4.6 Uvjetne razdiobe

(X,Y) sluč. vektor na $(\Omega,\mathcal{F})$

- $\forall y \in \mathbb{R}$ t.d. $\boxed{\mathbb{P}(Y=y)>0}$ (BSOMP $y \in D_Y$),

uvjetna distr. sl. var. X uz dano $Y=y$

je uvjetna distr. od X uz dano $\boxed{A:=\{Y=y\}}$,

te $\mathbb{E}[X\,|\,Y=y] := \mathbb{E}[X\,|\,A]$.

(vidi poglavlje 3.5)

- Definiramo $g: D_Y \to \mathbb{R}$ s

$$\boxed{g(y) := \mathbb{E}[X\,|\,Y=y]}, \quad y \in D_Y,$$

a $\boxed{\text{slučajnu varijablu}}$ $\mathbb{E}[X\,|\,Y]$ defin. sa

$$\mathbb{E}[X\,|\,Y] := g(Y), \tag{4.23}$$

zovemo uvjetno očekivanje od X uz dano Y.

Pr. 4.35 Neka su $X,Y \sim G_0(p)$ nezavisne, te $Z := X+Y$.

$$\Big[\underbrace{N\ N\ N\ N\ \boxed{U}}_{X}\ \underbrace{N\ N\ \boxed{U}}_{Y}\Big] \qquad \begin{bmatrix} N = \text{"neuspjeh"} \\ U = \text{"uspjeh"} \end{bmatrix}$$

(a) Za $m \in D_Z$, odredite uvjetn. distr. od X uz dano $Z=m$, te $\mathbb{E}[X\,|\,Z=m]$.

(b) Odredite $\mathbb{E}[X\,|\,Z]$.

"negativna binomna" iz (3.12)

Rj. $X,Y \in \mathbb{N}_0 \Rightarrow Z \in \underset{\overset{\shortparallel}{D_Z}}{\mathbb{N}_0}$, te zapravo $Z \sim NB(2,p)$

(a) $\forall m \in \mathbb{N}_0 = R_Z$,

$$(3.12) \Rightarrow \mathbb{P}(Z=m) = \binom{\overset{m+2-1}{\overset{\|}{m+1}}}{\underset{2-1}{\underset{\|}{1}}} q^m \cdot p^2$$

- za $k \in \mathbb{N}_0$, $k > m$, $\mathbb{P}(X=k \mid Z=m) \overset{X \leq Z}{=} 0$,
- za $k \in \mathbb{N}_0$, $k \leq m$

$$\mathbb{P}(X=k \mid Z=m) = \frac{\mathbb{P}(X=k, \overset{X+Y}{\overset{\|}{Z}}=m)}{\mathbb{P}(Z=m)} = \frac{\mathbb{P}(X=k, Y=m-k)}{\mathbb{P}(Z=m)}$$

$$\underset{\text{nez. od } X \text{ i } Y}{=} \frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=\overset{\in\{0,\dots,m\}}{m-k})}{\mathbb{P}(Z=m)} \underset{X,Y \sim G_0(p)}{=} \frac{\cancel{q^k} \cancel{p} \cdot \cancel{q^{m-k}} \cdot \cancel{p}}{(m+1) \cdot \cancel{q^m} \cdot \cancel{p^2}} = \boxed{\frac{1}{m+1}}\,)$$

tj.

$$\text{"}X \mid Z=m\text{"} \sim \begin{pmatrix} 0 & 1 & \cdots & m \\ \frac{1}{m+1} & \frac{1}{m+1} & \cdots & \frac{1}{m+1} \end{pmatrix}$$ [Intuicija?]

distr. od X uz dano $Z=m$

uniformna razdioba na $\{0,\dots,m\}$

Specijalno,

$$g(m) := \mathbb{E}[X \mid Z=m] = \frac{m}{2}, \forall m \in \mathbb{N}_0$$

(b) $$\mathbb{E}[X \mid Z] = g(Z) = \frac{Z}{2}$$ ■

Prip. 4.36 Ako oba očekivanja postoje i vrijedi

$$\mathbb{E}[\,\mathbb{E}[X|Y]\,] = \mathbb{E}[X]. \qquad (4.24)$$

Dokaz Slijedi odmah iz Tm. 3.42 jer

$$\mathbb{E}[\,\underbrace{\mathbb{E}[X|Y]}_{=g(Y)}\,] = \sum_{y\in D_Y} g(y)\,\mathbb{P}(Y=y)$$

$$= \sum_{y\in D_Y} \mathbb{E}[X|Y=y]\,\mathbb{P}(Y=y),$$

iz (4.23)

a $\{\{Y=y\},\ y\in D_Y\}$ je PS-D. ∎

[dakle, (4.24) je zapravo formula ukupnog očekivanja iz Tm.-e 3.42!]