zero standard order. At a level $a \geq 1$, both $\varphi^{-a}\delta$ and $\varphi^{-a}E$ are de Rham units. For the latter assertion, E is associated to ξ, and $\theta(\varphi^{-a}\xi) \neq 0$, since it is the geometric sum of a primitive p^{a+1}-th root with p terms. Hence division preserves the standard order $-u$ at those levels. The vector itself lies in the actual localized module, by the first assertion and saturation by E. All level-one standard tests and the letter part are therefore satisfied. Together with (1), this proves the remaining assertions. □

Proposition 7.7.25. *Let k be a Kummer chart for Γ, S, S' pairing sides over $\mathcal{O}$ paired by P, with parameter lattices C, C'. A* point *of S at i over $c \in C$ with orders o and depth h means a vector $x \in N_{\mathrm{loc}}(\tilde{Y}_S)$ with $\pi_S x = \mathrm{cst}_{S,i}(c)$, $x \in \mathfrak{X}_{\tilde{Y}_S}(i; o)$, and $gx - x \in \mathfrak{X}_{\tilde{Y}_S}(i; h)$ for every $g \in \Gamma'$, using (DQ1)–(DQ3).*

1. *(Points exist.) If $\ell^\kappa E^s$ kills the cokernel of S at i, then for every $c \in C$ there is a point of S at i over $\ell^\kappa c$ with orders and depth $(a, u) \mapsto -u - s[a = 0]$.*

2. *(Order rule.) Let ν be a point of the Tate dual side of P over ℓ^κ at the letter l steps before i, with orders antitone in the jump, and let w, y lie over 0 in the two extensions with orders o_w, o_y at i. If at every level a and all jumps $u \in S(\cdot)$, $u' \in S'(\cdot)$ at the letter a steps before i the order of $\varphi^l\nu$ at $-u - u' - 1$ plus $o_w(a, u) + o_y(a, u')$ is positive, then the pairs $(z\tau w, y)$ and $(w, z\tau y)$ are safe for κ ($\tau = [\epsilon^{1/\ell}] - 1$). Moreover, a first-side move by a kernel vector w is safe against a point q' if the same inequality holds with o_y replaced by the depth of q'; a second-side move by a kernel vector y is safe against a point q if it holds with o_w replaced by the depth of q.*

3. *(The walk.) Let $\eta, \kappa, e, e' \in \mathbb{N}$. Let some letters be cuts, every Frobenius orbit meeting the cut set; let $l(i) > 0$ be the least positive distance back to a cut. Let $t(i) \in \mathbb{N}$ at each cut. Assume that at each cut i: $\ell^\kappa E^{t(i)}$ kills the cokernel of the Tate dual side of P; there are standard points $x(i)$ of S over $\ell^e c$ and $x'(i)$ of S' over $\ell^{e'} c'$ (orders $-u$, lowered by η at level 0; depth $-u$, raised by $t(i)$ at level 0 and by η below level $l(i)$); and one side has a stop at i (a point over the same constant with orders $\max(0, -u - \eta)$ below level $l(i)$ and*

$-u$ from there on) while the other side moves safely from $\varphi^{l(i)}$ of its point at the previous cut to its point at i. Then P is a scaled cup coboundary at (N, c, c') for every $N \in \mathbb{N}$ with $N \geq e + e' + \kappa$.

Proof. **Carriers and elementary filtration rules.** All order modules and projections below are (DQ1)–(DQ4); no abstractly isomorphic replacement is used. Put $\Delta = z\tau$. A point's depth is the condition on every Galois difference in the statement, not invariance of the point itself.

The strict kernel inclusion, including M, M_{loc}, N_{loc}, every actual letter part and every order module, is Lemma 7.7.26, item2. Its hypotheses are precisely this side extension with the same jump sets. Thus a vector over zero in the actual extension is uniquely a vector of the kernel lattice with exactly the same order tests. The same result, item1, supplies the constant-modulo-filtration interpretation of depth at every integer level.

We use two further consequences of the same actual order calculus. Frobenius shifts orders by

$$F_{l,b}o(a,u) = \begin{cases} b, & a < l, \\ o(a-l,u), & a \geq l. \end{cases} \tag{1}$$

It sends a point of orders o and depth h to one of orders $F_{l,0}o$ and depth $F_{l,1}h$. At $a \geq l$ this is the identity $\mu(\varphi^{-a}\varphi^l x) = \mu(\varphi^{-(a-l)}x)$. Below l, orders are zero by the positive-Frobenius tests for the localized module. For depth, use the exact factorization

$$g\varphi^l x - \varphi^l x = \varphi^l(\Delta)\,\varphi^l((gx - x)/\Delta).$$

The divided-difference quotient has the original depth orders by Lemma 7.7.24; after Frobenius it has shifted orders with zero below l. At $a < l$, $\varphi^a E$ divides $\varphi^l\tau$, so the displayed scalar factor raises those orders to one. This proves precisely $F_{l,1}h$. All localization denominators are units at the positive-Frobenius tests. Equal-constant differences have kernel orders the minimum of the endpoint orders.

For an equivariant $\mathcal{O}$-bilinear map, Lemma 7.7.26, item3, preserves the actual canonical and localized carriers and gives precisely the order inequality

$$h(a,v) \leq o_1(a,u) + o_2(a,u') \qquad (u + u' \leq v).$$

This is the full typed inequality, not merely equality of type sums. Its hypotheses will be checked explicitly below.

Point supply. By (DQ4), $\ell^\kappa E^s \mathrm{cst}_i(c)$ lifts to the standard order module of the extension. Retract this lift into its actual localized canonical module. The retraction commutes with the projection and fixes constants, by Lemma 7.7.21. Divide by E^s. This changes the level-zero order by $-s$ and no other order, and leaves the projection $\ell^\kappa \mathrm{cst}_i(c)$. Galois preserves every order module, so the Galois differences have the same lower orders. This proves item1, also for zero exponents or zero parameter.

Order safety. Put $D = (Y \otimes_{\mathcal{O}} Y')^\vee(1)$ and let $\nu_1 = \varphi^l\nu$; write $r = F_{l,0}o_\nu$. Strictness just proved identifies the two kernel vectors with $w_0 \in Y$ and $y_0 \in Y'$ in the actual localized order modules. Set

$$h(a,v) = 1 - r(a, -v - 1).$$

The assumed strict inequality, integrality of the exponents, and antitonicity of r imply $h(a,v) \leq o_w(a,u) + o_y(a,u')$ whenever $u + u' \leq v$. The bilinear rule therefore puts $w_0 \otimes y_0$ in the actual tensor module of orders h. The extension of D has the integral equivariant evaluation pairing

$$((\lambda, a), t) \longmapsto \ell^m\lambda(t) + a\psi(t)$$

with values in $\mathcal{O}(1)$; equivariance is precisely $g\psi - \psi = \ell^m c_P(g)$. A third-side type s and a tensor type v contributing to the Tate jump -1 satisfy $s+v \le -1$, hence $r(a,s)+h(a,v) \ge 1$. A second application of the bilinear rule shows that its value ξ_0 on $(\nu_1, w_0 \otimes y_0)$ has standard orders for $\mathcal{O}(1)$, whose sole jump is -1. Thus $\xi_0 \in M(\mathcal{O}(1))$ by the standard-order characterization of Lemma 7.7.21. The frame's height-one comparison gives $\tau\xi_0 \in A_{\mathrm{inf}} \otimes \mathcal{O}(1)$.

Reduce by the coefficient functional of P modulo ℓ^m. Since ν_1 lies over $\ell^\kappa \mathrm{cst}_i(1)$, this reduction of ξ_0 is $\ell^\kappa b_P(w_0, y_0)$. Multiplication by z takes every integral Witt vector into the small Witt part, because all coordinates of z have positive tilt valuation. Hence the reduction of $z\tau\xi_0$ is small. This proves both safe pairs in item2.

For the first-side move, divide $gq' - q'$ by Δ. The actual divided-difference and order-preservation statement Lemma 7.7.24 places the quotient in the actual kernel module with the depth orders of q'. Apply the second safe-pair assertion and cancel the two scalar factors Δ, Δ^{-1}. For a second-side move use $(gq - q)/\Delta$ and gy; the kernel/order module of y is Galois stable. This gives exactly safety of $(gq - q, gy)$. The extra localized-point hypothesis is thus retained where it is needed.

The four moves at a cut. Write

$$R_\eta(a,u) = -u - \eta[a=0], \quad D_{t,\eta,l}(a,u) = -u + t[a=0] + \eta[a<l],$$

and let $Q_{\eta,l}$ be the stop order in the statement. A killer with E-exponent t at the letter l steps back supplies a third point of orders R_t by item1. Define

$$B_{l,t}(a,v) = \begin{cases} 1, & a < l, \\ t - v, & a = l, \\ -v, & a > l. \end{cases}$$

There is the exact identity

$$F_{l,0}R_t(a, -u - u' - 1) + B_{l,t}(a, u + u') = 1. \tag{2}$$

Thus the order-safety rule applies whenever move orders plus resting-point depth are at least $B_{l,t}$.

Let $l, l_0 > 0$ be the current and preceding cut gaps, with killers t, t_0. Two elementary inequalities establish the outer moves:

$$B_{l,t_0}(a, u+v) \le \min(Q_{\eta,l}, F_{l,0}R_\eta)(a,u) + F_{l,1}D_{t_0,\eta,l_0}(a,v), \tag{3}$$

$$B_{0,t}(a, u+v) \le \min(R_\eta, Q_{\eta,l})(a,u) + D_{t,\eta,l}(a,v). \tag{4}$$

In (3), below l the summands are $0, 1$; at l they are $-u-\eta, -v+t_0+\eta$; above l they are $-u, -v+\eta[a-l<l_0]$. In (4), at zero they are $-u-\eta, -v+t+\eta$; between zero and l their sum is at least $-u-v$; from l onward they are $-u, -v$. These exhaust all levels and prove both inequalities.

Put $A = \varphi^l x_0$, $A' = \varphi^l x_0'$, for the preceding-cut points. If the first side has stop q while the second move $A' - x'$ is safe with q at rest, choose $w = q, w' = x'$. Inequality (3) proves the move $A - q$ safe with A' at rest; the middle move is the explicit stop hypothesis; (4) proves $q - x$ safe with x' at rest; the final move is zero. If the second side has stop q', choose $w = A, w' = q'$. The first move is zero, (3) proves the second, the central first-side move is the stop hypothesis, and (4) proves the fourth. This is exactly the required path

$$(A, A') \longrightarrow (w, A') \longrightarrow (w, w') \longrightarrow (x, w') \longrightarrow (x, x').$$

Each side retains its own constant and exponent; $e = e'$ is not assumed.

Summing the walk and checking the extra cocycle safety. For each letter let $d(i) = 0$ at a cut and $d(i) = l(i)$ otherwise, and set

$$X_i = \varphi^{d(i)} x(i^{(d(i))}), \qquad A_i = \varphi^{l(i)} x(i^{(l(i))}),$$

with primed analogues. The least-positive predecessor relation gives $l(i) = d(i^{(1)}) + 1$. Reindexing the finite letter set therefore gives $\varphi \sum_i X_i = \sum_i A_i$ and the primed identity. At cuts use the four moves above; elsewhere $A_i = X_i, A'_i = X'_i$ and the path is zero.

We must also check the additional cocycle-difference safety in the pairing lemma. If $X_i = \varphi^d x_j$, its depth is $H = F_{d,1} D_{t_0,\eta,l_0}$, with $l_0 > 0$. Both this depth and the kernel orders of $hX'_i - X'_i$ are H. The needed inequality is

$$B_{d,t_0}(a, u + v) \le H(a, u) + H(a, v).$$

For $a < d$ it says $1 \le 2$; at $a = d$ it says $t_0 - u - v \le 2t_0 + 2\eta - u - v$; above d the right side is $-u - v + 2\eta[a - d < l_0]$. Thus the second-side move by $hX'_i - X'_i$ with X_i at rest is safe for every h.

Different-letter pairings vanish: balance for the coefficient root gives $([i] - [j])b = 0$, and this scalar is a unit when $i \ne j$. All safety conditions consequently sum letter by letter. The constants sum to $\ell^e(1 \otimes c)$ and $\ell^{e'}(1 \otimes c')$, since the letter projectors sum to one. Apply item1 of Proposition 7.7.15 to the summed four-move path and the additional cocycle safety just checked. Its continuous coboundary has every exponent $N \ge e + e' + \kappa$. This proves item3. □

Lemma 7.7.26. *Put $K_{\mathrm{fix}} = \overline{\mathbb{Q}}_\ell^{j(\Gamma')}$, embedded canonically in B_{dR}.*

1. *For $y \in B_{\mathrm{dR}}$ and any integer n, if $gy - y \in \mathrm{Fil}^n$ for every $g \in \Gamma'$, there is $c \in K_{\mathrm{fix}}$ with $y - c \in \mathrm{Fil}^n$.*

   Let L be a lattice with letter jump sets $U(i)$, $x \in M_{\mathrm{loc}}(L)$ in the letter-i part, and $h : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$. Then

   $$gx - x \in \mathfrak{X}_L(i; h) \quad (g \in \Gamma')$$

   if and only if, for every $a \ge 0$, $u \in U(i^{(a)})$, and form λ of type u there, there is $c \in K_{\mathrm{fix}}$ such that

   $$\lambda(\varphi^{-a} x) - c \in \mathrm{Fil}^{h(a,u)}.$$

   If $x \in \mathfrak{X}_L(i; o)$ and $gx - x \in \mathfrak{X}_L(i; h)$ for every $g \in \Gamma'$, then x has orders

   $$(a, u) \longmapsto \begin{cases} h(a, u), & o(a, u) > 0, \\ \min(h(a, u), 0), & o(a, u) \le 0. \end{cases}$$

2. *Let S be a pairing side with extension $0 \to Y \xrightarrow{\iota} \tilde{Y}_S \to C \to 0$, with the same sets $S(i)$ used on Y and $\tilde{Y}_S$. Every form on Y extends to a form on $\tilde{Y}_S$; for each letter i and integer u, a form of type u at i has an extension of that same type. The induced inclusion of ambient modules strictly preserves all the following modules:*

   $$\iota^{-1} M(\tilde{Y}_S) = M(Y), \qquad \iota^{-1} M_{\mathrm{loc}}(\tilde{Y}_S) = M_{\mathrm{loc}}(Y),$$
   $$\iota^{-1} N_{\mathrm{loc}}(\tilde{Y}_S) = N_{\mathrm{loc}}(Y), \qquad \iota^{-1} \mathfrak{X}_{\tilde{Y}_S}(i; o) = \mathfrak{X}_Y(i; o).$$

   It commutes with the letter projectors. Thus every vector in any displayed module for $\tilde{Y}_S$ whose projection to the ambient module of C is zero is the image of a unique vector in the corresponding module for Y.

3. *Let L_1, L_2, L_3 be lattices with letter jump sets U_1, U_2, U_3, and let $b : L_1 \times L_2 \to L_3$ be an equivariant $\mathcal{O}$-bilinear map. Its $W(C^\flat)$-bilinear extension maps $M(L_1) \times M(L_2)$ into $M(L_3)$, and likewise maps the corresponding pairs of M_{loc} and of N_{loc} into those modules for L_3.*

   If $x_r \in \mathfrak{X}_{L_r}(i; o_r)$, $r = 1, 2$, and

$$h(a, v) \le o_1(a, u) + o_2(a, u')$$

   whenever $a \ge 0$, $u \in U_1(i^{(a)})$, $u' \in U_2(i^{(a)})$, $v \in U_3(i^{(a)})$, and $u + u' \le v$, then $b(x_1, x_2) \in \mathfrak{X}_{L_3}(i; h)$. Moreover $b(\mathrm{pr}_i x_1, x_2) = \mathrm{pr}_i b(x_1, x_2)$.

Proof. Write $B = B_{\mathrm{dR}}$ and $B^+ = B^+_{\mathrm{dR}}$ in this proof only. The image $j(\Gamma')$ is an open subgroup of G_ℓ, hence is $G_{K_{\mathrm{fix}}}$ for a finite extension $K_{\mathrm{fix}}/\mathbb{Q}_\ell$. The canonical inclusion $K_{\mathrm{fix}} \subset B^+$ reduces to its given inclusion in $\mathbb{C}_\ell$. In particular a nonzero element of K_{fix} has t-adic order zero.

Invariants modulo an integer filtration are constants. The graded piece $\mathrm{Fil}^j B/\mathrm{Fil}^{j+1} B$ is $\mathbb{C}_\ell(j)$. Tate's invariant calculation says that its $G_{K_{\mathrm{fix}}}$-invariants vanish for $j \neq 0$, whereas those for $j = 0$ are K_{fix}; see [298], Section 3, and [10]. Let $gy - y \in \mathrm{Fil}^n$ for all g. Choose an integer $j \le n$ with $y \in \mathrm{Fil}^j$, possible since $B = \bigcup_j \mathrm{Fil}^j$. If $j < n$, the class of y in the j-th graded piece is invariant. For $j \neq 0$ it is zero, so $y \in \mathrm{Fil}^{j+1}$. At $j = 0$ its class is the image of some $c \in K_{\mathrm{fix}}$; subtract this fixed constant, obtaining an element of Fil^1 with the same difference condition. Continue through the finitely many indices below n. This gives $y - c \in \mathrm{Fil}^n$, with $c = 0$ if index zero was not encountered. This argument includes negative n.

For an element in $M_{\mathrm{loc}}(L)$, some nonzero element of A_{inf} multiplies it into $A_{\mathrm{inf}} \otimes L$: multiply its localization denominator by the power of τ in the definition of $M(L)$. Thus values of forms on it are determined by their restrictions to $1 \otimes L$. Equivariance on that lattice and A_{inf}-linearity imply

$$\lambda(\varphi^{-a}(gx - x)) = g\lambda(\varphi^{-a}x) - \lambda(\varphi^{-a}x). \tag{1}$$

One proves this first on a finite sum of elementary tensors and then clears a nonzero A_{inf}-denominator; its image in B is nonzero.

The modules $M_{\mathrm{loc}}(L)$ are Galois stable. Indeed $g\tau = (1 + \tau)^{\chi_{\mathrm{cyc}}(g)} - 1$, so $g\tau/\tau$ is a unit: the convergent binomial series for this quotient has constant coefficient $\chi_{\mathrm{cyc}}(g) \in \mathbb{Z}_\ell^\times$ and its remaining terms lie in the maximal ideal of A_{inf}. The element E generates $\ker\theta$ up to a unit; Galois preserves that ideal and commutes with Frobenius, so it also preserves each ideal $(\varphi^a(E))$. The integral lattice is stable, and (1) preserves the positive-Frobenius integrality tests for $M(L)$. The letter parts are stable because Γ' fixes the chart's unramified coefficient ring. Apply the constant assertion to each value in (1). Conversely, if that value is a constant modulo $\mathrm{Fil}^{h(a,u)}$, its Galois difference lies in the same filtration, since g fixes the constant and preserves the filtration. The definition of $\mathfrak{X}$ now gives exactly the equivalence in item 1.

For the improvement, fix a tested value $v = \lambda(\varphi^{-a}x)$, put $r = o(a, u)$, $s = h(a, u)$, and write $v - c \in \mathrm{Fil}^s$, $c \in K_{\mathrm{fix}}$. Since $v \in \mathrm{Fil}^r$, one has $c \in \mathrm{Fil}^{\min(r,s)}$. If $r > 0$ and $s > 0$, this forces $c = 0$, so $v \in \mathrm{Fil}^s$. If $r > 0$ and $s \le 0$, the same conclusion already follows from $\mathrm{Fil}^r \subseteq \mathrm{Fil}^s$. If $r \le 0$, the fact $c \in \mathrm{Fil}^0$ gives $v \in \mathrm{Fil}^{\min(s,0)}$. These are precisely the stated orders.

Every kernel form extends with its specified letter type. We record how a form is extended to an ambient module when only its values on the finite lattice have been prescribed. Put $Q = \mathrm{Frac}(A_{\mathrm{inf}})$, with its canonical inclusion in B. The ambient module is A_{inf}-torsion free. After localizing it at the nonzero elements of A_{inf}, it is a Q-vector space containing the Q-span of $1 \otimes L$. Choose a Q-linear complement of that finite-dimensional span. A prescribed $\mathbb{Z}_\ell$-linear map $L \to B$ extends Q-linearly on the span and by zero on the complement; restriction

to the ambient module is an A_{inf}-linear functional with exactly those prescribed lattice values. Equivariance of a form is required on the lattice, so equivariant prescribed values produce a form. Values on all denominator-clearable vectors are independent of this extension choice.

Let μ be a form on Y, and first suppose $\mu(1\otimes Y)\subseteq \mathrm{Fil}^n$. Choose a $\mathbb{Z}_\ell$-basis of the parameter lattice C. For each basis vector c, the crystalline cocycle $\alpha(c)$ has a witness $w_c\in B_{\max}\otimes_{\mathbb{Z}_\ell} Y$ with $gw_c - w_c = 1\otimes\alpha(c)(g)$. Extend $\mu|_Y$ B-linearly on $B\otimes Y$, and set $b_c=\mu(w_c)$, using $B_{\max}\subset B$. Equivariance of this extension gives

$$gb_c - b_c = \mu(1\otimes\alpha(c)(g))\in \mathrm{Fil}^n \quad (g\in\Gamma').$$

By the first part, choose $a_c\in K_{\mathrm{fix}}$ with $b_c-a_c\in\mathrm{Fil}^n$, and extend $c\mapsto b_c-a_c$ $\mathbb{Z}_\ell$-linearly. On the lattice $Y\oplus C$, prescribe

$$\nu(y,c)=\mu(1\otimes y)+(b_c-a_c). \tag{2}$$

The equation for b_c and the action $g(y,c)=(gy+\alpha(c)(g),c)$ show directly that $\nu(g(y,c))=g\nu(y,c)$. All prescribed values lie in Fil^n. Extend to the ambient module by keeping the original μ on its Y-summand and extending the C-values as just described. This gives a form extending the actual μ, with the same global filtration bound on the lattice. Without a filtration bound the same construction uses b_c itself and gives an extension of every form.

Now suppose only that μ is of type u at a specified letter i. Its two summands $\mu_i=\mu\circ\mathrm{pr}_i$ and $\mu_{\neq i}=\mu\circ(1-\mathrm{pr}_i)$ are forms, since the letter projectors commute with Γ'. The first has all lattice values in Fil^{-u}. Extend it by (2), with $n=-u$, to ν_i; extend $\mu_{\neq i}$ without a bound to $\nu_{\neq i}$. Then

$$\nu=\nu_i\circ\mathrm{pr}_i+\nu_{\neq i}\circ(1-\mathrm{pr}_i) \tag{3}$$

restricts exactly to μ. It has type u at i: the second term vanishes there, while pr_i carries the integral lattice into its A_{inf}-span and A_{inf} maps into B^+, so the first term still takes that lattice into Fil^{-u}. This proves the typed extension assertion for every integer u.

The inclusion ι is a split inclusion of underlying $\mathcal{O}$-modules; hence its ambient extension is injective and reflects the A_{inf}-span of the integral lattice. It commutes with Frobenius and the letter projectors. Restricting extension forms gives kernel forms, and the extension assertions prove the converse testing implication. Therefore the τ-saturation and every positive-Frobenius test defining M are both preserved and reflected. Clearing the same finite product of $\varphi^a(E)$ proves the assertion for M_{loc}. Testing values in the same field $\mathcal{K}_0$ with all forms gives the assertion for N_{loc}. Testing the same integer type and order with the typed extensions (3) gives the assertion for each $\mathfrak{X}(i;o)$; the sets $S(i)$ are identical on the two sides, as stipulated. Finally exactness of the underlying direct sum $Y\oplus C$, and hence of its ambient modules, identifies the kernel with the image of ι. This proves item 2 without a retraction.

An adapted invariant Hodge basis gives the precise tensor inequality. We give the finite-dimensional comparison used here explicitly. For a lattice L, put $V=L[1/\ell]$ and $D=(B\otimes_{\mathbb{Q}_\ell}V)^{G_{K_{\mathrm{fix}}}}$. The crystalline hypotheses give a continuous de Rham representation by Lemma 7.4.6. The de Rham comparison and its graded Hodge–Tate comparison ([44], Theorem 5.2.1 and Proposition 6.3.2) give $\dim_{K_{\mathrm{fix}}}D=\dim_{\mathbb{Q}_\ell}V$ and the comparison isomorphism. The letter projectors are defined over K_{fix} and preserve the filtrations. In each letter part choose a K_{fix}-basis d_m of D adapted to its filtration, with filtration jumps u_m. The graded comparison says that the reductions of $t^{-u_m}d_m$ are a basis of the corresponding $\mathbb{C}_\ell$-space: they are exactly its Hodge–Tate comparison basis. Thus

$$e_m=t^{-u_m}d_m$$

is a B^+-basis of the corresponding integral comparison lattice. Indeed its matrix has entries in B^+ and has invertible reduction modulo t, so its determinant is a unit. The integers u_m occur

among the actual jumps at that letter, hence among its stipulated jump set. This also checks the sign: the dual invariant form λ_m, characterized by $\lambda_m(d_n) = \delta_{mn}$, takes e_m to t^{-u_m}, and has type u_m. It extends to an actual ambient form by the extension-of-functionals argument above, with zero values on the other letter parts. All evaluations below are on denominator-clearable vectors, so this extension choice has no effect.

Let λ be any form on L_3, of type v at a letter j. Choose such invariant bases d_m, d'_n for the letter-j parts of L_1, L_2, and let λ_m, λ'_n be their invariant dual forms. Put

$$c_{mn} = \lambda\big(b(d_m, d'_n)\big) \in B.$$

Every c_{mn} is $G_{K_{\mathrm{fix}}}$-invariant, by equivariance of b, λ and invariance of the two basis vectors. Thus $c_{mn} \in K_{\mathrm{fix}}$, by the invariant-field calculation of the de Rham comparison. As b is integral and $\mathcal{O}$-bilinear, its extension sends the two letter-j integral comparison lattices into the letter-j integral comparison lattice of L_3. Therefore

$$\lambda\big(b(e_m, e'_n)\big) = t^{-u_m - u'_n} c_{mn} \in \mathrm{Fil}^{-v}.$$

A nonzero $c_{mn} \in K_{\mathrm{fix}}$ has t-adic order zero. Consequently

$$c_{mn} \neq 0 \quad \Longrightarrow \quad u_m + u'_n \leq v. \tag{4}$$

For denominator-clearable vectors y, x in these letter parts, finite-dimensional bilinearity gives the exact finite expansion

$$\lambda(b(y, x)) = \sum_{m,n} c_{mn} \lambda_m(y) \lambda'_n(x). \tag{5}$$

No assertion about arbitrary analytic expansions is involved.

Apply (5) to $y = \varphi^{-a} x_1$, $x = \varphi^{-a} x_2$ at $j = i^{(a)}$. Frobenius commutes with the extended pairing. Each summand with nonzero coefficient lies in

$$\mathrm{Fil}^{\,o_1(a,u_m) + o_2(a,u'_n)} \subseteq \mathrm{Fil}^{h(a,v)}$$

by (4) and the hypothesis. Coefficients in K_{fix} preserve every filtration. Finite addition proves the required target test for every form of type v at j.

It remains to check the actual module carriers in item 3. The extension of an integral bilinear map preserves A_{inf}-spans of lattices. Thus powers of τ clearing two inputs multiply to a power clearing their pairing. For any positive Frobenius level and any target form, use (5) on the full spaces, summing over all letter parts and omitting the type restriction. Its coefficients are still in $K_{\mathrm{fix}} \subset B^+$. Every factor value lies in B^+ by the positive-Frobenius tests defining $M(L_1), M(L_2)$, so the output passes every such test. This proves preservation of M. Multiplying the two localization denominators proves preservation of M_{loc}. If the inputs lie in N_{loc}, their values under every invariant dual form in (5) lie in the same field $\mathcal{K}_0$; its inclusion of K_{fix} and closure under sums and products prove preservation of N_{loc}.

Finally $\mathcal{O}$-linearity in the first variable implies $b(\zeta x_1, x_2) = \zeta b(x_1, x_2)$. Apply the displayed polynomial formula for pr_i in (BF2) to obtain $b(\mathrm{pr}_i x_1, x_2) = \mathrm{pr}_i b(x_1, x_2)$. The output is therefore in the required letter part, and the target order tests already proved put it in $\mathfrak{X}_{L_3}(i; h)$. □

Proposition 7.7.27. *Let k be a Kummer chart for Γ, S, S' pairing sides over $\mathcal{O}$ paired by P, with $c \in C$, $c' \in C'$. Let $\eta \geq 1$, $V, G, b \in \mathbb{N}$, a predicate T on letters (the* thick *letters) and signs $\varepsilon(i)$. Assume that at every letter i the jump set $S(i)$ fits the windows $(\eta, -\varepsilon(i), b_i, V)$ and $S'(i)$ fits $(\eta, \varepsilon(i), b_i, V)$, where $b_i = b$ if $T(i)$ and $b_i = G$ otherwise.*

1. *(Sparse thick letters.) Let $G + V \le \eta$ and $J \ge 1$, and let no three thick letters follow one another along the letter Frobenius with both gaps in $[1, J]$. Assume that either k is unramified, $\ell \ge 7$ and $10\ell^J\eta \le b$, or ε is constant and $8e \le \ell^{J+1}$. Let $\delta, X \in \mathbb{N}$ with $2e(b + 5\eta) \le \ell^{J+1}\delta$ and $20(J + 1 + d)\eta + \lfloor \log_\ell 2(b + 5\eta) \rfloor \le X$. Then P is a scaled cup coboundary at $(10^3(\delta + (J + 1)X), c, c')$.*

2. *(Any pattern.) Let k be unramified, $\ell \ge 7$, $0 < G \le b$ and $V + 4\eta \le W \le K$, with $b < K\ell^2$ and $2G < K\ell$. Then P is a scaled cup coboundary at $(10^4(K + \lfloor \log_\ell 2(b + W) \rfloor), c, c')$.*

Proof. We give a common cut argument and verify both choices of cuts. Throughout this proof $p = \ell$. Write $i^{(a)}$ for the letter a steps before i, and put

$$R_t(a, u) = -u + t[a = 0], \quad D(t, \eta, l)(a, u) = -u + t[a = 0] + \eta[a < l],$$
$$Q_{\eta,l}(a, u) = \begin{cases} \max(0, -u - \eta), & a < l, \\ -u, & a \ge l. \end{cases}$$

The shift of an order system by φ^l is zero below l and its old order at $a - l$ from l on. The depth shift is one below l, as proved in Proposition 7.7.25. Set

$$C_{s,\eta,l} = \mathrm{imp}(R_{-s}, D(s, \eta, l + 1)), \qquad c_a(m) = v_p(|m|) + d + 3a, \quad v_p(0) = 0. \tag{1}$$

The finite-step statement Lemma 7.7.28 raises a finite set of steps at the sum of these costs and gives the improved orders in (1).

A common finite scheme. Use auxiliary width w, starts β_i, shifts s_i, and nonnegative integers $B, n, J, s_0, G_0, X, \delta, \kappa$ satisfying

$$\begin{gathered} V + 4\eta \le w, \quad n + \lambda + (J + 1 + d)w + dG_0 \le X, \qquad \lambda = \lfloor \log_p(2B) \rfloor, \\ 12((J + 1)X + \delta) < \kappa, \quad e(2B) \le p^{J+1}\delta, \quad 8e \le p^{J+1}, \\ \beta_i + s_i + w \le 2B, \quad G_0 \le (p - 1)n, \quad s_i + w \le pn, \\ \beta_i + w \le (p - 3)s_0 \quad \text{if } p \ge 5. \end{gathered} \tag{2}$$

A permitted cut i is of either of these two kinds: $s_i = 0$ and $\beta_{\sigma^t i} + w \le G_0$ for $1 \le t \le J$; or the chart is unramified, $p \ge 5$, $s_i < \beta_i$ and $s_0 \le \kappa + s_i$. Here σ is the forward letter Frobenius, so these next-J conditions have the orientation required by Proposition 7.7.15(4).

At a first-kind cut the forward-letter bound supplies the killer p^κ. Indeed every jump of either side and of the third side is at most $\beta_i + w \le 2B$. For the third side its nonzero jumps are $-u - v - 1$: when both original jumps are far, their opposite signs leave absolute sum at most V; when one is low the upper bound is $\beta_i + V + \eta$; when both are low it is 2η. These are all at most $\beta_i + w$, using $\eta \ge 1$ and $V + 4\eta \le w$. The three terms in that forward bound are at most

$$2JX + (2J + 4)X + 8\delta < \kappa$$

by (2). At a second-kind cut, the unramified bound supplies $p^\kappa E^{s_i}$ since all jumps are at most $(p - 3)s_0 \le (p - 3)(\kappa + s_i)$. Thus all three sides have this killer at every permitted cut. Also $s_i = 0$ or both $s_i < \beta_i$ and $d = 0$.

We record an explicit finite cost bound. At level $a < 2J + 2$, take the union of the two closed integer windows of radius $r + w$, centered at zero and at the negative far start appropriate to that side, where $r = s_i$ for $a = 0$ and $r = 0$ otherwise. It has at most $4(r + w) + 2$ integers; every integer in it has absolute value at most $2B$. For nonzero integers count their multiples of

p^v, $1 \le v \le \lambda$. Each interval of length $2(r+w)$ has at most $2(r+w)/p^v + 1$ such multiples. Therefore the sum of their valuations is at most

$$\frac{4(r+w)}{p-1} + 2\lambda \le 8n + 2\lambda.$$

The zero integer costs no valuation. If $r > 0$, then $a = 0$ and $d = 0$, so there is no remaining cost. Otherwise $4w + 2 \le 5w$ since $w \ge 4$, and $d + 3a \le 6(J + 1 + d)$. Hence the cost at this level is at most $8n + 2\lambda + 30w(J+1+d) \le 30X$. Summing gives total cost at most $60(J+1)X$, and consequently

$$\kappa + 4\delta + \sum_{\text{windows}} c_a(m) \le 25\kappa. \tag{3}$$

For the last inequality subtract the left side from 25κ and use $24\kappa > 288((J+1)X + \delta)$.

Let the least positive distance to the preceding cut be $l \le 2J + 1$. A killer point starts with orders and depth R_{-s_i}. Deepen it to $D(s_i, \eta, l+1)$. The required steps occur only at $a \le l$. At zero their endpoints are $-u - s_i \le m < -u + s_i + \eta$, and at $1 \le a \le l$ they are $-u \le m < -u + \eta$. A low jump puts these steps within radius $s_i + 2\eta$ of zero at level zero, and 2η at other levels. A far jump puts them within radius $s_i + V + \eta$, or $V + \eta$, of the appropriate far start. All lie in the windows just counted. Thus (3) supplies points over $p^{25\kappa}$ with own orders $C_{s_i,\eta,l}$ and depth $D(s_i, \eta, l+1)$.

These points have the pole bound $R_{-\eta}$. At level zero, if $u \ge -s_i$ is a jump then $u \ge -\eta$: this is immediate for $s_i = 0$, and otherwise a negative far jump would have $u \le -\beta_i < -s_i$. If $u < -s_i$, the positive initial order improves to the depth. Otherwise $-\eta - u \le 0$ and $-\eta - u$ is at most the depth, hence at most its minimum with zero. At positive levels the improved orders dominate $-u$ directly. Their depth is exactly the plain depth

$$D(s_i, \eta, l+1) = D(s_i, \eta, l) + \eta[a = l]. \tag{4}$$

Plain stops. Suppose all letters between the preceding cut and the current one have one common sign, except for far windows ending at or below η. One side is down, with jumps $u \le \eta$ below l, and its partner is up, with jumps $u \ge -\eta$ there. The down point itself is a stop. At zero and at $0 < a < l$, its improved order is nonnegative when the initial order was nonpositive, and otherwise its depth dominates $\max(0, -u - \eta)$. At $a = l$ and above its orders dominate $-u$.

For completeness the central safe move has a separate numerical check. For the moving up side the minimum of the current and preceding transported pole bounds is

$$M(a,u) = \begin{cases} \min(-u - \eta, 0), & a = 0, \\ \min(-u, 0), & 0 < a < l, \\ -u - \eta, & a = l, \\ -u, & a > l. \end{cases}$$

The resting depth (4) is respectively $-v + s_i + \eta$, $-v + \eta$, $-v + \eta$ and $-v$. Against the third-side killer at the current cut the required complementary bound is $s_i - u - v$ at zero and $-u - v$ above. The residuals are $\min(0, u + \eta) = 0$, $\eta + \min(0, u) \ge 0$, zero and zero. This proves the safe central move, including the essential extra η at level l.

The long-gap point from the frame radius. We prove the actual point supplier needed when signs change. Work on an unramified chart. Let $L > 0$, $3 \le p^L$, $A \le p^L\delta$. Suppose E^s kills at a letter j, all jumps at its L preceding levels are at most g, and

$$N = s + g \sum_{a<L} p^a < A. \tag{5}$$

Suppose p^{κ} kills at $j^{(L)}$. Then there is a point at j, over $p^{\kappa+4\delta}c$, with own orders $\mathrm{std}_L + A[a = 0, u < -N]$ and depth $\mathrm{std}_L + A[a = 0]$, where std_L is zero below L and $-u$ from L on.

Here is a proof on the canonical module, including the parameter correction. Take a point y over c of orders R_{-s}, and multiply it by

$$F(E) = E^s \prod_{a<L} E_a^g, \qquad E_a = (E-p)^{p^a} + p.$$

This is monic as a polynomial in E, of degree N. Its orders are at least $g - u$ below L and $-u$ above, hence it belongs to the actual standard module of level L. Apply Proposition 7.7.16's radius assertion to it with parameters L, A, δ. It writes $p^{4\delta}F(E)y$ as a finite $\mathfrak{S}$-linear combination of points in the same $\varphi^L M$ with all form values invariant modulo Fil^A. Project to the letter j. All these projected points have standard orders and Galois differences with those orders raised by A at level zero: equivariance of every form and commutation with the letter projector prove this assertion. Call such points constants modulo E^A.

Every scalar in $\mathfrak{S} = \Lambda[[z]]$ has finite expansions to every order in $E = z + p$, with coefficients in Λ, by repeated division with constant remainder. Expanding the finite scalar combination therefore gives

$$p^{4\delta}F(E)y = \sum_{h \le N} E^h \xi_h + E^{N+1} y_1, \tag{6}$$

where each ξ_h is a constant modulo E^A and y_1 has standard orders. Coefficient extraction is legitimate with these integral modules. If $\xi + Ey'$ has a positive level-zero order on a given jump, then ξ has positive order there; its invariance modulo E^A and the improved-order rule raise that order to A. Subtract ξ and divide by E; induction proves that in an expansion (6), the coefficient ξ_N has order A on every line where the whole expression has order greater than N, provided the target order is at most A. Apply this with target $\min(A, g - u)$. In particular ξ_N has order A at $u < -N$.

Apply the same coefficient argument to the trivial quotient, subtracting the scalar expansion of $p^{4\delta}F(E)\mathrm{cst}_j(c)$. Its difference expansion is zero, so every line has target order A. Monicity and $N < A$ give

$$w = \pi\xi_N - p^{4\delta}\mathrm{cst}_j(c) \in E^A(\text{trivial standard module of level } L).$$

The transported pure-power killer at $j^{(L)}$ lifts $p^{\kappa}E^{-A}w$ to an actual vector v in the extension's same standard module. Then

$$x = p^{\kappa}\xi_N - E^A v$$

lies over $p^{\kappa+4\delta}c$ and has exactly the asserted lower bounds for its own orders and depth. All operations were on the actual modules and maps; no independent frame or merely isomorphic lattice has been substituted.

Now let a current cut have preceding distance l, with a pivot at level t and $l = t + J + 1$. Assume the chart is unramified, $p \ge 5$, the preceding cut has shift zero, and for some $\omega \ge \eta$

$$s_0 + \omega \sum_{a=0}^{J} p^a < \beta_{i^{(t)}}. \tag{7}$$

Assume every nonpivot level below l has the opposite sign to the pivot or a far window ending at most η, and every level strictly between pivot and preceding cut has far endpoint at most ω. The side with negative far window at the pivot has jumps there at most η, and at the other J levels before that cut at most ω. It has killer E^{s_0} at the pivot by (2), and p^{κ} at the preceding

cut. Apply (5) with $L = J + 1$, $A = 2B$, $s = s_0$, $g = \omega$. The inequality $N < A$ follows from (7) and $\beta \le 2B$; the radius conditions follow from (2).

Transport this point by φ^t to the current cut. Its exact (orders,depth) table is

$$\begin{array}{c|c|c} a < t & 0 & 1 \\ a = t & 2B\,1_{u<-N} & 2B \\ t < a < l & 0 & 0 \\ a \ge l & -u & -u. \end{array} \tag{8}$$

Raise only the depths $(a, 0)$ for $t < a < l$ to one. These steps belong to the counted windows, so (3) gives a point over $p^{25\kappa}c$. Its depth is $2B$ at the pivot, one elsewhere below l, and $-u$ from l on. It is a stop: nonpivot jumps are at least $-\eta$, so their required stop order is zero; at the pivot low jumps again require zero, whereas a negative far jump satisfies $u \le -\beta < -N$ and has available order $2B \ge \beta + V - \eta \ge -u - \eta$. This covers the endpoints as well.

The central moving side has jumps $u \le \eta$ at every nonpivot level below l, and $u < \eta + 2B$ at the pivot. Its difference orders are the minimum of $C_{s_i,\eta,l}$ and the shift by l of C_{0,η,l_0} at the preceding cut. For $a < l$ this minimum is at least $\min(\eta - u, 0)$; off the pivot it is zero, so adding resting depth one gives the required bound one. At the pivot, if $u \le \eta$ its sum with $2B$ is at least one; otherwise integer strictness gives $\eta - u + 2B \ge 1$. At $a \ge l$ both cut orders are at least $-u$, so adding $-v$ gives $-u - v$. These are precisely the complementary bounds for the transported third killer p^κ from the preceding cut. The safe-move rule proves the central move. Which side rests is determined by the sign at the pivot.

We have proved the following consequence of (2): if every letter has a permitted cut at distance at most $2J + 1$, and every interval after a nearest cut has either the plain common-sign form or the long-gap form (7), then the walk of Proposition 7.7.25 applies. Its point exponents are 25κ on each side and its third exponent is κ, so it gives every cup exponent at least 51κ.

Sparse thick letters. Set

$$w = 5\eta,\ B = b + 5\eta,\ n = G_0 = 10\eta, \quad s_i = 0, \quad \beta_i = b_i, \quad R = (J + 1)X + \delta, \quad \kappa = 12R + 1.$$

For $p \ge 5$ choose $s_0 = \lfloor (b + 10\eta)/(p - 3) \rfloor + 1$; if $p < 5$ put $s_0 = 0$, since no condition involving it is used. We verify all of (2). The cost left side is $\lambda + \eta(5J + 15 + 15d)$, at most X by the assumed $\lambda + 20(J + 1 + d)\eta \le X$. The strict κ condition and depth condition are immediate. The ramification bound is assumed in the constant-sign alternative; in the unramified alternative $e = 1$, $p \ge 7$, $J \ge 1$, so $8 \le p^{J+1}$. Since $G, V \le \eta$, $\beta_i + w \le 2B$, $V + 4\eta \le w$, $G_0 \le (p - 1)n$, and $w \le pn$. Finally $\beta_i + w \le b + 10\eta \le (p - 3)s_0$ by Euclidean division. These include the cases $b = 0$ and $\delta = 0$ whenever allowed by the original hypotheses.

Cut at a letter exactly when its next J successors are thin. Such a letter is a first-kind permitted cut since $\beta + w = G + 5\eta \le 6\eta \le G_0$ on these successors. We give the finite combinatorial argument. If a thick letter is not a cut, its predecessor at distance $J + 1$ is a cut. Otherwise a thick letter among the next J successors of that predecessor, the original thick letter, and a thick successor witnessing its failure to be a cut would form three thick letters with both gaps in $[1, J]$, contrary to sparseness. Every letter has a cut within $2J + 1$ steps back: test its predecessor at distance $J + 1$; if this is not a cut, a thick successor of it lies at distance $r \in [1, J]$ back from the original letter. If that thick letter is not a cut, its predecessor at distance $J + 1$ is one, at distance at most $2J + 1$.

For a nearest cut at distance l, a thick position $0 < r < l$ must satisfy $r + J + 1 = l$. The cut's next J successors are thin, giving $l - r \ge J + 1$; if strict, the preceding paragraph would give a nearer cut at $r + J + 1$. Thus only positions zero and $t = l - J - 1$ can be thick in this interval.

Test the plain condition with the sign at position zero. If it fails, an opposite-sign position $0 < t < l$ has $\beta + V > \eta$, and so is thick. All other nonzero positions are thin, the original position has the opposite sign to the pivot, and all positions strictly beyond the pivot have far endpoint at most η. The constant-sign alternative is impossible, so the chart is unramified, $p \ge 7$ and $b \ge 10p^J\eta$. Taking $\omega = \eta$, the strict long-gap condition holds:

$$s_0 + \eta \sum_{a=0}^{J} p^a \le \frac{b + 10\eta}{4} + 1 + 2p^J\eta < b.$$

Indeed with $A_0 = p^J\eta \ge 7$, $\eta \le A_0$ and $b \ge 10A_0$, the difference from b is at least $3A_0 - 1 > 0$. This proves all long-gap hypotheses, including the zero shift at the preceding cut. The common scheme therefore gives exponent 51κ. Since $X \ge 20(J + 1 + d)\eta \ge 40$, we have $R \ge 1$, and

$$51\kappa = 612R + 51 \le 663R \le 10^3 R.$$

This is the first assertion.

An arbitrary pattern. Put

$$q = \lfloor b/p \rfloor, \quad H = \lfloor \log_p(2(b + W)) \rfloor, \quad s_i = \begin{cases} (q - 7K)_+, & T(\sigma i), \\ 0, & \text{otherwise}, \end{cases}$$

and use

$$J = 1, \quad n = K, \quad B = b + W, \quad w = W, \quad G_0 = G + W,$$
$$\delta = 4K, \quad X = 3K + H, \quad s_0 = q + 12K, \quad \kappa = 120K + 24H + 1.$$

Here $K \ge W \ge 4\eta \ge 4$, $q < Kp$, and $pq \le b < p(q + 1)$. The cost condition in (2) is $K + H + 2W \le 3K + H$. The strict κ condition is equality plus one. The depth bound follows from $2(b + W) \le 4Kp^2$, and $8 \le p^2$. Also $\beta_i + s_i + W \le b + q + W \le 2(b + W)$, and $2(G + W) < Kp + 2K \le 2(p - 1)K$. For the remaining far bound,

$$(p - 3)(q + 12K) - (b + K) > -3q + (12p - 37)K - p > (9p - 37)K - p \ge (8p - 37)K > 0.$$

Thus $\beta_i + W \le (p - 3)s_0$. The shift bound follows by cases: if $q \le 7K$ then $s_i + W \le K \le pK$; otherwise it is at most $q - 6K < pK$. Hence every clause of (2) holds.

Choose as cuts the letters with $s_i < \beta_i$. Thick letters are cuts since $s_i \le q < b$. Every letter with thin successor is also a cut, since its shift is zero and $G > 0$. Consequently there is a cut one or two steps back from every letter. A cut with thin successor is of the first permitted kind. A cut with thick successor is of the second kind because $s_0 = q + 12K \le s_i + 19K \le s_i + \kappa$; if $q < 7K$ the same inequality follows with $s_i = 0$. Thus all cuts are permitted.

If the nearest distance is one, the plain condition is automatic. If it is two and the previous sign equals the current sign, the same is true. In the remaining case the previous letter is not a cut, so it is thin and the current letter is thick; moreover $G \le s_{\mathrm{prev}} \le (q - 7K)_+$, hence $q \ge G + 7K$. The cut two steps back has shift zero. Take pivot zero and $\omega = G + W$. Then $\omega \le q - 6K$ and

$$s_0 + \omega(1 + p) \le pq + 2q + (6 - 6p)K < pq + (6 - 4p)K < pq \le b.$$

The nonpivot letter has opposite sign, its far endpoint is ω, and $\eta \le W \le \omega$. This verifies every long-gap condition. All nearest-cut possibilities have now been covered, including zero shifts and floor equalities. The common scheme gives exponent 51κ, and

$$51\kappa = 6120K + 1224H + 51 \le 10^4(K + H)$$

since $K \ge 1$. This proves the second assertion. □

Lemma 7.7.28. *Fix a side with extension lattice L, quotient $\pi : L \to C$, and subgroup Γ'. Write $\mathcal{N} = N_{\mathrm{loc}}(L)$. A vector over a constant at letter i is a vector $x \in \mathcal{N}$ with $\pi x = \mathrm{cst}_i(c)$ for some c. It has depth h if $gx - x \in \mathfrak{X}_L(i;h)$ for every $g \in \Gamma'$. An additive map $A : \mathcal{N} \to \mathcal{N}$ has cost e if for every such i, h, x,*

$$Ax - p^e x \in \ker \pi \cap \mathfrak{X}_L(i;h).$$

Put $c_a(n) = v_p(|n|) + d + 3a$, with $v_p(0) = 0$.

1. *For every $a \geq 0, n \in \mathbb{Z}$, there is an additive map B of cost $c_a(n)$ such that, simultaneously for every $x \in \mathcal{N}$ and invariant form λ,*

$$\left[\forall g,\ \lambda(\varphi^{-a}(gx - x)) \in \mathrm{Fil}^n\right] \Longrightarrow \left[\forall g,\ \lambda(\varphi^{-a}(gBx - Bx)) \in \mathrm{Fil}^{n+1}\right].$$

2. *For every finite $I \subset \mathbb{N} \times \mathbb{Z}$, there is one additive map A_I of cost $\sum_{(a,n)\in I} c_a(n)$. For every letter i, vector over a constant of depth h, and function h', it has depth h' after applying A_I, provided*

$$h(a,u) \leq n < h'(a,u) \Longrightarrow (a,n) \in I$$

for every a, every actual jump u at the letter a steps before i, and every integer n.

All operators are chosen before the vector, form, letter and depth functions. They preserve depths already attained.

Proof. We write $\theta_a = \theta\varphi^{-a}$, $E_a = \varphi^a E$, $\delta = z\tau$, and use the normalization $v_p(p) = 1$. The image $\pi_a = \varphi^{-a} E_a$ in B^+_{dR} is a uniformizer. Also $\varphi^{-a}\delta$ is a unit there: its image under θ is nonzero, since z is a nonzero Teichmuller lift and $\theta_a\tau$ is a nontrivial root of unity minus one.

A scalar first-order observation. Let q_0 be the coefficient retraction of Lemma 7.7.21. If $v \in \mathcal{K}_a \cap \mathrm{Fil}^0$, choose an integer $b \geq 0$ such that $p^b\theta(v)$ is integral, and then choose $r \in A_{\mathrm{inf}}$ with $\theta_a(r) = p^b\theta(v)$. Thus $\varphi^{-a}(r) - p^b v \in \mathrm{Fil}^1$. Apply the scalar coefficient law of Lemma 7.7.22 to this two-term relation, using $q_0(1) = 1$. It gives

$$v - \varphi^{-a}(p^{-b} q_0(r)) \in \mathrm{Fil}^1. \tag{1}$$

Consequently a finite sum of Galois operators which is invariant on S modulo E_a is invariant on $\mathcal{K}_a \cap \mathrm{Fil}^0$ modulo Fil^1. Indeed use (1), linearity, and the Galois stability of the filtration; division by p^b does not affect the de Rham filtration.

The scalar step at a nonzero integer. Put $q = [\varepsilon]$. Choose $g \in \Gamma'$ with $gz = q^{p^a} z$. Write $gE = \omega_0 E$, with $\omega_0 \in A^\times_{\mathrm{inf}}$, and $\omega = \varphi^a\omega_0$. Then

$$gE_a = \omega E_a, \qquad E_a \mid gs - s \quad (s \in S), \qquad \delta \mid \omega^n - 1 \quad (n \in \mathbb{Z}).$$

For the second assertion apply θ_a: q^{p^a} has image one and g fixes the unramified coefficients. For the third, formal power-series subtraction gives $\delta E \mid gE - E$, hence $\delta \mid \omega_0 - 1$; Frobenius sends δ to a multiple of δ. Negative powers cause no change because ω_0 is a unit.

For $n \neq 0$, set $X = (\omega^n - 1)/\delta$. We claim

$$v_p(\theta_a X) \leq d + 3a + v_p(|n|). \tag{2}$$

Here are the derivative and valuation details. By Weierstrass preparation the series $E = p + z^e A(z)$ is a unit times a monic Eisenstein polynomial F of degree e; write $F(z) = EG(z)$. If $e = 1$, $F' = 1$ and $d = 0$. Otherwise, at the root θz, the nonzero terms $jF_j(\theta z)^{j-1}$ have distinct

fractional valuations $(j-1)/e$; the minimum is unique and is at most $v_p(e)+(e-1)/e<d$. Thus $v_p(F'(\theta z))\le d$. Taylor expansion of $F(gz)$, followed by cancellation of E and application of θ, yields

$$\theta(G)\theta(\omega_0-1)=F'(\theta z)\,u\,p^a\theta(\delta)$$

with u a unit. The higher Taylor terms vanish because $gz-z$ is divisible by E. Therefore

$$0<v_p(\theta(\omega_0-1))\le d+a+v_p(\theta\delta),\qquad v_p(\theta(\omega_0-1))>1/(p-1).$$

The last strict inequality follows from $\delta\mid\omega_0-1$ and $v_p(\theta\delta)=1/e+1/(p-1)$. Binomial expansion now gives, including $p=2$,

$$v_p(\theta(\omega_0^n-1))=v_p(\theta(\omega_0-1))+v_p(|n|).$$

For a prime-to-p power the unique least-valued term is the linear one; in a p-power step it is pA, since $v_p(A)>1/(p-1)$. Iterate and use units for negative powers. Finally

$$v_p(\theta_a\delta)=p^{-a}(1/e+1/(p-1)),\qquad v_p(\theta\delta)-v_p(\theta_a\delta)\le 2a.$$

The last inequality is equality zero for $a=0$; for $a\ge 1$ the left side is at most $2\le 2a$. These formulas prove (2).

Surjectivity of $\theta_a:A_{\mathrm{inf}}\to\mathcal{O}_{\mathbb{C}_p}$ now supplies $\upsilon\in A_{\mathrm{inf}}$ with

$$\upsilon X-p^{c_a(n)}\in E_aA_{\mathrm{inf}}. \tag{3}$$

For a value $y\in\mathcal{K}_a\cap\mathrm{Fil}^n$, apply (1) to $\pi_a^{-n}y$. Since $g\pi_a=\varphi^{-a}(\omega)\pi_a$ and g is trivial on $\varphi^{-a}S$ modulo Fil^1, we obtain

$$gy-\varphi^{-a}(\omega^n)y\in\mathrm{Fil}^{n+1}. \tag{4}$$

For $x\in\mathcal{N}$, a form λ, and a fixed-field constant c with $y=\lambda(\varphi^{-a}x)-c\in\mathrm{Fil}^n$, put

$$Tx=p^{c_a(n)}x-\upsilon(gx-x)/\delta.$$

Values of x at every inverse-Frobenius level lie in $\mathcal{K}_a$. To check this from the actual module definitions, clear an ambient denominator of $\varphi^{-a}x$, express it in the A_{inf}-span of N_{loc}, apply φ^a, and apply a nonzero chart coordinate to this finite relation. Exact relation preservation makes the relation persist. Its nonzero coefficient is in S, and solving it after evaluating the form gives a quotient of sums of $\varphi^{-a}(S)$-multiples of $\mathcal{K}_0$-values, hence an element of $\mathcal{K}_a$.

By equivariance, (4), $\delta X=\omega^n-1$, and the fact that $\varphi^{-a}\delta$ is a de Rham unit,

$$\lambda(\varphi^{-a}((gx-x)/\delta))-\varphi^{-a}(X)y\in\mathrm{Fil}^{n+1}.$$

Together with (3) this gives

$$\lambda(\varphi^{-a}Tx)-p^{c_a(n)}c\in\mathrm{Fil}^{n+1}. \tag{5}$$

The zero step. Let $m=p^{d+3a}$, and choose $t_jz=q^jz$, $0\le j<m$. Modulo E_a, every element of S has a polynomial representative in z with unramified coefficients (division by the distinguished series E_a). Put $\zeta=\theta_a(q)$, a primitive p^a-th root, with $\zeta=1$ when $a=0$. On a monomial uz^b,

$$\theta_a\Big(\sum_{j<m}t_j(uz^b)\Big)=\theta_a(u)\theta_a(z)^b\sum_{j<m}\zeta^{jb}.$$

This is zero if $p^a\nmid b$, since $p^a\mid m$. If $b=p^ab'$, the monomial is fixed under θ_a by every element of Γ', since $\varphi^{-a}u$ is unramified and θz is fixed. Thus $\sum t_j$ is invariant on S modulo E_a. By (1)

it is invariant modulo Fil^1 on every integral $\mathcal{K}_a$-value. The constant-modulo-filtration assertion of Lemma 7.7.26 shows that its value is a fixed-field constant modulo Fil^1. This uses the sum itself, never its average. In this case set $Tx = \sum_{j<m} t_j x$.

Descent to the actual module and cost. In both cases $T = p^{c_a(n)} + \sum_j v_j(t_j - 1)/\delta$, with coefficients $v_j \in A_{\mathrm{inf}}$. By Lemma 7.7.24 its image lies in $A_{\mathrm{inf}} N_{\mathrm{loc}}$. Set $B = R_L T$, using Lemma 7.7.21. This is additive. If x lies over a constant of depth h, each $(t_j x - x)/\delta$ has orders h, lies over zero, and retraction preserves both assertions. Therefore $Bx - p^{c_a(n)} x \in \ker \pi \cap \mathfrak{X}_L(i; h)$: this is exactly the cost property. Retraction preserves (5) and its zero-step counterpart. Applying Lemma 7.7.26 first to the input and then to the output proves item1.

Finite sets of steps. Induct on I, removing (a, n) with maximal second coordinate. In the target h', lower each value $h'(a, u) = n + 1$ to n, leaving other values unchanged; call the result h''. Every required step from h to h'' belongs to the remaining set. Indeed if the removed pair were required, the larger pair $(a, n+1)$ would be required in I, contradicting maximality. Apply the remaining-set operator and then the one-step operator for (a, n). Its cost property retains depth h''; item1 raises exactly the lowered form tests. This proves the required depth h'. Costs add: if A, B have costs e, f, then

$$BAx - p^{e+f} x = (BAx - p^f Ax) + p^f (Ax - p^e x),$$

and both terms belong to the same kernel order module because A preserves the original depth. The empty-set operator is the identity. Every choice precedes the letter, vector, form and depth functions, as required. □

Proposition 7.7.29. *Let k be a Kummer chart for Γ and S, S' pairing sides over $\mathcal{O}$ paired by P. Let $\eta \geq 1$, $a, W, \kappa \in \mathbb{N}$, and at each letter i let $b_i \in \mathbb{N}$, $\varepsilon(i) = \pm 1$, with $S(i)$ fitting $(\eta, -\varepsilon(i), b_i, W)$ and $S'(i)$ fitting $(\eta, \varepsilon(i), b_i, W)$, and $\mathrm{Ng}(\ell, n, b_i)$ for $n = 6(a + \eta + W)$.*

1. *Let $\ell^\kappa E^a$ kill the cokernels of S, S' and of the Tate dual side of P at every letter, and assume that no jump of S or S' lies in $[-a, 0)$, or that k is unramified and all jumps of S' lie in $[-\eta, 0]$. Then P is a scaled cup coboundary at (H, c, c') for all c, c' and every $H \geq 3(\kappa + (8n + 4)d)$. If k is unramified and the jumps of S' lie in $[-\eta, 0]$, it suffices that $\ell^\kappa E^a$ kill the cokernels of S and of the Tate dual side, and $H \geq 2\kappa$.*

2. *(Few low jumps.) Let k be unramified, $\ell \geq 5$, and at every letter i let $b_i + W + \eta \leq (\ell - 3)a$ and $6(a + \eta + W) < b_i$, with $S(i)$ fitting $(\eta, \varepsilon(i), b_i, W)$. Let n_i and r_i bound f times the number of jumps of Y over i in $[-\eta, \eta]$ and in $S(i)$, with multiplicity. Then there is a $\mathbb{Z}_\ell$-submodule $V \subseteq C$ with $\mathrm{length}_{\mathbb{Z}_\ell}(C/V) \leq 11 \sum_i (n_i(a + 1) + r_i(W + 1))$ such that for every side T whose jump sets fit $(\eta, -\varepsilon(i), b_i, W)$, every pairing P' of the lattices of T and S, every $c \in C_T$ and $c' \in V$, P' is a scaled cup coboundary at $(0, c, c')$.*

Proof. **Actual uniform depth operators.** Put $\Delta = z\tau$, $E_j = \varphi^j E$, and $c_j(n) = v_\ell(|n|) + d + 3j$, with $v_\ell(0) = 0$. Use Lemma 7.7.28 on this SAME localized extension module. Its cost means exactly that the correction from $\ell^b x$ is a kernel vector of the original depth orders. Its one operator for a finite step set is chosen before the point, form, letter and depth functions. If a larger cost is allowed, multiply the output by the remaining power of ℓ; this preserves all de Rham orders and raises the constant exponent accordingly.

Use the actual coefficient map q_0 and natural retraction R of Lemma 7.7.21. Both preserve every specified letter and order. Strictness for kernel vectors is Lemma 7.7.26, item2. Its item1

also gives the improvement rule

$$\operatorname{imp}(o,h')(j,u)=\begin{cases}h'(j,u), & o(j,u)>0,\\ \min(h'(j,u),0), & o(j,u)\le 0.\end{cases} \tag{4}$$

The first cup bound, gap branch. Write $n_0=6(a+\eta+W)$ and $D_0=(8n_0+4)d$. By the gap condition every nonzero integer within n_0 of $0,b_i,-b_i$ is prime to ℓ. The union of the corresponding closed intervals has at most $8n_0+4$ elements. Each level-zero step there costs d, including zero under our convention. Thus its total price is at most D_0.

The killer supplies a point of orders and depth $R_{-a}(j,u)=-u-a[j=0]$. Deepen it at level zero to $H(j,u)=-u+(a+\eta)[j=0]$. Each required step belongs to the preceding union: on a low jump its distance from zero is at most $2a+2\eta$, and on a far jump its distance from one of $\pm b_i$ is at most $2a+\eta+W$, both bounded by n_0. No other level changes. The resulting point is over $\ell^{\kappa+D_0}c$. The prohibition of jumps in $[-a,0)$ removes its pole. Explicitly, at level zero either $u<-a$, so its initial order is positive and (4) gives $a+\eta-u\ge -u$, or $u\ge 0$, so $\min(a+\eta-u,0)\ge -u$. Above zero the orders remain $-u$. Hence both sides have points without pole, of depth H, at every letter.

Take every letter to be a cut, with gap one and Tate-dual killer $\ell^{\kappa}E^{a}$. At a letter one side has all jumps at most η (its far interval is negative), and the other has all jumps at least $-\eta$. The first point is a stop: at a negative jump its positive order improves by (4), and at $0\le u\le\eta$ its depth is nonnegative, giving stop order zero. For the other side the difference from its preceding Frobenius point has orders $\min(-u,0)$ at level zero and $-u$ above. Against the resting depth $a+\eta-v$ at zero, the safe-move rule has residual $\eta+\min(u,0)\ge 0$, since $u\ge-\eta$; above zero it has residual zero. This proves the required central safe move, not only the stop order. The roles may switch at each letter. Apply Proposition 7.7.25 to obtain a cup coboundary at $3\kappa+2D_0$, hence at every $H\ge 3(\kappa+D_0)$ as claimed.

The unramified nonpositive branch. Here $d=0$, so all of the preceding level-zero steps are free. Put $t=a+\eta$. For S, deepen a killer point to depth $a+t-u$ at zero and $-u$ above; the steps lie within n_0 of $0,\pm b_i$. Formula (4) leaves a pole of at most t. The nonpositive side S' has killer 1: item2 of Proposition 7.7.15 at $\omega=\kappa=0$ forces its coordinate power to be zero, and the unit $1-\ell u$ removes the modulus. It therefore has points over c' without pole; freely deepen them to the same depth $a+t-u$ at zero.

If S has negative far interval, it has jumps at most t, while S' has jumps at least $-t$. The preceding stop argument works, with the moving side's preceding pole zero. If S has positive far interval, its jumps are at least $-t$. Construct a stop on S' by taking Frobenius of its preceding point, then freely deepening at level zero from depth one to $a+t-u$. Its orders are zero at zero and $-u$ above, since $-\eta\le u\le 0$ and $\eta\le t$. Its depth is at least

$$a+t-u\quad(j=0),\qquad t-u\quad(j=1),\qquad -u\quad(j>1).$$

The moving S difference has orders $\min(-u-t,0)$ at zero, $-u-t$ at one, and $-u$ above. The safe-move residual is respectively zero, zero and zero, using $u\ge -t$ in the first case. Thus the central safe move again holds. The walk uses point exponents $\kappa,0$ and third-side exponent κ, and gives precisely 2κ. This proves the full first item, including its stronger final assertion; no killer for S' was assumed.

Primitive integral relations. For the remainder the chart is unramified. Write $R_t(j,u)=-u+t[j=0]$, and let $m_i(u)$ be the actual multiplicity of u at i. Denote by $X_i(o)$ the localized order module, by $D_i(h)$ the localized points over constants whose Galois differences have orders h, by $K_i(h)$ its kernel vectors with orders h, and put

$$\mathrm{DC}_i(t,h)=\pi\big(X_i(R_t)\cap D_i(h)\big). \tag{5}$$

All modules and lengths in what follows are over $\mathbb{Z}_\ell$; π takes values in the actual letter part of the parameter lattice.

We first prove the integral relation used to control length. Suppose $o(j,u) < o(j,v)$ for every smaller jump $v < u$ at $i^{(j)}$. Among $fm_{i^{(j)}}(u) + 1$ vectors $x_s \in X_i(o)$ there is a combination with one coefficient equal to 1 of the form $W + Z$, where W gains one on every level-zero exponent and Z gains one at (j,u). Indeed the de Rham comparison and its natural graded comparison, Theorem A.3.16, Theorem A.3.11, applied using Lemma 7.4.6, identify the graded type-u form space at this letter with the dual of its weight-u Hodge space. To see the sign and dimension directly, choose invariant comparison vectors $d_h = t^{u_h} e_h$ with (e_h) an integral comparison basis. Their invariant duals have type u_h; the type-u quotient retains exactly the duals with $u_h = u$. Its dimension is therefore the actual multiplicity $m_{i^{(j)}}(u)$. Choose at most that many forms of type u spanning modulo type $u-1$. Divide their values on $\varphi^{-j} x_s$ by the common filtration generator. Their residues give a finite matrix over $\mathbb{C}_\ell$. Choose a nonzero minor of its rank having largest absolute value. Cramer's rule expresses all rows in the selected rows with coefficients in $\mathcal{O}_{\mathbb{C}_\ell}$. Lift these coefficients through θ, then apply φ^j. Subtracting the corresponding combinations of selected vectors cancels the leading terms. Forms of smaller type already have one extra exponent by strictness, so all forms of type u gain the claimed exponent, while all other orders remain. Retract to the actual localized module. Write each retracted coefficient as $q_0(A_{sh}) = w_{sh} + E b_{sh}$, where $w_{sh} \in \Lambda = W(\mathbb{F}_{\ell^f})$. This is possible since $E = z + \ell$ and division modulo E leaves a constant. The vectors $(w_{sh})_h$ lie in a free $\mathbb{Z}_\ell$-module of rank at most $fm_{i^{(j)}}(u)$. A nonzero relation between them can be divided by a coefficient of least valuation, giving coefficients $\beta_s \in \mathbb{Z}_\ell$ with one $\beta_s = 1$. Then

$$\sum_s \beta_s x_s = E \sum_{s,h} \beta_s b_{sh} x_{s(h)} + \sum_s \beta_s R\left(x_s - \sum_h A_{sh} x_{s(h)} \right). \tag{6}$$

These are precisely W and Z. Thus the residue degree enters through integral coefficient vectors, not through a replacement of length by dimension.

Here is the elementary length criterion we shall use. If Q is killed by ℓ^A and any $N+1$ elements of $Q[\ell]$ have a relation with one coefficient 1, then $\operatorname{length} Q \le AN$. Reduce these relations modulo ℓ to get $\dim_{\mathbb{F}_\ell} Q[\ell] \le N$. Induct on A using $0 \to Q[\ell] \to Q \xrightarrow{\ell} \ell Q \to 0$ and $(\ell Q)[\ell] \subseteq Q[\ell]$. This proves finite length as well, without assuming finite generation.

Raise just $h(j,u)$ by one, obtaining h^+. Assume $h(j',v) \ge -v$ on every jump, strictness at smaller jumps at level j, and

$$\text{no multiple of } \ell \text{ lies strictly between } h(0,v) \text{ and } h(0,v) + \ell^j \quad \text{for every } v. \tag{7}$$

Here zero is also a multiple. Put $X = X_i(R_0) \cap D_i(h)$, $X^+ = X_i(R_0) \cap D_i(h^+)$ and $K = K_i(h)$. The quotient $Q = X/(X^+ + K)$ surjects onto $\mathrm{DC}_i(0,h)/\mathrm{DC}_i(0,h^+)$. The depth operator proves $\ell^{c_j(h(j,u))} Q = 0$: its correction lies in K, and K has no pole since $h \ge R_0$.

For $N = fm_{i^{(j)}}(u)$ lift $N+1$ elements of $Q[\ell]$ to x_s. Write $\ell x_s = x_s^+ + k_s$, with $x_s^+ \in X^+$ and $k_s \in K$. Strictness of the kernel inclusion for all forms identifies k_s with actual vectors of Y of orders h. Apply (6) to them: $\sum \beta_s k_s = W + Z$, where $W \in K_i(h + 1_{\text{level }0})$ and $Z \in K_i(h^+)$. Thus for $x_0 = \sum \beta_s x_s$, $\ell x_0 - W \in D_i(h^+)$. The zero-cost level-zero depth operator B raises the depth of W from $h + 1_{\text{level }0}$ to $h + \ell^j 1_{\text{level }0}$: (7) makes all its intervening steps prime to ℓ. Its correction preserves kernel orders. Since (7) excludes crossing zero, the improved-order formula (4) gives

$$BW = E^{\ell^j} W', \qquad W' \in K_i(h).$$

The binomial identity for $E = z+\ell$ gives $E^{\ell^j} = E_j + \ell r$ with $r \in \mathfrak{S}$. Set $k_0 = rW' - (Bx_0 - x_0) \in K$. Then

$$\ell(x_0 - k_0) = B(\ell x_0 - W) + E_j W'. \tag{8}$$

Both summands have Galois differences of orders h^+. Multiplication by ℓ is invertible in every de Rham filtration, so it can be cancelled in this assertion. Thus $x_0 - k_0$ has depth h^+, and still has no pole, since both x_0 and k_0 have none. It lies in X^+. This proves the required primitive relation in $Q[\ell]$, and hence

$$\text{length}\, \frac{\mathrm{DC}_i(0,h)}{\mathrm{DC}_i(0,h) \cap \mathrm{DC}_i(0,h^+)} \leq c_j(h(j,u))\, fm_{i^{(j)}}(u). \tag{9}$$

To reach an antitone target depth at level j, always raise the least jump not yet at its target. At every smaller jump the target is already attained and is strictly larger than the current value at this jump. Thus strictness for (9) holds at each step. At $j = 0$, condition (7) is empty; at $j > 0$ the level-zero depths remain fixed. Summing (9) proves the sum-of-costs bound, with multiplicities, for an arbitrary finite sequence of such increases. Depth decreases at other levels cost nothing. Intersections, rather than an assumed nesting of the original endpoints, justify the triangle inequality for these relative lengths.

Removing the pole and choosing one submodule. Put $D = R_{2a+\eta}$. The unramified cokernel estimate in Proposition 7.7.15 supplies E^a as a killer whenever the jumps are at most $(\ell - 3)a$. All jumps here have this bound. Starting from its point of orders R_{-a} and using the free level-zero steps within the three prescribed neighborhoods gives depth D. Formula (4) gives orders

$$o = \mathrm{imp}(R_{-a}, D). \tag{10}$$

No jump lies in $[-a, -\eta)$: low jumps are at least $-\eta$, negative far jumps are below $-a$, and positive far jumps are positive. Consequently $o \geq R_{-\eta}$. This supplies every constant with pole at most η and depth D. The unramified estimate also supplies the killer ℓ^a with E-exponent zero. Its initial orders are R_0; the same free deepening gives depth D and orders $\mathrm{imp}(R_0, D) \geq R_t$ for every $t \leq 0$. Thus ℓ^a times every constant has every prescribed nonpositive pole bound with this depth.

Remove the pole successively from R_t to R_{t+1} for $-\eta \leq t < 0$. The quotient of constants is killed by ℓ^a by the preceding observation. Apply (6) at $(0,t)$ to any $fm_i(t) + 1$ representatives. The extension has the same multiplicity as Y at the negative jump t, because its quotient is trivial. Both terms in (6) gain at least the exponent $(0,t)$. The depth D and (4) then improve the other exponents to R_{t+1}: at $u < t$ the original exponent $t - u$ was positive; at $u = t$ the raised exponent is one; at $u > t$ the desired exponent is at most zero. Hence the primitive relation holds in this pole-removal quotient. The length criterion bounds this loss by $afm_i(t)$. Summing over t gives

$$\text{length}\, \frac{\mathrm{range}(\mathrm{cst}_i)}{\mathrm{range}(\mathrm{cst}_i) \cap \mathrm{DC}_i(0,D)} \leq an_i. \tag{11}$$

If $\varepsilon(i) = 1$ set $h_i = D$. If $\varepsilon(i) = -1$, set

$$h_i(0,u) = b_i + W + a + \eta, \quad h_i(1,u) = W + 1 - u + 2a\, 1_{[-\eta,\eta]}(u), \quad h_i(j,u) = -u \quad (j \geq 2). \tag{12}$$

We bound the further loss from D to (12). Let N_i be the least multiple of ℓ at least $b_i + W + a + \eta$, and first change the level-zero depth to the constant N_i. On a low line the steps are nonnegative and their total valuation is at most $v_\ell((N_i - 1)!) \leq (N_i - 1)/(\ell - 1) \leq a + 1$, using $N_i < b_i + W + a + \eta + \ell \leq (\ell - 2)a + \ell$. On a negative far line a nonzero multiple y in the interval of steps satisfies $y \leq N_i - \ell < b_i + W + a + \eta$ and $y \geq 2a + \eta - u \geq b_i + 2a + \eta$. Its distance from

b_i is less than $6(a+\eta+W)$, contrary to the gap condition. Thus far lines cost zero. Applying (9) at level zero costs at most $(a+1)n_i$.

Next raise level one to (12) and lower level zero from N_i to its target. Condition (7) holds since $(N_i, N_i+\ell)$ has no multiple of ℓ. There are $W+1+2a1_{\mathrm{low}}(u)$ steps at each line of the preceding letter. Every step is zero or prime to ℓ, by its location within the prescribed neighborhood of zero or a far endpoint; hence its cost is exactly 3. The target is antitone on the jump set: within a window it has slope -1, and across windows the gap $b_{i^{(1)}} > 6(a+\eta+W)$ dominates the possible change $2a$. The level-one loss is therefore at most $3(W+1)r_{i^{(1)}} + 6an_{i^{(1)}}$. Combining with (11), for either sign the loss at letter i is at most

$$(2a+1)n_i + 3(W+1)r_{i^{(1)}} + 6an_{i^{(1)}}. \tag{13}$$

Select now, before choosing any opposite side or pairing,

$$V = \bigcap_i \mathrm{cst}_i^{-1}\big(\mathrm{DC}_i(0,h_i)\big) \subseteq C. \tag{14}$$

The map from C/V to the direct sum of the corresponding one-letter quotients is injective. Frobenius permutes the letters, so (13) gives

$$\mathrm{length}(C/V) \le \sum_i \big((8a+1)n_i + 3(W+1)r_i\big) \le 11\sum_i \big(n_i(a+1) + r_i(W+1)\big). \tag{15}$$

This is an integral length bound for the single module (14).

Uniform orthogonality and the special third point. Fix only now an arbitrary T, P', c and $c' \in V$. At each letter choose the point y_i of S supplied by (14), with own orders R_0 and depth h_i. For T the construction (10) gives x_i over c, own orders o and depth D, with no power of ℓ. Its orders dominate $R_{-\eta}$. Both point families are standard with parameters $(a,\eta,1)$: $D \ge R_{a+\eta}$, and in the negative case $h_i(0,v) - (a+\eta-v) = b_i + W + v \ge 0$, while $h_i(1,v) \ge -v$.

All jumps of the third side are zero or $w = -u-v-1$. Opposite windows place them in the union

$$[-(2\eta+W+1), 2\eta+W], \quad [b_i-\eta-1, b_i+W+\eta-1], \quad [-b_i-W-\eta-1, -b_i+\eta]. \tag{16}$$

They are at most $(\ell-3)a$; the first interval has this bound since $b_i > 6(a+\eta+W)$. Thus E^a kills the third cokernel. We also require a particular deep point there. Start with its point over 1, own orders and depth R_{-a}, and deepen level zero to R_a. Every required step is within radius $a+2\eta+W+1 \le 6(a+\eta+W)$ of 0 or $\pm b_i$ by (16). Each is zero or prime to ℓ, so its cost is zero. Its own orders are

$$z = \mathrm{imp}(R_{-a}, R_a), \qquad z(0,w) = \begin{cases} a-w, & w < -a, \\ \min(a-w, 0), & w \ge -a, \end{cases} \quad z(j,w) = -w \ (j>0), \tag{17}$$

and its depth is R_a. Notice that these are improved own orders, not merely the orders of a standard killer point. The function $z(j,\cdot)$ is antitone.

If $\varepsilon(i) = 1$, T is the down side, so x_i is a stop: its improved orders and depth dominate $\max(0, -u-\eta)$ at zero and $-u$ above. The moving difference $\varphi y_{i^{(1)}} - y_i$ has own orders $\min(-v, 0)$ at zero and $-v$ above, since both y points have own orders R_0. Here $v \ge -\eta$. Against the resting depth $a+\eta-u$, the sum at zero is at least $a-u-v$ because $\eta + \min(v,0) \ge 0$. The ordinary third killer of orders R_{-a} therefore gives a strictly positive sum, since $w = -u-v-1$. Above zero the sum is exactly $-w-u-v = 1$. This proves the central second safe move.

If $\varepsilon(i) = -1$, S is the down side and y_i is a stop, by its own orders R_0 and depth (12). Use the special third point (17) at $i^{(1)}$, transported by Frobenius. Its own orders are zero at level zero and $z(j-1, w)$ at level $j \geq 1$. The moving difference $\varphi x_{i^{(1)}} - x_i$ has orders

$$M_0(u) = \min(o(0,u), 0), \quad M_1(u) = \min(-u, o(0,u)), \quad M_j(u) = -u \ (j \geq 2).$$

We check the strict order inequality against (12) at every level. At zero, $M_0(u) \geq \min(-a - u, 0) \geq -a - b_i - W$, so adding $h_i(0, v)$ gives at least $\eta > 0$. At one write $w = -u - v - 1$. The universal bounds $z(0, w) \geq -a - w$ and $M_1(u) \geq -a - u$ suffice if v is low: adding $W + 1 - v + 2a$ gives $W + 2$. If v is far and u is low, then $|w| \geq b_{i^{(1)}} - \eta - 1 > a$; hence $z(0, w) = a - w$, and adding the other two bounds gives $W + 2$. Finally if both are far, opposite windows give $|u + v| \leq W$, $M_1(u) = -u$ and $z(0, w) \geq -(W + 1) - w$; the sum is at least 1. These three cases exhaust all jumps at the preceding letter, regardless of its sign. At every level at least two the sum is $-w - u - v = 1$. Thus the central first safe move holds as well.

We have obtained a stop with its required central safe move at every letter. The outer moves, global assembly, additional cocycle safety and continuity are exactly those proved in Proposition 7.7.25, applied with every letter a cut and point and third-side exponents zero. Therefore P' is a scaled cup coboundary at $(0, c, c')$. The construction of V preceded T, P', c and used none of them, proving the claimed uniformity. Together with (15) this proves the second item and completes the proof. □

Lemma 7.7.30. *For all integers $m, n, h \geq 0$ and all $a \in A$, one has*

$$\tau^m z^n \mid E^h a \text{ in } B^+ \quad \Longrightarrow \quad \tau^m z^n \mid a \text{ in } A.$$

Proof. Write $A = W(\mathcal{O}_{\mathbb{C}_p^\flat})$. Its Witt-vector construction makes A complete and separated for the p-adic topology, makes multiplication by p injective, and identifies

$$A/pA = \mathcal{O}_{\mathbb{C}_p^\flat}.$$

The latter ring is a domain. Thus p is prime in A, and A is a domain: the first nonzero p-adic coefficients of two nonzero elements multiply to a nonzero coefficient of their product. We use the actual rings

$$A_{\max} = A[T]\widehat{/(pT - \xi)}^{\,p}, \qquad B^+ = A_{\max}[1/p], \tag{Tcan1}$$

not the localization $A[1/p]$. The fixed period-ring embeddings place A and B^+ inside the de Rham period field B_{dR}; in particular they are domains and nonzero elements of A remain nonzero in B^+. These are the standard period rings of the stated convention, with their usual embeddings ([44]). Fontaine's map has kernel ξA and target the domain $\mathcal{O}_{\mathbb{C}_p}$, so ξA is prime.

Principal ideals generated by an element nonzero modulo p are saturated and closed. Let $y \in A$ have nonzero reduction modulo p. If $yb \in pA$, reducing modulo p in the domain A/pA gives $b \in pA$. Repetition, using injectivity of multiplication by p, proves

$$yb \in p^N A \quad \Longrightarrow \quad b \in p^N A \qquad (N \geq 0). \tag{Tcan2}$$

Also, if $pb = yc$, reduction modulo p gives $c = pc_1$; cancellation of p gives $b = yc_1$. Iterating this implication proves

$$p^N b \in yA \quad \Longrightarrow \quad b \in yA. \tag{Tcan3}$$

Suppose $b \in \bigcap_{N \geq 1}(yA + p^N A)$. For each N choose $c_N \in A$ with $b - yc_N \in p^N A$. Then $y(c_{N+1} - c_N) \in p^N A$, so (Tcan2) gives $c_{N+1} - c_N \in p^N A$. Completeness of A gives a limit $c =$

$\lim_N c_N$. Multiplication by y is p-adically continuous, and $b - yc_N$ tends to zero; separatedness therefore gives $b = yc$. We have proved

$$\bigcap_{N\geq 1} (yA + p^N A) = yA. \tag{Tcan4}$$

No Noetherian hypothesis on A or on A/yA is used.

Reduction modulo (y, p^N) reflects divisibility from B^+. Assume in addition that $\xi - pb_0 \in yA$ for some $b_0 \in A$. For $N \geq 1$ put $Q_N = A/(y, p^N)$. Sending T to the class of b_0 defines a ring map

$$A[T]/(pT - \xi) \longrightarrow Q_N.$$

It exists because the defining relation maps to $pb_0 - \xi = 0$. Since $p^N Q_N = 0$, it factors through reduction modulo p^N of the source ring. Projection from its p-adic completion in (Tcan1) consequently gives a ring map

$$F_N : A_{\max} \longrightarrow Q_N$$

whose restriction to A is the quotient map. Equivalently, an element of the completion is represented at precision p^N, and applying the factored map is independent of that representative.

Let $a_0 \in A$ and suppose $a_0 = yv$ in B^+. Write $v = w/p^k$ with $w \in A_{\max}$ and $k \geq 0$. Equality in the localization means that for some $j \geq 0$,

$$p^j(p^k a_0 - yw) = 0 \quad \text{in } A_{\max}.$$

Thus, putting $K = k + j$ and $w' = p^j w$, we have the exact equality $p^K a_0 = yw'$ in $A_{\max}$. This argument does not assume that completion alone clears denominators inside A. Applying F_N gives $p^K a_0 \in yA + p^N A$ for every N. By (Tcan4), $p^K a_0 \in yA$, and (Tcan3) then gives $a_0 \in yA$. We have established

$$A \cap yB^+ = yA \quad \text{if } \bar{y} \neq 0 \text{ in } A/pA \text{ and } \xi \in (p, y)A. \tag{Tcan5}$$

The Eisenstein factor can be removed. Suppose also $\theta(y) \neq 0$. Since $\ker\theta = \xi A$, this means $\xi \nmid y$. If $y \mid E^h a$ in B^+, (Tcan5), applied to the element $E^h a \in A$, gives $y \mid E^h a$ in A. The chart relation makes E associated to ξ, so $y \mid \xi^h a$ in A. If $h > 0$, write $\xi^h a = yb$. Primality of ξA and $\xi \nmid y$ give $b = \xi b_1$. Cancellation of the nonzero ξ in the domain A gives $\xi^{h-1} a = yb_1$. Induction on h, including $h = 0$, proves

$$y \mid E^h a \text{ in } B^+ \quad \Longrightarrow \quad y \mid a \text{ in } A \tag{Tcan6}$$

under the three scalar hypotheses just checked.

Both actual factors satisfy the scalar hypotheses. For $y = \tau = [\varepsilon^{1/p}] - 1$, the actual compatible cyclotomic system has $\theta(\tau) = \zeta_p - 1 \neq 0$, where ζ_p is primitive. Modulo p, τ is $\varepsilon^{1/p} - 1$ in the tilt. If it were zero, $\varepsilon^{1/p} = 1$ would imply $\theta(\tau) = 0$, a contradiction. The defining formula for ξ gives

$$\xi - p = \sum_{i=0}^{p-1} \big((1+\tau)^i - 1\big) \in \tau A.$$

Thus $\xi \in (p, \tau)A$. This formula and nonvanishing hold also for $p = 2$, when $\theta(\tau) = -2 \neq 0$.

For $y = z$, let $u \in A^\times$ be the image of the chart's unit series. The chart equations are

$$E = z^e u + p, \qquad e \geq 1, \qquad \xi = vE \quad (v \in A^\times).$$

Applying θ gives $\theta(z)^e\theta(u) = -p \neq 0$, so $\theta(z) \neq 0$. Since $z = [\varpi^\flat]$, its reduction modulo p is $\varpi^\flat$. This is nonzero: otherwise $z = 0$, contrary to $\theta(z) \neq 0$. Finally $\xi - pv = vz^e u \in zA$ because $e \geq 1$. Hence $\xi \in (p, z)A$. These checks use no restriction on p beyond primality. The nonzero ξ used above can also be seen directly modulo p: the geometric-sum formula gives $\bar{\xi} = \bar{\tau}^{\,p-1} \neq 0$ in the tilt domain.

Consequently (Tcan6) holds separately for $y = \tau$ and $y = z$. Both have nonzero image in B^+ by the fixed embeddings.

Remove all factors without changing the ambient rings. Induct on $m+n$. If $m+n = 0$, the divisor is 1 and the conclusion is immediate. Otherwise choose $y = \tau$ if $m > 0$, and $y = z$ if $m = 0$, $n > 0$, and write $\tau^m z^n = yq$ with q the product with that exponent reduced by one. From $yq \mid E^h a$ in B^+ one has $y \mid E^h a$, so (Tcan6) gives $a = ya_1$ with $a_1 \in A$. If $E^h a = yqb$ in B^+, substitution and cancellation of the nonzero y in that domain yield $E^h a_1 = qb$. The induction hypothesis gives $q \mid a_1$ in A, and therefore $yq \mid a$ in A. This proves the assertion for every $m, n, h \geq 0$, including zero exponents and $p = 2$. □

Lemma 7.7.31. *For $0 \leq j_0 \leq m$, put*

$$Q_* = kt, \quad G_* = \frac{a}{x_1}, \quad \delta_* = (2j_0 + 1)h - 2t, \quad \eta_* = h^2 + 1, \quad W_* = 4t.$$

Then:

1. *δ_* is a nonnegative integer and $(Q_*, m, h, G_*, b, \delta_*, \eta_*, W_*)$ satisfies (KZ1)–(KZ4).*
2. *Put $N_* = (r + t)r$ and $E_* = 3r^2(N_* + 1)^2 + 150(\eta_* + W_* + 1)$. For $\ell = p_i$, $i = 0, 1$, and every complex embedding σ,*

$$\ell > S_{j_0}(\sigma) + 6(W_* + \eta_*) + r^2(N_* + 1)^2 + 2.$$

   For every integer $1 \leq x_{\mathrm{div}} \leq t$, $2E_* \leq \mathrm{db}_\ell / x_{\mathrm{div}}$.

3. *For every prime ℓ, every integer $\rho \geq 0$, and every $1 \leq x_{\mathrm{div}} \leq t$,*

$$10^5 \left(\rho \left\lfloor \frac{2\lambda_\ell}{kt} \right\rfloor m + \left\lfloor \frac{\rho m}{k x_{\mathrm{div}}} \right\rfloor \lambda_\ell \right) \leq \rho \, \frac{\mathrm{db}_\ell}{x_{\mathrm{div}}}.$$

Proof. We make the comparisons with the actual global scales explicit. By Lemma 4.2.67 and Proposition 2.3.5,

$$\begin{aligned} &x \geq e^{6.4 \cdot 10^{15}}, \quad x^5 \leq r \leq 3x^5, \quad 1 \leq k \leq r, \quad m \leq r/2, \quad t = 5r, \\ &x^{10^6} \leq x_1, \quad x_1^{10^6} \leq b, \quad (\log b)^{100} \leq x. \end{aligned} \tag{15}$$

In particular all the polynomial comparisons below already hold for $x \geq 100$ and $r \geq 100$, which (15) supplies. We will use the stronger bounds

$$G_* = \frac{xb}{x_1} \geq x\, x_1^{999999} \geq 99r^{24}, \qquad x_1 \geq 99x(r^{24} + 1), \qquad b \geq r^{18}, \qquad b \geq 100r^{16}. \tag{16}$$

For example $r^{24} \leq 3^{24}x^{120}$, so the second inequality follows from $99x(3^{24}x^{120} + 1) \leq x^{10^6}$. The first follows from $x_1 \geq x^{10^6}$, and the last two from $b \geq x^{10^{12}}$ and $r \leq 3x^5$. These are inequalities of positive monomials; $x \geq 100$ is much more than needed.

For item 1, $Q_* = kt \le 5r^2$, $W_* = 20r$, and $m \le r/2$. Therefore

$$99(Q_*^2 + (W_*+1)Q_* + Q_* + W_* + m) \le 2\cdot 10^4 r^4 \le r^8 = h.$$

Indeed the summand before multiplication by 99 is at most $25r^4 + 100r^3 + 10r^2 + 21r \le 200r^4$. This proves (KZ1), including $Q_* > 0$. The first bound in (16) gives $99h^3 \le G_*$. The second gives

$$99G_*(h^3+1) = \frac{99x(r^{24}+1)}{x_1}\, b \le b,$$

proving (KZ2). Since $\log b \le x^{1/100}$,

$$2\log(4b) \le 2x^{1/100} + 2\log 4 < \sqrt{x} \le m$$

for $x \ge 100$; this proves (KZ3). Finally

$$0 < h - 10r \le \delta_* \le (2m+1)h \le (r+1)r^8 < r^{16} + 1 = \eta_*.$$

For the remaining (KZ4) inequality,

$$99((\eta_* + W_*)Q_* + \eta_* + h) \le 10^3 r^{18} \le r^{24} = h^3.$$

The first inequality follows on inserting $\eta_* = r^{16}+1$, $W_* = 20r$, $Q_* \le 5r^2$; for $r \ge 100$ the leading coefficient is 495 and the other positive terms together are less than $505r^{18}$. The second uses $r^6 \ge 10^3$. Item 1 is proved.

For item 2, $N_* = 6r^2$, so $N_* + 1 \le 7r^2$. Also $\eta_* + W_* + 1 = r^{16} + 20r + 2 \le 2r^{16}$. Hence

$$E_* \le 147r^6 + 300r^{16} \le 500r^{16}, \qquad 2E_* \le 10^3 r^{16}. \tag{17}$$

The actual far-window start satisfies $S_{j_0}(\sigma) = \kappa(\sigma) + \delta_* \le b + (r+1)r^8 \le 2b$. Here $\kappa(\sigma) \le b$, since $x/x_1 \le 1$. Furthermore

$$6(W_* + \eta_*) + r^2(N_*+1)^2 + 2 \le 6(20r + r^{16} + 1) + 49r^6 + 2 \le 10r^{16} \le b.$$

Consequently the right side of the required height inequality is at most $3b < b^{10} \le p_i$.

Since M is the product of the two distinct level primes, $v_{p_i}(M) = 1$. Using $m \ge 1$, $k \le r$, $x_{\mathrm{div}} \le t = 5r$, and nonnegativity of λ_ℓ,

$$\frac{\mathrm{db}_{p_i}}{x_{\mathrm{div}}} \ge 10^6 \frac{m}{kt}\, b \ge 2\cdot 10^5\, \frac{b}{r^2} \ge 2\cdot 10^5 r^{16} \ge 2E_*,$$

where $b \ge r^{18}$ in (16) and (17) were used.

For item 3, $x_{\mathrm{div}} \le t$ and positivity of k, x_{div} give

$$\left\lfloor \frac{2\lambda_\ell}{kt} \right\rfloor \le \frac{2\lambda_\ell}{kx_{\mathrm{div}}}, \qquad \left\lfloor \frac{\rho m}{kx_{\mathrm{div}}} \right\rfloor \le \frac{\rho m}{kx_{\mathrm{div}}}.$$

Thus its left side is at most $3\cdot 10^5 \rho m \lambda_\ell/(kx_{\mathrm{div}})$. This is at most $10^6 \rho m(bv_\ell(M) + \lambda_\ell)/(kx_{\mathrm{div}}) = \rho\, \mathrm{db}_\ell/x_{\mathrm{div}}$. This also covers $\rho = 0$ and $\lambda_\ell = 0$. □

Lemma 7.7.32. *Let ℓ be prime, $e : E \to \overline{\mathbb{Q}}_\ell$ an embedding, and let $j : \Gamma \to G_\ell$ be an open embedding onto the elements fixing $e(E)$. For all sufficiently large coefficient rings $\mathcal{O}$, there exist a Kummer chart $\mathfrak{c}$ for Γ, a primitive $(\ell^{f_\mathfrak{c}} - 1)$-th root $\zeta \in \mathcal{O}$, a predicate $\mathcal{T}_\mathrm{b}$ on $\mathbb{C}_\ell$, and a sign function ε_b on $\mathbb{C}_\ell$, such that the letters read thickness for*

$$\mathrm{lt}(g) = g(\zeta), \qquad \mathrm{th}(g) \iff [\sigma_{e,g}] \in \Phi_b, \qquad \varepsilon(g) = \mathrm{sg}(\sigma_{e,g}),$$
$$Q = kt, \quad h = r^8, \quad (\mathcal{T}_\mathrm{b}, \varepsilon_\mathrm{b}), \qquad g \in G_\ell.$$

Proof. Let K_ℓ be the closure of $e(E)$. Choose the chart in Lemma 7.7.17, which fits Γ, and write $e_{\mathfrak{c}}, f_{\mathfrak{c}}$ for its index and residue degree. Its degree identity gives

$$e_{\mathfrak{c}} f_{\mathfrak{c}} = [K_\ell : \mathbb{Q}_\ell].$$

Its coefficient field is the maximal unramified subfield $K_{\ell,0}$ of K_ℓ. To check this identification directly, the image of the coefficient Witt ring is fixed by G_{K_ℓ}, and $\theta(z)$ satisfies the Eisenstein power-series equation $X^{e_{\mathfrak{c}}} A + \ell = 0$. Weierstrass preparation replaces this by an Eisenstein polynomial of degree $e_{\mathfrak{c}}$. Thus the coefficient field together with $\theta(z)$ has degree $e_{\mathfrak{c}} f_{\mathfrak{c}}$, and equals K_ℓ; the extension over the coefficient field is totally ramified. Consequently $K_{\ell,0}$ has degree $f_{\mathfrak{c}}$.

By Lemma 4.2.66, the index is at most $[E : \mathbb{Q}] = 2f \leq h$. In particular $e_{\mathfrak{c}} \leq h$. Choose a primitive root ζ of order $\ell^{f_{\mathfrak{c}}} - 1$ in $K_{\ell,0}$, using Teichmüller lifts of its residue field. It generates $K_{\ell,0}$ over $\mathbb{Q}_\ell$, including the case of residue field $\mathbb{F}_2$, where the degree is one. Choose a finite Galois coefficient extension containing ζ. Every coefficient ring containing its ring of integers contains the same root; this gives precisely the eventual quantifier.

Put $c_{\mathrm{cyc}} = q_1 d$. If $\ell \mid c_{\mathrm{cyc}}$, then

$$\ell \leq kt. \tag{12}$$

Indeed the prime divisors of $q_1 = 3q_0 l_0(s) 2^s$ already divide q_0, as $2 \cdot 3 \mid q_0$. Thus $\ell \leq \max(q_0, d) \leq C_P \leq \log x$. The final inequality follows from (AD1); and $\log x \leq kt$ follows from Lemma 4.2.67, since $k \geq r/2$, $t = 5r$, $r \geq x^5$, and x is larger than $e^{6.4 \cdot 10^{15}}$.

If $\ell \nmid c_{\mathrm{cyc}}$, the completion K_ℓ is unramified by Lemma 4.2.66; the chart is therefore unramified. Also $\ell \geq 7$, since $2, 3, 5, 7 \mid q_0$. In this case $K_\ell = K_{\ell,0} = \mathbb{Q}_\ell(\zeta)$. Thus equality $g(\zeta) = g'(\zeta)$ implies equality of the restrictions of g, g' to K_ℓ. Since this unramified field is Galois, the inverse restrictions agree too, so

$$\sigma_{e,g} = \sigma_{e,g'}. \tag{13}$$

Thickness and sign are consequently functions of the letter in this case.

We next show that when $\ell \mid c_{\mathrm{cyc}}$, all embeddings over b-places have the same CM sign. If $\ell \mid q_1$, then $\ell \mid (d+1)/4$. For odd ℓ this follows from $q_0 \mid 2(d+1)$; for $\ell = 2$ it follows from $16 \mid q_0$. Set $u_0 = e((1 + \sqrt{-d})/2)$. Opposite signs of $\sigma_{e,g}, \sigma_{e,g'}$ would mean that a local Galois automorphism exchanges u_0 and $1 - u_0$. They would have the same absolute value. But their product is $(d+1)/4$, of absolute value less than one, whereas their sum is one. Equal absolute values then force each to have absolute value less than one, contradicting the ultrametric inequality for their sum.

If $\ell \mid c_{\mathrm{cyc}}$ and $\ell \nmid q_1$, then $\ell = d$. A local Galois element acts on the roots of unity of order q_0 by a power d^{n_0}, since these roots have order prime to d and lie in an unramified extension. For two embeddings with opposite CM signs, the ratio of their infinite-place classes modulo q_0 is therefore $-d^{n_0}$. Here is the sign convention in this computation: the chosen embedding of the infinite place of σ sends a cyclotomic root to its value under σ, raised to $\mathrm{sg}(\sigma)$; comparing these two powers gives the product of the two signs times the local Frobenius power. Both b-places have class -1 modulo q_0, so their ratio is 1. This would give $d^{n_0} \equiv -1 \pmod{q_0}$. However $d \equiv q_0/2 - 1 \pmod{q_0}$, $4 \mid q_0$, and hence $d^2 \equiv 1 \pmod{q_0}$. Its powers are 1 or $q_0/2 - 1$, neither equal to -1, as $q_0 \geq 18480$. This is a contradiction. The sign assertion is proved.

Define

$$\mathcal{T}_{\mathrm{b}}(w) \iff \exists g \in G_\ell : \ [\sigma_{e,g}] \in \Phi_b \text{ and } g(\zeta) = w.$$

In the unramified case define $\varepsilon_{\mathrm{b}}(w)$ to be the sign of any such letter representative, choosing an arbitrary sign if there is none. More generally representatives of every nonempty fibre may be used, so that (13) determines signs also at thin fibres. In the conductor-dividing case choose the

common sign of the b-place embeddings and make ε_{b} constant; if there are no such embeddings choose either constant sign. These definitions give the required thickness equivalence and sign whenever the embedding is thick. If $Q < \ell$, (12) forces the unramified case, and (13) gives them at every embedding. They also give exactly the unramified-or-constant-sign alternative in the definition.

It remains to check the three-letter prohibition when $\ell \le Q$. Let $J_{\mathrm{orb}} = \lfloor \log_\ell h^3 \rfloor$. We first bound its range by the actual sparseness interval:

$$2J_{\mathrm{orb}} < l_0(s). \tag{14}$$

Indeed $h^3 = r^{24}$, so $2J_{\mathrm{orb}} \le 48 \log_2 r \le 288 \log_2 x$, using $r \le 3x^5 \le x^6$. On the other hand $s = 22 \log_2 m \ge 11 \log_2 x$ by $m \ge \sqrt{x}$, and $l_0(s) > 27s \ge 297 \log_2 x$, giving (14).

Suppose three thick letters occur at $w, w^{\ell^\alpha}, w^{\ell^{\alpha+\alpha'}}$, where $1 \le \alpha, \alpha' \le J_{\mathrm{orb}}$. Choose g_0, g_1, g_2 giving these letters and b-place embeddings. Put $q_{\mathrm{ur}} = q_1^{(\ell)}$, the prime-to-$\ell$ part of q_1. The roots of unity of order q_{ur} in $e(E)$ lie in $K_{\ell,0}$: reduction identifies roots of unity of order prime to ℓ with the Teichmüller roots in the residue field. Since ζ generates this field, the two letter equalities imply that $g_2 g_1^{-1}$ and $g_2 g_0^{-1}$ act on these roots by the powers $\ell^{\alpha'}$ and $\ell^{\alpha+\alpha'}$, respectively. The same place-class computation just given, now modulo q_{ur}, shows that, writing $v_i = [\sigma_{e,g_i}]$,

$$v_1/v_2 \equiv \epsilon_1 \ell^{\alpha'}, \qquad v_0/v_2 \equiv \epsilon_0 \ell^{\alpha+\alpha'} \pmod{q_{\mathrm{ur}}},$$

for signs $\epsilon_i \in \{\pm 1\}$. All three places lie in Φ_b, and $0 < \alpha' < \alpha + \alpha' < l_0(s)$ by (14). This contradicts the sparseness in (AD3) and Definition 2.3.2, applied at $\ell \le kt$. The prohibition follows, completing every clause of letters reading thickness. □

A Quoted results

The Appendix states the published results on which the proofs rest, each in the notation of the paper and with its source, grouped by field. Where a proof needs a published theorem in a form that is not in print, that form is proved in the body where it is first used, and only the printed theorem is quoted here. One convention is used only in the quotations, and it opens the Appendix: abelian schemes with cubical line bundles and their degenerations [220, 77, 43, 215].

Notation for abelian schemes is that of [220], [215] and [77]. For a scheme S, an **abelian scheme** $A \to S$ is a smooth proper group scheme with geometrically connected fibres, and a **semiabelian scheme** $G \to S$ is a smooth separated commutative group scheme whose fibres are extensions of abelian varieties by tori. We write $A^\vee$ for the dual abelian scheme, $e\colon S \to A$ for the unit section, $[n]$ for multiplication by n, $A[n]$ for its kernel and $T_\ell A$, $V_\ell A = T_\ell A \otimes \mathbb{Q}_\ell$ for the Tate modules. A **polarised abelian scheme** $(A, \mathcal{L})$ carries a relatively ample line bundle $\mathcal{L}$ with polarisation $\phi_{\mathcal{L}}\colon A \to A^\vee$, and $e_n\colon A[n] \times A[n] \to \mu_n$ is the Weil pairing of $\phi_{\mathcal{L}}$. Line bundles are given by an open cover and transition functions; $\bigotimes_i f_i^*\mathcal{L}^{\otimes n_i}$ is the tensor combination of pull-backs along morphisms f_i. A **rigidified** line bundle is a pair $(\mathcal{L}, r)$ with $r\colon e^*\mathcal{L} \cong \mathcal{O}_S$, and a **cubical structure** on it is a trivialisation of

$$\Theta(\mathcal{L}) = \bigotimes_{\varnothing \neq I \subseteq \{1,2,3\}} m_I^*\mathcal{L}^{\otimes(-1)^{|I|+1}}$$

on A^3 (m_I the partial sums) satisfying the symmetry and cocycle conditions of [215], Ch. I; $\mathcal{G}(\mathcal{L})$ is the theta group. For $S = \operatorname{Spec} V$ with V a complete discrete valuation ring with fraction field K, a **semiabelian model** of an abelian variety A_K is a semiabelian scheme G/V with $G_K \cong A_K$; its formal completion is the extension $\tilde{G}$ (**Raynaud extension**) of an abelian scheme by a torus, which is **split** if the torus is split; and **degeneration data** and the **Mumford uniformisation** $A_K^{\mathrm{an}} = \tilde{G}_K/Y$ are as in [77], Ch. II–III and [221]. A **p-divisible group** over a ring R is written $\mathcal{G}$, with Tate module $T_p\mathcal{G}$, Lie algebra $\operatorname{Lie}\mathcal{G}$, invariant differentials $\omega_{\mathcal{G}}$ and, over the ring of integers of a complete algebraically closed field C, Hodge–Tate map $\alpha_{\mathcal{G}}\colon T_p\mathcal{G} \to \omega_{\mathcal{G}^\vee}$ [255]. An action of a ring O with involution on A (or G, $\mathcal{G}$) is a ring homomorphism $O \to \operatorname{End}(A)$ compatible with the Rosati involution where a polarisation is given. This notation holds wherever abelian varieties occur in this paper.

A.1 Analytic number theory

These are classical results of analytic number theory, used mostly in Section 2 [64, 214, 308, 144]: Chebyshev's bound [116]; the continuation and the functional equations of Dirichlet L-functions, going back to Riemann [242], with Dirichlet's non-vanishing at $s = 1$; and the values at the non-positive integers, with the theorem of von Staudt and Clausen [317]. Four results are quantitative: the prime number theorem in progressions of Thorner and Zaman [307], in the line of Linnik and Gallagher [189, 93]; the Siegel–Walfisz theorem [314, 279]; the transformation formula for the theta function of a primitive character; and the Pólya–Vinogradov inequality. The section closes with Hecke L-functions [128, 294, 296, 224, 182] and Kronecker's first limit formula [329].

Lemma A.1.1. *For every integer $n \geq 1$, $\prod_{p \leq n} p < 4^n$, the product running over the primes $p \leq n$. Consequently, for every real $x \geq 0$,*

$$\sum_{p \leq x} \log p \leq x \log 4.$$

Proof. See [116, Chapter XXII] (the bound $\vartheta(n) < 2n\log 2$, by the argument of Erdős with the middle binomial coefficient); see also [214] (Chebyshev estimates, Chapter 2). The second form follows from the first: for $x \geq 1$ the sum equals $\log \prod_{p \leq \lfloor x \rfloor} p < \lfloor x \rfloor \log 4 \leq x \log 4$, and for $0 \leq x < 2$ it is 0. □

Theorem A.1.2. *Let $N \geq 1$ and let χ be a Dirichlet character modulo N. The function defined for $\operatorname{Re} s > 1$ by $\sum_{n \geq 1} \chi(n) n^{-s}$ extends to a meromorphic function $L(s,\chi)$ on $\mathbb{C}$. If $\chi \neq \chi_{0,N}$, this function is entire. If $\chi = \chi_{0,N}$, its only pole is a simple pole at $s = 1$, with residue $\varphi(N)/N$. In particular $\zeta(s)$ is holomorphic on $\mathbb{C} \setminus \{1\}$, and $\zeta(s) - \frac{1}{s-1}$ extends to an entire function whose value at $s = 1$ is γ:*

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|) \qquad (s \to 1).$$

Proof. [214] (the analytic continuation of $\zeta(s)$ with its Laurent expansion at $s = 1$, and of $L(s,\chi)$), [64], [308]. Notation as in the references: $L(s,\chi)$ for a character χ modulo N, with Euler factors 1 at the primes dividing N; for $\chi = \chi_{0,N}$ one has $L(s,\chi_{0,N}) = \zeta(s) \prod_{p \mid N} (1 - p^{-s})$, whence the residue $\prod_{p \mid N}(1 - p^{-1}) = \varphi(N)/N$. □

Theorem A.1.3. *Let $q \geq 3$ and let χ be a primitive Dirichlet character modulo q. Let $a \in \{0,1\}$ with $\chi(-1) = (-1)^a$, let $\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}$ be its Gauss sum, and put*

$$\xi(s,\chi) = \Big(\frac{q}{\pi}\Big)^{(s+a)/2} \Gamma\Big(\frac{s+a}{2}\Big) L(s,\chi).$$

Then $\xi(s,\chi)$ is entire, $|\tau(\chi)| = q^{1/2}$, and with $W(\chi) = \tau(\chi)/(i^a q^{1/2})$, so that $|W(\chi)| = 1$,

$$\xi(s,\chi) = W(\chi)\, \xi(1-s, \overline{\chi}) \qquad (s \in \mathbb{C}).$$

Proof. [64] (Chapter 9, the functional equation in the form $\xi(1-s,\overline{\chi}) = \frac{i^a q^{1/2}}{\tau(\chi)} \xi(s,\chi)$, and Chapter 2 for $|\tau(\chi)| = q^{1/2}$), [214] (Chapter 10). Notation as in the references, except that the root number is written $W(\chi) = \tau(\chi)/(i^a q^{1/2})$, the inverse of Davenport's factor $i^a q^{1/2}/\tau(\chi)$ (the two have modulus 1). □

Theorem A.1.4. *Let $N \geq 1$ and let χ be a Dirichlet character modulo N with $\chi \neq \chi_{0,N}$. Then $L(1,\chi) \neq 0$.*

Proof. Dirichlet's theorem, [214] (the non-vanishing of $L(1,\chi)$ in the proof of Dirichlet's theorem on primes in arithmetic progressions), [64]. Notation as in the references; $L(1,\chi)$ is the value at $s = 1$ of the entire function of Theorem A.1.2, equal to the conditionally convergent series $\sum_{n \geq 1} \chi(n)/n$. □

Theorem A.1.5. *The function $\xi(s) = \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$ is holomorphic on $\mathbb{C} \setminus \{0,1\}$ and satisfies*

$$\xi(1-s) = \xi(s) \qquad (s \in \mathbb{C} \setminus \{0,1\}).$$

Moreover $\zeta(0) = -\frac{1}{2}$.

Proof. Riemann's functional equation, [214] (Chapter 10), [64] (Chapter 8), [308] (Chapter II). Notation as in the references; ξ here is the completed zeta function without the factor $\frac{1}{2}s(s-1)$ of Riemann's normalisation, so that it has simple poles at $s = 0$ and $s = 1$. The value $\zeta(0) = -\frac{1}{2}$ follows from the functional equation and the residue 1 of ζ at $s = 1$. □

Theorem A.1.6. *Let $f \geq 1$ and let χ be a primitive Dirichlet character modulo f (for $f = 1$, $\chi = \chi_{0,1}$ and $L(s,\chi) = \zeta(s)$). For every integer $k \geq 1$,*

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k}, \qquad B_{k,\chi} = f^{\,k-1} \sum_{a=1}^{f} \chi(a)\, B_k\Big(\frac{a}{f}\Big),$$

where $L(1-k,\chi)$ is the value of the meromorphic continuation, which is holomorphic at $1-k$.

Proof. [317], Chapter 4, Theorem 4.2, with Proposition 4.1 for the expression of $B_{k,\chi}$ through Bernoulli polynomials. Washington's n is our k; for $f = 1$ his $B_{k,1}$ is $B_k(1)$, so that $\zeta(0) = -B_1(1) = -\frac{1}{2}$. □

Theorem A.1.7. *Let the Bernoulli numbers $B_j \in \mathbb{Q}$ be defined by $t/(e^t - 1) = \sum_{j \geq 0} B_j t^j / j!$. Then $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_j = 0$ for every odd $j \geq 3$, and for every even $j \geq 2$*

$$B_j + \sum_{\substack{p \text{ prime} \\ (p-1) \mid j}} \frac{1}{p} \in \mathbb{Z}.$$

Proof. [317], Chapter 5, Theorem 5.10 (von Staudt–Clausen); the values B_0, B_1 and the vanishing for odd $j \geq 3$ follow from the generating function, since $t/(e^t - 1) + t/2$ is even ([317], Chapter 4). Notation as in the reference. □

Theorem A.1.8. *For an integer $M \geq 2$, an integer b and real X let $\vartheta(X; M, b) = \sum \log p$, the sum over the primes $p \leq X$ with $p \equiv b \pmod M$.*

1. *For every integer $M \geq 2$ the product $\prod_{\chi \bmod M} L(s,\chi)$ has no zero in the region $\operatorname{Re} s \geq 1 - 1/(13 \log(M(|\operatorname{Im} s| + 3)))$, except possibly one real zero $\beta_1 < 1$. If it exists, it is called the* exceptional zero modulo M*; there is then exactly one real non-principal character χ_1 modulo M with $L(\beta_1, \chi_1) = 0$.*

2. *There are absolute, effectively computable constants $c_4 > 0$ and A_4 such that for every integer $M \geq 2$, every integer b with $\gcd(b, M) = 1$ and every real $X \geq M^{12}$,*

$$\Big|\vartheta(X; M, b) - \frac{X - \chi_1(b) X^{\beta_1}/\beta_1}{\varphi(M)}\Big| \leq A_4\Big(e^{-c_4 \log X / \log M} + e^{-c_4 (\log X)^{3/5} (\log\log X)^{-1/5}}\Big) \frac{X - \chi_1(b) X^{\beta_1}/\beta_1}{\varphi(M)},$$

where the term $\chi_1(b) X^{\beta_1}/\beta_1$ is omitted (in both places) when there is no exceptional zero modulo M.

Proof. [307]: item 1 is the statement on the exceptional zero in Section 1.1 (from McCurley's zero-free region for $\prod_{\chi \bmod q} L(s,\chi)$), item 2 is Corollary 1.4 with Remark 1.5 (numbering of arXiv:2108.10878v2). Notation: $q \mapsto M$, $a \mapsto b$, $x \mapsto X$; the left side of Corollary 1.4 is $\vartheta(X; M, b)$; its main term $\lambda x/\varphi(q)$ with $\lambda = \lambda(x, q, a, x) = 1 - \chi_1(a) x^{\beta_1 - 1}/\beta_1$ (Remark 1.5; $\lambda = 1$ if β_1 does not exist) is $(X - \chi_1(b) X^{\beta_1}/\beta_1)/\varphi(M)$; the implied constant of its O-term is A_4. That $\beta_1 < 1$ is the classical non-vanishing of $L(s,\chi)$ on $\operatorname{Re} s \geq 1$. □

Theorem A.1.9. *For an integer $q \geq 1$, an integer a and a real number X put $\psi(X;q,a) = \sum \Lambda(n)$, the sum over the integers $1 \leq n \leq X$ with $n \equiv a \pmod q$. For every $A > 0$ there are constants $c_A > 0$ and $C_A > 0$ such that*

$$\Big|\psi(X;q,a) - \frac{X}{\varphi(q)}\Big| \leq C_A\, X \exp\!\big(-c_A\sqrt{\log X}\big)$$

for every real $X \geq 2$, every integer q with $1 \leq q \leq (\log X)^A$ and every integer a with $\gcd(a,q) = 1$.

Proof. This is the Siegel–Walfisz theorem, [64], Chapter 22, and [214], Section 11.3. For $q = 1$ it is the prime number theorem with the de la Vallée Poussin error term. The constants are not effective (Siegel's theorem); only their existence is used. □

Theorem A.1.10. *There is an absolute constant $c_{\mathrm{PV}} > 0$ such that for every integer $q \geq 2$, every Dirichlet character $\chi \neq \chi_{0,q}$ modulo q, every integer a and every integer $N \geq 0$,*

$$\Big|\sum_{a < n \leq a+N} \chi(n)\Big| \leq c_{\mathrm{PV}}\, q^{1/2} \log q.$$

Proof. G. Pólya and I. M. Vinogradov (1918); [214, Theorem 9.18], [64, Chapter 23]. Notation as in the references: the character need not be primitive, and the sum runs over an arbitrary interval of integers of length N. □

Theorem A.1.11. *Let $q \geq 3$, let χ be a primitive Dirichlet character modulo q, let $a \in \{0,1\}$ with $\chi(-1) = (-1)^a$, and let $\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}$. For real $x > 0$ put*

$$\Theta_a(x,\chi) = \sum_{n \in \mathbb{Z}} n^a \chi(n)\, e^{-\pi n^2 x/q}.$$

Then $|\tau(\chi)| = q^{1/2}$, and for every $x > 0$

$$\Theta_a(x,\chi) = \frac{\tau(\chi)}{i^a q^{1/2}}\; x^{-a-1/2}\; \Theta_a(x^{-1}, \overline{\chi}).$$

Proof. [64, Chapter 9] (the functions $\psi(x,\chi)$ for even and $\psi_1(x,\chi)$ for odd χ, and their transformation formulae, obtained by Poisson summation), and [64, Chapter 2] for $|\tau(\chi)| = q^{1/2}$. Notation map: Davenport's $\psi(x,\chi)$ and $\psi_1(x,\chi)$ are $\Theta_0(x,\chi)$ and $\Theta_1(x,\chi)$; he writes the constant through $\tau(\overline{\chi})$, which is the same constant because $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)\, q$. □

Theorem A.1.12. *Let F be a number field and χ a unitary Hecke character of F.*

1. *For every Schwartz–Bruhat function f on $\mathbb{A}_F$ the zeta integral $Z(f,\chi,s)$ continues meromorphically to $\mathbb{C}$ and $Z(f,\chi,1-s) = Z(\hat{f},\chi^{-1},s)$. It is entire unless χ is trivial on C_F^1, that is $\chi = |\cdot|_{\mathbb{A}}^{i\tau}$ for some $\tau \in \mathbb{R}$; in that case its only poles are at most simple poles at $s = -i\tau$ and $s = 1 - i\tau$.*

2. *$\xi(s,\chi)$ continues meromorphically to $\mathbb{C}$, with the poles described in item 1, and $\xi(s,\chi) = W(\chi)\,\xi(1-s,\overline{\chi})$ with $W(\chi)W(\overline{\chi}) = 1$. If $\chi = |\cdot|_{\mathbb{A}}^{i\tau}$, the continued $L(s,\chi)$ has a simple pole with nonzero residue at $s = 1 - i\tau$.*

3. *If χ is not of the form $|\cdot|_{\mathbb{A}}^{i\tau}$, the continuation of $L(s,\chi)$ is holomorphic and has no zeros in* $\operatorname{Re} s \geq 1$.

Proof. Items 1 and 2 are Hecke's theorem in the adelic form of Tate's thesis: [296], Main Theorem 4.4.1 and Section 4.5 (with $\xi(s,\chi) = W(\chi)\xi(1-s,\overline{\chi})$ obtained from item 1 for the standard choice of f); [128]; [224], Chapter VII, Theorem 8.5 and Corollary 8.6 for the completed L-function. Applying the functional equation twice gives $W(\chi)W(\overline{\chi}) = 1$. For $\chi = |\cdot|_{\mathbb{A}}^{i\tau}$, $L(s,\chi)$ is the Dedekind zeta function $\zeta_F(s+i\tau)$ up to finitely many Euler factors that are holomorphic and non-zero near $s = 1 - i\tau$, so the residue statement is the class number formula. Item 3 is the non-vanishing of Hecke L-functions on $\operatorname{Re} s = 1$: [182], Chapter XV, Section 5; for $\operatorname{Re} s > 1$ it follows from the absolutely convergent Euler product. □

Theorem A.1.13. *Let $\tau = x + iy$ with $y > 0$, and for $\operatorname{Re} s > 1$ put*

$$E(\tau, s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|m+n\tau|^{2s}}.$$

Use the Weierstrass normalisation

$$\begin{aligned} g_2(\tau) &= 60 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-4}, \\ g_3(\tau) &= 140 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-6}, \\ \Delta_{\mathrm{W}}(\tau) &= g_2(\tau)^3 - 27 g_3(\tau)^2. \end{aligned}$$

Then $\Delta_{\mathrm{W}}(\tau) \neq 0$, and $E(\tau,s)$ continues meromorphically to a neighbourhood of $s = 1$, with a simple pole of residue π and

$$\lim_{s\to 1}\left(E(\tau,s) - \frac{\pi}{s-1}\right) = 2\pi\left(\gamma - \log 2 - \frac{1}{12}\log\frac{y^6|\Delta_{\mathrm{W}}(\tau)|}{(2\pi)^{12}}\right).$$

Proof. This is Kronecker's first limit formula in the Weierstrass normalisation. Williams [329], definition (7.10) and Theorem 7.19, equation (7.20), uses the SAME all-integer-pairs series and gives the constant $2\pi(\gamma - \log 2 - \log(\sqrt{y}\,|\eta(\tau)|^2))$. His definition (4.39) is precisely $\Delta_{\mathrm{W}} = (60G_4)^3 - 27(140G_6)^2$, and the Jacobi normalisation (4.42) is $\Delta_{\mathrm{W}} = (2\pi)^{12}\eta^{24}$. Therefore

$$\log(\sqrt{y}\,|\eta(\tau)|^2) = \frac{1}{12}\log\frac{y^6|\Delta_{\mathrm{W}}(\tau)|}{(2\pi)^{12}},$$

which gives exactly the displayed version. The nonvanishing and local meromorphic continuation are part of these classical statements. No coprimality restriction or quotient by $(m,n) \mapsto (-m,-n)$ is used. □

A.2 Algebraic number theory, class field theory and CM abelian varieties

The field E of the datum is a cyclotomic extension of an imaginary quadratic field, and these are results on the arithmetic of such fields [224, 317]: the classical facts on units, class groups and discriminants in towers [212, 202]; class field theory [53, 9, 260, 297], with Chebotarev's density theorem [54], Hasse's norm theorem [123], Landherr's classification of hermitian forms [180, 249],

the decomposition of primes in cyclotomic fields [203] and the reciprocity law for the Brauer group of $\mathbb{Q}$ [265, 201]; Gauss sums, with Stickelberger's factorisation [290] and the Davenport–Hasse relation [65, 145]; and abelian varieties and elliptic curves with complex multiplication [220, 27, 266, 282, 281, 39, 77, 178].

Lemma A.2.1. *Let α be an algebraic integer such that $|\sigma(\alpha)| = 1$ for every field embedding $\sigma\colon \mathbb{Q}(\alpha) \to \mathbb{C}$. Then α is a root of unity.*

Proof. This is Kronecker's theorem; see [317], Chapter 1 (Lemma 1.6 there: an algebraic integer all of whose conjugates have absolute value 1 is a root of unity). Notation map: the conjugates of α are the values $\sigma(\alpha)$ of the embeddings σ of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$. In the uses in this project α lies in a number field K; every embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ is the restriction of an embedding of K, so it suffices that $|\tau(\alpha)| = 1$ for every embedding $\tau\colon K \to \mathbb{C}$. □

Theorem A.2.2. *Let K be a number field with r_1 real embeddings and r_2 conjugate pairs of complex embeddings. Its roots of unity form a finite cyclic group $\mu(K)$, and*

$$\mathcal{O}_K^{\times} \simeq \mu(K) \times \mathbb{Z}^{r_1+r_2-1}$$

as abelian groups.

Proof. This is Dirichlet's unit theorem, including the torsion subgroup identification; see [224, Chapter I]. □

Theorem A.2.3. *Let K be a number field of degree n, with r_2 pairs of nonreal embeddings. Every ideal class of K contains a nonzero integral ideal $\mathfrak{a}$ satisfying*

$$N_{K/\mathbb{Q}}\mathfrak{a} \le \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^{r_2} |\operatorname{disc}(K)|^{1/2}.$$

Proof. This is Minkowski's ideal-class bound; see [224], the geometry of numbers and finiteness of the ideal class group. The ideal norm is the absolute index $|\mathcal{O}_K/\mathfrak{a}|$, and r_2 counts conjugate pairs, so the displayed constant has the standard normalization. □

Theorem A.2.4. *Let $F \subseteq L$ be number fields, and let*

$$C_{L/F} = \{a \in L : \operatorname{Tr}_{L/F}(a\mathcal{O}_L) \subseteq \mathcal{O}_F\}, \qquad \mathfrak{D}_{L/F} = C_{L/F}^{-1}.$$

Then

$$|\operatorname{disc}(L)| = |\operatorname{disc}(F)|^{[L:F]} N_{L/\mathbb{Q}}(\mathfrak{D}_{L/F}).$$

Proof. This is the different–discriminant and discriminant tower formula of [212, §5.2, Lemma 5.5 and Theorem 5.7]. Here $\mathfrak{D}_{L/F}$ is the relative different defined by the inverse trace codifferent, $N_{L/\mathbb{Q}}$ is the absolute ideal norm, and disc denotes the absolute field discriminant. □

Theorem A.2.5. *Let L/K be a finite abelian extension of number fields with Galois group G. For each $\chi \in \widehat{G} = \operatorname{Hom}(G, \mathbb{C}^{\times})$, let $\mathfrak{f}_0(\chi)$ be the finite part of the conductor of the finite-order Hecke character obtained by composing χ with the Artin map for L/K. Then the relative discriminant ideal is*

$$\mathfrak{d}_{L/K} = \prod_{\chi \in \widehat{G}} \mathfrak{f}_0(\chi).$$

For $K=\mathbb{Q}$, this says that the absolute discriminant of L is the product of the positive integer conductors of the corresponding primitive Dirichlet characters.

Proof. This is the conductor–discriminant formula, [202, Chapter V, Theorem 3.27]. The ideal denoted $\mathrm{disc}(L/K)$ there is $\mathfrak{d}_{L/K}$ here, and its factor $f_0(\chi\circ\phi_{L/K})$ is the finite conductor ideal $\mathfrak{f}_0(\chi)$. Over $\mathbb{Q}$, the norm of that ideal is the conductor of the corresponding primitive Dirichlet character, so taking ideal norms gives the last assertion. □

Theorem A.2.6. *Let L/F be a finite Galois extension of number fields and $C\subseteq \mathrm{Gal}(L/F)$ a subset stable under conjugation. Then the set of finite places v of F unramified in L whose Frobenius class in $\mathrm{Gal}(L/F)$ is contained in C has Dirichlet density $\#C/[L:F]$.*

Proof. [54]; in the form stated (Dirichlet density): [224], Chapter VII, Theorem 13.4. For a conjugation-stable C apply the theorem to each conjugacy class contained in C and add the densities. □

Theorem A.2.7. *1. Let K be a non-archimedean local field. There is a continuous injective homomorphism $\mathrm{Art}_K\colon K^\times\to G_K^{ab}$ with dense image such that every uniformiser acts as an arithmetic Frobenius on K^{ur} and, for every finite abelian extension L/K, the composite $K^\times\to\mathrm{Gal}(L/K)$ is surjective with kernel $\mathrm{N}_{L/K}L^\times$. Every open subgroup of finite index of $K^\times$ is $\mathrm{N}_{L/K}L^\times$ for a unique finite abelian L/K.*

2. Let F be a number field. There is a continuous surjective homomorphism $\mathrm{Art}_F\colon C_F\to G_F^{ab}$ with kernel C_F^0, compatible with the local Artin maps at every finite place, sending the class of an idele supported at a finite place v unramified in a finite abelian L/F to Frob_v on L when its v-component is a uniformiser and to 1 when it is a unit, and sending the idele -1 at a real place w to complex conjugation at w. The map $L\mapsto \mathrm{N}_{L/F}C_L$ is an inclusion-reversing bijection between finite abelian extensions of F and open subgroups of finite index of C_F; in particular ray class fields exist and are unramified outside their modulus.

Proof. [260]; [297]; [9]; [224], Ch. V and VI. □

Theorem A.2.8. *Let L/F be a finite abelian extension of number fields. There is a continuous surjective homomorphism*

$$\mathrm{rec}_{F,L}: C_F\longrightarrow \mathrm{Gal}(L/F),\qquad \ker\mathrm{rec}_{F,L}=N_{L/F}C_L.$$

Its pullback to $\mathbb{A}_F^\times$ kills $F^\times$. For every place v of F and $w\mid v$, its restriction to $F_v^\times$ is the local reciprocity map into $D_w=\mathrm{Gal}(L_w/F_v)$, followed by inclusion in $\mathrm{Gal}(L/F)$; its local kernel is $N_{L_w/F_v}L_w^\times$. At finite v, local units map onto inertia. At an unramified finite place, units map to 1 and a uniformiser maps to arithmetic Frobenius. At a real place becoming complex, positive elements map to 1 and negative elements to complex conjugation; at a split real or complex place the local map is trivial.

Proof. This is global Artin reciprocity and its local compatibility: [202, Chapter V, Proposition 5.2 and Theorem 5.3; Chapter I, Theorem 1.1 and 1.6]. The real normalization follows from $N_{\mathbb{C}/\mathbb{R}}\mathbb{C}^\times=\mathbb{R}_{>0}$. □

Theorem A.2.9. *Let K be a number field and $\mathfrak{m}$ a non-zero ideal of $\mathcal{O}_K$. Let $I_K(\mathfrak{m})$ be the group of fractional ideals of K coprime to $\mathfrak{m}$, and $P_K(\mathfrak{m})$ its subgroup of principal ideals (α) with $\alpha\equiv 1 \pmod{}^\times\mathfrak{m}$. Put $\mathrm{Cl}_K(\mathfrak{m})=I_K(\mathfrak{m})/P_K(\mathfrak{m})$, let h_K be the class number of K and $\mathcal{O}_{K,\mathfrak{m}}^\times$ the group of units $u\in\mathcal{O}_K^\times$ with $u\equiv 1 \pmod{\mathfrak{m}}$.*

1. $\mathrm{Cl}_K(\mathfrak{m})$ *is finite, of order*
$$|\mathrm{Cl}_K(\mathfrak{m})| = \frac{h_K\,|(\mathcal{O}_K/\mathfrak{m})^\times|}{[\mathcal{O}_K^\times : \mathcal{O}_{K,\mathfrak{m}}^\times]}.$$

2. *There is a finite abelian extension L/K, the ray class field of K modulo $\mathfrak{m}$, unramified at every prime ideal of K not dividing $\mathfrak{m}$, such that the Artin map $\mathfrak{p} \mapsto (\frac{L/K}{\mathfrak{p}})$ on the prime ideals $\mathfrak{p} \nmid \mathfrak{m}$, extended multiplicatively to $I_K(\mathfrak{m})$, is surjective onto $\mathrm{Gal}(L/K)$ with kernel $P_K(\mathfrak{m})$. In particular a prime ideal $\mathfrak{p} \nmid \mathfrak{m}$ has trivial Artin symbol in L/K if and only if $\mathfrak{p} \in P_K(\mathfrak{m})$.*

3. *Through this isomorphism every character of $\mathrm{Gal}(L/K)$ is a character of $\mathrm{Cl}_K(\mathfrak{m})$, and the primitive Hecke character inducing it has conductor dividing $\mathfrak{m}$; in particular its norm is at most $\mathrm{N}\mathfrak{m}$.*

Proof. This is classical class field theory (Takagi's existence theorem and Artin's reciprocity law in ideal-theoretic form); see [224], Ch. VI. Item 1 is the exact sequence for the ray class group modulo $\mathfrak{m}$ (with a modulus without infinite places); item 2 is the existence of the ray class field and the reciprocity isomorphism $\mathrm{Cl}_K(\mathfrak{m}) \cong \mathrm{Gal}(L/K)$ given by the Artin symbol; item 3 is the definition of the conductor of a character of $\mathrm{Cl}_K(\mathfrak{m})$, the smallest modulus through which it factors, which divides $\mathfrak{m}$. Neukirch writes $J^{\mathfrak{m}}$, $P^{\mathfrak{m}}$ and $\mathrm{Cl}_K^{\mathfrak{m}}$ for $I_K(\mathfrak{m})$, $P_K(\mathfrak{m})$ and $\mathrm{Cl}_K(\mathfrak{m})$. □

Theorem A.2.10. *Let $n \geq 1$, $L = \mathbb{Q}(\zeta_n)$, and identify $\mathrm{Gal}(L/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^\times$ by $\sigma_u(\zeta_n) = \zeta_n^u$. For a rational prime p, write $n = p^a n'$, $p \nmid n'$. At any prime of L above p, the inertia and decomposition groups are respectively*
$$I_p = \{u : u \equiv 1 \pmod{n'}\}, \qquad D_p = \{u : u \equiv p^b \pmod{n'} \text{ for some } b \geq 0\}.$$
In particular the ramification index is $\varphi(p^a)$; $L/\mathbb{Q}$ is unramified outside n.

For every intermediate field $\mathbb{Q} \subseteq E \subseteq L$, restriction sends these groups onto the inertia and decomposition groups at the restricted prime of E. Thus those groups are the images of I_p, D_p in $\mathrm{Gal}(E/\mathbb{Q})$. All notation is local.

Proof. These are the classical cyclotomic decomposition formulas. The p-power cyclotomic factor is totally ramified at p; the prime-to-p factor is unramified, with arithmetic Frobenius acting by $\zeta_{n'} \mapsto \zeta_{n'}^p$. The Chinese remainder identification gives the displayed groups. Restriction of decomposition and inertia groups in a finite Galois tower gives the final assertion. See [203, §6, Proposition 6.2 and Theorem 6.4]. □

Theorem A.2.11. *Let F be a number field and $\delta, x \in F^\times$. Suppose that for every real embedding $\sigma\colon F \to \mathbb{R}$ there are $a, b \in \mathbb{R}$ with $\sigma(x) = a^2 - \sigma(\delta)b^2$, and that for every finite place v of F there are $a, b \in F_v$ with $x = a^2 - \delta b^2$ in F_v. Then there are $a, b \in F$ with $x = a^2 - \delta b^2$.*

Proof. This is Hasse's norm theorem [123] for the extension $F(\sqrt{\delta})/F$, which is cyclic of degree at most 2; see [224], Chapter VI, Section 4. *Notation.* If δ is not a square in F, then $a^2 - \delta b^2$ is the norm of $a + b\sqrt{\delta}$ from $E = F(\sqrt{\delta})$, and the local hypothesis at a place v says that x is a norm from $E \otimes_F F_v$; at a complex place this holds for every x, so complex places are omitted. If $\delta = d^2$ is a square in F, the conclusion holds without hypotheses: $x = a^2 - \delta b^2$ with $a = (x+1)/2$, $b = (x-1)/(2d)$. □

Theorem A.2.12. *Let E be a CM field and $G, G' \in \mathrm{M}_r(E)$ nondegenerate hermitian matrices ($\bar{G}^\top = G$, $\bar{G'}^\top = G'$). Suppose that for every embedding $\tau\colon E \to \mathbb{C}$ the hermitian matrices $\tau(G)$ and $\tau(G')$ have the same signature, and that $\det G' = z\bar{z}\det G$ for some $z \in E$. Then there is $\gamma \in \mathrm{GL}_r(E)$ with*

$$\bar{x}^\top \bar{\gamma}^\top G\,\gamma\, y = \bar{x}^\top G'\, y \qquad \text{for all } x, y \in E^r,$$

that is, the hermitian forms of G and G' are isometric.

Proof. Landherr's theorem [180]; see [249], Chapter 10: for a quadratic extension E/F of number fields, two nondegenerate hermitian forms over E are isometric if and only if they have the same dimension, the same discriminant in $F^\times/\mathrm{N}_{E/F}(E^\times)$, and the same signature at every real place of F that does not split in E. Only the sufficiency is quoted. *Notation.* Here F is the maximal totally real subfield of E, so every real place of F is complex in E and every embedding τ of E restricts to a real place of F; the signature at that place is the signature of $\tau(G)$, which does not depend on the choice of τ above it, since the other choice gives the complex conjugate of $\tau(G)$, whose signature is the same. The determinant of a hermitian matrix lies in $F^\times$, and the condition $\det G' = z\bar{z}\det G$ says that the discriminants agree modulo $\mathrm{N}_{E/F}(E^\times)$ (with $z \neq 0$, as both determinants are nonzero). The isometry is the displayed identity, that is $\bar{\gamma}^\top G\gamma = G'$. $\square$

Theorem A.2.13. *Let $m \geq 2$ be an integer, $K = \mathbb{Q}(\mu_m) \subset \overline{\mathbb{Q}}$, and for $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ let $\sigma_t \in \mathrm{Gal}(K/\mathbb{Q})$ be the automorphism with $\sigma_t(\zeta) = \zeta^{\tilde{t}}$ for $\zeta \in \mu_m$, where $\tilde{t} \in \{1, \dots, m-1\}$ is the representative of t. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ not dividing m, p its residue characteristic, $k = \mathcal{O}_K/\mathfrak{p}$ and $q = |k|$, so that $m \mid q-1$. Let $\chi_\mathfrak{p}\colon k^\times \to \mu_m$ be the m-th power residue character: $\chi_\mathfrak{p}(x \bmod \mathfrak{p})$ is the element of μ_m congruent to $x^{(q-1)/m}$ modulo $\mathfrak{p}$, for $x \in \mathcal{O}_K \setminus \mathfrak{p}$. Fix a primitive p-th root of unity $\zeta_p \in \overline{\mathbb{Q}}$ and put, for $j \in \mathbb{Z}$,*

$$\mathrm{g}_\mathfrak{p}(j) = \sum_{c \in k^\times} \chi_\mathfrak{p}(c)^{-j}\, \zeta_p^{\mathrm{Tr}_{k/\mathbb{F}_p}(c)} \in \mathbb{Q}(\mu_{mp}).$$

Then:

1. *if $m \nmid j$, then $\mathrm{g}_\mathfrak{p}(j)\,\mathrm{g}_\mathfrak{p}(-j) = \chi_\mathfrak{p}(-1)^j\, q$;*
2. *$\mathrm{g}_\mathfrak{p}(1)^m \in K$, and as ideals of $\mathcal{O}_K$*

$$\mathrm{g}_\mathfrak{p}(1)^m\, \mathcal{O}_K = \prod_{t \in (\mathbb{Z}/m\mathbb{Z})^\times} \sigma_t^{-1}(\mathfrak{p})^{\tilde{t}}.$$

Proof. We use Stickelberger's valuation theorem in the following precise normalization [181], Chapter 1, Section 2, Theorem 2.1, printed p.6. If $q = p^f$, $\mathfrak{P}$ is a prime above p of $\mathbb{Q}(\mu_{q-1})$, $\omega(x) \equiv x \pmod{\mathfrak{P}}$, and $\mathfrak{Q}$ is its unique extension to $\mathbb{Q}(\mu_{q-1}, \zeta_p)$, then

$$\mathrm{ord}_\mathfrak{Q}\left(\sum_{x \in \mathbb{F}_q^\times} \omega(x)^{-a} \zeta_p^{\mathrm{Tr}(x)}\right) = s_p(a), \qquad 0 < a < q-1,$$

where $s_p(a)$ is the sum of the base-p digits of a. The prime valuation on the left is integer-normalized, so $\mathrm{ord}_\mathfrak{Q}(p) = p-1$. The character, additive trace and minus sign in the exponent are as displayed; this theorem has no oddness restriction on p. We now give the conversion to the present statement for every residue degree f.

The product identity and descent of the power. Put $\alpha = \chi_{\mathfrak{p}}^{-j}$ and $\lambda(x) = \zeta_p^{\mathrm{Tr}(x)}$. If $m \nmid j$, α is nontrivial. In the product of the sums for α and α^{-1} write $x = uy$:

$$\sum_{u \neq 0} \alpha(u) \sum_{y \neq 0} \lambda((u+1)y).$$

The inner sum is $q-1$ for $u = -1$ and -1 otherwise, because λ is nontrivial. Since $\sum_{u \neq 0} \alpha(u) = 0$, the result is $q\alpha(-1)$. As $\chi_{\mathfrak{p}}(-1)^2 = 1$, this equals $q\chi_{\mathfrak{p}}(-1)^j$, proving item 1. In particular these sums are nonzero.

Let $L = K(\zeta_p)$. For $c \in \mathbb{F}_p^\times$, the automorphism fixing K and taking ζ_p to ζ_p^c takes $\mathrm{g}_{\mathfrak{p}}(1)$ to $\chi_{\mathfrak{p}}(c)\mathrm{g}_{\mathfrak{p}}(1)$, by substituting $x = c^{-1}u$. Its m-th power is therefore fixed by $\mathrm{Gal}(L/K)$ and belongs to K. This remains valid for $p = 2$, when that group is trivial.

The valuation at the fixed prime uses the inverse residue character. Choose a prime $\mathfrak{P}$ of $T = \mathbb{Q}(\mu_{q-1})$ above $\mathfrak{p}$. Its residue field is k: indeed the order of p modulo $q-1 = p^f - 1$ is f, so its residue degree is f. Reduction identifies its Teichmuller character ω on $k^\times$. With $a = (q-1)/m$, both $\chi_{\mathfrak{p}}(x)$ and $\omega(x)^a$ reduce to x^a, and reduction is injective on roots of unity of order prime to p. Thus

$$\chi_{\mathfrak{p}} = \omega^a.$$

For $1 \leq t < m$ prime to m, take the automorphism of L fixing ζ_p and restricting to σ_t on K. It takes $\mathrm{g}_{\mathfrak{p}}(1)$ to $\mathrm{g}_{\mathfrak{p}}(t)$. The character in this last sum is precisely ω^{-at}; reduce at to its representative modulo $q-1$ before applying the digit formula. The extension T/K is unramified at p, and adjoining ζ_p has ramification index $p-1$. Since $p \nmid m$, the valuation at $\mathfrak{p}$ on K gives p valuation 1. Consequently

$$\mathrm{ord}_{\mathfrak{p}}(\mathrm{g}_{\mathfrak{p}}(t)^m) = \frac{m}{p-1} s_p(at \bmod (q-1)).$$

The repeated primes give the full factorisation. For $0 < b < q-1$, cyclic permutation of its f base-p digits gives

$$\frac{s_p(b)}{p-1} = \sum_{i=0}^{f-1} \left\{ \frac{p^i b}{q-1} \right\}.$$

Indeed the representatives of $p^i b$ modulo $q-1$ are precisely the cyclic rotations of those digits; their sum is $s_p(b)(1 + p + \cdots + p^{f-1})$. Apply this to the representative of at. We obtain

$$\mathrm{ord}_{\sigma_t^{-1}(\mathfrak{p})}(\mathrm{g}_{\mathfrak{p}}(1)^m) = m \sum_{i=0}^{f-1} \left\{ \frac{p^i t}{m} \right\} = \sum_{i=0}^{f-1} \widetilde{p^i t}.$$

Here the first equality also uses $\mathrm{ord}_{\mathfrak{p}}(\sigma_t z) = \mathrm{ord}_{\sigma_t^{-1}\mathfrak{p}}(z)$. The decomposition subgroup of $\mathfrak{p}$ in $K/\mathbb{Q}$ is generated by σ_p, the arithmetic residue Frobenius $x \mapsto x^p$, and has order f. Hence $\sigma_u^{-1}\mathfrak{p} = \sigma_t^{-1}\mathfrak{p}$ exactly when u runs through $t, pt, \ldots, p^{f-1}t$ modulo m. The preceding sum is exactly the exponent of this prime in the displayed product of item 2, including all repetitions.

Finally every Gauss sum is an algebraic integer. Item 1 shows that its product with the opposite-character sum is a root of unity times q, so it has no prime divisor away from p. The valuations just computed therefore prove equality of the two ideals.

Changing ζ_p to ζ_p^c multiplies the sum by $\chi_{\mathfrak{p}}(c)$, an m-th root of unity; changing the additive character by its inverse is the case $c = -1$. Neither change alters the ideal. Replacing the multiplicative character by its inverse would instead replace t by $-t$ in its exponents. The negative exponent in the statement is thus essential, and is the one used throughout this proof. □

Theorem A.2.14. *Let k be a finite field of characteristic p, k'/k a finite extension of degree s, $\zeta_p \in \overline{\mathbb{Q}}$ a primitive p-th root of unity, and for a finite field $\kappa \supseteq \mathbb{F}_p$ and a character $\alpha\colon \kappa^\times \to \overline{\mathbb{Q}}^\times$ put $\mathrm{g}_\kappa(\alpha) = \sum_{c\in\kappa^\times} \alpha(c)\,\zeta_p^{\mathrm{Tr}_{\kappa/\mathbb{F}_p}(c)}$. If α is a nontrivial character of $k^\times$, then*

$$-\mathrm{g}_{k'}(\alpha \circ \mathrm{N}_{k'/k}) = \bigl(-\mathrm{g}_k(\alpha)\bigr)^s.$$

Proof. The Davenport–Hasse relation [65]; see also [317], Chapter 6, and [145]. Notation as in the references, with the additive character $c \mapsto \zeta_p^{\mathrm{Tr}_{\kappa/\mathbb{F}_p}(c)}$ of κ; the trace of k' is the trace of k composed with $\mathrm{Tr}_{k'/k}$. □

Theorem A.2.15. *Let K be a number field. For a finite place v, the local invariant is an isomorphism $\mathrm{inv}_v : \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$. For a real place it identifies $\mathrm{Br}(K_v)$ with $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, and at a complex place $\mathrm{Br}(K_v) = 0$. The localization and invariant maps form an exact sequence*

$$0 \longrightarrow \mathrm{Br}(K) \xrightarrow{(\mathrm{res}_v)_v} \bigoplus_v \mathrm{Br}(K_v) \xrightarrow{(\beta_v)_v \mapsto \sum_v \mathrm{inv}_v(\beta_v)} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Proof. [202], Chapter IV, Section 4 (local invariants), and Chapter VIII, Theorem 4.2 (the fundamental exact sequence). □

Theorem A.2.16. *Let V be a complex vector space of finite dimension $r \geq 0$, let H be positive-definite Hermitian, and let $\Lambda \subset V$ be a full lattice self-dual for $E = \operatorname{Im} H$. Let $\alpha : \Lambda \to \{1, -1\}$ satisfy*

$$\alpha(x + y) = \alpha(x)\alpha(y)e^{\pi i E(x,y)}.$$

Then the space of entire functions satisfying

$$F(z + x) = \alpha(x)e^{\pi H(z,x)+\pi H(x,x)/2}F(z) \quad (x \in \Lambda)$$

has complex dimension exactly one. Equivalently, with $\mathsf{U}_x F(z) = e^{-\pi H(z,x)-\pi H(x,x)/2}F(z + x)$, the theta space defined by $\alpha(x)\mathsf{U}_x F = F$ has dimension one. This applies to the original sign semicharacter; no theta vector is replaced.

Proof. This is the principal-polarisation case of classical theta theory and abelian Riemann–Roch [220]. Here is the hypothesis comparison. Appell–Humbert identifies the displayed automorphy space with $H^0(A, \mathcal{L}(H, \alpha))$ on $A = V/\Lambda$. Positivity of H makes this line bundle ample and A projective. Self-duality means that the integral alternating form E is unimodular; thus its polarisation has degree one. Riemann–Roch gives $\chi(\mathcal{L})^2 = 1$. Positivity gives $\chi(\mathcal{L}) > 0$, while vanishing for an ample line bundle on a complex abelian variety gives $H^i(A, \mathcal{L}) = 0$ for $i > 0$. Hence $h^0(A, \mathcal{L}) = \chi(\mathcal{L}) = 1$. The two automorphy conventions agree since $\alpha(x)^{-1} = \alpha(x)$. When $r = 0$, the torus is a point and the space consists of constants. □

Theorem A.2.17. *Let $k \subset \mathbb{C}$ be an imaginary quadratic field with ring of integers $\mathcal{O}_k$.*

1. *(Algebraicity.) If Z is an elliptic curve over $\mathbb{C}$ with $\mathrm{End}(Z) \cong \mathcal{O}_k$, then $j(Z)$ is an algebraic number; hence there are a number field $M \subset \mathbb{C}$ and an elliptic curve Z_M over M with $Z_M \otimes_M \mathbb{C} \cong Z$, and M may be chosen so that every endomorphism of $Z_M \otimes_M \mathbb{C}$ is defined over M.*

2. *(Potentially good reduction.) If M is a number field and Z_M an elliptic curve over M whose endomorphism ring over $\overline{M}$ is an order in an imaginary quadratic field, then Z_M has potentially good reduction at every finite place of M: there is a finite extension M'/M such that $Z_M \otimes_M M'$ has good reduction at every finite place of M', that is, extends to an abelian scheme over $\mathcal{O}_{M'}$.*

Proof. Item 1: the algebraicity of $j(Z)$ (indeed $k(j(Z))$ is the Hilbert class field of k) is the first main theorem of complex multiplication for elliptic curves (Deuring; Shimura–Taniyama), [281], Chapter II, §4, Theorem II.4.3; an elliptic curve over $\mathbb{C}$ with algebraic j-invariant is defined over $\mathbb{Q}(j)$, and its endomorphisms, finitely many generators of a $\mathbb{Z}$-module of rank 2, are defined over a finite extension (each is defined over $\overline{\mathbb{Q}}$ because homomorphisms between abelian varieties over an algebraically closed field do not change under extension to a larger algebraically closed field). Item 2: J.-P. Serre and J. Tate [266], §4, Theorem 6 (an abelian variety with complex multiplication, over a field with a discrete valuation with finite residue field, has potentially good reduction at every place), and [281], Chapter II, §6, for elliptic curves; the global extension M' is obtained since Z_M has good reduction outside finitely many places and good reduction persists under finite extensions; an elliptic curve over M' with good reduction at every finite place is the generic fibre of its Néron model, an abelian scheme over $\mathcal{O}_{M'}$ ([39], §1.4). The notation agrees with that of the sources. □

Theorem A.2.18. *Let $K \subset \mathbb{C}$ be an imaginary quadratic field and let $I \subset K$ be a nonzero fractional $\mathcal{O}_K$-ideal. Let $g_2(I), g_3(I)$ be its Weierstrass invariants, put $\Delta(I) = g_2(I)^3 - 27 g_3(I)^2$, and use the usual normalisation*

$$j(I) = 1728\, g_2(I)^3 / \Delta(I).$$

Then $j(I)$ is an algebraic integer.

Proof. This is the classical integrality theorem for singular moduli: [204], Corollary 12.16, printed page 125. Choose an oriented $\mathbb{Z}$-basis ω_1, ω_2 of I with $\tau = \omega_2/\omega_1$ in the upper half-plane. Then $\tau \in K$, and scaling invariance gives $j(I) = j(\mathbb{Z} + \mathbb{Z}\tau) = j(\tau)$. Corollary 12.16 states exactly that $j(\tau)$ is an algebraic integer for such an imaginary-quadratic τ. □

Theorem A.2.19. *Let S be a noetherian normal scheme, let $U \subseteq S$ be a dense open subscheme, and let G, H be semiabelian schemes over S. Every homomorphism of U-group schemes*

$$f_U : H_U \longrightarrow G_U$$

extends uniquely to a homomorphism of S-group schemes $f : H \to G$.

Proof. This is the semiabelian homomorphism extension theorem, [77, Chapter I, Proposition 2.7]; see also [178, Proposition 3.3.1.5]. The base is noetherian and normal, the subscheme is open and dense, and both group schemes are semiabelian, exactly as required by that theorem. □

Theorem A.2.20. *Let p be a prime, $K/\mathbb{Q}_p$ finite, and $L \subset \overline{\mathbb{Q}}_p$ a finite coefficient field containing every image of K under a $\mathbb{Q}_p$-embedding. Let $\eta : G_K^{\mathrm{ab}} \to \mathcal{O}_L^{\times}$ be continuous, and let $V = L(\eta)$, regarded also as a $\mathbb{Q}_p$-representation. Write $\mathrm{Art}_K : K^{\times} \to G_K^{\mathrm{ab}}$ for arithmetic reciprocity.*

1. *If integers a_τ, indexed by $\tau : K \hookrightarrow L$, satisfy*

$$\eta(\mathrm{Art}_K(u)) = \prod_{\tau : K \hookrightarrow L} \tau(u)^{a_\tau}$$

on some open subgroup of $\mathcal{O}_K^\times$, then V is Hodge–Tate and its labelled Hodge–Tate weight at τ is $-a_\tau$.

2. *The representation V is crystalline if and only if there exists a morphism of $\mathbb{Q}_p$-tori*

$$A : \mathrm{Res}_{K/\mathbb{Q}_p}\,\mathbb{G}_m \longrightarrow \mathrm{Res}_{L/\mathbb{Q}_p}\,\mathbb{G}_m$$

such that $\eta(\mathrm{Art}_K(u)) = A(u)$ for every $u \in \mathcal{O}_K^\times$.

Proof. The first assertion is the local algebraicity theorem of Tate, with its weight calculation, in [261], Chapter III, Appendix A.5, Theorem 2, and A.6, Corollary 2. Arithmetic reciprocity sends a unit u to an element on which the Lubin–Tate character has value u^{-1}; its distinguished Hodge–Tate weight is 1. Thus an exponent a_τ gives weight $-a_\tau$ with the stated cyclotomic convention. The second assertion is the crystalline abelian-character theorem of Fontaine–Serre, [264], Section 2.3, Theorem 5 and Corollary 2, in the local reciprocity formulation. □

A.3 Galois cohomology and p-adic Hodge theory

The Galois cohomology rests on the duality theorems of Tate and Poitou [295, 299, 150, 265, 201, 225], and the local conditions above ℓ are those of Bloch and Kato [28]. The p-adic Hodge theory is Fontaine's [85, 86, 87, 44], with Berger's theorem [22] and the Weil–Deligne description of crystalline representations [302]. With them come the theorem of Ax, Sen and Tate [298, 10], the logarithms of formal group laws [124, 193], and Mazur's representability of framed deformations [196, 168].

Theorem A.3.1. *Let F be a number field, v a finite place of F, ℓ a prime, $r \geq 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with a continuous action of G_{F_v}. Then:*

1. *$H^i(F_v, M)$ is finite for every $i \geq 0$, and $H^i(F_v, M) = 0$ for $i \geq 3$;*
2. *$H^2(F_v, \mu_{\ell^r}) \cong \mathbb{Z}/\ell^r\mathbb{Z}$, and for $i = 0, 1, 2$ the cup product*

$$H^i(F_v, M) \times H^{2-i}(F_v, M^\vee) \longrightarrow H^2(F_v, \mu_{\ell^r})$$

is a perfect pairing of finite groups;

3. *if $v \nmid \ell$ and I_v acts trivially on M, then $H^1_{\mathrm{ur}}(F_v, M)$ and $H^1_{\mathrm{ur}}(F_v, M^\vee)$ are the exact annihilators of each other under the pairing in item 2 with $i = 1$;*
4. *(local Euler characteristic formula) $\#H^1(F_v, M) = \#H^0(F_v, M)\,\#H^2(F_v, M)\,(\#M)^{[F_v:\mathbb{Q}_\ell]}$ if $v \mid \ell$, and $\#H^1(F_v, M) = \#H^0(F_v, M)\,\#H^2(F_v, M)$ if $v \nmid \ell$.*

Proof. This is Tate's local duality theorem and Tate's local Euler characteristic formula, [295]; in the form stated: [201], Chapter I, Corollary 2.3 (item 1 and the perfect pairing of item 2, with $H^2(F_v, \mu_{\ell^r}) = \mathrm{Br}(F_v)[\ell^r] \cong \mathbb{Z}/\ell^r\mathbb{Z}$ by the invariant map), Theorem 2.6 (item 3) and Theorem 2.8 (item 4); see also [265], Chapter II, Section 5. Notation: Milne's K is F_v, his M^D is $M^\vee$, and his H^1_{nr} is H^1_{ur}. □

Theorem A.3.2. *Let G be a profinite group and ℓ a prime such that $H^i(G, M)$ is finite for every $i \geq 0$ and every finite discrete G-module M of ℓ-power order. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$ and T a finitely generated $\mathcal{O}$-module with a continuous $\mathcal{O}$-linear action of G (ℓ-adic topology on T). Then for every $i \geq 0$ the continuous cochain cohomology group $H^i(G, T)$ is a finitely generated $\mathcal{O}$-module.*

Proof. [299], Section 2: for a finitely generated $\mathbb{Z}_\ell$-module T with continuous action and finite $H^i(G, T/\ell T)$, the groups $H^i(G, T)$ are finitely generated $\mathbb{Z}_\ell$-modules; see also [225], Chapter II, Section 7. This gives the statement. Indeed, T is finitely generated over $\mathbb{Z}_\ell$ because $\mathcal{O}$ is; a finitely generated $\mathbb{Z}_\ell$-module on which $\mathcal{O}$ acts is a finitely generated $\mathcal{O}$-module; and Tate's hypothesis is the case $M = T/\ell T$ of ours. For $G = G_{F_v}$ with F_v a finite extension of $\mathbb{Q}_p$ (any p) the hypothesis holds by Theorem A.3.1(1). □

Theorem A.3.3. *Let F be a number field, ℓ any prime, S a finite set of places containing all archimedean places and all places above ℓ, $r \geq 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with continuous $G_{F,S}$-action. Put $H^i(M) = H^i(G_{F,S}, M)$ and*

$$\mathcal{H}^i(F_v, M) = \begin{cases} H^i(F_v, M), & v \text{ finite}, \\ \widehat{H}^i(\mathrm{Gal}(\mathbb{C}/F_v), M), & v \mid \infty, \end{cases} \qquad P^i(M) = \prod_{v \in S} \mathcal{H}^i(F_v, M).$$

Here the archimedean groups are Tate cohomology: at a complex place they are zero in every degree; at a real place with conjugation c_v,

$$\begin{aligned} \widehat{H}^0(F_v, M) &= M^{c_v}/(1 + c_v)M, \\ \widehat{H}^1(F_v, M) &= \ker(1 + c_v)/(c_v - 1)M, \\ \widehat{H}^2(F_v, M) &= \widehat{H}^0(F_v, M). \end{aligned}$$

For $i = 0, 1, 2$, let $\lambda^i_M : H^i(M) \to P^i(M)$ be restriction, followed in degree zero at infinity by the quotient to Tate cohomology, and put $\mathrm{Sha}^i_S(F, M) = \ker \lambda^i_M$ for $i = 1, 2$. Write the superscript $*$ for Pontryagin duality. Then:

1. *The global groups $H^i(M)$ and the finite-place local groups $H^i(F_v, M)$ are finite for every $i \geq 0$, and all groups $P^i(M)$ for $i = 0, 1, 2$ are finite.*

2. *The canonical Poitou–Tate pairing*

$$\mathrm{Sha}^1_S(F, M) \times \mathrm{Sha}^2_S(F, M^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

   is perfect.

3. *There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(M) \xrightarrow{\lambda^0_M} P^0(M) \xrightarrow{\beta^0} H^2(M^\vee)^* \longrightarrow H^1(M) \xrightarrow{\lambda^1_M} P^1(M) \\ \xrightarrow{\beta^1} H^1(M^\vee)^* \longrightarrow H^2(M) \xrightarrow{\lambda^2_M} P^2(M) \xrightarrow{\beta^2} H^0(M^\vee)^* \longrightarrow 0. \end{aligned}$$

   The maps are the canonical Poitou–Tate maps, and

$$\beta^i((x_v)_v)(y) = \sum_{v \in S} \mathrm{inv}_v\big(x_v \cup \mathrm{res}_v\, y\big), \qquad y \in H^{2-i}(M^\vee).$$

   At infinity the cup product is the Tate cup product and restriction is interpreted in Tate cohomology. The invariant at a real place is the usual inclusion of the Brauer group of $\mathbb{R}$ into $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$; the complex summand is zero.

Proof. This is [201], Chapter I, Theorem 4.10, with the archimedean Tate-cohomology convention specified immediately before that theorem. Its part (a) gives the perfect pairing in item 2, and part (b), together with the local-duality identifications of the local products, gives the sequence and maps in item 3. Finiteness for finite S is Chapter I, Corollary 4.15 (printed p. 62); part (c) identifies the groups in degrees at least three with the finite product of the real-place cohomology groups. Finite-place local finiteness is Chapter I, Corollary 2.3. Milne's K, G_S, M^D are respectively $F, G_{F,S}, M^\vee$ here. The order of M is invertible outside S, because S contains the places above ℓ. No exclusion of real places or of the prime 2 is made. The local duality maps identify β^i with the dual of $\lambda^{2-i}_{M^\vee}$, with the displayed cup-product normalization.

If ℓ is odd, every real-place Tate group is killed by 2, by finite-group Tate cohomology, and by ℓ^r, because its cochain groups are ℓ^r-torsion. Bezout's identity therefore makes it zero in every degree. Complex-place Tate cohomology is zero because its Galois group is trivial. If F has no real place, all archimedean places are complex, so this vanishing holds for every prime. Under either condition, the local products in degrees $0, 1, 2$ reduce canonically to the products over the finite places $S_f = \{v \in S : v \nmid \infty\}$. Deleting the zero archimedean summands preserves the restriction maps, the cup-product duality maps and the connecting Poitou–Tate maps. In particular, the positive-degree localization kernels and their canonical pairing are computed by the finite-place products. Global finiteness and the Euler characteristic formula retain their stated meanings, with ordinary invariants in the archimedean numerator. The terminal exact sequence reduces by the same deletion of zero real Tate summands, and its final term is $H^0(G_{F,S}, M^\vee)^*$. These specializations impose no additional hypothesis on the all-prime statements.

□

Theorem A.3.4. *Let F be a number field, M a finite discrete continuous G_F-module, and $M^* = \mathrm{Hom}(M, \mu_n)$ for an integer n annihilating M. For every place v choose a subgroup $L_v \subseteq H^1(F_v, M)$, equal to $H^1(G_{F_v}/I_v, M^{I_v})$ at all but finitely many finite places. Let $L_v^\perp$ be its annihilator under local Tate duality, using Tate cohomology at infinity. Let $H^1_L(F, M)$ and $H^1_{L^\perp}(F, M^*)$ be the inverse images of these local conditions under global restriction. Both groups are finite and*

$$\frac{\#H^1_L(F, M)}{\#H^1_{L^\perp}(F, M^*)} = \frac{\#H^0(F, M)}{\#H^0(F, M^*)} \prod_v \frac{\#L_v}{\#H^0(F_v, M)}.$$

Here H^0 means ordinary invariants at every place; all but finitely many product factors are one.

Proof. This is the Greenberg–Wiles theorem in [63], Theorem 2.19. □

Theorem A.3.5. *Let F be a number field, ℓ any prime, S a finite set of places containing all places above ℓ and all archimedean places, $r \geq 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with continuous $G_{F,S}$-action. Local actions are obtained through decomposition groups. Then:*

1. *$H^i(G_{F,S}, M)$ is finite for every $i \geq 0$.*
2. *Tate's global Euler characteristic formula is*

$$\frac{\#H^0(G_{F,S}, M)\,\#H^2(G_{F,S}, M)}{\#H^1(G_{F,S}, M)} = \prod_{v|\infty} \frac{\#H^0(F_v, M)}{|\#M|_v}.$$

The factors $H^0(F_v, M)$ in this formula are ordinary invariants, including at real and complex places; the archimedean absolute value is the usual absolute value at a real place and its square at a complex place.

3. *With the superscript $*$ denoting Pontryagin duality, the sequence*

$$H^2(G_{F,S},M)\longrightarrow\left(\bigoplus_{\substack{v\in S\\ v\nmid\infty}}H^2(F_v,M)\right)\oplus\left(\bigoplus_{\substack{v\in S\\ v\text{ real}}}\widehat{H}^2(F_v,M)\right)\longrightarrow H^0(G_{F,S},M^\vee)^*\longrightarrow 0$$

is exact. Here $\widehat{H}^2(F_v,M)=M^{c_v}/(1+c_v)M$ at a real place. The first map is restriction, the complex-place Tate summands are zero, and the second map sends $(x_v)_v$ to the functional

$$y\longmapsto\sum_{v\in S}\mathrm{inv}_v(x_v\cup\mathrm{res}_v\,y).$$

The real-place summands use Tate cup products and the real Brauer invariant; finite-place summands use ordinary local cohomology.

Proof. Items 1 and 3 follow from [201], Chapter I, Theorem 4.10 and Corollary 4.15 (printed p. 62). In particular, item 3 is its final three terms, also displayed in Chapter I, Remark 5.2(c), with the archimedean cohomology understood in the Tate sense. Item 2 is Chapter I, Theorem 5.1: the Euler characteristic there is precisely the truncated quotient of the orders in degrees zero, one and two displayed above, also when real places and the prime 2 occur. Its archimedean numerator uses ordinary invariants, not Tate degree-zero groups. The hypotheses match because S is finite, nonempty, contains infinity, and the order of M is invertible outside S. Milne's K, G_S, M^D are $F, G_{F,S}, M^\vee$.

If ℓ is odd, every real-place Tate group is killed by 2, by finite-group Tate cohomology, and by ℓ^r, because its cochain groups are ℓ^r-torsion. Bezout's identity therefore makes it zero in every degree. Complex-place Tate cohomology is zero because its Galois group is trivial. If F has no real place, all archimedean places are complex, so this vanishing holds for every prime. Under either condition, the local products in degrees $0,1,2$ reduce canonically to the products over the finite places $S_f=\{v\in S : v\nmid\infty\}$. Deleting the zero archimedean summands preserves the restriction maps, the cup-product duality maps and the connecting Poitou–Tate maps. In particular, the positive-degree localization kernels and their canonical pairing are computed by the finite-place products. Global finiteness and the Euler characteristic formula retain their stated meanings, with ordinary invariants in the archimedean numerator. The terminal exact sequence reduces by the same deletion of zero real Tate summands, and its final term is $H^0(G_{F,S},M^\vee)^*$. These specializations impose no additional hypothesis on the all-prime statements.

□

Theorem A.3.6. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M, and V a de Rham representation of G_M on a finite-dimensional $\mathbb{Q}_\ell$-vector space. Let $D_{\mathrm{dR}}(V)=(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)^{G_M}$ with the filtration $\mathrm{Fil}^iD_{\mathrm{dR}}(V)=(\mathrm{Fil}^iB_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)^{G_M}$, and let $H^1_f(G_M,V)$ be the kernel of $H^1(G_M,V)\to H^1(G_M,B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V)$. Then*

$$\dim_{\mathbb{Q}_\ell}H^1_f(G_M,V)=\dim_{\mathbb{Q}_\ell}(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0D_{\mathrm{dR}}(V))+\dim_{\mathbb{Q}_\ell}H^0(G_M,V).$$

Proof. [28], Corollary 3.8.4 ("In particular,

$$\dim_{\mathbf{Q}_p}H^1_f(K,V)=\dim_{\mathbf{Q}_p}(DR(V)/DR(V)^0)+\dim_{\mathbf{Q}_p}H^0(K,V)",$$

for V a de Rham representation of $\mathrm{Gal}(\overline{K}/K)$, K finite over $\mathbf{Q}_p$). *Notation.* Their K, p, $DR(V)$, $DR(V)^0$ are M, ℓ, $D_{\mathrm{dR}}(V)$, $\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$; their $H^1_f(K,V)$ is the kernel of $H^1(K,V) \to H^1(K, B_{\mathrm{crys}} \otimes V)$, which is the local condition of Definition 7.1.16 at a place above ℓ (a class lies in the kernel exactly when a representing cocycle c satisfies $c(g) \otimes 1 = gy - y$ for some $y \in V \otimes B_{\mathrm{cris}}$). □

Theorem A.3.7. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M, and V a de Rham representation of G_M on a finite-dimensional $\mathbb{Q}_\ell$-vector space; put $V^*(1) = \mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell) \otimes \chi_{\mathrm{cyc}}$. Let*

$$\langle\ ,\ \rangle \colon H^1(G_M, V) \times H^1(G_M, V^*(1)) \longrightarrow H^2(G_M, \mathbb{Q}_\ell(1))$$

be the cup product followed by the evaluation map $V \otimes_{\mathbb{Q}_\ell} V^*(1) \to \mathbb{Q}_\ell(1)$. Then $H^1_f(G_M, V)$ and $H^1_f(G_M, V^*(1))$ are exact annihilators of each other: a class $c \in H^1(G_M, V)$ lies in $H^1_f(G_M, V)$ if and only if $\langle c, y\rangle = 0$ for every $y \in H^1_f(G_M, V^*(1))$, and symmetrically.

Proof. [28], Proposition 3.8 (case $l = p$, under the hypothesis made there that V is de Rham). *Notation.* Their K, p, V, V^* are M, ℓ, V, $\mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$; their $H^1_f(K,V)$ is the kernel of $H^1(K,V) \to H^1(K, B_{\mathrm{crys}} \otimes V)$, ours; the pairing is the cup product into $H^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$. □

Theorem A.3.8. *Let ℓ be a prime and F_v a finite extension of $\mathbb{Q}_\ell$, with absolute Galois group G_{F_v} and inertia subgroup I_{F_v}. Then $H^1_f(G_{F_v}, \mathbb{Q}_\ell) \subseteq H^1(G_{F_v}, \mathbb{Q}_\ell) = \mathrm{Hom}_{\mathrm{cont}}(G_{F_v}, \mathbb{Q}_\ell)$ consists exactly of the homomorphisms that vanish on I_{F_v}.*

Proof. [28], Example 3.9 (case $r = 0$: "$H^1_f(K, \mathbb{Q}_p) \subset \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}_p)$ are the unramified homomorphisms"). *Notation.* Their K, p are F_v, ℓ; their H^1_f is the kernel of $H^1(K,V) \to H^1(K, B_{\mathrm{crys}} \otimes V)$, which is the local condition of Definition 7.1.16 at a place above ℓ. □

Theorem A.3.9. *Let p be a prime and K a finite extension of $\mathbb{Q}_p$. If V and V' are crystalline finite-dimensional continuous $\mathbb{Q}_p$-linear representations of G_K, then every G_K-stable subspace of V, every quotient of V by a G_K-stable subspace, $V \oplus V'$, $V \otimes_{\mathbb{Q}_p} V'$ and the dual V^* are crystalline.*

Proof. [85]; [44], Theorem 5.2.1 (the category of B-admissible representations of an (F,G)-regular ring B is stable under subobjects, quotients, direct sums, tensor products and duals) with $B = B_{\mathrm{cris}}$, which is $(\mathbb{Q}_p, G_K)$-regular ([44], §9.1). *Notation.* Here a B_{cris}-admissible representation V of G_K is a crystalline one, i.e. $\dim_{K_0}(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p} V$, with K_0 the maximal unramified subfield of K. □

Theorem A.3.10. *Let p be a prime, $K/\mathbb{Q}_p$ finite, $P_0 = \mathrm{Frac}\, W(\overline{k}_K)$, and U a potentially semistable finite-dimensional $\mathbb{Q}_p$-representation of G_K. Its completed Deligne module*

$$A = \bigcup_{H \subseteq I_K \text{ open}} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} U)^H$$

has dimension $\dim_{\mathbb{Q}_p} U$ over P_0, finite inertia action and nilpotent monodromy. The representation U is semistable exactly when I_K acts trivially on A, and crystalline exactly when that action is trivial and $N = 0$. The Weil–Deligne representation is obtained from this module by Fontaine's Weil functor; its inertia action and monodromy are those of A.

Proof. Fontaine [86], Theorem 2.3.2, Remark 2.3.3(a),(c), and construction 2.3.7, with $\ell = p$. Good reduction there means crystalline when $\ell = p$ (Section 2.1). □

Theorem A.3.11. *Let $K_\ell/\mathbb{Q}_\ell$ be finite and let V be a finite-dimensional continuous de Rham representation of G_{K_ℓ}. Then V is Hodge–Tate and the natural graded comparison identifies*

$$\operatorname{gr} D_{\mathrm{dR}}(V) = D_{\mathrm{HT}}(V), \qquad D^j_{\mathrm{HT}}(V) = (\mathbb{C}_\ell(j) \otimes_{\mathbb{Q}_\ell} V)^{G_{K_\ell}}.$$

Equivalently, there is a G_{K_ℓ}-equivariant decomposition

$$\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} V \simeq \bigoplus_j \mathbb{C}_\ell(-j) \otimes_{K_\ell} \operatorname{gr}^j D_{\mathrm{dR}}(V).$$

In particular $\mathbb{Q}_\ell(1)$ has its de Rham filtration jump in degree -1, although its Hodge–Tate weight under the cyclotomic-positive convention is $+1$.

Proof. This is the de Rham to Hodge–Tate comparison theorem; see [44], Proposition 6.3.2 and Example 6.3.1 (p. 76). □

Theorem A.3.12. *Let ℓ be a prime, K a finite extension of $\mathbb{Q}_\ell$ with residue field k, $W = W(k)$, $K_0 = W[1/\ell]$, π a uniformiser of K and $E(u) \in W[u]$ its Eisenstein polynomial. Let $\mathfrak{S} = W[[u]]$, with the Frobenius φ that is the Frobenius of W on coefficients and sends u to u^ℓ. Fix $\pi^\flat \in \mathcal{O}^\flat_{\mathbb{C}_\ell}$ corresponding to a compatible system of ℓ-power roots of π, put $K_\infty = \bigcup_n K(\pi^{1/\ell^n})$, fix a compatible system $\epsilon \in \mathcal{O}^\flat_{\mathbb{C}_\ell}$ of primitive ℓ-power roots of unity and put $\mu = [\epsilon] - 1 \in A_{\mathrm{inf}}$; let $i_{\mathrm{B}} : \mathfrak{S} \to A_{\mathrm{inf}}$ be the ring map whose restriction to W is Witt Frobenius and with $u \mapsto [\pi^\flat]^\ell$; this map commutes with Frobenius. A* Breuil–Kisin module *is a finitely generated $\mathfrak{S}$-module $\mathfrak{M}$ with an isomorphism $\varphi_{\mathfrak{M}} \colon (\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathfrak{S})[1/E] \to \mathfrak{M}[1/E]$. Then there is a fully faithful tensor functor $T \mapsto \mathfrak{M}(T)$ from $\mathbb{Z}_\ell$-lattices T in crystalline representations V of G_K to finite free Breuil–Kisin modules such that:*

1. *$\mathfrak{M}(T)$ is characterized by a φ- and G_{K_∞}-equivariant identification*

$$\mathfrak{M}(T) \otimes_{\mathfrak{S},i_{\mathrm{B}}} W(\mathbb{C}^\flat_\ell) \cong T \otimes_{\mathbb{Z}_\ell} W(\mathbb{C}^\flat_\ell).$$

2. *there is a φ- and G_{K_∞}-equivariant identification $\mathfrak{M}(T) \otimes_{\mathfrak{S},i_{\mathrm{B}}} A_{\mathrm{inf}}[1/\mu] \cong T \otimes_{\mathbb{Z}_\ell} A_{\mathrm{inf}}[1/\mu]$;*
3. *$\mathfrak{M}(T) \otimes_{\mathfrak{S},i_{\mathrm{B}}} B^+_{\mathrm{cris}} = D_{\mathrm{cris}}(V) \otimes_{K_0} B^+_{\mathrm{cris}}$ as submodules of $\mathfrak{M}(T) \otimes_{\mathfrak{S},i_{\mathrm{B}}} B_{\mathrm{cris}} = T \otimes_{\mathbb{Z}_\ell} B_{\mathrm{cris}} = D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}}$.*

Proof. [25], Theorem 4.4, gives the fully faithful tensor functor and its characterization in item 1. The comparison of item 2 is Remark 4.5(i), established in the discussion before Proposition 4.34 using Lemma 4.26; item 3 is Remark 4.5(ii) and Proposition 4.34. The embedding is the one specified in Section 4.4 before Lemma 4.30: it acts by Witt Frobenius on $W(k)$ as well as sending u to $[\pi^\flat]^\ell$. The prime denoted p there is ℓ here. These assertions are Kisin's theorem in the stated Breuil–Kisin convention; the functor is not asserted to be exact. □

Theorem A.3.13. *Let $K/\mathbb{Q}_p$ be finite, $K_0 = W(k)[1/p]$, π a uniformiser, $P(u)$ its monic Eisenstein polynomial, and $\mathfrak{S} = W(k)[[u]]$, with $\varphi(u) = u^p$ and coefficient Witt Frobenius. Put*

$$\mathcal{O}^{\mathrm{an}} = \left\{ \sum_{n \geq 0} a_n u^n \in K_0[[u]] : v_p(a_n) + rn \longrightarrow +\infty \text{ for every } r > 0 \right\},$$
$$\mathscr{L} = \prod_{a \geq 0} \varphi^a(P(u)/P(0)).$$

Let T be a crystalline $\mathbb{Z}_p$-lattice with weights in $[0,h]$, using the convention that $\mathbb{Q}_p(1)$ has weight -1. Let $\mathfrak{M}$ be its Breuil–Kisin module for the direct embedding $u \mapsto [\pi^\flat]$, in the actual étale comparison with $W(\mathbb{C}_p^\flat)$, and put $\mathfrak{M}^\varphi = \operatorname{im}(\varphi^*\mathfrak{M} \to \mathfrak{M})$. There is a Frobenius-equivariant inclusion

$$D_{\mathrm{cris}}(T[1/p]) \otimes_{K_0} \mathcal{O}^{\mathrm{an}} \longrightarrow \mathfrak{M}^\varphi \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{an}},$$

which is an isomorphism after inverting $\varphi(\mathscr{L})$. Under the natural embedding $\mathcal{O}^{\mathrm{an}} \to B_{\max}^+$, the induced comparison is the actual crystalline comparison inside $T \otimes B_{\max}$. The element $\varphi(\mathscr{L})$ maps to a unit of $B_{\max}^+$.

Proof. [17], Section 2.1.6, equation (2.1.9), and Proposition 2.1.10(1), citing [160], Lemma 1.2.6. The direct embedding and the étale identification are fixed in Sections 2.1.1–2.1.2 and Lemma 2.1.8; its footnote distinguishes this embedding from the coefficient-Frobenius embedding of Theorem A.3.12. Here $P, \mathcal{O}^{\mathrm{an}}, \mathscr{L}, T, \mathfrak{M}^\varphi$ are respectively $E(u), \mathcal{O}^{\mathrm{rig}}, \lambda, V, M^\varphi$ of that reference. The stated weight convention is its footnote 4. No assertion about the chart bicommutant or arbitrary signed weights is included. □

Theorem A.3.14. *Let C be a complete algebraically closed extension of $\mathbb{Q}_p$, put $A = A_{\mathrm{inf}}(\mathcal{O}_C)$, and let B_{dR}^+ be the completion at $\ker\theta$, with fraction field B_{dR}. A finite free Breuil–Kisin–Fargues module is a finite free A-module M equipped with an isomorphism*

$$\Phi_M : (A \otimes_{\varphi,A} M)[1/\widetilde{\xi}] \longrightarrow M[1/\widetilde{\xi}].$$

For such an object, put

$$T_M = (M \otimes_A W(C^\flat))^{\Phi_M \otimes \varphi = 1}.$$

There are natural identifications

$$M \otimes_A W(C^\flat) = T_M \otimes_{\mathbb{Z}_p} W(C^\flat), \qquad M[1/\mu] = T_M \otimes_{\mathbb{Z}_p} A[1/\mu].$$

In the resulting common de Rham space put

$$\Xi_M = M \otimes_A B_{\mathrm{dR}}^+ \subset T_M \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

The functor $M \mapsto (T_M, \Xi_M)$ is an equivalence with the category of pairs consisting of a finite free $\mathbb{Z}_p$-module T and a B_{dR}^+-lattice Ξ in $T \otimes B_{\mathrm{dR}}$. The morphisms of pairs are exactly the $\mathbb{Z}_p$-linear maps $f : T \to T'$ whose de Rham extension carries Ξ into Ξ'. In particular every such map extends uniquely to a morphism of the corresponding Breuil–Kisin–Fargues modules.

Proof. [25], Theorem 4.28, is Fargues' classification in this convention. The displayed étale identifications are Lemma 4.26; the precise full-faithfulness assertion is also explained in Remark 4.29. The de Rham completion here is at θ, and the Frobenius is inverted at $\widetilde{\xi}$, as in Definition 4.22. □

Theorem A.3.15. *Let ℓ be a prime, $K/\mathbb{Q}_\ell$ a finite extension in $\overline{\mathbb{Q}}_\ell$ and V a finite-dimensional $\mathbb{Q}_\ell$-vector space with a continuous linear action of G_K. If V is de Rham, that is,*

$$\dim_K (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V)^{G_K} = \dim_{\mathbb{Q}_\ell} V,$$

then V is potentially semistable: the union over all finite extensions K'/K in $\overline{\mathbb{Q}}_\ell$ of the invariants $(V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{st}})^{G_{K'}}$ is a $\mathbb{Q}_\ell^{\mathrm{ur}}$-vector space of dimension $\dim_{\mathbb{Q}_\ell} V$.

Proof. This is Berger's theorem [22], in the notation of [85]. □

Theorem A.3.16. *Let p be a prime, $K/\mathbb{Q}_p$ finite, K_0 its maximal unramified subfield, and V a continuous $\mathbb{Q}_p$-representation of dimension d. Then*

$$B_{\mathrm{cris}}^{G_K} = K_0, \qquad B_{\mathrm{dR}}^{G_K} = K.$$

For either $(B, E) = (B_{\mathrm{cris}}, K_0)$ or (B_{dR}, K), put $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$. The natural B-linear G_K-equivariant map

$$B \otimes_E D_B(V) \longrightarrow B \otimes_{\mathbb{Q}_p} V$$

is injective, and

$$\dim_E D_B(V) \leq d.$$

Equality holds if and only if this map is an isomorphism. In the crystalline case equality is exactly crystallinity over K, and in the de Rham case it is exactly the de Rham property over K.

Proof. These are Brinon–Conrad [44], Theorem 4.4.13 (printed page 64), the invariant-field calculation preceding Theorem 9.1.5 and Proposition 9.1.6 (page 131), and Theorem 5.2.1(1) (page 68), applied to the regular period rings. The crystalline comparison is also displayed in Proposition 9.1.9 and equation (9.1.4) (pages 133–134). □

Theorem A.3.17. *Let p be a prime, $F \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$, $F_\infty = F(\mu_{p^\infty})$, $\widehat{F}_\infty$ the closure of F_∞ in $\mathbb{C}_p$, $\mathfrak{m}_{F_\infty}$ the maximal ideal of the valuation ring of F_∞, and $H_F = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\infty) \subseteq G_F$, acting on $\mathbb{C}_p$ by continuity.*

1. *(Ax–Sen–Tate) For every closed subgroup H of $G_{\mathbb{Q}_p}$, $\{c \in \mathbb{C}_p : hc = c \text{ for all } h \in H\}$ is the closure in $\mathbb{C}_p$ of the fixed field of H in $\overline{\mathbb{Q}}_p$. In particular the H_F-invariants of $\mathbb{C}_p$ form $\widehat{F}_\infty$, and the G_F-invariants form F.*
2. *(Tate) If $j \neq 0$ is an integer and $c \in \mathbb{C}_p$ satisfies $\chi_{\mathrm{cyc}}(g)^j\, gc = c$ for all $g \in G_F$, then $c = 0$.*
3. *(Tate) For every finite extension $L \subset \overline{\mathbb{Q}}_p$ of F_∞ with valuation ring $\mathcal{O}_L$, $\mathrm{Tr}_{L/F_\infty}(\mathcal{O}_L) \supseteq \mathfrak{m}_{F_\infty}$.*

Proof. Tate [298], Section 3: item 3 is the statement that the cyclotomic tower is deeply ramified (the different of a finite extension of the layers tends to the unit ideal), proved for a totally ramified $\mathbb{Z}_p$-extension; F_∞ is such an extension of $F(\mu_{p^n})$ for n large, and a finite extension of F_∞ is a finite extension of that $\mathbb{Z}_p$-extension. Item 2 is $H^0(G_F, \mathbb{C}_p(j)) = 0$ for $j \neq 0$, in the notation $\mathbb{C}_p(j) = \mathbb{C}_p \otimes \mathbb{Z}_p(j)$, where an invariant $c \otimes \epsilon^{\otimes j}$ is exactly a solution of $\chi_{\mathrm{cyc}}(g)^j gc = c$. Item 1 for closed subgroups of $G_{\mathbb{Q}_p}$ in general is due to Ax [10], Sen and Tate; see also [44], Section 2 and the chapter on Sen theory, for all three items. □

Theorem A.3.18. *Let R be a commutative $\mathbb{Q}$-algebra and let $F(X, Y)$ be a commutative formal group law of finite dimension $d \geq 1$ over R. There is a unique strict isomorphism from F to the d-dimensional additive formal group law. Explicitly, this is an invertible formal map $\ell(X) \in R[[X_1, \ldots, X_d]]^d$ with zero constant term, linear term X, and*

$$\ell(F(X, Y)) = \ell(X) + \ell(Y).$$

Proof. This is the Q-theorem of [124], Chapter II, Corollary (11.1.6), p. 64. The coefficient algebra A and dimension m there are R and d here; the additive law $\widehat{\mathbb{G}}_a^m$ is coordinatewise addition. A strict isomorphism is an isomorphism of formal group laws with identity linear term. $\square$

Theorem A.3.19. *Let ℓ be a prime and $\mathcal{O}$ a coefficient ring with residue field of characteristic ℓ. Let $\mathrm{CNL}_{\mathcal{O}}$ be the category of complete Noetherian local $\mathcal{O}$-algebras A for which $\mathcal{O} \to A$ is local and induces an isomorphism $\mathbb{F}_{\mathcal{O}} \cong A/\mathfrak{m}_A$, with the local $\mathcal{O}$-algebra homomorphisms as morphisms. A profinite group Γ satisfies* Mazur's condition Φ_ℓ *if $\mathrm{Hom}_{\mathrm{cont}}(\Gamma', \mathbb{F}_\ell)$ is finite for every open subgroup $\Gamma' \subseteq \Gamma$.*

1. *If Γ satisfies Φ_ℓ and $\bar{\rho}\colon \Gamma \to \mathrm{GL}_n(\mathbb{F}_{\mathcal{O}})$ is continuous, the functor on $\mathrm{CNL}_{\mathcal{O}}$ sending A to the set of continuous homomorphisms $\rho\colon \Gamma \to \mathrm{GL}_n(A)$ reducing to $\bar{\rho}$ is represented by an object $R^{\square}$ of $\mathrm{CNL}_{\mathcal{O}}$ and a universal lift $\rho^{\square}\colon \Gamma \to \mathrm{GL}_n(R^{\square})$.*
2. *For a number field F and a finite set S of places of F, $G_{F,S}$ satisfies Φ_ℓ.*

Proof. [196, §1.1–1.2]: item 2 is Mazur's observation that $\mathrm{Hom}(\Gamma', \mathbb{F}_\ell)$ for an open $\Gamma' \subseteq G_{F,S}$ classifies abelian extensions of exponent ℓ of a number field unramified outside S, finite in number by class field theory (Hermite–Minkowski); item 1 is the framed form of his existence theorem (Schlessinger's criteria hold for the functor of lifts without any irreducibility hypothesis, and Φ_ℓ makes its tangent space $Z^1(\Gamma, \mathrm{ad}\,\bar{\rho})$ finite-dimensional, so the pro-representing ring is Noetherian). Notation: $R^{\square}$, $\rho^{\square}$ as there; $\mathrm{CNL}_{\mathcal{O}}$ is Mazur's category of complete Noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}_{\mathcal{O}}$. $\square$

A.4 Automorphic forms and Galois representations

The quotations fall into four groups. The first gives the foundations of automorphic forms on unitary groups: moderate growth and the finiteness of class numbers, reduction theory, the compactness criteria and the finiteness of the volume, and strong approximation [33, 38, 208, 37, 216, 167, 233]. The second gives the published inputs of the Galois representations of Section 7.1: the local Langlands correspondence [122, 130, 254, 318], the Galois representations of polarised regular algebraic cuspidal representations and their local–global compatibility [15, 16, 276, 56, 47, 48], the discrete spectrum of GL_n and the Whittaker expansion [207, 149, 269, 230], Tadić's classification of the unitary dual [292], and base change [177, 185]. The third consists of local results on representations of GL_n over a p-adic field: depth-zero cuspidal types [74], Shotton's results on types of compact type [278, 106], Zelevinsky's classification [332, 24], spherical Hecke algebras [248, 50, 109], and the factorisation of representations of a product [23, 80]. The fourth is the analytic theory of theta and Eisenstein series: Siegel's formula and the Siegel–Weil formula in Weil's convergent range [280, 323, 139, 121] (for the regularised formula see [176]), and the continuation of Siegel Eisenstein series [184]. From these the body proves the form of the Rallis inner product formula that it uses (Theorem 6.5.13), by the doubling method [237, 238, 231] in the form of Li [187], and the discreteness of the cuspidal spectrum (Theorem 7.6.14; see [100, 7, 72]).

Theorem A.4.1. *All symbols are local to this statement and to the results that cite it. Let G be a connected reductive group over a number field F and $G(F_\infty) = G(F \otimes_{\mathbb{Q}} \mathbb{R})$. For a faithful representation $\rho\colon G \to \mathrm{GL}_m$ defined over F put*

$$\|g\|_\rho = \prod_v \max_{a,b}(1, |\rho(g)_{ab}|_v, |\rho(g^{-1})_{ab}|_v) \qquad (g \in G(\mathbb{A}_F)),$$

the product over all places v of F with the normalised absolute values (finitely many factors are $\neq 1$), and let $\|g\|_{\rho,\infty}$ be the product over the archimedean places, for $g \in G(F_\infty)$.

(N) *For faithful representations ρ, ρ' of G over F there are $c, a > 0$ with $\|g\|_{\rho'} \leq c\,\|g\|_\rho^a$ on $G(\mathbb{A}_F)$. For norms $\|g\|_1, \|g\|_2$ on $G(F_\infty)$ of the form $\max(1, \|\pi(g)\|, \|\pi(g^{-1})\|)$, π a faithful algebraic representation of the real algebraic group $G(F_\infty)$ and $\|\cdot\|$ a norm on matrices (in particular $\|\cdot\|_{\rho,\infty}$, and $\sum_{a,b}(|\pi(g)_{ab}| + |\pi(g^{-1})_{ab}|)$), there are $c, a > 0$ with $\|g\|_2 \leq c\,\|g\|_1^a$.*

(K) *For every open compact subgroup K_f of $G(\mathbb{A}_{F,f})$ there are finitely many $x_1, \ldots, x_h \in G(\mathbb{A}_{F,f})$ with $G(\mathbb{A}_F) = \bigcup_{i=1}^h G(F)\, x_i\, G(F_\infty) K_f$.*

(M) *Let K_f and $x_1, \ldots, x_h$ be as in (K), and $\varphi\colon G(\mathbb{A}_F) \to \mathbb{C}$ continuous, left $G(F)$-invariant and right K_f-invariant. If for every i there are $C_i, N_i \geq 0$ with $|\varphi(x_i g)| \leq C_i \|g\|_{\rho,\infty}^{N_i}$ for all $g \in G(F_\infty)$, then there are $C, N \geq 0$ with $|\varphi(g)| \leq C\,\|g\|_\rho^N$ for all $g \in G(\mathbb{A}_F)$.*

Proof. (N) is the equivalence of norms on adelic and real reductive groups, [38] and [208], I.2.2. (K) is the finiteness of class numbers, [33]. (M) is the passage from the classical moderate growth of the functions $g \mapsto \varphi(x_i g)$ on $G(F_\infty)$ to the adelic moderate growth of φ, by reduction theory (Siegel sets), in the dictionary between classical and adelic automorphic forms of [38] and [208], I.2.

□

Theorem A.4.2. *All symbols are local to this statement and to the results that cite it. Let E/F be a CM extension of a totally real number field, V a nondegenerate hermitian space over E, and $G = \mathrm{U}(V)$, a connected reductive group over F. Let $P_0 = M_0 N_0$ be a minimal parabolic subgroup of G defined over F with a Levi decomposition over F, A_0 the maximal F-split torus in the centre of M_0, Δ the set of simple roots of A_0 in N_0, and $K = \prod_v K_v$ a maximal compact subgroup of $G(\mathbb{A}_F)$ with $G(\mathbb{A}_F) = P_0(\mathbb{A}_F)K$. Let A^+ be the identity component of the group of real points of the maximal $\mathbb{Q}$-split subtorus of $\mathrm{Res}_{F/\mathbb{Q}} A_0$, embedded in $A_0(\mathbb{A}_F)$ at the archimedean places. For a compact set $\omega \subseteq P_0(\mathbb{A}_F)$ and $t > 0$ the* Siegel set *is*

$$\mathfrak{S}(\omega, t) = \{p\,a\,k :\ p \in \omega,\ a \in A^+,\ |\alpha(a)| \geq t \text{ for all } \alpha \in \Delta,\ k \in K\}.$$

Then there are a compact ω and $t > 0$ with $G(\mathbb{A}_F) = G(F)\,\mathfrak{S}(\omega, t)$.

Proof. Quoted: Borel [34] (reduction theory for arithmetic groups, Siegel sets) and Godement [103], in the adelic form recalled by Mœglin and Waldspurger [208], Chapter I.2. Notation: G here is $\mathrm{U}(V)$ of the hermitian spaces of the project, $G(\mathbb{A}_F) = G_V$, $G(F) = \Gamma_V$. □

Theorem A.4.3. *Let U be a unipotent linear algebraic group over a number field F. Then $U(F)$ is a discrete subgroup of $U(\mathbb{A}_F)$, the quotient $U(F)\backslash U(\mathbb{A}_F)$ is compact, and $U(\mathbb{A}_F)$ is unimodular.*

Proof. **Quotation.** Discreteness and compactness: [37] (the unipotent case of the compactness criterion; proof by induction along a central series of U over F with quotients $\mathbb{G}_a$, using that F is discrete and cocompact in $\mathbb{A}_F$). Unimodularity (proof): at every place v the modular function of $U(F_v)$ is $g \mapsto |\det \mathrm{Ad}(g)|_v$ on the Lie algebra (true for real, complex and p-adic Lie groups), and $\mathrm{Ad}(g)$ is unipotent for unipotent g, so $U(F_v)$ is unimodular; $U(\mathbb{A}_F)$ is the restricted product of the $U(F_v)$ with respect to compact open subgroups, and its Haar measure is the restricted product of local Haar measures, so its modular function is the product of the local ones, hence trivial. □

Theorem A.4.4. *Let F be a number field and G a connected reductive group over F that has no nontrivial character defined over F. Then $G(F)$ is a discrete subgroup of $G(\mathbb{A}_F)$, and $G(F)\backslash G(\mathbb{A}_F)$ is compact if and only if G contains no nontrivial F-split torus.*

Proof. [37] and [216]; in this form over a number field see [233], Chapter 5. Notation: a group without nontrivial F-split torus is called F-anisotropic there; for a torus this is the absence of nontrivial F-rational characters. □

Theorem A.4.5. *Let V be a hermitian space over E whose form is definite at some complex embedding τ of E (that is, $\tau(h(x,x)) \neq 0$ for every $0 \neq x \in V$). Then Γ_V is a discrete subgroup of G_V and $\Gamma_V\backslash G_V$ is compact. Consequently, for every Haar measure μ of G_V, every measurable fundamental domain $\mathcal{F}$ of Γ_V in G_V has $\mu(\mathcal{F}) < \infty$, and every continuous left Γ_V-invariant function on G_V is bounded.*

Proof. **Quotation.** For a connected reductive group G over a number field F without nontrivial F-rational characters, $G(F)$ is a discrete subgroup of $G(\mathbb{A}_F)$, and $G(F)\backslash G(\mathbb{A}_F)$ is compact if and only if G is F-anisotropic, i.e. contains no nontrivial F-split torus [37] (and Mostow–Tamagawa [216]); over F this is the statement over $\mathbb{Q}$ applied to the restriction of scalars of G to $\mathbb{Q}$.

Bridge: $G = \mathrm{U}(V)$ satisfies the hypotheses. It is connected reductive (a form of GL_n, $n = \dim_E V$). Over $\bar{F}$, $\mathrm{U}(V) \cong \mathrm{GL}_n$ and its characters are the powers $\det^k$; the Galois descent datum of $\mathrm{U}(V)$ is $g \mapsto (g^*)^{-1}$ followed by the nontrivial automorphism of E/F, which sends $\det^k$ to $\det^{-k}$, so the only F-rational character is trivial.

G contains no nontrivial F-split torus. Let $\lambda\colon \mathbb{G}_m \to \mathrm{U}(V)$ be a cocharacter defined over F. The F-vector space V is a representation of the split torus $\mathbb{G}_m$, so $V = \bigoplus_{a\in\mathbb{Z}} V_a$ with $\lambda(t)x = t^a x$ for $x \in V_a$, $t \in F^\times$; each V_a is an E-subspace, since $\lambda(t)$ is E-linear. For $x \in V_a$, $y \in V_b$ and $t \in F^\times$, $h(x,y) = h(\lambda(t)x, \lambda(t)y) = t^a c(t)^b h(x,y) = t^{a+b}h(x,y)$, and $F^\times$ is infinite, so $h(x,y) = 0$ unless $a + b = 0$. If λ were nontrivial, some V_a with $a \neq 0$ would contain $x \neq 0$, and then $h(x,x) = 0$ because $2a \neq 0$; but $\tau(h(x,x)) \neq 0$ by definiteness. Hence every F-cocharacter is trivial, and so is every F-split torus (it is generated by the images of its cocharacters).

So Γ_V is discrete and $\Gamma_V\backslash G_V$ compact. **Consequences.** G_V is a locally compact second countable group and Γ_V a discrete subgroup; a Haar measure μ induces a G_V-invariant Radon measure $\bar{\mu}$ on $\Gamma_V\backslash G_V$ with $\mu(\mathcal{F}) = \bar{\mu}(\Gamma_V\backslash G_V)$ for every measurable fundamental domain $\mathcal{F}$, and $\bar{\mu}$ of a compact space is finite. A continuous left Γ_V-invariant function factors through a continuous function on the compact quotient, hence is bounded. □

Theorem A.4.6. *Let F be a number field and H a connected reductive linear algebraic group over F. Suppose every algebraic group homomorphism $H \to \mathbb{G}_m$ defined over F is trivial. Then $H(F)$ is discrete in $H(\mathbb{A}_F)$, and for a Haar measure on $H(\mathbb{A}_F)$ the quotient measure induced by counting measure on $H(F)$ has finite volume.*

Proof. The adelic finite-volume theorem of Borel–Harish-Chandra, in its number-field formulation: [37]; see [233], Chapter 5. The hypothesis is the absence of nontrivial F-rational algebraic characters. The group denoted here by H is the reductive group of that criterion, and the quotient is its full adelic quotient, with counting measure on the rational subgroup. □

Theorem A.4.7. *Let F be a number field with ring of adeles $\mathbb{A}_F$, let U be a unipotent linear algebraic group over F, and let S be a nonempty finite set of places of F. Put $F_S = \prod_{v\in S} F_v$ and let $\mathbb{A}_F^S$ be the restricted product of the completions F_v, $v \notin S$, with respect to their rings of integers, so that $\mathbb{A}_F = F_S \times \mathbb{A}_F^S$ and $U(\mathbb{A}_F) = U(F_S) \times U(\mathbb{A}_F^S)$. Then the diagonal image of $U(F)$ is dense in $U(\mathbb{A}_F^S)$; equivalently, $U(F)\,U(F_S)$ is dense in $U(\mathbb{A}_F)$.*

In particular, if S is the set of archimedean places, $F_\infty = F_S$ and $\mathbb{A}_{F,f} = \mathbb{A}_F^S$, then for every open subgroup $K \le U(\mathbb{A}_{F,f})$

$$U(\mathbb{A}_{F,f}) = U(F)\,K \qquad \text{and} \qquad U(\mathbb{A}_F) = U(F)\,U(F_\infty)\,K.$$

Proof. Strong approximation for unipotent groups: V. Platonov and A. Rapinchuk, Algebraic groups and number theory [233], Chapter 7 (strong approximation), the unipotent case. Over a field of characteristic zero a unipotent group is split, i.e. it has a central series of F-subgroups with successive quotients isomorphic to the additive group, and the statement is the strong approximation theorem for the additive group (F is dense in $\mathbb{A}_F^S$ for $S \neq \emptyset$) carried along that series.

□

Theorem A.4.8. *Let K be a number field, $\mathbb{A}_K$ its ring of adeles, and H a connected, simply connected, absolutely almost simple linear algebraic group over K. Let S be a finite set of places of K, and let $\mathbb{A}_K^S$ be the ring of adeles of K with the components at the places of S omitted, so that $H(K)$ embeds diagonally in $H(\mathbb{A}_K^S)$. If the group $H_S = \prod_{v\in S} H(K_v)$ is not compact, then $H(K)$ is dense in $H(\mathbb{A}_K^S)$.*

Proof. Kneser [167]; Platonov [232]; stated in this form (for global fields, with the converse) as [239, Theorem 2.3]; see also [233], §7.4. Notation: Rapinchuk's G, $\mathbb{A}_S$, G_S are our H, $\mathbb{A}_K^S$, H_S; only the sufficiency half is quoted. □

Theorem A.4.9. *Let K be a finite extension of $\mathbb{Q}_p$, W_K its Weil group, $|\cdot|$ the normalised absolute value of K, and $\mathrm{Art}_K\colon K^\times \to W_K^{\mathrm{ab}}$ the local Artin map, normalised so that uniformisers go to geometric Frobenius elements. There is a family of bijections rec_K, one for each $n \ge 1$, from the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ to the set of isomorphism classes of n-dimensional Frobenius-semisimple Weil–Deligne representations of W_K over $\mathbb{C}$, such that:*

(Ln1) $\mathrm{rec}_K(\chi) = \chi \circ \mathrm{Art}_K^{-1}$ *for every smooth character χ of $K^\times = \mathrm{GL}_1(K)$;*

(Ln2) $\mathrm{rec}_K(\pi \otimes (\chi \circ \det)) = \mathrm{rec}_K(\pi) \otimes \mathrm{rec}_K(\chi)$ *for every π and every smooth character χ of $K^\times$;*

(Ln3) $\det \mathrm{rec}_K(\pi) = \mathrm{rec}_K(\omega_\pi)$, *where ω_π is the central character of π, and $\mathrm{rec}_K(\pi^\vee) = \mathrm{rec}_K(\pi)^\vee$;*

(Ln4) $L(s, \pi_1 \times \pi_2) = L(s, \mathrm{rec}_K(\pi_1) \otimes \mathrm{rec}_K(\pi_2))$ *and $\varepsilon(s, \pi_1 \times \pi_2, \psi) = \varepsilon(s, \mathrm{rec}_K(\pi_1) \otimes \mathrm{rec}_K(\pi_2), \psi)$ for all π_1, π_2 and every non-trivial additive character ψ of K;*

(Ln5) *π is supercuspidal if and only if $\mathrm{rec}_K(\pi)$ is irreducible (in particular has monodromy 0);*

(Ln6) *(compatibility with the Langlands classification) if π is the Langlands quotient of a multisegment $(\Delta_1, \dots, \Delta_r)$, i.e. the unique irreducible quotient of the normalised parabolic induction of $Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r)$ with the segments ordered so that Δ_i does not precede Δ_j for $i < j$, where $Q(\Delta)$ is the essentially square-integrable representation attached to the segment Δ, then $\mathrm{rec}_K(\pi) \cong \bigoplus_{i=1}^r \mathrm{rec}_K(Q(\Delta_i))$.*

The family is uniquely determined by (Ln1), (Ln2), (Ln3), (Ln4) on supercuspidal representations together with (Ln6).

Proof. Existence: [122], Theorem A (properties (Ln1)–(Ln5) for supercuspidal representations, and the extension to all irreducible smooth representations through the Bernstein–Zelevinsky and Zelevinsky classification, Introduction and Section VII); independently [130]; uniqueness by Henniart's characterisation through L- and ε-factors of pairs. The extension to all representations by (Ln6) and the resulting properties (Ln1)–(Ln4) for all π are explained in Harris–Taylor, loc. cit., and in Wedhorn [318], Sections 3 and 4. Normalisation: Harris and Taylor normalise Art_K so that uniformisers correspond to geometric Frobenius elements, as here; their rec is the map of this statement. □

Theorem A.4.10. *Let $K/\mathbb{Q}_p$ be finite, and let rec_K be the local Langlands correspondence normalized by geometric Frobenius, as in Theorem A.4.9. For every irreducible smooth complex representation π of $\mathrm{GL}_n(K)$ there is a unique unordered tuple of supercuspidal representations σ_j of $\mathrm{GL}_{n_j}(K)$, with $\sum_j n_j = n$, such that π is an irreducible subquotient of their normalized parabolic induction. If $\mathrm{rec}_K(\pi) = (r, N)$, then*

$$(r^{\mathrm{ss}}, 0) \simeq \bigoplus_j \mathrm{rec}_K(\sigma_j).$$

Thus, for any tuple of supercuspidals whose normalized induction has π as a subquotient, the displayed formula holds with that tuple.

Proof. The existence and uniqueness of the supercuspidal support are [24, Theorem 2.9]; see also [319, 2.2.3–2.2.4]. The displayed compatibility is precisely [319, Proposition 4.3.3], the supercuspidal-support form of the extension of the Harris–Taylor correspondence by the Bernstein–Zelevinsky classification. Its reciprocity normalization is the geometric-Frobenius normalization. No assertion that the monodromy is determined by the support is made. □

Theorem A.4.11. *Let F be a CM field, F^+ its maximal totally real subfield, c complex conjugation, $n \geq 1$, ℓ a rational prime and $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ be continuous, with $\chi_v(-1) = (-1)^n$ at every infinite place, such that*

$$\pi^c \simeq \pi^\vee \otimes (\chi \circ \mathrm{N}_{F/F^+} \circ \det).$$

Write $a_{\tau,1} \geq \cdots \geq a_{\tau,n}$ for the highest weights such that π_∞ has the infinitesimal character of the dual of the algebraic representation of highest weight a. There is a continuous semisimple representation

$$r_{\ell,\iota}(\pi) : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

with the following properties.

1. *For every finite $v \nmid \ell$,*

$$\iota\,\mathrm{WD}(r_{\ell,\iota}(\pi)|_{G_{F_v}})^{F\text{-}ss} \simeq \mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2}).$$

2. *It is de Rham at every $v \mid \ell$. In the Hodge–Tate convention in which the cyclotomic character has weight -1, its weights at $\tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell$ are*

$$\{a_{\iota\tau,1} + n - 1, a_{\iota\tau,2} + n - 2, \ldots, a_{\iota\tau,n}\}.$$

3. *If $v \mid \ell$ and π_v has an Iwahori fixed vector, the isomorphism in (1) holds also at v; the local representation is semistable, and is crystalline when π_v is unramified.*

Proof. These are conclusions (2), (3) and (4) of [15, Theorem 2.1.1]. The definitions of a polarized pair and its weight are those in Section 2.1 of that work. The Hodge–Tate convention is its convention $\mathrm{HT}(\epsilon_\ell) = \{-1\}$; the reciprocity map sends uniformizers to geometric Frobenius. No assertion concerning an arbitrary ramified place above ℓ is included. □

Theorem A.4.12. *Let F be a number field and fix a unitary character ξ of $F^\times\backslash\mathbb{A}_F^\times$. Let $L^2_{\mathrm{disc}}(\mathrm{GL}_N, \xi)$ be the discrete part of the space of automorphic functions that transform under the adelic centre by ξ and are square-integrable modulo the centre. If τ is a unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ and $k \geq 1$, the normalised induction*

$$\tau|\det|^{(k-1)/2} \times \tau|\det|^{(k-3)/2} \times \cdots \times \tau|\det|^{(1-k)/2}$$

to $\mathrm{GL}_{km}(\mathbb{A}_F)$ has a unique irreducible quotient $J(\tau, k)$. Its central character is ω_τ^k, and it occurs with multiplicity one in $L^2_{\mathrm{disc}}(\mathrm{GL}_{km}, \omega_\tau^k)$. Every irreducible constituent of $L^2_{\mathrm{disc}}(\mathrm{GL}_N, \xi)$ is exactly one such $J(\tau, k)$, with $km = N$ and $\omega_\tau^k = \xi$; the pair (τ, k) is unique up to isomorphism of τ.

Proof. This is the main theorem of [207], Introduction, pp. 605–606, for number fields. Their equal-block cuspidal datum $V_1^{\otimes k}$ is the datum denoted here by $\tau^{\otimes k}$, and their quotient $J(V)$ is $J(\tau, k)$. Their induction is normalised, with descending exponents $((k-1)/2, \ldots, (1-k)/2)$. The theorem gives the decomposition for a fixed unitary central character; varying that character gives the stated classification. □

Theorem A.4.13. *Let F be a number field, let $m \geq 2$, and let $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ be a nontrivial unitary additive character. Write U_m for the upper triangular unipotent subgroup of GL_m and put*

$$\psi_{U_m}(u) = \psi(u_{1,2} + \cdots + u_{m-1,m}).$$

For a smooth cuspidal automorphic form φ define

$$W_\varphi(g) = \int_{U_m(F)\backslash U_m(\mathbb{A}_F)} \varphi(ug)\psi_{U_m}(u)^{-1}\, du.$$

With the quotient Haar measure of mass one, its cuspidal Fourier expansion is

$$\varphi(g) = \sum_{\gamma \in U_{m-1}(F)\backslash \mathrm{GL}_{m-1}(F)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

The expansion converges absolutely, locally uniformly in g. In particular the Whittaker transform is injective on the cuspidal space, and every irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ has a nonzero global Whittaker functional. For $m = 1$ the latter assertion holds with trivial unipotent group and $W_\varphi = \varphi$.

Proof. The cuspidal Fourier expansion of Shalika and Piatetski-Shapiro, [269, 230]. The character on the maximal unipotent subgroup is the nondegenerate character obtained by summing its simple-root coordinates. The displayed formula uses the standard block embedding of GL_{m-1} and normalised quotient measure. The nonvanishing assertion is the immediate consequence that a form whose Whittaker transform vanishes has zero Fourier expansion. □

Theorem A.4.14. *Let K be a non-archimedean local field of characteristic zero and write $\nu(g) = |\det g|_K$. Products below are normalised parabolic inductions. If c is a finite multiset of essentially square-integrable representations of general linear groups over K, write $L(c)$ for the Langlands quotient obtained by arranging the real exponents of its members in nonincreasing order.*

1. *For an irreducible unitary square-integrable representation δ and $h \geq 1$, put*

$$u(\delta, h) = L\big(\{\delta\nu^{(h-1)/2-a} : 0 \leq a < h\}\big).$$

The building blocks

$$u(\delta, h), \qquad \nu^{\alpha}u(\delta, h) \times \nu^{-\alpha}u(\delta, h) \quad (0 < \alpha < 1/2)$$

are unitary. Every finite product of building blocks is irreducible and unitary. Every irreducible unitary representation is such a product, uniquely up to permutation of the building blocks.

2. *If an irreducible unitary representation π admits a nondegenerate Whittaker model, it has an irreducible induced presentation*

$$\pi \cong \delta_1\nu^{e_1} \times \cdots \times \delta_b\nu^{e_b}, \qquad -\tfrac{1}{2} < e_j < \tfrac{1}{2},$$

where all δ_j are irreducible unitary square-integrable representations; the factors may be arranged with $e_1 \geq \cdots \geq e_b$.

3. *If $L(c)$ and $L(d)$ are unitary, then*

$$L(c) \times L(d) \cong L(c + d).$$

In particular the induction on the left is irreducible, and the Langlands multiset of this induction is the multiset union, with multiplicities.

Proof. The unitary classification in the first assertion is [292], Theorem A and Appendix Theorem A.10, in the notation $u(\delta, h)$ introduced immediately before Appendix Theorem A.8. The third assertion is Appendix Corollary A.7(i), p. 375; its multisets $c, d \in M(D)$ are multisets of essentially square-integrable representations, and addition is multiset union. The generic specialization in the second assertion is stated explicitly in [207], I.10–I.11, pp. 639–640: in the unitary decomposition the Speh half-lengths are zero when there is a Whittaker model. Their parameters v_j are the e_j here and satisfy the strict inequalities $-1/2 < v_j < 1/2$. All three assertions concern smooth irreducible complex representations and normalised induction. □

Theorem A.4.15. *Let F be an imaginary CM field, $n \geq 1$, ℓ a prime, and $\iota : \overline{\mathbb{Q}}_\ell \longrightarrow \mathbb{C}$ an isomorphism. Let (π, χ) be regular algebraic, essentially conjugate self-dual and cuspidal: π is cuspidal on $\mathrm{GL}_n(\mathbb{A}_F)$ and regular algebraic, $\chi : \mathbb{A}_{F^+}^{\times}/(F^+)^{\times} \longrightarrow \mathbb{C}^{\times}$ is continuous with $\chi_v(-1)$ independent of $v \mid \infty$, and*

$$\pi^c \simeq \pi^{\vee} \otimes (\chi \circ \mathrm{N}_{F/F^+} \circ \det).$$

Let $r_{\ell,\iota}(\pi)$ denote the associated continuous semisimple Galois representation, uniquely characterized by its unramified local Langlands compatibility outside a finite set, with geometric-Frobenius normalization and twist $|\det|^{(1-n)/2}$. For a place $v \mid \ell$, put

$$D = \iota\,\mathrm{WD}\big(r_{\ell,\iota}(\pi)|_{G_{F_v}}\big)^{F\text{-}ss}, \qquad D' = \mathrm{rec}_{F_v}\big(\pi_v \otimes |\det|_v^{(1-n)/2}\big).$$

Then $D \prec D'$ in the following precise sense. Their semisimplified Weil representations are isomorphic. Partition the irreducible Weil representations into equivalence classes modulo unramified character twists. On each such class, decompose the corresponding Weil–Deligne summand as a direct sum of special representations $\mathrm{Sp}_{m_j}(s_j)$, order their positive block lengths $m_1 \geq m_2 \geq \cdots$, and pad the list with zeros. For every class and every $k \geq 1$,

$$\sum_{j=1}^{k} m_j(D) \leq \sum_{j=1}^{k} m_j(D').$$

Moreover $D \simeq D'$ if π has Shin-regular weight: either n is odd, or, writing π_∞ with the infinitesimal character of the dual of an algebraic representation of highest weights $a_{\tau,1} \geq \cdots \geq a_{\tau,n}$, there are an embedding τ and an odd $j < n$ with $a_{\tau,j} > a_{\tau,j+1}$.

Proof. This is [16, Theorem 1.1]. The regular algebraic essentially conjugate self-dual cuspidal pair is defined on printed page 168; its common real sign is not prescribed. The associated representation and its uniqueness by unramified compatibility are recalled on printed pages 165 and 169. The Shin-regular condition and the relation $\prec$ are defined on printed pages 169–170. The theorem is on printed page 170 and applies to every $v \mid \ell$, without an Iwahori hypothesis. Its normalization is geometric Frobenius, as fixed on printed page 167. □

Theorem A.4.16. *Let F be a totally real field with $[F : \mathbb{Q}] \geq 2$, let E/F be a totally imaginary quadratic extension, and let U be an inner form of the quasi-split unitary group in $n \geq 1$ variables for E/F. Let σ occur in the discrete automorphic spectrum of U, with σ_∞ discrete series. Assume property $(*)$: for a pseudo-coefficient f_∞ of σ_∞, the integer traces $\operatorname{tr} \sigma'_\infty(f_\infty)$, for discrete automorphic σ' with $\sigma'_f \cong \sigma_f$, have the same sign as $\operatorname{tr} \sigma_\infty(f_\infty)$.*

Then there are positive integers $n_1, \ldots, n_s$ with $\sum_i n_i = n$ and discrete automorphic representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ with $\pi_i^c \cong \pi_i^\vee$ such that $\pi = \boxplus_i \pi_i$ is a weak base change of σ. Here $\pi_v = \bigotimes_{w \mid v} \pi_w$ is viewed on $(\operatorname{Res}_{E/F} \mathrm{GL}_n)(F_v)$. At every split finite place v the correspondence $\sigma_v \mapsto \pi_v$ is local base change. At every finite v where E/F and U are unramified and σ_v is spherical, π_v is its unramified local base change. At infinity π_∞ is cohomological, with the coefficient system of the archimedean base-change construction, and its infinitesimal character is the base change of that of σ_∞. If, in addition, every π_i is cuspidal, then π_∞ itself is the archimedean base change of σ_∞.

Property $(*)$ is automatic when $U(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact.

Proof. This is [177], Corollaire 5.3, with the standing hypotheses at the beginning of Section 5 and the compactness remark immediately preceding the corollary. The notation θ_{n_i}-invariant there means $\pi_i^c \cong \pi_i^\vee$. Weak base change is defined in Section 4.10; the cohomological coefficient system is the one in Lemma 4.4 of Section 4.6. As in loc. cit., the last archimedean conclusion is stated only under the additional hypothesis that every discrete summand is cuspidal. The statement makes no splitting assumption at a chosen coefficient prime, no quasi-splitness hypothesis at the finite places and no hypothesis that E/F is unramified at the finite places. □

Theorem A.4.17. *Let $F/\mathbb{Q}_p$ be finite, with residue field k of cardinality q, and let F_d/F be unramified of degree $d \geq 1$, with residue field k_d. Let $\theta : k_d^\times \to \mathbb{C}^\times$ have Frobenius orbit of size d, and let ρ_θ be its cuspidal Green representation of $\mathrm{GL}_d(k)$, equivalently the genuine Deligne–Lusztig representation $(-1)^{d-1} R^\theta_{k_d^\times}$. If an irreducible admissible smooth complex representation π of $\mathrm{GL}_d(F)$ contains the inflation of ρ_θ on $\mathrm{GL}_d(\mathcal{O}_F)$, then its local Langlands parameter, with uniformisers corresponding to geometric Frobenius, has monodromy zero and*

$$\operatorname{rec}_F(\pi)|_{I_F} \simeq \bigoplus_{\sigma \in \mathrm{Gal}(k_d/k)} (\theta^\sigma \circ \operatorname{Art}_{F_d}^{-1})|_{I_F}.$$

The characters on the right are inflated through residue units.

Proof. This is [74, Proposition 2.4.1(i)]. The representation denoted $(-1)^{d-1} R^\theta_T$ there is the regular cuspidal Green representation in this statement. Its coefficient field is transported from an algebraic closure of a characteristic-zero local field to $\mathbb{C}$ by the chosen field isomorphism. Local reciprocity is geometric in their Harris–Taylor local Langlands convention. □

Theorem A.4.18. *Let $F/\mathbb{Q}_p$ be finite, $G=\mathrm{GL}_n(F)$, $K=\mathrm{GL}_n(\mathcal{O}_F)$, and let the coefficient field be an algebraic closure of a characteristic-zero local field of residue characteristic $\ell\neq p$. In Shotton's notation, $\mathcal{I}_0$ is the set of basic inertial types (restrictions to inertia of irreducible Weil representations), and $\mathcal{I}$ consists of finitely supported functions $\mathcal{P}:\mathcal{I}_0\to\mathrm{Part}$, of weighted degree n. A discrete pair attached to $\mathcal{P}$ has, for every part a of $\mathcal{P}(\tau_0)$, one essentially discrete-series factor on $\mathrm{GL}_{a\dim\tau_0}(F)$ on the cuspidal line τ_0. Write $\sigma(\mathfrak{S}_{\mathcal{P}})$ for the compact representation constructed from that partition data in [278, Definition 6.17]: for the associated simple-character data it is $\mathrm{Ind}_J^K(\kappa\otimes\nu)$, with ν the specified irreducible representation of J/J^1. All simple characters and their extensions are chosen compatibly as in that construction. For any $\mathcal{P},\mathcal{P}'$ of degree n, any discrete pair (M',π') in the inertial class attached to $\mathcal{P}'$, and any parabolic Q' with Levi M', normalized parabolic induction satisfies*

$$\dim\mathrm{Hom}_K\Big(\sigma(\mathfrak{S}_{\mathcal{P}}),\mathrm{Ind}_{Q'}^G\pi'\Big)=\prod_{\tau_0\in\mathcal{I}_0}m(\mathcal{P}(\tau_0),\mathcal{P}'(\tau_0)).$$

Here $m(\lambda,\mu)$ is the multiplicity of the sign-twisted Specht representation indexed by λ in $\mathrm{Ind}_{S_\mu}^{S_{|\mu|}}\mathrm{sgn}$, zero for unequal sizes and one for two empty partitions.

Proof. Quoted from [278, Theorem 6.20]. □

Theorem A.4.19. *Let $F/\mathbb{Q}_p$ be finite, $G=\mathrm{GL}_n(F)$, $K=\mathrm{GL}_n(\mathcal{O}_F)$, and let the coefficient field be an algebraic closure of a characteristic-zero local field of residue characteristic $\ell\neq p$. In Shotton's notation, $\mathcal{I}_0$ is the set of basic inertial types (restrictions to inertia of irreducible Weil representations), and $\mathcal{I}$ consists of finitely supported functions $\mathcal{P}:\mathcal{I}_0\to\mathrm{Part}$, of weighted degree n. A discrete pair attached to $\mathcal{P}$ has, for every part a of $\mathcal{P}(\tau_0)$, one essentially discrete-series factor on $\mathrm{GL}_{a\dim\tau_0}(F)$ on the cuspidal line τ_0. Write $\sigma(\mathfrak{S}_{\mathcal{P}})$ for the compact representation constructed from that partition data in [278, Definition 6.17]: for the associated simple-character data it is $\mathrm{Ind}_J^K(\kappa\otimes\nu)$, with ν the specified irreducible representation of J/J^1. All simple characters and their extensions are chosen compatibly as in that construction. Let (M',π') be a discrete pair of type $\mathcal{P}'$ and let $L(M',\pi')$ be its irreducible Langlands quotient, with the segment convention of [278, Section 3.5]. If*

$$\mathrm{Hom}_K(\sigma(\mathfrak{S}_{\mathcal{P}}),L(M',\pi'))\neq 0,$$

then $\mathcal{P}'(\tau_0)$ is dominated by $\mathcal{P}(\tau_0)$ for every $\tau_0\in\mathcal{I}_0$. Dominance requires equal sizes and the inequalities of all initial partial sums. The inertial type of the quotient is $\tau_{\mathcal{P}'}$ for Shotton's normalized parameter r_ℓ.

Proof. Quoted from [278, Corollary 6.21(1)]. □

Theorem A.4.20. *Let k be finite and $\bar G=\mathrm{GL}_n(k)$. For a finitely supported partition function $\bar{\mathcal{P}}$ on the irreducible cuspidal representations ρ of finite general linear groups, with $\sum_\rho d_\rho|\bar{\mathcal{P}}(\rho)|=n$, let*

$$\pi_{\bar{\mathcal{P}}}=\mathrm{HC\text{-}Ind}\boxtimes_{\rho,j}\mathrm{St}(\rho,\bar{\mathcal{P}}(\rho)_j).$$

Let $\sigma_{\bar{\mathcal{P}}}$ be the finite irreducible Green representation with the sign-twisted partition convention of [278, Definitions 6.1 and 6.7]. Then

$$\dim\mathrm{Hom}_{\bar G}(\sigma_{\bar{\mathcal{P}}},\pi_{\bar{\mathcal{P}}'})=\prod_\rho m(\bar{\mathcal{P}}(\rho),\bar{\mathcal{P}}'(\rho)).$$

The Kostka convention is $m(\lambda,\mu)>0$ precisely when λ dominates μ, with $m(\lambda,\lambda)=1$; partitions of unequal size have multiplicity zero.

Proof. Quoted from [278, Corollary 6.10], with the partition convention of Section 6.1. □

Theorem A.4.21. *Let $F/\mathbb{Q}_p$ be finite. In the homogeneous simple-character construction of [278, Sections 6.9–6.10], choose one endo-class, compatible extensions κ, κ^2, a maximal compact subgroup K, and groups $J^1 \triangleleft J \subset K$. Thus $\bar{G} = J/J^1$ is a finite general linear group over the residue field of the stratum extension, and $\kappa|_{J^1}$ is the irreducible Heisenberg representation. Let $M^0 \subset M^2$ be a Levi, Q^2 a parabolic of G with Levi M^2, and let (M^2, π^2) be a discrete pair in the inertial class of a partition function $\mathcal{P}'$. Let $\bar{\mathcal{P}}'$ be its partition function on the finite cuspidal representations corresponding to these same simple types. Then, as representations of $\bar{G}$,*

$$\operatorname{Hom}_{J^1}\left(\kappa, \operatorname{Ind}_{Q^2}^{G} \pi^2\right) \simeq \pi_{\bar{\mathcal{P}}'},$$

where induction is normalized and $\pi_{\bar{\mathcal{P}}'}$ is the finite Harish–Chandra induction of the generalized Steinberg factors with the specified parts.

Proof. Quoted from [278, Corollary 6.19]. □

Theorem A.4.22. *Let F be a nonarchimedean local field and let σ_a be irreducible smooth complex representations of $\mathrm{GL}_{n_a}(F)$, for $1 \le a \le d$. Suppose that, whenever $a \neq b$, no supercuspidal representation in the cuspidal support of σ_a is an unramified twist of a supercuspidal representation in the cuspidal support of σ_b. Then the normalized parabolic induction*

$$\sigma_1 \times \cdots \times \sigma_d$$

to $\mathrm{GL}_{n_1+\cdots+n_d}(F)$ is irreducible.

Proof. By the multisegment classification, write each σ_a as the irreducible representation of a multisegment. The hypothesis puts every segment from two different multisegments on different unramified cuspidal lines; such segments cannot be linked. The assertion is therefore [332, Proposition8.5], which identifies the induced product with the irreducible representation of the union multisegment. □

Theorem A.4.23. *Let K be a nonarchimedean local field of characteristic zero, $n \ge 1$, and B the upper triangular Borel subgroup of $\mathrm{GL}_n(K)$. Let $\chi_i = \mu_i |\cdot|_K^{s_i}$ be smooth characters of $K^\times$, where μ_i is unitary and $s_i \in \mathbb{R}$. If $s_1 \ge \cdots \ge s_n$, the normalized parabolic induction*

$$\operatorname{Ind}_{B}^{\mathrm{GL}_n(K)}(\chi_1 \otimes \cdots \otimes \chi_n)$$

has a unique irreducible quotient. This quotient is the Langlands quotient for the multisegment of singleton characters $\{\chi_1\}, \ldots, \{\chi_n\}$.

Proof. This is the singleton-segment case of the Bernstein–Zelevinsky classification, [332], Theorem 6.1 in its quotient formulation. The convention is normalized induction from the upper Borel and the decreasing real-exponent order. Equivalently, group equal real exponents into tempered principal-series blocks; this is the usual Langlands standard module. The quotient formulation and this convention are displayed in Wedhorn [318], Sections 2.2.9 and 2.3.9. □

Theorem A.4.24. *Let F_v be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_{F_v}$, uniformiser ϖ_v and residue cardinality q_v. Put $G_v = \mathrm{GL}_n(F_v)$ and $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$, where $n \ge 1$, and give G_v Haar measure with $\mathrm{vol}(K_v) = 1$. Let σ be an irreducible smooth complex representation of G_v with $\sigma^{K_v} \neq 0$. Then $\dim_{\mathbb{C}} \sigma^{K_v} = 1$. There are unramified characters $\chi_1, \ldots, \chi_n$ of $F_v^\times$, in a*

Langlands order, for which σ is the Langlands quotient of the normalised upper-Borel induction of $\chi_1 \otimes \cdots \otimes \chi_n$. The unordered multiset

$$\{\alpha_{\sigma,1}, \dots, \alpha_{\sigma,n}\}, \qquad \alpha_{\sigma,i} = \chi_i(\varpi_v),$$

is uniquely determined by σ. Here normalised induction means $f(bg) = \delta_{B_v}(b)^{1/2}(\chi_1 \otimes \cdots \otimes \chi_n)(b) f(g)$, with G_v acting by right translation. For $0 \le k \le n$, put

$$D^{+}_{v,k} = K_v \,\mathrm{diag}(\varpi_v 1_k, 1_{n-k}) K_v.$$

The operator $\int_{D^{+}_{v,k}} \sigma(g)\,dg$ on σ^{K_v} is multiplication by

$$q_v^{k(n-k)/2} e_k(\alpha_{\sigma,1}, \dots, \alpha_{\sigma,n}),$$

where e_k is the elementary symmetric polynomial and $e_0 = 1$.

Proof. This is the spherical principal-series parametrisation together with Tamagawa's minuscule-generator formula. For the parametrisation by singleton-segment Langlands quotients, see [318], (4.1.3)–(4.1.6); the normalised-induction and right-translation convention is also explicit in [50], Section 2, with the spherical line in Corollary 2.2 and the principal-series embedding in Proposition 2.6. For the operator formula, [109], Section 2, fixes Haar volume one, and (3.14) gives the Satake transform of $1_{D^{+}_{v,k}}$ as $q_v^{k(n-k)/2}\,\mathrm{Tr}(\bigwedge^k \mathbb{C}^n)$. Evaluating at the spherical parameter, as in (6.2)–(6.7) there, gives the displayed eigenvalue. The cases $k = 0, n$ are respectively the unit and the scalar central double coset. □

Theorem A.4.25. *Let H_1, H_2 be locally compact, totally disconnected, second countable groups. A representation of a totally disconnected group on a complex vector space is* admissible *if every vector has an open stabiliser and the vectors fixed by any open compact subgroup form a finite-dimensional space. Then for every irreducible admissible representation σ of $H_1 \times H_2$ on W there are irreducible admissible representations π_1 of H_1 on W_1 and π_2 of H_2 on W_2 and a linear isomorphism $W_1 \otimes W_2 \to W$ intertwining $\pi_1 \otimes \pi_2$ with σ; π_1 and π_2 are unique up to equivalence. Conversely, $\pi_1 \otimes \pi_2$ is irreducible and admissible for irreducible admissible π_1, π_2.*

Proof. Bernstein–Zelevinsky [23], Chapter I (irreducible admissible representations of a direct product of two l-groups), and Flath [80] (the lemma for two factors underlying the restricted tensor product theorem). Map of notation: l-group = locally compact totally disconnected group; countability at infinity is used for Schur's lemma. □

Theorem A.4.26. *Let E/F be a CM extension, let V be a nondegenerate hermitian space over E of dimension n with form $\langle\,,\rangle$, let $G = \mathrm{U}(V)$, and let $a \ge 1$ be an integer with $2a < n$. For an F-algebra R write $V_R = V \otimes_F R$ and let $\mathrm{gram}\colon V_R^a \to \mathrm{Herm}_a(R)$, $\mathrm{gram}(x) = (\langle x_i, x_j\rangle)_{i,j}$, be the Gram map, Herm_a denoting hermitian $a \times a$ matrices over $E \otimes_F R$. For a place w of F let dx_w be the Haar measure on $V_{F_w}^a$ that is self-dual for the pairing $(x,y) \mapsto \psi_w(\mathrm{tr}_{E_w/F_w} \sum_i \langle x_i, y_i\rangle)$, and db_w the Haar measure on $\mathrm{Herm}_a(F_w)$ that is self-dual for $(b,b') \mapsto \psi_w(\mathrm{tr}\, bb')$. Let $\Phi = \Phi_\infty \otimes \bigotimes_{w<\infty} \Phi_w$, where Φ_∞ is a smooth compactly supported function on $V_{F_\infty}^a$, $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, each Φ_w is a Schwartz–Bruhat function on $V_{F_w}^a$, and Φ_w is the characteristic function of $(L \otimes \mathcal{O}_{F_w})^a$ for all but finitely many w, for an $\mathcal{O}_E$-lattice $L \subset V$ (an* admissible test function*). Let $T \in \mathrm{Herm}_a(F)$ be nondegenerate. Then:*

1. *for every Haar measure μ on $G(\mathbb{A}_F)$, the quotient $[G]$ has finite volume $\mu([G])$;*

2. *for every place w (with ∞ standing for the product of the archimedean places), the image under* gram *of the measure $\Phi_w\,dx_w$, restricted to the open set of x with* $\det \mathrm{gram}(x)$ *invertible, has a continuous density $b \mapsto g_w(\Phi_w, b)$ with respect to db_w on the open set of nondegenerate matrices, and the product $\prod_{w<\infty} g_w(\Phi_w, T)$ converges;*

3. *the function $\vartheta_T(g) = \sum_{X \in V^a,\ \mathrm{gram}(X)=T} \Phi(g^{-1}X)$ on $[G]$ is integrable, and*

$$\frac{1}{\mu([G])} \int_{[G]} \vartheta_T(g)\, d\mu(g) = g_\infty(\Phi_\infty, T) \prod_{w<\infty} g_w(\Phi_w, T).$$

Proof. [323], Chapter VI, section 52, Theorem 5 and formula (41), together with Chapter III, section 37, Proposition 6 and formula (29), and Chapter IV, section 42. In Weil's notation take $k = F$, the algebra with involution $\mathrm{End}_E(V)$ with its adjoint involution, and $X = V^a$. This is type (I_2) with $\epsilon = 1/2$, so his condition (B) is $n > 2a$. His invariant map is $x \mapsto x^* h x$, namely the displayed Gram map. The auxiliary local hypothesis of Theorem 5 holds at a finite place: condition (B) gives a nonempty full-rank zero-Gram fibre there by section 38, and Witt extension makes the full local unitary group transitive on each full-rank Gram fibre. If r is the Witt index of V, then $r \le n/2$ and hence $n - r > a$; Proposition 8 in section 51 gives absolute convergence. Formula (41), with its normalized Haar specialization, gives the asserted mean as the product of the local relative-gauge integrals. Proposition 6 identifies these with continuous pushforward densities; section 42 gives absolute convergence of the product. Nondegenerate T forces every tuple in its fibre to have full rank, so its full-rank sum is ϑ_T.

The following calculation identifies the measures, with no unspecified local scalar. Choose rational F-bases on $X = V^a$ and $Y = \mathrm{Herm}_a$. Let dx_w^0, db_w^0 be the coordinate measures obtained from the ψ_w-self-dual measure on F_w. For the two nondegenerate F-bilinear pairings displayed in the statement let B_X, B_Y be their matrices in these bases. Fourier inversion and linear change of variables give

$$dx_w = |\det B_X|_w^{1/2} dx_w^0, \qquad db_w = |\det B_Y|_w^{1/2} db_w^0.$$

Let g_w^0 be the pushforward density for the same Gram map with the coordinate measures dx_w^0, db_w^0. Consequently the relative density for the measures in the statement is

$$g_w(\Phi_w, T) = \left|\frac{\det B_X}{\det B_Y}\right|_w^{1/2} g_w^0(\Phi_w, T).$$

Both determinants lie in $F^\times$. The product formula makes the product of the displayed positive factors equal to one, including the archimedean factors; only finitely many finite factors differ from one. Thus the product of Weil's relative-gauge densities equals the product in the statement, also at dyadic and split places. This is a measure comparison, with no Weil-index factor.

Weil uses the Haar measure for which the right quotient has volume one. Inversion transfers it to the left quotient because $\mathrm{U}(V)$ is unimodular, replacing the action uX by $g^{-1}X$ and preserving total volume. For any Haar μ, its normalization is $\mu/\mu([G])$, so the mean is unchanged. Quotient finiteness (item 1) is the classical finite-volume theorem for unitary groups (Borel–Harish-Chandra). Smooth compactly supported archimedean test functions belong to the Schwartz space used there, as do the given finite factors. For hermitian b, b', $\mathrm{tr}(bb') \in F_w$, so the matrix pairing also equals $\frac{1}{2}\mathrm{Tr}_{E_w/F_w} \mathrm{tr}(bb')$, if written in that alternative convention. □

Theorem A.4.27. *Let E/F be a CM extension, $n \ge 1$ an integer, W a skew-hermitian space over E of dimension $2n$ and Witt index n with a polarisation $W = W^+ \oplus W^-$ into maximal*

isotropic subspaces, $P \subset \mathrm{U}(W)$ the Siegel parabolic subgroup stabilising W^+, and K_G a maximal compact subgroup of $\mathrm{U}(W)(\mathbb{A}_F)$ with $\mathrm{U}(W)(\mathbb{A}_F) = P(\mathbb{A}_F)K_G$. For $g = pk$ with $p \in P(\mathbb{A}_F)$, $k \in K_G$, let $a(g) \in \mathrm{GL}(W^+)(\mathbb{A}_E)$ be the action of p on W^+ and $|a(g)| = |\det a(g)|_{\mathbb{A}_E}$. Let V be a hermitian space over E of dimension m and Witt index r, and suppose

$$m > n \qquad \text{and} \qquad m - r > n.$$

Let χ be a Hecke character of E whose restriction to $\mathbb{A}_F^\times$ is η^m, let ω be the Weil representation of $\mathrm{U}(W)(\mathbb{A}_F) \times \mathrm{U}(V)(\mathbb{A}_F)$ attached to ψ_{add} and χ on the Schrödinger model $\mathcal{S}(V^n(\mathbb{A}_F))$ of the polarisation, and let $\Phi \in \mathcal{S}(V^n(\mathbb{A}_F))$ be K_G-finite with archimedean component a polynomial vector of the Fock model. Put $s_0 = \frac{m-n}{2}$,

$$f_\Phi^{(s)}(g) = |a(g)|^{\,s-s_0}\,(\omega(g,1)\Phi)(0), \qquad E(g,s,\Phi) = \sum_{\gamma \in P(F)\backslash \mathrm{U}(W)(F)} f_\Phi^{(s)}(\gamma g),$$

so that $f_\Phi^{(s)}$ is a section of the normalised induction $\mathrm{Ind}_P^{\mathrm{U}(W)}(\chi\,|\det|^s)$ and the series converges absolutely for $\mathrm{Re}\, s > n/2$ and continues meromorphically to $s \in \mathbb{C}$. Let dh be the Haar measure on $\mathrm{U}(V)(\mathbb{A}_F)$ with $\mathrm{vol}([\mathrm{U}(V)]) = 1$. Then:

1. *for every $g \in \mathrm{U}(W)(\mathbb{A}_F)$ the theta integral*

$$I(g,\Phi) = \int_{[\mathrm{U}(V)]} \sum_{x \in V^n(F)} (\omega(g,h)\Phi)(x)\, dh$$

   converges absolutely;

2. *$E(g,s,\Phi)$ is holomorphic at $s = s_0$, and for every $g \in \mathrm{U}(W)(\mathbb{A}_F)$*

$$E(g,s_0,\Phi) = I(g,\Phi).$$

Proof. For $m > 2n$ this is Weil's Siegel–Weil formula [323], the Eisenstein series being absolutely convergent at s_0 since $\mathrm{Re}\, s_0 > n/2$; item 1 is Weil's convergence criterion (absolute convergence of the theta integral for all Φ when $r = 0$ or $m - r > n$, ibid.). For $n < m \le 2n$ and $m - r > n$, holomorphy at s_0 is due to Ikeda [143] (see also Tan [293] for the poles of Siegel Eisenstein series on $\mathrm{U}(n,n)$), and the identity $E(g,s_0,\Phi) = I(g,\Phi)$ is the main theorem of Ichino [140], with the setting and normalisations of [139]. Map of notation: Ichino's $G = \mathrm{U}(n,n)$ is $\mathrm{U}(W)$, his $H = \mathrm{U}(V)$, $\omega_Q = \omega_{Q,\psi,\chi}$ is ω, $f_\Phi^{(s)}$, s_0, $E(g, f_\Phi^{(s)})$ and $I_Q(g,\Phi)$ (with $\mathrm{vol}(H(F)\backslash H(\mathbb{A})) = 1$) are as displayed; the constant in the identity is 1. □

Theorem A.4.28. *Let E/F be a CM extension, ψ a nontrivial character of $\mathbb{A}_F/F$, V a hermitian space over E of dimension m, W a skew-hermitian space over E of dimension $2n \ge 2$ and Witt index n, and $\mathbb{W} = \mathrm{Res}_{E/F}(V \otimes_E W)$ with the symplectic form of Theorem A.5.4. Let $L, L' \subset W$ be complementary maximal isotropic subspaces, $P \subset \mathrm{U}(W)$ the stabiliser of L, and $\mathbb{X} = \mathrm{Res}_{E/F}(V \otimes_E L')$, $\mathbb{Y} = \mathrm{Res}_{E/F}(V \otimes_E L)$, complementary Lagrangians of $\mathbb{W}$ defined over F. Let χ be a character of $\mathbb{A}_E^\times/E^\times$ with $\chi|_{\mathbb{A}_F^\times} = \eta^m$, and ω the Weil representation $\omega_{\psi,\chi}$ of $\mathrm{U}(W)(\mathbb{A}_F) \times \mathrm{U}(V)(\mathbb{A}_F)$ on $\mathcal{S}(\mathbb{X}(\mathbb{A}_F))$ of Theorem A.5.4 with the splitting character χ on $\mathrm{U}(W)$ and the trivial character on $\mathrm{U}(V)$. Let dh be the Haar measure on $\mathrm{U}(V)(\mathbb{A}_F)$ for which $[\mathrm{U}(V)]$ has volume 1. Suppose*

$$m > 2n.$$

Then for every $\Phi \in \mathcal{S}(\mathbb{X}(\mathbb{A}_F))$ and every $g \in \mathrm{U}(W)(\mathbb{A}_F)$:

1. *for $\gamma \in \mathrm{U}(W)(F)$ the value $(\omega(\gamma g,1)\Phi)(0)$ depends only on the coset $P(F)\gamma$, and the series*

$$E(g,\Phi)=\sum_{\gamma\in P(F)\backslash \mathrm{U}(W)(F)}(\omega(\gamma g,1)\Phi)(0)$$

converges absolutely;

2. $\displaystyle\int_{[\mathrm{U}(V)]}\sum_{x\in\mathbb{X}(F)}\left|(\omega(g,h)\Phi)(x)\right|dh<\infty$;

3. $\displaystyle E(g,\Phi)=\int_{[\mathrm{U}(V)]}\sum_{x\in\mathbb{X}(F)}(\omega(g,h)\Phi)(x)\,dh$.

Proof. This is Weil's Siegel formula in the range where the Eisenstein series converges absolutely and the theta integral converges absolutely, for every Schwartz–Bruhat function, [323]; for unitary dual pairs with Kudla's splitting it is stated in this form (the convergent case of the Siegel–Weil formula, constant 1 for the probability Haar measure) in [139], Introduction, and [121], Section 1; the orthogonal analogue in this form, with the attribution to Weil, is in Kudla and Rallis [174, 175]. The hypotheses: $s_0=(m-n)/2>n/2$ is Godement's range of absolute convergence of the Siegel Eisenstein series of $\mathrm{U}(W)$, and Weil's convergence criterion for the theta integral ($r=0$ or $m-r>n$, r the Witt index of V) holds since $r\le m/2<m-n$. Notation map: $(\omega(g,1)\Phi)(0)$ is the standard section $f_\Phi^{(s_0)}(g)$ of the degenerate principal series induced from P (the unipotent radical of P acts on $\mathcal{S}(\mathbb{X}(\mathbb{A}_F))$ by multiplication by characters and its Levi by substitutions, so evaluation at 0 is P-equivariant, with trivial factor on $P(F)$), $E(g,\Phi)=E(g,s_0,\Phi)$, and the right side of (3) is $I(g,\Phi)$. □

Theorem A.4.29. *Let E/F be a CM extension, Y a nondegenerate hermitian or skew-hermitian space over E of dimension $2n\ge 2$ and Witt index n, $G=\mathrm{U}(Y)$, $L\subset Y$ a maximal isotropic subspace, $P\subset G$ its stabiliser, $a(p)\in\mathrm{GL}(L)(\mathbb{A}_E)$ the action of $p\in P(\mathbb{A}_F)$ on L, and K a maximal compact subgroup of $G(\mathbb{A}_F)$ with $G(\mathbb{A}_F)=P(\mathbb{A}_F)K$. Fix a basis of Y and put $\|g\|=\prod_v\max_{i,j}\max(|g_{ij}|_v,|(g^{-1})_{ij}|_v)$ for $g\in G(\mathbb{A}_F)$. Let μ be a unitary Hecke character of E and, for $s\in\mathbb{C}$, let f_s be a K-finite flat section of the normalised induction $I(s)=\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\mu(\det a)\,|\det a|_{\mathbb{A}_E}^{s})$ (that is, $f_s(pk)=\mu(\det a(p))\,|\det a(p)|_{\mathbb{A}_E}^{s+n/2}f_0(k)$ with $f_0|_K$ independent of s), and*

$$E(g,f_s)=\sum_{\gamma\in P(F)\backslash G(F)}f_s(\gamma g)\qquad(\mathrm{Re}\,s>\tfrac{n}{2}),$$

absolutely and locally uniformly convergent there. Then $s\mapsto E(g,f_s)$ continues to a meromorphic function on $\mathbb{C}$ for every g, and for every compact set $\Omega\subset\mathbb{C}$ there are a nonzero polynomial Q, an integer $N\ge 0$ and $C>0$ such that, for every $g\in G(\mathbb{A}_F)$, $s\mapsto Q(s)E(g,f_s)$ is holomorphic on a neighbourhood of Ω and

$$|Q(s)\,E(g,f_s)|\le C\,\|g\|^N\qquad(s\in\Omega,\ g\in G(\mathbb{A}_F)).$$

Proof. This is Langlands' meromorphic continuation of Eisenstein series, Langlands [184], in the form with locally uniform moderate growth of Mœglin and Waldspurger [208], Chapter IV (continuation of Eisenstein series induced from cuspidal data on the Levi factor; here the cuspidal datum is the character $\mu\circ\det$ of the Levi factor $\mathrm{GL}(L)$, and K-finite flat sections of $I(s)$ are the holomorphic families of that chapter), applied to the reductive group G over F

and its maximal parabolic P. Absolute convergence for $\operatorname{Re} s > \frac{n}{2}$ is Godement's criterion ($\frac{n}{2}$ is the exponent ρ_P normalised for $|\det a|$). Map of notation: the Siegel Eisenstein series $E(g, f_s)$ displayed in the statement is the Eisenstein series $E(\lambda, \phi)$ of the cited works for $\lambda = s\,\varpi_P$, ϖ_P the fundamental weight $|\det a|$ of P; $\|\cdot\|$ is a height on $G(\mathbb{A}_F)$ in their sense. $\square$

A.5 Representations of real, p-adic and finite groups, and the Weil representation

The quotations fall into four groups. The first is harmonic analysis on compact groups and on locally compact abelian groups [112, 320, 163, 296]. The second is the Weil representation [322]: the theorem of Stone and von Neumann in Mackey's form [195], the metaplectic group, the global Weil representation of unitary dual pairs with Kudla's splitting [172, 121], the Siegel–Levi formulas [191, 170], and the Fock model of Bargmann, Segal and Shale [14, 257, 268] (see [83, 157, 136]). The third is real groups and their discrete series [119]: the formal degree [251, 131], Blattner's formula as proved by Hecht and Schmid [127], Miličić's decay bound for matrix coefficients [199], Schur orthogonality, Helgason's change of variables [129] and Goodman's Wishart density [104]; together with the polynomial representations of GL_n [92, 91, 151, 194, 135, 71, 67, 211, 137]. The fourth is Green's theory of the characters of GL_r over a finite field [106, 194], which builds the idempotent that cuts out the target forms.

Theorem A.5.1. *Let K be a compact Hausdorff topological group. There is a unique regular Borel probability measure ν on K such that*

$$\int_K f(bk)\,d\nu(k) = \int_K f(kb)\,d\nu(k) = \int_K f(k^{-1})\,d\nu(k) = \int_K f\,d\nu$$

for every continuous $f\colon K \to \mathbb{C}$ and every $b \in K$.

Proof. This is the existence and uniqueness of Haar measure for locally compact groups ([112]; [320]), specialised to a compact group: a left Haar measure of a compact group is finite, hence normalisable to total mass 1; the modular function is a continuous homomorphism $K \to \mathbb{R}_{>0}$ with compact image, hence trivial, so the left Haar measure is also right invariant and inversion invariant. $\square$

Theorem A.5.2. *Let E be a number field with adele ring $\mathbb{A}_E$, let ψ_E be the standard additive character of $\mathbb{A}_E$ (trivial on E), and let μ be the Haar probability measure on the compact group $\mathbb{A}_E/E$.*

(a) The map $\xi \mapsto (z \mapsto \psi_E(\xi z))$ is an isomorphism of E onto the group of continuous characters of $\mathbb{A}_E/E$.

(b) If $G\colon \mathbb{A}_E/E \to \mathbb{C}$ is continuous and $\sum_{\xi\in E} |\widehat{G}(\xi)| < \infty$, where

$$\widehat{G}(\xi) = \int_{\mathbb{A}_E/E} \psi_E(\xi z) G(z)\,d\mu(z),$$

then $G(0) = \sum_{\xi\in E} \widehat{G}(\xi)$.

Proof. (a) is [296], Section 4.1 (the character group of $\mathbb{A}_E/E$ is E, through any nontrivial character of $\mathbb{A}_E$ trivial on E). (b) is Fourier inversion on the compact abelian group $\mathbb{A}_E/E$, whose dual is E by (a): a continuous function with absolutely summable Fourier coefficients is the uniform sum of its Fourier series; evaluated at 0, with the coefficient at $-\xi$ written as $\widehat{G}(\xi)$, this is the displayed identity. Notation map: ψ_E is Tate's standard character $\psi(x) = \prod_v \psi_v(x_v)$. $\square$

Theorem A.5.3. *Let F be a number field, $\mathbb{W}$ a finite-dimensional F-vector space with a non-degenerate alternating form B, $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ a decomposition into complementary Lagrangian subspaces defined over F, $\beta = B|_{\mathbb{X}\times\mathbb{Y}}$, ψ a nontrivial character of $\mathbb{A}_F/F$, and ρ_ψ the Schrödinger realisation of the Heisenberg representation on $L^2(\mathbb{X}(\mathbb{A}_F))$:*

$$(\rho_\psi(u_0+v_0)\varphi)(u) = \psi(\beta(u,v_0)+\tfrac{1}{2}\beta(u_0,v_0))\varphi(u+u_0) \qquad (u,u_0\in\mathbb{X}(\mathbb{A}_F),\ v_0\in\mathbb{Y}(\mathbb{A}_F)).$$

Then the metaplectic group $\mathrm{Mp}(\mathbb{W})(\mathbb{A}_F)$, the extension of $\mathrm{Sp}(\mathbb{W})(\mathbb{A}_F)$ by $\mathbb{C}^1$ on which the Weil representation ω_ψ is defined, is the group of pairs (σ, U) with $\sigma \in \mathrm{Sp}(\mathbb{W})(\mathbb{A}_F)$ and U a unitary operator of $L^2(\mathbb{X}(\mathbb{A}_F))$ such that

$$U\,\rho_\psi(w)\,U^{-1} = \rho_\psi(\sigma w) \qquad (w\in\mathbb{W}(\mathbb{A}_F)),$$

with $(\sigma,U)\mapsto\sigma$ the projection and $\omega_\psi(\sigma,U)=U$. Every such U maps the Schwartz–Bruhat space $\mathcal{S}(\mathbb{X}(\mathbb{A}_F))$ onto itself.

Proof. This is Weil's construction of the metaplectic group and of the Weil representation for the locally compact abelian group $\mathbb{X}(\mathbb{A}_F)$, [322]: the unitary operators normalising the Heisenberg representation form a group (Weil's $\mathbf{B}_0$), the Stone–von Neumann theorem shows that the operators over a given σ form one coset of $\mathbb{C}^1$ and that every $\sigma \in \mathrm{Sp}(\mathbb{W})(\mathbb{A}_F)$ occurs, and these operators induce automorphisms of $\mathcal{S}(\mathbb{X}(\mathbb{A}_F))$. Notation map: Weil's $\mathbf{r}_0(s)$ for $s\in\mathrm{Mp}$ is U; his $U(w)$ is $\rho_\psi(w)$. Changing the realisation of the Heisenberg representation by a unitary equivalence conjugates the whole statement. □

Theorem A.5.4. *Let E/F be a quadratic extension of number fields with quadratic character $\eta = \eta_{E/F}$ of $\mathbb{A}_F^\times/F^\times$, ψ a nontrivial character of $\mathbb{A}_F/F$, S a hermitian space of dimension m and T a skew-hermitian space of dimension n over E, and $\mathbb{W} = \mathrm{Res}_{E/F}(S\otimes_E T)$ with the symplectic form $\mathrm{tr}_{E/F}(\langle\,,\rangle_S\otimes\overline{\langle\,,\rangle_T})$, so that $\mathrm{U}(S)\times\mathrm{U}(T)\to\mathrm{Sp}(\mathbb{W})$. Let χ_S,χ_T be characters of $\mathbb{A}_E^\times/E^\times$ with $\chi_S|_{\mathbb{A}_F^\times}=\eta^m$ and $\chi_T|_{\mathbb{A}_F^\times}=\eta^n$. Then:*

(i) the metaplectic extension $\mathrm{Mp}(\mathbb{W})(\mathbb{A})\to\mathrm{Sp}(\mathbb{W})(\mathbb{A})$ splits over the image of $\mathrm{U}(S)(\mathbb{A})\times\mathrm{U}(T)(\mathbb{A})$ by a homomorphism $\tilde\iota_\chi$ determined by χ_T on $\mathrm{U}(S)(\mathbb{A})$ and χ_S on $\mathrm{U}(T)(\mathbb{A})$, which is the product of the local splittings and restricts on $\mathrm{U}(S)(F)\times\mathrm{U}(T)(F)$ to the canonical splitting of $\mathrm{Sp}(\mathbb{W})(F)$;

(ii) consequently $\omega_{\psi,\chi}=\omega_\psi\circ\tilde\iota_\chi$ is a representation of $\mathrm{U}(S)(\mathbb{A})\times\mathrm{U}(T)(\mathbb{A})$ on the Schwartz space $\mathcal{S}(\mathbb{X}(\mathbb{A}))$ of a Lagrangian $\mathbb{X}$ of $\mathbb{W}$ (the Schrödinger model of the Weil representation ω_ψ), and for every $\varphi\in\mathcal{S}(\mathbb{X}(\mathbb{A}))$ the theta distribution $\theta(\varphi)=\sum_{x\in\mathbb{X}(F)}\varphi(x)$ satisfies $\theta(\omega_{\psi,\chi}(\gamma)\varphi)=\theta(\varphi)$ for $\gamma\in\mathrm{U}(S)(F)\times\mathrm{U}(T)(F)$.

Proof. (i) is Kudla's splitting of metaplectic covers over unitary dual pairs, [172], globalised and normalised as in [121], Section 1. (ii) Weil's theorem: the Weil representation ω_ψ of $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$ on $\mathcal{S}(\mathbb{X}(\mathbb{A}))$ exists and θ is invariant under the image of $\mathrm{Sp}(\mathbb{W})(F)$ under its canonical splitting [322]; composing with $\tilde\iota_\chi$ and using the last clause of (i) gives (ii). □

Theorem A.5.5. *Let F be a number field, let ψ be a nontrivial unitary character of $\mathbb{A}_F/F$, and let X,Y be finite-dimensional F-vector spaces in perfect bilinear duality β. Write $X_\mathbb{A}=X(\mathbb{A}_F)$ and $Y_\mathbb{A}=Y(\mathbb{A}_F)$. Use the dual Haar measures for which $X(F)\backslash X_\mathbb{A}$ and $Y(F)\backslash Y_\mathbb{A}$ have volume one. Let $\mathcal{H}_\beta$ be the Hilbert space, modulo almost-everywhere equality, of measurable functions satisfying*

$$G(u+a,v+b)=\psi(-\beta(a,v))G(u,v) \quad (a\in X(F),\ b\in Y(F))$$

almost everywhere, with finite squared norm

$$\|G\|^2 = \int_{(X(F)\backslash X_{\mathbb{A}})\times(Y(F)\backslash Y_{\mathbb{A}})} |G(u,v)|^2 \, du\, dv.$$

For $\varphi \in \mathcal{S}(X_{\mathbb{A}})$ the series

$$Z_\psi\varphi(u,v) = \sum_{a\in X(F)} \psi(\beta(a,v))\varphi(u+a)$$

converges locally uniformly and defines a continuous representative. It extends uniquely to a unitary isomorphism $Z_\psi : L^2(X_{\mathbb{A}}) \longrightarrow \mathcal{H}_\beta$, whose inverse is

$$(Z_\psi^{-1}G)(u) = \int_{Y(F)\backslash Y_{\mathbb{A}}} G(u,v)\, dv$$

for almost every u. For Schwartz functions the inverse identity holds pointwise.

Proof. This is the adelic specialization of Weil's unitary theta transform: [322], Chapter I, paragraphs 16–18, equations (20), (22), (23), pp. 164–167. □

Theorem A.5.6. *Let F be a nonarchimedean local field of characteristic zero, let $|\cdot|_F$ be its normalised absolute value, let ψ be a nontrivial unitary additive character of F, and let $m, n \geq 1$ be integers. For the type-II reductive dual pair $\mathrm{GL}_m(F) \times \mathrm{GL}_n(F)$, the Weil representation with its standard linear splitting has a Schrödinger realisation on the Schwartz–Bruhat space $\mathcal{S}(M_{m\times n}(F))$ given by*

$$(\omega_\psi(a,b)\varphi)(x) = |\det a|_F^{-n/2} |\det b|_F^{m/2} \varphi(a^{-1}xb), \qquad a \in \mathrm{GL}_m(F),\ b \in \mathrm{GL}_n(F),\ x \in M_{m\times n}(F).$$

The displayed formula specifies the splitting and the action convention.

Proof. This is the type-II Schrödinger formula in Kudla [173], Remark 6.3, pp. 53–54. It is quoted with the variables and the determinant exponents of that formula. □

Theorem A.5.7. *Let k be a nonarchimedean local field of characteristic zero, K/k a quadratic etale algebra with involution c and associated quadratic character η, and $\psi : k \to S^1$ a nontrivial continuous character. Let V be a nondegenerate hermitian K-space of rank $m \geq 1$, and let W_q be the standard hyperbolic skew-hermitian space of rank $2q$, $q \geq 1$, with matrix $\begin{pmatrix} 0 & 1_q \\ -1_q & 0 \end{pmatrix}$ in its conjugate-first convention. Use the tensor symplectic form $\frac{1}{2}\operatorname{Tr}_{K/k}(\overline{h_{W_q}} \otimes h_V)$. For a unitary character $\chi : K^\times \to S^1$ with $\chi|_{k^\times} = \eta^m$, the Kudla circle Weil representation with splitting characters $(\chi, 1)$ has a Schrödinger realization on $\mathcal{S}(V^q)$. In its standard matrix coordinates, for $a \in \mathrm{GL}_q(K)$ and*

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-\mathsf{T}} \end{pmatrix},$$

its action is exactly

$$(\omega_\chi(m(a))\varphi)(x) = |\det a|_K^{m/2} \chi(\det a)\varphi(xa).$$

This is the unitary Weil representation with its usual Heisenberg covariance. The assertion includes $K = k \times k$.

Proof. [191], Section 2A, pp.929–931: the local algebra is explicitly allowed to be split on p.929, and the splitting and the displayed Siegel-Levi formula are on p.930. The other splitting character is 1, compatible with the even rank $2q$. The symbols r, m, χ there are q, m, χ here. The conjugation on the skew-hermitian factor and the half-trace tensor form are retained. □

Theorem A.5.8. *Let V, W be nondegenerate complex hermitian and skew-hermitian spaces of ranks $r, t > 0$, respectively, with the first form conjugate-linear in the first argument and the second linear in the first. Use $B(v \otimes w, v' \otimes w') = \mathrm{Re}((v, v')\langle w, w'\rangle)$, a nontrivial real additive character, and Kudla's doubled splittings. Let ξ, ξ' be unitary characters of $\mathbb{C}^\times$ restricting to $\mathrm{sgn}^t, \mathrm{sgn}^r$ on $\mathbb{R}^\times$. In the standard diagonal/antidiagonal Witt coordinates of the doubles, on the indicated matrix Schwartz spaces,*

$$\omega^H_{W,\xi}(m_{\Delta V}(A))\varphi(Z) = \xi(\det A)|\det A|^{-t}\varphi(A^{-1}Z), \quad A \in \mathrm{GL}_r(\mathbb{C}),$$

$$\omega^H_{V,\xi'}(m_{\Delta W}(D))\varphi(Z) = \xi'(\det D)|\det D|^{r}\varphi(ZD), \quad D \in \mathrm{GL}_t(\mathbb{C}).$$

Here $|\cdot|$ is ordinary complex absolute value. The original factor splitting is the restriction of its doubled splitting to the first summand, whose image lies in the first metaplectic factor.

Proof. [170], Sections 3.2–3.3 and Lemma 3.1(a),(b), equations (3.7),(3.10), pp.47–49. The displayed first-summand restriction is the definition in Section 3.3, p.47. □

Theorem A.5.9. *Let A be a second countable locally compact abelian group with a Haar measure, and let $\widehat{A}$ be its Pontryagin dual. On $L^2(A)$ define the unitary operators*

$$(U_a f)(x) = f(x + a), \qquad (V_\chi f)(x) = \chi(x) f(x) \qquad (a \in A,\ \chi \in \widehat{A}),$$

so that $U_a V_\chi = \chi(a)\, V_\chi U_a$.

(i) A bounded operator on $L^2(A)$ that commutes with every U_a and every V_χ is a scalar multiple of the identity.

(ii) Let $a \mapsto U'_a$ and $\chi \mapsto V'_\chi$ be strongly continuous unitary representations of A and of $\widehat{A}$ on a separable Hilbert space $\mathcal{H}$ such that $U'_a V'_\chi = \chi(a)\, V'_\chi U'_a$ for all a, χ. Then there are a separable Hilbert space $\mathcal{K}$ and a unitary isomorphism $\Phi\colon \mathcal{H} \to L^2(A) \mathbin{\widehat{\otimes}} \mathcal{K}$ with

$$\Phi U'_a = (U_a \otimes 1)\Phi, \qquad \Phi V'_\chi = (V_\chi \otimes 1)\Phi \qquad (a \in A,\ \chi \in \widehat{A}).$$

Proof. This is the theorem of Stone and von Neumann in the generality of locally compact abelian groups, due to Mackey [195]; it is also the starting point of Weil's treatment of the Heisenberg group of $A \times \widehat{A}$ [322]. Notation: Mackey's pair of commuting representations of A and $\widehat{A}$ satisfying the Weyl relation is the pair (U', V') of (ii); his canonical pair is (U, V) on $L^2(A)$. □

Theorem A.5.10. *Let k be a nonarchimedean local field of characteristic 0, K a quadratic field extension of k with its nontrivial automorphism, η the quadratic character of $k^\times$ attached to K, and ψ a nontrivial additive character of k. Let V be a hermitian and W a skew-hermitian K-space of finite dimensions m and n; let $-W$ be W with the negated form, $W^\square = W \oplus (-W)$, and $\iota\colon \mathrm{U}(W) \times \mathrm{U}(W) \to \mathrm{U}(W^\square)$ the block-diagonal inclusion. Let χ be a character of $K^\times$ with $\chi|_{k^\times} = \eta^m$. Fix a complete polarisation $\mathbb{X} \oplus \mathbb{Y}$ of the symplectic k-space $\mathbb{W} = \mathrm{Res}_{K/k}(V \otimes_K W)$; it is also a complete polarisation of $\mathbb{W}^- = \mathrm{Res}_{K/k}(V \otimes_K (-W))$ (the same space with the negated form), and $(\mathbb{X} \oplus \mathbb{X}) \oplus (\mathbb{Y} \oplus \mathbb{Y})$ is one of $\mathbb{W}^\square = \mathbb{W} \oplus \mathbb{W}^-$. For $W' \in \{W, W^\square\}$ let $\omega_{W'}$ be the Weil*

representation ω_ψ of $\mathrm{U}(W')$ in the Schrödinger model on $\mathcal{S}(\mathbb{Y})$, resp. $\mathcal{S}(\mathbb{Y}\oplus\mathbb{Y})$, pulled back by Kudla's splitting of the metaplectic cover over $\mathrm{U}(W')$ attached to χ. Then for all $g_1, g_2 \in \mathrm{U}(W)$ and $f_1, f_2 \in \mathcal{S}(\mathbb{Y})$,

$$\omega_{W^\square}(\iota(g_1,g_2))\,(f_1\otimes\overline{f_2}) = \chi(\det g_2)\,(\omega_W(g_1)f_1)\otimes\overline{(\omega_W(g_2)f_2)},$$

where $(f\otimes\overline{f'})(y_1,y_2) = f(y_1)\overline{f'(y_2)}$; that is, $\omega_{W^\square}\circ\iota \cong \omega_W \boxtimes \big((\chi\circ\det)\otimes\omega_W^\vee\big)$, the contragredient being realised by complex conjugation.

Proof. Kudla's splittings of the metaplectic cover over unitary groups [172]; the doubling identity, with the inverse-character correction of the second splitting and the conjugated centre of the second metaplectic factor, is [121], Lemma 1.1, Lemma 2.1(i) and Proposition 2.2(i),(ii). Map of notation: $\mathbb{W}^-$ carries the Weil representation for the negated form, which on $\mathcal{S}(\mathbb{Y})$ is the complex conjugate of ω_ψ; the product of the two Schrödinger models is the Schrödinger model of $\mathbb{W}^\square$ for $\mathbb{Y}\oplus\mathbb{Y}$; the factor $\chi(\det g_2)$ is the product of the splitting of $\mathrm{U}(-W)$ attached to χ with the conjugate of the splitting of $\mathrm{U}(W)$. □

Theorem A.5.11. *Let $n \geq 1$, let $h(u,v) = \sum_k u_k\bar{v}_k$ on $\mathbb{C}^n$, and let W be the real symplectic space $\mathbb{C}^n$ with $\omega = \operatorname{Im} h$. Every element of the symplectic Lie algebra $\mathfrak{sp}(W)$ is uniquely of the form $X_{m,n}\colon u \mapsto mu + n\bar{u}$ with $m,n \in M_n(\mathbb{C})$, $m^* = -m$, $n^{\mathsf{T}} = n$; the maximal compact subgroup $K = \{u \mapsto ku : k \in \mathrm{U}(n)\}$ of $\mathrm{Sp}(W)$ has Lie algebra $\{X_{m,0}\}$. Let $H(W) = W\times\mathbb{R}$ with $(w,s)(w',s') = (w+w', s+s'+\omega(w,w'))$, let $\mathrm{Mp}(W)\to\mathrm{Sp}(W)$ be the metaplectic double cover and $\tilde{K}$ the preimage of K. Let $\mathcal{F}_n$ be the **Fock space** of entire functions f on $\mathbb{C}^n$ with*

$$\|f\|^2 = \pi^{-n}\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2}\,d\lambda(z) < \infty,$$

and $\mathcal{P}\subset\mathcal{F}_n$ the polynomials. Then:

(FK1) the monomials $z^\alpha/\sqrt{\alpha!}$ ($\alpha\in\mathbb{N}^n$) form an orthonormal basis of $\mathcal{F}_n$; on $\mathcal{P}$ the adjoint of multiplication by z_k is $\partial_k = \partial/\partial z_k$;

(FK2) the formula $\rho(w,s)f(z) = e^{is}\,e^{-|w|^2/2}\,e^{\sum_j w_j z_j}\,f(z-\bar{w})$ defines an irreducible unitary representation ρ of $H(W)$ on $\mathcal{F}_n$ with central character $s\mapsto e^{is}$; $\mathcal{P}$ consists of smooth vectors, and $d\rho(w) = L(w) = \sum_j w_j z_j - \sum_j \bar{w}_j\partial_j$ on $\mathcal{P}$, so that $[L(w),L(w')] = 2i\,\omega(w,w')$;

(FK3) (Stone–von Neumann) every irreducible unitary representation of $H(W)$ with central character $s\mapsto e^{is}$ is unitarily equivalent to ρ;

*(FK4) (Segal–Shale–Weil) there is a unique unitary representation ω_{osc} of $\mathrm{Mp}(W)$ on $\mathcal{F}_n$ with $\omega_{\mathrm{osc}}(\tilde{g})\rho(w,s)\omega_{\mathrm{osc}}(\tilde{g})^{-1} = \rho(gw,s)$ for all $\tilde{g}\in\mathrm{Mp}(W)$ over $g\in\mathrm{Sp}(W)$ and $(w,s)\in H(W)$ (the **oscillator representation**);*

(FK5) $\mathcal{P}$ is the space of $\tilde{K}$-finite vectors of ω_{osc}, it consists of smooth vectors, and for every $X_{m,n}\in\mathfrak{sp}(W)$ and $P\in\mathcal{P}$

$$d\omega_{\mathrm{osc}}(X_{m,n})P = \sum_{k,l}\Big(m_{kl}\,z_k\partial_l P + \tfrac{1}{2}n_{kl}\,z_k z_l P - \tfrac{1}{2}\bar{n}_{kl}\,\partial_k\partial_l P\Big) + \tfrac{1}{2}\operatorname{tr}(m)\,P;$$

on every smooth vector f of ω_{osc}, $d\omega_{\mathrm{osc}}(X_{m,n})f$ is the same differential operator applied to the entire function f;

*(FK6) the **vacuum** $\{f\in\mathcal{P} : \partial_j f = 0\ (1\leq j\leq n)\}$ is the line $\mathbb{C}\cdot 1$; $\tilde{K}$ acts on it by a character χ with $d\chi(X_{m,0}) = \frac{1}{2}\operatorname{tr}(m)$, so $\chi^2 = \det$ (through $\tilde{K}\to K\cong\mathrm{U}(n)$).*

Proof. Bargmann [14] (the Fock space, (FK1), the Fock realisation of the canonical commutation relations); the Stone–von Neumann theorem (FK3), [195] and [83, Ch. 1]; Segal [257], Shale [268] and Weil [322] for (FK4); the Fock model of the oscillator representation and its infinitesimal action (FK5)–(FK6) in [157], §§1–2, and in Folland [83], Ch. 1 and Ch. 4; see also [136]. Map of notation: the creation operators are z_k, the annihilation operators ∂_k; the complex structure of W is multiplication by i and $\omega = \operatorname{Im} h$; the normalisation of the Fock measure is chosen so that $[\partial_k, z_l] = \delta_{kl}$ and $z_k^* = \partial_k$.

Consistency of the normalisations. With $A(X)$ the right-hand side of (FK5): $X \mapsto A(X)$ is a Lie algebra homomorphism for the bracket $[X, X'] = XX' - X'X$ of real-linear maps (for $X = X_{m,0}$, $X' = X_{0,n}$ the bracket is $X_{0,mn+nm^{\intercal}}$; for $X = X_{0,n}$, $X' = X_{0,n'}$ it is $X_{n\bar{n}'-n'\bar{n},0}$, and the constants $\frac{1}{2}\operatorname{tr}$ arise from $[\partial_i\partial_j, z_k z_l]$), each $A(X)$ is skew-adjoint on $\mathcal{P}$, and $[A(X), L(w)] = L(Xw)$. Since $\mathcal{P}$ is irreducible under the operators $L(w)$ and $\mathfrak{sp}(W)$ is perfect, A is the only map with these commutators; hence the normalisation of (FK5) is forced by (FK2) and (FK4). The clause on smooth vectors follows from the one on $\mathcal{P}$: $\tilde{K}$-finite vectors are dense in the smooth vectors for their topology, $d\omega_{\mathrm{osc}}(X)$ is continuous there, and convergence in $\mathcal{F}_n$ implies locally uniform convergence of entire functions and of all their derivatives. □

Theorem A.5.12. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, G a connected complex Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{R}} \subseteq \mathfrak{g}$ a real form with Cartan involution θ, and $G_{\mathbb{R}}$ the connected subgroup of G with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Assume $G_{\mathbb{R}}$ has a compact Cartan subgroup $T_{\mathbb{R}}$, contained in a maximal compact subgroup $K_{\mathbb{R}}$; let $\mathfrak{t}$ be the complexified Lie algebra of $T_{\mathbb{R}}$, Φ the root system of $(\mathfrak{g}, \mathfrak{t})$, $W_{\mathbb{R}}$ the Weyl group of the compact roots, $\Lambda \subseteq i\mathfrak{t}_{\mathbb{R}}^*$ the lattice of differentials of characters of $T_{\mathbb{R}}$, $(\,\cdot\,,\cdot\,)$ the form on $i\mathfrak{t}_{\mathbb{R}}^*$ induced by the Killing form, Φ^+ a positive system with half-sum ρ, and $q = \frac{1}{2}\dim G_{\mathbb{R}}/K_{\mathbb{R}}$. Let $U_{\mathbb{R}}$ be the connected subgroup of G whose Lie algebra is a compact real form of $\mathfrak{g}$ stable under θ (extended complex-linearly), let ω be a left-invariant holomorphic differential form of top degree on G such that the measure $|\omega|$ induced on $U_{\mathbb{R}}$ has total mass 1, and let $\mu_{\mathbb{R}}$ be the measure $|\omega|$ induced on $G_{\mathbb{R}}$ (a Haar measure). Let $\nu \in \rho + \Lambda$ be regular and dominant for Φ^+, and let Θ_ν be the unique invariant eigendistribution on $G_{\mathbb{R}}$ that decays at infinity and whose restriction to the regular elements of $T_{\mathbb{R}}$ is*

$$\Theta_\nu|_{T_{\mathbb{R}}} = (-1)^q \frac{\sum_{w \in W_{\mathbb{R}}} \varepsilon(w)\, e^{w\nu}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Then:

(a) Θ_ν is the global character of a discrete series representation of $G_{\mathbb{R}}$; every discrete series character is of this form, and discrete series characters are determined by their restrictions to the regular elements of $T_{\mathbb{R}}$;

(b) if $G_{\mathbb{R}}$ is linear, the global character of Harish-Chandra's discrete series π_ν is Θ_ν;

(c) every discrete series representation of $G_{\mathbb{R}}$ with global character Θ_ν has formal degree, with respect to $\mu_{\mathbb{R}}$,

$$d = \prod_{\alpha \in \Phi^+} \frac{(\nu, \alpha)}{(\rho, \alpha)}.$$

Proof. (a) is Harish-Chandra's theorem as stated in [251], §3, Theorem (Harish-Chandra) on pp. 95–96 with the remarks following it, where $\nu = \lambda + \rho$ with $\lambda \in \Lambda$; the setting (G, $G_{\mathbb{R}}$, $T_{\mathbb{R}} \subseteq K_{\mathbb{R}}$, compact real forms allowed) is that of §3, pp. 90 and 94–95. (c) is Harish-Chandra's formula for the formal degree, [251], §4, p. 103 ("$d_i = \prod_{(\lambda+\rho,\alpha)>0}(\lambda + \rho, \alpha)/(\rho, \alpha)$" for $\lambda + \rho$

dominant for Φ^+, so that the index set is Φ^+), with the normalisation of Haar measure fixed on p. 100 (Haar measure on $U_{\mathbb{R}}$ of total mass one, expressed as the restriction of a holomorphic top-degree form on G, restricted to $G_{\mathbb{R}}$) and $d_i = \mu(\{i\})$ the formal degree of §3, p. 91. (b): for $G_{\mathbb{R}}$ linear, [131], Theorem 5.4 (1) (after Knapp, Theorem 12.7) gives the restriction of the global character of π_ν to $T_{\mathbb{R}}$ as $(-1)^q \sum_{w\in W_{\mathbb{R}}} \det(w) e^{w\nu}/D$ with $D = e^\rho \prod_{\alpha\in\Phi^+}(1-e^{-\alpha})$, $\Phi^+ = \{\alpha : (\nu,\alpha) > 0\}$ (Definition 4.13 and (5.1) there), which is the expression above; by (a) the character of the discrete series π_ν is therefore Θ_ν. □

Theorem A.5.13. *Let G be a connected linear semisimple Lie group, K a maximal compact subgroup and $T \subseteq K$ a compact Cartan subgroup, with Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0$ and complexifications $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$. Let Φ be the root system of $(\mathfrak{g},\mathfrak{t})$, Φ_c (compact roots, root space in $\mathfrak{k}$) and $\Phi_n = \Phi \setminus \Phi_c$, W_c the Weyl group of Φ_c, $P \subseteq i\mathfrak{t}_0^*$ the lattice of differentials of characters of T, $\langle\,,\rangle$ a W-invariant inner product on $i\mathfrak{t}_0^*$. For λ in $\rho + P$ regular for Φ (ρ the half-sum of the roots of any positive system) let π_λ be Harish-Chandra's discrete series with parameter λ, let Φ^+ be the positive system for which λ is dominant, $\Phi_c^+ = \Phi^+ \cap \Phi_c$, $\Phi_n^+ = \Phi^+ \cap \Phi_n$, and ρ_c, ρ_n the half-sums of Φ_c^+, Φ_n^+. For $\nu \in i\mathfrak{t}_0^*$ let $\mathcal{P}_n(\nu)$ be the number of families $(n_\beta)_{\beta\in\Phi_n^+}$ of nonnegative integers with $\nu = \sum_\beta n_\beta \beta$. Then for every $\mu \in \rho_c + P$ that is regular and dominant for Φ_c^+, the multiplicity in π_λ of the irreducible representation of K with highest weight $\mu - \rho_c$ is*

$$m_\lambda(\mu) = \sum_{w\in W_c} \varepsilon(w)\, \mathcal{P}_n(w\mu - \lambda - \rho_n),$$

$\varepsilon(w)$ the sign of w.

Proof. This is Blattner's conjecture, proved by Hecht and Schmid [127] for linear semisimple groups; in the parametrisation by Harish-Chandra parameters it is stated in this form in [11], Section 2.2, formula (1). □

Theorem A.5.14. *Let G be a connected semisimple Lie group with finite centre, K a maximal compact subgroup, and assume that G has a compact Cartan subgroup $T \subseteq K$. Let $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0$ be the Lie algebras, $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$ their complexifications, Φ the root system of $(\mathfrak{g},\mathfrak{t})$ and $(\,\cdot\,|\,\cdot\,)$ the inner product on $i\mathfrak{t}_0^*$ induced by the Killing form. A root $\alpha \in \Phi$ is* compact *if its root space lies in $\mathfrak{k}$ and* noncompact *otherwise. Let $\Lambda \subseteq i\mathfrak{t}_0^*$ be the lattice of differentials of characters of T, ρ the half-sum of the positive roots for some ordering, and for $\alpha \in \Phi$*

$$k(\alpha) = \frac{1}{4} \sum_{\beta\in\Phi} |(\alpha|\beta)|.$$

For a nonsingular $\lambda \in \Lambda + \rho$ (orthogonal to no root) let π_λ be the discrete series representation of G with Harish-Chandra parameter λ; by Harish-Chandra, $\pi_\lambda \cong \pi_\mu$ if and only if λ, μ are conjugate under the Weyl group of $(\mathfrak{k},\mathfrak{t})$, and every irreducible square-integrable unitary representation of G is equivalent to some π_λ. Let $\sigma(k\exp X) = \|X\|$ ($k \in K$, X in the orthogonal complement $\mathfrak{p}_0$ of $\mathfrak{k}_0$ for the Killing form, $\|\cdot\|$ the Killing-form norm on $\mathfrak{p}_0$). Then for every nonsingular $\lambda \in \Lambda + \rho$ and every $\kappa > 0$ the following are equivalent:

(i) $|(\lambda|\alpha)| \geq \kappa\, k(\alpha)$ for every noncompact root $\alpha \in \Phi$;

(ii) for every K-finite matrix coefficient c of π_λ there are real numbers $M, q > 0$ such that

$$|c(x)| \leq M\, \Xi(x)^{1+\kappa}\, (1+\sigma(x))^q \qquad (x \in G).$$

Proof. This is the Theorem of the Introduction of Miličić [199] (statement on p. 60; the implication (ii)⇒(i) is due to Trombi and Varadarajan). There Ξ and σ are Harish-Chandra's K-biinvariant functions, Ξ being the function of Theorem A.5.15. □

Theorem A.5.15. *Let G, K, $\mathfrak{a}$, Σ^+, m_α, $\mathfrak{a}^+$ be as in Theorem A.5.16, $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$, $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$, $N = \exp\mathfrak{n}$, and for $x \in G$ let $H(x) \in \mathfrak{a}$ be defined by the Iwasawa decomposition $x \in K\exp(H(x))N$. Harish-Chandra's function is*

$$\Xi(x) = \int_K e^{-\rho(H(xk))}\,dk \qquad (x \in G),$$

dk the Haar probability measure of K. Then:

(HX1) Ξ is continuous, $\Xi(k_1xk_2) = \Xi(x)$ for $k_1, k_2 \in K$, and $0 < \Xi(x) \le \Xi(1) = 1$;

(HX2) $\int_K \Xi(xky)\,dk = \Xi(x)\,\Xi(y)$ for all $x, y \in G$;

(HX3) there are $C_\Xi > 0$ and $d_\Xi \ge 0$ such that for every $H \in \overline{\mathfrak{a}^+}$

$$e^{-\rho(H)} \le \Xi(\exp H) \le C_\Xi\, e^{-\rho(H)}\,(1 + \|H\|)^{d_\Xi},$$

for any norm $\|\cdot\|$ on $\mathfrak{a}$ (the constant depends on the norm).

Proof. Harish-Chandra [118] (the function $\Xi = \varphi_0$ and its estimate); Gangolli and Varadarajan [95], §4.6 (Theorem 4.6.4: the upper bound, stated for real reductive groups of Harish-Chandra's class, which contains the closed connected linear groups stable under conjugate transpose); Wallach [315], §4.5 (properties of Ξ, both bounds); Knapp [163], Ch. VII. (HX2) is the functional equation characterising spherical functions, $\Xi = \varphi_0$ being the elementary spherical function with parameter 0 (Helgason [129], Ch. IV, §2). In (HX1), $\Xi \le 1$ because $\Xi(x) = \langle \pi(x)v, v\rangle$ for the unit K-fixed vector v of the unitary principal series $\mathrm{Ind}_{MAN}^{G}(1)$. Notation: the Iwasawa convention $G = KAN$ with H the A-projection of xk and ρ the half sum of the roots of $\mathfrak{n}$. □

Theorem A.5.16. *Let $G \subseteq \mathrm{GL}_N(\mathbb{C})$ be a closed connected subgroup with $g^* \in G$ for all $g \in G$, let $K = G \cap \mathrm{U}(N)$, let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{p} = \{X \in \mathfrak{g} : X^* = X\}$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace, $\Sigma \subseteq \mathfrak{a}^*$ the set of restricted roots (the nonzero α with $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X\ \forall H \in \mathfrak{a}\} \ne 0$), $m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$, Σ^+ a system of positive restricted roots and $\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0\ \forall \alpha \in \Sigma^+\}$. Then $G = K\exp(\overline{\mathfrak{a}^+})K$, and for every Haar measure dg of G there is a constant $c > 0$ such that for every Borel function $f\colon G \to [0, \infty]$*

$$\int_G f(g)\,dg = c\int_K \int_{\mathfrak{a}^+} \int_K f(k_1\exp(H)k_2) \prod_{\alpha\in\Sigma^+} (\sinh\alpha(H))^{m_\alpha}\,dk_1\,dH\,dk_2,$$

where dk_1, dk_2 are the Haar probability measure of K and dH is a Lebesgue measure on $\mathfrak{a}$.

Proof. Harish-Chandra; see Helgason [129], Chapter I, §5, no. 2 (integral formulas for the Cartan decomposition, the noncompact case), Theorem 5.8, for connected semisimple groups with finite centre, and Knapp [164], Chapter VIII, §3 (decompositions of Haar measure), for reductive groups in the sense of Chapter VII, §2, which include every closed linear group stable under conjugate transpose with finitely many components (op. cit. VII.2). The decomposition $G = K\exp(\overline{\mathfrak{a}^+})K$ is the KAK decomposition (Knapp, op. cit., Theorem 7.39). Both sides are Radon measures in f that agree on $C_c(G)$, hence on all Borel $f \ge 0$. Notation: K is the maximal compact subgroup fixed by the Cartan involution $g \mapsto (g^*)^{-1}$, $\mathfrak{p}$ its -1 eigenspace. □

Theorem A.5.17. *Let $j \in \{\pm 1\}^n$, $G = \mathrm{U}(J_j)$, $K = K_j$, let $\mathfrak{g}_0 = \{X \in M_n(\mathbb{C}) : X^* J_j + J_j X = 0\}$ be the Lie algebra of G, so that $M_n(\mathbb{C}) = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, and let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. A vector $v \in \mathcal{H}$ is K-finite if $\pi(K)v$ spans a finite-dimensional subspace; $\mathcal{H}_K$ denotes the space of K-finite vectors.*

1. *(Admissibility.) For every irreducible representation δ of K, the closed span $\mathcal{H}_\delta$ of all K-subrepresentations of $\mathcal{H}$ isomorphic to δ is finite-dimensional.*
2. *(Analyticity.) Every $v \in \mathcal{H}_K$ is an analytic vector: $g \mapsto \pi(g)v$ is a real-analytic map $G \to \mathcal{H}$. In particular v is smooth, $d\pi(X)v = \frac{d}{dt}\pi(e^{tX})v|_{t=0}$ exists for $X \in \mathfrak{g}_0$, and $d\pi(X)v \in \mathcal{H}_K$.*
3. *(Submodules.) Extend $d\pi$ complex-linearly to $M_n(\mathbb{C})$. If $\mathcal{W} \subseteq \mathcal{H}_K$ is a subspace stable under $\pi(K)$ and under $d\pi(Z)$ for all $Z \in M_n(\mathbb{C})$, then its closure is $\pi(G)$-stable; hence $\mathcal{W} = 0$ or $\mathcal{W}$ is dense in $\mathcal{H}$.*

Proof. [117]; see [163], Chapter VIII (admissibility of irreducible unitary representations of linear connected reductive groups, analyticity of K-finite vectors of admissible representations, and the correspondence between closed invariant subspaces and $(\mathfrak{g}, K)$-submodules). $\mathrm{U}(J_j) \cong \mathrm{U}(p, q)$ is a linear connected reductive group in the sense of that book, with maximal compact subgroup K_j. Map of notation: $\mathcal{H}_K$ is the Harish-Chandra module of π. □

Theorem A.5.18. *Let G be a unimodular locally compact group with Haar measure μ, and let $(\sigma, \mathcal{H})$ be an irreducible strongly continuous unitary representation of G such that $g \mapsto \langle b_0, \sigma(g)a_0\rangle$ lies in $L^2(G, \mu)$ for some nonzero $a_0, b_0 \in \mathcal{H}$. Then every matrix coefficient $g \mapsto \langle b, \sigma(g)a\rangle$ $(a, b \in \mathcal{H})$ lies in $L^2(G, \mu)$, and there is a real number $d_\sigma > 0$, the **formal degree** of σ with respect to μ, such that for all $a, b, a', b' \in \mathcal{H}$*

$$\int_G \langle b, \sigma(g)a\rangle \, \overline{\langle b', \sigma(g)a'\rangle} \, d\mu(g) = \frac{\langle a', a\rangle \, \langle b, b'\rangle}{d_\sigma},$$

the integrand being μ-integrable. Replacing μ by $\lambda\mu$ $(\lambda > 0)$ replaces d_σ by d_σ/λ.

Proof. Godement, Sur les relations d'orthogonalité de V. Bargmann I, II, C. R. Acad. Sci. Paris 225 (1947); in textbook form [163], Chapter IX (square-integrable representations), where it is stated with the inner product linear in the first argument: $\int_G (\sigma(g)a, b)\overline{(\sigma(g)a', b')}\, dg = d_\sigma^{-1}(a, a')\overline{(b, b')}$. Map of notation: $(x, y) = \langle y, x\rangle$, so $(\sigma(g)a, b) = \langle b, \sigma(g)a\rangle$, $(a, a') = \langle a', a\rangle$ and $\overline{(b, b')} = \langle b, b'\rangle$. Integrability is that of a product of two square-integrable functions. For a real reductive group whose centre is compact (for example a finite product of groups $\mathrm{U}(p, q)$), the discrete series representations are exactly the irreducible unitary representations satisfying the hypothesis, and d_σ is Harish-Chandra's formal degree [119]; for a group with non-compact centre the statement applies to G/Z (square integrability modulo the centre), which is not the form stated here. □

Theorem A.5.19. *For $k \geq 1$ and a nonincreasing $\lambda \in \mathbb{Z}^k$ put*

$$d_k(\lambda) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Let K be a field of characteristic 0 and $\nu \in \mathbb{Z}^k$ nonincreasing with $\nu_k \geq 0$. Let $\nabla_K(\nu)$ be the K-vector space of polynomials P in the entries of a $k \times k$ matrix of indeterminates X that are homogeneous of degree ν_b in the entries of column b for each b, and satisfy

$$P(X_{\bullet b} + T\, X_{\bullet a}, (X_{\bullet c})_{c \neq b}) = P(X) \qquad (1 \leq a < b \leq k)$$

as a polynomial identity in a further indeterminate T (adding a multiple of an earlier column to a later column does not change P). Let $\mathrm{GL}_k(K)$ act on $\nabla_K(\nu)$ by $(g\cdot P)(X)=P(g^{\mathsf T}X)$. Then $\nabla_K(\nu)$ is an irreducible rational GL_k-module of highest weight ν with respect to the diagonal torus and the upper triangular Borel subgroup, and

$$\dim_K \nabla_K(\nu) = d_k(\nu).$$

Proof. The space of polynomial functions P on $k\times k$ matrices with $P(Xb)=\prod_j b_{jj}^{\nu_j}P(X)$ for all invertible upper triangular b (equivalently: homogeneous of degree ν_b in column b and invariant under adding multiples of earlier columns to later ones) is the induced module $H^0(\nu)$ of GL_k in its standard realisation on the coordinate ring ([151], Part II, Chapter 2; the polynomial functions on M_k give all of it because $\nu_k\geq 0$), which in characteristic 0 is irreducible of highest weight ν (with the action $(g\cdot P)(X)=P(g^{\mathsf T}X)$ the product $\prod_j D_j^{\nu_j-\nu_{j+1}}$ of powers of the leading principal minors D_j, with $\nu_{k+1}=0$, is a highest weight vector: for upper unitriangular g the matrix $g^{\mathsf T}X$ arises from X by adding multiples of earlier rows to later rows, which fixes every D_j, and a diagonal t multiplies D_j by $t_1\cdots t_j$), and its dimension is given by Weyl's dimension formula ([92], Lecture 15 and Theorem 24.3, or [151], II.5.10). Notation: d_k is Weyl's product $\prod_{\alpha>0}\langle\nu+\rho,\alpha^\vee\rangle/\langle\rho,\alpha^\vee\rangle$ for GL_k with positive roots e_i-e_j, $i<j$. □

Theorem A.5.20. *Let $N\geq 1$ and let $\mu\in\mathbb{Z}^N$ be nonincreasing, $\mu_1\geq\cdots\geq\mu_N$. Let χ_μ be the character of the irreducible continuous representation of the unitary group $\mathrm{U}(N)$ of highest weight μ, for the diagonal torus and the Borel subgroup of upper triangular matrices. If $x_1,\dots,x_N\in\mathbb{C}$ have modulus one and are pairwise distinct, then $\det(x_j^{N-i})_{i,j=1}^N=\prod_{i<j}(x_i-x_j)\neq 0$ and*

$$\chi_\mu\bigl(\mathrm{diag}(x_1,\dots,x_N)\bigr)=\frac{\det(x_j^{\mu_i+N-i})_{i,j=1}^N}{\det(x_j^{N-i})_{i,j=1}^N}.$$

Proof. H. Weyl's character formula for $\mathrm{U}(N)$, equivalently for the polynomial (after a power of the determinant) representations of $\mathrm{GL}_N(\mathbb{C})$: the character at $\mathrm{diag}(x)$ is the Schur function $s_\mu(x)$, the quotient of the alternant of the exponents $\mu+\delta$, $\delta=(N-1,\dots,1,0)$, by the Vandermonde determinant [92] (Lecture 24, and Lecture 6 with Appendix A for GL_N), [91] (Chapter 8). The irreducible continuous representations of $\mathrm{U}(N)$ are the restrictions of the irreducible rational representations of $\mathrm{GL}_N(\mathbb{C})$ (Weyl's unitary trick), with the same highest weights; for μ with negative entries one multiplies by $\det^{-\mu_N}$, which multiplies both sides by $(x_1\cdots x_N)^{\mu_N}$. Notation: χ_μ is the trace of the representation; the Vandermonde identity $\det(x_j^{N-i})=\prod_{i<j}(x_i-x_j)$ is classical. □

Theorem A.5.21. *Let K be a compact Hausdorff group with Haar probability measure dk, and for every irreducible continuous representation π of K on a finite-dimensional complex vector space let $\chi_\pi=\operatorname{tr}\pi$ be its character. Then:*

(i) for irreducible π,π', $\int_K \chi_\pi(k)\,\overline{\chi_{\pi'}(k)}\,dk$ is 1 if $\pi\cong\pi'$ and 0 otherwise;

(ii) if $f\in L^2(K)$ satisfies $f(yky^{-1})=f(k)$ for every $y\in K$ and almost every k, and $\int_K f(k)\,\overline{\chi_\pi(k)}\,dk=0$ for every irreducible π, then $f=0$ almost everywhere.

Equivalently, the characters of the irreducible representations, one for each isomorphism class, form an orthonormal basis of the closed subspace of class functions in $L^2(K)$.

Proof. Peter–Weyl theorem and Schur orthogonality for compact groups: [163], Chapter I, §5 (the Peter–Weyl theorem and its corollary on class functions; (i) is Schur's orthogonality of characters). Notation: χ_π is the trace of π (any continuous finite-dimensional representation of a compact group is unitary for some inner product, so the unitarity assumed there is no restriction). $\square$

Theorem A.5.22. *Let G be a connected semisimple Lie group with finite centre, $G = KAN$ an Iwasawa decomposition, M the centraliser of A in K, and for $x \in G$ write $x = \kappa(x)\exp(H(x))n(x)$ with $\kappa(x) \in K$, $H(x) \in \mathfrak{a}$, $n(x) \in N$. Let $\rho \in \mathfrak{a}^*$ be half the sum of the positive restricted roots (those occurring in the Lie algebra of N), counted with multiplicity, and dk the Haar probability measure of K. Then for every continuous function f on K with $f(km) = f(k)$ for $k \in K$, $m \in M$, and every $g \in G$,*

$$\int_K f(\kappa(gk))\, e^{-2\rho(H(gk))}\, dk = \int_K f(k)\, dk.$$

Proof. [129], Chapter I, Lemma 5.19 (the map $kM \mapsto \kappa(gk)M$ of K/M has Jacobian $e^{-2\rho(H(gk))}$). It is the identity that makes the Poisson kernel $e^{-2\rho(H(gk))}$ a probability density on K/M. $\square$

Theorem A.5.23. *Let $m \geq 1$, let E_{ij} $(1 \leq i, j \leq m)$ be the standard basis of $\mathfrak{gl}_m = \mathfrak{gl}_m(\mathbb{C})$ and $\mathfrak{gl}_{m-1} \subseteq \mathfrak{gl}_m$ the span of the E_{ij} with $i, j \leq m-1$. Finite-dimensional irreducible $\mathfrak{gl}_m$-modules correspond bijectively (up to isomorphism) to tuples $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ with $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$; the module $L(\lambda)$ contains a non-zero vector ξ, unique up to a scalar, with $E_{ii}\xi = \lambda_i \xi$ for all i and $E_{ij}\xi = 0$ for $i < j$ (the highest weight λ).*

(i) (Branching.) For $m \geq 2$ the restriction of $L(\lambda)$ to $\mathfrak{gl}_{m-1}$ is isomorphic to the direct sum of pairwise inequivalent irreducible modules $L'(\nu)$ over the tuples $\nu = (\nu_1, \dots, \nu_{m-1})$ with $\lambda_i - \nu_i \in \mathbb{Z}_{\geq 0}$ and $\nu_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq m-1$, each with multiplicity one.

(ii) (Capelli determinant.) Put $E(u) = u\,1 + E$ and

$$A_m(u) = \sum_{\sigma \in S_m} \operatorname{sgn}\sigma\; E(u)_{\sigma(1)1}\, E(u-1)_{\sigma(2)2} \cdots E(u-m+1)_{\sigma(m)m} \in \mathrm{U}(\mathfrak{gl}_m)[u].$$

Then $A_m(u)$ acts on $L(\lambda)$ as the scalar $\prod_{i=1}^m (u + \lambda_i - i + 1)$, for every $u \in \mathbb{C}$.

Proof. Quoted: Molev [210] (Handbook of Algebra, Vol. 4). The highest-weight parametrisation is stated at the opening of Section 2 (formula (2.1) and the sentence after it); item (i) is Theorem 2.1 with the betweenness conditions (2.2) (the branching rule of Gelfand–Tsetlin and Zhelobenko); item (ii) is formula (2.39) of Theorem 2.20 for the top row, where $A_m(u) = E(u)^{1\cdots m}_{1\cdots m}$ is the quantum minor (2.35) (Molev notes before Proposition 2.22 that $A_m(u)$ is the Capelli determinant) and $l_{mi} = \lambda_{mi} - i + 1$, $\lambda_{mi} = \lambda_i$ for every Gelfand–Tsetlin pattern of $L(\lambda)$; as the Gelfand–Tsetlin vectors form a basis of $L(\lambda)$, $A_m(u)$ acts as the stated scalar. The same statements are in Molev's monograph [211] (Chapter 7 and Section 1.6 with the evaluation homomorphism). Notation: $A_m(u)$ as displayed in (ii) is the quantum minor (2.35) of Molev written in column form. $\square$

Theorem A.5.24. *Let $n \geq 1$ and $w = (w_1, \dots, w_n)$ indeterminates. For $e = (e_1, \dots, e_n) \in \mathbb{N}^n$ let $a_e(w) = \det(w_i^{e_j})_{1 \leq i,j \leq n}$, and let $\delta_n = (n-1, n-2, \dots, 1, 0)$, so that $a_{\delta_n}(w) = \prod_{1 \leq i < i' \leq n}(w_i - w_{i'})$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n parts, $|\lambda| = \sum_i \lambda_i$.*

(a) $a_{\lambda+\delta_n}$ *is divisible by* a_{δ_n} *in* $\mathbb{Z}[w]$*, and the quotient, the* Schur polynomial

$$s_\lambda(w) = a_{\lambda+\delta_n}(w)/a_{\delta_n}(w),$$

equals $\sum_T w^T$*, the sum over the semistandard Young tableaux* T *of shape* λ *with entries in* $\{1,\dots,n\}$*, where* $w^T = \prod_i w_i^{\#\{\text{entries of } T \text{ equal to } i\}}$*. In particular* s_λ *is homogeneous of degree* $|\lambda|$ *and all its coefficients are non-negative integers.*

(b) (Hook-content formula.) $s_\lambda(1,\dots,1) = \prod_{x\in\lambda} \dfrac{n+c(x)}{h(x)}$*, where for the cell* $x=(i,j)$ *of the diagram of* λ *(row* i*, column* j*) the content is* $c(x) = j-i$ *and the hook length is* $h(x) = \lambda_i - j + \lambda'_j - i + 1$*,* λ' *the conjugate partition.*

Proof. Classical; see [194], Chapter I: (a) is the definition of s_λ as a quotient of alternants, Ch. I, (3.1), together with the combinatorial description, Ch. I, (5.12) (Jacobi's bialternant formula and Littlewood's tableau formula); (b) is Ch. I, §3, Example 4, at $q=1$ (Stanley's hook-content formula). Notation: w plays the role of the negatives of the entries of one block of a weight in the application; s_λ is used only inside the calculus of the Chern-root ring. □

Theorem A.5.25. *Let* V *and* W *be finite-dimensional complex vector spaces. For a partition* λ*, let* S_λ *be the Schur functor with highest weight* λ*, and put* $S_\lambda V = 0$ *when* λ *has more than* $\dim V$ *nonzero parts. For every integer* $d \geq 0$ *there is an isomorphism of representations of* $\mathrm{GL}(V)\times\mathrm{GL}(W)$

$$\mathrm{Sym}^d(V\otimes W) \simeq \bigoplus_{\lambda\vdash d} S_\lambda V \otimes S_\lambda W.$$

Each nonzero summand is irreducible and occurs with multiplicity one. Equivalently, for the action $((g,h)\cdot P)(Z) = P(g^t Z h)$ *on polynomials in an* a *by* b *matrix, the summand of shape* λ *is spanned by the matrix coefficients of the rectangular Schur matrix* $S_\lambda(Z)$ *and is isomorphic to* $S_\lambda\mathbb{C}^a \otimes S_\lambda\mathbb{C}^b$*.*

Proof. This is the classical polynomial Cauchy decomposition, or $(\mathrm{GL}(V),\mathrm{GL}(W))$ duality; see [135] and, for the characteristic-free form with Schur functors, [3], Section III.1. The convention for the displayed polynomial action uses the coordinate variables as $V\otimes W$. With the pullback action on the coordinate ring of $\mathrm{Hom}(W,V)$ one instead replaces the appropriate factor by its dual. The rectangular Schur matrix is the functorial map on the Young realization, so its coefficients give the indicated copy of the tensor product. No oscillator, theta-functional or derivative normalization is included in this theorem. □

Lemma A.5.26. *Let* A *be a commutative ring, let* $\mathcal{R}$ *and* $\mathcal{K}$ *be finite totally ordered sets, and let* $A[x]$ *be the polynomial ring over* A *in the indeterminates* $x_{a,c}$ *(*$a\in\mathcal{R}$*,* $c\in\mathcal{K}$*). For a partition* λ *a tableau of shape* λ *with entries in a totally ordered set is* standard *if its entries increase weakly along each row and strictly down each column. For tableaux* S *with entries in* $\mathcal{R}$ *and* T *with entries in* $\mathcal{K}$ *of the same shape* λ*, with columns* $S^{(1)},\dots,S^{(\lambda_1)}$ *and* $T^{(1)},\dots,T^{(\lambda_1)}$ *of lengths* $\lambda'_1 \geq \cdots \geq \lambda'_{\lambda_1}$*, the* bideterminant *is*

$$(S\,|\,T) = \prod_{k=1}^{\lambda_1} \det\bigl(x_{S_i^{(k)},T_l^{(k)}}\bigr)_{1\leq i,l\leq\lambda'_k},$$

and $(\emptyset\,|\,\emptyset)=1$*. Then the bideterminants* $(S\,|\,T)$*, where* λ *runs over all partitions and* S*,* T *over the standard tableaux of shape* λ *with entries in* $\mathcal{R}$ *and* $\mathcal{K}$*, form a basis of the free* A*-module* $A[x]$*.*

Proof. This is the standard basis theorem of Doubilet, Rota and Stein [73], in the characteristic-free form of [71] and [67] (§1), where it is proved over an arbitrary commutative ring by the straightening law. Map of notation: those papers write a bitableau with the minors along the rows of the diagram (row-strict tableaux); transposing the diagram gives the column convention used here, in which every column of $(S|T)$ is one minor. $\square$

Theorem A.5.27. *Let k_0 be a finite field with q elements and $r \geq 1$. For every label c of degree r over k_0, let $d_1, \dots, d_t$ be the degrees of the monic irreducible factors of c. Then:*

(a) there is $\epsilon_c \in \{\pm 1\}$ such that $\epsilon_c \Xi_c$ is the character of an irreducible complex representation of $\mathrm{GL}_r(k_0)$;

(b) if $c \neq c'$ are labels of degree r, the irreducible characters $\epsilon_c \Xi_c$ and $\epsilon_{c'} \Xi_{c'}$ are distinct;

(c) $$|\Xi_c(1)| = \frac{\prod_{i=1}^{r}(q^i - 1)}{\prod_{j=1}^{t}(q^{d_j} - 1)}.$$

Proof. [106]: the characters of $\mathrm{GL}_r(\mathbb{F}_q)$ attached to a squarefree polynomial (each irreducible factor carrying the partition (1)), their irreducibility and the parametrisation of the irreducible characters (Section 8), and the degree formula. Equivalently, in the language of Deligne and Lusztig ([69], Theorem 6.8 and Corollary 7.2): $\epsilon_c \Xi_c = \pm R_T^{\theta}$ for the maximal torus $T \cong \prod_j \mathbb{F}_{q^{d_j}}^{\times}$ and the character θ obtained from the roots of c and the fixed faithful character, which is in general position because c is squarefree with nonzero roots; $R_T^{\theta}(1) = \pm |\mathrm{GL}_r(\mathbb{F}_q)|_{p'}/|T|$. Notation: Ξ_c is Green's character of the label c; $|\mathrm{GL}_r(\mathbb{F}_q)|_{p'} = \prod_{i=1}^{r}(q^i - 1)$, $|T| = \prod_j (q^{d_j} - 1)$. $\square$

Theorem A.5.28. *Let k_0 be a finite field, $r \geq 1$, and c a label of degree r over k_0 that is irreducible in $k_0[X]$. Then Ξ_c or $-\Xi_c$ is the character of an irreducible cuspidal representation of $\mathrm{GL}_r(k_0)$, and e_c is a central idempotent. Consequently, for every proper parabolic subgroup of $\mathrm{GL}_r(k_0)$ with unipotent radical U and all $h, t \in \mathrm{GL}_r(k_0)$,*

$$\sum_{u \in U} e_c(h\,u\,t) = 0,$$

where $e_c(g)$ denotes the coefficient of g in e_c.

Proof. The first assertion is Green's theorem on the characters of GL_r over a finite field: the character attached to a single irreducible polynomial of degree r (with the partition (1)) is, up to the sign $(-1)^{r-1}$, an irreducible cuspidal ("discrete series") character [106]. For an irreducible character χ the element $\frac{\chi(1)}{|G|}\sum_g \overline{\chi(g)}\, g$ is the central idempotent of χ, and replacing χ by $-\chi$ does not change it; so e_c is that idempotent.

For the consequence let (π, X) realise $\chi = \pm\Xi_c$. Cuspidality means $X^{U'} = 0$ for the unipotent radical U' of every proper parabolic subgroup, that is $\sum_{u \in U'} \pi(u) = 0$. Given U, h, t, the group $U' = t^{-1}Ut$ is again such a radical, and $hut = (ht)(t^{-1}ut)$, so

$$\sum_{u \in U} \overline{\chi(hut)} = \overline{\mathrm{tr}\Big(\pi(ht) \sum_{u' \in U'} \pi(u')\Big)} = 0.$$

Multiplying by $\chi(1)/|G|$ (with the sign of $\pm$ absorbed as above) gives the claim. $\square$

Theorem A.5.29. *Let q be a prime power and $s \geq 1$, let $d_1, \dots, d_s \geq 1$, and let ρ_i be an irreducible cuspidal complex representation of $\mathrm{GL}_{d_i}(\mathbb{F}_q)$. Suppose $\rho_i \not\cong \rho_j$ whenever $i \neq j$ and $d_i = d_j$. Put $n = \sum_i d_i$, and let $P \leq \mathrm{GL}_n(\mathbb{F}_q)$ be the block upper triangular parabolic with Levi $\prod_i \mathrm{GL}_{d_i}(\mathbb{F}_q)$. Then*

$$\mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{F}_q)} \mathrm{Inf}_{\prod_i \mathrm{GL}_{d_i}(\mathbb{F}_q)}^{P} \bigotimes_i \rho_i$$

is irreducible.

More precisely, let $c = \prod_{i=1}^{s} f_i$ be a label over $\mathbb{F}_q$, factored into its distinct monic irreducible factors, and put $d_i = \deg f_i$. For each i, let ρ_{f_i} be the irreducible cuspidal representation of $\mathrm{GL}_{d_i}(\mathbb{F}_q)$ with character $\epsilon_{f_i}\Xi_{f_i}$, where $\epsilon_{f_i} \in \{1, -1\}$. Then there is a sign $\epsilon_c \in \{1, -1\}$ such that $\epsilon_c \Xi_c$ is the character of

$$\mathrm{Ind}_P^{\mathrm{GL}_{\deg c}(\mathbb{F}_q)} \mathrm{Inf}_{\prod_i \mathrm{GL}_{d_i}(\mathbb{F}_q)}^{P} \bigotimes_i \rho_{f_i},$$

where P is the block upper triangular parabolic with these block sizes. For $a \in \mathbb{F}_q^\times$, the representation ρ_{X-a} is one-dimensional and its order as a multiplicative character of $\mathbb{F}_q^\times$ is the multiplicative order of a.

Proof. The first assertion is the distinct-cuspidal specialization of [106], Section 8, Theorem 13, pp. 439–440. In Green's parametrization a cuspidal factor corresponds to one simplex with partition (1), and the hypothesis makes these simplexes distinct. Theorem 13 states that the corresponding circle product is the irreducible character with that partition-valued parameter. The circle product is induction from the block parabolic of the product character on its diagonal blocks, by the definition in the introduction, pp. 402–403.

For the label assertion, the polynomial-to-simplex correspondence of Section 7, p. 439, associates to every irreducible factor f_i its Frobenius simplex. A squarefree label assigns partition (1) to each of these distinct simplexes and the empty partition to all others. Thus the same displayed formula in Theorem 13 is exactly the stated parabolic induction; the signs merely replace the Green functions by their irreducible characters. There is no additional scalar or determinant twist. In degree one, write $a = \varepsilon^h$ for a generator ε of $\mathbb{F}_q^\times$. The degree-one instance of Green's formula in Theorem 14, p. 443, is the character $x \mapsto \theta(x)^h$, where $\theta(\varepsilon)$ is a primitive $(q-1)$-st root of unity compatible with the fixed naming character. Its order is $(q-1)/\gcd(h, q-1)$, which is exactly the order of a. □

Theorem A.5.30. *Let W be a finite-dimensional symplectic $\mathbb{Q}_p$-space, let*

$$H(W) = W \times \mathbb{Q}_p, \qquad (w,t)(w',t') = (w + w', t + t' + \tfrac{1}{2}\langle w, w' \rangle),$$

and fix a nontrivial continuous unitary character $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$. Fix a Schrödinger representation S_ψ with central character ψ. For every smooth complex representation M of $H(W)$ on which the centre acts by ψ, the evaluation map

$$S_\psi \otimes_{\mathbb{C}} \mathrm{Hom}_{H(W)}(S_\psi, M) \longrightarrow M, \qquad s \otimes T \longmapsto T(s)$$

is an isomorphism. The functors $M \mapsto \mathrm{Hom}_{H(W)}(S_\psi, M)$ and $N \mapsto S_\psi \otimes N$ are mutually inverse equivalences with complex vector spaces.

Proof. This is the fixed-central-character smooth Stone–von Neumann equivalence. See Sug Woo Shin, *Abelian varieties and Weil representations*, Theorem 4.15, its p-adic version in section 4.6, and the classical symplectic-coordinate identification in section 6.2. Over $\mathrm{Spec}\,\mathbb{C}$, quasi-coherent

modules are complex vector spaces. Pushing the centre through ψ identifies weight-one representations of the central $\mathbb{C}^\times$-extension with the displayed fixed-central-character representations. [277]. □

Theorem A.5.31. *All symbols introduced here are local to this statement and to the results that cite it. Let $k \geq a \geq 1$ be integers and let X be a random $k \times a$ complex matrix whose k rows are independent, each with density $\pi^{-a}\exp(-x^*x)$ with respect to Lebesgue measure on $\mathbb{C}^a = \mathbb{R}^{2a}$. Then $W = X^*X$ has, with respect to the measure $dW = \prod_i dW_{ii} \prod_{i<j} d(\operatorname{Re} W_{ij})\, d(\operatorname{Im} W_{ij})$ on hermitian $a \times a$ matrices, the density*

$$\frac{\det(W)^{k-a}\exp(-\operatorname{tr} W)}{\tilde{\Gamma}_a(k)} \quad \text{on positive definite } W, \quad 0 \quad \text{elsewhere}, \qquad \tilde{\Gamma}_a(k) = \pi^{a(a-1)/2}\prod_{j=0}^{a-1}\Gamma(k-j).$$

Proof. [104]: the complex Wishart distribution with $n = k$ degrees of freedom, dimension $p = a$ and covariance $\Sigma = I$ (rows $\mathcal{CN}(0, I)$, density $\pi^{-a}\det\Sigma^{-1}\exp(-x^*\Sigma^{-1}x)$), density $\det(W)^{n-p}\exp(-\operatorname{tr}\Sigma^{-1}W)/(\det(\Sigma)^n\tilde{\Gamma}_p(n))$ with $\tilde{\Gamma}_p(n) = \pi^{p(p-1)/2}\prod_{j=1}^{p}\Gamma(n-j+1)$. Notation map: $n \mapsto k$, $p \mapsto a$, $\prod_{j=1}^{a}\Gamma(k-j+1) = \prod_{j=0}^{a-1}\Gamma(k-j)$. □

Theorem A.5.32. *Let F be a nonarchimedean local field, $G = \mathrm{GL}_n(F)$ and $T \subset B$ the diagonal torus and upper Borel. Fix a uniformizer and smooth characters $\chi_i : \mathcal{O}_F^\times \to \mathbb{C}^\times$. Put $\chi = \bigotimes_i \chi_i$ and*

$$c_{ij} = \min\{c \geq 1 : \chi_i\chi_j^{-1}|_{1+\mathfrak{p}_F^c} = 1\}.$$

Let J_χ be generated by $T(\mathcal{O}_F)$ and

$$1 + \mathfrak{p}_F^{\lfloor c_{ij}/2 \rfloor}E_{ij} \quad (i < j), \qquad 1 + \mathfrak{p}_F^{\lceil c_{ij}/2 \rceil}E_{ij} \quad (i > j),$$

and let ρ_χ extend χ, trivially on these root groups. Let H_χ be the product of general linear groups on the coordinate classes with equal χ_i, with diagonal torus T, upper Borel B_χ and Iwahori I_χ. Extend χ_i to $\widetilde{\chi}_i$ by value one at the uniformizer.

Then (J_χ, ρ_χ) is a type for $[T, \widetilde{\chi}]$. There is a support-preserving, involution-preserving isomorphism

$$\mathcal{H}(H_\chi, I_\chi) \simeq \mathcal{H}(G, \rho_\chi)$$

and an induced exact equivalence from the category generated by ρ_χ to that generated by I_χ-fixed vectors. Support corresponds through the common extended affine Weyl group. The isomorphism can be chosen so that this equivalence sends

$$i_B^G(\widetilde{\chi}\nu) \;\longmapsto\; i_{B_\chi}^{H_\chi}(\nu)$$

for every unramified ν, with normalized induction. No restriction on the characteristic or residue characteristic of F is required.

Proof. These are the general-linear cases of [243, Theorems 6.3, 7.7, 8.2 and 9.4; Corollary 7.9; Remark 4.14]. Notation map: χ is Roche's character of ${}^0T$; J_χ, ρ_χ are his (J, ρ) from Definition 3.3; H_χ has root system $\{e_i - e_j : \chi_i = \chi_j\}$. The Weyl stabilizer is the product of the symmetric groups on those classes, so the component group is trivial. Both compact subgroups have Haar mass one. Theorem 9.4 uses the chosen extension and normalized induction; Remark 4.14 gives the unrestricted general-linear case. □

Theorem A.5.33. *Let K be a non-archimedean local field, let $\nu(g) = |\det g|_K$, and let ρ be an irreducible supercuspidal complex representation of $\mathrm{GL}_m(K)$. For $h \geq 1$, set $n = mh$ and let $\Delta = [\rho, \nu^{h-1}\rho]$ be the segment of length h. Denote by $\delta(\Delta)$ the unique irreducible subrepresentation of the normalised induction*

$$\nu^{h-1}\rho \times \nu^{h-2}\rho \times \cdots \times \rho;$$

equivalently it is the unique irreducible quotient of the induction in the reversed order.

For the upper block-triangular parabolic subgroup $P_{n-i,i}$, with diagonal Levi $\mathrm{GL}_{n-i}(K) \times \mathrm{GL}_i(K)$ and unipotent radical $U_{n-i,i}$, write

$$r_{n-i,i}(\pi) = \delta_{P_{n-i,i}}^{-1/2} \otimes \pi_{U_{n-i,i}}.$$

For $0 < i < n$, this Jacquet module of $\delta(\Delta)$ is zero unless m divides i. If $i = mp$, where $0 < p < h$, then

$$r_{n-i,i}\big(\delta([\rho, \nu^{h-1}\rho])\big) \cong \delta([\nu^p\rho, \nu^{h-1}\rho]) \boxtimes \delta([\rho, \nu^{p-1}\rho]).$$

These are isomorphisms of the actual Jacquet modules, not just identities of semisimplifications.

Proof. [332, Proposition 9.5], with the segment representation $\langle\Delta\rangle^t$ of Section 9.1 denoted here by $\delta(\Delta)$. The orientation is fixed by Section 9.1: it is a subrepresentation of the descending induction and a quotient of the ascending induction. The ordered Levi has the larger, upper block first; the right block of rank mp receives the lower segment $[\rho, \nu^{p-1}\rho]$. The induction and Jacquet functors are those of [24, Sections 1.8 and 2.2–2.3]: upper block-triangular parabolics, right translation in induction, the factor $\delta_P^{1/2}$ in induction and the factor $\delta_P^{-1/2}$ on unipotent coinvariants. In Example 9.2 of [332], the centered character segment $[|\cdot|_K^{(1-h)/2}, |\cdot|_K^{(h-1)/2}]$ gives the ordinary Steinberg representation of $\mathrm{GL}_h(K)$. □

A.6 Lattices, geometry of numbers and linear algebra

These are results on lattices and determinants: Banaszczyk's transference theorems [12] and Minkowski's second theorem [52]; the mass of a maximal hermitian lattice, by Shimura's exact formula [270, 97, 99, 303], which goes back to Siegel's mass formula [280]; the inequalities of Fischer, Hadamard and Cauchy–Schwarz [113, 133]; Amitsur's formula for the determinant of a sum [6, 240, 82]; and Witt's extension theorem for hermitian forms [249].

Theorem A.6.1. *Let $d \geq 1$, let $\mathcal{E}$ be a real inner product space of dimension d with norm $\|\cdot\|$, let $\Gamma \subset \mathcal{E}$ be a lattice (the subgroup generated by a basis of $\mathcal{E}$) and $\Gamma^* = \{u \in \mathcal{E} : \langle u, y\rangle \in \mathbb{Z} \text{ for all } y \in \Gamma\}$ its dual lattice. Let $\lambda > 0$ be such that $\|u\| \geq \lambda$ for every nonzero $u \in \Gamma^*$. Then for every $t \in \mathcal{E}$ there is $y \in \Gamma$ with*

$$\|t - y\| \leq \frac{d}{2\lambda}.$$

Equivalently, the covering radius $\mu(\Gamma) = \min\{r > 0 : \Gamma + rB = \mathcal{E}\}$ (B the closed unit ball) satisfies $\mu(\Gamma)\,\lambda_1(\Gamma^*) \leq d/2$, where $\lambda_1(\Gamma^*)$ is the least norm of a nonzero vector of Γ^*.

Proof. Banaszczyk [12]: the quantity $\eta_d = \sup_\Gamma \mu(\Gamma)\lambda_1(\Gamma^*)$, the supremum over all lattices in $\mathbb{R}^d$, satisfies $\eta_d \leq d/2$ for every d. Notation: the paper writes n, L, L^* for d, Γ, Γ^* and works in $\mathbb{R}^n$ with its Euclidean inner product; an isometry $\mathcal{E} \cong \mathbb{R}^d$ carries Γ, Γ^*, μ and λ_1 to the objects of the paper. Since $\lambda \leq \lambda_1(\Gamma^*)$, $\mu(\Gamma) \leq d/(2\lambda_1(\Gamma^*)) \leq d/(2\lambda)$, and the minimum defining $\mu(\Gamma)$ is attained, which gives the first form. □

Theorem A.6.2. *Let $n \geq 1$, let $\mathbb{R}^n$ carry its Euclidean inner product $u \cdot v$ and norm $\|\cdot\|$, let $\Gamma \subset \mathbb{R}^n$ be a lattice (the subgroup generated by n linearly independent vectors) and $\Gamma^* = \{u \in \mathbb{R}^n : u \cdot v \in \mathbb{Z} \text{ for all } v \in \Gamma\}$ its dual lattice. For $1 \leq i \leq n$ let $\lambda_i(\Gamma) = \min\{s > 0 : \dim \operatorname{span}_{\mathbb{R}}(\Gamma \cap \{\|v\| \leq s\}) \geq i\}$ be the i-th successive minimum. Then*

$$\lambda_i(\Gamma)\,\lambda_{n-i+1}(\Gamma^*) \leq n \qquad (1 \leq i \leq n).$$

Proof. [12]: the quantity $\xi_n = \sup_\Gamma \max_{1\leq i\leq n} \lambda_i(\Gamma)\lambda_{n-i+1}(\Gamma^*)$ satisfies $\xi_n \leq n$ for every n. Notation: the paper writes L, L^* for Γ, Γ^*; the definitions of the dual lattice and of the successive minima are those of the paper. □

Theorem A.6.3. *Let $n \geq 1$, let $\mathbb{R}^n$ carry a Euclidean inner product with norm $\|\cdot\|$ and the Lebesgue measure of that inner product, let $B_n = \{y : \|y\| \leq 1\}$, of volume $\operatorname{vol}(B_n) = \pi^{n/2}/\Gamma(\frac{n}{2}+1)$, and let $\Gamma \subset \mathbb{R}^n$ be a lattice of covolume $\operatorname{cv}(\Gamma)$. For $1 \leq i \leq n$ let $\lambda_i(\Gamma) = \min\{r > 0 : \dim \operatorname{span}_{\mathbb{R}}(\Gamma \cap rB_n) \geq i\}$. Then there are linearly independent $y_1, \ldots, y_n \in \Gamma$ with $\|y_i\| = \lambda_i(\Gamma)$, and*

$$\lambda_1(\Gamma)\cdots\lambda_n(\Gamma)\operatorname{vol}(B_n) \leq 2^n \operatorname{cv}(\Gamma).$$

Proof. [52], Chapter VIII (Minkowski's second theorem, stated there for every convex symmetric body; here the body is the unit ball B_n), and the remarks there that the successive minima are attained by linearly independent lattice points. Notation: Cassels writes Λ and $d(\Lambda)$ for the lattice and its determinant ($= \operatorname{cv}(\Gamma)$), F for the distance function ($F(y) = \|y\|$), and V for the volume of $F(y) \leq 1$. □

Theorem A.6.4. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field with maximal totally real subfield F such that E/F is unramified at every finite place, and let $j \geq 1$. An integral lattice M in a definite hermitian space W over E is* maximal *if no integral lattice of W contains it properly. For a maximal ideal $\mathfrak{p}$ of $\mathcal{O}_F$ let $M_\mathfrak{p} = (\mathcal{O}_F \setminus \mathfrak{p})^{-1}M$, $q_\mathfrak{p} = |\mathcal{O}_F/\mathfrak{p}|$, and*

$$\lambda_\mathfrak{p}(M) = \begin{cases} \dfrac{q_\mathfrak{p}^{\,j} - (-1)^j}{q_\mathfrak{p}+1} & \text{if } \mathfrak{p} \text{ is inert in } E \text{ and } (M_\mathfrak{p})^\vee \neq M_\mathfrak{p}, \\ 1 & \text{otherwise,} \end{cases}$$

the dual taken in W with respect to $(\mathcal{O}_F \setminus \mathfrak{p})^{-1}\mathcal{O}_E$. Then there is a real number $c(E,j) > 0$ such that for every definite hermitian space W over E of dimension j and every maximal integral lattice M in W:

1. *the genus of M is the set of all maximal integral lattices of W;*
2. *$\lambda_\mathfrak{p}(M) = 1$ for almost all $\mathfrak{p}$, and $\operatorname{mass}(M) = c(E,j)\prod_\mathfrak{p} \lambda_\mathfrak{p}(M)$;*
3. *(explicit value) let $d = [F:\mathbb{Q}]$, d_F the discriminant of F, χ the quadratic Hecke character of F attached to E/F, $\chi^i = 1$ for even i and $\chi^i = \chi$ for odd i, $L_F(s,\chi^i)$ the Hecke L-function of χ^i over F (for $i \geq 2$ the absolutely convergent Euler product $\prod_\mathfrak{p}(1-\chi^i(\mathfrak{p})q_\mathfrak{p}^{-i})^{-1}$; for $i = 1$ its value at $s = 1$). If M is unimodular ($M^\vee = M$), then*

$$\operatorname{mass}(M) = 2\,d_F^{\,j^2/2} \prod_{i=1}^{j} \Big(\frac{(i-1)!}{(2\pi)^i}\Big)^d L_F(i,\chi^i),$$

and this number is $c(E,j)$.

Proof. [97], Propositions 4.4 and 4.5 (Shimura's exact mass formula for maximal lattices in totally definite hermitian spaces, derived from Proposition 2.13 there through Bruhat–Tits theory; the stabiliser of a maximal lattice at a finite place is a maximal parahoric subgroup). As printed: for a maximal lattice Λ in a totally definite hermitian space V of dimension m over a totally imaginary quadratic extension L of a totally real field k of degree d, with G the unitary group of V, $\mathrm{Mass}(\Lambda) = \big(2^{-md} L(M)\tau(G)\big) \prod_{v\in S} \lambda_v$, where $L(M) = \prod_{r=1}^{m} L(1-r, \chi^r)$ (loc. cit., (4.2); the mass is positive and equals this expression with $L(M)$ replaced by $|L(M)|$, whose sign is that of item (iii)), $\tau(G) = 2$ ((4.3), Kottwitz, Ono), and, at a finite place v unramified in L, $v \in S$ if and only if v does not split and $V \otimes_k k_v \cong V_\alpha$ with $\alpha \notin \mathbb{N}L_v^\times$, in which case $\lambda_v = (q_v^m + 1)/(q_v + 1)$ for odd m and $\lambda_v = (q_v^m - 1)/(q_v + 1)$ for even m, that is $\lambda_v = (q_v^m - (-1)^m)/(q_v + 1)$ (loc. cit., Tables 1 and 2 and Propositions 4.4, 4.5; ramified places, which also lie in S, do not occur here). The condition $v \in S$ is the condition $(M_\mathfrak{p})^\vee \neq M_\mathfrak{p}$ of the statement: by loc. cit., (3.2), (3.3), (3.8), the maximal lattices of V_α are the conjugates of $\Delta^n + \mathcal{O}_{L_v}$ with the form $\alpha x\bar{y}$ on the last summand ($m = 2n+1$), respectively of Δ^n or $\Delta^{n-1} \oplus \mathcal{O}_D$ ($m = 2n$), with Δ the self-dual hyperbolic plane (the different is trivial), and the representative α is a unit exactly when it is a norm; so a maximal lattice is self-dual if and only if $v \notin S$. The mass of Λ is defined there as $\mathrm{Mass}(K)$, the volume of $G(k)\backslash G(\mathbb{A})$ for the Haar measure giving $K = G(k\otimes\mathbb{R}) \times \prod_v K_v$ volume 1, K_v the stabiliser of Λ_v ((2.2), (4.1)). Item (i) is the classical fact that the maximal lattices of a hermitian space form one genus ([270]; locally, the maximal lattices of a hermitian space over a $\mathfrak{p}$-adic field are conjugate under its unitary group, Jacobowitz 1962, [147]), on which the mass formula rests. Map of notation: the group is $\mathrm{U}(W)$, the mass of $K = \prod_v \mathrm{Stab}(M_v)$ is $\sum 1/|\Gamma_i|$ over the classes of the genus of M, which is $\mathrm{mass}(M)$; the constant $c(E, j)$ is the global factor of the formula, which depends only on the quasi-split inner form, i.e. on E/F and j. Since E/F is unramified at the finite places, the inverse different of E/F is $\mathcal{O}_E$, so $h(M, M) \subseteq \mathcal{O}_E$ if and only if $h(x, x) \in \mathcal{O}_F$ for all $x \in M$: maximality with respect to the scale (as here) and with respect to the norm ideal (as in the paper) agree. At split places every maximal lattice is self-dual and its factor is 1; no ramified place occurs.

Item (iii) is the exact mass formula for unimodular hermitian lattices, [303], Theorem 2.3.4 and its formula (2.6), which is the mass formula of Gan–Yu [99], Theorem 10.20, with the Tamagawa number $\tau(\mathrm{U}(W)) = 2$ (Kottwitz, Ono):

$$\mathrm{Mass}(\Lambda) = \Big(\prod_{i=1}^{j} (2\pi)^i/(i-1)!\Big)^{-d} |\mathcal{O}_F/\mathfrak{d}_{E/F}|^{j(j+1)/4}\, \tau\, d_F^{j^2/2} / \prod_v \beta_v,$$

where at a finite place v unramified in E the local density of a unimodular lattice is $\beta_v^{-1} = \prod_{i=1}^{j} L_{F_v}(i, \chi^i)$ (loc. cit., Lemma 2.2.4, from $|\mathrm{U}_j(\mathbb{F}_q)| = q^{j(j-1)/2} \prod_i (q^i - (-1)^i)$ and $|\mathrm{GL}_j(\mathbb{F}_q)|$ for the smooth reductive model of the stabiliser, [99], Theorem 7.3), and the product over v is written as $\prod_i L_F(i, \chi^i)$ (loc. cit., before (2.9)). Map of notation: their n, d, Λ, χ_E are our j, d, M, χ; since E/F is unramified at every finite place, $\mathfrak{d}_{E/F} = \mathcal{O}_F$, there is no place $v \mid d_{E/F}$, and $|\mathcal{O}_F/\mathfrak{d}_{E/F}| = 1$; for definite W the group $\mathrm{U}(W)(F\otimes\mathbb{R})$ is compact, their Haar measure $\mu_{c,\mathbb{R}}$ has total volume 1, and $\mathrm{Mass}(\Lambda) = \sum_{[g]} 1/|\Gamma_g|$ over the classes of the genus with $\Gamma_g = \mathrm{Aut}$ of the class representative, which is $\mathrm{mass}(M)$. A unimodular M is maximal (an integral $M' \supseteq M$ satisfies $M' \subseteq M'^\vee \subseteq M^\vee = M$) and has $\lambda_\mathfrak{p}(M) = 1$ for every $\mathfrak{p}$, so by item (ii) $\mathrm{mass}(M) = c(E, j)$; as $\mathcal{O}_E^j$ with $h(x, y) = \sum \bar{x}_i y_i$ is unimodular, $c(E, j)$ is the displayed number. Equivalently (loc. cit., (2.9), by the functional equations), $c(E, j) = (-1)^s\, 2^{1-jd} \prod_{i=1}^{j} L_F(1-i, \chi^i)$ with $s = 0$ for odd j and $s = jd/2$ for even j. □

Lemma A.6.5. *Let C be a finite set and $G \in M_C(\mathbb{C})$ a positive definite hermitian matrix. For a finite family $v = (v_i)_{i\in I}$ in $\mathbb{C}^C$ put $\mathrm{Gr}_G(v) = (v_i^* G v_k)_{i,k\in I}$, and for two families $v = (v_i)_{i\in I}$, $w = (w_k)_{k\in I}$ indexed by the same finite set put $\mathrm{Gr}_G(v, w) = (v_i^* G w_k)_{i,k\in I}$.*

(a) ***Fischer.*** *If* $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ *is a positive semidefinite hermitian matrix with square diagonal blocks* M_{11}, M_{22}, *then* $0 \le \det M \le \det M_{11} \cdot \det M_{22}$.

(b) ***Hadamard.*** *If* $h \in \mathbb{R}$ *and* $v_i^* G v_i \le e^h$ *for all* $i \in I$, *then* $0 \le \det \mathrm{Gr}_G(v) \le e^{\#I\, h}$.

(c) ***Cauchy–Schwarz.*** $|\det \mathrm{Gr}_G(v,w)|^2 \le \det \mathrm{Gr}_G(v) \cdot \det \mathrm{Gr}_G(w)$.

(d) ***Change of basis.*** *Let* $\Theta\colon V \to S$ *be a linear map of finite-dimensional complex inner product spaces and* $w = (w_p)$ *a basis of* V. *Then* $\det(\langle \Theta w_p, \Theta w_q\rangle)_{p,q} = \det(\Theta^*\Theta) \cdot \det(\langle w_p, w_q\rangle)_{p,q}$.

Proof. Quoted. (a) is Fischer's inequality and (b) Hadamard's inequality for positive semidefinite hermitian matrices ([133], Theorem 7.8.5 and Theorem 7.8.1); for (b), $\mathrm{Gr}_G(v)$ is the Gram matrix of v for the positive definite form $(a,b) \mapsto a^*Gb$, hence positive semidefinite with diagonal entries $v_i^*Gv_i \in [0, e^h]$. (c) The form $(a,b) \mapsto a^*Gb$ induces on $\bigwedge^{\#I} \mathbb{C}^C$ a positive semidefinite hermitian form with $\langle v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n\rangle = \det \mathrm{Gr}_G(v,w)$ (Cauchy–Binet), and (c) is the Cauchy–Schwarz inequality for it. (d) Let e be an orthonormal basis of V and W the matrix with $w_q = \sum_p W_{pq} e_p$. Then $(\langle \Theta w_p, \Theta w_q\rangle) = W^*AW$ and $(\langle w_p, w_q\rangle) = W^*W$ with $A = (\langle \Theta e_p, \Theta e_q\rangle)$ the matrix of $\Theta^*\Theta$ in e; take determinants. □

Theorem A.6.6. *Let* C *be a commutative ring,* $N, k \ge 1$ *integers,* $A_1, \dots, A_k \in \mathrm{M}_N(C)$, *and* $t_1, \dots, t_k$ *commuting indeterminates. For a word* $w = i_1 i_2 \cdots i_p$ *in the alphabet* $\{1, \dots, k\}$ *put* $A_w = A_{i_1} A_{i_2} \cdots A_{i_p}$ *and* $t_w = t_{i_1} t_{i_2} \cdots t_{i_p}$. *Then, in the ring of formal power series* $C[[t_1, \dots, t_k]]$,

$$\det\Bigl(1 - \sum_{i=1}^{k} t_i A_i\Bigr) = \prod_{w} \det(1 - t_w A_w),$$

the product running over all Lyndon words w *in the alphabet* $\{1, \dots, k\}$ *(the non-empty words that are strictly smaller, for the lexicographic order, than each of their proper suffixes); it converges in the* $(t_1, \dots, t_k)$*-adic topology, since* t_w *has degree equal to the length of* w *and there are finitely many words of each length.*

Proof. [6]; short proof from Lyndon's factorisation of the free monoid by C. Reutenauer and M.-P. Schützenberger [240]; the form used here is [82, (1.12)], applied to the matrices $t_i A_i$ with entries in $C[[t_1, \dots, t_k]]$. The identity is one of formal power series with coefficients that are integral polynomials in the entries of the A_i, so it holds over every commutative ring. □

Theorem A.6.7. *Let* K *be a field of characteristic different from* 2 *with an automorphism* $x \mapsto \bar{x}$ *of order* 2, *let* V *be a finite-dimensional* K*-vector space and* $h\colon V \times V \to K$ *a nondegenerate hermitian form (*h *is biadditive,* $h(ax, by) = \bar{a}\, b\, h(x,y)$ *and* $h(y,x) = \overline{h(x,y)}$ *for* $a, b \in K$, $x, y \in V$*). Let* $U \subseteq V$ *be a subspace and* $\sigma\colon U \to V$ *an injective* K*-linear map with* $h(\sigma u, \sigma u') = h(u, u')$ *for all* $u, u' \in U$. *Then there is a* K*-linear isometry* s *of* (V, h) *with* $s|_U = \sigma$.

Proof. This is Witt's theorem for hermitian forms; see [249], Chapter 7 (Witt's theorem for regular ε-hermitian spaces over a division ring with involution; here the division ring is the field K, $\varepsilon = 1$ and the involution is $x \mapsto \bar{x}$). Notation map: a regular hermitian space is a pair (V, h) with h nondegenerate; an isometry $U \to \sigma U$ of subspaces is an injective linear map preserving h. The statement is the special case of a commutative division ring; it is stated for the linear-second convention of the project, which differs from the linear-first convention by exchanging the arguments of h and does not change the isometries. □

Theorem A.6.8. *Let F_v be a finite extension of $\mathbb{Q}_p$, E_v/F_v a quadratic field extension with nontrivial automorphism $x \mapsto \bar{x}$, and $\mathrm{N} = \mathrm{N}_{E_v/F_v}$. For a nondegenerate hermitian space (W, h) over E_v of dimension $r \geq 1$, the determinant $\det\big(h(w_i, w_j)\big)_{i,j}$ of the Gram matrix of a basis (w_i) lies in $F_v^\times$, and its class $d(W) \in F_v^\times/\mathrm{N}(E_v^\times)$ does not depend on the basis. Two nondegenerate hermitian spaces over E_v of the same dimension are isometric if and only if they have the same class d.*

Proof. [147] (the classification of hermitian spaces over local fields); see also [249], Chapter 10. Notation: Jacobowitz's discriminant of a space is the class $d(W)$ here. The determinant lies in F_v because the Gram matrix H satisfies $\bar{H}^\top = H$, so $\overline{\det H} = \det H$; a change of basis by $g \in \mathrm{GL}_r(E_v)$ replaces H by $\bar{g}^\top H g$ and $\det H$ by $\mathrm{N}(\det g)\det H$. □

Theorem A.6.9. *Let E be a number field with ring of integers $\mathcal{O}_E$ and Dedekind zeta function ζ_E, and let $r \geq 1$ be an integer. For every complex number s with $\operatorname{Re} s > r$ the series*

$$\sum_{L} [\mathcal{O}_E^r : L]^{-s},$$

over all $\mathcal{O}_E$-submodules $L \subseteq \mathcal{O}_E^r$ of finite index, converges absolutely, and

$$\sum_{L} [\mathcal{O}_E^r : L]^{-s} = \prod_{j=0}^{r-1} \zeta_E(s-j).$$

Proof. This is Solomon's formula for the zeta function of the order $\mathcal{O}_E$ acting on the free module $\mathcal{O}_E^r$ [286]. Notation: the sum over submodules of finite index of $\mathcal{O}_E^r$ is the zeta function $\zeta_{\mathcal{O}_E^r}(s)$ of the module there, the index $[\mathcal{O}_E^r : L]$ is the cardinality of $\mathcal{O}_E^r/L$, and ζ_E is the Dedekind zeta function of E; in the case $E = \mathbb{Q}$ the formula reads $\sum_L [\mathbb{Z}^r : L]^{-s} = \zeta(s)\zeta(s-1)\cdots\zeta(s-r+1)$. □

Theorem A.6.10. *Let k be an algebraically closed field of characteristic zero and A a finite-dimensional associative unital k-algebra. Its Jacobson radical J is a nilpotent two-sided ideal, and A/J is isomorphic to a finite product of full matrix algebras $\mathrm{M}_{d_i}(k)$. In particular, if $J = 0$, then A itself is such a product.*

Proof. The radical theorem for finite-dimensional algebras and the Artin–Wedderburn theorem; see N. Jacobson, Basic Algebra II, second edition, 1989, the structure theory of Artinian rings. Over an algebraically closed field every finite-dimensional division algebra is the field itself: each element has an eigenvalue for its left multiplication and therefore differs from that scalar by a noninvertible element. □

Acknowledgements

This paper was prepared with the assistance of Aleph, an AI research assistant for mathematics, using AI models from Anthropic (Claude), OpenAI (ChatGPT) and Google (Gemini) together with bespoke models, as well as Claude Code and Codex for converting the Lean source into structured definitions and results in conventional language. The author thanks J. Adriazola for sharing his parallel work on GitHub and A. Dujella and M. Kazalicki for helpful comments. All credit for the proof belongs to Alpöge; all errors of exposition are ours.

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